

Generalized Gibbs Ensembles for Quantum Field Theories

F.H.L. Essler,¹ G. Mussardo,^{2,3} and M. Panfil²

¹ *The Rudolf Peierls Centre for Theoretical Physics
University of Oxford, Oxford, OX1 3NP, United Kingdom*

² *SISSA and INFN, Sezione di Trieste, via Beirut 2/4, I-34151, Trieste, Italy*

³ *International Centre for Theoretical Physics (ICTP), I-34151, Trieste, Italy*

We consider the non-equilibrium dynamics in quantum field theories (QFTs). After being prepared in a density matrix that is not an eigenstate of the Hamiltonian, such systems are expected to relax *locally* to a stationary state. In presence of local conservation laws, these stationary states are believed to be described by appropriate generalized Gibbs ensembles. Here we demonstrate that in order to obtain a correct description of the stationary state, it is necessary to take into account conservation laws that are not (ultra-)local in the usual sense of QFT, but fulfil a significantly weaker form of locality. We discuss implications of our results for integrable QFTs in one spatial dimension.

PACS numbers:

Introduction. The last decade has witnessed dramatic progress in realizing and analyzing isolated many-particle quantum systems out of equilibrium [1–6]. Key questions that emerged from these experiments is why and how observables relax towards time independent values, and what principles underlie a possible statistical description of the latter [7–29]. It was demonstrated early on that non-equilibrium dynamics is strongly affected by dimensionality, and that conservation laws play an important role. In particular, the experiments of [2] on trapped ⁸⁷Rb atoms established that three-dimensional condensates rapidly relax to a stationary state characterized by an effective temperature, whereas constraining the motion of atoms to one dimension greatly reduces the relaxation rate and dramatically changes the nature of the stationary state. The suggestion that this unusual steady state is a consequence of (approximate) conservation laws motivated a host of theoretical studies investigating the role played by conservation laws. We may summarize the results of these works as follows: given an initial state $|\Psi\rangle$ and a translationally invariant system with Hamiltonian $H \equiv I_0$ and conservation laws I_n such that $[I_n, I_m] = 0$, the stationary behaviour of n -point functions of local operators $\mathcal{O}_a(x)$ in the thermodynamic limit is described by a *generalized Gibbs ensemble*, as proposed by Rigol et al in a seminal paper [9]

$$\lim_{t \rightarrow \infty} \langle \Psi(t) | \prod_{j=1}^n \mathcal{O}_j(x_j) | \Psi(t) \rangle = \text{Tr}[\rho_{\text{GGE}} \prod_{j=1}^n \mathcal{O}_j(x_j)]. \quad (1)$$

Here $|\Psi(t)\rangle = \exp(-iHt)|\Psi\rangle$ and

$$\rho_{\text{GGE}} = \frac{1}{Z} \exp\left(-\sum_n \lambda_n I_n\right), \quad (2)$$

where the values of the Lagrange multipliers λ_n are fixed by the requirement that the expectation values of the conserved charges must be the same at time zero and in the stationary state, i.e. $\lim_{V \rightarrow \infty} \langle \Psi | I_n | \Psi \rangle / V = \lim_{V \rightarrow \infty} \text{Tr}[\rho_{\text{GGE}} I_n] / V$. Very recently it has become

clear that the question which conservation laws I_n need to be included in the definition of (2) is quite subtle [30–34]. Here we address this issue for continuum Quantum Field Theories (QFTs), in both relativistic and non-relativistic cases. This is of fundamental importance as a problem in QFT per se. It is also a pressing concern due to the crucial role QFT has played in establishing the current theoretical understanding of the non-equilibrium dynamics of isolated quantum systems, providing key insights [35–37] of experimental relevance [6, 38]. We show that it is in general necessary to include “quasi-local” charges in the definition of the GGE. This can already be seen for the simplest possible example, namely non-interacting QFTs, to which we turn next.

Free Majorana fermion. Let us consider a general quantum quench in the free Majorana fermion theory with Hamiltonian density

$$\mathcal{H} = \frac{iv}{2} [R(x) \partial_x R(x) - L(x) \partial_x L(x)] + imR(x)L(x), \quad (3)$$

where R and L are real chiral fermions, and v is the velocity. This theory describes the scaling limit of the transverse field Ising chain, where the mass term is a measure of the distance to the quantum critical point. The initial state $|\Psi(0)\rangle$ of the quench process could be, for example, the ground state at a particular, but different, value m_0 of the mass [39, 40]. The Hamiltonian is diagonalized through a mode expansion and takes the form

$$H = \int \frac{dk}{2\pi} \sqrt{m^2 + v^2 k^2} Z^\dagger(k) Z(k), \quad (4)$$

where $\{Z^\dagger(k), Z(q)\} = 2\pi\delta(k - q)$. Clearly the mode occupation operators $N(k) = Z^\dagger(k)Z(k)$ commute with H and are therefore conserved. In cases like this, the GGE density matrix in a large, finite volume L is most conveniently constructed in terms of the charges $N(k)$ [9]

$$\rho_{\text{GGE}} = \frac{1}{Z} \exp\left(-\sum_{n \in \mathbb{Z}} \lambda(k_n) N(k_n)\right), \quad k_n = \frac{2\pi n}{L}. \quad (5)$$

The Lagrange multipliers $\lambda(k)$ are related to the mode occupation numbers $n_\Psi(k) = \langle \Psi(0) | N(k) | \Psi(0) \rangle$ by $\lambda(k) = \ln(n(k)) - \ln(1 - n(k))$. In practice it is more convenient to work with the “microcanonical” version of the GGE [41, 42]. This is defined by the density matrix $\rho_{\text{GMC}} = |\Phi\rangle\langle\Phi|$, where the state $|\Phi\rangle$ is an eigenstate of all $N(k_n)$ with eigenvalues equal to $n_\Psi(k_n)$. By construction, the knowledge of the eigenvalues $n_\Psi(k_n)$ of the conserved charges $N(k_n)$ is sufficient to construct ρ_{GMC} .

