Strain-limiting viscoelasticity

Victoria Patel
Corpus Christi College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2022

This thesis is dedicated to my
two wonderful sisters.
A constant source of inspiration.

Acknowledgements

First and foremost, the greatest thanks have to go to my supervisor Endre Süli. Throughout the ups and downs that came with producing this thesis, his endless positivity and optimism is something I will forever be grateful for. Honestly, I do not have the words to express my gratitude properly because they would not do justice to how incredible he has been. His enthusiasm for mathematics is inspirational and I could not have asked for a better supervisor to work with over the past four years, or a more patient one to listen to whatever problem I am currently facing. I truly appreciate the support both academically and personally.

Furthermore, I would like to extend my gratitude to the rest of the OxPDE group at Oxford. It has been a pleasure being a part of the final cohort of the PDE CDT and I am very thankful for the guidance of all of the faculty that I have interacted with throughout my time at Oxford. This research degree is by no means easy and the support of those around me has been invaluable. I want to thank my collaborators: Miroslav Bulíček, Alexei Gazca-Orozco, Yasemin Şengül and Endre Süli. I gained so much from working with you and I thank you for your patience with me.

I must also thank those at Corpus Christi, both staff and students, as well as the support that I have had through the Garside scholarship there. Teaching at Corpus has been a pleasure and a large factor in my decision over my plans for post-postgraduate life. I can only hope that my students have been able to enjoy undergraduate Mathematics as much as I did and they took inspiration from my passion for the subject.

The next thanks must go to my friends within the Mathematical institute. I would not have made it through this process without your support, encouragement and friendship. I have made many fond memories (mostly involving food) and I look forward to make many more, even as we go our separate ways in the world, whether that be academically or geographically. Special thanks in particular must go to my office mates, Catalina and Eliana, and my favourite coffee catch-up, Jeremy. The three of you have brought so much to my life over the past few years and I cannot wait to see all of the success that you achieve in the future. To my other friends, both in Oxford, from Warwick and back home, thank you for being there for me and believing that I could even when I didn't.

A huge appreciation must also go to those I have met throughout my rowing career. It was a pleasure to row with my college clubs - CCCBC and QCBC, as well as

Accies. Introducing me to the sport I love is something that has changed me forever and I would not be the person that I currently am without it - I'd definitely have a lot more free time and a lot fewer pre-5am alarm clocks. To the friends that I've picked up along the way, particularly Andre, Denise, Harrison and Sophia - I am not sure what I'd do without you letting me go on about training/trialling/life for endless amounts of time. Even though I am officially retired (excluding BUCS, Eights and maybe some Summer regattas...), I will never not have time for some good old rowing chat. This year with OUWLRC has been one of the absolute highlights of my four years at Oxford. I have never met such an incredible group of women, dedicated to pushing themselves to the limit at every opportunity, breaking barriers that they didn't think would ever be possible. Every single person at the club has inspired me in such a big way. To my Tethys crew, it was a pleasure to send it down the course with you. For the team.

Finally, hoping that they've managed to read at least this far through the thesis, I need to mention the appreciation that must go to my family for their endless support and love. Hannah and Kath, if you are reading this then you have already read the dedication so I am not sure that there is too much to add, other than I hope I get some crazy graduation present from Mum that you can complain about for the next four years. To my wonderful parents, I really cannot describe what I'd do without you. Dad, in every presentation and speech I have done throughout the past four years, the only way I could get past my nerves was to pretend that I was you and channel your confidence. It's never failed me yet. I promise that we can actually play some golf now. Mum, you have been there for me through all the ups and downs, and there is so much more I could say but I think you already know it.

Abstract

This thesis is concerned with PDEs that have a strain-limiting constitutive relation, originating in the framework of continuum mechanics and implicit constitutive theory. The work presented here is broadly split into two halves. In the first half, we consider the balance of linear momentum

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \quad \text{in } (0,T) \times \Omega =: Q, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a domain under the action of some external body force $\boldsymbol{f}$ and $\boldsymbol{u} : Q \to \mathbb{R}^d$ denotes the displacement with Cauchy stress tensor $\mathbf{T} : Q \to \mathbb{R}^{d\times d}$. This is supplemented with the strain-limiting constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}, \tag{2}$$

where $\boldsymbol{\varepsilon}$ is the symmetric gradient operator and $a > 0$ is a fixed parameter. The linearised strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is assumed to be small. Provided that the strain at the initial time $t = 0$ is sufficiently small, in fact it follows from (2) that the linearised strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is uniformly bounded. This is the motivation for the terminology *strain-limiting*.

We prove the existence of unique weak solutions to (1), (2) under periodic boundary conditions, Dirichlet boundary conditions and mixed Dirichlet–Neumann boundary conditions. The theory is first developed in the simplest case, namely, the periodic setting. The techniques are built upon so that we can eventually investigate the most challenging case of mixed Dirichlet–Neumann boundary conditions. The main ideas used in the existence proofs are as follows. First, we introduce an elliptic regularisation term into the constitutive relation (2) on the right-hand side, replacing F with a function F_n that is a bijection on $\mathbb{R}^{d\times d}$ to itself. Existence of a solution to the regularised problem is proven via a Galerkin approximation in space. Then n-uniform estimates are constructed. The main technical difficulty here is that the sequence of approximate stress tensors is, at best, bounded in $L^\infty(0,T;L^1(\Omega)^{d\times d})$, which has poor compactness properties. We must construct weighted higher regularity estimates that allow the deduction of a pointwise convergence result. However, in the case that the parameter a is small, a higher integrability estimate can in fact be proven. The limit in the regularisation parameter n can then be taken and a weak solution of the strain-limiting problem constructed.

In the second half of the thesis, the focus is on nonlinear dynamic fracture problems. We consider a domain that is experiencing damage and cracking under the action of an external force. As in the previous half of the thesis, there is an unknown displacement $\boldsymbol{u}$ and Cauchy stress tensor $\mathbf{T}$, but now we also deal with an unknown phase-field function $v : Q \to [0,1]$ which approximates an *a priori* unknown time-dependent crack set to a certain approximation level ϵ. The function v takes the value 1 'away' from the crack set and the value 0 'near' the crack set. The balance of linear momentum is replaced with

$$\boldsymbol{u}_{tt} = \operatorname{div}\big((v^2 + \eta)\mathbf{T}\big) \qquad \text{in } Q, \tag{3}$$

where η is a fixed parameter that prevents degeneracy near the crack set. This is supplemented with a nonlinear constitutive relation, either a strain-limiting relation of the form (2), where the strain is necessarily small, or the relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F_p(\mathbf{T}) := |\mathbf{T}|^{p-2}\mathbf{T}, \tag{4}$$

where $p \in (1, \infty)$. First, we consider the problem (3), (4) and develop a general theory of nonlinear dynamic fracture problems of this kind. The existence of weak energy solutions is proven, meaning that they solve (3) weakly, (4) holds pointwise, a minimisation problem which ensures growth in the crack set wherever possible (eliminating the case of a stationary crack) and the satisfaction of an energy-dissipation equality. Then we combine the techniques developed with those from the first half of the thesis to investigate dynamic fracture problems with a strain-limiting constitutive relation. We solve (3) coupled with (2). A partial existence result is proven in the case of mixed Dirichlet–Neumann boundary conditions and a full existence result in the case of fully Dirichlet boundary conditions.

Contents

1 Introduction **1**

1.1 Kinematics and equations of motion . . . 2

1.2 Implicit constitutive equations and elasticity . . . 4

1.3 Elasticity . . . 9

1.4 Viscoelasticity . . . 15

1.5 Fracture Problems . . . 22

1.6 Notation and auxiliary results . . . 28

1.7 Chapter Summaries . . . 30

2 Strain-limiting problem with periodic boundary conditions **32**

2.1 The regularised problem . . . 34

2.2 Higher regularity estimates . . . 43

2.3 The limit in the regularisation parameter . . . 48

3 Strain-limiting problem with Dirichlet boundary conditions **56**

3.1 Higher regularity estimates . . . 61

3.2 Existence of solutions to the strain-limiting problem in the case that a is small . 66

4 Strain-limiting problem with mixed Dirichlet–Neumann boundary conditions **70**

4.1 The regularised problem . . . 72

4.2 Higher regularity estimates . . . 79

4.2.1 Regularity in the time variable . . . 80

4.2.2 Regularity in the spatial variable . . . 88

4.2.3 The limit as $n \to \infty$ . . . 94

4.3 Lower regularity initial data . . . 102

4.4 Higher integrability estimates . . . 103

5 Nonlinear fracture problem with growth **105**

5.1 The discrete problem . . . 109

5.2 The limit as $M \to \infty$ and the time-continuous Galerkin problem . . . 116

5.3 The limit as $N \to \infty$ . . . 120

5.3.1 Uniform estimates . . . 120

6 Nonlinear fracture problem with strain-limiting constitutive relation **129**

6.1 A solution to the regularised problem 140

6.2 Limit in the regularisation parameter 164

7 Conclusions and open problems **178**

Bibliography **181**

Chapter 1

Introduction

The aim of this thesis is to investigate systems of partial differential equations (PDEs) that model problems from continuum mechanics for viscoelastic solids that exhibit a strain-limiting property, from the point of view of mathematical analysis. The term *viscoelastic* is used to describe a body that can display typical properties of both elastic and viscous materials. The term *elastic* is used to mean a body that does not dissipate energy under any process, for example, a deformation. A *viscous* material is one that resists shear force. The term *strain-limiting* describes materials that can be accurately modelled using a constitutive law that enforces bounded strains for arbitrarily large stresses. The concept has been investigated much in recent years in the context of elasticity but significantly less so for viscoelasticity. Hence, viscoelasticity is the focus here.

A classical example of a viscoelastic system is that of a linear spring and a linear viscous dashpot in parallel [17]. We refer also to Figure 2 in [16] and Figure 1(a) in [31]. However, it is well-known that linear models are insufficient to describe the real world. A simple generalisation that could be more appropriate is a system with a nonlinear spring and a nonlinear viscous dashpot. For the systems of PDEs considered here, nonlinear properties are expressed through a constitutive relation, which encodes specific material properties for a body. While balance equations are needed to describe the general motion of a body, the resulting problem is closed by the relevant constitutive laws. We note that the system of PDEs used to model a viscoelastic body here is identical to that of an elastic solid and a viscoelastic fluid, except for the constitutive relation.

The term implicit constitutive theory will be used here to refer to the general framework introduced by Rajagopal in [88]. His idea was to expand upon the previous theory that was used to describe elastic and viscoelastic bodies since experimental data showed that the classical models were insufficient. The aim was to do this in a systematic way while ensuring that the resulting models are thermodynamically consistent. By this, it is meant that the second law of thermodynamics is satisfied. While implicit theories are common in viscoelasticity, Rajagopal explains that many are not truly implicit: in particular, explicit relations can be obtained from them by basic manipulations. However, here implicit will be understood in the most general sense.

The theory has been used to carefully explain phenomena that were not fully understood previously. Particular instances include the response of rock [65], Gum Metal and other titanium alloys (e.g. [104], [53]), stainless steel alloys [100], biological matter [50] and soft materials such as rubber [30]. Experimental observations of these materials show nonlinear behaviour in the small strain range. In particular, some demonstrate characteristics of strain-limiting materials. For additional examples, we refer to [32], [82], [69], and the references therein.

One of the reasons for such interest in strain-limiting materials is their potential use in the analysis of fracture problems. It is well-known that if there is a linear relationship between the stress and strain, as in the case of linearised elasticity, the strain has magnitude of order $r^{-\frac{1}{2}}$, where r is the distance to the crack tip [95]. The strain experiences a singularity around the damage. However, we are working in the setting of linearised elasticity, for which the standing assumption is that the strain is *a priori* small. Various *ad hoc* methods have been suggested to deal with this issue. However, they are not necessarily thermodynamically justified, nor systematic in their presentation. In this work, we develop a framework of nonlinear dynamic fracture problems in which we are able to choose a strain-limiting constitutive relation. This ensures that the strain remains *a priori* bounded by a chosen constant, thus avoiding a singularity near the crack tip while still being a thermodynamically consistent model.

The remainder of this section is dedicated to investigating implicit constitutive theory, dynamic fracture problems and discussing the relevant literature on these topics. First, the definitions and notation that are required in order to understand this new class of implicit relations will be introduced. Next, a review of the theory for elastic solids is undertaken with the view that this is a stepping stone to the new results presented here on viscoelastic solids. Of particular interest are existence results for boundary-value problems (BVPs) that result from these models for elastic bodies. Then, a review of the current results for viscoelastic bodies is presented. The existing literature on the subject is very limited, especially in the context of a time-dependent problem. This thesis aims to fill some of these gaps. Next, we present an overview of quasi-static and dynamic fracture problems. We begin with quasi-static problems because this is the basis of the dynamic theory. Then we investigate dynamic problems as a basis for the study in Chapters 5 and 6. In particular, we formulate the nonlinear fracture problem of interest that uses a phase-field approximation to model the damage in the body.

1.1 Kinematics and equations of motion

The notation introduced here is standard in the literature. For simplicity, throughout the remainder of this section a three-dimensional setting is considered unless stated otherwise. However, the later mathematical analysis takes place in an arbitrary spatial dimension $d \geq 2$. Let $\Omega_0 \subset \mathbb{R}^3$ denote the *initial configuration* of a body and $\Omega_t \subset \mathbb{R}^3$ the *current configuration* of the body at time $t \in \mathbb{R}$. If $\mathbf{X}$ denotes a point in Ω_0, denote the position of that point at time

t in Ω_t by $\mathbf{x} = \boldsymbol{\chi}(t, \mathbf{X})$. The *displacement* is defined by $\boldsymbol{u}(t, \mathbf{X}) := \mathbf{x} - \mathbf{X}$. The *velocity* $\boldsymbol{v}$ and *deformation gradient* $\mathbf{F}$ are defined by

$$\boldsymbol{v} := \frac{\partial \boldsymbol{\chi}}{\partial t}, \quad \mathbf{F} := \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}.$$

The balance of mass, linear momentum and angular momentum are the equations of motion used to model the kinematics of a body. They are not specific to a particular material - constitutive relations are used to make this specification. With respect to the current configuration, they are expressed as the following system:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho \boldsymbol{v}) &= 0, \\ \frac{\partial (\rho \boldsymbol{v})}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) &= \operatorname{div}_{\mathbf{x}} \mathbf{T} + \rho \boldsymbol{f}, \\ \mathbf{T} &= \mathbf{T}^{\mathrm{T}}, \end{aligned} \tag{1.1}$$

for every $(t, \mathbf{x}) \in \mathbb{R} \times \Omega_t$, where ρ is the *current density*, $\mathbf{T}$ is the *Cauchy stress tensor*, and $\boldsymbol{f}$ is the *density of body forces*. We refer to (1.1) as the Eulerian formulation of the balance equations. The *deviatoric part* $\mathbf{T}^\delta$ of the stress is defined by $\mathbf{T}^\delta = \mathbf{T} - \frac{1}{d}(\operatorname{tr} \mathbf{T})\mathbf{I}$, where d is the dimension and $\mathbf{I}$ denotes the identity matrix in $\mathbb{R}^{d \times d}$. Thus for $d = 3$, the three-dimensional case, we have $\mathbf{T}^\delta = \mathbf{T} - \frac{1}{3}(\operatorname{tr} \mathbf{T})\mathbf{I}$. The *volumetric part* of the stress is the trace $\operatorname{tr} \mathbf{T}$.

The system of equations can also be expressed with respect to the initial configuration. This gives the following Lagrangean formulation of the balance equations:

$$\begin{aligned} \rho &= \rho_0 \det \mathbf{F}, \\ \rho_0 \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} &= \operatorname{div}_{\mathbf{X}} \mathbf{S} + \rho_0 \boldsymbol{f}, \\ \mathbf{S}\mathbf{F}^{\mathrm{T}} &= \mathbf{F}\mathbf{S}^{\mathrm{T}}, \end{aligned} \tag{1.2}$$

for every $(t, \mathbf{X}) \in \mathbb{R} \times \Omega_0$, where $\mathbf{S} := (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-\mathrm{T}}$ is the *first Piola–Kirchhoff stress tensor* and ρ_0 is the *initial density.*

As mentioned previously, the set of balance equations is insufficient to describe the motion of a body. Specific constitutive relations are needed in order to ensure that individual properties of the body are encoded into the system of equations. Typically, the relation will involve the stress and one or more of the kinematic variables, for example, the displacement or the velocity. As discussed in [16], an equation for the balance of energy is also required in the case that fields (such as thermal or magnetic fields) are acting on the body. Then further constitutive relations are also needed. However, for simplicity, it is assumed that the energy is constant throughout this thesis.

The stretch tensors that are of interest are the *left and right Cauchy–Green stretch tensors*, defined respectively by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}.$$

Furthermore, we consider the *Green strain tensor* defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}).$$

The *linearised strain* plays a key role in the subsequent analysis, particularly when we introduce the small displacement gradient assumption that is stated in (1.5) below. Consider the symmetric displacement gradient defined by

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla_{\mathbf{X}}\boldsymbol{u} + (\nabla_{\mathbf{X}}\boldsymbol{u})^{\mathrm{T}}),$$

referred to as the linearised strain. It is related to the left Cauchy–Green stretch tensor by

$$\mathbf{B} = \mathbf{I} + 2\boldsymbol{\varepsilon} + (\nabla_{\mathbf{X}}\boldsymbol{u})(\nabla_{\mathbf{X}}\boldsymbol{u})^{\mathrm{T}}.$$

This follows by noting that $\mathbf{F} = \nabla_{\mathbf{X}}\boldsymbol{\chi} = \nabla_{\mathbf{X}}(\mathbf{X} + \boldsymbol{u}) = \mathbf{I} + \nabla_{\mathbf{X}}\boldsymbol{u}$. We write $\mathbf{L}$ for the velocity gradient $\nabla_{\mathbf{x}}\boldsymbol{v}$. The *symmetric part of the velocity gradient* is then denoted by $\mathbf{D}$, i.e., $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})$. This is related to the deformation gradient by

$$\frac{\partial \mathbf{F}}{\partial t} = \mathbf{LF}.$$

Indeed, using the definitions of $\mathbf{F}$ and $\mathbf{L}$, we have that

$$\mathbf{LF} = \frac{\partial^2 \boldsymbol{u}}{\partial t \partial \mathbf{x}}\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} = \frac{\partial^2 \boldsymbol{u}}{\partial t \partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial^2}{\partial t \partial \mathbf{X}}\Big(\mathbf{x} - \mathbf{X}\Big) = \frac{\partial^2 \mathbf{x}}{\partial t \partial \mathbf{X}} = \frac{\partial \mathbf{F}}{\partial t}.$$

From now on, we are only interested in the Lagrangean formulation of the equations of motion.

1.2 Implicit constitutive equations and elasticity

With the equations of motion in mind, we now consider the constitutive relations that encode specific material properties, with a particular interest in the theory of implicit constitutive relations. Terms that are traditionally used in the context of elasticity are Cauchy elastic body and Green elastic body. A *Cauchy elastic body* is defined as one in which the stress can be given as a function of the deformation gradient. A *Green elastic* (or *hyperelastic*) *body* is one in which the stress is defined in terms of a strain energy density function. This takes the form $\mathbf{T} = (\det \mathbf{F})^{-1}\frac{\partial W}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}$ where $W(\mathbf{F})$ is the *strain energy density function.* In particular, Green elastic bodies bodies are a subset of Cauchy elastic bodies.

The notion of elasticity is discussed in [90] and [93] in considerable detail. To summarise in a line, an elastic body is defined as one which does not dissipate energy under any process. However, a question that is raised is whether or not the class of Cauchy elastic bodies is in fact the whole class of elastic bodies. Rajagopal claims that this is not the case [93]. By definition, the constitutive relation for a Cauchy elastic body is

$$\mathbf{T} = \mathbf{h}(\mathbf{F}). \tag{1.3}$$

Assuming that the body is an isotropic, compressible, homogeneous, Cauchy elastic solid, from standard representation theory we deduce that (1.3) must be of the following form:

$$\mathbf{T} = \gamma_0\mathbf{I} + \gamma_1\mathbf{B} + \gamma_2\mathbf{B}^2. \tag{1.4}$$

The functions γ_i for $0 \le i \le 2$ are material moduli that depend on ρ and the principal invariants of $\mathbf{B}$, i.e., $I_{\mathbf{B}} = \mathrm{tr}(\mathbf{B})$, $II_{\mathbf{B}} = \mathrm{tr}(\mathbf{B}^2)$ and $III_{\mathbf{B}} = \mathrm{tr}(\mathbf{B}^3)$ [92]. These invariants are considered since we are considering the problem in three-dimensions. By *isotropic*, it is meant that a body has the same property irrespective of the direction. *Homogeneous* is used to mean that the body has the same properties throughout its entirety.

Suppose that the *small displacement gradient assumption* holds. By this, we mean that for some $\delta \ll 1$ we have that

$$\max_{\mathbf{X}\in\Omega_0, t\in\mathbb{R}} \|\nabla_{\mathbf{X}} \boldsymbol{u}\| = O(\delta). \tag{1.5}$$

This is sometimes referred to as the *small strain assumption*. In this case, we have that $\mathbf{B} = \mathbf{I} + 2\boldsymbol{\varepsilon} + O(\delta^2)$ and it follows that, up to an error of order δ^2, the relation (1.4) reduces to

$$\mathbf{T} = \mathcal{H}\boldsymbol{\varepsilon},$$

where $\mathcal{H}$ is a fourth order tensor that is independent of $\boldsymbol{\varepsilon}$, although it may vary and depend on $(\mathbf{X}, t)$ [16]. In particular, there is a linear relationship between the stress and the linearised strain when assuming that we are in the small strain range. As stated in [92], even if the conditions stated above (isotropy and homogeneity) do not hold, a linear relationship between the stress and linearised strain is always obtained. In particular, there is no justification for Cauchy elastic bodies to have a nonlinear relationship between the stress and the strain in the small strain range. However, as shown in [104], for example, it is possible for an elastic body to have a nonlinear stress-strain relationship under condition (1.5). This suggests that the class of Cauchy elastic materials is perhaps insufficient to model all elastic bodies. In particular, as discussed in more detail below, experimental evidence shows that nonlinear relationships model certain materials in the small strain range particularly well, motivating the study of nonlinear models.

Introduced in [88], and subsequently studied in a large number of further papers by Rajagopal and his co-workers, rather than (1.3) we instead consider the following implicit relation:

$$\mathbf{h}(\mathbf{T}, \mathbf{B}) = \mathbf{0}. \tag{1.6}$$

Suppose that the material under consideration is isotropic. Then the relation (1.6) must be of the following form:

$$\begin{aligned} &\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{B} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2) \\ &\quad + \alpha_7(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2) + \alpha_8(\mathbf{B}^2\mathbf{T}^2 + \mathbf{T}^2\mathbf{B}^2) = \mathbf{0}, \end{aligned} \tag{1.7}$$

where α_i for $0 \le i \le 8$ are material moduli depending only on

$$\rho,\ \mathrm{tr}\,\mathbf{T},\ \mathrm{tr}\,\mathbf{B},\ \mathrm{tr}\,\mathbf{T}^2,\ \mathrm{tr}\,\mathbf{B}^2,\ \mathrm{tr}\,\mathbf{T}^3,\ \mathrm{tr}\,\mathbf{B}^3,\ \mathrm{tr}\,\mathbf{TB},\ \mathrm{tr}\,\mathbf{T}^2\mathbf{B},\ \mathrm{tr}\,\mathbf{B}^2\mathbf{T},\ \mathrm{tr}\,\mathbf{T}^2\mathbf{B}^2.$$

We refer to [92] and the references therein for details of this calculation. A subclass of this model is (1.4). However, an analogous subclass to consider is

$$\mathbf{B} = \tilde{\beta}_0\mathbf{I} + \tilde{\beta}_1\mathbf{T} + \tilde{\beta}_2\mathbf{T}^2, \tag{1.8}$$

where the material moduli $\tilde{\beta}_i$ for $0 \le i \le 2$ depend only on the principal invariants of $\mathbf{T}$ as described previously. Under the small displacement gradient assumption (1.5), the relation (1.8) reduces to

$$\boldsymbol{\varepsilon} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2. \tag{1.9}$$

For certain choices of β_i, by arguing in such a way we see that a nonlinear relationship between the stress and linearised strain is justified. The fact that this is a nonlinear relationship is key, since this was the main shortcoming of (1.3). In [104], it was shown that 90% cold worked Gum Metal displays an elastic response up to a strain of 2.5% and the stress-strain relationship in this region is nonlinear. An attempt to fit the experimental data to the model is presented in [95]. The relationship that was considered is

$$\boldsymbol{\varepsilon} = \lambda_1 (\operatorname{tr} \mathbf{T}) \mathbf{I} + 2\lambda_2 \mathrm{e}^{\eta \operatorname{tr} \mathbf{T}} \mathbf{T}, \tag{1.10}$$

where λ_1, λ_2, η are positive constants. Clearly this is a subclass of (1.9). It was shown in [95] that this relationship can be fitted to the data 'admirably well.' Although further experiments would be required to show the validity of the model, Rajagopal states that this is a 'compelling reason' to consider such implicit models.

Similarly, the goal of [69] was to fit experimental data for cold worked Gum Metal and other titanium alloys to a power-law model that forms a subclass of (1.9). An interesting aspect of the analysis is that the trace part $\operatorname{tr} \mathbf{T}$ and the deviatoric part $\mathbf{T}^\delta$ of the stress are separated and are allowed to have different polynomial growth. We refer to [16] for a mathematical study concerning the strain-limiting problem where the trace and traceless parts are separated. The authors of [69] state that this model outperforms or is at least as good as (1.10). We note that Gum Metal is of importance due to its use in applications, in particular in the biomedical industry [53]. Hence it is necessary to have an understanding of its elastic properties. However, it is clear that the strain $\boldsymbol{\varepsilon}$ is unbounded for large stresses in models such as (1.10). This contradicts the standing assumption under which the model is constructed, i.e., the small strain assumption (1.5). Thus one must be very careful when working with such models.

Consider a material in which the strain grows faster than linearly with respect to the stress. We call such a body *superlinear*. This is the case for stainless steel alloys [100], Gum Metal and certain titanium alloys. For the model considered in [95], the strain grows exponentially with respect to the stress. However, the materials of interest in this thesis exhibit the opposite behaviour. They are *sublinear*, the strain grows significantly slower than the stress and, in particular, slower than linearly. Examples of sublinear elastic bodies include rubber and biological matter; see [58], [59] and [60], for example. An extreme case of this is when the strain is bounded. Such bodies are called *strain-limiting* and are the focus of this thesis. Irrespective of the magnitude of the stress, the strain remains bounded. The expected response of such materials has been demonstrated by certain biological matter [28] and is further investigated in [85], where an attempt is made to identify appropriate parameters to fit a strain-limiting relation to such materials. In practice, a material will fracture before infinite stress is applied to it. However, strain-limiting constitutive relations are a good fit for experimental data compared to certain linear models and hence are worth investigating in further detail. The relationship to fracture will be explored in more detail later on.

It is worth noting that the type of material investigated in [85] is in the context of *strain-locking materials*. This is a type of Green elastic body, first introduced by Prager [87] and expanded upon in [86]. Variational problems are studied in the context of strain-locking materials in [37]. We refer also to [6], where the existence of minimisers is studied in the context of elastic ideal locking materials. We mention also [5] where a discussion on strain-limiting materials is made from the point of view of a variational setting.

The idea that Prager had was to classify materials from soft to hard. As explained in [86], an extreme soft material is one in which the stress increases until some yield value, without any increase in the strain. At the yield point, the strain then increases arbitrarily without any increase in stress. An extreme hard material is opposite in that the roles of the stress and strain are reversed. Hence, an extreme hard material is an ideal strain-limiting material. The stress can increase without any increase in strain, at least up until some yield value. It is worth noting that Rajagopal initially overlooked such materials in the development of his implicit constitutive theory. However, in [88], he considered bodies where the stress is not a function of the deformation gradient. Such a body cannot be Green elastic and so falls outside the class of strain-locking materials. In particular, strain-limiting theory is in some sense a generalisation of strain-locking theory.

An extremely important possible application of strain-limiting models is the study of fracture mechanics and crack propagation. When the classical linear model is used to study crack problems, the strain behaves like $O(r^{-\frac{1}{2}})$, where r is the distance to the crack tip [95]. In particular, the model predicts that the strain blows up at the crack tip. *Anti-plane shear* (or *anti-plane strain*) is used to mean that the only nonzero component of the displacement of a three-dimensional body is perpendicular to the plane. In particular, the strain takes the following particular form:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & 0 & \epsilon_{13} \\ 0 & 0 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix}.$$

The stress tensor has an analogous structure. It is common to study the problems of interest here in the anti-plane setting first due to the simplified structure that is available.

The fact that the strain blows up contradicts the very assumption under which the model is constructed, i.e., the small displacement gradient assumption (1.5). A better model for considering fracture in brittle materials might ensure that the strains are bounded. Indeed, in brittle materials fracture can occur for small strains. This is the motivation for the second half of this thesis where the analysis of strain-limiting materials is combined with dynamic fracture problems.

Historically, a variety of approaches have been used to try to deal with the singularity at the crack tip. Specific methods can be found in [95], [18], and the references within. In [95], it is stated that even 'fully nonlinear theories of elasticity are inadequate in describing the response in the neighbourhood of cracks in that the singularity yet persists.' An advantage of the strain-limiting theory is that the strains are a priori bounded so no singularity is present, without the use of an *ad hoc* method to essentially 'smooth out' the singularity. Furthermore, the approach is systematic as well as being thermodynamically justified. Rajagopal states that this is a significant advantage over the *ad hoc* approaches that had been used previously [82].

The first analysis of fracture mechanics using strain-limiting elasticity from the point of view of implicit constitutive theory was in [99]. Only the anti-plane shear setting was considered. The constitutive relation was assumed to be of the form $\boldsymbol{\varepsilon} = \phi(|\mathbf{T}|)\mathbf{T}$, where ϕ is a monotone decreasing function such that $r \mapsto \phi(r)r$ is a bounded map on $[0, \infty)$. We note that a similar relationship is considered here. The authors concluded that using such a model negates the need for techniques that 'remove' the singularity that occurs in the linear case. Similar problems have been considered in some specific geometries. In [71] and [70], anti-plane strain is considered in the context of a plate with a V-notch. The simplifications allow the reduction of the problem via the Airy stress function, resulting in a problem in one-dimension. This problem was then studied numerically using the finite element method and the stress was seen to concentrate around the tip of the V-notch. This contradicts the asymptotic analysis performed in [99], where the stress was found to vanish in the vicinity of the crack tip. The conflict may be due to the fact that solutions to nonlinear PDEs can exhibit very different behaviour to what is suggested by asymptotic analysis. In [81], a plate with a hole was studied numerically. No singularity was found to occur in this case and the strain was shown to grow significantly more slowly than the stress. In [18], an infinite cylinder with constant planar cross-section was considered in the case of anti-plane shear. It was possible to reduce the problem to one that is similar to the minimal surface equation. A complete mathematical analysis was performed and the authors proved the existence of a weak solution under certain parameter conditions on the constitutive relation. Further investigations of crack problems can be found in [61], [62] and [54]. The specific geometric setting and anti-plane setting make these works significantly different from the study made here.

Another problem that can be studied in the context of strain-limiting theory is that of concentrated loads. The idea is to consider large stresses being applied to a body. Under a linear constitutive relation, the strain will become large due to the magnitude of the stress and the small strain assumption will be violated. Hence, a theory where the strain remains bounded a priori could be useful in obtaining an understanding of such a problem. We note that the loading could be inside the body or on the boundary, leading to two very different problems. Furthermore, motivating the work in Chapter 2, if the loading is sufficiently far away from the boundary, such problems may be approximated by a system with periodic boundary conditions. More general boundary conditions will be the cause of some issue in Chapter 4.

1.3 Elasticity

In this thesis, we are interested in the rigorous mathematical analysis of the initial-boundary-value problems (IBVPs) that model problems from continuum mechanics for a class of strain-limiting viscoelastic bodies. The known results for the elastic case must be discussed first since the problems are deeply related. We note that this has, in general, only been studied in the steady case by which we mean that there is no time dependence in the problem at hand.

The majority of the aforementioned papers deal with only the anti-plane strain case. However, mathematical analysis has also been performed in the general case on strain-limiting elasticity. To this end, we consider the effect of the small strain assumption (1.5) on the balance equations (1.2). The density is explicitly given by $(1.2)_1$. The balance of angular momentum $(1.2)_3$ reduces to $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. This restriction will be encoded by the constitutive relation so we need no longer include it in the system of balance equations. Finally, for the balance of momentum $(1.2)_2$, we notice that the first Piola–Kirchhoff stress tensor $\mathbf{S}$ can be replaced by the Cauchy stress tensor $\mathbf{T}$ up to an error of order δ^2. Thus we have that

$$\begin{aligned} \rho(t,\cdot) &= \rho_0(1 + \operatorname{tr}\boldsymbol{\varepsilon}(t,\cdot)), \\ \rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= \rho_0 \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbf{T} + \rho_0 \boldsymbol{f}, \end{aligned} \tag{1.11}$$

in $\mathbb{R} \times \Omega_0$. The definitions of $\boldsymbol{\chi}$ and $\boldsymbol{u}$ were used to obtain the first equality in $(1.11)_2$. Since ρ is given explicitly as a function of ρ_0 and the linearised strain $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$, we no longer consider it as one of our unknowns. In particular, we obtain a system of PDEs with unknowns $(\boldsymbol{u}, \mathbf{T})$. To obtain such a system, we have had to make various approximations. However, such simplifications have to be made in order to make the model usable with regards to mathematical analysis, at the cost of a loss of modelling accuracy.

Thus, in this thesis, we focus on

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \quad \text{ in } Q := (0,T) \times \Omega, \tag{1.12}$$

where $(\boldsymbol{u}, \mathbf{T})$ are unknown and $\boldsymbol{f}$ is a given external force. In relation to the previous discussion, we have $\Omega := \Omega_0$ and $T \in (0, \infty)$ is the final time. Furthermore, for simplicity the initial density ρ_0 is taken to be constant. However, under suitable regularity assumptions, a non-constant density can be included in the analysis of Chapters 2, 3 and 4. We couple (1.12) with the requirement that $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$ and a constitutive relation of the form

$$\boldsymbol{\varepsilon} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2. \tag{1.13}$$

In the particular case that $\beta_2 \equiv 0$, which we do indeed later assume, it follows that the angular momentum balance is naturally included in (1.13) and so will not be specified.

At the time of writing, as far as I am aware, full analysis of the BVP for the strain-limiting elasticity problem has only been done in the static case. We note that specific problems have been studied in [94], for example, extension, simple shear, and torsion among others. However, as with the anti-plane shear examples discussed previously, the authors of [94] assume a specific structure on at least one of $\mathbf{T}$ and $\boldsymbol{\varepsilon}$. Thus we do not investigate them further here. Furthermore, the problems under consideration in [94] are not analysed in the context of a BVP.

Consider the problem of finding a couple $(\boldsymbol{u}, \mathbf{T}) : \Omega \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ such that

$$\begin{aligned} -\mathrm{div}(\mathbf{T}) &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{\varepsilon}(\boldsymbol{u}) &= F(\mathbf{T}) && \text{in } \Omega, \\ \boldsymbol{u} &= \tilde{\boldsymbol{u}} && \text{on } \partial\Omega_D, \\ \mathbf{Tn} &= \boldsymbol{g} && \text{on } \partial\Omega_N, \end{aligned} \tag{1.14}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with sufficiently smooth boundary and $\partial\Omega_D$, $\partial\Omega_N$ are relatively open, disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$. The body forces $\boldsymbol{f}$, boundary traction $\boldsymbol{g}$, and boundary displacement $\tilde{\boldsymbol{u}}$ are given functions of sufficient regularity. The function $F : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is assumed to be bounded, with a prototypical example given by

$$F(\mathbf{T}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}, \tag{1.15}$$

for a fixed parameter $a > 0$. We refer to Figure 1.1 to see how F varies with the parameter a. Another prototypical example that is commonly seen in the literature is the following. The

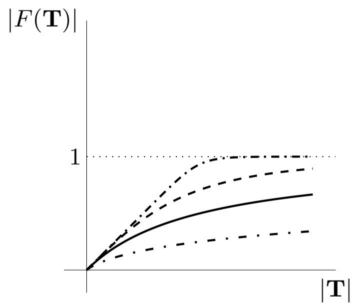

Figure 1.1: Dependence of $|F(\mathbf{T})|$ on $|\mathbf{T}|$ for the prototype model $F(\mathbf{T}) = (1 + |\mathbf{T}|^a)^{-\frac{1}{a}}\mathbf{T}$. The four curves correspond to $a = \frac{1}{2}$ (loosely dash-dotted curve), $a = 1$ (solid curve), $a = 2$ (dashed curve) and $a = 10$ (dash-dotted curve). Clearly, $|\mathbf{T}|$ tends to 1 more rapidly with increasing a. In particular, the graph becomes 'sharper' as a increases, suggesting why we are able to obtain improved results in the case that a is small. However, it is not clear why $a < \frac{2}{d}$ is, at present, optimal.

trace part and deivatoric part of the stress are decoupled to give the constitutive relation

$$F(\mathbf{T}) = \lambda(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\mathbf{T})\mathbf{I} + \mu(|\mathbf{T}^\delta|)\mathbf{T}^\delta, \tag{1.16}$$

where the functions λ and μ are given. A typical example for both λ and μ is

$$\frac{1}{(1 + |s|^a)^{\frac{1}{a}}}, \tag{1.17}$$

where the parameter a need not be the same for λ and μ. The following general assumptions on λ and μ are from [16] and are based on the fact that (1.17) is a typical choice. We have the coercivity and boundedness assumptions

$$\frac{C_1 s^2}{\kappa + |s|} \leq \lambda(s)s^2 \leq C_2 s \qquad s \in \mathbb{R}, \tag{A1}$$

$$\frac{C_1 s^2}{\kappa + s} \leq \mu(s)s^2 \leq C_2 s \qquad s \in \mathbb{R}_+, \tag{A2}$$

and growth assumptions

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}s}(\lambda(s)s) \qquad s \in \mathbb{R}, \tag{A3}$$

$$\frac{C_1}{(\kappa + s)^{a+1}} \leq \frac{\mathrm{d}}{\mathrm{d}s}(\mu(s)s) \qquad s \in \mathbb{R}_+, \tag{A4}$$

where C_1, C_2, κ and a are positive constants. In this thesis, we focus on relations of the form (1.15) but the analysis can be extended to constitutive relations like (1.16) under suitable assumptions such as (A1)–(A4).

For the completeness of this review of the literature, although it only deals with the anti-plane strain case, we include a result from [18]. It was the first result of its kind, concerning problems of the form (1.14), (1.15). However, the geometry is specific. An infinite cylinder with planar cross-section is considered. The proof is quite different to those in this thesis. Due to the simplified geometry, the authors of [18] reduce the problem by means of the Airy stress function. This is the reason for the geometric restrictions on the domain, in comparison to the results that we prove here which have no such restrictions.

Theorem 1.1. *Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected, Lipschitz domain. Define the spatial domain $\Omega = \Omega_0 \times \mathbb{R}$ with $\partial\Omega_D = \emptyset$, $\partial\Omega_N = \partial\Omega_0 \times \mathbb{R}$. Assume that $\boldsymbol{f} = \mathbf{0}$ with boundary traction $\boldsymbol{g}(\mathbf{x}) = (0, 0, g(x_1, x_2))$ where $g \in C^\infty(\partial\Omega_0)$ and $\int_{\partial\Omega_0} g \,\mathrm{d}S = 0$. Consider (1.14) with F given by (1.16). Suppose that (A1)–(A4) hold. Then there exists a unique weak solution to (1.14) provided that either*

(i) $a \in (0, \infty)$ is arbitrary and Ω_0 is uniformly convex, or

(ii) $a \in (0, 2)$ and $\partial\Omega = \Gamma_{con} \cup \Gamma_{flat}$, where Γ_{con} is piecewise uniformly convex, Γ_{flat} is piecewise flat, and $\mathbf{g}$ vanishes on Γ_{flat}.

Something of note is the fact that Theorem 1.1 proves the existence of weak solutions under fully Neumann boundary conditions. This is of contrast to the results of this thesis. We have full existence results in the case of fully Dirichlet boundary conditions but if part of the boundary is supplied with a Neumann boundary condition, we are not able to obtain existence of a weak solution in the usual sense. An error term will be present on the Neumann part of the boundary. This is discussed in much greater detail in Chapter 4.

Our main interest is in existence results that are similar to that of Theorem 1.1 but for the case when Ω is an arbitrary domain in $\mathbb{R}^3$, rather than an infinite cylinder. We note that a strong existence result holds in the case that $a \in (0, 2)$ in Theorem 1.1. The restriction on a

that appears in this thesis for higher integrability estimates is $a \in (0, \frac{2}{d})$. This leads to the next theorem of interest.

The following result is proven in [19]. It deals with the system (1.14) in the spatially periodic setting when the function F is defined by (1.15). The existence of a unique solution is proven in the case $a < \frac{2}{d}$ when working on a periodic domain in $\mathbb{R}^d$. For larger values of a, only the existence of a renormalised solution is proven. However, later work, in particular that of [5], shows that the existence of weak solutions in fact holds for every positive value of a. Furthermore, the renormalised solution constructed in [19] is exactly the weak solution of interest that is constructed in [5]. For the subscripts $*$, $\#$ that are used in the statement, if $\mathcal{F}$ is a local Lebesgue or Sobolev space $L^p_{loc}(\mathbb{R}^d)$ or $W^{k,p}_{loc}(\mathbb{R}^d)$, $\mathcal{F}_\#$ denotes the set of all functions from $\mathcal{F}$ that are 1-periodic in each co-ordinate direction. $\mathcal{F}_*$ represents the subset of this class of functions having zero integral average on Ω.

Theorem 1.2. *Let $\Omega = (0,1)^d$, $\partial\Omega_D = \partial\Omega$ and $\partial\Omega_N = \emptyset$, and replace the Dirichlet boundary condition in (1.14) by a periodic boundary condition. Define F by (1.15) and consider the resulting periodic problem (P). Suppose that $\boldsymbol{f} \in W^{1,t}_*(\Omega)^d$ for some $t > 1$. Then, there exists a unique weak solution $(\boldsymbol{u}, \mathbf{T})$ to (P), provided that either $a \in (0, \frac{2}{d})$ if $d > 2$ or $a \in (0, 1]$ if $d = 2$. By a weak solution, it is meant that*

$$\int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x,$$

for every test function $\boldsymbol{v} \in W^{1,1}_*(\Omega)^d$ such that $\boldsymbol{\varepsilon}(\boldsymbol{v}) \in L^\infty_\#(\Omega)^{d\times d}$, and the constitutive relation $\boldsymbol{\varepsilon}(\boldsymbol{u}) = F(\mathbf{T})$ holds pointwise a.e. in Ω.

The proof uses the Galerkin method with a Fourier basis, taking advantage of the periodic structure of the problem. As mentioned earlier, this periodic setting could be used to model the problem of concentrated loads acting on a body provided that the concentration occurs sufficiently far away from the boundary. This motivates our extension of Theorem 1.2 to the dynamic setting, albeit for viscoelastic materials rather than elastic ones. Furthermore, periodic structure removes any issues that may arise due to the boundary conditions, such as those in Theorem 1.4, and so from the point of view of the mathematical analysis of time-dependent strain-limiting problems is the natural place to start.

The strain-limiting problem considered in Theorem 1.2 is extended to a general domain in [16]. Indeed, the existence result is proven for a strain-limiting problem on an arbitrary bounded domain with Dirichlet boundary conditions. Furthermore, the result is extended to a more general constitutive relation, where F is decomposed into a volumetric and deviatoric part as in (1.16). However, the same result applies in the case that F takes the form (1.15).

Theorem 1.3. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open, bounded set with Lipschitz boundary such that $\partial\Omega_D = \partial\Omega$, $\partial\Omega_N = \emptyset$. Assume that λ and μ satisfy (A1)–(A4), with $a \in (0, \frac{1}{d})$. Suppose that $\boldsymbol{f} = -\mathrm{div}(\mathbf{F})$ for a given $\mathbf{F} \in W^{\beta,1}(\Omega)^{d\times d}$ where $\beta \in (ad, 1)$. Then, there exists a couple $(\boldsymbol{u}, \mathbf{T})$ such that*

- $\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym}$,
- $\boldsymbol{u} \in W^{1,p}_0(\Omega)^d$ *for every* $p \in [1, \infty)$, *and*
- $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}$,

such that the couple is a weak solution of (1.14), (1.16) in that, for every $\boldsymbol{v} \in C^\infty_c(\Omega)^d$,

$$\int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \mathbf{F} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x,$$

and the constitutive relation $\boldsymbol{\varepsilon}(\boldsymbol{u}) = F(\mathbf{T})$ *holds pointwise a.e. in* Ω.

Further regularity results are also proven in [16]. The proofs of existence in Chapters 2, 3 and 4 for the viscoelastic dynamic problem rely on similar methods to those used in the proof of Theorem 1.3. Namely, an elliptic regularisation is introduced in the constitutive relation and the strain-limiting problem is approximated by one with growth.

One of the most general results in the literature regarding strain-limiting problems is from [5].[1] A significant feature of the result is the fact that the restriction on the parameter a is removed, as well as the radial nature of the constitutive relation. The function F need only be asymptotically radial. Another important extension is that mixed boundary conditions are considered. However, a consequence of this choice is the presence of a penalisation term on the Neumann part of the boundary which cannot be removed.

We introduce the following assumptions on the structure on F. We suppose that F is a C^1 function and define a fourth-order tensor[2] function $\mathcal{A}$ by

$$\mathcal{A}_{ijkl}(\mathbf{T}) = \frac{\partial F_{ij}}{\partial T_{kl}}(\mathbf{T}) \quad \forall\, i,\, j,\, k,\, l \in \{1, ..., d\}.$$

We also assume that F is uniformly h-monotone, i.e., there exists a positive, non-increasing, continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for every $\mathbf{T}$, $\mathbf{S} \in \mathbb{R}^{d\times d}_{sym}$,

$$h(|\mathbf{T}|)|\mathbf{S}|^2 \leq \sum_{i,j,k,l=1}^{d} \mathcal{A}_{ijkl}(\mathbf{T})\mathbf{S}_{ij}\mathbf{S}_{kl} \leq \frac{C_2|\mathbf{S}|^2}{1+|\mathbf{T}|}.$$

Furthermore, $\mathcal{A}$ must be asymptotically symmetric, i.e., there exists a positive constant C_2 such that, for every $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$,

$$\frac{|\mathcal{A}^s(\mathbf{T}) - \mathcal{A}(\mathbf{T})|^2}{h(|\mathbf{T}|)} \leq \frac{C_2}{1+|\mathbf{T}|},$$

where h is as above and $\mathcal{A}^s(\mathbf{T}) := \frac{1}{2}(\mathcal{A}(\mathbf{T}) + \mathcal{A}^{\mathrm{T}}(\mathbf{T}))$. Finally, F needs to have an asymptotic radial structure, by which we mean there exists a non-negative continuous function g defined on $\mathbb{R}_+$ with

$$g(t) \leq C_2(1+t),$$

[1] We note that the restriction on a in Theorem 1.2 (and also Theorem 1.3) is purely technical. The work from [5] and, in particular, the proof of Theorem 1.4 can be used to extend the results for the periodic and Dirichlet case to existence for any positive value of a.

[2] We use such tensors repeatedly throughout the thesis. In particular, we use them to define an inner product on $\mathbb{R}^{d\times d}$ by $(\mathbf{S},\mathbf{U})_{\mathcal{A}(\mathbf{T})} = \sum_{i,j,k,l} \mathcal{A}_{ijkl}(\mathbf{T})S_{ij}U_{kl}$. This simplifies the notation significantly and provides information such as non-negativity of $(\mathbf{S},\mathbf{S})_{\mathcal{A}(\mathbf{T})}$, as well as use of the Cauchy–Schwarz inequality. However, the most important part to note is that $(\partial\mathbf{T},\partial\mathbf{T})_{\mathcal{A}(\mathbf{T})} = \partial\mathbf{T}\cdot\partial F(\mathbf{T})$. This is vital for higher regularity estimates.

for every $t \in \mathbb{R}_+$ and there exists a constant C_2 such that, for every $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$, the following holds:

$$\frac{|g(|\mathbf{T}|)F(\mathbf{T}) - \mathbf{T}|^2}{h(|\mathbf{T}|)} \le C_2(1 + |\mathbf{T}|^3).$$

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and suppose the initial data $\boldsymbol{u}_0$ and external data $(\boldsymbol{f}, \boldsymbol{g})$ satisfy a safety strain condition and compatibility condition as described in [5]. Suppose that F is a C^1, uniformly h-monotone function with asymptotic radial structure that is also asymptotically symmetric, with the coercivity and growth conditions*

- $F(\mathbf{T}) \cdot \mathbf{T} \ge C_1|\mathbf{T}| - C_0$, *and*
- $|F(\mathbf{T})| \le C_2$,

for every $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$. There exists a triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in W^{1,1}(\Omega)^d \times L^1(\Omega)^{d\times d} \times (C^1_0(\partial\Omega_N)^d)^*$ solving (1.14) weakly, up to Neumann part of the boundary. The triple satisfies the regularity conditions

- $\boldsymbol{u} \in W^{1,p}(\Omega)^d$ *for every* $p \in [1, \infty)$,
- $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}$,
- $\boldsymbol{u} = \boldsymbol{u}_0$ *on* $\partial\Omega_D$,

with the constitutive relation $\boldsymbol{\varepsilon}(\boldsymbol{u}) = F(\mathbf{T})$ holding pointwise a.e. in Ω. The elastodynamic equation holds weakly up to a penalisation of the Neumann part of the boundary in the following sense:

$$\int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x + \langle \boldsymbol{g} - \tilde{\boldsymbol{g}}, \boldsymbol{v}\rangle|_{\partial\Omega_N},$$

for every $\boldsymbol{v} \in C^1_D(\overline{\Omega})^d$. Furthermore, the triple can be chosen so that the following inequality holds:

$$\int_\Omega \mathbf{T} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))\,\mathrm{d}x \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{v})\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{v})\,\mathrm{d}S,$$

for every $\boldsymbol{v} \in W^{1,1}(\Omega)^d$ such that $\boldsymbol{v} = \boldsymbol{u}_0$ on $\partial\Omega_D$ and $\boldsymbol{\varepsilon}(\boldsymbol{v}) = F(\mathbf{S})$ in Ω for some function $\mathbf{S} \in L^1(\Omega)^{d\times d}$. If $\partial\Omega_D \ne \emptyset$ or the integral mean value of $\boldsymbol{u}$ is fixed, the triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}})$ is unique in the above class of solutions.

We note that Theorem 1.4 gives a full existence result to the strain-limiting problem with fully Dirichlet boundary conditions, under weak assumptions on the constitutive function F.

For a further discussion of the measure $\tilde{\boldsymbol{g}}$, we refer to [5]. We prove a result similar to Theorem 1.4 in this thesis in Chapter 4. In particular, we obtain an existence result for a dynamic strain-limiting problem with mixed boundary conditions up to a correction term concentrated on the Neumann part of the boundary. The penalisation only occurs on the Neumann part of the boundary and it is for this reason that we do not need to extend the class of weak solutions to the space of Radon measures on $\overline{\Omega}$. In particular, each component of the solution couple

$(\boldsymbol{u}, \mathbf{T})$ is a Lebesgue function over the entire space-time domain. The presence of the penalisation term is also the reason that we are unable to obtain the existence of a weak energy solution to the fracture problem in Chapter 6. We cannot obtain a suitable energy equality due to the penalty term.

We note that there seems to be a gap in the current literature for the rigorous analysis of numerical methods for strain-limiting problems. As far as I am aware, the only studies in this area seem to be [8] and [51]. Although the problem is studied numerically in special geometries for the case of anti-plane strain in several papers, (for example, see [81], [82], [71], [70]) none give general error estimates. The strain-limiting problem (1.14) is studied in [8] with F given by (1.16) and homogeneous Dirichlet boundary conditions. Under some regularity assumptions, it is shown that the finite element approximations converge at a given rate. Numerical experiments are used to demonstrate these theoretical results in two-dimensions.

In a similar way to how [8] extended the work presented in [16], a spectral approximation of the problem from [19] is discussed in [51]. A nonlinear, finite-dimensional problem was constructed to approximate the original problem, without requiring an additional regularisation parameter. An iterative method was used to approximate this finite-dimensional problem and was shown to converge linearly to the solution of the approximate problem. The periodic structure permitted the method used in [51] and is significantly different to the ideas used in [8]. Furthermore, in the periodic case, it is worth noting that there was no restriction on the parameter in the constitutive relation. This was due to the existence of renormalised solutions for any positive value of a (although we now know that weak solutions exist irrespective of the value of a). An area of significant research for the future would be to extend the work in this thesis in a similar way to that of [8] and [51]. Of particular interest would be to consider the fracture problems discussed in Chapters 5 and 6. However, this would likely require significant computational effort.

1.4 Viscoelasticity

In this thesis, we are concerned with dynamic problems in viscoelasticity. The reason for studying viscoelasticity rather than elasticity is due to the smoothing effect that results from the presence of the viscous term. The presence of the strain rate $\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ is vital to the construction of weak solutions. Without this term, we are unable to obtain sufficient regularity on the sequence of approximations for the strain-limiting problem. We note that the presence of this 'damping' term is common in fracture problems, as is discussed in Section 1.5. It is out of the scope of this work to remove the viscous term so from now on we assume the presence of the strain rate term in the constitutive relation.

The standard models that are generally mentioned in the context of viscoelasticity are the Maxwell, Kelvin–Voigt, Zener and Burgers models. The focus here is on a Kelvin–Voigt type model. In the one-dimensional, linear case, the constitutive relation for a linear Kelvin–Voigt body is given by

$$\sigma = E\epsilon + \eta\epsilon_t, \tag{1.18}$$

where E and η are constants. The classical physical representation of this model is a linear elastic spring and a linear viscous dashpot aligned in parallel. Conversely, the Maxwell model is traditionally represented by a spring and dashpot in series leading to a relation that involves σ, σ_t and ϵ_t. The Zener model involves the stress, strain and both of their first order derivatives with respect to the time variable. A generalisation of this model includes the use of fractional derivatives in time rather than first order derivatives. For more details, see [3], [105] and the references therein, for example. The Burgers model is even more complicated and generally contains a second-order derivative in time.

