

# DEFICIENCY, RELATION GAP AND TWO DIMENSIONAL GROUPS

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ABSTRACT. Let  $G$  be a finitely presented, residually finite group and let  $\delta(G)$  denote the deficiency of  $G$ . Assume that every subgroup  $H$  of finite index in  $G$  satisfies  $\delta(H) - 1 = |G : H|(\delta(G) - 1)$ . We conjecture that  $G$  has a two-dimensional finite classifying space  $K(G, 1)$ . This conjecture is motivated by an open question about the deficiency gradient of groups and their  $L^2$ -Betti numbers. In this note we relate this conjecture to the relation gap problem for group presentations. We verify the pro- $p$  version of the conjecture, as well as its higher dimensional abstract analogues.

Given a finitely presented group  $G$  we define its deficiency  $\delta(G)$  to be the maximum of  $|X| - |R|$  over all presentations  $G = \langle X | R \rangle$ . For example the deficiency of a direct product of a free group of rank  $n$  with a free group of rank  $m$  is  $n + m - nm$ , while the deficiency of a one relator group defined on  $d$  generators is  $d - 1$ .

Let  $d(G)$  be the cardinality of a minimal generating set of  $G$  and let  $b_i(G) = \dim_{\mathbb{Q}} H_i(G, \mathbb{Q})$ . Note the well-known inequalities (e.g. using [2, Lemma 1.2])

$$\delta(G) \leq b_1(G) - b_2(G) \leq d(G).$$

When  $G$  has a finite classifying space  $X = K(G, 1)$  we define its Euler characteristic to be

$$\chi(G) := \chi(X) = \sum_{i=0}^{\dim X} (-1)^i e_i(X) = \sum_{i=0}^{\dim X} (-1)^i b_i(G),$$

where  $e_i(X)$  is the number of cells of dimension  $i$  in  $X$ .

Starting with a presentation  $\langle X | R \rangle$  for  $G$  and a subgroup  $H$  of finite index in  $G$  one obtains a Schreier presentation for  $H$  with  $[G : H](|X| - 1) + 1$  generators and  $[G : H]|R|$  relations. Therefore

$$\delta(H) - 1 \geq [G : H](\delta(G) - 1).$$

We are interested in the situation when the above inequality is in fact equality for every finite index subgroup  $H$  of  $G$ . This is the case for groups which have finite two-dimensional classifying spaces. Examples of such groups are surface groups or more generally, torsion-free one relator groups and direct products of two free groups.

**Lemma 1.** *If a group  $G$  has a finite two-dimensional space  $K(G, 1)$ , then  $\delta(G) = 1 - \chi(G)$  and consequently,  $\delta(H) - 1 = [G : H](\delta(G) - 1)$  for every subgroup  $H$  of finite index in  $G$ .*

In [7] we formulated a conjecture that the converse of Lemma 1 holds.

**2D conjecture** ([7]). *Let  $G$  be a residually finite, finitely presented group such that  $\delta(H) - 1 = [G : H](\delta(G) - 1)$  for every subgroup  $H$  of finite index in  $G$ . Then  $G$  has a finite 2-dimensional classifying space  $K(G, 1)$ .*

**Motivation.** The original motivation of the 2D conjecture in [7] comes from deficiency gradients of groups and their  $L^2$ -cohomology. For a finitely presented, residually finite group  $G$  we denote by  $b_i^{(2)}(G)$  its  $L^2$ -Betti number in dimension  $i$ , see [11, Definition 1.30]. The *deficiency gradient* of  $G$  is defined as

$$\Delta(G) := \sup \left\{ \frac{\delta(H) - 1}{|G : H|} \mid H <_f G \right\},$$

where the supremum is taken over all subgroups  $H$  of finite index in  $G$ . It is easy to see that  $\Delta(H) = |G : H|\Delta(G)$  for all subgroups  $H$  of finite index in  $G$ . Lemma 2 below, together with  $b_i^{(2)}(H) = |G : H|b_i^{(2)}(G)$  gives

$$(1) \quad \Delta(G) \leq b_1^{(2)}(G) - b_2^{(2)}(G)$$

and it is an open question if the inequality can be strict. A counterexample to the 2D conjecture will answer this affirmatively, using the following.

