The Arithmetic Geometry of Mirror Symmetry and the Conifold Transition

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To the teachers in my life.

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Abstract

The central theme of this thesis is the application of mirror symmetry to the study of the arithmetic geometry of Calabi-Yau threefolds. More specifically, we will study two important types of singular fibers that occur in the mirror family: the large complex structure limit and the conifold singularity. Given a mirror pair (M, W) of Calabi-Yau threefolds, we conjecture that the limit mixed Hodge structure on $H^3(W, \mathbb{Q})$ at the large complex structure limit of the complex moduli space of W splits into

$$\mathbf{M} \oplus \mathbb{Q}(-1)^{h^{12}(W)} \oplus \mathbb{Q}(-2)^{h^{12}(W)} \ ,$$

where $\mathbf{M}$ is a mixed Hodge-Tate structure that is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. The dual of $\mathbf{M}$, denoted by $\mathbf{M}^\vee$, is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$, whose image in $\mathrm{Ext}^1_{\mathbf{MHS}_\mathbb{Q}}\big(\mathbb{Q}(0), \mathbb{Q}(3)\big)$ (which is isomorphic to $\mathbb{C}/(2\pi i)^3\mathbb{Q}$) is conjectured to be the coset of a rational multiple of $(2\pi i)^3 Y_{000}$, where Y_{000} is the constant term of the prepotential $\mathcal{F}$ on the complexified Kähler moduli space of M. The proof of this conjecture is potentially very hard, so instead we will give an interesting example in support of it, which also illuminates the general case. More precisely, we will study the mirror symmetry of a self-mirror Calabi-Yau threefold $X_\mathfrak{G}$ with Hodge numbers $h^{11} = h^{12} = 2$, and we will compute the limit mixed Hodge structure on $H^3(X_\mathfrak{G}, \mathbb{Q})$ at the large complex structure limit and show that it satisfies the predictions of this conjecture.

If further the mirror family, which is a deformation of W, is defined over $\mathbb{Q}$ and the large complex structure limit is a rational point, then the limit mixed Hodge structure is conjectured to be computed by the Hodge realisation of the limit motive constructed by Ayoub's motivic nearby cycle functor. In all examples of mirror pairs where Y_{000} has been computed, it is always of the form

$$-3\,\chi(M)\,\zeta(3)/(2\pi i)^3 + r,\ r \in \mathbb{Q}$$

where $\chi(M)$ is the Euler characteristic of M. In this thesis we will observe the interesting connections between this conjecture and periods of mixed Tate motives, which also provides a motivic explanation for the occurrence of $\zeta(3)$ in Y_{000}.

We will also study the interesting connections between the conifold transition and Beilinson's conjecture on the central values of L-functions, and we will focus on the conifold transition in the mirror family of the quintic, but our constructions work more generally. The conifold $\mathcal{W}_1$ of the mirror family of the quintic is a threefold defined over $\mathbb{Q}$ with singularity that consists of a single rational double point. Its zeta functions are

$$\frac{(1-(p/5)\,p\,T)(1-a_f(p)\,T+p^3\,T^2)}{(1-T)\,(1-p\,T)^{101}\,(1-p^2\,T)^{101}\,(1-p^3\,T)},\quad p\neq 5$$

where $a_f(p)$ is the p-th coefficient in the q-expansion of a modular form f of weight 4 and level 25. Suppose $\widetilde{\mathcal{W}_1}$ is the blow up of $\mathcal{W}_1$ at its double point, we will show that the L-function of the pure motive $h^3(\widetilde{\mathcal{W}_1})$ is $L(f,s)$. By restriction of base field, the variety $\widetilde{\mathcal{W}_1}\times\mathbb{Q}(5^{1/2})$ defines a variety over $\mathbb{Q}$ whose L-function is $L(f,s)\cdot L(f\otimes(\cdot/5),s)$. Beilinson's conjecture predicts that

$$\dim_{\mathbb{Q}}\mathrm{CH}^2\big(\widetilde{\mathcal{W}_1}\times\mathbb{Q}(5^{1/2})\big)_0\otimes_{\mathbb{Z}}\mathbb{Q}=\mathrm{ord}_{s=2}\,L\big(h^3\big(\widetilde{\mathcal{W}_1}\times\mathbb{Q}(5^{1/2})\big),s\big)$$

where $\mathrm{CH}^2\big(\widetilde{\mathcal{W}_1}\otimes\mathbb{Q}(5^{1/2})\big)_0$ is the subgroup of $\mathrm{CH}^2\big(\widetilde{\mathcal{W}_1}\otimes\mathbb{Q}(5^{1/2})\big)$ that consists of cycle classes homologous to zero. We will show that $L\big(f\otimes(\cdot/5),s\big)$ vanishes at $s=2$, hence the right hand side is ≥ 1. To test Beilinson's conjecture, we want to construct a non-torsion algebraic cycle of $\widetilde{\mathcal{W}_1}\times\mathbb{Q}(5^{1/2})$ that is homologous to zero. Over $\mathbb{Q}(5^{1/2})$, $\mathcal{W}_1$ admits two small resolutions $\widehat{\mathcal{W}_1^i}, i=1,2$, which are constructed from the contractions of the exceptional divisor of $\widetilde{\mathcal{W}_1}$ in the category of algebraic spaces. The exceptional curve C_i of the small resolution $\widehat{\mathcal{W}_1^i}$ is a conic that is homologous to zero, and it pulls back to a cycle class $[\widetilde{C}_i]$ of $\mathrm{CH}^2(\widetilde{\mathcal{W}_1}\otimes\mathbb{Q}(5^{1/2}))$ that is homologous to zero. The involution of $\mathrm{Gal}(\mathbb{Q}(5^{1/2})/\mathbb{Q})$ induces an involution of $\widetilde{\mathcal{W}_1}\times\mathbb{Q}(5^{1/2})$, which maps $[\widetilde{C}_i]$ to $-[\widetilde{C}_i]$. This property corresponds to the fact that it is the twisted L-function $L(f\otimes(\cdot/5),s)$, not $L(f,s)$, that vanishes at $s=2$. To test Beilinson's conjecture, we have to show $[\widetilde{C}_i]$ is not torsion, but our argument depends on a conjecture of Beilinson and Bloch, and a property of the periods of the threeform defined on the smooth locus of $\mathcal{W}_1$.

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Chapter 1

Introduction

The mirror symmetry conjecture, roughly speaking, predicts the existence of mirror pairs (M, W) of Calabi-Yau threefolds such that the complexified Kähler moduli space of M is isomorphic to an open subset of the complex moduli space of W, which is a neighbourhood of a special boundary point called the large complex structure limit. The complexified Kähler moduli space of M, denoted by $\mathscr{M}_K(M)$, is essentially the space whose points represent the Kähler structures of M, while the complex moduli space of W, denoted by $\mathscr{M}_C(W)$, is the space whose points represent the complex structures of W. The isomorphism between $\mathscr{M}_K(M)$ and a neighbourhood of the large complex structure limit in $\mathscr{M}_C(W)$ is called the mirror map, which is constructed by identifying certain functions on $\mathscr{M}_K(M)$ with those on $\mathscr{M}_C(W)$ [23, 30, 51]. Mirror symmetry is a very powerful tool in the study of algebraic geometry, as it allows one to transfer questions that are very difficult on one side to considerably easier questions on the other side. For example, the Yukawa couplings on the Kähler side which encodes enumerative invariants are very difficult to compute, but on the complex side the computations of Yukawa couplings are considerably much easier. In the past three decades, we have witnessed the great success of mirror symmetry. At the same time a vague question has come to the minds of both physicists and mathematicians:

Can mirror symmetry be applied to the study of arithmetic geometry and number theory?

There is much positive evidence that this should be so, but still we are at a very early stage and very little is known. The main purpose of this thesis is to provide more compelling evidence to support that this is indeed a right question to ask.

In algebraic geometry, given a family of varieties, usually the singular fibers encode a lot of essential information about the properties of the family. In this thesis, we will follow this philosophy and study the singular fibers that occur in the mirror family, which is a deformation of W. More precisely, we will focus on the following two important types of singular fibers that occur in the mirror family,

1. the large complex structure limit.
2. the conifold singularity of one parameter mirror family.

We now give a brief overview of each of these two cases in the introduction.

1.1 The large complex structure limit and mixed motives

Suppose X is an algebraic variety defined over $\mathbb{Q}$, then it defines an analytic variety that will be denoted by $X(\mathbb{C})$. Now let us recall how to construct the singular cohomology groups $H^i(X(\mathbb{C}),\mathbb{Q})$: first construct a complex

$$\mathbf{C}^\bullet_{\mathrm{B}} := \cdots \longrightarrow \mathbf{C}^r_{\mathrm{B}} \longrightarrow \mathbf{C}^{r+1}_{\mathrm{B}} \longrightarrow \mathbf{C}^{r+2}_{\mathrm{B}} \longrightarrow \cdots ,$$

where each object $\mathbf{C}^r_{\mathrm{B}}$ is a rational vector space. Then the singular cohomology group $H^i(X(\mathbb{C}),\mathbb{Q})$ is given by the cohomology of this complex, $H^i(\mathbf{C}^\bullet_{\mathrm{B}})$, which is also called the Betti cohomology of X. The constructions of the algebraic de Rham cohomology and the étale cohomology of X have similar philosophy: first construct a complex of vector spaces over a field and then take the cohomology of this complex. Grothedieck's vision, supplemented by Deligne, Beilinson and many other mathematicians, is that there should exist an abelian category $\mathbf{MM}_{\mathbb{Q}}$ over $\mathbb{Q}$ which acts as a universal cohomology theory of varieties defined over $\mathbb{Q}$. More precisely, for every variety X defined over $\mathbb{Q}$, there is a complex of objects of $\mathbf{MM}_{\mathbb{Q}}$, whose cohomology groups act as the universal cohomology of X, in the sense that all the other cohomology theories of X are the realisations of this master cohomology theory, like the different incarnations of the avatar.

Currently, however, the construction of the abelian category $\mathbf{MM}_{\mathbb{Q}}$ is still far beyond our reach, and the best thing we have is a triangulated category $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q},\mathbb{Q})$ which has almost all the expected properties of the derived category of the conjectured $\mathbf{MM}_{\mathbb{Q}}$, except for those properties that need $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q},\mathbb{Q})$ to be realised as the derived category of an abelian category, such as a motivic t-structure. Intuitively, a triangluated category is a category which have similar properties as the category of complexes of an abelian category (which is modelled on the category of vector spaces).

A t-structure on a triangulated category is a way to realise it as the complex of an abelian category. For a variety X, we have an object $M_{\mathrm{gm}}(X)$ in $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q}, \mathbb{Q})$ which looks like a complex, but we do not know how to take its cohomology. However, inside $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q}, \mathbb{Q})$ there is a triangulated subcategory which does have a t-structure, and the abelian category obtained from it is the category of mixed Tate motives $\mathbf{TM}_{\mathbb{Q}}$. The reader is referred to Appendix A and Appendix B for more detail.

One of our motivations is the question whether mirror symmetry can be applied to the study of mixed motives, and whether the theory of mixed motives can offer new insights into the study of mirror symmetry. In this thesis, we will observe the interesting connections between the limit mixed Hodge structure at the large complex structure limit and the following conjecture.

Period Conjecture: *suppose a mixed Hodge-Tate structure* $\mathbf{P}$ *occurs as a direct summand of* $H^q(\Re(\mathcal{N})), q \in \mathbb{Z}$, *where* $\mathcal{N}$ *is an object of* $\boldsymbol{DM}_{gm}(\mathbb{Q}, \mathbb{Q})$ *and* $\Re$ *is the Hodge realisation functor*

$$\Re : \boldsymbol{DM}_{gm}(\mathbb{Q}, \mathbb{Q}) \to D^b(\boldsymbol{MHS}_{\mathbb{Q}}).$$

See Appendix B for more detail. Suppose further that it is an extension of $\mathbb{Q}(0)$ *by* $\mathbb{Q}(n), n \geq 3$, *then the class of* $\mathbf{P}$ *in*

$$\mathbb{C}/(2\pi i)^n\mathbb{Q} \simeq Ext^1_{\boldsymbol{MHS}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big)$$

(see equation B.18) is the coset of a rational multiple of $\zeta(n)$.

Remark 1.1.1. *If* $\mathcal{N}$ *is isomorphic to an object of* $\boldsymbol{DTM}_{\mathbb{Q}}$ *(the derived category of* $\boldsymbol{TM}_{\mathbb{Q}}$*), then it follows from the computations in algebraic* K*-theory that this conjecture is true. See Appendix B for more detail.*

1.1.1 The limit MHS at the large complex structure limit

For the mirror pair (M, W) of Calabi-Yau threefolds, the large complex structure limit of the mirror family (a deformation of W) is a special point in the base space where the irreducible components of the discriminant locus crossing normally, and moreover the monodromy matrices satisfy certain properties. When the base space of the mirror family is one dimensional, i.e. $h^{12}(W) = 1$, it just means that the monodromy around the large complex structure limit is maximally unipotent [23, 30]. In this case we have shown in [72] that the limit mixed Hodge structure on $H^3(W,\mathbb{Q})$ at the large complex structure limit splits into

$$\mathbf{M} \oplus \mathbb{Q}(-1) \oplus \mathbb{Q}(-2),$$

where $\mathbf{M}$ is a mixed Hodge-Tate object that is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. Therefore the dual of $\mathbf{M}$, denoted by $\mathbf{M}^\vee$, is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$. The extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ form a group which is given by

$$\mathrm{Ext}^1_{\mathbf{MHS}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(3)\big) = \mathbb{C}/(2\pi i)^3\mathbb{Q}.$$

The image of $\mathbf{M}^\vee$ in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is shown to be a rational multiple of $(2\pi i)^3 Y_{000}$, where Y_{000} is the constant term of the perturbative part of the prepotential $\mathcal{F}$ on the complexified Kähler moduli space of M.

Remark 1.1.2. *For the mirror pair (M,W), the prepotential $\mathcal{F}$ on the complexified Kähler moduli space of M has an expansion of the form*

$$\mathcal{F} = -\frac{1}{6}\sum Y_{ijk}t^i t^j t^k - \frac{1}{2}\sum Y_{0ij}t^i t^j - \frac{1}{2}\sum Y_{00i}t^i - \frac{1}{6}Y_{000} + \mathcal{F}^{np},$$

where $\{t_i\}$ are the flat coordinates of $\mathscr{M}_K(M)$, and $\mathcal{F}^{np}$ is the non-perturbative instanton correction.

So it is a natural question to ask what is the limit mixed Hodge structure on $H^3(W,\mathbb{Q})$ at the large complex structure limit of a general mirror pair (M,W), and we have the following conjecture.

Conjecture LMHS: *for a general mirror pair (M,W), the limit mixed Hodge structure on $H^3(W,\mathbb{Q})$ at the large complex structure limit splits into*

$$\mathbf{M} \oplus \mathbb{Q}(-1)^{h^{12}(W)} \oplus \mathbb{Q}(-2)^{h^{12}(W)},$$

where $\mathbf{M}$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. Therefore its dual $\mathbf{M}^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$, whose image in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is the coset of a rational multiple of $(2\pi i)^3 Y_{000}$, where Y_{000} is the constant term of the perturbative part of the prepotential $\mathcal{F}$ on the complexified Kähler moduli space of M.

Suppose now the mirror family is defined over $\mathbb{Q}$ and the large complex structure limit is a rational point, then from Ayoub's motivic nearby cycle functor, there exists a limit mixed motive $\mathcal{M}$ at the large complex structure limit, which is an object of $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q},\mathbb{Q})$. Its Hodge realisation $\mathfrak{R}(\mathcal{M})$ is an object of the bounded derived category of the mixed Hodge structures over $\mathbb{Q}$, $D^b(\mathbf{MHS}_{\mathbb{Q}})$. From **Conjecture**

C.2.1, $\mathfrak{R}(\mathcal{M})$ is expected to compute the limit mixed Hodge structures of W, in particular $H^3(\mathfrak{R}(\mathcal{M}))$ is the limit mixed Hodge structure on $H^3(W, \mathbb{Q})$ at the large complex structure limit. On the other hand, in all mirror pairs where the coefficient Y_{000} has been computed, it is always of the form

$$Y_{000} = -3\,\chi(M)\,\frac{\zeta(3)}{(2\pi i)^3} + r,\ r \in \mathbb{Q}. \tag{1.1}$$

Conclusion: assuming mirror symmetry conjecture, **Conjecture LMHS** and **Conjecture** C.2.1, then the limit mixed motive $\mathcal{M}$ is an interesting example of the **Period Conjecture**. On the other hand, if we assume mirror symmetry conjecture, **Conjecture LMHS**, **Conjecture** C.2.1 and **Period Conjecture**, then the coset of the constant term Y_{000} in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is the coset of a rational multiple of $\zeta(3)$, which provides a motivic explanation of the occurrence of $\zeta(3)$ in Y_{000}.

1.1.2 Mirror symmetry of a self-mirror Calabi-Yau threefold

The proof of **Conjecture LMHS** is potentially very hard, so instead we will study an illuminating example that supports this conjecture. More precisely we will study the mirror symmetry a self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$ with Hodge numbers

$$h^{11}(X_{\mathfrak{G}}) = \dim H^{11}(X_{\mathfrak{G}}) = 2, \quad h^{12}(X_{\mathfrak{G}}) = \dim H^{12}(X_{\mathfrak{G}}) = 2.$$

After that we will compute the limit mixed Hodge structure on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit, and show it satisfies the predictions of **Conjecture LMHS**. The construction of $X_{\mathfrak{G}}$ is very interesting in its own right. We start with a complete intersection Calabi-Yau threefold Y, which is a subvariety of $(\mathbb{P}^2_{\mathbb{C}})^4$ cut out by five multi-homogeneous polynomials. The threefold Y admits a free action by a finite group $\mathfrak{G}$ (the dicyclic group of order 12), and the quotient $Y_{\mathfrak{G}} := Y/\mathfrak{G}$ is a Calabi-Yau threefold with Hodge numbers

$$h^{11}(Y_{\mathfrak{G}}) = 1,\ h^{12}(Y_{\mathfrak{G}}) = 4.$$

The threefold $Y_{\mathfrak{G}}$ has a deformation given by

$$\pi_{Y_{\mathfrak{G}}} : \mathcal{Y}_{\mathfrak{G}} \to \mathbb{P}^4_{\mathbb{C}}.$$

Moreover, there exists a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{Y}^{ss}_{\mathfrak{G}} \arrow[r, hook] \arrow[d, "\pi^{ss}_{Y_{\mathfrak{G}}}"] & \mathcal{Y}^{s}_{\mathfrak{G}} \arrow[r, hook] \arrow[d, "\pi^{s}_{Y_{\mathfrak{G}}}"] & \mathcal{Y}_{\mathfrak{G}} \arrow[d, "\pi_{Y_{\mathfrak{G}}}"] \\
\mathbb{P}^1_{\mathbb{C}} \arrow[r, hook] & \mathbb{P}^2_{\mathbb{C}} \arrow[r, hook] & \mathbb{P}^4_{\mathbb{C}}
\end{tikzcd}$$

the terms of which will be explained now. The $\mathbb{P}^2_{\mathbb{C}}$ is part of the discriminant locus of $\pi_{Y_{\mathfrak{G}}}$, and over it we have a subfamily $\pi^s_{Y_{\mathfrak{G}}}$. A general fiber of $\pi^s_{Y_{\mathfrak{G}}}$ is denoted by $Y^s_{\mathfrak{G}}$, which is a singular threefold with singularity that consists of three double points. While inside this $\mathbb{P}^2_{\mathbb{C}}$, there is a one dimensional discriminant locus $\mathbb{P}^1_{\mathbb{C}}$, over which we have a subfamily $\pi^{ss}_{Y_{\mathfrak{G}}}$. A general fiber of $\pi^{ss}_{Y_{\mathfrak{G}}}$, denoted by $Y^{ss}_{\mathfrak{G}}$, is a singular threefold with singularity that consists of 6 double points. The 6 double points of $Y^{ss}_{\mathfrak{G}}$ can be resolved by a projective small resolution, and after resolution we get a Calabi-Yau threefold $Y^{\vee}_{\mathfrak{G}}$ with Hodge numbers

$$h^{11}(Y^{\vee}_{\mathfrak{G}}) = 4,\ h^{12}(Y^{\vee}_{\mathfrak{G}}) = 1$$

and furthermore $Y^{\vee}_{\mathfrak{G}}$ is the mirror threefold of $Y_{\mathfrak{G}}$. The three double points of $Y^s_{\mathfrak{G}}$ can also be resolved by a projective small resolution, and after resolution we get a Calabi-Yau threefold $X_{\mathfrak{G}}$ with Hodge numbers $h^{11} = h^{12} = 2$. The intuitive observation is that the conifold $Y^s_{\mathfrak{G}}$ falls in the middle ground between $Y_{\mathfrak{G}}$ and $Y^{ss}_{\mathfrak{G}}$, therefore it is expected that its resolution $X_{\mathfrak{G}}$ is self-mirror. What's more, the mirror family of $X_{\mathfrak{G}}$ is defined over $\mathbb{Q}$.

The mirror symmetry of $X_{\mathfrak{G}}$ will be studied carefully in this thesis. We will construct two divisors ϵ and δ, which form a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsion) and more importantly, they will be shown to lie in the closure of the Kähler cone of $X_{\mathfrak{G}}$. Hence they define a framing of $X_{\mathfrak{G}}$ and the complexified Kähler moduli space of $X_{\mathfrak{G}}$ is given by $\{t_1\,\epsilon + t_2\,\delta : t_i \in \mathbb{H}\}$, where $\mathbb{H}$ is the upper half plane of $\mathbb{C}$. The rank of the homology group $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ is also two, and we will construct two one-dimensional algebraic cycles Γ_ϵ and Γ_δ which form a basis of $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). Furthermore $\{\Gamma_\epsilon, \Gamma_\delta\}$ is dual to $\{\epsilon, \delta\}$, and this will be important when we study the Gromov-Witten invariants of $X_{\mathfrak{G}}$.

From toric geometry, there is a threeform $\Omega_{\mathfrak{G}}$ defined on the mirror family of $X_{\mathfrak{G}}$, which is defined over $\mathbb{Q}$. By integrating $\Omega_{\mathfrak{G}}$ over an explicit cycle of $H_3(X_{\mathfrak{G}}, \mathbb{Z})$, we obtain a period ϖ_0 that is called the fundamental period of $X_{\mathfrak{G}}$. The other five linearly independent periods of $X_{\mathfrak{G}}$ will be found by solving Picard-Fuchs equations, but first we need to find the Picard-Fuchs operators of $X_{\mathfrak{G}}$. For the threefold $X_{\mathfrak{G}}$, the complexity of Griffiths-Dwork method is too great for a computer, therefore we have to have recourse to the method introduced in the original paper [23]. Even though it is not mathematically rigorous, but it nevertheless works. More precisely, we look for partial differential operators $\mathcal{L}$ such that

$$\mathcal{L}\,\varpi_0 = 0,$$

and since we have got a power series expansion of ϖ_0, we can solve this equation to find $\mathcal{L}$. This method can be programmed by Mathematica, but the drawback is that from a mathematical point of view we have not proved they are indeed Picard-Fuchs operators of $X_{\mathfrak{G}}$. Assuming they are, we face the challenging task of solving these equations, which are PDEs in two variables, and with much labour, we solve these PDEs by the Frobenius method. Suppose (φ, ν) form a local coordinate of the base space of the mirror family, then there exist canonical solutions of the Picard-Fuchs equations of $X_{\mathfrak{G}}$ given by

$$\begin{aligned}
\varpi_0 &= h_0 \\
\varpi_1 &= \frac{1}{2\pi i}\left(\varpi_0 \log\varphi + h_1\right) \\
\varpi_2 &= \frac{1}{(2\pi i)^2}\left(\varpi_0 \log^2\varphi + 2\,h_1 \log\varphi + h_2\right) \\
\varpi_3 &= \frac{1}{(2\pi i)^3}\left(\varpi_0 \log^3\!\varphi + (3\,h_1 + g_4)\log^2\!\varphi + (3\,h_2 + 2\,g_5)\log\varphi + h_3\right) + \frac{1}{2\pi i}\,\varpi_2 \log\nu \\
\varpi_4 &= \frac{1}{2\pi i}\left(\varpi_0 \log\nu + g_4\right) \\
\varpi_5 &= \frac{1}{(2\pi i)^2}\left(\varpi_0 \log\varphi \log\nu + g_4 \log\varphi + h_1 \log\nu + g_5\right) \\
&= \frac{1}{2\pi i}\,\varpi_1 \log\nu + \frac{1}{(2\pi i)^2}\left(g_4 \log\phi + g_5\right) \\
&= \frac{1}{2\pi i}\,\varpi_4 \log\phi + \frac{1}{(2\pi i)^2}\left(h_1 \log\nu + g_5\right)
\end{aligned}$$

where h_0, h_1, h_2, h_3, g_4 and g_5 are power series in φ and ν that satisfy the conditions

$$h_0(0,0) = 1; \quad h_i(0,0) = 0,\ \forall\ i > 0\ ;\ g_j\,(0,0) = 0, \forall\ j.$$

The coefficients of the power series expansions of h_i and g_j are determined uniquely by recursion relations that follow from the Picard-Fuchs equations, and using Mathematica, we have computed the first ten thousand terms of each of these functions.

The monodromy of the above canonical periods can be easily found from their form, from which we find that $\varphi = \nu = 0$ is the large complex structure limit of $X_{\mathfrak{G}}$. Moreover the mirror map is given by the identifications

$$t_1 \equiv \varpi_1/\varpi_0, \quad t_2 \equiv \varpi_4/\varpi_0.$$

We will compute the Yukawa couplings of $X_{\mathfrak{G}}$ and their instanton expansions, from which we find the Gromov-Witten invariants of $X_{\mathfrak{G}}$. It is very interesting that the mirror symmetry of $X_{\mathfrak{G}}$ is closely related to the mirror symmetry of the mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$, which will be discussed later in Section 7.7.

Based on the study of the mirror symmetry of $X_{\mathfrak{G}}$, we will compute the limit mixed Hodge structure on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit and show it satisfies the predictions of **Conjecture LMHS**. The method used in the computations of the limit mixed Hodge structure is constructive and it works even for a general mirror pair. Furthermore, this method sheds light on the nature of **Conjecture LMHS**.

1.2 The conifold transition and Beilinson's conjecture

The study of L-functions has been one of the central areas in number theory, while this study is very substantial, many parts are still largely conjectural. There are many important conjectures regarding the behaviour of L-functions. Among these are Beilinson's conjectures on the values of L-functions at integral points, which can be considered as far-reaching generalisations of the Birch and Swinnerton-Dyer conjecture on the L-functions of elliptic curves. Beilinson's conjectures have also been crucial driving forces in the developments of many areas, including mixed motives, algebraic K-theory etc. However Beilinson's conjectures still lie far beyond our reach, and one of our motivations is whether mirror symmetry could be applied to the study of them. In this thesis, we will observe interesting connections between the conifold transition and Beilinson's conjectures. Ever since the paper [23], conifold transition has attracted the interests of both physicists and mathematicians. In recent years, Candelas, de la Ossa, Rodriguez-Villegas, van Staten and others have studied the zeta functions of conifolds in one-parameter mirror families, and they possess very interesting properties. Here, we will focus on the conifold of the mirror family of the quintic, but our constructions work more generally for the conifolds of any other one-parameter mirror family.

The mirror family of quintic, denoted by $\mathcal{W} \to \mathbb{P}^1_{\mathbb{Q}}$, is a fibration defined over $\mathbb{Q}$. Following conventions of [23, 30], the conifold in this family is the singular fiber of $\mathcal{W}$ over the rational point $1 \in \mathbb{P}^1_{\mathbb{Q}}$, hence it will be denoted by $\mathcal{W}_1$, while $1 \in \mathbb{P}^1_{\mathbb{Q}}$ will be called the conifold point. We will call $\mathcal{W}_1$ the **quintic mirror conifold**, which is a singular variety with singularity that consists of a rational double point. The local structure of the double point of $\mathcal{W}_1$ is isomorphic to that of the double point O of the hypersurface in $\mathbb{A}^4_{\mathbb{Q}}$ defined by the rational polynomial

$$u_1^2 + 3\,u_2^2 + \frac{2}{3}\,u_3^2 + \frac{5}{2}\,u_4^2 + g(u_1, u_2, u_3, u_4) = 0, \tag{1.2}$$

where $\{u_i\}_{i=1}^4$ are the affine coordinates of $\mathbb{A}^4_{\mathbb{Q}}$ and the monomials of g have degrees ≥ 3. The mirror family of the quintic is constructed as a family of hypersurfaces in a toric fourfold, therefore there is a natural choice of an integral model of $\mathcal{W}_1$ which only has bad reduction at $p = 5$. In the pioneering work of [24], the zeta functions of $\mathcal{W}_1$ was computed which are given by

$$\zeta_{\mathcal{W}_1}(p, T) = \frac{(1 - \chi(p)\,p\,T)\,(1 - a_f(p)\,T + p^3\,T^2)}{(1 - T)\,(1 - p\,T)^{101}\,(1 - p^2\,T)^{101}\,(1 - p^3\,T)},\ \ p \neq 5, \tag{1.3}$$

where χ is the real Dirichlet character $(\cdot/5)$ and $a_f(p)$ is the p-th coefficient in the q-expansion of a newform f with weight 4 and level 25 given by

$$f = \eta(5\tau)^4 \Big[\eta(\tau)^4 + 5\,\eta(\tau)^3\,\eta(25\tau) + 20\,\eta(\tau)^2\,\eta(25\tau)^2 + 25\,\eta(\tau)\,\eta(25\tau)^3 + 25\,\eta(25\tau)^4\Big]. \tag{1.4}$$

Here $\eta(\tau)$ is Dedekind's η-function

$$\eta(\tau) = \exp\big(\frac{2\pi i\,\tau}{24}\big) \prod_{n=1}^{\infty} \big(1 - \exp(2\pi i\,\tau)\big),\ \text{with}\ q := \exp(2\pi i\,\tau).$$

1.2.1 The zeta functions of the blow up of the quintic mirror conifold

It is very interesting that the numerator of 1.3 factors as the product of a linear polynomial and a quadratic polynomial, while the quadratic polynomial gains modularity. But why does the Dirichlet character χ appear in the linear polynomial and where does modularity come from? The smooth models of a singular variety usually contain lots of essential information, so let us first look at the blow up of the quintic mirror conifold.

Let $\widetilde{\mathcal{W}_1}$ be the blow up of $\mathcal{W}_1$ at its double point, which is a smooth variety defined over $\mathbb{Q}$. The exceptional divisor D^1 of $\widetilde{\mathcal{W}_1}$ is isomorphic to the quadratic surface of $\mathbb{P}^3_{\mathbb{Q}}$ defined by the equation

$$U_1^2 + 3\,U_2^2 + \frac{2}{3}\,U_3^2 + \frac{5}{2}\,U_4^2 = 0, \tag{1.5}$$

where $(U_1, \cdots, U_4)$ are the projective coordinates of $\mathbb{P}^3_{\mathbb{Q}}$. The determinant of the quadratic polynomial in equation 1.5 is 5, and the zeta functions of the pure motive $h^3(D^1)$ are

$$(1 - p\,T)(1 - \chi(p)\,p\,T),\ p \neq 5.$$

The blow up $\widetilde{\mathcal{W}_1}$ induces a homomorphism of étale cohomology groups

$$H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

which will be shown to be surjective with kernel that is isomorphic to $\mathbb{Q}_\ell(0) \otimes \chi$. The smooth variety $\widetilde{\mathcal{W}_1}$ also only has bad reduction at $p = 5$. The zeta functions of the pure motive $h^3(\widetilde{\mathcal{W}_1})$ are given by

$$1 - a_f(p)\, T + p^3\, T^2, \;\; p \neq 5, \tag{1.6}$$

hence the L-function $L(h^3(\widetilde{\mathcal{W}_1}), s)$ is just the L-function associated to the modular form f, i.e. $L(f, s)$. However it turns out to be more natural to consider everything over the quadratic field $F := \mathbb{Q}(5^{1/2})$. By extension of base field, $\widetilde{\mathcal{W}_1}$ defines a smooth projective variety

$$\widetilde{\mathcal{W}}_{1,F} := \widetilde{\mathcal{W}_1} \times_{\mathbb{Q}} F,$$

over F, then by restriction of base field, $\widetilde{\mathcal{W}}_{1,F}$ defines a variety $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ over $\mathbb{Q}$ by

$$\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} : \widetilde{\mathcal{W}}_{1,F} \to \operatorname{Spec} F \to \operatorname{Spec} \mathbb{Q}.$$

The involution ι of $\operatorname{Gal}(F/\mathbb{Q})$ which maps $5^{1/2}$ to $-5^{1/2}$ defines an involution of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ (as a $\mathbb{Q}$-variety), which further gives two idempotent correspondences

$$(1 + \iota)/2, \;\; (1 - \iota)/2.$$

The two correspondences decompose the pure motive $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ into the direct sum

$$\begin{aligned} h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}) &= \big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), (1+\iota)/2\big) \oplus \big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), (1-\iota)/2\big) \\ &= h^3(\widetilde{\mathcal{W}_1}) \oplus \big(h^3(\widetilde{\mathcal{W}_1}) \otimes \chi\big), \end{aligned}$$

where χ also means the pure motive associated to the Dirichlet character χ, hence the L-function of $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ is given by

$$L\big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), s\big) = L(f, s) \cdot L(f \otimes \chi, s).$$

1.2.2 A test of Beilinson's conjecture

Beilinson's conjecture on central values of L-functions predicts that

$$\dim_{\mathbb{Q}} \mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0 \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{ord}_{s=2} L\big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), s\big). \tag{1.7}$$

We will prove that $L(f \otimes \chi, s)$ vanishes at $s = 2$, which immediately shows that the right hand side is ≥ 1. To test Beilinson's conjecture, we need to show that the left hand side is also ≥ 1, i.e. we want to construct a non-torsion codimension-2 algebraic cycle that is homologous to zero.

It has been noticed in [22] that the exceptional curve of a small resolution of $\mathcal{W}_1(\mathbb{C})$, the singular complex analytical variety defined by $\mathcal{W}_1$, is homologous to zero. This property is already very interesting from the physics point of view for its potential connection to supersymmetry breaking. But we are studying the arithmetic geometry of $\mathcal{W}_1$, hence we would want to construct the **algebraic** small resolutions. However since a blow up is always projective, this property eliminates the possibility of constructing an algebraic small resolution of $\mathcal{W}_1$ by blowing up a smooth surface that contains the double point.

The strategy to construct algebraic small resolutions of $\mathcal{W}_1$ is to contract the exceptional divisor D^1 of $\widetilde{\mathcal{W}_1}$ to its Hilbert scheme in the category of algebraic spaces. The Hilbert scheme of lines in D^1 has two disconnected components $C_i, i = 1, 2$, both of which are conics defined over $\mathbb{Q}(5^{1/2})$. Moreover, $D^1_F := D^1 \times_{\mathbb{Q}} F$ is fibered over C_i, and the fibration morphism is given by

$$p_i : D^1_F \to C_i,\ i = 1, 2,$$

whose fibers are conics. From Artin's work [2], there exist algebraic spaces $\widehat{\mathcal{W}}^i_1$ and morphisms λ_i

$$\lambda_i : \widetilde{\mathcal{W}}_{1,F} \to \widehat{\mathcal{W}}^i_1,\ i = 1, 2,$$

such that the restriction of λ_i to D^1_F is p_i, while being an isomorphism outside D^1_F. We can further contract the conic C_i to the double point of $\mathcal{W}_{1,F}$, therefore the algebraic space $\widehat{\mathcal{W}}^i_1$ is indeed an algebraic small resolution of $\mathcal{W}_1$. The analytification of C_i is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ which is homologous to zero.

By restriction of base field, $\widehat{\mathcal{W}}^i_1$ defines an algebraic space over $\mathbb{Q}$

$$\widehat{\mathcal{W}}^i_{1/\mathbb{Q}} : \widehat{\mathcal{W}}^i_1 \to F \to \mathbb{Q},$$

and the involution ι of $\mathrm{Gal}(F/\mathbb{Q})$ yields a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \arrow[r, "\iota"] \arrow[d, "\lambda_1"] & \widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \arrow[d, "\lambda_2"] \\
\widehat{\mathcal{W}}^1_{1/\mathbb{Q}} \arrow[r, "\iota"] & \widehat{\mathcal{W}}^2_{1/\mathbb{Q}}
\end{tikzcd}.$$

Under the morphism λ_i, the cycle class $[C_i]$ pulls back to a cycle class in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})$, and by intersection theory [45], the pull back is represented by an algebraic cycle $\widetilde{C}_i$

with support that lies in D_F^1. From the construction of the cycle class map, $[\widetilde{C}_i]$ is also homologous to zero, hence it lies in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0$. The Chow group is invariant under restriction of base field, and there is a canonical isomorphism

$$\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0 = \mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0,$$

therefore $[\widetilde{C}_i]$ also lies in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0$. By explicit computation, we will show that $[\widetilde{C}_1] = -[\widetilde{C}_2]$, while the involution ι maps $[C_1]$ to $-[C_1]$, so we have

$$\frac{1}{2}(1+\iota)([\widetilde{C}_1]) = 0, \quad \frac{1}{2}(1-\iota)([\widetilde{C}_1]) = [\widetilde{C}_1].$$

Thus in a non-rigorous sense, $[C_i]$ is an algebraic cycle associated to the pure motive $h^3(\widetilde{\mathcal{W}}_1) \otimes \chi$, and this corresponds to the fact that it is $L(f \otimes \chi, s)$, not $L(f, s)$, that vanishes at $s = 2$.

But we still do not know whether the cycle class $[\widetilde{C}_i]$ is non-torsion, i.e. whether it is nonzero in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. To proceed, we will need a conjecture of Beilinson and Bloch [13, 14]: the general Abel-Jacobi map is injective up to torsions. From this conjecture, we will show that $[\widetilde{C}_i]$ is non-torsion if and only if the following short exact sequence does not split

$$0 \longrightarrow \ker \beta^* \longrightarrow H^3\big(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}\big) \xrightarrow{\beta^*} H^3\big(\widetilde{\mathcal{W}}_1(\mathbb{C}), \mathbb{Q}\big) \longrightarrow 0, \tag{1.8}$$

where $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$ means the mixed Hodge structure on the singular cohomology group $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$, etc. So we need to find a way to compute the mixed Hodge structure on $\mathcal{W}_1$ and the short exact sequence 1.8 explicitly. Our strategy is to compute the limit MHS of the mirror family of the quintic at $\mathcal{W}_1$, then from local invariant cycle theorem we will show that the MHS on $\mathcal{W}_1$ is isomorphic to the sub-MHS of the limit MHS that is the kernel of the monodromy operator [88, 95].

Conclusion: there exist three cycles $A_1|_{\psi=1}$, $B_1|_{\psi=1}$ and $B_2|_{\psi=1}$ that form a basis of $H_3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$. The short exact sequence 1.8 does not split if and only if

$$\int_{B_2|_{\psi=1}} \Omega_{\mathcal{W}_1} \neq r \int_{A_1|_{\psi=1}} \Omega_{\mathcal{W}_1} + r' \int_{B_1|_{\psi=1}} \Omega_{\mathcal{W}_1}, \quad \forall\, r,\, r' \in \mathbb{Q},$$

where $\Omega_{\mathcal{W}_1}$ is a nowhere vanishing threeform defined on the smooth locus of $\mathcal{W}_1$. This show partial evidence that the extension 1.8 is non-trivial.

1.3 Layout of the thesis

The structure of the body of this thesis is as follows:

- Chapter 2 discusses briefly Beilinson's conjecture on the central values of L-functions.
- Chapter 3 discusses Schmid's and Steenbrink's constructions of limit mixed Hodge structures.
- Chapter 4 is a brief overview of mirror symmetry.
- Chapter 5 discusses the construction of the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$.
- Chapter 6 describes the construction of two divisors $\{\epsilon, \delta\}$ of $X_{\mathfrak{G}}$, which lies in the closure of the Kähler cone of $X_{\mathfrak{G}}$ and form a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). It also computes the triple intersection products of the two divisors, which are important for the computations of Yukawa couplings. Furthermore, it describes the construction of two one dimensional curves which form a basis of $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions).
- Chapter 7 computes the canonical periods of $X_{\mathfrak{G}}$ by solving Picard-Fuchs equations. We will find the large complex structure limit of $X_{\mathfrak{G}}$ and compute the instanton expansion of its Yukawa couplings, from which we find the Gromov-Witten invariants of $X_{\mathfrak{G}}$.
- Chapter 8 is concerned with the computations of the limit mixed Hodge structure on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit. We will show it satisfies all the predictions of **Conjecture LMHS**.
- Chapter 9 is concerned with the construction of the mirror family of the quintic Calabi-Yau threefold.
- Chapter 10 studies the small resolutions of the conifold in the mirror family of the quintic Calabi-Yau threefold (which is called quintic mirror conifold).
- Chapter 11 studies the étale cohomology groups of the blow up and the small resolutions of the quintic mirror conifold. It also studies the extensions induced by the exceptional curves of the small resolutions.

- Chapter 12 studies the interesting connections between the blow up of the quintic mirror conifold and Beilinson's conjecture on the central values of L-functions.

- Chapter 13 studies the limit mixed Hodge structure of the mirror family of the quintic at the quintic mirror conifold. It also studies whether the exceptional curves of the small resolutions induce non-trivial extensions.

Chapter 2

Beilinson's Conjecture on the central values of L-functions

In this chapter, we will introduce Beilinson's conjecture on the central values of L-functions. The reader to whom this is familar can skip this chapter completely. We will follow the papers [83] and [89], which are also strongly recommended. This chapter also includes some elementary arithmetic geometry, which prepares the background for later discussions. The structure of this chapter is as follows:

- Section 2.1 is a short discussion of some elementary properties of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that will be needed in the definition of the L-function of a variety.

- Section 2.2 contains a brief discussion of pure motives through their realisations. We will also briefly talk about three important Weil cohomology theories of varieties: Betti cohomology, de Rham cohomology and étale cohomology.

- Section 2.3 introduces the L-function associated to a pure motive. It also briefly discusses the expected (conjectured) properties of L-functions.

- Section 2.4 discusses the order of vanishing of L-functions at integer points. It also introduces Deligne's period map and Deligne's period.

- Finally, Section 2.5 discusses Beilinson's conjecture on the central values of L-functions.

2.1 The absolute Galois group Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)

In this section, we will briefly talk about some elementary properties of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, all of which can be found from [84, 93, 97]. Suppose K is a number field, i.e. it is a finite extension of the field of rational numbers $\mathbb{Q}$. An element of K is called an algebraic integer if it is a solution to an integral monic polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \text{ with } a_i \in \mathbb{Z}. \tag{2.1}$$

The set of algebraic integers forms a subring of K that will be denoted by $\mathcal{O}_K$, and it includes $\mathbb{Z}$ as a subring. The ring $\mathcal{O}_K$ is a Dedekind domain, i.e. it is an integral domain in which every non-zero proper ideal has a unique factorisation as a product of prime ideals. In particular the principal ideal $(p) \subset \mathcal{O}_K$, generated by a prime number $p \in \mathbb{Z}$, has a factorisation

$$(p) = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}, \text{ with } \mathfrak{P}_i \neq \mathfrak{P}_j \text{ when } i \neq j, \tag{2.2}$$

where each e_i is ≥ 1 and are called the ramification indices. Every nonzero prime ideal of $\mathcal{O}_K$ is maximal, therefore $\mathcal{O}_K/\mathfrak{P}_i$ is a field that is a finite extension of the finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ [93]. The degree of this extension is denoted by

$$f(\mathfrak{P}_i/p) := [\mathcal{O}_K/\mathfrak{P}_i : \mathbb{F}_p], \tag{2.3}$$

and it is called the residue class degree. Furthermore we have a relation

$$\sum_{i=1}^{g} e_i\, f(\mathfrak{P}_i/p) = [K : \mathbb{Q}] \ . \tag{2.4}$$

If any of the numbers e_i is > 1, we say p is ramified in $\mathcal{O}_K$, otherwise we say p is unramified. If a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ occurs in the factorisation 2.2, we say $\mathfrak{P}$ divides p and write $e(\mathfrak{P}/p)$ (resp. $f(\mathfrak{P}/p)$) as its ramification index (resp. residue class degree $[\mathcal{O}_K/\mathfrak{P} : \mathbb{F}_p]$). The prime number p is said to split in $\mathcal{O}_K$ if $e_i = f(\mathfrak{P}_i/p) = 1$ for every i in the factorisation 2.2, while p is said to be inert in $\mathcal{O}_K$ if (p) is a prime ideal of $\mathcal{O}_K$.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $\overline{\mathbb{Z}}$ be the subring of $\overline{\mathbb{Q}}$ that consists of algebraic integers, then the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the group of automorphisms of $\overline{\mathbb{Q}}$. It is a profinite group that is also given by the following inverse limit

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \varprojlim_{L} \mathrm{Gal}(L/\mathbb{Q}), \tag{2.5}$$

where L runs over all the finite Galois extensions of $\mathbb{Q}$. This inverse limit endows $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with a topology called the Krull topology, which is compact, Hausdorff and totally disconnected. For a prime number $p \in \mathbb{Z}$, let $\mathbb{Q}_p$ be the field of p-adic numbers, whose algebraic closure will be denoted by $\overline{\mathbb{Q}}_p$. From the construction of completion, an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is determined by the choice of a prime ideal $\overline{p} \subset \overline{\mathbb{Z}}$ such that $\overline{p} \cap \mathbb{Z} = (p)$ [93]. The elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that admit an extension to a continuous automorphism of $\overline{\mathbb{Q}}_p$ form a subgroup D_p that is called decomposition group, which is isomorphic to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The definition of D_p depends on the chosen embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, or equivalently the chosen prime ideal $\overline{p}$, hence it is only well-defined modulo a conjugation induced by an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The quotient of $\overline{\mathbb{Z}}_p$ (the ring of integers of $\overline{\mathbb{Q}}_p$) by its maximal ideal is isomorphic to $\overline{\mathbb{F}}_p$ (the algebraic closure of $\mathbb{F}_p$) [93]. Moreover we have a short exact sequence [93]

$$0 \longrightarrow I_p \longrightarrow D_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 0 \ , \tag{2.6}$$

where I_p is called the inertia group. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is a profinite group

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p), \tag{2.7}$$

which is topologically generated by the Frobenius map

$$\phi_p : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p, \ x \mapsto x^p. \tag{2.8}$$

The map ϕ_p is also called the arithmetic Frobenius element, while its inverse ϕ_p^{-1} is called the geometric Frobenius element.

Remark 2.1.1. *For a dicussion of the absolute Galois group Gal($\overline{K}/K$) and its decomposition groups, inertia groups, Frobenius elements, etc, the reader is referred to the text [93].*

2.2 Pure motives

In this section, we will talk about pure motives through their realisations, and a more rigorous treatment may be found in Appendix A. A pure motive is somewhat like an avatar, which only influences the human world through its incarnations. As human, we do not really need to know what an avatar is, instead we only need to know its incarnations. Suppose X is a smooth projective variety defined over a number field K, and let M be the following pure motive associated to X

$$M := h^i(X)(n), \ i, n \in \mathbb{Z}, \tag{2.9}$$

which has three important realisations [67]:

1. The Betti realisation. Suppose $\sigma : K \hookrightarrow \mathbb{C}$ is an embedding of K into $\mathbb{C}$, then by extension of the base field X defines a variety over $\mathbb{C}$, namely $X \times_\sigma \mathbb{C}$. The $\mathbb{C}$-valued points (classical points) of $X \times_\sigma \mathbb{C}$, denoted by $(X \times_\sigma \mathbb{C})(\mathbb{C})$, form a smooth projective complex manifold. The Betti realisation of M associated to the embedding σ is the singular cohomology group

   $$M_{B,\sigma} := H^i\big((X \times_\sigma \mathbb{C})(\mathbb{C}),\ \mathbb{Q}(n)\big) = H^i\big((X \times_\sigma \mathbb{C})(\mathbb{C}),\ \mathbb{Q}\big) \otimes \mathbb{Q}(n), \tag{2.10}$$

   where $\mathbb{Q}(n)$ is $(2\pi i)^n\,\mathbb{Q}$ which has a pure Hodge structure of Hodge type $(-n, -n)$ [85]. From Hodge theory, there exists a pure Hodge structure on $M_{B,\sigma}$ with weight $w := i - 2\,n$, i.e. it has a Hodge decomposition

   $$M_{B,\sigma} \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=w}\, H^{p,q}, \quad h^{p,q} := \dim_{\mathbb{C}} H^{p,q}. \tag{2.11}$$

   Moreover if σ is real, i.e. the image of σ is contained in $\mathbb{R}$, then the complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the points of $(X \times_\sigma \mathbb{C})(\mathbb{C})$, which induces an involution c^* of $M_{B,\sigma}$. Define ϕ_σ to be the involution on $M_{B,\sigma}$ induced by the action of c on both the points $(X \times_\sigma \mathbb{C})(\mathbb{C})$ and the coefficient ring $\mathbb{Q}(n)$. Then the conjugate-linear involution $\phi_\sigma \otimes c$ preserves the Hodge decomposition of $M_{B,\sigma} \otimes \mathbb{C}$, i.e. it sends $H^{p,q}$ to $H^{p,q}$.

2. The de Rham realisation. Over the variety X, there exists a complex of sheaves of algebraic differential forms [98]

   $$\Omega^*_{X/K} : 0 \to \mathcal{O}_{X/K} \xrightarrow{d} \Omega^1_{X/K} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim(X)}_{X/K} \to 0. \tag{2.12}$$

   However in order to define a 'reasonable' cohomology theory, we have to choose an injective resolution $\Omega^*_{X/K} \to I^*$ in the category of the complex of sheaves on X, then the de Rham cohomology of X is defined to be [100]

   $$\mathbb{H}^i(X_{\mathrm{Zar}}, \Omega^*_{X/K}) = H^i(\Gamma(X, I^*)), \tag{2.13}$$

   which is also called the hypercohomology of $\Omega^*_{X/K}$. Here X_{Zar} means the Zariski topology on X. The de Rham realisation of M is the hypercohomology of the shifted complex $\Omega^*_{X/K}[n]$

   $$M_{\mathrm{dR}} := \mathbb{H}^i(X_{\mathrm{Zar}}, \Omega^*_{X/K}[n]), \text{ where } (\Omega^*_{X/K}[n])^l = \Omega^{l+n}_{X/K}, \tag{2.14}$$

   which is a finite dimensional K-vector space [100]. The de Rham realisation M_{dR} has a decreasing filtration $F^p M_{\mathrm{dR}}$ given by

   $$F^p M_{\mathrm{dR}} := \mathbb{H}^i(X_{\mathrm{Zar}}, F^p\Omega^*_{X/K}[n]), \tag{2.15}$$

where the complex $F^p\Omega^*_{X/K}[n]$ is

$$F^p\,\Omega^*_{X/K}[n] : 0 \to \cdots \to 0 \to \Omega^{p+n}_{X/K} \xrightarrow{d} \Omega^{p+1+n}_{X/K} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim X}_{X/K} \to 0. \tag{2.16}$$

3. The ℓ-adic realisation. Suppose ℓ is a prime number, then the ℓ-adic cohomology of X is defined by the inverse limit

   $$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) := \varprojlim_n H^i((X \times_K \overline{K})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \tag{2.17}$$

   where $(X \times_K \overline{K})_{\text{ét}}$ means the étale topology on the $\overline{K}$-variety $X_{\overline{K}} := X \times_K \overline{K}$ and $\mathbb{Z}/\ell^n\mathbb{Z}$ means the constant étale torsion sheaf on $(X \times_K \overline{K})_{\text{ét}}$. The ℓ-adic cyclotomic character $\mathbb{Q}_\ell(1)$ is defined by the following inverse limit

   $$\mathbb{Q}_\ell(1) := \varprojlim_n \mu_{\ell^n}(\overline{K}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \tag{2.18}$$

   where $\mu_{\ell^n}(\overline{K})$ consists of the ℓ^n-th roots of unity which admits an action by $\mathbb{Z}/\ell\mathbb{Z}$. Let $\mathbb{Q}_\ell(n)$ be the tensor product $\mathbb{Q}_\ell(1)^{\otimes n}$, which is a continuous representation of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ [97]. The ℓ-adic realisation of M is

   $$M_\ell := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(n), \tag{2.19}$$

   which is also a continuous representation of $\mathrm{Gal}(\overline{K}/K)$ [78].

There are standard comparison isomorphisms between the three realisations [83]:

1. There is an isomorphism I_σ between the Betti realisation and de Rham realisation of M

   $$I_\sigma : M_{B,\sigma} \otimes_{\mathbb{Q}} \mathbb{C} \to M_{\text{dR}} \otimes_\sigma \mathbb{C}, \tag{2.20}$$

   which sends $\oplus_{k \geq p} H^{k,w-k}$ to $F^p M_{\text{dR}} \otimes_\sigma \mathbb{C}$. The isomorphism I_σ clearly depends on the choice of σ. If further σ is a real embedding, I_σ sends the involution $\phi_\sigma \otimes c$ on the left hand side to the involution $1 \otimes c$ on the right hand.

2. Suppose $\overline{\sigma} : \overline{K} \hookrightarrow \mathbb{C}$ is an extension of σ, then there is an isomorphism $I_{\ell,\overline{\sigma}}$ between the Betti realisation and ℓ-adic realisation of M

   $$I_{\ell,\overline{\sigma}} : M_{\text{B},\sigma} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to M_\ell, \tag{2.21}$$

   which clearly depends on the choice of $\overline{\sigma}$. If further σ is a real embedding, the complex conjugation c defines an element $\overline{\sigma}^*(c)$ in $\mathrm{Gal}(\overline{K}/K)$. Then $I_{\ell,\overline{\sigma}}$ sends the involution $\phi_\sigma \otimes 1$ on the left hand side to the involution $\overline{\sigma}^*(c)$ on the right hand side.

The two comparison isomorphisms imply

$$\dim_{\mathbb{Q}}(M_{B,\sigma}) = \dim_K(M_{\mathrm{dR}}) = \dim_{\mathbb{Q}_\ell}(M_\ell), \tag{2.22}$$

and the common dimension is denoted by $\mathrm{rk}(M)$, which is called the rank of M.

Example 2.2.1. *The pure Tate motive $\mathbb{Q}(1)$ is by definition the dual of the Lefschetz motive $h^2(\mathbb{P}^1_K)$, whose realisations are:*

1. $\mathbb{Q}(1)_B = 2\pi i\,\mathbb{Q}$*, which has a pure Hodge structure of type* $(-1,-1)$.
2. $\mathbb{Q}(1)_{dR} = K$*, with filtrations* $F^0 = 0$ *and* $F^{-1} = K$.
3. $\mathbb{Q}(1)_\ell = \mathbb{Q}_\ell(1)$.

The pure Tate motive $\mathbb{Q}(m)$ is the tensor product $\mathbb{Q}(1)^{\otimes m}$.

The twist of the pure motive M by the Tate motive $\mathbb{Q}(m)$ will be denoted by

$$M(m) := M \otimes \mathbb{Q}(m), \tag{2.23}$$

so M can also be written as

$$M = h^i(X) \otimes \mathbb{Q}(n). \tag{2.24}$$

There is a Poincaré duality and a hard Lefschetz theorem in each of the three realisations above, which are compatible with each other under the standard comparison isomorphisms, therefore the dual of M is given by

$$M^\vee = h^i(X)^\vee(-n) = h^{2\dim X - i}(X)(\dim X - n) = h^i(X)(i-n) = M(w), \tag{2.25}$$

where $w = i - 2\,n$ is the weight of M.

2.3 The L-functions of pure motives

From last section, the ℓ-adic realisation M_ℓ of the pure motive M is a continuous representation of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Recall that a non-archimedean prime v of $\mathcal{O}_K$ is given by a prime ideal of $\mathcal{O}_K$ [84, 93]. Suppose I_v is the inertia group of v in $\mathrm{Gal}(\overline{K}/K)$, then we say M_ℓ is unramified at v if the action of I_v on M_ℓ is trivial, in which case the geometric Frobenius element has a well defined action on M_ℓ that will be denoted by Fr_v [93, 97]. Since X is a smooth projective variety, the ℓ-adic realisation M_ℓ is pure of weight w. Here 'pure' means that there exists a set S

consists of finitely many primes such that for a nonarchimedean prime $v \notin S$ which does not divide ℓ, the representation M_ℓ is unramified at v and all the eigenvalues of Fr_v are algebraic numbers with absolute values $\mathrm{Nm}(v)^{w/2}$ [40]. Here Nm is the norm map defined on the fractional ideals of $\mathcal{O}_K$ [84, 93]. For a nonarchimedean prime v of $\mathcal{O}_K$ such that $\ell \nmid \mathrm{Nm}(v)$, let $M_\ell^{I_v}$ be the subspace of M_ℓ that is invariant under the action of I_v. Then the geometric Frobenius element has a well-defined action on $M_\ell^{I_v}$ that will also be denoted by Fr_v. The characteristic polynomial of M at v is defined by

$$P_v(M,T) = \det\big(1 - T\,\mathrm{Fr}_v|M_\ell^{I_v}\big), \;\; \ell \nmid \mathrm{Nm}(v). \tag{2.26}$$

From Deligne's proof of Weil conjectures [40], if X has good reduction at the nonarchimedean prime v, we have:

1. $P_v(M,T)$ is an integral polynomial of $\mathbb{Z}[T]$ and it is independent of the choice of ℓ.

2. $P_v(M,T)$ has a factorisation of the form

$$P_v(M,T) = \prod_{j=1}^{\mathrm{rk}(M)} (1 - \alpha_j T), \tag{2.27}$$

where α_j is an algebraic integer with $|\alpha_j| = \mathrm{Nm}(v)^{w/2}$ for every j.

The variety X has bad reduction at only finitely many primes, and Serre has a conjecture about the behaviour of $P_v(M,T)$ at these bad primes [94].

Conjecture 2.3.1. *For an arbitrary non-archimedean prime v, the characteristic polynomial $P_v(M,T)$ is in $\mathbb{Z}[T]$, and it does not depend on the choice of ℓ. The integral polynomial $P_v(M,T)$ has a factorisation*

$$P_v(M,T) = \prod_{j=1}^{dim(M_\ell^{I_v})} (1 - \alpha_j T), \tag{2.28}$$

where for every j, α_j is an algebraic integer with absolute value

$$|\alpha_j| = Nm(v)^{w_j/2}, \;\; \text{with } 0 \le w_j \le w. \tag{2.29}$$

Assuming this conjecture, the local L-factor of M at v is defined by

$$L_v(M,s) := P_v^{-1}(M, \mathrm{Nm}(v)^{-s}), \tag{2.30}$$

while the L-function of M is defined by

$$L(M,s) := \prod_{v} L_v(M,s), \tag{2.31}$$

where the product is over all non-archimedean primes of $\mathcal{O}_K$. The local L-factor $L_v(M,s)$ satisfies the following properties [83]

$$L_v(M(m),s) = L_v(M,m+s), \quad L_v(M_1 \oplus M_2, s) = L_v(M_1,s)L_v(M_2,s), \tag{2.32}$$

and the L-function of M satisfies similar properties. Deligne's theorem and Conjecture 2.3.1 imply that the infinite product occurs in the definition of $L(M,s)$ 2.31 converges absolutely when $\mathrm{Re}(s) > w/2+1$, hence $L(M,s)$ is a nowhere vanishing holomorphic function in this region.

Conjecture 2.3.2. *The L-function $L(M,s)$ has a meromorphic extension to the whole complex plane, and the only possible pole occurs at $s = w/2+1$ when w is an even integer. Moreover, when it is well-defined, the value $L(M,w/2+1)$ is non-zero [35, 89].*

The archimedean primes of $\mathcal{O}_K$ are given by real embeddings and conjugate pairs of complex embeddings, and they are also associated with local L-factors [35, 84]. For simplicity, let us define

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \cdot \Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s) \cdot \Gamma_{\mathbb{R}}(s+1) = 2 \cdot (2\pi)^{-s} \cdot \Gamma(s). \tag{2.33}$$

Suppose v is the archimedean prime of $\mathcal{O}_K$ that corresponds to $\sigma : K \to \mathbb{C}$, then the associated local L-factor $L_v(M,s)$ only depends on the real pure Hodge structure on $M_{\mathrm{B},\sigma} \otimes_{\mathbb{Q}} \mathbb{R}$, which is carefully discussed in [94] and section 5.2 of [35]:

1. If σ is a complex embedding, $L_v(M,s)$ is defined by

$$L_v(M,s) = \prod_{p \leq q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{q,p}} \ . \tag{2.34}$$

2. If σ is a real embedding, then if w is odd, $L_v(M,s)$ is defined by

$$L_v(M,s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \ . \tag{2.35}$$

If w is even, the subspace $H^{w/2,w/2}$ decomposes into the direct sum

$$H^{w/2,w/2} = H^{w/2,+} \oplus H^{w/2,-}, \tag{2.36}$$

where the two subspaces are defined by

$$\phi_\sigma | H^{w/2,+} = (-1)^{w/2}, \;\; \phi_\sigma | H^{w/2,-} = (-1)^{w/2+1}, \tag{2.37}$$

then $L_v(M,s)$ is defined by

$$L_v(M,s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \Gamma_{\mathbb{R}}(s-w/2)^{\dim H^{w/2,+}} \cdot \Gamma_{\mathbb{R}}(s-w/2+1)^{\dim H^{w/2,-}}. \tag{2.38}$$

The local L-factor of M at an archimedean prime v also satisfies [35]

$$L_v(M(m),s) = L_v(M, m+s), \;\; L_v(M_1 \oplus M_2, s) = L_v(M_1,s) \cdot L_v(M_2,s). \tag{2.39}$$

The total L-factor at infinity is defined by

$$L_\infty(M,s) = \prod_{v|\infty} L_v(M,s), \tag{2.40}$$

where the product is over all the archimedean primes of $\mathcal{O}_K$. The full L-function of M is defined by

$$\Lambda(M,s) = L(M,s) \cdot L_\infty(M,s). \tag{2.41}$$

Conjecture 2.3.3. *$\Lambda(M,s)$ satisfies the following functional equation [35, 94],*

$$\Lambda(M,s) = \varepsilon(M,s)\, \Lambda(M^\vee, 1-s), \tag{2.42}$$

where $\varepsilon(M,s)$ is of the form $a \cdot b^s$ with a and b as non-zero complex numbers. From equation 2.25, this functional equation can also be rewritten as

$$\Lambda(M,s) = \varepsilon(M,s)\, \Lambda(M, w+1-s). \tag{2.43}$$

There is an operation on pure motives called restriction of scalars which maps a motive M, defined over K, to a motive $R_{K/\mathbb{Q}}(M)$ defined over $\mathbb{Q}$ [83]

$$R_{K/\mathbb{Q}}(M) := h^i(X_{/\mathbb{Q}}) \otimes \mathbb{Q}(n), \tag{2.44}$$

where $X_{/\mathbb{Q}}$ means X viewed as a variety over $\mathbb{Q}$ by

$$X \to \operatorname{Spec} K \to \operatorname{Spec} \mathbb{Q}. \tag{2.45}$$

The local L-factors of the pure motive $R_{K/\mathbb{Q}}(M)$ satisfy [83]

$$L_p(R_{K/\mathbb{Q}}(M),s) = \prod_{v|p} L_v(M,s), \quad L_\infty(R_{K/\mathbb{Q}}(M),s) = \prod_{v|\infty} L_v(M,s), \tag{2.46}$$

which immediately implies

$$L(M,s) = L(R_{K/\mathbb{Q}}(M),s). \tag{2.47}$$

Therefore in order to study the L-functions of pure motives defined over K, we only need to study the L-functions of pure motives defined over $\mathbb{Q}$. The field $\mathbb{Q}$ only has one archimedean prime ∞, which corresponds to the unique embedding of $\mathbb{Q}$ into $\mathbb{C}$.

2.4 Properties of the L-functions at integral points

Suppose X is a smooth projective variety defined over $\mathbb{Q}$, then we are interested in the order of vanishing and leading coefficient of $L(h^i(X)(n), s)$ at an integral point.

Definition 2.4.1. *Suppose f is a holomorphic function and its order of vanishing at $z = a$ is r, then the leading coefficient $f^*(a)$ is defined by*

$$f^*(a) := \lim_{z \to a} (z-a)^{-r} f(z). \tag{2.48}$$

From equation 2.32, after a possible Tate twist we only need to study $L(h^i(X)(n), s)$ at $s = 0$. While from the functional equation 2.42, we can further assume 0 is on the right side of the central point $(w+1)/2$ or equals it, i.e. $w \leq -1$. Therefore from now on we will assume that the weight of $M = h^i(X)(n)$ satisfies $w \leq -1$. There are three different cases where the behaviours of $L(M, s)$ are of quite different natures:

1. $w = -1$, and $s = 0$ is the central point.
2. $w = -2$, and $s = 0$ is the near the central point.
3. $w \leq -3$, and $s = 0$ is in the absolutely convergent region.

In this thesis, we will be mostly interested in the case where $w = -1$, and $s = 0$ is the central point. The Gamma function $\Gamma(s)$ does not have any zero point, but it has a simple pole at non-positive integral points. From the construction of the total L-factor $L_\infty(M, s)$ at infinity in Section 2.3, we have [83, 89]

$$\begin{aligned} \mathrm{ord}_{s=0} L_\infty(M, s) &= 0, \\ -\mathrm{ord}_{s=0} L_\infty(M, s) &= \dim_{\mathbb{Q}} M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}} - \dim_{\mathbb{Q}} M_{\mathrm{B}}^+, \\ L_\infty^*(M, 0)/L_\infty^*(M^\vee(1), 0) &\in (2\pi)^{w\,\mathrm{rk}(M)/2 + \dim M_{\mathrm{B}}^-} \cdot \mathbb{Q}^*, \end{aligned} \tag{2.49}$$

where $M_{\mathrm{B}}^\pm$ means the subspace of M_{B} on which ϕ_∞ acts as ± 1. The standard comparison isomorphism I_∞ between the Betti realisation and de Rham realisation is

$$I_\infty : M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \to M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.50}$$

Definition 2.4.2. *The period $\delta(M)$ is defined as the determinant of I_∞ computed with respect to the two rational structures M_B and M_{dR}. More precisely we choose a rational basis of M_B and a rational basis of M_{dR}, then $\delta(M)$ is the determinant of I_∞ computed with respect to these two bases. Therefore $\delta(M)$ is only determined up to a rational multiple, or equivalently it is uniquely determined in $\mathbb{C}^*/\mathbb{Q}^*$.*

Under the comparison isomorphism I_∞, the involution $\phi_\infty \otimes c$ of $M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ is sent to the involution $1 \otimes c$ of $M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$. Therefore the real subspace of $M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ that is invariant under the action of $\phi_\infty \otimes c$ is mapped isomorphically to the real subspace of $M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ that is invariant under the action of $1 \otimes c$, i.e.

$$I_\infty : M_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R} \oplus M_{\mathrm{B}}^- \otimes_{\mathbb{Q}} \mathbb{R}(-1) \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R}, \tag{2.51}$$

where $\mathbb{R}(n) := (2\pi i)^n \mathbb{R}$. Hence I_∞ sends $M_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R}$ into $M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R}$, and Deligne's period map is defined by [35]

$$\alpha_M : M_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R} \to M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R} \to (M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{2.52}$$

Since I_∞ preserves filtrations on both sides, the kernel of α_M lies in

$$\ker \alpha_M \subset F^0 \cap \overline{F}^0 = H^{0,0} = 0, \tag{2.53}$$

where we have used the property that the weight w is < 0, therefore Deligne's period map α_M is injective.

Remark 2.4.3. *The injectivity of* α_M *is equivalent to*

$$I_\infty(M_B^+ \otimes_{\mathbb{Q}} \mathbb{R}) \cap F^0 M_{dR} \otimes_{\mathbb{Q}} \mathbb{R} = 0. \tag{2.54}$$

So the first two equations in 2.49 can be rewritten into

$$\begin{aligned} \mathrm{ord}_{s=0} L_\infty(M,s) &= \dim_{\mathbb{R}} \ker \alpha_M = 0, \\ -\mathrm{ord}_{s=0} L_\infty(M,s) &= \dim_{\mathbb{R}} \operatorname{coker} \alpha_M. \end{aligned} \tag{2.55}$$

The isomorphism 2.51 also induces the following homomorphism

$$M_{\mathrm{B}}^- \otimes_{\mathbb{Q}} \mathbb{R}(-1) \xrightarrow{\simeq} M_{\mathrm{dR}} \otimes \mathbb{R}/I_\infty(M_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R}) \to M_{\mathrm{dR}} \otimes \mathbb{R}/\big(I_\infty(M_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R}) + F^0 M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R}\big), \tag{2.56}$$

where the last term is just $\operatorname{coker} \alpha_M$. Together with the isomorphism

$$M_{\mathrm{B}}^- \otimes_{\mathbb{Q}} \mathbb{R}(-1) \simeq M(-1)_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R}, \tag{2.57}$$

the homomorphism in equation 2.56 induces a homomorphism β_M from $\ker(\alpha_{M(-1)})$ to $\operatorname{coker} \alpha_M$

$$\beta_M : \ker(\alpha_{M(-1)}) \subset M(-1)_{\mathrm{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R} \to \operatorname{coker} \alpha_M. \tag{2.58}$$

If the weight of M is not -2, then the weight of $M(-1)$ is not zero, hence equation 2.53 is still true, therefore $\alpha_{M(-1)}$ is still injective, and the domain of β_M is 0. On the other hand, if the weight of M is -2, then the weight of $M(-1)$ is 0, so we have

$$\ker \alpha_{M(-1)} \subset F^0 M(-1)_{\mathrm{B}} \cap \overline{F}^0 M(-1)_{\mathrm{B}} = H^{0,0}(M(-1)_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}) = H^{-1,-1}(M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}). \tag{2.59}$$

For weight reasons, the intersection of $I_\infty(H^{-1,-1}(M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}))$ and $F^0 M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ is zero, hence the homomorphism β_M is injective.