Combining the ideas of implicit constitutive theory with the Kelvin–Voigt model of the form (1.18), following [97] we consider the following implicit relation:

$$\mathbf{h}(\mathbf{T}, \mathbf{B}, \mathbf{D}) = \mathbf{0}. \tag{1.19}$$

An interesting subclass of (1.19) to consider is

$$\mathbf{T} = \mathbf{h}(\mathbf{B}, \mathbf{D}). \tag{1.20}$$

Under the condition that $\mathbf{h}$ is isotropic, a representation similar to (1.7) can be deduced. We refer to [74] for the details. Under the small displacement gradient assumption (1.5), the isotropic representation of (1.20) reduces to

$$\mathbf{T} = \mathcal{H}\boldsymbol{\varepsilon} + \mathcal{G}\boldsymbol{\varepsilon}_t, \tag{1.21}$$

where $\mathcal{H}$, $\mathcal{G}$ are fourth-order tensors that are independent of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_t$. In particular, the relation (1.21) is linear in $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_t)$. However, the linearity leaves open the possibility that the underlying assumption (1.5) of the model could be contradicted in the case of large stresses. Hence, as in the elastic case, we look for a system of equations in which the strain is bounded *a priori* by a constant that we can choose to be arbitrarily small. For a further study of equations of the type (1.20), we refer to [96] and the references therein.

An aspect of constitutive relations repeatedly discussed by Rajagopal is that, philosophically, the force (hence stress) is the cause of the deformation (and consequently changes in the kinematic variables including strain and strain rate). Mathematically, this suggests that the kinematic variables should be given as a function of the stress. However, the converse, with the stress as a function of the strain, seems to have been the focus of historical studies. Not least, this appears to be due to the fact that it provides mathematical simplicity because the stress can be substituted into the balance equation for momentum. This results in an equation in just the displacement variable, albeit of a higher order than the original problem. However, without such a substitution being available, the problem at hand is fully coupled and thus potentially more challenging to solve.

Despite this, we consider the case where the kinematic variables, namely the displacement and thus strain, are functions of the stress, but not necessarily invertible. The subclass of implicit relationships of the form (1.19) that we study here is a generalisation of the Kelvin–Voigt model. In particular, we investigate models of the form

$$\alpha \mathbf{B} + \nu \mathbf{D} = \mathbf{h}(\mathbf{T}), \tag{1.22}$$

where $\mathbf{h}$ is a nonlinear function and α, ν are non-negative constants. This form is justified by the theory of Rajagopal. Under the linearisation procedure, we do not remove the nonlinear aspect on the right-hand side of (1.22). Thus we obtain a nonlinear constitutive relation between the strain, strain rate, and stress tensor. Furthermore, the procedure is systematic and can be shown to be thermodynamically justified [32].

As in [97], under the condition of isotropy, the relation (1.22) reduces to the form

$$\alpha \mathbf{B} + \nu \mathbf{D} = \tilde{\beta}_0 \mathbf{I} + \tilde{\beta}_1 \mathbf{T} + \tilde{\beta}_2 \mathbf{T}^2, \tag{1.23}$$

where $\tilde{\beta}_i$ for $0 \leq i \leq 2$ are functions of the density and the principal invariants of $\mathbf{T}$. Applying the small strain assumption (1.5), we obtain

$$\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}) + \nu \boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2. \tag{1.24}$$

With regards to the constants α and ν, as mentioned at the beginning of this section, we require the presence of the strain rate $\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$.[3] Thus, without loss of generality, from now on we assume that $\nu = 1$. For the constant α, the corresponding results are more flexible. Indeed, the proofs here are valid for any non-negative value $\alpha \geq 0$. However, if $\alpha = 0$ we move into the class of implicit fluid models [88, 89] so our main interest is the case that $\alpha > 0$.

It should be noted that the theory of implicitly constituted fluids is very rich. Rigorous mathematical analysis has been done on many of the systems of PDEs that result from such a model. Both the steady and unsteady cases have been investigated. See [13], [14] and the references therein, for example. An elegant part of the analysis is that truly implicit relations can be considered. The relation $\mathbf{h}(\mathbf{T}, \mathbf{D}) = \mathbf{0}$ is identified with a maximal monotone graph $\mathcal{A}$, where $(\mathbf{T}, \mathbf{D}) \in \mathcal{A}$ if and only if $\mathbf{h}(\mathbf{T}, \mathbf{D}) = \mathbf{0}$. Properties of the graph are used in the analysis rather than those of $\mathbf{h}$. In particular, in [13] and [14], the graph is assumed to be an M-graph, where M is some N-function. We note that this means that $\boldsymbol{\varepsilon}_t$ grows without bound as $\mathbf{T}$ grows. As a result, properties of Orlicz spaces are used as they are the natural function space setting, in a similar way to how they are employed in [56]. This growth condition differentiates the results on fluids from the results presented here. See also [57] for similar work dealing with monotonic operators that have an Orlicz growth condition. Furthermore, we note that problems in the context of fluids have different balance equations to those that are considered here because the Eulerian formulation is used rather than the Lagrangean formulation.

[3] The inclusion of the viscous term $\boldsymbol{\varepsilon}(\ bu_t)$ reflects a physical change to the model and is completely valid. However, the analysis in this thesis is not possible in the case that we do not have the viscous term and the analysis of dynamic strain-limiting elastic models is an open problem.

We must also mention the studies in [40] and [41] where similar problems to those considered in this thesis are discussed and analysed. In [40], for an unknown scalar function u, the following abstract problem is investigated:

$$u'' + Au' + Bu = f.$$

The convergence of numerical solutions is used to prove existence of a weak solution. The following related problem is studied in [41]:

$$u_{tt} - \nabla \cdot F(\nabla u_t) - \Delta u = f.$$

The problem is supplemented with a homogeneous Dirichlet boundary condition. Here, F is any continuous, monotone function such that there exists an N-function M with convex conjugate $M^*(\boldsymbol{v}) := \sup_{\boldsymbol{u}\in\mathbb{R}^d}\{\boldsymbol{v}\cdot\boldsymbol{u} - M(\boldsymbol{v})\}$ such that, for every $\boldsymbol{v} \in \mathbb{R}^d$, we have that

$$F(\boldsymbol{v}) : \boldsymbol{v} \geq c(M(\boldsymbol{v}) + M^*(F(\boldsymbol{v})).$$

The coercivity condition above distinguishes the problem from the ones discussed here. In particular, the system studied in [41] is more similar to the regularised problem that we use to approximate the strain-limiting problem. Furthermore, the additional damping term Δu guarantees the solution extra spatial regularity, which is not necessarily available for the problems presented in this thesis. The problem in [41] is somewhat similar to those studied in the context of implicitly constituted fluids, although the proof is related to some results in this thesis. A discretisation in time is used to prove existence of a weak solution. Such a method is used when we consider fracture problems in Chapters 5 and 6, but we have an extra variable due to the presence of the phase-field function which approximates the fracture set.

It is worth mentioning the analysis in [107], where the following system of equations is investigated:

$$\boldsymbol{u}_{tt} - \operatorname{div}(\mathbf{S}(\nabla\boldsymbol{u}_t, \nabla\boldsymbol{u})) = \mathbf{f}.$$

Although, the restrictions on the function $\mathbf{S}$ exclude any physically realistic constitutive functions, Tvedt suggests that the methods used could be extended to more realistic cases. However, we note that the analysis does not cover the strain-limiting problem. Furthermore, the full gradient is considered rather than the symmetric part of the gradient.

Let us now return to strain-limiting solids. In [97], the authors assume that the solid is incompressible. It is shown that, under the additional assumption that $\beta_2 = 0$, the constitutive relation (1.24) must be of the form $\alpha\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_t = \beta_1\mathbf{T}^\delta$. The authors suppose that β_1 depends only on $\operatorname{tr}((\mathbf{T}^\delta)^2)$. The propagation of circularly polarised transverse stress waves, standing shear stress waves, and oscillatory shear waves in this class of viscoelastic solids is investigated in the context of the time-dependent problem. The authors conclude that the solutions found provide examples of global existence. Further analysis on problems with the relation (1.24) is currently limited as far as I am aware, particularly in the case that there is no assumption on the specific structure of the deformation. However, in our study we do not assume a dependence of the strain on only the deviatoric part $\mathbf{T}^\delta$. Equivalently, we do not assume that $\mathrm{div}(\boldsymbol{u}) = 0$.

The one-dimensional strain-limiting viscoelastic problem is considered in [42] in the context of travelling wave solutions. The right-hand side of (1.24) is assumed to be a general nonlinear function of the stress. The problem is reduced to a one-dimensional PDE in only the stress variable. The work [42] was followed by [44], in which the authors investigate an IBVP based on the system

$$\begin{aligned} u_{tt} &= \sigma_x, && \text{in } (0,\infty)\times\mathbb{R}, \\ \nu\epsilon_t + \epsilon &= h(\sigma), && \text{in } (0,\infty)\times\mathbb{R}, \end{aligned} \tag{1.25}$$

where u and σ are scalar-valued unknowns. The function h is non-linear, increasing and bounded, with $h(0) = 0$. The authors use a substitution argument based on the fact that $\epsilon = u_x$ to obtain the equivalent problem

$$\eta_{tt} = g(\eta_x) + \nu g(\eta_x)_{xt} \quad \text{in } (0,\infty)\times\mathbb{R}, \tag{1.26}$$

where η is a potential that satisfies $\epsilon + \nu\epsilon_t = \eta_x$ and g is the inverse of h. We note that g is well-defined by the assumptions on h. The authors use results based on the variable coefficient heat equation and the theory of elliptic operators, alongside a linearisation and Banach's fixed point theorem, to conclude that a solution to (1.26) exists. Using this, they obtain an existence result for the original problem (1.25). Although the final existence result is similar to that presented in Chapter 3, we note that the one-dimensional problem is a very different situation to the multi-dimensional setting. The equality of the full gradient with the symmetric gradient is a significant simplification and allows the eradication of the stress from the problem, despite the strain-limiting setting. No such equality is available in higher spatial dimensions. Furthermore, such substitutions increase the order of the problem, which we are keen to avoid wherever possible. Hence the work here is remarkably distinct to the aforementioned results, despite being based on the same problem originally.

Returning to the topic of travelling wave solutions, which was the basis of [42], we note that they have been considered in different viscoelastic settings other than for the strain-limiting problem that is the main interest here. In [38], the stress was assumed to be a nonlinear function of $\mathbf{B}$ and $\mathbf{D}$, with no small-strain assumption. A shearing motion in a three-dimensional solid was considered, out of which a PDE with a scalar unknown was obtained. Compact travelling waves were then investigated in this setting. In [66], a relation where the stress is given as a nonlinear function of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_t$ was considered. A third-order PDE in a scalar function was deduced and travelling wave solutions were considered. We note that all of these results use substitution methods which increase the order of the original problem.

Another context in which a constitutive relation of the form (1.24) has been studied is in the quasi-static setting, by which we mean that the inertial term $\boldsymbol{u}_{tt}$ is assumed to be sufficiently small so that it is negligible. In [63], the authors consider the system

$$-\mathrm{div}(\mathbf{T}) = \boldsymbol{f},$$
$$\boldsymbol{\varepsilon}_t + \nu\boldsymbol{\varepsilon} = F(\mathbf{T}),$$

in $Q_c^T = (0,T) \times \Omega_c$, where Ω_c is a domain with a crack, supplemented with mixed Dirichlet–Neumann boundary conditions. In the case that F is a bounded, monotone, hemi-continuous, semi-coercive function, it is shown that a weak solution to the problem exists. However, the constitutive relation only holds in the form of an integral equality. In this thesis, we will demand that the constitutive relation holds pointwise a.e. in the domain of interest. We note that in the fracture problems of Chapters 5 and 6, we do not work directly in a domain with a crack. Instead we approximate the crack with a phase-field function based off an approximation parameter ϵ. For a further discussion, we refer to Section 1.5.

A similar problem to that studied in [63] is considered in [64]. The constitutive relation that is of interest in this work is $\boldsymbol{\varepsilon}(t) = \mathcal{I}(F(\mathbf{T}), t)$, where the prototypical form of F is given by $F(\mathbf{T}) = \mathbf{T}/(1+|\mathbf{T}|^r)^{\frac{2-p}{r}}$ for a given $p \in [1,\infty)$ and $r > 0$. The function $\mathcal{I}$ is an integral operator defined by

$$\mathcal{I}(\mathbf{T}, t) = J(0)\mathbf{T}(t) + \int_0^t J'(t-s)\mathbf{T}(s)\,\mathrm{d}s,$$

where J is a generalised creep operator. The existence of solutions is studied for both the case $p > 1$, i.e., when there is growth in the constitutive relation, and when $p = 1$, the strain-limiting case. Creep is a phenomenon displayed by viscoelastic materials. It means that when stress is applied to a body at a constant level, the strain continues to increase gradually over time. The prototypical form of a creep function is $J(t) = J_0 + J_1(1 - \mathrm{e}^{-t/\tau_1})$ where J_0, J_1, τ_1 are positive constants. In the case that $J_0 = 0$, the constitutive relation reduces to $\boldsymbol{\varepsilon} + \frac{1}{\tau_1}\boldsymbol{\varepsilon}_t = F(\mathbf{T})$, which is exactly the prototype of the constitutive relation that is the focus in this thesis if $p = 1$.

A related model is studied in [77]. It is a one-dimensional 'quasi-linear' viscoelastic model but the strain is given in terms of the stress. The history of the stress up to the current time is used to define the strain. If σ is the one-dimensional stress, the authors of [77] suppose that

$$\epsilon(t) = f(\sigma(0), t) + \int_0^t \frac{\partial f(\sigma(s), t-s)}{\partial \sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\,\mathrm{d}s,$$

where $f(\sigma(t), t) = G(\sigma)J(t)$ for a nonlinear function G and a generalised creep function J, as mentioned previously. Further mathematical analysis for such a problem can be found in [4]. We also note the similarities of this problem to the one which is studied in [64].

The Kelvin–Voigt model can be generalised in other ways than the relation discussed above. An interesting study of a Kelvin–Voigt type model has been undertaken in [91], with rigorous mathematical analysis and regularity theory of the corresponding IBVP in [17, 15], respectively. The body considered is a mixture of an elastic solid and a viscous fluid with no motion between the fluid and the solid. This is an exact generalisation of a system with a spring and dashpot set up in parallel. The stress is decomposed into the part coming from the elastic solid $\mathbf{T}_e$ and the part coming from the fluid $\mathbf{T}_f$. This leads us to consider the following implicit relationships:

$$\mathbf{h}_1(\mathbf{T}_e, \mathbf{B}) = \mathbf{0}, \quad \mathbf{h}_2(\mathbf{T}_f, \mathbf{D}) = \mathbf{0}, \quad \mathbf{T} = \mathbf{T}_e + \mathbf{T}_f.$$

In the event that the strain is small, consider

$$\mathbf{h}_1(\mathbf{T}_e, \boldsymbol{\varepsilon}) = \mathbf{0}, \quad \mathbf{h}_2(\mathbf{T}_f, \boldsymbol{\varepsilon}_t) = \mathbf{0}, \quad \mathbf{T} = \mathbf{T}_e + \mathbf{T}_f.$$

In [17], the assumption is that these relations can be rewritten in the specific form

$$\mathbf{T}_e = \tilde{\mathbf{h}}_1(\boldsymbol{\varepsilon}), \quad \mathbf{T}_f = \tilde{\mathbf{h}}_2(\boldsymbol{\varepsilon}_t), \quad \mathbf{T} = \mathbf{T}_e + \mathbf{T}_f.$$

We note that $\tilde{\mathbf{h}}_1$, $\tilde{\mathbf{h}}_2$ would need to be invertible in order for the presence of nonlinear functions to be justified under the small strain assumption (1.5). In the context of the balance equation for linear momentum, $\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T})$, we can replace $\mathbf{T}$ by $\mathbf{h}_1(\boldsymbol{\varepsilon}) + \mathbf{h}_2(\boldsymbol{\varepsilon}_t)$. A proof of the existence of a weak solution to the corresponding problem is given in [17] under the assumptions that $\mathbf{h}_2$ satisfies

$$(\mathbf{h}_2(\mathbf{T}) - \mathbf{h}_2(\mathbf{S})) \cdot (\mathbf{T} - \mathbf{S}) \geq C|\mathbf{T} - \mathbf{S}|^2.$$

In particular, $\mathbf{h}_2$ must be an unbounded function. Hence the model is not strain-limiting and the methods used for the proof cannot be directly adapted to the problems studied here. However, it would be interesting to consider the effect of either $\mathbf{h}_1$ or $\mathbf{h}_2$ being a bounded function.

A relation similar to (1.19) is studied in [98]. However, it is a generalisation of the Maxwell model. Suppose that

$$\mathbf{h}(\mathbf{T}, \mathbf{T}_t, \mathbf{L}) = \mathbf{0}. \tag{1.27}$$

The authors develop a general framework for describing a certain class of rate-type viscoelastic bodies that is a subclass of (1.27) and ensure that it is thermodynamically justified. We refer to [98] for further details. A highly related problem, referred to as a stress-rate model, has been studied in [43]. The authors introduce a thermodynamically consistent model of the form

$$\epsilon = h(\sigma) - \gamma \sigma_t, \tag{1.28}$$

where h is a nonlinear function and γ is a non-negative constant. The requirement on γ is to ensure that the first and second laws of thermodynamics are satisfied, and so the model can be thermodynamically justified. The analysis is currently limited to the one-dimensional setting. This stress-rate type model is compared to the strain-rate type model $\epsilon + \nu\epsilon_t = h(\sigma)$ in the one-dimensional setting from the point of view of travelling wave solutions. The extension of stress-rate problems of this form to higher dimensions does not appear to be possible at the moment in the case that h is uniformly bounded. This is due to a lack of *a priori* bounds in the supposedly thermodynamic case that γ is non-negative.

To summarise the discussion so far, we are interested in IBVP of the form

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \quad \text{ in } Q := \Omega \times (0, T),$$

where $(\boldsymbol{u}, \mathbf{T})$ is an unknown couple that satisfies the constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} =: F(\mathbf{T}) \quad \text{pointwise on } Q.$$

The function F is bounded above by a fixed constant. The focus of Chapters 2, 3 and 4 is to provide mathematical analysis of such problems under various boundary conditions for the prototypical choice of F above.

1.5 Fracture Problems

In the second half of the thesis (namely Chapters 5 and 6), we are interested in combining the theory of dynamic fracture with the theory of nonlinear, implicit constitutive relations. To my knowledge, at present the mathematical analysis of dynamic fracture problems has only been done in the setting of linear elasticity and linear viscoelasticity. We aim to extend such fracture studies and examine time-dependent damage problems with a constitutive relation from nonlinear viscoelasticity. Of particular interest is the strain-limiting case when the linearised strain is bounded *a priori*. The key motivation for considering strain-limiting models in the context of fracture is that such a defining relation ensures no singularity is experienced in the strain. Indeed, we cannot contradict the small strain assumption which is essential to the construction of the model here. This contrasts the known results for damage models with a linear constitutive relation where a singularity occurs at the crack tip [95]. In this section, we introduce the theory of dynamic fracture based on the principles in [76, 72]. Then we combine this with implicit constitutive theory to obtain a fracture problem with a nonlinear constitutive relation between the Cauchy stress tensor, linearised strain, and strain rate. The most important contribution of this thesis is in Chapter 6 where we consider a dynamic fracture problem with strain-limiting constitutive relation and prove the existence of weak energy solutions.

Consider a body with a crack which can grow in time due to the action of an external force. In the case of a linear constitutive relation in the small strain setting, the strain has magnitude of order $r^{-\frac{1}{2}}$ where r is the distance to the crack tip [95]. The strain experiences a singularity at the crack tip, contradicting the standing assumption that the strain is small. A more reliable model might be one in which the magnitude of the strain is bounded *a priori* so such contradictions cannot arise. This motivates considering a strain-limiting constitutive relation. In particular, consider the prototype

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}, \tag{1.29}$$

for fixed parameter $a > 0$. Other choices for the function F are available, but for simplicity of the presentation, we make this specific choice as suggested in [23, 92]. For further details of the possible generalisations, we refer to the discussion at the end of Chapter 2.

We consider (1.29) coupled with (1.12) in the context of dynamic fracture. We now give a brief introduction to the modern theory of fracture. The principles that drive the mathematical analysis of fracture in brittle bodies can be traced back to the seminal work of Griffith [55]. In the quasi-static setting, he stipulated that a crack can only grow if the increase in surface energy is exactly equal to the resulting decrease in elastic potential energy. In other words, the newly created area is proportional to the loss of stored energy. In the anti-plane setting with a linear constitutive relationship where u is the scalar displacement, we write this mathematically as

$$\frac{\mu}{2}\int_{\Omega\setminus C(t)} |\nabla u(t)|^2 \,\mathrm{d}x + G_c\mathcal{H}^{d-1}(C(t)) = \frac{\mu}{2}\int_{\Omega\setminus C(0)} |\nabla u(0)|^2 \,\mathrm{d}x + G_c\mathcal{H}^{d-1}(C(0)), \tag{1.30}$$

where $C(t) \subset \Omega$ is the crack set, μ is the elasticity constant and G_c is the fracture toughness. We use $\mathcal{H}^{d-1}$ to denote the $(d-1)$-dimensional Hausdorff measure. The first term in (1.30) corresponds to the elastic energy and the second term to the surface energy. This formulation was first introduced by Francfort and Marigo [49], based on the Mumford–Shah functional that is used in image segmentation [78].

Alongside this energy balance, the balance of linear momentum must hold. Essentially we require $\mathrm{div}(u(t)) = 0$ to hold away from the crack set at any point in time. We formulate this as a variational problem. The couple $(u(t), C(t))$ must satisfy

$$\mathcal{A}(u(t), C(t)) := \frac{\mu}{2}\int_{\Omega\setminus C(t)} |\nabla u(t)|^2 \,\mathrm{d}x + G_c\mathcal{H}^{d-1}(C(t)) \leq \frac{\mu}{2}\int_{\Omega\setminus \tilde{C}} |\nabla \tilde{u}|^2 \,\mathrm{d}x + G_c\mathcal{H}^{d-1}(\tilde{C}), \tag{1.31}$$

for every suitable comparison couple $(\tilde{u}, \tilde{C})$ with $C(t) \subset \tilde{C}$. The principle of irreversibility is also required to hold, i.e., the crack is non-decreasing in time so $C(s) \subset C(t)$ for every $s < t$. Problems of the form (1.30), (1.31) where we work directly with the crack set are called sharp interface problems. Mathematical analysis of this anti-plane shear problem takes place in [49], with higher dimensional settings treated in [36] and [48]. Both the crack set and displacement are treated as unknowns. This causes significant analytical difficulties. The analysis takes places on the set $\cup_{t\geq 0}\{t\} \times (\Omega \setminus C(t))$ which is not only a domain that changes in time but is *a priori* unknown because the crack set $(C(t))_{t\geq 0}$ is unknown. Even if the crack set is known, the analysis is still delicate because the appropriate function spaces change at every point in time.

An approach to avoid working on time dependent cracking domains is to approximate the crack set by a phase-field function. The Ambrosio–Tortorelli functional [1, 2] takes the following form in the context of quasi-static fracture for linear elasticity:

$$\mathcal{A}_\epsilon(u, v) = \frac{\mu}{2}\int_\Omega (v^2 + \eta_\epsilon)|\nabla u|^2 \,\mathrm{d}x + G_c\int_\Omega \epsilon|\nabla v|^2 + \frac{1}{4\epsilon}(1-v)^2 \,\mathrm{d}x, \tag{1.32}$$

where $\epsilon > 0$ is an approximation parameter, η_ϵ is a numerical stability parameter such that $0 < \eta_\epsilon \ll \epsilon$, and v is the unknown phase-field function, also referred to in the literature as the material parameter. The function v approximates the crack set in the sense that v takes values close to 1 away from the crack and values close to 0 near the crack. Although ϵ acts as a length scale for the approximation, we must be careful not to assign a physical meaning to this parameter [10]. For the mathematical analysis of this approximation in the quasi-static setting, we refer to [68] and [52], for example.

The approximation (1.32) is a sensible choice because $\mathcal{A}_\epsilon$ converges to the sharp-interface functional $\mathcal{A}$ in the sense of Γ-convergence. We refer to [11] for the details but the essence is as follows. Let $((u^\epsilon, v^\epsilon))_\epsilon$ be a sequence such that at any point in time v^ϵ minimises $v \mapsto \mathcal{A}_\epsilon(u^\epsilon, v)$ over a set of suitable test functions such that $v \leq v_\epsilon$. Then there exists a subsequence in ϵ and a solution couple (u, C) to (1.30), (1.31) such that for a.e. time t we have

$$\begin{aligned}
&v^\epsilon \nabla u^\epsilon(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega)^d,\\
&\lim_{\epsilon\to 0+} \int_\Omega ((v^\epsilon)^2 + \eta_\epsilon)|\nabla u^\epsilon|^2 \,\mathrm{d}x = \int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x,\\
&\lim_{\epsilon\to 0+} \int_\Omega \epsilon|\nabla v|^2 + \frac{1}{4\epsilon}(1-v)^2 \,\mathrm{d}x = \mathcal{H}^{d-1}(C(t)).
\end{aligned}$$

Hence, the sequence of solutions for the approximate problem represents a suitable approximation to the solution of the sharp-interface functional.

We remark that there are other choices of approximation available. For example, the fourth order functional

$$\tilde{\mathcal{A}}_\epsilon(u, v) = \frac{\mu}{2}\int_\Omega (v^2 + \eta_\epsilon)|\nabla u|^2 \,\mathrm{d}x + \frac{G_c}{4}\int_\Omega \frac{1}{\epsilon}(1-v)^2 + 2\epsilon|\nabla v|^2 + \epsilon^3|\Delta v|^2 \,\mathrm{d}x,$$

is investigated in [9] and [79]. This functional also Γ-converges to $\mathcal{A}$. An advantage of this choice is that the higher order spatial derivatives improve the convergence rate of numerical solutions. In particular, when studying these fracture problems from the point of view of computational analysis, the Ambrosio–Tortorelli functional is not necessarily the best choice of approximation for the sharp-interface functional. On the other hand, from the point of view of mathematical analysis, the Ambrosio–Tortorelli functional and its generalisations to higher dimensions is sufficient for good existence results.

The aforementioned literature all concern the quasi-static problem. Here, we are interested in the adaptation of Griffith's theory to the dynamic problem, i.e., we do not assume that the term $\boldsymbol{u}_{tt}$ is negligible. An extension of quasi-static fracture theory to dynamic fracture theory was first introduced by Mott [76] and relies on the following three principles:

(i) elastodynamics, the balance of linear momentum should hold away from the crack set;

(ii) energy balance, an energy-dissipation balance that includes the kinetic energy should hold;

(iii) irreversibility, the crack cannot heal itself in time.

However, this is not sufficient for a well-posed problem. The case of a stationary crack should be ruled out. The following additional principle was introduced by Larsen [72]:

(iv) maximal dissipation, if the crack can grow while satisfying (i)–(iii) then it should.

As with the quasi-static problem, the study of dynamic fracture problems broadly follows one of two approaches: working directly in the sharp-interface setting, or an approximation by means of a phase-field function, generally based on the Ambrosio–Tortorelli functional. The focus in this thesis is on the use of a phase-field approximation, avoiding the technicalities of working in a time dependent domain. However, for completeness, we first discuss the few known results concerning the sharp-interface case.

For sharp-interface problems, the analysis is significantly more complicated compared to the quasi-static model due to the presence of the intertial term. Hence, a first step in the analysis is to solve the elastodynamic equation for an unknown displacement when the crack set $(C(t))_{t>0}$ is known *a priori*. Consider the problem of looking for a function $\boldsymbol{u}$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}\big(\mathbb{E}(t,x)\nabla\boldsymbol{u}\big) + \boldsymbol{f}(t,x) \quad \text{in } \Omega\setminus C(t), \quad t\in[0,T], \tag{1.33}$$

where $\mathbb{E}$ is an elasticity tensor and $(C(t))_{t\geq 0}$ is a given sequence of subsets of Ω satisfying some regularity requirements. In an attempt to better understand (1.33), in [33] for the scalar-valued case and [106] for the vector-valued case, first consider the following simplified problem. We look for a function $\boldsymbol{u}$ solving the wave equation

$$\boldsymbol{u}_{tt} - \Delta\boldsymbol{u} - \gamma\Delta\boldsymbol{u}_t = \boldsymbol{f},$$

where $\gamma \in \{0,1\}$. The system is damped if $\gamma = 1$ and undamped if $\gamma = 0$. The viscous term $\Delta\boldsymbol{u}_t$ from the damping provides a certain smoothing effect. A result of this is that, if $\gamma = 1$, uniqueness and an energy-dissipation balance can be shown. However, uniqueness and satisfaction of an appropriate energy-dissipation balance is an open problem in the undamped setting. We note the similarities between this and the existence analysis available for the elastic and viscoelastic problems on an undamaged domain.

In both [33] and [106], the existence proofs use a discretisation in the time variable. This is the standard approach to tackle such fracture problems, and we use a similar approach here. However, different approaches to prove the existence of solutions are available under stronger regularity requirements on the crack set, assuming that it is *a priori* known. In [80], [35] and, for the vector-valued case, [25], a co-ordinate change is used to transform the problem from one on the cracking domain $\cup_{t\geq 0}\{t\}\times(\Omega\setminus C(t))$ to one on the reference domain $[0,\infty)\times(\Omega\setminus C(0))$. A solution to a suitable transformation of the original problem is found on the reference domain. The existence of a solution to the transformed problem is equivalent to the existence of a solution to the original problem on the cracking domain. However, it is unlikely we can expect such good regularity from crack sets in practice.

The next step in the analysis is to deal with the case where both the crack set and displacement are *a priori* unknown. As far as I am aware, the only known result in the multi-dimensional setting is that which is presented in [34]. Under some assumptions on the regularity and geometry of the crack set, the authors show that among all possible couples $(u(t), C(t))_{t\geq 0}$, there exists at least one which satisfies the maximal dissipation condition so the crack grows whenever it is possible for it to do so. There is no damping term present, an advantage of the analysis compared to the aforementioned works. Although a significant amount remains unknown at this stage, such a result suggests that the maximal dissipation condition is the correct principle to impose on the dynamic problem.

An important aspect of all of the aforementioned literature is that only linear elasticity or linear viscoelasticity is considered, i.e., a linear relationship between the Cauchy stress tensor and linearised strain. This is a significant gap in the literature. To my knowledge, there has been no study of dynamic fracture problems in the context of nonlinear constitutive relations except for the work presented in this thesis. In Chapters 5 and 6, we aim to extend some of the known results for linear problems to such nonlinear models, considering both a general case of polynomial growth and a strain-limiting constitutive relation. To avoid the technicalities that come with the sharp-interface model, we consider a dynamic problem with phase-field approximation based on the Ambrosio–Tortorelli functional. The main focus is on the technicalities that arise due to the highly nonlinear nature of the underlying PDE.

A regularised model of the type that we study in this thesis was first introduced in the works [10, 72, 73]. The formulation is based on the principles of elastodynamics, energy-balance and irreversibility, but not maximal dissipation as in [34]. This is because maximal dissipation is naturally guaranteed by the formulation. The minimisation problem (1.35) introduced below ensures that the 'crack' grows whenever it is possible for it to do so. In [73], the authors look for an unknown displacement $\boldsymbol{u}$ and phase-field function v that solve

$$\boldsymbol{u}_{tt} = \operatorname{div}\big((v^2 + \eta_\epsilon)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u} + \boldsymbol{u}_t)\big) + \boldsymbol{f} \quad \text{ in } (0,T) \times \Omega, \tag{1.34}$$

subject to the minimisation problem

$$\mathcal{A}_\epsilon(\boldsymbol{u}(t), v(t)) = \inf_{v \leq v(t)} \mathcal{A}_\epsilon(\boldsymbol{u}(t), v), \tag{1.35}$$

where we define the approximate energy functional

$$\mathcal{A}_\epsilon(\boldsymbol{u}, v) = \frac{1}{2}\int_\Omega (v^2 + \eta_\epsilon)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) \,\mathrm{d}x + \int_\Omega \frac{1}{4\epsilon}(1-v)^2 + \epsilon|\nabla v|^2 \,\mathrm{d}x. \tag{1.36}$$

The first integral on the right-hand side of (1.36) represents an approximation of the elastic energy and the second integral is an approximation of the surface energy, with fixed approximation parameter ϵ. The minimisation problem is identical to that for the quasi-static model. In the literature, it is sometimes referred to as unilateral minimality.

An issue occurs as a result of the minimisation problem when taking the limit in the approximation parameter ϵ. As stated in [73], "the unilateral minimality of the v variable has no obvious counterpart in a sharp-interface model." This means that there is nothing ensuring maximal dissipation in the limiting model. A further deficiency of the model (1.34), (1.35) is the presence of the viscous term $\boldsymbol{u}_t$. When taking the limit in the approximation parameter, the viscoelastic paradox can occur [27]. That is, the only possible solution of the corresponding sharp-interface problem is constant in time. However, in such dynamic fracture models, our interest lies in non-stationary solutions. This paradox is a potential issue with including the damping term in (1.34). However, in the problems that we study here, we include a damping term. It is out of the scope of this thesis to remove this term. As mentioned in the setting of problems without damage, the viscoelastic term provides a smoothing effect that is currently required in order to perform suitable mathematical analysis.

Such a shortcoming is overcome in [26] where the author considers a variation of (1.34), (1.35) that has no such damping term in (1.34). Rather than include a dissipative term in (1.34), a rate-dependent term is included in the minimisation problem (1.35). The author of [26] considers

$$\boldsymbol{u}_{tt} = \operatorname{div}\big((v^2 + \eta_\epsilon)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u})\big) + \boldsymbol{f}, \tag{1.37}$$

coupled with the rate-dependent minimisation problem

$$\mathcal{A}_\epsilon(\boldsymbol{u}(t), v(t)) + (v_t(t), v(t))_{k,2} = \inf_{v \le v(t)} \Big\{\mathcal{A}_\epsilon(\boldsymbol{u}(t), v) + (v_t(t), v)_{k,2}\Big\}, \tag{1.38}$$

where the functional $\mathcal{A}_\epsilon$ is as in (1.36) and $(\cdot,\cdot)_{k,2}$ is the standard inner product in the Sobolev space $H^k(\Omega)$. For every $k \in \mathbb{N}_0$, it is shown that a weak solution to (1.37), (1.38) exists that also satisfies an energy-dissipation inequality. If $k > \frac{d}{2}$, it is possible to prove that the energy-dissipation inequality is in fact an equality. This is essentially a consequence of the Sobolev embedding theorem, from which it follows that $H^k(\Omega)$ is continuously embedded into $C(\overline{\Omega})$ for $k > \frac{d}{2}$.

The proof of the existence of solutions to the dynamic problem in both [73] and [26] involves introducing a discretisation in the time variable. The time-discrete formulation involves an alternating procedure. The elastodynamic equation is solved for the displacement given the phase-field function from the previous time step. The minimisation problem for an unknown phase-field function is then solved, given the displacement function at the current time step. A similar procedure is used here, combined with a Galerkin approximation in the spatial domain.

A key novelty of the presentation in Chapter 5 is the formulation of the problem and the extension of the analysis of the linear case to a model of nonlinear viscoelasticity with a Kelvin–Voigt rheology, based on the implicit constitutive framework in the small strain setting. In particular, we note the derivation of uniform estimates on the sequence of solutions to the time discrete problem. However, the case of the strain-limiting material is the real highlight of this thesis. We make a significant step towards the use of strain-limiting constitutive relations in sharp-interface fracture models, which is the ultimate goal.

The nonlinear nature of the problem adds significant analytical challenges to the problem, compared to the equivalent problem in the linear setting. Not only does the problem require careful formulation, but we have to work hard to obtain uniform estimates and to improve the regularity of approximate solution sequences. In the strain-limiting case, we have the added complication that the stress tensor $\mathbf{T}$ is only bounded *a priori* in $L^1(Q)^{d\times d}$. At best, this gives weak-* compactness in the space of Radon measures on $\overline{Q}$. However, we want to stay out of the space of measure-valued solutions. Inspired by the analysis in the case where there is no fracture, we carefully improve the regularity of the approximations of the stress. This allows the deduction of a pointwise convergence result despite the poor integrability. We later improve this to strong convergence in $L^1(0,T;L^1_{loc}(\Omega)^{d\times d})$ under specific boundary conditions.

1.6 Notation and auxiliary results

Throughout this thesis, for a domain $\Omega \subset \mathbb{R}^d$, let $L^p(\Omega)$ and $W^{k,p}(\Omega)$ denote the standard Lebesgue and Sobolev spaces, respectively, for $k \in \mathbb{N}$ and $p \in [1,\infty]$. The norms on these spaces are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$. If we are working with a function defined on some space Ω_0 that is different to Ω, we denote the norms in $L^p(\Omega_0)$ and $W^{k,p}(\Omega_0)$ by $\|\cdot\|_{L^p(\Omega_0)}$ and $\|\cdot\|_{W^{k,p}(\Omega_0)}$, respectively.

Let $W^{k,p}_0(\Omega)$ denote the closure of the set of smooth, compactly supported functions $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{k,p}$ for $p \in [1,\infty)$ and for $p = \infty$ we will denote $W^{k,\infty}_0(\Omega) = W^{k,1}_0(\Omega) \cap W^{k,\infty}(\Omega)$. If the boundary $\partial\Omega$ has a Dirichlet part $\partial\Omega_D$ and a Neumann part $\partial\Omega_N$, let $C^k_D(\overline{\Omega})$ be the set of all k-times differentiable functions up to the boundary of Ω such that they vanish in a neighbourhood of $\partial\Omega_D$. Then $W^{k,p}_D(\Omega)$ is the closure of $C^k_D(\overline{\Omega})$ with respect to $\|\cdot\|_{k,p}$ for $p \in [1,\infty)$ and denote $W^{k,\infty}_D(\Omega) = W^{k,\infty}(\Omega) \cap W^{k,1}_D(\Omega)$. If $p = 2$, we write $H^k(\Omega) := W^{k,2}(\Omega)$ and $H^k_0(\Omega) := W^{k,2}_0(\Omega)$. $H^k_D(\Omega)$ is defined analogously. Furthermore, we define $H^k_{D+1}(\Omega)$ to be the closure with respect to $\|\cdot\|_{k,2}$ of the set of all functions from $C^k(\overline{\Omega})$ that are equal to 1 in a neighbourhood of the Dirichlet part of the boundary $\partial\Omega_D$. The Sobolev spaces $W^{k,p}_D(\Omega)$ behave how one would expect them to. In particular, the Sobolev embedding theorem holds. Suppose that p, $q \in [1,\infty)$ and k, $l \in \mathbb{N}_0$ such that $p < q$, $k > l$, $p < d$ and $\frac{1}{p} - \frac{k}{d} = \frac{1}{q} - \frac{l}{d}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open domain with Lipschitz boundary and $v \in W^{k,p}_D(\Omega)$. Then $v \in W^{l,q}(\Omega)$ by the standard Sobolev embedding. There exists $(v_n)_n \subset C^\infty_D(\overline{\Omega})$ such that $v_n \to v$ in $W^{k,p}(\Omega)$. Applying the Sobolev embedding theorem to $v_n - v$, it follows that $v_n \to v$ in $W^{l,q}(\Omega)$. Hence, by definition, $v \in W^{l,q}_D(\Omega)$.

We define $C^1_0(\partial\Omega_N)$ to be the set of all functions from $C^1(\partial\Omega_N)$ such that there exists an extension to $\overline{\Omega}$ that lies in $C^1_D(\overline{\Omega})$. The dual space is denoted by $(C^1_0(\partial\Omega_N))^*$.

Given a space $\mathcal{F}$ of real-valued functions, let $\mathcal{F}^d$ and $\mathcal{F}^{d\times d}$ be the sets of $\mathbb{R}^d$- and $\mathbb{R}^{d\times d}$-valued functions such that each component defines an element of $\mathcal{F}$. We denote by $W^{-k,p'}(\Omega)$ the dual space of $W^{k,p}_0(\Omega)$, where p' is the Hölder conjugate of p. We let $W^{-k,p'}_D(\Omega)$ be the dual space of $W^{k,p}_D(\Omega)$.

We require various Korn-type inequalities throughout this work, the several variations depending on the boundary conditions under consideration. The proof of the periodic result (Theorem 1.5) can be found in [19]. The proof of the general case (Theorem 1.6) can be found in [29]. For the remaining results, they correspond to test functions in the setting of Dirichlet and mixed Dirichlet–Neumann boundary conditions. We use Korn's inequality from [39] and [67], respectively, and then apply the Poincaré inequality to obtain the corresponding Korn–Poincaré inequality.

Theorem 1.5. *Let $p \in (1,\infty)$ and $\Omega = (0,1)^d$, the d-dimensional flat torus. There exists a positive constant $C = C(p,d)$ such that, for every $\boldsymbol{u} \in W^{1,p}_*(\Omega)^d$,*

$$\|\boldsymbol{u}\|_{1,p} \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p.$$

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^d$ be a domain with $d \geq 2$. There exists a constant $C = C(\Omega)$ such that*

$$\|\boldsymbol{u}\|_{1,2} \leq C\left(\|\boldsymbol{u}\|_2^2 + \|\varepsilon(\boldsymbol{u})\|_2^2\right)^{\frac{1}{2}}.$$

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Let $p \in (1,\infty)$. There exists a positive constant C depending on p, d and Ω such that, for every $\boldsymbol{v} \in W^{1,p}_0(\Omega)^d$,*

$$\|\nabla \boldsymbol{v}\|_p \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_p.$$

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Let $p \in (1,\infty)$. There exists a positive constant C depending on p, d and Ω such that, for every $\boldsymbol{v} \in W^{1,p}_0(\Omega)^d$,*

$$\|\boldsymbol{v}\|_{1,p} \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_p.$$

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary $\partial\Omega$ such that there are open, disjoint subsets $\partial\Omega_D$, $\partial\Omega_N \subset \partial\Omega$ with $\overline{\partial\Omega_D \cup \partial\Omega_N} = \partial\Omega$ and $\partial\Omega_D \neq \emptyset$. For every $p \in (1,\infty)$, there exists a constant C depending only on Ω, $\partial\Omega_D$ and p such that, for every $\boldsymbol{u} \in W^{1,p}_D(\Omega)^d$,*

$$\|\boldsymbol{u}\|_{1,p} \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p.$$

In this thesis, the prototype function that is considered in the strain-limiting constitutive relation is $F(\mathbf{T}) = (1 + |\mathbf{T}|^a)^{-\frac{1}{a}}\mathbf{T}$ where $a > 0$ is a positive constant. We make use of the following property of the function F, the proof of which can be found in [19].

Lemma 1.10. *Let $a \in (0,1)$. There exists a positive constant C, depending only on a, such that*

$$(\mathbf{T} - \mathbf{S}) \cdot (F(\mathbf{T}) - F(\mathbf{S})) \geq C\left|(1 + |\mathbf{T}|)^{\frac{1-a}{2}} - (1 + |\mathbf{S}|)^{\frac{1-a}{2}}\right|^2,$$

for every $\mathbf{T}$, $\mathbf{S} \in \mathbb{R}^{d\times d}$.

In Chapter 3, we require knowledge of the fractional Nikol'skiĭ spaces and an appropriate embedding theorem [16]. For parameters $\beta \in [0,1)$ and $p \in [1,\infty)$, for an open, bounded domain $\Omega \subset \mathbb{R}^d$, we define the local fractional Nikol'skiĭ space to be the set of $f \in L^p_{loc}(\Omega)$ such that, for every subset Ω_0 that is compactly contained in Ω,

$$\limsup_{h\to 0} \sup_{1\leq i\leq d} \left\|\frac{\Delta_i^h f}{h^\beta}\right\|_{L^p(\Omega_0)} < \infty,$$

where Δ_i^h denotes the difference quotient with increment h in the i-th coordinate direction. Let $\Omega_1 \subset \Omega$ be a compactly contained subset such that Ω_0 is compactly contained in Ω_1. We define

$$\|f\|_{\mathcal{N}^{\beta,p}(\Omega_1)} := C\Big[\|f\|_{L^p(\Omega_1)} + \limsup_{h\to 0} \sup_{1\le i\le d} \Big\| \frac{\Delta_i^h f}{h^\beta} \Big\|_{L^p(\Omega_1)}\Big].$$

We have the following embedding result.

Lemma 1.11 (Nikol'skiĭ embedding). *Let $\beta \in [0,1)$ and $p \in [1,\infty)$ be such that $\beta p < d$. For every $\delta > 0$, sufficiently small, there exists a positive constant C, depending on β, p, d Ω_0 and Ω_1, such that*

$$\|f\|_{L^{\frac{dp}{d-\beta p}-\delta}(\Omega_0)} \le C\|f\|_{\mathcal{N}^{\beta,p}(\Omega_1)},$$

for every $f \in \mathcal{N}^{\beta,p}_{loc}(\Omega)$.

With this, we are ready to begin our investigation into the mathematical analysis of strain-limiting viscoelastic solids.

1.7 Chapter Summaries

In Chapters 2, 3 and 4, we consider (1.12) coupled with $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F(\mathbf{T})$, and supplemented with periodic, homogeneous Dirichlet and mixed Dirichlet–Neumann boundary conditions, respectively. In the periodic and Dirichlet cases, we prove the existence of a unique weak solution of the problem for any $a > 0$. This is done by approximating the strain-limiting problem with a regularised one, i.e., one that has slightly better integrability of the stress tensor. For the mixed Dirichlet–Neumann case, we are only able to show the existence of a weak solution up to a penalisation term on the Neumann part of the boundary. Furthermore, we can show higher integrability estimates on the stress tensor in the case that a is small. We build up throughout the three chapters and the optimal result is proven in Chapter 4.

Now we discuss the main results from Chapters 5 and 6. The first, presented in Chapter 5, concerns a problem that has constitutive relation of the form

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F(\mathbf{T}) := |\mathbf{T}|^{p-2}\mathbf{T}, \tag{1.39}$$

where $p > 1$. For $p \in (1,2]$, we prove the existence of a weak solution $(\boldsymbol{u}, v)$ to the elastodynamic equation

$$\boldsymbol{u}_{tt} = \operatorname{div}(b(v)\mathbf{T}) + \boldsymbol{f}, \tag{1.40}$$

coupled with a minimisation problem analogous to (1.35) and an energy-dissipation balance. For $p > 2$, we also obtain an existence result. However, the energy-dissipation balance only holds at almost every point in time. This results from a lack of regularity of the phase-field function that is not present in the case that $p \in (1,2]$.

The second result, presented in Chapter 6, concerns strain-limiting solids with the constitutive relation (1.29). The elastodynamic equation (1.40) is coupled with (1.29), a rate-dependent minimisation problem of the type (1.38), and an energy-dissipation balance. We supplement this with mixed Dirichlet–Neumann boundary conditions. We restrict to $k > \frac{d}{2} + 1$ to ensure good regularity of the phase-field function. This allows the derivation of higher regularity estimates on approximations of the displacement and stress tensor, replicating the ideas of Chapter 4.

If the Neumann part of the boundary is non-empty, due to a lack of global estimates on the stress tensor, the elastodynamic equation is only shown to hold weakly up to a penalisation term on the Neumann part of the boundary as in Chapter 4. An important consequence of this is that we are only able to show that an energy-dissipation inequality holds. However, if we have fully Dirichlet boundary conditions, there is no such penalisation term and we are able to obtain a full existence result (cf. Chapter 3). A weak form of (1.40) holds, the constitutive relation is satisfied pointwise a.e., the minimisation problem holds at every point in time and we have an energy-dissipation equality.

Chapter 2

Strain-limiting problem with periodic boundary conditions

In this chapter, we investigate the strain-limiting problem in a periodic setting. The results presented here are the basis of the paper [21]. Throughout this chapter, we will denote $\Omega = (0,1)^d$, the d-dimensional flat torus with spatial dimension $d \geq 2$. Let $Q = (0,T) \times \Omega$, the space-time domain. We look for a couple $(\boldsymbol{u}, \mathbf{T}) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{2.1a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{in } Q, \tag{2.1b}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1 \qquad \text{in } \Omega, \tag{2.1c}$$

$$\boldsymbol{u}(t,\cdot) \in L^1_*(\Omega)^d \qquad \text{on } [0,T]. \tag{2.1d}$$

The restriction (2.1d) refers to periodic boundary conditions. We are interested in studying the existence and uniqueness of weak solutions to the strain-limiting problem (2.1) in this periodic setting. We introduce the necessary analysis required to study such viscoelastic problems without the technicalities when we have more complex boundary conditions.

As is standard in the periodic setting, we fix the integral mean value to be $\mathbf{0}$ so that we can obtain a uniqueness result. The constants $a > 0$ and $\alpha \geq 0$ are fixed material parameters. We are given the body force $\boldsymbol{f}$, as well as initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ for the displacement and velocity, respectively. The exact regularity requirements are introduced in Theorem 2.10 and Theorem 2.11. We are concerned with the existence of long-time, large-data weak solutions of (2.1) defined in the following sense.

Definition 2.1. *Given data $\boldsymbol{u}_0 \in W^{1,2}_*(\Omega)^d$, $\boldsymbol{u}_1 \in L^2_*(\Omega)^d$, and $\boldsymbol{f} \in L^2(0,T; W^{-1,2}_*(\Omega)^d)$, we say that a couple $(\boldsymbol{u}, \mathbf{T})$ with regularity*

- $\boldsymbol{u} \in W^{2,2}(0,T; L^2_*(\Omega)^d) \cap W^{1,2}(0,T; W^{1,2}_*(\Omega)^d)$,
- $\boldsymbol{u}_t + \alpha \boldsymbol{u} \in L^p(0,T; W^{1,p}_*(\Omega)^d)$ *for every* $p \in [1,\infty)$ *with* $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) \in L^\infty(Q)^{d\times d}$,
- $\mathbf{T} \in L^1(0,T; L^1_{\#}(\Omega)^{d\times d})$,

is a weak solution of the strain-limiting problem (2.1) *if*

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{v} + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle, \tag{2.2}$$

for every $\boldsymbol{v} \in W^{1,\infty}_*(\Omega)^d$ *and a.e.* $t \in (0,T)$*, with constitutive relation*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) \quad \textit{pointwise a.e. in } Q, \tag{2.3}$$

and initial conditions holding in the sense that

$$\lim_{t \to 0+} \left(\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_2 \right) = 0. \tag{2.4}$$

Recalling standard regularity results concerning Bochner spaces (see [45, Chapter 5], for example) the space $W^{1,p}(0,T;X)$ is continuously embedded into $C([0,T];X)$ for every $p \in [1,\infty]$ up to a continuous representative where X is an arbitrary, fixed Banach space. In this thesis, we always assume that we are working with the continuous representative if one is available. It follows that the initial conditions (2.4) are well-defined under the regularity conditions on $\boldsymbol{u}$ in Definition 2.1.

The standard procedure for proving the existence of a weak solution to such a nonlinear problem is to use a Galerkin approximation in space. However, this is not possible in the strain-limiting setting due to the coupling of the PDE (2.1a) with the constitutive relation (2.1b). On the other hand, if we can formulate the problem in terms of only the displacement, we can use a Galerkin approximation to show that a solution exists to this alternative problem. From this, we then try to find a solution to the strain-limiting problem. We cannot obtain a formulation in terms of only $\boldsymbol{u}$ at present due to the fact that F is not a bijection from $\mathbb{R}^{d\times d}$ to itself. Indeed, F is bounded because this encapsulates the strain-limiting property.

In the spirit of [16, 5], we approximate (2.1) by means of an elliptic regularisation. We replace the function F by a function F_n with approximation parameter n where F_n is a bijection from $\mathbb{R}^{d\times d}$ to itself, but retains certain properties of F. In particular, F_n is strictly monotonic and continuously differentiable. We use a Galerkin approximation by means of trigonometric polynomials to prove the existence of a unique weak solution to the regularised problem. The resulting weak solution is denoted by $(\boldsymbol{u}^n, \mathbf{T}^n)$. Careful higher regularity estimates are then derived in order to obtain suitable convergence results in the limit with respect to n. We use these uniform estimates to show that the limiting couple is in fact a weak solution to the original strain-limiting problem (2.1).

Let us discuss the lack of integrability of the stress tensor $\mathbf{T}^n$ in some detail. Generally, the best uniform estimate on $\mathbf{T}^n$ and *a priori* estimate on $\mathbf{T}$ is in $L^1(0,T;L^1_\#(\Omega)^{d\times d})$. In particular, suppose that $(\boldsymbol{u}, \mathbf{T})$ is a sufficiently smooth solution to (2.1) so that the following manipulations are justified. Taking the inner-product of (2.1a) with $\boldsymbol{u}_t + \alpha\boldsymbol{u}$ and integrating over Ω, we use integration by parts to see that

$$\int_\Omega \frac{\partial}{\partial t}\left(\frac{|\boldsymbol{u}_t|^2}{2} + \alpha \boldsymbol{u}_t \cdot \boldsymbol{u}\right) - \alpha|\boldsymbol{u}_t|^2 + \mathbf{T}\cdot F(\mathbf{T})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}_t + \alpha\boldsymbol{u})\,\mathrm{d}x$$
$$\leq C\|\boldsymbol{f}\|_{-1,2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})\|_2$$
$$\leq C\|\boldsymbol{f}\|_{-1,2}.$$

We used the Korn–Poincaré inequality on a periodic domain (Theorem 1.5) to obtain the first inequality and the strain-limiting property to obtain the second. Next, we note that

$$\mathbf{T}\cdot F(\mathbf{T}) \geq \frac{|\mathbf{T}|}{2} - C_a, \tag{2.5}$$

for every $\mathbf{T}\in\mathbb{R}^{d\times d}$, where C_a is a fixed positive constant depending only on a. We integrate with respect to the time variable and use Gronwall's inequality to deduce that

$$\sup_{t\in[0,T]}\left(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{u}_t(t)\|_2^2\right) + \int_Q |\mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t \leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_{-1,2}^2\,\mathrm{d}t\right),$$

with the full reasoning to be shown in the proof of Theorem 2.4. The constant C on the right-hand side depends only on α, a, T and d. Boundedness in $L^1(0,T;L^1_\#(\Omega)^{d\times d})$ at best gives weak-* compactness in the space of Radon measures $\mathcal{M}(\overline{\Omega})^{d\times d}$. We do not have a convergence result that gives a limit in the space of Lebesgue functions. Instead, we construct higher regularity results and prove a pointwise convergence result.