**Lemma 2** ([7], Lemma 4). *Let  $G$  be an infinite finitely presented group. Then  $\delta(G) - 1 \leq b_1^{(2)}(G) - b_2^{(2)}(G)$  with equality if and only if  $G$  has a two dimensional classifying space.*

Suppose that the 2D conjecture is false with a group  $G$  being a counterexample. Then by Lemma 2 we have  $\Delta(G) = \delta(G) - 1 < b_1^{(2)}(G) - b_2^{(2)}(G)$ , proving that the inequality (1) is strict for  $G$ .

We remark that the condition that  $G$  is residually finite in the 2D conjecture is necessary. Without it, any finitely presented infinite simple group of cohomological dimension greater than 2 is a counterexample (e.g. let  $G$  be Thompson's group  $T$ ). Moreover, we cannot in general weaken the requirement  $\delta(H) - 1 = |G : H|(\delta(G) - 1)$  to apply just to some proper infinite subset of the subgroups of  $G$ . Here is a family of examples. Let  $S$  be a finitely presented group such that  $\delta(S) = b_1(S) - b_2(S)$  and  $S$  has cohomological dimension greater than 2. For example we can take  $S$  to be Thompson's group  $F$  or any finite group of deficiency zero. Let  $G = \mathbb{Z} * S$  be the free product of the infinite cyclic group  $\mathbb{Z}$  with  $S$  and note that  $\delta(G) = 1 + \delta(S)$  (since  $b_1(G) = 1 + b_1(S)$  and  $b_2(G) = b_2(S)$ ). For an integer  $n$  let  $f_n : G \rightarrow \mathbb{Z}/n\mathbb{Z}$  be a surjective homomorphism such that  $S \leq \ker f_n$ . Thus  $|G : H_n| = n$  and  $H_n$  is the free product of  $n\mathbb{Z} \simeq \mathbb{Z}$  with  $n$  conjugates of  $S$ . We deduce that  $b_1(H_n) = 1 + nb_1(S)$  and  $b_2(H_n) = nb_2(S)$ . From the

obvious presentation of  $H_n$  we obtain

$$1 + n\delta(S) \leq \delta(H_n) \leq b_1(H_n) - b_2(H_n) = 1 + n(b_1(S) - b_2(S)) = 1 + n\delta(S).$$

Therefore  $\delta(H_n) - 1 = n\delta(S) = n(\delta(G) - 1)$  for each  $n$ . However  $G$  is not 2-dimensional because  $S$  is not 2-dimensional by assumption. If in addition  $S$  can be chosen such every finite image of  $S$  is trivial, then every subgroup of finite index in  $G$  is equal to  $H_n$  for some  $n$ .

Note that the 1-dimensional analogue of the 2D Conjecture is true as shown by R. Strebel [14] (see also [1, Theorem 7] for a different perspective):

**Proposition 3** ([14]). *Let  $G$  be a finitely generated residually finite group. Then  $G$  is a free group if and only if  $d(H) - 1 = |G : H|(d(G) - 1)$  for every subgroup  $H$  of finite index in  $G$ .*

Strebel proved Proposition 3 as an answer to a question of Lubotzky and van den Dries [9], who had shown that its analogue does not hold in the class of profinite groups. At the same time Lubotzky [8, Proposition 4.2] proved that the analogue of Proposition 3 is true in the class of pro- $p$  groups. In section 2 below we will prove the pro- $p$  analogue of the 2D conjecture.

**The relation gap problem.** In this paper, we relate the 2D conjecture with the *relation gap problem* in combinatorial group theory. A finitely presented group  $G = \langle X | R \rangle$  is isomorphic to the quotient  $F/N$ , where  $F$  is the free group on  $X$  and  $N$  is normally generated in  $F$  by the relators  $R \subset F$ . The action of  $F$  by conjugation on  $N$  induces an action of  $G$  on the abelianisation  $N^{ab}$  of  $N$ . This makes  $N^{ab}$  into a  $G$ -module called the relation module of the presentation. Evidently, the  $G$ -module  $N^{ab}$  can be generated by  $|R|$  elements and so the  $G$ -rank of  $N^{ab}$ , written  $d_G(N^{ab})$ , satisfies  $d_G(N^{ab}) \leq d_F(N)$ , where  $d_F(N)$  is the minimum number of normal generators required for  $N$ . A presentation is said to have a relation gap if  $d_G(N^{ab}) \neq d_F(N)$  and the *relation gap problem* asks, if there exists a finitely presented group with a relation gap. Very little is known about the relation gap problem and most proposed counterexamples are not torsion-free, see [5].