The existence of conserved mode occupation operators in a large, finite volume is a particular property of free theories and does not generalize to the interacting case (see below). In contrast, no such problem arises for local conservation laws, which are therefore the appropriate charges to consider in the general case. Following the standard approach in a relativistic QFT (which we recall in the Supplementary Material) one can construct the following set of ultra-local conserved charges for the free Majorana theory

$$\begin{aligned} I_n^- &= \frac{iv}{2} \int dx [R(x) \partial_x^{2n+1} R(x) + L(x) \partial_x^{2n+1} L(x)] , \\ I_n^+ &= \frac{i}{2} \int dx [R(x) v \partial_x^{2n+1} R(x) - L(x) v \partial_x^{2n+1} L(x) \\ &\quad + 2m R(x) \partial_x^{2n} L(x)] . \end{aligned} \quad (6)$$

A widely held belief is that the GGE (2) constructed from these charges is the same as the one built from the mode occupation operators (5). However, in the infinite volume, this can not be generally the case, simply because there is a mismatch between the *countable* number of conserved charges and the *continuum* number of degrees of freedom in the field theory. In order to see this, we express the charges (6) in momentum space. This gives

$$I_n^\pm = (-1)^n \int_{-\infty}^{\infty} \frac{dk}{2\pi} \epsilon_n^\pm(k) N(k), \quad (7)$$

where $\epsilon_n^+(k) = \sqrt{m^2 + v^2 k^2} k^{2n}$ and $\epsilon_n^-(k) = v k^{2n+1}$. The question is then whether the knowledge of $i_n^\pm = \lim_{L \rightarrow \infty} \langle \Psi | I_n^\pm | \Psi \rangle / L$ is sufficient to reconstruct the function $n_\Psi(k)$ and hence the density matrix ρ_{GMC} . The answer is negative: as is shown in the Supplementary Material, one can explicitly construct functions $f(k)$ such that $i_n^\pm = (-1)^n \int \frac{dk}{2\pi} \epsilon_n^\pm(k) [n_\Psi(k) + f(k)]$ are independent of f . This suggests that there are additional local conservation laws that need to be taken into account in the construction of the GGE. How to find such charges? We recall that (3) is obtained as the scaling limit of a model of lattice Majorana fermions a_n , for which a complete set of local conservation laws is [43]

$$\begin{aligned} I_n^+ &= \frac{iJ}{2} \sum_{j, \sigma=\pm 1} a_{2j} [a_{2j+2n\sigma+1} - h a_{2j+2n\sigma-1}] \\ I_{n-1}^- &= -\frac{iJ}{2} \sum_j [a_{2j} a_{2j+2n} + a_{2j-1} a_{2j+2n-1}] . \end{aligned}$$

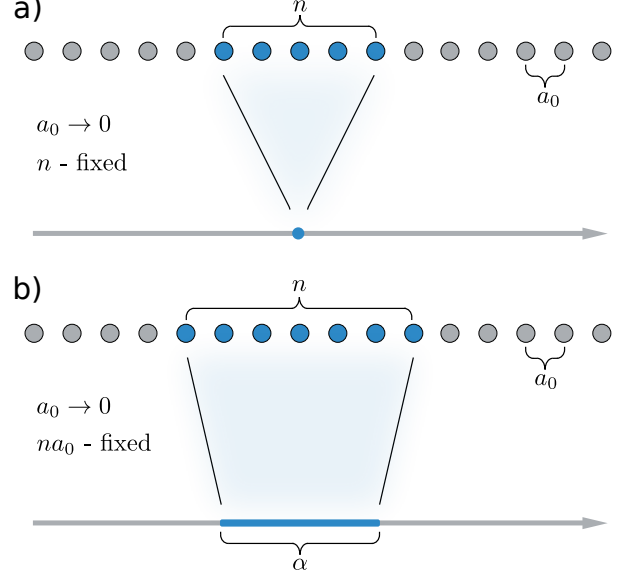


FIG. 1: Construction of ultra-local and quasi-local charges by taking the continuum limit of an integrable lattice model with conservation laws $\mathcal{I}_n$, whose densities act on n consecutive lattice sites. a) Ultra-local charges are obtained by taking the lattice spacing a_0 to zero, while keeping the index n fixed. b) Quasi-local charges are obtained by taking the double scaling limit $a_0 \rightarrow 0$, $n \rightarrow \infty$, while keeping $na_0 = \alpha$ fixed.