2.1 The regularised problem

Throughout this section, let $n\in\mathbb{N}$ be a fixed approximation parameter. We define the 'regularised' function F_n on $\mathbb{R}^{d\times d}$ by

$$F_n(\mathbf{T}) = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{\mathbf{T}}{n(1+|\mathbf{T}|^{1-\frac{1}{n}})}.$$

The term regularised refers to the presence of the second-term on the left-hand side, commonly referred to as an elliptic regularisation term. By the Browder–Minty Theorem, we immediately see that the function F_n is a continuous bijection from $\mathbb{R}^{d\times d}$ to itself. In particular, we note the coercivity property

$$F_n(\mathbf{T})\cdot\mathbf{T} = \frac{|\mathbf{T}|^2}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{|\mathbf{T}|^2}{n(1+|\mathbf{T}|^{1-\frac{1}{n}})} \geq \frac{|\mathbf{T}|^{1+\frac{1}{n}}}{2n} + \frac{|\mathbf{T}|}{2} - C_a.$$

The exact statement of the Browder–Minty Theorem is as follows [12].

Theorem 2.2 (Browder–Minty)**.** *Let S be a bounded, continuous, coercive and monotone function from a real, separable Banch space X into its continuous dual X^*. Then S is surjective. If S is additionally strictly monotonic, then S is a bijection.*

In fact, the function F_n is a C^1-diffeomorphism from $\mathbb{R}^{d\times d}$ to itself. This is due to the following result from [103]. The function F_n satisfies the hypotheses of Lemma 2.3 by viewing F_n as a map on $\mathbb{R}^{d^2}$ and directly calculating the derivative.

Lemma 2.3. *A C^1-map $f : \mathbb{R}^l \to \mathbb{R}^l$ is a C^1-diffeomorphism if and only if the Jacobian* $\det(Df)$ *never vanishes and* $|f(\boldsymbol{v})| \to \infty$ *whenever* $|\boldsymbol{v}| \to \infty$.

Later on in this thesis, we consider a different regularisation term, namely, $n^{-1}\mathbf{T}$. The linear choice in fact provides more optimal results. However, we consider this slightly more complicated term here for two reasons. First, we want to extend the work that has been done in the steady case and this choice of regularisation was used more frequently. Second, the nonlinear term ensures an improved convergence result for the approximation sequence $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$, where $\boldsymbol{u}^n$ is defined below. We have that $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$ converges to $\boldsymbol{u}_t + \alpha \boldsymbol{u}$ weakly in $L^p(0, T; W_*^{1,p}(\Omega)^d)$ for every $p \in [1, \infty)$. In fact, by the Aubin–Lions lemma, we have strong convergence in $L^p(Q)^d$ for every $p \in [1, \infty)$. In comparison, we can at best take $p = 2$ in the case of the linear regularisation term. We also note that such a regularisation term suggests how one might prove the existence of a solution to a nonlinear viscoelastic problem when the constitutive function provides similar growth to F_n.

We consider the following regularised problem: find a couple $(\boldsymbol{u}^n, \mathbf{T}^n) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that (2.1a), (2.1c)–(2.1d) hold with

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n) = F_n(\mathbf{T}^n) \quad \text{in } Q. \tag{2.6}$$

The function F_n is a bijection so we can rewrite the momentum balance (2.1a) and the constitutive equation (2.6) as a single equation, namely,

$$\boldsymbol{u}_{tt}^n = \operatorname{div}(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n))) + \boldsymbol{f}.$$

We define a weak solution of the regularised problem analogously to Definition 2.1 for a weak solution of (2.1). However, the regularity requirements are replaced by asking that $\boldsymbol{u}^n \in W^{2,2}(0, T; L_*^2(\Omega)^d)$ and $\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n \in L^{n+1}(0, T; W_*^{1,n+1}(\Omega)^d)$, with the approximate stress satisfying $\mathbf{T}^n \in L^{1+\frac{1}{n}}(0, T; L_{\#}^{1+\frac{1}{n}}(\Omega)^{d\times d})$. In the weak form of the (2.1a), we consider test functions $\boldsymbol{v} \in W_*^{1,n+1}(\Omega)^d$.

For the remainder of this section, we focus on proving the existence of a solution to the regularised problem by means of a Galerkin approximation. Let $(\phi_i)_{i\geq 1}$ be a sequence of trigonometric polynomials from $C_*^\infty(\overline{\Omega})$ such that the functions form an orthonormal basis of $L_*^2(\Omega)$ and, for every $m \in \mathbb{N}$, there exists an $M_m \in \mathbb{N}$ such that the linear span of $(\phi_i)_{i=1}^{M_m}$ is exactly the vector space of trigonometric polynomials of degree at most m with integral over Ω equal to 0. We define $V_m = (\operatorname{span}\{\phi_1, \dots, \phi_{M_m}\})^d$. A basis of V_m is $(\phi_i \boldsymbol{e}_j)_{i,j=1}^{M_m,d}$. We use $\boldsymbol{e}_j$ to denote the j-th standard basis vector in $\mathbb{R}^d$. This basis is orthonormal with respect to the inner product in $L_*^2(\Omega)^d$. We denote by P^m the orthogonal projection operator from $L_{\#}^2(\Omega)^d$ to the space of $\mathbb{R}^d$-valued trigonometric polynomials of degree at most m. The restriction of P^m to $L_*^2(\Omega)^d$ is exactly the orthogonal projection from $L_*^2(\Omega)^d$ to V_m.

For the remainder of this section, we suppress dependencies on n since the parameter is fixed throughout. However, we do indicate if a constant C depends on the approximation parameter

n. In Theorem 2.4, we introduce the Galerkin approximation of the regularised problem, prove that a solution exists to the finite-dimensional problem and deduce some uniform estimates on the Galerkin solution.

Theorem 2.4. *Suppose that we are given data $\boldsymbol{u}_0,\ \boldsymbol{u}_1 \in L^2_*(\Omega)^d$ and $\boldsymbol{f} \in L^2(0,T;L^2_*(\Omega)^d)$. There exists a function $\boldsymbol{u}^m \in W^{2,2}(0,T;L^2_*(\Omega)^d)$ of the form*

$$\boldsymbol{u}^m(t,x) = \sum_{i=1}^{M_m}\sum_{j=1}^{d} \beta^m_{ij}(t)\phi_i(x)\boldsymbol{e}_j$$

such that, for every $\boldsymbol{v}^m \in V_m$ and a.e. $t \in (0,T)$,

$$\int_\Omega \boldsymbol{u}^m_{tt}(t)\cdot \boldsymbol{v}^m + \mathbf{T}^m(t)\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}^m)\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot \boldsymbol{v}^m\,\mathrm{d}x, \tag{2.7}$$

where $\mathbf{T}^m$ is defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t+\alpha\boldsymbol{u}^m) = F_n(\mathbf{T}^m)$ in Q, with initial conditions given by $\boldsymbol{u}^m(0,\cdot) = P^m\boldsymbol{u}_0$ and $\boldsymbol{u}^m_t(0,\cdot) = P^m\boldsymbol{u}_1$. Furthermore, there exists a constant C, independent of n and m, such that

$$\begin{aligned}\sup_{t\in[0,T]}\|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T]}\|\boldsymbol{u}^m_t(t)\|_2^2 + \int_Q |\mathbf{T}^m| + \frac{|\mathbf{T}^m|^{1+\frac{1}{n}}}{n}\,\mathrm{d}x\,\mathrm{d}t\\ \le C\left(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\int_0^T\|\boldsymbol{f}(t)\|_2^2\,\mathrm{d}t\right).\end{aligned} \tag{2.8}$$

Proof. For an arbitrary $\boldsymbol{v} \in L^2_*(\Omega)^d$, we may identify $P^m\boldsymbol{v}$ with a vector $(v_{ij})_{i,j=1}^{M_m,d}$ such that

$$P^m\boldsymbol{v} = \sum_{i=1}^{M_m}\sum_{j=1}^{d} v_{ij}\phi_i\boldsymbol{e}_j.$$

With this in mind, the Galerkin approximation from V_m is equivalent to the following system of first order ODEs:

$$\begin{gathered}(\boldsymbol{\beta}^m_t(t),\boldsymbol{\gamma}^m_t(t)) = (\boldsymbol{\gamma}^m(t),\mathbf{g}(t,\boldsymbol{\beta}^m(t),\boldsymbol{\gamma}^m(t))),\\ \boldsymbol{\beta}^m(0) = P^m\boldsymbol{u}_0,\quad \boldsymbol{\gamma}^m(0) = P^m\boldsymbol{u}_1,\end{gathered} \tag{2.9}$$

where $\mathbf{g} = (g_{ij})_{i,j=1}^{M_m,d}$ is defined on $(0,T)\times\mathbb{R}^{M_m\times d}\times\mathbb{R}^{M_m\times d}$ by

$$g_{ij}(t,\boldsymbol{\beta},\boldsymbol{\gamma}) = -\int_\Omega F_n^{-1}\left(\sum_{k=1}^{M_m}\sum_{l=1}^{d}\boldsymbol{\varepsilon}\left((\gamma_{kl}+\alpha\beta_{kl})\,\phi_k\boldsymbol{e}_l\right)\right)\cdot\boldsymbol{\varepsilon}(\phi_i\boldsymbol{e}_j)\,\mathrm{d}x + \int_\Omega \boldsymbol{f}(t)\cdot(\phi_i\boldsymbol{e}_j)\,\mathrm{d}x.$$

The function $\boldsymbol{g}$ is measurable with respect to t by the assumptions on $\boldsymbol{f}$ and is continuous in $(\boldsymbol{\beta},\boldsymbol{\gamma})$, recalling the continuity of F_n^{-1}. Hence, applying standard Carathéodory theory (see, for example, Theorem 2.4.1 from [108]) we deduce that a solution $\boldsymbol{u}^m$ exists on a possibly short time interval $[0,T_*)$ for some $T_* \le T$, where T_* may depend on n and m. Given a solution $\boldsymbol{u}^m$, define $\mathbf{T}^m = F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t+\alpha\boldsymbol{u}^m))$ which is justified because F_n is a bijection from $\mathbb{R}^{d\times d}_{sym}$ to itself.

To extend the solution to the whole of $(0,T)$, we show that the coefficients $\boldsymbol{\beta}^m$ and $\boldsymbol{\beta}^m_t$ remain bounded as the time approaches T_*. First, we note that, by the orthonormality of the basis,

$$\|\boldsymbol{u}^m(t)\|_2^2 + \|\boldsymbol{u}^m_t(t)\|_2^2 = \sum_{i=1}^{M_m}\sum_{j=1}^{d}\left(|\beta^m_{ij}(t)|^2+|\gamma^m_{ij}(t)|^2\right). \tag{2.10}$$

We test in (2.7) against $\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m$ and see that

$$
\begin{aligned}
0 &= \int_\Omega \boldsymbol{u}_{tt}^m \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) + \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) - \boldsymbol{f} \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\,\mathrm{d}x \\
&= \int_\Omega \frac{\partial}{\partial t}\left(\frac{|\boldsymbol{u}_t^m|^2}{2} + \alpha \boldsymbol{u}_t^m \cdot \boldsymbol{u}^m\right) - \alpha |\boldsymbol{u}_t^m|^2 + \mathbf{T}^m \cdot F_n(\mathbf{T}^m) - \boldsymbol{f} \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\,\mathrm{d}x.
\end{aligned}
$$

For an arbitrary $t \in (0, T_*)$, we integrate over $(0,t)$ to get

$$
\begin{aligned}
&\frac{\|\boldsymbol{u}_t^m(t)\|_2^2}{2} + \int_0^t \int_\Omega \Big[\frac{|\mathbf{T}^m|^2}{(1+|\mathbf{T}^m|^a)^{\frac{1}{a}}} + \frac{|\mathbf{T}^m|^2}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})}\Big] \\
&= \frac{\|P^m \boldsymbol{u}_1\|_2^2}{2} + \alpha \int_\Omega [P^m \boldsymbol{u}_1 \cdot P^m \boldsymbol{u}_0 - \boldsymbol{u}_t^m(t) \cdot \boldsymbol{u}^m(t)] + \int_0^t \int_\Omega \big[\alpha |\boldsymbol{u}_t^m|^2 + \boldsymbol{f} \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\big] \\
&\le C(\alpha)\left(\|P^m \boldsymbol{u}_1\|_2^2 + \|P^m \boldsymbol{u}_0\|_2^2 + \int_0^t \int_\Omega \big[|\boldsymbol{u}_t^m|^2 + |\boldsymbol{u}^m|^2 + |\boldsymbol{f}|^2\big]\right) + \frac{\|\boldsymbol{u}_t^m(t)\|_2^2}{4},
\end{aligned} \tag{2.11}
$$

where we have used the fact that

$$
\boldsymbol{u}^m(t) = P^m \boldsymbol{u}_0 + \int_0^t \boldsymbol{u}_t^m(s)\,\mathrm{d}s. \tag{2.12}
$$

Using (2.12) again, we see that

$$
\begin{aligned}
\|\boldsymbol{u}^m(t)\|_2^2 &\le 2\|P^m \boldsymbol{u}_0\|_2^2 + 2\Big\|\int_0^t \boldsymbol{u}_t^m(s)\,\mathrm{d}s\Big\|_2^2 \\
&\le 2\|P^m \boldsymbol{u}_0\|_2^2 + 2t^{\frac{1}{2}} \int_0^t \int_\Omega |\boldsymbol{u}_t^m(s)|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Adding this inequality to (2.11), we deduce that

$$
\begin{aligned}
&\|\boldsymbol{u}_t^m(t)\|_2^2 + \|\boldsymbol{u}^m(t)\|_2^2 + \int_0^t \int_\Omega \Big[\frac{|\mathbf{T}^m|^2}{(1+|\mathbf{T}^m|^a)^{\frac{1}{a}}} + \frac{|\mathbf{T}^m|^2}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})}\Big] \\
&\le C(\alpha, T_*)\left(\|P^m \boldsymbol{u}_1\|_2^2 + \|P^m \boldsymbol{u}_0\|_2^2 + \int_0^t \int_\Omega \big[|\boldsymbol{u}_t^m|^2 + |\boldsymbol{u}^m|^2 + |\boldsymbol{f}|^2\big]\right).
\end{aligned}
$$

Applying Gronwall's inequality yields

$$
\begin{aligned}
&\sup_{t\in[0,T_*)} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T_*)} \|\boldsymbol{u}_t^m(t)\|_2^2 + \int_0^{T_*} \int_\Omega |\mathbf{T}^m| + \frac{|\mathbf{T}^m|^{1+\frac{1}{n}}}{n}\,\mathrm{d}x\,\mathrm{d}t \\
&\le C(\alpha, a, d, T_*)\left(1 + \|P^m \boldsymbol{u}_0\|_2^2 + \|P^m \boldsymbol{u}_1\|_2^2 + \int_0^{T_*} \|\boldsymbol{f}\|_2^2\,\mathrm{d}t\right),
\end{aligned} \tag{2.13}
$$

where C is a positive constant that is finite for any finite value of T_*. The dependence on a comes from applying (2.5). It follows from (2.13) and (2.10) that

$$
\sup_{t\in[0,T_*)} \max_{1\le i,j\le d} \left(|\beta_{ij}^m(t)|^2 + |\gamma_{ij}^m(t)|^2\right) \le C.
$$

Thus there exists a solution to the Galerkin approximation from V_m on the whole of $[0,T]$. Furthermore, the asserted bound (2.8) follows from (2.13), using that $\|P^m \boldsymbol{v}\|_2 \le \|\boldsymbol{v}\|_2$ for every $\boldsymbol{v} \in L_*^2(\Omega)^d$ by standard properties of orthogonal projection operators. □

Corollary 2.5. *Let $\boldsymbol{u}^m$ be the solution of the Galerkin approximation from V_m constructed in Theorem 2.4. There exists a constant C, independent of n and m, such that*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\|_{L^{n+1}(Q)} \le C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 \,\mathrm{d}t\right).$$

Proof. From the definition of $\mathbf{T}^m$, we have that

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)| \le 1 + \frac{|\mathbf{T}^m|^{\frac{1}{n}}}{n}.$$

It immediately follows that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\|_{L^{n+1}(Q)} \le C\left(1 + \frac{\|\mathbf{T}^m\|_{L^{1+\frac{1}{n}}(Q)}^{\frac{1}{n}}}{n}\right) \le C\left(1 + \frac{\|\mathbf{T}^m\|_{L^{1+\frac{1}{n}}(Q)}}{n}\right).$$

Applying the bound of Theorem 2.4, we deduce the asserted result. □

Next, we look for a bound on $\boldsymbol{u}_{tt}^m$. Considering the previous estimates, the best (n,m)-independent bound we currently have on $\boldsymbol{u}_{tt}^m$ is in $L^1(0,T;W_*^{1,-1}(\Omega)^d))$ where $W_*^{1,-1}(\Omega)^d$ denotes the dual space of $W_*^{1,\infty}(\Omega)^d$. This space clearly has poor compactness properties. We need to look for a bound in a reflexive Sobolev space (or the dual of one) because then, by the Banach–Alaoglu theorem, the weak convergence of a subsequence follows. To do this, we test in (2.7) against $\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m$ to obtain a bound on $\boldsymbol{u}_{tt}^m$ in $L^2(Q)^d$ that is independent of n and m. However, we require stronger requirements on the initial data. The safety strain condition (2.14) stated below must hold. We need the image of $\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)$ on Ω to be compactly contained in the open unit ball so that $F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))$ is bounded uniformly on Ω with respect to n. Otherwise, we are unable to bound the initial stress $\mathbf{T}^m(0)$ independent of m and n. Furthermore, we ask that $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0$ is in the space $W_*^{k+1,2}(\Omega)^d$ for some $k > \frac{d}{2}$. This is so that the projected data $P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)$ uniformly approximates $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0$ in $W^{1,\infty}(\Omega)^d$. Combined with the safety strain assumption, we are able to deduce that $F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))$ is uniformly bounded in $L^\infty(\Omega)^{d\times d}$ with respect to both n and m, provided that m is sufficiently large. In Section 4.3, we discuss how the requirement on k can be weakened.

We note that $F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))$ must exist a.e. in Ω because otherwise the initial stress could not be defined. Hence we must have that $|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)| < 1$ a.e. in Ω. However, the stronger requirement of the safety strain condition is required in the proof in order that the approximations $(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))_n$ can be bounded uniformly in L^∞. At present, the safety strain condition is required in the steady case also, suggesting that this condition is necessary for an existence proof.

Lemma 2.6. *Let the assumptions of Theorem 2.4 hold and let $\boldsymbol{u}^m$ be the solution of Galerkin approximation. Suppose additionally that $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W_*^{k+1,2}(\Omega)^d$ for some $k > \frac{d}{2}$ and we have the safety strain condition*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_\infty = C_* < 1. \tag{2.14}$$

There exist an $m_0 \in \mathbb{N}$ and a constant C, independent of m and n, such that, for every $m \geq m_0$,

$$\int_0^T \|\boldsymbol{u}_{tt}^m\|_2^2 + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^m(t)|^{1-a}\chi_{\{|\mathbf{T}^m(t)|\geq 1\}} \leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2\right),$$

where χ_A denotes the indicator function of a set A.

Proof. Testing in (2.7) against $\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m$, it follows that

$$0 = \int_\Omega |\boldsymbol{u}_{tt}^m|^2 + \frac{\partial}{\partial t}\left(\frac{\alpha}{2}|\boldsymbol{u}_t^m|^2 + \frac{h_n(|\mathbf{T}^m|^2)}{2}\right) - \boldsymbol{f}\cdot(\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m)\,\mathrm{d}x, \tag{2.15}$$

where $h_n : [0,\infty) \to [0,\infty)$ is defined by

$$h_n(s) = \int_0^s \frac{1}{(1+t^{\frac{a}{2}})^{1+\frac{1}{a}}} + \frac{1}{n^2(1+t^{\frac{1}{2}-\frac{1}{2n}})^2} + \left(1-\frac{1}{n}\right)\frac{1}{n(1+t^{\frac{1}{2}-\frac{1}{2n}})^2}\,\mathrm{d}t.$$

This can be seen by noticing that

$$\begin{aligned}
&\mathbf{T}^m\cdot\frac{\partial}{\partial t}\left(F_n(\mathbf{T}^m)\right)\\
&= \frac{\mathbf{T}^m\cdot\mathbf{T}_t^m}{(1+|\mathbf{T}^m|^a)^{1+\frac{1}{a}}} + \frac{\mathbf{T}^m\cdot\mathbf{T}_t^m}{n^2(1+|\mathbf{T}^m|^{1-\frac{1}{n}})} + \left(1-\frac{1}{n}\right)\frac{\mathbf{T}^m\cdot\mathbf{T}_t^m}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})^2}\\
&= h_n'(|\mathbf{T}^m|^2)\mathbf{T}^m\cdot\mathbf{T}_t^m\\
&= \frac{\partial}{\partial t}\left(\frac{h_n(|\mathbf{T}^m|^2)}{2}\right).
\end{aligned}$$

There exist positive constants c_a, C_a depending only on a such that

$$c_a s^{\frac{1}{2}-\frac{a}{2}}\chi_{\{s\geq 1\}} - C_a \leq h_n(s) \leq s^{\frac{1}{n}} + C_a\left(s^{\frac{1}{2}-\frac{a}{2}}\chi_{\{s\geq 1\}} + 1\right),$$

for every $s \in [0,\infty)$. Integrating (2.15) over $(0,t)$, it follows that

$$\begin{aligned}
&\int_0^t\int_\Omega |\boldsymbol{u}_{tt}^m|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\alpha}{2}\|\boldsymbol{u}_t^m(t)\|_2^2 + \int_\Omega c_a|\mathbf{T}^m(t)|^{1-a}\chi_{\{|\mathbf{T}^m(t)\geq 1\}}\,\mathrm{d}x\\
&\leq C\left(1 + \|\boldsymbol{u}_1\|_2^2 + \int_\Omega |\mathbf{T}^m(0)|^2\,\mathrm{d}x + \int_0^t\int_\Omega |\boldsymbol{f}|^2 + |\boldsymbol{u}_t^m|^2\,\mathrm{d}x\,\mathrm{d}s\right) + \frac{1}{2}\int_0^t\int_\Omega |\boldsymbol{u}_{tt}^m|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

where $C = C(\alpha, a)$ is independent of m and n. It remains to bound $\mathbf{T}^m(0)$ in $L^2(\Omega)^{d\times d}$, independent of m and n. In fact, we show that $\mathbf{T}^m(0)$ is uniformly bounded in $L^\infty(Q)^{d\times d}$ for every $m \geq m_0$, where m_0 depends only on the choice of initial data.

By definition, we have that $\mathbf{T}^m(0) = F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))$. Suppose that there exist constants $m_0 \in \mathbb{N}$ and $C_1 \in (0,1)$ such that, for every $m \geq m_0$, $\|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\|_\infty \leq C_1$. By definition of F_n and the radial property, we see that $|F^{-1}(\mathbf{T})| \geq |F_n^{-1}(\mathbf{T})|$ for every $\mathbf{T} \in \mathbb{R}^{d\times d}$ with $|\mathbf{T}| < 1$. Furthermore, F^{-1} is a radial function that increases in absolute value as $|\mathbf{T}|$ increases. It follows that

$$|\mathbf{T}^m(0)| = |F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))| \leq |F^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))| \leq f^{-1}(C_1) < \infty,$$

where f is defined on $[0,\infty)$ by $f(s) = \frac{s}{(1+s^a)^{\frac{1}{a}}}$. It follows that $\mathbf{T}^m(0)$ is bounded in $L^\infty(\Omega)^{d\times d}$ independent of m and n, with bound depending only on the data and parameters of the problem, for every $m \geq m_0$.

It remains to prove the existence of such an m_0 and C_1. From standard properties of projection operators, we have $P^m \boldsymbol{v} \to \boldsymbol{v}$ strongly in $L^2_*(\Omega)^d$ as $m \to \infty$, for every $\boldsymbol{v} \in L^2_*(\Omega)^d$. However, the projection operator also commutes with derivation [24], i.e., $\partial_i(P^m \boldsymbol{v}) = P^m(\partial_i \boldsymbol{v})$ for every $\boldsymbol{v} \in W^{1,2}_{\#}(\Omega)^d$ and $1 \leq i \leq d$. It follows that

$$\lim_{m\to\infty} \|P^m \boldsymbol{v} - \boldsymbol{v}\|_{k+1,2} = 0 \quad \forall \boldsymbol{v} \in W^{k+1,2}_*(\Omega)^d.$$

However, $W^{k+1,2}_*(\Omega)^d$ embeds continuously into $C^1(\overline{\Omega})^d$ by the Sobolev embedding theorem, recalling that $k > \frac{d}{2}$. In particular, for an arbitrary $\boldsymbol{v} \in W^{k+1,2}_*(\Omega)^d$, we have that

$$\|\boldsymbol{\varepsilon}(P^m \boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_\infty \leq C\|\boldsymbol{\varepsilon}(P^m \boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_{k,2} \leq C\|P^m \boldsymbol{v} - \boldsymbol{v}\|_{k+1,2},$$

where C is independent of m and $\boldsymbol{v}$. It follows that $(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)))_m$ converges strongly in $L^\infty(\Omega)^{d\times d}$ to $\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)$. Hence, by the safety strain condition (2.14), there exists m_0 such that, for every $m \geq m_0$,

$$\begin{aligned}
\|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\|_\infty &\leq \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_\infty + \|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_\infty \\
&= C_* + \|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_\infty \\
&\leq C_* + \frac{1 - C_*}{2} = \frac{1 + C_*}{2} =: C_1 < 1.
\end{aligned}$$

Thus we conclude that the stated bound holds. □

Due to the growth of the function F_n and, in particular, the good integrability of the sequence $(\mathbf{T}^m)_m$, for each fixed n we can take the limit as $m \to \infty$ and prove the existence of a weak solution to the regularised solution.

Theorem 2.7. *Let the assumptions of Lemma 2.6 hold and denote by $\boldsymbol{u}^m$ the solution of the Galerkin approximation from V_m of the regularised problem. There exists a couple $(\boldsymbol{u}, \mathbf{T})$, the unique weak solution of the regularised problem, such that the following convergence results hold as $m \to \infty$:*

- $\boldsymbol{u}^m \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0, T; L^2_*(\Omega)^d)$;
- $\mathbf{T}^m \rightharpoonup \mathbf{T}$ *weakly in* $L^{1+\frac{1}{n}}(0, T; L^{1+\frac{1}{n}}_{\#}(\Omega)^{d\times d})$;
- $\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m \rightharpoonup \boldsymbol{u}_t + \alpha\boldsymbol{u}$ *weakly in* $L^{n+1}(0, T; W^{1,n+1}_*(\Omega)^d)$.

Furthermore, there exists a constant C, independent of n, such that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|_2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_t(t)\|_2 + \int_Q |\boldsymbol{u}_{tt}|^2 + |\mathbf{T}| + \frac{|\mathbf{T}|^{1+\frac{1}{n}}}{n} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sup_{t\in[0,T]} \left(\int_\Omega |\mathbf{T}(t)|^{1-a} \chi_{\{|\mathbf{T}(t)|\geq 1\}} \,\mathrm{d}x \right) \leq C \left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 \,\mathrm{d}t \right).
\end{aligned} \tag{2.16}$$

Proof. Considering the bounds in Theorem 2.4, Corollary 2.5 and Lemma 2.6, along with the Korn–Poincaré inequality and standard compactness results for Bochner spaces, the convergence results follow immediately up to a subsequence in m that we do not relabel. The subsequence restriction can be removed once both existence and uniqueness have been proven. Consider arbitrary but fixed test functions $\boldsymbol{v} \in C_*^\infty(\overline{\Omega})^d$ and $\psi \in C([0,T])$. Then we must have that

$$\int_Q \boldsymbol{u}_{tt}^m \cdot (\psi P^m \boldsymbol{v}) + \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\psi P^m \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f} \cdot (\psi P^m \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t. \tag{2.17}$$

By the previously discussed properties of the projection operator with the smoothness of $\boldsymbol{v}$, we have that $P^m \boldsymbol{v} \to \boldsymbol{v}$ strongly in $C^1(\overline{\Omega})^d$ as $m \to \infty$. Combining this with the weak convergence results, we take $m \to \infty$ in (2.17) to deduce that

$$\int_Q \boldsymbol{u}_{tt} \cdot (\psi \boldsymbol{v}) + \mathbf{T} \cdot \boldsymbol{\varepsilon}(\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f} \cdot (\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t.$$

Since ψ is arbitrary, we can use a standard density argument with respect to the time variable and deduce that the required weak form of (2.1a) is satisfied by the limiting couple which we denote by $(\boldsymbol{u}, \mathbf{T})$.

We know that $\boldsymbol{u} \in C^1([0,T]; L_*^2(\Omega)^d)$ by standard regularity results for Bochner functions. Hence, we must have that

$$\lim_{t \to 0+} \left(\|\boldsymbol{u}(t) - \boldsymbol{u}(0)\|_2 + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_t(0)\|_2 \right) = 0. \tag{2.18}$$

However, by the Aubin–Lions lemma, up to a possible further subsequence that we do not relabel, the sequences $(\boldsymbol{u}^m)_m$ and $(\boldsymbol{u}_t^m)_m$ converge strongly to $\boldsymbol{u}$ and $\boldsymbol{u}_t$, respectively, in $C([0,T]; W_*^{-1,2}(\Omega)^d)$ where $W_*^{-1,2}(\Omega)$ is the dual space of $W_*^{1,2}(\Omega)$. In particular, we have that $\boldsymbol{u}^m(0) \to \boldsymbol{u}(0)$ as $m \to \infty$. However, $\boldsymbol{u}^m(0) = P^m \boldsymbol{u}_0 \to \boldsymbol{u}_0$ in $L_*^2(\Omega)^d$ as $m \to \infty$. Thus $\boldsymbol{u}(0) = \boldsymbol{u}_0$. By similar reasoning, we also have that $\boldsymbol{u}_t(0) = \boldsymbol{u}_1$. Using (2.18), we conclude that the initial conditions are satisfied in the required sense.

To see that the constitutive relation (2.6) is satisfied by the limiting couple, we use a variant of Minty's method [75]. First, we notice that $(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)_m$ converges weakly in $L^2(0,T; W_*^{1,2}(\Omega)^d)$ and $(\boldsymbol{u}_{tt}^m + \alpha \boldsymbol{u}_t^m)_m$ converges weakly in $L^2(0,T; L_*^2(\Omega)^d)$. Applying the Aubin–Lions lemma, we see that $(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)_m$ converges strongly in $L^2(0,T; L_*^2(\Omega)^d)$ to $\boldsymbol{u}_t + \alpha \boldsymbol{u}$. Testing in the Galerkin approximation from V_m against $\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m$, integrating over $(0,T)$ and using this strong convergence, we see that

$$\begin{aligned}
&\lim_{m \to \infty} \int_Q \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) \,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{m \to \infty} \int_Q \boldsymbol{f} \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) - \boldsymbol{u}_{tt}^m \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \boldsymbol{f} \cdot (\boldsymbol{u}_t + \alpha \boldsymbol{u}) - \boldsymbol{u}_{tt} \cdot (\boldsymbol{u}_t + \alpha \boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.19}$$

Let $\mathbf{S} \in L^{1+\frac{1}{n}}(Q)^{d\times d}$ be arbitrary but fixed. By the monotonicity of F_n, the final line of (2.19) and the previous convergence results, we get the following:

$$\begin{aligned} 0 &\leq \lim_{m\to\infty} \int_Q (\mathbf{T}^m - \mathbf{S}) \cdot (F_n(\mathbf{T}^m) - F_n(\mathbf{S}))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q (\mathbf{T} - \mathbf{S}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) - F_n(\mathbf{S}))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

We now choose $\mathbf{S} = \mathbf{T} \pm \gamma\mathbf{U}$ for some $\gamma > 0$ and $\mathbf{U} \in L^\infty(Q)^{d\times d}$. Dividing the resulting inequality through by γ and letting $\gamma \to 0+$, we deduce that

$$0 \leq \mp \int_Q \mathbf{U} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) - F_n(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t.$$

Setting $\mathbf{U} = \frac{\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})-F_n(\mathbf{T})}{1+|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})-F_n(\mathbf{T})|}$, we conclude that the constitutive relation (2.6) holds.

It remains to prove that (2.16) holds. The bounds on the displacement follow from the weak convergence results and weak lower semi-continuity of norms. For the $L^1(Q)$ and $L^{1+\frac{1}{n}}(Q)$ bounds on $\mathbf{T}$, we apply weak lower semi-continuity again and notice that if $\mathbf{T}^n \rightharpoonup \mathbf{T}$ weakly in $L^{1+\frac{1}{n}}(Q)^{d\times d}$ then clearly the weak convergence holds in $L^p(Q)^{d\times d}$ for every $p \in [1, 1+\frac{1}{n})$. For the final term, we use Fatou's lemma. However, we first need to prove the pointwise convergence of $(\mathbf{T}^m)_m$. In fact, we need to show that $\mathbf{T}^m(t) \to \mathbf{T}(t)$ pointwise a.e. on Ω, for a.e. $t \in (0, T)$.

First, we mimic an argument from [19] to show that $\mathbf{T}^m \to \mathbf{T}$ strongly in $L^1(Q)^{d\times d}$. For each $k > 0$, define the set

$$Q_k^m = \{(t,x) \in Q \,:\, 1 + |\mathbf{T}| + |\mathbf{T}^m| > k\}.$$

From Theorem 2.4 and weak lower semi-continuity, there exists a constant $C = C(n)$, independent of m, such that

$$\int_Q |\mathbf{T}|^{1+\frac{1}{n}} + |\mathbf{T}^m|^{1+\frac{1}{n}}\,\mathrm{d}x\,\mathrm{d}t \leq C(n).$$

It follows that $|Q_k^m| \leq C(n)k^{-(1+\frac{1}{n})}$. Splitting the domain Q into Q_k^m and its complement, we have

$$\begin{aligned} &\left(\int_Q |\mathbf{T}^m - \mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t\right)^2 \\ &\leq C\|\mathbf{T}^m - \mathbf{T}\|^2_{L^{1+\frac{1}{n}}(Q_k^m)} |Q_k^m|^{\frac{2}{n+1}} + Ck^{1+a}\int_{Q\setminus Q_k^m} \frac{|\mathbf{T}^m - \mathbf{T}|^2}{(1+|\mathbf{T}^m|+|\mathbf{T}|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq Ck^{-\frac{2}{n}} + Ck^{1+a}\int_Q (\mathbf{T}^m - \mathbf{T}) \cdot (F(\mathbf{T}^m) - F(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t \\ &\leq Ck^{-\frac{2}{n}} + Ck^{1+a}\int_Q (\mathbf{T}^m - \mathbf{T}) \cdot (F_n(\mathbf{T}^m) - F_n(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using (2.19), it follows that

$$
\begin{aligned}
&\lim_{m\to\infty}\int_Q (\mathbf{T}^m-\mathbf{T})\cdot(F_n(\mathbf{T}^m)-F_n(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t\\
&=\lim_{m\to\infty}\int_Q \mathbf{T}^m\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)-\mathbf{T}^m\cdot F_n(\mathbf{T})-\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)+\mathbf{T}\cdot F_n(\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_Q \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})-\mathbf{T}\cdot F_n(\mathbf{T})-\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})+\mathbf{T}\cdot F_n(\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t\\
&=0.
\end{aligned}
$$

Hence, we deduce that

$$
\lim_{m\to\infty}\int_Q |\mathbf{T}^m-\mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t\le Ck^{-\frac{1}{n}},
$$

for arbitrary $k\in\mathbb{N}$. Thus $\mathbf{T}^m\to\mathbf{T}$ strongly in $L^1(Q)^{d\times d}$. Hence, up to a further subsequence that we do not relabel, we have that $\mathbf{T}^m\to\mathbf{T}$ pointwise a.e. on Q and so also $\mathbf{T}^m(t)\to\mathbf{T}(t)$ pointwise on Ω for a.e. $t\in(0,T)$.

Now we prove the uniqueness of solutions. Suppose that $(\boldsymbol{u}_1,\mathbf{T}_1)$ and $(\boldsymbol{u}_2,\mathbf{T}_2)$ are weak solutions of the regularised problem emanating from the same data. Let $\boldsymbol{v}:=\boldsymbol{u}_1-\boldsymbol{u}_2$, $\mathbf{S}:=\mathbf{T}_1-\mathbf{T}_2$. Testing in the weak form of (2.1a) for $(\boldsymbol{u}_i,\mathbf{T}_i)$ against $\boldsymbol{v}_t+\alpha\boldsymbol{v}$ for $i\in\{1,2\}$ and subtracting the resulting equations, we integrate the result over $(0,t)$ to obtain

$$
\begin{aligned}
0&=\int_0^t\int_\Omega \boldsymbol{v}_{tt}\cdot(\boldsymbol{v}_t+\alpha\boldsymbol{v})+\mathbf{S}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_t+\alpha\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}s\\
&=\int_\Omega \frac{|\boldsymbol{v}_t(t)|^2}{2}+\alpha\boldsymbol{v}_t(t)\cdot\boldsymbol{v}(t)\,\mathrm{d}x+\int_0^t\int_\Omega \mathbf{S}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}_t+\alpha\boldsymbol{v})-\alpha|\boldsymbol{v}_t|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
$$

where we use the fact that $\boldsymbol{v}(0)=\mathbf{0}$, $\boldsymbol{v}_t(0)=\mathbf{0}$ and $\boldsymbol{v}$, $\boldsymbol{v}_t$ are continuous functions from $[0,T]$ to $L^2_*(\Omega)^d$. It follows that

$$
\begin{aligned}
0&\le\int_\Omega \frac{|\boldsymbol{v}_t(t)|^2}{2}\,\mathrm{d}x+\int_0^t\int_\Omega (\mathbf{T}_1-\mathbf{T}_2)\cdot(F_n(\mathbf{T}_1)-F_n(\mathbf{T}_2))\,\mathrm{d}x\,\mathrm{d}s\\
&\le\int_0^t\int_\Omega (2\alpha^2+\alpha)|\boldsymbol{v}_t|^2\,\mathrm{d}x\,\mathrm{d}s+\int_\Omega\frac{|\boldsymbol{v}_t(t)|^2}{4}\,\mathrm{d}x.
\end{aligned}
$$

We absorb the last term on the right-hand side into the left and apply Gronwall's inequality to deduce that, for every $t\in(0,T)$,

$$
0=\int_\Omega \frac{|\boldsymbol{v}_t(t)|^2}{4}\,\mathrm{d}x+\int_0^t\int_\Omega (\mathbf{T}_1-\mathbf{T}_2)\cdot(F_n(\mathbf{T}_1)-F_n(\mathbf{T}_2))\,\mathrm{d}x\,\mathrm{d}s.
$$

It follows that $\boldsymbol{u}_1=\boldsymbol{u}_2$ a.e. in Q and, by the strict monotonicity of F_n, we have that $\mathbf{T}_1=\mathbf{T}_2$. □

2.2 Higher regularity estimates

In this section, we focus on obtaining higher regularity estimates that allow us to take the limit as $n\to\infty$ and obtain a weak solution of the strain-limiting problem (2.1). As discussed previously, these regularity results are vital in order to obtain a suitable convergence result on the sequence of approximate stress tensors, bypassing the poor integrability estimates.

We first consider the case that $a > 0$ takes any arbitrary value and construct estimates that enable us to prove the pointwise convergence of the sequence of approximate stress tensors. We work at the level of the Galerkin approximation. The time regularity estimates must be constructed in this setting so that they are rigorous. We can also obtain the spatial regularity estimates at the level of the Galerkin approximation because in this periodic setting, spatial derivatives of test functions are also valid test functions in (2.7). We note that this is not the case outside of the periodic setting.

Lemma 2.8. *Let the assumptions of Lemma 2.6 hold and let $\boldsymbol{u}^m$ denote the solution of the Galerkin approximation of the regularised problem from V_m. Suppose additionally that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_*(\Omega)^d$ and $\boldsymbol{f} \in L^2(0,T;W^{1,2}_*(\Omega)^d)$. There exists a constant $C = C(\alpha, a, T, d)$, independent of m and n, such that*

$$\begin{aligned}&\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T]} \|\nabla \boldsymbol{u}^m_t(t)\|_2^2 + \int_Q \frac{|\nabla \mathbf{T}^m|^2}{(1+|\mathbf{T}^m|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left(\|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{u}_1\|_2^2 + \|\nabla \boldsymbol{f}\|^2_{L^2(Q)}\right).\end{aligned} \tag{2.20}$$

If we additionally have that $\boldsymbol{f} \in W^{1,2}(0,T;L^2_*(\Omega)^d)$, then

$$\begin{aligned}&\sup_{t\in[0,T]} \|\boldsymbol{u}^m_{tt}(t)\|_2^2 + \int_Q \frac{|\mathbf{T}^m_t|^2}{(1+|\mathbf{T}^m|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left(\|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{f}_t\|^2_{L^2(Q)} + \|\boldsymbol{f}(0)\|_2^2 + \|\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0\|^2_{2,2}\right),\end{aligned} \tag{2.21}$$

for a constant $C = C(\alpha, a, T, d, C_*)$, for every $m \geq m_0$ where m_0 is from Lemma 2.6.

Proof. First we prove the spatial estimate (2.20). We take $\nabla\cdot\nabla(\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m)$ as a test function in (2.7), a valid choice because differentiation does not increase the degree of a trigonometric polynomial and has integral over Ω equal to $\mathbf{0}$ due to periodicity. Integration by parts yields

$$\begin{aligned}0 &= -\int_\Omega (\boldsymbol{u}^m_{tt} - \boldsymbol{f})\cdot(\nabla\cdot\nabla(\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m)) + \mathbf{T}^m\cdot\boldsymbol{\varepsilon}\left(\nabla\cdot\nabla(\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m)\right)\,\mathrm{d}x \\ &= \int_\Omega \frac{\partial}{\partial t}\left(\frac{|\nabla \boldsymbol{u}^m_t|^2}{2} + \alpha\nabla\boldsymbol{u}^m_t\cdot\nabla\boldsymbol{u}^m\right) - \alpha|\nabla\boldsymbol{u}^m_t|^2 + \nabla\mathbf{T}^m\cdot\nabla F_n(\mathbf{T}^m) \\ &\quad - \nabla\boldsymbol{f}\cdot\nabla(\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m)\,\mathrm{d}x.\end{aligned} \tag{2.22}$$

In the above, we use $\nabla\mathbf{S}$ to denote the third-order tensor $(\partial_k S_{ij})_{i,j,k}$ and $\nabla\cdot\mathbf{S}$ to denote the vector $(\partial_j S_{ij})_i$. We need justify the equality

$$\int_\Omega (\nabla\cdot\mathbf{T}^m)\cdot(\nabla\cdot\nabla(\boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m))\,\mathrm{d}x = \int_\Omega \nabla\mathbf{T}^m\cdot\nabla F_n(\mathbf{T}^m)\,\mathrm{d}x.$$

We proceed as follows, denoting $\mathbf{S} := \mathbf{T}^m$ and $\boldsymbol{v} := \boldsymbol{u}^m_t + \alpha\boldsymbol{u}^m$ for simplicity of notation. Using integration by parts twice and the symmetry of $\mathbf{S}$, we have that

$$\int_\Omega (\nabla\cdot\mathbf{S})\cdot(\nabla\cdot\nabla\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \frac{\partial S_{ij}}{\partial x_j}\frac{\partial^2 v_i}{\partial x_k^2}\,\mathrm{d}x = \int_\Omega \frac{\partial S_{ij}}{\partial x_k}\frac{\partial^2 v_i}{\partial x_k\partial x_j}\,\mathrm{d}x = \int_\Omega \nabla\mathbf{S}\cdot\nabla\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x.$$

The boundary integrals from the integration by parts vanish due to periodicity. Furthermore, trigonometric functions are smooth so the manipulations are all justified. Proceeding as in the proof of Theorem 2.4, we integrate over an arbitrary time interval. Applying Young's inequality and Gronwall's inequality, we deduce that

$$\begin{aligned}\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}^m(t)\|_2^2+\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}_t^m(t)\|_2^2+\int_Q \nabla\mathbf{T}^m\cdot\nabla F_n(\mathbf{T}^m)\,\mathrm{d}x\,\mathrm{d}t\\ \leq C\left(\|\nabla \boldsymbol{u}_0\|_2^2+\|\nabla \boldsymbol{u}_1\|_2^2+\|\nabla \boldsymbol{f}\|_{L^2(Q)}^2\right).\end{aligned}$$

The function $\mathbf{T}\mapsto n^{-1}(1+|\mathbf{T}|^{1-\frac{1}{n}})^{-1}\mathbf{T}$ is monotonic so we have that

$$\nabla\mathbf{T}^m\cdot\nabla F_n(\mathbf{T}^m)\geq \nabla\mathbf{T}^m\cdot\nabla F(\mathbf{T}^m).$$

Next, we apply the chain rule to F to see that

$$\nabla\mathbf{T}^m\cdot\nabla F(\mathbf{T}^m)\geq \frac{|\partial_k\mathbf{T}^m|^2}{(1+|\mathbf{T}^m|^a)^{\frac{1}{a}}}-\frac{(\mathbf{T}^m\cdot\partial_k\mathbf{T}^m)^2|\mathbf{T}^m|^{a-2}}{(1+|\mathbf{T}^m|^a)^{1+\frac{1}{a}}}\geq \frac{|\nabla\mathbf{T}^m|^2}{(1+|\mathbf{T}^m|^a)^{1+\frac{1}{a}}}.$$

It follows immediately that (2.20) is satisfied.[1]

Now we focus on (2.21). If the Galerkin solution was smooth in time, we could differentiate (2.7) with respect to the time variable and test in the resulting equation against $\boldsymbol{u}_{tt}^m+\alpha\boldsymbol{u}_t^m$. However, we do not have sufficient regularity to justify this. Instead, we use difference quotients in the time variable to mimic this idea. We define by Δ_t^h the difference quotient with increment h in the time variable, i.e.,

$$\Delta_t^h g(t):=g(t+h)-g(t), \tag{2.23}$$

for any t and h such that (2.23) is defined. Fix $t\in(0,T)$ and let $h>0$ be sufficiently small such that $t+h\leq T$. Using (2.7), we have

$$\int_\Omega \Delta_t^h\boldsymbol{u}_{tt}^m(t)\cdot\boldsymbol{v}+\Delta_t^h\mathbf{T}^m(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x=\int_\Omega \Delta_t^h\boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x,$$

for every $\boldsymbol{v}\in V_m$. Setting $\boldsymbol{v}=\Delta_t^h(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)(t)$, replacing t by $\tau\in(0,t)$ and integrating with respect to τ over $(0,t)$, we obtain

$$\begin{aligned}&\frac{\|\Delta_t^h\boldsymbol{u}_t^m(t)\|_2^2}{h^2}+\int_0^t\int_\Omega\frac{\Delta_t^h\mathbf{T}^m}{h}\cdot\frac{\Delta_t^hF_n(\mathbf{T}^m)}{h}\,\mathrm{d}x\,\mathrm{d}s\\ &\leq C(a,\alpha,d,T)\left(\frac{\|\Delta_t^h\boldsymbol{u}_t^m(0)\|_2^2}{h^2}+\frac{\|\Delta_t^h\boldsymbol{u}^m(0)\|_2^2}{h^2}+\frac{\|\Delta_t^h\boldsymbol{f}\|_{L^2((0,t)\times\Omega)}^2}{h^2}\right).\end{aligned}$$

Using the regularity properties of $(\phi_i)_{i\geq1}$ and the coefficients $\boldsymbol{\beta}^m$, we can apply Lebesgue's dominated convergence theorem when taking the limit as $h\to0+$. Optimising the resulting inequality over $t\in(0,T)$, we deduce that

$$\sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^m(t)\|_2^2+\int_Q\frac{|\mathbf{T}_t^m|^2}{(1+|\mathbf{T}^m|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t\leq C\left(\|\boldsymbol{u}_{tt}^m(0)\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{f}_t\|_{L^2(Q)}^2\right), \tag{2.24}$$

[1]This arguement will be used repeatedly throughout the thesis and referred to as 'monotonicity and the chain rule.'

using the fact that (as a consequence of monotonicity and the chain rule)

$$\frac{|\mathbf{T}^m_t|^2}{(1+|\mathbf{T}^m|)^{1+a}} \leq \mathbf{T}^m_t \cdot F_n(\mathbf{T}^m)_t.$$

We note that $\boldsymbol{u}^m_{tt}$ is continuous on $\overline{Q}$ because $\boldsymbol{f} \in C([0,T]; L^2_*(\Omega)^d)$ under the additional assumptions on the data. Now, we will show that $\|\boldsymbol{u}^m_{tt}(0)\|_2$ is bounded above by a constant that is independent of m and n. Considering (2.7) at time $t = 0$, which we can do because of the continuity of $\boldsymbol{u}^m_{tt}$ on $[0,T]$, and testing against $\boldsymbol{u}^m_{tt}(0)$, we see that

$$\begin{aligned}
\|\boldsymbol{u}^m_{tt}(0)\|_2^2 &= \int_\Omega -\mathbf{T}^m(0)\cdot \boldsymbol{\varepsilon}(\boldsymbol{u}^m_{tt}(0)) + \boldsymbol{f}(0)\cdot \boldsymbol{u}^m_{tt}(0)\,\mathrm{d}x \\
&= \int_\Omega \left[\operatorname{div}\left(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\right) + \boldsymbol{f}(0)\right]\cdot \boldsymbol{u}^m_{tt}(0)\,\mathrm{d}x \\
&\leq \left[\|\operatorname{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))))\|_2 + \|\boldsymbol{f}(0)\|_2\right]\|\boldsymbol{u}^m_{tt}(0)\|_2.
\end{aligned} \tag{2.25}$$

Using the fact that F_n is a C^1-diffeomorphism (recall Lemma 2.3), we get

$$\|\operatorname{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))))\|_2 \leq \|D(F_n^{-1})(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_\infty \|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}.$$

Considering properties of symmetric positive definite matrices, direct calculation shows that there exists a constant C_a depending only on a such that

$$|D(F_n^{-1})(\mathbf{S})| = |(DF_n)^{-1}(F_n^{-1}(\mathbf{S}))| \leq C_a(1+|F_n^{-1}(\mathbf{S})|^{1+a}) \quad \forall\, \mathbf{S}\in\mathbb{R}^{d\times d},$$

so the application of Lemma 2.3 is justified. Returning to (2.25), it follows that, for every $m \geq m_0$,

$$\begin{aligned}
\|\boldsymbol{u}^m_{tt}(0)\|_2 &\leq \|(DF_n)^{-1}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))))\|_\infty\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2} + \|\boldsymbol{f}(0)\|_2 \\
&\leq C_a(1+f^{-1}(C_1)^{1+a})\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2} + \|\boldsymbol{f}(0)\|_2 \\
&\leq C(a,C_*)\left(\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2} + \|\boldsymbol{f}(0)\|_2^2\right),
\end{aligned}$$

where $m_0 \in \mathbb{N}$ and $C_1 \in (0,1)$ are from Lemma 2.6. Substituting this into (2.24), the required bound (2.21) follows. □

Next, we show that, in the case that a is small, where smallness depends on the dimension d, higher integrability estimates can be proven for the sequence $(\mathbf{T}^m)_m$. We prove this result in three spatial dimensions, since this is the physically relevant case. However, in Chapter 4, we will see that this result can be improved, but only under the stronger assumptions on the data than are used in Lemma 2.8. We only require minimal regularity assumptions on the data for the following result.

Lemma 2.9. *Let the assumptions of Lemma 2.6 hold. Suppose additionally that* $\boldsymbol{u}_0,\ \boldsymbol{u}_1 \in W^{1,2}_*(\Omega)^d$ *and* $\boldsymbol{f} \in L^2(0,T;W^{1,2}_*(\Omega)^d)$. *Furthermore, assume that* $d=3$ *and* $a \in (0,\frac{2}{7}]$. *Setting* $\delta = \frac{2a}{3}$, *there exists a constant* $C>0$, *independent of* n *and* m, *such that*

$$\int_Q |\mathbf{T}^m|^{1+\delta}\,\mathrm{d}x\,\mathrm{d}t \leq C\left(1+\|\nabla\boldsymbol{u}_0\|_2^2 + \|\nabla\boldsymbol{u}_1\|_2^2 + \|\nabla\boldsymbol{f}\|_{L^2(Q)}^2\right),$$

for every $m \geq m_0$.