We will show that any counterexample to the 2D conjecture above leads to the existence of a group with relation gap. More precisely in section 1 we prove.

**Theorem 4.** *Suppose that  $G$  is a counterexample to the 2D conjecture. There exists a finite index subgroup  $H$  of  $G$  with a presentation which has a relation gap. Moreover this presentation realizes the deficiency of  $H$ .*

**Wall's D2 problem and higher dimensions.** We next introduce and establish the higher dimensional analogues of the 2D Conjecture. Deficiency can be viewed as one of the partial Euler characteristics, which are defined as follows. Let  $n \geq 2$  be an integer and let  $G$  be a group of type  $F_n$ . Define  $\nu_n(G)$  to be the minimum of  $(-1)^n \chi(X)$ , where  $X$  is a finite CW-complex of dimension  $n$  such that  $\pi_1(X) = G$  and  $\pi_i(X) = \{0\}$  for  $i = 2, 3, \dots, n-1$

(i.e its universal cover  $\tilde{X}$  is  $(n-1)$ -connected. Note that  $\nu_2(G) = 1 - \delta(G)$  and for completeness we define  $\nu_1(G) = d(G) - 1$ .

A closely related invariant to  $\nu_n(G)$  is the invariant  $\mu_n(G)$  of Swan. A partial free resolution of  $\mathbb{Z}$  of length  $n$  is an exact sequence

$$\mathcal{F}: (\mathbb{Z}G)^{f_n} \rightarrow (\mathbb{Z}G)^{f_{n-1}} \rightarrow \cdots \rightarrow (\mathbb{Z}G)^{f_0} \rightarrow \mathbb{Z} \rightarrow 0$$

and we define  $\mu_n(\mathcal{F}) = \sum_{i=0}^n (-1)^{n-i} f_i$ . R. Swan [15] defined  $\mu_n(G)$  to be the minimum of  $\mu_n(\mathcal{F})$  as  $\mathcal{F}$  ranges over all partial free resolutions  $\mathcal{F}$  of  $\mathbb{Z}$  of length  $n$ . Note that by the Morse inequality

$$\sum_{i=0}^n (-1)^{n-i} b_i(G) \leq \mu_n(\mathcal{F})$$

the minimum of  $\mu_n(\mathcal{F})$  does exist.

From the definition of  $\nu_n$  and  $\mu_n$  we have  $\nu_n(G) \geq \mu_n(G)$  for all  $n$ . The equality of  $\mu_n(G)$  and  $\nu_n(G)$  is connected to the geometric realization problem for free algebraic resolutions and Wall's  $Dn$  problem.

A finite CW-complex  $X$  is said to be a *D2-complex* if it has cohomological dimension 2. The D2 problem for a finitely presented group  $G$  asks if every finite D2-complex with fundamental group  $G$  is homotopy equivalent to a finite 2-complex. If the answer is affirmative we shall say that  $G$  has the *D2 property*. The problem was proposed by C.T.C. Wall in 1965 [16] and little is known about it except in the case when  $G$  is finite, free or abelian, see [6].

By the results of Wall [16] (see also [6]) proving the higher dimensional analogues of the D2 property, the geometric realization problem has a positive solution when  $n > 2$  and hence  $\nu_n(G) = \mu_n(G)$  when  $n > 2$ . We note that if  $G$  has the D2 property then

$$(2) \quad 1 - \delta(G) = \nu_2(G) = \mu_2(G)$$

from [12] (or Remark 1.3 in [3]). As a consequence we have

**Proposition 5.** *Let  $G$  be a finitely presented group.*

(i) *Suppose that  $G$  has the D2 property. Then any presentation which realizes the deficiency of  $G$  has no relation gap.*

(ii) *If all finitely presented residually finite groups have the D2 property then the 2D conjecture is true.*

Part (i) is a well-known consequence of the equality (2) and for completeness we give a short proof of Proposition 5 at the end of section 1.

Finally in section 3 we prove the higher dimensional analogue of the 2D conjecture.