The lattice Hamiltonian itself is $\mathcal{I}_0^+$. The $\mathcal{I}_n^\pm$ have the important property that their densities have strictly finite ranges: the density of $\mathcal{I}_n^\pm$ involves only $n+2$ neighbouring sites. The scaling limit is defined as $J \rightarrow \infty$, $h \rightarrow 1$, $a_0 \rightarrow 0$ while keeping $J|h-1| = m$ and $Ja_0 = v$ fixed. In this limit, upon taking appropriate linear combinations of the lattice charges $\mathcal{I}_n^\pm$, one recovers the QFT charges (6). However, in the process of taking the scaling limit, we can also scale the index n in such a way that the combination $na_0 = \alpha$ is kept fixed, obtaining in this way conserved charges of the form (see Fig. 1)

$$\begin{aligned} I^+(\alpha) &= \frac{i}{4} \int_0^L dx [R(x) + L(x)] (v \partial_x - m) \\ &\quad \times [R(x + \alpha) - L(x + \alpha) + (\alpha \rightarrow -\alpha)] , \\ I^-(\alpha) &= \frac{iv}{2} \int_0^L dx [R(x) R(x + \alpha) + L(x) L(x + \alpha)] . \end{aligned} \quad (8)$$

Here the index α is by construction a real positive number such that $0 < \alpha < L$, where L is the system size and we have imposed periodic boundary conditions on the fields. The charges $I^\pm(\alpha)$ are no longer local quantities in the usual QFT sense, but have densities with support on a finite interval of size α . We will call such operators *quasi-local*. In momentum space we have $I^\pm(\alpha) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \epsilon^\pm(k, \alpha) N(k)$, where $\epsilon^+(k, \alpha) =$

$\sqrt{m^2 + v^2 k^2} \cos(\alpha k)$ and $\epsilon^-(k, \alpha) = \sin(\alpha k)$. This establishes $\{I^\pm(\alpha)\}$ of conserved charges is complete in the sense that the initial data $\langle \Psi | I^\pm(\alpha) | \Psi \rangle$ suffices to fix any given occupation number distribution $n_\Psi(k)$. Hence the appropriate GGE for the free Majorana theory is

$$\rho_{\text{GGE}} = \frac{1}{Z} \exp \left(- \sum_{\sigma=\pm} \int_0^\infty d\alpha \lambda^\sigma(\alpha) I^\sigma(\alpha) \right). \quad (9)$$

We stress for the lattice model itself the conservation laws that give rise to the quasi-local charges in the scaling limit are both unnatural and unimportant: the goal is to describe finite subsystems of arbitrary size in the thermodynamic limit, and here truncated GGEs [43] involving only $\mathcal{I}_n^\pm$ with fixed n in the $L \rightarrow \infty$ limit are required.

Interacting integrable QFTs (IQFTs). We next turn to the case of integrable QFTs with non-trivial S-matrices. The scattering in IQFTs is purely elastic [45, 46, 49], and concomitantly a convenient way to describe their Hilbert spaces in the infinite volume is in terms of the Faddeev-Zamolodchikov algebra [46]. In the scalar case the latter reads

$$\begin{aligned} Z(\theta_1)Z(\theta_2) &= S(\theta_1 - \theta_2)Z(\theta_2)Z(\theta_1), \\ Z(\theta_1)Z^\dagger(\theta_2) &= 2\pi\delta(\theta_1 - \theta_2) \\ &\quad + S(\theta_2 - \theta_1)Z^\dagger(\theta_2)Z(\theta_1), \end{aligned} \quad (10)$$

where $Z^\dagger(\theta)$, $Z(\theta)$ are creation and annihilation operators of elementary excitations with rapidity θ (related to momentum by $vq = M \sinh \theta$), and $S(\theta)$ is the two-particle S-matrix. In the infinite volume the quantities $N(\theta) = Z^\dagger(\theta)Z(\theta)$ are integrals of motion and can be viewed as appropriate generalizations of the mode occupation numbers in free field theories. Unfortunately, in contrast to the special case of free fields, the occupation numbers $N(\theta)$ cannot be used for the construction of the GGE [52]. The reason is that while for free fields the possible values for rapidities are simply given by $m \sinh \theta_n = 2\pi n/L$ and can be independently occupied, in the IQFT case the quantization conditions are given by the Bethe Ansatz equations

$$e^{iLm \sinh \theta_n} = \prod_{m \neq n} S(\theta_n - \theta_m), \quad n = 1, \dots, N. \quad (11)$$

Hence the allowed values of θ_n depend on the entire set $\{\theta_m\}$ specifying the particular eigenstate under consideration. Due to this complication it is not clear how to define a finite volume version of $N(\theta)$ in an operator sense [52]. We therefore want to construct GGE using local conservation laws. The standard *ultra-local* conserved charges are related to conserved currents $\partial_\mu j_n^\mu(t, x)$ by $I_n = \int dx j_n^0(t, x)$. In relativistic IQFTs there is a standard method for constructing I_n [45, 47–49]. There, the index n is related to the Lorentz spin of the operator I_n . As n can take only discrete values, ultra-local conserved charges are insufficient for constructing

GGEs for general initial states. To see this we recall that their action on eigenstates can be represented in the form $I_n^\pm = \int d\theta \varepsilon_n^\pm(\theta) N(\theta)$ with $\varepsilon_n^+(\theta) = \cosh n\theta$ and $\varepsilon_n^-(\theta) = \sinh(n\theta)$ [45]. It is again convenient to consider the microcanonical version $\rho_{\text{GMC}} = |\Phi\rangle\langle\Phi|$, which describes the saddle-point of the GGE [42]. In the infinite volume limit we require the knowledge of the function $n(\theta) = \langle \Phi | N(\theta) | \Phi \rangle$ [51] in order to specify ρ_{GMC} . The knowledge of the countable set $\{\langle \Phi | I_m | \Phi \rangle\}$ does not suffice to uniquely determine $n(\theta)$. Indeed, let us consider the family of states $|\Phi_f\rangle$ characterized by the macroscopically distinct mode occupations $n(\theta) + f(\theta)$, where $f(\theta)$ is an analytic function, whose Fourier transform has an infinite number of zeroes at $z_m = im$. Using the explicit expression for the eigenvalues of I_m given above, one finds that $\langle \Phi_f | I_m | \Phi_f \rangle = \langle \Phi | I_m | \Phi \rangle$. This establishes that the countable set $\{I_m\}$ of charges is in general insufficient to fully characterize ρ_{GMC} .