The injectivity of Deligne's period map α_M 2.52 induces a short exact sequence

$$0 \longrightarrow M_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\alpha_M} (M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \operatorname{coker} \alpha_M \longrightarrow 0, \tag{2.60}$$

which defines an isomorphism

$$\det((M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \det(M_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R}) \cdot \det(\operatorname{coker} \alpha_M), \tag{2.61}$$

where $\det(\cdot)$ means the highest exterior power of vector spaces. The domain of α_M has a rational structure M_{B}^{+}, while the target of α_M has a rational structure $M_{\mathrm{dR}}/F^0 M_{\mathrm{dR}}$, hence the isomorphism 2.61 defines a rational structure $\mathcal{D}(M)$ on the $\mathbb{R}$-vector space $\det(\operatorname{coker} \alpha_M)$.

Definition 2.4.4. *The pure motive M is called critical (by Deligne [35]) if Deligne's period map α_M is an isomorphism, in which case the Deligne's period of M is defined by*

$$c^{+}(M) = det(\alpha_M) \in \mathbb{R}^{*}/\mathbb{Q}^{*}. \tag{2.62}$$

where the determinant of α_M is computed with respect to the rational structures M_B^{+} and $M_{dR}/F^0 M_{dR}$.

If the pure motive M is critical, i.e. α_M is an isomorphism, then $\operatorname{coker} \alpha_M$ is 0, so $\det(\operatorname{coker} \alpha_M)$ is canonically isomorphic to $\mathbb{R}$. In this case, the rational structure $\mathcal{D}(M)$ on $\det(\operatorname{coker} \alpha_M)$ is given by

$$\mathcal{D}(M) = c^{+}(M)^{-1} \cdot \mathbb{Q}. \tag{2.63}$$

The weight of the pure motive $M^\vee(1)$ is $-w-2$, which can be positive, but let us still define Deligne's period map $\alpha_{M^\vee(1)}$ by

$$\alpha_{M^\vee(1)} : (M^\vee(1))_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R} \to (M^\vee(1))_{\mathrm{dR}}/F^0(M^\vee(1))_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R}. \tag{2.64}$$

The dual of $\alpha_{M^\vee(1)}$ is given by [83]

$$F^0 M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R} \to M_{\mathrm{B}}^{-}(-1) \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{R}/M_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R}, \tag{2.65}$$

which is also the homomorphism induced by the inverse of I_∞. Remark 2.4.3 immediately implies that the map in equation 2.65 is injective, hence $\alpha_{M^\vee(1)}$ is surjective. Therefore we get a short exact sequence

$$0 \to \ker(\alpha_{M^\vee(1)}) \to (M^\vee(1))_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\alpha_{M^\vee(1)}} \big((M^\vee(1))_{\mathrm{dR}}/F^0(M^\vee(1))_{\mathrm{dR}}\big) \otimes_{\mathbb{Q}} \mathbb{R} \to 0. \tag{2.66}$$

The dual of the short exact sequence 2.66 fits into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (F^0 M_{\mathrm{dR}} + M_{\mathrm{B}}^{+}) \otimes_{\mathbb{Q}} \mathbb{R} \arrow[r] \arrow[d, "\mathrm{Id}"] & (M_{\mathrm{B}}^{-}(-1) \oplus M_{\mathrm{B}}^{+}) \otimes_{\mathbb{Q}} \mathbb{R} \arrow[r] \arrow[d, "I_\infty"] & \ker(\alpha_{M^\vee(1)})^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (F^0 M_{\mathrm{dR}} + M_{\mathrm{B}}^{+}) \otimes_{\mathbb{Q}} \mathbb{R} \arrow[r] & M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{R} \arrow[r] & \operatorname{coker}(\alpha_M) \arrow[r] & 0
\end{tikzcd} \tag{2.67}$$

which induces an isomorphism

$$\ker(\alpha_{M^\vee(1)})^\vee \xrightarrow{\simeq} \operatorname{coker}(\alpha_M). \tag{2.68}$$

Under this isomorphism, the rational structure on $\det(\ker(\alpha_{M^\vee(1)})^\vee)$ (induced by the top short exact sequence of 2.67) is mapped to another rational structure $\mathcal{B}(M)$ on $\det(\operatorname{coker}(\alpha_M))$ given by

$$\mathcal{B}(M) = (2\pi i)^{-\dim_{\mathbb{Q}} M_{\mathrm{B}}^{-}}\, \delta(M)\, \mathcal{D}(M), \tag{2.69}$$

where the factor $(2\pi i)^{-\dim_{\mathbb{Q}} M_{\mathrm{B}}^{-}}$ comes from the direct summand $M_{\mathrm{B}}^{-}(-1)$ in the term $(M_{\mathrm{B}}^{-}(-1) \oplus M_{\mathrm{B}}^{+}) \otimes_{\mathbb{Q}} \mathbb{R}$. In the paper [35], under some general assumptions Deligne shows that

$$(2\pi i)^{-\dim_{\mathbb{Q}} M_{\mathrm{B}}^{-}} \delta(M) \in (2\pi i)^{-\dim_{\mathbb{Q}} M_{\mathrm{B}}^{-} - w\cdot \mathrm{rk}(M)/2} \varepsilon(M,0)\, \mathbb{Q}^*, \tag{2.70}$$

which depends on a conjectural description of rank-1 pure motives. Then equations 2.70, 2.49 together with the functional equation 2.42 tell us

$$L^*(M^\vee(1),0)\, \mathcal{B}(M) = L^*(M,0)\, \mathcal{D}(M). \tag{2.71}$$

2.5 Beilinson's conjecture on the central values of L-functions

In this section, we will talk about Beilinson's conjecture on the central value of $L(h^i(X),s)$ where i is an odd integer and X is a smooth projective variety defined over $\mathbb{Q}$. For simplicity, let n be

$$n := (i+1)/2, \; i = 2n-1. \tag{2.72}$$

The weight w of the pure motive $M := h^{2n-1}(X)(n)$ is -1, which immediately implies

$$\dim_{\mathbb{R}} M_{\mathrm{B}}^{+} \otimes_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{R}} (M_{\mathrm{dR}}/F^0) \otimes_{\mathbb{Q}} \mathbb{R} = \frac{1}{2} \operatorname{rk} M. \tag{2.73}$$

Therefore the domain and image of α_M have the same dimension, so the injectivity of α_M implies it is an isomorphism, hence we can still define Deligne's period, $c^+(h^{2n-1}(X)(n))$. Another crucial ingredient in Beilinson's conjecture is the height pairing h

$$h : \left(\mathrm{CH}^n(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}\right) \otimes_{\mathbb{Q}} \left(\mathrm{CH}^{\mathrm{dim}X+1-n}(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}\right) \to \mathbb{R}, \tag{2.74}$$

where $\mathrm{CH}^*(X)_0$ is the subgroup of $\mathrm{CH}^*(X)$ that consists of cycle classes which are homologous to zero, i.e. the elements of $\mathrm{CH}^*(X)$ which are mapped to zero by cycle class map [78].

Remark 2.5.1. *Currently, height pairing is still not well understood, and the readers are referred to [13, 14] for careful treatments.*

Beilinson's Conjecture: *Suppose X is a smooth projective variety defined over $\mathbb{Q}$ and n is an integer such that $0 < 2\,n - 1 < 2\,dimX$, then*

1. *The height pairing h 2.74 is non-degenerate.*
2. *The dimension of $CH^n(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite, which equals the order of vanishing of $L(h^{2n-1}(X), s)$ at $s = n$.*
3. *The leading coefficient $L^*(h^{2n-1}(X), n)$ is in $c^+(h^{2n-1}(X)(n)) \cdot det(h) \cdot \mathbb{Q}^*$, where the determinant of h is computed with respect to arbitrary bases of $CH^n(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $CH^{dimX+1-n}(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}$.*

For a discussion of Chow groups, the readers are referred to Appendix A. The following property of Chow group will be needed later in this thesis.

Lemma 2.5.2. *Suppose X is a variety defined over a number field K and $X_{/\mathbb{Q}}$ means X viewed as a variety defined over $\mathbb{Q}$ by restriction of base field, then we have*

$$CH^*(X) = CH^*(X_{/\mathbb{Q}}),\ \ CH^*(X)_0 = CH^*(X_{/\mathbb{Q}})_0. \tag{2.75}$$

Proof. The prime cycles of X are in one to one bijection with irreducible closed subsets of X, so the group $C^*(X)$ of algebraic cycles of X is isomorphic to the group $C^*(X_{/\mathbb{Q}})$ of algebraic cycles of $X_{/\mathbb{Q}}$. From Section 1.3 of [45], rational equivalence is defined using rational functions on varieties, which also does not change under restriction of base field. Therefore we have a canonical isomorphism

$$\mathrm{CH}^*(X) = \mathrm{CH}^*(X_{/\mathbb{Q}}). \tag{2.76}$$

The $\mathbb{C}$-valued points of $X_{/\mathbb{Q}}$ is

$$X_{/\mathbb{Q}}(\mathbb{C}) = \amalg_\sigma \ (X \times_\sigma \mathbb{C})(\mathbb{C}), \tag{2.77}$$

where the disjoint union is over all embeddings $\sigma : K \to \mathbb{C}$. So the Betti cohomology of $X_{/\mathbb{Q}}$ is given by

$$H^*(X_{/\mathbb{Q}}(\mathbb{C}), \mathbb{Q}) = \oplus_\sigma \ H^*((X \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q}). \tag{2.78}$$

This immediately shows that for a cycle z of $\mathrm{CH}^*(X)$ $(= \mathrm{CH}^*(X_{/\mathbb{Q}}))$ we have

$$cl_X(z) = 0 \iff cl_{X_{/\mathbb{Q}}}(z) = 0, \tag{2.79}$$

where cl_X (resp. $cl_{X_{/\mathbb{Q}}}$) is the cycle class map of $\mathrm{CH}^*(X)$ (resp. $\mathrm{CH}^*(X_{/\mathbb{Q}})$). So we have a canonical isomorphism

$$\mathrm{CH}^*(X)_0 = \mathrm{CH}^*(X_{/\mathbb{Q}})_0. \tag{2.80}$$

□

Summary: in this chapter, we have introduced Beilinson's conjecture on the central values of L-functions, which is a far-reaching generalisation of the Birch and Swinnerton-Dyer conjecture. However, currently this conjecture is still beyond our reach, therefore it is very important to provide interesting examples to it, which can further help us understand this conjecture better. From Chapter 9 to Chapter 13, we will study the arithmetic geometry of the conifold transition in the mirror family of the quintic, and show that it provides a compelling example that supports **Beilinson's Conjecture**.

Chapter 3

Limit mixed Hodge structure

In this chapter, we will discuss the constructions of limit mixed Hodge structures by Schmid and Steenbrink. We will follow [28, 85, 88, 95] closely, which are also recommended to the reader unfamiliar with the construction. The structure of this chapter is as follows:

- Section 3.1 briefly discusses the variations of Hodge structures and the Gauss-Manin connection.
- Section 3.2 is an overview of mixed Hodge structures and their extensions.
- Section 3.3 discusses Deligne's canonical extension and Schmid's construction of limit mixed Hodge structures.
- Section 3.4 discusses Steenbrink's construction of limit mixed Hodge structures.
- Finally, Section 3.5 discusses the connections between Steenbrink's construction of limit MHS and Ayoub's motivic nearby cycle functor. It also includes a conjecture of Ayoub that says limit mixed Hodge structure is motivic.

3.1 Variations of Hodge structures

In this section, we will briefly talk about the variations of Hodge structures, and for the purpose of this thesis, we will focus on the geometric situation in the sense of [88]. Suppose $\mathscr{X}$ and S are smooth quasi-projective varieties defined over $\mathbb{C}$, and there is a smooth fibration

$$\pi_S : \mathscr{X} \to S \tag{3.1}$$

such that the fibers are smooth projective varieties of dimension n. Let $\mathscr{X}(\mathbb{C})$ (resp. $S(\mathbb{C})$) be the $\mathbb{C}$-valued points (classical points) of $\mathscr{X}$ (resp. S), which is a smooth quasi-projective manifold called the analytification of $\mathscr{X}$ (resp. S). The analytification of π_S is a smooth fibration between quasi-projective complex manifolds [98]

$$\pi_S^{\mathrm{an}}: \mathscr{X}(\mathbb{C}) \to S(\mathbb{C}), \tag{3.2}$$

the fibers of which are smooth projective manifolds. The fibration π_S^{an} defines a local system on $S(\mathbb{C})$ given by [85, 101]

$$V_{\mathbb{Z}} := \mathrm{R}^q\, \pi_{S,*}^{\mathrm{an}}\, \mathbb{Z}, \tag{3.3}$$

where $\mathbb{Z}$ means the constant sheaf on $\mathscr{X}(\mathbb{C})$. The fiber of $V_{\mathbb{Z}}$ over a point $\varphi \in S(\mathbb{C})$ is the singular cohomology group $H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{Z})$ (modulo torsions).

Remark 3.1.1. *The local system $V_{\mathbb{Z}}$ is a way to glue the lattice $H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{Z})$ (modulo torsions) together in a compatible way. In this thesis torsions of singular cohomology groups are not important, and frequently they will be ignored.*

The dual of $V_{\mathbb{Z}}$, denoted by $V_{\mathbb{Z}}^{\vee}$, is a local system over $S(\mathbb{C})$ whose fiber over $\varphi \in S(\mathbb{C})$ is the singular homology group $H_q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{Z})$ (modulo torsions). Similarly let $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{C}}$) be the local systems

$$V_{\mathbb{Q}} := \mathrm{R}^q\, \pi_{S,*}^{\mathrm{an}}\, \mathbb{Q} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q},\ V_{\mathbb{C}} := \mathrm{R}^q\, \pi_{S,*}^{\mathrm{an}}\, \mathbb{C} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}, \tag{3.4}$$

whose fiber over φ is $H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{Q})$ (resp. $H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{C})$). The local system $V_{\mathbb{Z}}$ defines a bundle $\mathcal{V}$ over $S(\mathbb{C})$

$$\mathcal{V} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{S(\mathbb{C})}, \tag{3.5}$$

where $\mathcal{O}_{S(\mathbb{C})}$ is the sheaf of holomorphic functions on $S(\mathbb{C})$. Since the complex manifold $\mathscr{X}(\mathbb{C})_\varphi$ is projective, there is a Hodge decomposition on $H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{C})$ [48, 88]

$$H^q(\mathscr{X}(\mathbb{C})_\varphi, \mathbb{C}) = \oplus_{0 \leq k \leq q} H^{k,q-k}(\mathscr{X}(\mathbb{C})_\varphi), \tag{3.6}$$

which induces a Hodge filtration

$$F_\varphi^p := \oplus_{k \geq p} H^{k,q-k}(\mathscr{X}(\mathbb{C})_\varphi). \tag{3.7}$$

The complex vector space F_φ^p varies holomorphically with respect to φ, and their union forms a holomorphic vector bundle $\mathcal{F}^p$ that defines a sub-bundle filtration of $\mathcal{V}$ [48, 88]. There exists a Gauss-Manin connection ∇ on $\mathcal{V}$ such that the local sections of the local system $V_{\mathbb{C}}$ are flat

$$\nabla : \mathcal{V} \to \Omega^1_{S(\mathbb{C})} \otimes_{\mathcal{O}_{S(\mathbb{C})}} \mathcal{V}. \tag{3.8}$$

Moreover, ∇ satisfies Griffiths transversality [48, 88]

$$\nabla \mathcal{F}^p \subset \Omega^1_{S(\mathbb{C})} \otimes_{\mathcal{O}_{S(\mathbb{C})}} \mathcal{F}^{p-1}. \tag{3.9}$$

3.1.1 Algebraizations

The constructions above have algebraic counterparts. More precisely, there is a complex of sheaves of algebraic forms $\Omega^*_{\mathscr{X}/S}$ [1, 70, 98]

$$\Omega^*_{\mathscr{X}/S} : 0 \to \mathcal{O}_{\mathscr{X}/S} \xrightarrow{d} \Omega^1_{\mathscr{X}/S} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathscr{X}/S} \to 0, \tag{3.10}$$

and the relative de Rham cohomology sheaf $\mathscr{V}_S$ is defined by

$$\mathscr{V}_S := \mathrm{R}^q\, \pi_{S,*}(\Omega^*_{\mathscr{X}/S}). \tag{3.11}$$

Since π_S is a smooth fibration between smooth varieties, $\mathscr{V}_S$ is a locally free sheaf on S. The fiber of $\mathscr{V}_S$ over a closed point $\varphi \in S$ is the q-th algebraic de Rham cohomology $\mathbb{H}^q(\mathscr{X}_\varphi, \Omega^*_{\mathscr{X}_\varphi})$ of $\mathscr{X}_\varphi$ [53, 70]. The complex $\Omega^*_{\mathscr{X}/S}$ is naively filtered by the complex $F^p\,\Omega^*_{\mathscr{X}/S}$ given by

$$F^p\,\Omega^*_{\mathscr{X}/S} : 0 \to \cdots \to 0 \to \Omega^p_{\mathscr{X}/S} \xrightarrow{d} \Omega^{p+1}_{\mathscr{X}/S} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathscr{X}/S} \to 0, \tag{3.12}$$

which defines a locally free subsheaf $\mathscr{F}^p_S$ on S

$$\mathscr{F}^p_S := \mathrm{Im}\left(\mathrm{R}^q\, \pi_{S,*}(F^p\,\Omega^*_{\mathscr{X}/S}) \to \mathrm{R}^q\, \pi_{S,*}(\Omega^*_{\mathscr{X}/S})\right) \tag{3.13}$$

that induces a filtration of $\mathscr{V}_S$. The analytification of the complex 3.10 is

$$\Omega^*_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} : 0 \to \mathcal{O}_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \xrightarrow{d} \Omega^1_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \to 0, \tag{3.14}$$

which is the usual complex of sheaves of holomorphic forms in complex geometry [52]. The complex $\Omega^*_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})}$ is filtered by the analytification of the complex 3.12

$$F^p\,\Omega^*_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} : 0 \to \cdots \to 0 \to \Omega^p_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \xrightarrow{d} \Omega^{p+1}_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})} \to 0. \tag{3.15}$$

Similarly, there are locally free sheaves $\mathscr{V}^{\mathrm{an}}_S$ and $\mathscr{F}^{p,\mathrm{an}}_S$ on $S(\mathbb{C})$ defined by

$$\mathscr{V}^{\mathrm{an}}_S := \mathrm{R}^q\, \pi^{\mathrm{an}}_{S,*}\,(\Omega^*_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})}),\ \mathscr{F}^{p,\mathrm{an}}_S := \mathrm{R}^q\, \pi^{\mathrm{an}}_{S,*}\,(F^p\,\Omega^*_{\mathscr{X}(\mathbb{C})/S(\mathbb{C})}), \tag{3.16}$$

which, as the notations have suggested, are just the analytifications of $\mathscr{V}_S$ and $\mathscr{F}^p_S$ respectively. There exists a unique Gauss-Manin connection ∇ on $\mathscr{V}_S$, the construction of which is left to [70], and it is an integrable algebraic connection

$$\nabla : \mathscr{V}_S \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathscr{V}_S \tag{3.17}$$

that satisfies Griffiths transversality

$$\nabla(\mathscr{F}^p_S) \subset \Omega^1_S \otimes_{\mathcal{O}_S} \mathscr{F}^{p-1}_S. \tag{3.18}$$

The Gauss-Manin connection 3.8 is the analytification of the algebraic Gauss-Manin connection 3.17, and both of them have been denoted by ∇ by abuse of notation.

3.1.2 Do Hodge structures have a limit?

Since the field $\mathbb{C}$ admits resolution of singularities, there exists a compactification of the family $\mathscr{X}$ 3.1 given by the commutative diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[r, hook] \arrow[d, "\pi_S"] & \overline{\mathscr{X}} \arrow[d, "\overline{\pi}_S"] \\
S \arrow[r, hook] & \overline{S}
\end{tikzcd} \tag{3.19}$$

where $\overline{S}$ is a complete smooth variety which contains S as an open sub-variety, while $\overline{\mathscr{X}}$ is a complete variety that contains $\mathscr{X}$ as an open smooth sub-variety. The fiber of $\overline{\mathscr{X}}$ over a point in $\overline{S} - S$ can be singular.

Question 3.1.2. *Suppose pt is a point in $\overline{S} - S$ and $\{pt_n\}_{n=1}^{\infty}$ is a sequence of points in $S(\mathbb{C})$ whose limit is pt. Over each point pt_n we have a pure Hodge structure on*

$$\mathcal{V}_{pt_n} = H^q(\mathscr{X}(\mathbb{C})_{pt_n}, \mathbb{C}). \tag{3.20}$$

Let $n \to \infty$, does the pure Hodge structures on $\{\mathcal{V}_{pt_n}\}$ have a limit?

The limit does exist, but it is not a pure Hodge structure any more [88].

3.2 Mixed Hodge Structure

In this section we will give a very brief introduction to mixed Hodge structures (MHS), while the readers are referred to [26, 85] for more systematic and complete treatments. Throughout this section, the ring R will be either $\mathbb{Z}$ or $\mathbb{Q}$.

3.2.1 Definition of Mixed Hodge Structure

An (pure) R-Hodge structure H of weight $l \in \mathbb{Z}$ consists of the data [85]:

1. An R-module H_R of finite rank.
2. A decreasing filtration F^*H of the complex vector space $H_{\mathbb{C}} := H_R \otimes_R \mathbb{C}$.

such that $H_{\mathbb{C}}$ admits a decomposition

$$H_{\mathbb{C}} = \oplus_{p+q=l} H^{p,q}, \text{ with } H^{p,q} := F^p \cap \overline{F}^q. \tag{3.21}$$

Here the complex conjugation is defined with respect to the real structure $H_R \otimes_R \mathbb{R}$ of $H_{\mathbb{C}}$. The definition immediately implies that [85]

$$F^k = \oplus_{p \geq k} H^{p,\, l-p}. \tag{3.22}$$

The simplest example of an R-Hodge structure is the Hodge-Tate object $R(n), n \in \mathbb{Z}$ of weight $-2\,n$.

Definition 3.2.1. *The R module of the Hodge-Tate object $R(n)$ is $(2\pi i)^n R \subset \mathbb{C}$, while its Hodge filtrations are uniquely determined by the Hodge decomposition*

$$R(n)^{-n,-n} = \mathbb{C}$$

An R-mixed Hodge structure (MHS) consists of the data:

1. An R-module H_R of finite rank.
2. An increasing weight filtration W_* on $H_{\mathbb{Q}} := H_R \otimes_R \mathbb{Q}$.
3. A decreasing Hodge filtration F^* on $H_{\mathbb{C}} := H_R \otimes_R \mathbb{C}$.

such that the Hodge filtration F^* induces a pure Hodge structure of weight l on each graded piece $\mathrm{Gl}_l^W\, W := W_l \,/\, W_{l-1}$[85]. Morphisms between two R-MHS are given by linear maps that preserve both weight filtrations and Hodge filtrations [26, 85].

Definition 3.2.2. *For two R-MHS A and B, a morphism from A to B is given by a homomorphism $\phi : A_R \to B_R$ such that*

$$\begin{aligned} \phi\,(W_l\, A) &\subset W_l\, B, \ \forall\, l, \\ \phi\,(F^p\, A) &\subset F^p\, B, \ \forall\, p. \end{aligned} \tag{3.23}$$

The category of R-MHS will be denoted by $\mathbf{MHS}_R$, which is an abelian category [85]. In $\mathbf{MHS}_R$, there exists an internal Hom operation [26, 85].

Definition 3.2.3. *For two R-MHS A and B, there exists an R-MHS $Hom(A, B)$ with R-module given by*

$$Hom(A, B)_R := Hom(A_R, B_R). \tag{3.24}$$

Its weight filtrations and Hodge filtrations are given by

$$\begin{aligned} W_l\,(Hom(A, B)) &= \{\phi : \phi\,(W_r\, A) \subset W_{r+l}\, B, \forall\, r\}, \\ F^p\,(Hom(A, B)) &= \{\phi : \phi\,(F^r\, A) \subset F^{r+p}\, B, \forall\, r\}. \end{aligned} \tag{3.25}$$

In fact, $\mathbf{MHS}_R$ is a rigid tensor abelian category [85].

3.2.2 Extensions of MHS

Given two R-MHS A and B, an extension of B by A in $\mathbf{MHS}_R$ is given by a short exact sequence

$$0 \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow 0, \tag{3.26}$$

and two extensions are said to be isomorphic if there exists a commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & H \arrow[r] \arrow[d, "\simeq"] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & H' \arrow[r] & B \arrow[r] & 0. \end{tikzcd} \tag{3.27}$$

The extension 3.26 is said to split if it is isomorphic to the trivial extension which is defined by [26, 85]

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{j} B \longrightarrow 0, \tag{3.28}$$

where i is the natural inclusion map and j is the natural projection map.

Definition 3.2.4. *The abelian category of mixed Hodge-Tate structures $\boldsymbol{MHT}_R$ is defined to be the smallest full abelian subcategory of $\boldsymbol{MHS}_R$ that contains Hodge-Tate objects $R(n), n \in \mathbb{Z}$ and is also closed under extension.*

The set of isomorphism classes of extensions of B by A, denoted by $\mathrm{Ext}^1_{\mathbf{MHS}_R}(B, A)$, has a group structure induced by Baer summation, while the zero object is the trivial extension in 3.28 [26, 85]. Two R-MHS A and B are said to be separated if the highest weight of A is lower than the lowest weight of B, in which case the extension 3.26 is said to be separated. When A and B are separated, there is a canonical and functorial description of the group $\mathrm{Ext}^1_{\mathbf{MHS}_R}(B, A)$ given by [26, 85]

$$\mathrm{Ext}^1_{\mathbf{MHS}_R}(B, A) = \mathrm{Hom}(B, A)_R \otimes_R \mathbb{C}/(F^0 \,\mathrm{Hom}(B, A) + \mathrm{Hom}(B, A)_R), \tag{3.29}$$

and in particular we have the following important lemma.

Lemma 3.2.5. *When $n \geq 1$, $\mathbb{Q}(n)$ and $\mathbb{Q}(0)$ are separated and we have*

$$Ext^1_{\boldsymbol{MHS}_\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathbb{C}/(2\pi i)^n\,\mathbb{Q}. \tag{3.30}$$

Proof. The Hodge-Tate object $\mathbb{Q}(n)$ is pure of weight $-2\,n$ and the object $\mathbb{Q}(0)$ is pure of weight 0, so they form a separated pair. The rational vector spaces of $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$ are respectively

$$\mathbb{Q}(0) : \mathbb{Q} \subset \mathbb{C}, \quad \mathbb{Q}(n) : (2\pi i)^n\,\mathbb{Q} \subset \mathbb{C}. \tag{3.31}$$

From the definition of internal Hom in Definition 3.2.3, we have

$$F^0 \operatorname{Hom}(B, A) = 0. \tag{3.32}$$

There also exists an isomorphism

$$\operatorname{Hom}(\mathbb{Q}(0), \mathbb{Q}(n))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}, \tag{3.33}$$

which sends $\operatorname{Hom}(\mathbb{Q}(0), \mathbb{Q}(n))_{\mathbb{Q}}$ to $(2\pi i)^n \, \mathbb{Q}$. Then this lemma follows from 3.29. □

Given an element $\bar{s}$ of $\mathbb{C}/(2\pi i)^n \, \mathbb{Q}, n \geq 1$, we want to construct an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ whose image under the isomorphism 3.30 is $\bar{s}$. The complex vector space $\mathbb{C}^2$ has a natural basis $\{e_j\}_{j=1}^2$ given by

$$e_1 = (1, 0), \;\; e_2 = (0, 1). \tag{3.34}$$

Let us denote this extension by H, and its rational vector space is

$$H_{\mathbb{Q}} := \mathbb{Q}\,(2\pi i)^n \, e_1 + \mathbb{Q}\, e_2 \subset \mathbb{C}^2. \tag{3.35}$$

The weight filtration of H is chosen to be

$$\begin{aligned} &W_{-2\,n-1}\, H = W_{-2\,n-2}\, H = \cdots = 0, \\ &W_{-2\,n} = \cdots = W_{-1} = \mathbb{Q}\,(2\pi i)^n \, e_1, \\ &W_0\, H = W_1\, H = \cdots = H_{\mathbb{Q}}. \end{aligned} \tag{3.36}$$

Let s be an arbitrary complex number whose coset in $\mathbb{C}/(2\pi i)^n \, \mathbb{Q}$ is $\bar{s}$, then the Hodge filtration on $H_{\mathbb{C}}$ is chosen to be

$$\begin{aligned} &F^1 = F^2 = \cdots = 0, \\ &F^0 = \cdots = F^{-(n-1)} = \mathbb{C}\,(s\, e_1 + e_2), \\ &F^{-n} = F^{-n-1} = \cdots = \mathbb{C}^2. \end{aligned} \tag{3.37}$$

There exists an inclusion map of $\mathbb{Q}(n)$ into H, which defines a short exact sequence

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow H \longrightarrow \mathbb{Q}(0) \longrightarrow 0, \tag{3.38}$$

hence H gives an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. From the proof of equation 3.29 in [26, 85], H is sent to $\bar{s}$ by the isomorphism 3.30. The construction of H immediately shows that the isomorphism class of the extension given by H does not depend on the choice of s.

3.3 Limit mixed Hodge structure

In this section, we will give a briefly overview of the construction of a limit MHS over a point pt $\in \overline{S}(\mathbb{C}) - S(\mathbb{C})$, and our treatments are completely from the papers [28, 29, 88]. The unit disc Δ and punctured unit disc Δ^* are defined by

$$\Delta := \{\ z \in \mathbb{C} : |z| < 1\ \},\ \ \Delta^* := \{\ z \in \mathbb{C} : 0 < |z| < 1\ \}. \tag{3.39}$$

Suppose Δ^m is a local neighbourhood of pt in $\overline{S}(\mathbb{C})$, where m is the dimension of $\overline{S}(\mathbb{C})$, and we will assume that

$$S \cap \Delta^m = (\Delta^*)^m. \tag{3.40}$$

Namely the discriminant locus of Δ^m is the union of m hyperplanes H_i

$$H_i := \Delta^m \cap \{z_i = 0\}, \tag{3.41}$$

which cross each other normally at the origin. Choose a point $\varphi_0 := (\varphi_{i,0}) \in (\Delta^*)^m$, then the fundamental group of $(\Delta^*)^m$ is

$$\pi_1((\Delta^*)^m, \varphi_0) = \oplus_{i=1}^m \pi_1(\Delta^*, \varphi_{i,0}) = \mathbb{Z}^m. \tag{3.42}$$

Here $\pi_1(\Delta^*, \varphi_{i,0})$ is the fundamental group of the i-th punctured disc in the direct product, and let T_i be a generator of it. The sub-family of $\mathscr{X}(\mathbb{C})$ over $(\Delta^*)^m$ will be denoted by

$$\pi_{(\Delta^*)^m} : (\pi_S^{\mathrm{an}})^{-1}((\Delta^*)^m) \to (\Delta^*)^m. \tag{3.43}$$

As before, there is a local system $V'_{\mathbb{Z}}$ over $(\Delta^*)^m$ defined by

$$V'_{\mathbb{Z}} := \mathrm{R}^q\, \pi_{(\Delta^*)^m,*}\, \mathbb{Z} = V_{\mathbb{Z}}|_{(\Delta^*)^m}, \tag{3.44}$$

whose fiber over φ_0 is $H^q(\mathscr{X}(\mathbb{C})_{\varphi_0}, \mathbb{Z})$. Similarly we have

$$V'_{\mathbb{Q}} = V'_{\mathbb{Z}} \otimes \mathbb{Q} = V_{\mathbb{Q}}|_{(\Delta^*)^m},\ V'_{\mathbb{C}} = V'_{\mathbb{Z}} \otimes \mathbb{C} = V_{\mathbb{C}}|_{(\Delta^*)^m}. \tag{3.45}$$

The local system $V'_{\mathbb{Z}}$ defines a bundle $\mathcal{V}'$, which has a sub-bundle filtration $\mathcal{F}'^p$

$$\mathcal{V}' := V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{(\Delta^*)^m} = \mathcal{V}|_{(\Delta^*)^m},\ \ \mathcal{F}'^p := \mathcal{F}^p|_{(\Delta^*)^m}. \tag{3.46}$$

The monodromy of the local system $V'_{\mathbb{Z}}$ defines a representation

$$\Psi : \pi_1((\Delta^*)^m, \varphi_0) \to \mathrm{Aut}(H^q(\mathscr{X}(\mathbb{C})_{\varphi_0}, \mathbb{Z})), \tag{3.47}$$

which is determined by the images $\{\Psi(T_i)\}_{i=1}^m$. The fundamental group $\pi_1((\Delta^*)^m, \varphi_0)$ is abelian, hence $\Psi(T_i)$ commutes with $\Psi(T_j)$ for different i and j. The operator $\Psi(T_i)$ is quasi-unipotent in general [28, 88], but for the purpose of this thesis, we will assume $\Psi(T_i)$ is unipotent for every i, i.e.

$$(\Psi(T_i) - \mathrm{Id})^{k_i} = 0, \text{ for some } k_i \in \mathbb{N}_+. \tag{3.48}$$

Remark 3.3.1. *This is not a strong restriction, see [27] for more detail.*

The monodromy operator N_i is defined by

$$N_i := \log\, \Psi(T_i), \tag{3.49}$$

which is nilpotent, and N_i commutes with N_j for different i and j.

Remark 3.3.2. *Some literatures define N_i by $-\log \Psi(T_i)/(2\pi i)$, but the definition 3.49 will make the computations of limit MHS simpler.*

Definition 3.3.3. *Suppose we have an element $\xi \in H^q(\mathscr{X}(\mathbb{C})_{\varphi_0}, \mathbb{C})$, then locally it extends to a section of the local system $V'_{\mathbb{C}}$, and the value of this section at φ will be denoted by $\xi(\varphi)$.*

However globally this extension will only yield a multi-valued section of $V'_{\mathbb{C}}$ because of monodromy. The regularised section associated to ξ is defined by

$$\widehat{\xi}(\varphi) := \exp\,(-\frac{\sum_i \log \varphi_i\, N_i}{2\pi i})\, \xi(\varphi), \tag{3.50}$$

which is a single-valued section of the bundle $\mathcal{V}'$ over $(\Delta^*)^m$. Suppose $\{\sigma^a\}$ is a basis of $H^q(\mathscr{X}(\mathbb{C})_{\varphi_0}, \mathbb{C})$, then the regularised sections $\{\widehat{\sigma}^a(\varphi)\}$ form a frame of $\mathcal{V}'$, which defines a trivialisation of $\mathcal{V}'$ over $(\Delta^*)^m$. This trivialisation induces an extension of $\mathcal{V}'$ to a bundle $\widetilde{\mathcal{V}}'$ over $(\Delta)^m$, which is called **Deligne's canonical extension**. Moreover, the bundle $\mathcal{F}'^p$ is also extended to a bundle $\widetilde{\mathcal{F}}'^p$ over $(\Delta)^m$ which gives a filtration of $\widetilde{\mathcal{V}}'$ [37, 38, 56, 88, 101].

From the construction of Deligne's canonical extension, the section $\widehat{\xi}(\varphi)$ of $\mathcal{V}'$ is extended to a section $\widetilde{\xi}(\varphi)$ of $\widetilde{\mathcal{V}}'$ over Δ^m, whose value at 0 will be denoted by $\widetilde{\xi}(0)$. In particular the frame $\{\widehat{\sigma}^a(\varphi)\}$ of $\mathcal{V}'$ is extended to a frame $\{\widetilde{\sigma}^a(\varphi)\}$ of $\widetilde{\mathcal{V}}'$, while $\{\widetilde{\sigma}^a(0)\}$ forms a basis of the fiber $\widetilde{\mathcal{V}}'|_0$. Hence there is an isomorphism given by

$$\rho_{\varphi_0} : H^q(\mathscr{X}(\mathbb{C})_{\varphi_0}, \mathbb{C}) \to \widetilde{\mathcal{V}}'|_0, \quad \rho_{\varphi_0}(\xi) = \widetilde{\xi}(0), \tag{3.51}$$

under which the lattice $H^q(\mathscr{X}(\mathbb{C})_{\varphi_0},\mathbb{Z})$ is sent to a lattice of $\widetilde{\mathcal{V}}'|_0$ that will be denoted by $\widetilde{\mathcal{V}}'|_{0,\mathbb{Z}}$ [37, 56, 88]

$$\widetilde{\mathcal{V}}'|_{0,\mathbb{Z}} := \rho_{\varphi_0}\left(H^q(\mathscr{X}(\mathbb{C})_{\varphi_0},\mathbb{Z})\right). \tag{3.52}$$

Moreover there is a rational structure on $\widetilde{\mathcal{V}}'|_0$ given by

$$\widetilde{\mathcal{V}}'|_{0,\mathbb{Q}} = \widetilde{\mathcal{V}}'|_{0,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.53}$$

The monodromy operators $\{N_i\}$ define a weight filtration W_* on $\widetilde{\mathcal{V}}'|_{0,\mathbb{Q}}$, while the fibers $\{\widetilde{\mathcal{F}}'^p|_0\}$ define a Hodge filtration F^* on $\widetilde{\mathcal{V}}'|_0$, and we have the following important theorem.

Theorem 3.3.4. *The following data forms a MHS*

$$(\widetilde{\mathcal{V}}'|_{0,\mathbb{Z}}, (\widetilde{\mathcal{V}}'|_{0,\mathbb{Q}}, W_*), (\widetilde{\mathcal{V}}'|_0, W_*, F^*)). \tag{3.54}$$

Proof. See [28, 88]. ☐

3.4 Steenbrink's construction of limit MHS

In this section, we will talk about Steenbrink's construction of limit MHS, which is crucial in showing that the limit MHS is motivic. In the compactification 3.19, $\overline{S}$ is a smooth variety, hence locally there exists a smooth algebraic curve $C \subset \overline{S}$ such that

$$\mathrm{pt} \in C,\ C - \mathrm{pt} \subset S, \tag{3.55}$$

and C is not required to be complete. We further assume C intersects with the discriminant locus of $\overline{S}$ transversely at pt, i.e. the complex curve $C(\mathbb{C})$ intersects with H_i in 3.41 transversely at origin. The restriction of the family $\overline{\mathscr{X}}$ to C is denoted by

$$\pi_C : \mathscr{Z} \to C, \tag{3.56}$$

and the only singular fiber of this family is over pt. For simplicity, let us define

$$C^* := C - \mathrm{pt},\ Z := \pi_C^{-1}(\mathrm{pt}),\ \mathscr{Z}^* := \mathscr{Z} - Z. \tag{3.57}$$

There is an open sub-family over C^* given by

$$\pi_{C^*} : \mathscr{Z}^* \to C^*, \tag{3.58}$$

which is a smooth fibration between smooth varieties. The analytifications of π_C and π_{C^*} will be denoted by

$$\pi_C^{\mathrm{an}} : \mathscr{Z}(\mathbb{C}) \to C(\mathbb{C}),\ \pi_{C^*}^{\mathrm{an}} : \mathscr{Z}^*(\mathbb{C}) \to C^*(\mathbb{C}). \tag{3.59}$$

Suppose Δ is a small neighbourhood of pt in $C(\mathbb{C})$ and the coordinate of pt is 0. For simplicity, let us denote the restriction of π_C^{an} to Δ by

$$\pi_\Delta : \mathcal{Z} \to \Delta, \text{ with } \mathcal{Z} := (\pi_C^{\mathrm{an}})^{-1}(\Delta) \subset \mathscr{X}(\mathbb{C}), \tag{3.60}$$

while let π_{Δ^*} be the restriction of π_Δ to Δ^*

$$\pi_{\Delta^*} : \mathcal{Z}^* \to \Delta^*, \text{ with } \mathcal{Z}^* := (\pi_C^{\mathrm{an}})^{-1}(\Delta^*) = \mathcal{Z} - Z(\mathbb{C}). \tag{3.61}$$

There is a local system $V_{\Delta^*,\mathbb{Z}}$ over Δ^* defined by

$$V_{\Delta^*,\mathbb{Z}} := \mathrm{R}^q\, \pi_{\Delta^*,*}\, \mathbb{Z}, \tag{3.62}$$

which defines a bundle over Δ^*

$$\mathcal{V}_{\Delta^*} := V_{\Delta^*,\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}, \tag{3.63}$$

and its Deligne's canonical extension to Δ will be denoted by $\widetilde{\mathcal{V}}_\Delta$. As before, the bundle $\mathcal{V}_{\Delta^*}$ has a filtration $\mathcal{F}^p_{\Delta^*}$, the Deligne's canonical extension of which to Δ is denoted by $\widetilde{\mathcal{F}}^p_\Delta$. Suppose φ_0 is a point of Δ^*, then the monodromy of $V_{\Delta^*,\mathbb{Z}}$ induces a representation of the fundamental group $\pi_1(\Delta^*, \varphi_0)$

$$\Psi : \pi_1(\Delta^*, \varphi_0) \to \mathrm{Aut}(H^q(\mathcal{Z}_{\varphi_0}, \mathbb{Z})), \tag{3.64}$$

which is determined by the image $\Psi(T)$, where T is a generator of $\pi_1(\Delta^*, \varphi_0)$. From the assumption that the monodromy matrices $\{\Psi(T_i)\}$ in Section 3.3 are unipotent, we deduce that $\Psi(T)$ is also unipotent. From the papers [28, 29], the limit MHS on the fiber $\widetilde{\mathcal{V}}_\Delta|_0$ defined by the family over C does not depend on the choice of C and it is the same as the limit MHS 3.54 constructed in Section 3.3.

3.4.1 Semi-stable reduction

The family $\mathscr{X}$ has a semi-stable reduction (Chapter II of [71])

$$\begin{tikzcd}
\mathscr{X}' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right, "\pi_{C'}"'] & & \\
& C' \times_C \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\pi_C"] \\
& C' \arrow[r] & C
\end{tikzcd} \tag{3.65}$$

and we now explain the terms in this commutative diagram:

1. C' is a smooth curve and the morphism $C' \to C$ is finite, moreover the inverse image of $\mathrm{pt} \in C$ is a single point $\mathrm{pt}' \in C'$.
2. $\mathscr{X}'$ is a smooth variety and the morphism $\mathscr{X}' \to C' \times_C \mathscr{X}$ is the blow up of an ideal sheaf $\mathscr{I}$ that is trivial outside the fiber of $C' \times_C \mathscr{X}$ over pt'. Therefore over $C' - \mathrm{pt}'$, the morphism $\mathscr{X}' \to C' \times_C \mathscr{X}$ is an isomorphism.
3. The fiber $\pi_{C'}^{-1}(\mathrm{pt}')$ is a reduced divisor with smooth irreducible components crossing normally.

Since the matrix $\Psi(T)$ is already unipotent, it follows from the construction in Section 3.3 that the limit MHS of the family $\mathscr{X}'(\mathbb{C})$ over pt' is the same as the limit MHS of the family $\mathscr{X}(\mathbb{C})$ over pt. Hence from now on, we will assume that the singular fiber $Z = \pi_C^{-1}(\mathrm{pt})$ of $\mathscr{X}$ is reduced with smooth irreducible components crossing normally.

Remark 3.4.1. *However currently there is no semi-stable reduction theorem for fibrations where the dimension of the base variety is ≥ 2 [27].*

3.4.2 Extensions of the de Rham cohomology sheaves

Similarly as before there is a complex of sheaves of algebraic forms $\Omega^*_{\mathscr{X}^*/C^*}$ and the relative de Rham cohomology sheaf $\mathscr{V}_{C^*}$ 3.56 is given by [1]

$$\mathscr{V}_{C^*} := \mathrm{R}^q\, \pi_{C^*,*}(\Omega^*_{\mathscr{X}^*/C^*}) \tag{3.66}$$

which is a locally free sheaf over C^* since π_{C^*} is a smooth fibration between smooth varieties. The fiber of $\mathscr{V}_{C^*}$ over a closed point $\varphi \in C^*$ is the q-th algebraic de Rham cohomology $\mathbb{H}^q(\mathscr{X}^*_\varphi, \Omega^*_{\mathscr{X}^*_\varphi})$ of $\mathscr{X}^*_\varphi$ [53, 70, 95]. The complex $\Omega^*_{\mathscr{X}^*/C^*}$ is naively filtered by the complex $F^p\,\Omega^*_{\mathscr{X}^*/C^*}$, whose cohomology yields a locally free sheaf $\mathscr{F}^p_{C^*}$ that filters $\mathscr{V}_{C^*}$. Let $\Omega^*_{\mathscr{X}/C}(\log Z)$ be the complex of sheaves of algebraic forms over C with at worst logarithmic poles along the divisor Z, which also has a naive filtration $F^p\,\Omega^*_{\mathscr{X}/C}(\log Z)$ [95]. The sheaves $\widetilde{\mathscr{V}}_C$ and $\widetilde{\mathscr{F}}^p_C$ are defined by

$$\widetilde{\mathscr{V}}_C := \mathrm{R}^q\, \pi_{C,*}(\Omega^*_{\mathscr{X}/C}(\log Z)), \quad \widetilde{\mathscr{F}}^p_C := \mathrm{R}^q\, \pi_{C,*}(F^p\,\Omega^*_{\mathscr{X}/C}(\log Z)), \tag{3.67}$$

which are locally free over C, and moreover they are the extensions of $\mathscr{V}_{C^*}$ and $\mathscr{F}^p_{C^*}$ to C respectively. The Gauss-Manin connection ∇ of $\mathscr{V}_{C^*}$ is canonically extended to a connection ∇ of $\widetilde{\mathscr{V}}_C$ which has a logarithmic pole at pt with a nilpotent residue. On the other hand this property also determines the extension of $\mathscr{V}_{C^*}$ (resp. $\mathscr{F}^p_{C^*}$) to C uniquely [37, 70, 85, 95]. The connection ∇ also satisfies Griffiths transversality

$$\nabla(\widetilde{\mathscr{F}}^p_C) \subset \Omega^1_C(\log \mathrm{pt}) \otimes \widetilde{\mathscr{F}}^{p-1}_C. \tag{3.68}$$

Let the analytification of $\mathscr{V}_{C^*}$ (resp. $\mathscr{F}^p_{C^*}$) be $\mathscr{V}^{\mathrm{an}}_{C^*}$ (resp. $\mathscr{F}^{p,\mathrm{an}}_{C^*}$), and let the analytification of $\widetilde{\mathscr{V}}_C$ (resp. $\widetilde{\mathscr{F}}^p_C$) be $\widetilde{\mathscr{V}}^{\mathrm{an}}_C$ (resp. $\widetilde{\mathscr{F}}^{p,\mathrm{an}}_C$), then we have [95]

$$\begin{aligned}\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_\Delta &= \mathrm{R}^q\,\pi_{\Delta,*}(\Omega^*_{\mathcal{Z}/\Delta}(\log\,Z(\mathbb{C})),\\ \widetilde{\mathscr{F}}^{p,\mathrm{an}}_C|_\Delta &= \mathrm{R}^q\,\pi_{\Delta,*}(F^p\,\Omega^*_{\mathcal{Z}/\Delta}(\log\,Z(\mathbb{C})).\end{aligned} \tag{3.69}$$

Under comparison isomorphism, the extension $\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_\Delta$ (resp. $\widetilde{\mathscr{F}}^{p,\mathrm{an}}_C|_\Delta$) is isomorphic to the Deligne's canonical extension $\widetilde{\mathcal{V}}_\Delta$ (resp. $\widetilde{\mathcal{F}}^p_\Delta$) [37, 56, 88]. The fiber $\widetilde{\mathscr{V}}_C|_{\mathrm{pt}}$ has an explicit description [95]

$$\widetilde{\mathscr{V}}_C|_{\mathrm{pt}} := (\widetilde{\mathscr{V}}_C)_{\mathrm{pt}} \otimes_{\mathcal{O}_{C,\mathrm{pt}}} \mathcal{O}_{C,\mathrm{pt}}/\mathfrak{m}_{C,\mathrm{pt}} = \mathbb{H}^q\,(Z, \Omega^*_{\mathscr{X}/C}(\log\,Z)|_Z), \tag{3.70}$$

which is just the hypercohomology of the restriction of $\Omega^*_{\mathscr{X}/C}(\log\,Z)$ to Z. Similarly, $\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}}$ has a description given by [85, 95]

$$\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}} := \mathbb{H}^q(Z(\mathbb{C}), \Omega^*_{\mathcal{Z}/\Delta}(\log\,Z(\mathbb{C}))|_{Z(\mathbb{C})}). \tag{3.71}$$

Given a point $\varphi \in \Delta^*$, the fiber $\widetilde{\mathscr{V}}_C|_\varphi$ satisfies [70, 85]

$$\widetilde{\mathscr{V}}_C|_\varphi = \mathbb{H}^q\,(\mathscr{X}_\varphi, \Omega^*_{\mathscr{X}/C}(\log\,Z)|_{\mathscr{X}_\varphi}) = \mathbb{H}^q(\mathscr{X}_\varphi, \Omega^*_{\mathscr{X}_\varphi}), \tag{3.72}$$

while the fiber $\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_\varphi$ satisfies [85, 95]

$$\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_\varphi = \mathbb{H}^q(\mathcal{Z}_\varphi, \Omega^*_{\mathcal{Z}/\Delta}(\log\,Z(\mathbb{C}))|_{\mathcal{Z}_\varphi}) = H^q_{\mathrm{dR}}(\mathcal{Z}_\varphi, \Omega^*_{\mathcal{Z}_\varphi}) \simeq H^q(\mathcal{Z}_\varphi, \mathbb{C}). \tag{3.73}$$

We have used the following canonical isomorphisms

$$\mathbb{H}^q(\mathscr{X}_\varphi, \Omega^*_{\mathscr{X}_\varphi}) = H^q_{\mathrm{dR}}(\mathcal{Z}_\varphi, \Omega^*_{\mathcal{Z}_\varphi}) \simeq H^q(\mathcal{Z}_\varphi, \mathbb{C}). \tag{3.74}$$

3.4.3 Limit Mixed Hodge Structure

The action of $\Psi(T)$ on $H^q(\mathcal{Z}_{\varphi_0}, \mathbb{Z})$ in equation 3.64 is extended to an automorphism of the sheaf $\widetilde{\mathcal{V}}_\Delta$ ($\simeq \widetilde{\mathscr{V}}^{\mathrm{an}}_C|_\Delta$), which further induces an automorphism T_0 of the fiber $\widetilde{\mathcal{V}}_\Delta|_{\mathrm{pt}}$ ($\simeq \widetilde{\mathscr{V}}^{\mathrm{an}}_C|_0$) (Proposition 11.2 of [85]). There exists an endomorphism N_0 of $\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}}$ defined by the Gauss-Manin connection ∇ [95]. Let Res_0 be the residue map from $\Omega^1_C(\log\,\mathrm{pt})$ to $\mathbb{C}$ defined by

$$\mathrm{Res}_0\,(\,g(t)\,dt/t\,) = g(0). \tag{3.75}$$

There is a homomorphism from the germ $(\widetilde{\mathscr{V}}^{\mathrm{an}}_C)_{\mathrm{pt}}$ to the fiber $\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}}$ defined by

$$\big(\mathrm{Res}_0 \otimes (\otimes_{\mathcal{O}_{C,\mathrm{pt}}}\mathcal{O}_{C,\mathrm{pt}}/\mathfrak{m}_{C,\mathrm{pt}})\big) \circ \nabla : (\widetilde{\mathscr{V}}^{\mathrm{an}}_C)_{\mathrm{pt}} \to \widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}}, \tag{3.76}$$

which vanishes on $(\widetilde{\mathscr{V}}_C^{\mathrm{an}})_{\mathrm{pt}} \otimes_{\mathcal{O}_{C,\mathrm{pt}}} \mathfrak{m}_{C,\mathrm{pt}}$, hence it defines an endomorphism N_0 of $\widetilde{\mathscr{V}}_C^{\mathrm{an}}|_{\mathrm{pt}}$ that is called the residue of ∇ at 0. From Theorem II 3.11 of [38] or Corollary 11.17 of [85] we have

$$T_0 = \exp(-2\pi i\, N_0). \tag{3.77}$$

The upper half plane $\mathbb{H}$ of $\mathbb{C}$ forms a universal cover of Δ^* by the holomorphic map $\varphi \mapsto \exp(2\pi i\, \varphi)$, then we get a holomorphic map from $\mathbb{H}$ to Δ given by the composition

$$\mathbb{H} \to \Delta^* \hookrightarrow \Delta. \tag{3.78}$$

From this map, we obtain a commutative diagram

$$\begin{tikzcd}
\mathscr{Z} \times_\Delta \mathbb{H} \arrow[r, "p"] \arrow[d] & \mathscr{Z} \arrow[d, "\pi_\Delta"] & Z(\mathbb{C}) \arrow[l, hook', "i_Z"'] \arrow[d] \\
\mathbb{H} \arrow[r] & \Delta & 0 \arrow[l, hook']
\end{tikzcd} . \tag{3.79}$$

Definition 3.4.2. *Let Λ be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$, and it is also considered as the constant sheaf on $\mathscr{Z}$, then the nearby cycle sheaf $R\Psi_{\pi_\Delta}(\Lambda)$ is defined by*

$$R\Psi_{\pi_\Delta}(\Lambda) := i_Z^* Rp_* p^* \Lambda, \tag{3.80}$$

which is a complex of sheaves on $Z(\mathbb{C})$ that lies in the derived category $D^+(Z(\mathbb{C}), \Lambda)$. See Exposé XIV of [55] for more detail.

In the paper [95], Steenbrink constructs the following data:

1. A representative of $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Z}$ in the derived category $D^+(Z(\mathbb{C}), \mathbb{Z})$.

2. A representative of $(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}, W_*)$ in the filtered derived category $D^+F(Z(\mathbb{C}), \mathbb{Q})$, where W_* is an increasing filtration on $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}$ and

$$\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q} \simeq \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Z} \otimes \mathbb{Q}, \text{ in } D^+(Z(\mathbb{C}), \mathbb{Q}). \tag{3.81}$$

3. A representative of the object $(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}, W_*, F^*)$ in the bifiltered derived category $D^+F_2(Z(\mathbb{C}), \mathbb{C})$, where W_* is an increasing filtration on $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}$ and F^* is a decreasing filtration on $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}$, such that

$$(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}, W_*) \simeq (\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q} \otimes \mathbb{C}, W_*) \text{ in } D^+F(Z(\mathbb{C}), \mathbb{C}). \tag{3.82}$$

These constructions depend on the choice of φ and $\log \varphi$ on Δ^* [63, 95], both of which will be fixed in this thesis.

Theorem 3.4.3. *The following data*

$$(R\Psi_{\pi_\Delta}\mathbb{Z}, (R\Psi_{\pi_\Delta}\mathbb{Q}, W_*), (R\Psi_{\pi_\Delta}\mathbb{C}, W_*, F^*)) \tag{3.83}$$

forms a cohomological mixed Hodge complex of sheaves in the sense of Deligne [39].

Proof. Chapter 11 of [85]. □

Let us denote the triangulated category of $\mathbb{Z}$-mixed Hodge complexes by $D^*_{\mathbf{MHS}_\mathbb{Z}}$, where $*$ is a boundedness condition, i.e. $*$ could be $\emptyset$, $+$, $-$ or b, which is also called the derived category of mixed Hodge complexes over $\mathbb{Z}$ [12]. From Proposition 8.1.7 of [39], the mixed Hodge complex of sheaves in equation 3.83 yields an object of $D^+_{\mathbf{MHS}_\mathbb{Z}}$

$$(\mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Z}), \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}, W_*), \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}, W_*, F^*)). \tag{3.84}$$

For every $q \in \mathbb{Z}$, there exists a cohomological functor $\underline{H}^q$ from $D^+_{\mathbf{MHS}_\mathbb{Z}}$ to $\mathbf{MHS}_\mathbb{Z}$ [12]

$$\underline{H}^q : D^+_{\mathbf{MHS}_\mathbb{Z}} \to \mathbf{MHS}_\mathbb{Z}, \tag{3.85}$$

which sends the mixed Hodge complex 3.84 to the following MHS

$$(\underline{H}^q \circ \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Z}), \underline{H}^q \circ \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}, W_*), \underline{H}^q \circ \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}, W_*, F^*)). \tag{3.86}$$

This MHS is also denoted by

$$(\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Z}), (\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}), W_*), (\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}), W_*, F^*))\,. \tag{3.87}$$

Steenbrink proves the following important proposition in [85, 95].

Proposition 3.4.4. *There exists a quasi-isomorphism between the complex of sheaves* $R\Psi_{\pi_\Delta}\mathbb{C}$ *and* $\Omega^*_{\mathcal{Z}/\Delta}(\log\, Z(\mathbb{C}))|_{Z(\mathbb{C})}$ *in the derived category* $D^+(Z(\mathbb{C}), \mathbb{C})$*, which depends on the choice of* φ *and* $\log\,\varphi$.

Proof. Chapter 11 of [85]. □

From this proposition we have

$$\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_{\mathrm{pt}} = \mathbb{H}^q(Z(\mathbb{C}), \Omega^*_{\mathcal{Z}/\Delta}(\log\, Z(\mathbb{C}))|_{Z(\mathbb{C})}) \simeq \mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}(\mathbb{C})). \tag{3.88}$$

Steenbrink constructs an endomorphism ν_N of the complex of sheaves $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}$, which induces the endomorphism N_0 after taking the hypercohomology of $\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}$ [85, 95]. The endomorphism ν_N also induces a morphism

$$\nu_N : \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q} \to \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}(-1), \tag{3.89}$$

where (-1) means Tate twist. After taking hypercohomology, the morphism 3.89 becomes

$$N_0 : \mathbb{H}^n(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}(\mathbb{Q})) \to \mathbb{H}^n(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}(\mathbb{Q}))(-1), \tag{3.90}$$

which determines the weight filtration W_* in 3.87 uniquely [63, 95, 96]. The Hodge filtration F^* 3.87 on $\widetilde{\mathscr{V}}_C^{\mathrm{an}}|_{\mathrm{pt}}$ is given by the fiber $\widetilde{\mathscr{F}}_C^{p,\mathrm{an}}|_{\mathrm{pt}}$ [95, 96]

$$F^p = \widetilde{\mathscr{F}}_C^{p,\mathrm{an}}|_{\mathrm{pt}} = \widetilde{\mathscr{F}}_C^{p}|_{\mathrm{pt}}. \tag{3.91}$$

It has been proved by Steenbrink that Schmid's construction and his construction are compatible with each other [85, 95, 96].

3.5 Limit MHS is motivic

The $\mathbb{Z}$-mixed Hodge complex 3.84 gives us a $\mathbb{Q}$-mixed Hodge complex by forgetting the integral structure

$$\mathbf{Y}^\bullet := \big(\mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}), \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}, W_*), \mathrm{R}\Gamma(\mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}, W_*, F^*)\big), \tag{3.92}$$

whose hypercohomology is the underlying rational MHS of 3.87

$$\underline{H}^q(\mathbf{Y}^\bullet) = \big(\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}), (\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q}), W_*), (\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{C}), W_*, F^*)\big). \tag{3.93}$$

Now we need the following lemma from [54].

Lemma 3.5.1. $\mathbb{H}^q(Z(\mathbb{C}), R\Psi_{\pi_\Delta}\Lambda)$ *is isomorphic to* $H^q(\mathcal{Z}_\varphi, \Lambda)$ *when* Λ *is* $\mathbb{Z}$, $\mathbb{Q}$ *or* $\mathbb{C}$.

Therefore $\mathbb{H}^q(Z(\mathbb{C}), \mathrm{R}\Psi_{\pi_\Delta}\mathbb{Q})$ is 0 when $q < 0$ or $q > 2\dim \mathcal{Z}_\varphi$, which immediately implies that $\underline{H}^q(\mathbf{Y}^\bullet)$ is 0 when $q < 0$ or $q > 2\dim \mathcal{Z}_\varphi$, hence $\mathbf{Y}^\bullet$ is essentially an object of $D^b_{\mathbf{MHS}_\mathbb{Q}}$, which is the bounded derived category of mixed Hodge complexes over $\mathbb{Q}$ [12]. From Theorem 3.4 of [12], the natural functor $D^b(\mathbf{MHS}_\mathbb{Q}) \hookrightarrow D^b_{\mathbf{MHS}_\mathbb{Q}}$ is an equivalence of categories, and under this equivalence $\mathbf{Y}^\bullet$ defines a complex $\mathbf{M}^\bullet$ of $D^b(\mathbf{MHS}_\mathbb{Q})$ such that

$$H^q(\mathbf{M}^\bullet) = \underline{H}^q(\mathbf{Y}^\bullet), \forall\, q \in \mathbb{Z}. \tag{3.94}$$

If the deformation $\pi_S : \mathscr{X} \to S$ is defined over $\mathbb{Q}$ and pt is $\mathbb{Q}$-valued point, then the curve C can also be chosen to be defined over $\mathbb{Q}$ and the family $\pi_C : \mathscr{Z} \to C$ is also defined over $\mathbb{Q}$. In this case, the locally free sheaves $\widetilde{\mathscr{V}}_C$ and $\widetilde{\mathscr{F}}_C^p$ are also rationally defined, and the rational vector space $\widetilde{\mathscr{F}}_C^p|_{\mathrm{pt}}$ filters the rational vector space $\widetilde{\mathscr{V}}_C|_{\mathrm{pt}}$, i.e. the limit Hodge filtration F^p in equation 3.91 is rationally defined. Moreover, from Ayoub's construction of motivic nearby cycle functor, which is briefly talked in Appendix C, there exists a limit mixed motive $\mathcal{M}$ whose Hodge realisation is conjectured to be isomorphic to $\mathbf{M}^\bullet$ by Ayoub.

Remark 3.5.2. *The limit mixed motive* $\mathcal{M}$ *is an object of* $\boldsymbol{DM}_{gm}(\mathbb{Q},\mathbb{Q})$*, which is Voevodsky's triangulated category of mixed motives. The readers who are not familiar with it are referred to Appendix B for a brief overview.*

Summary: in this chapter, we have discussed Deligne's canonical extension and Schmid's construction of the limit MHS at a singular fiber. Later in Chapter 8, we will apply the method in this chapter to compute the limit MHS at the large complex structure limit of the mirror family of the self-mirror Calabi-Yau threefold $X_{\mathfrak{S}}$. While in Chapter 13, we will apply the method in this chapter to compute the limit MHS at the conifold of the mirror family of the quintic.

We have also discussed Steenbrink's construction of the limit MHS, and the crucial object in this construction is a complex in the derived category $D^b(\mathbf{MHS}_\mathbb{Q})$, whose cohomology computes the limit MHS constructed by Schmid. On the other hand, Ayoub has constructed the motivic nearby cycle functor in his thesis [4]. Given an algebraic family defined over $\mathbb{Q}$ and a singular fiber over a rational point of the base variety, Ayoub's construction yields a limit motive which is an object of $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q},\mathbb{Q})$ (Voevodsky's triangulated category of mixed motives). It is conjectured by Ayoub that the Hodge realisation of the limit motive is just the complex in $D^b(\mathbf{MHS}_\mathbb{Q})$ constructed by Steenbrink. This is very crucial when we study the limit MHS at the large complex structure limit.

Chapter 4

Mirror symmetry

In this chapter, we will give a brief overview of mirror symmetry. We will follow the books [30, 51] closely, which are also strongly recommended to the reader unfamiliar with mirror symmetry. The statement of mirror symmetry conjecture in this chapter is from [22, 23, 25]. The structure of this chapter is as follows:

- Section 4.1 is a very short discussion of the complexified Kähler moduli space of Calabi-Yau threefolds.
- Section 4.2 is a very intuitive (non-rigorous) discussion of the Gromov-Witten invariants of a Calabi-Yau threefold, which should not be read seriously.
- Section 4.3 discusses the prepotential of the complexified Kähler moduli space of a Calabi-Yau threefold.
- Section 4.4 discusses the integral periods of the threeform of the mirror family, and it also introduces the definition of the large complex structure limit.
- Section 4.5 briefly discusses the canonical periods of the threeform of the mirror family, which are the canonical solutions of Picard-Fuchs equations.
- Finally, Section 4.6 introduces the mirror symmetry conjecture.

4.1 Complexified Kähler moduli space

Given a mirror pair (M, W) of Calabi-Yau threefolds, the complexified Kähler moduli space of M, denoted by $\mathscr{M}_K(M)$, is essentially the space whose point represents the Kähler structures on M. We will now give a brief description of $\mathscr{M}_K(M)$ which follows from the books [30, 51]. Recall that the Hodge diamond of the Calabi-Yau threefold M is of the form

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{12} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

where $h^{11} = \dim H^{1,1}(M)$ and $h^{12} = \dim H^{1,2}(M)$, in particular we have

$$H^1(M,\mathbb{R}) = H^5(M,\mathbb{R}) = 0. \tag{4.1}$$

The real vector space $H^{1,1}(M,\mathbb{R})$ is defined by

$$H^{1,1}(M,\mathbb{R}) := H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C}), \tag{4.2}$$

i.e. it consists of the element of $H^2(M,\mathbb{R})$ that can be represented by a real closed form of type $(1,1)$. The Kähler cone of M is defined by

$$\mathcal{K}_M := \{\omega \in H^{1,1}(M,\mathbb{R}) \mid \omega \text{ can be represented by a Kähler form of } M\}, \tag{4.3}$$

which is an open cone in $H^{1,1}(M,\mathbb{R})$ [51]. The complexified Kähler moduli space of M is defined by [51]

$$\mathscr{M}_K(M) := (H^2(M,\mathbb{R}) + i\,\mathcal{K}_M)/H^2(M,\mathbb{Z}). \tag{4.4}$$

We have two important examples.

Example 4.1.1. *Given a one-parameter mirror pair (M,W), i.e. $h^{1,1}(M) = 1$, the Kähler cone $\mathcal{K}_M$ is the open ray $\mathbb{R}_{>0}$ in $\mathbb{R}$, hence $\mathscr{M}_K(M)$ has a very simple description [51]*

$$\mathscr{M}_K(M) = (\mathbb{R} + i\,\mathbb{R}_{>0})/\mathbb{Z} = \mathbb{H}/\mathbb{Z}, \tag{4.5}$$

where $\mathbb{H}$ is the upper half plane of $\mathbb{C}$. Now let e be a basis of $H^2(M,\mathbb{Z})$ (modulo torsion) that lies in the Kähler cone $\mathcal{K}_M$ [51], then every point of $\mathscr{M}_K(M)$ is represented by $t\,e$ with $t \in \mathbb{H}$, while $e\,t$ is equivalent to $e\,(t+1)$. Conventionally t is called the flat coordinate of $\mathscr{M}_K(M)$ [22, 23].

Example 4.1.2. *When $h^{11}(M) > 1$, a framing of M is a basis $\{e_i\}_{i=1}^{h^{11}(M)}$ of $H^2(M,\mathbb{Z})$ (modulo torsions) such that each e_i lies in $\overline{\mathcal{K}}_M$ (the closure of $\mathcal{K}_M$ in $H^{1,1}(M,\mathbb{R})$). A framing defines a cone in $H^2(M,\mathbb{R})$ by*

$$\Sigma := \{x \in H^2(M,\mathbb{R}) : x = \sum_{i=1}^{h^{11}(M)} t_i\, e_i, \textit{ with } t_i > 0\}. \tag{4.6}$$

In this case, the complexified Kähler moduli space of M *is set to be*

$$\mathscr{M}_{K,\Sigma}(M) := (H^2(M,\mathbb{R}) + i\,\Sigma)/H^2(M,\mathbb{Z}), \tag{4.7}$$

which has an explicit description: $\{\sum_i t_i\, e_i,\ t_i \in \mathbb{H}\}/\mathbb{Z}^{h^{11}(M)}$. *The coordinates* $\{t_i\}$ *are called flat coordinates of the complexified Kähler moduli space.*

4.2 Gromov-Witten invariants and Yukawa couplings

This section is a leisurely discussion of Gromov-Witten invariants, which should not be taken seriously, and the readers are referred to books [30, 51] for more rigorous treatment. Suppose η is an element of $H_2(M,\mathbb{Z})$, then we would want to count the 'number' of holomorphic maps (up to equivalence)

$$\mathbb{P}^1_{\mathbb{C}} \to M, \tag{4.8}$$

whose image lies in the homology class η. However even for a general complex structure of M, this number is not always well defined [51]. To cure this disease, it is necessary to deform the complex structure of M to a general almost complex structure J, then we can count the number of J-holomorphic curves.

Definition 4.2.1. *Suppose* J *is an almost complex structure on* M *and* Σ_g *is a genus-*g *Riemann surface with complex structure* j. *A smooth map* $u : \Sigma_g \to M$ *is called* J*-holomorphic if the push-forward map* u_* *satisfies*

$$J \circ u_* = u_* \circ j, \tag{4.9}$$

where the composition is a map from the tangent bundle of Σ_g *to the tangent bundle of* M.

Let $\mathscr{M}(\eta, J, \Sigma_g)$ be the moduli space of J-holomorphic maps $\Sigma_g \to M$ whose image lies in the homology class η. For an arbitrary almost complex structure J, this moduli space can be very bad, therefore it is necessary to restrict our attention to **nice** J, where the definition of nice is left to the books [30, 51]. For a 'nice' J, the moduli space $\mathscr{M}(\eta, J, \Sigma_g)$ is a smooth manifold of real dimension $3(2-2g)$, which also has a natural orientation. Here we have used the properties that the dimension of M is 3 and its first Chern class is zero since M is Calabi-Yau. See Theorem 15.4 of [51] for a more careful discussion.

If the Riemann surface Σ_g is $\mathbb{P}^1_{\mathbb{C}}$ of genus $g = 0$, then the real dimension of $\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})$ is 6. On the other hand, the real dimension of the automorphism group of $\mathbb{P}^1_{\mathbb{C}}$, i.e. $\mathrm{PGL}(2,\mathbb{C})$, is also 6. Furthermore, for $x \in \mathrm{PGL}(2,\mathbb{C})$, $u \circ x$ is also a J-holomorphic curve, hence this defines an action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})$, and the dimension of the quotient space $\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})$ is expected to be zero. After compactification, $\overline{\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})}$ consists of finitely many points, hence the point-counting of it is well defined, which will be denoted by

$$\#\overline{\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})}. \tag{4.10}$$

It is very subtle that this point-counting process actually includes the orientations of the points of $\overline{\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})}$, so the result can be negative. The number 4.10 is essentially independent of the choice of the 'nice' almost complex structure J, and the readers are referred to [30, 51] for more rigorous and thorough treatments.