**Theorem 6.** *Let  $n > 2$  be an integer and let  $G$  be a residually finite group of type  $F_n$ . Then  $G$  has finite classifying space of dimension  $n$  if and only if  $\nu_n(H) = \nu_n(G)[G : H]$  for every subgroup  $H$  of finite index in  $G$ .*

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## 1. PROOF OF THEOREM 4

Suppose that  $G$  is a finitely presented, residually finite group such that  $\delta(H) - 1 = |G : H|(\delta(G) - 1)$  for every subgroup  $H$  of finite index in  $G$ . Assume that  $X$  is a presentation 2-complex for  $G$  realising the deficiency  $\delta(G)$ . If  $X$  is not aspherical, then by Whitehead's Theorem,  $H_2(\tilde{X}) \neq 0$ . Let  $e_i$  denote the number of  $i$ -cells in  $X$ . So  $\delta(G) - 1 = e_1 - e_2 - 1$ . The cellular chain complex of  $X$  gives rise to the exact sequence of  $\mathbb{Z}G$ -modules

$$0 \longrightarrow H_2(\tilde{X}) \longrightarrow \mathbb{Z}G^{e_2} \xrightarrow{\partial_2} \mathbb{Z}G^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where  $H_2(\tilde{X}) = \ker \partial_2$ . The relation module  $R$  associated to  $X$  is isomorphic to  $\ker \partial_1 = \text{im } \partial_2 \cong \mathbb{Z}G^{e_2}/H_2(\tilde{X})$ . Take a non-zero element  $\rho$  of  $H_2(\tilde{X})$ . As an element of  $\mathbb{Z}G^{e_2}$ ,  $\rho$  has a representation as a non-zero tuple  $(a_1, \dots, a_{e_2})$ , where each  $a_i$  is a linear combination in  $\mathbb{Z}G$  with support  $C_i \subseteq G$  as follows:

$$a_i = \sum_{g \in C_i} a_g^i g$$

Let  $C = \cup_i C_i$ ; this is a finite collection of elements of  $G$ . As  $G$  is residually finite there exists a finite index subgroup of  $G$ , say  $H$  such that the elements of  $C$  project to distinct cosets in  $G/H$ . The natural structure of  $\mathbb{Z}G$  as a  $\mathbb{Z}H$ -module makes  $\mathcal{F}$  into the chain complex for the action of  $H$  on  $\tilde{X}$ . Let  $E$  be a collection of coset representatives for  $H$  in  $G$  such that  $C \subseteq E$ . Consider

$$\mathbb{Z}G^{e_2} = \left( \bigoplus_{g \in E} \mathbb{Z}H.g \right)^{e_2} \cong \mathbb{Z}H^{e_2[G:H]}.$$

Let  $d$  be the greatest common divisor of the set of integers

$$\{a_g^i \mid i = 1, 2, \dots, e_2, g \in C_i\}.$$

We have  $\rho = d\rho'$ , where  $\rho' \in \mathbb{Z}G^{e_2}$  and all its coefficients are setwise co-prime. As  $\rho$  is an element of  $\ker \partial_2$  and  $\partial_2$  is a homomorphism of torsion-free abelian groups, we deduce that  $\rho'$  is also an element of  $\ker \partial_2$ . Therefore, we can assume that  $d = 1$ .

Consider the presentation for  $H$  arising from the action of  $H$  on  $\tilde{X}$ : this presentation has  $(e_1 - 1)[G : H] + 1$  generators and  $e_2[G : H]$  relations and realizes the deficiency of  $H$  since  $\delta(H) = 1 + |G : H|(\delta(G) - 1)$ .

The relation module  $R'$  for this presentation of  $H$  is the restriction  $R \downarrow_H^G$  of the relation module  $R$ , wherein  $\rho$  represents the zero element. We have assumed that the coefficients of  $\rho$  are co-prime and so  $\rho$  is a primitive element in the abelian group  $(\mathbb{Z}E)^{e_2}$  containing its support in  $\mathbb{Z}G^{e_2} \cong \mathbb{Z}H^{e_2[G:H]}$ . Consequently

$$R' \cong \frac{\mathbb{Z}H^{e_2[G:H]}}{H_2(\tilde{X})}$$

can be generated by fewer than  $e_2[G : H]$  elements as an  $H$ -module. If the above presentation of  $H$  has no relation gap then it needs strictly fewer than

$e_2[G : H]$  relations and hence

$$\delta(H) - 1 > [G : H](e_1 - e_2 - 1) = [G : H](\delta(G) - 1),$$

a contradiction to the assumption  $\delta(H) - 1 = |G : H|(\delta(G) - 1)$ .

Therefore, if  $X$  is not aspherical, some finite index subgroup of  $G$  has a presentation realising its deficiency and with a relation gap. Theorem 4 follows.

We note that the argument above gives the following general criterion for freeness of  $\mathbb{Z}G$ -modules.