In order to construct the GGE we therefore follow the procedure used for free fields: (i) find an integrable lattice discretization of the field theory (with lattice spacing a_0); (ii) follow the standard procedure [47] for constructing local integrals of motion $\mathcal{I}_n$ for integrable lattice models. Here the index n roughly speaking sets a number of lattice sites the density of $\mathcal{I}_n$ acts on; (iii) take a *double scaling limit* $a_0 \rightarrow 0$, $n \rightarrow \infty$, while keeping $\alpha = na_0$ fixed. This procedure generates a continuous family of conserved charges $I(\alpha)$ (labelled by a real positive number α), which are quasi-local. In cases like the one considered below, it is known that the $\mathcal{I}_n$ form a complete set of integrals of motion on the lattice. Concomitantly the set $\{I(\alpha)\}$ is sufficient to construct the GGE in a large finite volume, and hence in the thermodynamic limit.

We now illustrate this programme for the example of the nonlinear Schrödinger model, also known as the Lieb-Liniger delta-function Bose gas [53], which is a key theory for the description of ultra-cold quantum gases [54]. In particular, it underlies seminal experiments probing thermalization in such systems [2, 3].

Nonlinear Schrödinger model (NLS). The Hamiltonian density of the NLS is [55]

$$\mathcal{H} = \varphi^\dagger(x) \left[-\frac{\partial_x^2}{2m} - \mu \right] \varphi(x) + \lambda |\varphi^2(x)|^2, \quad (12)$$

where $\varphi(x, t)$ is a complex bosonic field and μ is a chemical potential. Quenches to the NLS have been previously considered by several groups [25, 26, 56–65]. A key issue in many of these works has been how to construct the appropriate GGE describing the stationary state at late times after the quench. Let us now address this question using the framework introduced above. The ultra-local integrals of motion for the NLS can be constructed by the Quantum Inverse Scattering Method [47, 66] through an appropriate expansion of the quantum transfer matrix. This provides a countable number of $\mathcal{I}_n$, which by the above argument are insufficient for constructing the

GGE describing the stationary behavior after a quench from a general initial state. Moreover, as was discussed in detail in Ref. [65], the expectation values i_n in fact do not exist for many initial states due to ultraviolet divergences. These problems can be overcome by using quasi-local charges. To construct them, we utilize an integrable lattice regularization [67–69] of the NLS in terms of so-called q -boson operators fulfilling commutation relations

$$B_j^\dagger B_k - q^2 B_k B_j^\dagger = \delta_{jk}. \quad (13)$$

The q -bosons are related to canonical lattice bosons b_j by the relation $B_j = \sqrt{\frac{[N_j+1]_q}{N_j+1}} b_j$, where $[x]_q = \frac{1-q^{-2x}}{1-q^{-2}}$. The Hamiltonian of the lattice model is

$$H_q = -\frac{1}{a_0^2} \sum_{j=1}^L \left(B_j^\dagger B_{j+1} + B_{j+1}^\dagger B_j - 2N_j \right), \quad (14)$$

where $N_j = b_j^\dagger b_j$. The lattice conserved charges $\mathcal{I}_n^\pm$ are known and their eigenvalues are [70]

$$i_n^\pm(p_1, \dots, p_N) = \frac{1 - q^{-2|n|}}{|n|a_0} \sum_{j=1}^N f^\pm(np_j), \quad (15)$$

where n is an integer, $f^+(x) = \cos(x)$, $f^-(x) = \sin(x)$, and $\{p_1, \dots, p_N\}$ are solutions to the Bethe Ansatz equations for the q -boson model. The NLS is recovered taking the scaling limit is $a_0 \rightarrow 0$ and $q \rightarrow 1$ with $c = 2 \ln(q)/a_0$ fixed. The continuum field $\varphi(x)$ is related to the canonical lattice bosons by $\varphi(ja_0) = a_0^{-1/2} b_j$. In this limit the appropriate rapidity variables are $\lambda_j = p_j/a_0$. The ultra-local conserved charges of the NLS are obtained by considering appropriate linear combinations of the $I_n^\pm$ and then taking the continuum limit, see e.g. [65]. In contrast, the quasi-local charges $I^\pm(\alpha)$ are constructed by keeping $na_0 = \alpha$ fixed in the scaling limit. Their eigenvalues on Bethe Ansatz states are then found to be

$$i^\pm(\alpha; \lambda_1, \dots, \lambda_N) = \frac{1 - e^{-c|\alpha|}}{|\alpha|} \sum_{j=1}^N f^\pm(\alpha \lambda_j). \quad (16)$$

Let us now show that the set $\{I^\pm(\alpha)\}$ is sufficient for constructing the microcanonical version of the GGE, i.e. the density matrix $\rho_{\text{GMC}} = |\Phi\rangle\langle\Phi|$. Here $|\Phi\rangle$ is a particular Bethe eigenstate [42]. In a large, finite volume L it is characterized by rapidities $\{\lambda_1, \dots, \lambda_N\}$, and we are interested in the thermodynamic limit $N, L \rightarrow \infty$ with N/L fixed. In this limit the state is described by a root density $\rho_\Phi(\lambda)$, which arises from the finite volume quantity $\rho_L(\lambda_j) = \frac{1}{L(\lambda_{j+1} - \lambda_j)}$. The expectation values of the quasi-local charges are then