Given homology classes $D_1, D_2, D_3 \in H^2(M,\mathbb{C})$, the Gromov-Witten invariant is defined by [30, 51]

$$\Psi_\eta(D_1, D_2, D_3) := (D_1 \cdot \eta)\,(D_2 \cdot \eta)\,(D_3 \cdot \eta)\,\#\overline{\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})}. \tag{4.11}$$

If there is a framing $\{e_j\} \subset \overline{\mathcal{K}}_M$, then $\mathscr{M}_{K,\Sigma}(M)$ is constructed in Example 4.1.2, and the coordinate q_j is defined by

$$q_j := \exp 2\pi i\, t_j. \tag{4.12}$$

Now suppose $H_2(M,\mathbb{Z})$ is torsion free, then let us define $< D_1, D_2, D_3 >$ by

$$< D_1, D_2, D_3 >:= D_1 \cdot D_2 \cdot D_3 + \sum_{0 \neq \eta \in H_2(M,\mathbb{Z})} \Psi_\eta(D_1, D_2, D_3) \frac{\prod_j q_j^{e_j \cdot \eta}}{1 - \prod_j q_j^{e_j \cdot \eta}}, \tag{4.13}$$

where $D_1 \cdot D_2 \cdot D_3$ is the intersection number $\int_M D_1 \wedge D_2 \wedge D_3$.

Remark 4.2.2. *With this definition of Yukawa coupling, we are only counting generically injective maps.*

Notice that in order for the number $\#\overline{\mathscr{M}(\eta, J, \mathbb{P}^1_{\mathbb{C}})/\mathrm{PGL}(2,\mathbb{C})}$ 4.10 to be non-zero, the pairing $e_j \cdot \eta$ must be ≥ 0, since e_j lies in the closure of Kähler cone of M. Therefore $< D_1, D_2, D_3 >$ lies in the ring $\mathbb{C}[[q_1, \cdots, q_{h^{11}(M)}]]$ of power series in q_j, however it is unclear whether this power series converges on $\mathscr{M}_{K,\Sigma}(M)$ or not, which is in fact still an unsolved question [30, 51]. The tangent space of a point of $\mathscr{M}_{K,\Sigma}(M)$ is naturally $H^2(M,\mathbb{C})$, hence modulo the convergence issue, we have got a cubic form defined on the tangent bundle of $\mathscr{M}_{K,\Sigma}(M)$, which is called Yukawa coupling. The component $\mathcal{Y}^K_{ijk}$ of Yukawa coupling is defined by

$$\mathcal{Y}^K_{ijk} = Y_{ijk} + \sum_{0\neq\eta\in H_2(M,\mathbb{Z})} \Psi_\eta(e_i, e_j, e_k) \frac{\prod_l q_l^{e_l\cdot\eta}}{1-\prod_l q_l^{e_l\cdot\eta}}, \tag{4.14}$$

where Y_{ijk} is the intersection number given by

$$Y_{ijk} := \int_M e_i \wedge e_j \wedge e_k, \tag{4.15}$$

which is a non-negative integer.

4.3 Prepotential

There exists a prepotential $\mathcal{F}$ defined on the complexified Kähler moduli space $\mathscr{M}_K(M)$, and when $t_k \to i\,\infty$, it has an expansion of the form [22, 23]

$$\mathcal{F} = -\frac{1}{6}\sum Y_{ijk}t^it^jt^k - \frac{1}{2}\sum Y_{0ij}t^it^j - \frac{1}{2}\sum Y_{00i}t^i - \frac{1}{6}Y_{000} + \mathcal{F}^{\text{np}}, \tag{4.16}$$

where $\mathcal{F}^{\text{np}}$ is the non-perturbative instanton correction which is invariant under the operation $t_k \to t_k + 1$ and is exponentially small when $t_k \to i\,\infty$, i.e. it admits a series expansion in $q_k = \exp(2\pi i\, t_k)$ with no constant term [22]. The coefficients Y_{0ij} and Y_{00i} will be determined by mirror symmetry, which are rational numbers. The coefficient Y_{000} is certainly the most mysterious one and in all examples of mirror pairs where it has been computed, it is always of the form [23]

$$Y_{000} = -3\,\chi(M)\,\frac{\zeta(3)}{(2\pi i)^3} + r,\ r \in \mathbb{Q}, \tag{4.17}$$

where $\chi(M)$ is the Euler characteristic of M. It is believed that equation 4.17 is valid for an arbitrary mirror pair but currently there isn't a proof available. The mirror period vector Π on $\mathscr{M}_{K,\Sigma}(M)$ is a column vector defined by [22, 23]

$$\Pi = (\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_{h^{11}}, 1, t^1, \cdots, t^{h^{11}})^t, \text{ with } \mathcal{F}_0 = 2\,\mathcal{F} - \sum_i t_i\,\frac{\partial\mathcal{F}}{\partial t_i},\ \mathcal{F}_i = \frac{\partial\mathcal{F}}{\partial t^i}, \tag{4.18}$$

where ^{t} means transpose. The i-th monodromy of Π is induced by the operation

$$t_i \to t_i + 1,\ t_k \to t_k,\ \text{for } k \neq i, \tag{4.19}$$

under which $\mathcal{F}^{\mathrm{np}}$ is invariant. The i-th monodromy matrix $T_{K,i}$ of Π under the operation 4.19 is defined by

$$\Pi_a \to \sum_{b=0}^{2h^{11}+1} (T_{K,i})_{b\,a}\,\Pi_b. \tag{4.20}$$

The matrix $T_{K,i}$ can be easily computed from the expansion of $\mathcal{F}$ in equation 4.16, which is a $(2\,h^{11}+2)\times(2\,h^{11}+2)$ matrix with entries as linear functions of the coefficients Y_{ijk}, Y_{0ij} and Y_{00i}, see Section 4 of [22] for more detail.

4.4 Periods of the threeform of the mirror family

In this section, we will talk about the integral periods of the threeform Ω of the mirror family and definition of large complex structure limit. Suppose locally there is a deformation of the mirror threefold W of the form

$$\pi : \mathcal{W} \to (\Delta^*)^{h^{12}},\ h^{12} := h^{12}(W), \tag{4.21}$$

which is a smooth fibration between smooth manifolds. There are local systems over $(\Delta^*)^{h^{12}}$ defined by

$$V_{\mathbb{Z}} := \mathrm{R}^3\,\pi_*\mathbb{Z},\ V_{\mathbb{Q}} := \mathrm{R}^3\,\pi_*\mathbb{Q},\ V_{\mathbb{C}} := \mathrm{R}^3\,\pi_*\mathbb{C}. \tag{4.22}$$

4.4.1 Integral symplectic basis

Given a point $\varphi_0 \in (\Delta^*)^{h^{12}}$, we now introduce a construction of an integral symplectic basis of $H_3(\mathcal{W}_{\varphi_0},\mathbb{Z})$ (modulo torsions).

Lemma 4.4.1. *There exists an integral symplectic basis of $H_3(\mathcal{W}_{\varphi_0},\mathbb{Z})$ (modulo torsions)*

$$A_0^s,\, A_1^s, \cdots,\, A_{h^{12}}^s,\, B_0^s,\, B_1^s, \cdots,\, B_{h^{12}}^s, \tag{4.23}$$

which satisfies the following intersection pairings

$$A_a^s\cdot A_b^s = 0,\quad B_a^s\cdot B_b^s = 0,\quad A_a^s\cdot B_b^s = \delta_{ab}. \tag{4.24}$$

Proof. Let $[\mathcal{W}_{\varphi_0}]$ be the fundamental class associated to the manifold $\mathcal{W}_{\varphi_0}$, which is an element of $H_6(\mathcal{W}_{\varphi_0},\mathbb{Z})$. From Poincaré duality, there exists an isomorphism given by [59]

$$[\mathcal{W}_{\varphi_0}] : \phi \in H^3(\mathcal{W}_{\varphi_0},\mathbb{Z}) \mapsto [\mathcal{W}_{\varphi_0}] \frown \phi \in H_3(\mathcal{W}_{\varphi_0},\mathbb{Z}), \tag{4.25}$$

while $[\mathcal{W}_{\varphi_0}] \frown \phi$ is called the Poincaré dual of ϕ, and vice versa. Given two cycles $A, B \in H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$, their Poincaré duals $A^*, B^* \in H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ satisfy

$$A^* \smile B^* = (A \cdot B)^* \in H^6(\mathcal{W}_{\varphi_0}, \mathbb{Z}). \tag{4.26}$$

The integral symplectic basis will be constructed by induction. First choose an element A_0^s of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ such that it is not a $k \in \mathbb{Z}, k \geq 2$ multiple of any other element in $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$. We choose a sublattice V_0 of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ such that

$$H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z}) = \mathbb{Z}A_0^s \oplus V_0. \tag{4.27}$$

Let α_s^0 be the element of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ such that

$$\alpha_s^0(A_0^s) = A_0^s \frown \alpha_s^0 = 1, \quad \alpha_s^0(V_0) = 0. \tag{4.28}$$

Moreover, let B_0 be the Poincaré dual of α_s^0 and let $-\beta_s^0$ be the Poincaré dual of A_0^s, then we have

$$A_0^s \cdot B_0^s = [\mathcal{W}_{\varphi_0}] \frown (-\beta_s^0 \smile \alpha_s^0) = ([\mathcal{W}_{\varphi_0}] \frown -\beta_s^0) \frown \alpha_s^0 = A_0^s \frown \alpha_s^0 = 1, \tag{4.29}$$

which also implies $B_0^s \cdot A_0^s = -1$. Similarly we also have

$$\begin{aligned} A_0^s \frown \beta_s^0 = ([\mathcal{W}_{\varphi_0}] \frown -\beta_s^0) \frown \beta_s^0 \quad &= [\mathcal{W}_{\varphi_0}] \frown (-\beta_s^0 \smile \beta_s^0) \quad &&= 0, \\ B_0^s \frown \alpha_s^0 = ([\mathcal{W}_{\varphi_0}] \frown \alpha_s^0) \frown \alpha_s^0 \quad &= [\mathcal{W}_{\varphi_0}] \frown (\alpha_s^0 \smile \alpha_s^0) \quad &&= 0, \\ B_0 \frown \beta_s^0 = ([\mathcal{W}_{\varphi_0}] \frown \alpha_s^0) \frown \beta_s^0 \quad &= [\mathcal{W}_{\varphi_0}] \frown (\alpha_s^0 \smile \beta_s^0) \quad &&= 1. \end{aligned} \tag{4.30}$$

Now let V_1 be the sublattice $\ker \alpha_s^0 \cap \ker \beta_s^0$ of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$. For a cycle $C \in V_1$, let ϕ_C be its Poincaré dual, and we have

$$\begin{aligned} A_0 \cdot C &= [\mathcal{W}_{\varphi_0}] \frown (-\beta_s^0 \smile \phi_C) = [\mathcal{W}_{\varphi_0}] \frown (\phi_C \smile \beta'^0) \\ &= ([\mathcal{W}_{\varphi_0}] \frown \phi_C) \frown \beta_s^0 = C \frown \beta_s^0 = 0, \\ B_0 \cdot C &= [\mathcal{W}_{\varphi_0}] \frown (\alpha_s^0 \smile \phi_C) = -[\mathcal{W}_{\varphi_0}] \frown (\phi_C \smile \alpha_s^0) \\ &= (-[\mathcal{W}_{\varphi_0}] \frown \phi_C) \smile \alpha^0 = -C \frown \alpha^0 = 0. \end{aligned} \tag{4.31}$$

A straightforward application of induction will finish the proof. □

From the proof of Lemma 4.4.1, the basis $\{\alpha_s^a, \beta_s^b\}$ of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ satisfies

$$\alpha_s^a(A_b^s) = \delta_{ab}, \ \beta_s^a(B_b^s) = \delta_{ab}, \ \alpha_s^a(B_b^s) = \beta_s^a(A_b^s) = 0, \tag{4.32}$$

from which we get the following pairings

$$[\mathcal{W}_{\varphi_0}] \frown (\alpha_s^a \smile \beta_s^b) = \delta_{ab}, \ [\mathcal{W}_{\varphi_0}] \frown (\alpha_s^a \smile \alpha_s^b) = 0, \ [\mathcal{W}_{\varphi_0}] \frown (\beta_s^a \smile \beta_s^b) = 0, \tag{4.33}$$

therefore $\{\alpha_s^a, \beta_s^b\}$ forms an integral symplectic basis of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$. For later convenience, we also order the two integral symplectic bases in the way

$$\begin{aligned} (A_0, \cdots, A_{2h^{12}+1}) &:= (B_0^s, \cdots, B_{h^{12}}^s, A_0^s, \cdots, A_{h^{12}}^s), \\ (\alpha^0, \cdots, \alpha^{2h^{12}+1}) &:= (\beta_s^0, \cdots, \beta_s^{h^{12}}, \alpha_s^0, \cdots, \alpha_s^{h^{12}}). \end{aligned} \tag{4.34}$$

4.4.2 Integral periods of the threeform

In a simply connected neighbourhood of φ_0 in $(\Delta^*)^{h^{12}}$, the integral symplectic basis $\{A^s_a, B^s_b\}$ of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ extends to an integral symplectic basis $\{A^s_a(\varphi), B^s_b(\varphi)\}$ of $H_3(\mathcal{W}_\varphi, \mathbb{Z})$. While the integral symplectic basis $\{\alpha^a_s, \beta^b_s\}$ of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ extends to an integral symplectic basis $\{\alpha^a_s(\varphi), \beta^b_s(\varphi)\}$ of $H^3(\mathcal{W}_\varphi, \mathbb{Z})$, which is the dual basis of $\{A^s_a(\varphi), B^s_b(\varphi)\}$ [101]. Here we have used the property that the unimodular skew symmetric intersection pairing in Lemma 4.4.1 is preserved by extension. The threeform Ω of the mirror family is a section of the bundle $\mathrm{R}^3\,\pi_*\mathbb{Z} \otimes \mathcal{O}_{(\Delta^*)^{h^{12}}}$ over $(\Delta^*)^{h^{12}}$, i.e. for every φ in $(\Delta^*)^{h^{12}}$, $\Omega(\varphi)$ is a nowhere vanishing holomorphic threeform on $\mathcal{W}_\varphi$. The integral periods of $\Omega(\varphi)$ with respect to the basis $\{A^s_a(\varphi), B^s_b(\varphi)\}$ are defined by

$$z_a(\varphi) = \int_{A^s_a(\varphi)} \Omega(\varphi)\ ,\quad \mathcal{G}_b(\varphi) = \int_{B^s_b(\varphi)} \Omega(\varphi). \tag{4.35}$$

which are multi-valued holomorphic functions on $(\Delta^*)^{h^{12}}$ [22, 23, 51]. The integral periods vector Π is a column vector given by

$$\Pi\,(\varphi) := (\mathcal{G}_0(\varphi), \cdots, \mathcal{G}_{h^{12}}(\varphi), z_0(\varphi), \cdots, z_{h^{12}}(\varphi))^t. \tag{4.36}$$

Under the comparison isomorphism between de Rham cohomology and singular cohomology, $\Omega(\varphi)$ has an expansion of the form

$$\Omega(\varphi) = \sum_{a=0}^{h^{12}} z_a(\varphi)\,\alpha^a_s(\varphi) + \mathcal{G}_a(\varphi)\,\beta^a_s(\varphi) = \sum_{a=0}^{2h^{12}+1} \Pi_a(\varphi)\alpha^a(\varphi). \tag{4.37}$$

The sections $\{A^s_a(\varphi), B^s_b(\varphi)\}$ of $V_\mathbb{Z}$ are usually multi-valued over the punctured polydiscs $(\Delta^*)^{h^{12}}$, and its monodromy induces a representation of the fundamental group $\pi_1((\Delta^*)^{h^{12}}, \varphi_0)$ [101]

$$\Phi : \pi_1((\Delta^*)^{h^{12}}, \varphi_0) \to \mathrm{Aut}(H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})). \tag{4.38}$$

The fundamental group $\pi_1((\Delta^*)^{h^{12}}, \varphi_0)$ is isomorphic to $\mathbb{Z}^{h^{12}}$ with generators $\{T_i\}_{i=1}^{h^{12}}$, so the representation Φ is uniquely determined by the images $\{\Psi(T_i)\}$. Since unimodular intersection pairing is preserved by extension, the image of Φ lies in $\mathrm{Sp}(2h^{12}+2, \mathbb{Z})$ (with respect to the basis $\{A_a, B_b\}$) [22, 23, 30]. Let $T_{C,i}$ be the matrix of $\Phi(T_i)$ with respect to the basis $\{A_a\}_{a=0}^{2h^{12}+1}$ 4.34

$$\Phi(T_i).A_a = \sum_{b=0}^{2h^{12}+1} A_b\,(T_{C,i})_{b\,a}, \tag{4.39}$$

and so $T_{C,i}$ is an integral symplectic matrix. The i-th monodromy of the period Π_a is given by

$$\Pi_a\,(\varphi_0) = \int_{A_a} \Omega(\varphi_0) \to \sum_b (T_{C,i})_{ba} \int_{A_b} \Omega(\varphi_0) = \sum_b\,(T_{C,i})_{ba}\ \Pi_b\,(\varphi_0). \tag{4.40}$$

4.4.3 Definition of the large complex structure limit

The monodromy of the local system $V_{\mathbb{Z}}$ 4.22 induces a representation Ψ which is dual to Φ [101]

$$\Psi : \pi_1((\Delta^*)^{h^{12}}, \varphi_0) \to \mathrm{Aut}(H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})). \tag{4.41}$$

Definition 4.4.2. *The origin of the polydiscs $\Delta^{h^{12}}$ is the large complex structure limit of the mirror family if the matrices $\{\Psi(T_i) - Id\}$ are linearly independent and they satisfy [21, 25, 82]:*

1. $\Psi(T_i) - Id$ *is unipotent.*

2. $(\Psi(T_i) - Id)\,(\Psi(T_j) - Id)\,(\Psi(T_k) - Id) = Y_{ijk}\,R$*, where R is a non-zero constant matrix.*

3. $(\Psi(T_i) - Id)\,(\Psi(T_j) - Id)\,(\Psi(T_k) - Id)\,(\Psi(T_l) - Id) = 0$ *for arbitrary i, j, k and l.*

4. *For every i, there exist j and k such that $Y_{ijk} \neq 0$, which is a non-degeneracy condition.*

Therefore a general linear combination $\sum_i a_i(\Psi(T_i) - \mathrm{Id}),\ a_i \in \mathbb{Q}$ will satisfy

$$\big(\sum_i a_i(\Psi(T_i) - \mathrm{Id})\big)^3 \neq 0,\ \big(\sum_i a_i(\Psi(T_i) - \mathrm{Id})\big)^4 = 0. \tag{4.42}$$

The monodromy operator N_i is defined by

$$N_i := \log \Psi(T_i), \tag{4.43}$$

which are also linearly independent over $\mathbb{Q}$. A general linear combination of the form $N = \sum_i a_i\, N_i,\ a_i \in \mathbb{Q}$ satisfies

$$N^3 \neq 0,\ N^4 = 0. \tag{4.44}$$

4.5 Canonical solutions of Picard-Fuchs equations

From Griffiths transversality, the integral periods $\{\Pi_a\}$ are solutions to Picard-Fuchs equations

$$\mathcal{L}\,\Pi_a = 0, \tag{4.45}$$

where $\mathcal{L}$ is a Picard-Fuchs operator which is a partial differential operator that has regular singularities at the singular locus $\{\varphi_i = 0\}$ of $\Delta^{h^{12}}$ [38, 69]. Now we assume 0 is the large complex structure limit of mirror family, then there are $h^{12}+1$ canonical solutions of the Picard-Fuchs equations of the form

$$\begin{aligned} \varpi_0 &= f_0, \\ \varpi_1 &= \frac{1}{2\pi i}\left(f_0 \log \varphi_1 + f_1\right), \\ &\cdots, \\ \varpi_{h^{12}} &= \frac{1}{2\pi i}\left(f_0 \log \varphi_{h^{12}} + f_{h^{12}}\right), \end{aligned} \tag{4.46}$$

where $\{f_j\}_{j=0}^{h^{12}}$ are holomorphic functions on $\Delta^{h^{12}}$ that satisfy the conditions

$$f_0(0) = 1, \quad f_1(0) = \cdots = f_{h^{12}}(0) = 0. \tag{4.47}$$

There exist another $h^{21}+1$ solutions $\varpi_{h^{12}+1}, \ \cdots, \ \varpi_{2h^{12}+1}$ of the Picard-Fuchs equations which are of the form

$$\sum_{\substack{0\le l\le 3 \\ 1\le k\le h^{12}}} f_{k,l}(\varphi) \, \log^l \varphi_k, \tag{4.48}$$

where $f_{k,l}(\varphi)$ are holomorphic functions on $\Delta^{h^{12}}$, however unlike the one-parameter case, we do not know their precise form in general. The canonical period vector ϖ is the column vector given by

$$\varpi = \left(\varpi_0, \ \varpi_1, \ \cdots, \ \varpi_{2h^{12}+1}\right)^t. \tag{4.49}$$

Since the integral period vector Π and canonical period vector ϖ are two different bases of the solution space of Picard-Fuchs equations, we have

$$\Pi = S\,\varpi, \ S \in \mathrm{GL}(2h^{12}+2, \mathbb{C}), \tag{4.50}$$

and the matrix S will be determined by mirror symmetry. The expansion 4.37 now becomes

$$\Omega(\varphi) = \sum_{a=0}^{2h^{12}+1} \alpha^a(\varphi) \ \Pi_a\,(\varphi) = \sum_{a,b} \alpha^a(\varphi)\, S_{a\,b}\, \varpi_b(\varphi). \tag{4.51}$$

Let us now define the canonical basis $\{\gamma^a\}_{a=0}^{2h^{12}+1}$ of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{C})$ by

$$\gamma^a = \sum_{b=0}^{2h^{12}+1} \alpha^b\, S_{b\,a}, \tag{4.52}$$

then the expansion 4.51 becomes

$$\Omega(\varphi) = \sum_{a=0}^{2h^{12}+1} \gamma^a(\varphi)\, \varpi_a(\varphi), \tag{4.53}$$

where $\gamma^a(\varphi)$ is the extension of γ^a to a section of the local system $V_{\mathbb{C}}$. Let $\{C_a\}_{a=0}^{2h^{12}+1}$ be the dual of $\{\gamma^a\}_{a=0}^{2h^{12}+1}$, and it forms a basis of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{C})$. The canonical period ϖ_a is also given by

$$\varpi_a(\varphi) = \int_{C_a(\varphi)} \Omega(\varphi), \tag{4.54}$$

where $C_a(\varphi)$ is the extension of C_a to a section of $V_{\mathbb{C}}^{\vee}$. Suppose $T_{\mathrm{Can},i}$ is the matrix of $\Phi(T_i)$ with respect to the basis $\{C_a\}_{a=0}^{2h^{12}+1}$

$$\Phi(T_i)\, C_a = \sum_b \, (T_{\mathrm{Can},i})_{b\,a}\, C_b, \tag{4.55}$$

then the i-th monodromy of ϖ_a is given by

$$\varpi_a(\varphi_0) = \int_{C_a} \Omega(\varphi_0) \to \sum_b (T_{\mathrm{Can},i})_{b\,a} \int_{C_b} \Omega(\varphi_0) = \sum_b (T_{\mathrm{Can},i})_{b\,a}\, \varpi_b(\varphi_0). \tag{4.56}$$

The matrix $T_{\mathrm{Can},i}$ is related to the matrix $T_{C,i}$ by

$$T_{\mathrm{Can},i} = S^t\, T_{C,i}\, (S^t)^{-1}, \tag{4.57}$$

therefore at the large complex structure limit, the matrix $T_{\mathrm{Can},i}$ satisfies similar conditions as those in Definition 4.4.2, hence we can check whether a point is the large complex structure limit by looking at the monodromy matrix $T_{\mathrm{Can},i}$.

4.6 Mirror symmetry conjecture

In this section we will talk about the mirror symmetry conjecture and our treatments follow from the papers [22, 23].

4.6.1 Prepotential and Yukawa coupling on the complex side

From [19, 48–50], the integral periods $(z_0, \cdots, z_{h^{12}})$ cannot all vanish at a point φ and locally they define a projective coordinate system of $\mathscr{M}_C(W)$, with respect to which $\mathcal{G}_a$ is a homogeneous function of degree one. The expansion of the threeform Ω becomes

$$\Omega(z) = \sum_a z_a\, \alpha_s^a(z) + \mathcal{G}_a(z)\, \beta_s^a(z). \tag{4.58}$$

From Griffiths transversality, we have

$$\int_W \Omega(z) \wedge \frac{\partial\, \Omega(z)}{\partial\, z_a} = 0, \tag{4.59}$$

which yields the following relation

$$\mathcal{G}_a(z) = \frac{\partial\, \mathcal{G}(z)}{\partial\, z_a}, \text{ where } \mathcal{G}(z) := \frac{1}{2} \sum_b z_b\, \mathcal{G}_b(z). \tag{4.60}$$

The function $\mathcal{G}(z)$ is called the prepotential, and it is a homogeneous function of degree two with respect to the coordinates $(z_0, \cdots, z_{h^{12}})$. The Yukawa couplings $\mathcal{Y}_{abc}$ are defined by

$$\mathcal{Y}_{abc} = \int_W \Omega(z) \wedge \frac{\partial^3\, \Omega(z)}{\partial z_a \partial z_b \partial z_c}, \tag{4.61}$$

and from equation 4.33 and 4.60, we also have

$$\mathcal{Y}_{abc} = \sum_d z_d\, \frac{\partial^3\, \mathcal{G}_d(z)}{\partial z_a \partial z_b \partial z_c} = -\frac{\partial^3\, \mathcal{G}(z)}{\partial z_a \partial z_b \partial z_c}. \tag{4.62}$$

There is a symmetric tensor $\mathcal{Y}$ on $\mathscr{M}_C(W)$ defined by

$$\mathcal{Y} := \sum_{a,b,c} \mathcal{Y}_{abc}\, dz_a\, dz_b\, dz_c. \tag{4.63}$$

From 4.61, the component $\mathcal{Y}_{abc}$ is also given by

$$\mathcal{Y}_{abc} = -\Pi^T\, \Sigma\, \partial_{abc}\, \Pi, \tag{4.64}$$

where Σ is the block matrix

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{4.65}$$

and from equation 4.50, $\mathcal{Y}_{abc}$ becomes

$$\mathcal{Y}_{abc} = -\varpi^T\, S^T\, \Sigma\, S\, \partial_{abc}\, \varpi. \tag{4.66}$$

4.6.2 Mirror map

In all examples of mirror pairs, there exists an integral symplectic basis $\{A^s_a, B^s_b\}$ of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ such that

$$z_a(\varphi) = \lambda\, \varpi_a(\varphi),\ a = 0, \cdots, h^{12}, \tag{4.67}$$

where λ is a nonzero constant. Define t'^i by the quotient ϖ_i/ϖ_0

$$t'^i := \frac{z_i}{z_0} = \frac{\varpi_i}{\varpi_0} = \frac{1}{2\pi i} \log \varphi_i + \frac{f_i(\varphi)}{f_0(\varphi)},\ i = 1, \cdots, h^{12}. \tag{4.68}$$

Definition 4.6.1. *The* ***mirror map*** *is induced by the identification of the flat coordinate $\{t^i\}$ of the complexified Kähler moduli space $\mathscr{M}_{K,\Sigma}(M)$ (see Example 4.1.2) with $\{\varpi_i/\varpi_0\}$ in a local neighbourhood of the large complex structure limit, i.e.*

$$t^i \equiv t'^i. \tag{4.69}$$

Remark 4.6.2. *It is very important that the periods 4.46 satisfy the boundary conditions 4.47.*

4.6.3 Mirror symmetry conjecture

On the complex side, the period vector II is homogeneous of degree 1 with respect to the coordinates $(z_0, \cdots, z_{h^{12}})$. After a rescaling by z_0, II is normalised to

$$\mathrm{II}_{\mathrm{A}} = (\mathcal{G}_0/z_0, \cdots, \mathcal{G}_{h^{12}}/z_0,\ 1, \cdots,\ z_{h^{12}}/z_0)^t, \tag{4.70}$$

which further can be written as

$$\mathrm{II}_{\mathrm{A}} = (2\,\mathcal{G} - \sum_i t'^i \frac{\partial \mathcal{G}}{\partial t'^i}, \frac{\partial \mathcal{G}}{\partial t'^1}, \cdots, \frac{\partial \mathcal{G}}{\partial t'^{h^{12}}}.\ 1,\ t'^1, \cdots, t'^{h^{12}})^t. \tag{4.71}$$

From equation 4.61, the normalisations of $\mathcal{Y}_{abc}$ and $\mathcal{Y}$ are

$$\mathcal{Y}^A_{abc} = \mathcal{Y}_{abc}/z_0^2 = \mathcal{Y}_{abc}/\lambda^2\varpi_0^2,\ \mathcal{Y}^A = \mathcal{Y}/z_0^2 = \mathcal{Y}/\lambda^2\varpi_0^2. \tag{4.72}$$

Mirror Symmetry Conjecture: *given a mirror pair (M, W) of Calabi-Yau threefolds, there exists an integral symplectic basis $\{A_a, B_b\}$ of $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$ such that 4.67 is true. The mirror map in Definition 4.6.1 induces an isomorphism between $\mathscr{M}_{K,\Sigma}(M)$ and a neighbourhood of the large complex structure limit in $\mathscr{M}_C(W)$, under which the mirror period vector Π of $\mathscr{M}_{K,\Sigma}(M)$ is identified with the normalised integral period vector II_A of $\mathscr{M}_C(W)$ [22, 23].*

Take the third partial derivatives of $\mathcal{F}$, then we get the Yukawa coupling $\mathcal{Y}^K_{ijk}$ on the Kähler side given by

$$\mathcal{Y}^K_{ijk} := -\frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_k}. \tag{4.73}$$

From mirror symmetry conjecture, the normalised Yukawa coupling $\mathcal{Y}^A_{ijk}$ on the complex side is identified with the Yukawa coupling $\mathcal{Y}^K_{ijk}$ on the Kähler side. The identification of Π with II_{A} in mirror symmetry implies that the monodromy matrices of Π equals those of II [17, 22]

$$T_{C,i} = T_{K,i}, \tag{4.74}$$

therefore $T_{K,i}$ is also an integral symplectic matrix which immediately implies the following property in mirror symmetry.

Corollary 4.6.3. *The coefficients Y_{0ij} and Y_{00i} are rational numbers.*

Summary: in this chapter, we have briefly discussed the mirror symmetry conjecture, and our treatment is from the original paper [23]. It should be noticed that in the mathematics literature, it is the Yukawa complings on the Kähler side and complex side that are identified under the mirror map, which is a weaker version of the mirror symmetry conjecture. From Chapter 5 to Chapter 7, we will apply the constructions in this chapter to the study of the mirror symmetry of a self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$. In Chapter 8, we will compute the limit MHS at the large complex structure limit of the mirror family of $X_{\mathfrak{G}}$, and it will be clear that the stronger version of the mirror symmetry conjecture in this chapter is crucial in the computations.

Chapter 5

The construction of a self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$

In this chapter, we will discuss the construction of a self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$ with Hodge numbers $h^{11} = h^{12} = 2$. All the results in this section are from [15], but we will rewrite them in a more mathematical way. The structure of this chapter is as follows:

- Section 5.1 introduces a complete intersection Calabi-Yau (CICY) threefold Y which admits a free action by a finite group $\mathfrak{G}$ of order 12. The quotient of Y by $\mathfrak{G}$, denoted by $Y_{\mathfrak{G}}$, is a Calabi-Yau threefold with Hodge numbers $h^{11} = 1$ and $h^{12} = 4$. Moreover, $Y_{\mathfrak{G}}$ has a deformation over $\mathbb{P}^4$.
- Section 5.2 discusses the geometry of del Pezzo surface dP_6, which is the blow up of $\mathbb{P}^2$ at three points in general positions.
- Section 5.3 discusses the construction of a $\mathfrak{G}$-invariant ample divisor of Y, which defines an ample divisor of $Y_{\mathfrak{G}}$.
- Section 5.4 discusses the toric geometric constructions of Y and $Y_{\mathfrak{G}}$.
- Section 5.5 studies a two parameter subfamily of conifolds in the deformation of $Y_{\mathfrak{G}}$ over $\mathbb{P}^4$, whose resolution yields the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$.
- Finally, Section 5.6 discusses the discriminant locus of this two parameter family of conifolds, which will be important when we study the mirror symmetry of $X_{\mathfrak{G}}$.

5.1 A complete intersection Calabi-Yau threefold

Let Y be a complete intersection Calabi-Yau (CICY) threefold with configuration matrix

$$\begin{array}{cc} & \begin{array}{ccccc} p^1 & q^1 & r & q^2 & p^2 \end{array} \\ \begin{array}{cc} x_{1j} & \mathbb{P}^2 \\ x_{2j} & \mathbb{P}^2 \\ x_{3j} & \mathbb{P}^2 \\ x_{4j} & \mathbb{P}^2 \end{array} & \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}, \tag{5.1}$$

where the projective coordinates of the i-th $\mathbb{P}^2$ in the product $(\mathbb{P}^2)^4$ are $(x_{ij})_{j=0}^2$. Namely Y is the sub-variety of $(\mathbb{P}^2)^4$ cut out by five multi-homogeneous polynomials p^1, q^1, p^2, q^2 and r with degrees shown in the configuration matrix 5.1. When the coefficients of p^1, q^1, p^2, q^2 and r are general, Y is a smooth Calabi-Yau threefold with Hodge numbers $h^{11} = 8$ and $h^{12} = 44$, hence the Euler number of Y is -72. Now let us define the following polynomials

$$\begin{aligned} p(w,z) &:= w_0\, z_0 + w_1\, z_2 + w_2\, z_1, \qquad q(w,z) := w_1\, z_1 + w_0\, z_2 + w_2\, z_0, \\ m_{ijkl} &:= \frac{1}{4} \sum_{\alpha} x_{\alpha,\, i}\, x_{\alpha+1,\, j}\, x_{\alpha+2,\, k}\, x_{\alpha+3,\, l}, \end{aligned} \tag{5.2}$$

where for simplicity the index α (resp. j) of $x_{\alpha j}$ takes value in $\mathbb{Z}/4\mathbb{Z}$ (resp. $\mathbb{Z}/3\mathbb{Z}$). Let the special p^1, q^1, p^2, q^2 and r be given by

$$\begin{aligned} p^1 \ &= \ p(x_1, x_3)\ , \quad p^2 \ = \ p(x_2, x_4)\ , \quad q^1 \ = \ q(x_1, x_3)\ , \quad q^2 \ = \ q(x_2, x_4), \\ r \ &= \ c_0\, m_{0000} + c_1\, m_{1111} + c_2\, m_{2222} + c_3\, m_{0011} + c_4\, m_{0212}. \end{aligned} \tag{5.3}$$

When the coefficients $\{c_i\}_{i=0}^4$ are general, the threefold Y cut out by the polynomials in 5.3 is still a smooth Calabi-Yau threefold, but now Y admits a free action by a group of order 12. More precisely, let $\mathfrak{G}$ be the group with generators g_3 and g_4 that are subjected to the relations

$$g_3^3 = 1 = g_4^4, \quad g_4\, g_3 = g_3^2\, g_4, \tag{5.4}$$

and in fact it is the dicyclic group of order 12. This group $\mathfrak{G}$ has an action on the ambient space $(\mathbb{P}^2)^4$ given by

$$g_3 \,:\, x_{\alpha,j} \to \zeta_3^{(-1)^\alpha j} x_{\alpha,j}, \quad g_4 \,:\, x_{\alpha,j} \to x_{\alpha+1,j}, \text{ with } \zeta_3 = \exp(2\pi i/3), \tag{5.5}$$

which further defines an action on the CICY Y cut out by the polynomials in 5.3. Moreover, for general coefficients $\{c_i\}_{i=0}^4$, this action of $\mathfrak{G}$ on Y is free, so the quotient $Y_{\mathfrak{G}} := Y/\mathfrak{G}$ is also a Calabi-Yau threefold. Let $(c_0, \cdots, c_4)$ be the projective coordinates of $\mathbb{P}^4$, then the five polynomials in equation 5.3 cut out a variety $\mathcal{Y}$ of $(\mathbb{P}^2)^4 \times \mathbb{P}^4$. The natural projection map $(\mathbb{P}^2)^4 \times \mathbb{P}^4 \to \mathbb{P}^4$ defines a fibration of $\mathcal{Y}$ over $\mathbb{P}^4$ that will be denoted by π_Y

$$\pi_Y : \mathcal{Y} \to \mathbb{P}^4. \tag{5.6}$$

From now on, Y will mean a general smooth fiber of this family.

Remark 5.1.1. *The fibers of the family $\mathcal{Y}$ in 5.6 over the following two points are isomorphic*

$$(c_0, c_1, c_2, c_3, c_4), \quad (c_0, \zeta_3 c_1, \zeta_3^2 c_2, \zeta_3^2 c_3, \zeta_3^2 c_4). \tag{5.7}$$

From geometric invariant theory (GIT), the quotient of $\mathcal{Y}$ by $\mathfrak{G}$ exists and it defines a fibration over $\mathbb{P}^4$

$$\pi_{Y_{\mathfrak{G}}} : \mathcal{Y}_{\mathfrak{G}} \to \mathbb{P}^4, \text{ with } \mathcal{Y}_{\mathfrak{G}} := \mathcal{Y}/\mathfrak{G}. \tag{5.8}$$

The action of $\mathfrak{G}$ on a general fiber of the family $\mathcal{Y}$ is free, so a general fiber of the family $\mathcal{Y}_{\mathfrak{G}}$ is a smooth Calabi-Yau threefold, and from now on a general smooth fiber of this family will be denoted by $Y_{\mathfrak{G}}$ $(= Y/\mathfrak{G})$. Since the order of $\mathfrak{G}$ is 12, the Euler characteristic of $Y_{\mathfrak{G}}$ is -6.

5.2 The del Pezzo Surface

In this section, we will discuss some properties of the del Pezzo surface dP_6, which is the blow-up of $\mathbb{P}^2$ at three points that are in general position. It is important to notice that the polynomials $p^1(x_{1j}, x_{3j})$ and $q^1(x_{1j}, x_{3j})$ in equation 5.3 only depend on variables x_{1j} and x_{3j}, hence they cut out a surface $\mathcal{S}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (the product of the first and third $\mathbb{P}^2$ in $(\mathbb{P}^2)^4$) which is isomorphic to the del Pezzo surface dP_6.

Remark 5.2.1. *The cohomology groups of del Pezzo surface $\mathcal{S}$ are given by*

$$H^0(\mathcal{S}, \mathbb{Z}) \simeq H^0(\mathcal{S}, \mathbb{Z}) = \mathbb{Z}, \; H^1(\mathcal{S}, \mathbb{Z}) \simeq H^3(\mathcal{S}, \mathbb{Z}) = 0, \; H^2(\mathcal{S}, \mathbb{Z}) = \mathbb{Z}^4. \tag{5.9}$$

The projection of $\mathbb{P}^2 \times \mathbb{P}^2$ to the first $\mathbb{P}^2$ induces a map $\mathcal{S} \to \mathbb{P}^2$, which realises $\mathcal{S}$ as the blow up of $\mathbb{P}^2$ at the following three points

$$\mathrm{pd}_1 = (1,1,1), \; \mathrm{pd}_2 = (1, \zeta_3, \zeta_3^2), \; \mathrm{pd}_3 = (1, \zeta_3^2, \zeta_3). \tag{5.10}$$

Let E_i be the exceptional divisor of $\mathcal{S}$ over the point pd_i. Let L_{ij} be the unique line of $\mathbb{P}^2$ that passes through pd_i and pd_j, and by abuse of notation the proper transform of this line in $\mathcal{S}$ will also be denoted by L_{ij}. The lines L_{ij} and E_i are listed in Table 5.1 [15].

-1 line	x_{1j}	x_{3j}
E_1	$(1,1,1)$	$x_{30} + x_{31} + x_{32} = 0$
L_{12}	$x_{10} + \zeta_3\, x_{11} + \zeta_3^2\, x_{12} = 0$	$(1, \zeta_3^2, \zeta_3)$
E_2	$(1, \zeta_3, \zeta_3^2)$	$x_{30} + \zeta_3^2\, x_{31} + \zeta_3\, x_{32} = 0$
L_{23}	$x_{10} + x_{11} + x_{12} = 0$	$(1,1,1)$
E_3	$(1, \zeta_3^2, \zeta_3)$	$x_{30} + \zeta_3\, x_{31} + \zeta_3^2\, x_{32} = 0$
L_{31}	$x_{10} + \zeta^2\, x_{11} + \zeta_3\, x_{12} = 0$	$(1, \zeta_3, \zeta_3^2)$

Table 5.1: The six -1 lines of $\mathcal{S}$

On the other hand, the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ to the second $\mathbb{P}^2$ induces another map $\mathcal{S} \to \mathbb{P}^2$ which realises $\mathcal{S}$ as the blow up of $\mathbb{P}^2$ at the following three points

$$\mathrm{pd}_{1,2} = (1, \zeta_3^2, \zeta_3),\ \mathrm{pd}_{2,3} = (1,1,1),\ \mathrm{pd}_{3,1} = (1, \zeta_3, \zeta_3^2). \tag{5.11}$$

Now the line L_{ij} of $\mathcal{S}$ is the exceptional divisor over the point $\mathrm{pd}_{i,j}$ and the line E_i of $\mathcal{S}$ is the proper transform of the line in $\mathbb{P}^2$ that passes through the points $\mathrm{pd}_{i,j}$ and $\mathrm{pd}_{j',i}$. From properties of blow up [98], the self-intersection number (in $\mathcal{S}$) of each of the six lines above is -1. For later convenience, the six lines are ordered in the following way

$$D_a = (E_1, L_{12}, E_2, L_{23}, E_3, L_{31}). \tag{5.12}$$

Now let H be a general line of $\mathbb{P}^2$, which is a hyperplane divisor of $\mathbb{P}^2$, and its proper transform in $\mathcal{S}$, which will also be denoted by H, is the hyperplane divisor of $\mathcal{S}$. From [52], the self-intersection number of H in $\mathcal{S}$ is 1, and the four divisors $\{H, E_1, E_2, E_3\}$ form a basis of $H^2(\mathcal{S}, \mathbb{Z})$, moreover their intersection numbers in $\mathcal{S}$ are given by [52]

$$H \cdot H = 1,\ H \cdot E_i = 0,\ E_i \cdot E_j = -\delta_{ij}. \tag{5.13}$$

The only non-zero intersection products between $\{L_{ij}\}$ and $\{E_i\}$ are

$$L_{ij} \cdot E_i = L_{ij} \cdot E_j = 1. \tag{5.14}$$

Poincaré duality implies the intersection pairing of $H^2(\mathcal{S}, \mathbb{Z})$ with itself is not degenerated, so we find [15]

$$L_{ij} = H - E_i - E_j. \tag{5.15}$$

Similarly, the polynomials p^2 and q^2 in 5.3 only depend on the variables x_{2j} and x_{4j}, hence they cut out a surface $\widetilde{\mathcal{S}}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (the product of the second and fourth

$\mathbb{P}^2$ in $(\mathbb{P}^2)^4$). There is a symmetry between p^1 (resp. q^1) and p^2 (resp. q^2) induced by the swaps coordinates

$$x_{1j} \leftrightarrow x_{3j},\ x_{2j} \leftrightarrow x_{4j}, \tag{5.16}$$

therefore $\widetilde{\mathcal{S}}$ is isomorphic to $\mathcal{S}$. So parallel constructions work for the surface $\widetilde{\mathcal{S}}$ and its six -1-lines will be denoted correspondingly by

$$\widetilde{D}_a = (\widetilde{E}_1, \widetilde{L}_{12}, \widetilde{E}_2, \widetilde{L}_{23}, \widetilde{E}_3, \widetilde{L}_{31}), \tag{5.17}$$

which have the same intersection products as in equation 5.13, etc. So the polynomials p^1, q^1, p^2 and q^2 cut out a smooth fourfold $\mathcal{S} \times \widetilde{\mathcal{S}}$ in $(\mathbb{P}^2)^4$, and Y is a hypersurface cut out by the polynomial r.

5.3 A $\mathfrak{G}$-invariant divisor of Y and the Hodge numbers of the quotient $Y_{\mathfrak{G}}$

From equation 5.9 and Künneth formula, we find that

$$H^2(\mathcal{S} \times \widetilde{\mathcal{S}}, \mathbb{Z}) = H^2(\mathcal{S}, \mathbb{Z}) \oplus H^2(\widetilde{\mathcal{S}}, \mathbb{Z}). \tag{5.18}$$

The polynomial r in equation 5.3 defines a section of an ample line bundle of $(\mathbb{P}^2)^4$, therefore the Lefschetz hyperplane theorem yields an isomorphism [52]

$$H^2(Y, \mathbb{Z}) \simeq H^2(\mathcal{S} \times \widetilde{\mathcal{S}}, \mathbb{Z}). \tag{5.19}$$

The quotient map $Y \to Y/\mathfrak{G} = Y_{\mathfrak{G}}$ induces homomorphisms of cohomology groups

$$H^2(Y_{\mathfrak{G}}, \mathbb{Z}) \to H^2(Y, \mathbb{Z}), \quad H^2(Y_{\mathfrak{G}}, \mathbb{Q}) \to H^2(Y, \mathbb{Q}). \tag{5.20}$$

The action of $\mathfrak{G}$ on Y induces an action of $\mathfrak{G}$ on $H^2(Y, \mathbb{Q})$, and the standard Leray-Cartan spectral sequence gives us

$$H^2(Y_{\mathfrak{G}}, \mathbb{Q}) = H^2(Y, \mathbb{Q})^{\mathfrak{G}}, \tag{5.21}$$

where $H^2(Y, \mathbb{Q})^{\mathfrak{G}}$ consists of $\mathfrak{G}$-invariant elements of $H^2(Y, \mathbb{Q})$. From the (-1)-lines of del Pezzo surface, there are 12 divisors on Y that are given by

$$D_a^Y = (D_a \times \widetilde{\mathcal{S}}) \cap Y, \quad \widetilde{D}_a^Y = (\mathcal{S} \times \widetilde{D}_a) \cap Y. \tag{5.22}$$

In [15], the authors have shown that the only $\mathfrak{G}$-invariant element of $H^2(Y, \mathbb{Q})$ (up to a multiplication) is given by

$$\sum_{a=1}^{6} (D_a^Y + \widetilde{D}_a^Y), \tag{5.23}$$

hence the Hodge number $h^{11}(Y_{\mathfrak{G}})$ must be 1. The Euler characteristic of $Y_{\mathfrak{G}}$ is -6, thus the Hodge number $h^{12}(Y_{\mathfrak{G}})$ is 4.

Remark 5.3.1. *After quotient by* $\mathfrak{G}$, $\sum_{a=1}^{6}(D_a^Y + \widetilde{D}_a^Y)$ *defines an ample divisor of* $Y_{\mathfrak{G}}$ *which generates its Kähler cone. The divisor* $\sum_{a=1}^{6}(D_a^Y + \widetilde{D}_a^Y)$ *is the anti-canonical divisor of the toric variety* $\mathcal{S} \times \widetilde{\mathcal{S}}$*, which will be discussed later.*

5.4 The toric geometric constructions of Y and $Y_{\mathfrak{G}}$

The del Pezzo surface $\mathcal{S}$ is a toric variety whose polytope $P_{\mathcal{S}}$ is the convex hull of the following seven lattice points

$$\{\mu_a\}_{a=0}^{6} = \{(0,0),\ (1,0),\ (0,1),\ (-1,1),\ (-1,0),\ (0,-1),\ (1,-1)\}, \tag{5.24}$$

which are the only lattice points of $P_{\mathcal{S}}$. The fan of $\mathcal{S}$ (in the real vector space spanned by the lattice of one-parameter subgroups of $\mathcal{S}$) will be denoted by $\Sigma_{\mathcal{S}}$, and it has six one-dimensional cones with ray generators given by

$$(1,0),\ (1,1),\ (0,1),\ (-1,0),\ (-1,-1),\ (0,-1), \tag{5.25}$$

while the two dimensional cones of $\Sigma_{\mathcal{S}}$ are generated by adjacent ray generators. From toric geometry [31], there exist six divisors associated to the six one dimensional cones of $\Sigma_{\mathcal{S}}$, which are just the six (-1) lines $\{D_a\}_{a=1}^{6}$. The polytope of $\widetilde{\mathcal{S}}$ will be denoted by $P_{\widetilde{\mathcal{S}}}$, which is isomorphic to $P_{\mathcal{S}}$, while the fan of $\widetilde{\mathcal{S}}$ will be denoted by $\Sigma_{\widetilde{\mathcal{S}}}$, which is isomorphic to $\Sigma_{\mathcal{S}}$.

Therefore the fourfold $\mathcal{S} \times \widetilde{\mathcal{S}}$ is also a toric variety whose polytope $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ is the Minkowski sum of $P_{\mathcal{S}}$ and $P_{\widetilde{\mathcal{S}}}$

$$P_{\mathcal{S}\times\widetilde{\mathcal{S}}} = P_{\mathcal{S}} \times P_{\widetilde{\mathcal{S}}}, \tag{5.26}$$

thus $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ is the convex hull of the following lattice points

$$\{\mu_a \times \mu_b\}_{a,b=0}^{6}, \tag{5.27}$$

which are in fact the only lattice points of $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$. The fan of $\mathcal{S} \times \widetilde{\mathcal{S}}$, denoted by $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$, is given by

$$\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}} = \Sigma_{\mathcal{S}} \times \Sigma_{\widetilde{\mathcal{S}}}, \tag{5.28}$$

hence $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ has 12 one dimensional cones with ray generators $\{\rho_i\}_{i=1}^{12}$

$$\begin{gathered}\{(1,0,0,0),(1,1,0,0),(0,1,0,0),(-1,0,0,0),(-1,-1,0,0),(0,-1,0,0)\\(0,0,1,0),(0,0,1,1),(0,0,0,1),(0,0,-1,0),(0,0,-1,-1),(0,0,0,-1)\}.\end{gathered} \tag{5.29}$$

There exist 12 divisors associated to the 12 rays of $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$, which are given by

$$\{D_a \times \widetilde{\mathcal{S}},\ \mathcal{S} \times \widetilde{D}_a\}_{a=1}^{6}. \tag{5.30}$$

The group $\mathfrak{G}$ acts transitively on these 12 divisors 5.30, which induces an action of $\mathfrak{G}$ on the lattice of one-parameter subgroups of $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$, and moreover it determines the action of $\mathfrak{G}$ on $\mathcal{S} \times \widetilde{\mathcal{S}}$ uniquely [31].

The action of $\mathfrak{G}$ on the lattice of one-parameter subgroups of $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ induces an action of $\mathfrak{G}$ on the lattice of characters of $\Sigma_{\mathcal{S}\times\widetilde{\mathcal{S}}}$. Under this action, the 49 lattice points 5.27 of $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ fall into five orbits: the trivial orbit that consists of the origin $(0,0,0,0)$ and four other orbits each consists of 12 lattice points. The toric polynomial of the trivial orbit is 1 and the toric polynomials of the other four orbits are given by

$$\begin{aligned}
f_1(t) &= t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{t_3}{t_4} + \frac{t_4}{t_3},\\
f_2(t) &= (t_2 + \frac{1}{t_2})(t_4 + \frac{1}{t_4}) + (t_1 + \frac{1}{t_1})(\frac{t_3}{t_4} + \frac{t_4}{t_3}) + (\frac{t_1}{t_2} + \frac{t_2}{t_1})(t_3 + \frac{1}{t_3}),\\
f_3(t) &= (t_1 + \frac{1}{t_1})(t_3 + \frac{1}{t_3}) + (t_4 + \frac{1}{t_4})(\frac{t_1}{t_2} + \frac{t_2}{t_1}) + (t_2 + \frac{1}{t_2})(\frac{t_3}{t_4} + \frac{t_4}{t_3}),\\
f_4(t) &= (t_2 + \frac{1}{t_2})(t_3 + \frac{1}{t_3}) + (t_1 + \frac{1}{t_1})(t_4 + \frac{1}{t_4}) + (\frac{t_1}{t_2} + \frac{t_2}{t_1})(\frac{t_3}{t_4} + \frac{t_4}{t_3}).
\end{aligned} \tag{5.31}$$

Therefore the most general $\mathfrak{G}$-invariant toric polynomial of $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ is given by

$$f(t) = \gamma_0 + \gamma_1\, f_1(t) + \gamma_2\, f_2(t) + \gamma_3\, f_3(t) + \gamma_4\, f_4(t), \tag{5.32}$$

where $\{\gamma_i\}_{i=0}^{4}$ are free complex parameters. Hence the toric polynomial $f(t)$ 5.32 cuts out a four-parameter family of hypersurfaces in $\mathcal{S}\times\widetilde{\mathcal{S}}$, while each hypersurface admits an action of $\mathfrak{G}$, so we get a fibration over $\mathbb{P}^4$

$$\pi_Y^{\mathrm{tor}} : \mathcal{Y}^{\mathrm{tor}} \to \mathbb{P}^4, \tag{5.33}$$

where the projective coordinates of $\mathbb{P}^4$ are $(\gamma_0, \cdots, \gamma_4)$.

Remark 5.4.1. *From toric geometry [31], the sections of the anti-canonical bundle of $\mathcal{S} \times \widetilde{\mathcal{S}}$ are in one-to-one bijection with the toric polynomials of $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$. Since the polytope $P_{\mathcal{S}\times\widetilde{\mathcal{S}}}$ is reflexive, the vanishing locus of a general section of the anti-canonical bundle of $\mathcal{S}\times\widetilde{\mathcal{S}}$ is a Calabi-Yau threefold. Furthermore the vanishing locus of a general section that is given by a $\mathfrak{G}$-invariant toric polynomial 5.32 is a Calabi-Yau threefold that admits a free action of $\mathfrak{G}$.*

Proposition 5.4.2. *The family π_Y^{tor} 5.33 is isomorphic to the family π_Y 5.6*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\simeq"] \arrow[d, "\pi_Y"] & \mathcal{Y}^{tor} \arrow[d, "\pi_Y^{tor}"] \\
\mathbb{P}^4 \arrow[r, "\simeq"] & \mathbb{P}^4
\end{tikzcd} . \tag{5.34}$$

The isomorphism between the base space $\mathbb{P}^4$ is given by a projective linear transformation

$$\begin{aligned}
\gamma_0 &= c_2 + c_3 + c_4, & c_0 &= 3\,(\zeta_3\,\gamma_2 + \zeta_3^2\,\gamma_3 + \gamma_4),\\
\gamma_1 &= \frac{1}{12}\,(4\,c_2 - 2\,c_3 + c_4), & c_1 &= 3\,(\zeta_3^2\,\gamma_2 + \zeta_3\,\gamma_3 + \gamma_4),\\
\gamma_2 &= \frac{1}{36}\,\left(4\,\zeta_3^2\,c_0 + 4\,\zeta_3\,c_1 + (4\,c_2 + c_3 - 2\,c_4)\right), & c_2 &= \frac{1}{9}\,(\gamma_0 + 12\gamma_1 + 12(\gamma_2 + \gamma_3 + \gamma_4))\,,\\
\gamma_3 &= \frac{1}{36}\,\left(4\,\zeta_3\,c_0 + 4\,\zeta_3^2\,c_1 + (4\,c_2 + c_3 - 2\,c_4)\right), & c_3 &= \frac{4}{9}\,(\gamma_0 - 6\,\gamma_1 + 3\,(\gamma_2 + \gamma_3 + \gamma_4))\,,\\
\gamma_4 &= \frac{1}{36}\,(4\,c_0 + 4\,c_1 + (4\,c_2 + c_3 - 2\,c_4))\,, & c_4 &= \frac{4}{9}\,(\gamma_0 + 3\,\gamma_1 - 6\,(\gamma_2 + \gamma_3 + \gamma_4))\,.
\end{aligned} \tag{5.35}$$

Proof. The projective linear transformation is obtained by studying the functions on the torus of $\mathcal{S} \times \widetilde{\mathcal{S}}$ induced by the polynomial r 5.3 and the toric polynomial $f(t)$ 5.32 [15]. ☐

The quotient of $\mathcal{Y}^{\mathrm{tor}}$ by $\mathfrak{G}$ defines a fibration over $\mathbb{P}^4$

$$\pi_{Y_{\mathfrak{G}}}^{\mathrm{tor}} : \mathcal{Y}_{\mathfrak{G}}^{\mathrm{tor}} \to \mathbb{P}^4, \text{ with } \mathcal{Y}_{\mathfrak{G}}^{\mathrm{tor}} = \mathcal{Y}^{\mathrm{tor}}/\mathfrak{G}, \tag{5.36}$$

which is isomorphic to the family 5.8

$$\begin{tikzcd}
\mathcal{Y}_{\mathfrak{G}} \arrow[r, "\simeq"] \arrow[d, "\pi_{Y_{\mathfrak{G}}}^{\mathrm{tor}}"] & \mathcal{Y}_{\mathfrak{G}}^{\mathrm{tor}} \arrow[d, "\pi_{Y_{\mathfrak{G}}}"] \\
\mathbb{P}^4 \arrow[r, "\simeq"] & \mathbb{P}^4
\end{tikzcd} . \tag{5.37}$$

5.5 The conifold transition and the mirror of $Y_{\mathfrak{G}}$

In this section, we will discuss the two parameter subfamily of conifolds of the family $\mathcal{Y}_{\mathfrak{G}}$ 5.8 and the construction of the mirror of $Y_{\mathfrak{G}}$.

5.5.1 A two parameter subfamily of conifolds

Over the subvariety $\mathbb{P}^2$ of $\mathbb{P}^4$ defined by $c_0 = c_1 = 0$, we get a two parameter subfamily of $\mathcal{Y}_{\mathfrak{G}}$ 5.8 that will be denoted by

$$\pi^s_{Y_{\mathfrak{G}}} : \mathcal{Y}^s_{\mathfrak{G}} \to \mathbb{P}^2, \tag{5.38}$$

where the projective coordinates of $\mathbb{P}^2$ are (c_2, c_3, c_4). A general fiber of $\mathcal{Y}^s_{\mathfrak{G}}$ will be denoted by $Y^s_{\mathfrak{G}}$, which is a conifold with singularity consists of three double points (this will be shown in next chapter). From equation 5.35, the $\mathbb{P}^2$ defined by $c_0 = c_1 = 0$ corresponds to the $\mathbb{P}^2$ defined by $\gamma_2 = \gamma_3 = \gamma_4$, and for the two $\mathbb{P}^2$, the linear transformation between their coordinates (c_2, c_3, c_4) and $(\gamma_0, \gamma_1, \gamma_2)$ becomes

$$\begin{aligned}
\gamma_0 &= c_2 + c_3 + c_4, & c_2 &= \frac{1}{9}\,(\gamma_0 + 12\,\gamma_1 + 36\,\gamma_2),\\
\gamma_1 &= \frac{1}{12}\,(4\,c_2 - 2\,c_3 + c_4), & c_3 &= \frac{4}{9}\,(\gamma_0 - 6\,\gamma_1 + 9\,\gamma_2),\\
\gamma_2 &= \frac{1}{36}\,(4\,c_2 + c_3 - 2\,c_4), & c_4 &= \frac{4}{9}\,(\gamma_0 + 3\,\gamma_1 - 18\,\gamma_2).
\end{aligned} \tag{5.39}$$

From Section 5 of [15], the three double points of $Y^s_{\mathfrak{G}}$ can be resolved by a projective small resolution, and the smooth variety after resolution is a Calabi-Yau threefold with Hodge numbers $h^{11} = h^{12} = 2$ that will be denoted by $X_{\mathfrak{G}}$. When c_0 and c_1 approach 0 in a general way, there are three spheres S^3 in the smooth fiber that shrink to three double points. In the projective small resolution of $Y^s_{\mathfrak{G}}$, each double point is resolved by a $\mathbb{P}^1$. This process is called conifold transition by mathematical physicists [23]

$$\begin{tikzcd}
 & X_{\mathfrak{G}} \arrow[d, "\text{small resolution}\ \cdot"] \\
Y_{\mathfrak{G}} \arrow[r, "c_0\to 0", "c_1\to 0"'] & Y^s_{\mathfrak{G}}
\end{tikzcd} \tag{5.40}$$

Remark 5.5.1. *In next chapter, we will give a construction of a projective small resolution of $Y^s_{\mathfrak{G}}$ that will be convenient for the computations in this thesis.*

5.5.2 The mirror Calabi-Yau threefold of $Y_{\mathfrak{G}}$

The mirror Calabi-Yau threefold of $Y_{\mathfrak{G}}$ has been constructed in Section 6 of [15]. The equations $\gamma_2 = \gamma_3 = \gamma_4 = 0$ define a one dimensional subvariety of $\mathbb{P}^4$ that is isomorphic to $\mathbb{P}^1$, and over this $\mathbb{P}^1$, we get a subfamily of $\mathcal{Y}_{\mathfrak{G}}$ 5.8

$$\pi^{ss}_{Y_{\mathfrak{G}}} : \mathcal{Y}^{ss}_{\mathfrak{G}} \to \mathbb{P}^1, \tag{5.41}$$

a general fiber of which is a singular variety with singularity consists of 6 double points. The 6 double points can be resolved by a projective small resolution, and after resolution we get a Calabi-Yau threefold $Y_{\mathfrak{G}}^{\vee}$ with Hodge numbers $h^{11}(Y_{\mathfrak{G}}^{\vee}) = 4$ and $h^{12}(Y_{\mathfrak{G}}^{\vee}) = 1$, which is the mirror of $Y_{\mathfrak{G}}$. So we get a fibered diagram

$$\begin{tikzcd}
\mathcal{Y}_{\mathfrak{G}}^{ss} \arrow[r, hook] \arrow[d, "\pi_{Y_{\mathfrak{G}}}^{ss}"] & \mathcal{Y}_{\mathfrak{G}}^{s} \arrow[r, hook] \arrow[d, "\pi_{Y_{\mathfrak{G}}}^{s}"] & \mathcal{Y}_{\mathfrak{G}} \arrow[d, "\pi_{Y_{\mathfrak{G}}}"] \\
\mathbb{P}_{\mathbb{C}}^{1} \arrow[r, hook] & \mathbb{P}_{\mathbb{C}}^{2} \arrow[r, hook] & \mathbb{P}_{\mathbb{C}}^{4}
\end{tikzcd} . \tag{5.42}$$

The observation is that $\mathcal{Y}_{\mathfrak{G}}^{s}$ falls in the middle ground between $\mathcal{Y}_{\mathfrak{G}}^{ss}$ and $\mathcal{Y}_{\mathfrak{G}}$, since the resolution of $\mathcal{Y}_{\mathfrak{G}}^{ss}$ is the mirror of $\mathcal{Y}_{\mathfrak{G}}$, so the resolution of $\mathcal{Y}_{\mathfrak{G}}^{s}$ is expected to be the mirror of itself, i.e. $X_{\mathfrak{G}}$ is expected to be self-mirror.

5.6 The discriminant locus of the conifold $Y_{\mathfrak{G}}^{s}$

The singularity of $Y_{\mathfrak{G}}^{s}$, which is a general fiber of $\mathcal{Y}_{\mathfrak{G}}^{s}$ 5.38, consists of three double points, but over some special locus in $\mathbb{P}^2$, the fibers can have more severe singularities, and this special locus is called discriminant locus of the deformation 5.38. Based on Gröbner basis computations, the discriminant locus of the deformation 5.38 has been found which are listed in Table 5.2 [15].

Component	Equation	Component	Equation
$\Gamma^{(\mathrm{o})}$	$4\,c_2\,c_3 - c_4^2 = 0$	$\Gamma^{(\mathrm{vi})}$	$4\,c_2 - 8\,c_3 + 7\,c_4 = 0$
$\Gamma^{(\mathrm{i})}$	$4\,c_2 + c_3 - 2\,c_4 = 0$	$\Gamma^{(\mathrm{vii})}$	$8\,c_3 + c_4 = 0$
$\Gamma^{(\mathrm{ii})}$	$c_2 + 16\,c_3 + 4\,c_4 = 0$	$\Gamma^{(\mathrm{viii})}$	$4\,c_2 - c_4 = 0$
$\Gamma^{(\mathrm{iii})}$	$c_2 = 0$	$\Gamma^{(\mathrm{ix})}$	$c_3 - c_4 = 0$
$\Gamma^{(\mathrm{iv})}$	$c_3 = 0$	$\Gamma^{(\mathrm{x})}$	$c_2 + 2\,c_4 = 0$
$\Gamma^{(\mathrm{v})}$	$c_4 = 0$		

Table 5.2: Discriminant locus

It is very interesting that there exists a $(\mathbb{Z}/2\mathbb{Z})^2$ symmetry of $\mathbb{P}^2$ generated by the projective transformations S and T of the form

$$S = \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} \to \begin{pmatrix} \frac{8}{9} & \frac{2}{9} & -\frac{4}{9} \\ \frac{1}{18} & \frac{8}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{8}{9} & -\frac{7}{9} \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix}, \ T : \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} \to \begin{pmatrix} \frac{1}{9} & \frac{16}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{7}{9} \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix}, \tag{5.43}$$

which preserves the discriminant locus in Table 5.2 [15]. To see this symmetry more clearly, let us define new projective coordinates

$$\begin{aligned} u &:= -2\,c_2 + 4\,c_3 + c_4, \\ v &:= 2\,(c_2 - 2\,c_3 + 4\,c_4), \\ w &:= 6\,(c_2 + 2\,c_3), \end{aligned} \tag{5.44}$$

with respect to which, the actions of S and T become simpler

$$\begin{aligned} S &: u \to u, \quad v \to -v, \quad w \to w, \\ T &: u \to -u, \quad v \to v, \quad w \to w. \end{aligned} \tag{5.45}$$

On the open affine subvariety of $\mathbb{P}^2$ where $w \neq 0$, let us define coordinates ξ and η by

$$\xi := \frac{u}{w}, \quad \eta := \frac{v}{w}. \tag{5.46}$$

The coordinates of the quotient $\{w \neq 0\}/(\mathbb{Z}/2\mathbb{Z})^2$ is given by

$$X_1 = \xi^2, \quad X_2 = \eta^2, \tag{5.47}$$

which will be convenient when we compute the Yukawa couplings of $X_{\mathfrak{G}}$.

Summary: in this chapter, we have discussed the constructions of the Calabi-Yau threefold $X_{\mathfrak{G}}$, which will be important in Chapter 6 and Chapter 7. We also give some intuition why $X_{\mathfrak{G}}$ is expected to be self-mirror, which is supported by the computations in Chapter 7.

Chapter 6

Divisors of the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$

In this chapter, we will study the homology group $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ and the cohomology group $H^2(X_{\mathfrak{G}}, \mathbb{Z})$, which will be important when we study the mirror symmetry of $X_{\mathfrak{G}}$. The structure of this chapter is as follows:

- Section 6.1 discusses a projective small resolution of the conifold $Y^s_{\mathfrak{G}}$ 5.38, while the smooth Calabi-Yau threefold after resolution is $X_{\mathfrak{G}}$.
- Section 6.2 constructs two divisors ϵ and δ of $X_{\mathfrak{G}}$, which form a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). It also shows that the two divisors lie in the closure of the Kähler cone $\mathcal{K}_{X_{\mathfrak{G}}}$, hence they define a framing of $X_{\mathfrak{G}}$ and its complexified Kähler moduli space is given by

  $$\mathscr{M}_{K,\Sigma}(X_{\mathfrak{G}}) = \{t_1\, \epsilon + t_2\, \delta : t_i \in \mathbb{H}\},$$

  where $\mathbb{H}$ is the upper half plane of $\mathbb{C}$.
- Section 6.3 discusses the intersection product of a curve and a surface in a Calabi-Yau threefold.
- Section 6.4 computes the intersection products of ϵ and δ with themselves, which are shown to be

  $$[\epsilon]^3 = 18,\ [\epsilon]^2 \cdot [\delta] = 6,\ [\delta]^2 = 0, \tag{6.1}$$

  where $[\epsilon]$ means the algebraic cycle class represented by ϵ, etc.
- Finally, Section 6.5 constructs two one dimensional algebraic cycles Γ_ϵ and Γ_δ which form a basis of $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). Moreover Γ_ϵ and Γ_δ are

dual to ϵ and δ, i.e. their intersection products are given by

$$
\begin{aligned}
&[\epsilon] \cdot [\Gamma_\epsilon] = 1, \quad [\delta] \cdot [\Gamma_\epsilon] = 0, \\
&[\epsilon] \cdot [\Gamma_\delta] = 0, \quad [\delta] \cdot [\Gamma_\delta] = 1.
\end{aligned}
$$

6.1 A projective small resolution of the conifold

In this section, we will introduce a projective small resolution of the conifold $Y^s_{\mathfrak{G}}$. Over the locus in $\mathbb{P}^4$ defined by $c_0 = c_1 = 0$, we have a sub-family of $\mathcal{Y}$ 5.6 that will be denoted by

$$
\pi^s_Y : \mathcal{Y}^s \to \mathbb{P}^2, \tag{6.2}
$$

and the quotient of $\mathcal{Y}^s$ by $\mathfrak{G}$ is just the family $\mathcal{Y}^s_{\mathfrak{G}}$ in equation 5.38. It is important to notice that when $c_0 = c_1 = 0$, the polynomial r in equation 5.3 can be written as

$$
r = \frac{1}{4}\,(A_r\, D_r - B_r\, C_r), \tag{6.3}
$$

where A_r, B_r, C_r and D_r are given by

$$
\begin{aligned}
A_r \;&=\; x_{12}\, x_{32}\ , & B_r \;&= x_{10}\, x_{31} + x_{11}\, x_{30}, \\
C_r \;&= -c_3(x_{21}\, x_{40} + x_{20}\, x_{41}) - c_4\, x_{22}\, x_{42}\ , & D_r \;&=\; c_4(x_{21}\, x_{40} + x_{20}\, x_{41}) + 4\, c_2\, x_{22}\, x_{42}.
\end{aligned} \tag{6.4}
$$

A general fiber of π^s_Y is a singular variety with singularity given by

$$
\begin{aligned}
&\{A_r = B_r = C_r = D_r = 0\}\ \cap (\mathcal{S} \times \widetilde{\mathcal{S}}) \\
=&(\{A_r = B_r = 0\}\ \cap \mathcal{S}) \times (\{C_r = D_r = 0\}\ \cap \widetilde{\mathcal{S}}).
\end{aligned} \tag{6.5}
$$

The intersection $\{A_r = B_r = 0\} \cap \mathcal{S}$ is transverse and it consists of six points $\{\mathrm{pt}_a\}_{a=1}^6$

$$
\begin{aligned}
\mathrm{pt}_1 &: (x_{1j}) = (1,\ 1,\ 1), & (x_{3j}) &= (1,\ -1,\ 0), \\
\mathrm{pt}_2 &: (x_{1j}) = (1, -\zeta_3^2,\ 0), & (x_{3j}) &= (1,\ \zeta_3^2,\ \zeta_3), \\
\mathrm{pt}_3 &: (x_{1j}) = (1,\ \zeta_3,\ \zeta_3^2), & (x_{3j}) &= (1, -\zeta_3,\ 0), \\
\mathrm{pt}_4 &: (x_{1j}) = (1,\ -1,\ 0), & (x_{3j}) &= (1,\ 1,\ 1), \\
\mathrm{pt}_5 &: (x_{1j}) = (1,\ \zeta_3^2,\ \zeta_3), & (x_{3j}) &= (1, -\zeta_3^2,\ 0), \\
\mathrm{pt}_6 &: (x_{1j}) = (1, -\zeta_3,\ 0), & (x_{3j}) &= (1,\ \zeta_3,\ \zeta_3^2).
\end{aligned} \tag{6.6}
$$

For general $\{c_i\}_{i=2}^4$, the equations $C_r = D_r = 0$ are equivalent to

$$
\widetilde{C}_r = x_{22}\, x_{42} = 0, \quad \widetilde{D}_r = x_{21}\, x_{40} + x_{20}\, x_{41} = 0. \tag{6.7}
$$

Similarly the intersection $\{\widetilde{C}_r = \widetilde{D}_r = 0\} \cap \widetilde{\mathcal{S}}$ is also transverse and it consists of six points $\{\widetilde{\mathrm{pt}}_b\}_{b=1}^6$, therefore the singularity of a general fiber of $\mathcal{Y}^s$ 6.2, which will be denoted by Y^s, consists of 36 double points $\mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b$. Under the action of $\mathfrak{G}$, the 36 double points of Y^s fall into three orbits, hence the singularity of $Y^s_{\mathfrak{G}} = Y^s/\mathfrak{G}$ consists of three double points.