**Proposition 7.** *Let  $G$  be a residually finite group and let  $M$  be a finitely generated  $\mathbb{Z}G$ -module. Assume that  $M$  is torsion free as an abelian group and let  $f : (\mathbb{Z}G)^r \rightarrow M$  be a surjective homomorphism of  $\mathbb{Z}G$ -modules. Then  $f$  is an isomorphism if and only if  $d_H(M) = r[G : H]$  for each subgroup  $H$  of finite index in  $G$ .*

*In particular  $M$  is a free  $\mathbb{Z}G$ -module if and only if  $d_H(M) = [G : H]d_G(M)$  for each subgroup  $H$  of finite index in  $G$ .*

*Proof.* If  $f$  is not injective we can find an element  $\rho = (a_1, \dots, a_r) \in \ker f$  with support  $C = \cup_{i=1}^r C_i$  and coefficients  $a_g^i \in \mathbb{Z}$  defined by  $a_i = \sum_{g \in C_i} a_g^i g$ . Since  $M$  is torsion free we can assume that the greatest common divisor of all integers  $a_g^i$  is 1. There is a finite index subgroup  $H$  of  $G$  such that  $C$  projects injectively into  $G/H$  and arguing in the same way as in the proof of Theorem 4 we deduce that  $d_H(M) < r[G : H]$ , a contradiction. Therefore  $f$  is a bijection and  $M$  is a free  $\mathbb{Z}G$ -module.  $\square$

*Proof of Proposition 5.* (i) Let  $G$  be a group with the D2 property and in particular the equation (2) holds.

Take a presentation  $\langle X | R \rangle$  for  $G$  with  $e_1$  generators and  $e_2$  relations such that  $e_1 - e_2 = \delta(G)$ . We have  $G \cong F/N$  where  $F$  is the free group of rank  $e_1$  on  $X$  and  $N$  is the normal closure of the relations  $R$ . Since  $e_1 - e_2$  realises the deficiency of  $G$  it follows that  $e_2 = d_F(N)$ . Let  $M = N^{ab}$  be the relation module of this presentation. Recall the chain complex

$$\mathbb{Z}G^{e_2} \xrightarrow{\partial_2} \mathbb{Z}G^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

associated to the presentation  $\langle X | R \rangle$ . We have  $M \cong \ker \partial_1 = \text{im } \partial_2$ . If  $M$  has relation gap then  $u := d_G(M) < e_2$  and in particular there is a surjection of  $\mathbb{Z}G$  modules  $f : (\mathbb{Z}G)^u \rightarrow \ker \partial_1$ . Therefore we can amend the partial resolution above to

$$(\mathbb{Z}G)^u \xrightarrow{f} (\mathbb{Z}G)^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \rightarrow 0.$$

This gives

$$\mu_2(G) \leq 1 - e_1 + u < 1 - e_1 + e_2 = 1 - \delta(G),$$

contradicting (2). Therefore presentations of  $G$  which realize  $\delta(G)$  have no relation gap.

(ii) Let  $G$  be a finitely presented residually finite group which satisfies the assumptions of the 2D conjecture. Assume further that every subgroup  $H$  of finite index in  $G$  has the D2 property. By (i) the presentations of  $H$  which realize its deficiency have no relation gap. Theorem 4 then implies that the 2D conjecture holds for  $G$ .  $\square$

## 2. THE 2D CONJECTURE FOR PRO- $p$ GROUPS.

Let  $G$  be a finitely presented pro- $p$  group, where we consider presentations in the category of pro- $p$  groups. We keep the notation  $\delta(G)$  for the maximum of  $|X| - |R|$  over all pro- $p$  presentations  $\langle X|R \rangle$  of  $G$ . As usual  $cd_p(G)$  denotes the  $p$ -cohomological dimension of  $G$ , defined in [13, I §3.1].

Below we prove the analogue of the 2D conjecture for  $G$ .

**Theorem 8.** *Let  $G$  be a finitely presented pro- $p$  group. The following are equivalent:*

- (i)  $\delta(H) - 1 = [G : H](\delta(G) - 1)$  for every open subgroup  $H$  of  $G$ .
- (ii)  $cd_p(G) \leq 2$ .