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{\langle \Phi | I^\pm(\alpha) | \Phi \rangle}{\langle \Phi | \Phi \rangle} = \frac{1 - e^{-c|\alpha|}}{|\alpha|} \int_{-\infty}^{\infty} d\lambda f^\pm(\alpha \lambda) \rho_\Phi(\lambda). \quad (17)$$

This shows that $\rho_\Phi(\lambda)$ can be determined by Fourier transform from the expectation values of the $I^\sigma(\alpha)$. Inspection of (16) shows that in contrast to the ultra-local charges [65], there are no ultra-violet divergences in the expectation values (17) (the integral over $\rho_\Phi(\lambda)$ is equal to the density and must be finite).

Discussion. The main lesson to be drawn from our work is that understanding the non-equilibrium evolution in QFTs requires one to go beyond the usual concept of locality. More precisely, we have shown that the construction of generalized Gibbs ensembles in QFTs requires integrals of motion $I^\pm(\alpha)$ that are not strictly local. In the cases we have considered, the densities of the $I^\pm(\alpha)$ act non-trivially only on intervals of length α , and are different from known non-local conserved charges related to Yangian or quantum group symmetries [71, 72]. We stress that locality of the charges required to build a GEE is a different matter from the locality of the quantity $\lambda_n I_n$ entering the definition of the GGE density matrix [73]. We have presented a general argument showing that GGEs built from the usual local conservation laws I_m are generally insufficient for describing the stationary state at late times after quantum quenches (this does not preclude the possibility that they may do so in particular examples). In analogy to observations made for the transverse field Ising chain [43], we expect that in order to obtain an accurate description of the stationary values of local observables acting on a subsystem of size ℓ , only charges with $\alpha \lesssim \ell + \xi$ will be required. Here ξ is a constant related to the correlation length in the stationary state.

Our work raises a number of open problems. First, our construction should be employed to determine the expectation values of local observables for particular quenches to the NLS model directly from the GGE. This requires the generalization of the method developed in Refs [30] to the q -boson model. Second, it would be interesting to consider quantum quenches in other QFTs such as the sine-Gordon or $SU(2)$ Thirring models. Here an additional complication arises, because the conservation laws obtained by standard methods for the corresponding lattice regularizations are no longer complete [30–34], and charges such as those constructed in [74–76] should be taken into account. Third, we expect quasi-local charges to be of importance for certain non-integrable models in the context of prethermalization [77–85]. For a number of examples it has been found that quenching to lattice models with weak integrability breaking terms, which includes the case of weakly interacting systems, leads to relaxation of local observables to non-thermal values at intermediate time scales. It has been suggested and substantiated in particular cases that almost conserved charges are the underlying cause of these prethermalization plateaux. It would be interesting to investigate this issue for QFTs in light of our findings. Finally, quasi-local charges may also be of importance for understanding the

equilibration of QFTs in large-N limits [86].

This work was supported by the EPSRC under grants EP/I032487/1 and EP/J014885/1.