Proof. Using Lemma 2.6 and (2.20) from Lemma 2.8, there exists a constant C, independent of n and m, such that

$$\sup_{t\in[0,T]}\left(\int_\Omega |\mathbf{T}^m(t)|^{1-a}\,\mathrm{d}x\right) + \int_Q \frac{|\nabla \mathbf{T}^m|^2}{(1+|\mathbf{T}^m|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t \le C, \tag{2.26}$$

for every $m \ge m_0$. We can remove the indicator function from the bound in Lemma 2.6 (and so the first term on the left-hand side of (2.26)) because $a < 1$. Applying the Sobolev embedding theorem to the function $(1+|\mathbf{T}^m|)^{\frac{1-a}{2}}$, we see that

$$\int_0^T \left(\int_\Omega |\mathbf{T}^m|^{\frac{p(1-a)}{2}}\,\mathrm{d}x\right)^{\frac{2}{p}} \mathrm{d}t \le C(p,a)\left(\int_Q \frac{|\nabla \mathbf{T}^m|^2}{(1+|\mathbf{T}^m|)^{1+a}}\,\mathrm{d}x\,\mathrm{d}t + 1\right),$$

for every $p \in (2,6]$. In particular, setting $p=6$, we get

$$\sup_{t\in[0,T]}\left(\int_\Omega |\mathbf{T}^m(t)|^{1-a}\,\mathrm{d}x\right) + \int_0^T \left(\int_\Omega |\mathbf{T}^m|^{3(1-a)}\,\mathrm{d}x\right)^{\frac{1}{3}} \mathrm{d}t \le C. \tag{2.27}$$

Fix some $q \in (1,\infty)$ that is to be determined later. We denote by $q' = \frac{q}{q-1}$ the Hölder conjugate of q. Applying Hölder's inequality with exponents (q,q') and considering the first term on the left-hand side of (2.27), we see that, for a.e. $t \in (0,T)$,

$$\int_\Omega |\mathbf{T}^m|^{1+\frac{a}{q'}}\,\mathrm{d}x \le \||\mathbf{T}^m|^{1-a}\|_1^{\frac{1}{q'}} \|\mathbf{T}^m\|_{1+2a(q-1)}^{\frac{1+2a(q-1)}{q}} \le C\left(\int_\Omega |\mathbf{T}^m|^{1+2a(q-1)}\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

We claim that q can be chosen such that

$$\frac{1}{q} \le \frac{2}{p} = \frac{1}{3}, \quad 1+2a(q-1) \le \frac{p(1-a)}{2} = 3(1-a). \tag{2.28}$$

The existence of such a $q \in (1,\infty)$ is possible in the case that $a \le \frac{2}{7}$. Choose $q = 3$. Then the first inequality in (2.28) holds trivially. To see that the second holds under the restriction that $a \le \frac{2}{7}$, we simply notice that

$$1 + 2a(q-1) = 1 + 4a \le 3(1-a) \quad \Longleftrightarrow \quad 7a \le 2.$$

Thus such a q exists. Given this choice, we use the second inequality of (2.28) to deduce that

$$\begin{aligned}\int_Q |\mathbf{T}^m|^{1+\frac{a}{q'}}\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \left(\int_\Omega |\mathbf{T}^m|^{1+2a(q-1)}\,\mathrm{d}x\right)^{\frac{1}{q}} \mathrm{d}t \\ &\le C(d,T) + \int_0^T \left(\int_\Omega |\mathbf{T}^m|^{\frac{p(1-a)}{2}}\,\mathrm{d}x\right)^{\frac{2}{p}} \mathrm{d}t \\ &\le C\left(1 + \|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{u}_1\|_2^2 + \|\nabla \boldsymbol{f}\|_{L^2(Q)}^2\right).\end{aligned}$$

Setting $\delta = \frac{a}{q'} = \frac{2a}{3}$, the asserted bound follows. ☐

We remark that the choice of p is likely optimised in the above argument as it enables a large choice of q in the first inequality in (2.28) while maximising the choice in the second inequality. However, this result is not optimal. Under stronger conditions on the time regularity of $\boldsymbol{f}$, we can show a similar result in any spatial dimension $d \ge 2$ when $a \in (0, \frac{2}{d})$. This is explored further in Chapter 4.

2.3 The limit in the regularisation parameter

In this section, we use the higher regularity and integrability estimates of Section 2.2 to take the limit in the regularisation parameter n. Provided that $m \geq m_0$, the order in which the limit in m and n in fact does not matter. However, for simplicity, we take $m \to \infty$ followed by $n \to \infty$. We denote by $(\boldsymbol{u}^n, \mathbf{T}^n)$ the weak solution of the regularised problem with approximation parameter n and $\boldsymbol{u}^{n,m}$ the solution of the Galerkin approximation from V_m of the regularised problem with parameter n.

Theorem 2.10. *Suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_*(\Omega)^d$ and $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W^{k+1,2}_*(\Omega)^d$ for a $k > \frac{d}{2}$, such that the following safety strain condition holds:*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)\|_\infty = C_* < 1.$$

Furthermore, assume that $\boldsymbol{f} \in L^2(0,T; W^{1,2}_*(\Omega)^d) \cap W^{1,2}(0,T; L^2_*(\Omega)^d)$. There exists a couple $(\boldsymbol{u}, \mathbf{T})$ that is the unique weak solution of the strain-limiting problem (2.1) *in the sense of Definition 2.1.*

Moreover, if $((\boldsymbol{u}^n, \mathbf{T}^n))_n$ is the sequence of solutions to the regularised problem, the following convergence results hold:

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ *weakly-* in $W^{2,\infty}(0,T; L^2_*(\Omega)^d) \cap W^{1,\infty}(0,T; W^{1,2}_*(\Omega)^d)$;*
- $\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u}$ *weakly in $L^p(0,T; W^{1,p}_*(\Omega)^d)$ for every $p \in [1, \infty)$;*
- $\mathbf{T}^n \to \mathbf{T}$ *pointwise a.e. in Q.*

Proof. The proof of uniqueness can be argued almost exactly as in the proof of Theorem 2.7 so we do not include the details here. We focus on the existence of weak solutions. Summarising the bounds from Theorem 2.4, Corollary 2.5, Lemma 2.6 and Lemma 2.8, there exists a constant C, independent of m and n, such that

$$\begin{aligned}\sup_{t\in[0,T]} \|\boldsymbol{u}^{n,m}(t)\|_{1,2}^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_t^{n,m}(t)\|_{1,2}^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_{tt}^{n,m}(t)\|_2^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^{n,m} + \alpha \boldsymbol{u}^{n,m})\|_{L^{n+1}(Q)} \\ + \int_Q |\mathbf{T}^{n,m}| + \frac{|\mathbf{T}^{n,m}|^{1+\frac{1}{n}}}{n} + \frac{|\nabla \mathbf{T}^{n,m}|^2}{(1+|\mathbf{T}^{n,m}|)^{1+a}} + \frac{|\mathbf{T}_t^{n,m}|^2}{(1+|\mathbf{T}^{n,m}|)^{1+a}} \,\mathrm{d}x\,\mathrm{d}t \leq C, \end{aligned} \tag{2.29}$$

assuming from now on that $m \geq m_0$. First, we prove the pointwise convergence of $(\mathbf{T}^n)_n$. Define sequences

$$\mathbf{S}^{n,m} = \frac{\mathbf{T}^{n,m}}{(1+|\mathbf{T}^{n,m}|)^{1+a}}, \quad s^{n,m} = \frac{1}{(1+|\mathbf{T}^{n,m}|)^{1+a}}.$$

By definition, the modulus of each function is bounded above by 1. Using (2.29), we immediately see that

$$\int_Q |\nabla \mathbf{S}^{n,m}|^2 + |\mathbf{S}_t^{n,m}|^2 + |\nabla s^{n,m}|^2 + |s_t^{n,m}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C, \tag{2.30}$$

where C is independent of n and m. By standard compactness properties and the Aubin–Lions lemma, it follows that $(\mathbf{S}^{n,m})_m$ converges strongly in $L^2(Q)^{d\times d}$ to a limit $\mathbf{S}^n$, say. However, by the pointwise convergence of $(\mathbf{T}^{n,m})_m$ a.e. on Q, shown in the proof of Theorem 2.7, we can identify the limit $\mathbf{S}^n$ by

$$\mathbf{S}^n = \frac{\mathbf{T}^n}{(1+|\mathbf{T}^n|)^{1+a}}.$$

An analogous result holds for $(s^{n,m})_m$ and the corresponding limit s^n. By weak lower semi-continuity, it follows from (2.30) that

$$\|\mathbf{S}^n\|_{L^\infty(Q)} + \|s^n\|_{L^\infty(Q)} + \int_Q |\nabla \mathbf{S}^n|^2 + |\mathbf{S}^n_t|^2 + |\nabla s^n|^2 + |s^n_t|^2 \,\mathrm{d}x\,\mathrm{d}t \le C,$$

where C is independent of n. Applying the Aubin–Lions lemma again, the sequences $(\mathbf{S}^n)_n$ and $(s^n)_n$ converge strongly in $L^2(Q)$ and so pointwise a.e. on Q as $n\to\infty$ to limits that we denote by $\mathbf{S}$ and s, respectively, up to a subsequence in n, not relabelled.[2]

By Fatou's lemma, we see that

$$\int_Q s^{-\frac{1}{1+a}} \,\mathrm{d}x\,\mathrm{d}t \le C + \liminf_{n\to\infty} \int_Q |\mathbf{T}^n| \,\mathrm{d}x\,\mathrm{d}t \le C.$$

Thus, $s^{-\frac{1}{1+a}}$ is integrable so $s > 0$ a.e. in Q. By elementary analysis, we deduce that $\mathbf{T}^n = \mathbf{S}^n(s^n)^{-1}$ converges pointwise a.e. on Q as $n\to\infty$. We denote the limit by $\mathbf{T}$. By Fatou's lemma and the uniform L^1-bound on $(\mathbf{T}^n)_n$, the limit $\mathbf{T}$ is an element of $L^1(Q)^{d\times d}$. The remaining asserted convergence results from the statement of the theorem follow easily from (2.29) and standard compactness results for Bochner spaces.

Next, we recall that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n) = F_n(\mathbf{T}^n) = F(\mathbf{T}^n) + \frac{\mathbf{T}^n}{n(1+|\mathbf{T}^n|^{1-\frac{1}{n}})}.$$

Using the pointwise convergence of $(\mathbf{T}^n)_n$ with the fact that $\mathbf{T}$ is finite a.e. in Q, we see that

$$\frac{\mathbf{T}^n}{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}} \to \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \quad \text{and} \quad \frac{\mathbf{T}^n}{n(1+|\mathbf{T}^n|^{1-\frac{1}{n}})} \to \mathbf{0},$$

pointwise a.e. on Q as $n\to\infty$. It follows that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n) = F_n(\mathbf{T}^n) \to F(\mathbf{T}) \quad \text{pointwise a.e. in } Q.$$

However, we have that $(\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n))_n$ converges weakly in $L^2(Q)^{d\times d}$ to $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$. Weak limits in $L^2(Q)$ and pointwise limits must coincide.[3] Thus the constitutive relation (2.1b) holds pointwise a.e. in Q. Satisfaction of the initial conditions (2.4) can be argued by standard

[2]As when we took the limit in m, any convergence results that only hold up to subsequence in n later improve to convergence of the entire sequence due to the uniqueness of the limit.

[3] By the Banach–Saks theorem [101], there is a subsequence of $(\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n))_n$ whose Cesàro averages converge strongly to $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$. The sequence of averages converges pointwise a.e. on Q up to a further subsequence. On the other hand, the pointwise convergence of $(F_n(\mathbf{T}^n))_n$ is preserved by taking any combination of Cesàro averages and subsequences. Thus we must also have that the same sequence of averages converges pointwise to $F(\mathbf{T})$. However, pointwise limits are unique and so $F(\mathbf{T}) = \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$ as required.

reasoning and using the initial conditions for the sequence of regularised solutions, so we do not include the details here.

It remains to show that the weak form (2.2) of the momentum equation is satisfied by the limiting couple $(\boldsymbol{u}, \mathbf{T})$. Let $\boldsymbol{v} \in C_*^\infty(\overline{\Omega})^d$ and $\psi \in C([0,T])$ be arbitrary but fixed test functions. Given $\tau \in C_c^1([0,\infty))$, we consider the function $\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi$. We note that we cannot use $\boldsymbol{v}\psi$ directly as a test function because the pointwise convergence is not sufficient to evaluate the limit

$$\lim_{n\to\infty} \int_Q \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi)\, \mathrm{d}x\, \mathrm{d}t.$$

Instead we introduce that cut-off function τ. This allows application of the pointwise convergence result. Then the integrability of $\mathbf{T}$ is used to evaluate what remains after taking the limit in n.

The function $\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi$ is a valid test function in the weak form of (2.1a) provided that $\tau(|\mathbf{T}^n|)$ is weakly differentiable with weak derivative in $L^2(\Omega)$. To see why this is true, we note that we can write

$$\tau(|\mathbf{T}^n|) = \tau\left(f_n\left((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}\right)\right) \qquad \text{where } f_n(s) = s^{\frac{2}{1+1/n}} - 1.$$

First, we claim that $(1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}$ is an element of $L^2(0,T;W^{1,2}(\Omega)^d)$. Applying the reasoning from Lemma 2.8, there exists a constant C, independent of m and n, such that

$$\begin{aligned} C &\geq \int_Q \mathbf{T}_t^{n,m} \cdot F_n(\mathbf{T}^{n,m})_t + \nabla \mathbf{T}^{n,m} \cdot \nabla F_n(\mathbf{T}^{n,m})\, \mathrm{d}x\, \mathrm{d}t \\ &\geq \frac{1}{n}\int_Q \left|\left((1+|\mathbf{T}^{n,m}|)^{\frac{1+1/n}{2}}\right)_t\right|^2 + \left|\nabla(1+|\mathbf{T}^{n,m}|)^{\frac{1+1/n}{2}}\right|^2 \, \mathrm{d}x\, \mathrm{d}t. \end{aligned}$$

Hence $(1+|\mathbf{T}^{n,m}|)^{\frac{1+1/n}{2}}$ is bounded in $W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$. Reasoning as we did with the sequence $(\mathbf{S}^{n,m})_m$, we deduce that $(1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}$ is an element of $W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$. By the chain rule for weak derivatives and noting that $\tau\circ f_n \in C_c^1([0,\infty))$, we deduce that the function $\tau(|\mathbf{T}^n|) = (\tau\circ f_n)((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}})$ is weakly differentiable in space for a.e. $t\in(0,T)$ with weak derivative given by

$$\nabla\tau(|\mathbf{T}^n|) = \tau'(|\mathbf{T}^n|)\, f_n'\left((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}\right)\nabla\left((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}\right) \in L^2(Q)^d.$$

Thus $\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi$ is indeed a valid test function and the following is justified:

$$\begin{aligned} &\int_Q \boldsymbol{u}_{tt}^n \cdot (\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) + \tau(|\mathbf{T}^n|)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi)\, \mathrm{d}x\, \mathrm{d}t \\ &= -\int_Q \psi\mathbf{T}^n \cdot (\nabla\tau(|\mathbf{T}^n|)\otimes\boldsymbol{v})\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{2.31}$$

Next, we replace τ by a function $\tau_k \in C_c^1([0,\infty))$ such that

$$\tau_k(s) = \begin{cases} 1 & \text{if } s \leq k, \\ 0 & \text{if } s \geq 2k, \end{cases}$$

and $|\tau_k'(s)| \leq Ck^{-1}$ for a constant C that is independent of k. Using Lebesgue's dominated convergence theorem and the integrability of $\mathbf{T}$, we deduce that

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} -\int_Q \psi\mathbf{T}^n \cdot \Big(\nabla\left(\tau_k \circ f_n\right)\Big((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}\Big)\otimes \boldsymbol{v}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \int_Q \boldsymbol{u}_{tt}^n \cdot (\tau_k(|\mathbf{T}^n|)\boldsymbol{v}\psi) + \tau_k(|\mathbf{T}^n|)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau_k(|\mathbf{T}^n|)\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_Q \boldsymbol{u}_{tt} \cdot (\tau_k(|\mathbf{T}|)\boldsymbol{v}\psi) + \tau_k(|\mathbf{T}|)\mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau_k(|\mathbf{T}|)\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \boldsymbol{u}_{tt} \cdot (\boldsymbol{v}\psi) + \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

To prove that (2.2) holds, it suffices to show that the limit on the left-hand side vanishes. First, we notice that

$$\begin{aligned}
&\int_Q \psi\mathbf{T}^n \cdot \Big(\nabla\left(\tau_k \circ f_n\right)\Big((1+|\mathbf{T}^n|)^{\frac{1+1/n}{2}}\Big)\otimes \boldsymbol{v}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \lim_{m\to\infty}\int_Q \psi\mathbf{T}^{n,m}\cdot\left(\nabla\tau_k(|\mathbf{T}^{n,m}|)\otimes\boldsymbol{v}\right)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where the integral on the right-hand side is well-defined by the regularity of the approximate stress tensor $\mathbf{T}^{n,m}$. Thus it is sufficient to show that

$$\lim_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\int_Q \psi\mathbf{T}^{n,m}\cdot\left(\nabla\tau_k(|\mathbf{T}^{n,m}|)\otimes\boldsymbol{v}\right)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Expanding out the derivative, justified because we are working with the Galerkin approximation, we see that

$$\begin{aligned}
&\int_Q \psi\mathbf{T}^{n,m}\cdot\left(\nabla\tau_k(|\mathbf{T}^{n,m}|)\otimes\boldsymbol{v}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi\tau_k'(|\mathbf{T}^{n,m}|)(1+|\mathbf{T}^{n,m}|^a)^{\frac{1}{a}}F(\mathbf{T}^{n,m})\cdot\left(\nabla|\mathbf{T}^{n,m}|\otimes\boldsymbol{v}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi F(\mathbf{T}^{n,m})\cdot\left(\nabla B_k(|\mathbf{T}^{n,m}|)\otimes\boldsymbol{v}\right)\mathrm{d}x\,\mathrm{d}t \\
&= -\int_Q \psi B_k(|\mathbf{T}^{n,m}|)\frac{\partial}{\partial x_i}\left(F(\mathbf{T}^{n,m})_{ij}\right)\boldsymbol{v}_j + \psi B_k(|\mathbf{T}^{n,m}|)F(\mathbf{T}^{n,m})_{ij}\frac{\partial \boldsymbol{v}_j}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where B_k is defined by $B_k(s) := \int_0^s (1+t^a)^{\frac{1}{a}}\tau_k'(t)\,\mathrm{d}t$. The boundary integrals coming from the integration by parts vanish due to periodicity. For each $\mathbf{T}\in\mathbb{R}^{d\times d}$, we define the fourth-order tensor $\mathcal{A}(\mathbf{T})$ by

$$\mathcal{A}_{ijpq}(\mathbf{T}) = \frac{\partial}{\partial \mathbf{T}_{pq}}\left(\frac{\mathbf{T}_{ij}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}\right).$$

We can easily check that, for every $\mathbf{T}\in\mathbb{R}^{d\times d}$, the bilinear functional

$$(\mathbf{S},\mathbf{U})_{\mathcal{A}(\mathbf{T})} := \sum_{i,j,p,q=1}^{d}\mathcal{A}_{ijpq}(\mathbf{T})\mathbf{S}_{ij}\mathbf{U}_{pq}$$

defines an inner product on $\mathbb{R}^{d\times d}$.[4] In fact, this inner product satisfies

$$\begin{aligned}\int_Q (\partial_l \mathbf{T}^{n,m}, \partial_l \mathbf{T}^{n,m})_{\mathcal{A}(\mathbf{T}^{n,m})}\,\mathrm{d}x\,\mathrm{d}t &= \int_Q \partial_l \mathbf{T}^{n,m}\cdot \partial_l F(\mathbf{T}^{n,m})\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_Q \nabla \mathbf{T}^{n,m}\cdot \nabla F_n(\mathbf{T}^{n,m})\,\mathrm{d}x\,\mathrm{d}t \\ &\le C,\end{aligned} \tag{2.32}$$

where C is a constant that is independent of n and m, using Lemma 2.8. By integration by parts, the inner product property of $\mathcal{A}(\mathbf{T})$ and the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}&\left|\int_Q \psi \mathbf{T}^{n,m}\cdot (\nabla \tau_k(|\mathbf{T}^{n,m}|)\otimes \boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \int_Q \left| \left(\left(\delta_{il} B_k(|\mathbf{T}^{n,m}|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \psi|\boldsymbol{v}|\partial_l \mathbf{T}^{n,m}\right)_{\mathcal{A}(\mathbf{T}^{n,m})}\right| + |\psi B_k(|\mathbf{T}^{n,m}|)||F(\mathbf{T}^n)||\nabla \boldsymbol{v}|\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\int_Q \left(\left(\delta_{il} B_k(|\mathbf{T}^{n,m}|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\delta_{il} B_k(|\mathbf{T}^{n,m}|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^{n,m})}\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_Q (\psi|\boldsymbol{v}|\partial_l \mathbf{T}^{n,m}, \psi|\boldsymbol{v}|\partial_l \mathbf{T}^{n,m})_{\mathcal{A}(\mathbf{T}^{n,m})}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \int_Q |\psi B_k(|\mathbf{T}^{n,m}|)||F(\mathbf{T}^n)||\nabla \boldsymbol{v}|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.33}$$

For the second term on the right-hand side, by the boundedness of F, we have that

$$\int_Q |\psi B_k(|\mathbf{T}^{n,m}|)||F(\mathbf{T}^n)||\nabla \boldsymbol{v}|\,\mathrm{d}x\,\mathrm{d}t \le C(\psi, \boldsymbol{v})\int_Q |B_k(\mathbf{T}^{n,m})|\,\mathrm{d}x\,\mathrm{d}t.$$

For the second factor of the first term on the right-hand side of (2.33), we use (2.32) to see that

$$\int_Q (\psi|\boldsymbol{v}|\partial_l \mathbf{T}^{n,m}, \psi|\boldsymbol{v}|\partial_l \mathbf{T}^{n,m})_{\mathcal{A}(\mathbf{T}^{n,m})}\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \psi^2|\boldsymbol{v}|^2 \nabla \mathbf{T}^{n,m}\cdot \nabla F(\mathbf{T}^{n,m})\,\mathrm{d}x\,\mathrm{d}t \le C,$$

where C is independent of n and m. For the first factor of the first term on the right-hand side of (2.33), we have

$$\int_Q \left(\left(\delta_{il} B_k(|\mathbf{T}^{n,m}|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\delta_{il} B_k(|\mathbf{T}^{n,m}|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^{n,m})} \le C\int_Q \frac{|B_k(|\mathbf{T}^{n,m}|)|^2}{(1+|\mathbf{T}^{n,m}|^a)^{\frac{1}{a}}},$$

where the constant C depends only on the dimension d. By the choice of τ_k, we have $B_k(t) = 0$ if $t \le k$ and if $t \ge k$ then

$$|B_k(t)| = \left|\int_k^t \tau_k'(s)(1+s^a)^{\frac{1}{a}}\,\mathrm{d}s\right| \le \frac{C}{k}\int_k^{2k}(1+s^a)^{\frac{1}{a}}\,\mathrm{d}s \le Ck \le Ct.$$

In particular, B_k is uniformly bounded for each fixed $k \in \mathbb{N}$. Using the weaker bound that

[4] We use inner product arguments of this vein throughout the thesis. The notation is simplifying and allows use of standard inner product results, in particular, the Cauchy–Schwarz inequality. Furthermore, we can directly relate $(\partial_i \mathbf{T}, \partial_i \mathbf{T})_{\mathcal{A}(\mathbf{T})}$ to the derivative $\partial_i \mathbf{T}\cdot \partial_i F(\mathbf{T})$, which we have spent time in deriving estimates for.

$|B_k(t)| \leq Ct$, we see that

$$\begin{aligned}
&\int_Q \left(\left(\delta_{il} B_k(|\mathbf{T}^{n,m}|) \frac{\boldsymbol{v}_j}{|\boldsymbol{v}|} \right)_{i,j}, \left(\delta_{il} B_k(|\mathbf{T}^{n,m}|) \frac{\boldsymbol{v}_j}{|\boldsymbol{v}|} \right)_{i,j} \right)_{\mathcal{A}(\mathbf{T}^{n,m})} \mathrm{d}x\, \mathrm{d}t \\
&\leq C \int_Q \frac{|B_k(|\mathbf{T}^{n,m}|)|^2}{(1+|\mathbf{T}^{n,m}|^a)^{\frac{1}{a}}} \mathrm{d}x\, \mathrm{d}t \\
&\leq C \int_Q |B_k(|\mathbf{T}^{n,m}|)| \frac{|\mathbf{T}^{n,m}|}{(1+|\mathbf{T}^{n,m}|^a)^{\frac{1}{a}}} \mathrm{d}x\, \mathrm{d}t \\
&\leq C \int_Q |B_k(|\mathbf{T}^{n,m}|)| \,\mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

Using the uniform boundedness of B_k on $[0,\infty)$ for each fixed k with the dominated convergence theorem and the pointwise convergence of $(\mathbf{T}^{n,m})_m$ and $(\mathbf{T}^n)_n$, it follows that

$$\begin{aligned}
&\lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \left| \int_Q \psi \mathbf{T}^{n,m} \cdot (\nabla \tau_k(|\mathbf{T}^{n,m}|) \otimes \boldsymbol{v}) \,\mathrm{d}x\, \mathrm{d}t \right| \\
&\leq C \lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \int_Q |B_k(|\mathbf{T}^{n,m}|)| \,\mathrm{d}x\, \mathrm{d}t \\
&= C \lim_{k\to\infty} \int_Q |B_k(|\mathbf{T}|)| \,\mathrm{d}x\, \mathrm{d}t \leq C \lim_{k\to\infty} \int_{\{|\mathbf{T}|>k\}} |\mathbf{T}| \,\mathrm{d}x\, \mathrm{d}t = 0,
\end{aligned}$$

where the final line follows from the fact that $\mathbf{T} \in L^1(Q)^{d\times d}$. Hence we have proven the existence of a weak solution to the strain-limiting problem (2.1). □

Next, using the higher integrability result from Lemma 2.9 we prove an existence result under weaker conditions on the data, provided that the parameter a in the strain-limiting relation is small. In particular, we use the uniform bound on $(\mathbf{T}^n)_n$ in $L^{1+\delta}(Q)^{d\times d}$ to prove that $(\mathbf{T}^n)_n$ converges strongly to $\mathbf{T}$ in $L^1(Q)^{d\times d}$.

Theorem 2.11. *Suppose that we have $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_*(\Omega)^d$ and $\boldsymbol{f} \in L^2(0,T;W^{1,2}_*(\Omega)^d)$, such that $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W^{k+1,2}_*(\Omega)^d$ for some $k > \frac{d}{2}$ and the safety strain condition (2.14) holds. Furthermore, assume that $d = 3$ and $a \in (0, \frac{2}{7})$. There exists a unique weak solution $(\boldsymbol{u}, \mathbf{T})$ of the strain-limiting problem (2.1) in the sense of Definition 2.1. Furthermore, if $((\boldsymbol{u}^n, \mathbf{T}^n))_n$ is the sequence of solutions to the regularised problem,*

$$\mathbf{T}^n \to \mathbf{T} \qquad \textit{strongly in } L^1(0,T;L^1_{\#}(\Omega)^{d\times d}).$$

Proof. From Lemma 2.9, we immediately deduce that

$$\int_Q |\mathbf{T}^n|^{1+\delta} \,\mathrm{d}x\, \mathrm{d}t \leq C,$$

where C is independent of n and $\delta = \frac{2a}{3} > 0$. From Theorem 2.4, Lemma 2.6 and Theorem 2.7, we have the following bound:

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_2 + \sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{1,2} + \sup_{t\in[0,T]} \|\boldsymbol{u}^n_t(t)\|_2 + \int_Q |\boldsymbol{u}^n_{tt}|^2 + |\mathbf{T}^n| \,\mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \|\boldsymbol{u}^n_t(t)\|_2^2 \,\mathrm{d}t + \|\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)\|_{L^{n+1}(Q)} \leq C,
\end{aligned}$$

where C is independent of n. It follows that, up to a subsequence in n that we do not relabel, the following convergence results hold:

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $W^{1,\infty}(0,T;L^2_*(\Omega)^d) \cap L^\infty(0,T;W^{1,2}_*(\Omega)^d)$;
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ weakly in $W^{2,2}(0,T;L^2_*(\Omega)^d) \cap W^{1,2}(0,T;W^{1,2}_*(\Omega)^d)$;
- $\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha\boldsymbol{u}$ weakly in $L^p(0,T;W^{1,p}_*(\Omega)^d)$;
- $\mathbf{T}^n \rightharpoonup \mathbf{T}$ weakly in $L^{1+\delta}(0,T;L^{1+\delta}_\#(\Omega)^{d\times d})$.

Reasoning exactly as in the proof of Theorem 2.7, but using the bound

$$\int_Q |\mathbf{T}^n|^{1+\delta} + |\mathbf{T}|^{1+\delta}\,\mathrm{d}x\,\mathrm{d}t \leq C,$$

we have that

$$\mathbf{T}^n \to \mathbf{T} \quad \text{strongly in } L^1(0,T;L^1_\#(\Omega)^{d\times d}).$$

Thus, $\mathbf{T}^n \to \mathbf{T}$ pointwise a.e. on Q up to a further subsequence that we do not relabel and so also $F_n(\mathbf{T}^n) \to F(\mathbf{T})$ pointwise a.e. on Q. By the uniqueness of pointwise and weak limits in $L^2(Q)$, we deduce that $F(\mathbf{T}) = \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$ pointwise a.e. in Q. By standard arguments, we can show that the limiting couple $(\boldsymbol{u},\mathbf{T})$ satisfies the initial conditions in the required sense. Finally, it remains to show that the weak form (2.2) of (2.1a) holds. However, using the strong L^1-convergence of $(\mathbf{T}^n)_n$, this can be done in a standard way. Thus we have existence of a weak solution. The proof of uniqueness is unchanged from the case of a general a. $\square$

Remark. *Adapting the proofs in this section, we can generalise the results to a larger class of strain-limiting constitutive relations. In particular, consider*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = G(\mathbf{T}),$$

where G is a function such that

$$(G(\mathbf{T}) - G(\mathbf{S}))\cdot(\mathbf{T}-\mathbf{S}) \geq 0,$$
$$G(\mathbf{T})\cdot\mathbf{T} \geq c_0|\mathbf{T}| - c_1,$$
$$|G(\mathbf{T})| \leq c_1,$$

for positive constants c_0, c_1. Furthermore, we assume that G has the Uhlenbeck form

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = \varphi'(|\mathbf{T}|)\frac{\mathbf{T}}{|\mathbf{T}|},$$

where $\varphi \in C^2(\mathbb{R}_+;\mathbb{R}_+)$ is a strictly convex function such that

$$\varphi(0) = \varphi'(0) = 0, \quad |\varphi''(s)| \leq \frac{c_1}{1+s} \quad \forall s \in \mathbb{R}_+.$$

Applying the reasoning in [20], this is sufficient to obtain a pointwise convergence result on the sequence of approximations to the stress tensor. To obtain results like those in Lemma 2.9 and Theorem 2.11, we ask that

$$(\mathbf{S},\mathbf{S})_{\mathcal{A}(\mathbf{T})} \geq c_0 \frac{|\mathbf{S}|^2}{(1+|\mathbf{T}|)^{1+a}} \quad \forall \mathbf{T},\, \mathbf{S} \in \mathbb{R}^{d\times d},$$

for a constant $a \in (0, \frac{2}{7})$, *in spatial dimensions* $d = 3$, *where the inner product* $(\cdot,\cdot)_{\mathcal{A}(\mathbf{T})}$ *is defined as in the proof of Theorem 2.10. In Chapter 4, we see that this can be improved to* $a \in (0, \frac{2}{d})$ *in an arbitrary spatial dimension* $d \geq 2$. *Throughout the rest of the thesis we will focus on the prototype form* $F(\mathbf{T}) = (1+|\mathbf{T}|^a)^{-\frac{1}{a}}\mathbf{T}$. *However, the results are still valid for this Uhlenbeck form.*

Similarly, in the spirit of [16], we can also consider relations of the form

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = \lambda(|\mathrm{tr}\mathbf{T}|)(\mathrm{tr}\mathbf{T})\mathbf{I} + \mu(|\mathbf{T}^\delta|)\mathbf{T}^\delta,$$

where λ *and* μ *satisfy similar conditions to those on* φ'.

We do not consider the periodic problem again in this thesis. However, the techniques developed in this chapter are vital to the results proven for problems on a general domain. The physically relevant problem has mixed Dirichlet–Neumann boundary conditions so the final aim is to deal with this more complicated example, making use of the theory built here.

Chapter 3

Strain-limiting problem with Dirichlet boundary conditions

In this chapter, we extend the work of Chapter 2 and consider the strain-limiting problem on a general domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions. We prove an existence result for homogeneous Dirichlet boundary conditions here, but the case of inhomogeneous Dirichlet boundary conditions can be found in [20]. Indeed, the proof here can be easily extended to inhomogeneous boundary conditions but complicates the notation and data restrictions. Thus we tackle only the homogeneous problem here.

As discussed previously, the periodic setting has potential application in strain-limiting problems where there is a concentrated load sufficiently far away from the boundary. This is so that there is negligible effect of the load on the boundary. However, the final aim is to consider strain-limiting problems in the context of fracture problems. Thus we need to investigate strain-limiting problems in a non-periodic setting. The problem with mixed Dirichlet–Neumann boundary conditions experiences additional issues, which will be discussed in Chapter 4. Instead, we start with the simpler case of fully Dirichlet boundary conditions. We note that if part of the boundary has a homogeneous Dirichlet boundary condition, this refers to the physical constraint that this part of the boundary is clamped which is always the case in practice.

Throughout this chapter, we fix $\Omega \subset \mathbb{R}^d$, an open, bounded domain with Lipschitz boundary where $d \geq 2$. The assumption of a Lipschitz domain is to ensure that we can apply Korn's inequality (Theorem 1.7) and, in particular, the Korn–Poincaré inequality (Theorem 1.8). The boundary is Lipschitz and hence continuous so we have the following density result concerning the choice of test functions [16].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with continuous boundary. Given a $\boldsymbol{w} \in W_0^{1,2}(\Omega)^d$ such that $\boldsymbol{\varepsilon}(\boldsymbol{w}) \in L^\infty(\Omega)^{d\times d}$, there exists an approximation sequence $(\boldsymbol{w}^n)_n \subset C_c^\infty(\Omega)^d$ such that $\boldsymbol{w}^n \to \boldsymbol{w}$ strongly in $L^2(\Omega)^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{w}^n) \overset{*}{\rightharpoonup} \boldsymbol{\varepsilon}(\boldsymbol{w})$ weakly-* in $L^\infty(\Omega)^{d\times d}$ as $n \to \infty$.*

Consider the following strain-limiting IBVP. Given a finite final time horizon T, define the space-time domain $Q := (0,T) \times \Omega$. Consider the problem of finding an unknown couple

$(\boldsymbol{u}, \mathbf{T}) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{3.1a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{in } Q, \tag{3.1b}$$

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0, \cdot) = \boldsymbol{u}_1 \qquad \text{in } \Omega, \tag{3.1c}$$

$$\boldsymbol{u}(t, \cdot) = \mathbf{0} \qquad \text{on } (0, T] \times \partial\Omega, \tag{3.1d}$$

for fixed constants $a > 0$, $\alpha \geq 0$, initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ and an external body force $\boldsymbol{f}$. The notion of a weak solution is defined as follows.

Definition 3.2. *Given data $\boldsymbol{u}_0 \in W^{1,2}_0(\Omega)^d$, $\boldsymbol{u}_1 \in L^2(\Omega)^d$ and $\boldsymbol{f} \in L^2(0, T; W^{-1,2}(\Omega)^d)$, we say that the couple $(\boldsymbol{u}, \mathbf{T})$ with regularity*

- $\boldsymbol{u} \in W^{2,2}(0, T; L^2(\Omega)^d) \cap W^{1,2}(0, T; W^{1,2}_0(\Omega)^d)$,
- $\boldsymbol{u}_t + \alpha \boldsymbol{u} \in L^p(0, T; W^{1,p}_0(\Omega)^d)$ *for every* $p \in [1, \infty)$ *with* $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) \in L^\infty(Q)^{d\times d}$,
- $\mathbf{T} \in L^1(Q)^{d\times d}$,

is a weak solution of the strain-limiting problem (3.1) *if*

$$\int_\Omega \boldsymbol{u}_{tt} \cdot \boldsymbol{v} + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle, \tag{3.2}$$

for every $\boldsymbol{v} \in W^{1,2}_0(\Omega)^d$ such that $\boldsymbol{\varepsilon}(\boldsymbol{v}) \in L^\infty(\Omega)^{d\times d}$ and a.e. $t \in (0, T)$, with constitutive relation (3.1b) *holding pointwise a.e. in Q, and initial conditions*

$$\lim_{t\to 0+} \left(\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_2\right) = 0. \tag{3.3}$$

As in Chapter 2, to prove the existence of a weak solution to this problem, we cannot consider a Galerkin approximation directly. We must first introduce a regularised problem and replace the function F in the constitutive relation by a function F_n that is a bijection from $\mathbb{R}^{d\times d}$ to itself. Thus we obtain a problem that can be formulated in terms of the displacement $\boldsymbol{u}$ only. Consider the following:

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{3.4a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F_n(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{\mathbf{T}}{n(1 + |\mathbf{T}|^{1-\frac{1}{n}})} \qquad \text{in } Q, \tag{3.4b}$$

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0, \cdot) = \boldsymbol{u}_1 \qquad \text{in } \Omega, \tag{3.4c}$$

$$\boldsymbol{u}(t, \cdot) = \mathbf{0} \qquad \text{on } (0, T] \times \partial\Omega. \tag{3.4d}$$

The notion of a weak solution of (3.4) is defined in an analogous way to Definition 3.2, but with $\boldsymbol{u}_t + \alpha \boldsymbol{u} \in L^{n+1}(0, T; W^{1,n+1}_0(\Omega)^d)$ and $\mathbf{T} \in L^{1+\frac{1}{n}}(Q)^{d\times d}$. The test functions in the weak form of (3.4a) are from $W^{1,n+1}_0(\Omega)^d$.

To show the existence of a weak solution to the regularised problem, we use a Galerkin approximation. An appropriate finite dimensional function space with which we approximate $W_0^{1,2}(\Omega)^d$ is defined as follows. Let $(\phi_i)_{i\geq 1}$ be a basis of eigenfunctions from $W_0^{d+1,2}(\Omega)$ corresponding to the problem

$$\lambda(\phi, \psi) = a(\phi, \psi) := (\phi, \psi) + \sum_{|\beta|=d+1} (\partial^\beta \phi, \partial^\beta \psi) \quad \forall \psi \in W_0^{d+1,2}(\Omega),$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, $a(\cdot, \cdot)$ is the given bilinear form and $\lambda \in \mathbb{R}$ is a scalar corresponding to an eigenvalue of $a(\cdot, \cdot)$. The basis is chosen so that it is orthogonal in $W_0^{d+1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$. We refer to Lemma 5.1 in [47] and the references therein for the details on this construction. Letting $\boldsymbol{e}_j$ be the j-th standard basis vector in $\mathbb{R}^d$, it follows that $(\phi_i \boldsymbol{e}_j)_{i,j=1}^{\infty,d}$ is a basis for $W_0^{d+1,2}(\Omega)^d$. We define V_m to be $\text{span}\{\phi_1, \ldots, \phi_m\}^d$. By the Sobolev embedding theorem, we have $W_0^{d+1,2}(\Omega) \subset C^1(\overline{\Omega})$. Hence, for every $p \in [1, \infty)$, the linear span of the eigenfunctions $(\phi_i \boldsymbol{e}_j)_{i,j=1}^{\infty,d}$ is dense in $W_0^{1,p}(\Omega)^d$.

Let P^m denote the orthogonal projection from $L^2(\Omega)$ onto $\text{span}\{\phi_1, \ldots, \phi_m\}$. In an abuse of notation, we also denote by P^m the function that applies P^m component-wise on $L^2(\Omega)^d$ and $L^2(\Omega)^{d\times d}$. By standard results on orthogonal projection operators in Hilbert spaces, we have that

$$\|P^m \boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_2 \quad \forall \boldsymbol{v} \in L^2(\Omega)^d,$$

and, by construction, there exists a constant C independent of m such that, for every $\boldsymbol{v} \in W_0^{d+1,2}(\Omega)^d$,

$$\|P^m \boldsymbol{v}\|_{d+1,2} \leq C\|\boldsymbol{v}\|_{d+1,2}. \tag{3.5}$$

To see why (3.5) is true, note that it is sufficient to prove the result for functions from $W_0^{d+1,2}(\Omega)$. Let $W_m = \text{span}\{\phi_1, \ldots, \phi_m\}$. For every i, $j \in \mathbb{N}$, we have $a(\phi_i, \phi_j) = \delta_{ij}\lambda_i$. Clearly W_m is an orthonormal subspace of $W_0^{d+1,2}(\Omega)$. For $w \in W_0^{d+1,2}(\Omega)$, we can write $w = P^m w + w^\perp$ where $w^\perp \in \overline{\text{span}\{\phi_{m+1}, \ldots\}}^{\|\cdot\|_2}$. Since w, $P^m w \in W_0^{d+1,2}(\Omega)$, then $w^\perp \in W_0^{d+1,2}(\Omega)$ and so $w^\perp \in \overline{\text{span}\{\phi_{m+1}, \ldots\}}^{\|\cdot\|_{d+1,2}}$. By orthogonality we have that

$$a(w, P^m w) = a(P^m w, P^m w). \tag{3.6}$$

Let $\|\cdot\|_a$ be the norm on $W_0^{d+1,2}(\Omega)$ that is induced by the inner product $a(\cdot, \cdot)$. From (3.6), we deduce that

$$\|P^m w\|_a \leq \|w\|_a.$$

However, $\|\cdot\|_a$ is equivalent to the usual norm $\|\cdot\|_{d+1,2}$ on $W^{d+1,2}(\Omega)$. The existence of a constant C such that (3.5) holds now follows from combining these two facts.

It follows from standard interpolation theory [7] that, for every $k \in \{1, \ldots, d\}$, there exists a constant C such that, for every $\boldsymbol{v} \in W_0^{k,2}(\Omega)^d$,

$$\|P^m \boldsymbol{v}\|_{k,2} \leq C\|\boldsymbol{v}\|_{k,2}.$$

Since P^m is a projection operator, for every $\boldsymbol{v} \in W_0^{d+1,2}(\Omega)^d$, we have $P^m \boldsymbol{v} \to \boldsymbol{v}$ strongly in $W^{d+1,2}(\Omega)^d$. It follows that, for every $k \in \{0, \dots, d\}$ and every $\boldsymbol{v} \in W_0^{k,2}(\Omega)^d$, we have the approximation property

$$P^m \boldsymbol{v} \to \boldsymbol{v} \quad \text{strongly in } W^{k,2}(\Omega)^d \text{ as } m \to \infty.$$

With these results in mind, we focus on proving the following result concerning the existence of solutions to the regularised problem.

Theorem 3.3. *Let $a > 0$, $\alpha \geq 0$ and $n \in \mathbb{N}$ be fixed constants. Let the initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in L^2(\Omega)^d$ and body force $\boldsymbol{f} \in L^2(Q)^{d \times d}$ be given such that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_0^{k+1,2}(\Omega)^d$ for some $k > \frac{d}{2}$ and we have the safety strain condition*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)\|_\infty = C_* < 1.$$

There exists a unique weak solution $(\boldsymbol{u}, \mathbf{T})$ of the regularised problem (3.4) *with approximation parameter n. Furthermore, there exists a constant C, independent of n, such that*

$$\begin{aligned}\sup_{t \in [0,T]} \left[\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{u}_t(t)\|_2^2\right] + \int_Q |\boldsymbol{u}_{tt}|^2 + |\mathbf{T}| + \frac{|\mathbf{T}|^{1+\frac{1}{n}}}{n} \,\mathrm{d}x\,\mathrm{d}t + \sup_{t \in [0,T]} \int_\Omega |\mathbf{T}(t)|^{1-a} \chi_{\{|\mathbf{T}(t)| \geq 1\}} \\ + \sup_{t \in [0,T]} \int_\Omega \frac{|\mathbf{T}(t)|^{1+\frac{1}{n}}}{n(n+1)} \,\mathrm{d}x \leq C \left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 \,\mathrm{d}t\right).\end{aligned} \tag{3.7}$$

Proof. For simplicity of notation, we do not explicitly include the dependence on n of the regularised solution or Galerkin approximation in this proof. However, we do indicate the dependence of constants on n where appropriate. The Galerkin approximation from V_m of the regularised problem that we use to prove the existence of a solution is formulated as follows. We look for a function $\boldsymbol{u}^m$ of the form

$$\boldsymbol{u}^m(t,x) = \sum_{i=1}^m \sum_{j=1}^d \beta_{ij}^m(t) \phi_i(x) \boldsymbol{e}_j,$$

such that, for every test function $\boldsymbol{v} \in V_m$ and a.e. $t \in (0,T)$, we have

$$\int_\Omega \boldsymbol{u}_{tt}^m(t) \cdot \boldsymbol{v} + \mathbf{T}^m(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \,\mathrm{d}x, \tag{3.8}$$

where the approximation stress tensor $\mathbf{T}^m$ is defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) = F_n(\mathbf{T}^m) \quad \text{pointwise in } Q, \tag{3.9}$$

with initial conditions

$$\boldsymbol{u}^m(0,\cdot) = P^m \boldsymbol{u}_0, \quad \boldsymbol{u}_t^m(0,\cdot) = P^m \boldsymbol{u}_1.$$

We note that (3.9) is well-defined because F_n is a bijection from $\mathbb{R}^{d \times d}_{sym}$ to itself. We reason as in the proof of Theorem 2.4 to transform the problem into a system of first order ODEs. We apply Caratheodory theory to yield the existence of a solution to the Galerkin approximation from V_m on a possibly short time interval $[0, T_*)$ where $T_* \leq T$. To ensure that the solution can be defined on the whole of $[0,T]$, we show that the coefficients $\boldsymbol{\beta}^m$ and the time derivatives $\boldsymbol{\beta}^m_t$ do not blow up as $t \to T_*$. Testing in (3.8) against $\boldsymbol{v} = \boldsymbol{u}^m_t + \alpha \boldsymbol{u}^m$, we see that

$$\int_\Omega \frac{\partial}{\partial t}\left(\frac{|\boldsymbol{u}^m_t|^2}{2} + \alpha \boldsymbol{u}^m_t \cdot \boldsymbol{u}^m\right) - \alpha |\boldsymbol{u}^m_t|^2 + \mathbf{T}^m \cdot F_n(\mathbf{T}^m)\,\mathrm{d}x \leq C\|\boldsymbol{f}\|_2 \|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t + \alpha \boldsymbol{u}^m)\|_2, \quad (3.10)$$

applying the Korn–Poincaré inequality. Next, we write

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t + \alpha \boldsymbol{u}^m)| \leq 1 + \frac{|\mathbf{T}^m|}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})},$$

from which it follows that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t + \alpha \boldsymbol{u}^m)\|_2 \leq C\left(1 + \left(\int_\Omega \frac{|\mathbf{T}^m|^2}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\right).$$

We integrate (3.10) over $(0,t)$ and use Young's inequality to deduce that

$$\begin{aligned}
&\|\boldsymbol{u}^m_t(t)\|_2^2 + \int_0^t \int_\Omega \frac{|\mathbf{T}^m|^2}{(1+|\mathbf{T}^m|^a)^{\frac{1}{a}}} + \frac{|\mathbf{T}^m|^2}{n(1+|\mathbf{T}^m|^{1-\frac{1}{n}})}\,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^t \int_\Omega |\boldsymbol{u}^m_t|^2 + |\boldsymbol{f}|^2\,\mathrm{d}x\,\mathrm{d}s + \|\boldsymbol{u}^m(t)\|_2^2\right) \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_0^t \int_\Omega |\boldsymbol{u}^m_t|^2 + |\boldsymbol{f}|^2\,\mathrm{d}x\,\mathrm{d}s\right),
\end{aligned}$$

arguing as in the proof of Theorem 2.4 to absorb the $\|\boldsymbol{u}^m(t)\|_2^2$ term. By Gronwall's inequality, it follows that

$$\begin{aligned}
&\sup_{t\in[0,T_*)} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T_*)} \|\boldsymbol{u}^m_t(t)\|_2^2 + \int_0^{T_*}\int_\Omega |\mathbf{T}^m| + \frac{|\mathbf{T}^m|^{1+\frac{1}{n}}}{n}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \|\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t + \alpha \boldsymbol{u}^m)\|_{L^{n+1}(Q_*)} \leq C\left(1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_2^2 + \int_0^T \|\boldsymbol{f}\|_2^2\,\mathrm{d}t\right),
\end{aligned} \quad (3.11)$$

for any $T_* \leq T$, where $Q_* = (0, T_*) \times \Omega$, for a constant $C = C(T, a, \alpha, d)$ that is independent of m and n. Recalling that the basis is orthonormal with respect to the inner product in $L^2(\Omega)^d$, it follows that

$$\max_{i,j} \sup_{t\in[0,T_*)} |\beta^m_{ij}(t)|^2 + \max_{i,j} \sup_{t\in[0,T_*)} |(\beta^m_{ij})_t(t)|^2 \leq \sup_{t\in[0,T_*)} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T_*)} \|\boldsymbol{u}^m_t(t)\|_2^2 \leq C.$$

In particular, $\boldsymbol{\beta}^m$ and $\boldsymbol{\beta}^m_t$ do not blow up in any finite time interval and so we have existence of a solution to the Galerkin approximation from V_m on the whole of $[0,T]$. Furthermore, we can replace T_* by T in (3.11).

Next, test in (3.8) against $\boldsymbol{u}^m_{tt} + \alpha \boldsymbol{u}^m_t$ and reason as in Lemma 2.6 to obtain

$$\begin{aligned}
&\int_Q |\boldsymbol{u}^m_{tt}|^2\,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^m(t)|^{1-a} \chi_{\{|\mathbf{T}^m(t)|\geq 1\}}\,\mathrm{d}x \\
&+ \sup_{t\in[0,T]} \int_\Omega \frac{|\mathbf{T}^m(t)|^{1+\frac{1}{n}}}{n(n+1)}\,\mathrm{d}x \leq C\left(1 + \|\boldsymbol{u}_1\|_2^2 + \int_Q |\boldsymbol{f}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\mathbf{T}^m(0)|^2\,\mathrm{d}x\right).
\end{aligned}$$

Under the assumptions on $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0$, we know that $P^m(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0) \to \boldsymbol{u}_1 + \alpha \boldsymbol{u}_0$ strongly in $W_0^{k,2}(\Omega)^d$ and thus the convergence holds in $C^1(\overline{\Omega})^d$. Hence, there exists $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$,

$$\|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\|_\infty \leq \frac{1 + C_*}{2} =: C_1 < 1.$$

We conclude that there exists a constant C, independent of n and m, such that, for every $m \geq m_0$,

$$\int_Q |\boldsymbol{u}_{tt}^m|^2 \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^m(t)|^{1-a} \chi_{\{|\mathbf{T}^m(t)|\geq 1\}} \,\mathrm{d}x + \sup_{t\in[0,T]} \int_\Omega \frac{|\mathbf{T}^m(t)|^{1+\frac{1}{n}}}{n(n+1)} \,\mathrm{d}x$$
$$\leq C \left(1 + \|\boldsymbol{u}_1\|_2^2 + \int_Q |\boldsymbol{f}|^2 \,\mathrm{d}x\,\mathrm{d}t\right).$$

With this, (3.11) and using the proof of Theorem 2.7, we deduce that the following convergence results hold up to a subsequence in m, not relabelled:

- $\boldsymbol{u}^m \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $W^{1,\infty}(0,T; L^2(\Omega)^d)$ and weakly in $W^{2,2}(0,T; L^2(\Omega)^d)$;
- $\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m \rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u}$ weakly in $L^{n+1}(0,T; W_0^{1,n+1}(\Omega)^d)$;
- $\mathbf{T}^m \rightharpoonup \mathbf{T}$ weakly in $L^{1+\frac{1}{n}}(Q)^{d\times d}$ and strongly in $L^1(Q)^{d\times d}$.

We know that $P^m \boldsymbol{v} \to \boldsymbol{v}$ strongly in $C^1(\overline{\Omega})^d$ for every $\boldsymbol{v} \in W_0^{d+1,2}(\Omega)^d$ and so we deduce from (3.8) and standard arguments that

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{v} + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \,\mathrm{d}x, \tag{3.12}$$

for every $\boldsymbol{v} \in W_0^{1,n+1}(\Omega)^d$ and a.e. $t \in (0,T)$. Using the strong convergence of $(\mathbf{T}^m)_m$, we deduce that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) = F_n(\mathbf{T}^m) \to F_n(\mathbf{T}) \quad \text{pointwise a.e. in } Q.$$

Pointwise and weak limits in $L^2(Q)$ coincide. Thus the constitutive relation must hold pointwise in Q. The initial conditions hold by standard arguments. Hence, we have the existence of a weak solution to the regularised problem (3.4). To prove the uniqueness of such solutions, we notice that $\boldsymbol{u}_t + \alpha \boldsymbol{u} \in L^{n+1}(0,T; W_0^{n+1}(\Omega)^d)$ and argue as in the proof of Theorem 2.7. By Fatou's lemma and weak lower semi-continuity of norms, we deduce that (3.7) holds. □

3.1 Higher regularity estimates

In this section, we prove results analogous to those from Section 2.2. We focus on obtaining higher regularity estimates for the solution of the regularised problem. As in Theorem 3.3, we denote the solution of the regularised problem with approximation parameter n by $(\boldsymbol{u}, \mathbf{T})$. The major difference here compared to the periodic problem is that we cannot use $\nabla \cdot \nabla(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)$ as a test function in the Galerkin problem. Instead, we use difference quotients in the spatial variable at the level of the regularised solution in order to ensure that we are working with valid test functions. We will use Lemma 1.10 regarding the monotonicity of F as well as the Nikol'skiĭ embedding, Lemma 1.11, to obtain a higher integrability estimate, provided that a is small.

Lemma 3.4. *Suppose that the assumptions of Theorem 3.3 hold. Assume additionally that the data satisfies $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_0^{1,2}(\Omega)^d$ and $\boldsymbol{f} \in L^2(0,T;W_{loc}^{1,2}(\Omega)^d)$, and we have that $a \in (0,\frac{1}{6})$ and $d=3$. There exists $\delta = \delta(a) > 0$ such that, for every compact subset $\Omega_0 \subset \Omega$, there exists a constant $C = C(\Omega_0)$, independent of n, such that*

$$\int_0^T \int_{\Omega_0} |\mathbf{T}|^{1+\delta} \,\mathrm{d}x\,\mathrm{d}t \leq C,$$

for n sufficiently large.