*Proof of Theorem 8.* Let  $G$  be a finitely presented pro- $p$  group. We have  $\delta(G) = \dim_{\mathbb{F}_p} H^1(G) - \dim_{\mathbb{F}_p} H^2(G)$ , where we write  $H^i(G) = H^i(G, \mathbb{F}_p)$ , see [13, I §4.2 & §4.3]. Hence, if  $cd_p(G) \leq 2$  then

$$1 - \delta(G) = \sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(G)$$

equals the Euler-Poincare characteristic of  $G$  and therefore (i) follows from the results in [13, I §4.1].

Conversely, suppose that (i) holds and let  $e_i = \dim_{\mathbb{F}_p} H^i(G)$  for  $i = 1, 2$ . We have the partial free resolution

$$(3) \quad \mathbb{F}_p[[G]]^{e_2} \xrightarrow{d_2} \mathbb{F}_p[[G]]^{e_1} \xrightarrow{d_1} \mathbb{F}_p[[G]] \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

arising from the presentation of  $G$  with  $e_1$  generators and  $e_2$  relations. We claim that  $J := \ker d_2$  must be zero. Suppose not. Then we can find an open normal subgroup  $N$  of  $G$  such that the image  $\bar{J}$  of  $J$  under the reduction  $(\mathbb{F}_p[[G]])^{e_2} \rightarrow (\mathbb{F}_p[G/N])^{e_2}$  is non-zero.

Note that the free  $\mathbb{F}_p[[G]]$ -resolution above is also a partial free resolution of  $\mathbb{F}_p[[N]]$ -modules. We apply the functor  $\text{Hom}_N(-, \mathbb{F}_p)$  to the above resolution, using  $\text{Hom}_N(\mathbb{F}_p G, \mathbb{F}_p) \simeq (\mathbb{F}_p[G/N])^*$ , where by  $V^*$  we denote the dual of the vector space  $V$  over  $\mathbb{F}_p$ . We obtain the chain complex

$$0 \longleftarrow \bar{J}^* \longleftarrow (\mathbb{F}_p[G/N]^*)^{e_2} \xleftarrow{d_2^*} (\mathbb{F}_p[G/N]^*)^{e_1} \xleftarrow{d_1^*} \mathbb{F}_p[G/N]^* \longleftarrow 0,$$

which is exact at  $\bar{J}^*$  and whose homology group in degree  $i$  is  $H^i(N)$ . Therefore

$$\begin{aligned} \delta(N) - 1 &= \dim_{\mathbb{F}_p} H^1(N) - \dim_{\mathbb{F}_p} H^2(N) - \dim_{\mathbb{F}_p} H^0(N) = \\ &= (e_1 - e_2 - 1)[G : N] + \dim_{\mathbb{F}_p} \bar{J}^* > [G : N](\delta(G) - 1), \end{aligned}$$

since  $\bar{J}^* \neq \{0\}$ , a contradiction to (i).

Therefore  $J = \{0\}$  and (3) is a free resolution of  $\mathbb{F}_p$ , hence  $cd_p(G) \leq 2$ .  $\square$

From now on let  $G$  be a finitely presented profinite group. Recall that its cohomological dimension,  $cd(G)$  is the supremum of  $cd_p(G)$  for all primes  $p$ . It would be interesting to find a characterization of the finitely presented profinite groups  $G$  such that  $\delta(H) - 1 = |G : H|(\delta(G) - 1)$  for all open subgroups  $H$  of  $G$ . The direct analogue of Theorem 8 would suggest that these are the profinite groups  $G$  with  $cd(G) \leq 2$ , however this is false: There exist procyclic groups  $G$  such that  $\delta(H) - 1 > |G : H|(\delta(G) - 1)$  for all of their proper open subgroups  $H$ . Indeed, let  $p \neq q$  be two different primes and let  $G = \mathbb{Z}_p \times \mathbb{Z}_q$  be the product of the free pro- $p$  and pro- $q$  procyclic groups. Observe first, that any open subgroup of  $G$  is isomorphic to  $G$  and second, that  $cd(G) = 1$ . Moreover  $G$  can be presented with 1 generator and 1 relation and so  $\delta(G) \geq 0$ . On the other hand  $\delta(G) \leq 0$  since every profinite group with positive deficiency maps onto  $\hat{\mathbb{Z}} = \prod_{l \text{ prime}} \mathbb{Z}_l$ . Therefore  $\delta(G) = 0$  and if  $H$  is an open subgroup of index  $n > 1$  in  $G$  then  $\delta(H) - 1 = -1 > -n = |G : H|(\delta(G) - 1)$ .

It might still be the case that the reverse implication of the profinite analogue of Theorem 8 is true.