-
- [1] M. Greiner, O. Mandel, T. W. Hänsch, and I. Bloch, *Nature* **419** 51 (2002).
 - [2] T. Kinoshita, T. Wenger, D. S. Weiss, *Nature* **440**, 900 (2006).
 - [3] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, *Nature* **449**, 324 (2007).
 - [4] S. Trotzky Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, *Nature Physics* **8**, 325 (2012).
 - [5] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauss, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, *Nature* **481**, 484 (2012).
 - [6] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. Adu Smith, E. Demler, and J. Schmiedmayer, *Science* **337**, 1318 (2012).
 - [7] J. M. Deutsch, *Phys. Rev. A* **43**, 2046 (1991); M. Srednicki, *Phys. Rev. E* **50**, 888 (1994).
 - [8] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Rev. Mod. Phys.* **83**, 863 (2011).
 - [9] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, *Phys. Rev. Lett.* **98**, 050405 (2007).
 - [10] M. Rigol, V. Dunjko, and M. Olshanii, *Nature* **452**, 854 (2008).
 - [11] A. Iucci and M. A. Cazalilla, *Phys. Rev. A* **80**, 063619 (2009).
 - [12] T. Barthel and U. Schollwöck, *Phys. Rev. Lett.* **100**, 100601 (2008).
 - [13] S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, *Phys. Rev. B* **79**, 155104 (2009).
 - [14] D. Fioretto and G. Mussardo, *New J. Phys.* **12**, 055015 (2010).
 - [15] G. Mussardo, *Phys. Rev. Lett.* **111**, 100401 (2013).
 - [16] G. Biroli, C. Kollath, and A.M. Läuchli, *Phys. Rev. Lett.* **105**, 250401 (2010).
 - [17] M. C. Bañuls, J. I. Cirac, and M. B. Hastings, *Phys. Rev. Lett.* **106**, 050405 (2011).
 - [18] C. Gogolin, M. P. Müller, and J. Eisert, *Phys. Rev. Lett.* **106**, 040401 (2011).
 - [19] P. Calabrese, F.H.L. Essler, and M. Fagotti, *Phys. Rev. Lett.* **106**, 227203 (2011).
 - [20] P. Calabrese, F.H.L. Essler, and M. Fagotti, *J. Stat. Mech.* P07016 (2012).
 - [21] P. Calabrese, F.H.L. Essler, and M. Fagotti, *J. Stat. Mech.* P07022 (2012).
 - [22] F.H.L. Essler, S. Evangelisti and M. Fagotti, *Phys. Rev. Lett.* **109**, 247206 (2012).
 - [23] P. Barmettler, M. Punk, V. Gritsev, E. Demler, and E. Altman, *New J. Phys.* **12**, 055017 (2010).
 - [24] J. Mossel and J.-S. Caux, *New J. Phys.* **12**, 055028 (2010).
 - [25] J. Mossel and J.-S. Caux, *New J. Phys.* **14**, 075006 (2012).
 - [26] J.-S. Caux and R. M. Konik, *Phys. Rev. Lett.* **109**, 175301 (2012).
 - [27] B. Pozsgay, *J. Stat. Mech.: Th. Exp.* P01011 (2011).
 - [28] A. Mitra and T. Giamarchi, *Phys. Rev. Lett.* **107**, 150602 (2011).
 - [29] C. Gramsch and M. Rigol, *Phys. Rev. A* **86**, 053615 (2012).
 - [30] M. Fagotti and F. H. L. Essler, *J. Stat. Mech.* (2013) P07012.
 - [31] M. Fagotti, M. Collura, F.H.L. Essler and P. Calabrese, *Phys. Rev. B* **89**, 125101 (2014).
 - [32] B. Wouters, M. Brockmann, J. De Nardis, D. Fioretto, M. Rigol and J.-S. Caux, *Phys. Rev. Lett.* **113**, 117202 (2014).
 - [33] B. Pozsgay, M. Mestyán, M. A. Werner, M. Kormos, G. Zaránd and G. Takács, *Phys. Rev. Lett.* **113**, 117203 (2014).
 - [34] G. Goldstein and N. Andrei, arXiv:1405.4224.
 - [35] J. Berges, S. Borsanyi, C. Wetterich, *Phys. Rev. Lett.* **93**, 142002, (2004).
 - [36] P. Calabrese and J. Cardy, *Phys. Rev. Lett.* **96**, 136801 (2006).
 - [37] P. Calabrese and J. Cardy, *J. Stat. Mech.* P06008 (2007).
 - [38] T. Kitagawa, A. Imambekov, J. Schmiedmayer and E. Demler, *New J. Phys.* **13**, 073018 (2011).
 - [39] D. Rossini, S. Suzuki, G. Mussardo, G. E. Santoro and A. Silva, *Phys. Rev. B* **82**, 144302 (2010).
 - [40] D. Schuricht and F.H.L. Essler, *J. Stat. Mech.* P04017 (2012).
 - [41] A. C. Cassidy, C. W. Clark, and M. Rigol, *Phys. Rev. Lett.* **106**, 140405 (2011).
 - [42] J.-S. Caux and F. H. L. Essler, *Phys. Rev. Lett.* **110**, 257203 (2013).
 - [43] M. Fagotti and F.H.L. Essler, *Phys. Rev. B* **87**, 245107 (2013).
 - [44] For a two-dimensional relativistic field theory, this assumption is sufficiently general to essentially include all known integrable theories, simply because the underlying CFT describe their ultraviolet behavior.
 - [45] A.B. Zamolodchikov, *Adv. Study of Pure Math.* 19 (1989), 641.
 - [46] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys.* **120**, 253 (1979); L. D. Faddeev, *Sov. Sci. Rev. Math. Phys. C* **1**, 107 (1980).
 - [47] V.E. Korepin, A.G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*, Cambridge University Press, 1993.
 - [48] C. Gomez, M. Altaba, G. Sierra, *Quantum Groups in Two-dimensional Physics* (Cambridge Univ. Press, 1996).
 - [49] G. Mussardo, *Statistical Field Theory*. Oxford Univ. Press, Oxford (2010).
 - [50] A. Zamolodchikov, *Nucl. Phys.* **B342**, 695 (1990).
 - [51] As we are discussing the infinite volume limit, we are strictly speaking considering only zero density states.
 - [52] B. Bertini, D. Schuricht and F.H.L. Essler, *J. Stat. Mech.* (2014) P10035.
 - [53] E.H. Lieb, W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
 - [54] M.A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac and M. Rigol, *Rev. Mod. Phys.* **83** (2011).
 - [55] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons* (Springer, New York, 1987).
 - [56] V. Gritsev, T. Rostunov and E. Demler, *J. Stat. Mech.* (2010) P05012.
 - [57] M. Collura, S. Sotiriadis and P. Calabrese, *Phys. Rev. Lett.* **110**, 245301 (2013).
 - [58] M. Collura, S. Sotiriadis and P. Calabrese, *J. Stat. Mech.* (2013) P09025.
 - [59] M. Kormos, M. Collura and P. Calabrese, *Phys. Rev.*