For the variety $\mathbb{P}^1 \times (\mathbb{P}^2)^4 \times \mathbb{P}^2$, let (t_1, t_2) be the projective coordinates of $\mathbb{P}^1$, and let (c_2, c_3, c_4) be the projective coordinates of the last $\mathbb{P}^2$, while as before (x_{ij}) are the projective coordinates of $(\mathbb{P}^2)^4$. Let us first define polynomials r_1 and r_2 by

$$r^1 := t_1 A_r + t_2\, B_r, \quad r^2 := t_1\, C_r + t_2\, D_r. \tag{6.8}$$

Now let $\mathcal{X}$ be the subvariety of $\mathbb{P}^1 \times (\mathbb{P}^2)^4 \times \mathbb{P}^2$ cut out by the polynomials

$$p^1 = q^1 = p^2 = q^2 = r^1 = r^2 = 0. \tag{6.9}$$

The projection of $\mathbb{P}^1 \times (\mathbb{P}^2)^4 \times \mathbb{P}^2$ to the last $\mathbb{P}^2$ yields a fibration of $\mathcal{X}$ over $\mathbb{P}^2$

$$\pi_X : \mathcal{X} \to \mathbb{P}^2, \tag{6.10}$$

a general fiber of which is smooth and will be denoted by X. The configuration matrix of the fibers of this family is

$$\begin{array}{cc} & & p^1 & q^1 & r^1 & r^2 & q^2 & p^2 \\ t_j & \mathbb{P}^1 \\ x_{1j} & \mathbb{P}^2 \\ x_{2j} & \mathbb{P}^2 \\ x_{3j} & \mathbb{P}^2 \\ x_{4j} & \mathbb{P}^2 \end{array} \left[\begin{array}{cccccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right], \tag{6.11}$$

so a general fiber of $\mathcal{X}$ is a smooth Calabi-Yau threefold. The natural projection morphism

$$\mathbb{P}^1 \times (\mathbb{P}^2)^4 \times \mathbb{P}^2 \to (\mathbb{P}^2)^4 \times \mathbb{P}^2 \tag{6.12}$$

induces a morphism between the two fibrations $\mathcal{X}$ and $\mathcal{Y}^s$ which gives us a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr, "\pi_X"'] & & \mathcal{Y}^s \arrow[dl, "\pi^s_Y"] \\ & \mathbb{P}^2 & \end{tikzcd} . \tag{6.13}$$

Lemma 6.1.1. *For a general parameter* $\boldsymbol{c} := \{c_i\}_{i=2}^4$, *the fiber* $\mathcal{X}_{\boldsymbol{c}}$ (X) *is a projective small resolution of the conifold* $\mathcal{Y}^s_{\boldsymbol{c}}$ (Y^s).

Proof. For a general parameter $\mathbf{c}$, $\mathcal{X}_{\mathbf{c}}$ is a smooth Calabi-Yau threefold from the configuration matrix 6.11, and it is projective as it is a subvariety of $\mathbb{P}^1 \times (\mathbb{P}^2)^4$. The equations $r^1 = r^2 = 0$ can also be written as

$$\begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 0. \tag{6.14}$$

In order for (t_1, t_2) to have a nontrivial solution, the rank of the coefficient matrix in equation 6.14 must be ≤ 1. If the rank of the coefficient matrix is 1, then (t_1, t_2) has a unique solution (up to a multiplication). If the rank of the coefficient matrix is zero, i.e. $A_r = B_r = C_r = D_r = 0$, then equation 6.14 is always true, which means the solution space is $\mathbb{P}^1$, hence the a double point of $\mathcal{Y}^s_{\mathbf{c}}$ is resolved by a $\mathbb{P}^1$. Since $\mathbb{P}^1$ has codimension 2 in $\mathcal{X}_{\mathbf{c}}$, therefore it is a small resolution. □

The action of $\mathfrak{G}$ on $\mathcal{Y}^s$ can be extended to an action of $\mathfrak{G}$ on $\mathcal{X}$. Let g_3 and g_4 (the generators of $\mathfrak{G}$) act on $\mathbb{P}^1 \times (\mathbb{P}^2)^4 \times \mathbb{P}^2$ through

$$\begin{aligned} &g_3 : t_i \to t_i,\, x_{\alpha,j} \to \zeta_3^{(-1)^\alpha j} x_{\alpha,j},\, c_i \to c_i, \\ &g_4 : t_1 \to -c_4\, t_1 + 4\, c_2\, t_2,\, t_2 \to -c_3\, t_1 + c_4\, t_2,\, x_{\alpha,j} \to x_{\alpha+1,j}, c_i \to c_i. \end{aligned} \tag{6.15}$$

Under this action, the polynomials p^1, q^1, p^2 and q^2 transform in the same way as before, while r_1 and r_2 transform in the following way

$$\begin{aligned} g_3 :\ & r^1 \to \zeta_3^2\, r^1,\ r^2 \to \zeta_3\, r^2, \\ g_4 :\ & r^1 \to r^2,\ r^2 \to (c_4^2 - 4\, c_2\, c_3)\, r^1. \end{aligned} \tag{6.16}$$

The action of $\mathfrak{G}$ on $\{c_i\}_{i=2}^4$ is trivial, hence this action preserves the fibers of π_X 6.10. For parameters $\mathbf{c} = \{c_i\}_{i=2}^4$ which are not in the discriminant locus listed in Table 5.2, the action of $\mathfrak{G}$ on the fiber $\mathcal{X}_{\mathbf{c}}$ is free. Let us define the variety $\mathcal{X}_{\mathfrak{G}}$ by

$$\mathcal{X}_{\mathfrak{G}} := \mathcal{X}/\mathfrak{G}, \tag{6.17}$$

and there is a fibration over $\mathbb{P}^2$

$$\pi_{X_{\mathfrak{G}}} : \mathcal{X}_{\mathfrak{G}} \to \mathbb{P}^2, \tag{6.18}$$

a smooth fiber of which is a Calabi-Yau threefold with Hodge numbers $h^{11} = h^{12} = 2$ [15].

Lemma 6.1.2. *The resolution 6.13 defines a commutative diagram*

$$\begin{tikzcd} \mathcal{X}_{\mathfrak{G}} \arrow[rr] \arrow[dr, "\pi_{X_{\mathfrak{G}}}"'] & & \mathcal{Y}^s_{\mathfrak{G}} \arrow[dl, "\pi_{Y^s_{\mathfrak{G}}}"] \\ & \mathbb{P}^2 & \end{tikzcd}, \tag{6.19}$$

moreover over a parameter $\mathbf{c}$ *not in the discriminant locus listed in Table 5.2, the fiber* $\mathcal{X}_{\mathfrak{G},\mathbf{c}}$ *is a small resolution of* $\mathcal{Y}^s_{\mathfrak{G},\mathbf{c}}$*.*

Remark 6.1.3. *Furthermore, there is a* $\mathbb{Z}/3\mathbb{Z}$*-action on* $\mathcal{X}$ *generated by* h_3

$$h_3 : t_i \to t_i,\, x_{\alpha,j} \to \zeta_3^j\, x_{\alpha,j},\, c_i \to c_i, \tag{6.20}$$

which commutes with the actions of g_3 *and* g_4*, hence it defines a* $\mathbb{Z}/3\mathbb{Z}$ *action on* $\mathcal{X}_{\mathfrak{G}}$ *that preserves its fibers* $\mathcal{X}_{\mathfrak{G},\mathbf{c}}$*. Moreover, this* $\mathbb{Z}/3\mathbb{Z}$ *action permutes the three exceptional curves of* $\mathcal{X}_{\mathfrak{G},\mathbf{c}}$*.*

6.2 The $\mathfrak{G}$-invariant divisors of X

In this section, we will construct two important divisors ϵ and δ of $X_{\mathfrak{G}}$ which will be shown to form a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). The notation X will mean a general smooth fiber of the family $\mathcal{X}$ 6.10, whose quotient by $\mathfrak{G}$ is $X_{\mathfrak{G}}$, and the quotient map will be denoted by

$$q_X : X \to X_{\mathfrak{G}}. \tag{6.21}$$

The action of $\mathfrak{G}$ on X induces an action of $\mathfrak{G}$ on its divisors, and the divisors of $X_{\mathfrak{G}}$ are in one-to-one bijection with the $\mathfrak{G}$-invariant divisors of X. More precisely a divisor of $X_{\mathfrak{G}}$ pulls back to a $\mathfrak{G}$-invariant divisor of X, while a $\mathfrak{G}$-invariant divisor of X induces a divisor of $X_{\mathfrak{G}}$ after quotient by $\mathfrak{G}$. We will construct two $\mathfrak{G}$-invariant divisors $\widehat{\epsilon}$ and $\widehat{\delta}$ of X such that their quotients are the divisors of $X_{\mathfrak{G}}$ that we are looking for. From intersection theory [45, 103], we have the following relations between intersection products

$$[\epsilon]^3 = [\widehat{\epsilon}]^3/12, \quad [\epsilon]^2 \cdot [\delta] = [\widehat{\epsilon}]^2 \cdot [\widehat{\delta}]/12, \quad [\delta]^2 = [\widehat{\delta}]^2/12, \tag{6.22}$$

where we have used the property that X is a degree-12 cover of $X_{\mathfrak{G}}$.

Remark 6.2.1. *In this thesis, the notation* $[C]$ *means the rational equivalence class of an algebraic cycle* C *in the Chow group. Intersection product is only well-defined for equivalence classes of cycles, so when we compute intersection products, we will freely replace a cycle by one in the same equivalence class if needed.*

6.2.1 The divisor $\widehat{\delta}$

The threefold X is a sub-variety of $\mathbb{P}^1 \times (\mathbb{P}^2)^4$, and the natural projection map

$$\mathbb{P}^1 \times (\mathbb{P}^2)^4 \to \mathbb{P}^1 \tag{6.23}$$

induces a fibration of X over $\mathbb{P}^1$

$$X \to \mathbb{P}^1. \tag{6.24}$$

Recall that the del Pezzo surface $\mathcal{S}$ is defined by $p^1 = q^1 = 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$, while $\widetilde{\mathcal{S}}$ is defined by $p^2 = q^2 = 0$ in the other $\mathbb{P}^2 \times \mathbb{P}^2$. Let $\widehat{\delta}_t$ be the fiber of X over a point $t \in \mathbb{P}^1$, which is given by

$$\widehat{\delta}_t := (t \times \mathcal{S} \times \widetilde{\mathcal{S}}) \cap X = t \times \mathcal{E}_t \times \widetilde{\mathcal{E}}_t, \tag{6.25}$$

where $\mathcal{E}_t$ (resp. $\widetilde{\mathcal{E}}_t$) is the subvariety of $(\mathbb{P}^2)^2$ defined by

$$\mathcal{E}_t : p^1 = q^1 = r^1|_t = 0, \ \widetilde{\mathcal{E}}_t : p^2 = q^2 = r^2|_t = 0. \tag{6.26}$$

The configuration matrices of $\mathcal{E}_t$ and $\widetilde{\mathcal{E}}_t$ are given by

$$\mathcal{E}_t : \begin{array}{cc} & \\ x_{1j} & \mathbb{P}^2 \\ x_{3j} & \mathbb{P}^2 \end{array} \begin{array}{c} \begin{array}{ccc} p^1 & q^1 & r^1|_t \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}, \quad \widetilde{\mathcal{E}}_t : \begin{array}{cc} & \\ x_{2j} & \mathbb{P}^2 \\ x_{4j} & \mathbb{P}^2 \end{array} \begin{array}{c} \begin{array}{ccc} p^2 & q^2 & r^2|_t \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}. \tag{6.27}$$

When t is general, $\mathcal{E}_t$ and $\widetilde{\mathcal{E}}_t$ are CICY of dimension 1, hence they must be elliptic curves. From equation 6.15, the action of g_3 on (t_1, t_2) is trivial, hence the fiber $\widehat{\delta}_t$ is preserved by g_3

$$g_3 : \widehat{\delta}_t \to \widehat{\delta}_t. \tag{6.28}$$

But the action of g_4 on (t_1, t_2) is not trivial, even though the action of g_4^2 on (t_1, t_2) is trivial, hence we have

$$g_4.\widehat{\delta}_t = \widehat{\delta}_{g_4.t}, \ (g_4)^2.\widehat{\delta}_t = \widehat{\delta}_t. \tag{6.29}$$

Let t_e be a point of $\mathbb{P}^1$ that is invariant under the action of g_4, which always exists for general parameters $\{c_i\}_{i=2}^4$, then the divisor $\widehat{\delta}_{t_e}$ will be $\mathfrak{G}$-invariant, and $\widehat{\delta}$ is defined by

$$\widehat{\delta} := \widehat{\delta}_{t_e}, \tag{6.30}$$

while the divisor δ is given by

$$\delta = \widehat{\delta}_{t_e}/\mathfrak{G}. \tag{6.31}$$

Lemma 6.2.2. *The divisor* $\widehat{\delta}$ *lies in the closure of Kähler cone of* X*.*

Proof. The ample line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ of $\mathbb{P}^1$ pulls back to a line bundle on $\mathbb{P}^1 \times (\mathbb{P}^2)^4$ which lies in the closure of the Kähler cone of $\mathbb{P}^1 \times (\mathbb{P}^2)^4$. The restriction of this line bundle to X is a line bundle L that lies in the closure of the Kähler cone of X. The divisor $\widehat{\delta}$ is the vanishing locus of a section of L. $\square$

Remark 6.2.3. *Since different fibers of the fibration $X \to \mathbb{P}^1$ are rationally equivalent to each other, later when we compute intersection products, we will replace $\widehat{\delta}$ with the $\mathfrak{G}$-invariant divisor $(\widehat{\delta}_t + \widehat{\delta}_{g_4.t})/2$ sometimes, which is rationally equivalent to $\widehat{\delta}_t$. This should be clear from context.*

6.2.2 The divisor $\widehat{\epsilon}$

From Section 5.3, the $\mathfrak{G}$-invariant divisor $\sum_{a=1}^{6}(D_a^Y + \widetilde{D}_a^Y)$ of Y^s is ample. In the $\mathfrak{G}$ equivariant projective small resolution of Y^s in equation 6.13, this ample divisor of Y^s pulls back to a $\mathfrak{G}$-invariant divisor of X that is defined to be $\widehat{\epsilon}$. From computations using Mathematica, the divisor $\widehat{\epsilon}$ is shown to be of the form

$$\widehat{\epsilon} = \sum_a \left((\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathcal{S}}) \cap X\right) + \sum_b \left((\mathbb{P}^1 \times \mathcal{S} \times \widetilde{\mathrm{pt}}_b) \cap X\right) + \widehat{\delta}_{(1,1)} + \widehat{\delta}_{(4c_2-c_4,c_4-c_3)}. \quad (6.32)$$

The divisor $\widehat{\epsilon}$ is rationally equivalent to the following $\mathfrak{G}$-invariant divisor of X

$$\sum_a \left((\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathcal{S}}) \cap X\right) + \sum_b \left((\mathbb{P}^1 \times \mathcal{S} \times \widetilde{\mathrm{pt}}_b) \cap X\right) + 2\,\widehat{\delta}_{t_e}. \quad (6.33)$$

Lemma 6.2.4. *The divisor $\widehat{\epsilon}$ is ample.*

Proof. The divisor $\sum_{a=1}^{6}(D_a^Y + \widetilde{D}_a^Y)$ of Y^s is ample hence its intersection product with every algebraic curve of Y^s is positive. The threefold X is a projective small resolution of Y, and the 36 double points of Y^s is resolved by 36 $\mathbb{P}^1$. So to show $\widehat{\epsilon}$ is ample we only need to show its intersection products with the exceptional curves of X are positive, and this is done later in this chapter. $\square$

For simplicity, let us define the divisor $\widehat{\epsilon}_0$ by

$$\widehat{\epsilon}_0 := \sum_a \left((\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathcal{S}}) \cap X\right) + \sum_b \left((\mathbb{P}^1 \times \mathcal{S} \times \widetilde{\mathrm{pt}}_b) \cap X\right), \quad (6.34)$$

which is a $\mathfrak{G}$-invariant divisor of X and its quotient by $\mathfrak{G}$ will be denoted by ϵ_0

$$\epsilon_0 = \Big\{ \big(\cup_a (\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathcal{S}}) \cap X \big) \cup \big(\cup_b (\mathbb{P}^1 \times \mathcal{S} \times \widetilde{\mathrm{pt}}_b) \cap X \big) \Big\} / \mathfrak{G}, \quad (6.35)$$

which is a divisor of $X_{\mathfrak{G}}$. Moreover, the divisor ϵ is rationally equivalent to $\epsilon_0 + 2\,\delta$.

Remark 6.2.5. *The 12 irreducible components of the divisor* $\widehat{\epsilon}_0$*, e.g.*

$$(\mathbb{P}^1 \times pt_a \times \widetilde{\mathcal{S}}) \cap X,\ \ (\mathbb{P}^1 \times \mathcal{S} \times \widetilde{pt}_b) \cap X, \tag{6.36}$$

are permuted by the action of $\mathfrak{G}$*, and so* ϵ_0 *is also the pushforward of one irreducible component of* $\widehat{\epsilon}_0$*.*

Definition 6.2.6. *To simplify notations, the irreducible components of* $\widehat{\epsilon}_0$ *will be denoted by* Y_a *and* $\widetilde{Y}_b$ *respectively*

$$Y_a := (\mathbb{P}^1 \times pt_a \times \widetilde{\mathcal{S}}) \cap X, \quad \widetilde{Y}_b := (\mathbb{P}^1 \times \mathcal{S} \times \widetilde{pt}_b) \cap X. \tag{6.37}$$

Before compute the intersection products of the divisors $\widehat{\epsilon}$ and $\widehat{\delta}$ with themselves, we first need to discuss the intersection product of a curve and a surface in a Calabi-Yau threefold [45, 52].

6.3 The intersection product of a curve and a surface in a Calabi-Yau threefold

In this section, suppose M_1 is a Calabi-Yau threefold, N_1 is an irreducible smooth surface and Σ_g is an irreducible smooth curve of genus g, moreover there are embeddings

$$i_{N_1} : N_1 \hookrightarrow M_1, \quad i_{\Sigma_g} : \Sigma_g \hookrightarrow M_1. \tag{6.38}$$

If the intersection of Σ_g and N_1 inside M_1 is transverse, i.e. Σ_g does not lie in N_1, then their intersection will be finitely many points, the cardinality of which is their intersection product. However if Σ_g lies in N_1, their set-theoretic intersection is Σ_g itself, whose cardinality is infinite, so their intersection product has to be defined otherwise [45]. The embedding of N_1 into M_1 yields a short exact sequence of bundles on N_1

$$0 \to \mathcal{T}_{N_1} \to \mathcal{T}_{M_1}|_{N_1} \to \mathcal{N}_{N_1/M_1} \to 0, \tag{6.39}$$

where $\mathcal{T}_{N_1}$ is the tangent bundle of N_1, $\mathcal{T}_{M_1}|_{N_1}$ is the restriction of the tangent bundle of M_1 to N_1 and $\mathcal{N}_{N_1/M_1}$ is the normal bundle of N_1 in M_1. The intersection product of N_1 and Σ_g, denoted by $[N_1] \cdot [\Sigma_g]$, is given by [45, 52]

$$[N_1] \cdot [\Sigma_g] = \int_{\Sigma_g} i^*_{\Sigma_g} \big(c_1(N_1, \mathcal{N}_{N_1/M_1})\big), \tag{6.40}$$

where $c_1(\mathcal{N}_{N_1/M_1})$ is the first Chern class of $\mathcal{N}_{N_1/M_1}$. From the short exact sequence 6.39, we have

$$i^*_1(c_1(M_1, \mathcal{T}_{M_1})) = c_1(N_1, \mathcal{T}_{N_1}) + c_1(N_1, \mathcal{N}_{N_1/M_1}). \tag{6.41}$$

Since M_1 is a Calabi-Yau threefold, so its first Chern class vanishes, which implies

$$c_1(\mathcal{N}_{N_1/M_1}) = -c_1(N_1, \mathcal{T}_{N_1}) = c_1(N_1, \Omega_{N_1}) = c_1(N_1, \Omega_{N_1} \wedge \Omega_{N_1}) = K_{N_1}, \tag{6.42}$$

where K_{N_1} is the canonical class of N_1. Hence the intersection product becomes

$$[N_1] \cdot [\Sigma_g] = \int_{\Sigma_g} i^*_{\Sigma_g}\,(K_{N_1}) = [K_{N_1}] \cdot [\Sigma_g], \tag{6.43}$$

then adjunction formula shows [52]

$$[N_1] \cdot [\Sigma_g] = [K_{N_1}] \cdot [\Sigma_g] = 2g - 2 - [\Sigma_g] \cdot [\Sigma_g], \tag{6.44}$$

where $[\Sigma_g] \cdot [\Sigma_g]$ is the self-intersection of Σ_g in N_1.

6.4 Intersection products

In this section, we will compute the following intersection products

$$[\widehat{\epsilon}_0]^3, \quad [\widehat{\epsilon}_0]^2 \cdot [\widehat{\delta}], \quad [\widehat{\epsilon}_0] \cdot [\widehat{\delta}]^2, \quad [\widehat{\delta}]^3. \tag{6.45}$$

First since $\widehat{\delta}_t$ is rationally equivalent to $\widehat{\delta}_{t'}$ for any other $t' \in \mathbb{P}^1$, we immediately get

$$[\widehat{\delta}]^2 = [\widehat{\delta}_t]^2 = [\widehat{\delta}_t] \cdot [\widehat{\delta}_{t'}] = 0, \tag{6.46}$$

which shows

$$[\widehat{\epsilon}_0] \cdot [\widehat{\delta}]^2 = 0, \quad [\widehat{\delta}]^3 = 0. \tag{6.47}$$

To compute the other intersection products, it's important to notice that the surfaces Y_a and $\widetilde{Y}_b$ are complete intersections in $\mathbb{P}^1 \times (\mathbb{P}^2)^2$ given by

$$Y_a \ : \ \begin{array}{cc} & \begin{array}{ccc} p^2 & q^2 & r^2 \end{array} \\ \begin{array}{cc} t_j & \mathbb{P}^1 \\ x_{2j} & \mathbb{P}^2 \\ x_{4j} & \mathbb{P}^2 \end{array} & \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}, \qquad \widetilde{Y}_b \ : \ \begin{array}{cc} & \begin{array}{ccc} p^1 & q^1 & r^1 \end{array} \\ \begin{array}{cc} t_j & \mathbb{P}^1 \\ x_{1j} & \mathbb{P}^2 \\ x_{3j} & \mathbb{P}^2 \end{array} & \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}. \tag{6.48}$$

The natural projection map

$$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^1 \tag{6.49}$$

induces a fibration Y_a (resp. $\widetilde{Y}_b$) over $\mathbb{P}^1$, whose fiber over $t \in \mathbb{P}^1$ is given by

$$\begin{aligned} Y_a|_t &= \{t\} \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t, \\ \widetilde{Y}_b|_t &= \{t\} \times \mathcal{E}_t \times \widetilde{\mathrm{pt}}_b, \end{aligned} \tag{6.50}$$

where the elliptic curves $\mathcal{E}_t$ and $\widetilde{\mathcal{E}}_t$ have already been defined in 6.27. The surface Y_a is the blow up of $\mathrm{pt}_a \times \widetilde{\mathcal{S}}$ at six points

$$\mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b, \quad b = 1, \cdots, 6, \tag{6.51}$$

while the surface $\widetilde{Y}_b$ is the blow-up of $\mathcal{S} \times \widetilde{\mathrm{pt}}_b$ at six points

$$\mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b, \quad a = 1, \cdots, 6. \tag{6.52}$$

Lemma 6.4.1. *The intersection product* $[\widehat{\epsilon}_0] \cdot [\widehat{\epsilon}_0] \cdot [\widehat{\delta}]$ *is 72, hence the intersection product* $[\epsilon_0] \cdot [\epsilon_0] \cdot [\delta]$ *is 6.*

Proof. We have $\widehat{\epsilon}_0 = \sum_a Y_a + \sum_b \widetilde{Y}_b$, and in order to compute $[\widehat{\epsilon}_0] \cdot [\widehat{\epsilon}_0] \cdot [\widehat{\delta}]$, there are four different cases of intersection products that need to be computed

$$[Y_a] \cdot [Y_b] \cdot [\widehat{\delta}_t]\ , \quad [Y_a] \cdot [\widetilde{Y}_b] \cdot [\widehat{\delta}_t]\ , \quad [\widetilde{Y}_a] \cdot [Y_b] \cdot [\widehat{\delta}_t]\ , \quad [\widetilde{Y}_a] \cdot [\widetilde{Y}_b] \cdot [\widehat{\delta}_t]. \tag{6.53}$$

Let us first do the case $[Y_a] \cdot [Y_b] \cdot [\widehat{\delta}_t]$:

1. If $a \neq b$, we have $Y_a \cap Y_b = \emptyset$, which shows

$$[Y_a] \cdot [Y_b] \cdot [\widehat{\delta}_t] = 0, \ \forall\ a \neq b. \tag{6.54}$$

2. If $a = b$, since intersection product of cycles are associative, let us compute the intersection product of Y_a and $\widehat{\delta}_t$ first. They intersect with each other transversely, and the intersection is $Y_a|_t = t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t$, so the intersection product $[Y_a] \cdot [Y_a] \cdot [\widehat{\delta}_t]$ becomes

$$[Y_a] \cdot [Y_a] \cdot [\widehat{\delta}_t] = [Y_a] \cdot [t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] = [t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] \cdot [Y_a]. \tag{6.55}$$

For general t, $t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t$ is a smooth elliptic curve of genus $g = 1$, hence from equation 6.44 we have

$$\begin{aligned} [t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] \cdot [Y_a] &= -[t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] \cdot [t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] \\ &= -[t \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_t] \cdot [t' \times \mathrm{pt}_a \times \widetilde{\mathcal{E}}_{t'}] \\ &= 0, \end{aligned} \tag{6.56}$$

where t' is another general point of $\mathbb{P}^1$ that is different from t. Here we have used the property that different fibers of the fibration $Y_a \to \mathbb{P}^1$ are rationally equivalent to each other.

So for every a and b, we have

$$[Y_a] \cdot [Y_b] \cdot [\widehat{\delta_t}] = 0, \tag{6.57}$$

and by parallel computations, we also have

$$[\widetilde{Y_a}] \cdot [\widetilde{Y_b}] \cdot [\widehat{\delta_t}] = 0, \ \forall\, a, b. \tag{6.58}$$

The intersection products of the remaining two cases are transverse, and we have

$$\begin{aligned} [Y_a] \cdot [\widetilde{Y_b}] \cdot [\widehat{\delta_t}] &= [\{t\} \times \mathrm{pt}_a \times \widetilde{\mathcal{E}_t}] \cdot [\widetilde{Y_b}] = [\{t\} \times \mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b], \\ [\widetilde{Y_a}] \cdot [Y_b] \cdot [\widehat{\delta_t}] &= [\{t\} \times \mathcal{E}_t \times \widetilde{\mathrm{pt}}_a] \cdot [Y_b] = [\{t\} \times \mathrm{pt}_b \times \widetilde{\mathrm{pt}}_a], \end{aligned} \tag{6.59}$$

so we have got

$$[\widehat{\epsilon_0}] \cdot [\widehat{\epsilon_0}] \cdot [\widehat{\delta_t}] = 72. \tag{6.60}$$

Since the order of $\mathfrak{G}$ is 12, we have

$$[\epsilon_0] \cdot [\epsilon_0] \cdot [\delta] = 72/12 = 6. \tag{6.61}$$

$\square$

Lemma 6.4.2. *The intersection product* $[\widehat{\epsilon_0}]^3$ *is -216, hence the intersection product* $[\epsilon_0]^3$ *is -18.*

Proof. To compute $[\widehat{\epsilon_0}]^3$, there are eight different cases that we need to study

$$\begin{aligned} &[Y_a] \cdot [Y_b] \cdot [Y_c],\ [Y_a] \cdot [Y_b] \cdot [\widetilde{Y_c}],\ [Y_a] \cdot [\widetilde{Y_b}] \cdot [Y_c],\ [Y_a] \cdot [\widetilde{Y_b}] \cdot [\widetilde{Y_c}], \\ &[\widetilde{Y_a}] \cdot [Y_b] \cdot [Y_c],\ [\widetilde{Y_a}] \cdot [Y_b] \cdot [\widetilde{Y_c}],\ [\widetilde{Y_a}] \cdot [\widetilde{Y_b}] \cdot [Y_c],\ [\widetilde{Y_a}] \cdot [\widetilde{Y_b}] \cdot [\widetilde{Y_c}]. \end{aligned} \tag{6.62}$$

First let us compute $[Y_a] \cdot [Y_b] \cdot [Y_c]$. When $a \neq b$, the intersection $Y_a \cap Y_b$ is empty, hence when a, b and c are not all the same, this intersection product is 0. When $a = b = c$, the triple self-intersection of $[Y_a]$ is given by [32, 45]

$$[Y_a] \cdot [Y_a] \cdot [Y_a] = \int_{Y_a} c_1^2 \left(\mathcal{N}_{Y_a/X}\right) = \int_{Y_a} c_1^2 \left(\mathcal{T}_{Y_a}\right) = \int_{Y_a} c_1^2 \left(\Omega_{Y_a}\right) = \int_{Y_a} K_{Y_a}^2, \tag{6.63}$$

where we have used the property 6.42. It is important to notice that the surface Y_a is the blow up of $\mathrm{pt}_a \times \widetilde{\mathcal{S}}$ at six points $\{\mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b\}_{b=1}^6$. To simplify notations, let $\widehat{\mathcal{S}}$ be the blow up of $\mathcal{S}$ at the six points $\{\mathrm{pt}_a\}_{a=1}^6$

$$\beta_{\mathcal{S}} : \ \widehat{\mathcal{S}} \to \mathcal{S}, \tag{6.64}$$

and let us denote the exceptional divisor over pt_a by D'_a. The surfaces Y_a and $\widetilde{Y}_b$ are both isomorphic to $\widehat{\mathcal{S}}$. To proceed, we need to compute the integration $\int_{\widehat{\mathcal{S}}} K^2_{\widehat{\mathcal{S}}}$, which is given by a standard fact in algebraic geometry [52]

$$[K_{\widehat{\mathcal{S}}}] \cdot [K_{\widehat{\mathcal{S}}}] = 9 - 9 = 0, \tag{6.65}$$

where we have used the property that $\widehat{\mathcal{S}}$ is the blow up of $\mathbb{P}^2$ at 9 points. In conclusion, we find that

$$[Y_a] \cdot [Y_b] \cdot [Y_c] = 0,\ [\widetilde{Y}_a] \cdot [\widetilde{Y}_b] \cdot [\widetilde{Y}_c] = 0,\ \forall\ a, b, c. \tag{6.66}$$

Now let us compute $[Y_a] \cdot [Y_b] \cdot [\widetilde{Y}_c]$. If $a \neq b$, the intersection $Y_a \cap Y_b$ is empty, hence this intersection product is 0, so we only need to compute the case where $a = b$. First Y_a intersects with $\widetilde{Y}_b$ transversely, and their intersection is given by

$$Y_a \cap \widetilde{Y}_b = \mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b, \tag{6.67}$$

which is an exceptional curve of Y_a, hence its self-intersection in Y_a is -1 [98]. The genus of $\mathbb{P}^1$ is 0, therefore adjunction formula 6.44 yields

$$\begin{aligned}[Y_a] \cdot [Y_a] \cdot [\widetilde{Y}_b] &= [Y_a] \cdot [\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b] \\ &= -2 - [\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b] \cdot [\mathbb{P}^1 \times \mathrm{pt}_a \times \widetilde{\mathrm{pt}}_b] \\ &= -1.\end{aligned} \tag{6.68}$$

From parallel computations, we have

$$[\widetilde{Y}_a] \cdot [\widetilde{Y}_a] \cdot [Y_b] = -1, \tag{6.69}$$

therefore we have found

$$[\widehat{\epsilon}_0]^3 = -216,\ [\epsilon_0]^3 = -18. \tag{6.70}$$

□

Thus we have got the intersection products

$$[\widehat{\epsilon}]^3 = ([\widehat{\epsilon}_0] + 2\,[\widehat{\delta}])^3 = [\widehat{\epsilon}_0]^3 + 6\,[\widehat{\epsilon}_0]^2 \cdot [\widehat{\delta}] = 216,\quad [\widehat{\epsilon}]^2 \cdot [\widehat{\delta}] = [\widehat{\epsilon}_0]^2 \cdot [\widehat{\delta}] = 72, \tag{6.71}$$

from which we find

$$[\epsilon]^3 = 18,\ [\epsilon]^2 \cdot [\delta] = 6,\ [\delta]^2 = 0. \tag{6.72}$$

6.5 The algebraic cycles dual to the divisors ϵ, δ

In this section, we will construct two one dimensional algebraic cycles Γ_ϵ and Γ_δ of $X_{\mathfrak{G}}$, which form a basis of $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions), moreover, their intersection products with ϵ and δ are shown to be

$$\begin{aligned} &[\epsilon] \cdot [\Gamma_\epsilon] = 1, \quad [\delta] \cdot [\Gamma_\epsilon] = 0, \\ &[\epsilon] \cdot [\Gamma_\delta] = 0, \quad [\delta] \cdot [\Gamma_\delta] = 1, \end{aligned} \tag{6.73}$$

which shows ϵ and δ from a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsions). First, the one dimensional algebraic cycle $(1,1) \times D_1 \times \widetilde{\mathrm{pt}}_a$, which is isomorphic to $\mathbb{P}^1$, lies in the surface $\widetilde{Y}_a$, from which we construct the following $\mathfrak{G}$-invariant cycle

$$\widehat{\Gamma}_\epsilon := \sum_{g \in \mathfrak{G}} g\big((1,1) \times D_1 \times \widetilde{\mathrm{pt}}_1\big). \tag{6.74}$$

The cycle Γ_ϵ is defined to be the quotient of $\widehat{\Gamma}_\epsilon$ by $\mathfrak{G}$

$$\Gamma_\epsilon := \{\cup_{g \in \mathfrak{G}} g\big((1,1) \times D_1 \times \widetilde{\mathrm{pt}}_1\big)\}/\mathfrak{G}, \tag{6.75}$$

which is a cycle of $X_{\mathfrak{G}}$ that is isomorphic to $\mathbb{P}^1$.

Lemma 6.5.1. *The intersection products of* $\widehat{\Gamma}_\epsilon$ *and* $\widehat{\epsilon}_0$, $\widehat{\delta}_t$ *are given by*

$$[\widehat{\epsilon}_0] \cdot [\widehat{\Gamma}_\epsilon] = 12, \ [\widehat{\delta}_t] \cdot [\widehat{\Gamma}_\epsilon] = 0, \tag{6.76}$$

hence Γ_ϵ *satisfies the intersection products in equation 6.73.*

Proof. Choose a general $t \in \mathbb{P}^1$ and $\widehat{\Gamma}_\epsilon$ does not intersect with $\widehat{\delta}_t$, hence we get

$$[\widehat{\delta}_t] \cdot [\widehat{\Gamma}_\epsilon] = 0. \tag{6.77}$$

The surface $\widetilde{Y}_1$ is the blow up of $\mathcal{S} \times \widetilde{\mathrm{pt}}_1$ at the six points

$$\mathrm{pt}_b \times \widetilde{\mathrm{pt}}_1, \quad b = 1, \cdots, 6, \tag{6.78}$$

where point $\mathrm{pt}_1 \times \widetilde{\mathrm{pt}}_1$ lies in the cycle $D_1 \times \widetilde{\mathrm{pt}}_1$. The cycle $(1,1) \times D_1 \times \widetilde{\mathrm{pt}}_1$ is the proper transform of $D_1 \times \widetilde{\mathrm{pt}}_1$ in $\widetilde{Y}_1$, hence from the proof of Lemma 6.4.2, its self-intersection in $\widetilde{Y}_1$ is -2. Thus adjunction formula 6.44 tells us

$$[\widetilde{Y}_1] \cdot [(1,1) \times D_1 \times \widetilde{\mathrm{pt}}_1] = 0. \tag{6.79}$$

Since the intersection of $D_1 \times \widetilde{\mathrm{pt}}_1$ and $\widetilde{Y}_b, b \neq 1$ is empty, so we have found

$$[\widetilde{Y}_b] \cdot [(1,1) \times D_1 \times \widetilde{\mathrm{pt}}_1] = 0, \ \forall\ b. \tag{6.80}$$

The cycle $(1,1)\times D_1\times \widetilde{\mathrm{pt}}_1$ intersects transversely with Y_1 at the point $(1,1)\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1$, while its intersection with $Y_b, b\neq 1$ is empty, so we get

$$[\widehat{\epsilon}_0]\cdot[(1,1)\times D_1\times \widetilde{\mathrm{pt}}_1]=1, \tag{6.81}$$

which immediately proves this lemma. □

The cycle $\widehat{\Gamma}_\delta$ is defined by

$$\widehat{\Gamma}_\delta := \sum_{g\in\mathfrak{G}} g\big(\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1\big), \tag{6.82}$$

which is the union of 12 exceptional curves of X, and the cycle Γ_δ is given by

$$\Gamma_\delta := \{\cup_{g\in\mathfrak{G}} g\big(\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1\big)\}/\mathfrak{G}, \tag{6.83}$$

therefore Γ_δ is one of the three exceptional curves of $X_{\mathfrak{G}}$.

Lemma 6.5.2. *The intersection products of* $\widehat{\Gamma}_\delta$ *and* $\widehat{\epsilon}_0$, $\widehat{\delta}_t$ *are given by*

$$[\widehat{\epsilon}_0]\cdot[\widehat{\Gamma}_\delta]=-24,\ [\widehat{\delta}_t]\cdot[\widehat{\Gamma}_\delta]=12, \tag{6.84}$$

from which we get

$$[\epsilon_0]\cdot[\Gamma_\delta]=-2,\ [\delta]\cdot[\Gamma_\delta]=1. \tag{6.85}$$

Proof. The curve $\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1$ is an exceptional divisor of $\widetilde{Y}_1$ (viewed as a blow up of $\mathcal{S}\times \widetilde{\mathrm{pt}}_1$), hence from the proof of Lemma 6.4.2 and formula 6.44, we have

$$[\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1]\cdot[\widetilde{Y}_1]=-1, \tag{6.86}$$

and similarly we also have

$$[\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1]\cdot[Y_1]=-1. \tag{6.87}$$

The intersection of $\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1$ with Y_b, $\widetilde{Y}_b$ is empty when $b\neq 1$, so we immediately have $[\widehat{\epsilon}_0]\cdot[\widehat{\Gamma}_\delta]=-24$. On the other hand, $\mathbb{P}^1\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1$ intersects with $\widetilde{\delta}_t$ transversely at the point $t\times \mathrm{pt}_1\times \widetilde{\mathrm{pt}}_1$, therefore we have $[\widehat{\delta}_t]\cdot[\widehat{\Gamma}_\delta]=12$. □

Since $[\epsilon]=[\epsilon_0]+2\,[\delta]$, so Γ_ϵ and Γ_δ satisfy the intersection products in equation 6.73.

Remark 6.5.3. *The* $\mathbb{Z}/3\mathbb{Z}$*-action on* $X_{\mathfrak{G}}$ *in Remark 6.1.3 sends* δ *to itself, while it permutes the three exceptional curves of* $X_{\mathfrak{G}}$*, hence* $[\delta]\cdot[\Gamma_\delta]=1$ *immediately implies that the three exceptional curves of* $X_{\mathfrak{G}}$ *are homologous to each other. So Theorem 3.2 [87] also proves that the Hodge numbers of* $X_{\mathfrak{G}}$ *are* $h^{11}=h^{12}=2$.

Summary: in this chapter, we have constructed two divisors ϵ and δ of $X_{\mathfrak{G}}$, whose Poincaré duals form a basis of $H^2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsion). The two divisors lie in the closure of the Kähler cone of $X_{\mathfrak{G}}$, hence they define a framing of $X_{\mathfrak{G}}$, from which the complexified Kähler moduli space of $X_{\mathfrak{G}}$ is constructed. We have also computed the triple intersection products of the two divisors, i.e. $[\epsilon]^3$, $[\epsilon]^2 \cdot [\delta]$, $\cdots$, which will be important when we compute the Yukawa couplings of $X_{\mathfrak{G}}$ in Chapter 7. We have also constructed two one dimensional algebraic cycles Γ_ϵ and Γ_δ, which form a basis of $H_2(X_{\mathfrak{G}}, \mathbb{Z})$ (modulo torsion). We have shown that $\{\Gamma_\epsilon, \Gamma_\delta\}$ are dual to $\{\epsilon, \delta\}$, which will be important when we study the Gromov-Witten invariants of $X_{\mathfrak{G}}$ in Chapter 7.

Chapter 7

The mirror symmetry of the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$

In this chapter, we will study the mirror symmetry of the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$. The structure of this chapter is as follows:

- Section 7.1 discusses the construction of a threeform on the mirror family.
- Section 7.2 computes the fundamental period ϖ_0 of $X_{\mathfrak{G}}$ by explicitly evaluating the integral of the threeform over a cycle in $H_3(X_{\mathfrak{G}}, \mathbb{Z})$.
- Section 7.3 studies the discriminant locus of the mirror family and points out which is the large complex structure limit.
- Section 7.4 computes the Picard-Fuchs operators using a non-rigorous method, as the Griffiths-Dwork method turns out to be too complicated for $X_{\mathfrak{G}}$. This section also solves the Picard-Fuchs equations of $X_{\mathfrak{G}}$ by Frobenius method, which yields six canonical periods of $X_{\mathfrak{G}}$.
- Section 7.5 constructs the mirror map of $X_{\mathfrak{G}}$ and discusses its mirror symmetry.
- Section 7.6 computes the Yukawa couplings of $X_{\mathfrak{G}}$ and their instanton expansions, from which the Gromov-Witten invariants of $X_{\mathfrak{G}}$ are found.
- Finally, Section 7.7 studies the connections between the mirror symmetry of $X_{\mathfrak{G}}$ and that of the one parameter mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$.

7.1 The threeform of $X_{\mathfrak{G}}$

In this section, we will discuss the construction of a threeform $\Omega_{X_{\mathfrak{G}}}$ for the mirror family of $X_{\mathfrak{G}}$. From Chapter 5, the lattice of characters of the toric fourfold $\mathcal{S} \times \widetilde{\mathcal{S}}$ has a basis $\{e_i\}_{i=1}^4$ given by

$$e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1). \tag{7.1}$$

The fan of $\mathcal{S} \times \widetilde{\mathcal{S}}$ has 12 rays generated by $\{\rho_i\}_{i=1}^{12}$ 5.29, and let x_i be the homogeneous coordinate (which is in the sense of Chapter 5 of [31]) associated to the ray generated by ρ_i. For each subset

$$I = \{i_1, \cdots, i_4\} \subset \{1, \cdots, 12\}\ , \tag{7.2}$$

let us define

$$\det(e_I) := \det(< e_k, \rho_{i_l} >_{1 \leq k,l \leq 4}),\quad dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_4},\quad \widehat{x}_I := \prod_{i \notin I} x_i\ . \tag{7.3}$$

There exists a homogeneous fourform Ω_0 on $\mathcal{S} \times \widetilde{\mathcal{S}}$ given by

$$\Omega_0 := \sum_I \det(e_I)\, \widehat{x}_I\, dx_I\ , \tag{7.4}$$

the degree of which is 12. The homogenization of the toric polynomial $f(t)$ in equation 5.32 will be denoted by $f_h(x)$, which is a homogeneous polynomial of degree 12 in variables x_i. The Poincaré residue of Ω_0 defines a threeform on Y by

$$\Omega_Y := \frac{1}{2\pi i} \operatorname{Res}_{f_h(x_i)=0} \frac{\Omega_0}{f_h(x_i)}\ . \tag{7.5}$$

The action of $\mathfrak{G}$ on $\mathcal{S} \times \widetilde{\mathcal{S}}$ induces an action of $\mathfrak{G}$ on the homogeneous coordinates $\{x_i\}_{i=1}^{12}$ which are given by permutations. Both Ω_0 and $f_h(x_i)$ invariant under this action of $\mathfrak{G}$, hence the threeform Ω_Y is also invariant under the action of $\mathfrak{G}$, thus it induces a threeform defined on the quotient $Y_{\mathfrak{G}} = Y/\mathfrak{G}$.

The formula 7.5 even works for the conifold Y^s, and now the threeform Ω_{Y^s} is defined on the smooth locus of Y^s which is nowhere vanishing and invariant under the action of $\mathfrak{G}$. Hence after quotient we get a threefrom $\Omega_{Y^s_{\mathfrak{G}}}$ defined on the smooth locus of the conifold $Y^s_{\mathfrak{G}}$ that is also nowhere vanishing. The projective small resolution of $Y^s_{\mathfrak{G}}$ in Lemma 6.1.2 is crepant, so $\Omega_{Y^s_{\mathfrak{G}}}$ pulls back to a nowhere vanishing threeform $\Omega_{X_{\mathfrak{G}}}$ on $X_{\mathfrak{G}}$.

Remark 7.1.1. *From its construction, the threeform $\Omega_{X_{\mathfrak{G}}}$ is defined over $\mathbb{Q}$.*

7.2 Fundamental Period

In this section, we will compute the fundamental period of $X_{\mathfrak{G}}$ by explicitly evaluating the integration of $\Omega_{X_{\mathfrak{G}}}$ over a cycle of $H_3(X_{\mathfrak{G}}, \mathbb{Z})$. There is a three cycle T_Y on the conifold Y^s given by

$$T_Y : |t_1| = |t_2| = |t_3| = \varepsilon,\ f(t_1, t_2, t_3, t_4) = 0,\ 0 < \varepsilon < 1, \tag{7.6}$$

which lies in the torus of $\mathcal{S} \times \widetilde{\mathcal{S}}$ and is isomorphic to the three dimensional real torus $(S^1)^3$. The cycle T_Y defines a cycle class of $H_3(Y^s, \mathbb{Z})$, whose image in $Y^s_{\mathfrak{G}}$ defines a cycle class of $H_3(Y^s_{\mathfrak{G}}, \mathbb{Z})$. Since T_Y does not contain any of the double points of Y^s, therefore the quotient $T_Y/\mathfrak{G}$ does not contain any of the double points of $Y^s_{\mathfrak{G}}$, thus after resolution it defines a cycle class of $H_3(X_{\mathfrak{G}}, \mathbb{Z})$. From the construction of the threeform $\Omega_{X_{\mathfrak{G}}}$, we deduce that the integration of $\Omega_{X_{\mathfrak{G}}}$ on $T_Y/\mathfrak{G}$ is a rational multiple of the following integration

$$\varpi_0 = \int_{T_Y} \Omega_{Y^s}. \tag{7.7}$$

Remark 7.2.1. *This is very similar to the quintic case.*

For simplicity, let us define affine coordinates (φ, ρ) by

$$\varphi = -\gamma_1/\gamma_0,\ \rho = -\gamma_2/\gamma_0, \tag{7.8}$$

then the period ϖ_0 is given by the integration [30]

$$\begin{aligned} \varpi_0 &= \frac{1}{(2\pi i)^4} \int_{|t_i|=\varepsilon} \frac{d^4 t}{t_1\, t_2\, t_3\, t_4 \Big(1 - \varphi\, f_1(t) - \rho\, \big(f_2(t) + f_3(t) + f_4(t)\big)\Big)} \\ &= \sum_{n=0}^{\infty} \Big[\Big(\varphi\, f_1(t) + \rho\, \big(f_2(t) + f_3(t) + f_4(t)\big)\Big)^n \Big]_0, \end{aligned} \tag{7.9}$$

where we have used Cauchy's residue theorem and $[*]_0$ means the constant term of the Laurent polynomial $*$. From equation 5.31, the toric polynomials $f_i(t)$ satisfy the property [15]

$$f_1(t) = T(t_1, t_2) + T(t_3, t_4), \quad f_2(t) + f_3(t) + f_4(t) = T(t_1, t_2)\, T(t_3, t_4) \tag{7.10}$$

where T is the Laurent polynomial defined by

$$T(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + \frac{x}{y} + \frac{y}{x}. \tag{7.11}$$

By evaluating equation 7.9, we have found a power series expansion of the fundamental period ϖ_0 given by [18]

$$\varpi_0 \;=\; \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{n!}{(n-r-s)!\,r!\,s!}\, T_{n-s}\, T_{r+s}\, \varphi^{n-r} \rho^{r} \ , \tag{7.12}$$

where the number T_k is

$$T_k \;=\; \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \;\sum_{s=\max\left(0,\lceil \frac{k-3r}{2} \rceil\right)}^{\min\left(k-2r,\lfloor \frac{k-r}{2} \rfloor\right)} \frac{k!\,(2r)!}{(r!)^2\, s!\,(3r+2s-k)!\,(k+r-2s)!\,(k-2r-s)!} \ . \tag{7.13}$$

The first several terms of ϖ_0 are

$$\begin{aligned} \varpi_0 = {} & 1 + 36\,\rho^2 + 12\,\varphi^2 + 144\,\rho^3 + 432\,\rho^2\,\varphi + 216\,\rho\,\varphi^2 + 24\,\varphi^3 + 8100\,\rho^4 + 8640\,\rho^3\,\varphi \\ & + 8208\,\rho^2\,\varphi^2 + 1728\,\rho\,\varphi^3 + 396\,\varphi^4 + 129600\,\rho^5 + 324000\,\rho^4\,\varphi + 248400\,\rho^3\,\varphi^2 \\ & + 108000\,\rho^2\,\varphi^3 + 25920\,\rho\,\varphi^4 + 2160\,\varphi^5 + \cdots \ . \end{aligned} \tag{7.14}$$

The procedure to study the mirror symmetry of $X_{\mathfrak{G}}$ is as follows:

1. Find the Picard-Fuchs operators of $X_{\mathfrak{G}}$. However the Griffiths-Dwork method turns out to be too complicated in this example, therefore we will use a method from [23], i.e. we will compute the PDE operators which satisfy the following equation

   $$\mathcal{L}\,\varpi_0 = 0, \tag{7.15}$$

   where $\mathcal{L}$ is an unknown PDE operator in variables φ and ρ. This method is not mathematically rigorous, but nevertheless it works.

2. Compute the other five canonical periods by solving the Picard-Fuchs equations

   $$\mathcal{L}\,\varpi = 0, \tag{7.16}$$

   which is certainly very very challenging and involves a huge amount of work.

3. Find the large complex structure limit and the mirror map of $X_{\mathfrak{G}}$, which is another very difficult part.

4. Compute the Yukawa couplings of $X_{\mathfrak{G}}$ and their instanton expansions near the large complex structure limit.

7.3 The discriminant locus and the large complex structure limit

In this section, we will study the discriminant locus of $X_{\mathfrak{G}}$ and point out which point is the large complex structure limit. In the affine coordinate $(\varphi,\, \rho)$, the irreducible components of the discriminant locus in Table 5.2 are listed in Table 7.1.

Component	Equation	Component	Equation
$\Gamma^{(\mathrm{o})}$	$\rho + \varphi^2 = 0$	$\Gamma^{(\mathrm{vi})}$	$2\,\rho - \varphi = 0$
$\Gamma^{(\mathrm{i})}$	$\rho = 0$	$\Gamma^{(\mathrm{vii})}$	$-6\,\rho + 5\,\varphi + 1 = 0$
$\Gamma^{(\mathrm{ii})}$	$-4\,\rho + 4\,\varphi + 1 = 0$	$\Gamma^{(\mathrm{viii})}$	$6\,\rho + \varphi = 0$
$\Gamma^{(\mathrm{iii})}$	$-36\,\rho - 12\,\varphi + 1 = 0$	$\Gamma^{(\mathrm{ix})}$	$\varphi - 3\,\rho = 0$
$\Gamma^{(\mathrm{iv})}$	$-9\,\rho + 6\,\varphi + 1 = 0$	$\Gamma^{(\mathrm{x})}$	$12\,\rho - 4\,\varphi + 1 = 0$
$\Gamma^{(\mathrm{v})}$	$18\,\rho - 3\,\varphi + 1 = 0$		

Table 7.1: Discriminant locus in affine coordinate

From this table, the conic $\Gamma^{(\mathrm{o})}$ intersects with four lines $\Gamma^{(\mathrm{i})}$, $\Gamma^{(\mathrm{vi})}$, $\Gamma^{(\mathrm{viii})}$, $\Gamma^{(\mathrm{ix})}$ at the point $\varphi = \rho = 0$, which is shown in the top left figure of Figure 7.1, and to make the mirror family less singular, the first step is to blow the parameter space up at the point $\varphi = \rho = 0$. The blow up of $\mathbb{C}^2$ at $\varphi = \rho = 0$ is the subvariety of $\mathbb{C}^2 \times \mathbb{P}^1$ given by

$$\varphi\, b = \rho\, a, \tag{7.17}$$

where (a, b) are the projective coordinates of $\mathbb{P}^1$. In the open subset of $\mathbb{C}^2 \times \mathbb{P}^1$ where $a \neq 0$, let us define σ by

$$\sigma := b/a, \tag{7.18}$$

then the proper transform of $\Gamma^{(\mathrm{o})}$ is given by

$$\rho = \varphi\, \sigma,\ \ \varphi + \sigma = 0. \tag{7.19}$$

The natural projection given by

$$(\varphi,\, \rho,\, \sigma) \to (\varphi,\, \sigma), \tag{7.20}$$

induces an isomorphism between the proper transform of $\Gamma^{(\mathrm{o})}$ and the following line in $\mathbb{C}^2$ (whose coordinates are given by (φ, σ)) defined by

$$\mathrm{B}_0\, \Gamma^{(\mathrm{o})} : \varphi + \sigma = 0. \tag{7.21}$$

Similarly, the proper transforms of the other four lines are given by

$$\mathrm{B}_0\,\Gamma^{\mathrm{i}} : \sigma = 0 \;;\quad \mathrm{B}_0\,\Gamma^{(\mathrm{vi})} : \sigma = \frac{1}{2} \;;\quad \mathrm{B}_0\,\Gamma^{(\mathrm{viii})} : \sigma = -\frac{1}{6} \;;\quad \mathrm{B}_0\,\Gamma^{(\mathrm{ix})} : \sigma = \frac{1}{3}, \tag{7.22}$$

while the exceptional divisor of this blow up is given by

$$\mathrm{F}_0 : \varphi = 0. \tag{7.23}$$

These six lines are shown in top right figure of Figure 7.1.

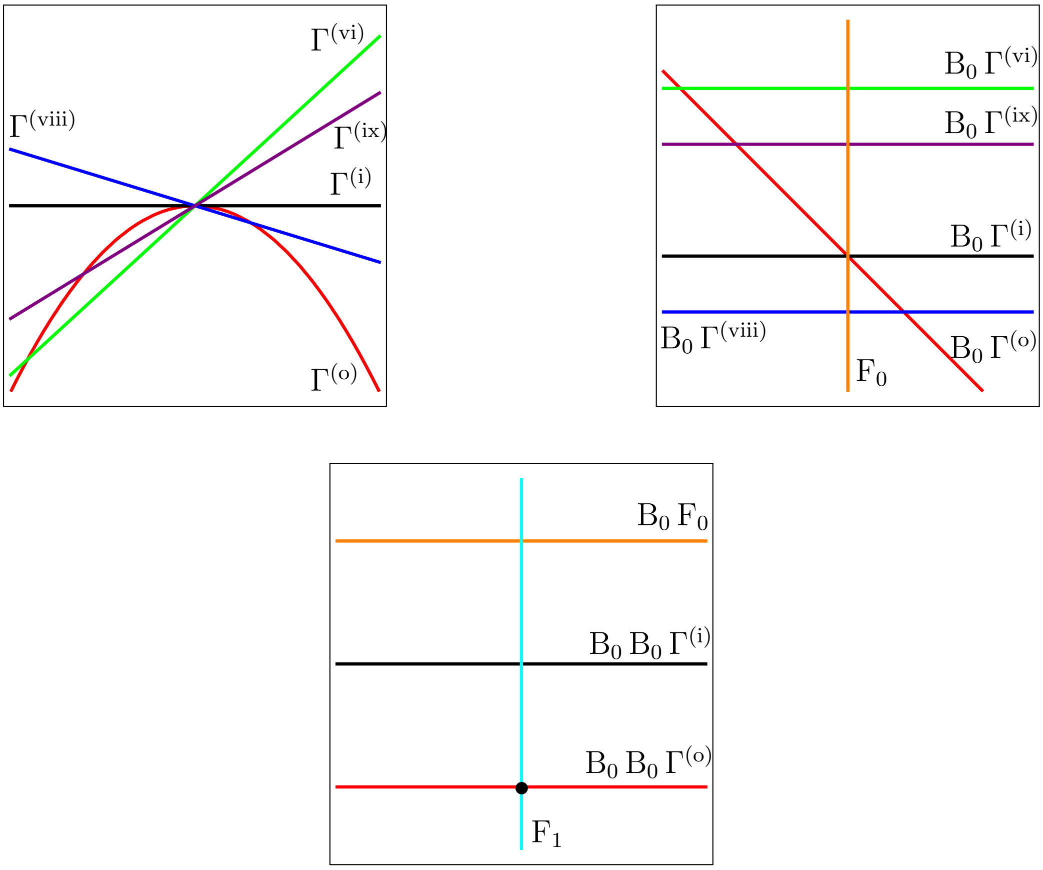

Figure 7.1: *Discriminant locus and their blow ups*

However there are still three lines $\mathrm{B}_0\,\Gamma^{(\mathrm{o})}$, $\mathrm{B}_0\,\Gamma^{(\mathrm{i})}$ and F_0 crossing normally at the point $\varphi = \sigma = 0$, so we need to blow this parameter space up again at the point $\varphi = \sigma = 0$. By the same method, the blow up of $\mathbb{C}^2$ at $\varphi = \sigma = 0$ is the subvariety of $\mathbb{C}^2 \times \mathbb{P}^1$ given by

$$\varphi\, b_1 = \sigma\, a_1, \tag{7.24}$$

where (a_1, b_1) are the projective coordinates of this new $\mathbb{P}^1$. Similarly in the affine open set where $a_1 \neq 0$, let us define μ by

$$\mu := b_1/a_1. \tag{7.25}$$

Over an open subvariety, σ satisfies

$$\sigma = \mu\ \varphi. \tag{7.26}$$

The natural projection defined by

$$(\varphi, \sigma, \mu) \to (\varphi, \mu) \tag{7.27}$$

maps the proper transforms of $\mathrm{B}_0\,\Gamma^{(\mathrm{o})}$, $\mathrm{B}_0\,\Gamma^{(\mathrm{i})}$ and F_0 to the following three lines respectively

$$\mathrm{B}_0\,\mathrm{B}_0\,\Gamma^{(\mathrm{o})}\ : \mu = -1\ ; \quad \mathrm{B}_0\,\mathrm{B}_0\,\Gamma^{(\mathrm{i})}\ :\ \mu = 0\ ; \quad \mathrm{B}_0\,\mathrm{F}_0\ :\ \mu = \infty, \tag{7.28}$$

while the exceptional divisor of this second blow up is given by

$$\mathrm{F}_1\ : \varphi = 0. \tag{7.29}$$

The four lines are shown in the bottom figure of Figure 7.1. In an open subvariety, the coordinate μ satisfies

$$\mu = \sigma/\varphi = \rho/\varphi^2. \tag{7.30}$$

Later we will show that the point ($\varphi = 0$, $\mu = -1$) (the point $\bullet$ in the bottom figure of Figure 7.1) is the large complex structure limit of the mirror family, at which $\mathrm{B}_0\,\mathrm{B}_0\,\Gamma^{(\mathrm{o})}$ and F_1 cross with each other transversely. Now we define ν by

$$\nu = \mu + 1 = \rho/\varphi^2 + 1, \tag{7.31}$$

so the large complex structure limit is given by $\varphi = \nu = 0$.

7.4 Picard-Fuchs equations and canonical periods

First, we need to find the Picard-Fuchs operators of $X_{\mathfrak{G}}$, however the Griffiths-Dwork method turns out to be too complicated for $X_{\mathfrak{G}}$ [30], therefore we have to use a non-rigorous method from [23]. From equation 7.9, in a neighbourhood of $\varphi = \nu = 0$, the fundamental period ϖ_0 has a power series expansion in φ and ν with first several terms given by

$$\begin{aligned}\varpi_0 = 1 &+ 12\,\varphi^2 + 24\,\varphi^3 + 216\,\varphi^4 + 144\,\nu\,\varphi^4 + 864\,\varphi^5 + 36\,\nu^2\,\varphi^4 + 864\,\nu\,\varphi^5 + 5304\,\varphi^6 \\ &+ 432\,\nu^2\,\varphi^5 + 9936\,\nu\,\varphi^6 + 26640\,\varphi^7 + 7776\,\nu^2\,\varphi^6 + 69120\,\nu\,\varphi^7 + 150840\,\varphi^8 \\ &+ 144\,\nu^3\,\varphi^6 + 82080\,\nu^2\,\varphi^7 + 552960\,\nu\,\varphi^8 + 816000\,\varphi^9 + \cdots .\end{aligned} \tag{7.32}$$

Define the operators ϑ_φ and ϑ_ν by

$$\vartheta_\varphi := \varphi \frac{\partial}{\partial \varphi}, \quad \vartheta_\nu := \nu \frac{\partial}{\partial \nu}. \tag{7.33}$$

Suppose $\mathcal{L}$ is a PDE operator of the form

$$\mathcal{L} = \sum_{k,\, l \geq 0}^{k+l \leq 3} R_{k,\, l}\, \vartheta_\varphi^k\, \vartheta_\nu^l, \tag{7.34}$$

where $R_{k,l}$ is an unknown polynomial in variables φ and ν, and now we plug $\mathcal{L}$ into the following equation

$$\mathcal{L}\, \varpi_0 = 0, \tag{7.35}$$

which will yield linear equations of the unknown coefficients of the polynomials $R_{k,l}$. By solving these linear equations, we get the polynomials $R_{k,l}$. Using Mathematica, we find two special operators $\mathcal{L}_1$, $\mathcal{L}_2$ of the form

$$\mathcal{L}_1 = \sum_{k,\, l \geq 0}^{k+l \leq 2} R^1_{k,\, l}\, \vartheta_\varphi^k\, \vartheta_\nu^l, \quad \mathcal{L}_2 = \sum_{k,\, l \geq 0}^{k+l \leq 3} R^2_{k,\, l}\, \vartheta_\varphi^k\, \vartheta_\nu^l, \tag{7.36}$$

where the polynomials $R^1_{k,\, l}$ and $R^2_{k,\, l}$ are listed in Appendix D. We have checked that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the following equations to a very high order

$$\mathcal{L}_i\, \varpi_0 = 0\,; \;\; i = 1,\, 2. \tag{7.37}$$

But the drawback of this method is that first $\mathcal{L}_i$ is only shown to satisfy the equation 7.37 to some order, and second it is hugely difficult to prove that they are indeed Picard-Fuchs operators of $X_{\mathfrak{G}}$.

7.4.1 The Frobenius methods

In this section, we will use Frobenius method [102] to solve the PDE equations

$$\mathcal{L}_i\, \varpi = 0\,; \;\; i = 1,\, 2. \tag{7.38}$$

The basic idea is that we first try solutions of the form

$$\varpi = \varphi^\epsilon\, \nu^\delta \sum_{k,l} a_{k,l}\,(\epsilon, \delta)\, \varphi^k\, \nu^l, \tag{7.39}$$

where $a_{k,l}\,(\epsilon, \delta)$ are unknown coefficients that depend on ϵ and δ. Next we plug this ϖ into the PDEs 7.38, which will yield equations of ϵ, δ and $a_{k,l}\,(\epsilon, \delta)$. The lowest recursion relations are given by

$$\delta^2 = 0, \quad 3\,\delta\,\epsilon^2 - \epsilon^3 = 0, \tag{7.40}$$

which must be satisfied in order to have a non-trivial solution. Multiplying the second equation by ϵ and δ respectively we get

$$\epsilon^3\,\delta = 3\,\delta^{\,2}\epsilon^2 = 0, \quad \epsilon^4 = 3\,\delta\,\epsilon^3 = 0, \tag{7.41}$$

from which we deduce that the algebra generated by ϵ and δ has a basis given by

$$1,\ \epsilon,\ \epsilon^2,\ \epsilon^3,\ \epsilon\ \delta,\ \epsilon^3 = 3\ \epsilon^2\,\delta. \tag{7.42}$$

For simplicity, let us define $h(\epsilon,\delta)$ by

$$h(\epsilon,\delta) := \sum_{k,l} a_{k,l}\,(\epsilon,\delta)\,\varphi^k\,\nu^l. \tag{7.43}$$

From complex analysis, we have

$$\left.\frac{\partial}{\partial\epsilon}\,\varphi^\epsilon\right|_{\epsilon=0} = \log\varphi\,, \qquad \left.\frac{\partial}{\partial\epsilon}\,\nu^\epsilon\right|_{\epsilon=0} = \log\nu, \tag{7.44}$$

and we also formally define

$$\varpi_{i,j} := \left.\frac{\partial^i}{\partial\epsilon^i}\,\frac{\partial^j}{\partial\delta^j}\,\varpi\right|_{\epsilon=\delta=0}, \qquad h_{i,j} := \left.\frac{\partial^i}{\partial\epsilon^i}\,\frac{\partial^j}{\partial\delta^j}\,h\right|_{\epsilon=\delta=0}, \tag{7.45}$$

while $\varpi_{i,j}$ is given by

$$\begin{aligned}
\varpi_{0,0} &= h_{0,0},\\
\varpi_{1,0} &= h_{0,0}\,\log\varphi + h_{1,0},\\
\varpi_{0,1} &= h_{0,0}\,\log\nu + h_{0,1},\\
\varpi_{2,0} &= h_{0,0}\,\log^2\varphi + 2\,h_{1,0}\,\log\varphi + h_{2,0},\\
\varpi_{1,1} &= h_{0,0}\,\log\varphi\,\log\nu + h_{0,1}\,\log\varphi + h_{1,0}\,\log\nu + h_{1,1},\\
\varpi_{0,2} &= h_{0,0}\,\log^2\nu + 2\,h_{0,1}\,\log\nu + h_{0,2},\\
\varpi_{3,0} &= h_{0,0}\,\log^3\varphi + 3\,h_{1,0}\,\log^2\varphi + 3\,h_{2,0}\,\log\varphi + h_{3,0},\\
\varpi_{2,1} &= \big(h_{0,0}\,\log^2\varphi + 2\,h_{1,0}\,\log\varphi + h_{2,0}\big)\,\log\nu\\
&\quad + \big(h_{0,1}\,\log^2\varphi + 2\,h_{1,1}\,\log\varphi + h_{2,1}\big)\,,\\
\varpi_{1,2} &= \big(h_{0,0}\,\log^2\nu + 2\,h_{0,1}\,\log\nu + h_{0,2}\big)\,\log\varphi\\
&\quad + \big(h_{1,0}\,\log^2\nu + 2\,h_{1,1}\,\log\nu + h_{1,2}\big)\,,\\
\varpi_{0,3} &= h_{0,0}\,\log^3\nu + 3\,h_{0,1}\,\log^2\nu + 3\,h_{0,2}\,\log\nu + h_{0,3}.
\end{aligned} \tag{7.46}$$

Furthermore, $h_{i,j}$ is a power series in variables φ and ν, the coefficients of which is determined by the recursion relations given by the Picard-Fuchs equations 7.38. The formal Taylor series expansion of ϖ is given by

$$\begin{aligned}\varpi =& \sum_{i,j} \frac{1}{i!\,j!}\,\epsilon^i\,\delta^j\,\varpi_{i,j} \\ =&\varpi_{0,0} + \epsilon\,\varpi_{1,0} + \delta\,\varpi_{0,1} + \frac{1}{2}\,\epsilon^2\,\varpi_{2,0} + \epsilon\,\delta\,\varpi_{1,1} + \frac{1}{2}\,\epsilon^2\,\delta\,(\varpi_{2,1} + \varpi_{3,0}),\end{aligned} \tag{7.47}$$

where we have used the equation $\epsilon^3 = 3\,\epsilon^2\,\delta$ to combine the two terms together. The coefficients in this expansion, i.e. $\varpi_{0,0}$, $\varpi_{1,0}$, $\varpi_{2,0}$, $\varpi_{3,0} + \varpi_{2,1}$, $\varpi_{0,1}$ and $\varpi_{1,1}$, give us six independent solutions of the Picard-Fuchs equations 7.38. For simplicity let us define

$$h_0 := h_{0,0},\; h_1 := h_{1,0},\; h_2 := h_{2,0},\; h_3 := h_{3,0},\; g_4 := h_{0,1},\; g_5 := h_{1,1}. \tag{7.48}$$

Multiply the six solutions by suitable powers of $2\pi i$, and re-order them into

$$\begin{aligned}\varpi_0 \;=& h_0,\\ \varpi_1 \;=& \frac{1}{2\pi i}\,(\varpi_0 \log\varphi + h_1),\\ \varpi_2 \;=& \frac{1}{(2\pi i)^2}\,(\varpi_0\,\log^2\varphi + 2\,h_1\,\log\varphi + h_2),\\ \varpi_3 \;=& \frac{1}{(2\pi i)^3}\,(\varpi_0\,\log^3\!\varphi + (3\,h_1 + g_4)\log^2\!\varphi + (3\,h_2 + 2\,g_5)\log\varphi + h_3) + \frac{1}{2\pi i}\,\varpi_2\,\log\nu,\\ \varpi_4 \;=& \frac{1}{2\pi i}\,(\varpi_0\,\log\nu + g_4),\\ \varpi_5 \;=& \frac{1}{(2\pi i)^2}\,(\varpi_0\,\log\varphi\,\log\nu + g_4\,\log\varphi + h_1\,\log\nu + g_5)\\ =& \frac{1}{2\pi i}\,\varpi_1\,\log\nu + \frac{1}{(2\pi i)^2}\,(g_4\,\log\phi + g_5)\\ =& \frac{1}{2\pi i}\,\varpi_4\,\log\phi + \frac{1}{(2\pi i)^2}\,(h_1\,\log\nu + g_5).\end{aligned} \tag{7.49}$$

We further impose the following conditions on h_0, h_1, h_2, h_3, g_4 and g_5

$$h_0(0,0) = 1;\quad h_i(0,0) = 0,\; \forall\; i > 0\;;\; g_j\,(0,0) = 0, \forall\; j, \tag{7.50}$$

and then the coefficients of h_i and g_j are uniquely determined by the recursion relations given by Picard-Fuchs equations 7.38, which can be solved order by order. Use Mathematica, we have computed the first ten thousands terms of them.