Proof. Fix $h_0 > 0$ and let $h \in (0,h_0)$. Let $\boldsymbol{v} \in W_0^{1,n+1}(\Omega)^d$ be such that $\boldsymbol{v}(x) = \mathbf{0}$ for every $x \in \Omega$ such that $d(x,\partial\Omega) < h_0$. Then we have

$$\int_\Omega \Delta_i^h \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v} + \Delta_i^h \mathbf{T}(t)\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \Delta_i^h \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x, \tag{3.13}$$

for a.e. $t \in (0,T)$. Let $\tau \in C_c^\infty(\Omega)$ be a cut-off function such that

$$\tau(x) = \begin{cases} 1 & \text{if } d(x,\partial\Omega) \geq 2h_0, \\ 0 & \text{if } d(x,\partial\Omega) \leq h_0. \end{cases}$$

Then $\boldsymbol{v} = \tau(\boldsymbol{u}_t + \alpha\boldsymbol{u})(t)$ is a valid test function in (3.13) and we get

$$\begin{aligned}\int_\Omega \tau\Delta_i^h\boldsymbol{f}\cdot\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\,\mathrm{d}x &= \int_\Omega \tau\Delta_i^h\boldsymbol{u}_{tt}\cdot\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u}) + \Delta_i^h\mathbf{T}\cdot\boldsymbol{\varepsilon}\left(\tau(\boldsymbol{u}_t+\alpha\boldsymbol{u})\right)\,\mathrm{d}x \\ = \int_\Omega \frac{\partial}{\partial t}\left(\frac{\tau|\Delta_i^h\boldsymbol{u}_t|^2}{2} + \alpha\tau\Delta_i^h\boldsymbol{u}_t\cdot\Delta_i^h\boldsymbol{u}\right) &- \alpha\tau|\Delta_i^h\boldsymbol{u}|^2 \\ + \tau\Delta_i^h\mathbf{T}\cdot\Delta_i^h\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u}) &+ \Delta_i^h\mathbf{T}\cdot\Big(\nabla\tau\otimes\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\Big)\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

Using the constitutive relation and Lemma 1.10, there exists a constant C depending only on a such that

$$\int_\Omega \tau\Delta_i^h\mathbf{T}\cdot\Delta_i^h\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})\,\mathrm{d}x \geq C\int_\Omega \tau\left|\Delta_i^h\left((1+|\mathbf{T}|)^{\frac{1-a}{2}}\right)\right|^2\,\mathrm{d}x.$$

Recalling (3.7), we can bound $\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})$ in $L^\infty(0,T;L^{n+1}(\Omega)^{d\times d})$ independent of n by the following reasoning. From the constitutive relation, we have that

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})| \leq 1 + \frac{|\mathbf{T}|^{\frac{1}{n}}}{n},$$

and so, for a.e. $t \in (0,T)$,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})(t)\|_{n+1} \leq C(\Omega) + \frac{1}{n}\left(\int_\Omega |\mathbf{T}(t)|^{1+\frac{1}{n}}\,\mathrm{d}x\right)^{\frac{1}{n+1}} \leq C,$$

where C is a constant that is independent of n. This is a slightly stronger result than the previous bound in $L^{n+1}(Q)$. We note that in the static setting (cf. [5]) we do not need such a bound.

Next, we mimic an argument from [16], adapted to this time-dependent setting. Fix $\beta \in (0,1)$, to be determined later. We note that $2\beta < d = 3$. This is required in order to justify an application of the Nikol'skiĭ embedding (Lemma 1.11). We define $\tilde{\beta} := \beta - \frac{d}{n+1}$ and assume that n is sufficiently large so that $\tilde{\beta} > 0$. Returning to (3.14) and dividing by $h^{\tilde{\beta}}$, we have that

$$\begin{aligned}\frac{1}{h^{\tilde{\beta}}}\int_\Omega \Delta_i^h \mathbf{T}\cdot\Big(\nabla\tau\otimes\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\Big)\,\mathrm{d}x &\le C\|\mathbf{T}(t)\|_1\frac{\||\nabla\tau||\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})|\|_\infty}{h^{\tilde{\beta}}}\\ &\le C(\tau)\|\mathbf{T}(t)\|_1\|\boldsymbol{u}_t+\alpha\boldsymbol{u}\|_{C^{0,\tilde{\beta}}(\overline{\Omega}_1)}\\ &\le C(\tau)\|\mathbf{T}(t)\|_1\|\boldsymbol{u}_t+\alpha\boldsymbol{u}\|_{1,\frac{d}{1-\tilde{\beta}}}\\ &\le C(h_0)\|\mathbf{T}(t)\|_1\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})\|_{\frac{d}{1-\tilde{\beta}}},\end{aligned}\tag{3.15}$$

using Morrey's inequality followed by the Korn–Poincaré inequality (Theorem 1.7). By definition of $\tilde{\beta}$, we have that

$$d \le \frac{d}{1-\tilde{\beta}} \le \frac{d}{1-\beta} \le n+1,$$

for n sufficiently large. The set Ω_1 denotes $\operatorname{supp}(\tau)$, a compact subset of Ω. The constant C in (3.15) is independent of n. The application of Morrey's inequality is independent of n because of the upper bound on $\frac{d}{1-\tilde{\beta}}$ by $\frac{d}{1-\beta}$. As a result of these bounds, we also have that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})(t)\|_{\frac{d}{1-\tilde{\beta}}} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})(t)\|_{n+1} \le C\sup_{s\in[0,T]}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})(s)\|_{n+1} \le C,$$

for a.e. $t \in (0,T)$. It follows that

$$\frac{1}{h^{\tilde{\beta}}}\int_\Omega \Delta_i^h \mathbf{T}\cdot\Big(\nabla\tau\otimes\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\Big)\,\mathrm{d}x \le C(h_0)\|\mathbf{T}^n(t)\|_1,$$

for a.e. $t \in (0,T)$. Returning to (3.14), we get

$$\begin{aligned}&\frac{1}{h^{\tilde{\beta}}}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\tau|\Delta_i^h\boldsymbol{u}_t|^2}{2}+\alpha\tau\Delta_i^h\boldsymbol{u}_t\cdot\Delta_i^h\boldsymbol{u}\right)-\alpha\tau|\Delta_i^h\boldsymbol{u}|^2+C\tau\left|\Delta_i^h\left((1+|\mathbf{T}|)^{\frac{1-a}{2}}\right)\right|^2\,\mathrm{d}x\\ &\le C(h_0)\|\mathbf{T}(t)\|_1+\frac{1}{h^{\tilde{\beta}}}\int_\Omega\tau\Delta_i^h\boldsymbol{f}\cdot\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\,\mathrm{d}x\\ &\le C(h_0)\|\mathbf{T}(t)\|_1+\frac{\|\Delta_i^h\boldsymbol{f}\|_{L^2(\Omega_1)}^2}{2h^{\tilde{\beta}}}+\frac{\|\tau\Delta_i^h(\boldsymbol{u}_t+\alpha\boldsymbol{u})\|_2^2}{2h^{\tilde{\beta}}}.\end{aligned}$$

Integrating over $(0,t)$ and rearranging the terms involving $\Delta_i^h\boldsymbol{u}$ and $\Delta_i^h\boldsymbol{u}_t$, as in the proof of Theorem 3.3, we see that

$$\begin{aligned}&\int_\Omega\tau\frac{|\Delta_i^h\boldsymbol{u}_t(t)|^2}{2h^{\tilde{\beta}}}\,\mathrm{d}x+\int_\Omega\tau\frac{|\Delta_i^h\boldsymbol{u}(t)|^2}{2h^{\tilde{\beta}}}\,\mathrm{d}x+\frac{C}{h^{\tilde{\beta}}}\int_0^t\int_\Omega\tau\left|\Delta_i^h\left((1+|\mathbf{T}|)^{\frac{1-a}{2}}\right)\right|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\le C\Big(\int_0^t\int_\Omega|\mathbf{T}|+\tau\frac{|\Delta_i^h\boldsymbol{u}_t|^2}{h^{\tilde{\beta}}}+\tau\frac{|\Delta_i^h\boldsymbol{u}|^2}{h^{\tilde{\beta}}}+\tau\frac{|\Delta_i^h\boldsymbol{f}|^2}{h^{\tilde{\beta}}}\,\mathrm{d}x\,\mathrm{d}s+\int_\Omega\tau\frac{|\Delta_i^h\boldsymbol{u}_t(0)|^2}{h^{\tilde{\beta}}}\\ &\quad+\tau\frac{|\Delta_i^h\boldsymbol{u}(0)|^2}{h^{\tilde{\beta}}}\,\mathrm{d}x\Big)+\int_\Omega\tau\frac{|\Delta_i^h\boldsymbol{u}_t(t)|^2}{4h^{\tilde{\beta}}}\,\mathrm{d}x.\end{aligned}$$

We absorb the final term on the right-hand side into the left. Next we apply Gronwall's inequality. Ignoring the displacement terms on the left-hand side, we deduce that

$$\int_Q \frac{\tau}{h^{\tilde{\beta}}} \left| \Delta_i^h \left((1+|\mathbf{T}|)^{\frac{1-a}{2}} \right) \right|^2 \le C \left(\int_Q |\mathbf{T}| + \int_0^T \int_{\Omega_1} |\nabla \boldsymbol{f}|^2 + \|\nabla \boldsymbol{u}_0\|^2_{L^2(\Omega_1)} + \|\nabla \boldsymbol{u}_1\|^2_{L^2(\Omega_1)} \right)$$
$$\le C \left(1 + \|\boldsymbol{u}_0\|^2_{1,2} + \|\boldsymbol{u}_1\|^2_{1,2} + \|\boldsymbol{f}\|^2_{L^2(Q)} + \|\nabla \boldsymbol{f}\|^2_{L^2((0,T)\times\Omega_1)} \right)$$

where C is independent of n, using standard properties of difference quotients (see [45], for example). Using the uniform L^1-bound on $\mathbf{T}$, we know that $(1+|\mathbf{T}|)^{\frac{1-a}{2}}$ is an element of $L^2(Q)$. It follows that $\mathbf{T} \in \mathcal{N}^{\frac{\tilde{\beta}}{2}}_{loc}(\Omega)$, defining the local Nikol'skiĭ spaces analogously to those on the whole of $\mathbb{R}^d$. Applying the embedding result from Lemma 1.11, we see that, for any $\tilde{\delta} > 0$, sufficiently small,

$$C \ge \int_0^T \|(1+|\mathbf{T}|)^{\frac{1-a}{2}}\|^2_{\mathcal{N}^{\frac{\tilde{\beta}}{2},2}(\Omega_1)} \,\mathrm{d}t \ge c_{\tilde{\delta}} \int_0^T \|(1+|\mathbf{T}|)^{\frac{1-a}{2}}\|^2_{L^{\frac{2d}{d-\tilde{\beta}}-\tilde{\delta}}(\Omega_0)} \,\mathrm{d}t, \tag{3.16}$$

where C, $c_{\tilde{\delta}}$ are positive constants that may depend on h_0 and $\tilde{\delta}$, respectively, but are independent of n. Here, Ω_0 and Ω_1 are open subsets of Ω such that Ω_0 is compactly contained in Ω_1 and Ω_1 is compactly contained in Ω.

At this point in [16], the argument was essentially complete. There was no integral in time so the exponent on the $L^{\frac{2d}{d-\tilde{\beta}}-\tilde{\delta}}(\Omega_0)$-norm did not matter. However, here, we have an exponent $1-a$, followed by an integral in time. To see why this is the case, we rewrite (3.16) as

$$\int_0^T \|\mathbf{T}\|^{1-a}_{L^{\frac{d(1-a)}{d-\tilde{\beta}}-\tilde{\delta}}(\Omega_0)} \,\mathrm{d}t \le C(h_0, \tilde{\delta}), \tag{3.17}$$

relabelling $\tilde{\delta}$ but still choosing it sufficiently small. The factor of $1-a$ complicates matters because the Bochner space $L^{1-a}(0,T;X)$ is not a Banach space for $a \in (0,1)$. Hence we must carry out further manipulations in order to deduce useful information from (3.17).

For n sufficiently large, we have

$$\frac{d(1-a)}{d-\tilde{\beta}} = \frac{d(1-a)}{d-\beta+\frac{d}{n+1}} > 1 + \frac{1}{2}\frac{\beta - ad}{d-\beta} = 1 + \frac{p}{2},$$

where $p := \frac{\beta-ad}{d-\beta}$. Under the assumptions on d and a, we have $ad < 1$. From now on, choose β from the interval $(ad, 1)$ so that $\beta - ad > 0$. Clearly $\beta < d$. Hence we must have that $p > 0$. Next, notice that $\frac{d(1-a)}{d-\beta+\frac{d}{n+1}}$ is an increasing sequence in n. We fix $\tilde{\delta} > 0$ such that

$$\frac{d(1-a)}{d-\beta} - \tilde{\delta} > 1 + \frac{p}{2}.$$

Then there exists $n_0 \in \mathbb{N}$ sufficiently large such that, for every $n \ge n_0$,

$$\frac{d(1-a)}{d-\beta+\frac{d}{n+1}} - \tilde{\delta} > 1 + \frac{p}{2}. \tag{3.18}$$

From now on, we assume that n is sufficiently large so that (3.18) is true. It follows that

$$\|\mathbf{T}(t)\|_{L^{1+\frac{p}{2}}(\Omega_0)} \leq C\|\mathbf{T}(t)\|_{L^{\frac{d(1-a)}{d-\beta}-\tilde{\delta}}(\Omega_0)},$$

where $C = C(\Omega_0, d, a)$, $\beta = \beta(a, d)$ and $\tilde{\delta} = \tilde{\delta}(a, d)$ are positive constants, independent of n. From (3.17), we have that

$$\int_0^T \|\mathbf{T}(t)\|^{1-a}_{L^{1+\frac{p}{2}}(\Omega_0)} \,\mathrm{d}t \leq C.$$

We use this bound with (3.7) to show that $\mathbf{T} \in L^{1+\delta}(0, T; L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for a $\delta = \delta(a) > 0$, with bound in this space independent of n. We proceed in a similar way to the periodic case. However, the admissible values of a are smaller due to the use of the Nikol'skiĭ embedding, rather than the stronger Sobolev embedding. Let $q \in (1, \infty)$, to be determined later. By Hölder's inequality, we have

$$\begin{aligned}
\int_0^T \int_{\Omega_0} |\mathbf{T}|^{1+\frac{a}{q'}} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\Omega_0} |\mathbf{T}|^{\frac{1}{q'}-\frac{a}{q'}} |\mathbf{T}|^{\frac{1}{q}+\frac{2a}{q'}} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \left(\int_{\Omega_0} |\mathbf{T}|^{1-a}\,\mathrm{d}x\right)^{\frac{1}{q'}} \left(\int_{\Omega_0} |\mathbf{T}|^{1+2a(q-1)}\,\mathrm{d}x\right)^{\frac{1}{q}} \mathrm{d}t \\
&\leq C\int_0^T \left(\int_{\Omega_0} |\mathbf{T}|^{1+2a(q-1)}\,\mathrm{d}x\right)^{\frac{1}{q}} \mathrm{d}t,
\end{aligned} \tag{3.19}$$

using (3.7) in the transition to the final line. Suppose $q \in (1, \infty)$ can be chosen so that

$$\frac{1}{q} \leq \frac{1-a}{1+\frac{p}{2}}, \quad 1 + 2a(q-1) \leq 1 + \frac{p}{2}. \tag{3.20}$$

Since $d = 3$, we notice that $a < \frac{5}{27}$ is equivalent to $2d + 9ad < 11$. Then, $\beta = \beta(a)$ can be chosen sufficiently large from the interval $(ad, 1)$ so that $2d + 9ad < 11\beta$. Basic algebra shows that this is equivalent to $\frac{p}{2} = \frac{\beta - ad}{2(d-\beta)} > \frac{1}{9}$. Next, we set $q = \frac{1+\frac{p}{2}}{1-a} > 1 + \frac{p}{2} > 1$. In this case, the first statement of (3.20) trivially holds. To see that the second statement holds for this choice of q, the required statement is equivalent to $2a\left(\frac{a+\frac{p}{2}}{1-a}\right) < \frac{p}{2}$, which is equivalent to $4a^2 + 2ap < p - ap$. Thus, it is enough to show that, for the given choice of a, we have $4a^2 < p(1-3a)$. Because $\frac{p}{2} > \frac{1}{9}$, a sufficient condition is $2a^2 < \frac{1}{9}(1-3a)$. However, $18a^2 + 3a - 1$ is strictly negative when $a \in (0, \frac{1}{6})$. In particular, (3.20) can be satisfied provided that $a \in (0, \frac{1}{6})$, noticing that $\frac{1}{6} < \frac{5}{27}$.

For this choice of q and using (3.20), we see that

$$\begin{aligned}
\int_0^T \left(\int_{\Omega_0} |\mathbf{T}|^{1+2a(q-1)}\,\mathrm{d}x\right)^{\frac{1}{q}} \mathrm{d}t &\leq C\left(\int_0^T \left(\int_{\Omega_0} |\mathbf{T}|^{1+\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{1}{q}} \mathrm{d}t + 1\right) \\
&\leq C\left(\int_0^T \left(\int_{\Omega_0} |\mathbf{T}|^{1+\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{1-a}{1+\frac{p}{2}}} \mathrm{d}t + 1\right) \leq C,
\end{aligned}$$

where C is independent of n, but depends on Ω_0, and the parameters and data of the original problem. Returning to (3.19) and setting $\delta = \frac{a}{q'}$, it follows that

$$\int_0^T \int_{\Omega_0} |\mathbf{T}|^{1+\delta} \,\mathrm{d}x\,\mathrm{d}t \leq C,$$

where $C = C(\Omega_0)$ is independent of n, thus concluding the proof. □

3.2 Existence of solutions to the strain-limiting problem in the case that a is small

With the previous integrability result and (3.7), we have sufficient information to show the existence of a weak solution to the strain-limiting problem (3.1) in the case that $a \in (0, \frac{1}{6})$ and $d = 3$. We prove the existence of a solution for an arbitrary $a > 0$ and spatial dimension d in Chapter 4. However, this is under much stronger regularity assumptions on the data. Under the current assumptions, namely those of Lemma 3.4, we are unable to obtain a bound of the form (2.20), due to a lack of regularity of the approximate stress tensors.

Theorem 3.5. *Suppose that we have initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_0^{1,2}(\Omega)^d$ and a body force $\boldsymbol{f} \in L^2(Q)^{d\times d} \cap L^2(0,T; W_{loc}^{1,2}(\Omega)^d)$ such that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_0^{k+1,2}(\Omega)^d$ for some $k > \frac{d}{2}$ with the safety strain condition*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)\|_\infty = C_* < 1.$$

Furthermore, suppose that $a \in (0, \frac{1}{6})$ and $d = 3$. There exists a unique weak solution $(\boldsymbol{u}, \mathbf{T})$ of the strain-limiting problem (3.1) in the sense of Definition 3.2. Furthermore, if $((\boldsymbol{u}^n, \mathbf{T}^n))_n$ is the sequence of solutions to the regularised problem, the following convergence results hold:

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ *weakly-* in* $W^{1,\infty}(0,T; L^2(\Omega)^d) \cap L^\infty(0,T; W_0^{1,2}(\Omega)^d)$;
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0,T; L^2(\Omega)^d) \cap W^{1,2}(0,T; W_0^{1,2}(\Omega)^d)$;
- $\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u}$ *weakly in* $L^p(0,T; W_0^{1,p}(\Omega)^d)$ *for every* $p \in [1, \infty)$;
- $\mathbf{T}^n \rightharpoonup \mathbf{T}$ *weakly in* $L^{1+\delta}(0,T; L_{loc}^{1+\delta}(\Omega)^{d\times d})$ *and strongly in* $L^1(0,T; L_{loc}^1(\Omega)^{d\times d})$.

Proof. Using Lemma 3.4 and (3.7) from Theorem 3.3 alongside standard compactness results, we immediately deduce that, up to a subsequence in n not relabelled,

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $W^{1,\infty}(0,T; L^2(\Omega)^d)$ and weakly in $W^{2,2}(0,T; L^2(\Omega)^d)$,
- $\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u}$ weakly in $L^p(0,T; W_0^{1,p}(\Omega)^d)$ for every $p \in [1, \infty)$, and
- $\mathbf{T}^n \rightharpoonup \mathbf{T}$ weakly in $L^{1+\delta}(0,T; L^{1+\delta}(\Omega)^{d\times d})$.

Before proving the higher regularity convergence results for $(\boldsymbol{u}^n)_n$ and the strong convergence result for $(\mathbf{T}^n)_n$, we first consider the weak form of (3.1a). For every $\boldsymbol{v} \in C_c^1(\overline{\Omega})^d$ and every $\psi \in C([0,T])$, we have that

$$\int_Q \boldsymbol{u}_{tt}^n \cdot (\psi \boldsymbol{v}) + \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f} \cdot (\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t.$$

Letting $n \to \infty$ and using the above weak convergence results, we see that

$$\int_Q \boldsymbol{u}_{tt} \cdot (\psi \boldsymbol{v}) + \mathbf{T} \cdot \boldsymbol{\varepsilon}(\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f} \cdot (\psi \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t.$$

Using the regularity of $(\boldsymbol{u}, \mathbf{T})$, it follows that (3.2) holds. The initial conditions hold in the required sense as a result of standard arguments.

To show that the constitutive relation is satisfied, we first derive the stronger convergence results. Let $\boldsymbol{u}^{n,m}$ denote the Galerkin approximation from V_m of the regularised problem with approximation parameter n. Using the constitutive relation and initial condition, we see that

$$\boldsymbol{u}^{n,m}(t) = \mathrm{e}^{-\alpha t} P^m \boldsymbol{u}_0 + \int_0^t \mathrm{e}^{\alpha(s-t)} (\boldsymbol{u}_t^{n,m} + \alpha \boldsymbol{u}^{n,m})(s) \, \mathrm{d}s.$$

Taking the symmetric gradient of both sides, we have the memory kernel property

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,m})(t) = \mathrm{e}^{-\alpha t} \boldsymbol{\varepsilon}(P^m \boldsymbol{u}_0) + \int_0^t \mathrm{e}^{\alpha(s-t)} F_n(\mathbf{T}^{n,m})(s) \, \mathrm{d}s.$$

It follows that

$$\begin{aligned}
|\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,m})(t)|^2 &\leq C \left(|\boldsymbol{\varepsilon}(P^m \boldsymbol{u}_0)|^2 + \int_0^t |F_n(\mathbf{T}^{n,m})(s)|^2 \, \mathrm{d}s \right) \\
&\leq C \left(1 + |\boldsymbol{\varepsilon}(P^m \boldsymbol{u}_0)|^2 + \int_0^t \frac{|\mathbf{T}^{n,m}(s)|^{\frac{2}{n}}}{n^2} \, \mathrm{d}s \right).
\end{aligned}$$

Integrating over Ω and applying Fubini's theorem, we see that

$$\begin{aligned}
\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,m})(t)\|_2 &\leq C \left(1 + \|\boldsymbol{\varepsilon}(P^m \boldsymbol{u}_0)\|_2 + \left(\int_0^t \int_\Omega \frac{|\mathbf{T}^{n,m}(s)|^{\frac{2}{n}}}{n^2} \, \mathrm{d}x \, \mathrm{d}s \right)^{\frac{1}{2}} \right) \\
&\leq C \left(1 + \|\boldsymbol{\varepsilon}(P^m \boldsymbol{u}_0)\|_2 + \left(\int_0^t \|\mathbf{T}^{n,m}(s)\|_1 \, \mathrm{d}s \right)^{\frac{1}{2}} \right) \\
&\leq C \left(1 + \|P^m \boldsymbol{u}_0\|_{1,2} \right),
\end{aligned}$$

where C is a constant that is independent of n and m. By the Korn–Poincaré inequality, it follows that $(\boldsymbol{u}^{n,m})_m$ is bounded in $L^\infty(0, T; W_0^{1,2}(\Omega)^d)$, independent of n and m. Hence $\boldsymbol{u}^n \in L^\infty(0, T; W_0^{1,2}(\Omega)^d)$ and is bounded in this space, independent of n. Next, we write

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^{n,m})(t)| \leq \alpha |\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,m})(t)| + 1 + \frac{|\mathbf{T}^{n,m}(t)|^{\frac{1}{n}}}{n},$$

but the right-hand side is an element of $L^2(Q)$ and uniformly bounded in this space with respect to n. Applying the Korn–Poincaré inequality, it follows that $\boldsymbol{u}_t^{n,m} \in L^2(0, T; W_0^{1,2}(\Omega)^d)$ and is bounded in this space, independent of n and m. Hence $\boldsymbol{u}^n \in W^{1,2}(0, T; W_0^{1,2}(\Omega)^d) \cap L^\infty(0, T; W_0^{1,2}(\Omega)^d)$, bounded uniformly in this space with respect to n and so

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $L^\infty(0, T; W_0^{1,2}(\Omega)^d)$, and
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ weakly in $W^{1,2}(0, T; W_0^{1,2}(\Omega)^d)$.

Next, we adapt the proof of Theorem 2.7 to show that $\mathbf{T}^n$ converges strongly to $\mathbf{T}$ in $L^1(0, T; L^1_{loc}(\Omega)^{d \times d})$. If this is true, it follows that $\mathbf{T}^n \to \mathbf{T}$ pointwise a.e. on Q and so the constitutive relation is satisfied by the uniqueness of weak and pointwise limits. Fix a non-empty, compact subset $\Omega_0 \subset \Omega$ and denote $Q_0 = (0,T) \times \Omega_0$. For every $k > 0$, we define the set Q_k^n by

$$Q_k^n = \{(t,x) \in Q \,:\, 1 + |\mathbf{T}| + |\mathbf{T}^n| > k\}.$$

Using Lemma 3.4 and Markov's inequality, we have that

$$|Q_0 \cap Q_k^n| \le Ck^{-(1+\delta)},$$

where C is independent of n and k but depends on Ω_0. Reasoning as in the proof of Lemma 3.4, we see that

$$\left(\int_{Q_0} |\mathbf{T}^n - \mathbf{T}| \,\mathrm{d}x\,\mathrm{d}t\right)^2 \le C\left(k^{-2\delta} + k^{1+a}\int_{Q_0} (\mathbf{T}^n - \mathbf{T}) \cdot (F(\mathbf{T}^n) - F(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.21}$$

For any fixed $p \in [1,\infty)$, taking n sufficiently large such that $n+1 \ge p$, we have that

$$\begin{aligned}\int_Q \left|\frac{\mathbf{T}^n}{n(1+|\mathbf{T}^n|^{1-\frac{1}{n}})}\right|^p \mathrm{d}x\,\mathrm{d}t &= \int_Q \frac{|\mathbf{T}^n|^{\frac{p}{n}}}{n^p}\,\mathrm{d}x\,\mathrm{d}t \le \frac{C}{n^p}\left(\int_Q |\mathbf{T}^n|^{1+\frac{1}{n}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{n+1}} \\ &= \frac{C}{n^{\frac{np}{n+1}}}\left(\int_Q \frac{|\mathbf{T}^n|^{1+\frac{1}{n}}}{n}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{n+1}} \le \frac{C}{n^{\frac{p}{2}}}.\end{aligned}$$

It follows that

$$\frac{\mathbf{T}^n}{n(1+|\mathbf{T}^n|^{1-\frac{1}{n}})} \to \mathbf{0} \quad \text{strongly in } L^p(Q)^{d\times d},$$

for every $p \in [1,\infty)$. Let $\tau \in C_c^\infty(\Omega)$ be a cut-off function such that $\tau = 1$ on Ω_0. By the monotonicity of F, we see that

$$\begin{aligned}0 &\le \lim_{n\to\infty}\int_{Q_0} (\mathbf{T}^n - \mathbf{T}) \cdot (F(\mathbf{T}^n) - F(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t \\ &\le \lim_{n\to\infty}\int_Q \tau(\mathbf{T}^n - \mathbf{T}) \cdot (F(\mathbf{T}^n) - F(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t \\ &\le \lim_{n\to\infty}\int_Q \tau(\mathbf{T}^n - \mathbf{T}) \cdot (F_n(\mathbf{T}^n) - F(\mathbf{T}))\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{n\to\infty}\int_Q \tau(\mathbf{T}^n - \mathbf{T}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n) - \tau(\mathbf{T}^n - \mathbf{T}) \cdot F(\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.22}$$

Since F is bounded, using the weak convergence of $(\mathbf{T}^n)_n$ and compact support of τ, we have that

$$\lim_{n\to\infty}\int_Q \tau(\mathbf{T}^n - \mathbf{T}) \cdot F(\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t = 0.$$

For the other term on the right-hand side of (3.22), we see that

$$\begin{aligned}&\int_Q \tau(\mathbf{T}^n - \mathbf{T}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q (\mathbf{T}^n - \mathbf{T}) \cdot \boldsymbol{\varepsilon}\left(\tau(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\right) - (\mathbf{T}^n - \mathbf{T}) \cdot ((\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n) \otimes \nabla\tau)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q (\boldsymbol{u}_{tt} - \boldsymbol{u}_{tt}^n) \cdot (\tau(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)) - (\mathbf{T}^n - \mathbf{T}) \cdot ((\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n) \otimes \nabla\tau)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

using (3.2) in the transition to the last line, noting that the body force terms cancel. Next, we note that

$$\begin{aligned} \boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n &\rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u} && \text{weakly in } L^2(0,T;W_0^{1,2}(\Omega)^d), \\ \boldsymbol{u}_{tt}^n + \alpha \boldsymbol{u}_t^n &\rightharpoonup \boldsymbol{u}_{tt} + \alpha \boldsymbol{u}_t && \text{weakly in } L^2(0,T;L^2(\Omega)^d). \end{aligned}$$

By the Aubin–Lions lemma, $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$ converges strongly in $L^2(Q)^d$ and so

$$\lim_{n\to\infty} \int_Q (\boldsymbol{u}_{tt} - \boldsymbol{u}_{tt}^n) \cdot (\tau(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

For the other term, we note that

$$\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha \boldsymbol{u} \quad \text{weakly in } L^{(1+\delta)'}(0,T;W_0^{1,(1+\delta)'}(\Omega)^d),$$

where $(1+\delta)'$ is the Hölder conjugate of $1+\delta$. Since $\delta < 1$, we have that $W_0^{1,(1+\delta)'}(\Omega) \subset L^{(1+\delta)'}(\Omega) \subset L^2(\Omega)$, where each embedding is dense and the first embedding is compact. Using the Aubin–Lions lemma again, $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$ converges strongly in $L^{(1+\delta)'}(Q)^d$. Combining this with the weak convergence of $(\mathbf{T}^n)_n$, we get that

$$\lim_{n\to\infty} \int_Q (\mathbf{T}^n - \mathbf{T}) \cdot ((\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n) \otimes \nabla \tau) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

and so

$$\lim_{n\to\infty} \int_{Q_0} (\mathbf{T}^n - \mathbf{T}) \cdot (F(\mathbf{T}^n) - F(\mathbf{T})) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Returning to (3.21), we deduce that, for every fixed k,

$$\lim_{n\to\infty} \left(\int_{Q_0} |\mathbf{T}^n - \mathbf{T}| \, \mathrm{d}x \, \mathrm{d}t \right)^2 \leq C k^{-2\delta}.$$

Letting $k \to \infty$, the required convergence result follows, concluding the proof. □

To summarise this chapter, we have shown the existence of a unique weak solution of the strain-limiting viscoelastic problem in the case of homogeneous boundary conditions provided that the parameter a in the constitutive relation is sufficiently small. However, under the regularity assumptions on the data, at this stage we are unable to prove the existence of a solution for an arbitrary value of a. We require the methods from Chapter 4 in order to obtain such a result.

Chapter 4

Strain-limiting problem with mixed Dirichlet–Neumann boundary conditions

In this chapter, we prove an existence result that generalises the work of Chapter 3. We consider mixed Dirichlet–Neumann boundary conditions for the strain-limiting problem. Furthermore, we extend the proof of the previous section to show the existence of a unique weak solution of the Dirichlet problem in any dimension and for any $a > 0$ by adapting the reasoning from Chapter 2. Furthermore, we show that the integrability of the stress tensor can be improved to $L^{1+\delta}(0,T;L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for some $\delta > 0$ in any dimension $d \geq 2$, provided that $a \in (0, \frac{2}{d})$. This significantly improves the results of both Chapter 2 and Chapter 3. The work in this chapter has not been submitted as a stand-alone paper but is combined in [21], [20] and [84].

However, we do not prove a full existence result as in the previous chapters when the Neumann part of the boundary is non-empty. Due to the lack of integrability of approximations of the stress tensor near the Neumann part of the boundary, an error term appears there. In particular, for a sequence $(\mathbf{T}^n)_n$ that is bounded in $L^1(Q)^{d\times d}$, the best global convergence result that we have is weak-* convergence in $\mathcal{M}(\overline{Q})^{d\times d}$ to some limit $\overline{\mathbf{T}}$, say. On the other hand, using higher regularity estimates, we prove that the sequence under consideration converges pointwise a.e. on Q. This is done via higher regularity estimates as in Chapter 2. We use this to show that the singular part of the limit $\overline{\mathbf{T}}$ can be localised to the boundary $\partial\Omega$. The test functions in the weak form of the PDE vanish on the Dirichlet part of the boundary so we only see singularities on the Neumann part. This reflects what is known about the problem in the steady case [5] where there is no time dependence.

The structure of the proof is similar to previous chapters. However, we now use a linear regularisation term. This is because the approximations of the stress tensor have much higher regularity in this case, compared to the regularisation term used in Chapters 2 and 3. The uniform estimates on the displacement approximations are slightly weaker but are still sufficient for our purposes. We prove the existence of a weak solution to the regularised problem using a Galerkin approximation. Higher regularity estimates with respect to the time variable are constructed at the level of the Galerkin approximation to ensure that the estimates are fully rigorous and to deal with the lack of continuity of $\boldsymbol{u}_{tt}^n$ in time, where $(\boldsymbol{u}^n, \mathbf{T}^n)$ is the solution of the regularised problem. As in Chapter 3, the spatial estimates are constructed using difference quotients in space and the solution of the regularised problem. As a result they are only local in space.

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary so that the outward unit normal $\mathbf{n}$ is well-defined a.e. on the boundary. We assume that we have relatively open subsets $\partial\Omega_D$, $\partial\Omega_N$ of $\partial\Omega$ such that $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $\overline{\partial\Omega_D \cup \partial\Omega_N} = \partial\Omega$. We assume that the Dirchlet part is non-empty, $\partial\Omega_D \neq \emptyset$. This is because in a true physical experiment the body under investigation must be clamped in some part, corresponding mathematically to a partial homogeneous Dirichlet boundary condition. Furthermore, under these assumptions, we can apply the Korn–Poincaré inequality for mixed boundary conditions, Theorem 1.9, which is vital to the analysis. We denote the space-time domain by $Q = (0,T) \times \Omega$, for a fixed final time $T \in (0, \infty)$. We look for a couple $(\boldsymbol{u}, \mathbf{T}) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{4.1a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{in } Q, \tag{4.1b}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1 \qquad \text{in } \Omega, \tag{4.1c}$$

$$\boldsymbol{u}(t,\cdot) = \mathbf{0} \qquad \text{on } (0,T] \times \partial\Omega_D, \tag{4.1d}$$

$$\mathbf{T}\mathbf{n} = \boldsymbol{g} \qquad \text{on } (0,T] \times \partial\Omega_N, \tag{4.1e}$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega_N$ and $\boldsymbol{f}$, $\boldsymbol{g}$ are a given external body force and boundary traction, respectively. The parameters $a > 0$ and $\alpha \geq 0$ are fixed but arbitrary. Concerning the assumptions on the initial data, we ask that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,2}(\Omega)^d$. Furthermore, we assume that $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W_D^{k+1,2}(\Omega)^d$ for a $k > \frac{d}{2}$ and we have the safety strain condition

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_\infty = C_* < 1. \tag{4.2}$$

This is so that the convergence of the approximations of the initial data is sufficiently strong. We also require a compatibility condition between the initial data and the boundary traction $\boldsymbol{g}$. We ask that

$$\boldsymbol{g}(0,\cdot) = F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} \quad \text{on } \partial\Omega_N. \tag{4.3}$$

This is well-defined because F^{-1} is defined on the open unit ball in $\mathbb{R}^{d\times d}$ and we have the safety strain condition (4.2). This compatibility condition (4.3) is a vital assumption in the existence proof. It is used to obtain time regularity estimates which are needed for the pointwise convergence of $(\mathbf{T}^n)_n$. We must take care to ensure that similar compatibility conditions hold at the level of the regularised and Galerkin approximation (where the function F and the initial data are approximated). Although the compatibility is not needed for the existence of solutions at the Galerkin or regularised level, it is required for the n-independent estimates.

4.1 The regularised problem

For a fixed approximation parameter $n \in \mathbb{N}$, we consider the problem of finding a couple $(\boldsymbol{u}, \mathbf{T})$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{4.4a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F_n(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{\mathbf{T}}{n} \qquad \text{in } Q, \tag{4.4b}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1 \qquad \text{in } \Omega, \tag{4.4c}$$

$$\boldsymbol{u}(t,\cdot) = \mathbf{0} \qquad \text{on } (0,T] \times \partial\Omega_D, \tag{4.4d}$$

$$\mathbf{T}\mathbf{n} = \boldsymbol{g}_n \qquad \text{on } (0,T] \times \partial\Omega_N. \tag{4.4e}$$

By the Browder–Minty theorem (Theorem 2.2), the function F_n is a bijection from $\mathbb{R}^{d\times d}$ to itself. Furthermore, by Lemma 2.3, F_n is a C^1-diffeomorphism. We use a linear regularisation term from now on because it simplifies the reasoning concerning higher regularity of the regularised solution, which is then used to construct uniform higher regularity estimates. Furthermore, we can show optimal (to my knowledge) higher integrability estimates on the stress tensor. The arguments can be applied directly to Chapters 2 and 3 to improve the results there.

The boundary traction $\boldsymbol{g}$ is replaced by an approximation $\boldsymbol{g}_n$ defined by

$$\boldsymbol{g}_n(t,x) = \chi(nt) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\mathbf{n} + (1 - \chi(nt))\boldsymbol{g}(t,x), \tag{4.5}$$

where $\chi \in C_c^\infty([0,\infty))$ is a smooth cut-off function such that $\chi = 1$ on $[0, \frac{1}{2}]$, $\chi = 0$ on $[1,\infty)$. We note that there exists a constant C such that $|\chi'(t)| \le C$ and $|\chi''(t)| \le C$ for every $t \in [0,\infty)$. This approximation is used in order to ensure that a condition analogous to (4.3) holds, namely,

$$\boldsymbol{g}_n(0,\cdot) = F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\mathbf{n} \quad \text{on } \partial\Omega_N. \tag{4.6}$$

This approximation choice is vital in the higher regularity estimate Lemma 4.5. Before proceeding further, we consider properties of the approximation sequence $(\boldsymbol{g}_n)_n$ and its derivatives in time. From the safety strain condition (4.2), we notice that the approximation sequence $(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)))_n$ converges uniformly on Ω to $F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))$ at a rate of order $\frac{1}{n}$. Indeed, define scalar valued functions f and f_n on $[0,\infty)$ by

$$f(t) = \frac{t}{(1+t^a)^{\frac{1}{a}}}, \quad f_n(t) = f(t) + \frac{t}{n}.$$

On the open unit ball and the whole of $\mathbb{R}^{d\times d}$, respectively, we have that

$$F^{-1}(\mathbf{T}) = f^{-1}(|\mathbf{T}|)\frac{\mathbf{T}}{|\mathbf{T}|}, \quad F_n^{-1}(\mathbf{T}) = f_n^{-1}(|\mathbf{T}|)\frac{\mathbf{T}}{|\mathbf{T}|}.$$

It follows that

$$|F^{-1}(\mathbf{T}) - F_n^{-1}(\mathbf{T})| = |f^{-1}(|\mathbf{T}|) - f_n^{-1}(|\mathbf{T}|)|,$$

for any $\mathbf{T} \in \mathbb{R}^{d\times d}$ with $|\mathbf{T}| < 1$. Consider the subset $[0, C_*] \subset [0, 1)$ for some $C_* \in (0, 1)$. From explicit calculation we have that

$$f^{-1}(t) = \frac{t}{(1-t^a)^{\frac{1}{a}}},$$

which is continuously differentiable on $[0, C_*]$ with bounded derivative there. Define a constant C_1 to be the maximum of $f^{-1}(C_*)$ and the supremum of $|(f^{-1})'|$ over $[0, C_*]$. It follows that for every x, $y \in [0, C_*]$ we have that

$$|f^{-1}(x) - f^{-1}(y)| \leq C_1|x - y|, \tag{4.7}$$

applying the mean value theorem. For every $y \in [0, f^{-1}(C_*)]$, we also have

$$|f_n(y) - f(y)| \leq \frac{C_1}{n}.$$

Fix $n \in \mathbb{N}$ and let $x \in [0, C_*]$. Assuming that $x \pm \frac{C_1}{n} \in [0, C_*]$, we deduce that

$$\left| f_n\left(f^{-1}\left(x \pm \frac{C_1}{n}\right)\right) - \left(x \pm \frac{C_1}{n}\right)\right| = \left| f_n\left(f^{-1}\left(x \pm \frac{C_1}{n}\right)\right) - f\left(f^{-1}\left(x \pm \frac{C_1}{n}\right)\right)\right| \leq \frac{C_1}{n}.$$

This yields

$$f_n\left(f^{-1}\left(x - \frac{C_1}{n}\right)\right) \leq x \leq f_n\left(f^{-1}\left(x + \frac{C_1}{n}\right)\right).$$

By the monotonicity of f_n^{-1} and (4.7),

$$f^{-1}(x) - \frac{C_1^2}{n} \leq f^{-1}\left(x - \frac{C_1}{n}\right) \leq f_n^{-1}(x) \leq f^{-1}\left(x + \frac{C_1}{n}\right) \leq f^{-1}(x) + \frac{C_1^2}{n},$$

and so

$$|f_n^{-1}(x) - f^{-1}(x)| \leq \frac{C_1^2}{n}. \tag{4.8}$$

In the case that $x - \frac{C_1}{n} < 0$, we replace $x - \frac{C_1}{n}$ with 0 and analogously for $x + \frac{C_1}{n}$ with C_*. Thus, for every $x \in [0, C_*]$ and every n, (4.8) holds. With (4.8) in mind, we return to the approximation $\boldsymbol{g}_n$. Suppose that $\boldsymbol{g} \in W^{1,2}(0, T; L^2(\partial\Omega_N)^d)$. Then

$$\begin{aligned}
|\boldsymbol{g}_n(t, x) - \boldsymbol{g}(t, x)| &= \chi(nt)|F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} - \boldsymbol{g}(t, x)| \\
&\leq |F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)) - F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))| + \chi(nt)|\boldsymbol{g}(0, x) - \boldsymbol{g}(t, x)| \\
&\leq |F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)) - F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))| + \chi(nt)\int_0^t |\boldsymbol{g}_t(s, x)|\,\mathrm{d}s.
\end{aligned}$$

We square and integrate over $\partial\Omega_N$ to obtain

$$\|\boldsymbol{g}_n(t) - \boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)} \leq \frac{C}{n} + \chi(nt)t^{\frac{1}{2}}\left(\int_0^t \|\boldsymbol{g}_t\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}s\right)^{\frac{1}{2}} \leq \frac{C}{n^{\frac{1}{2}}}.$$

Thus $\boldsymbol{g}_n \to \boldsymbol{g}$ in $L^\infty(0, T; L^2(\partial\Omega_N)^d)$ as $n \to \infty$ and, for a constant C, we have that

$$\|\boldsymbol{g}_n\|_{L^\infty(L^2(\partial\Omega_N))} \leq C(1 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_t\|_{L^2(L^2(\partial\Omega_N))}).$$

For the first derivative in time, from the definition of $\boldsymbol{g}_n$, we have that

$$\begin{aligned}(\boldsymbol{g}_n)_t(t,x) &= n\chi'(nt)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n} + (1-\chi(nt))\boldsymbol{g}_t(t,x) - n\chi'(nt)\boldsymbol{g}(t,x)\\ &= n\chi'(nt)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n} - F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n}\right]\\ &\quad + n\chi'(nt)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n} - \boldsymbol{g}(t,x)\right] + (1-\chi(nt))\boldsymbol{g}_t(t,x).\end{aligned}$$

For the first term on the right-hand side, we have

$$\begin{aligned}&\int_0^T\int_{\partial\Omega_N}\left|n\chi'(nt)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n} - F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n}\right]\right|^2\,\mathrm{d}S\,\mathrm{d}t\\ &\le C\int_0^T|\chi'(nt)|^2\,\mathrm{d}t = C\int_{\frac{1}{2n}}^{\frac{2}{n}}|\chi'(nt)|^2\,\mathrm{d}t \le C\int_{\frac{1}{2n}}^{\frac{2}{n}}|nt|^4\,\mathrm{d}t \le \frac{C}{n},\end{aligned}$$

where C is a constant that is independent of n. Similarly, for the second term on the right-hand side, we get

$$\int_0^T\int_{\partial\Omega_N}\left|n\chi'(nt)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\mathbf{n} - \boldsymbol{g}(t,x)\right]\right|^2\,\mathrm{d}S\,\mathrm{d}t \le \frac{C}{n}.$$

It follows that

$$\int_0^T\|(\boldsymbol{g}_n)_t(t)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}t \le C\left(1+\int_0^T\|\boldsymbol{g}_t(t)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}t\right).$$

Thus $\boldsymbol{g}_n \rightharpoonup \boldsymbol{g}$ weakly in $W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$. It also follows that if we have $\boldsymbol{g}\in W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$, then $\boldsymbol{g}_n \overset{*}{\rightharpoonup} \boldsymbol{g}$ weakly-* in $W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$ with

$$\|(\boldsymbol{g}_n)_t\|_{L^\infty(L^2(\partial\Omega_N))} \le C(1+\|\boldsymbol{g}_t\|_{L^\infty(L^2(\partial\Omega_N))}).$$

Finally, suppose that $\boldsymbol{g}\in W^{2,1}(0,T;L^2(\partial\Omega_N)^d)$. From direct calculation, we have that

$$\begin{aligned}(\boldsymbol{g}_n)_{tt}(t,x) &= n^2\chi''(nt)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)) + (1-\chi(nt))\boldsymbol{g}_{tt} - 2n\chi'(nt)\boldsymbol{g}_t - n^2\chi''(nt)\boldsymbol{g}\\ &= n^2\chi''(nt)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)) - F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\right]\\ &\quad + n^2\chi''(nt)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)) - \boldsymbol{g}\right] + (1-\chi(nt))\boldsymbol{g}_{tt} - 2n\chi'(nt)\boldsymbol{g}_t.\end{aligned}$$

Taking the norm in $L^1(0,T;L^2(\partial\Omega_N)^d)$, we deduce that

$$\|(\boldsymbol{g}_n)_{tt}\|_{L^1(L^2(\partial\Omega_N))} \le C\left[1+\|\boldsymbol{g}_t\|_{L^\infty(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_{tt}\|_{L^1(L^2(\partial\Omega_N))}\right].$$

To summarise, we have the following result.

Lemma 4.1. *Suppose $\boldsymbol{g}\in W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$ with approximation sequence $(\boldsymbol{g}_n)_n$ defined as above. Then $\boldsymbol{g}_n \rightharpoonup \boldsymbol{g}$ weakly in $W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$ and there exists a constant $C=C(C_*,a)$ such that*

$$\|\boldsymbol{g}_n\|_{L^\infty(L^2(\partial\Omega_N))} \le C(1+\|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_t\|_{L^2(L^2(\partial\Omega_N))}),$$

and

$$\|(\boldsymbol{g}_n)_t\|_{L^2(L^2(\partial\Omega_N))} \leq C(1 + \|\boldsymbol{g}_t\|_{L^2(L^2(\partial\Omega_N))}).$$

If $\boldsymbol{g} \in W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$*, the convergence holds weakly-* in this space and*

$$\|(\boldsymbol{g}_n)_t\|_{L^\infty(L^2(\partial\Omega_N))} \leq C(1 + \|\boldsymbol{g}_t\|_{L^\infty(L^2(\partial\Omega_N))}).$$

If $\boldsymbol{g} \in W^{2,1}(0,T;L^2(\partial\Omega_N)^d)$*, the convergence holds weakly in this space and*

$$\|(\boldsymbol{g}_n)_{tt}\|_{L^1(L^2(\partial\Omega_N))} \leq C(1 + \|\boldsymbol{g}_{tt}\|_{L^1(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_t\|_{L^\infty(L^2(\partial\Omega_N))}).$$

With this in mind, we focus on proving the existence of a weak solution to the regularised problem (4.4), defined in the following sense.

Definition 4.2. *Given data* $\boldsymbol{u}_0 \in L^2(\Omega)^d$*,* $\boldsymbol{u}_1 \in W^{-1,2}(\Omega)^d$*, body force* $\boldsymbol{f} \in L^2(Q)^d$ *and boundary traction* $\boldsymbol{g}_n \in L^2(0,T;L^2(\partial\Omega_N)^d)$*, we say that a couple* $(\boldsymbol{u}, \mathbf{T})$ *with regularity*

- $\boldsymbol{u} \in W^{2,2}(0,T;W_D^{-1,2}(\Omega)^d) \cap W^{1,2}(0,T;L^2(\Omega)^d)$,
- $\boldsymbol{u}_t + \alpha\boldsymbol{u} \in L^2(0,T;W_D^{1,2}(\Omega)^d)$*, and*
- $\mathbf{T} \in L^2(Q)^{d\times d}$,

is a weak solution of the regularised problem (4.4) *with mixed Dirichlet–Neumann boundary conditions if*

$$\langle \boldsymbol{u}_{tt}(t), \boldsymbol{v}\rangle + \int_\Omega \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}_n(t)\cdot\boldsymbol{v}\,\mathrm{d}S, \tag{4.9}$$

for a.e. $t \in (0,T)$ *and every* $\boldsymbol{v} \in W_D^{1,2}(\Omega)^d$*, with constitutive relation*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F_n(\mathbf{T}) \quad \textit{pointwise in } Q, \tag{4.10}$$

and initial conditions satisfied in the sense that

$$\lim_{t\to0+} (\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_2 + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_{-1,2}) = 0. \tag{4.11}$$

Theorem 4.3. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded, Lipschitz domain with Dirichlet and Neumann parts of the boundary as before. Let* $a > 0$ *and* $\alpha \geq 0$ *be fixed problem parameters. Suppose that we are given initial data* $\boldsymbol{u}_0$*,* $\boldsymbol{u}_1 \in L^2(\Omega)^d$*, body force* $\boldsymbol{f} \in L^2(Q)^d$ *and boundary traction* $\boldsymbol{g} \in W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$*. Define the approximate boundary traction* $\boldsymbol{g}_n \in W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$ *by* (4.5)*. Suppose further that* $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W_D^{k+1,2}(\Omega)^d$ *for some* $k > \frac{d}{2}$ *and the safety strain condition* (4.2) *holds. There exists a unique weak solution* $(\boldsymbol{u}, \mathbf{T})$ *of the regularised problem* (4.4) *with approximation parameter* n *in the sense of Definition 4.2 and there exists a constant* C *depending only on* Ω*,* T*,* α*,* a *and* C_* *such that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|_2^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_t(t)\|_2^2 + \int_Q |\mathbf{T}| + \frac{|\mathbf{T}|^2}{n}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \|(\boldsymbol{u}_t + \alpha\boldsymbol{u})(t)\|_{1,2}^2 + \frac{\|\boldsymbol{u}_{tt}(t)\|_{-1,2}^2}{n}\,\mathrm{d}t \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t\right).
\end{aligned} \tag{4.12}$$

Proof. Let $(\phi_i)_{i\geq 1}$ be a basis of eigenfunctions $W_D^{d+1,2}(\Omega)$ corresponding to the problem

$$\lambda(\phi,\psi) = a(\phi,\psi) := (\phi,\psi) + \sum_{|\beta|=d+1} (\partial^\beta\phi, \partial^\beta\psi) \quad \forall\psi \in W_D^{d+1,2}(\Omega),$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $a(\cdot,\cdot)$ is the given bilinear form and $\lambda \in \mathbb{R}$ is a scalar. The basis is chosen so that it is orthogonal in $W_D^{d+1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$, reasoning as in Chapter 3 to deduce the existence of such a basis. By the equivalence of the norm induced by the inner product $a(\cdot,\cdot)$ and the usual norm on $W_D^{d+1,2}(\Omega)$, if P^m is the projection operator from $L^2(\Omega)$ to $\text{span}\{\phi_1,\dots,\phi_m\}$ extending component-wise to vector-valued functions where appropriate, we have that

$$\|P^m v\|_{d+1,2} \leq C\|v\|_{d+1,2} \quad \forall v \in W_D^{d+1,2}(\Omega),$$

where C is independent of m. Also $(\phi_i \boldsymbol{e}_j)_{i,j=1}^{\infty,d}$ is a basis for $W_D^{d+1,2}(\Omega)^d$ that is orthonormal in $L^2(\Omega)^d$ and is orthogonal in $W_D^{d+1,2}(\Omega)^d$. We define $V_m = \text{span}\{\phi_1,\dots,\phi_m\}^d$. Then $P^m : L^2(\Omega)^d \to V_m$ and consequently there exists a constant C, independent of m, such that

$$\|P^m \boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_2 \quad \forall \boldsymbol{v} \in L^2(\Omega)^d,$$

and

$$\|P^m \boldsymbol{v}\|_{d+1,2} \leq C\|\boldsymbol{v}\|_{d+1,2} \quad \forall \boldsymbol{v} \in W_D^{d+1,2}(\Omega)^d.$$

By standard interpolation theory [7], it follows that, for every $k \in \{1,\dots,d\}$, there exists a constant C, independent of m, such that

$$\|P^m \boldsymbol{v}\|_{k,2} \leq C\|\boldsymbol{v}\|_{k,2} \quad \forall \boldsymbol{v} \in W_D^{k,2}(\Omega)^d.$$

By the density of the linear span of $(\phi_i \boldsymbol{e}_j)_{i,j}$ in $W_D^{d+1,2}(\Omega)^d$, the span must also be dense in $W_D^{1,p}(\Omega)^d$ for every $p \in [1,\infty)$.