- A**89**, 013609 (2014).
- [60] M. Collura, M. Kormos and P. Calabrese, J. Stat. Mech. (2014) P01009.
 - [61] J. De Nardis, B. Wouters, M. Brockmann, J.-S. Caux, Phys. Rev. A **89**, 033601 (2014).
 - [62] P.P. Mazza, M. Collura, M. Kormos and P. Calabrese, arXiv:1407.1037.
 - [63] G. P. Brandino, J.-S. Caux, R. M. Konik, arXiv:1407.7167.
 - [64] J. De Nardis and J.-S. Caux, arXiv:1410.0620.
 - [65] M. Kormos, A. Shashi, Y.-Z. Chou, J.-S. Caux and A. Imambekov, Phys. Rev. B **88**, 205131 (2013).
 - [66] B. Davies and V.E. Korepin, arXiv:1109.6604 (2011).
 - [67] N.M. Bogoliubov and R.K. Bullough, J. Phys. **A 25**, 4057 (1992).
 - [68] N. M. Bogoliubov, R. K. Bullough, and G. D. Pang, Phys. Rev. B**47**, 11495 (1993).
 - [69] V. V. Cheianov, H. Smith, and M. B. Zvonarev, J. Stat. Mech. (2006) P08015.
 - [70] B. Pozsgay, arXiv:1407.8344 (2014).
 - [71] M. Lüscher, Nucl. Phys. **B135**, 1 (1978).
 - [72] D. Bernard and A. LeClair, Comm. Math. Phys. **142**, 99 (1991).
 - [73] B. Doyon, A. Lucas, K. Schalm and M.J. Bhaseen, arXiv:1409.6660.
 - [74] T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011).
 - [75] T. Prosen and E. Ilievski, Phys. Rev. Lett. **111**, 057203 (2013);
 - [76] R. G. Pereira, V. Pasquier, J. Sirker and I. Affleck, J. Stat. Mech. (2014) P09037.
 - [77] M. Moeckel and S. Kehrein, Phys. Rev. Lett. **100**, 175702 (2008).
 - [78] M. Moeckel and S. Kehrein, Ann. Phys. **324**, 2146 (2009).
 - [79] M. Moeckel and S. Kehrein, New J. Phys. **12**, 055016 (2010).
 - [80] M. Kollar, F.A. Wolf, and M. Eckstein, Phys. Rev. **B84**, 054304 (2011).
 - [81] M. Marcuzzi, J. Marino, A. Gambassi, and A. Silva, Phys. Rev. Lett. **111**, 197203 (2013).
 - [82] F.H.L. Essler, S. Kehrein, S.R. Manmana, N.J. Robinson, Phys. Rev. **B89**, 165104 (2014).
 - [83] M. Fagotti, J. Stat. Mech. (2014) P03016.
 - [84] G. Brandino, J.-S. Caux, and R. M. Konik , arXiv:1407.7167.
 - [85] M. Fagotti, arXiv:1408.1950.
 - [86] A. Chandran, A. Nanduri, S.S. Gubser and S.L. Sondhi, Phys. Rev. B **88**, 024306 (2013).

Supplementary Material

Here we provide some technical details underlying the discussion in the main text. We start with a derivation of the ultra-local conserved charges $I_n^\pm$ for the free Majorana fermion field theory. Then, by transforming $I_n^\pm$ to the rapidity space, we show that their expectation values cannot uniquely determine the mode occupation numbers $n_\Psi(k)$.

Ultra-local conservation laws

Our starting point is the Hamiltonian (3)

$$\mathcal{H} = \int dx [iv(R(x)\partial_x R(x) - L(x)\partial_x L(x)) + 2miR(x)L(x)], \quad (\text{S1})$$

where R and L are real chiral fermions, v plays the role of the speed of light and m is the mass. The equations of motion are

$$(v\partial_x - \partial_t) R(x, t) = -mL(x, t) \quad (\text{S2})$$

$$(v\partial_x + \partial_t) L(x, t) = -mR(x, t). \quad (\text{S3})$$

To simplify notations we set the velocity $v = 1$ from now on and only restore it in the final expressions. Ultra-local conserved charges I_n are given as spatial integrals

$$I_n = \frac{im}{2} \int dx (T_{n+1} + \Theta_n) \quad (\text{S4})$$

involving the local densities $T_{n+1}(x, t)$ and $\Theta_n(x, t)$. Here the factor $im/2$ has been introduced for convenience. In order to guarantee that $\partial_t I_n = 0$ the local densities must fulfil one of the divergence-free conditions

$$\partial_\tau T_{n+1}^\sigma = \partial_\sigma \Theta_n^\sigma, \quad (\text{S5})$$

$$\partial_\sigma T_{n+1}^\tau = \partial_\tau \Theta_n^\tau. \quad (\text{S6})$$

Here $\tau = x + t$ and $\sigma = x - t$ are the light-cone coordinates. Using equations of motions one verifies that the following local densities obey this condition ($n \geq 0$)

$$\Theta_n^\sigma = (\partial_\sigma^n R) (\partial_\sigma^n L), \quad (\text{S7})$$

$$\Theta_n^\tau = (\partial_\tau^n R) (\partial_\tau^n L), \quad (\text{S8})$$

$$T_n^\alpha = m^{-2} \Theta_n^\alpha, \quad \alpha = \sigma, \tau. \quad (\text{S9})$$

In this way we have constructed two infinite denumerable sets $\{I_n^\sigma\}$, $\{I_n^\tau\}$ of conserved charges. Explicit expressions for the first few local densities are easily written down

$$\Theta_0^\sigma = \Theta_0^\tau = RL, \quad (\text{S10})$$

$$\Theta_1^\sigma = m^2 RL - 2mL\partial_x L, \quad (\text{S11})$$

$$\Theta_1^\tau = m^2 RL + 2mR\partial_x R, \quad (\text{S12})$$

$$\Theta_2^\sigma = m^4 RL - 2m^3 (R\partial_x R + L\partial_x L) - 4m^2 R\partial_x^2 L - 4m^2 \partial_x L\partial_x R + 8m\partial_x^2 R\partial_x R, \quad (\text{S13})$$

$$\Theta_2^\tau = m^4 RL + 2m^3 (R\partial_x R + L\partial_x L) - 4m^2 \partial_x L\partial_x R - 4m^2 (\partial_x^2 R) L - 8m\partial_x^2 L\partial_x L. \quad (\text{S14})$$