7.4.2 The monodromy of the canonical periods

The monodromies of the canonical periods $\{\varpi_i\}_{i=0}^5$ in equation 7.49 are induced by the following two operations

$$\{\log\varphi \to \log\varphi + 2\pi i,\ \log\nu \to \log\nu\} \text{ and } \{\log\varphi \to \log\varphi,\ \log\nu \to \log\nu + 2\pi i\}. \quad (7.51)$$

The canonical period vector φ is defined by

$$\varpi = (\varpi_0, \varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)^t, \quad (7.52)$$

and its monodromy matrix around $\varphi = 0$ (resp. $\nu = 0$) will be denoted by $T_{\text{Can},\varphi}$ (resp. $T_{\text{Can},\nu}$)

$$\begin{aligned} &\varpi^t \to \varpi^t\, T_{\text{Can},\varphi} \text{ when } \log\varphi \to \log\varphi + 2\pi i,\ \ \log\nu \to \log\nu, \\ &\varpi^t \to \varpi^t\, T_{\text{Can},\nu} \text{ when } \log\varphi \to \log\varphi,\ \ \log\nu \to \log\nu + 2\pi i. \end{aligned} \quad (7.53)$$

From the form of the canonical periods in equation 7.49, the matrices $T_{\text{Can},\varphi}$ and $T_{\text{Can},\nu}$ can be easily computed

$$T_{\text{Can},\varphi} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}, \quad T_{\text{Can},\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7.54)$$

Let the matrices R_φ and R_ν be defined by

$$R_\varphi := T_{\text{Can},\varphi} - \text{Id},\ \ R_\nu := T_{\text{Can},\nu} - \text{Id}, \quad (7.55)$$

which satisfy the following relations

$$R_\varphi\, R_\nu = R_\nu\, R_\varphi\ ,\quad R_\varphi^3 = 3\, R_\varphi^2\, S_\nu \neq 0\ ,\quad R_\varphi^4 = R_\nu^2 = R_\varphi^3\, R_\nu = 0. \quad (7.56)$$

Therefore $T_{\text{Can},\varphi}$ and $T_{\text{Can},\nu}$ satisfy the conditions in the definition of the large complex structure limit (Definition 4.4.2), hence the point $\varphi = \nu = 0$ is found to be a large complex structure limit of $X_{\mathfrak{G}}$.

7.5 The mirror map and the mirror symmetry

In this section, we will construct the mirror map of $X_{\mathfrak{G}}$ and study its mirror symmetry. Since the Hodge numbers of $X_{\mathfrak{G}}$ are $h^{11} = h^{12} = 2$, from Poincaré duality, there exists an integral symplectic basis of $H_3(X_{\mathfrak{G}}, \mathbb{Z})$

$$A_0^s,\ A_1^s,\ A_2^s,\ B_0^s,\ B_1^s,\ B_2^s, \quad (7.57)$$

which satisfy the following intersection pairings

$$A^s_a \cdot A^s_b = 0, \quad B^s_a \cdot B^s_b = 0, \quad A^s_a \cdot B^s_b = \delta_{ab}. \tag{7.58}$$

The integral periods of $X_{\mathfrak{G}}$ are defined by

$$z_a(\varphi,\nu) = \int_{A^s_a(\varphi,\nu)} \Omega(\varphi,\nu)\ , \quad \mathcal{G}_b(\varphi,\nu) = \int_{B^s_b(\varphi,\nu)} \Omega(\varphi,\nu), \tag{7.59}$$

and the integral period vector Π is defined by

$$\Pi = (\mathcal{G}_0,\, \mathcal{G}_1,\, \mathcal{G}_2,\, z_0,\, z_1,\, z_2), \tag{7.60}$$

From Section 4.6, there exists a prepotential $\mathcal{G}$ such that

$$\mathcal{G}_a = \frac{\partial \mathcal{G}}{\partial z^a},\ a = 0, 1, 2. \tag{7.61}$$

From the form of canonical periods in 7.49, we have the following assumption.

Assumption: *there exists an integral symplectic basis of $H_3(X_{\mathfrak{G}}, \mathbb{Z})$ such that*

$$z_0 = \lambda\, \varpi_0,\ z_1 = \lambda\, \varpi_1,\ z_2 = \lambda\, \varpi_4;\ \lambda \in \mathbb{C}^*, \tag{7.62}$$

Now we define t'_1 and t'_2 by

$$t'_1 := z_1/z_0 = \varpi_1/\varpi_0, \quad t'_2 := z_2/z_0 = \varpi_4/\varpi_0. \tag{7.63}$$

Under the operation $\{\log\varphi \to \log\varphi + 2\pi i,\, \log\nu \to \log\nu\}$, they transform in the way

$$t'_1 \to t'_1 + 1, \quad t'_2 \to t'_2, \tag{7.64}$$

while under the operation $\{\log\varphi \to \log\varphi,\, \log\nu \to \log\nu + 2\pi i\}$, they transform in the way

$$t'_1 \to t'_1, \quad t'_2 \to t'_2 + 1. \tag{7.65}$$

Let Π_{A} be the normalised integral period vector

$$\Pi_{\mathrm{A}} := (\mathcal{G}_0/z_0, \mathcal{G}_1/z_0, \mathcal{G}_3/z_0, 1, t'_1, t'_2)^t. \tag{7.66}$$

The i-th monodromy matrix of Π_{A} is denoted by $T_{C,i}$

$$(\Pi_{\mathrm{A}})_a = \sum_b (T_{C,i})_{ba}\, (\Pi_{\mathrm{A}})_b,\ i = 1, 2, \tag{7.67}$$

where $T_{C,i}$ is an integral symplectic matrix. Comparing the intersection products 6.1 with the relations of monodromy matrices in equation 7.56, the mirror map is constructed by the following identifications

$$t'_1 \equiv t_1,\ t'_2 \equiv t_2, \tag{7.68}$$

and from now on, t_i will also mean t'_i. From the intersection products 6.1, we deduce that the numbers Y_{ijk} that appear in the prepotential $\mathcal{F}$ of $\mathscr{M}_{K,\Sigma}(X_{\mathfrak{G}})$ are given by

$$Y_{111} = 18\,,\quad Y_{112} = 6\,,\quad Y_{122} = Y_{222} = 0, \tag{7.69}$$

so $\mathcal{F}$ is of the form

$$\begin{aligned}\mathcal{F} =& -3\,t_1^3 - 3\,t_1^2\,t_2 - \frac{1}{2}\,Y_{011}\,t_1^2 - Y_{012}\,t_1\,t_2 - \frac{1}{2}\,Y_{022}\,t_2^2\\ &- \frac{1}{2}\,Y_{001}\,t_1 - \frac{1}{2}\,Y_{002}\,t_2 - \frac{1}{6}\,Y_{000} + \mathcal{F}^{\mathrm{np}}.\end{aligned} \tag{7.70}$$

The mirror period vector Π on the complexified Kähler moduli space $\mathscr{M}_{K,\Sigma}(X_{\mathfrak{G}})$ is defined by

$$\Pi = (\mathcal{F}_0,\, \mathcal{F}_1,\, \mathcal{F}_2,\, 1,\, t_1,\, t_2)^t,\ \mathcal{F}_0 = 2\,\mathcal{F} - \sum_{i=1}^{2} t_i\,\frac{\partial \mathcal{F}}{\partial t_i},\ \mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial t^i},\ i = 1, 2 \tag{7.71}$$

From **Mirror Symmetry Conjecture**, the mirror map 7.68 identifies the normalised integral period vector Π_{A} on the complex side with the mirror period vector Π on the Kähler side.

7.5.1 The transformation matrix between Π and ϖ

The non-perturbative part $\mathcal{F}^{\mathrm{np}}$ of the prepotential $\mathcal{F}$ in equation 7.70 has a series expansion in $q_1 := \exp(2\pi i\,t_1)$ and $q_2 := \exp(2\pi i\,t_2)$. The constant term of $\mathcal{F}^{\mathrm{np}}$ is zero, hence in the limit $t_1, t_2 \to i\,\infty$, $\mathcal{F}^{\mathrm{np}}$ goes to zero, and the leading part of the mirror period Π is determined by the perturbative part of $\mathcal{F}$ 7.70. The leading part of the canonical period vector ϖ under the limit $t'_1, t'_2 \to i\,\infty$ (or equivalently $\varphi \to 0$ and $\nu \to 0$) is given by

$$\varpi^t \sim (1,\, t'_1,\, t'^2_1,\, t'^3_1 + t'^2_1\,t'_2,\, t'_2,\, t'_1\,t'_2). \tag{7.72}$$

Since both Π and ϖ form bases of the solution space of Picard-Fuchs equations of $X_{\mathfrak{G}}$, there exists a matrix S in $\mathrm{GL}(6,\mathbb{C})$ such that

$$\Pi = \lambda\,S\,\varpi. \tag{7.73}$$

Since the mirror map 7.68 identifies Π_{A} with Π in mirror symmetry, so we have

$$\Pi \equiv \Pi_A = \frac{1}{\varpi_0} S \, \varpi. \tag{7.74}$$

From the leading part of Π and ϖ near the large complex structure limit, S is found to be

$$S = \begin{pmatrix} -\frac{1}{3}Y_{0,0,0} & -\frac{1}{2}Y_{0,0,1} & 0 & 3 & -\frac{1}{2}Y_{0,0,2} & 0 \\ -\frac{1}{2}Y_{0,0,1} & -Y_{0,1,1} & -9 & 0 & -Y_{0,1,2} & -6 \\ -\frac{1}{2}Y_{0,0,2} & -Y_{0,1,2} & -3 & 0 & -Y_{0,2,2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7.75}$$

7.5.2 The monodromy of the integral period vector Π

The monodromies of the mirror period Π are induced by the following two operations

$$t_1 \to t_1 + 1, \;\; t_2 \to t_2 \text{ and } t_1 \to t_1, \;\; t_2 \to t_2 + 1. \tag{7.76}$$

Under the operation $t_1 \to t_1 + 1$, $t_2 \to t_2$, the mirror period Π transforms in the way

$$\Pi_a \to \sum_b (T_{K,1})_{ba} \, \Pi_b, \tag{7.77}$$

where the matrix $T_{K,1}$ is found to be

$$T_{K,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 3 - Y_{0,0,1} & -Y_{0,1,1} - 9 & -Y_{0,1,2} - 3 & 1 & 1 & 0 \\ 9 - Y_{0,1,1} & -18 & -6 & 0 & 1 & 0 \\ 3 - Y_{0,1,2} & -6 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.78}$$

Similarly, under the operation $t_1 \to t_1$, $t_2 \to t_2 + 1$, Π transforms in the way

$$\Pi_a \to \sum_b (T_{K,2})_{ba} \, \Pi_b, \tag{7.79}$$

where the matrix $T_{K,2}$ is found to be

$$T_{K,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -Y_{0,0,2} & -Y_{0,1,2} & -Y_{0,2,2} & 1 & 0 & 1 \\ -Y_{0,1,2} & -6 & 0 & 0 & 1 & 0 \\ -Y_{0,2,2} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.80}$$

Mirror Symmetry Conjecture implies that $T_{K,i}$ is equal to $T_{C,i}$, therefore $T_{K,i}$ is an integral symplectic matrix, which shows the following property.

Corollary 7.5.1. *The numbers Y_{011}, Y_{012}, Y_{022}, Y_{001} and Y_{002} are all integers.*

7.6 The Yukawa couplings and Gromov-Witten invariants

In this section, we will compute the Yukawa couplings of $X_{\mathfrak{G}}$ and their instanton expansion. From Section 4.6, the Yukawa coupling tensor of $X_{\mathfrak{G}}$ has components $\mathcal{Y}_{abc}$ where the indices a, b and c run over φ and ν 4.66. The normalised Yukawa coupling $\mathcal{Y}^A_{abc}$ on the complex side is given by

$$\mathcal{Y}^A_{abc} = -\varpi^T \, S^T \, \Sigma \, S \, \partial_{abc} \, \varpi / \varpi_0^2. \tag{7.81}$$

The matrix $S^T \, \Sigma \, S$ is found to be

$$S^T \, \Sigma \, S = \begin{pmatrix} 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 6 \\ 0 & -9 & 0 & 0 & -3 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.82}$$

which has no dependence on the coefficients Y_{0ij}, Y_{00i} and Y_{000} that appear in $\mathcal{F}$ 7.70, and this corresponds to the fact that Yukawa couplings are the third derivatives of the prepotential $\mathcal{F}$. The coordinates t_1 and t_2 are given by

$$\begin{aligned} t_1 &= \frac{\varpi_1}{\varpi_0} = \frac{1}{2\pi i} \, \log \varphi + \frac{h_1}{\varpi_0} = \frac{1}{2\pi i} \, \log \varphi + O(\varphi, \, \nu), \\ t_2 &= \frac{\varpi_4}{\varpi_0} = \frac{1}{2\pi i} \, \log \nu + \frac{g_4}{\varpi_0} = \frac{1}{2\pi i} \, \log \nu + O(\varphi, \, \nu). \end{aligned} \tag{7.83}$$

The Yukawa coupling of $X_{\mathfrak{G}}$ is a symmetric tensor, and its normalisation is given by

$$\mathcal{Y}^A = \mathcal{Y}^A_{\varphi\varphi\varphi} \, d\varphi^3 \, + 3 \, \mathcal{Y}^A_{\varphi\varphi\nu} \, d\varphi^2 \, d\nu \, + 3 \, \mathcal{Y}^A_{\varphi\nu\nu} \, d\varphi \, d\nu^2 + \mathcal{Y}^A_{\nu\nu\nu} \, d\nu^3. \tag{7.84}$$

With respect to the natural coordinate $(t_1, \, t_2)$, the component $\mathcal{Y}^A_{ijk}$ is given by

$$\mathcal{Y}^A_{ijk} = \sum_{a,b,c} \mathcal{Y}^A_{abc} \, \frac{d\varphi^a}{dt_i} \, \frac{d\varphi^b}{dt_j} \, \frac{d\varphi^c}{dt_k}. \tag{7.85}$$

In order to compute the instanton expansion of $\mathcal{Y}^A_{ijk}$, we need to invert the coordinates transformation between $(t_1, \, t_2)$ and $(\varphi, \, \nu)$. Since q_1 and q_2 are defined by

$$\begin{aligned} q_1 &= \exp\big(2\pi i \, t_1\big) = \varphi \, \exp\big(2 \, \pi \, i \, \frac{h_1}{\varpi_0}\big), \\ q_2 &= \exp\big(2\pi i \, t_2\big) = \nu \, \exp\big(2 \, \pi \, i \, \frac{g_4}{\varpi_0}\big), \end{aligned} \tag{7.86}$$

and from the power series expansion of h_1 and g_4, the inverse of this coordinate transformation is determined recursively. So the coordinates $(\varphi,\, \nu)$ admit power series expansions in $(q_1,\, q_2)$, the first several terms of which are given by

$$\begin{aligned} \varphi &= q_1 - q_1^2 - 5\, q_1^3 + q_1^3\, (7\, q_1 - 12\, q_2) + 3\, q_1^3\, (10\, q_1^2 + q_2^2) + \cdots , \\ \nu &= q_2 - 18\, q_1^2\, q_2 - 8\, q_1^2\, q_2\, (\, q_1 - 3\, q_2) + 3\, q_1^2\, q_2\, (63\, q_1^2 + 8\, q_2\, q_1 - 2\, q_2^2) + \cdots . \end{aligned} \tag{7.87}$$

The normalised Yukawa coupling $\mathcal{Y}^A_{ijk}$ has a power series expansion in $(q_1,\, q_2)$ which is of the form

$$\begin{pmatrix} \mathcal{Y}_{111} \\ \mathcal{Y}_{112} \\ \mathcal{Y}_{122} \\ \mathcal{Y}_{222} \end{pmatrix} = \begin{pmatrix} 18 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \sum_{j,k} \begin{pmatrix} j^3 \\ j^2 k \\ j k^2 \\ k^3 \end{pmatrix} \frac{n_{j,k}\, q_1^j\, q_2^k}{1 - q_1^j\, q_2^k}. \tag{7.88}$$

The first several instanton numbers n_{jk} of $X_{\mathfrak{S}}$ are listed in Table 7.2.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	3	0	0	0	0	0	0	0	0
$j=1$	0	6	0	0	0	0	0	0	0	0
$j=2$	0	21	-6	0	0	0	0	0	0	0
$j=3$	0	48	-24	6	0	0	0	0	0	0
$j=4$	0	117	-72	81	-12	0	0	0	0	0
$j=5$	0	240	-72	540	-216	30	0	0	0	0
$j=6$	0	525	336	3060	-1968	660	-84	0	0	0
$j=7$	0	1008	2976	15120	-12120	7560	-2160	252	0	0
$j=8$	0	1998	13896	71145	-56112	62070	-30024	7587	-816	0
$j=9$	0	3696	52272	314100	-178992	410634	-293184	121740	-27888	2790

Table 7.2: Instanton numbers

It is very difficult to prove that $H_2(X_{\mathfrak{S}}, \mathbb{Z})$ is torsion free, so in this thesis we will assume this fact. Then by mirror symmetry conjecture, $n_{j,k}$ is the number of generically injective J-holomorphic curves (modulo equivalence), $\mathbb{P}^1 \to X_{\mathfrak{S}}$, whose image falls into the homology class $j\, \Gamma_\epsilon + k\, \Gamma_\delta$ of $H_2(X_{\mathfrak{S}}, \mathbb{Z})$. From Section 6.5, the three exceptional curves of $X_{\mathfrak{S}}$ are homologous to each other, and Γ_δ is one of the exceptional curves. Then it is very interesting to notice that $n_{0,1}$ is 3, which corresponds to the three exceptional curves of $X_{\mathfrak{S}}$. Therefore the computations in this section provides evidence to the assumption that $X_{\mathfrak{S}}$ is self-mirror.

Remark 7.6.1. *The assumption that $H_2(X_{\mathfrak{S}}, \mathbb{Z})$ is torsion free is not needed when we compute the limit MHS on $H^3(X_{\mathfrak{S}}, \mathbb{Q})$ at the large complex structure limit in Chapter 13.*

7.7 The connections between $X_{\mathfrak{G}}$ and $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$

In [15], the mirror symmetry of the mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$ has been studied. In this section, we will discuss the interesting connections between the mirror symmetry of $X_{\mathfrak{G}}$ and that of the one parameter mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$. From Section 5.5, the mirror threefold $Y_{\mathfrak{G}}^{\vee}$ is a projective small resolution of the subfamily of $\mathcal{Y}_{\mathfrak{G}}^{\mathrm{ss}}$ 5.41 over the discriminant locus $\mathbb{P}^1_{\mathbb{C}}$ defined by $\gamma_2 = \gamma_3 = \gamma_4 = 0$. In affine coordinate, this discriminant locus $\mathbb{P}^1$ is also defined by $\rho = 0$ or $\nu = 1$. The restrictions $\varpi_0|_{\nu=1}$, $\varpi_1|_{\nu=1}$, $\varpi_2|_{\nu=1}$ and $\varpi_3|_{\nu=1}$ are in fact the four canonical periods of $Y_{\mathfrak{G}}^{\vee}$ which have been computed in [15]. Furthermore, we have

$$\varpi_4|_{\nu=1} = 0, \tag{7.89}$$

which shows that over the locus $\nu = 1$

$$t_2|_{\nu=1} = \varpi_4/\varpi_0|_{\nu=1} = 0,\ q_2|_{\nu=1} = 1. \tag{7.90}$$

Remark 7.7.1. *The restriction of ϖ_5 to $\nu = 1$ is not well-defined.*

In [15], the Yukawa coupling of the mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$ and its instanton expansion have been computed. The restriction of the Yukawa coupling tensor $\mathcal{Y}$ of $X_{\mathfrak{G}}$ to $\nu = 1$ is given by

$$\mathcal{Y}|_{\nu=1} = \mathcal{Y}_{\varphi\varphi\varphi}|_{\nu=1}\, d\varphi^3, \tag{7.91}$$

and it is very interesting that $\mathcal{Y}|_{\nu=1}$ is equal to the Yukawa coupling of the mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$. Together with equation 7.90, this shows that the Gromov-Witten invariants of the mirror pair $(Y_{\mathfrak{G}}, Y_{\mathfrak{G}}^{\vee})$, denoted by $n_j(Y_{\mathfrak{G}})$, satisfies

$$n_j(Y_{\mathfrak{G}}) = \sum_k n_{jk}(X_{\mathfrak{G}}). \tag{7.92}$$

See Section 2.4 of [15] for more detail. Therefore, the computations of canonical periods and Yukawa couplings of $X_{\mathfrak{G}}$ in this chapter also supports that $X_{\mathfrak{G}}$ is self-mirror.

Summary: in this chapter, we first compute the series expansion of the fundamental period of $X_{\mathfrak{G}}$, from which we can compute PDE operators which are expected to be the Picard-Fuchs operators of $X_{\mathfrak{G}}$. Using Frobenius method, we have solved the Picard-Fuchs equations and found the six canonical periods of $X_{\mathfrak{G}}$, from which we identify the large complex structure limit of the mirror family of $X_{\mathfrak{G}}$. Then we construct the mirror map, from which we find the instanton expansions of the Yukawa couplings of $X_{\mathfrak{G}}$. This chapter is very crucial to the computation of the limit MHS at the large complex structure limit of the mirror family of $X_{\mathfrak{G}}$ in Chapter 8.

Chapter 8

The limit MHS at the large complex structure limit of $X_{\mathfrak{G}}$

In this chapter we will compute the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit of the mirror family of $X_{\mathfrak{G}}$. The structure of this chapter is as follows:

- Section 8.1 discusses Deligne's Canonical extension.
- Section 8.2 computes the weight filtration of the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit of $X_{\mathfrak{G}}$.
- Section 8.3 computes the limit Hodge filtration of the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit of $X_{\mathfrak{G}}$.
- Section 8.4 shows that the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit splits into
  $$\mathbf{M} \oplus \mathbb{Q}(-1)^2 \oplus \mathbb{Q}(-2)^2,$$
  where $\mathbf{M}$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$.
- Finally, Section 8.5 introduces a conjecture which states that for a general mirror pair (M, W), the limit MHS on $H^3(W, \mathbb{Q})$ at the large complex structure limit splits into
  $$\mathbf{M} \oplus \mathbb{Q}(-1)^{h^{12}(W)} \oplus \mathbb{Q}(-2)^{h^{12}(W)},$$
  where $\mathbf{M}$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$.

8.1 Deligne's canonical extension

It is important to notice that the deformation of $X_{\mathfrak{G}}$, i.e. $\pi_{X_{\mathfrak{G}}}: \mathcal{X}_{\mathfrak{G}} \to \mathbb{P}^2$ in equation 6.18, is defined over $\mathbb{Q}$, moreover, all the irreducible components of the discriminant locus listed in Table 7.1 are also defined over $\mathbb{Q}$. There exists a neighbourhood of the large complex structure limit which is isomorphic to an open affine subvariety of $\operatorname{Spec}\mathbb{Q}[\varphi,\nu]$ such that its discriminant locus are given by $\varphi=0$ and $\nu=0$. After analytification, suppose Δ^2 is a local (analytic) neighbourhood of the large complex structure limit $\varphi=\nu=0$, then we get a smooth fibration between smooth manifolds

$$\pi_{(\Delta^*)^2}: \pi^{-1}_{X_{\mathfrak{G}}}((\Delta^*)^2) \to (\Delta^*)^2. \tag{8.1}$$

There are local systems defined over $(\Delta^*)^2$ which are given by

$$V_{\mathbb{Z}} := \mathrm{R}^3\,\pi_{(\Delta^*)^2,*}\mathbb{Z},\ V_{\mathbb{Q}} := \mathrm{R}^3\,\pi_{(\Delta^*)^2,*}\mathbb{Q},\ V_{\mathbb{C}} := \mathrm{R}^3\,\pi_{(\Delta^*)^2,*}\mathbb{C}, \tag{8.2}$$

which defines a bundle $\mathcal{V}$ over $(\Delta^*)^2$ by

$$\mathcal{V} := V_{\mathbb{Z}} \otimes \mathcal{O}_{(\Delta^*)^2}. \tag{8.3}$$

Choose a point $(\varphi_0,\nu_0) \in (\Delta^*)^2$, and from Section 7.5 an integral symplectic basis of $H_3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Z})$ has been chosen, which is denoted by

$$A^s_0,\ A^s_1,\ A^s_2,\ B^s_0,\ B^s_1,\ B^s_2. \tag{8.4}$$

Let $\{\alpha^a_s, \beta^b_s\}$ be its dual, which form a basis of $H^3(X_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Z})$, and their pairings are given by

$$\alpha^a_s(A^s_b)=\delta_{ab},\ \beta^a_s(B^s_b)=\delta_{ab},\ \alpha^a_s(B^s_b)=\beta^a_s(A^s_b)=0. \tag{8.5}$$

For simplicity, let us also order the two bases in the following way

$$\begin{aligned}(A_0,\cdots,A_5) &:= (B^s_0,B^s_1,B^s_2,A^s_0,A^s_1,A^s_2),\\ (\alpha^0,\cdots,\alpha^5) &:= (\beta^0_s,\beta^1_s,\beta^2_s,\alpha^0_s,\alpha^1_s,\alpha^2_s).\end{aligned} \tag{8.6}$$

The monodromy of the local system $V^\vee_{\mathbb{Z}}$ (the dual of $V_{\mathbb{Z}}$) induces a representation

$$\Phi: \pi_1((\Delta^*)^2,(\varphi_0,\nu_0)) \to \operatorname{Aut}(H_3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Z})). \tag{8.7}$$

The fundamental group $\pi_1((\Delta^*)^2,(\varphi_0,\nu_0))$ is isomorphic to $\mathbb{Z}^2$ which has two generators T_1 and T_2, then the representation Φ is determined by the images of T_1 and T_2. For simplicity, let us denote the matrix of $\Phi(T_i)$ with respect to the basis $(A_a)^5_{a=0}$ by $T_{C,i}$. From equation 4.40, the i-th monodromy of the integral period vector Π is given by $T_{C,i}$, see equation 7.67 for more detail. From mirror symmetry conjecture in Section 4.6, we have

$$T_{C,i} = T_{K,i}, \tag{8.8}$$

where the matrix $T_{K,i}$ has been computed in Section 7.5.2. The monodromy of the local system $V_{\mathbb{Z}}$ induces a representation Ψ

$$\Psi : \pi_1((\Delta^*)^2, (\varphi_0, \nu_0)) \to \mathrm{Aut}(H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Z})), \tag{8.9}$$

which is the dual representation of Φ. For simplicity, let $T_{\alpha,i}$ be the matrix of $\Psi(T_i)$ with respect to the basis $(\alpha^a)_{a=0}^5$, i.e.

$$\Psi(T_i) : \alpha_a \to \sum_b \alpha_b \, (T_{\alpha,i})_{ba}, \ i = 1, 2. \tag{8.10}$$

Since Ψ is the dual representation of Φ and $(\alpha^a)_{a=0}^5$ is the dual basis of $(A_a)_{a=0}^5$, then the matrix $T_{\alpha,i}$ is given by

$$T_{\alpha,i} = (T_{C,i}^t)^{-1} = (T_{K,i}^t)^{-1}, \ i = 1, 2. \tag{8.11}$$

The monodromy operators $N_{\alpha,i}, i = 1, 2$ are defined by

$$N_{\alpha,i} := \log T_{\alpha,i}, \ i = 1, 2, \tag{8.12}$$

and from equation 7.56, they satisfies

$$N_{\alpha,1}^3 \neq 0, \ N_{\alpha,1}^4 = 0, \ N_{\alpha,2}^2 = 0. \tag{8.13}$$

Given an element ξ of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{C})$, its regularisation is given by

$$\widehat{\xi}(\varphi, \nu) := \exp(-\frac{\log \varphi \, N_{\alpha,1} + \log \nu \, N_{\alpha,2}}{2\pi i})\xi, \tag{8.14}$$

which a single-valued section of $\mathcal{V}$ over $(\Delta^*)^2$. Then the regularisations $\{\widehat{\alpha}^a\}_{a=0}^5$ form a frame of $\mathcal{V}$, which define a trivialisation of $\mathcal{V}$ over $(\Delta^*)^2$. This trivialisation induces an extension of $\mathcal{V}$ to a bundle $\widetilde{\mathcal{V}}$ over Δ^2, which is the Deligne's canonical extension of $\mathcal{V}$, see Section 3.3 for more detail. Follow the conventions in Section 3.3, the section $\widehat{\xi}$ of $\mathcal{V}$ extends to a section $\widetilde{\xi}$ of $\widetilde{\mathcal{V}}$ over Δ^2 under Deligne's canonical extension, and its value at origin will be denoted by $\widetilde{\xi}(0)$. In this way, we have got an isomorphism

$$\rho_{(\varphi_0,\nu_0)} : H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{C}) \to \widetilde{\mathcal{V}}|_0, \tag{8.15}$$

and see Section 3.3 for more detail. Under this isomorphism, the rational vector space $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q})$ defines a rational structure on $\widetilde{\mathcal{V}}|_0$ that will be denoted by $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$

$$\widetilde{\mathcal{V}}|_{0,\mathbb{Q}} := \oplus_{a=0}^5 \mathbb{Q} \, \widetilde{\alpha}^a(0). \tag{8.16}$$

8.2 Weight Filtration

In this section, we will compute the weight filtration on $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$, which is given by the weight filtration on $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$ induced by the monodromy operator $N_{\alpha,1}$ [51] (since $N_{\alpha,1}$ satisfies the conditions in equation 8.13). To compute the weight filtration on $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$ induced by $N_{\alpha,1}$, it is more convenient to choose a new basis $\{\beta^a\}_{a=0}^5$ of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$ given by

$$\beta^a = \sum_{b=0}^{5} (S_1)_{b\,a}\, \alpha^b, \tag{8.17}$$

where the transformation matrix S_1 is given by

$$S_1 = \begin{pmatrix} 0 & \frac{1}{2}Y_{0,0,1} & 0 & -18 & 0 & \frac{1}{6}Y_{0,0,1} - \frac{1}{2}Y_{0,0,2} \\ -\frac{1}{2}Y_{0,0,1} & Y_{0,1,1} & -18 & 0 & 0 & \frac{1}{3}Y_{0,1,1} - Y_{0,1,2} \\ -\frac{1}{2}Y_{0,0,2} & Y_{0,1,2} & -6 & 0 & -2 & \frac{1}{3}Y_{0,1,2} - Y_{0,2,2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.18}$$

From Corollary 7.5.1, Y_{0ij} and Y_{00i} are all rational numbers, while the determinant of S_1 is -648, hence the matrix S_1 lies in $\mathrm{GL}(6,\mathbb{Q})$ and $\{\beta^a\}_{a=0}^5$ is indeed a basis of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$. Let us now define the canonical basis $\{\gamma^a\}_{a=0}^5$ of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{C})$ by

$$\gamma^a = \sum_{b=0}^{5} \alpha^b\, S_{b\,a}, \tag{8.19}$$

where the matrix S is given in equation 7.75. Then the expansion of $\Omega_{X_{\mathfrak{G}}}$ becomes

$$\Omega_{X_{\mathfrak{G}}}(\varphi,\nu) = \sum_{a=0}^{5} \gamma^a(\varphi,\nu)\, \varpi_a(\varphi,\nu), \tag{8.20}$$

where $\gamma^a(\varphi,\nu)$ is the extension of γ^a to a section of the local system $V_{\mathbb{C}}$. Let $\{C_a\}_{a=0}^5$ be the dual of $\{\gamma^a\}_{a=0}^5$, and it forms a basis of $H_3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{C})$. The canonical period ϖ_a is also given by

$$\varpi_a(\varphi,\nu) = \int_{C_a(\varphi,\nu)} \Omega_{X_{\mathfrak{G}}}(\varphi,\nu), \tag{8.21}$$

where $C_a(\varphi,\nu)$ is the extension of C_a to a section of $V_{\mathbb{C}}^{\vee}$. With respect to the new basis $\{\beta^a\}_{a=0}^5$, the basis $\{\gamma^a\}_{a=0}^5$ of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{C})$ is expressed as

$$\gamma^a = \sum_{b=0}^{5} (S_2)_{b\,a}\, \beta^b, \tag{8.22}$$

where the transformation matrix S_2 is given by

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{54} Y_{0,0,0} & 0 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{8.23}$$

The action of the operator $N_{\alpha,1}$ on the basis $\{\beta^a\}_{a=0}^5$ is very simple

$$N_{\alpha,1}\beta^0 = \beta^1, N_{\alpha,1}\beta^1 = \beta^2, N_{\alpha,1}\beta^2 = \beta^3, N_{\alpha,1}\beta^3 = 0, N_{\alpha,1}\beta^4 = 0, N_{\alpha,1}\beta^5 = \beta^4. \tag{8.24}$$

The weight filtration on $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q})$ induced by $N_{\alpha,1}$ can be computed inductively [51, 56, 88]. First, W_{-1} and W_6 are given by

$$W_{-1}H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) = 0, \quad W_6 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) = H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}), \tag{8.25}$$

then W_0 and W_5 are given by

$$\begin{aligned} W_0 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) &= \operatorname{im} N_{\alpha,1}^3 = \mathbb{Q}\,\beta^3, \\ W_5 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) &= \ker N_{\alpha,1}^3 = \oplus_{i=1}^5 \mathbb{Q}\,\beta^i. \end{aligned} \tag{8.26}$$

Now form the quotient space W_5/W_0, which is isomorphic to

$$W_5 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q})/W_0 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) \simeq \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2 + \mathbb{Q}\,\beta^4 + \mathbb{Q}\,\beta^4. \tag{8.27}$$

As the operator $N_{\alpha,1}^2$ induces a zero map on it, so we have

$$\begin{aligned} W_1 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) &= W_0 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}), \\ W_4 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) &= W_5 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}). \end{aligned} \tag{8.28}$$

Then form the quotient space W_4/W_1, which is isomorphic to

$$W_4 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q})/W_1 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) \simeq \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2, \tag{8.29}$$

and the operator $N_{\alpha,1}$ induces a map on it given by

$$N_{\alpha,1}\beta^1 = \beta^2, N_{\alpha,1}\beta^2 = 0, N_{\alpha,1}\beta^4 = 0, N_{\alpha,1}\beta^5 = \beta^4, \tag{8.30}$$

therefore we have

$$W_2 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) = W_3 H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)}, \mathbb{Q}) = \mathbb{Q}\,\beta^2 + \mathbb{Q}\,\beta^3 + \mathbb{Q}\,\beta^4. \tag{8.31}$$

The isomorphism $\rho_{(\varphi_0,\nu_0)}$ 8.15 sends the basis $\{\beta^a\}_{a=0}^5$ of $H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$ to the basis $\{\widetilde{\beta}^a(0)\}_{a=0}^5$ of $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$, hence we have

$$\widetilde{\beta}^a(0)=\sum_{b=0}^5 (S_1)_{b\,a}\,\widetilde{\alpha}^b(0),\quad \widetilde{\gamma}^a(0)=\sum_{b=0}^5 (S_2)_{b\,a}\,\widetilde{\beta}^b(0). \tag{8.32}$$

The isomorphism $\rho_{(\varphi_0,\nu_0)}$ 8.15 sends the weight filtration $W_*H^3(\mathcal{X}_{\mathfrak{G},(\varphi_0,\nu_0)},\mathbb{Q})$ to a weight filtration on $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$

$$\begin{aligned}
&W_0(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=W_1(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=\mathbb{Q}\,\widetilde{\beta}^3(0),\\
&W_2(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=W_3(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=\mathbb{Q}\,\widetilde{\beta}^2(0)+\mathbb{Q}\,\widetilde{\beta}^3(0)+\mathbb{Q}\,\widetilde{\beta}^4(0),\\
&W_4(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=W_5(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=\oplus_{i=1}^5\mathbb{Q}\,\widetilde{\beta}^i(0),\\
&W_6(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})=\widetilde{\mathcal{V}}|_{0,\mathbb{Q}},
\end{aligned} \tag{8.33}$$

which is the weight filtration on $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$ in the limit MHS.

8.3 Limit Hodge Filtration

In this section, we will compute the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$. From Section 3.4, the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$ is actually **rationally** defined. There are plenty of smooth lines that contains the origin which satisfy the conditions in Section 3.4, e.g. we can choose a general line given by

$$C:k_1\,\varphi+k_2\,\nu=0,\ k_i\in\mathbb{Q}, \tag{8.34}$$

which is defined over $\mathbb{Q}$. By abuse of notations, let us denote the pullback of the family $\mathcal{X}_{\mathfrak{G}}$ to the line C by

$$\pi_C:\mathscr{X}\to C. \tag{8.35}$$

See Section 3.4 for the definition of $\mathscr{V}_C$, $\widetilde{\mathscr{V}}_C$, $\mathscr{F}^p_C$, $\widetilde{\mathscr{F}}^p_C$ and their analytifications. The fiber of $\widetilde{\mathscr{F}}^p_C$ at 0, denoted by $\widetilde{\mathscr{F}}^p_C|_0$, defines a decreasing filtration on $\widetilde{\mathscr{V}}_C|_0$, whose complexification is the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$ [56, 85, 88, 95, 96]

$$F^p(\widetilde{\mathcal{V}}|_0)=F^p(\widetilde{\mathscr{V}}^{\mathrm{an}}_C|_0)=F^p(\widetilde{\mathscr{V}}_C|_0)\otimes_{\mathbb{Q}}\mathbb{C}=\widetilde{\mathscr{F}}^p_C|_0\otimes_{\mathbb{Q}}\mathbb{C}. \tag{8.36}$$

Remark 8.3.1. *The limit Hodge filtration $F^p(\widetilde{\mathcal{V}}|_0)$ does not depend on the choice of C. See [28, 29] for more detail.*

See Section 7.1 for the construction of the threeform $\Omega_{X_{\mathfrak{G}}}$ on the mirror family $\pi_{X_{\mathfrak{G}}}:\mathcal{X}_{\mathfrak{G}}\to\mathbb{P}^2$ in equation 6.18, which is defined over $\mathbb{Q}$. The threeform $\Omega_{X_{\mathfrak{G}}}$

has logarithmic poles along the smooth components of the singular fiber of $\mathscr{Z}$ over the large complex structure limit, so it can be extended to a global section of $\widetilde{\mathscr{F}}^3_C$, therefore we have

$$F^3(\widetilde{\mathscr{V}}_C|_0) \supset \mathbb{Q}\,\Omega_{X_{\mathfrak{G}}}|_0. \tag{8.37}$$

The tangent sheaf of C has a section of the form

$$\vartheta = k_1\,\vartheta_1 + k_2\,\vartheta_2, \text{ with } \vartheta_1 := \varphi\, d/d\varphi,\ \vartheta_2 := \nu\, d/d\nu. \tag{8.38}$$

The Gauss-Manin connection of $\mathscr{V}_C$ canonically extends to a connection ∇ of $\widetilde{\mathscr{V}}_C$ which has a logarithmic pole along the large complex structure limit $0 \in C$, see Section 3.4 for more detail. From Griffiths transversality, $\nabla_\vartheta\,\Omega_{X_{\mathfrak{G}}}$ is a section of $\widetilde{\mathscr{F}}^2_C$. From [28, 29], the limit Hodge filtration does not depend on the choice of C. By a change of k_1 and k_2 in the definition of C, we find that [37, 96]

$$F^2(\widetilde{\mathscr{V}}_C|_0) \otimes \mathbb{C} \supset \mathbb{C}\,\Omega_{X_{\mathfrak{G}}}|_0 + \sum_{i=1}^{2} \mathbb{C}\,(\nabla_{\vartheta_i}\,\Omega_{X_{\mathfrak{G}}})|_0. \tag{8.39}$$

Similarly $\nabla^2_\vartheta\,\Omega_{X_{\mathfrak{G}}}$ is a section of $\widetilde{\mathscr{F}}^1_C$ and $\nabla^3_\vartheta\,\Omega_{X_{\mathfrak{G}}}$ is a section of $\widetilde{\mathscr{F}}^0_C$, and again by a change of k_1 and k_2, we find that

$$\begin{aligned} &F^1(\widetilde{\mathscr{V}}_C|_0) \otimes \mathbb{C} \supset \mathbb{C}\,\Omega_{X_{\mathfrak{G}}}|_0 + \sum_{i=1}^{2} \mathbb{C}\,(\nabla_{\vartheta_i}\,\Omega_{X_{\mathfrak{G}}})|_0 + \sum_{i,j=1}^{2} \mathbb{C}\,(\nabla_{\vartheta_i}\nabla_{\vartheta_i}\,\Omega_{X_{\mathfrak{G}}})|_0, \\ &F^0(\widetilde{\mathscr{V}}_C|_0) \otimes \mathbb{C} = \widetilde{\mathscr{V}}_C|_0 \otimes \mathbb{C}. \end{aligned} \tag{8.40}$$

With respect to the regularised frame $\{\widehat{\gamma}^a(\varphi,\nu)\}_{a=0}^5$ of $\mathcal{V}$ over $(\Delta^*)^2$, the threeform $\Omega_{X_{\mathfrak{G}}}$ has an expansion given by

$$\begin{aligned} I_\infty^{-1}(\Omega_{X_{\mathfrak{G}}})|_{(\varphi,\nu)} &= \sum_{a=0}^{5} \gamma^a(\varphi,\nu)\,\varpi_a(\varphi,\nu) \\ &= \sum_{a,b,c} \gamma^a(\varphi,\nu)\,\big(\exp(-\frac{\log\varphi\, N_{\alpha,1} + \log\nu\, N_{\alpha,2}}{2\pi i})\big)_{ab}\cdot \\ &\qquad \big(\exp(\frac{\log\varphi\, N_{\alpha,1} + \log\nu\, N_{\alpha,2}}{2\pi i})\big)_{bc}\,\varpi_c(\varphi,\nu) \\ &= \sum_{a,b} \widehat{\gamma}^a(\varphi,\nu)\big(\exp(\frac{\log\varphi\, N_{\alpha,1} + \log\nu\, N_{\alpha,2}}{2\pi i})\big)_{ab}\,\varpi_b(\varphi,\nu), \end{aligned} \tag{8.41}$$

where $\big(\exp(-\frac{\log\varphi\, N_{\alpha,1}+\log\nu\, N_{\alpha,2}}{2\pi i})\big)_{ab}$ is the matrix with respect to the basis $\{\gamma^a(\varphi,\nu)\}_{a=0}^5$. From this expansion we find that

$$I_\infty^{-1}(\Omega_{X_{\mathfrak{G}}})|_0 = \sum_{a,b} \lim_{\varphi,\nu\to 0} \widehat{\gamma}^a(\varphi,\nu)\big(\exp(\frac{\log\varphi\, N_{\alpha,1} + \log\nu\, N_{\alpha,2}}{2\pi i})\big)_{ab}\,\varpi_b(\varphi,\nu) = \widehat{\gamma}^0(0), \tag{8.42}$$

hence we have

$$F^3(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0). \tag{8.43}$$

To compute $I_\infty^{-1}(\nabla_\vartheta^p\,\Omega_{X_{\mathfrak{G}}})|_0$, we will need the following equation [30]

$$I_\infty^{-1}(\nabla^p_{\vartheta_i}\,\Omega_{X_{\mathfrak{G}}}) = \sum_{a=0}^{5}\gamma^a(\varphi,\nu)\int_{C_a(\varphi)}\nabla^p_{\vartheta_i}\,\Omega_{X_{\mathfrak{G}}} = \sum_{a=0}^{5}\gamma^a(\varphi,\nu)\,\vartheta_i^p\,\varpi_a(\varphi,\nu), \tag{8.44}$$

from which we find that

$$\begin{aligned}
I_\infty^{-1}(\nabla_{\vartheta_1}\,\Omega_{X_{\mathfrak{G}}})|_0 =& \sum_{a,b}\lim_{\varphi,\nu\to 0}\widehat{\gamma}^a\big(\exp(\frac{\log\varphi\,N_{\alpha,1}+\log\nu\,N_{\alpha,2}}{2\pi i})\big)_{ab}\,\vartheta_1\,\varpi_b = \frac{\widetilde{\gamma}^1(0)}{2\pi i},\\
I_\infty^{-1}(\nabla_{\vartheta_2}\,\Omega_{X_{\mathfrak{G}}})|_0 =& \sum_{a,b}\lim_{\varphi,\nu\to 0}\widehat{\gamma}^a\big(\exp(\frac{\log\varphi\,N_{\alpha,1}+\log\nu\,N_{\alpha,2}}{2\pi i})\big)_{ab}\,\vartheta_2\,\varpi_b = \frac{\widetilde{\gamma}^4(0)}{2\pi i},
\end{aligned} \tag{8.45}$$

so we have got

$$F^2(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0) + \mathbb{C}\,\frac{1}{2\pi i}\widetilde{\gamma}^1(0) + \mathbb{C}\,\frac{1}{2\pi i}\widetilde{\gamma}^4(0). \tag{8.46}$$

By similar method we have

$$\begin{aligned}
I_\infty^{-1}(\nabla^2_{\vartheta_1}\,\Omega_{X_{\mathfrak{G}}})|_0 =& \sum_{a,b}\lim_{\varphi,\nu\to 0}\widehat{\gamma}^a\big(\exp(\frac{\log\varphi\,N_{\alpha,1}+\log\nu\,N_{\alpha,2}}{2\pi i})\big)_{ab}\,\vartheta_1^2\,\varpi_b = \frac{2\,\widetilde{\gamma}^2(0)}{(2\pi i)^2},\\
I_\infty^{-1}(\nabla_{\vartheta_1}\nabla_{\vartheta_2}\Omega_{X_{\mathfrak{G}}})|_0 =& \sum_{a,b}\lim_{\varphi,\nu\to 0}\widehat{\gamma}^a\big(\exp(\frac{\log\varphi N_{\alpha,1}+\log\nu N_{\alpha,2}}{2\pi i})\big)_{ab}\vartheta_1\vartheta_2\varpi_b = \frac{\widetilde{\gamma}^5(0)}{(2\pi i)^2},\\
I_\infty^{-1}(\nabla^3_{\vartheta_1}\,\Omega_{X_{\mathfrak{G}}})|_0 =& \sum_{a,b}\lim_{\varphi,\nu\to 0}\widehat{\gamma}^a\big(\exp(\frac{\log\varphi\,N_{\alpha,1}+\log\nu\,N_{\alpha,2}}{2\pi i})\big)_{ab}\,\vartheta_1^3\,\varpi_b = \frac{6\,\widetilde{\gamma}^3(0)}{(2\pi i)^3}.
\end{aligned} \tag{8.47}$$

The limits are linearly independent from each other, therefore we have found the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$, which is given by

$$\begin{aligned}
&F^3(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0),\\
&F^2(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0) + \mathbb{C}\,\frac{\widetilde{\gamma}^1(0)}{2\pi i} + \mathbb{C}\,\frac{\widetilde{\gamma}^4(0)}{2\pi i},\\
&F^1(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0) + \sum_{j=1,4}\mathbb{C}\,\frac{\widetilde{\gamma}^j(0)}{2\pi i} + \sum_{j=2,5}\mathbb{C}\,\frac{\widetilde{\gamma}^j(0)}{(2\pi i)^2},\\
&F^0(\widetilde{\mathcal{V}}|_0) = \mathbb{C}\,\widetilde{\gamma}^0(0) + \sum_{j=1,4}\mathbb{C}\,\frac{\widetilde{\gamma}^j(0)}{2\pi i} + \sum_{j=2,5}\mathbb{C}\,\frac{\widetilde{\gamma}^j(0)}{(2\pi i)^2} + \mathbb{C}\,\frac{\widetilde{\gamma}^3(0)}{(2\pi i)^3}.
\end{aligned} \tag{8.48}$$

8.4 Properties of the limit MHS

To study the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit, it is more convenient to choose a new basis $\{x^j\}_{j=0}^5$ of $\widetilde{\mathcal{V}}|_0$ given by

$$\begin{aligned} & x^j := (2\pi i)^{3-j}\, \widetilde{\beta}^j(0),\ j = 0, 1, 2, 3, \\ & x^4 := (2\pi i)\, \widetilde{\beta}^4(0),\ x^5 := (2\pi i)^2\, \widetilde{\beta}^5(0). \end{aligned} \tag{8.49}$$

With respect to this new basis, the basis $\{\widetilde{\gamma}^a(0)\}_{a=0}^5$ 8.22 is expressed as

$$\begin{aligned} \widetilde{\gamma}^0(0) &= \frac{\lambda}{(2\,\pi\,i)^3}\, x^0 + \frac{\lambda\, Y_{000}}{54} x^3, \\ \widetilde{\gamma}^1(0) &= -\frac{\lambda}{(2\pi i)^2}\, x^1, \\ \widetilde{\gamma}^2(0) &= \frac{1}{2}\frac{\lambda}{(2\pi i)}\, x^2, \\ \widetilde{\gamma}^3(0) &= -\frac{\lambda}{6}\, x^3, \\ \widetilde{\gamma}^4(0) &= -\frac{1}{3}\frac{\lambda}{(2\pi i)^2}\, x^1 + \frac{\lambda}{(2\pi i)^2}\, x^5, \\ \widetilde{\gamma}^5(0) &= \frac{1}{3}\frac{\lambda}{(2\pi i)}\, x^2 - \frac{\lambda}{(2\pi i)}\, x^4. \end{aligned}$$

The rational vector space $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$ is spanned by the following vectors

$$\{(2\pi i)^{j-3}\, x^j\}_{j=0}^3 \cup \{(2\pi i)^{-1} x^4, (2\pi i)^{-2} x^5\} \tag{8.50}$$

The weight filtration $W_*(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})$ 8.33 becomes

$$\begin{aligned} & W_0(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = W_1(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q}\, x^3, \\ & W_2(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = W_3(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q}\, x^3 + \sum_{j=2,4} \mathbb{Q}\, \frac{x^j}{2\pi i}, \\ & W_4(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = W_5(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \sum_{j=1,5} \mathbb{Q}\, \frac{x^j}{(2\pi i)^2} + \sum_{j=2,4} \mathbb{Q}\, \frac{x^j}{2\pi i} + \mathbb{Q}\, x^3, \\ & W_6(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q}\, \frac{x^0}{(2\pi i)^3} + \sum_{j=1,5} \mathbb{Q}\, \frac{x^j}{(2\pi i)^2} + \sum_{j=2,4} \mathbb{Q}\, \frac{x^j}{2\pi i} + \mathbb{Q}\, x^3, \end{aligned} \tag{8.51}$$

while the Hodge filtration $F^*(\widetilde{\mathcal{V}}|_0)$ 8.48 becomes

$$\begin{aligned}
F^3(\widetilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,\mathrm{span}\{x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3\} \otimes_{\mathbb{Q}} \mathbb{C},\\
F^2(\widetilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,\mathrm{span}\{x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3, x^1, x^5\} \otimes_{\mathbb{Q}} \mathbb{C},\\
F^1(\widetilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,\mathrm{span}\{x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3, x^1, x^2, x^4, x^5\} \otimes_{\mathbb{Q}} \mathbb{C},\\
F^0(\widetilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,\mathrm{span}\{x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3, x^1, x^2, x^3, x^4, x^5\} \otimes_{\mathbb{Q}} \mathbb{C}.
\end{aligned} \tag{8.52}$$

The crucial observation is the following theorem.

Theorem 8.4.1. *Assuming mirror symmetry conjecture, the limit MHS on $H^3(X_{\mathfrak{G}}, \mathbb{Q})$ at the large complex structure limit splits into the direct sum*

$$\mathbf{M} \oplus \mathbb{Q}(-1)^2 \oplus \mathbb{Q}(-2)^2, \tag{8.53}$$

where $\mathbf{M}$ is a two-dimensional MHS with rational vector space $\mathbf{M}_{\mathbb{Q}}$

$$\mathbf{M}_{\mathbb{Q}} = \mathbb{Q}\,\frac{1}{(2\pi i)^3}\,x^0 + \mathbb{Q}\,x^3. \tag{8.54}$$

The weight filtration $W_*\,\mathbf{M}$ is

$$\begin{aligned}
&W_{-1}\,\mathbf{M} = W_{-2}\,\mathbf{M} = \cdots = 0,\\
&W_0\,\mathbf{M} = W_1\,\mathbf{M} = \cdots = W_5\,\mathbf{M} = \mathbb{Q}\,x^3,\\
&W_6\,\mathbf{M} = W_7\,\mathbf{M} = \cdots = \mathbb{Q}\,\frac{1}{(2\pi i)^3}\,x^0 + \mathbb{Q}\,x^3,
\end{aligned} \tag{8.55}$$

and the Hodge filtration $F^*\,\mathbf{M}$ is

$$\begin{aligned}
&F^4\,\mathbf{M} = F^5\,\mathbf{M} = \cdots = 0,\\
&F^3\,\mathbf{M} = F^2\,\mathbf{M} = F^1\,\mathbf{M} = \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,(x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3) \otimes_{\mathbb{Q}} \mathbb{C},\\
&F^0\,\mathbf{M} = \frac{\lambda}{(2\pi i)^3}\big(\mathbb{Q}\,(x^0 + \frac{(2\pi i)^3\,Y_{000}}{54}\,x^3) + \mathbb{Q}\,x^3\big) \otimes_{\mathbb{Q}} \mathbb{C},\\
&F^{-1}\,\mathbf{M} = F^{-2}\,\mathbf{M} = \cdots = F^0\,\mathbf{M}.
\end{aligned} \tag{8.56}$$

Proof. This follows directly from the weight filtration $W_*(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})$ in equation 8.51 and the limit Hodge filtration $F^*(\widetilde{\mathcal{V}}|_0)$ in equation 8.52. ☐

Since $W_{-1}\,\mathbf{M}$ is 0, $\mathrm{Gr}_0^W\,\mathbf{M}$ equals $W_0\,\mathbf{M}$, and the Hodge filtration $F^*\,\mathbf{M}$ induces a pure Hodge structure on $W_0\,\mathbf{M}$ which is isomorphic to $\mathbb{Q}(0)$. The inclusion map $W_0\,\mathbf{M} \hookrightarrow \mathbf{M}$ induces an injective homomorphism from $\mathbb{Q}(0)$ to $\mathbf{M}$, the quotient of which is the pure Hodge structure on $\mathrm{Gr}_6^W\,\mathbf{M}$ that is isomorphic to $\mathbb{Q}(-3)$. Therefore we have obtained a short exact sequence in $\mathbf{MHS}_{\mathbb{Q}}$

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \mathbf{M} \longrightarrow \mathbb{Q}(-3) \longrightarrow 0, \tag{8.57}$$

which shows $\mathbf{M}$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. In the abelian category $\mathbf{MHS}_{\mathbb{Q}}$, the dual of an object $\mathbf{H}$ is defined by [26, 85]

$$\mathbf{H}^\vee := \mathrm{Hom}_{\mathbf{MHS}_{\mathbb{Q}}}\big(\mathbf{H}, \mathbb{Q}(0)\big), \tag{8.58}$$

moreover, the dual operation is exact [85], i.e. it sends a short exact sequence to a short exact sequence, therefore the dual of 8.57 is a short exact sequence

$$0 \longrightarrow \mathbb{Q}(3) \longrightarrow \mathbf{M}^\vee \longrightarrow \mathbb{Q}(0) \longrightarrow 0. \tag{8.59}$$

Theorem 8.4.2. *Assuming mirror symmetry conjecture, the dual object* $\mathbf{M}^\vee$ *is an extension of* $\mathbb{Q}(0)$ *by* $\mathbb{Q}(3)$ *whose image in* $\mathbb{C}/(2\pi i)^3\,\mathbb{Q}$ *is the coset of* $-(2\pi i)^3\,Y_{000}/54$.

Proof. The short exact sequence 8.59 immediately shows $\mathbf{M}^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$. We now compute the object $\mathbf{M}^\vee$ from Definition 3.2.3. Let $\{x_j\}_{j=0}^5$ be the basis of $(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})^\vee$ that is dual to $\{x^j\}_{j=0}^5$, and their pairings are given by

$$x_j(x^k) = \delta_j^k. \tag{8.60}$$

The rational vector space of $\mathbf{M}^\vee$ is the subspace of $(\widetilde{\mathcal{V}}|_{0,\mathbb{Q}})^\vee$ spanned by $\{(2\pi i)^3\,x_0, x_3\}$

$$(\mathbf{M}^\vee)_{\mathbb{Q}} := \mathbb{Q}\,(2\pi i)^3\,x_0 + \mathbb{Q}\,x_3. \tag{8.61}$$

The weight filtration $W_*\,\mathbf{M}^\vee$ is defined by

$$W_l\,\mathbf{M}^\vee := \{\phi : \phi\,(W_r\,\mathbf{M}) \subset W_{r+l}\,\mathbb{Q}(0)\}, \tag{8.62}$$

so we find that

$$\begin{aligned} &W_{-7}\,\mathbf{M}^\vee = W_{-8}\,\mathbf{M}^\vee = \cdots = 0,\\ &W_{-6}\,\mathbf{M}^\vee = \cdots = W_{-1}\,\mathbf{M}^\vee = \mathbb{Q}\,(2\pi i)^3\,x_0,\\ &W_0\,\mathbf{M}^\vee = W_1\,\mathbf{M}^\vee = \cdots = \mathbb{Q}\,(2\pi i)^3\,x_0 + \mathbb{Q}\,x_3. \end{aligned} \tag{8.63}$$

The Hodge filtration $F^*\,\mathbf{M}^\vee$ is defined by

$$F^p\,\mathbf{M}^\vee := \{\phi : \phi\,(F^r\,\mathbf{M}) \subset F^{r+p}\,\mathbb{Q}(0)\}, \tag{8.64}$$

so we find that

$$
\begin{aligned}
&F^1\,\mathbf{M}^\vee = F^2\,\mathbf{M}^\vee = \cdots = 0,\\
&F^0\,\mathbf{M}^\vee = F^{-1}\,\mathbf{M}^\vee = F^{-2}\,\mathbf{M}^\vee = (2\pi i)^3\,\mathbb{Q}\,(-\frac{(2\pi i)^3\,Y_{000}}{54}\,x_0 + x_3)\otimes_{\mathbb{Q}}\mathbb{C},\\
&F^{-3}\,\mathbf{M}^\vee = F^{-4}\,\mathbf{M}^\vee = \cdots = (2\pi i)^3\,\big(\mathbb{Q}\,(-\frac{(2\pi i)^3\,Y_{000}}{54}\,x_0 + x_3) + \mathbb{Q}\,x_0\big)\otimes_{\mathbb{Q}}\mathbb{C}.
\end{aligned}
\tag{8.65}
$$

From Section 3.2.2, the image of $\mathbf{M}^\vee$ in $\mathbb{C}/(2\pi i)^3\,\mathbb{Q}$ is the coset of $-(2\pi i)^3\,Y_{000}/54$. □

8.5 The limit MHS at the large complex structure limit of a general mirror pair

From the study of one parameter mirror pairs in [72] and the two parameter example $X_{\mathfrak{G}}$ in this thesis, we have a conjecture about the limit MHS at the large complex structure limit of a general mirror pair (M, W).

Conjecture LMHS: *Given a mirror pair (M, W) of Calabi-Yau threefolds, the limit MHS on $H^3(W, \mathbb{Q})$ at the large complex structure limit of W splits into the direct sum*

$$
\mathbf{M} \oplus \mathbb{Q}(-1)^{h^{12}(W)} \oplus \mathbb{Q}(-2)^{h^{12}(W)},
\tag{8.66}
$$

where $\mathbf{M}$ is a two-dimensional MHS with rational vector space given by

$$
\mathbf{M}_{\mathbb{Q}} = \mathbb{Q}\,\frac{1}{(2\pi i)^3}\,x^0 + \mathbb{Q}\,x^3.
\tag{8.67}
$$

Furthermore, its weight filtration $W_*\,\mathbf{M}$ is

$$
\begin{aligned}
&W_{-1}\,\mathbf{M} = W_{-2}\,\mathbf{M} = \cdots = 0,\\
&W_0\,\mathbf{M} = W_1\,\mathbf{M} = \cdots = W_5\,\mathbf{M} = \mathbb{Q}\,x^3,\\
&W_6\,\mathbf{M} = W_7\,\mathbf{M} = \cdots = \mathbb{Q}\,\frac{1}{(2\pi i)^3}\,x^0 + \mathbb{Q}\,x^3,
\end{aligned}
\tag{8.68}
$$

and its Hodge filtration $F^*\,\boldsymbol{M}$ is

$$
\begin{aligned}
&F^4\,\mathbf{M} = F^5\,\mathbf{M} = \cdots = 0,\\
&F^3\,\mathbf{M} = F^2\,\mathbf{M} = F^1\,\mathbf{M} = \frac{\lambda}{(2\pi i)^3}\mathbb{Q}\,(x^0 + \frac{(2\pi i)^3\,Y_{000}}{3\,Y_{eee}}\,x^3)\otimes_{\mathbb{Q}}\mathbb{C},\\
&F^0\,\mathbf{M} = \frac{\lambda}{(2\pi i)^3}\big(\mathbb{Q}\,(x^0 + \frac{(2\pi i)^3\,Y_{000}}{3\,Y_{eee}}\,x^3) + \mathbb{Q}\,x^3\big)\otimes_{\mathbb{Q}}\mathbb{C},\\
&F^{-1}\,\mathbf{M} = F^{-2}\,\mathbf{M} = \cdots = F^0\,\mathbf{M}.
\end{aligned}
\tag{8.69}
$$

Here Y_{eee} *is a positive integer given by*

$$Y_{eee} := \int_M e \wedge e \wedge e, \tag{8.70}$$

where e *is an element of* $H^2(M, \mathbb{Z})$ *that lies in the Kähler cone of* M*.*

Summary: in this chapter, we have applied the methods in Chapter 3 to the computation of the limit MHS at the large complex structure limit of the mirror family of $X_{\mathfrak{G}}$. It is clear in the computation that the **Mirror Symmetry Conjecture** plays a very essential role. Moreover we have studied this limit MHS and shown that it splits into the direct sum $\mathbf{M} \oplus \mathbb{Q}(-1)^2 \oplus \mathbb{Q}(-2)^2$, where the dual $\mathbf{M}^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ whose image in $\mathbb{C}/(2\pi i)^3\,\mathbb{Q}$ is the coset of $-(2\pi i)^3\,Y_{000}/54$. Based on the computations in this chapter, we formulate the **Conjecture LMHS** which is about the properties of the limit MHS at the large complex structure limit of an arbitrary mirror pair. The interesting connections between this conjecture and the theory of mixed motives, i.e. the **Period Conjecture**, have been discussed in Section 1.1 of **Introduction**. From next chapter, we will study the arithmetic geometry of the conifold in the mirror family of the quintic.

Chapter 9

The mirror family of the quintic Calabi-Yau threefold

From this chapter to Chapter 13, we will study the quintic mirror conifold and its interesting connections to **Beilinson's Conjecture**. First in this chapter, we will discuss the construction of the mirror family of quintic Calabi-Yau threefold. We will follow [30, 31] closely, which are also recommended to the reader unfamiliar with the construction, while the reader to whom this is familar can skip this chapter completely. The structure of this chapter is as follows:

- Section 9.1 discusses the toric geometric construction of the quintic Calabi-Yau threefold.
- Section 9.2 discusses the toric geometric construction of the mirror family of quintic Calabi-Yau threefold.
- Finally, Section 9.3 discusses the construction of the threeform on the mirror family of quintic Calabi-Yau threefold.

9.1 The quintic Calabi-Yau threefold

In this section, we will talk about the toric geometric construction of quintic Calabi-Yau threefold. Suppose $(x_0, \cdots, x_4)$ are the projective coordinates of the projective space $\mathbb{P}^4_{\mathbb{C}}$, then a section of the anti-canonical bundle $\mathscr{O}_{\mathbb{P}^4_{\mathbb{C}}}(5)$ is given by a quintic polynomial in variables $\{x_i\}_{i=0}^4$ [52, 98]. Then adjunction formula tells us that the vanishing locus of a quintic polynomial, if it is smooth, has trivial canonical bundle. So a smooth hypersurface of $\mathbb{P}^4_{\mathbb{C}}$ that is given as the vanishing locus of a quintic polynomial is a Calabi-Yau threefold, which is usually called quintic for short. Its Hodge diamond is [30]

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}.$$

9.1.1 The polytope of $\mathbb{P}^4_{\mathbb{C}}$

Let M (resp. N) be the lattice of characters (resp. the lattice of one-parameter subgroups) of the complex torus $(\mathbb{C}^*)^4$, both of which are isomorphic to $\mathbb{Z}^4$. The tensor product of N (resp. M) with $\mathbb{R}$ is a four dimensional real linear space $N_{\mathbb{R}}$ (resp. $M_{\mathbb{R}}$)

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^4, \;\; M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^4. \tag{9.1}$$

There is a canonical pairing between M and N given by

$$M \times N \to \mathbb{Z}, \quad \langle (a_i), (b_i) \rangle \mapsto \sum_{i=1}^{4} a_i b_i. \tag{9.2}$$

Let $\{e_i\}_{i=1}^4$ be the basis of $N \simeq \mathbb{Z}^4$ given by

$$e_1 = (1,0,0,0), \;\; e_2 = (0,1,0,0), \;\; e_3 = (0,0,1,0), \;\; e_4 = (0,0,0,1). \tag{9.3}$$

The projective space $\mathbb{P}^4_{\mathbb{C}}$ is a toric variety, whose fan Σ in $N_{\mathbb{R}}$ has ray generators

$$u_0 = -\sum_{i=1}^{4} e_i, \;\; u_i = e_i, \;\; \text{for } i = 1,2,3,4, \tag{9.4}$$

while the cones of Σ are generated by the proper subsets of $\{u_i\}_{i=0}^4$ [31]. The standard simplex Δ_4 of $M_{\mathbb{R}}$ is the convex hull of the following lattice points of M

$$(0,0,0,0), \;\; (1,0,0,0), \;\; (0,1,0,0), \;\; (0,0,1,0), \;\; (0,0,0,1), \tag{9.5}$$

and the normal fan of Δ_4 is just Σ. Let P be the polytope of $M_{\mathbb{R}}$ given by

$$P := 5\,\Delta_4 - (1,1,1,1), \tag{9.6}$$

whose vertices are the lattice points

$$\begin{aligned} &v_0 = (-1,-1,-1,-1), \quad v_1 = (4,-1,-1,-1), \;\; v_2 = (-1,4,-1,-1) \\ &v_3 = (-1,-1,4,-1), \qquad v_4 = (-1,-1,-1,4). \end{aligned} \tag{9.7}$$

The normal fan of P is also Σ, and the toric variety associated to P, denoted by X_P, is the same as the toric variety associated to the fan Σ, i.e. $\mathbb{P}^4_{\mathbb{C}}$.

9.1.2 Anti-canonical bundle

The dual polytope of P, denoted by P°, is a lattice polytope in $N_\mathbb{R}$ with vertices $\{u_i\}_{i=0}^4$, i.e. P° is the convex hull of the lattice points $\{u_i\}_{i=0}^4$ in $N_\mathbb{R}$. The dual of P° is just P, which shows P is reflexive. Let D_i be the divisor of X_P associated to the i-th surface of P, then the anti-canonical divisor of X_P is given by [31]

$$D_P = \sum_{i=0}^4 D_i. \tag{9.8}$$

The polytope P has 126 lattice points and the toric polynomials associated to them form a basis of the sections of the anti-canonical bundle $\mathcal{O}_{X_P}(D_P)$ [31]. Since the polytope P is reflexive, the vanishing locus of a section of $\mathcal{O}_{X_P}(D_P)$, if smooth, is a Calabi-Yau threefold [31]. By abuse of notations, let x_i also be the homogeneous coordinate associated to the ray of Σ generated by u_i, then the total coordinate ring S_P of X_P is given by

$$S_P = \mathbb{C}[x_0, \cdots, x_4]. \tag{9.9}$$

The class group of X_P, denoted by $\mathrm{Cl}(X_P)$, is isomorphic to $\mathbb{Z}$, and the divisor D_i is a generator of $\mathrm{Cl}(X_P)$. Moreover D_i is rationally equivalent to every other D_j, therefore the degree of D_i is 1, and the degree of D_P is 5. The toric polynomials associated to the 126 lattice points of P homogenize to the 126 quintic monomials of S_P, hence the sections of $\mathcal{O}_{X_P}(D_P)$ are given by quintic polynomials of S_P.