Since we replace the initial data in the regularised problem with the projections $P^m \boldsymbol{u}_0$, $P^m \boldsymbol{u}_1$ in the Galerkin approximation of the problem, we require a further approximation of the boundary traction $\boldsymbol{g}_n$. Using the cut-off function χ as before, we define an approximate boundary traction $\boldsymbol{g}_{n,m}$ by

$$\boldsymbol{g}_{n,m}(t,x) = \chi(mt) F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))\mathbf{n} + (1-\chi(mt))\boldsymbol{g}_n(t,x).$$

Using the above approximation results and the regularity of $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0$, we have that

$$P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0) \to \boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \quad \text{strongly in } C^1(\overline{\Omega})^d.$$

Next, we write

$$\begin{aligned}|\boldsymbol{g}_{n,m}(t,x) - \boldsymbol{g}_n(t,x)| \leq \chi(mt)|F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))) - F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))| \\ + \chi(mt)|\boldsymbol{g}_n(0) - \boldsymbol{g}_n(t)|.\end{aligned}$$

It follows that

$$\begin{aligned}\|\boldsymbol{g}_{n,m}(t) - \boldsymbol{g}_n(t)\|_{L^2(\partial\Omega_N)} &\leq \|F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))) - F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\|_{L^2(\partial\Omega_N)} \\ &\quad + \frac{C}{m^{\frac{1}{2}}}\left(\int_{\frac{1}{2m}}^{\frac{2}{m}} \|(\boldsymbol{g}_n)_t(s)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}s\right)^{\frac{1}{2}} \leq O(1),\end{aligned} \tag{4.13}$$

where $O(1) \to 0$ as $m \to \infty$. Hence $\boldsymbol{g}_{n,m} \to \boldsymbol{g}_n$ strongly in $L^\infty(0,T;L^2(\partial\Omega_N)^d)$ as $m \to \infty$. Furthermore, for a constant C depending only on the problem parameters, we have

$$\|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)} \leq C\left(1 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_t\|_{L^2(L^2(\partial\Omega_N))}\right).$$

The Galerkin approximation from V_m of the regularised problem (4.4) is formulated as follows. We look for a function $\boldsymbol{u}^m$ of the form

$$\boldsymbol{u}^m(t,x) = \sum_{i=1}^m\sum_{j=1}^d \beta_{ij}^m(t)\phi_i(x)\boldsymbol{e}_j,$$

such that, for every $\boldsymbol{v} \in V_m$ and a.e. $t \in (0,T)$,

$$\int_\Omega \boldsymbol{u}_{tt}^m(t)\cdot\boldsymbol{v} + \mathbf{T}^m(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N}\boldsymbol{g}_{n,m}(t)\cdot\boldsymbol{v}\,\mathrm{d}S, \tag{4.14}$$

where the approximate stress tensor $\mathbf{T}^m$ is defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m) = F_n(\mathbf{T}^m)$ pointwise in Q, and with the initial conditions

$$\boldsymbol{u}^m(0) = P^m\boldsymbol{u}_0, \quad \boldsymbol{u}_t^m(0) = P^m\boldsymbol{u}_1.$$

Reasoning as in the proof of Theorem 2.4, a solution exists to the Galerkin approximation from V_m on a possibly short time interval $[0,T_*)$ where $T_* \leq T$. To prove that the existence result can be extended to the entire time interval of interest, we find an energy estimate which shows that the coefficients $\boldsymbol{\beta}^m$ and $\boldsymbol{\beta}_t^m$ do not blow up as t approaches T_*. Testing in (4.14) against $(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)$, we have that

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial t}\left(\frac{|\boldsymbol{u}_t^m|^2}{2} + \alpha\boldsymbol{u}_t^m\cdot\boldsymbol{u}^m\right) - \alpha|\boldsymbol{u}_t^m|^2 + \mathbf{T}^m\cdot F(\mathbf{T}^m) + \frac{|\mathbf{T}^m|^2}{n}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}x + \int_{\partial\Omega_N}\boldsymbol{g}_{n,m}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}S \\ &\leq \|\boldsymbol{f}(t)\|_2\|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_2 + C\|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)}\|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_{1,2} \\ &\leq C\left(1 + \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}(t)\|^2_{L^2(\partial\Omega_N)} + \|\boldsymbol{g}_t\|^2_{L^2(L^2(\partial\Omega_N))}\right) + \frac{\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_2^2}{4},\end{aligned} \tag{4.15}$$

using the trace theorem, Young's inequality and the Korn–Poincaré inequality (Theorem 1.9). To bound the final term on the right-hand side, we note that

$$\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)|^2\,\mathrm{d}x = \int_\Omega |F_n(\mathbf{T}^m)|^2\,\mathrm{d}x \leq 2|\Omega| + 2\int_\Omega \frac{|\mathbf{T}|^2}{n}\,\mathrm{d}x.$$

Substituting this into (4.15) and absorbing the stress tensor term into the left-hand side, we reason as in the proof of Theorem 2.4 to see that

$$\begin{aligned}
&\sup_{t\in[0,T_*)} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T_*)} \|\boldsymbol{u}_t^m(t)\|_2^2 + \int_0^{T_*} \|\mathbf{T}^m\|_1 + \frac{\|\mathbf{T}^m\|_2^2}{n} + \|\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m\|_{1,2}^2 \,\mathrm{d}t \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^{T_*} \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t\right) \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^{T_*} \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t\right),
\end{aligned}$$

where C is independent of n and m, and is finite for any finite value of T_*. By the orthonormality of the basis functions, we have that

$$\sup_{t\in[0,T_*)} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T_*)} \|\boldsymbol{u}_t^m(t)\|_2^2 = \sup_{t\in[0,T_*)} \sum_{i=1}^m \sum_{j=1}^d |\beta_{ij}^m(t)|^2 + \sup_{t\in[0,T_*)} \sum_{i=1}^m \sum_{j=1}^d |(\beta_{ij}^m)_t(t)|^2.$$

Thus we conclude the existence of a solution to the Galerkin approximation from V_m on $[0,T]$. Furthermore, there exists a constant C, independent of n and m, such that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}^m(t)\|_2^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_t^m(t)\|_2^2 + \int_Q |\mathbf{T}^m| + \frac{|\mathbf{T}^m|^2}{n} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_{1,2}^2 \,\mathrm{d}t \\
&\leq C\left(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t\right).
\end{aligned} \tag{4.16}$$

Next, we notice from (4.14) that, for every $\boldsymbol{v} \in W_D^{1,2}(\Omega)^d$,

$$\begin{aligned}
\langle \boldsymbol{u}_{tt}^m(t), \boldsymbol{v}\rangle &= \int_\Omega \boldsymbol{u}_{tt}^m(t)\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_{tt}^m(t)\cdot P^m\boldsymbol{v}\,\mathrm{d}x \\
&= \int_\Omega -\mathbf{T}^m(t)\cdot\boldsymbol{\varepsilon}(P^m\boldsymbol{v}) + \boldsymbol{f}(t)\cdot P^m\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}_{n,m}(t)\cdot P^m\boldsymbol{v}\,\mathrm{d}S \\
&\leq \|\mathbf{T}^m(t)\|_2\|\boldsymbol{\varepsilon}(P^m\boldsymbol{v})\|_2 + \|\boldsymbol{f}(t)\|_2\|P^m\boldsymbol{v}\|_2 + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)}\|P^m\boldsymbol{v}\|_{L^2(\partial\Omega_N)} \\
&\leq C\left(1 + \|\mathbf{T}^m(t)\|_2 + \|\boldsymbol{f}(t)\|_2 + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)}\right)\|P^m\boldsymbol{v}\|_{1,2} \\
&\leq C\left(1 + \|\mathbf{T}^m(t)\|_2 + \|\boldsymbol{f}(t)\|_2 + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)}\right)\|\boldsymbol{v}\|_{1,2}.
\end{aligned}$$

It follows that

$$\int_0^T \|\boldsymbol{u}_{tt}^m\|_{W^{-1,2}}^2 \leq C\left(1 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^T \left[\|\mathbf{T}^m\|_2^2 + \|\boldsymbol{f}\|_2^2 + \|\boldsymbol{g}_t\|_{L^2(\partial\Omega_N)}^2\right]\right).$$

Hence $\boldsymbol{u}_{tt}^m \in L^2(0,T;W_D^{-1,2}(\Omega)^d)$ with bound in this space independent of m, but depending on n. With this and (4.16), the following convergence results hold up to a subsequence in m that we do not relabel:

- $\boldsymbol{u}^m \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $W^{1,\infty}(0,T;L^2(\Omega)^d)$;
- $\boldsymbol{u}^m \rightharpoonup \boldsymbol{u}$ weakly in $W^{2,2}(0,T;W_D^{-1,2}(\Omega)^d)$;
- $\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m \rightharpoonup \boldsymbol{u}_t + \alpha\boldsymbol{u}$ weakly in $L^2(0,T;W_D^{1,2}(\Omega)^d)$;

- $\mathbf{T}^m \rightharpoonup \mathbf{T}$ weakly in $L^2(Q)^{d\times d}$.

By standard arguments, we see that

$$\lim_{t\to 0+} (\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|_2 + \|\boldsymbol{u}_t(t)-\boldsymbol{u}_1\|_{-1,2}) = 0.$$

Reasoning as in the proof of Theorem 2.7, we further have that $\mathbf{T}^m \to \mathbf{T}$ strongly in $L^1(Q)^{d\times d}$ and thus pointwise a.e. in Q. It follows from the constitutive relation that $\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u}) = F_n(\mathbf{T})$ pointwise a.e. in Q. Next, we fix arbitrary test functions $\boldsymbol{w}\in C_D^\infty(\overline{\Omega})^d$ and $\psi\in C([0,T])$. We test in (4.14) against $\boldsymbol{v}=\psi P^m\boldsymbol{w}$ and integrate over $(0,T)$. Using that $P^m\boldsymbol{w}\to\boldsymbol{w}$ strongly in $W^{d+1,2}(\Omega)^d$, we take $m\to\infty$ in the equation resulting from (4.14) and use the arbitrariness of ψ and $\boldsymbol{w}$ to obtain

$$\langle \boldsymbol{u}_{tt}(t),\boldsymbol{w}\rangle + \int_\Omega \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x + \int_{\partial\Omega_N}\boldsymbol{g}_n(t)\cdot\boldsymbol{w}\,\mathrm{d}S,$$

for every $\boldsymbol{w}\in W_D^{1,2}(\Omega)^d$ and a.e. $t\in(0,T)$. Hence we conclude the proof of existence of a weak solution to the regularised problem.

For the proof of uniqueness, suppose that $(\boldsymbol{u}_i,\mathbf{T}_i)$ for $i\in\{1,2\}$ are weak solutions of the regularised problem emanating from the same initial data. Let $\boldsymbol{v}=\boldsymbol{u}_1-\boldsymbol{u}_2$. First, we note that

$$\begin{aligned}\langle \boldsymbol{v}_{tt},\boldsymbol{v}_t+\alpha\boldsymbol{v}\rangle &= \langle \boldsymbol{v}_{tt}+\alpha\boldsymbol{v}_t,\boldsymbol{v}_t+\alpha\boldsymbol{v}\rangle - \alpha\langle \boldsymbol{v}_t,\boldsymbol{v}_t+\alpha\boldsymbol{v}\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\boldsymbol{v}_t+\alpha\boldsymbol{v}\|_2^2}{2} - \frac{\alpha^2\|\boldsymbol{v}\|_2^2}{2}\right) - \alpha\int_\Omega |\boldsymbol{v}_t|^2\,\mathrm{d}x,\end{aligned}$$

using the regularity of $\boldsymbol{v}_t+\alpha\boldsymbol{v}$ to justify the transition to the second line. We refer to [102] concerning Gelfand triples for the details. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\boldsymbol{v}_t+\alpha\boldsymbol{v}\|_2^2}{2} - \frac{\alpha^2\|\boldsymbol{v}\|_2^2}{2}\right) - \alpha\int_\Omega |\boldsymbol{v}_t|^2\,\mathrm{d}x + \int_\Omega (\mathbf{T}_1-\mathbf{T}_2)\cdot(F_n(\mathbf{T}_1)-F_n(\mathbf{T}_2))\,\mathrm{d}x = 0.$$

Integrating over $(0,t)$ for some $t<T$, we deduce that

$$\frac{\|\boldsymbol{v}_t(t)\|_2^2}{2} + \int_0^t\int_\Omega (\mathbf{T}_1-\mathbf{T}_2)\cdot(F_n(\mathbf{T}_1)-F_n(\mathbf{T}_2))\,\mathrm{d}x\,\mathrm{d}s \le C\int_0^t \|\boldsymbol{v}_t(s)\|_2^2\,\mathrm{d}s.$$

Applying Gronwall's inequality, it follows that

$$\frac{\|\boldsymbol{v}_t(t)\|_2^2}{2} + \int_0^t\int_\Omega (\mathbf{T}_1-\mathbf{T}_2)\cdot(F_n(\mathbf{T}_1)-F_n(\mathbf{T}_2))\,\mathrm{d}x\,\mathrm{d}s = 0,$$

which implies that $\boldsymbol{u}_1=\boldsymbol{u}_2$, $\mathbf{T}_1=\mathbf{T}_2$ and so we have uniqueness of solutions. □

4.2 Higher regularity estimates

Now, we focus on proving higher regularity estimates for the solution of the regularised problem that are independent of n. We suppress the dependence of the regularised solution on n throughout this section for simplicity of notation.

4.2.1 Regularity in the time variable

Lemma 4.4. *Let the assumptions of Theorem 4.3 hold and let* $(\boldsymbol{u}, \mathbf{T})$ *be the solution of the regularised problem* (4.4) *constructed there. There exists a constant* C*, independent of* n*, such that*

$$\int_Q |\boldsymbol{u}_{tt}|^2 \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}(t)|^{1-a} + \frac{|\mathbf{T}(t)|^2}{n} \,\mathrm{d}x \leq C\Big(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{1,2}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2\Big).$$

Proof. Let $(\boldsymbol{u}^m, \mathbf{T}^m)$ denote the solution of the Galerkin approximation from V_m. Testing in (4.14) against $\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m$ we obtain

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_{tt}^m|^2 + \frac{\partial}{\partial t}\left(\frac{\alpha|\boldsymbol{u}_t^m|^2}{2} + \frac{|\mathbf{T}^m|^2}{2n}\right) + \mathbf{T}^m \cdot F(\mathbf{T}^m)_t \,\mathrm{d}x \\
&= \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m) \,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}_{n,m} \cdot (\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m) \,\mathrm{d}S \\
&\leq \|\boldsymbol{f}\|_2 \|\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m\|_2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\partial\Omega_N} \boldsymbol{g}_{n,m} \cdot (\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m) \,\mathrm{d}S\right) \\
&\quad - \int_{\partial\Omega_N} (\boldsymbol{g}_{n,m})_t \cdot (\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m) \,\mathrm{d}S.
\end{aligned}$$

Integrating over $(0, t)$ yields the following:

$$\begin{aligned}
&\int_0^t \int_\Omega |\boldsymbol{u}_{tt}^m|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \frac{|\mathbf{T}^m(t)|^2}{n} + |\mathbf{T}^m(t)|^{1-a}\chi_{\{|\mathbf{T}^m(t)|\geq 1\}} \,\mathrm{d}x \\
&\leq C\Big(1 + \int_0^t \|\boldsymbol{f}(\tau)\|_2 \|(\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m)(\tau)\|_2 + \|(\boldsymbol{g}_{n,m})_t(\tau)\|_{L^2(\partial\Omega_N)} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(\tau)\|_2 \,\mathrm{d}\tau \\
&\quad + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)} \|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_{1,2} + \|\boldsymbol{g}_{n,m}(0)\|_{L^2(\partial\Omega_N)} \|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(0)\|_{1,2} \\
&\quad + \|P^m\boldsymbol{u}_1\|_2^2 + \int_\Omega |\mathbf{T}^m(0)|^2 \,\mathrm{d}x\Big) \\
&\leq \frac{1}{2}\int_0^t \int_\Omega |\boldsymbol{u}_{tt}^m|^2 \,\mathrm{d}x\,\mathrm{d}\tau + C\Big(1 + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_2 \\
&\quad + \int_0^t \|\boldsymbol{f}(\tau)\|_2^2 + \|\boldsymbol{u}_t^m(\tau)\|_2^2 + \|(\boldsymbol{g}_{n,m})_t(\tau)\|_{L^2(\partial\Omega_N)}^2 + \|(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(\tau)\|_{1,2}^2 \,\mathrm{d}\tau \\
&\quad + \|\boldsymbol{g}_{n,m}(0)\|_{L^2(\partial\Omega_N)} \|P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_{1,2} + \|P^m\boldsymbol{u}_1\|_2^2 + \int_\Omega |\mathbf{T}^m(0)|^2 \,\mathrm{d}x\Big).
\end{aligned} \tag{4.17}$$

Next, notice that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\|_2 \leq \|F(\mathbf{T}^m)(t)\|_2 + \frac{\|\mathbf{T}^m(t)\|_2}{n} \leq |\Omega|^{\frac{1}{2}} + \frac{\|\mathbf{T}^m(t)\|_2}{n}. \tag{4.18}$$

To bound the $\boldsymbol{g}_{n,m}(0)$ term on the right-hand side of (4.17), we recall that, for every $m \geq m_0$, we have $\|\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\|_\infty \leq C_1 < 1$. Hence, there exists a constant C depending only on a, C_* and Ω such that

$$\|\boldsymbol{g}_{n,m}(0)\|_{L^2(\partial\Omega_N)} \leq \|F^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))\|_{L^2(\partial\Omega_N)} \leq C. \tag{4.19}$$

Reasoning as in the proof of Lemma 2.6, for every $m \geq m_0$, we have that

$$|\mathbf{T}^m(0)| \leq C. \tag{4.20}$$

Using the bounds in (4.16) with (4.18), (4.19), (4.20) and Young's inequality, it follows that for a constant C, independent of n and m, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \int_\Omega |\boldsymbol{u}_{tt}^m|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \frac{|\mathbf{T}^m(t)|^2}{n} + |\mathbf{T}^m(t)|^{1-a}\chi_{\{|\mathbf{T}^m(t)|\geq 1\}}\,\mathrm{d}x \\
&\leq C\Big(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{1,2}^2 + \int_0^t \|\boldsymbol{f}(\tau)\|_2^2 + \|\boldsymbol{g}(\tau)\|_{L^2(\partial\Omega_N)}^2 \\
&\quad + \|(\boldsymbol{g}_{n,m})_t(\tau)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}\tau + \|\boldsymbol{g}_{n,m}(t)\|_{L^2(\partial\Omega_N)}^2\Big) + \frac{1}{2}\int_\Omega \frac{|\mathbf{T}^m(t)|^2}{n}\,\mathrm{d}x,
\end{aligned}$$

for a constant C, independent of n and m, from which we deduce that

$$\begin{aligned}
&\int_Q |\boldsymbol{u}_{tt}^m|^2 \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]}\int_\Omega \frac{|\mathbf{T}^m(t)|^2}{n} + |\mathbf{T}^m(t)|^{1-a}\chi_{\{|\mathbf{T}^m(t)|\geq 1\}}\,\mathrm{d}x \\
&\leq C\Big(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{1,2}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t \\
&\quad + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}\Big).
\end{aligned} \tag{4.21}$$

Hence, up to a subsequence in m that can be chosen independent of n, we have that

$$\begin{aligned}
\boldsymbol{u}_{tt}^m &\rightharpoonup \boldsymbol{u}_{tt} && \text{weakly in } L^2(Q)^d, \\
\mathbf{T}^m &\overset{*}{\rightharpoonup} \mathbf{T} && \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^{d\times d}).
\end{aligned}$$

Furthermore, using weak lower semi-continuity, Fatou's lemma and the pointwise convergence of $(\mathbf{T}^m)_m$, taking $m \to \infty$ in (4.21) we have that

$$\begin{aligned}
&\int_Q |\boldsymbol{u}_{tt}|^2 \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]}\int_\Omega \frac{|\mathbf{T}(t)|^2}{n} + |\mathbf{T}(t)|^{1-a}\chi_{\{|\mathbf{T}(t)|\geq 1\}}\,\mathrm{d}x \leq C\Big(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 \\
&\quad + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{1,2}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}\Big),
\end{aligned}$$

as required. □

We now prove that the bound in Lemma 4.4 can be improved under further assumptions on the data. Indeed, we obtain higher regularity estimates with respect to the time variable. As in Chapter 2, we do this at the level of the Galerkin approximation because the regularised solution does not necessarily have continuity of $\boldsymbol{u}_{tt}$ at the initial time. In particular, we cannot bound it in terms of the initial data.

Lemma 4.5. *Let the assumptions of Theorem 4.3 hold and suppose additionally that $\boldsymbol{f} \in W^{1,2}(0,T;L^2(\Omega)^d)$. Then $\mathbf{T} \in W^{1,2}(0,T;L^2(\Omega)^{d\times d})$ and there exists a constant C, independent of n, such that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}_{tt}(t)\|_2 + \int_Q \frac{|\mathbf{T}_t|^2}{(1+|\mathbf{T}|)^{1+a}} + \frac{|\mathbf{T}_t|^2}{n}\,\mathrm{d}x\,\mathrm{d}t \leq C\Big(1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2}^2 \\
&\quad + \|\boldsymbol{f}(0)\|_2^2 + \int_0^T \|\boldsymbol{f}_t(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2\Big).
\end{aligned}$$

Proof. Let $\boldsymbol{u}^m$ denote the solution of the Galerkin approximation from V_m of the regularised problem with approximate stress tensor $\mathbf{T}^m$. Under the given assumptions, we have that $\boldsymbol{f} \in C([0,T];L^2(\Omega)^d)$ and $\boldsymbol{g}_{n,m} \in C([0,T];L^2(\partial\Omega_N)^d)$. Hence, $\boldsymbol{u}^m_{tt} \in C([0,T];V_m)$. In particular, for every $\boldsymbol{v} \in V_m$, taking $t=0$ and using integration by parts, we see that

$$\begin{aligned}
&\int_\Omega \boldsymbol{u}^m_{tt}(0)\cdot\boldsymbol{v}\,\mathrm{d}x \\
&= -\int_\Omega \mathbf{T}^m(0)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{f}(0)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}_{n,m}(0)\cdot\boldsymbol{v}\,\mathrm{d}S \\
&= \int_\Omega \big(\mathrm{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))) + \boldsymbol{f}(0)\big)\cdot\boldsymbol{v}\,\mathrm{d}x \\
&\quad - \int_{\partial\Omega_N} F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\mathbf{n}\cdot\boldsymbol{v}\,\mathrm{d}S \\
&\quad + \int_{\partial\Omega_N} F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\mathbf{n}\cdot\boldsymbol{v}\,\mathrm{d}S \\
&= \int_\Omega \big(\mathrm{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))) + \boldsymbol{f}(0)\big)\cdot\boldsymbol{v}\,\mathrm{d}x.
\end{aligned}$$

We see that we can identify

$$\boldsymbol{u}^m_{tt}(0) = P^m[\mathrm{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))) + \boldsymbol{f}(0)].$$

The function F_n^{-1} is continuously differentiable by Lemma 2.3. Using the chain rule, we can write

$$\begin{aligned}
&\|P^m[\mathrm{div}(F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))))]\|_2 \\
&\le \|D(F_n^{-1})(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_\infty \|\nabla(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_2 \\
&\le \|D(F_n^{-1})(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_\infty \|P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)\|_{2,2} \\
&\le C\|D(F_n^{-1})(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_\infty \|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}.
\end{aligned}$$

We apply the inverse function theorem to see that, for every $\mathbf{S} \in \mathbb{R}^{d\times d}$,

$$|D(F_n^{-1})(\mathbf{S})| = |(DF_n)^{-1}(F_n^{-1}(\mathbf{S}))| \le C_a(1+|F_n^{-1}(\mathbf{S})|^{a+1}).$$

If additionally $|\mathbf{S}| < 1$ then

$$|D(F_n^{-1})(\mathbf{S})| \le C_a(1+|F_n^{-1}(\mathbf{S})|^{a+1}) \le C_a(1+|F^{-1}(\mathbf{S})|^{a+1}).$$

Hence, for every $m \ge m_0$, it follows that

$$\|D(F_n^{-1})(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\|_\infty \le C,$$

where C is a constant depending only on a and C_*. Thus we deduce that

$$\|\boldsymbol{u}^m_{tt}(0)\|_2 \le C\left(\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2} + \|\boldsymbol{f}(0)\|_2\right). \tag{4.22}$$

This fact will be vital later in the proof when we look for a bound on $\boldsymbol{u}^m_{tt}$ in $L^\infty(0,T;L^2(\Omega)^d)$.

We denote by Δ_t^h the difference quotient of length h in the time variable. Fix an arbitrary $t \in (0,T)$ and let $h > 0$ be sufficiently small so that $t + h \leq T$. On $(0,t)$, for a test function $\boldsymbol{v} \in V_m$, we have that

$$\int_\Omega \Delta_t^h \boldsymbol{u}_{tt}^m \cdot \boldsymbol{v} + \Delta_t^h \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x = \int_\Omega \Delta_t^h \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\partial\Omega_N} \Delta_t^h \boldsymbol{g}_{n,m} \cdot \boldsymbol{v} \, \mathrm{d}S.$$

Choosing $\boldsymbol{v} = \Delta_t^h(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)$, it follows that

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t} \left(\frac{|\Delta_t^h \boldsymbol{u}_t^m|^2}{2} + \alpha \Delta_t^h \boldsymbol{u}_t^m \cdot \Delta_t^h \boldsymbol{u}^m \right) - \alpha |\Delta_t^h \boldsymbol{u}_t^m|^2 + \Delta_t^h \mathbf{T}^m \cdot \Delta_t^h F_n(\mathbf{T}^m) \, \mathrm{d}x \\
&= \int_\Omega \Delta_t^h \boldsymbol{f} \cdot \Delta_t^h (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) \, \mathrm{d}x + \int_{\partial\Omega_N} \Delta_t^h \boldsymbol{g}_{n,m} \cdot \Delta_t^h(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) \, \mathrm{d}S.
\end{aligned}$$

Replacing t with $\tau \in (0,t)$ and integrating over $(0,t)$ with respect to τ, we manipulate terms as in Lemma 2.8 to see that

$$\begin{aligned}
&\|\Delta_t^h \boldsymbol{u}^m(t)\|_2^2 + \|\Delta_t^h \boldsymbol{u}_t^m(t)\|_2^2 + \int_0^t \int_\Omega \Delta_t^h \mathbf{T}^m \cdot \Delta_t^h F_n(\mathbf{T}^m) \, \mathrm{d}x \, \mathrm{d}\tau \\
&\leq C(\epsilon) \left(\|\Delta_t^h \boldsymbol{u}^m(0)\|_2^2 + \|\Delta_t^h \boldsymbol{u}_t^m(0)\|_2^2 + \int_0^t \|\Delta_t^h \boldsymbol{f}\|_2^2 + \|\Delta_t^h \boldsymbol{g}_{n,m}\|_{L^2(\partial\Omega_N)}^2 \, \mathrm{d}\tau \right) \qquad (4.23) \\
&\quad + \epsilon \int_0^t \|\Delta_t^h (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\|_{1,2}^2 \, \mathrm{d}\tau,
\end{aligned}$$

where $\epsilon > 0$ is some constant which is to be fixed sufficiently small later in the proof. We apply the Korn–Poincaré inequality and use the constitutive relation to deduce that

$$\|\Delta_t^h (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\|_{1,2}^2 \leq C \|\Delta_t^h \boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m)\|_2^2 = C \int_\Omega |\Delta_t^h F(\mathbf{T}^m)|^2 + \frac{|\mathbf{T}^m|^2}{n^2} \, \mathrm{d}x. \qquad (4.24)$$

We claim that, for every $\mathbf{T}, \mathbf{S} \in \mathbb{R}^{d \times d}$,

$$|F(\mathbf{T}) - F(\mathbf{S})|^2 \leq (\mathbf{T} - \mathbf{S}) \cdot (F(\mathbf{T}) - F(\mathbf{S})). \qquad (4.25)$$

Without loss of generality, suppose that $|\mathbf{T}| \geq |\mathbf{S}|$. Then

$$\frac{1}{(1 + |\mathbf{S}|^a)^{\frac{1}{a}}} \geq \frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}},$$

and we also note that

$$\left(1 - \frac{1}{(1 + |\mathbf{S}|^a)^{\frac{1}{a}}} \right) |\mathbf{S}| \leq \left(1 - \frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} \right) |\mathbf{T}|.$$

Equivalently,

$$|F(\mathbf{T})| - |F(\mathbf{S})| = \frac{|\mathbf{T}|}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} - \frac{|\mathbf{S}|}{(1 + |\mathbf{S}|^a)^{\frac{1}{a}}} \leq |\mathbf{T}| - |\mathbf{S}|. \qquad (4.26)$$

Next, we write

$$\begin{aligned}
&(\mathbf{T} - \mathbf{S}) \cdot (F(\mathbf{T}) - F(\mathbf{S})) - |F(\mathbf{T}) - F(\mathbf{S})|^2 \\
&= \left(\frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} - \frac{1}{(1 + |\mathbf{T}|^a)^{\frac{2}{a}}} \right) |\mathbf{T}|^2 + \left(\frac{1}{(1 + |\mathbf{S}|^a)^{\frac{1}{a}}} - \frac{1}{(1 + |\mathbf{S}|^a)^{\frac{2}{a}}} \right) |\mathbf{S}|^2 \\
&\quad - \left(\frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{1}{(1 + |\mathbf{S}|^a)^{\frac{1}{a}}} - 2 \frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}} (1 + |\mathbf{S}|^a)^{\frac{1}{a}}} \right) \mathbf{T} \cdot \mathbf{S}.
\end{aligned}$$

Notice that for any $\xi,\ \zeta \in [0,1)$, we have that $\xi + \zeta - 2\xi\zeta > 0$. Using this with the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
&(\mathbf{T}-\mathbf{S})\cdot(F(\mathbf{T})-F(\mathbf{S})) - |F(\mathbf{T})-F(\mathbf{S})|^2\\
&\geq \left(\frac{1}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} - \frac{1}{(1+|\mathbf{T}|^a)^{\frac{2}{a}}}\right)|\mathbf{T}|^2 + \left(\frac{1}{(1+|\mathbf{S}|^a)^{\frac{1}{a}}} - \frac{1}{(1+|\mathbf{S}|^a)^{\frac{2}{a}}}\right)|\mathbf{S}|^2\\
&\quad - \left(\frac{1}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{1}{(1+|\mathbf{S}|^a)^{\frac{1}{a}}} - 2\frac{1}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}(1+|\mathbf{S}|^a)^{\frac{1}{a}}}\right)|\mathbf{T}||\mathbf{S}|\\
&= (|F(\mathbf{T})|-|F(\mathbf{S})|)\cdot(|\mathbf{T}|-|\mathbf{S}|-|F(\mathbf{T})|+|F(\mathbf{S})|) \geq 0,
\end{aligned}$$

where the final line follows from (4.24). Hence (4.25) holds as required.

Returning to (4.24), we use (4.25) to yield

$$\|\Delta_t^h(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\|_{1,2}^2 \leq C\int_\Omega \Delta_t^h\mathbf{T}^m \cdot \Delta_t^h F_n(\mathbf{T}^m)\,\mathrm{d}x.$$

Fixing ϵ appropriately small so that the final term on the right-hand side of (4.23) can be absorbed into the left, it follows that

$$\begin{aligned}
&\|\Delta_t^h\boldsymbol{u}^m(t)\|_2^2 + \|\Delta_t^h\boldsymbol{u}_t^m(t)\|_2^2 + \int_0^t\int_\Omega \Delta_t^h\mathbf{T}^m\cdot\Delta_t^h F_n(\mathbf{T}^m)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq C\left(\|\Delta_t^h\boldsymbol{u}^m(0)\|_2^2 + \|\Delta_t^h\boldsymbol{u}_t^m(0)\|_2^2 + \int_0^t \|\Delta_t^h\boldsymbol{f}\|_2^2 + \|\Delta_t^h\boldsymbol{g}_{n,m}\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}\tau\right).
\end{aligned}$$

We divide through by h^2 and let $h \to 0+$ to deduce that

$$\begin{aligned}
&\|\boldsymbol{u}_{tt}^m(t)\|_2^2 + \int_0^t\int_\Omega \mathbf{T}_t^m\cdot F(\mathbf{T}^m)_t + \frac{|\mathbf{T}_t^m|^2}{n}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq C(\epsilon)\left(\|\boldsymbol{u}_{tt}^m(0)\|_2^2 + \|\boldsymbol{u}_t^m(0)\|_2^2 + \int_0^t \|\boldsymbol{f}_t\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau\right),
\end{aligned}$$

using the dominated convergence theorem to justify taking the limit inside the integral. We apply (4.22) to obtain

$$\begin{aligned}
&\|\boldsymbol{u}_{tt}^m(t)\|_2^2 + \int_0^t\int_\Omega \mathbf{T}_t^m\cdot F(\mathbf{T}^m)_t + \frac{|\mathbf{T}_t^m|^2}{n}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq C(\epsilon)\left(\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{f}(0)\|_2^2 + \int_0^t \|\boldsymbol{f}_t\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau\right).
\end{aligned} \tag{4.27}$$

Recalling that $((\boldsymbol{g}_{n,m})_t)_m$ is bounded in $L^2(0,T;L^2(\partial\Omega_N)^d)$, we deduce that

$$\begin{aligned}
\boldsymbol{u}_{tt}^m &\overset{*}{\rightharpoonup} \boldsymbol{u}_{tt} && \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^d),\\
\mathbf{T}_t^m &\rightharpoonup \mathbf{T}_t && \text{weakly in } L^2(Q)^{d\times d}.
\end{aligned}$$

Performing the explicit differentiation, we note that there exists a constant C, depending only on a, such that

$$\mathbf{T}_t^m\cdot F(\mathbf{T}^m)_t \geq C\frac{|\mathbf{T}_t^n|^2}{(1+|\mathbf{T}^m|)^{1+a}}.$$

Using the pointwise convergence result of $(\mathbf{T}^m)_m$ and the dominated convergence theorem, $((1+|\mathbf{T}^m|)^{-(1+a)})_m$ converges strongly in $L^p(Q)^{d\times d}$ to $(1+|\mathbf{T}|)^{-(1+a)}$ for any $p\in[1,\infty)$. Combining this strong convergence with the weak convergence of $(\mathbf{T}^m_t)_m$, it follows that

$$\frac{\mathbf{T}^m_t}{(1+|\mathbf{T}^m|)^{\frac{1+a}{2}}} \rightharpoonup \frac{\mathbf{T}_t}{(1+|\mathbf{T}|)^{\frac{1+a}{2}}} \quad \text{weakly in } L^2(Q)^{d\times d}.$$

Hence taking $m\to\infty$ in (4.27) and using weak lower semi-continuity, we get the required result. □

As a consequence of this, we are able to prove the following result concerning the regularity of $\boldsymbol{u}_{tt}$.

Corollary 4.6. *Let the assumptions of Lemma 4.5 hold and suppose additionally that $\boldsymbol{u}_0$, $\boldsymbol{u}_1\in W^{1,2}_D(\Omega)^d$. Then $\boldsymbol{u}\in W^{2,2}(0,T;W^{1,2}_D(\Omega)^d)$ and there exists a constant C, independent of n, such that*

$$\begin{aligned}\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{1,2}^2 + \sup_{t\in[0,T]}\|\boldsymbol{u}_t(t)\|_{1,2}^2 + \int_0^T \|\boldsymbol{u}_{tt}(t)\|_{1,2}\,\mathrm{d}t \le C\Big(1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_1\|_{1,2}^2 \\ + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{f}_t(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}t + \sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|^2_{L^2(\partial\Omega_N)}\Big).\end{aligned}$$

Proof. Recalling (4.25), there exists a constant C, independent of n, such that

$$|F_n(\mathbf{T})-F_n(\mathbf{S})|^2 \le C(\mathbf{T}-\mathbf{S})\cdot(F_n(\mathbf{T})-F_n(\mathbf{S})),$$

for every $\mathbf{T}$, $\mathbf{S}\in\mathbb{R}^{d\times d}$. From the constitutive relation, it follows that

$$|\Delta^h_t\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t+\alpha\boldsymbol{u}^m)|^2 = |\Delta^h_t F_n(\mathbf{T}^m)|^2 \le C\Delta^h_t\mathbf{T}^m\cdot\Delta^h_t F_n(\mathbf{T}^m).$$

Using the proof of Lemma 4.5, for every $0<t_1<t_2<T$ and $h>0$ sufficiently small, we see that

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega \frac{|\Delta^h_t\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t+\alpha\boldsymbol{u}^m)|^2}{h^2}\,\mathrm{d}x\,\mathrm{d}t \le C\int_{t_1}^{t_2}\int_\Omega \frac{\Delta^h_t\mathbf{T}^m}{h}\cdot\frac{\Delta^h_t F_n(\mathbf{T}^m)}{h}\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\Big(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2+\int_0^T\|\boldsymbol{f}_t(t)\|_2^2+\|\boldsymbol{g}_t(t)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}t \\ &\quad+\sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|^2_{L^2(\partial\Omega_N)}\Big)+Cn\|F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))-F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\|_\infty,\end{aligned}$$

where C is a positive constant that is independent of m and n. We let $h\to 0+$ and apply the Korn–Poincaré inequality (Theorem 1.9) to deduce that

$$\begin{aligned}\int_0^T\|(\boldsymbol{u}^m_{tt}+\alpha\boldsymbol{u}^m_t)(t)\|_{1,2}^2\,\mathrm{d}t \le C\Big(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2 \\ +\int_0^T\|\boldsymbol{f}_t(t)\|_2^2+\|\boldsymbol{g}_t(t)\|^2_{L^2(\partial\Omega_N)}\,\mathrm{d}t+\sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|^2_{L^2(\partial\Omega_N)}\Big) \\ +Cn\|F_n^{-1}(\boldsymbol{\varepsilon}(P^m(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))-F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\|_\infty.\end{aligned}$$

The right-hand side is bounded above, independent of m, and so $\boldsymbol{u}_{tt}+\alpha\boldsymbol{u}_t \in L^2(0,T;W_D^{1,2}(\Omega)^d)$. Taking the limit as $m\to\infty$ in the above yields

$$\int_0^T \|(\boldsymbol{u}_{tt}+\alpha\boldsymbol{u}_t)(t)\|_{1,2}^2\,\mathrm{d}t \le C\Big(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2 + \int_0^T \|\boldsymbol{f}_t(t)\|_2^2+\|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t + \sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2\Big).$$

Next, recall that $\boldsymbol{u}_t+\alpha\boldsymbol{u}\in L^2(0,T;W_D^{1,2}(\Omega)^d)$. Since $\boldsymbol{u}_{tt}+\alpha\boldsymbol{u}_t\in L^2(0,T;W_D^{1,2}(\Omega)^d)$, we have that $\boldsymbol{u}_t+\alpha\boldsymbol{u}\in L^\infty(0,T;W_D^{1,2}(\Omega)^d)$. Now, we show that $\boldsymbol{u}\in L^\infty(0,T;W_D^{1,2}(\Omega)^d)$. It will follow from this that $\boldsymbol{u}_t\in L^\infty(0,T;W_D^{1,2}(\Omega)^d)$ and thus also $\boldsymbol{u}_{tt}\in L^2(0,T;W_D^{1,2}(\Omega)^d)$ as required.

We reason at the level of the Galerkin approximation, as usual, in order to ensure the arguments are fully rigorous. First, we notice that

$$\frac{\partial}{\partial t}(\mathrm{e}^{\alpha t}\boldsymbol{u}^m)=\mathrm{e}^{\alpha t}(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}_m).$$

Integrating this over $(0,t)$ yields

$$\boldsymbol{u}^m(t)=\mathrm{e}^{-\alpha t}\boldsymbol{u}^m(0)+\int_0^t \mathrm{e}^{\alpha(\tau-t)}(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)(\tau)\,\mathrm{d}\tau. \tag{4.28}$$

Since $\boldsymbol{u}^m(0)=P^m\boldsymbol{u}_0\in W_D^{1,2}(\Omega)^d$ and $\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m\in L^2(0,T;W_D^{1,2}(\Omega)^d)$, the right-hand side of (4.28) is an element of $W_D^{1,2}(\Omega)^d$ for every $t\in(0,T)$. Hence the left-hand side is also an element of this space. It follows that

$$\begin{aligned}\|\boldsymbol{u}^m(t)\|_{1,2}^2 &\le C\left(\|\boldsymbol{u}^m(0)\|_{1,2}^2+\int_0^t \|(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)(\tau)\|_{1,2}^2\,\mathrm{d}\tau\right)\\ &\le C\left(\|\boldsymbol{u}_0\|_{1,2}^2+\int_0^t \|(\boldsymbol{u}_t^m+\alpha\boldsymbol{u}^m)(\tau)\|_{1,2}^2\,\mathrm{d}\tau\right).\end{aligned}$$

Taking the supremum over $(0,T)$ and using Theorem 4.3, it follows that $(\boldsymbol{u}^m)_m$ is bounded uniformly in $L^\infty(0,T;W_D^{1,2}(\Omega)^d)$. Thus $\boldsymbol{u}\in L^\infty(0,T;W_D^{1,2}(\Omega)^d)$ and satisfies the bound

$$\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{1,2}\le C\Big(1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2+\|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t\Big).$$

Hence $\boldsymbol{u}_t\in L^2(0,T;W_D^{1,2}(\Omega)^d)$ and so also $\boldsymbol{u}_{tt}\in L^2(0,T;W_D^{1,2}(\Omega)^d)$. Using this and the bound on $\boldsymbol{u}_{tt}+\alpha\boldsymbol{u}_t$ in $L^2(0,T;W^{1,2}(\Omega)^d)$, we see that

$$\begin{aligned}\sup_{t\in[0,T]}\|\boldsymbol{u}_t(t)\|_{1,2}^2 &\le C\left(\sup_{t\in[0,T]}\|(\boldsymbol{u}_t+\alpha\boldsymbol{u})(t)\|_{1,2}^2+\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{1,2}^2\right)\\ &\le C\left(\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2+\int_0^T\|(\boldsymbol{u}_{tt}+\alpha\boldsymbol{u}_t)(t)\|_{1,2}^2\,\mathrm{d}t+\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{1,2}^2\right)\\ &\le C\left(1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_1\|_{1,2}^2+\int_0^T\|\boldsymbol{f}\|_2^2+\|\boldsymbol{f}_t\|_2^2+\|\boldsymbol{g}_t\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t+\sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2\right).\end{aligned}$$

Finally, we have that

$$\int_0^T \|\boldsymbol{u}_{tt}(t)\|_{1,2}^2 \,\mathrm{d}t \le C\Big(1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_{1,2}^2 + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{f}_t(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t\Big),$$

as required and so the desired bound follows. □

Using the improved regularity of $\mathbf{T}$, we can also improve the bound from Lemma 4.4 and show that $\mathbf{T} \in L^\infty(0,T;L^1(\Omega)^{d\times d})$ with bound in this space that is independent of n. This is a significant improvement because boundedness in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ implies weak-* compactness in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$, rather than $\mathcal{M}(\overline{Q})^{d\times d}$.

Lemma 4.7. *Suppose that the assumptions of Corollary 4.6 hold. There exists a constant C, independent of n, such that*

$$\sup_{t\in[0,T]} \int_\Omega |\mathbf{T}(t)| + \frac{|\mathbf{T}(t)|^2}{n} \,\mathrm{d}x \le C\Big(1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_{1,2}^2 + \sup_{t\in[0,T]} \|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2 + \int_0^T \|\boldsymbol{f}(t)\|_2^2 + \|\boldsymbol{f}_t(t)\|_2^2 + \|\boldsymbol{g}_t(t)\|_{L^2(\partial\Omega_N)}^2 \,\mathrm{d}t\Big).$$

Proof. Using difference quotients in the time variable at the level of the Galerkin approximation, we see that

$$\int_\Omega \Delta_t^h \boldsymbol{u}_{tt}^m \cdot (\boldsymbol{u}_t^m + \alpha \boldsymbol{u}^m) + \Delta_t^h \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}x = \int_\Omega \Delta_t^h \boldsymbol{f}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}x + \int_{\partial\Omega_N} \Delta_t^h \boldsymbol{g}_{n,m}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}S.$$

For the inertial term, that is the one involving $\boldsymbol{u}_{tt}^m$, we write

$$\int_\Omega \Delta_t^h \boldsymbol{u}_{tt}^m \cdot (\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial t}\Big(\Delta_t^h \boldsymbol{u}_t^m \cdot (\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\Big) - \Delta_t^h \boldsymbol{u}_t^m \cdot (\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m)\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}
&\int_0^t \int_\Omega \Delta_t^h \mathbf{T}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^t \int_\Omega \Delta_t^h \boldsymbol{f}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m) + \Delta_t^h \boldsymbol{u}_t^m \cdot (\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int_{\partial\Omega_N} \Delta_t^h \boldsymbol{g}_{n,m}\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}S\,\mathrm{d}\tau \\
&\quad + \int_\Omega \Delta_t^h \boldsymbol{u}_t^m(0)\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(0) - \Delta_t^h \boldsymbol{u}_t^m(t)\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(t)\,\mathrm{d}x.
\end{aligned}$$

Dividing through by h and letting $h \to 0$ yields

$$\begin{aligned}
\int_0^t \int_\Omega \mathbf{T}_t^m \cdot F(\mathbf{T}^m)\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t \int_\Omega \boldsymbol{f}_t\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m) + \boldsymbol{u}_{tt}^m\cdot(\boldsymbol{u}_{tt}^m + \alpha\boldsymbol{u}_t^m)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int_{\partial\Omega_N} (\boldsymbol{g}_{n,m})_t\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)\,\mathrm{d}S\,\mathrm{d}\tau - \Big[\int_\Omega \boldsymbol{u}_{tt}^m(s)\cdot(\boldsymbol{u}_t^m + \alpha\boldsymbol{u}^m)(s)\,\mathrm{d}x\Big]_{s=0}^{s=t} \qquad (4.29) \\
&\le C\Big(\int_0^t \|\boldsymbol{f}_t(\tau)\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau + \sup_{\tau\in[0,t]}\Big[\|\boldsymbol{u}^m\|_{1,2}^2 + \|\boldsymbol{u}_t^m\|_{1,2}^2 + \|\boldsymbol{u}_{tt}^m\|_2^2\Big]\Big).
\end{aligned}$$

To deal with the left-hand side, we notice that

$$\mathbf{T}_t^m \cdot F_n(\mathbf{T}^m) = \left(\frac{1}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{1}{n}\right)\frac{\partial}{\partial t}\left(\frac{|\mathbf{T}^m|^2}{2}\right) = \frac{\partial}{\partial t}\left(\frac{g_n(|\mathbf{T}^m|^2)}{2}\right),$$

where g_n is defined on $[0,\infty)$ by

$$g_n(s) = \int_0^s \frac{1}{(1+u^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}u + \frac{s}{n}.$$

Suppose $s \geq 2$. Then

$$g_n(s) \geq \int_{\frac{s}{2}}^s \frac{1}{(1+u^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}u + \frac{s}{n} \geq \frac{s}{2}\frac{1}{2^{\frac{1}{a}}s^{\frac{1}{2}}} + \frac{s}{n} = \frac{s^{\frac{1}{2}}}{2^{1+\frac{1}{a}}} + \frac{s}{n}.$$

Furthermore, we easily see that $g_n(s) \leq 2s$ for every $s \geq 0$. Returning to (4.29), we obtain the following:

$$\begin{aligned}
&\int_\Omega |\mathbf{T}^m(t)| + \frac{|\mathbf{T}^m(t)|^2}{n}\,\mathrm{d}x \leq C\int_\Omega g_n(|\mathbf{T}^m(t)|^2)\,\mathrm{d}x \\
&\leq C\Big(\int_\Omega g_n(|\mathbf{T}^m(0)|^2)\,\mathrm{d}x + \int_0^t \|\boldsymbol{f}_t(\tau)\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau + \sup_{\tau\in[0,t]}\|\boldsymbol{u}^m(\tau)\|_{1,2}^2 \\
&\quad + \sup_{\tau\in[0,t]}\|\boldsymbol{u}_t^m(\tau)\|_{1,2}^2 + \sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^m(\tau)\|_2^2\Big) \\
&\leq C\Big(\int_\Omega |\mathbf{T}^m(0)|^2\,\mathrm{d}x + \int_0^t \|\boldsymbol{f}_t(\tau)\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau + \sup_{\tau\in[0,t]}\|\boldsymbol{u}^m(\tau)\|_{1,2}^2 \\
&\quad + \sup_{\tau\in[0,t]}\|\boldsymbol{u}_t^m(\tau)\|_{1,2}^2 + \sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^m(\tau)\|_2^2\Big) \\
&\leq C\Big(1 + \int_0^t \|\boldsymbol{f}_t(\tau)\|_2^2 + \|(\boldsymbol{g}_{n,m})_t\|_{L^2(\partial\Omega_N)}\,\mathrm{d}\tau + \sup_{\tau\in[0,t]}\|\boldsymbol{u}^m(\tau)\|_{1,2}^2 + \sup_{\tau\in[0,t]}\|\boldsymbol{u}_t^m(\tau)\|_{1,2}^2 \\
&\quad + \sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^m(\tau)\|_2^2\Big),
\end{aligned}$$

for every $m \geq m_0$, where m_0 is as in Lemma 4.4, and C is independent of m and n. Applying the estimates from Theorem 4.3, Lemma 4.5 and Corollary 4.6 with the pointwise convergence of $(\mathbf{T}^m)_m$ and Fatou's lemma, the required result follows. □

4.2.2 Regularity in the spatial variable

In this section, we concentrate on improving the spatial regularity of the solution of the regularised problem $(\boldsymbol{u}, \mathbf{T})$. First, we show that $\mathbf{T} \in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$, possible due to the linear regularisation term. It follows that $\boldsymbol{u}_{tt} = \mathrm{div}(\mathbf{T}) + \boldsymbol{f}$ is satisfied pointwise in Q. This is used to show that a bound analogous to that of Lemma 4.5 holds but with respect to spatial derivatives rather than time derivatives. However, all bounds are only local in space because we use the difference quotient method.

Lemma 4.8. *Suppose that the assumptions of Lemma 4.4 hold. Suppose additionally that the data satisfies $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_{loc}(\Omega)^d$ and $\boldsymbol{f} \in L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$. Let $\Omega_0 \subset \Omega_1 \subset \Omega$ be open subsets* *such that each containment is compact so* $\overline{\Omega_0} \subset \Omega_1$ *and* $\overline{\Omega_1} \subset \Omega$. *Then* $\mathbf{T} \in L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$ *and there exists a constant* C, *depending* n, Ω_0 *and* Ω_1 *such that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{W^{1,2}(\Omega_0)} + \sup_{t\in[0,T]} \|\boldsymbol{u}_t(t)\|^2_{W^{1,2}(\Omega_0)} + \int_0^T \int_{\Omega_0} \frac{|\nabla \mathbf{T}|^2}{n} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C(n)\Big(1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2 + \|\nabla \boldsymbol{u}_0\|^2_{L^2(\Omega_1)} + \|\nabla \boldsymbol{u}_1\|^2_{L^2(\Omega_1)} \\
&\quad + \int_0^T \|\boldsymbol{f}\|_2^2 + \|\nabla \boldsymbol{f}\|^2_{L^2(\Omega_0)} + \|\boldsymbol{g}(t)\|^2_{L^2(\partial\Omega_N)} \,\mathrm{d}t\Big).
\end{aligned}$$

Proof. Let Ω_0 and Ω_1 be as in the hypotheses of the lemma. Fix a smooth cut-off function $\tau \in C_c^\infty(\Omega)$ with $\tau = 1$ in Ω_0 and $\mathrm{supp}(\tau) \subset \Omega_1$. Let Δ_i^h denote the difference quotient of length h in the i-th co-ordinate direction. For h sufficiently small, depending on Ω_0 and Ω_1, for an arbitrary $\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)^d$, we have that

$$\int_\Omega \Delta_i^h \boldsymbol{u}_{tt} \cdot \left(\tau^2 \Delta_i^h \boldsymbol{v}\right) + \Delta_i^h \mathbf{T} \cdot \boldsymbol{\varepsilon}\left(\tau^2 \Delta_i^h \boldsymbol{v}\right) \,\mathrm{d}x = \int_\Omega \Delta_i^h \boldsymbol{f} \cdot \left(\tau^2 \Delta_i^h \boldsymbol{v}\right) \,\mathrm{d}x,$$

where the boundary traction term $\boldsymbol{g}_n$ vanishes due to the compact support of τ. Replacing $\boldsymbol{v}$ with $\boldsymbol{u}_t + \alpha\boldsymbol{u}$, it follows that

$$\begin{aligned}
&\int_\Omega \Delta_i^h \boldsymbol{u}_{tt} \cdot \left(\tau^2 \Delta_i^h (\boldsymbol{u}_t + \alpha\boldsymbol{u})\right) + \Delta_i^h \mathbf{T} \cdot \boldsymbol{\varepsilon}\left(\tau^2 \Delta_i^h (\boldsymbol{u}_t + \alpha\boldsymbol{u})\right) \,\mathrm{d}x \\
&= \int_\Omega \frac{\partial}{\partial t}\left(\frac{|\tau \Delta_i^h \boldsymbol{u}_t|^2}{2} + \alpha\tau^2 \Delta_i^h \boldsymbol{u}_t \cdot \Delta_i^h \boldsymbol{u}\right) - \alpha\tau^2 |\Delta_i^h \boldsymbol{u}_t|^2 + \tau^2 \Delta_i^h \mathbf{T} \cdot \Delta_i^h F_n(\mathbf{T}) \\
&\quad + 2\tau \Delta_i^h \mathbf{T} \cdot \left(\Delta_i^h (\boldsymbol{u}_t + \alpha\boldsymbol{u}) \otimes \nabla\tau\right) \,\mathrm{d}x \\
&\geq \int_\Omega \frac{\partial}{\partial t}\left(\frac{|\tau \Delta_i^h \boldsymbol{u}_t|^2}{2} + \alpha\tau^2 \Delta_i^h \boldsymbol{u}_t \cdot \Delta_i^h \boldsymbol{u}\right) - \alpha\tau^2 |\Delta_i^h \boldsymbol{u}_t|^2 + \tau^2 \frac{|\Delta_i^h \mathbf{T}|^2}{2n} \,\mathrm{d}x \\
&\quad - C(n) \||\nabla\tau| \Delta_i^h (\boldsymbol{u}_t + \alpha\boldsymbol{u})\|_2^2.
\end{aligned}$$

Integrating the resulting inequality over $(0,t)$ and arguing as in the proof of Theorem 2.4, there exists a constant C, depending on n, such that

$$\begin{aligned}
&\|\tau \Delta_i^h \boldsymbol{u}(t)\|_2^2 + \|\tau \Delta_i^h \boldsymbol{u}_t(t)\|_2^2 + \int_0^t \int_\Omega \tau^2 \frac{|\Delta_i^h \mathbf{T}|^2}{2n} \,\mathrm{d}x\,\mathrm{d}s \\
&\leq C(n)\left(\|\tau \Delta_i^h \boldsymbol{u}(0)\|_2^2 + \|\tau \Delta_i^h \boldsymbol{u}_t(0)\|_2^2 + \int_0^t \|\tau \Delta_i^h \boldsymbol{f}\|_2^2 + \||\nabla\tau| \Delta_i^h (\boldsymbol{u}_t + \alpha\boldsymbol{u})\|_2^2 \,\mathrm{d}s\right).
\end{aligned}$$

We divide through by h^2 and let $h \to 0+$ to see that the approximate stress tensor $\mathbf{T}$ is weakly differentiable in each spatial co-ordinate direction, by standard results on difference quotients.