**Question 9.** *Let  $G$  be a finitely presented profinite group such that*

$$\delta(H) - 1 = |G : H|(\delta(G) - 1)$$

*for all open subgroups  $H$  of  $G$ . Is it true that  $cd(G) \leq 2$ ?*

Answering this question is likely to require additional techniques for study of relation modules of profinite groups.

Let  $G$  be as in the question above and let  $M$  be the profinite relation module of a presentation realizing the deficiency of  $G$ . We note that profinite presentations have no relation gap, see [10, Proposition 2.3.4]. Therefore the argument in the proof of Theorem 4 shows that  $d_H(M) = |G : H|d_G(M)$  for any open subgroup  $H$  of  $G$ . However the analogue of Proposition 7 is false: such a module  $M$  may not be free or even a projective  $\hat{\mathbb{Z}}[[G]]$ -module. Here is a general counterexample. Take a profinite group  $G$  and a sequence  $p_1 < p_2 < \dots$  of distinct prime numbers such that  $G$  has an infinite Sylow pro- $p_i$  subgroup for each  $i$ . For example we can take  $G = \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ . Let  $G > G_1 > G_2 > \dots$  be any sequence of open normal subgroups  $G_i$  in  $G$  such that  $\cap_i G_i = \{1\}$  and consider the  $\hat{\mathbb{Z}}[[G]]$ -module  $M := \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}[G/G_i]$ . Then  $d_G(M) = 1$  and for each open subgroup  $H$  of  $G$  there is some  $G_i$  with  $H \geq G_i$ . It follows that

$$|G : H| \geq d_H(M) \geq d_H(\mathbb{Z}_{p_i}[G/G_i]) \geq |G : H|,$$

where the last inequality follows since every orbit of  $H$  on  $\mathbb{Z}_{p_i}[G/G_i]$  has size at most  $|H : G_i|$ . Therefore  $d_H(M) = |G : H|d_G(M)$  for every open subgroup  $H$  of  $G$  as required. We claim that the direct summands



$M_i := \mathbb{Z}_{p_i}[G/G_i]$  of  $M$  are not projective  $\hat{\mathbb{Z}}[[G]]$ -modules and so  $M$  cannot be projective.

*Proof of claim:* Let  $L_i$  be a Sylow pro- $p_i$  subgroup of  $G_i$ . Since the Sylow pro- $p_i$  subgroups of  $G$  are infinite it follows that  $L_i$  is infinite and in particular nontrivial. If we assume that  $M_i$  is a projective  $\hat{\mathbb{Z}}[[G]]$ -module then  $M_i$  is also a projective  $\hat{\mathbb{Z}}[[L_i]]$ -module. Note that the action of  $L_i$  on  $M_i$  is trivial. Reducing mod  $p_i$  it follows that the trivial module  $\mathbb{F}_{p_i}$  is a projective  $\mathbb{F}_{p_i}[[L_i]]$ -module and hence  $cd_p(L_i) = 0$ . This is impossible since  $L_i \neq \{1\}$  and  $cd_p(L) > 0$ , see [13, I §3.3 Corollary 2]. The claim follows.

### 3. PROOF OF THEOREM 6

Let  $G$  be a residually finite group of type  $F_n$  ( $n > 2$ ), and let  $H$  be a subgroup of finite index of  $G$ . Suppose that  $X$  is an  $n$ -dimensional  $K(G, 1)$ -complex for  $G$ , then  $\nu_n(G) \leq (-1)^n \chi(X)$  from the definition of  $\nu_n(G)$ . On the other hand the Morse inequalities give  $\nu_n(G) \geq \sum_{i=0}^n (-1)^{n-i} b_i(G) = (-1)^n \chi(X)$ . Therefore  $\nu_n(G) = (-1)^n \chi(X)$  and in the same way  $\nu_n(H) = (-1)^n \chi(X')$ , where  $X'$  is the cover of  $X$  corresponding to  $H$ . Since  $\chi(X') = [G : H] \chi(X)$  the equality  $\nu_n(H) = \nu_n(G)[G : H]$  follows.

For the other direction we could use the equality of  $\mu_n(G) = \nu_n(G)$  but instead we take a more elementary approach and argue directly using Proposition 7.