Remark 9.1.1. *The line bundle $\mathcal{O}_{X_P}(D_P)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^4_\mathbb{C}}(5)$, therefore the toric geometry picture is completely compatible with the classical algebraic geometry picture.*

9.2 The mirror family of the quintic

In this section, we will talk about the construction of the mirror family of quintic Calabi-Yau threefold. We will follow the Example 5.4.10 of [31] and Section 4.2 of [30] closely, which are also recommended to the readers unfamiliar with the construction.

9.2.1 The singular mirror family of the quintic

The normal fan of P°, denoted by Σ°, is a fan in the real vector space $M_\mathbb{R}$ with ray generators $\{v_i\}_{i=0}^4$, i.e. the vertices of P; while the proper subsets of $\{v_i\}_{i=0}^4$ generate all the cones of Σ°. The toric variety associated to P°, which is the same as the toric variety associated to the fan Σ°, will be denoted by X_{P°, which is called the dual toric variety of X_P. Let y_i be the homogeneous coordinate associated to the ray of Σ° generated by v_i, then the total coordinate ring of X_{P° is given by

$$S^\circ = \mathbb{C}\,[y_0, \cdots, y_4]. \tag{9.10}$$

Remark 9.2.1. *The lattice of characters of X_{P° is N, while the lattice of one-parameter subgroup of X_{P° is M, which are swapped compared to that of X_P.*

The points $\{v_i\}_{i=0}^4$ generate a lattice M_1, which is a sublattice of M, while its dual lattice, denoted by N_1, canonically contains N as a sublattice. The index of M_1 in M is 125, therefore the index of N in N_1 is also 125. There is an injective homomorphism from M to $\mathbb{Z}^5$ defined by

$$m \in M \mapsto (\langle m, u_0\rangle, \langle m, u_1\rangle, \langle m, u_2\rangle, \langle m, u_3\rangle, \langle m, u_4\rangle) \in \mathbb{Z}^5. \tag{9.11}$$

This homomorphism induces an isomorphism between the quotient abelian group M/M_1 and the group G defined by

$$G := \{(a_0, \cdots, a_4) \in (\mathbb{Z}/5\,\mathbb{Z})^5 |\ \sum a_i = 0\}/\{(a, a, a, a, a)|\, a \in \mathbb{Z}/5\,\mathbb{Z}\}, \tag{9.12}$$

which is isomorphic to $(\mathbb{Z}/5\,\mathbb{Z})^3$. The lattice points $\{v_i\}_{i=0}^4$ satisfy a linear equation

$$v_0 + v_1 + v_2 + v_3 + v_4 = 0. \tag{9.13}$$

If Σ° is viewed as a fan with respect to the sublattice M_1, then the toric varietry associated to it, denoted by X_{Σ°, M_1}, is $\mathbb{P}^4_{\mathbb{C}}$. On the other hand, X_{Σ°, M_1} is also the toric variety associated to the polytope P° with respect to the lattice N_1. From Proposition 3.3.7 of [31], we have

$$X_{P^\circ} \simeq X_{\Sigma^\circ, M_1}/(M/M_1) = \mathbb{P}^4_{\mathbb{C}}/G. \tag{9.14}$$

Remark 9.2.2. *Notice that the lattice of characters of X_{P°, M_1} is N_1, while the lattice of one-parameter subgroups of X_{P°, M_1} is M_1.*

From Example 5.4.10 of [31], the class group $\mathrm{Cl}(X_{P^\circ})$ is isomorphic to the abelian group $\mathbb{Z} \oplus N_1/N$. From Exercise 5.4.7 of [31], the character group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_{P^\circ}), \mathbb{C}^*)$, as a subgroup of $(\mathbb{C}^*)^5$, has an explicit description given by

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_{P^\circ}), \mathbb{C}^*) = \{(\lambda\, \zeta_0', \cdots, \lambda\, \zeta_4') : \lambda \in \mathbb{C}^*,\ (\zeta_i')^5 = 1, \prod_{i=0}^4 \zeta_i' = 1\} \simeq \mathbb{C}^* \oplus G. \tag{9.15}$$

The toric variety X_{P° is also constructed as the geometric quotient of $\mathbb{C}^5 - 0$ by the character group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_{P^\circ}), \mathbb{C}^*)$ [31]. After quotient by $\mathbb{C}^*$, X_{P° is described as the quotient of $\mathbb{P}^4_{\mathbb{C}}$ by G, while the action of an element $(a_0, \cdots, a_4)$ of G on $\mathbb{P}^4_{\mathbb{C}}$ is

$$(y_0, \cdots, y_4) \to (\zeta_5^{a_0}\, y_0, \cdots, \zeta_5^{a_4}\, y_4), \tag{9.16}$$

where ζ_5 is $\exp(2\pi i/5)$. Let D_i° be the divisor of X_{P° associated to the i-th surface of P°, then the anti-canonical divisor of X_{P°, denoted by D_{P°, is given by

$$D_{P^\circ} := \sum_i D_i^\circ. \tag{9.17}$$

The sections of the anti-canonical bundle $\mathcal{O}_{X_{P^\circ}}(D_{P^\circ})$ has a basis given by the six lattice points of P°: the origin and the vertices $\{u_i\}_{i=0}^4$. The homogenisations of the six lattice points in the total coordinate ring S° are

$$\text{Origin} \to \prod_{i=0}^4 y_i,\ u_i \to y_i^5, \tag{9.18}$$

hence a general section of $\mathcal{O}_{X_{P^\circ}}(D_{P^\circ})$ is of the form

$$\sum_{i=0}^4 c_i\, y_i^5 + c_5 \prod_{i=0}^4 y_i. \tag{9.19}$$

If $\prod_{i=0}^4 c_i \neq 0$, the vanishing locus of this section is an irreducible hypersurface of X_{P° that is isomorphic to

$$Y_\psi:\ \sum_{i=0}^4 y_i^5 - 5\,\psi \prod_{i=0}^4 y_i = 0. \tag{9.20}$$

Now we also allow ψ to take the value ∞, and Y_∞ is given by

$$\prod_{i=0}^4 y_i = 0, \tag{9.21}$$

which is singular and reducible. So we have got an algebraic family Y_ψ fibrated over $\mathbb{P}^1_{\mathbb{C}}$, however the fiber Y_ψ is singular even for a general parameter ψ.

9.2.2 The singularities of the singular mirror family

Now we will look at the singularities of the fiber Y_ψ carefully. Let X_ψ be the hypersurface of $\mathbb{P}^4_{\mathbb{C}}$ defined by

$$X_\psi : f_\psi = \sum_{i=0}^4 x_i^5 - 5\,\psi \prod_{i=0}^4 x_i = 0, \tag{9.22}$$

which is smooth when $\psi^5 \neq 1, \infty$. The singularities of the singular fibers are:

1. When $\psi^5 = 1$, the singularity of X_ψ consists of 125 double points given by

$$(\zeta_5^{a_0}, \cdots, \zeta_5^{a_4}) \text{ with } a_i \in \mathbb{Z}/5\,\mathbb{Z} \text{ and } \sum a_i = 0. \tag{9.23}$$

2. When $\psi = \infty$, X_∞ is the union of five $\mathbb{P}^3_{\mathbb{C}}$

$$X_\infty : \prod_{i=0}^{4} x_i = 0. \tag{9.24}$$

The action of G on $\mathbb{P}^4_{\mathbb{C}}$ is given by

$$(x_0, \cdots, x_4) \to (\zeta_5^{a_0}\, x_0, \cdots, \zeta_5^{a_4}\, x_4), \tag{9.25}$$

which preserves X_ψ, and the quotient variety is just Y_ψ

$$X_\psi/G = Y_\psi. \tag{9.26}$$

However this action of G on X_ψ is not free and the quotient process creates singularities: if the stabilizer of a point is nontrivial, then its image in Y_ψ is a singular point. When $\psi \neq \infty$, the points of X_ψ that possess nontrivial stabilizers are [51]:

1. 10 curves $C_{ij} := \{x_i = x_j = 0\} \cap X_\psi$, the stabilizer of which is of order 5. For example C_{01} is given by

$$C_{01} : x_0 = x_1 = x_2^5 + x_3^5 + x_4^5 = 0, \tag{9.27}$$

   and its stabilizer is $\{(\zeta_5^a, \zeta_5^{-a}, 1, 1, 1) : a \in \mathbb{Z}/5\mathbb{Z}\}$.

2. 10 points $P_{ijk} := \{x_i = x_j = x_k = 0\} \cap X_\psi$, the stabilizer of which is of order 25. For example P_{012} is given by

$$P_{ijk} : x_0 = x_1 = x_2 = x_3^5 + x_4^5 = 0, \tag{9.28}$$

   and its stabilizer is $\{(\zeta_5^a, \zeta_5^b, \zeta_5^{-a-b}, 1, 1) : a, b \in \mathbb{Z}/5\mathbb{Z}\}$.

The variety $\mathbb{P}^4_{\mathbb{C}}$ is covered by five open affine subspaces $U_i := \{x_i \neq 0\}$, which are also the affine toric varieties associated to the maximal cones of its fan Σ. The action of G on $\mathbb{P}^4_{\mathbb{C}}$ preserves U_i, hence the quotient variety $\mathbb{P}^4_{\mathbb{C}}/G$ is covered by affine toric varieties $\{U_i/G\}_{i=0}^4$. Now let us look at the quotient variety U_4/G carefully: define $x_{4,i} := x_i/x_4$, then the G-invariant elements of $\mathbb{C}[x_{4,i}]$ form a subring $\mathbb{C}[x_{4,i}]^G$ that is given by

$$\mathbb{C}[x_{4,i}]^G = \mathbb{C}[x_{4,0}^5, x_{4,1}^5, x_{4,2}^5, x_{4,3}^5, x_{4,0}x_{4,1}x_{4,2}x_{4,3}]. \tag{9.29}$$

By sending $x_{4,i}^5 \mapsto t_i$ and $x_{4,0}x_{4,1}x_{4,2}x_{4,3} \mapsto t_4$, we get an isomorphism

$$\mathbb{C}[x_{4,i}]^G \simeq \mathbb{C}[t_0, t_1, t_2, t_3, t_4]/\langle\, t_4^5 - t_0\, t_1\, t_2\, t_3\, \rangle. \tag{9.30}$$

So the quotient variety U_4/G is isomorphic to the affine toric variety

$$U_4/G \simeq \operatorname{Spec} \mathbb{C}[t_0, t_1, t_2, t_3, t_4]/\langle\, t_4^5 - t_0\, t_1\, t_2\, t_3\, \rangle, \tag{9.31}$$

while the open subvariety $(X_\psi \cap U_4)/G$ of Y_ψ is a hypersurface of U_4/G defined by the equation

$$1 + t_0 + t_1 + t_2 + t_3 - 5\,\psi\, t_4 = 0. \tag{9.32}$$

The images of C_{ij} and P_{ijk} in $Y_\psi, \psi \neq \infty$ are singular points, therefore even for a smooth X_ψ, its quotient Y_ψ is still singular.

Remark 9.2.3. *When $\psi \neq \infty$, the singular locus of Y_ψ is of codimension 2, hence its dualizing sheaf ω_{Y_ψ} is well defined, which is trivial because P° is reflexive [30].*

9.2.3 The resolution of the singularities of the mirror family

Now we will talk about the resolution of the singularities of Y_ψ. Let us choose a maximal projective subdivision $\widehat{\Sigma}^\circ$ of Σ°, which is a refinement of Σ° [30]. There are lots of maximal projective subdivisions of Σ° and a special one is given in the appendix of [81]. The toric variety $X_{\widehat{\Sigma}^\circ}$ associated to $\widehat{\Sigma}^\circ$ satisfies the following properties:

1. $X_{\widehat{\Sigma}^\circ}$ is projective.
2. The map of fans $\widehat{\Sigma}^\circ \to \Sigma^\circ$ induces a birational toric map
$$\phi : X_{\widehat{\Sigma}^\circ} \to X_{P^\circ}. \tag{9.33}$$
3. When $\psi \neq \infty$, the proper transform of Y_ψ in $X_{\widehat{\Sigma}^\circ}$ resolves the singularities C_{ij}/G and P_{ijk}/G.
4. When $\psi^5 \neq 0, 1, \infty$, the proper transform of Y_ψ in $X_{\widehat{\Sigma}^\circ}$ is a smooth Calabi-Yau threefold. While when $\psi^5 = 1$, the proper transform of Y_ψ in $X_{\widehat{\Sigma}^\circ}$ is singular with singularity consists of a single double point, which is the quotient of the 125 double points of X_ψ by G.

The readers are referred to [30] for the detail and proofs. Denote the proper transform of Y_ψ in $X_{\widehat{\Sigma}^\circ}$ be $\mathcal{W}_\psi$, and we get the mirror family of quintic

$$\pi : \mathcal{W} \to \mathbb{P}^1_{\mathbb{C}}, \tag{9.34}$$

a smooth fiber of which is a Calabi-Yau threefold with Hodge diamond [30]

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{9.35}$$

Remark 9.2.4. *When $\psi \neq \infty$, the resolution morphism*

$$\phi_\psi : \mathcal{W}_\psi \to Y_\psi \tag{9.36}$$

is crepant [30]. Since the dualizing sheaf of Y_ψ is trivial, so the dualizing sheaf of $\mathcal{W}_\psi$ is also trivial. The proof that $\mathcal{W}_\psi$ is Calabi-Yau (when $\psi^5 \neq 0, 1, \infty$) can be found in the book [30].

The mirror family in equation 9.34 has the advantage that it is more symmetric, which is important when doing computations, however it suffers a defect that $\mathcal{W}_\psi$ is isomorphic to $\mathcal{W}_{\zeta_5 \psi}$, therefore the true moduli of this family should be parametrised by ψ^5. This defect can be cured by a similar construction. Rescale y_0 by the factor $1/\psi$ and define $\phi := 1/\psi^5$, then equation 9.20 becomes

$$\phi\, y_0^5 + \sum_{i=1}^{4} y_i^5 - 5 \prod_{i=0}^{4} y_i = 0. \tag{9.37}$$

Parallel construction as before yields a new mirror family

$$\pi : \mathcal{W} \to \mathbb{P}^1_{\mathbb{C}}, \tag{9.38}$$

which has three singular fibers over the points $0, 1, \infty$. The point $\phi = 0$ is the large complex structure limit, which is an essential ingredient in mirror symmetry [23]. The singular fiber over 1, denoted by $\mathcal{W}_1$, is called conifold. To avoid potential confusions, we will call it quintic mirror conifold in this thesis.

9.3 The threeform of the mirror family

Since toric varieties and toric morphisms are actually defined over $\operatorname{Spec}\mathbb{Z}$ [31], the families in equations 9.20, 9.34 and 9.38 are in fact fibrations defined over $\mathbb{Z}$, a priory the three families are defined over $\mathbb{Q}$. This is also shown in the explicit description of U_i/G 9.31 and Y_ψ 9.32. Since X_ψ is a hypersurface in $\mathbb{P}^4_{\mathbb{C}}$, there is well-known method to construct a threeform on it, and over the affine open subset $X_\psi \cap U_4$, the threeform is given by [51]

$$\Omega_{X_\psi} = 5\,\psi\, \frac{dx_{4,0}\wedge dx_{4,1}\wedge dx_{4,2}}{\partial f_\psi/\partial x_{4,3}}\bigg|_{X_\psi} = \psi\, \frac{dx_{4,0}\wedge dx_{4,1}\wedge dx_{4,2}}{x_{4,3}^4 - \psi\, x_{4,0}x_{4,1}x_{4,2}}\bigg|_{X_\psi}. \tag{9.39}$$

When X_ψ is smooth, the denominator is nowhere vanishing, hence Ω_{X_ψ} is a well-defined threeform on X_ψ that is also nowhere vanishing [51]. The threeform Ω_{X_ψ} is G-invariant, hence it induces a threeform on the smooth locus of Y_ψ that is nowhere vanishing, which also shows that the dualizing sheaf of Y_ψ is trivial [51]. More explicitly, the affine open subset $(X_\psi \cap U_4)/G$ is given by equation 9.32, and the induced threeform over its smooth locus is given by

$$\Omega_{Y_\psi} = \frac{\psi}{125}\frac{t_3\, dt_0\wedge dt_1\wedge dt_2}{t_4^4\,(t_3-\psi\, t_4)}\bigg|_{Y_\psi}. \tag{9.40}$$

From Remark 9.2.4, the resolution morphism 9.36 is crepant, hence the threeform Ω_{Y_ψ} will pull back to a nowhere vanishing threeform $\Omega(\psi)$ on $\mathcal{W}_\psi$ if it is smooth. The constructions of threeforms immediately show that Ω_{X_ψ}, Ω_{Y_ψ} and $\Omega(\psi)$ are all defined over $\mathbb{Q}$.

Summary: in this chapter, we have discussed the construction of the mirror family of the quintic, which in fact is an algebraic fibration defined over $\mathbb{Q}$. We also discuss the construction of the threeform for the mirror family, which is algebraically defined over $\mathbb{Q}$. This chapter will be important when we discuss the conifold transition in Chapter 10 and when we compute the limit MHS at the quintic mirror conifold later in Chapter 13. This chapter is totally from the literature, and it is included here in order to make this thesis as self-contained as possible.

Chapter 10

The small resolutions of the conifold in the mirror family of the quintic

In this chapter we will construct the small resolutions of the conifold in the mirror family of quintic, which is called quintic mirror conifold, in the category of algebraic spaces. The structure of this chapter is as follows:

- Section 10.1 discusses the construction of quintic mirror conifold and the local structure of its double point. It also shows there exists a nowhere vanishing threeform defined on the smooth locus of quintic mirror conifold.
- Section 10.2 studies the analytic small resolutions of quintic mirror conifold, i.e. the small resolutions of the analytic variety defined by quintic mirror conifold in the category of analytic varieties.
- Section 10.3 discusses the conifold transition in the mirror family of quintic.
- Section 10.4 studies the Hilbert scheme of lines in a quadratic surface of $\mathbb{P}^3_{\mathbb{Q}}$.
- Finally, Section 10.5 constructs the algebraic small resolutions of quintic mirror conifold in the category of algebraic spaces.

10.1 Properties of the conifold in the mirror family of the quintic

In this section, we will look at the construction of the quintic mirror conifold $\mathcal{W}_1$ which is a variety defined over $\mathbb{Q}$ and study the local structure of its double point.

The quintic conifold X_1 is given by

$$\sum_{i=0}^{4} x_i^5 - 5\prod_{i=0}^{4} x_i = 0, \tag{10.1}$$

whose singularity consists of 125 double points

$$\{\ (\zeta_5^{a_0}, \zeta_5^{a_1}, \zeta_5^{a_2}, \zeta_5^{a_3}, \zeta_5^{a_4}) : a_i \in \mathbb{Z}/5\mathbb{Z},\ \sum_{i=0}^{4} a_i = 0 \bmod 5\ \}, \tag{10.2}$$

and G acts transitively on the 125 double points. From last Chapter, the singular locus of $Y_1 = X_1/G$ consists of C_{ij}/G, P_{ijk}/G and a double point. From Section 9.2, the affine open subvariety $(U_4/G) \cap Y_1$ is isomorphic to the subvariety of $\operatorname{Spec} \mathbb{Q}[t_0, \cdots, t_4]$ defined by

$$t_4^5 - t_0\, t_1\, t_2\, t_3 = 0, \quad 1 + t_0 + t_1 + t_2 + t_3 - 5\, t_4 = 0, \tag{10.3}$$

and moreover it is isomorphic to the hypersurface of $\operatorname{Spec} \mathbb{Q}[t_1, \cdots, t_4]$ defined by

$$t_4^5 - t_1\, t_2\, t_3\, (5\, t_4 - t_1 - t_2 - t_3 - 1) = 0. \tag{10.4}$$

The coordinates of the double point of Y_1 are given by

$$t_1 = \cdots = t_4 = 1. \tag{10.5}$$

Let s_i be $t_i - 1$ and define the new coordinates u_i by

$$\begin{aligned} u_1 &:= s_1 + \frac{1}{2}\, s_2 + \frac{1}{2}\, s_3 - \frac{5}{2}\, s_4, \\ u_2 &:= \frac{1}{2}\, s_2 + \frac{1}{6}\, s_3 - \frac{5}{6}\, s_4, \\ u_3 &:= s_3 - \frac{5}{4}\, s_4, \\ u_4 &:= \frac{1}{2}\, s_4, \end{aligned} \tag{10.6}$$

then the equation 10.4 becomes

$$u_1^2 + 3\, u_2^2 + \frac{2}{3}\, u_3^2 + \frac{5}{2}\, u_4^2 + g(u_1, u_2, u_3, u_4) = 0, \tag{10.7}$$

where g is a rational polynomial whose monomials are of degree ≥ 3. With respect to the new coordinate u_i, the double point of Y_1 becomes the origin $O := (0, 0, 0, 0)$. Therefore the local structure of the double point of Y_1 is determined by equation 10.7. The singularities C_{ij}/G and P_{ijk}/G are resolved by the crepant resolution 9.36

$$\phi_1 : \mathcal{W}_1 \to Y_1, \tag{10.8}$$

hence the singularity of $\mathcal{W}_1$ is a double point defined over $\mathbb{Q}$ that will be called **dp** in this thesis. The double point of Y_1 does not lie in C_{ij}/G and P_{ijk}/G, therefore a local neighbourhood of dp in $\mathcal{W}_1$ is isomorphic to an open neighbourhood of the double point O of the hypersurface in $\mathbb{A}^4_{\mathbb{Q}}$ defined by equation 10.7.

From the adjunction formula [98], the dualizing sheaf ω_{X_1} of X_1 is trivial. There is a threeform Ω_{X_1} defined on the smooth locus of X_1 that is given by 9.39

$$\Omega_{X_1} = \left.\frac{dx_{4,0} \wedge dx_{4,1} \wedge dx_{4,2}}{x_{4,3}^4 - x_{4,0}\, x_{4,1}\, x_{4,2}}\right|_{X_1}, \tag{10.9}$$

which is nowhere vanishing, hence it is a global section of ω_{X_1} [51]. The threeform Ω_{X_1} is G-invariant, therefore it defines a threeform Ω_{Y_1} on the smooth locus of Y_1 that is also nowhere vanishing. Since the codimension of the singular locus of Y_1 is 2, so the existence of a nowhere vanishing threeform Ω_{Y_1} on its smooth locus also shows the dualizing sheaf ω_{Y_1} is trivial. In the affine open subvariety $(U_4/G) \cap Y_1$, the threeform Ω_{Y_1} is given by

$$\Omega_{Y_1} = \left.\frac{1}{125}\frac{t_3\, dt_0 \wedge dt_1 \wedge dt_2}{t_4^4\,(t_3 - t_4)}\right|_{Y_1}. \tag{10.10}$$

The resolution morphism ϕ_1 10.8 is crepant, hence the dualizing sheaf $\omega_{\mathcal{W}_1}$ of $\mathcal{W}_1$ is also trivial, and the threeform Ω_{Y_1} pulls back to a nowhere vanishing threeform $\Omega_{\mathcal{W}_1}$ on the smooth locus of $\mathcal{W}_1$.

In this thesis, the resolutions of $\mathcal{W}_1$ will play a very important role, and first let us look at its blow up. Let $\widetilde{\mathcal{W}_1}$ be the blow up of $\mathcal{W}_1$ at its double point dp, which is a smooth projective variety defined over $\mathbb{Q}$. So we get a fibered diagram [98]

$$\begin{tikzcd} D^1 \arrow[r, "i_{D^1}"] \arrow[d, "\beta'"] & \widetilde{\mathcal{W}_1} \arrow[d, "\beta"] \\ \mathrm{dp} \arrow[r, "i"] & \mathcal{W}_1 \end{tikzcd}, \tag{10.11}$$

where D^1 is the exceptional divisor of $\widetilde{\mathcal{W}_1}$. From the construction of blow up, D^1 is isomorphic to the quadratic surface of $\mathbb{P}^3_{\mathbb{Q}}$ defined by

$$U_1^2 + 3\,U_2^2 + \frac{2}{3}\,U_3^2 + \frac{5}{2}\,U_4^2 = 0, \tag{10.12}$$

where $\{U_i\}_{i=1}^4$ are the projective coordinates of $\mathbb{P}^3_{\mathbb{Q}}$.

Remark 10.1.1. *Under the blow up morphism β 10.11, the threeform $\Omega_{\mathcal{W}_1}$ pulls back to a threeform $\Omega_{\widetilde{\mathcal{W}_1}}$ on $\widetilde{\mathcal{W}_1}$, which is a section of the canonical bundle $\Omega^3_{\widetilde{\mathcal{W}_1}}$. Moreover $\Omega_{\mathcal{W}_1}$ only vanishes on D^1.*

10.2 The analytic small resolutions of the quintic mirror conifold

The $\mathbb{C}$-valued points of the quintic mirror conifold $\mathcal{W}_1$, denoted by $\mathcal{W}_1(\mathbb{C})$, is the analytic variety defined by $\mathcal{W}_1$. In this section, we will study the small resolutions of $\mathcal{W}_1(\mathbb{C})$ in the category of analytic varieties.

10.2.1 The analytic small resolutions of a double point

In $\mathcal{W}_1(\mathbb{C})$, there exists a local (analytic) neighbourhood of dp which is isomorphic to a local neighbourhood of the double point $O := (0, 0, 0, 0)$ of the hypersurface in $\mathbb{C}^4$ defined by

$$H : \; wz - xy = 0, \tag{10.13}$$

where (w, x, y, z) are the affine coordinates of $\mathbb{C}^4$. There exist two small resolutions of the double point O of H [98]:

1. The blow up of $\mathbb{C}^4$ at $w = x = 0$ is the hypersurface of $\mathbb{C}^4 \times \mathbb{P}^1_{\mathbb{C}}$ defined by

   $$w\,X - x\,W = 0, \tag{10.14}$$

   where (X, W) are the projective coordinates of $\mathbb{P}^1_{\mathbb{C}}$. The proper transform of H in this blow up, denoted by $\widehat{H}_1$, is given by

   $$\begin{aligned} w\,X - x\,W &= 0,\\ y\,X - z\,W &= 0. \end{aligned} \tag{10.15}$$

   The natural projection map $\mathbb{C}^4 \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}^4$ induces a morphism

   $$\kappa_{H,1} : \widehat{H}_1 \to H, \tag{10.16}$$

   which is an isomorphism on the open subset $\widehat{H}_1 - (\kappa_{H,1})^{-1}(O)$. The exceptional curve $(\kappa_{H,1})^{-1}(O)$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

2. The blow up of $\mathbb{C}^4$ at $w = y = 0$ is the hypersurface of $\mathbb{C}^4 \times \mathbb{P}^1_{\mathbb{C}}$ defined by

   $$w\,Y - y\,W = 0, \tag{10.17}$$

   where (Y, W) are the projective coordinates of $\mathbb{P}^1_{\mathbb{C}}$. The proper transform of H in this blow up, denoted by $\widehat{H}_2$, is given by

   $$\begin{aligned} w\,Y - y\,W &= 0,\\ x\,Y - z\,W &= 0. \end{aligned} \tag{10.18}$$

The natural projection map $\mathbb{C}^4 \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}^4$ induces a morphism

$$\kappa_{H,2} : \widehat{H}_2 \to H, \tag{10.19}$$

which is an isomorphism on the open subset $\widehat{H}_2 - (\kappa_{H,2})^{-1}(O)$. The exceptional curve $(\kappa_{H,2})^{-1}(O)$ is also isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Remark 10.2.1. *The two small resolutions $\widehat{H}_1$ and $\widehat{H}_2$ of H are not isomorphic, i.e. there does not exist an isomorphism from $\widehat{H}_1$ to $\widehat{H}_2$ such that the following diagram is commutative*

$$\begin{tikzcd} \widehat{H}_1 \arrow[rr, dotted] \arrow[dr, "\kappa_{H,1}"'] & & \widehat{H}_2 \arrow[dl, "\kappa_{H,2}"] \\ & H & \end{tikzcd} . \tag{10.20}$$

There does exist a birational map between $\widehat{H}_1$ and $\widehat{H}_2$ which is called Atiyah flop [98].

Let the blow up of H at O be $\widetilde{H}$, then the exceptional divisor of $\widetilde{H}$ is isomorphic to the quadratic surface in $\mathbb{P}^3_{\mathbb{C}}$ defined by [98]

$$WZ - XY = 0, \tag{10.21}$$

where (W, X, Y, Z) are the projective coordinates of $\mathbb{P}^3_{\mathbb{C}}$. Moreover, $\widetilde{H}$ is also the blow up of $\widehat{H}_i$ at its exceptional curve $(\kappa_{H,i})^{-1}(O)$, and the corresponding blow up morphism is denoted by

$$\lambda_{H,i} : \widetilde{H} \to \widehat{H}_i, \ i = 1, 2. \tag{10.22}$$

Therefore we get a commutative diagram

$$\begin{tikzcd} & \widetilde{H} \arrow[dl, "\lambda_{H,1}"'] \arrow[dr, "\lambda_{H,2}"] & \\ \widehat{H}_1 \arrow[dr, "\kappa_{H,1}"'] & & \widehat{H}_2 \arrow[dl, "\kappa_{H,2}"] \\ & H & \end{tikzcd} , \tag{10.23}$$

where $\kappa_{H,i} \circ \lambda_{H,i}$ is the blow up morphism $\widetilde{H} \to H$. The exceptional divisor of $\widetilde{H}$, i.e. the quadratic surface 10.21, is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, and the restriction of $\lambda_{H,i}$ to it is just the natural projection morphism

$$p_i : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}. \tag{10.24}$$

10.2.2 The analytic small resolutions of the quintic mirror conifold

Since the construction of (analytic) small resolution is local, parallel construction works for $\mathcal{W}_1(\mathbb{C})$, therefore we get two small resolutions of $\mathcal{W}_1(\mathbb{C})$ that will be denoted by $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$, $i = 1, 2$. Let the resolution morphism be

$$\kappa_i^{\mathrm{an}} : \widehat{\mathcal{W}}_1^{i,\mathrm{an}} \to \mathcal{W}_1(\mathbb{C}),\ i = 1, 2, \tag{10.25}$$

where the exceptional curve $(\kappa_i^{\mathrm{an}})^{-1}(\mathrm{dp})$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Here the notation "an" is to remind us that $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ is a complex manifold, not an algebraic variety. There also exists an Atiyah flop

$$\begin{tikzcd} \widehat{\mathcal{W}}_1^{1,\mathrm{an}} \arrow[rr, dotted] \arrow[dr, "\kappa_1^{\mathrm{an}}"'] & & \widehat{\mathcal{W}}_1^{2,\mathrm{an}} \arrow[dl, "\kappa_2^{\mathrm{an}}"] \\ & \mathcal{W}_1(\mathbb{C}) & \end{tikzcd} . \tag{10.26}$$

The $\mathbb{C}$-valued points of the blow up $\widetilde{\mathcal{W}}_1$, denoted by $\widetilde{\mathcal{W}}_1(\mathbb{C})$, is a projective complex manifold, and similarly $\widetilde{\mathcal{W}}_1(\mathbb{C})$ is also the blow up of $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ at its exceptional curves $(\kappa_i^{\mathrm{an}})^{-1}(\mathrm{dp})$. Let us denote this blow up morphism by

$$\lambda_i^{\mathrm{an}} : \widetilde{\mathcal{W}}_1(\mathbb{C}) \to \widehat{\mathcal{W}}_1^{i,\mathrm{an}},\ i = 1, 2, \tag{10.27}$$

and we get a commutative diagram

$$\begin{tikzcd} & \widetilde{\mathcal{W}}_1(\mathbb{C}) \arrow[dl, "\lambda_1^{\mathrm{an}}"'] \arrow[dr, "\lambda_2^{\mathrm{an}}"] & \\ \widehat{\mathcal{W}}_1^{1,\mathrm{an}} \arrow[dr, "\kappa_1^{\mathrm{an}}"'] & & \widehat{\mathcal{W}}_1^{2,\mathrm{an}} \arrow[dl, "\kappa_2^{\mathrm{an}}"] \\ & \mathcal{W}_1(\mathbb{C}) & \end{tikzcd} , \tag{10.28}$$

where the composition $\kappa_i^{\mathrm{an}} \circ \lambda_i^{\mathrm{an}}$ is the blow up morphism of $\mathcal{W}_1(\mathbb{C})$ at dp. The exceptional divisor of $\widetilde{\mathcal{W}}_1(\mathbb{C})$ is $D^1(\mathbb{C})$, i.e. the $\mathbb{C}$-valued points of D^1, which is isomorphic to the quadratic surface in $\mathbb{P}^3_{\mathbb{C}}$ defined by the equation 10.12. As before, $D^1(\mathbb{C})$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, and the restriction of λ_i^{an} to it is the natural projection map p_i in equation 10.24.

10.3 The conifold transition in the mirror family of the quintic

In order to study the arithmetic geometry of $\mathcal{W}_1$, we need to construct **algebraic** small resolutions of it such that their analytifications are $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$. At first one might hope to find a smooth surface in $\mathcal{W}_1$ (defined over a number field) that contains the double point dp, and then a small resolution of $\mathcal{W}_1$ can be constructed as the blow up of $\mathcal{W}_1$ at this smooth surface. However if a small resolution of $\mathcal{W}_1$ can be constructed in this way, then it must be projective as blow up is always projective [98]. But its analytification $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ cannot be projective as its exceptional curve is homologous to zero [23], so this method does not work. In this section, we will show why the exceptional curve of $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ is homologous to zero, which is an essential ingredient in Beilinson's conjecture.

Suppose Δ is a small (analytic) neighbourhood of the conifold point $\psi = 1$ in $\mathbb{P}^1_{\mathbb{C}}$, and let the mirror family 9.34 over Δ be denoted by

$$\pi_\Delta : \mathcal{W}(\mathbb{C})|_\Delta \to \Delta, \tag{10.29}$$

the only singular fiber of which is the quintic mirror conifold $\mathcal{W}_1$. Let the restriction of the mirror family to the punctured disc $\Delta^* := \Delta \setminus \{\psi = 1\}$ be denoted by

$$\pi_{\Delta^*} : \mathcal{W}(\mathbb{C})|_{\Delta^*} \to \Delta^*, \tag{10.30}$$

which is a smooth fibration between smooth complex manifolds. There exists a local system $V_\mathbb{Z}$ defined by π_{Δ^*}

$$V_\mathbb{Z} := \mathrm{R}^3\, \pi_{\Delta^*,*}\mathbb{Z}, \tag{10.31}$$

whose fiber over a point $\psi \in \Delta^*$ is $H^3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Z})$ (modulo torsion). The dual of $V_\mathbb{Z}$ will be denoted by $V_\mathbb{Z}^\vee$, whose fiber over ψ is $H_3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Z})$ (modulo torsion). There are also local systems given by

$$V_\mathbb{Q} := \mathrm{R}^3\, \pi_{\Delta^*,*}\mathbb{Q} = V_\mathbb{Z} \otimes \mathbb{Q},\ V_\mathbb{C} := \mathrm{R}^3\, \pi_{\Delta^*,*}\mathbb{C} = V_\mathbb{Z} \otimes \mathbb{C}, \tag{10.32}$$

whose fibers over ψ are $H^3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Q})$ and$H^3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{C}))$ respectively. From the Hodge diamond 9.35, the Betti number $b_3(\mathcal{W}_\psi(\mathbb{C}))$ of a smooth fiber is 4, hence by Poincaré duality there exists a symplectic basis $\{A_i(\psi), B_i(\psi)\}_{i=1}^2$ of $H_3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Z})$ (modulo torsion) with intersection pairings [23, 51]

$$A_a(\psi) \cdot B_b(\psi) = \delta_{ab}, \quad A_a(\psi) \cdot A_b(\psi) = 0, \quad B_a(\psi) \cdot B_b(\psi) = 0. \tag{10.33}$$

From the paper [23], the homology cycles $A_2(\psi)$ and $B_2(\psi)$ can be chosen in a way such that when $\psi \to 1$ we have:

1. The homology cycle $A_2(\psi)$, which is homeomorphic to S^3, shrinks to the double point of $\mathcal{W}_1(\mathbb{C})$.
2. The homology cycle $B_2(\psi)$, which is homeomorphic to the real torus $(S^1)^3$, becomes a homology cycle of $\mathcal{W}_1(\mathbb{C})$ that contains the double point dp.
3. The homology cycles $A_1(\psi)$ and $B_1(\psi)$ are chosen far away from $A_2(\psi)$ and $B_2(\psi)$ so that the limit $\psi \to 1$ will not affect them.

From Theorem 3.2 [87], the Betti number $b_3(\mathcal{W}_1(\mathbb{C}))$ is 3, and later we will show that

$$A_1|_{\psi=1},\ B_1|_{\psi=1},\ B_2|_{\psi=1} \tag{10.34}$$

form a basis of $H_3(\mathcal{W}_1(\mathbb{C}), \mathbb{Z})$ (modulo torsion). In the small resolution of $\mathcal{W}_1(\mathbb{C})$, the double point is resolved by a $\mathbb{P}^1_{\mathbb{C}}$, which is homeomorphic to S^2. This whole process is called conifold transition, which is summarised in the following diagram [23],

$$\begin{tikzcd} & \widehat{\mathcal{W}}_1^{i,\mathrm{an}} \arrow[d, "\kappa_i^{\mathrm{an}}"] \\ \mathcal{W}_\psi(\mathbb{C}) \arrow[r, "\psi\to 1"] & \mathcal{W}_1(\mathbb{C}) \end{tikzcd}. \tag{10.35}$$

Since the cycle $A_2(\psi)$ is nonzero in $H_3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Q})$, Theorem 3.2 of [87] implies that the exceptional curve of $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ must be homologous to zero. This interesting property has already been explained explicitly in the paper [23]. When $\psi \to 1$, the cycle $B_2(\psi)$ becomes a cycle $B_2|_{\psi=1}$ of $\mathcal{W}_1(\mathbb{C})$ which contains the double point. Then the exceptional curve $\mathbb{P}^1_{\mathbb{C}}$ is the boundary of the cycle $(\kappa_i^{\mathrm{an}})^{-1}(B_2|_{\psi=1})$. From this property we immediately deduce that $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ cannot be a Kähler manifold: suppose there exists a Kähler form J on $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$, then the "area" of the exceptional curve, i.e. $\int_{\mathbb{P}^1_{\mathbb{C}}} J$, cannot be zero. But this exceptional curve is a boundary, then Stokes theorem implies that this integration must be zero, thus we have reached a contradiction. This also shows that after small resolution, the homology cycle $(\kappa_i^{\mathrm{an}})^{-1}(B_2|_{\psi=1})$ is not closed any more. Theorem 3.2 of [87] shows the third Betti number of $\widehat{\mathcal{W}}_1^{i,\mathrm{an}}$ is

$$b_3(\widehat{\mathcal{W}}_1^{i,\mathrm{an}}) = 2, \tag{10.36}$$

while when $j \neq 3$, the Betti numbers satisfy

$$b_j(\mathcal{W}_\psi(\mathbb{C})) = b_j(\mathcal{W}_1(\mathbb{C})) = b_j(\widehat{\mathcal{W}}_1^{i,\mathrm{an}}),\ j \neq 3. \tag{10.37}$$

10.4 The Hilbert scheme of lines in a quadratic surface

The idea to construct an algebraic small resolution of $\mathcal{W}_1$ is to contract the exceptional divisor D^1 of $\widetilde{\mathcal{W}_1}$ to a conic, but first we need to study the Hilbert scheme of lines in a smooth quadratic surface of $\mathbb{P}^3_{\mathbb{Q}}$. The Hilbert scheme of lines in $\mathbb{P}^3_{\mathbb{Q}}$ is the Grassmannian $\mathbb{G}_{1,3}$, which is isomorphic to a quadratic hypersurface of $\mathbb{P}^5_{\mathbb{Q}}$ defined over $\mathbb{Q}$. A line L in $\mathbb{P}^3_{\mathbb{Q}}$ is determined by two distinct closed points, and suppose their coordinates are

$$(x_0, x_1, x_2, x_3), \quad (y_0, y_1, y_2, y_3), \tag{10.38}$$

then the Plücker coordinates p_{ij} of L are defined by

$$p_{ij} := x_i\, y_j - x_j\, y_i. \tag{10.39}$$

There are only 6 independent p_{ij} which are ordered in the way

$$(p_{01},\, p_{02},\, p_{03},\, p_{12},\, p_{13},\, p_{23}), \tag{10.40}$$

and they cannot all vanish because the two closed points are different. Moreover, L is uniquely determined by its Plücker coordinates 10.40, hence we get a well-defined injective map

$$\mathbb{G}_{1,3} \hookrightarrow \mathbb{P}^5_{\mathbb{Q}}, \tag{10.41}$$

which sends the Grassmannian $\mathbb{G}_{1,3}$ isomorphically onto the quadratic hypersurface of $\mathbb{P}^5_{\mathbb{Q}}$ defined by

$$p_{01}\, p_{23} - p_{02}\, p_{13} + p_{03}\, p_{12} = 0. \tag{10.42}$$

After a rational projective transformation, every smooth quadratic surface S of $\mathbb{P}^3_{\mathbb{Q}}$ that is defined over $\mathbb{Q}$ is of the form

$$S : X_0^2 + k_1 X_1^2 + k_2 X_2^2 + k_3 X_3^2 = 0, \text{ with } k_i \in \mathbb{Q} \text{ and } \prod_{i=1}^{3} k_i \neq 0, \tag{10.43}$$

where $(X_0, \cdots, X_3)$ are the projective coordinates of $\mathbb{P}^3_{\mathbb{Q}}$. We now show that the Hilbert scheme of lines in S has two disconnected components, both of which are conics defined over the quadratic field $F := \mathbb{Q}((k_1 k_2 k_3)^{1/2})$. The equation 10.43 can also be written into

$$(X_0 + i\, k_1^{1/2} X_1)(X_0 - i\, k_1^{1/2} X_1) + (k_2^{1/2} X_2 + i\, k_3^{1/2} X_3)(k_2^{1/2} X_2 - i\, k_3^{1/2} X_3) = 0. \tag{10.44}$$

Given two numbers a and b, the following two closed points

$$\Big(\frac{a+b}{2}, -\frac{i(a-b)}{2\,k_1}, \frac{b-a}{2\,k_2}, -\frac{i(a+b)}{2\,k_3}\Big),\ \Big(\frac{a-b}{2}, -\frac{i(a+b)}{2\,k_1}, \frac{a+b}{2\,k_2}, \frac{i(a-b)}{2\,k_3}\Big) \tag{10.45}$$

determines a line in S, the image of which under Plücker embedding 10.41 lies in the conic C_1 defined by

$$\begin{aligned} k_1\,p_{01} + (k_1k_2k_3)^{1/2}\,p_{23} = 0, \quad k_2\,p_{02} - (k_1k_2k_3)^{1/2}\,p_{13} &= 0,\\ k_3\,p_{03} + (k_1k_2k_3)^{1/2}\,p_{12} = 0, \quad k_1\,p_{01}^2 + k_2\,p_{02}^2 + k_3\,p_{03}^2 &= 0. \end{aligned} \tag{10.46}$$

This conic is defined over the quadratic field F, and it is a subvariety of the hypersurface 10.42 (over the field F).

Lemma 10.4.1. *Over the quadratic field F, there exists a map*

$$p_1 : S_F \to C_1, \text{ with } S_F := S \times_{\mathbb{Q}} F, \tag{10.47}$$

which is a fibration over C_1, and the fibers are conics.

Proof. We have a homomorphism between homogeneous rings

$$F[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}] \to F[X_0, X_1, X_2, X_3] \tag{10.48}$$

that is defined by

$$\begin{aligned} &p_{01} \to X_0^2 + k_1X_1^2, && p_{02} \to k_1X_1X_2 - \frac{(k_1k_2k_3)^{1/2}}{k_2}X_0X_3,\\ &p_{03} \to k_1X_1X_3 + \frac{(k_1k_2k_3)^{1/2}}{k_3}X_0X_2, && p_{12} \to -X_0X_2 - \frac{(k_1k_2k_3)^{1/2}}{k_2}X_1X_3,\\ &p_{13} \to -X_0X_3 + \frac{(k_1k_2k_3)^{1/2}}{k_3}X_1X_2, && p_{23} \to (k_1k_2k_3)^{1/2}(\frac{X_2^2}{k_3} + \frac{X_3^2}{k_2}). \end{aligned} \tag{10.49}$$

From equation 10.46, this homomorphism induces a map $p_1 : S_F \to C_1$. A fiber of p_1 is the intersection of a plane in $\mathbb{P}^3_{\mathbb{Q}}$ with S_F, which is a conic. □

By parallel construction, given two numbers a and b, the following two closed points

$$\Big(\frac{a+b}{2}, -\frac{i(a-b)}{2\,k_1}, \frac{b-a}{2\,k_2}, \frac{i(a+b)}{2\,k_3}\Big),\ \Big(\frac{a-b}{2}, -\frac{i(a+b)}{2\,k_1}, \frac{a+b}{2\,k_2}, -\frac{i(a-b)}{2\,k_3}\Big) \tag{10.50}$$

determines another line in S. The image of this line under the Plücker embedding 10.41 lies in the conic C_2 defined by

$$\begin{aligned} k_1\,p_{01} - (k_1k_2k_3)^{1/2}\,p_{23} = 0, \quad k_2\,p_{02} + (k_1k_2k_3)^{1/2}\,p_{13} &= 0,\\ k_3\,p_{03} - (k_1k_2k_3)^{1/2}\,p_{12} = 0, \quad k_1\,p_{01}^2 + k_2\,p_{02}^2 + k_3\,p_{03}^2 &= 0. \end{aligned} \tag{10.51}$$

This conic C_2 is defined over the field F, and it is a subvariety of the hypersurface 10.42 (over the field F).

Lemma 10.4.2. *Over the quadratic field F, there exists a map*

$$p_2 : S_F \to C_2, \tag{10.52}$$

which is a fibration over C_2 and the fibers are conics.

Proof. We have a homomorphism between homogeneous rings

$$F[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}] \to F[X_0, X_1, X_2, X_3], \tag{10.53}$$

that is defined by

$$\begin{aligned} &p_{01} \to X_0^2 + k_1X_1^2, && p_{02} \to k_1X_1X_2 + \frac{(k_1k_2k_3)^{1/2}}{k_2}X_0X_3, \\ &p_{03} \to k_1X_1X_3 - \frac{(k_1k_2k_3)^{1/2}}{k_3}X_0X_2, && p_{12} \to -X_0X_2 + \frac{(k_1k_2k_3)^{1/2}}{k_2}X_1X_3, \\ &p_{13} \to -X_0X_3 - \frac{(k_1k_2k_3)^{1/2}}{k_3}X_1X_2, && p_{23} \to -(k_1k_2k_3)^{1/2}(\frac{X_2^2}{k_3} + \frac{X_3^2}{k_2}). \end{aligned} \tag{10.54}$$

From equation 10.51, this homomorphism induces a map $p_2 : S_F \to C_2$. A fiber of p_2 is the intersection of a plane in $\mathbb{P}^3_{\mathbb{Q}}$ with S_F, which is a conic. □

Remark 10.4.3. *Over the algebraic closure $\overline{\mathbb{Q}}$, the quadratic surface $S_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}}$, hence its Hilbert scheme of lines is the disjoint union of two $\mathbb{P}^1_{\overline{\mathbb{Q}}}$. The fibration of $S_{\overline{\mathbb{Q}}}$ over $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ is just the natural projection $\mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$.*

The Galois group $\mathrm{Gal}(F/\mathbb{Q})$ consists of two elements $\{1, \iota\}$, while the involution ι sends $(k_1\, k_2\, k_3)^{1/2}$ to $-(k_1\, k_2\, k_3)^{1/2}$. By restriction of base field, S_F and C_i are viewed as varieties defined over $\mathbb{Q}$ by

$$S_{F/\mathbb{Q}} : S_F = S \times_{\mathbb{Q}} F \to F \to \mathbb{Q}, \quad C_{i/\mathbb{Q}} : C_i \to F \to \mathbb{Q}. \tag{10.55}$$

The involution ι defines an automorphism of $S_{F/\mathbb{Q}}$ and an isomorphism between $C_{1/\mathbb{Q}}$ and $C_{2/\mathbb{Q}}$, which fit into a diagram

$$\begin{tikzcd} S_{F/\mathbb{Q}} \arrow[r, "\iota"] \arrow[d, "p_1"] & S_{F/\mathbb{Q}} \arrow[d, "p_2"] \\ C_{1/\mathbb{Q}} \arrow[r, "\iota"] & C_{2/\mathbb{Q}} \end{tikzcd}, \tag{10.56}$$

the commutativity of which follows directly from the constructions of p_1 and p_2.

10.5 The algebraic small resolutions of the quintic mirror conifold

The exceptional divisor D^1 of $\widetilde{\mathcal{W}}_1$ is isomorphic to the quadratic surface of $\mathbb{P}^3_{\mathbb{Q}}$ defined by the equation 10.12, therefore the Hilbert scheme of lines in D^1 has two disconnected components C_1 and C_2, both of which are conics defined over the field $F := \mathbb{Q}(5^{1/2})$. Furthermore there are morphisms

$$p_i : D^1_F \to C_i,\ i = 1,2,\quad D^1_F := D^1 \times_{\mathbb{Q}} F, \tag{10.57}$$

which are fibrations over C_i, and the fibers are conics. By extension of base field to F, we got the following blow-up morphism over F

$$\beta : \widetilde{\mathcal{W}}_{1,F} \to \mathcal{W}_{1,F},\ \text{with}\ \widetilde{\mathcal{W}}_{1,F} := \widetilde{\mathcal{W}}_1 \times_{\mathbb{Q}} F,\ \mathcal{W}_{1,F} := \mathcal{W}_1 \times_{\mathbb{Q}} F. \tag{10.58}$$

The idea to construct algebraic small resolutions of $\mathcal{W}_1$ is to contract the exceptional divisor D^1_F to the conics $C_i, i = 1,2$, however such contractions only exist in the category of algebraic spaces over F.

Remark 10.5.1. *An algebraic variety is constructed by patching together affine varieties using their Zariski topologies. Intuitively and non-rigorously, if we patch together affine varieties using their étale topologies, we get algebraic spaces. We realise that it will be more confusing if we give the definitions of algebraic spaces here, so instead we will refer the readers to the chapters on algebraic spaces in the Stacks Project [104]. For the category of complete algebraic spaces defined over $\mathbb{C}$, the analytification functor is fully faithful. The essential image of this functor consists of compact analytic spaces whose complex dimension equals the transcendence degree of the field of meromorphic functions (these varieties are Moishezon spaces).*

Proposition 10.5.2. *Over the field F, there exist algebraic spaces $\widehat{\mathcal{W}}^i_1$, defining ideal $I_i \subset \mathscr{O}_{\widehat{\mathcal{W}}^i_1}$ whose vanishing locus is C_i, and morphisms*

$$\lambda_i : \widetilde{\mathcal{W}}_{1,F} \to \widehat{\mathcal{W}}^i_1 \tag{10.59}$$

such that the restriction of λ_i to D^1_F is the map p_i in equation 10.57, while λ_i is an isomorphism outside D^1_F.

Proof. The conormal bundle $\mathcal{N}^\vee_{D^1_F}$ of the exceptional divisor D^1_F is the restriction of the ample line bundle $\mathcal{O}_{\mathbb{P}^3_F}(1)$ to it, hence $\mathcal{N}^\vee_{D^1_F}$ is also ample. Over the algebraically closed field $\overline{\mathbb{Q}}$, $\mathcal{N}^\vee_{D^1_F}$ is isomorphic to [98]

$$\mathcal{N}^\vee_{D^1_F} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1), \tag{10.60}$$

where we have used the property that $D^1_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}}$. Over $\overline{\mathbb{Q}}$, the pushforward $(p_i)_*(\mathcal{N}^\vee_{D^1_F})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1)$, while the higher pushforward $\mathrm{R}^n(p_i)_*(\mathcal{N}^\vee_{D^1_F})$ is zero when $n \geq 1$. Then this proposition is an immediate consequence of Theorem 3.1 and 6.2 of Artin's paper [2], supplemented by Corollary 1 of Mazur's paper [77]. □

Moreover, $\widetilde{\mathcal{W}}_{1,F}$ is the blow up of $\widehat{\mathcal{W}}^i_1$ at its exceptional curve C_i. From the construction of contraction, the analytification of the algebraic space $\widehat{\mathcal{W}}^i_1$ is just the (non-Kähler) complex manifold $\widehat{\mathcal{W}}^{i,\mathrm{an}}_1$ (as the notation has already suggested), therefore we get the following crucial observation.

Lemma 10.5.3. *The exceptional curve C_i of $\widehat{\mathcal{W}}^i_1$ is homologous to zero.*

Proposition 10.5.4. *There exists a morphism κ_i in the category of algebraic spaces over F*

$$\kappa_i : \widehat{\mathcal{W}}^i_1 \to \mathcal{W}_{1,F}, \tag{10.61}$$

which contracts the exceptional curve C_i of $\widehat{\mathcal{W}}^i_1$ to the double point of $\mathcal{W}_{1,F}$.

Proof. Over the field $\overline{\mathbb{Q}}$, the conormal bundle of the defining ideal $I_i \subset \mathscr{O}_{\widehat{\mathcal{W}}^i_1}$ on $C_i \times \overline{\mathbb{Q}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is $\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(1)$, which is an ample vector bundle [58]. Then this proposition is an immediate consequence of Corollary 2 of [77]. □

Therefore we have obtained a commutative diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{W}}_{1,F} \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\
\widehat{\mathcal{W}}^1_1 \arrow[dr, "\kappa_1"'] & & \widehat{\mathcal{W}}^2_1 \arrow[dl, "\kappa_2"] \\
 & \mathcal{W}_{1,F} &
\end{tikzcd} , \tag{10.62}$$

whose analytification (which depends on the embedding $\sigma : F \to \mathbb{C}$) is the commutative diagram 10.28. From the construction of λ_i and κ_i, we have

$$\beta = \kappa_1 \circ \lambda_1 = \kappa_2 \circ \lambda_2. \tag{10.63}$$

By restriction of base field, we get algebraic spaces defined over $\mathbb{Q}$ given by

$$\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} : \widetilde{\mathcal{W}}_{1,F} \to F \to \mathbb{Q}, \quad \widehat{\mathcal{W}}^i_{1/\mathbb{Q}} : \widehat{\mathcal{W}}^i_1 \to F \to \mathbb{Q}. \tag{10.64}$$

The Galois group of $F = \mathbb{Q}(5^{1/2})$ has two elements $\{1,\ \iota\}$, where the involution ι sends $5^{1/2}$ to $-5^{1/2}$. The involution ι defines an automorphism of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ and an isomorphism between $\widehat{\mathcal{W}}^1_{1/\mathbb{Q}}$ and $\widehat{\mathcal{W}}^2_{1/\mathbb{Q}}$, hence we get a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \arrow[r, "\iota"] \arrow[d, "\lambda_1"] & \widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \arrow[d, "\lambda_2"] \\
\widehat{\mathcal{W}}^1_{1/\mathbb{Q}} \arrow[r, "\iota"] & \widehat{\mathcal{W}}^2_{1/\mathbb{Q}}
\end{tikzcd} . \qquad (10.65)$$

The commutativity of this diagram follows from the construction of λ_i and the commutativity of the diagram 10.56.

Summary: in this chapter, we have discussed the construction of the quintic mirror conifold, which is a threefold defined over $\mathbb{Q}$ whose singularity consists of a rational double point. We have also studied the local structure of this double point and its blow up. The main result of this chapter is the construction of the two small resolutions of the quintic mirror conifold, which are algebraic spaces defined over $\mathbb{Q}(5^{1/2})$. The two small resolutions are obtained from the contractions of the exceptional divisor of the blow up of the quintic mirror conifold at its rational double point. The exceptional curves of the two small resolutions are conics defined over $\mathbb{Q}(5^{1/2})$ that are homologous to zero, which will be crucial when we study the interesting connections to **Beilinson's Conjecture** in Chapter 12. Furthermore there also exists a rational algebraic threeform defined on the smooth locus of the quintic mirror conifold, whose periods will play an important role when we discuss **Beilinson's Conjecture**.

Chapter 11

The étale cohomology groups of the resolutions of the quintic mirror conifold

In this chapter, we will study the étale cohomology groups of the blow up $\widetilde{\mathcal{W}_1}$ and the small resolutions $\widehat{\mathcal{W}_1^i}$ of the quintic mirror conifold $\mathcal{W}_1$. We will discuss the extensions induced by the exceptional curves of the small resolutions $\widehat{\mathcal{W}_1^i}$, which is an essential ingredient in Beilinson's conjecture. The structure of this chapter is as follows:

- Section 11.1 is a brief overview of the étale h-topology and étale h-cohomology theory [46].
- Section 11.2 studies the long exact sequence of étale cohomology groups associated to the blow up $\beta : \widetilde{\mathcal{W}_1} \to \mathcal{W}_1$, and it will also discuss the zeta functions of $\widetilde{\mathcal{W}_1}$.
- Section 11.3 studies the compactly supported cohomology of the smooth locus of quintic mirror conifold.
- Finally, Section 11.4 discusses the extensions induced by the exceptional curves of the small resolutions $\widehat{\mathcal{W}_1^i}$, which lies in the group
$$\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(5^{1/2}))}\big(\mathbb{Q}_\ell(0), H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2)\big). \tag{11.1}$$
These extensions will be crucial in the test of Beilinson's conjecture.

11.1 The étale h-cohomology group

In this section, we will very briefly discuss the étale h-cohomology theory that will be needed in this thesis, and the readers are referred to the original paper [46] for proofs and further detail. First, the étale h-topology is a Grothendieck topology on the category of schemes which is finer than the usual étale topology, but nevertheless possesses many advantages.

Definition 11.1.1. *Given a suitable subcategory of the category of schemes, the eh-topology, short for étale h-topology, is the Grothendieck topology whose opens are the same as the opens of the étale topology, while its coverings are generated by the following types of coverings:*

1. *Usual étale coverings.*
2. *Abstract blow ups. More precisely, suppose we have a fibered diagram*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "\beta'"] & X' \arrow[d, "\beta"] \\ Z \arrow[r, "i"] & X \end{tikzcd}, \tag{11.2}$$

where i is a closed embedding and β is a proper morphism that is an isomorphism from $X'-Z'$ to $X-Z$. Then we say the morphisms

$$\{X' \xrightarrow{\beta} X, Z \xrightarrow{i} X\} \tag{11.3}$$

form a covering of X.

For a perfect field k, suppose Sch/k is the category that consists of separated schemes of finite type over k, then we equip Sch/k with the eh-topology and denote this site by $(\mathrm{Sch}/k)_{\mathrm{eh}}$. Let $(\mathrm{Sch}/k)^{\mathrm{sh}}_{\mathrm{eh}}$ be the topos of eh-sheaves on Sch/k, which has enough injectives. Given an eh-sheaf $\mathscr{F} \in (\mathrm{Sch}/k)^{\mathrm{sh}}_{\mathrm{eh}}$ and a scheme $X \in \mathrm{Sch}/k$, the eh-cohomology group $H^i_{\mathrm{eh}}(X,\mathscr{F}), i \in \mathbb{Z}$ is defined by the derived functors of the global-section functor

$$\mathscr{F} \to \Gamma(X,\mathscr{F}). \tag{11.4}$$

A very important property of eh-cohomology theory is that every abstract blow up 11.2 yields a long exact sequence of eh-cohomology groups.

Proposition 11.1.2. *For every abstract blow up 11.2, there exists a long exact sequence of eh-chomology groups*

$$\cdots \longrightarrow H^i_{eh}(X,\mathscr{F}) \xrightarrow{(i^*,\beta^*)} H^i_{eh}(Z,\mathscr{F}) \oplus H^i_{eh}(X',\mathscr{F}) \xrightarrow{\beta'^*-i'^*} H^i(Z',\mathscr{F}) \longrightarrow \cdots . \tag{11.5}$$

Proof. Proposition 3.2 of [46]. □

While for the locally constant torsion sheaves, the eh-cohomology groups are the same as the étale cohomology groups.

Lemma 11.1.3. *For a scheme $X \in Sch/k$, there is an isomorphism*

$$H^i_{et}(X, \mu_m^{\otimes n}) \simeq H^i_{eh}(X, \mu_m^{\otimes n}), \ n \in \mathbb{Z}, \tag{11.6}$$

where the characteristic of k does not divide m.

Proof. Theorem 3.6 of [46]. $\square$

11.2 The étale cohomology groups of the blow up

From Proposition 11.1.2 and Lemma 11.1.3, the blow up diagram 10.11 yields a long exact sequence of étale cohomology groups

$$\cdots \to H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \xrightarrow{(i^*,\beta^*)} H^n_{\text{ét}}(\text{dp}, \mathbb{Q}_\ell) \oplus H^n_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \xrightarrow{\beta'^* - i^*_{D^1}} H^n_{\text{ét}}(D^1_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to \cdots. \tag{11.7}$$

Since over the field $\overline{\mathbb{Q}}$, $D^1_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}}$, therefore we have

$$H^n_{\text{ét}}(D^1_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = 0, \ \text{when } n \neq 0, 2, 4, \tag{11.8}$$

therefore the long exact sequence 11.7 gives us an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\beta^*"] & H^2_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "-i^*_{D^1}"] & H^2_{\text{ét}}(D^1_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) node[below, pos=0.75]{\scriptstyle \delta} -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\beta^*"] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r] & 0
\end{tikzcd}. \tag{11.9}$$

When $p \neq 5$, both D^1 and $\widetilde{\mathcal{W}}_1$ have good reductions at it. From Proposition IV 5 of [92], the characteristic polynomial $P_p(h^2(D^1), T)$ is given by

$$P_p(h^2(D^1), T) = (1 - pT)(1 - \chi(p)\, pT), \ p \neq 5, \tag{11.10}$$

where χ is the real Dirichlet character $(\cdot/5)$. Therefore from the exact sequence 11.9 and the zeta functions of $\mathcal{W}_1$ in equation 1.3, we get

$$P_p(h^2(\widetilde{\mathcal{W}}_1), T) = (1 - pT)^{102}, \ \ P_p(h^3(\widetilde{\mathcal{W}}_1), T) = 1 - a_f(p)\, T + p^3 T^2, \ p \neq 5. \tag{11.11}$$

The exact sequence 11.7 also shows that the Betti numbers of $\widetilde{\mathcal{W}}_1$ are given by

$$b^0(\widetilde{\mathcal{W}}_1) = b^6(\widetilde{\mathcal{W}}_1) = 1, \ b^1(\widetilde{\mathcal{W}}_1) = b^5(\widetilde{\mathcal{W}}_1) = 0, \ b^2(\widetilde{\mathcal{W}}_1) = b^4(\widetilde{\mathcal{W}}_1) = 102, \ b^3(\widetilde{\mathcal{W}}_1) = 2, \tag{11.12}$$

while Remark 10.1.1 tells us

$$h^{3,0}(\widetilde{\mathcal{W}_1}) = h^{0,3}(\widetilde{\mathcal{W}_1}) = 1, \quad h^{2,1}(\widetilde{\mathcal{W}_1}) = h^{1,2}(\widetilde{\mathcal{W}_1}) = 0, \tag{11.13}$$

so $\widetilde{\mathcal{W}_1}$ is a rigid threefold, but it is not Calabi-Yau any more. The exact sequence 11.9 also shows that β^* is surjective, so its kernel, denoted by $\ker \beta^*$, is a one dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Later in this section we will show that as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(5^{1/2}))$, $\ker \beta^*$ is isomorphic to $\mathbb{Q}_\ell(-1)$, hence as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\ker \beta^*$ must be of the form

$$\ker \beta^* \simeq \mathbb{Q}_\ell(-1) \otimes \epsilon, \tag{11.14}$$

where ϵ is a real Dirichlet character. But there are only two possibilities: the trivial character and $\chi = (\cdot/5)$, hence the zeta functions of $\mathcal{W}_1$ in equation 1.3 immediately implies that ϵ must be χ.

Let U be the smooth open subvariety $\mathcal{W}_1 - \mathrm{dp}$ of $\mathcal{W}_1$, which also equals $\widetilde{\mathcal{W}_1} - D^1$. The inclusion morphism of U into $\mathcal{W}_1$ (resp. $\widetilde{\mathcal{W}_1}$) will be denoted by j (resp. $\widetilde{j}$), and they form a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\widetilde{j}"] \arrow[dr, "j"'] & \widetilde{\mathcal{W}_1} \arrow[d, "\beta"] \\ & \mathcal{W}_1 \end{tikzcd}, \tag{11.15}$$

which induces a commutative diagram of étale cohomology groups

$$\begin{tikzcd} H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\beta^*"] \arrow[dr, "j^*"] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[d, "\widetilde{j}^*"] \\ & H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \end{tikzcd}. \tag{11.16}$$

Lemma 11.2.1. *The homomorphism $\widetilde{j}^*$ in 11.16 is injective, hence the image of $H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ in $H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ under j^* is identified with $H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.*

Proof. The inclusion $\widetilde{j}$ induces a long exact sequence [78]

$$\cdots \to H^3_{\text{ét},D^1_{\overline{\mathbb{Q}}}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \xrightarrow{\widetilde{j}^*} H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to H^4_{\text{ét},D^1_{\overline{\mathbb{Q}}}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to \cdots. \tag{11.17}$$

The Gysin sequence yields an isomorphism [78]

$$H^3_{\text{ét},D^1_{\overline{\mathbb{Q}}}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq H^1_{\text{ét}}(D^1_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(-1) = 0, \tag{11.18}$$

hence the homomorphism $\widetilde{j}^*$ in this exact sequence is injective. □

11.3 The compactly supported cohomology groups

In this section we will discuss the compactly supported cohomology of the smooth open subvariety $U = \mathcal{W}_1 - \mathrm{dp}$. First, let us choose an embedding

$$\sigma : F = \mathbb{Q}(5^{1/2}) \to \overline{\mathbb{Q}}. \tag{11.19}$$

In Section 10.5, we have constructed a pair of small resolutions of $\mathcal{W}_1$ over F

$$\kappa_i : \widehat{\mathcal{W}_1^i} \to \mathcal{W}_{1,F},\ i = 1, 2,\ \text{with } \mathcal{W}_{1,F} = \mathcal{W}_1 \times_{\mathbb{Q}} F, \tag{11.20}$$

and the exceptional curve $C_i = \kappa_i^{-1}(\mathrm{dp})$ is a conic defined over F. Let U_F be $U \times_{\mathbb{Q}} F$, and it is a smooth open subvariety of $\widehat{\mathcal{W}_1^i}$ while the inclusion morphism is denoted by

$$U_F = \widehat{\mathcal{W}_1^i} - C_i, \quad j_i : U_F \hookrightarrow \widehat{\mathcal{W}_1^i}. \tag{11.21}$$

So we get a commutative diagram

$$\begin{tikzcd} U_F \arrow[r, "j_i"] \arrow[dr, "j"'] & \widehat{\mathcal{W}_1^i} \arrow[d, "\kappa_i"] \\ & \mathcal{W}_{1,F} \end{tikzcd}, \tag{11.22}$$

where by abuse of notation the inclusion morphism of U_F into $\mathcal{W}_{1,F}$ has also been denoted by j. From Section 10.5, we have a commutative diagram

$$\begin{tikzcd} D_F^1 \arrow[r, "i_{D_F^1}"] \arrow[d, "p_i"] & \widetilde{\mathcal{W}}_{1,F} \arrow[d, "\lambda_i"] \\ C_i \arrow[r, "i_{C_i}"] & \widehat{\mathcal{W}_1^i} \end{tikzcd}. \tag{11.23}$$

The following isomorphism will be important in this thesis.

Lemma 11.3.1.

$$H^n_{\acute{e}t,c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq H^n_{\acute{e}t}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),\ \forall\ n \geq 2. \tag{11.24}$$

Proof. The compactly supported étale cohomology group of U is given by [79]

$$H^n_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) := H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, j_!(\mathbb{Z}/\ell\mathbb{Z})) \tag{11.25}$$

where j is the inclusion morphism of U into $\mathcal{W}_1$. The following pair of maps

$$U \xrightarrow{j} \mathcal{W}_1 \xleftarrow{i} \mathrm{dp} \tag{11.26}$$

induces a pair of maps

$$U_{\overline{\mathbb{Q}}} \xrightarrow{j} \mathcal{W}_{1,\overline{\mathbb{Q}}} \xleftarrow{i} \mathrm{dp}, \tag{11.27}$$

which yields a short exact sequence of étale sheaves on $\mathcal{W}_{1,\overline{\mathbb{Q}}}$ [79]

$$0 \longrightarrow j_!j^*(\mathbb{Z}/\ell\mathbb{Z}) \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \longrightarrow i_*i^*(\mathbb{Z}/\ell\mathbb{Z}) \longrightarrow 0\ . \tag{11.28}$$

Since $i_*i^*(\mathbb{Z}/\ell\mathbb{Z})$ is a skyscraper sheaf on $\mathcal{W}_{1,\overline{\mathbb{Q}}}$, therefore the long exact sequence of cohomology groups produced by this short exact sequence induces an isomorphism

$$H^n_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, j_!j^*(\mathbb{Z}/\ell\mathbb{Z})) \simeq H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),\ \forall\ n \geq 2. \tag{11.29}$$

□

Both $H^n_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ and $H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ are continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the isomorphism in Lemma 11.3.1 is an isomorphism between Galois representations. The étale cohomology group $H^n_{\text{ét},c}(U_F \times_\sigma \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ is just $H^n_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ viewed as a representation of the closed subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. By abuse of notations, from now on $H^n_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ also means $H^n_{\text{ét},c}(U_F \times_\sigma \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$, which should clear from context. Similarly, $H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ (viewed as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$) also means $H^3_{\text{ét}}(\mathcal{W}_{1,F} \times_\sigma \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$.