Furthermore, we have that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}(t)\|_{L^2(\Omega_0)}^2+\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}_t(t)\|_{L^2(\Omega_0)}^2+\int_0^T\int_{\Omega_0}\frac{|\nabla \mathbf{T}|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq \sup_{t\in[0,T]}\|\tau\nabla \boldsymbol{u}(t)\|_2^2+\sup_{t\in[0,T]}\|\tau\nabla \boldsymbol{u}_t(t)\|_2^2+\int_Q\tau^2\frac{|\nabla \mathbf{T}|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C(n)\left(\|\tau\nabla \boldsymbol{u}_0\|_2^2+\|\tau\nabla \boldsymbol{u}_1\|_2^2+\int_0^T\|\tau\nabla \boldsymbol{f}\|_2^2+\||\nabla\tau|\nabla(\boldsymbol{u}_t+\alpha\boldsymbol{u})\|_2^2\,\mathrm{d}t\right)\\
&\leq C(n)\left(\|\nabla \boldsymbol{u}_0\|_{L^2(\Omega_1)}^2+\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega_1)}^2+\int_0^T\|\nabla \boldsymbol{f}\|_{L^2(\Omega_1)}^2\,\mathrm{d}t\right)\\
&\quad+C(n,\|\nabla\tau\|_\infty)\left(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\sup_{t\in[0,T]}\|\boldsymbol{g}\|_{L^2(\partial\Omega_N)}^2+\int_0^T\|\boldsymbol{f}\|_2^2+\|\boldsymbol{g}_t\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t\right).
\end{aligned}$$

Noticing that τ can be chosen such that $\|\nabla\tau\|_\infty$ depends only on Ω_0 and Ω_1, the required bound follows. ☐

Clearly the estimate in Lemma 4.8 has a dependence on n on the right-hand side so is not useful when taking the limit as $n\to\infty$. However, we will use the extra regularity of $\mathbf{T}$, and in particular the fact that (4.4a) holds pointwise, to prove an estimate that is independent of n.

Lemma 4.9. *Suppose that the assumptions of Lemma 4.8 hold, with compactly contained open subsets $\Omega_0\subset\Omega_1\subset\Omega$ such that Ω_0, Ω_1 have smooth boundaries. There exists a constant C, depending on Ω_0 and Ω_1 but independent of n, such that*

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}(t)\|_{L^2(\Omega_0)}^2+\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}_t(t)\|_{L^2(\Omega_0)}^2+\int_0^T\int_{\Omega_0}\frac{|\nabla \mathbf{T}|^2}{(1+|\mathbf{T}|)^{1+a}}+\frac{|\nabla \mathbf{T}|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\Big(1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2+\|\nabla \boldsymbol{u}_0\|_{L^2(\Omega_1)}^2+\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega_1)}^2\\
&\quad+\int_0^T\|\boldsymbol{f}\|_2^2+\|\nabla \boldsymbol{f}\|_{L^2(\Omega_1)}^2+\|\boldsymbol{g}_t\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t+\sup_{t\in[0,T]}\|\boldsymbol{g}(t)\|_{L^2(\partial\Omega_N)}^2\Big).
\end{aligned}$$

Proof. Using the weak form of (4.4a) with the regularity result of Lemma 4.8, we immediately see that

$$\boldsymbol{u}_{tt}=\mathrm{div}(\mathbf{T})+\boldsymbol{f}\qquad\text{pointwise a.e. in Q.}\tag{4.30}$$

By the continuity and boundedness of F and DF, $\mathbf{T}\in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$ implies that $F(\mathbf{T})\in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$, recalling the chain rule for weakly differentiable functions. Such a result is well-known in one spatial dimension but in higher dimensions is not freely available in the literature. Hence we prove the claim for this choice of F. Indeed, $F\in C^1(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})$ with derivative

$$\frac{\partial F(\mathbf{T})_{ij}}{\partial \mathbf{T}_{kl}}=\frac{\delta_{ik}\delta_{jl}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}-\frac{|\mathbf{T}|^{a-2}\mathbf{T}_{ij}\mathbf{T}_{kl}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}.$$

Thus we have that $|\partial F(\mathbf{T})_{ij}/\partial\mathbf{T}_{kl}|\leq 2$. Furthermore, from Lemma 3.2 in [19], we have that $|F(\mathbf{T})-F(\mathbf{S})|\leq 2|\mathbf{T}-\mathbf{S}|$ for every $\mathbf{T}$, $\mathbf{S}\in\mathbb{R}^{d\times d}$. Since $\mathbf{T}\in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$, we have $\mathbf{T}(t)\in W^{1,2}_{loc}(\Omega)^{d\times d}$ for a.e. $t\in(0,T)$. Then there exists a sequence $(\mathbf{S}_q)_q\subset C^1_c(\Omega)^{d\times d}$ such that $\mathbf{S}_q|_{\Omega_0} \to \mathbf{T}(t)$ strongly in $W^{1,2}(\Omega_0)^{d\times d}$ and pointwise a.e. on Ω_0 as $q \to \infty$. Using the chain rule for classically differentiable functions with integration by parts, for every test function $v \in C^1_c(\Omega_0)$ and every $p \in \{1, \dots, d\}$, we have that

$$\int_{\Omega_0} F(\mathbf{S}_q)_{ij} \frac{\partial v}{\partial x_p} \, \mathrm{d}x = -\int_{\Omega_0} \frac{\partial F(\mathbf{S}_q)_{ij}}{\partial \mathbf{S}_{kl}} \frac{\partial (\mathbf{S}_q)_{kl}}{\partial x_p} v \, \mathrm{d}x.$$

Applying the pointwise convergence result for $(\mathbf{S}_q)_q$ with the uniform bounds on F and its derivative, we see that

$$F(\mathbf{S}_q)_{ij} \to F(\mathbf{T}(t))_{ij} \text{ and } \frac{\partial F(\mathbf{S}_q)_{ij}}{\partial \mathbf{S}_{kl}} \to \frac{\partial F(\mathbf{T}(t))_{ij}}{\partial \mathbf{S}_{kl}} \quad \text{strongly in } L^p(\Omega_0),$$

for any $p \in [1, \infty)$. It follows that

$$\begin{aligned} \int_{\Omega_0} F(\mathbf{T}(t))_{ij} \frac{\partial v}{\partial x_p} \, \mathrm{d}x &= \lim_{q\to\infty} \int_{\Omega_0} F(\mathbf{S}_q)_{ij} \frac{\partial v}{\partial x_p} \, \mathrm{d}x \\ &= \lim_{q\to\infty} -\int_{\Omega_0} \frac{\partial F(\mathbf{S}_q)_{ij}}{\partial \mathbf{S}_{kl}} \frac{\partial (\mathbf{S}_q)_{kl}}{\partial x_p} v \, \mathrm{d}x = -\int_{\Omega_0} \frac{\partial F(\mathbf{T}(t))_{ij}}{\partial \mathbf{S}_{kl}} \frac{\partial \mathbf{T}(t)_{kl}}{\partial x_p} v \, \mathrm{d}x. \end{aligned}$$

Hence $F(\mathbf{T}(t))$ is weakly differentiable in Ω_0. As Ω_0 is arbitrary, $F(\mathbf{T}(t))$ is weakly differentiable in Ω with derivative in $L^2(0,T; L^2_{loc}(\Omega))$, given by what we would expect from the chain rule. Thus $F(\mathbf{T}) \in L^2(0,T; W^{1,2}_{loc}(\Omega)^{d\times d})$.

From the constitutive relation, it follows that $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) \in L^2(0,T; W^{1,2}_{loc}(\Omega)^{d\times d})$. We claim that $\boldsymbol{u}_t + \alpha\boldsymbol{u} \in L^2(0,T; W^{2,2}_{loc}(\Omega)^d)$. We prove this via a density result. Suppose that $\boldsymbol{v} \in W^{2,2}(\Omega_0)^d$. Applying Korn's inequality (Theorem 1.6), there exists a constant C, depending only on Ω_0, such that

$$\begin{aligned} \|\boldsymbol{v}\|_{W^{2,2}(\Omega_0)} &\le C\Bigg(\|\boldsymbol{v}\|_{W^{1,2}(\Omega_0)} + \sum_{i=1}^{d} \|\boldsymbol{\varepsilon}(\partial_i \boldsymbol{v})\|_{L^2(\Omega_0)} \Bigg) \\ &\le C\left(\|\boldsymbol{v}\|_{W^{1,2}(\Omega_0)} + \|\nabla(\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_{L^2(\Omega_0)} \right) \\ &\le C\left(\|\boldsymbol{v}\|_{W^{1,2}(\Omega_0)} + \|\nabla^2 \boldsymbol{v}\|_{L^2(\Omega_0)} \right) \le C\|\boldsymbol{v}\|_{W^{2,2}(\Omega_0)}. \end{aligned}$$

In particular, $\|\boldsymbol{v}\|_{W^{1,2}(\Omega_0)} + \|\nabla(\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_{L^2(\Omega_0)}$ is a norm on $W^{2,2}(\Omega_0)^d$ that is equivalent to the usual norm on $W^{2,2}(\Omega_0)$. It follows that if $\boldsymbol{v} \in W^{1,2}(\Omega_0)$ is such that $\boldsymbol{\varepsilon}(\boldsymbol{v}) \in W^{1,2}(\Omega_0)^{d\times d}$, then necessarily $\boldsymbol{v} \in W^{2,2}(\Omega_0)^d$. Thus the claim holds and $\boldsymbol{u}_t + \alpha\boldsymbol{u} \in L^2(0,T; W^{2,2}_{loc}(\Omega)^d)$.

Let $\tau \in C^\infty_c(\Omega)$ be a cut-off function between Ω_0 and Ω_1 as in the proof of Lemma 4.8. Then $\tau^2 \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})) \in L^2(Q)^d$ and has support in $(0,T)\times\Omega_1$. Taking the dot product of this function with (4.30), we integrate the resulting equality over Ω to obtain

$$\int_\Omega \tau^2 \boldsymbol{u}_{tt} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})) = \int_\Omega \Big[\tau^2 \mathrm{div}(\mathbf{T}) \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})) + \tau^2 \boldsymbol{f} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})) \Big].$$

For the first term on the right-hand side, write

$$\begin{aligned} &\int_\Omega \tau^2 \mathrm{div}(\mathbf{T}) \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})) \, \mathrm{d}x = \int_\Omega \tau^2 \frac{\partial \mathbf{T}_{ij}}{\partial x_j} \frac{\partial^2 (\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \, \mathrm{d}x \\ &= \int_\Omega \tau^2 \frac{\partial \mathbf{T}_{ij}}{\partial x_k} \frac{\partial^2 (\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} + 2\tau \mathbf{T}_{ij} \frac{\partial^2 (\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} \frac{\partial \tau}{\partial x_k} - 2\tau \mathbf{T}_{ij} \frac{\partial^2 (\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \frac{\partial \tau}{\partial x_j} \, \mathrm{d}x, \end{aligned}$$

using an integration by parts argument to obtain the final line. Indeed, we consider smooth approximations of the stress and displacement tensors in Ω_1, perform classical integration by parts and then take the limit in the approximation sequences.

Since the stress tensor $\mathbf{T}$ is symmetric, it follows that

$$\begin{aligned}\int_\Omega \tau^2 \mathrm{div}(\mathbf{T}) \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u}))\,\mathrm{d}x &= \int_\Omega \tau^2 \frac{\partial \mathbf{T}_{ij}}{\partial x_k} \frac{1}{2} \frac{\partial}{\partial x_k} \left(\frac{\partial(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j} + \frac{\partial(\boldsymbol{u}_t + \alpha\boldsymbol{u})_j}{\partial x_i} \right) \\ &+ 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} \frac{\partial \tau}{\partial x_k} - 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \frac{\partial \tau}{\partial x_j}\,\mathrm{d}x \\ &= \int_\Omega \tau^2 \frac{\partial \mathbf{T}_{ij}}{\partial x_k} \frac{\partial}{\partial x_k} (\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}))_{ij} + 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} \frac{\partial \tau}{\partial x_k} - 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \frac{\partial \tau}{\partial x_j}\,\mathrm{d}x.\end{aligned}$$

Recalling that the gradient of a tensor $\mathbf{S}$ is defined to be $\nabla \mathbf{S} = (\partial_i \mathbf{T}_{jk})_{i,j,k=1}^d$, we get

$$\begin{aligned}&\int_\Omega \tau^2 \mathrm{div}(\mathbf{T}) \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u}))\,\mathrm{d}x \\ &= \int_\Omega \tau^2 \nabla \mathbf{T} \cdot \nabla(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})) + 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} \frac{\partial \tau}{\partial x_k} - 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \frac{\partial \tau}{\partial x_j}\,\mathrm{d}x \\ &= \int_\Omega \tau^2 \nabla \mathbf{T} \cdot \nabla F_n(\mathbf{T}) + 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_j \partial x_k} \frac{\partial \tau}{\partial x_k} - 2\tau \mathbf{T}_{ij} \frac{\partial^2(\boldsymbol{u}_t + \alpha\boldsymbol{u})_i}{\partial x_k^2} \frac{\partial \tau}{\partial x_j}\,\mathrm{d}x.\end{aligned}$$

For each $k \in \{1, \dots, d\}$, define a tensor $\mathbf{B}^k = (\mathbf{B}^k_{ij})_{i,j=1}^d$ by

$$\mathbf{B}^k_{ij} = \mathbf{T}_{ij} \frac{\partial \tau}{\partial x_k} - 2\delta_{jk} \mathbf{T}_{il} \frac{\partial \tau}{\partial x_l} + \delta_{ij} \mathbf{T}_{kl} \frac{\partial \tau}{\partial x_l}.$$

By definition, it follows that

$$\begin{aligned}\frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{v})_{ij}}{\partial x_k} \mathbf{B}^k_{ij} &= \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{v})_{ij}}{\partial x_k} \mathbf{T}_{ij} \frac{\partial \tau}{\partial x_k} - 2 \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{v})_{ik}}{\partial x_k} \mathbf{T}_{il} \frac{\partial \tau}{\partial x_l} + \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{v})_{ii}}{\partial x_k} \mathbf{T}_{kl} \frac{\partial \tau}{\partial x_l} \\ &= \frac{\partial^2 \boldsymbol{v}_i}{\partial x_j \partial x_k} \mathbf{T}_{ij} \frac{\partial \tau}{\partial x_k} - \frac{\partial^2 \boldsymbol{v}_i}{\partial x_k \partial x_k} \mathbf{T}_{ij} \frac{\partial \tau}{\partial x_j}.\end{aligned}$$

Hence we have that

$$\begin{aligned}&\int_\Omega \tau^2 \nabla \mathbf{T} \cdot \nabla F_n(\mathbf{T}) - \tau^2 \boldsymbol{u}_{tt} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}))\,\mathrm{d}x \\ &\quad = - \int_\Omega 2\tau \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})_{ij}}{\partial x_k} \mathbf{B}^k_{ij} + \tau^2 \boldsymbol{f} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}))\,\mathrm{d}x.\end{aligned} \tag{4.31}$$

Integrating by parts and using the fact that τ has compact support in Ω, we rewrite the terms involving $\boldsymbol{u}_{tt}$ as

$$\begin{aligned}&- \int_\Omega \tau^2 \boldsymbol{u}_{tt} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}))\,\mathrm{d}x \\ &\quad = \int_\Omega \tau^2 \nabla \boldsymbol{u}_{tt} \cdot \nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}) + 2\tau \nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}) \cdot (\boldsymbol{u}_{tt} \otimes \nabla\tau)\,\mathrm{d}x,\end{aligned}$$

and the parts involving the body force as

$$- \int_\Omega \tau^2 \boldsymbol{f} \cdot \mathrm{div}(\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}))\,\mathrm{d}x = \int_\Omega \tau^2 \nabla \boldsymbol{f} \cdot \nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}) + 2\tau \nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u}) \cdot (\boldsymbol{f} \otimes \nabla\tau)\,\mathrm{d}x.$$

Returning to (4.31), we get

$$\int_\Omega \tau^2 \nabla \boldsymbol{u}_{tt} \cdot \nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u}) + \tau^2 \nabla \mathbf{T} \cdot \nabla F_n(\mathbf{T}) \, \mathrm{d}x = \int_\Omega \tau^2 \nabla \boldsymbol{f} \cdot \nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u})$$
$$+ 2\tau \nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u}) \cdot (\boldsymbol{f} \otimes \nabla \tau) - 2\tau \nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u}) \cdot (\boldsymbol{u}_{tt} \otimes \nabla \tau) - 2\tau \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})_{ij}}{\partial x_k} \mathbf{B}^k_{ij} \, \mathrm{d}x.$$

Integrating over $(0, t)$, we deduce that

$$\begin{aligned} &\|\tau \nabla \boldsymbol{u}(t)\|_2^2 + \|\tau \nabla \boldsymbol{u}_t(t)\|_2^2 + \int_0^t \int_\Omega \tau^2 \nabla \mathbf{T} \cdot \nabla F_n(\mathbf{T}) \, \mathrm{d}x \, \mathrm{d}s \\ &\le C\Big(\|\tau \nabla \boldsymbol{u}_0\|_2^2 + \|\tau \nabla \boldsymbol{u}_1\|_2^2 + \int_0^t \|\nabla \boldsymbol{f}\|^2_{L^2(\Omega_1)} + \|\nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{u}_{tt}\|^2_{L^2(\Omega_1)} \, \mathrm{d}s \\ &\quad + \int_0^t \int_\Omega \left| \tau \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})_{ij}}{\partial x_k} \mathbf{B}^k_{ij} \right| \mathrm{d}x \, \mathrm{d}s \Big). \end{aligned} \tag{4.32}$$

We now look more carefully at the final integral on the right-hand side of (4.32). Define a fourth-order tensor $\mathcal{A}_n(\mathbf{T})$ for each $\mathbf{T} \in \mathbb{R}^{d \times d}$ by

$$\mathcal{A}_n(\mathbf{T})_{ijkl} = \frac{\partial F_n(\mathbf{T})_{ij}}{\partial \mathbf{T}_{kl}},$$

which induces an inner product on $\mathbb{R}^{d \times d}$ defined by

$$(\mathbf{S}, \mathbf{U})_{\mathcal{A}_n(\mathbf{T})} = \sum_{i,j,k,l=1}^d \mathcal{A}_n(\mathbf{T})_{ijkl} \mathbf{S}_{ij} \mathbf{U}_{kl}.$$

It is easy to check by direct calculation that this is a well-defined inner product on $\mathbb{R}^{d \times d}$ for every $\mathbf{T} \in \mathbb{R}^{d \times d}$. Also, for every $\mathbf{S} \in \mathbb{R}^{d \times d}$, we have that

$$\frac{|\mathbf{S}|^2}{(1 + |\mathbf{T}|^a)^{1 + \frac{1}{a}}} + \frac{|\mathbf{S}|^2}{n} \le (\mathbf{S}, \mathbf{S})_{\mathcal{A}_n(\mathbf{T})} \le C \left(\frac{|\mathbf{S}|^2}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{|\mathbf{S}|^2}{n} \right). \tag{4.33}$$

Next, we notice that, for any $\mathbf{C} \in \mathbb{R}^{d \times d}_{sym}$,

$$\frac{\partial}{\partial x_k} (F_n(\mathbf{T}))_{ij} \mathbf{C}_{ij} = \mathcal{A}_n(\mathbf{T})_{ijlp} \frac{\partial \mathbf{T}_{lp}}{\partial x_k} \mathbf{C}_{ij} = (\partial_k \mathbf{T}, \mathbf{C})_{\mathcal{A}_n(\mathbf{T})}.$$

With this in mind, (4.32) can be rewritten as

$$\begin{aligned} &\|\tau \nabla \boldsymbol{u}(t)\|_2^2 + \|\tau \nabla \boldsymbol{u}_t(t)\|_2^2 + \int_0^t \int_\Omega (\tau \partial_k \mathbf{T}, \tau \partial_k \mathbf{T})_{\mathcal{A}_n(\mathbf{T})} \, \mathrm{d}x \, \mathrm{d}s \\ &\le C(\|\nabla \tau\|_\infty) \Big(\|\tau \nabla \boldsymbol{u}_0\|_2^2 + \|\tau \nabla \boldsymbol{u}_1\|_2^2 + \int_0^t \|\nabla \boldsymbol{f}\|^2_{L^2(\Omega_1)} + \|\nabla(\boldsymbol{u}_t + \alpha \boldsymbol{u})\|^2_{L^2(\Omega_1)} \\ &\quad + \|\boldsymbol{u}_{tt}\|^2_{L^2(\Omega_1)} \, \mathrm{d}s + \int_0^t \int_\Omega \left| (\tau \partial_k \mathbf{T}, \mathbf{B}^k)_{\mathcal{A}_n(\mathbf{T})} \right| \mathrm{d}x \, \mathrm{d}s \Big). \end{aligned}$$

Using the inner product property, the Cauchy–Schwarz inequality and (4.33), the final term on the right-hand side can be bounded as follows:

$$\begin{aligned}
&\int_0^t \int_\Omega \left|(\tau\partial_k \mathbf{T}, \mathbf{B}^k)_{\mathcal{A}_n(\mathbf{T})}\right| \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_0^t \int_\Omega (\tau\partial_k \mathbf{T}, \tau\partial_k \mathbf{T})^{\frac{1}{2}}_{\mathcal{A}_n(\mathbf{T})} (\mathbf{B}^k, \mathbf{B}^k)^{\frac{1}{2}}_{\mathcal{A}_n(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \frac{1}{2}\int_0^t \int_\Omega (\tau\partial_k \mathbf{T}, \tau\partial_k \mathbf{T})_{\mathcal{A}_n(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\Omega_1} (\mathbf{B}^k, \mathbf{B}^k)_{\mathcal{A}_n(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{1}{2}\int_0^t \int_\Omega (\tau\partial_k \mathbf{T}, \tau\partial_k \mathbf{T})_{\mathcal{A}_n(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \int_{\Omega_1} \frac{|\mathbf{B}^k|^2}{1+|\mathbf{T}|} + \frac{|\mathbf{B}^k|^2}{n} \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

From definition, we see that $|\mathbf{B}^k| \leq C|\mathbf{T}|$, where C is a positive constant depending only on the dimension d and $\|\nabla\tau\|_\infty$. Hence, we have that

$$\begin{aligned}
&\|\tau\nabla \boldsymbol{u}(t)\|_2^2 + \|\tau\nabla \boldsymbol{u}_t(t)\|_2^2 + \int_0^t \int_\Omega (\tau\partial_k \mathbf{T}, \tau\partial_k \mathbf{T})_{\mathcal{A}_n(\mathbf{T})} \,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\Big(\|\tau\nabla \boldsymbol{u}_0\|_2^2 + \|\tau\nabla \boldsymbol{u}_1\|_2^2 + \int_0^t \|\nabla \boldsymbol{f}\|^2_{L^2(\Omega_1)} + \|\nabla(\boldsymbol{u}_t + \alpha\boldsymbol{u})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{u}_{tt}\|^2_{L^2(\Omega_1)} \,\mathrm{d}s \\
&\quad + \int_0^t \int_{\Omega_1} \frac{|\mathbf{T}|^2}{1+|\mathbf{T}|} + \frac{|\mathbf{T}|^2}{n} \,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}$$

Applying the bounds from Theorem 4.3 and Lemma 4.4 , we conclude the required result. ☐

4.2.3 The limit as $n \to \infty$

We have sufficient information on the weak solution of the regularised problem in order to consider the limit as $n \to \infty$. From now on, we denote by $(\boldsymbol{u}^n, \mathbf{T}^n)$ the weak solution of the regularised problem (4.4) with approximation parameter n. First we focus on the convergence of $(\mathbf{T}^n)_n$ as $n \to \infty$. We show a pointwise convergence result using the higher regularity estimates of the previous sections and the Aubin–Lions lemma, bypassing the poor integrability bounds.

Lemma 4.10. *Suppose that* $\boldsymbol{u}_0 \in W^{1,2}_D(\Omega)^d$*,* $\boldsymbol{u}_1 \in W^{1,2}_D(\Omega)^d$ *and* $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W^{k+1,2}_D(\Omega)^d$ *for a* $k > \frac{d}{2}$ *such that the safety strain condition* (4.2) *holds. Suppose that we have* $\boldsymbol{g} \in W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$ *and* $\boldsymbol{f} \in W^{1,2}(0,T;L^2(\Omega)^d) \cap L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$ *such that the compatibility condition* (4.3) *holds. Let* $(\boldsymbol{u}^n, \mathbf{T}^n)$ *denote the weak solution of the regularised problem* (4.4)*. There exists a subsequence in* n*, not relabelled, and a limiting function* $\mathbf{T} \in L^\infty(0,T;L^1(\Omega)^{d\times d})$ *such that*

$$\mathbf{T}^n \to \mathbf{T} \quad \textit{pointwise a.e. on } Q.$$

Proof. Fix an open, compactly contained subset $\Omega_0 \subset \Omega$. Using the results of Lemma 4.5 and Lemma 4.9, there exists a constant $C = C(\Omega_0)$, independent of n, such that

$$\int_0^T \int_{\Omega_0} \frac{|\mathbf{T}^n_t|^2}{(1+|\mathbf{T}^n|)^{1+a}} + \frac{|\nabla \mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} \,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{4.34}$$

We argue using similar techniques to those used in the proof of Theorem 2.10. Define sequences $(\mathbf{S}^n)_n$ and $(s^n)_n$ by

$$\mathbf{S}^n = \frac{\mathbf{T}^n}{(1+|\mathbf{T}^n|)^{1+a}}, \quad s^n = \frac{1}{(1+|\mathbf{T}^n|)^{1+a}}.$$

Clearly the sequences are uniformly bounded on Q. Using (4.34), the weak differentiability of $\mathbf{T}^n$ in the time and space variables, and the chain rule for weak derivatives as discussed previously, there exists a constant $C = C(\Omega_0)$, independent of n, such that

$$\int_0^T \int_{\Omega_0} |\nabla \mathbf{S}^n|^2 + |\mathbf{S}^n_t|^2 + |\nabla s^n|^2 + |s^n_t|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C.$$

Thus the sequences are bounded in $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{loc}(\Omega))$. We use the Aubin–Lions lemma to deduce that there exists a subsequence in n, not relabelled, such that

$$\begin{aligned} \mathbf{S}^n &\to \mathbf{S} && \text{strongly in } L^2(0,T;L^2(\Omega_0)^{d\times d}), \\ s^n &\to s && \text{strongly in } L^2(0,T;L^2(\Omega_0)), \end{aligned}$$

and both sequences converge pointwise a.e. on $(0,T)\times\Omega_0$. Let $(\Omega_i)_{i\geq 1}$ be an increasing sequence of open, compactly contained subsets of Ω such that $\cup_i \Omega_i = \Omega$. Considering a diagonal subsequence with regards to pointwise convergence on sets of the form $(0,T)\times\Omega_i$, we find a further subsequence, not relabelled, such that $\mathbf{S}^n \to \mathbf{S}$ and $s^n \to s$ pointwise on $(0,T)\times\Omega_i$ for every i. Since $\cup_i\Omega_i = \Omega$, we conclude that the pointwise convergence holds a.e. on $(0,T)\times\Omega$. Next, recall that there is a constant C, independent of n, such that

$$\int_Q |\mathbf{T}^n| \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)| \,\mathrm{d}x \leq C. \tag{4.35}$$

Using Fatou's lemma and the pointwise convergence result, it follows that

$$\int_Q s^{-\frac{1}{1+a}} \,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_Q (s^n)^{-\frac{1}{1+a}} \,\mathrm{d}x\,\mathrm{d}t = \liminf_{n\to\infty} \int_Q 1 + |\mathbf{T}^n| \,\mathrm{d}x\,\mathrm{d}t \leq C.$$

Hence $s^{-\frac{1}{1+a}} \in L^1(Q)$ so $s > 0$ a.e. in Q. Hence, $(s^n)^{-1}$ converges pointwise a.e. on Q to s^{-1} and so $\mathbf{T}^n = (s^n)^{-1}\mathbf{S}^n$ converges pointwise a.e. on Q as $n\to\infty$ to a limit which we denote by $\mathbf{T}$. This implies that, for a.e. $t\in(0,T)$, $\mathbf{T}^n(t) \to \mathbf{T}(t)$ pointwise a.e. on Ω. Applying Fatou's lemma again and the second estimate in (4.35), we see that $\mathbf{T} \in L^\infty(0,T;L^1(\Omega)^{d\times d})$ as required. □

Theorem 4.11. *Suppose that $\boldsymbol{u}_0 \in W^{1,2}_D(\Omega)^d$, $\boldsymbol{u}_1 \in W^{1,2}_D(\Omega)^d$ and $\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0 \in W^{k+1,2}_D(\Omega)^d$ for a $k > \frac{d}{2}$ such that the safety strain condition (4.2) holds. Suppose that we have $\boldsymbol{g} \in W^{1,2}(0,T;L^2(\partial\Omega_N)^d)$ and $\boldsymbol{f} \in W^{1,2}(0,T;L^2(\Omega)^d) \cap L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$ such that the compatibility condition (4.3) holds. There exists a triple $(\boldsymbol{u},\mathbf{T},\tilde{\boldsymbol{g}})$ with regularity*

- $\boldsymbol{u} \in W^{2,2}(0,T;W^{1,2}_D(\Omega)^d) \cap W^{2,\infty}(0,T;L^2(\Omega)^d)$,
- $\mathbf{T} \in L^\infty(0,T;L^1(\Omega)^{d\times d})$,

- $\tilde{\boldsymbol{g}} \in L^\infty(0,T;(C_0^1(\partial\Omega_N)^d)^*)$,

that is a weak solution of the mixed boundary value problem (4.1) up to an error $\tilde{\boldsymbol{g}}$ on the Neumann part of the boundary in the following sense. For every test function $\boldsymbol{v} \in C_D^1(\overline{\Omega})^d$ and a.e. $t \in (0,T)$, we have

$$\int_\Omega \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v} + \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S - \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N}, \tag{4.36}$$

where $\boldsymbol{u}$ and $\mathbf{T}$ satisfy the constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \quad \text{pointwise a.e. in } Q.$$

The initial conditions are satisfied in the sense that

$$\lim_{t\to 0+}\Big[\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t)-\boldsymbol{u}_1\|_2\Big] = 0.$$

If the solution triple additionally satisfies

$$\int_\Omega \boldsymbol{u}_{tt}\cdot(\boldsymbol{u}_t+\alpha\boldsymbol{u}-\boldsymbol{v})+\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u}-\boldsymbol{v})\,\mathrm{d}x \le \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}_t+\alpha\boldsymbol{u}-\boldsymbol{v})\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}\cdot(\boldsymbol{u}_t+\alpha\boldsymbol{u}-\boldsymbol{v})\,\mathrm{d}S, \tag{4.37}$$

for a.e. $t \in (0,T)$ and every $\boldsymbol{v} \in W_D^{1,2}(\Omega)^d$ such that there exists $\mathbf{S} \in L^1(\Omega)^{d\times d}$ with $\boldsymbol{\varepsilon}(\boldsymbol{v}) = F(\mathbf{S})$ a.e. in Ω, then $(\boldsymbol{u},\mathbf{T},\tilde{\boldsymbol{g}})$ is unique in the class of all such solutions.[1]

The solution constructed by taking the limit in the sequence of solutions to the regularised problem (4.4) satisfies (4.37) and, in particular, is the unique solution in this class of weak solutions. Furthermore, the following convergence results hold:

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ *weakly-* in* $W^{1,\infty}(0,T;W_D^{1,2}(\Omega)^d) \cap W^{2,\infty}(0,T;L^2(\Omega)^d)$;
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0,T;W_D^{1,2}(\Omega)^d)$;
- $\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha\boldsymbol{u}$ *weakly in* $L^2(0,T;W_{loc}^{2,2}(\Omega)^d)$;
- $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n) \to \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$ *pointwise a.e. on* Q;
- $\mathbf{T}^n \to \mathbf{T}$ *pointwise a.e. on* Q;
- $\mathbf{T}^n \overset{*}{\rightharpoonup} \overline{\mathbf{T}}$ *weakly-* in* $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ *where we identify*

$$\langle\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))} = \int_\Omega \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N},$$

  for every $\boldsymbol{v} \in C_D^1(\overline{\Omega})^d$ and a.e. $t \in (0,T)$.

[1] We note that (4.37) is a reasonable ask. In particular, any true weak solution of the problem will satisfy the inequality. In fact, it will satisfy the relation as an equality. The inequality comes from the presence of the penalty on the Neumann part of the boundary.

Proof. Putting together the results of Theorem 4.3, Lemma 4.4, Lemma 4.5, Corollary 4.6, Lemma 4.7 and Lemma 4.9, there exists a constant C, independent of n, such that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{1,2} + \sup_{t\in[0,T]} \|\boldsymbol{u}^n_t(t)\|_{1,2} + \sup_{t\in[0,T]} \|\boldsymbol{u}^n_{tt}(t)\|_2 + \int_0^T \|\boldsymbol{u}^n_{tt}(t)\|^2_{1,2}\,\mathrm{d}t$$
$$+ \int_Q \frac{|\mathbf{T}^n_t|^2}{n}\,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)| + \frac{|\mathbf{T}^n(t)|^2}{n}\,\mathrm{d}x \le C$$

and for any open, compactly contained subset $\Omega_0 \subset \Omega$, there is a constant $C = C(\Omega_0)$, independent of n, such that

$$\int_0^T \frac{\|\nabla \mathbf{T}^n(t)\|^2_{L^2(\Omega_0)}}{n} + \|(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)(t)\|^2_{W^{2,2}(\Omega_0)}\,\mathrm{d}t \le C.$$

Hence, up to a subsequence in n that we do not relabel, we have the following convergences:

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ weakly-* in $W^{1,\infty}(0,T;W^{1,2}_D(\Omega)^d) \cap W^{2,\infty}(0,T;L^2(\Omega)^d)$;
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ weakly in $W^{2,2}(0,T;W^{1,2}_D(\Omega)^d)$;
- $\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}_t + \alpha\boldsymbol{u}$ weakly in $L^2(0,T;W^{2,2}_{loc}(\Omega)^d)$;
- $\mathbf{T}^n \overset{*}{\rightharpoonup} \overline{\mathbf{T}}$ weakly-* in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$.

Here $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ is the linear space of the equivalence classes[2] of functions $\mathbf{h} : (0,T) \to \mathcal{M}(\overline{\Omega})^{d\times d}$ such that $\mathbf{h}$ is w^*-measurable and there exists a constant C such that

$$|\langle \mathbf{h}(t), \boldsymbol{v}\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))}| \le C\|\boldsymbol{v}\|_{C(\overline{\Omega})} \quad \text{a.e. on } (0,T), \text{ for every } \boldsymbol{v} \in C(\overline{\Omega})^{d\times d}.$$

The bracket $\langle\cdot,\cdot\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))}$ denotes the duality pairing between $\mathcal{M}(\overline{\Omega})$ and $C(\overline{\Omega})$. The norm on this space is the infimum over all such possible C. By Theorem 10.1.16 of [83], $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ is the dual space of $L^1(0,T;C(\overline{\Omega})^{d\times d})$. Next, we note that $C(\overline{\Omega})^{d\times d}$ is a separable space with respect to the usual norm because $\overline{\Omega}$ is a compact metric space. Then, it is a well-known fact of Bochner spaces that therefore $L^1(0,T;C(\overline{\Omega})^{d\times d})$ is separable. Thus, by the sequential form of the Banach–Alaoglu Theorem, the closed unit ball of $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ is sequentially compact in the weak-* topology. In particular, a bounded sequence in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ has a weakly-* convergence subsequence. Since a bounded sequence in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ is bounded in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$, any bounded sequence in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ has a weakly-* convergence subsequence in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$. Hence, the sequence of approximate stress tensors $(\mathbf{T}^n)_n$ converges weakly-* in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$, up to a subsequence, to some limit $\overline{\mathbf{T}}$.

Let $\boldsymbol{v} \in C^1_D(\overline{\Omega})^d$ and $\psi \in C([0,T])$. Then, for every n, we have

$$\int_Q \boldsymbol{u}^n_{tt} \cdot (\boldsymbol{v}\psi) + \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f} \cdot (\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega_N} \boldsymbol{g}_n \cdot (\boldsymbol{v}\psi)\,\mathrm{d}S\,\mathrm{d}t.$$

[2]Equivalence defined to mean equality a.e. on $(0,T)$ as usual.

Taking $n \to \infty$ and using the above convergence results followed by a standard density argument with respect to the time variable, we deduce that

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \langle \overline{\mathbf{T}}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))} = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}(t) \cdot \boldsymbol{v} \, \mathrm{d}S, \tag{4.38}$$

for every $\boldsymbol{v} \in C^1_D(\overline{\Omega})^d$ and a.e. $t \in (0, T)$.

Furthermore, from Lemma 4.10, we have that $\mathbf{T}^n \to \mathbf{T}$ pointwise a.e. in Q as $n \to \infty$, where $\mathbf{T} \in L^\infty(0, T; L^1(\Omega)^{d \times d})$. Then $F(\mathbf{T}^n) \to F(\mathbf{T})$ pointwise a.e. on Q by continuity of F and $n^{-1}\mathbf{T}^n \to \mathbf{0}$ pointwise a.e. on Q because $\mathbf{T}$ is finite a.e. on Q. Hence $F_n(\mathbf{T}^n) \to F(\mathbf{T})$ pointwise a.e. on Q. It follows from the weak convergence result for $(\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n))_n$ and the constitutive relation for the regularised problem that $(\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n))_n$ converges pointwise a.e. on Q to $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})$ and also $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = F(\mathbf{T})$ pointwise a.e. in Q. We see that $\boldsymbol{u}$ satisfies the required initial conditions by standard arguments and using that $\boldsymbol{u}^n(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}^n_t(0) = \boldsymbol{u}_1$.

Using the pointwise convergence of $(\mathbf{T}^n)_n$ and arguing as in the proof of Theorem 2.10, we see that

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{v} + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x. \tag{4.39}$$

for a.e. $t \in (0, T)$ and every $\boldsymbol{v} \in W^{1,2}_0(\Omega)^d$.

The next step is to compare (4.38) and (4.39) in order to obtain the required error term $\tilde{\boldsymbol{g}}$. We mimic the arguments from [5] but adapt them to this more complicated time-dependent setting. We define the normal components of $\mathbf{T}(t)$ and $\overline{\mathbf{T}}(t)$ on $\partial\Omega_N$ by their action on elements of $C^1_0(\partial\Omega_N)^d$. We recall that this is defined to be the subset of those functions $\boldsymbol{v} \in C^1(\partial\Omega)^d$ such that $\boldsymbol{v}$ vanishes on $\overline{\partial\Omega_D}$. By Theorem 1 in [46], there exists a linear extension map $E : C^1(\partial\Omega) \to C^1(\overline{\Omega})$ such that $E\boldsymbol{v} = \boldsymbol{v}$ on $\partial\Omega$ for every $\boldsymbol{v} \in C^1(\partial\Omega)^d$. Furthermore, there exists a constant C, depending only on Ω, such that

$$\|E\boldsymbol{v}\|_{1,\infty} \leq C \|\boldsymbol{v}\|_{C^1(\partial\Omega)}. \tag{4.40}$$

The restriction of E to the functions from $C^1_0(\partial\Omega_N)^d$ (that are defined on the whole of $\partial\Omega$ via extension by $\mathbf{0}$) are mapped into $C^1_D(\overline{\Omega})^d$. Moreover, for every $\boldsymbol{v} \in C^1_0(\partial\Omega_N)^d$, there exists an extension $\tilde{\boldsymbol{v}} \in C^1_D(\overline{\Omega})^d$ such that (4.40) holds, noting that $\|\boldsymbol{v}\|_{C^1(\partial\Omega)} = \|\boldsymbol{v}\|_{C^1(\partial\Omega_N)}$ in this case.

For a.e. $t \in (0, T)$, we define the normal component of $\mathbf{T}(t)$ on $\partial\Omega_N$ by

$$\langle \mathbf{T}(t)\mathbf{n}, \boldsymbol{v} \rangle|_{\partial\Omega_N} := \int_\Omega \boldsymbol{u}_{tt}(t) \cdot \tilde{\boldsymbol{v}} + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}) - \boldsymbol{f}(t) \cdot \tilde{\boldsymbol{v}} \, \mathrm{d}x,$$

where $\boldsymbol{v} \in C^1_0(\partial\Omega_N)^d$ and $\tilde{\boldsymbol{v}}$ is a C^1-extension of $\boldsymbol{v}$ to $\overline{\Omega}$. First, we notice that $\mathbf{T}(t)\mathbf{n}$ is well-defined as an operator on $C^1_0(\partial\Omega_N)^d$ by the following reasoning. Suppose that $\tilde{\boldsymbol{v}}_1$, $\tilde{\boldsymbol{v}}_2 \in C^1_D(\overline{\Omega})^d$ are any two extensions of $\boldsymbol{v}$ to $\overline{\Omega}$. Clearly, we have that $\tilde{\boldsymbol{v}}_1 - \tilde{\boldsymbol{v}}_2 \in C^1_0(\overline{\Omega})^d$ and so the difference is a valid test function in (4.39). It follows that

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \tilde{\boldsymbol{v}}_1 + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_1) - \boldsymbol{f}(t) \cdot \tilde{\boldsymbol{v}}_1 \, \mathrm{d}x = \int_\Omega \boldsymbol{u}_{tt}(t) \cdot \tilde{\boldsymbol{v}}_2 + \mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_2) - \boldsymbol{f}(t) \cdot \tilde{\boldsymbol{v}}_2 \, \mathrm{d}x,$$

and so $\langle \mathbf{T}(t)\mathbf{n}, \boldsymbol{v}\rangle|_{\partial\Omega_N}$ is defined independently of the choice of extension. Thus it is well-defined. In particular, we can choose the extension defined by the operator E. Using (4.40), we see that

$$\begin{aligned}\left|\langle \mathbf{T}(t)\mathbf{n}, \boldsymbol{v}\rangle|_{\partial\Omega_N}\right| &= \left|\int_\Omega \boldsymbol{u}_{tt}(t)\cdot E\boldsymbol{v} + \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(E\boldsymbol{v}) - \boldsymbol{f}(t)\cdot E\boldsymbol{v}\,\mathrm{d}x\right| \\ &\le C\left(\|\boldsymbol{u}_{tt}(t)\|_2 + \|\mathbf{T}(t)\|_1 + \|\boldsymbol{f}(t)\|_2\right)\|E\boldsymbol{v}\|_{1,\infty} \\ &= C\left(\|\boldsymbol{u}_{tt}(t)\|_2 + \|\mathbf{T}(t)\|_1 + \|\boldsymbol{f}(t)\|_2\right)\|\boldsymbol{v}\|_{C_0^1(\partial\Omega_N)}.\end{aligned}$$

Hence, for a.e. $t\in(0,T)$, we have $\mathbf{T}(t)\mathbf{n}\in(C_0^1(\partial\Omega_N)^d)^*$. Since $\boldsymbol{u}_{tt}$, $\boldsymbol{f}\in L^\infty(0,T;L^2(\Omega)^d)$ and $\mathbf{T}\in L^\infty(0,T;L^1(\Omega)^{d\times d})$, it follows that $t\mapsto\mathbf{T}(t)\mathbf{n}$ is weakly-* measurable on $(0,T)$ and so $\mathbf{T}(t)\mathbf{n}$ defines an element of $L^\infty_{w^*}(0,T;(C_0^1(\partial\Omega_N)^d)^*)$.

In a similar manner, we define the normal component of $\overline{\mathbf{T}}(t)$ on $\partial\Omega_N$ by

$$\langle \overline{\mathbf{T}}(t)\mathbf{n}, \boldsymbol{v}\rangle|_{\partial\Omega_N} := \int_\Omega \boldsymbol{u}_{tt}(t)\cdot\tilde{\boldsymbol{v}} - \boldsymbol{f}(t)\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}x + \langle\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))}.$$

Using (4.38), we reason as with $\mathbf{T}(t)\mathbf{n}$ to see that $\overline{\mathbf{T}}(t)\mathbf{n}$ is well-defined and is an element of $(C_0^1(\partial\Omega_N)^d)^*$. Since $\overline{\mathbf{T}}\in L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$, we deduce that $\overline{\mathbf{T}}(t)\mathbf{n}$ is an element of $L^\infty_{w^*}(0,T;(C_0^1(\partial\Omega_N)^d)^*)$.

The error on the Neumann part of the boundary is the difference between these normal components. For a.e. $t\in(0,T)$, we define the penalisation $\tilde{\boldsymbol{g}}(t)$ on $\partial\Omega_N$ by

$$\langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} := \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle_{(C_0^1(\partial\Omega_N)^*,C_0^1(\partial\Omega_N))} := \langle\overline{\mathbf{T}}(t)\mathbf{n}-\mathbf{T}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N}.$$

By definition, we have that $\tilde{\boldsymbol{g}}\in L^\infty_{w^*}(0,T;(C_0^1(\partial\Omega_N)^d)^*)$. Let $\boldsymbol{v}\in C_D^1(\overline{\Omega})^d$. Then we see that

$$\begin{aligned}&\int_\Omega \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v} + \mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x - \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S \\ &= \langle\mathbf{T}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} - \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S \\ &= -\langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} + \langle\overline{\mathbf{T}}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} - \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S \\ &= -\langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} + \int_\Omega\boldsymbol{u}_{tt}\cdot\boldsymbol{v} - \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \langle\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))} - \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S \\ &= -\langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N},\end{aligned}$$

using (4.38) in the transition to the final line. It follows that, for every $\boldsymbol{v}\in C_D^1(\overline{\Omega})^d$ and a.e. $t\in(0,T)$,

$$\int_\Omega\boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v}+\mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega\boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}S - \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N},$$

which is exactly (4.36).

Now that we have an existence result, we prove that the triple that is constructed from the limit of the solutions to the regularised problems satisfies the inequality (4.37). Let $\boldsymbol{v}\in W_D^{1,2}(\Omega)^d$ and $\psi\in C([0,T])$ be fixed but arbitrary test functions such that $\psi\ge 0$ and there exists an $\mathbf{S}\in L^1(\Omega)^{d\times d}$ with $\boldsymbol{\varepsilon}(\boldsymbol{v}) = F(\mathbf{S})$ a.e. in Ω. Let $n\in\mathbb{N}$ and test in the weak form of the regularised problem against $\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}$, multiply by ψ and integrate over $(0,T)$. Taking the limit as $n \to \infty$ yields

$$\begin{aligned}
&\int_Q \psi \boldsymbol{f} \cdot (\boldsymbol{u}_t + \alpha \boldsymbol{u} - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega_N} \psi \boldsymbol{g} \cdot (\boldsymbol{u}_t + \alpha \boldsymbol{u} - \boldsymbol{v}) \,\mathrm{d}S\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_Q \psi \boldsymbol{f} \cdot (\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega_N} \psi \boldsymbol{g}_n \cdot (\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}S\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_Q \psi \boldsymbol{u}_{tt}^n \cdot (\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) + \psi \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we used the strong convergence of $(\boldsymbol{g}_n)_n$ in $L^\infty(0,T;L^2(\partial\Omega_N)^d)$ and the weak convergence of $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$ in $L^2(0,T;L^2(\partial\Omega_N)^d)$, a consequence of the weak convergence in $L^2(0,T;W^{1,2}(\Omega)^d)$. Using the weak convergence of $(\boldsymbol{u}_{tt}^n)_n$ and the strong convergence of $(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)_n$ from the Aubin–Lions lemma, we see that

$$\lim_{n\to\infty} \int_Q \psi \boldsymbol{u}_{tt}^n \cdot (\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \psi \boldsymbol{u}_{tt} \cdot (\boldsymbol{u}_t + \alpha \boldsymbol{u} - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t.$$

For the term involving the approximate stress tensor, we need to make use of the pointwise convergence of $(\mathbf{T}^n)_n$. It is clear that we cannot use the dominated convergence theorem or monotone convergence theorem directly. Instead, we rely on Fatou's lemma, the coercivity of F_n and the monotonicity of F. Using the assumption that $F(\mathbf{S}) = \boldsymbol{\varepsilon}(\boldsymbol{v})$ and the constitutive relation for the regularised problem, we can write

$$\begin{aligned}
&\int_Q \psi \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi \mathbf{T}^n \cdot (F_n(\mathbf{T}^n) - F(\mathbf{S})) \,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_Q \psi \mathbf{T}^n \cdot (F(\mathbf{T}^n) - F(\mathbf{S})) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi (\mathbf{T}^n - \mathbf{S}) \cdot (F(\mathbf{T}^n) - F(\mathbf{S})) + \mathbf{S} \cdot (F(\mathbf{T}^n) - F(\mathbf{S})) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using the boundedness of F and the pointwise convergence of $(\mathbf{T}^n)_n$, we have that $F(\mathbf{T}^n) \to F(\mathbf{T})$ strongly in $L^p(Q)^{d\times d}$ for every $p \in [1,\infty)$. Using this and Fatou's lemma, we see that

$$\begin{aligned}
&\liminf_{n\to\infty} \int_Q \psi \mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_Q \psi (\mathbf{T} - \mathbf{S}) \cdot (F(\mathbf{T}) - F(\mathbf{S})) + \mathbf{S} \cdot (F(\mathbf{T}) - F(\mathbf{S})) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi \mathbf{T} \cdot (F(\mathbf{T}) - F(\mathbf{S})) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \psi \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u} - \boldsymbol{v}) \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we have used the constitutive relation for the limiting couple in the transition to the final line. Putting the above together and recalling that ψ is arbitrary, we conclude that (4.37) is satisfied by the solution constructed from the sequence of solutions to the regularised problem.

Finally we show that weak solutions satisfying (4.37) are unique. Consider two triples $(\boldsymbol{u}^i, \mathbf{T}^i, \tilde{\boldsymbol{g}}^i)$, where $i \in \{1,2\}$, emanating from the same data such that they solve the strain-limiting problem in the stated weak sense up to the boundary and the inequality (4.37) is satisfied. We notice that $\boldsymbol{u}_t^2 + \alpha \boldsymbol{u}^2$ is a valid test function in the inequality (4.37) for the weak solution $(\boldsymbol{u}^1, \mathbf{T}^1, \tilde{\boldsymbol{g}}^1)$, and vice versa. Adding the resulting inequalities and denoting $\boldsymbol{w} = \boldsymbol{u}^1 - \boldsymbol{u}^2$, we see that

$$\int_\Omega \boldsymbol{w}_{tt} \cdot (\boldsymbol{w}_t + \alpha \boldsymbol{w}) + (\mathbf{T}^1 - \mathbf{T}^2) \cdot (F(\mathbf{T}^1) - F(\mathbf{T}^2)) \, \mathrm{d}x \leq 0,$$

for a.e. $t \in (0,T)$, where the right-hand side vanishes because the solutions have the same data. Integrating over $(0,t)$ and using that the initial data for the two solutions coincide, we deduce that

$$\sup_{t\in[0,T]} \|\boldsymbol{w}(t)\|_2^2 + \sup_{t\in[0,T]} \|\boldsymbol{w}_t(t)\|_2^2 + \int_Q (\mathbf{T}^1 - \mathbf{T}^2) \cdot (F(\mathbf{T}^1) - F(\mathbf{T}^2)) \, \mathrm{d}x \, \mathrm{d}t \leq 0.$$

Thus $\boldsymbol{u}^1 = \boldsymbol{u}^2$ and, by the strict monotonicity of F, $\mathbf{T}^1 = \mathbf{T}^2$. Taking the difference of (4.36) for the two solutions, it follows that

$$\langle \tilde{\boldsymbol{g}}^1(t), \boldsymbol{v} \rangle|_{\partial\Omega_N} = \langle \tilde{\boldsymbol{g}}^2(t), \boldsymbol{v} \rangle|_{\partial\Omega_N},$$

for a.e. $t \in (0,T)$ and every $\boldsymbol{v} \in C^1_D(\overline{\Omega})^d$. Hence $\tilde{\boldsymbol{g}}^1 = \tilde{\boldsymbol{g}}^2$ and thus we have uniqueness of solutions in this class. □

Corollary 4.12. *Suppose that the assumptions of Theorem 4.11 hold and suppose that* $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}})$ *is the weak solution of the strain-limiting problem constructed from the sequence of solutions for the regularised problem. Suppose also that there exists a weak solution* $(\check{\boldsymbol{u}}, \check{\mathbf{T}})$ *of the problem, without penalisation on the Neumann part of the boundary. Then* $\boldsymbol{u} = \check{\boldsymbol{u}}$, $\mathbf{T} = \check{\mathbf{T}}$ *and* $\tilde{\boldsymbol{g}} = \mathbf{0}$.

This corollary confirms that our notion of solution with a penalisation is in fact the correct notion. Indeed, if a weak solution to the strain-limiting problem exists, then the one we construct from the limit of solutions to the regularised problem coincides with it. We can also use Theorem 4.11 to extend the results of Section 3 because if the Neumann part of the boundary is empty we trivially have no error term present.

Corollary 4.13. *Suppose that* $\Omega \subset \mathbb{R}^d$ *is an open, bounded Lipschitz domain and let* $a > 0$, $\alpha \geq 0$ *be fixed problem parameters. Suppose that we have initial data* $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_0(\Omega)^d$ *such that* $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W^{k+1,2}_0(\Omega)^d$ *for some* $k > \frac{d}{2}$ *such that the safety strain condition* (4.2) *holds. Suppose that* $\boldsymbol{f} \in W^{1,2}(0,T; L^2(\Omega)^d) \cap L^2(0,T; W^{1,2}_{loc}(\Omega)^d)$. *There exists a unique weak solution to the strain-limiting problem* (3.1) *with homogeneous Dirichlet boundary condition in the sense of Definition 3.2. Furthermore, if* $(\boldsymbol{u}^n, \mathbf{T}^n)_n$ *is the sequence of solutions to the approximate problem with regularisation term* $n^{-1}\mathbf{T}^n$, *we have that*

$$\mathbf{T}^n \to \mathbf{T} \quad \textit{pointwise a.e. on } Q.$$

4.3 Lower regularity initial data

In this section, we show that, rather than requiring that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{k+1,2}(\Omega)^d$ for some $k > \frac{d}{2}$, in fact we only need $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{2,2}(\Omega)^d$. This is an improvement in every spatial dimension $d \geq 2$. We do this by approximating the problem with the linear regularisation term and by introducing an approximation of the initial data. To do this, we need an appropriate approximation result concerning functions from the relevant spaces which will allow us to use the previous results. First, we state a lemma concerning approximations in the case of Dirichlet boundary conditions. This result is proven by adapting the linear approximation that is used in the proof of Lemma 4.3 from [16].