Suppose that  $\nu_n(H) = \nu_n(G)[G : H]$  for every subgroup  $H$  of finite index in  $G$ . Let  $X$  be the  $n$ -dimensional CW-complex which realises  $\nu_n(G)$ . Let  $e_i$  be the number of  $i$ -dimensional cells of  $X$  and let  $F_i = (\mathbb{Z}G)^{e_i}$ . Consider the chain complex of the universal cover  $\tilde{X}$  in the top row of the following diagram.

$$\begin{array}{ccccccc}
 F_n & \xrightarrow{\partial_n} & F_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_1} & F_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \\
 & \searrow \partial_n & \uparrow & & & & \\
 & & M & & & & \\
 & & \uparrow & \searrow & & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Let  $M = \ker \partial_{n-1} = \text{im } \partial_n \subseteq F_{n-1}$ . By the Hurewicz theorem  $\pi_n(X) \simeq H_n(X) = \ker \partial_n$  and thus  $X$  is aspherical if and only if  $\partial_n$  is injective. Suppose  $\ker \partial_n \neq \{0\}$  and apply Proposition 7 to the  $\mathbb{Z}G$ -epimorphism  $\partial_n : F_n \rightarrow M$ . Since  $F_n = (\mathbb{Z}G)^{e_n}$  we deduce that  $u := d_H(M) < e_n[G : H]$  for some finite index subgroup  $H$  of  $G$ .

Choose a set of generators  $\alpha_1, \dots, \alpha_u$  of the  $\mathbb{Z}H$ -module  $M$ . Let  $Y$  be the cover of  $X$  with degree  $[Y : X] = [G : H]$  and  $\pi_1(Y) = H$ . Let  $p : \tilde{X} \rightarrow Y$  be the universal covering map. Denote by  $Y^{n-1}$  and  $\tilde{X}^{n-1}$  the  $(n-1)$ -skeleta of  $Y$  and  $\tilde{X}$  respectively and observe that  $\pi_{n-1}(Y^{n-1}) \simeq H_{n-1}(\tilde{X}^{n-1}) = \ker \partial_{n-1} = M$  by the Hurewicz theorem. Therefore for each

$i = 1, \dots, u$  we can find a cellular map  $j_i : S^{n-1} \rightarrow \tilde{X}^{n-1}$  representing  $\alpha_i$ . This means that  $H_{n-1}(j_i)$  sends the generator of  $H_{n-1}(S^{n-1})$  to the element  $\alpha_i \in H_{n-1}(\tilde{X}^{n-1}) = M$ .

We now attach  $n$ -dimensional cells  $\sigma_i^n$  to  $Y^{n-1}$  for  $i = 1, \dots, u$  with boundary attaching maps

$$S^{n-1} \xrightarrow{j_i} \tilde{X}^{n-1} \xrightarrow{p} Y^{n-1}$$

and define  $Z := Y^{n-1} \cup_{i=1}^u \sigma_i^n$ . Note that since  $Y^{n-1} = Z^{n-1}$  we have  $\pi_i(Z) = \pi_i(Y)$  for  $i = 1, \dots, n-2$ . We claim that  $\pi_{n-1}(Z) = \{0\}$ . It is sufficient to prove that  $H_{n-1}(\tilde{Z}) = \{0\}$  for the universal cover  $\tilde{Z}$  of  $Z$ . Since the  $(n-1)$ -skeleta of  $Z$  and  $X$  coincide, the boundary maps  $\partial_{n-1}$  on the chain complex of  $\tilde{Z}$  and  $\tilde{X}$  are the same and hence  $\ker \partial_{n-1} = M$ . On the other hand the boundary map  $\partial'_n : (\mathbb{Z}H)^u \rightarrow M$  of degree  $n$  of the chain complex of  $\tilde{Z}$  is surjective since by construction its image contains the generators  $\alpha_i$ . Therefore  $H_{n-1}(\tilde{Z}) = \{0\}$  and so  $\tilde{Z}$  is  $(n-1)$ -connected as claimed.

Note that  $Z$  has  $[G : H]e_i$  cells in dimension  $i$  for  $i = 0, 1, \dots, n-1$  and  $u$  cells in dimension  $n$ . Since  $u < e_n[G : H]$  it follows that

$$\nu_n(H) \leq (-1)^n \chi(Z) = u + \sum_{i=0}^{n-1} (-1)^{n-i} e_i[G : H] < \nu_n(G)[G : H],$$

a contradiction to  $\nu_n(H) = \nu_n(G)[G : H]$ . Therefore  $H_n(\tilde{X}) = \{0\}$  and  $X$  is a finite  $K(G, 1)$ -complex of dimension  $n$ .

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