Lemma 11.3.2. *The following diagram of continuous representations of Gal($\overline{\mathbb{Q}}/F$) is commutative*

$$\begin{tikzcd}
H^3_{\acute{e}t,c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\simeq"] \arrow[rd] & H^3_{\acute{e}t}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[d, "\kappa_i^*"] \\
& H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)
\end{tikzcd} \quad . \tag{11.30}$$

Proof. We just need to show the following diagram is commutative

$$\begin{tikzcd}
H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\simeq"] \arrow[rd] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) \arrow[d, "\kappa_i^*"] \\
& H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})
\end{tikzcd} \quad . \tag{11.31}$$

The inclusion map $j : U_{\overline{\mathbb{Q}}} \hookrightarrow \mathcal{W}_{1,\overline{\mathbb{Q}}}$ defines an injective homomorphism of étale sheaves on $\mathcal{W}_{1,\overline{\mathbb{Q}}}$

$$j_!j^*(\mathbb{Z}/\ell\mathbb{Z}) \hookrightarrow \mathbb{Z}/\ell\mathbb{Z}, \tag{11.32}$$

which induces a commutative diagram of complexes of étale sheaves on $\mathcal{W}_{1,\overline{\mathbb{Q}}}$

$$\begin{tikzcd}
j_!(\mathbb{Z}/\ell\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Z}/\ell\mathbb{Z} \arrow[d] \\
\mathrm{R}\kappa_{i,*}\kappa_i^* j_!(\mathbb{Z}/\ell\mathbb{Z}) \arrow[r] & \mathrm{R}\kappa_{i,*}(\mathbb{Z}/\ell\mathbb{Z})
\end{tikzcd} . \tag{11.33}$$

We have a fibered diagram

$$\begin{tikzcd}
U_F \arrow[r, "j_i"] \arrow[d, "\mathrm{Id}"] & \widehat{\mathcal{W}}_1^i \arrow[d, "\kappa_i"] \\
U_F \arrow[r, "j"] & \mathcal{W}_{1,F}
\end{tikzcd} , \tag{11.34}$$

and since extension by zero commutes with base change [78, 105], we get an isomorphism of étale sheaves on $\widehat{\mathcal{W}}_1^i$

$$\kappa_i^* j_!(\mathbb{Z}/\ell\mathbb{Z}) = j_{i,!}(\mathbb{Z}/\ell\mathbb{Z}). \tag{11.35}$$

So we get a commutative diagram

$$\begin{tikzcd}
H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, j_!(\mathbb{Z}/\ell\mathbb{Z})) \arrow[r] \arrow[d] & H^n_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) \arrow[d, "\kappa_i^*"] \\
H^n_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, j'_!(\mathbb{Z}/\ell\mathbb{Z})) \arrow[r] & H^n_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})
\end{tikzcd} . \tag{11.36}$$

From the proof of Proposition 3.1 in Chapter VI of [78], the following homomorphism between complex of étale sheaves

$$j_!(\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{R}\kappa_{i,*} j_{i,!}(\mathbb{Z}/\ell\mathbb{Z}) \tag{11.37}$$

is in fact a quasi-isomorphism. Hence the left vertical homomorphism in 11.36 is an isomorphism, which is the property used to show the definition of $H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ is independent of the choice of a complete variety which contains U_F. Thus the commutativity of the diagram 11.30 follows from the commutativity of the diagram in equation 11.36. □

11.4 The extensions induced by the exceptional curves of the small resolutions

In this section, we will discuss the extensions induced by the exceptional curves of the small resolutions of $\mathcal{W}_1$, which will be very crucial in the test of Beilinson's conjecture. There is a long exact sequence of étale cohomology groups associated to the pair $(\widehat{\mathcal{W}}^i_1, C_i)$, part of which is [78]

$$\begin{tikzcd} H^3_{\text{ét},C_{i,\overline{\mathbb{Q}}}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \arrow[r] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \arrow[r] & H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \arrow[dll, rounded corners] \\ H^4_{\text{ét},C_{i,\overline{\mathbb{Q}}}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \arrow[r, "\delta"] & H^4_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) & \end{tikzcd}. \tag{11.38}$$

The Gysin sequence gives us

$$H^3_{\text{ét},C_{i,\overline{\mathbb{Q}}}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) = 0, \;\; H^4_{\text{ét},C_{i,\overline{\mathbb{Q}}}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \simeq H^0_{\text{ét}}(C_{i,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(-2) = \mathbb{Q}_\ell(-2), \tag{11.39}$$

where we have used the property that C_i is a conic defined over the field F. From last chapter, we know the cycle C_i is homologous to zero, thus from the definition of cycle class map, the homomorphism δ is zero. Hence we get a short exact sequence

$$0 \longrightarrow H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \longrightarrow H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-2) \longrightarrow 0, \tag{11.40}$$

whose dual is

$$0 \longrightarrow \mathbb{Q}_\ell(2) \longrightarrow H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(3) \longrightarrow H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(3) \longrightarrow 0. \tag{11.41}$$

Here we have used Poincaré duality to deduce that [78]

$$H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \simeq H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^\vee(-3), \;\; H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \simeq H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^\vee(3) \tag{11.42}$$

Remark 11.4.1. *The surjective homomorphism from the étale cohomology group* $H^3_{\acute{e}t,c}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(3)$ *to* $H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)(3)$ *in the short exact sequence 11.41 is in fact the natural morphism given by*

$$H^3_{\acute{e}t,c}(U_{\overline{\mathbb{Q}}},\mathbb{Z}/\ell\mathbb{Z}) = H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},j_{i,!}(\mathbb{Z}/\ell\mathbb{Z})) = H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},j_{i,!}j_i^*(\mathbb{Z}/\ell\mathbb{Z})) \to H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Z}/\ell\mathbb{Z}). \tag{11.43}$$

After a Tate twist by $\mathbb{Q}_\ell(-3)$, the short exact sequence 11.41 becomes

$$0 \longrightarrow \mathbb{Q}_\ell(-1) \xrightarrow{v_i} H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \longrightarrow H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \longrightarrow 0. \tag{11.44}$$

From Lemma 11.3.2, we can replace the group $H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$ in the short exact sequence with the group $H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$, hence we get the following important result.

Proposition 11.4.2. *The small resolution κ_i induces a short exact sequence of representations of Gal$(\overline{\mathbb{Q}}/F)$*

$$0 \longrightarrow \mathbb{Q}_\ell(-1) \xrightarrow{v_i} H^3_{\acute{e}t}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \xrightarrow{\kappa_i^*} H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longrightarrow 0, \tag{11.45}$$

where by abuse of notations the injective homomorphism from $\mathbb{Q}_\ell(-1)$ to $H^3_{\acute{e}t}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ has also been denoted by v_i.

From the commutative diagram in equation 10.62, $\beta : \widetilde{\mathcal{W}}_{1,F} \to \mathcal{W}_{1,F}$ has a factorisation given by

$$\beta : \widetilde{\mathcal{W}}_{1,F} \xrightarrow{\lambda_i} \widehat{\mathcal{W}}^i_1 \xrightarrow{\kappa_i} \mathcal{W}_{1,F}, \tag{11.46}$$

which yields a commutative diagram of continuous representations of Gal$(\overline{\mathbb{Q}}/F)$

$$\begin{tikzcd}
H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "k_i^*"] \arrow[d, "\text{Id}"] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[d, "\lambda_i^*"] \\
H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\beta^*"] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)
\end{tikzcd} . \tag{11.47}$$

Since the homomorphisms κ_i^* and β^* are both surjective and the dimension of the group $H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ equals the dimension of $H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, we have the following important lemma.

Lemma 11.4.3. *The pull back homomorphism*

$$\lambda_i^* : H^3_{\acute{e}t}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \to H^3_{\acute{e}t}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \tag{11.48}$$

is an isomorphism.

So the vertical homomorphisms of the commutative diagram 11.47 are both isomorphisms, hence this commutative diagram can be extended to the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}_\ell(-1) \arrow[r, "v_i"] \arrow[d, "\text{Id}"] & H^3_{\text{ét},c}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r] \arrow[d, "\simeq"] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r] \arrow[d, "\text{Id}"] & 0 \\
0 \arrow[r] & \mathbb{Q}_\ell(-1) \arrow[r, "v_i"] \arrow[d] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "k_i^*"] \arrow[d, "\text{Id}"] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r] \arrow[d, "\lambda_i^*"] & 0 \cdot \\
0 \arrow[r] & L_i \arrow[r, "v'_i"] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r, "\beta^*"] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \arrow[r] & 0
\end{tikzcd} \tag{11.49}$$

Here the injective homomorphism $\upsilon_i' : L_i \to H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is constructed in the way such that this digram is commutative. Take the dual of 11.49 and then take a Tate twist by $\mathbb{Q}_\ell(-1)$, we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r] & H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r, "\upsilon_i^\vee"] & \mathbb{Q}_\ell(0) \arrow[r] & 0 \\
0 \arrow[r] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r, "(\kappa_i^*)^\vee"] \arrow[u, "\mathrm{Id}"] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^\vee(-1) \arrow[r, "\upsilon_i^\vee"] \arrow[u] & \mathbb{Q}_\ell(0) \arrow[r] \arrow[u, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r, "(\beta^*)^\vee"] \arrow[u] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^\vee(-1) \arrow[r, "\upsilon_i'^\vee"] \arrow[u, "\mathrm{Id}"] & L_i^\vee(-1) \arrow[r] \arrow[u] & 0
\end{tikzcd} , \tag{11.50}$$

where all the vertical homomorphisms in it are isomorphisms.

Remark 11.4.4. *Notice that all the rows of the commutative diagrams 11.49 and 11.50 are short exact sequences, while all the vertical homomorphisms are isomorphisms and the identity homomorphisms have been explicitly denoted by Id.*

Summary: in this chapter, we first study the étale cohomology of the blow up $\widetilde{\mathcal{W}}_1$ of the quintic mirror conifold, from which we find the zeta functions of the pure motive $h^3(\widetilde{\mathcal{W}}_1)$, which shows $h^3(\widetilde{\mathcal{W}}_1)$ is modular and the L-function $L(h^3(\widetilde{\mathcal{W}}_1), s)$ is just the L-function associated to the modular form f in equation 1.4. We have also studied the extensions induced by the exceptional curves of the small resolutions of the quintic mirror conifold, which will be crucial when we discuss the **Beilinson's Conjecture** in Chapter 12.

Chapter 12

The blow up of the quintic mirror conifold and Beilinson's conjecture

In this chapter, we will study the interesting connections between the blow up of quintic mirror conifold and Beilinson's conjecture. More precisely, we will show that the blow up of quintic mirror conifold provides a compelling example to Beilinson's conjecture. The structure of this chapter is as follows:

- Section 12.1 studies the L-function of the blow up of quintic mirror conifold and its connections to Beilinson's conjecture.
- Section 12.2 constructs the pull backs of the exceptional curves of small resolutions $\widehat{\mathcal{W}}_1^i$ under the morphism
$$\lambda_i : \widetilde{\mathcal{W}}_{1,F} \to \widehat{\mathcal{W}}_1^i.$$
- Finally, Section 12.3 studies the extensions induced by the pull backs of the exceptional curves of small resolutions $\widehat{\mathcal{W}}_1^i$, which lie in the group
$$\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}(\mathbb{Q}_\ell(0), H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2)),\ F = \mathbb{Q}(5^{1/2}).$$
This section also discusses whether these extensions split or not, which is very crucial in Beilinson's conjecture.

12.1 The blow up of the quintic mirror conifold and Beilinson's conjecture

In this section we will study the L-function of the blow up $\widetilde{\mathcal{W}}_1$ of the quintic mirror conifold $\mathcal{W}_1$ and its connection to Beilinson's conjecture. By extension of base field, we get a smooth projective variety $\widetilde{\mathcal{W}}_{1,F}$ defined over $F = \mathbb{Q}(5^{1/2})$

$$\widetilde{\mathcal{W}}_{1,F} := \widetilde{\mathcal{W}}_1 \times_{\mathbb{Q}} F. \tag{12.1}$$

Then by restriction of base field, we get a smooth projective variety $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ defined over $\mathbb{Q}$ given by

$$\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} : \widetilde{\mathcal{W}}_{1,F} \to F \to \mathbb{Q}. \tag{12.2}$$

Beilinson's Conjecture predicts that

$$\dim_{\mathbb{Q}} \mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0 \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{ord}_{s=2}\, L(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), s). \tag{12.3}$$

Let us look at the right hand side first. The variety $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ is the product of two smooth projective varieties defined over $\mathbb{Q}$, i.e. $\widetilde{\mathcal{W}}_1$ and $\operatorname{Spec} F$, hence Künneth formula yields

$$H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) = H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^0_{\text{ét}}(\operatorname{Spec} F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell). \tag{12.4}$$

The second factor in this tensor product is

$$H^0_{\text{ét}}(\operatorname{Spec} F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(0) \oplus \chi, \tag{12.5}$$

where χ also means the Galois representation associated to the Dirichlet character χ, so the L-function of the pure motive $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ is given by

$$\begin{aligned} L(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), s) &= L(h^3(\widetilde{\mathcal{W}}_1), s) \cdot L(h^3(\widetilde{\mathcal{W}}_1) \otimes \chi, s) \\ &= L(f, s) \cdot L(f \otimes \chi, s). \end{aligned} \tag{12.6}$$

Lemma 12.1.1. *The L-function of the twisted modular form $f \otimes \chi$ vanishes at $s = 2$, hence we have*

$$\mathit{ord}_{s=2}\ L(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), s) \geq 1. \tag{12.7}$$

Proof. From Magma, the eigenvalue of the twisted modular form $f \otimes \chi$ under Fricke involution is -1. While from Theorem 7.1 [64], the sign of the functional equation satisfied by a weight-4 newform is -1. So we immediately have

$$L(f \otimes \chi, 2) = 0. \tag{12.8}$$

The method here is from a post on MathOverflow, "Special Values of a L function". □

Remark 12.1.2. *Numerical method shows that* $L(f,2)\neq 0$ *and the derivative of* $L(f\otimes\chi,s)$ *also does not vanish at* $s=2$*, i.e.*

$$ord_{s=2}\, L(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}),s)=1. \tag{12.9}$$

Hence in order to show that the pure motive $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ provides interesting evidence to **Beilinson's Conjecture**, we will need

$$\dim_{\mathbb{Q}}\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0\otimes_{\mathbb{Z}}\mathbb{Q}=\dim_{\mathbb{Q}}\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0\otimes_{\mathbb{Z}}\mathbb{Q}\geq 1. \tag{12.10}$$

Namely we want to construct a codimension 2 algebraic cycle of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ that defines a nonzero element of $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0\otimes_{\mathbb{Z}}\mathbb{Q}$, i.e. it is homologous to zero and non-torsion. The crucial observation is that the exceptional curve C_i of the small resolution $\widehat{\mathcal{W}}_1^i$ is of codimension 2 and it is also homologous to zero, therefore the cycle class $[C_i]$ lies in $\mathrm{CH}^2(\widehat{\mathcal{W}}_1^i)_0$. So the pull back of the cycle class $[C_i]$ under the morphism

$$\lambda_i:\widetilde{\mathcal{W}}_{1,F}\to\widehat{\mathcal{W}}_1^i \tag{12.11}$$

is a cycle class in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0$ ($=\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})_0$)

12.2 The pull backs of the exceptional curves of the small resolutions

In this section, we will construct the pull back of the cycle class $[C_i]$ under the morphism λ_i. Since the scheme-theoretic inverse of C_i under λ_i is the exceptional divisor D_F^1 of $\widetilde{\mathcal{W}}_{1,F}$ which is of codimension 1, so we have to construct the pull back of $[C_i]$ using intersection theory [45], and we will need Proposition 6.7 of [45].

Remark 12.2.1. *Intersection theory has been defined for stacks, and in this section we will use the intersection theory for algebraic spaces freely.*

Let $\mathcal{N}_{C_i}$ be the normal bundle of C_i in $\widehat{\mathcal{W}}_1^i$, and from the construction of blow up, the exceptional divisor D_F^1 is given by

$$D_F^1=\mathbb{P}(\mathcal{N}_{C_i}). \tag{12.12}$$

Let $\mathcal{N}_{D_F^1}$ be the normal bundle of D_F^1 in $\widetilde{\mathcal{W}}_{1,F}$, then the excess normal bundle on D_F^1 is defined by

$$\mathcal{E}_i:=(p_i)^*\mathcal{N}_{C_i}/\mathcal{N}_{D_F^1}, \tag{12.13}$$

where p_i is the fibration map $D^1_F \to C_i$ in equation 11.23. From Proposition 6.7 of [45], the pull back of $[C_i]$ is given by the formula

$$\lambda_i^*([C_i]) = (i_{D^1_F})_*(c_1(\mathcal{E}_i)), \tag{12.14}$$

where $i_{D^1_F}$ is the inclusion morphism $D^1_F \hookrightarrow \widetilde{\mathcal{W}}_{1,F}$. Here $c_1(\mathcal{E}_i)$ is the first Chern class of the line bundle $\mathcal{E}_i$, which is a one dimensional cycle class of D^1_F. Now let us choose a one dimensional cycle $\widetilde{C}_i$ of D^1_F which represents $c_1(\mathcal{E}_i)$, then $\widetilde{C}_i$ is also a codimension-2 cycle of $\widetilde{\mathcal{W}}_{1,F}$ which represents the pull back $\lambda_i^*([C_i])$. Proposition 6.7 (b) of [45] shows

$$(\lambda_i)_*([\widetilde{C}_i]) = (\lambda_i)_*\big(\lambda_i^*([C_i])\big) = [C_i], \tag{12.15}$$

so the map λ_i sends the support of $\widetilde{C}_i$ onto the support of C_i.

Lemma 12.2.2. *In the Chow group $CH^2(\widetilde{\mathcal{W}}_{1,F})$, the algebraic cycle classes $\{[\widetilde{C}_i]\}_{i=1}^2$ satisfy the equation*

$$[\widetilde{C}_1] = -[\widetilde{C}_2]. \tag{12.16}$$

Proof. We will prove $[\widetilde{C}_1] = -[\widetilde{C}_2]$ in the Chow group $\mathrm{CH}^1(D^1_F)$, which will immediately prove this lemma. Given an embedding $\sigma : F \to \mathbb{C}$, the natural morphism

$$D^1_{\overline{\mathbb{Q}}} := D^1_F \times_\sigma \overline{\mathbb{Q}} \to D^1_F \tag{12.17}$$

induces an isomorphism

$$\mathrm{Pic}(D^1_F) = \mathrm{Pic}(D^1_{\overline{\mathbb{Q}}}), \tag{12.18}$$

so we only need to prove this equation over the algebraically closed field $\overline{\mathbb{Q}}$. Over $\overline{\mathbb{Q}}$, we have an isomorphism

$$D^1_{\overline{\mathbb{Q}}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}}, \tag{12.19}$$

so the Hilbert scheme of lines in $D^1_{\overline{\mathbb{Q}}}$ consists of two disjoint union of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, i.e.

$$C_i \times_\sigma \overline{\mathbb{Q}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}}. \tag{12.20}$$

Moreover, the two fibrations $p_i : D^1_F \to C_i$ become the two natural projections

$$p_i : D_{\overline{\mathbb{Q}}} \simeq \mathbb{P}^1_{\overline{\mathbb{Q}}} \times \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}. \tag{12.21}$$

The normal bundle $\mathcal{N}_{D^1_{\overline{\mathbb{Q}}}}$ is given by

$$\mathcal{N}_{D^1_{\overline{\mathbb{Q}}}} = p_1^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big) \otimes p_2^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big), \tag{12.22}$$

while the normal bundle $\mathcal{N}_{C_i}$ is given by

$$\mathcal{N}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}} = \mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1). \tag{12.23}$$

From properties of first Chern class, we have

$$c_1(\mathcal{E}_{i,\overline{\mathbb{Q}}}) = p_i^*\big(c_1(\mathcal{N}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}})\big) - c_1(\mathcal{N}_{D^1_{\overline{\mathbb{Q}}}}), \tag{12.24}$$

from which we get

$$\begin{aligned} c_1(\mathcal{E}_{1,\overline{\mathbb{Q}}}) &= c_1\Big(p_1^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big)\Big) - c_1\Big(p_2^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big)\Big), \\ c_1(\mathcal{E}_{2,\overline{\mathbb{Q}}}) &= c_1\Big(p_2^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big)\Big) - c_1\Big(p_1^*\big(\mathcal{O}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(-1)\big)\Big), \end{aligned} \tag{12.25}$$

which shows $[\widetilde{C}_1] = -[\widetilde{C}_2]$ over $\overline{\mathbb{Q}}$. □

Since the Chow group $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})$ is canonically isomorphic to $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$, the cycle class $[\widetilde{C}_i]$ of $\widetilde{\mathcal{W}}_{1,F}$ is canonically a cycle class of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$. The involution ι of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ induces an involution of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$

$$\iota : \widetilde{\mathcal{W}}_{1,F/\mathbb{Q}} \to \widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}, \tag{12.26}$$

which further defines an involution of $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$. From the commutative diagram 10.65, we find that ι maps $[\widetilde{C}_1]$ to $[\widetilde{C}_2]$ and vice versa, i.e.

$$\iota : [\widetilde{C}_1] \mapsto [\widetilde{C}_2] = -[\widetilde{C}_1]. \tag{12.27}$$

The involution ι 12.26 defines two idempotent correspondences of $\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}$ (see Appendix A for the definition of correspondences)

$$(1+\iota)/2,\ (1-\iota)/2, \tag{12.28}$$

which decomposes the pure motive $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ into the following direct sum

$$\begin{aligned} h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}) &= \big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), (1+\iota)/2\big) \oplus \big(h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}}), (1-\iota)/2\big) \\ &= h^3(\widetilde{\mathcal{W}}_1) \oplus \big(h^3(\widetilde{\mathcal{W}}_1) \otimes \chi\big). \end{aligned} \tag{12.29}$$

The two idempotent correspondences send the cycle class $[\widetilde{C}_1]$ to

$$\frac{1}{2}(1+\iota)([\widetilde{C}_1]) = 0,\ \ \frac{1}{2}(1-\iota)([\widetilde{C}_1]) = [\widetilde{C}_1], \tag{12.30}$$

and intuitively $[\widetilde{C}_1]$ should be considered as an element of $\mathrm{CH}^2(h^3(\widetilde{\mathcal{W}}_1) \otimes \chi)_0$, which corresponds to the property that it is the L-function of $h^3(\widetilde{\mathcal{W}}_1) \otimes \chi$, i.e. $L(f \otimes \chi, s)$, that vanishes at $s = 2$, while numerical results shows the L-function of $h^3(\widetilde{\mathcal{W}}_1)$, i.e. $L(f, s)$, does not vanish at $s = 2$.

12.3 The extensions induced by cycle classes homologous to zero

In this section, we will study the extensions induced by the pull backs of the exceptional curves of small resolutions, and we will also study whether these extensions split or not, which depends on a conjecture of Beilinson and Bloch.

12.3.1 Extensions

From Proposition 9.2 in Chapter VI of [78], the cycle class map commutes with pull back, therefore cycle $\widetilde{C}_i$ of $\widetilde{\mathcal{W}}_{1,F}$ is also homologous to zero. Let $\widetilde{U}^i$ be the open subvariety $\widetilde{\mathcal{W}}_{1,F} - |\widetilde{C}_i|$ defined over the field F, where $|\widetilde{C}_i|$ is the support of $\widetilde{C}_i$. The cycle $\widetilde{C}_i$ induces a long exact sequence with first several terms given by [66, 67, 78]

$$0 \to H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \to H^3_{\text{ét}}(\widetilde{U}^i_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \to H^4_{\text{ét},|\widetilde{C}_i|}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \xrightarrow{\delta} H^4_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2). \tag{12.31}$$

There exists a homomorphism

$$cl(\widetilde{C}_i) : \mathbb{Q}_\ell(0) \to H^4_{\text{ét},|\widetilde{C}_i|}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2), \tag{12.32}$$

whose composition with δ gives the image of $\widetilde{C}_i$ in $H^4_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2)$ under the cycle class map, which is zero. The image of the homomorphism $cl(\widetilde{C}_i)$ is a one dimensional vector space that is isomorphic to $\mathbb{Q}_\ell(0)$, whose inverse image in $H^3_{\text{ét}}(\widetilde{U}^i_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2)$ will be denoted by E_i. Thus we have obtained a short exact sequence

$$0 \longrightarrow H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \longrightarrow E_i \longrightarrow \mathbb{Q}_\ell(0) \longrightarrow 0. \tag{12.33}$$

The map λ_i sends the pair $(\widetilde{\mathcal{W}}_{1,F}, \widetilde{C}_i)$ to the pair $(\widehat{\mathcal{W}}^i_1, C_i)$, hence it induces a homomorphism from the following exact sequence associated to $(\widehat{\mathcal{W}}^i_1, C_i)$

$$0 \to H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_1, \mathbb{Q}_\ell)(2) \to H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \to H^4_{\text{ét},C_{i,\overline{\mathbb{Q}}}}(\widehat{\mathcal{W}}^i_1, \mathbb{Q}_\ell)(2) \xrightarrow{\delta=0} H^4_{\text{ét}}(\widehat{\mathcal{W}}^i_1, \mathbb{Q}_\ell)(2) \tag{12.34}$$

to the exact sequence 12.31 associated to the pair $(\widetilde{\mathcal{W}}_{1,F}, \widetilde{C}_i)$. Then Proposition 9.2 in Chapter VI of [78] yields a homomorphism between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^3_{\text{ét}}(\widehat{\mathcal{W}}^i_1, \mathbb{Q}_\ell)(2) \arrow[r] \arrow[d, "\lambda_i^*"] & H^3_{\text{ét}}(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r] \arrow[d] & \mathbb{Q}_\ell(0) \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r] & E_i \arrow[r] & \mathbb{Q}_\ell(0) \arrow[r] & 0
\end{tikzcd} . \tag{12.35}$$

In Section 11.4, we have shown λ_i^* is an isomorphism, so from the five lemma the central vertical homomorphism is also an isomorphism. Together with the commutative diagram 11.50, this yields an isomorphism between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r] \arrow[d, "\mathrm{Id}"] & E_i \arrow[r] \arrow[d] & \mathbb{Q}_\ell(0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \arrow[r, "(\beta^*)^\vee"] & H^3_{\text{ét}}(\mathcal{W}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^\vee(-1) \arrow[r, "\upsilon_i'^\vee"] & L_i^\vee(-1) \arrow[r] & 0
\end{tikzcd} . \tag{12.36}$$

Here is a subtle point, if we choose an isomorphism

$$L_i^\vee(-1) \simeq \mathbb{Q}_\ell(0), \tag{12.37}$$

then from Lemma 12.2.2, we will have

$$\upsilon_1'^\vee = -\upsilon_2'^\vee. \tag{12.38}$$

Therefore we have [66, 67]

$$E_1 = -E_2, \text{ in } \mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}(\mathbb{Q}_\ell(0), H^3_{\text{ét}}(\widetilde{\mathcal{W}}_{1,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2)). \tag{12.39}$$

12.3.2 A criterion for non-torsion

The codimension-2 cycle $\widetilde{C}_i$ of $\widetilde{\mathcal{W}}_{1,F}$ also induces an extension in $\mathbf{MHS}_{\mathbb{Q}}$ [66, 67]

$$0 \longrightarrow H^3((\widetilde{\mathcal{W}}_{1,F} \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q})(2) \longrightarrow H^3((\mathcal{W}_{1,F} \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q})^\vee(-1) \xrightarrow{\upsilon_i'^\vee} \mathbb{Q}(0) \longrightarrow 0, \tag{12.40}$$

where by abuse of notation $H^3((\widetilde{\mathcal{W}}_{1,F} \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q})$ also means the MHS on the Betti cohomology group $H^3((\widetilde{\mathcal{W}}_{1,F} \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q})$, etc. But we also have

$$\widetilde{\mathcal{W}}_{1,F} \times_\sigma \mathbb{C} = (\widetilde{\mathcal{W}}_1 \times_{\mathbb{Q}} F) \times_\sigma \mathbb{C} = \widetilde{\mathcal{W}}_1 \times_{\mathbb{Q}} \mathbb{C}, \tag{12.41}$$

hence take the dual of 12.40 and then twist it with $\mathbb{Q}(-1)$, we get a short exact sequence

$$0 \longrightarrow \mathbb{Q}(-1) \xrightarrow{\upsilon_i'} H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) \xrightarrow{\beta^*} H^3(\widetilde{\mathcal{W}}_1(\mathbb{C}), \mathbb{Q}) \longrightarrow 0. \tag{12.42}$$

Remark 12.3.1. *Notice that the short exact sequences associated to υ_1' is isomorphic to the short exact sequence associated to υ_2', but they are not isomorphic to each other as extensions of $H^3(\widetilde{\mathcal{W}}_1(\mathbb{C}), \mathbb{Q})$ by $\mathbb{Q}(-1)$. For more detail, see the definition of extensions in Section 3.2.*

In order to determine whether the cycle class $[\widetilde{C}_i]$ is zero or not in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F})_0 \otimes \mathbb{Q}$, we will need a conjecture of Beilinson and Bloch [13, 14]. Given a smooth projective variety X defined over a number field K, the generalised Abel-Jacobi map is given by

$$AJ_\sigma : \mathrm{CH}^c(X)_0 \to \mathrm{Ext}^1_{\mathbf{MHS}_{\mathbb{Q}}}\big(\mathbb{Q}(0), H^{2c-1}_{\mathrm{B},\sigma}(X)(c)\big) = \frac{H^{2c-1}((X \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{C})}{H^{2c-1}((X \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q}(c)) + F^c}, \tag{12.43}$$

where c is a positive integer, σ is an embedding of K into $\mathbb{C}$, and $H^{2c-1}_{\mathrm{B},\sigma}(X)(c)$ means the rational MHS on the cohomology group $H^{2c-1}((X \times_\sigma \mathbb{C})(\mathbb{C}), \mathbb{Q}(c))$.

Beilinson-Bloch's Conjecture *The Abel-Jacobi map AJ_σ 12.43 is injective up to torsions [13, 14, 67].*

From this conjecture, the cycle class $[\widetilde{C}_i]$ is non-torsion if and only if the extension 12.40 does not split, which is equivalent to the property that the short exact sequence 12.42 does not split. A trivial simplification is that the short exact sequence 12.42 does not split if and only if the following short exact sequence does not split

$$0 \longrightarrow \ker \beta^* \longrightarrow H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) \xrightarrow{\beta^*} H^3(\widetilde{\mathcal{W}}_1(\mathbb{C}), \mathbb{Q}) \longrightarrow 0. \tag{12.44}$$

To proceed, we need to compute the MHS on the quintic mirror conifold $\mathcal{W}_1$ and construct the short exact sequence 12.44 explicitly. Our strategy is that we first compute the limit MHS of the mirror family of quintic over $\mathcal{W}_1$, then from local invariant cycle theorem we will show the MHS on $\mathcal{W}_1$ will be given by the kernel of monodromy operator [88, 95]. Our conclusion is: the short exact sequence 12.44 does not split if and only if

$$\int_{B_2|_{\psi=1}} \Omega_{\mathcal{W}_1} \neq r \int_{A_1|_{\psi=1}} \Omega_{\mathcal{W}_1} + r' \int_{B_1|_{\psi=1}} \Omega_{\mathcal{W}_1}, \ \forall\ r,\, r' \in \mathbb{Q}. \tag{12.45}$$

Here $\Omega_{\mathcal{W}_1}$ is the nowhere vanishing threeform on the smooth locus $\mathcal{W}_1 - \mathrm{dp}$, and $\{A_1|_{\psi=1},\ B_1|_{\psi=1},\ B_2|_{\psi=1}\}$ are from Section 10.3, which will be shown to form a basis of $H_3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$.

Summary: in this chapter, we first study the L-function of the pure motive $h^3(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ and prove it vanishes at $s = 2$. Then **Beilinson's Conjecture** predicts the existence of a non-torsion cycle class in $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ that is homologous to zero. The crucial observation is that the exceptional curve of a small resolution of the quintic mirror conifold is homologous to zero, so it pulls back to a cycle class of $\mathrm{CH}^2(\widetilde{\mathcal{W}}_{1,F/\mathbb{Q}})$ that is homologous to zero. To test **Beilinson's Conjecture**, we will have to show that the pull back of the exceptional curve of a small resolution is non-torsion. By the **Beilinson-Bloch's Conjecture**, it is equivalent to the condition that the short exact sequence 12.44 does not split. In Chapter 13, we will compute the short exact sequence 12.44 by studying the limit MHS at the quintic mirror conifold. We further show that the short exact sequence 12.44 does not split if and only if one period of the threeform $\Omega_{\mathcal{W}_1}$ is not equal to the sum of rational multiples of the two other periods, which shows partial evidence that the pull back of the exceptional curve of a small resolution is not torsion.

Chapter 13

The MHS of the quintic mirror conifold

In this chapter, we will compute the limit MHS of the mirror family of quintic at the quintic mirror conifold $\mathcal{W}_1$. From local invariant cycle theorem, we will show that the MHS of quintic mirror conifold $\mathcal{W}_1$ is given by the kernel of the monodromy operator. Furthermore, we will also discuss the criterion given in Section 12.3 for the extensions to be non-trivial. The structure of this chapter is as follows:

- Section 13.1 computes the weight filtration of the limit MHS of the mirror family at quintic mirror conifold.
- Section 13.2 computes the limit Hodge filtration of the limit MHS of the mirror family at quintic mirror conifold.
- Section 13.3 studies the MHS on quintic mirror conifold.
- Finally, Section 13.4 discusses the criterion for the extensions constructed in Section 12.3 to be non-trivial.

13.1 The weight Filtration of limit MHS at the quintic mirror conifold

In this section, we will compute the weight filtration of the limit MHS of the mirror family to quintic over the conifold $\mathcal{W}_1$, and the method we will use is from Chapter 3. The notations in this chapter are from Section 10.3, and recall that Δ is a small neighbourhood of the conifold point $\psi = 1$ in $\mathbb{P}^1_{\mathbb{C}}$, and $V_{\mathbb{Z}}$ is the local system over Δ^*

defined in equation 10.31, whose fiber over a point $\psi \in \Delta^*$ is $H^3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{Z})$ (modulo torsions). This local system $V_\mathbb{Z}$ defines a bundle $\mathcal{V}$ over Δ^* by

$$\mathcal{V} := V_\mathbb{Z} \otimes \mathcal{O}_{\Delta^*}, \tag{13.1}$$

the fiber of which over a point $\psi \in \Delta^*$ is $H^3(\mathcal{W}_\psi(\mathbb{C}), \mathbb{C})$. The bundle $\mathcal{V}$ has a filtration given by sub-bundle $\mathcal{F}^p$, whose fiber over ψ is

$$\mathcal{F}^p_\psi := \oplus_{k \geq p} H^{k,3-k}(\mathcal{W}_\psi(\mathbb{C}), \mathbb{C}). \tag{13.2}$$

See Chapter 3 for more detail. There exists a Gauss-Manin connection ∇ on $\mathcal{V}$ such that the local sections of $V_\mathbb{C}$ are flat

$$\nabla : \mathcal{V} \to \Omega^1_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{V}. \tag{13.3}$$

Moreover, ∇ satisfies Griffiths transversality

$$\nabla \mathcal{F}^p \subset \Omega^1_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{F}^{p-1}. \tag{13.4}$$

In this chapter, it is more convenient to choose a new coordinate w of Δ defined by

$$w := \psi^{-5} - 1 = -5\,(\psi - 1)\,(\sum_{i=-5}^{-1} \psi^i), \tag{13.5}$$

and the coordinate of the conifold point $\psi = 1$ in this new coordinate is $w = 0$. Now let us repeat what has been discussed in Section 10.3 with respect to this new coordinate w. Choose a point w_0 in Δ^*, then from Poincaré duality, there exists an integral symplectic basis $\{A_i, B_i\}_{i=1}^2$ of $H_3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{Z})$ (modulo torsions) with intersection pairings

$$A_a \cdot A_b = 0, \quad B_a \cdot B_b = 0, \quad A_a \cdot B_b = \delta_{ab}. \tag{13.6}$$

By extension, A_a (resp. B_b) extends to a local section of $V_\mathbb{Z}^\vee$ (the dual of $V_\mathbb{Z}$) whose value at w will be denoted by $A_a(w)$ (resp. $B_b(w)$), furthermore, $\{A_i(w), B_i(w)\}_{i=1}^2$ form an integral symplectic basis of $H_3(\mathcal{W}_w(\mathbb{C}), \mathbb{Z})$ [30, 51]. From the paper [23], there is a special choice of $\{A_i, B_i\}_{i=1}^2$ such that when $w \to 0$ we have:

1. The homology cycle $A_2(w)$, which is homeomorphic to S^3, shrinks to the double point of $\mathcal{W}_1(\mathbb{C})$.
2. The homology cycle $B_2(w)$, which is homeomorphic to the real torus $(S^1)^3$, becomes a homology cycle of $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Z})$ that contains the double point dp.

3. The homology cycles $A_1(w)$ and $B_1(w)$ are far away from $A_2(w)$ and $B_2(w)$ so that the limit $w \to 0$ will not affect them.

Moreover, the monodromy of this basis around $w = 0$ is given by [23]

$$A_1 \to A_1,\ A_2 \to A_2,\ B_1 \to B_1,\ B_2 \to B_2 + A_2. \tag{13.7}$$

The integral periods of the mirror family of quintic are defined by

$$z_a(w) = \int_{A_a(w)} \Omega(w)\ ,\ \ \mathcal{G}_b(w) = \int_{B_b(w)} \Omega(w), \tag{13.8}$$

where Ω is the threeform of the mirror family of quintic constructed in Section 9.3. The integral period vector $\amalg$ is defined by

$$\amalg\,(w) := (\mathcal{G}_1(w), \mathcal{G}_2(w), z_1(w), z_2(w))^t. \tag{13.9}$$

From the monodromy of $\{A_i, B_i\}_{i=1}^2$ in equation 13.7, the monodromy of integral periods are found to be

$$z_1 \to z_1,\ z_2 \to z_2,\ \mathcal{G}_1 \to \mathcal{G}_1,\ \mathcal{G}_2 \to \mathcal{G}_2 + z_2. \tag{13.10}$$

Let the dual of $\{A_i, B_i\}_{i=1}^2$ be $\{\alpha^i, \beta^i\}_{i=1}^2$, which forms a basis of $H^3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{Z})$, and the pairings between them are

$$\alpha^i(A_j) = \delta_{ij},\ \alpha^i(B_j) = 0,\ \beta^i(A_j) = 0,\ \beta^i(B_j) = \delta_{ij}. \tag{13.11}$$

The monodromy of $\{\alpha^i, \beta^i\}_{i=1}^2$ around $w = 0$ is dual to that of $\{A_i, B_i\}_{i=1}^2$, which is given by

$$\alpha^1 \to \alpha^1, \alpha^2 \to \alpha^2 - \beta^2, \beta^1 \to \beta^1, \beta^2 \to \beta^2. \tag{13.12}$$

Let T_m be the monodromy matrix of the basis $\{\alpha^i, \beta^i\}_{i=1}^2$, and it is unipotent

$$(T_m - \mathrm{Id})^2 = 0, \tag{13.13}$$

and the monodromy operator N is defined by

$$N := \log T_m = T_m - \mathrm{Id},\ N^2 = 0. \tag{13.14}$$

The monodromy operator N induces a weight filtration on $H^3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{Q})$ that is constructed inductively [30, 51]. Because $N^2 = 0$, we have

$$W_0 = W_1 = 0,\ W_4 = W_5 = W_6 = H^3(\mathcal{W}_{\psi_0}(\mathbb{C}), \mathbb{Q}). \tag{13.15}$$

But $N \neq 0$, so we have

$$W_2 = \operatorname{Im} N = \mathbb{Q}\,\beta^2,\ W_3 = \ker N = \mathbb{Q}\,\alpha^1 + \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2. \tag{13.16}$$

The elements α^1, β^1 and β^2 are invariant under monodromy around $w = 0$, while α^2 isn't. The regularised section $\widehat{\alpha}^2$ is defined by

$$\widehat{\alpha}^2 := \exp\,(-\frac{\log\, w}{2\pi i} N)\,\alpha^2(w), \tag{13.17}$$

which is a single valued section of $\mathcal{V}$ over Δ^*. Then $\{\alpha^1,\, \widehat{\alpha}^2,\, \beta^1,\, \beta^2\}$ form a frame of $\mathcal{V}$, and defines a trivialisation of $\mathcal{V}$ over Δ^*. This trivialisation induces an extension of $\mathcal{V}$ to a bundle $\widetilde{\mathcal{V}}$ over Δ, which is called Deligne's canonical extension, and the readers are referred to Section 3.3 for more detail. Following the notations in Section 3.3, the frame $\{\alpha^1,\, \widehat{\alpha}^2,\, \beta^1,\, \beta^2\}$ extends to a frame $\{\alpha^1,\, \widetilde{\alpha}^2,\, \beta^1,\, \beta^2\}$ of $\widetilde{\mathcal{V}}$. Let their values at $w = 0$ be

$$\alpha^1,\ \widetilde{\alpha}^2(0),\ \beta^1,\ \beta^2, \tag{13.18}$$

which defines an integral structure on $\widetilde{\mathcal{V}}|_0$ that will be denoted by $\widetilde{\mathcal{V}}|_{0,\mathbb{Z}}$

$$\widetilde{\mathcal{V}}|_{0,\mathbb{Z}} := \mathbb{Z}\,\alpha^1 + \mathbb{Z}\,\widetilde{\alpha}^2(0) + \mathbb{Z}\,\beta^1 + \mathbb{Z}\,\beta^2. \tag{13.19}$$

Tensor it with $\mathbb{Q}$, we get a rational structure $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$ on $\widetilde{\mathcal{V}}|_0$

$$\widetilde{\mathcal{V}}|_{0,\mathbb{Q}} := \mathbb{Q}\,\alpha^1 + \mathbb{Q}\,\widetilde{\alpha}^2(0) + \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2. \tag{13.20}$$

From Section 3.3, the weight filtration on $H^3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{Q})$ induces a weight filtration on $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$, which is given by

$$W_0 = W_1 = 0,\ W_2 = \mathbb{Q}\,\beta^2,\ W_3 = \mathbb{Q}\,\alpha^1 + \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2,\ W_4 = W_5 = W_6 = \widetilde{\mathcal{V}}|_{0,\mathbb{Q}}. \tag{13.21}$$

13.2 The limit Hodge Filtration of limit MHS at the quintic mirror conifold

In this section, we will compute the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$, but first we need to study the canonical solutions of Picard-Fuchs equations of the mirror family of quintic near the conifold point $w = 0$.

13.2.1 The canonical periods near the conifold point

Suppose C is an element of $H_3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{C})$, then it extends to a (multi-valued) section of the local system $V_{\mathbb{Z}}^{\vee}$, and let the period $\rho(w)$ be defined by

$$\rho(w) := \int_{C(w)} \Omega(w), \tag{13.22}$$

which is a multi-valued holmorphic function over Δ^*. From Griffiths transversality, $\rho(w)$ satisfies Picard-Fuchs equations of the mirror family of quintic. From [23, 30, 51], one special Picard-Fuchs equation is of the form

$$\{\frac{d^4}{dz^4} - \frac{2(4\,z-3)}{z(1-z)}\frac{d^3}{dz^3} - \frac{(72\,z-35)}{5\,z^2(1-z)}\frac{d^2}{dz^2} - \frac{(24\,z-5)}{5\,z^3(1-z)}\frac{d}{dz} - \frac{24}{625\,z^3(1-z)}\}\,\rho = 0, \tag{13.23}$$

where the variable z is ψ^{-5}. In this thesis, we are interested in the 'canonical' solutions of the Picard-Fuchs equation 13.23 in the local neighbourhood Δ of the conifold point. With respect to w, the Picard-Fuchs equation 13.23 becomes

$$\{\frac{d^4}{dw^4} + \frac{2(4\,w+1)}{w(1+w)}\frac{d^3}{dw^3} + \frac{(72\,w+37)}{5\,w(1+w)^2}\frac{d^2}{dw^2} + \frac{(24\,w+19)}{5\,w(1+w)^3)}\frac{d}{dw} + \frac{24}{625\,w(1+w)^3}\}\,\rho = 0, \tag{13.24}$$

which has a regular singularity at the conifold point $w = 0$ [30, 37]. On the other hand, the monodromy operator N defined in equation 13.14 satisfies $N^2 = 0$, hence the solution space of the Picard-Fuchs equation 13.24 has a canonical basis of the form [30, 37, 38]

$$\rho_0,\ \rho_1,\ \rho_2,\ \rho_3 := \frac{1}{2\pi i}(\sum_{i=0}^{2} \lambda_i\,\rho_i\,\log w + h),\ \text{with } \lambda_i \in \mathbb{C} \text{ and } \lambda_1\lambda_2\lambda_3 \neq 0, \tag{13.25}$$

where $\{\rho_i\}_{i=0}^2$ and h are holomorphic functions in the disc Δ. The Picard-Fuchs equation 13.24 can be solved by Frobenius method, from which we get power series expansions of $\{\rho_i\}_{i=0}^2$ and h that are convergent on Δ. There exists a canonical choice of $\{\rho_i\}_{i=0}^2$ and h that are given by

$$\begin{aligned}
\rho_0 &= 1 - \frac{2}{625}w^3 + \frac{97}{18750}w^4 - \frac{2971}{468750}w^5 + O\left(w^6\right), \\
\rho_1 &= w - \frac{7}{10}w^2 + \frac{41}{75}w^3 - \frac{1133}{2500}w^4 + \frac{6089}{15625}w^5 + O\left(w^6\right), \\
\rho_2 &= w^2 - \frac{37}{30}w^3 + \frac{2309}{1800}w^4 - \frac{286471}{225000}w^5 + O\left(w^6\right), \\
h &= -\frac{23}{360}w^3 + \frac{6397}{60000}w^4 - \frac{333323}{2500000}w^5 + O\left(w^6\right),
\end{aligned} \tag{13.26}$$

moreover ρ_3 is of the form

$$\rho_3 = \frac{1}{2\pi i}(\rho_1 \log w + h). \tag{13.27}$$

These canonical solutions are also periods of the mirror family of quintic, hence there exists a basis $\{D_i\}_{i=0}^3$ of $H_3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{C})$ such that

$$\rho_i(w) = \int_{D_i(w)} \Omega(w), \tag{13.28}$$

where $D_i(w)$ is the extension of D_i to a (multi-valued) section of $V_{\mathbb{Z}}^\vee$. The monodromy of the canonical periods $\{\rho_i\}_{i=0}^3$ is induced by the operation $\log w \to \log w + 2\pi i$, which is easily found from their form in equation 13.26 and 13.27

$$\begin{pmatrix}\rho_0\\ \rho_1\\ \rho_2\\ \rho_3\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}\rho_0\\ \rho_1\\ \rho_2\\ \rho_3\end{pmatrix}. \tag{13.29}$$

Let the dual of $\{D_i\}_{i=0}^3$ be $\{\delta^i\}_{i=0}^3$, which form a basis of $H^3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{C})$. From the monodromy of $\{\rho_i\}_{i=0}^3$ in equation 13.29, the monodromy of $\{\delta^i\}_{i=0}^3$ is found to be

$$\begin{pmatrix}\delta^0 & \delta^1 & \delta^2 & \delta^3\end{pmatrix} \to \begin{pmatrix}\delta^0 & \delta^1 & \delta^2 & \delta^3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-1&0&1\end{pmatrix}. \tag{13.30}$$

So δ^0, δ^2 and δ^3 are invariant under monodromy, hence they define sections of $\widetilde{\mathcal{V}}$ over Δ, however δ^1 is not invariant under monodromy, and its regularisation is

$$\widehat{\delta}^1(w) = \exp\left(-\frac{\log w}{2\pi i} N\right) \delta^1(w), \tag{13.31}$$

which is a single valued section of $\mathcal{V}$ over Δ^*. Following the notations in Section 3.3, $\widehat{\delta}^1(w)$ extends to a section of $\widetilde{\mathcal{V}}$ over Δ that will be denoted by $\widetilde{\delta}^1(w)$.

13.2.2 The computation of the limit Hodge filtration

The limited Hodge filtration on $\widetilde{\mathcal{V}}|_0$ will be computed by the limit values of periods $\{\rho_i\}_{i=0}^3$ and their derivatives [72, 85, 88, 95]. From the standard comparison isomorphism I_∞ 2.20, the threeform Ω of the mirror family of quintic has an expansion

$$I_\infty^{-1}(\Omega)|_w = \sum_{a=0}^{3} \rho_a(w)\, \delta^a(w). \tag{13.32}$$

With respect to the frame $\{\delta^0, \widehat{\delta}^1, \delta^2, \delta^3\}$ of $\mathcal{V}$, $I_\infty^{-1}(\Omega)$ has an expansion

$$\begin{aligned} I_\infty^{-1}(\Omega)|_w &= \sum_{a=0}^{3} \delta^a(w)\, \rho_a(w) \\ &= \sum_{a,b,c} \delta^a(w) \big(\exp(-\frac{\log w}{2\pi i} N)\big)_{ab} \big(\exp(\frac{\log w}{2\pi i} N)\big)_{bc}\, \rho_c(w) \\ &= \sum_{a,b} \widehat{\delta}^a(w) \big(\exp(\frac{\log w}{2\pi i} N)\big)_{ab}\, \rho_b(w), \end{aligned} \tag{13.33}$$

where $\big(\exp(-\frac{\log w}{2\pi i}N)\big)_{ab}$ is the matrix of $\exp(-\frac{\log w}{2\pi i}N)$ with respect to $\{\delta^a(w)\}_{a=0}^3$. From this expansion we find that

$$I_\infty^{-1}(\Omega)|_0 = \sum_{a,b} \lim_{w\to 0} \widehat{\delta}^a(w) \big(\exp(\frac{\log w}{2\pi i} N)\big)_{ab}\, \rho_b(w) = \delta^0. \tag{13.34}$$

Since $I_\infty^{-1}(\Omega)|_\Delta$ is a section of the rank-1 $\widetilde{\mathcal{F}}^3$ (the extension of $\mathcal{F}^3$ to a bundle over Δ under Deligne's canonical extension), we deduce that

$$\widetilde{\mathcal{F}}^3|_0 = \mathbb{C}\, \delta^0. \tag{13.35}$$

From Griffiths transversality, $I_\infty^{-1}(\nabla_{d/dw}\,\Omega)$ is a section of $\widetilde{\mathcal{F}}^2$, and we also have

$$\int_{D_i(w)} \nabla_{d/dw}\,\Omega = \frac{d}{dw} \int_{D_i(w)} \Omega(w) = \frac{d}{dw} \rho_i(w), \tag{13.36}$$

therefore an expansion of $I_\infty^{-1}(\nabla_{d/dw}\,\Omega)$ is given by [30]

$$\begin{aligned} I_\infty^{-1}(\nabla_{d/dw}\,\Omega)|_w &= \sum_{a=0}^{3} \delta^a(w) \int_{D_i(w)} \nabla_{d/dw} \Omega(w) \\ &= \sum_{a,b} \widehat{\delta}^a(w) \big(\exp(\frac{\log w}{2\pi i} N)\big)_{ab} \frac{d}{dw} \rho_b(w). \end{aligned} \tag{13.37}$$

Its limit under $w \to 0$ is

$$\lim_{w\to 0} I_\infty^{-1}(\nabla_{d/dw}\,\Omega)|_w = \widetilde{\delta}^1(0) + \frac{1}{2\pi i}\, \delta^3, \tag{13.38}$$

where $\widetilde{\delta}^1(0)$ means the value of $\widetilde{\delta}^1$ at $w = 0$, thus we have

$$\widetilde{\mathcal{F}}^2|_0 = \mathbb{C}\, \delta^0 + \mathbb{C}\, (\widetilde{\delta}^1(0) + \frac{1}{2\pi i}\, \delta^3). \tag{13.39}$$

Similarly $I_\infty^{-1}(w \nabla^2_{d/dw} \Omega)$ is a section of $\widetilde{\mathcal{F}}^1$ whose limit at $w = 0$ is

$$\lim_{w\to 0} I_\infty^{-1}(w \nabla^2_{d/dw}\,\Omega)|_w = \frac{1}{2\pi i}\, \delta^3, \tag{13.40}$$

which implies that

$$\widetilde{\mathcal{F}}^1|_0 = \mathbb{C}\,\delta^0 + \mathbb{C}\,(\widetilde{\delta}^1(0) + \frac{1}{2\pi i}\,\delta^3) + \mathbb{C}\,\frac{1}{2\pi i}\,\delta^3. \tag{13.41}$$

Therefore we have found the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$

$$\begin{aligned} F^3\widetilde{\mathcal{V}}|_0 &= \mathbb{C}\,\delta^0, \\ F^2\widetilde{\mathcal{V}}|_0 &= \mathbb{C}\,\delta^0 + \mathbb{C}(\widetilde{\delta}^1(0) + \frac{1}{2\pi i}\,\delta^3), \\ F^1\widetilde{\mathcal{V}}|_0 &= \mathbb{C}\,\delta^0 + \mathbb{C}(\widetilde{\delta}^1(0) + \frac{1}{2\pi i}\,\delta^3) + \mathbb{C}\frac{1}{2\pi i}\,\delta^3, \\ F^0\widetilde{\mathcal{V}}|_0 &= \widetilde{\mathcal{V}}|_0. \end{aligned} \tag{13.42}$$

From Section 3.3, the weight filtration 13.21 and limit Hodge filtration 13.42 form a mixed Hodge structure on $\widetilde{\mathcal{V}}|_0$.

13.3 The MHS on the quintic mirror conifold

In this section, we will discuss the MHS on $\mathcal{W}_1$. We have constructed two bases of $H^3(\mathcal{W}_{w_0}(\mathbb{C}), \mathbb{C})$, the integral symplectic basis $\{\alpha^i, \beta^i\}_{i=1}^2$ and the canonical basis $\{\delta^i\}_{i=0}^3$, while the difference between them is a general linear transformation. Since the integral periods $\{\mathcal{G}_i, z_i\}_{i=1}^2$ and canonical periods $\{\rho_i\}_{i=0}^3$ both form bases of the solution space of Picard-Fuchs equation 13.24, their difference is a general linear transformation $S \in \mathrm{GL}(4, \mathbb{C})$, which must be of the form

$$\begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} S_{00} & S_{01} & S_{02} & 0 \\ S_{10} & S_{11} & S_{12} & \lambda \\ S_{20} & S_{21} & S_{22} & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}, \ \lambda \neq 0. \tag{13.43}$$

Remark 13.3.1. *From the paper [23],* λ *is* $-4\pi^2/5^{5/2}$.

The three form $I_\infty^{-1}(\Omega)$ has two different expansions

$$I_\infty^{-1}(\Omega) = \beta^1\,\mathcal{G}_1 + \beta^2\,\mathcal{G}_2 + \alpha^1\,z_1 + \alpha^2\,z_2 = \sum_{i=0}^{3} \delta^i\,\rho_i, \tag{13.44}$$

which implies that

$$\begin{pmatrix} \delta^0 & \delta^1 & \delta^2 & \delta^3 \end{pmatrix} = \begin{pmatrix} \beta^1 & \beta^2 & \alpha^1 & \alpha^2 \end{pmatrix} \begin{pmatrix} S_{00} & S_{01} & S_{02} & 0 \\ S_{10} & S_{11} & S_{12} & \lambda \\ S_{20} & S_{21} & S_{22} & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix}. \tag{13.45}$$

At the conifold point $w = 0$, the limit values of the canonical periods $\{\rho_i\}_{i=0}^3$ are

$$\rho_0(0) = 1,\ \rho_1(0) = \rho_2(0) = \rho_3(0) = 0, \tag{13.46}$$

which shows that the limit values of integral periods at $w = 0$ satisfy

$$\mathcal{G}_1|_{w=0} = S_{00},\ \mathcal{G}_2|_{w=0} = S_{10},\ z_1|_{w=0} = S_{20},\ z_2|_{w=0} = 0. \tag{13.47}$$

As $w \to 0$, the cycle $A_2(w)$ shrinks to the double point of $\mathcal{W}_1(\mathbb{C})$, while $A_1(w)$, $B_1(w)$ and $B_2(w)$ induce three cycles $A_1|_{w=0}$, $B_1|_{w=0}$ and $B_2|_{w=0}$ of $H_3(\mathcal{W}_1(\mathbb{C}), \mathbb{Z})$, and we have

$$S_{00} = \int_{B_1|_{w=0}} \Omega_{\mathcal{W}_1},\ S_{10} = \int_{B_2|_{w=0}} \Omega_{\mathcal{W}_1},\ S_{20} = \int_{A_1|_{w=0}} \Omega_{\mathcal{W}_1}. \tag{13.48}$$

The value of S_{10} has been computed in [23]

$$S_{10} = \left(\frac{2\pi i}{5}\right)^3 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5\, 5^{5n}}. \tag{13.49}$$

From [39], there exists a MHS on $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$, which will also be denoted by the same symbol. The monodromy operator N, defined in equation 13.14, defines an endomorphism of the limit MHS on $\widetilde{\mathcal{V}}|_0$, and the kernel of N, denoted by $\ker N$, is a sub-MHS of this limit MHS. From the local invariant cycle theorem [85, 88, 95], there is a surjective homomorphism $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) \to \ker N$ in $\mathbf{MHS}_{\mathbb{Q}}$. However both $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$ and $\ker N$ have dimension 3, hence this surjective homomorphism must be an isomorphism. The underlying rational vector space of $\ker N$ is

$$\ker N = \mathbb{Q}\,\alpha^1 + \mathbb{Q}\,\beta^1 + \mathbb{Q}\,\beta^2. \tag{13.50}$$

The weight filtration on $\ker N$ is induced by the weight filtration on $\widetilde{\mathcal{V}}|_{0,\mathbb{Q}}$

$$W_0 = W_1 = 0,\ W_2 = \mathbb{Q}\,\beta^2,\ W_3 = \ker N, \tag{13.51}$$

and it is crucial that the dimension of W_2 is one and the dimension of W_3/W_2 is two. While the Hodge filtration on $\ker N$ is induced by the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$

$$F^3 = F^2 = \mathbb{C}\,\delta^0,\ F^1 = \mathbb{C}\,\delta^0 + \mathbb{C}\,\frac{1}{2\pi i}\,\delta^3,\ F^0 = \ker N. \tag{13.52}$$

From equation 13.45, the element δ^0 is given by

$$\delta^0 = S_{00}\,\beta^1 + S_{10}\,\beta^2 + S_{20}\,\alpha^1. \tag{13.53}$$

Furthermore, the quotient homomorphism

$$\ker N = W_3 \to W_3/W_2 \tag{13.54}$$

defines a homomorphism in $\mathbf{MHS}_{\mathbb{Q}}$, where W_3/W_2 is a Hodge structure of weight 3.

13.4 A criterion for the non-torsion

The blow up $\beta : \widetilde{\mathcal{W}_1} \to \mathcal{W}_1$ in 10.11 induces a homomorphism of MHS [39]

$$\beta^* : H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) \to H^3(\widetilde{\mathcal{W}_1}(\mathbb{C}), \mathbb{Q}), \tag{13.55}$$

which is surjective from Section 11.2. From the same section, we know the dimension of $H^3(\widetilde{\mathcal{W}_1}(\mathbb{C}), \mathbb{Q})$ is 2 and the pure Hodge structure on $H^3(\widetilde{\mathcal{W}_1}(\mathbb{C}), \mathbb{Q})$ is of weight 3. From Section 13.3, the weight filtration on $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$ satisfies

$$\dim_{\mathbb{Q}} W_2\, H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) = 1,\ \dim_{\mathbb{Q}} W_3\, H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})/W_2\, H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q}) = 2. \tag{13.56}$$

Therefore we deduce that the pure Hodge structure on $H^3(\widetilde{\mathcal{W}_1}(\mathbb{C}), \mathbb{Q})$ is isomorphic to the pure Hodge structure on the graded piece W_3/W_2 of $H^3(\mathcal{W}_1(\mathbb{C}), \mathbb{Q})$, hence the homomorphism β^* in equation 13.55 is isomorphic to the quotient homomorphism in equation 13.54. Moreover the kernel of β^*, denoted by $\ker \beta^*$, is isomorphic to the pure Hodge structure on W_2.

Let $\overline{\alpha}^1$ (resp. $\overline{\beta}^1$) be the image of α^1 (resp. β^1) in the quotient space W_3/W_2, then the rational vector space W_3/W_2 is spanned by

$$W_3/W_2 = \mathbb{Q}\,\overline{\alpha}^1 + \mathbb{Q}\,\overline{\beta}^1, \tag{13.57}$$

while the induced Hodge filtration on W_3/W_2 is given by

$$F^3 = F^2 = F^1 = \mathbb{C}(S_{00}\,\overline{\beta}^1 + S_{20}\,\overline{\alpha}^1),\ F_0 = (W_3/W_2) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{13.58}$$

Notice that the transformation matrix S in equation 13.43 is non-degenerate, so S_{00} and S_{20} cannot both be zero. The induced Hodge filtration on W_2 is given by

$$F^2 = 0,\ F^1 = F^0 = W_2 \otimes_{\mathbb{Q}} \mathbb{C}, \tag{13.59}$$

hence the pure Hodge structure on W_2 is isomorphic to $\mathbb{Q}(-1)$, and we have the following important proposition.

Proposition 13.4.1. *The short exact sequence 12.44 is isomorphic to*

$$0 \longrightarrow W_2 \longrightarrow ker N \longrightarrow W_3/W_2 \longrightarrow 0, \tag{13.60}$$

which splits if and only if there exist two rational numbers r_1 and r_2 such that

$$S_{10} = r_1\, S_{00} + r_2\, S_{20}. \tag{13.61}$$

Proof. $\Longrightarrow$ Suppose there exist two rational numbers r_1 and r_2 such that this equation is satisfied, then let us define x and y by

$$x := \alpha^1 + r_2\,\beta^2,\ \ y := \beta^1 + r_1\,\beta^2. \tag{13.62}$$

So $\{\beta^2, x, y\}$ form a basis of W_3, while δ^0 in 13.53 is equal to

$$\delta^0 = S_{00}\,y + S_{20}\,x. \tag{13.63}$$

Suppose H' is a pure Hodge structure with underlying rational vector space

$$H'_{\mathbb{Q}} = \mathbb{Q}\,x + \mathbb{Q}\,y, \tag{13.64}$$

and the Hodge filtration

$$F^3H' = F^2H' = F^1H' = \mathbb{C}(S_{00}\,y + S_{20}\,x),\ F_0\,H' = H'_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{13.65}$$

By sending x (resp. y) to $\overline{\alpha}^1$ (resp. $\overline{\beta}^1$), we get an isomorphism between H' and the pure Hodge structure W_3/W_2. Therefore in this case, the short exact sequence 13.60 is isomorphic to

$$0 \longrightarrow W_2 \longrightarrow W_2 \oplus H' \longrightarrow H' \longrightarrow 0. \tag{13.66}$$

$\Longleftarrow$ Suppose the short exact sequence 13.60 splits, then there exists a homomorphism of mixed Hodge structures $j : W_3/W_2 \to \ker N$ whose composition with the quotient homomorphism 13.54 is identity. Let H' be the image of j, then the MHS $\ker N$ is isomorphic to the direct sum of W_2 and H'. Let x (reps. y) be the image of $\overline{\alpha}^1$ (resp. $\overline{\beta}^1$) under the homomorphism j, then there exist r_1 and r_2 in $\mathbb{Q}$ such that

$$x = j(\overline{\alpha}^1) = \alpha^1 + r_2\,\beta^2,\ \ y = j(\overline{\beta}^1) = \beta^1 + r_1\,\beta^2. \tag{13.67}$$

The image of $S_{00}\,\overline{\beta}^1 + S_{2,0}\,\overline{\alpha}^1$ under the homomorphism j is given by

$$\begin{aligned} j(S_{00}\,\overline{\beta}^1 + S_{2,0}\,\overline{\alpha}^1) &= S_{00}(\beta^1 + r_1\,\beta^2) + S_{20}(\alpha^1 + r_2\,\beta^2) \\ &= S_{00}\,\beta^1 + (r_1\,S_{00} + r_2\,S_{20})\,\beta^2 + S_{20}\,\alpha^1. \end{aligned} \tag{13.68}$$

Since j is a homomorphism between mixed Hodge structures, we must have

$$S_{10} = r_1\,S_{00} + r_2\,S_{20}. \tag{13.69}$$

□

Remark 13.4.2. *From equation 13.48 and this proposition, the short exact sequence 13.60 does not split if and only if*

$$\int_{B_2|_{\psi=1}} \Omega_{\mathcal{W}_1} \neq r \int_{A_1|_{\psi=1}} \Omega_{\mathcal{W}_1} + r' \int_{B_1|_{\psi=1}} \Omega_{\mathcal{W}_1}, \ \forall \ r, \, r' \in \mathbb{Q}. \tag{13.70}$$

Summary: in this chapter, we have applied the methods in Chapter 3 to the computation of the limit MHS at the conifold of the mirror family of the quintic, from which we can compute the short exact sequence 12.44 explicitly. Then we have shown that the short exact sequence 12.44 does not split if and only if the condition in Remark 13.4.2 is satisfied, so it shows partial evidence that the short exact sequence 12.44 does not split.

Appendix A

Pure motives

This appendix is a very brief overview of the pure motives, thus it is not meant to be complete and references will be given for a further reading. The structure of this appendix is as follows:

- Section A.1 briefly discusses the theory of algebraic cycles.
- Section A.2 describes the Weil cohomology theory and some of its axioms.
- Section A.3 is concerned with the construction of the category of pure motives using algebraic cycles.
- Section A.4 discusses some expected properties of the conjectured abelian category of mixed motives.

A.1 Algebraic cycles

An excellent reference to the theory of algebraic cycles is the book [45], which is strongly recommended to the readers. In this section, we will follow the notations of [80]. First let **SmProj**$/k$ be the category of non-singular projective varieties over a field k, which is a symmetric monoidal category with product of objects given by the fiber product of varieties. Moreover, the 'symmetry' of this category is given by the canonical isomorphism

$$X \times_k Y \to Y \times_k X. \tag{A.1}$$

A prime cycle Z of the non-singular projective variety X is defined to be an irreducible algebraic subvariety of X, whose codimension is defined to be $\dim X - \dim Z$. The set of all prime cycles of dimension r (resp. codimension r) generates a free abelian group that will be denoted by $C_r(X)$ (resp. $C^r(X)$), and elements of $C_r(X)$ (resp.

$C^r(X)$) will be called algebraic cycles of dimension r (resp. codimension r). Two prime cycles Z_1 and Z_2 are said to intersect with each other properly if

$$\operatorname{codim}(Z_1 \cap Z_2) = \operatorname{codim}(Z_1) + \operatorname{codim}(Z_2), \tag{A.2}$$

where $Z_1 \cap Z_2$ means the set-theoretic intersection of the underlying point sets of the subvarieties $\{Z_i\}_{i=1}^2$.

Remark A.1.1. *An irreducible closed subset of X has a natural algebraic variety structure induced from that of X [98].*

If two prime cycles Z_1 and Z_2 intersect with each other properly, then the intersection product $Z_1 \cdot Z_2$ is defined by

$$Z_1 \cdot Z_2 = \sum_T m(T;\, Z_1 \cdot Z_2)\, T, \tag{A.3}$$

where the sum is over all irreducible components of $Z_1 \cap Z_2$ and $m(T;\, Z_1 \cdot Z_2)$ is Serre's intersection multiplicity formula [45]. Extend this definition by linearity, then intersection products are defined for algebraic cycles $Z = \sum_j m_j\, Z_j$ and $W = \sum_l n_l\, W_l$ when Z_j and W_l intersect properly for all j and l. Therefore there exists a partially defined intersection product on the group of algebraic cycles

$$C^r(X) \times C^s(X) \dashrightarrow C^{r+s}(X). \tag{A.4}$$

Suppose $f : X \to Y$ is a morphism between two non-singular projective varieties X and Y, then the pushforward homomorphism f_* on algebraic cycles is defined by its action on an arbitrary prime cycle Z of X

$$f_*(Z) := \begin{cases} 0 & \text{if } \dim f(Z) < \dim Z \\ [k(Z) : k(f(Z))] \cdot f(Z) & \text{if } \dim f(Z) = \dim Z \end{cases}, \tag{A.5}$$

where $k(Z)$ (resp. $k(f(Z))$) is the function field of Z (resp. $f(Z)$) and $[k(Z) : k(f(Z))]$ is the degree of field extension [98]. The next step is to define the pullback homomorphism f^* on algebraic cycles. Given a prime cycle W of Y, the first attempt is to naively define f^* by [45]

$$f^*(W) := \sum_{T \subset f^{-1}(Z)} \ell_{\mathcal{O}_{X,T}}(\mathcal{O}_{f^{-1}(Z),T}) \cdot T, \tag{A.6}$$

where the sum is over all the irreducible components of $f^{-1}(Z)$ and $\ell_{\mathcal{O}_{X,T}}(\mathcal{O}_{f^{-1}(Z),T})$ is the length of the ring $\mathcal{O}_{f^{-1}(Z),T}$ in $\mathcal{O}_{X,T}$. However this definition is only partially defined and in general $f^*(W)$ does not make sense [45]. The solution to the above problems is to find an equivalence relation $\sim$ on the group of algebraic cycles which satisfies the property: for two arbitrary cycles Z_1 and Z_2, there exists a cycle Z_1' in the equivalence class of Z_1 such that Z_1' intersects with Z_2 properly and the equivalence class of $Z_1' \cdot Z_2$ is independent of the choice of Z_1'. The equivalence relation $\sim$ that satisfies this property will be called an adequate equivalence relation. For an adequate equivalence relation $\sim$, let $C^*_\sim(X)$ be the quotient group $C^*(X)/\sim$, etc. Then we have a well-defined intersection product

$$C^r(X)_\sim \times C^s(X)_\sim \to C^{r+s}(X)_\sim, \tag{A.7}$$

and a well-defined pushforward homomorphism $f_* : C_{r,\sim}(X) \to C_{r,\sim}(Y)$, while the pullback homomorphism $f^* : C^r_\sim(Y) \to C^r_\sim(X)$ can also be defined. Suppose W is a prime cycle of Y, then $f^*(W)$ is defined by [45]

$$f^*(W) := (\mathrm{pr}_X)_*\big(\Gamma_f \cdot (X \times W)\big), \tag{A.8}$$

where pr_X is the natural projection map from $X \times Y$ to X and Γ_f is the graph of f. The set of adequate equivalence relations are ordered in the way such that $\sim_1$ is said to be finer than $\sim_2$ if for every algebraic cycle Z, $Z \sim_1 0$ implies $Z \sim_2 0$. The two most important adequate equivalence relations are rational equivalence and numerical equivalence [80]. In fact rational equivalence is the finest adequate equivalence relation and numerical equivalence is the coarsest adequate equivalence relation [80].