Further simplifications occur if we consider even and odd combinations

$$\frac{I_n^\tau \pm I_n^\sigma}{2} = \frac{im}{4} \int dx \left[\frac{\Theta_{n+1}^\tau \pm \Theta_{n+1}^\sigma}{m^2} + \Theta_n^\tau \pm \Theta_n^\sigma \right]. \quad (\text{S15})$$

By taking suitable linear combinations of (S15) (and restoring the velocity v) we arrive at the following set of conserved charges (6)

$$I_n^+ = \frac{i}{2} \int dx (Rv\partial_x^{2n+1} R - Lv\partial_x^{2n+1} L + 2mR\partial_x^{2n} L), \quad (\text{S16})$$

$$I_n^- = \frac{iv}{2} \int dx (R\partial_x^{2n+1} R + L\partial_x^{2n+1} L). \quad (\text{S17})$$

Rapidity space and incompleteness of ultra-local charges

The Bogoliubov transformation used to diagonalize the Hamiltonian (S1) is given by

$$R(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\omega(k) + vk}{2\omega(k)}} [e^{i\frac{\pi}{4}} Z(k) e^{-ixk} + \text{h.c.}] , \quad (\text{S18})$$

$$L(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\omega(k) - vk}{2\omega(k)}} [e^{-i\frac{\pi}{4}} Z(k) e^{-ixk} + \text{h.c.}] . \quad (\text{S19})$$

where $\{Z(k), Z^\dagger(q)\} = 2\pi\delta(k - q)$ and the dispersion relation is $\omega(k) = \sqrt{m^2 + v^2 k^2}$. Applying the Bogoliubov transformation (S19) to the ultra-local charges (S17) leads to

$$I_n^\pm = (-1)^n \int_{-\infty}^{\infty} \frac{dk}{2\pi} \epsilon_n^\pm(k) Z^\dagger(k) Z(k), \quad \epsilon_n^+(k) = \omega(k) k^{2n}, \quad \epsilon_n^-(k) = v k^{2n+1}. \quad (\text{S20})$$

In order to proceed we now wish to change from momentum to rapidity variables θ defined by

$$k = \frac{m}{v} \sinh(\theta). \quad (\text{S21})$$

The dispersion relation now simply becomes $\omega = m \cosh \theta$. The eigenvalues ϵ_n^- become

$$\epsilon_n^-(\theta) = v k^{2n+1} = v \left(\frac{m}{v} \sinh \theta \right)^{2n+1} = v \left[\frac{m}{v} \right]^{2n+1} \sum_{j=0}^n \delta_j \sinh((2j+1)\theta), \quad (\text{S22})$$

where δ_j are known constants. This implies that by taking appropriate linear combinations of the I_n^- we can obtain an equivalent set of conservation laws

$$J_n^- = \int \frac{d\theta}{2\pi} \sinh[(2n+1)\theta] N(\theta). \quad (\text{S23})$$

Here $N(\theta) = Z^\dagger(\theta) Z(\theta)$ are mode occupation numbers in rapidity space. The creation and annihilation operators fulfil canonical anticommutation relations $\{Z(\theta), Z^\dagger(\theta')\} = 2\pi\delta(\theta - \theta')$ and are related to the corresponding operators in momentum space by $Z(\theta) = \sqrt{m/v \cosh \theta} Z(k)$. An analogous construction can be carried out for the I_n^+ charges. The only technical difference is that in forming linear combinations we also have to include I_n^- charges. This procedure results in conserved charges of the form

$$J_n^+ = \int \frac{d\theta}{2\pi} \cosh[(2n+1)\theta] N(\theta). \quad (\text{S24})$$

The two sets of ultra-local charges, $\{I_n^\pm\}$ and $\{J_n^\pm\}$, are completely equivalent. However, the functional form of the eigenvalues is clearly much simpler for the $J_n^\pm$. We may exploit this to establish that ultra-local charges are insufficient for specifying a general representative state $|\Phi\rangle$. By definition this is an eigenstate of all $N(\theta)$ with eigenvalues $n_\Psi(\theta)$. Let us assume for simplicity that this is an even function that decays sufficiently quickly at infinity for the integrals below to exist. Then we have $\langle \Phi | J_n^- | \Phi \rangle = 0$ and

$$j_n^+ = \langle \Phi | J_n^+ | \Phi \rangle = \int \frac{d\theta}{2\pi} \cosh[(2n+1)\theta] n_\Psi(\theta). \quad (\text{S25})$$

Let us now consider a different eigenstate $|\Phi_f\rangle$ of all $N(\theta)$, which we take to have eigenvalues $n_\Psi(\theta) + f(\theta)$, where

$$f(\theta) = A \exp(-\theta^2/4) \cos(\pi\theta/4). \quad (\text{S26})$$

The constant A should be (and can be) chosen such that $n_\Psi(\theta) + f(\theta) \geq 0$. By construction we have

$$\langle \Phi | J_n^+ | \Phi \rangle = \langle \Phi_f | J_n^+ | \Phi_f \rangle. \quad (\text{S27})$$

This shows that the ultra-local charges $J_n^\pm$ are insufficient for distinguishing between the rapidity distributions $n_\Psi(\theta)$ and $n_\Psi(\theta) + f(\theta)$, and are hence insufficient for constructing a generalized Gibbs ensemble.