A.2 Weil cohomology theory

In this section, we will introduce the definition of Weil cohomology theory, and our treatment is from the reference [33]. Let $\mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K$ be the rigid tensor abelian category of finite dimensional graded vector spaces over a field K such that $\operatorname{char} K = 0$ [42]. Every object V of $\mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K$ has a decomposition

$$V = \oplus_{r \geq 0} V_r, \tag{A.9}$$

where V_r consists of elements that are homogeneous of degree r. The tensor product of $\mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K$ is denoted by $\otimes_K$. The category $\mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K$ admits a graded symmetry defined by

$$v \otimes_K w \to (-1)^{\deg v \deg w}\, w \otimes_K v, \tag{A.10}$$

when v and w are both homogeneous elements. A Weil cohomology theory is a symmetric monoidal functor

$$H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K \tag{A.11}$$

that satisfies a list of axioms [33]. Examples of Weil cohomology theory include Betti cohomology, algebraic de Rham cohomology, étale cohomology, which have been discussed in Chapter 2. We will not give all the axioms in this section, but instead we leave them to the reference [33]. Among these axioms is the existence of the cycle class map

$$\mathrm{cl} : C^*_{\mathrm{rat}}(X)_{\mathbb{Q}} \to H^*(X), \tag{A.12}$$

which doubles the degree and sends the intersection product of cycles to the cup product of cohomology classes. Here $C^*_{\sim}(X)_{\mathbb{Q}}$ is defined by

$$C^*_{\sim}(X)_{\mathbb{Q}} := C^*_{\sim}(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{A.13}$$

where $\sim$ is an adequate equivalence relation. On the other hand, a Weil cohomology theory H^* yields an adequate equivalence relation $\sim_{H^*}$ defined by [80]

$$Z \sim_{H^*} 0 \Leftrightarrow \mathrm{cl}(Z) = 0. \tag{A.14}$$

However, it is not clear whether $\sim_{H^*}$ depends on the chosen Weil cohomolgy theory H^*.

A.3 Pure motives

The three examples of classical Weil cohomology theory given in Chapter 2 behave as if they all arise from a cohomology theory defined over $\mathbb{Q}$, however this is known to be false [80]. Grothedieck's idea to explain this phenomenon is that there exists a universal cohomology theory such that all Weil cohomology theories are the realisations of it. More precisely there exists a rigid tensor abelian category $\mathbf{M}_{\mathrm{hom}}$ over $\mathbb{Q}$ and a functor

$$M_{gm} : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathbf{M}_{\mathrm{hom}} \tag{A.15}$$

such that for every Weil cohomology theory H^*, there exists a functor H^*_M that makes the following diagram commute

$$\begin{tikzcd}
\mathbf{SmProj}/k^{\mathrm{op}} \arrow[r, "M_{gm}"] \arrow[dr, "H^*"'] & \mathbf{M}_{\mathrm{hom}} \arrow[d, dotted, "H^*_M"] \\
& \mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K
\end{tikzcd} . \tag{A.16}$$

Now suppose $\sim$ is rational equivalence or numerical equivalence [80], and we will construct the category of motives $\mathbf{M}_\sim$ for $\sim$. Given two non-singular projective varieties X and Y, the group of correspondences from X to Y of degree r is defined by

$$\mathrm{Corr}^r(X,Y) := C^{\dim X + r}(X \times Y). \tag{A.17}$$

Correspondences can be composed [91]

$$\mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^s(Y,Z) \to \mathrm{Corr}^{r+s}(X,Z), \tag{A.18}$$

which is given by

$$g \times h \to h \circ g := (p_{13})_* \big((p_{12})^* g \cdot (p_{23})^* h \big), \tag{A.19}$$

where p_{12} is the natural projection morphism from $X \times Y \times Z$ to $X \times Y$, etc. Given a morphism $f : Y \to X$, its graph Γ_f in $X \times Y$ is an algebraic variety that is isomorphic to Y, therefore Γ_f is an element of $\mathrm{Corr}^0(X,Y)$ [98]. On the other hand, a correspondence of $\mathrm{Corr}^0(X,Y)$ can be viewed as a multi-valued morphism from Y to X. A correspondence γ induces a homomorphism from $H^*(X)$ to $H^*(Y)$ through

$$\gamma_* : x \to p_{2,*}\,(p_1^* x \cup \mathrm{cl}(\gamma)), \tag{A.20}$$

where p_1 (resp. p_2) is the projection morphism from $X \times Y$ to X (resp. Y). The homomorphism $(\Gamma_f)_*$ induced by Γ_f is just the pullback homomorphism f^*. The category $\mathbf{M}_\sim$ is constructed by the following three steps [80]:

1. Construct a category whose objects are formal symbols

   $$\{M_{gm}(X) : X \in \mathbf{SmProj}/k\}, \tag{A.21}$$

   and let the morphisms between two objects be

   $$\mathrm{Hom}(M_{gm}(X), M_{gm}(Y)) := \mathrm{Corr}^0_\sim(X,Y)_{\mathbb{Q}}, \tag{A.22}$$

   where

   $$\mathrm{Corr}^r_\sim(X,Y) = \mathrm{Corr}^r(X,Y)/\sim,\ \mathrm{Corr}^r_\sim(X,Y)_{\mathbb{Q}} = \mathrm{Corr}^r_\sim(X,Y) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{A.23}$$

   This category can be viewed as the linearisation of $\mathbf{SmProj}/k^{\mathrm{op}}$.

2. Take the pseudo-abelianisation of the category constructed in step 1 and denote this new category by $\mathbf{M}^{\text{eff}}_{\sim}$. The objects of $\mathbf{M}^{\text{eff}}_{\sim}$ are formally given by

   $$\{(M_{gm}(X), e) : X \in \mathbf{SmProj}/k \,\text{and}\; e \in \operatorname{Corr}^0_{\sim}(X, X)_{\mathbb{Q}},\, e^2 = e\}, \tag{A.24}$$

   and the morphisms between two objects are

   $$\operatorname{Hom}((M_{gm}(X), e), (M_{gm}(Y), f)) := f \circ \operatorname{Corr}^0_{\sim}(X, Y)_{\mathbb{Q}} \circ e. \tag{A.25}$$

   Denote the graph of the identity morphism of $\mathbb{P}^1$ by $\Delta_{\mathbb{P}^1}$, then the object $(M_{gm}(\mathbb{P}^1), \Delta_{\mathbb{P}^1})$ has a decomposition [80]

   $$(M_{gm}(\mathbb{P}^1), \Delta_{\mathbb{P}^1}) = (M_{gm}(\mathbb{P}^1), \{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}) = M^0_{gm}(\mathbb{P}^1) \oplus M^2_{gm}(\mathbb{P}^1). \tag{A.26}$$

   The component $M^0_{gm}(\mathbb{P}^1)$ will be called $\mathbb{Q}(0)$ and the component $M^2_{gm}(\mathbb{P}^1)$ will be called $\mathbb{Q}(-1)$.

3. The category of motives $\mathbf{M}_{\sim}$ is constructed from $\mathbf{M}^{\text{eff}}_{\sim}$ by inverting the object $\mathbb{Q}(-1)$. The objects of $\mathbf{M}_{\sim}$ are formally given by

   $$\{(M_{gm}(X), e, m) : X \in \mathbf{SmProj}/k,\; e \in \operatorname{Corr}^0_{\sim}(X, X)_{\mathbb{Q}},\, e^2 = e, \;\text{and}\; m \in \mathbb{Z}\}, \tag{A.27}$$

   and the morphisms between two objects are

   $$\operatorname{Hom}((M_{gm}(X), e, m), (M_{gm}(Y), f, n)) := f \circ \operatorname{Corr}^{n-m}_{\sim}(X, Y)_{\mathbb{Q}} \circ e. \tag{A.28}$$

   The category $\mathbf{M}^{\text{eff}}_{\sim}$ is isomorphic to the full subcategory of $\mathbf{M}_{\sim}$ whose objects are of the form $(M_{gm}(X), e, 0)$.

The morphisms between two objects of $\mathbf{M}_{\sim}$ form a rational vector space whose dimension is finite when $\sim$ is the numerical equivalence. The direct sum in $\mathbf{M}_{\sim}$ is essentially defined by [91]

$$(M_{gm}(X), e, m) \oplus (M_{gm}(Y), f, m) := (M_{gm}(X \amalg Y), e \oplus f, m), \tag{A.29}$$

while the tensor product in $\mathbf{M}_{\sim}$ is essentially defined by

$$(M_{gm}(X), e, m) \otimes (M_{gm}(Y), f, n) := (M_{gm}(X \times Y), e \times f, m + n). \tag{A.30}$$

The object $\mathbb{Q}(0)$ is shown to be a unit [80, 91], and the dual operation in $\mathbf{M}_{\sim}$ is defined by

$$(M_{gm}(X), e, m)^{\vee} := (M_{gm}(X), e^t, \dim X - m), \tag{A.31}$$

where e^t means the transpose of e. From the construction of $\mathbf{M}_\sim$, there exists a functor

$$M_{gm} : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathbf{M}_\sim, \tag{A.32}$$

which sends X to $(M_{gm}(X), \Delta_X, 0)$ and $f : Y \to X$ to Γ_f. It is straightforward to see that every Weil cohomology theory H^* automatically factors through $\mathbf{M}_{\mathrm{rat}}$

$$\begin{tikzcd}
\mathbf{SmProj}/k^{\mathrm{op}} \arrow[r, "M_{gm}"] \arrow[dr, "H^*"'] & \mathbf{M}_{\mathrm{rat}} \arrow[d, dotted, "H^*_{\mathrm{rat}}"] \\
& \mathrm{Gr}^{\geq 0}\,\mathrm{Vec}_K
\end{tikzcd}, \tag{A.33}$$

however the category $\mathbf{M}_{\mathrm{rat}}$ is not abelian [80, 91]. On the other hand the category $\mathbf{M}_{\mathrm{num}}$ has been proved to be abelian and semi-simple [65, 80], but it is not known whether an arbitrary Weil cohomology theory H^* will factor through it. Given an algebraic cycle γ such that $\gamma \sim_{\mathrm{num}} 0$, then it induces a zero homomorphism in $\mathbf{M}_{\mathrm{num}}$. In order for H^* to factor through $\mathbf{M}_{\mathrm{num}}$, the induced homomorphism γ_* A.20 needs to be zero, which is equivalent to the property that $\mathrm{cl}(\gamma) = 0$. However this is not known currently, but it is conjectured to be true by Grothendieck.

Conjecture D *Given an algebraic cycle γ that is numerically equivalent to 0, then its cohomology class cl(γ) is zero for every Weil cohomology theory.*

This conjecture also implies that the homological equivalence relation $\sim_{H^*}$ defined by a Weil cohomology theory H^* is the same as the numerical equivalence. There are also several other important conjectures concerning the properties of $\mathbf{M}_{\mathrm{num}}$, together with this one, they are called the standard conjectures of Grothendieck [74].

A.4 The conjectured abelian category of mixed motives

The theory of pure motives can be viewed as the universal cohomology theory for non-singular projective varieties, so one might wonder what is the universal (Bloch-Ogus) cohomology theory for the arbitrary varieties. Beilinson conjectured that there exists a rigid tensor abelian category of mixed motives $\mathbf{MM}_k$ that has similar properties to that of $\mathbf{MHS}_{\mathbb{Q}}$, which forms the universal cohomology theory for the arbitrary varieties defined over k [75]. Here we will list several expected properties of the conjectured abelian category $\mathbf{MM}_k$:

1. $\mathbf{MM}_k$ is a rigid tensor abelian category whose Hom sets are vector spaces over $\mathbb{Q}$. It contains Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$ that satisfy

   $$\mathbb{Q}(m) \otimes \mathbb{Q}(n) = \mathbb{Q}(m+n), \tag{A.34}$$

   while $\mathbb{Q}(0)$ is a unit object.

2. There exists a contravariant functor M_{gm} from the category of varieties over k to the derived category of $\mathbf{MM}_k$

   $$M_{gm} : \mathbf{Var}/k^{\mathrm{op}} \to D^b(\mathbf{MM}_k). \tag{A.35}$$

3. The full subcategory of $\mathbf{MM}_k$ consisting of semi-simple objects is equivalent to the category of pure motives.

4. For every object $\mathcal{M}$ of $\mathbf{MM}_k$, there exists a finite weight filtration $W_*(\mathcal{M})$ such that all the graded pieces $\mathrm{Gr}_l^W(\mathcal{M})$ are pure motives.

5. If $\sigma : k \to \mathbb{C}$ is an embedding, then there exists a Hodge realisation functor

   $$\mathfrak{R}_\sigma : \mathbf{MM}_k \to \mathbf{MHS}_{\mathbb{Q}}, \tag{A.36}$$

   which is compatible with all the structures of $\mathbf{MM}_k$ and $\mathbf{MHS}_{\mathbb{Q}}$. For every variety X over k, $\mathfrak{R}_\sigma(H^q(M_{gm}(X)))$ is the q-th Betti cohomology $H^q_{B,\sigma}(X)$ together with the (natural) rational MHS on it [39].

6. The abelian category of mixed Tate motives $\mathbf{TM}_k$ is the smallest full abelian subcategory of $\mathbf{MM}_k$ that contains the Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$ and is also closed under extensions.

The construction of an abelian category $\mathbf{MM}_k$ that possesses all the expected properties is still beyond reach. However now there are several constructions of the triangulated tensor categories (e.g. Voevodsky's construction of $\mathbf{DM}(k, \mathbb{Q})$ [76, 99]) that satisfy nearly all the expected properties of the derived category of $\mathbf{MM}_k$, except those properties that need these triangulated categories to be realised as the derived category of an abelian category, like a motivic t-structure.

Appendix B

Voevodsky's triangulated category of mixed motives

This appendix is a very brief overview of Voevodsky's triangulated category of mixed motives and the abelian category of mixed Tate motives over $\mathbb{Q}$. The structure of this appendix is as follows:

- Section B.1 discusses some properties of Voevodsky's triangulated category of mixed motives.
- Section B.2 discusses some properties of the abelian category of mixed Tate motives.

B.1 Voevodsky's mixed motives

Let k be a field that admits resolution of singularities and Λ be a commutative ring with unit, Voevodsky's category of mixed motives, denoted by $\mathbf{DM}(k, \Lambda)$, is a rigid tensor triangulated category [76, 99]. The ring Λ is called the coefficient ring, while in this paper we are mostly interested in the case where it is $\mathbb{Q}$. The category $\mathbf{DM}(k, \mathbb{Q})$ has nearly all the expected properties of the derived category of the conjectured abelian category of mixed motives defined over k [75]. In this section, we will only list some properties of $\mathbf{DM}(k, \Lambda)$ and use them as a blackbox, while leave the construction of $\mathbf{DM}(k, \Lambda)$ and the proofs to these properties to the excellent references [76, 99]. Meanwhile the first section of [3] is also very helpful.

1. The category $\mathbf{DM}(k, \Lambda)$ is a rigid tensor triangulated category that contains pure Tate motives $\Lambda(n)$, $n \in \mathbb{Z}$. $\Lambda(0)$ is a unit object of it and $\Lambda(-1)$ is the dual of $\Lambda(1)$
$$\Lambda(-1) = \mathrm{Hom}\big(\Lambda(1), \Lambda(0)\big), \tag{B.1}$$

where Hom is the internal Hom operator. The Tate object $\Lambda(n)$ satisfies

$$\Lambda(n) = \begin{cases} \Lambda(1)^{\otimes n} & \text{if } n \geq 0 \\ \Lambda(-1)^{\otimes n} & \text{if } n < 0 \end{cases} . \tag{B.2}$$

For an object $\mathcal{N}$ of $\mathbf{DM}(k, \Lambda)$, its Tate twist $\mathcal{N}(n)$ is defined as $\mathcal{N} \otimes \Lambda(n)$, while its dual $\mathcal{N}^\vee$ is defined as $\mathrm{Hom}\big(\mathcal{N}, \Lambda(0)\big)$.

2. There exists a contravariant functor from the category of non-singular projective varieties over k to the category $\mathbf{DM}(k, \Lambda)$

$$M_{gm} : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathbf{DM}(k, \Lambda), \tag{B.3}$$

which sends a non-singular projective variety X to a constructible object of $\mathbf{DM}(k, \Lambda)$ such that fibered product in $\mathbf{SmProj}/k$ is sent to tensor product in $\mathbf{DM}(k, \Lambda)$, i.e.

$$M_{gm}(X \times_k Y) = M_{gm}(X) \otimes M_{gm}(Y). \tag{B.4}$$

The definition of constructibility could be found in [3], and the full triangulated subcategory of $\mathbf{DM}(k, \Lambda)$ consists of constructible objects will be denoted by $\mathbf{DM}_{\mathrm{gm}}(k, \Lambda)$, which is the smallest full pseudoabelian triangulated subcategory of $\mathbf{DM}(k, \Lambda)$ that contains the image of M_{gm} and is also closed under Tate twists.

3. When the field k admits an embedding into $\mathbb{C}$, say $\sigma : k \to \mathbb{C}$, there exists a Hodge realisation functor

$$\mathfrak{R}_\sigma : \mathbf{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to D^b(\mathbf{MHS}_{\mathbb{Q}}) \tag{B.5}$$

such that for every non-singular projective variety X, $\mathfrak{R}_\sigma\big(M_{gm}(X)\big)$ is a complex in $D^b(\mathbf{MHS}_{\mathbb{Q}})$ whose cohomology computes the singular cohomology of $X(\mathbb{C})$ with the natural rational MHS [61, 62, 86]. In this paper we are mostly interested in the case where $k = \mathbb{Q}$, since there is only one embedding of $\mathbb{Q}$ into $\mathbb{C}$, let us denote the Hodge realisation functor by $\mathfrak{R}$ for simplicity.

4. The composition of $\mathfrak{R}_\sigma$ with the forgetful functor from $D^b(\mathbf{MHS}_{\mathbb{Q}})$ to the derived category of rational vector spaces $D^b(\mathbf{Vec}_{\mathbb{Q}})$ is (up to an equivalence) the Betti realisation functor $\mathfrak{R}_{\mathrm{Betti}}$ [5]

$$\mathfrak{R}_{\mathrm{Betti}} : \mathbf{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \to D^b(\mathbf{Vec}_{\mathbb{Q}}). \tag{B.6}$$

B.2 The mixed Tate motives

We now briefly talk about the abelian category of mixed Tate motives $\mathbf{TM}_{\mathbb{Q}}$, while the readers are referred to the paper [73] for detail. Let $K_i(k)$ be the i-th algebraic K-group of the field k [44], there exists a family of Adams operators $\{\psi^l\}_{l\geq 1}$ which act on $K_i(k)$ as group homomorphisms [44]. These Adams operators induce linear maps on the rational vector space $K_i(k)\otimes_{\mathbb{Z}}\mathbb{Q}$, which induce a decomposition

$$K_i(k)\otimes_{\mathbb{Z}}\mathbb{Q} = \oplus_{j\geq 0}\, K_i(k)^{(j)}, \tag{B.7}$$

where the eigenspace $K_i(k)^{(j)}$ is defined by

$$K_i(k)^{(j)} := \{x \in K_i(k)\otimes_{\mathbb{Z}}\mathbb{Q} : \psi^l(x) = l^j\, x, \forall\, l \geq 1\}. \tag{B.8}$$

The strong version of Beilinson and Soulé's vanishing conjecture claims that [73]

Conjecture BS $K_{2q-p}(k)^{(q)} = 0$ *if* $p \leq 0$ *and* $q > 0$.

When the field k is $\mathbb{Q}$, Conjecture **BS** has been proved [41]. Let $\mathbf{DTM}_{\mathbb{Q}}$ be the full triangulated subcategory of $\mathbf{DM}_{\mathrm{gm}}(\mathbb{Q},\mathbb{Q})$ generated by Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$, from [73] there exists a motivic t-structure on $\mathbf{DTM}_{\mathbb{Q}}$ whose heart is defined to be the category of mixed Tate motives $\mathbf{TM}_{\mathbb{Q}}$. For two objects A and B of $\mathbf{TM}_{\mathbb{Q}}$, an extension of B by A is a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{B.9}$$

Two extensions are said to be isomorphic to each other if there exists a commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & E \arrow[r] \arrow[d, "\simeq"] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}, \tag{B.10}$$

where the central vertical morphism is an isomorphism. The extension B.9 is said to split if it is isomorphic to the trivial extension

$$0 \longrightarrow A \xrightarrow{i} A\oplus B \xrightarrow{j} B \longrightarrow 0, \tag{B.11}$$

where i is the natural inclusion and j is the natural projection. The set of isomorphism classes of extensions of B by A, denoted by $\mathrm{Ext}^1_{\mathbf{TM}_{\mathbb{Q}}}(B,A)$, has a group structure that is induced by Baer summation, whose zero element is the trivial extension B.11. The extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n), n \geq 3$ could be described explicitly by Corollary 4.3 of [73].

Lemma B.2.1. *There exists an isomorphism*

$$\tau_{n,1} : Ext^1_{\boldsymbol{TM}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big) \to K_{2n-1}(\mathbb{Q})^{(n)},\ n \geq 2. \tag{B.12}$$

The rank of the higher algebraic K-group $K_{2n-1}(\mathbb{Q})$ is well-known [47]

$$\operatorname{rank} K_{2n-1}(\mathbb{Q}) = \begin{cases} 0 & \text{if } n = 2k, k \geq 1 \\ 1 & \text{if } n = 2k+1, k \geq 1 \end{cases}, \tag{B.13}$$

which yields

$$K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} 0 & \text{if } n = 2\,k,\ k \geq 1 \\ \mathbb{Q} & \text{if } n = 2\,k+1,\ k \geq 1 \end{cases}. \tag{B.14}$$

As $K_{2n-1}(\mathbb{Q})^{(n)}$ is a linear subspace of $K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$, this immediately implies that

$$K_{2n-1}(\mathbb{Q})^{(n)} = 0, \text{ if } n = 2\,k,\ k \geq 1. \tag{B.15}$$

When $n = 2\,k+1,\ k \geq 1$, as a linear subspace of $K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $K_{2n-1}(\mathbb{Q})^{(n)}$ is either 0 or $\mathbb{Q}$. From [36, 41, 57], there exists a nontrivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ when $n = 2\,k+1,\ k \geq 1$, hence $\mathrm{Ext}^1_{\mathbf{TM}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big)$ is nonzero and we have

$$\mathrm{Ext}^1_{\mathbf{TM}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big) \simeq K_{2n-1}(\mathbb{Q})^{(n)} = \mathbb{Q}, \text{ if } n = 2\,k+1,\ k \geq 1. \tag{B.16}$$

The restriction of the Hodge realisation functor $\mathfrak{R}$ to $\mathbf{TM}_{\mathbb{Q}}$ is a functor

$$\mathfrak{R} : \mathbf{TM}_{\mathbb{Q}} \to \mathbf{MHT}_{\mathbb{Q}}, \tag{B.17}$$

where $\mathbf{MHT}_{\mathbb{Q}}$ is the full abelian subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ that consists of mixed Hodge-Tate structures, i.e. those mixed Hodge structures whose semi-simplifications are direct sums of Tate objects $\mathbb{Q}(n)$. From [41], the restriction of $\mathfrak{R}$ to $\mathbf{TM}_{\mathbb{Q}}$ B.17 is exact and full-faithful, hence it induces an injective homomorphism from $\mathrm{Ext}^1_{\mathbf{TM}_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(n))$ to $\mathrm{Ext}^1_{\mathbf{MHT}_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(n))$. Since $\mathbf{MHT}_{\mathbb{Q}}$ is a full abelian subcategory of $\mathbf{MHS}_{\mathbb{Q}}$, the isomorphism 3.30 immediately implies

$$\mathrm{Ext}^1_{\mathbf{MHT}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big) = \mathrm{Ext}^1_{\mathbf{MHS}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big) \simeq \mathbb{C}/(2\pi i)^n\,\mathbb{Q}. \tag{B.18}$$

Lemma B.2.2. *When* $n \geq 3$*, the image of* $Ext^1_{\boldsymbol{TM}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big)$ *in* $\mathbb{C}/(2\pi i)^n\,\mathbb{Q}$ *under Hodge realisation is the subgroup of* $\mathbb{C}/(2\pi i)^n\,\mathbb{Q}$ *consists of elements which are the cosets of rational multiples of* $\zeta(n)$*.*

Proof. When $n = 2\,k$, $k \geq 2$, $\zeta(n)$ is a rational multiple of $(2\,\pi\,i)^n$, therefore the coset of $\zeta(n)$ in $\mathbb{C}/(2\pi i)^n\,\mathbb{Q}$ is 0, hence this lemma is a direct result of Lemma B.2.1.

When $n = 2\,k + 1$, $k \geq 1$, from [36, 41, 57], there exists a mixed Tate motive that is a nontrivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$, whose Hodge realisation in $\mathrm{Ext}^1_{\mathbf{MHT}_{\mathbb{Q}}}\big(\mathbb{Q}(0), \mathbb{Q}(n)\big)$ is the coset of a nonzero rational multiple of $\zeta(n)$. As the coset of $\zeta(n)$ in $\mathbb{C}/(2\pi i)^n\,\mathbb{Q}$ is nonzero, this lemma is a direct result of Lemma B.2.1. $\square$

Appendix C

Limit mixed motive

This appendix is devoted to the construction of the limit mixed motive using Ayoub's motivic nearby cycle functor [4]. The structure of this appendix is as follows:

- Section C.1 discusses the construction of the category of étale motivic sheaves.
- Section C.2 discusses the construction of the limit mixed motive.

C.1 A naive construction of étale motivic sheaves

Let Λ be a commutative ring which will be the coefficients ring in the construction of étale motivic sheaves, and in this paper we are mostly interested in the case when Λ is $\mathbb{Q}$. In order to satisfy some technical assumptions, all schemes in this section are assumed to be separated, Noetherian and of finite Krull dimension. For a base scheme U, the category of étale motivic sheaves with coefficients ring Λ will be denoted by $\mathbf{DA}^{\text{ét}}(U, \Lambda)$. Here we will follow [10] and give an incorrect naive construction of a category $\mathbf{DA}^{\text{ét,naive}}(U, \Lambda)$, which nonetheless catches some essences of $\mathbf{DA}^{\text{ét}}(U, \Lambda)$ and suffices for this paper.

Let $\mathbf{Sm}/U$ be the category of smooth U-schemes endowed with étale topology [78] and let $\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda)$ be the abelian category of étale sheaves on $\mathbf{Sm}/U$ that take values in the abelian category of Λ-modules. A smooth U-scheme Z defines a presheaf through

$$V \in \mathbf{Sm}/U \to \Lambda \otimes \mathrm{Hom}_U(V, Z), \tag{C.1}$$

where $\Lambda \otimes \mathrm{Hom}_U(V, Z)$ is the Λ-module generated by the set $\mathrm{Hom}_U(V, Z)$, and the sheaf associated to this presheaf will be denoted by $\Lambda_{\text{ét}}(X)$. In this way we find a Yoneda functor

$$\Lambda_{\text{ét}} : \mathbf{Sm}/U \to \mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda), \tag{C.2}$$

which can be considered as the first-step linearisation of the category $\mathbf{Sm}/S$.

Lemma C.1.1. *The étale sheaf $\Lambda_{\acute{e}t}(U)$ associated to the identity morphism of U is the constant étale sheaf on $\boldsymbol{Sm}/U$.*

Proof. For the smooth U-scheme U given by identity morphism, $\mathrm{Hom}_U(V, U)$ consists of only one element for every scheme V of $\mathbf{Sm}/U$, i.e. the structure morphism $V \to U$. So we have

$$\Lambda_{\text{ét}}(U)(V) = \Lambda \otimes \mathrm{Hom}_U(V, U) \simeq \Lambda. \tag{C.3}$$

It is easy to check that the restriction homomorphism is the identity homomorphism of Λ, hence the sheaf $\Lambda_{\text{ét}}(U)$ is the constant sheaf on $\mathbf{Sm}/U$. □

The next step in the construction of the triangulated category of étale motivic sheaves is to take $\mathbb{A}^1$-localisation [10]. Suppose $D(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda))$ is the derived category of $\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda)$, and let $\mathcal{T}_{\mathbb{A}^1}$ be the smallest full triangulated subcategory of $D(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda))$ that is closed under arbitrary direct sums and also contains all the complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow \Lambda_{\text{ét}}(\mathbb{A}^1_U \times_U V) \longrightarrow \Lambda_{\text{ét}}(V) \longrightarrow 0 \longrightarrow \cdots, \tag{C.4}$$

where V is a smooth U-scheme and the morphism from $\Lambda_{\text{ét}}(\mathbb{A}^1_U \times_U V)$ to $\Lambda_{\text{ét}}(V)$ is induced by the projection $\mathbb{A}^1_U \times_U V \to V$. Define $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$ to be the Verdier quotient [10]

$$\mathbf{DA}^{\text{ét,eff}}(U, \Lambda) := D\big(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda)\big)/\mathcal{T}_{\mathbb{A}^1}, \tag{C.5}$$

whose objects are the same as that of $D\big(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda)\big)$, hence by abuse of notations, the objects of $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$ will be denoted by the same symbols as that of $D(\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U; \Lambda))$. As the name implies, objects of $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$ will be called effective U-motives. The effect of Verdier quotient is that morphisms of $D(\mathbf{Sh}_{\text{ét}}\big(\mathbf{Sm}/U; \Lambda)\big)$ whose cones lie in $\mathcal{T}_{\mathbb{A}^1}$ get inverted in $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$. For example, the cone of the morphism

$$\Lambda_{\text{ét}}(\mathbb{A}^1_U \times_U V) \to \Lambda_{\text{ét}}(V) \tag{C.6}$$

is in $\mathcal{T}_{\mathbb{A}^1}$, hence it becomes an isomorphism in $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$, i.e. in $\mathbf{DA}^{\text{ét,eff}}(U, \Lambda)$, the object $\Lambda_{\text{ét}}(\mathbb{A}^1_U \times_U V)$ is isomorphic to the object $\Lambda_{\text{ét}}(V)$.

Definition C.1.2. *Let $\boldsymbol{DA}^{\acute{e}t,eff}_{ct}(U, \Lambda)$ be the smallest full triangulated subcategory of the category $\boldsymbol{DA}^{\acute{e}t,eff}(U, \Lambda)$ that contains all the objects of the form $\Lambda_{\acute{e}t}(Z)$ with $Z \in \boldsymbol{Sm}/U$ of finite presentation and is also closed under taking direct summand. Objects of $\boldsymbol{DA}^{\acute{e}t,eff}_{ct}(U, \Lambda)$ will be called constructible U-motives.*

The last step in the construction is stabilisation, and we will follow Ayoub and only give a naive stabilisation in this paper [10]. The injection $\infty_U \hookrightarrow \mathbb{P}^1_U$ induces a morphism

$$\Lambda_{\text{ét}}(\infty_U) \to \Lambda_{\text{ét}}(\mathbb{P}^1_U), \tag{C.7}$$

whose cokernel in $\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U;\Lambda)$ will be denoted by $\Lambda_{\text{ét}}(\mathbb{P}^1_U, \infty_U)$. The motive in $\mathbf{DA}^{\text{ét,eff}}(U,\Lambda)$ defined by $\Lambda_{\text{ét}}(\mathbb{P}^1_U, \infty_U)$ will be called the Lefschetz motive

$$L := \Lambda_{\text{ét}}(\mathbb{P}^1_U, \infty_U). \tag{C.8}$$

The naive stabilisation is to invert the Lefschetz motive L

$$\mathbf{DA}^{\text{ét,naive}}(U,\Lambda) := \mathbf{DA}^{\text{ét,eff}}(U,\Lambda)[L^{-1}]. \tag{C.9}$$

More precisely, objects of $\mathbf{DA}^{\text{ét,naive}}(U,\Lambda)$ are formal pairs (M,m) where M is an object of $\mathbf{DA}^{\text{ét,eff}}(U,\Lambda)$ and m is an integer. Morphisms between two objects (M,m) and (N,n) are given by

$$\varinjlim_{r\geq -\min(m,n)} \mathrm{Hom}_{\mathbf{DA}^{\text{ét,eff}}(U,\Lambda)}(M \otimes L^{r+m}, N \otimes L^{r+n}). \tag{C.10}$$

This naive stabilisation has the merit of being very straightforward, but suffers many technical problems, e.g. $\mathbf{DA}^{\text{ét,naive}}(U,\Lambda)$ is not even a triangulated category. However it still catches some of the essences of the category of étale motivic sheaves $\mathbf{DA}^{\text{ét}}(U,\Lambda)$. More precisely, let $\mathbf{DA}^{\text{ét,naive}}_{\text{ct}}(U,\Lambda)$ be the full subcategory of $\mathbf{DA}^{\text{ét,naive}}(U,\Lambda)$ that consists of objects (M,m) such that M is an object of $\mathbf{DA}^{\text{ét,eff}}_{\text{ct}}(U,\Lambda)$. Under some technical assumptions, which are all satisfied when U is a quasiprojective variety over a field of characteristic 0 and Λ is $\mathbb{Q}$, the category $\mathbf{DA}^{\text{ét,naive}}_{\text{ct}}(U,\Lambda)$ is equivalent to the full triangulated subcategory $\mathbf{DA}^{\text{ét}}_{\text{ct}}(U,\Lambda)$ of $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ that consists of constructible objects, which is certainly the most important subcategory of $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ [10]. Since in this paper we will only be concerned with constructible objects of $\mathbf{DA}^{\text{ét}}(U,\Lambda)$, this naive stabilisation will suffice for this paper. The defect-free construction of $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ will not be talked here and is left to the paper [10].

Remark C.1.3. *From the construction of* $\mathbf{DA}^{\text{ét}}(U,\Lambda)$, *there exists a covariant functor from* $\mathbf{Sm}/U$ *to* $\mathbf{DA}^{\text{ét}}(U,\Lambda)$, *so the construction in this section is covariant. However in Section B, the functor which attaches a mixed motive to a variety is contravariant as we are interested in cohomology theory instead of homology theory, while the difference is just a dual operation [74].*

The category $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ satisfies Grothendieck's six operations formalism [4, 10], but here we will only mention one such operation. Given a morphism $g : U \to V$, there exists a pushforward functor g_*

$$g_* : \mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U;\Lambda) \to \mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/V;\Lambda), \tag{C.11}$$

which sends an étale sheaf $\mathcal{G}$ of $\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/U;\Lambda)$ to $g_*\mathcal{G}$ of $\mathbf{Sh}_{\text{ét}}(\mathbf{Sm}/V;\Lambda)$ such that

$$g_*\mathcal{G}(W) := \mathcal{G}(W \times_V U),\ W \in \mathbf{Sm}/V. \tag{C.12}$$

The functor g_* can be derived and it induces a functor $\mathrm{R}g_*$ [10]

$$\mathrm{R}g_* : \mathbf{DA}^{\text{ét,eff}}(U,\Lambda) \to \mathbf{DA}^{\text{ét,eff}}(V,\Lambda). \tag{C.13}$$

The functor g_* can be extended to the L-spectra, which can again be derived and yields a functor $\mathrm{R}g_*$ [4, 10]

$$\mathrm{R}g_* : \mathbf{DA}^{\text{ét}}(U,\Lambda) \to \mathbf{DA}^{\text{ét}}(V,\Lambda). \tag{C.14}$$

From [4], the functor $\mathrm{R}g_*$ sends constructible objects of $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ to constructible objects of $\mathbf{DA}^{\text{ét}}(V,\Lambda)$.

C.2 Ayoub's motivic nearby cycle functor

The category $\mathbf{DA}^{\text{ét}}(U,\Lambda)$ satisfies the nearby cycle formalism, which realises to the classical nearby cycle functors [4, 9]. Since 0 is a smooth point of C, the local ring $\mathscr{O}_{C,0}$ is a discrete valuation ring, and the affine scheme $\operatorname{Spec}\mathscr{O}_{C,0}$ admits an injection into C [98]

$$\operatorname{Spec}\mathscr{O}_{C,0} \hookrightarrow C. \tag{C.15}$$

Let the henselisation of $\mathscr{O}_{C,0}$ be $\mathscr{O}^{\mathrm{h}}_{C,0}$, then there is an injective local ring homomorphism from $\mathscr{O}_{C,0}$ to $\mathscr{O}^{\mathrm{h}}_{C,0}$ that induces a morphism [106]

$$\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,0} \to \operatorname{Spec}\mathscr{O}_{C,0}. \tag{C.16}$$

The affine scheme $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,0}$ consists of two points: a generic point η and a closed point s with residue field is $\mathbb{Q}$. For simplicity, let us denote $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,0}$ by B, and we have a henselian trait (B, s, η)

$$\eta \longrightarrow B \longleftarrow s. \tag{C.17}$$

The composition of C.15 and C.16 is a morphism $i : B \to C$ and let $f : X_B \to B$ be the pull-back of $\pi_{\mathbb{Q}}$ 3.1 along i

$$\begin{tikzcd} X_B \arrow[r] \arrow[d, "f"] & X \arrow[d, "\pi_{\mathbb{Q}}"] \\ B \arrow[r, "i"] & C \end{tikzcd}, \tag{C.18}$$

then the pull-backs of f along $\eta \to B$ and $s \to B$ form a commutative diagram

$$\begin{tikzcd} X_\eta \arrow[r] \arrow[d, "f_\eta"] & X_B \arrow[d, "f"] & X_s \arrow[l] \arrow[d, "f_s"] \\ \eta \arrow[r] & B & s \arrow[l] \end{tikzcd}, \tag{C.19}$$

where X_s is just Y. There exists a motivic nearby cycle functor $\mathrm{R}\Psi_f$

$$\mathrm{R}\Psi_f : \mathbf{DA}^{\text{ét}}(X_\eta, \mathbb{Q}) \to \mathbf{DA}^{\text{ét}}(X_s, \mathbb{Q}), \tag{C.20}$$

whose construction is left to [4, 9]. From Theorem 10.9 of [9], the functor $\mathrm{R}\Psi_f$ sends constructible objects of $\mathbf{DA}^{\text{ét}}(X_\eta, \mathbb{Q})$ to constructible objects of $\mathbf{DA}^{\text{ét}}(X_s, \mathbb{Q})$. From Lemma C.1.1, the identity morphism of X_η induces the constant étale sheaf $\mathbb{Q}_{\text{ét}}(X_\eta)$ on $\mathbf{Sm}/X_\eta$ which defines a constructible motive of $\mathbf{DA}^{\text{ét}}(X_\eta, \Lambda)$ that is also denoted by $\mathbb{Q}_{\text{ét}}(X_\eta)$. Therefore $\mathrm{R}\Psi_f(\mathbb{Q}_{\text{ét}}(X_\eta))$ is a constructible motive of $\mathbf{DA}^{\text{ét}}(X_s, \mathbb{Q})$, which is called nearby motivic sheaf by Ayoub. The nearby motivic sheaf $\mathrm{R}\Psi_f\big(\mathbb{Q}_{\text{ét}}(X_\eta)\big)$ realises to the classical nearby cycle sheaves by Théorème 4.9 of [5] in the Betti realisation case and by Théorème 10.11 of [9] in the ℓ-adic realisation case, also see Section 1.2 of [8]. The structure morphism f_s in C.19 induces a functor $\mathrm{R}(f_s)_*$ [4]

$$\mathrm{R}(f_s)_* : \mathbf{DA}^{\text{ét}}(X_s, \mathbb{Q}) \to \mathbf{DA}^{\text{ét}}(\mathbb{Q}, \mathbb{Q}), \tag{C.21}$$

which sends constructible objects of $\mathbf{DA}^{\text{ét}}(X_s, \mathbb{Q})$ to constructible objects of $\mathbf{DA}^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$ [4]. Now define the limit mixed motive $\mathscr{Z}$ to be

$$\mathscr{Z} := \mathrm{R}(f_s)_* \circ \mathrm{R}\Psi_f(\mathbb{Q}_{\text{ét}}(X_\eta)), \tag{C.22}$$

which will be a constructible object of $\mathbf{DA}^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$. From Theorem 4.4 of [10], the category $\mathbf{DA}^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$ is equivalent to $\mathbf{DM}^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$, a fact that is also discussed in Section 4.3 of [10]. From Theorem 14.30 of [76], $\mathbf{DM}^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$ is equivalent to the dual of $\mathbf{DM}(\mathbb{Q}, \mathbb{Q})$, therefore the dual $\mathscr{Z}^\vee$ of $\mathscr{Z}$ can be considered as a constructible object of $\mathbf{DM}(\mathbb{Q}, \mathbb{Q})$, i.e. an object of $\mathbf{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$. The motive $\mathrm{R}\Psi_f\big(\mathbb{Q}_{\text{ét}}(X_\eta)\big)$ realises to the classical nearby cycle sheave for the Betti realisation by Théorème 4.9

of [5] (when X_B is the base change of a finite type $\mathbb{Q}[\varphi]$-scheme) and for the ℓ-adic realisation by Théorème 10.11 of [9]. Therefore the realisations of $\mathscr{Z}^\vee$ compute the cohomologies of classical nearby cycle sheaves (in Betti case or ℓ-adic case) and Ayoub conjectures that more is true,

Conjecture C.2.1. *The Hodge realisation of $\mathscr{Z}^\vee$ C.22 is isomorphic to the complex $\mathbf{Z}^\bullet$ in 3.94, i.e. the Hodge realisation of $\mathscr{Z}^\vee$ computes the limit MHS in 3.93 (up to an isomorphism).*

However currently it is not available in literature and the proof of it, even though could be considered as 'routine', can be technically very hard.

Appendix D

The Picard-Fuchs operators of $X_{\mathfrak{G}}$

This appendix gives the two Picard-Fuchs operators 7.36 of $X_{\mathfrak{G}}$ introduced in Chapter 7, which are of the form

$$\mathcal{L}_1 \;=\; \sum_{k,\,l\,\geq 0}^{k+l\,\leq\,2} R^1_{k,\,l}\,\vartheta_\varphi^k\,\vartheta_\nu^l\;, \qquad \mathcal{L}_2 \;=\; \sum_{k,\,l\,\geq 0}^{k+l\,\leq\,3} R^2_{k,\,l}\,\vartheta_\varphi^k\,\vartheta_\nu^l, \tag{D.1}$$

where ϑ_φ and ϑ_ν are defined by

$$\vartheta_\varphi := \varphi\,\frac{\partial}{\partial\varphi}, \quad \vartheta_\nu := \nu\,\frac{\partial}{\partial\nu}. \tag{D.2}$$

The polynomial coefficients $R^1_{k,\,l}$ of the operator $\mathcal{L}_1$ are

$$\begin{aligned}
R^1_{2,0} &= \nu\,(36\,\nu^2\,\varphi^3 + 36\,\varphi^3 + 24\,\varphi^2 + \varphi - 1),\\
R^1_{1,1} &= -\,\nu\,(144\,\nu^2\,\varphi^3 - 144\,\nu\,\varphi^3 - 24\,\nu\,\varphi^2 + 24\,\varphi^2 + 2\,\varphi - 3),\\
R^1_{0,2} &= 2\,(\nu-1)(72\,\nu^2\,\varphi^3 - 108\,\nu\,\varphi^3 - 24\,\nu\,\varphi^2 + 36\,\varphi^3 + 24\,\varphi^2 + \varphi - 1),\\
R^1_{1,0} &= -\,\nu\,\varphi\,(36\,\nu^2\,\varphi^2 - 144\,\nu\,\varphi^2 - 24\,\nu\,\varphi - 108\,\varphi^2 - 48\,\varphi - 1),\\
R^1_{0,1} &= 24\,(\nu-1)\,\nu\,\varphi^2\,(3\,\nu\,\varphi - 9\,\varphi - 2),\\
R^1_{0,0} &= 48\,\nu\,\varphi^2(3\,\varphi + 1),
\end{aligned} \tag{D.3}$$

while the polynomial coefficients $R^2_{k,\,l}$ of the operator $\mathcal{L}_2$ are

$$\begin{aligned}
R^2_{3,0} = \;& -\,216\,\nu^5\,\varphi^3 + 33372\,\nu^4\,\varphi^4 + 2268\,\nu^4\,\varphi^3 - 114048\,\nu^3\,\varphi^5 - 16848\,\nu^3\,\varphi^4\\
& -\,1080\,\nu^3\,\varphi^3 - 3888\,\nu^2\,\varphi^6 - 2592\,\nu^2\,\varphi^5 - 432\,\nu^2\,\varphi^4 - 216\,\nu^2\,\varphi^3 - 81\,\nu^2\,\varphi^2\\
& +\,6\,\nu^2\,\varphi + 3\,\nu^2 + 32400\,\nu\,\varphi^5 + 35640\,\nu\,\varphi^4 + 11340\,\nu\,\varphi^3 + 210\,\nu\,\varphi^2 - 360\,\nu\,\varphi\\
& -\,30\,\nu - 1296\,\varphi^6 - 1728\,\varphi^5 - 648\,\varphi^4 + 24\,\varphi^3 + 47\,\varphi^2 + 2\,\varphi - 1,
\end{aligned}$$

$$\begin{aligned}
R^2_{2,1} &= 3\,(\,360\,\nu^5\,\varphi^3 - 55080\,\nu^4\,\varphi^4 - 4104\,\nu^4\,\varphi^3 - 48\,\nu^4\,\varphi^2 + 194976\,\nu^3\,\varphi^5 \\
&\quad + 76752\,\nu^3\,\varphi^4 + 12528\,\nu^3\,\varphi^3 + 456\,\nu^3\,\varphi^2 - 2\,\nu^3\,\varphi + 2\,\nu^3 + 4320\,\nu^2\,\varphi^6 \\
&\quad - 148608\,\nu^2\,\varphi^5 - 35760\,\nu^2\,\varphi^4 + 3408\,\nu^2\,\varphi^3 + 648\,\nu^2\,\varphi^2 - 315\,\nu^2\,\varphi - 25\,\nu^2 \\
&\quad - 5184\,\nu\,\varphi^6 - 59040\,\nu\,\varphi^5 - 55200\,\nu\,\varphi^4 - 13040\,\nu\,\varphi^3 + 1164\,\nu\,\varphi^2 + 650\,\nu\,\varphi \\
&\quad + 50\,\nu - 2592\,\varphi^6 - 2592\,\varphi^5 - 648\,\varphi^4 + 12\,\varphi^3 - \varphi + 1\,), \\
R^2_{1,2} &= -\,2\,(864\,\nu^5\,\varphi^3 - 130248\,\nu^4\,\varphi^4 - 10908\,\nu^4\,\varphi^3 - 216\,\nu^4\,\varphi^2 + 485568\,\nu^3\,\varphi^5 \\
&\quad + 339660\,\nu^3\,\varphi^4 + 57240\,\nu^3\,\varphi^3 + 2448\,\nu^3\,\varphi^2 - 3\,\nu^3\,\varphi + 9\,\nu^3 + 2592\,\nu^2\,\varphi^6 \\
&\quad - 895752\,\nu^2\,\varphi^5 - 376812\,\nu^2\,\varphi^4 - 60300\,\nu^2\,\varphi^3 - 2277\,\nu^2\,\varphi^2 - 1455\,\nu^2\,\varphi - 105\,\nu^2 \\
&\quad - 15552\,\nu\,\varphi^6 + 294192\,\nu\,\varphi^5 + 65412\,\nu\,\varphi^4 - 11874\,\nu\,\varphi^3 + 5658\,\nu\,\varphi^2 + 2121\,\nu\,\varphi \\
&\quad + 156\,\nu + 12960\,\varphi^6 + 115992\,\varphi^5 + 101844\,\varphi^4 + 24726\,\varphi^3 - 675\,\varphi^2 - 759\,\varphi - 59), \\
R^2_{0,3} &= 12\,(\,\nu - 1)\,(72\,\nu^4\,\varphi^3 - 10584\,\nu^3\,\varphi^4 - 936\,\nu^3\,\varphi^3 - 24\,\nu^3\,\varphi^2 + 42912\,\nu^2\,\varphi^5 \\
&\quad + 31284\,\nu^2\,\varphi^4 + 5904\,\nu^2\,\varphi^3 + 288\,\nu^2\,\varphi^2 + \nu^2 - 864\,\nu\,\varphi^6 - 88776\,\nu\,\varphi^5 \\
&\quad - 45024\,\nu\,\varphi^4 - 9072\,\nu\,\varphi^3 - 495\,\nu\,\varphi^2 - 165\,\nu\,\varphi - 11\,\nu + 864\,\varphi^6 \\
&\quad + 45864\,\varphi^5 + 24300\,\varphi^4 + 3978\,\varphi^3 + 1009\,\varphi^2 + 150\,\varphi + 10), \\
R^2_{2,0} &= 3\,(\,72\,\nu^5\,\varphi^3 - 9828\,\nu^4\,\varphi^4 - 1080\,\nu^4\,\varphi^3 - 48\,\nu^4\,\varphi^2 + 26208\,\nu^3\,\varphi^5 \\
&\quad + 52704\,\nu^3\,\varphi^4 + 11160\,\nu^3\,\varphi^3 + 504\,\nu^3\,\varphi^2 - 4752\,\nu^2\,\varphi^6 - 153792\,\nu^2\,\varphi^5 \\
&\quad - 48144\,\nu^2\,\varphi^4 - 5448\,\nu^2\,\varphi^3 - 285\,\nu^2\,\varphi^2 - 9\,\nu^2\,\varphi - 2\,\nu^2 - 5184\,\nu\,\varphi^6 \\
&\quad + 28080\,\nu\,\varphi^5 + 24120\,\nu\,\varphi^4 + 6780\,\nu\,\varphi^3 + 1374\,\nu\,\varphi^2 + 290\,\nu\,\varphi + 20\,\nu \\
&\quad - 3888\,\varphi^6 - 4320\,\varphi^5 - 1296\,\varphi^4 + 36\,\varphi^3 + 47\,\varphi^2 + \varphi), \\
R^2_{1,1} &= -\,3\,(\,216\,\nu^5\,\varphi^3 - 27648\,\nu^4\,\varphi^4 - 3312\,\nu^4\,\varphi^3 - 144\,\nu^4\,\varphi^2 + 71712\,\nu^3\,\varphi^5 \\
&\quad + 160128\,\nu^3\,\varphi^4 + 34272\,\nu^3\,\varphi^3 + 1776\,\nu^3\,\varphi^2 + 4\,\nu^3\,\varphi - 22464\,\nu^2\,\varphi^6 \\
&\quad - 460800\,\nu^2\,\varphi^5 - 267120\,\nu^2\,\varphi^4 - 65328\,\nu^2\,\varphi^3 - 4182\,\nu^2\,\varphi^2 - 69\,\nu^2\,\varphi \\
&\quad - 6\,\nu^2 + 10368\,\nu\,\varphi^6 + 394848\,\nu\,\varphi^5 + 214896\,\nu\,\varphi^4 + 34420\,\nu\,\varphi^3 + 4966\,\nu\,\varphi^2 \\
&\quad + 890\,\nu\,\varphi + 60\,\nu + 15552\,\varphi^6 + 12960\,\varphi^5 + 2592\,\varphi^4 - 36\,\varphi^3 + \varphi),
\end{aligned}$$

$$
\begin{aligned}
R^2_{0,2} \;&=\; 12\,\varphi\,(36\,\nu^5\,\varphi^2 - 3996\,\nu^4\,\varphi^3 - 540\,\nu^4\,\varphi^2 - 24\,\nu^4\,\varphi + 9648\,\nu^3\,\varphi^4 + 21420\,\nu^3\,\varphi^3 \\
&\quad + 5382\,\nu^3\,\varphi^2 + 312\,\nu^3\,\varphi + \nu^3 - 6480\,\nu^2\,\varphi^5 - 49248\,\nu^2\,\varphi^4 - 46824\,\nu^2\,\varphi^3 \\
&\quad - 13248\,\nu^2\,\varphi^2 - 945\,\nu^2\,\varphi - 16\,\nu^2 + 12960\,\nu\,\varphi^5 + 71280\,\nu\,\varphi^4 + 51864\,\nu\,\varphi^3 \\
&\quad + 12550\,\nu\,\varphi^2 + 1435\,\nu\,\varphi - 6480\,\varphi^5 - 31680\,\varphi^4 - 22488\,\varphi^3 - 4234\,\varphi^2), \\
R^2_{1,0} \;&=\; \varphi\,(-3888\,\nu^4\,\varphi^3 + 108\,\nu^4\,\varphi^2 + 35424\,\nu^3\,\varphi^4 - 432\,\nu^3\,\varphi^2 - 144\,\nu^3\,\varphi \\
&\quad + 18144\,\nu^2\,\varphi^5 - 63072\,\nu^2\,\varphi^4 + 69984\,\nu^2\,\varphi^3 + 27648\,\nu^2\,\varphi^2 + 1458\,\nu^2\,\varphi \\
&\quad + 3\,\nu^2 - 51840\,\nu\,\varphi^5 - 195264\,\nu\,\varphi^4 - 74880\,\nu\,\varphi^3 - 4680\,\nu\,\varphi^2 - 480\,\nu\,\varphi \\
&\quad - 30\,\nu - 33696\,\varphi^5 - 31968\,\varphi^4 - 7776\,\varphi^3 + 300\,\varphi^2 + 190\,\varphi + 1), \\
R^2_{0,1} \;&=\; -\,216\,(\,\nu - 1)\,\varphi^3\,(-36\,\nu^3\,\varphi + \nu^3 + 328\,\nu^2\,\varphi^2 - 36\,\nu^2\,\varphi + \nu^2 + 168\,\nu\,\varphi^3 \\
&\quad - 256\,\nu\,\varphi^2 - 36\,\nu\,\varphi + \nu - 312\,\varphi^3 - 224\,\varphi^2 - 36\,\varphi + 1), \\
R^2_{0,0} \;&=\; -\,96\,\varphi^2\,(3\,\varphi + 1)(6\,\varphi + 1)(18\,\varphi^2 + 6\,\varphi - 1).
\end{aligned}
\tag{D.4}
$$

Moreover, with respect to the coordinates $(\varphi,\,\mu)$ of the parameter space of $X_{\mathfrak{G}}$, there exist two Picard-Fuchs operators $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ of the form

$$
\widetilde{\mathcal{L}}_1 = \sum_{k,\,l\,\geq\,0}^{k+l\,\leq\,2} \widetilde{R}^1_{k,\,l}\,\vartheta^k_\varphi\,\vartheta^l_\mu\;, \qquad \widetilde{\mathcal{L}}_2 = \sum_{k,\,l\,\geq\,0}^{k+l\leq 3} \widetilde{R}^1_{k,\,l}\,\vartheta^k_\varphi\,\vartheta^l_\mu, \tag{D.5}
$$

where ϑ_φ and ϑ_μ are defined by

$$
\vartheta_\varphi := \varphi\,\frac{\partial}{\partial\varphi}, \quad \vartheta_\mu := \mu\,\frac{\partial}{\partial\mu}. \tag{D.6}
$$

The polynomial coefficients $\widetilde{R}^1_{i,j}$ of the operator $\widetilde{\mathcal{L}}_1$ are

$$
\begin{aligned}
\widetilde{R}^1_{2,0} \;&=\; \mu\,(36\,\mu^2\,\varphi^3 + 72\,\mu\,\varphi^3 + 72\,\varphi^3 + 24\,\varphi^2 + \varphi - 1), \\
\widetilde{R}^1_{1,1} \;&=\; -\,(\mu + 1)\,(144\,\mu^2\,\varphi^3 + 144\,\mu\,\varphi^3 - 24\,\mu\,\varphi^2 + 2\,\varphi - 3), \\
\widetilde{R}^1_{0,2} \;&=\; 2\,(\mu + 1)\,(72\,\mu^2\,\varphi^3 + 36\,\mu\,\varphi^3 - 24\,\mu\,\varphi^2 + \varphi - 1), \\
\widetilde{R}^1_{1,0} \;&=\; -\,\mu\,\varphi\,(36\,\mu^2\,\varphi^2 - 72\,\mu\,\varphi^2 - 24\,\mu\,\varphi - 216\,\varphi^2 - 72\,\varphi - 1), \\
\widetilde{R}^1_{0,1} \;&=\; 2\,(36\,\mu^3\,\varphi^3 - 108\,\mu^2\,\varphi^3 - 24\,\mu^2\,\varphi^2 - 108\,\mu\,\varphi^3 - \varphi + 1), \\
\widetilde{R}^1_{0,0} \;&=\; 48\,\mu\,\varphi^2\,(3\,\varphi + 1),
\end{aligned}
\tag{D.7}
$$

while the polynomial coefficients $\widetilde{R}^2_{i,j}$ of the operator $\widetilde{\mathcal{L}}_2$ are

$$\begin{aligned}
\widetilde{R}^2_{3,0} &= \mu^2\,(2\,\varphi+1)(3\,\varphi+1)(6\,\varphi-1),\\
\widetilde{R}^2_{2,1} &= \mu\,(72\,\mu^2\,\varphi^3+72\,\mu\,\varphi^3-72\,\mu\,\varphi^2-4\,\mu\,\varphi+5\,\mu+72\,\varphi^3-24\,\varphi^2-2\,\varphi+3),\\
\widetilde{R}^2_{1,2} &= -\,(\,\mu+1)\,(252\,\mu^2\,\varphi^3+144\,\mu\,\varphi^3-120\,\mu\,\varphi^2-5\,\mu\,\varphi+8\,\mu+2\,\varphi-1),\\
\widetilde{R}^2_{0,3} &= 2\,(\,\mu+1)\,(108\,\mu^2\,\varphi^3+36\,\mu\,\varphi^3-48\,\mu\,\varphi^2-\,\mu\,\varphi+2\,\mu+\,\varphi),\\
\widetilde{R}^2_{2,0} &= \mu^2\,(144\,\mu\,\varphi^3+324\,\varphi^3+72\,\varphi^2+\,\varphi+1),\\
\widetilde{R}^2_{1,1} &= -\,504\,\mu^3\,\varphi^3-396\,\mu^2\,\varphi^3-48\,\mu^2\,\varphi^2-3\,\mu^2\,\varphi-3\,\mu^2+216\,\mu\,\varphi^3\\
&\quad -\,72\,\mu\,\varphi^2-6\,\mu\,\varphi+5\,\mu+2\,\varphi-1,\\
\widetilde{R}^2_{0,2} &= 2\,(216\,\mu^3\,\varphi^3-36\,\mu^2\,\varphi^3-24\,\mu^2\,\varphi^2+\,\mu^2\,\varphi+\,\mu^2-108\,\mu\,\varphi^3\\
&\quad +\,72\,\mu\,\varphi^2+3\,\mu\,\varphi-3\,\mu-2\,\varphi),\\
\widetilde{R}^2_{1,0} &= 24\,\mu^2\,\varphi^2\,(24\,\varphi+5),\\
\widetilde{R}^2_{0,1} &= -\,2\,(252\,\mu^2\,\varphi^3+48\,\mu^2\,\varphi^2-72\,\mu\,\varphi^3+24\,\mu\,\varphi^2+2\,\mu\,\varphi-\,\mu-\,\varphi),\\
\widetilde{R}^2_{0,0} &= 48\,\mu^2\,\varphi^2\,(6\,\varphi+1).
\end{aligned} \tag{D.8}$$

Appendix E

The Yukawa couplings of $X_{\mathfrak{G}}$

In this appendix, we will give the Yukawa couplings of the self-mirror Calabi-Yau threefold $X_{\mathfrak{G}}$, which are introduced in Section 7.6. With respect to the coordinates φ and ν of the parameter space of $X_{\mathfrak{G}}$, the Yukawa coupling tensor $\mathcal{Y}$ of $X_{\mathfrak{G}}$ is of the form

$$\mathcal{Y} = \mathcal{Y}_{\varphi\varphi\varphi}\, d\varphi^3 \; + 3\, \mathcal{Y}_{\varphi\varphi\nu}\, d\varphi^2\, d\nu \; + 3\, \mathcal{Y}_{\varphi\nu\nu}\, d\varphi\, d\nu^2 + \mathcal{Y}_{\nu\nu\nu}\, d\nu^3. \tag{E.1}$$

The four components $(\mathcal{Y}_{\varphi\varphi\varphi}, \cdots, \mathcal{Y}_{\nu\nu\nu})$ are rational functions of φ and ν, which have been computed by studying the coefficients of the Picard-Fuchs operators of $X_{\mathfrak{G}}$ [23, 51]. From Section 7.3, over the open locus of the parameter space of $X_{\mathfrak{G}}$ where ρ, φ and ν are all well-defined, we have

$$\rho = (\nu - 1)\, \varphi^2. \tag{E.2}$$

Let Δ be

$$\begin{aligned} \Delta := \varphi^3\, &(9\,\rho - 6\,\varphi - 1)(6\,\rho - 5\,\varphi - 1)(4\,\rho - 4\,\varphi - 1)(12\,\rho - 4\,\varphi + 1) \\ &(18\,\rho - 3\,\varphi + 1)(2\,\rho - \varphi)(3\,\rho - \varphi)(6\,\rho + \varphi)(36\,\rho + 12\,\varphi - 1), \end{aligned} \tag{E.3}$$

then the four components of $\mathcal{Y}$ are given by

$$\begin{aligned} \mathcal{Y}_{\varphi\varphi\varphi} \;=\; & -\frac{6}{(2\pi i)^3\, \Delta}(\, 186624\,\rho^6 + 466560\,\rho^5\,\varphi^2 + 5184\,\rho^5 - 466560\,\rho^4\,\varphi^3 \\ & - 129600\,\rho^4\,\varphi^2 + 8640\,\rho^4\,\varphi + 3456\,\rho^4 + 129600\,\rho^3\,\varphi^4 + 45360\,\rho^3\,\varphi^3 \\ & - 7992\,\rho^3\,\varphi^2 - 2160\,\rho^3\,\varphi - 144\,\rho^3 - 8640\,\rho^2\,\varphi^5 - 4752\,\rho^2\,\varphi^4 - 288\,\rho^2\,\varphi^3 \\ & + 72\,\rho^2\,\varphi^2 + 96\,\rho^2\,\varphi + 216\,\rho\,\varphi^4 + 27\,\rho\,\varphi^3 - 4\,\rho\,\varphi^2 + 2\,\varphi^4 - 3\,\varphi^3\,), \end{aligned}$$

$$\mathcal{Y}_{\varphi\varphi\nu} \;=\; \frac{6\,\varphi^3}{(2\pi i)^3\,(\,\rho + \varphi^2)\,\Delta}(-93312\,\rho^6 - 513216\,\rho^5\,\varphi^2 - 62208\,\rho^5\,\varphi - 2592\,\rho^5$$

$$
\begin{aligned}
&\quad -466560\,\rho^4\,\varphi^4+311040\,\rho^4\,\varphi^3+106272\,\rho^4\,\varphi^2-4320\,\rho^4\,\varphi-1728\,\rho^4\\
&\quad +466560\,\rho^3\,\varphi^5+45360\,\rho^3\,\varphi^4-32400\,\rho^3\,\varphi^3+612\,\rho^3\,\varphi^2+1008\,\rho^3\,\varphi+72\,\rho^3\\
&\quad -129600\,\rho^2\,\varphi^6-34560\,\rho^2\,\varphi^5+8712\,\rho^2\,\varphi^4+2736\,\rho^2\,\varphi^3+12\,\rho^2\,\varphi^2-48\,\rho^2\,\varphi\\
&\quad +8640\,\rho\,\varphi^7+3024\,\rho\,\varphi^6-144\,\rho\,\varphi^5-167\,\rho\,\varphi^4-3\,\rho\,\varphi^3+2\,\rho\,\varphi^2-72\,\varphi^6\\
&\quad -24\,\varphi^5-\,\varphi^4+\,\varphi^3),\\
\mathcal{Y}_{\varphi\nu\nu} &= \frac{6\,\varphi^6}{(2\pi i)^3\,(\,\rho\,+\varphi^2)\;\Delta}(-46656\,\rho^5-373248\,\rho^4\,\varphi^2-62208\,\rho^4\,\varphi-1296\,\rho^4\\
&\quad -466560\,\rho^3\,\varphi^4+155520\,\rho^3\,\varphi^3+64800\,\rho^3\,\varphi^2-2160\,\rho^3\,\varphi-864\,\rho^3\\
&\quad +466560\,\rho^2\,\varphi^5+90720\,\rho^2\,\varphi^4-12960\,\rho^2\,\varphi^3-504\,\rho^2\,\varphi^2+468\,\rho^2\,\varphi+36\,\rho^2\\
&\quad -129600\,\rho\,\varphi^6-32400\,\rho\,\varphi^5+7236\,\rho\,\varphi^4+2280\,\rho\,\varphi^3+12\,\rho\,\varphi^2-24\,\rho\,\varphi\\
&\quad +8640\,\varphi^7+1296\,\varphi^6-684\,\varphi^5-59\,\varphi^4+22\,\varphi^3+\,\varphi^2),\\
\mathcal{Y}_{\nu\nu\nu} &= -\frac{3\,\varphi^9}{(2\pi i)^3\,\rho\,(\,\rho\,+\varphi^2)^2\;\Delta}(\,46656\,\rho^6+559872\,\rho^5\,\varphi^2+93312\,\rho^5\,\varphi+1296\,\rho^5\\
&\quad +1399680\,\rho^4\,\varphi^4-64800\,\rho^4\,\varphi^2+2160\,\rho^4\,\varphi+864\,\rho^4+933120\,\rho^3\,\varphi^6\\
&\quad -933120\,\rho^3\,\varphi^5-272160\,\rho^3\,\varphi^4+8640\,\rho^3\,\varphi^3+2160\,\rho^3\,\varphi^2-432\,\rho^3\,\varphi\\
&\quad -36\,\rho^3-933120\,\rho^2\,\varphi^7-12960\,\rho^2\,\varphi^6+51840\,\rho^2\,\varphi^5-11988\,\rho^2\,\varphi^4\\
&\quad -3912\,\rho^2\,\varphi^3-72\,\rho^2\,\varphi^2+24\,\rho^2\,\varphi+259200\,\rho\,\varphi^8+43200\,\rho\,\varphi^7\\
&\quad -15768\,\rho\,\varphi^6-2688\,\rho\,\varphi^5-61\,\rho\,\varphi^4-22\,\rho\,\varphi^3-\,\rho\,\varphi^2-17280\,\varphi^9\\
&\quad +864\,\varphi^8+2664\,\varphi^7+10\,\varphi^6-115\,\varphi^5-4\,\varphi^4+\,\varphi^3). \qquad \text{(E.4)}
\end{aligned}
$$

With respect to the coordinates $(\varphi,\,\rho)$ of the parameter space of $X_{\mathfrak{G}}$, the Yukawa coupling tensor $\mathcal{Y}$ is of the form

$$\mathcal{Y} = Y_{\varphi\varphi\varphi}\,d\varphi^3\;+3\,Y_{\varphi\varphi\rho}\,d\varphi^2\,d\rho\;+3\,Y_{\varphi\rho\rho}\,d\varphi\,d\rho^2+Y_{\rho\rho\rho}\,d\rho^3. \qquad \text{(E.5)}$$

From the transformation between (φ,ν) and (φ,ρ), we have

$$\nu = 1+\frac{\rho}{\varphi^2},\quad d\nu = \frac{1}{\varphi^2}\,d\rho\;-\frac{2\,\rho}{\varphi^3}\,d\varphi, \qquad \text{(E.6)}$$

from which the four components $(Y_{\varphi\varphi\varphi},\cdots,Y_{\rho\rho\rho})$ can be easily computed from the components $(Y_{\varphi\varphi\varphi},\cdots,Y_{\nu\nu\nu})$. From Section 5.6, the parameter space of $X_{\mathfrak{G}}$ has a

$(\mathbb{Z}/2\mathbb{Z})^2$ symmetry generated by the transformations S and T of the form

$$S\ :\ \begin{cases} \varphi \to \dfrac{36\,\rho+1}{6(-36\,\rho+12\,\varphi+1)} \\[2ex] \rho \to \dfrac{36\,\rho+12\,\varphi-1}{36(-36\,\rho+12\,\varphi+1)} \end{cases} , \qquad T\ :\ \begin{cases} \varphi \to -\,\varphi-\dfrac{1}{3} \\[2ex] \rho \to \rho-\dfrac{2}{3}\,\varphi-\dfrac{1}{9} \end{cases} . \tag{E.7}$$

From Section 5.6, the quotient of the parameter space of $X_{\mathfrak{G}}$ by $(\mathbb{Z}/2\mathbb{Z})^2$ has natural coordinates given by

$$X_1=\frac{(6\ \varphi+1)^2}{9\,(-12\,\rho+4\ \varphi+1)^2}\ ,\ X_2=\frac{(36\,\rho-12\ \varphi+1)^2}{9\,(-12\,\rho+\ 4\ \varphi+1)^2}. \tag{E.8}$$

For simplicity, let us also define η by

$$\eta:=\frac{12\,(\,3\,\rho-\varphi)+1}{3\,(-12\,\rho+4\,\varphi+1\,)}, \tag{E.9}$$

which is invariant under T and is sent to $-\eta$ under S. The Yukawa coupling tensor $\mathcal{Y}$ is also expressed as

$$\begin{aligned} \mathcal{Y}\ \ =\ \ &-27\,\frac{(1+\eta)^2}{(2\pi i)^3\,\widetilde{\mathrm{D}}}\times\Big\{dX_1^3X_2(1-9X_2)\Big[(9+8X_1-219X_2+648X_1X_2^2)- \\ &(1+9X_2)(512X_1^2+X_2^2)+1904X_1X_2+220X_2^2\Big]-dX_2^3X_1(1-9X_1) \\ &\Big[(9-219X_1+8X_2+648X_1^2X_2)-(1+9X_1)(512X_1^2+X_2^2)+2264X_1^2-140X_1X_2\Big]- \\ &3\,dX_1^2dX_2\,X_2(1-9X_1)(1-9X_2)\Big[\big(9-128X_1-2X_2+512X_1^2-72X_1X_2+X_2^2\big) \\ &+8(X_1-X_2)\Big]+3\,dX_1dX_2^2\,X_1(1-9X_1)(1-9X_2)\times \\ &\Big[\big(9-128X_1-2X_2+512X_1^2-72X_1X_2+X_2^2\big)-8(X_1-X_2)\Big]\Big\}, \end{aligned} \tag{E.10}$$

where $\widetilde{\mathrm{D}}$ is defined by

$$\begin{aligned} \widetilde{\mathrm{D}}\ :=\ &X_1\,X_2\,(1-9\,X_1)(1-9\,X_2)(X_1-X_2)(1-8\,X_1-X_2)^2\times \\ &\times\ \big(81\ -1152\,X_1-18\,X_2+4096\,X_1^2-128\,X_1X_2+X_2^2\big)\,. \end{aligned} \tag{E.11}$$

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