Lemma 4.14. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_0^{1,2}(\Omega)^d$ are given functions such that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_0^{2,2}(\Omega)^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0) \in L^\infty(\Omega)^{d\times d}$. There exists sequences $(\boldsymbol{u}_{0,k})_k$, $(\boldsymbol{u}_{1,k})_k \subset C_c^\infty(\Omega)^d$ such that*

- $\boldsymbol{u}_{0,k} \to \boldsymbol{u}_0$ *strongly in* $W_0^{1,2}(\Omega)^d$,
- $\boldsymbol{u}_{1,k} \to \boldsymbol{u}_1$ *strongly in* $W_0^{1,2}(\Omega)^d$,
- $\boldsymbol{u}_{1,k} + \alpha \boldsymbol{u}_{0,k} \to \boldsymbol{u}_1 + \alpha \boldsymbol{u}_0$ *strongly in* $W_0^{2,2}(\Omega)^d$,
- $\boldsymbol{\varepsilon}(\boldsymbol{u}_{1,k} + \alpha \boldsymbol{u}_{0,k}) \overset{*}{\rightharpoonup} \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)$ *weakly-* in* $L^\infty(\Omega)^{d\times d}$.

This can then be used to prove an analogous approximation result for initial data in the case of mixed Dirichlet–Neumann boundary conditions. The proof also employs similar methods to those used in the proof of Lemma A.3 from [5].

Theorem 4.15. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary such that there are open subsets of $\partial\Omega$, the Dirichlet $\partial\Omega_D$ and Neumann $\partial\Omega_N$ parts, such that $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $\overline{\partial\Omega_D \cup \partial\Omega_N} = \partial\Omega$. Let $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,2}(\Omega)^d$ be functions such that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{2,2}(\Omega)^d \cap W^{1,\infty}(\Omega)^d$. There exists sequences $(\boldsymbol{u}_{0,k})_k$, $(\boldsymbol{u}_{1,k})_k \subset C_D^\infty(\overline{\Omega})^d$ such that the following convergence results hold as $k \to \infty$:*

- $\boldsymbol{u}_{0,k} \to \boldsymbol{u}_0$ *strongly in* $W_D^{1,2}(\Omega)^d$;
- $\boldsymbol{u}_{1,k} \to \boldsymbol{u}_1$ *strongly in* $W_D^{1,2}(\Omega)^d$;
- $\boldsymbol{u}_{1,k} + \alpha \boldsymbol{u}_{0,k} \to \boldsymbol{u}_1 + \alpha \boldsymbol{u}_0$ *strongly in* $W_D^{2,2}(\Omega)^d$;
- $\boldsymbol{\varepsilon}(\boldsymbol{u}_{1,k} + \alpha \boldsymbol{u}_{0,k}) \overset{*}{\rightharpoonup} \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)$ *weakly-* in* $L^\infty(\Omega)^d$.

With this in mind, we consider the following problem. Suppose that we are given initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,2}(\Omega)^d$ such that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{2,2}(\Omega)^d$ and the safety strain condition (4.2) holds, with external body force $\boldsymbol{f} \in W^{1,2}(0,T;L^2(\Omega)^d) \cap L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$ and boundary traction $\boldsymbol{g} \in W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$ such that the compatibility condition (4.3) holds. Let $(\boldsymbol{u}_{0,k})_k$,

$(\boldsymbol{u}_{1,k})_k \subset C_D^\infty(\Omega)^d$ be the approximation sequences for $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, respectively, from Theorem 4.15. For approximation parameters n, $k \in \mathbb{N}$, we consider the following problem:

$$\begin{aligned} \boldsymbol{u}_{tt}^{n,k} &= \operatorname{div}(\mathbf{T}^{n,k}) + \boldsymbol{f} && \text{in } Q, \\ \boldsymbol{\varepsilon}(\boldsymbol{u}_t^{n,k} + \alpha \boldsymbol{u}^{n,k}) &= F(\mathbf{T}^{n,k}) + \frac{\mathbf{T}^{n,k}}{n} && \text{in } Q, \\ \boldsymbol{u}^{n,k}(0,\cdot) &= \boldsymbol{u}_{0,k},\ \boldsymbol{u}_t^{n,k}(0,\cdot) = \boldsymbol{u}_{1,k} && \text{on } \Omega, \\ \boldsymbol{u}^{n,k}(t,\cdot) &= \mathbf{0} && \text{on } (0,T] \times \partial\Omega_D, \\ \mathbf{T}^{n,k}\mathbf{n} &= \boldsymbol{g}_{n,k} && \text{on } (0,T] \times \partial\Omega_N, \end{aligned} \tag{4.41}$$

where $\boldsymbol{g}_{n,k}$ is defined by

$$\boldsymbol{g}_{n,k}(t,x) = \chi(kt) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{1,k} + \alpha \boldsymbol{u}_{0,k}))\mathbf{n} + (1 - \chi(kt))\boldsymbol{g}_n(t,x),$$

with $\boldsymbol{g}_n$ is defined as before. The existence of a solution to (4.41) can be proven using the Galerkin method. For each fixed n, we take the limit as $k \to \infty$. This leads us to obtain the existence of a solution to the regularised problem (4.4). Then we proceed exactly as before to get a solution of the strain-limiting problem up to penalisation on the Neumann part of the boundary.

4.4 Higher integrability estimates

In this section, we improve the estimates from Chapter 3 concerning the integrability of the stress tensor $\mathbf{T}$ provided that the parameter a is small, where smallness depends on the spatial dimension d. We prove the result in the case of stronger assumptions on the initial data. However, the result can easily be extended by using the aforementioned extra approximation level to the case that $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{2,2}(\Omega)^d$. We state the theorem in the setting of mixed boundary conditions, but it applies directly to the case of fully Dirichlet boundary conditions.

Theorem 4.16. *Suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,2}(\Omega)^d$ and $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W_D^{k+1,2}(\Omega)^d$ for a $k > \frac{d}{2}$ such that the safety strain condition (4.2) holds. Suppose that we have boundary traction $\boldsymbol{g} \in W^{1,\infty}(0,T; L^2(\partial\Omega_N)^d)$ and body force $\boldsymbol{f} \in W^{1,2}(0,T; L^2(\Omega)^d) \cap L^2(0,T; W_{loc}^{1,2}(\Omega)^d)$ such that the compatibility condition (4.3) holds. Furthermore, assume that $a \in (0, \frac{2}{d})$. Then, for every $\delta > 0$ such that $a + \delta < \frac{2}{d}$, for every compact subset $\Omega_0 \subset \Omega$, there exists a constant $C = C(\delta, \Omega_0)$ such that*

$$\int_0^T \int_{\Omega_0} |\mathbf{T}|^{1+\delta} + |\mathbf{T}^n|^{1+\delta} \,\mathrm{d}x\,\mathrm{d}t \le C,$$

for every $n \in \mathbb{N}$, where $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}})$ is the solution of the strain-limiting problem constructed in Theorem 4.11 and $(\boldsymbol{u}^n, \mathbf{T}^n)$ is the solution of the regularised problem with approximation parameter n. Furthermore, we have that

$$\mathbf{T}^n \rightharpoonup \mathbf{T} \quad \textit{weakly in } L^{1+\delta}(0,T; L_{loc}^{1+\delta}(\Omega)^{d\times d}).$$

Proof. From the proof of Lemma 4.7 and Lemma 4.9, there exists a constant $C = C(\Omega_0)$, independent of n, such that

$$\int_0^T \int_{\Omega_0} \frac{|\nabla \mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)| \,\mathrm{d}x \le C.$$

Applying the Sobolev embedding to the first term on the left-hand side to $(1+|\mathbf{T}^n(t)|)^{\frac{1-a}{2}}$ with parameter $p \in (2, \frac{2d}{d-2}]$ if $d \ge 3$, or $p \in (2,\infty)$ if $d = 2$, we see that

$$\int_0^T \|\mathbf{T}^n\|^{1-a}_{L^{\frac{p(1-a)}{2}}(\Omega_0)} \,\mathrm{d}t \le C.$$

From now on, we assume that $d \ge 3$ but the argument for $d = 2$ is similar. Letting $p = \frac{2d}{d-2}$, we have that

$$\sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)| \,\mathrm{d}x + \int_0^T \left(\int_{\Omega_0} |\mathbf{T}^n(t)|^{\frac{d(1-a)}{d-2}} \,\mathrm{d}x \right)^{\frac{d-2}{d}} \mathrm{d}t \le C,$$

where C is independent of n. Let $\delta = \delta(a,d) > 0$ be fixed so that $a+\delta \in (0, \frac{2}{d})$. A standard manipulation shows that

$$\frac{d(1+\delta)-2}{d-2} < \frac{d(1-a)}{d-2}.$$

We apply Hölder's inequality with parameters $\frac{d}{2}$ and $\frac{d}{d-2}$ to obtain

$$\begin{aligned}\int_{\Omega_0} |\mathbf{T}^n(t)|^{1+\delta} \,\mathrm{d}x &\le \left(\int_{\Omega_0} |\mathbf{T}^n(t)| \,\mathrm{d}x \right)^{\frac{2}{d}} \left(\int_{\Omega_0} |\mathbf{T}^n(t)|^{1+\frac{\delta d}{d-2}} \,\mathrm{d}x \right)^{\frac{d-2}{d}} \\ &\le C \left[\left(\int_{\Omega_0} |\mathbf{T}^n(t)|^{\frac{d(1-a)}{d-2}} \,\mathrm{d}x \right)^{\frac{d-2}{d}} + 1 \right].\end{aligned}$$

Integration over $(0,T)$ yields

$$\int_0^T \int_{\Omega_0} |\mathbf{T}^n(t)|^{1+\delta} \,\mathrm{d}x\,\mathrm{d}t \le C \left[\int_0^T \left(\int_{\Omega_0} |\mathbf{T}^n(t)|^{\frac{d(1-a)}{d-2}} \,\mathrm{d}x \right)^{\frac{d-2}{d}} \mathrm{d}t + 1 \right] \le C,$$

where C is a positive constant that is independent of n. It follows that $(\mathbf{T}^n)_n$ is uniformly bounded in $L^{1+\delta}(0,T; L^{1+\delta}_{loc}(\Omega)^{d\times d})$ and so converges weakly in this space. Since $\mathbf{T}^n \to \mathbf{T}$ pointwise, we conclude that the weak limit of $(\mathbf{T}^n)_n$ must be $\mathbf{T}$. □

This chapter marks the end of our study of strain-limiting problems on a general domain that is undamaged. We have proven the existence and uniqueness of weak solutions in the case of mixed Dirichlet-Neumann boundary conditions, up to a penalisation on the Neumann part of the boundary, and a full result in the periodic setting. The remaining two chapters use the methods developed in Chapters 2, 3 and 4 in order to investigate the more challenging problem of nonlinear dynamic fracture problems. The first step is to build the theory for a nonlinear constitutive relationship between the linearised strain and Cauchy stress tensor. This is the purpose of Chapter 5. Then we combine Chapters 4 and 5 to investigate strain-limiting dynamic fracture problems. We prove a partial existence result for mixed Dirichlet–Neumann boundary conditions and a full existence result for Dirichlet boundary conditions.

Chapter 5

Nonlinear fracture problem with growth

Before studying strain-limiting dynamic fracture problems and extending the work of the previous chapters concerning strain-limiting problems, we first introduce general nonlinear dynamic fracture problems where the strain is not *a priori* bounded. We consider constitutive relations of the form $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = |\mathbf{T}|^{p-2}\mathbf{T}$ where $p \in (1,\infty)$. This choice of constitutive relation provides similar properties to the regularised relation from Chapters 2 and 3. In this 'growth' setting, the stress tensor (and its approximations) are elements of a reflexive Bochner space so good compactness properties are available. This work is submitted and can be found in [84].

We approximate the unknown fracture set by a function called the phase-field function. We fix an approximation parameter ϵ. We refer back to Section 1.5 for further explanation of this parameter, but it will appear in the surface energy functional $\mathcal{H}$ and refers to the 'thickness' of the approximation of the phase-field approximation compared to the crack set. We obtain a system of PDEs similar to that of the previous chapters, with a couple of extra restrictions due to the presence of the unknown phase-field function. The system is supplemented with a minimisation problem. This ensures that the 'approximate crack' grows whenever it is able to do so, i.e., while balancing the appropriate energies. We further supplement the problem with an energy-dissipation balance.

Throughout this chapter, $\Omega \subset \mathbb{R}^d$ is an open, bounded set with Lipschitz boundary which is split into a Dirichlet part $\partial\Omega_D$ and a Neumann part $\partial\Omega_N$, as in Chapter 4. That is, $\partial\Omega_D$ and $\partial\Omega_N$ are open, disjoint measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$. We fix a final time horizon $T > 0$ and denote by $Q = (0,T) \times \Omega$ the space-time domain. Let $p \in (1,\infty)$ and $\alpha \geq 0$ be fixed problem parameters. Given a body force $\boldsymbol{f} : Q \to \mathbb{R}^d$, boundary traction $\boldsymbol{g} : (0,T) \times \partial\Omega_N \to \mathbb{R}^d$ and initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1 : \Omega \to \mathbb{R}^d$, $v_0 : \Omega \to \mathbb{R}$, we look for a triple

$(\boldsymbol{u}, \mathbf{T}, v) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}$ such that

$$\begin{aligned}
\boldsymbol{u}_{tt} &= \operatorname{div}(b(v)\mathbf{T}) + \boldsymbol{f} && \text{in } Q, && (5.1a)\\
\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) &= F(\mathbf{T}) := |\mathbf{T}|^{p-2}\mathbf{T} && \text{in } Q, && (5.1b)\\
\boldsymbol{u} &= \mathbf{0},\ v = 0 && \text{on } (0,T] \times \partial\Omega_D, && (5.1c)\\
b(v)\mathbf{T}\mathbf{n} &= \boldsymbol{g} && \text{on } (0,T] \times \partial\Omega_N, && (5.1d)\\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1,\ v(0,\cdot) = v_0 && \text{in } \Omega, && (5.1e)
\end{aligned}$$

subject to the crack non-healing property $v_t \le 0$, satisfaction of the minimisation problem

$$\mathcal{E}(\boldsymbol{u}(t), v(t)) + \mathcal{H}(v(t)) = \inf\left\{\mathcal{E}(\boldsymbol{u}(t), v) + \mathcal{H}(v) \,:\, v \in H^1_{D+1}(\Omega),\ v \le v(t)\right\}, \tag{5.2}$$

for every $t \in [0,T]$, and the energy dissipation balance

$$\begin{aligned}
&\mathcal{F}(t; \boldsymbol{u}(t), \boldsymbol{u}_t(t), v(t)) + \int_0^t \int_\Omega b(v)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\right] \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \boldsymbol{f}_t \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^t \int_{\partial\Omega_N} \boldsymbol{g}_t \cdot \boldsymbol{u}\,\mathrm{d}S\,\mathrm{d}s = \mathcal{F}(0; \boldsymbol{u}_0, \boldsymbol{u}_1, v_0),
\end{aligned} \tag{5.3}$$

for every $t \in [0,T]$. The functionals $\mathcal{F}$, $\mathcal{E}$, $\mathcal{H}$ denote the total, elastic and surface energy, respectively. They are defined mathematically by

$$\begin{aligned}
\mathcal{F}(t; \boldsymbol{u}, \boldsymbol{w}, v) &:= \mathcal{K}(\boldsymbol{w}) + \mathcal{E}(\boldsymbol{u}, v) + \mathcal{H}(v) - \langle l(t), \boldsymbol{u}\rangle,\\
\mathcal{E}(\boldsymbol{u}, v) &:= \int_\Omega \frac{b(v)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\,\mathrm{d}x, \quad \mathcal{H}(v) := \frac{1}{4\epsilon}\|1 - v\|_2^2 + \epsilon\|\nabla v\|_2^2,
\end{aligned}$$

where the kinetic energy $\mathcal{K}$ and external force functional l are defined, respectively, by

$$\mathcal{K}(\boldsymbol{w}) := \frac{\|\boldsymbol{w}\|_2^2}{2}, \quad \langle l(t), \boldsymbol{u}\rangle := \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{u}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}(t)\cdot\boldsymbol{u}\,\mathrm{d}S. \tag{5.4}$$

The function φ^* is the convex conjugate of φ, where φ is the anti-derivative of F. In particular, we have

$$\varphi(\mathbf{T}) = \int_0^{|\mathbf{T}|} s^{p-1}\,\mathrm{d}s.$$

We note that $F^{-1}(\mathbf{T}) = \frac{\partial\varphi^*}{\partial\mathbf{T}}$ on $\mathbb{R}^{d\times d}$ and, for every $\mathbf{T} \in \mathbb{R}^{d\times d}$, we have

$$\mathbf{T}\cdot F(\mathbf{T}) = \varphi(\mathbf{T}) + \varphi^*(F(\mathbf{T})), \quad \mathbf{T}\cdot F^{-1}(\mathbf{T}) = \varphi(F^{-1}(\mathbf{T})) + \varphi^*(\mathbf{T}). \tag{5.5}$$

In what follows, we are interested in weak solutions of (5.1)–(5.3), where (5.1a) holds in the sense that

$$\langle \boldsymbol{u}_{tt}(t), \boldsymbol{w}\rangle + \int_\Omega b(v(t))\mathbf{T}(t)\cdot\boldsymbol{w}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x + \int_{\partial\Omega_N} \boldsymbol{g}(t)\cdot\boldsymbol{w}\,\mathrm{d}S, \tag{5.6}$$

for a.e. $t \in (0,T)$ and every test function $\boldsymbol{w} \in W_D^{1,p'}(\Omega)^d$. The exact definition of a weak solution will be made clear in Theorem 5.1.

For the minimisation problem (5.2) to hold at the initial time $t = 0$, we require a compatibility condition between the initial data $\boldsymbol{u}_0$ and v_0. We ask that

$$\mathcal{E}(\boldsymbol{u}_0, v_0) + \mathcal{H}(v_0) = \inf\left\{\mathcal{E}(\boldsymbol{u}_0, v) + \mathcal{H}(v) \,:\, v \in H^1_{D+1}(\Omega),\, v \leq v_0\right\}. \tag{5.7}$$

The function $b : \mathbb{R} \to \mathbb{R}$ is defined by

$$b(v) = v^2 + \eta,$$

where $\eta > 0$ is a fixed approximation parameter. In practice, specifically in computational analysis, it is common to require that $\eta \ll \epsilon$. However, this is not required for the existence proofs here. The presence of η is to ensure that we do not experience degeneracy in the elastodynamic equation (5.1a) at points where $v = 0$.

The functional $\mathcal{A}_{\boldsymbol{u}}(v) := \mathcal{E}(\boldsymbol{u}, v) + \mathcal{H}(v)$ is well-defined on $H^1_{D+1}(\Omega)$ with values in $[0, \infty]$ for each fixed $\boldsymbol{u} \in W^{1,2}_D(\Omega)^d$. It may take the value positive infinity if v is not additionally an element of $L^\infty(\Omega)$. The framework of the problem ensures that, for any solution $(\boldsymbol{u}, v)$, $t \mapsto v(t)$ is a decreasing function so the crack non-healing property holds automatically. If $v_0 \in L^\infty(\Omega)$ then $v(t) \leq \|v_0\|_\infty$ for every $t \in (0, T]$. Furthermore, given any $v \in H^1_{D+1}(\Omega)$ and defining $v^+ = \max\{0, v\}$, we have that $v^+ \in H^1_{D+1}(\Omega)$ and $\mathcal{A}_{\boldsymbol{u}}(v^+) \leq \mathcal{A}_{\boldsymbol{u}}(v)$. It follows that any solution of the minimisation problem (5.2) and (5.7) is non-negative a.e. on Ω. Hence the phase-field function satisfies $v(t) \in L^\infty(\Omega)$ for every $t \in [0, T]$ and $t \mapsto \|v(t)\|_\infty$ is a decreasing function, provided that $v_0 \in L^\infty(\Omega)$.

As in previous chapters, the Dirichlet boundary data is taken to be $\mathbf{0}$ to simplify the presentation. However, the proof adapts easily to inhomogeneous Dirichlet boundary conditions. We refer to [20] for further details. In this chapter, because the stress tensor is an element of a reflexive space, we can show the existence of a weak energy solution to the fracture problem without any penalisation on the Neumann part of the boundary. This will not be the case for the strain-limiting dynamic fracture problem (cf. Chapter 6).

Theorem 5.1. *Suppose that $p \in (1, 2]$, and we are given $\boldsymbol{f} \in C^1([0, T]; W^{-1,p}_D(\Omega)^d)$ and $\boldsymbol{g} \in C^1([0, T]; L^{p'}(\partial\Omega_N)^d)$ with initial data $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,p'}_D(\Omega)^d$ and $v_0 \in H^1_{D+1}(\Omega)$ such that the compatibility condition* (5.7) *holds. Furthermore, assume that $v_0 \in [0, 1]$ a.e. on Ω. There exists a weak energy solution $(\boldsymbol{u}, \mathbf{T}, v)$ of* (5.1)–(5.3) *with regularity*

- $\boldsymbol{u} \in W^{2,2}(0, T; L^2(\Omega)^d) \cap W^{1,\infty}(0, T; W^{1,p'}_D(\Omega)^d)$,
- $\mathbf{T} \in L^\infty(0, T; L^p(\Omega)^{d\times d})$,
- $v \in W^{1,\infty}(0, T; H^1(\Omega))$ *with* $v(t) \in H^1_{D+1}(\Omega)$ *for every* $t \in [0, T]$,

such that it solves the dynamic fracture problem in the following sense. The elastodynamic equation (5.1a) *holds weakly in the sense that*

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{w} + b(v(t))\mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x = \langle l(t), \boldsymbol{w}\rangle, \tag{5.8}$$

for a.e. $t \in (0,T)$ *and every* $\boldsymbol{w} \in W_D^{1,p'}(\Omega)^d$*. The constitutive relation* (5.1b) *holds pointwise a.e. in* Q*, the minimisation problem* (5.2) *holds for every* $t \in [0,T]$ *and the energy-dissipation balance* (5.3) *holds for every* $t \in [0,T]$*. We have the non-healing property* $v_t \leq 0$ *a.e. in* Q *and the initial conditions are satisfied in the following sense:*

$$\lim_{t\to 0+} [\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_2 + \|v(t) - v_0\|_{1,2}] = 0. \tag{5.9}$$

Theorem 5.2. *Suppose that* $p \in (2,\infty)$*, and we are given* $\boldsymbol{f} \in C^1([0,T]; W_D^{-1,p}(\Omega)^d)$ *and* $\boldsymbol{g} \in C^1([0,T]; L^{p'}(\Omega)^d)$ *with initial data* $\boldsymbol{u}_0$*,* $\boldsymbol{u}_1 \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$*, and* $v_0 \in H^1_{D+1}(\Omega)$ *such that* $v_0 \in [0,1]$ *a.e. in* Ω *and the compatibility condition* (5.7) *holds. There exists a weak energy solution* $(\boldsymbol{u}, \mathbf{T}, v)$ *of* (5.1)–(5.3) *with regularity*

- $\boldsymbol{u} \in W^{2,2}(0,T; L^2(\Omega)^d) \cap W^{1,\infty}(0,T; W_D^{1,p'}(\Omega)^d)$*,*
- $\mathbf{T} \in L^\infty(0,T; L^p(\Omega)^{d\times d})$*,*
- $v : [0,T] \to H^1_{D+1}(\Omega)$ *with* $v \in L^\infty(0,T; H^1(\Omega))$ *and* v *is continuous a.e. as a map from* $[0,T]$ *to* $L^2(\Omega)$*,*

such that it solves the dynamic fracture problem in the following sense. The weak form (5.8) *of the elastodynamic equation* (5.1a) *holds for a.e.* $t \in (0,T)$ *and every test function* $\boldsymbol{w} \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$*, the constitutive relation* (5.1b) *is satisfied pointwise a.e. in* Ω*, the minimisation problem* (5.2) *holds for a.e.* $t \in [0,T]$ *and the energy-dissipation balance* (5.3) *holds for a.e.* $t \in [0,T]$*. The non-healing property holds in the sense that* $v(t) \leq v(s)$ *a.e. in* Ω*, for every* $0 \leq s \leq t \leq T$*. The initial conditions are satisfied in the sense that* $v(0) = v_0$ *and*

$$\lim_{t\to 0+} \left[\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,p'} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_2\right] = 0.$$

The proofs of Theorem 5.1 and Theorem 5.2 follow the same steps initially. We differentiate between the cases $p \leq 2$ and $p > 2$ for technical reasons at the final step. We introduce a two-level approximation with parameters N and M, where N refers to a discretisation in space and M to a discretisation in time. For simplicity, denote $\beta = (N, M)$. We solve this approximate problem which is discrete in both the space and time variables. For each fixed N, take the limit as $M \to \infty$ to obtain a solution to a semi-discrete problem which is continuous with respect to the time variable. Then derive suitable uniform estimates and take the limit as $N \to \infty$. It is at this stage that we differentiate between $p \in (1,2]$ and $p \in (2,\infty)$. We first prove the result when $p \in (1,2]$ and then show the necessary adaptations when $p > 2$. We remark that we do not need a regularisation in the constitutive relation because the function F is already a bijection from $\mathbb{R}^{d\times d}$ to itself.

The two-level approximation suggests how the analysis could be adapted to standard finite-element approximations on a Lipschitz polygonal domain. However, we choose approximations in space that enjoy high regularity in order to simplify the analysis and avoid technical arguments such as those in Section 3.6 of [73].

The exact structure of the proof of Theorem 5.1 is as follows. In Theorem 5.4, we prove the existence of a solution to a time discrete Galerkin approximation of (5.1). In Propositions 5.5 and 5.7, we show that the discrete solution satisfies a suitable discrete formulation of the minimisation problem (5.2) and an energy-dissipation inequality, analogous to (5.3). Then we derive M-independent bounds on the discrete solution. Taking the limit as $M \to \infty$, we obtain a solution to a Galerkin approximation of (5.1) that is continuous in the time variable. Take the limit as $N \to \infty$ to obtain a weak solution of (5.1)–(5.3) in the sense of Theorem 5.1. If $p > 2$, we apply a Helly type theorem to make use of the monotonicity of the approximate phase-field functions to deduce a suitable strong convergence result for $(v^N)_N$ since an Aubin–Lions result is not available.

5.1 The discrete problem

Let $(\boldsymbol{\phi}_i)_{i=1}^{\infty}$ be an orthogonal basis of $W_D^{k,2}(\Omega)^d$ for some $k > \frac{d}{2}+1$ such that it is orthonormal in $L^2(\Omega)^d$. By the bound on k and the Sobolev embedding theorem, $W_D^{k,2}(\Omega)^d$ continuously embeds into $C^1(\overline{\Omega})^d$. Construction of such a basis was given in Chapter 4. Define $V_N = \operatorname{span}\{\boldsymbol{\phi}_1, \dots \boldsymbol{\phi}_N\}$ and let $P_N : L^2(\Omega)^d \to V_N$ be the orthogonal projection operator. By construction, there exists a positive constant C, independent of N, such that

$$\|P_N \boldsymbol{u}\|_{l,2} \leq C\|\boldsymbol{u}\|_{l,2} \quad \forall\, \boldsymbol{u} \in W_D^{l,2}(\Omega)^d,$$

for every $0 \leq l \leq k$. A fact we use multiple times in the following is that, for every $q \in [1, \infty]$, the norms $\|\cdot\|_q$ and $\|\cdot\|_{1,q}$ are equivalent to the standard norm $\|\cdot\|_{V_N} := \|\cdot\|_{k,2}$ on V_N. However, the constants of equivalence depend on N so this fact can only be used in the construction of M-uniform estimates.

Fix an approximation parameter $\beta = (N, M) \in \mathbb{N}^2$ and define a time step $h = \frac{T}{M}$. For every $m \in \{0, \dots, M\}$, denote $t_m^\beta := mh$. Given problem data $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, v_0, $\boldsymbol{f}$ and $\boldsymbol{g}$, satisfying the regularity assumptions of Theorem 5.1, we define an initialisation for the β-approximation of (5.1) by

$$\boldsymbol{u}_0^\beta := \boldsymbol{u}_{0,N}, \quad \boldsymbol{u}_{-1}^\beta := \boldsymbol{u}_{0,N} - h\boldsymbol{u}_{1,N}, \quad v_0^\beta := v_0^N, \tag{5.10}$$

where $v_0^N \in H_{D+1}^1(\Omega)$ is the solution of the minimisation problem for $\boldsymbol{u}_{0,N}$, i.e.,

$$\mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N) + \mathcal{H}(v_0^N) = \inf\left\{\mathcal{E}(\boldsymbol{u}_{0,N}, v) + \mathcal{H}(v) \,:\, v \leq v_0,\, v \in H_{D+1}^1(\Omega)\right\}, \tag{5.11}$$

where the approximations $\boldsymbol{u}_{0,N}$, $\boldsymbol{u}_{1,N}$ are defined below. We impose (5.11) so that a condition analogous to (5.7) holds. This allows us to show that the minimisation problem holds at every time $t \in [0, T]$. The existence and uniqueness of the function $v_0^N \in H_{D+1}^1(\Omega)$ satisfying (5.11) can be proven using the direct method in the calculus of variations and strict convexity, the details of which can be found in the proof of Theorem 5.4. Under suitable regularity assumptions on the external forces, we define a time discrete approximation of $l = \boldsymbol{f} + \boldsymbol{g}$ by the discrete sequence $(l_m^\beta)_{m=1}^M$ where $l_m^\beta = l(t_m^\beta)$.

Given a sequence $(u_m^\beta)_{m=M_0}^{M_1}$ from some vector space X, where $M_0 < M_1$ are integers, with corresponding discrete time step h, we use δ and δ^2 to denote the first and second order difference quotients of the sequence. That is, denote

$$\delta u_m^\beta := \frac{u_m^\beta - u_{m-1}^\beta}{h}, \quad \delta^2 u_m^\beta := \frac{\delta u_m^\beta - \delta u_{m-1}^\beta}{h} = \frac{u_m^\beta - 2u_{m-1}^\beta + u_{m-2}^\beta}{h^2}, \tag{5.12}$$

for any index $M_0 \leq m \leq M_1$ such that the difference is well-defined.

We choose approximations $\boldsymbol{u}_{0,N}$, $\boldsymbol{u}_{1,N} \in V_N$ such that $(\|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_2)_N$, $(\|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_{1,p'})_N$ are non-increasing, null sequences with respect to N for $i \in \{0, -1\}$. In particular, we have that

$$\lim_{N\to\infty} [\|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_2 + \|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_{1,p}] = 0,$$

for $i \in \{0, -1\}$. Considering $\mathbf{0} \in V_N$ as a comparison with $\boldsymbol{u}_i$, we notice that

$$\|\boldsymbol{u}_i - \mathbf{0}\|_2 + \|\boldsymbol{u}_i - \mathbf{0}\|_{1,p'} = \|\boldsymbol{u}_i\|_2 + \|\boldsymbol{u}_i\|_{1,p'}.$$

In particular, choosing $\boldsymbol{u}_{i,N}$ as a better approximation of $\boldsymbol{u}_i$ than $\mathbf{0}$, the sequences can be chosen such that

$$\|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_2 + \|\boldsymbol{u}_i - \boldsymbol{u}_{i,N}\|_{1,p'} \leq \|\boldsymbol{u}_i\|_2 + \|\boldsymbol{u}_i\|_{1,p'} \quad \text{for } i \in \{0, 1\}.$$

The existence of such approximations follows from the fact that $\cup_N V_N$ is dense in $W_D^{k,2}(\Omega)^d$, which is dense in both $W_D^{1,p'}(\Omega)^d$ and $L^2(\Omega)^d$. Before introducing the β-approximation of the fracture problem, we consider the approximation properties of $(v_0^N)_N$.

Lemma 5.3. *Let $v_0 \in H^1_{D+1}(\Omega)$ with $v_0 \in [0,1]$ a.e. in Ω and $\boldsymbol{u}_0 \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$ such that the compatibility condition* (5.7) *holds. Let $(\boldsymbol{u}_0^N)_N$ be an approximation of $\boldsymbol{u}_0$ as above. Let $v_0^N \in H^1_{D+1}(\Omega)$ be the solution of the minimisation problem* (5.11)*. Then $v_0^N \in [0,1]$ a.e. in Ω and $v_0^N \to v_0$ strongly in $H^1(\Omega)$.*

Proof. By construction, we have $v_0^N \leq v_0 \leq 1$ and, as discussed previously, comparing v_0^N with the non-negative part $\max\{0, v_0^N\} = (v_0^N)^+$, we deduce that $v_0^N \geq 0$ by the uniqueness of minimisers. We refer to the proof of Theorem 5.4 for the uniqueness of minimisers. Hence $v_0^N \in [0,1]$ a.e. in Ω. Next, by construction, we have that

$$\begin{aligned}
\|v_0^N\|_{1,2}^2 &\leq C(\epsilon)\left[\int_\Omega \frac{b(v_0^N)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_{0,N}))\,\mathrm{d}x + \frac{1}{4\epsilon}\|1 - v_0^N\|_2^2 + \epsilon\|\nabla v_0^N\|_2^2 + 1\right] \\
&= C(\epsilon)\left[\mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N) + \mathcal{H}(v_0^N) + 1\right] \\
&\leq C(\epsilon)\left[\mathcal{E}(\boldsymbol{u}_{0,N}, v_0) + \mathcal{H}(v_0) + 1\right] \\
&\leq C(\epsilon)\left[\frac{b(1)}{\alpha}\int_\Omega \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_{0,N}))\,\mathrm{d}x + \|v_0\|_{1,2}^2 + 1\right] \\
&\leq C(\epsilon)\left[\|\boldsymbol{\varepsilon}(\boldsymbol{u}_{0,N})\|_{1,p'}^{p'} + \|v_0\|_{1,2}^2 + 1\right] \\
&\leq C(\epsilon)\left[\|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|\boldsymbol{u}_0\|_2^{p'} + \|v_0\|_{1,2}^2 + 1\right],
\end{aligned}$$

using the minimising property of v_0^N in the transition to the third line. Thus, up to a subsequence that we do not relabel, $(v_0^N)_N$ converges weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise a.e. in Ω to a limit $\tilde{v}_0 \in H^1_{D+1}(\Omega)$. Since $v_0^N \le v_0$ for every N, we have $\tilde{v}_0 \le v_0$ in Ω. By weak lower semi-continuity[1] and Fatou's lemma,

$$\begin{aligned}\mathcal{E}(\boldsymbol{u}_0, \tilde{v}_0) + \mathcal{H}(\tilde{v}_0) &\le \lim_{N\to\infty} \left[\mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N) + \mathcal{H}(v_0^N)\right] \le \lim_{N\to\infty} \left[\mathcal{E}(\boldsymbol{u}_{0,N}, v_0) + \mathcal{H}(v_0)\right] \\ &= \mathcal{E}(\boldsymbol{u}_0, v_0) + \mathcal{H}(v_0) \le \mathcal{E}(\boldsymbol{u}_0, \tilde{v}_0) + \mathcal{H}(\tilde{v}_0),\end{aligned}$$

where the final line follows from the compatibility condition (5.7). Hence $\tilde{v}_0$ is also a minimiser of (5.7) and so, by strict convexity, we deduce that $v_0 = \tilde{v}_0$.

For the strong convergence of $(v_0^N)_N$ in $H^1(\Omega)$, we notice that $(\mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N))_N$ converges to $\mathcal{E}(\boldsymbol{u}_0, v_0)$ by the dominated convergence theorem and the uniform boundedness of $(b(v_0^N))_N$. Inserting this into the above, it follows that $\mathcal{H}(v_0) = \lim_{N\to\infty} \mathcal{H}(v_0^N)$. However, from the strong convergence in $L^2(\Omega)$,

$$\frac{1}{4\epsilon}\|1 - v_0\|_2^2 + \epsilon\|\nabla v_0\|_2^2 = \mathcal{H}(v_0) = \lim_{n\to\infty} \mathcal{H}(v_0^N) = \frac{1}{4\epsilon}\|1 - v_0\|_2^2 + \lim_{N\to\infty} \epsilon\|\nabla v_0^N\|_2^2.$$

In particular, $(\nabla v_0^N)_N$ converges weakly and in norm in $L^2(\Omega)$. Thus the convergence is strong in $L^2(\Omega)$. ☐

Theorem 5.4. *Suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$, $v_0 \in H^1_{D+1}(\Omega)$ such that $v_0 \in [0,1]$ a.e. in Ω and the compatibility condition (5.7) holds. Let $\boldsymbol{f} \in C^1([0,T]; W_D^{-1,p}(\Omega)^d)$, $\boldsymbol{g} \in C^1([0,1]; L^{p'}(\partial\Omega_N)^d)$ and define l as in (5.4). Given $\beta = (N, M) \in \mathbb{N}^2$ and initialisations $\boldsymbol{u}_0^\beta$, $\boldsymbol{u}_{-1}^\beta$, v_0^β as above, there exists a unique solution sequence $(\boldsymbol{u}_m^\beta, v_m^\beta)_{m=1}^M$ to the β-approximation problem which we define recursively as follows. For $m \in \{1, \ldots, M\}$, let $\boldsymbol{u}_m^\beta \in V_N$ be such that*

$$\int_\Omega \delta^2 \boldsymbol{u}_m^\beta \cdot \boldsymbol{w} + b(v_{m-1}^\beta)\mathbf{T}_m^\beta \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x = \langle l_m^\beta, \boldsymbol{w}\rangle, \tag{5.13}$$

for every $\boldsymbol{w} \in V_M$, where we define $\mathbf{T}_m^\beta = F^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta + \alpha\boldsymbol{u}_m^\beta))$. We let $v_m^\beta \in H^1_{D+1}(\Omega)$ be the function that solves the minimisation problem

$$\mathcal{E}(\boldsymbol{u}_m^\beta, v_m^\beta) + \mathcal{H}(v_m^\beta) = \inf\left\{\mathcal{E}(\boldsymbol{u}_m^\beta, v) + \mathcal{H}(v) \;:\; v \in H^1_{D+1}(\Omega), v \le v_{m-1}^\beta\right\}. \tag{5.14}$$

Proof. Fix $m \in \{1, \ldots, M\}$ and suppose that the problem is solved at the $(m-1)$-st time step. We look to show that a solution $\boldsymbol{u}_m^\beta \in V_N$ to (5.13) exists and then we show that a solution $v_m^\beta \in H^1_{D+1}(\Omega)$ to (5.14) exists. Then we show that this solution couple is unique.

By the Browder–Minty Theorem (Theorem 2.2), S is a bijection and there exists a unique $\boldsymbol{u}_m^\beta \in V_N$ such that $S(\boldsymbol{u}_m^\beta) = l_m^\beta$ in V_N.

To solve (5.14) at the m-th time step, define a functional $\mathcal{A} : H^1_{D+1}(\Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{A}(v) := \mathcal{E}(\boldsymbol{u}_m^\beta, v) + \mathcal{H}(v) = \int_\Omega \frac{b(v)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta)) + \frac{1}{4\epsilon}(1-v)^2 + \epsilon|\nabla v|^2\,\mathrm{d}x.$$

[1] Throughout this chapter and the next, by weak lower semi-continuity we mean the weak lower semi-continuity of norms. In particular, if $(x_n)_n$ is a sequence in a normed space X that converges weakly to some $x \in X$, then $\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X$.

The functional $\mathcal{A}$ is bounded below by 0 so the infimum of $\mathcal{A}$ over functions from $H^1_{D+1}(\Omega)$ that are bounded above by v^β_{m-1} exists. Let $(v_k)_{k\geq 1}$ be a minimising sequence. It follows that

$$\mathcal{A}(v_k) \geq \frac{1}{4\epsilon}\|1 - v_k\|_2^2 + \epsilon\|\nabla v_k\|^2 \geq C(\epsilon)\|v_k - 1\|^2_{1,2}.$$

Hence the sequence $(v_k)_k$ is bounded in $H^1(\Omega)$. Up to a subsequence that we do not relabel, $(v_k)_k$ converges strongly in $L^2(\Omega)$, pointwise a.e. on Ω and weakly in $H^1(\Omega)$ to a limit v, say. Then $v \in H^1_{D+1}(\Omega)$ because $H^1_{D+1}(\Omega)$ is weakly closed. Applying weak lower semi-continuity and Fatou's lemma, we deduce that $\mathcal{A}(v) \leq \inf_{\tilde{v}\in H^1_{D+1}(\Omega)} \mathcal{A}(\tilde{v})$. Hence v is a minimiser. To see that v is the unique minimiser, we simply note that $\mathcal{A}$ is strictly convex. □

Proposition 5.5. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}^\beta_m, v^\beta_m)^M_{m=1}$ be the solution sequence constructed there. Then $v^\beta_m \in [0,1]$ and $\delta v^\beta_m \leq 0$ a.e. in Ω where δv^β_m is defined as in (5.12). Furthermore, for every $\tilde{\chi} \in H^1_{D+1}(\Omega)$ such that $\tilde{\chi} \leq v^\beta_{m-1}$, we have*

$$\begin{aligned} 0 &\leq \left[\partial_v \mathcal{E}(\boldsymbol{u}^\beta_m, v^\beta_m) + \mathcal{H}'(v^\beta_m)\right](\tilde{\chi} - v^\beta_m) \\ &= \int_\Omega \frac{b'(v^\beta_m)}{\alpha}(\tilde{\chi} - v^\beta_m)\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^\beta_m)) + \frac{1}{2\epsilon}(v^\beta_m - 1)(\tilde{\chi} - v^\beta_m) + 2\epsilon\nabla v^\beta_m \cdot \nabla(\tilde{\chi} - v^\beta_m)\,\mathrm{d}x. \end{aligned} \tag{5.15}$$

In particular, for every $\chi \in H^1_D(\Omega)$ such that $\chi \leq 0$, we have

$$0 \leq \left[\partial_v \mathcal{E}(\boldsymbol{u}^\beta_m, v^\beta_m) + \mathcal{H}'(v^\beta_m)\right](\chi). \tag{5.16}$$

Proof. The initialisation v^β_0 satisfies $v^\beta_0 \leq 1$ in Ω so by construction $v^\beta_m \leq 1$ in Ω. By previous reasoning, $v^\beta_m \geq 0$ a.e. in Ω since v^β_m is a minimiser. By definition, $v^\beta_m \leq v^\beta_{m-1}$ and so $\delta v^\beta_m \leq 0$ in Ω.

The first statement follows from the definition of v^β_m being a solution of the problem. In particular, consider $\tilde{v}_\epsilon = v^\beta_m + \epsilon(\tilde{\chi} - v^\beta_{m-1})$ as a comparison function for v^β_m. The function $\tilde{v}_\epsilon$ is a well-defined test function because $\tilde{v}_\epsilon \in H^1_{D+1}(\Omega)$ and $\tilde{v}_\epsilon \leq v^\beta_m \leq v^\beta_{m-1}$. We have $0 \leq \epsilon^{-1}(\mathcal{A}(\tilde{v}_\epsilon) - \mathcal{A}(v))$ for every $\epsilon > 0$. Expanding out the right-hand side and letting $\epsilon \to 0+$, we conclude the required result. For (5.16), given such a χ, set $\tilde{\chi} = \chi + v^\beta_m$. Then $\tilde{\chi} \leq v^\beta_m \leq v^\beta_{m-1}$ and $\tilde{\chi} \in H^1_{D+1}(\Omega)$. Substituting this $\tilde{\chi}$ into (5.15), we obtain (5.16). □

Corollary 5.6. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}^\beta_m, v^\beta_m)^M_{m=1}$ be the solution sequence constructed there. For every $1 \leq m \leq M$,*

$$0 = \left[\partial_v \mathcal{E}(\boldsymbol{u}^\beta_m, v^\beta_m) + \mathcal{H}'(v^\beta_m)\right](\delta v^\beta_m). \tag{5.17}$$

Proof. We reason as in the proof of Lemma 3.2 of [73]. Testing in (5.15) against v^β_{m-1}, it follows that $0 \geq \left[\partial_v \mathcal{E}(\boldsymbol{u}^\beta_m, v^\beta_m) + \mathcal{H}'(v^\beta_m)\right](\delta v^\beta_m)$. For the opposite inequality, we choose $\tilde{\chi} = 2v^\beta_m - v^\beta_{m-1}$ in (5.15). □

Proposition 5.7 (Discrete energy-dissipation inequality)**.** *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}_m^\beta, v_m^\beta)_{m=1}^M$ be the solution sequence constructed there. For every $m \in \{1, \dots M\}$, the following energy-dissipation inequality holds:*

$$\mathcal{F}(t_m^\beta; \boldsymbol{u}_m^\beta, \delta\boldsymbol{u}_m^\beta, v_m^\beta) + h\sum_{j=1}^{m}\int_\Omega b(v_{j-1}^\beta)\left[F^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_j^\beta + \alpha\boldsymbol{u}_j^\beta)) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_j^\beta))\right]\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_j^\beta)\,\mathrm{d}x$$
$$+ h\sum_{j=1}^{m}\langle\delta l_{j-1}^\beta, \boldsymbol{u}_j^\beta\rangle \le \mathcal{F}(0; \boldsymbol{u}_{0,N}, \boldsymbol{u}_{1,N}, v_0^N).$$

Proof. We test in the elastodynamic equation (5.13) against $h\delta\boldsymbol{u}_m^\beta$. Using the relation

$$\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = \frac{1}{2}\left(|\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2\right) \quad \forall\,\mathbf{a},\,\mathbf{b}\in\mathbb{R}^d, \tag{5.18}$$

we rewrite the inertial term (the one involving $\delta^2\boldsymbol{u}_m^\beta$) as

$$\int_\Omega \delta^2\boldsymbol{u}_m^\beta\cdot h\delta\boldsymbol{u}_m^\beta\,\mathrm{d}x = \frac{1}{2}\left(\|\delta\boldsymbol{u}_m^\beta\|_2^2 - \|\delta\boldsymbol{u}_{m-1}^\beta\|_2^2 + \|\delta\boldsymbol{u}_m^\beta - \delta\boldsymbol{u}_{m-1}^\beta\|_2^2\right). \tag{5.19}$$

For the term involving the external force and boundary traction, we write

$$\langle l_m^\beta, h\delta\boldsymbol{u}_m^\beta\rangle = \langle l_m^\beta, \boldsymbol{u}_m^\beta\rangle - \langle l_{m-1}^\beta, \boldsymbol{u}_{m-1}^\beta\rangle + h\langle\delta l_m^\beta, \boldsymbol{u}_{m-1}^\beta\rangle. \tag{5.20}$$

For the term involving the stress tensor, we have that

$$\int_\Omega b(v_{m-1}^\beta)\mathbf{T}_m^\beta\cdot\boldsymbol{\varepsilon}(h\delta\boldsymbol{u}_m^\beta)\,\mathrm{d}x = \int_\Omega b(v_{m-1}^\beta)F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\cdot\boldsymbol{\varepsilon}(h\delta\boldsymbol{u}_m^\beta)\,\mathrm{d}x$$
$$+ h\int_\Omega b(v_{m-1}^\beta)\left[F^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta + \alpha\boldsymbol{u}_m^\beta)) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\right]\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta)\,\mathrm{d}x.$$

The first term on the right-hand side is in the given energy-dissipation inequality. For the second term on the right, we use the minimisation problem to rewrite the expression in terms of the elastic energy and surface energy. By the monotonicity of F^{-1} and recalling that F^{-1} is the derivative of φ^* (cf. (5.5)), we get

$$\begin{aligned}
&\int_\Omega b(v_{m-1}^\beta)F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\cdot\boldsymbol{\varepsilon}(h\delta\boldsymbol{u}_m^\beta)\,\mathrm{d}x \\
&= \int_\Omega\int_0^1 b(v_{m-1}^\beta)F^{-1}(\alpha\boldsymbol{\varepsilon}(s\boldsymbol{u}_m^\beta + (1-s)\boldsymbol{u}_{m-1}^\beta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta - \boldsymbol{u}_{m-1}^\beta)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \int_\Omega\int_0^1 b(v_{m-1}^\beta)\left[F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta)) - F^{-1}(\alpha\boldsymbol{\varepsilon}(s\boldsymbol{u}_m^\beta + (1-s)\boldsymbol{u}_{m-1}^\beta))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta - \boldsymbol{u}_{m-1}^\beta)\,\mathrm{d}s\,\mathrm{d}x \\
&\ge \int_\Omega\int_0^1 b(v_{m-1}^\beta)F^{-1}(\alpha\boldsymbol{\varepsilon}(s\boldsymbol{u}_m^\beta + (1-s)\boldsymbol{u}_{m-1}^\beta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta - \boldsymbol{u}_{m-1}^\beta)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_\Omega \frac{b(v_{m-1}^\beta)}{\alpha}\left(\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta)) - \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_{m-1}^\beta))\right)\mathrm{d}x \\
&= \int_\Omega \frac{b(v_m^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta)) - \frac{b(v_{m-1}^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_{m-1}^\beta)) + \frac{b(v_{m-1}^\beta) - b(v_m^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\,\mathrm{d}x \\
&\ge \mathcal{E}(\boldsymbol{u}_m^\beta, v_m^\beta) - \mathcal{E}(\boldsymbol{u}_{m-1}^\beta, v_{m-1}^\beta) + \mathcal{H}(v_m^\beta) - \mathcal{H}(v_{m-1}^\beta).
\end{aligned} \tag{5.21}$$

The first inequality in the above follows from the monotonicity of F^{-1}, and the second inequality is from considering the final term and v_m^β being a solution of the minimisation problem with respect to $\boldsymbol{u}_m^\beta$. Putting together (5.19), (5.20) and (5.21), we deduce that

$$\mathcal{F}(t_m^\beta;\boldsymbol{u}_m^\beta,\delta\boldsymbol{u}_m^\beta,v_m^\beta)+h\int_\Omega b(v_{m-1}^\beta)\left[F^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta+\alpha\boldsymbol{u}_m^\beta))-F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\right]\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta)\,\mathrm{d}x$$
$$+h\langle\delta l_m^\beta,\boldsymbol{u}_{m-1}^\beta\rangle\le\mathcal{F}(t_{m-1}^\beta;\boldsymbol{u}_{m-1}^\beta,\delta\boldsymbol{u}_{m-1}^\beta,v_{m-1}^\beta).$$

Using this recursively, we obtain the required energy-dissipation inequality. □

Lemma 5.8. *Let the assumptions of Theorem 5.4 hold and let* $(\boldsymbol{u}_m^\beta,v_m^\beta)_{m=1}^M$ *be the solution sequence constructed there. There exists a constant* C*, independent of* N *and* M*, such that*

$$\max_{1\le m\le M}\|\boldsymbol{u}_m^\beta\|_2^2+\max_{1\le m\le M}\|\delta\boldsymbol{u}_m^\beta\|_2^2+\max_{1\le m\le M}\|\boldsymbol{u}_m^\beta\|_{1,p'}^{p'}+\max_{1\le m\le M}\|v_m^\beta\|_{1,2}^2$$
$$\le C\big[1+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_2^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|l\|_{L^\infty(W^{-1,p})}^p+\|l_t\|_{L^p(W^{-1,p})}^p\big].$$

Furthermore, there exists another constant $C=C(N)$*, independent of* M*, such that*

$$\max_{1\le m\le M}\|\boldsymbol{u}_m^\beta\|_{V_N}^2+\max_{1\le m\le M}\|\delta\boldsymbol{u}_m^\beta\|_{V_N}^2+\max_{1\le m\le M}\|v_m^\beta\|_{1,2}^2$$
$$\le C(N)\big[1+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_2^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|l\|_{L^\infty(W^{-1,p})}^p+\|l_t\|_{L^p(W^{-1,p})}^p\big].$$

Proof. We expand the total energy functional and use the monotonicity of F to obtain

$$\mathcal{K}(\delta\boldsymbol{u}_m^\beta)+\mathcal{E}(\boldsymbol{u}_m^\beta,v_m^\beta)+\mathcal{H}(v_m^\beta)$$
$$\le\mathcal{K}(\delta\boldsymbol{u}_0^\beta)+\mathcal{E}(\boldsymbol{u}_0^\beta,v_0^\beta)+\mathcal{H}(v_0^\beta)-\langle l_0^\beta,\boldsymbol{u}_0^\beta\rangle+\langle l_m^\beta,\boldsymbol{u}_m^\beta\rangle-h\sum_{j=1}^m\langle\delta l_j^\beta,\boldsymbol{u}_{j-1}^\beta\rangle.$$

Using the Korn–Poincaré inequality and Young's inequality, we see that

$$\begin{aligned}
&\|\delta\boldsymbol{u}_m^\beta\|_2^2+\|\boldsymbol{u}_m^\beta\|_{1,p'}^{p'}+\|v_m^\beta\|_{1,2}^2\\
&\le\|\delta\boldsymbol{u}_m^\beta\|_2^2+\int_\Omega\frac{b(v_m^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\,\mathrm{d}x+\|v_m^\beta\|_{1,2}^2\\
&\le C\Bigg[\|\boldsymbol{u}_{-1}^\beta\|_2^2+\int_\Omega\frac{b(v_0^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_0^\beta))\,\mathrm{d}x+\|v_0^N\|_{1,2}+\|l(0)\|_{-1,p}\|\boldsymbol{u}_0^\beta\|_{1,p'}\\
&\quad+\|l_m^\beta\|_{-1,p}\|\boldsymbol{u}_m^\beta\|_{1,p'}+h\sum_{j=1}^m\|\delta l_j^\beta\|_{-1,p}\|\boldsymbol{u}_{j-1}^\beta\|_{1,p'}\Bigg]\\
&\le C\Bigg[1+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|l(0)\|_{-1,p}^p+\|l_m^\beta\|_{-1,p}^p+h\sum_{j=1}^m\|\delta l_j^\beta\|_{-1,p'}^{p'}\\
&\quad+h\sum_{j=1}^m\|\boldsymbol{u}_{j-1}^\beta\|_{1,p'}^{p'}\Bigg]+\frac{1}{2}\|\boldsymbol{u}_m^\beta\|_{p'}^{p'},
\end{aligned}\tag{5.22}$$

where C is independent of M and N. We write

$$\boldsymbol{u}_m^\beta=\boldsymbol{u}_{m-1}^\beta+h\delta\boldsymbol{u}_m^\beta=\boldsymbol{u}_0^\beta+h\sum_{j=1}^m\delta\boldsymbol{u}_j^\beta,$$

so adding $\|\boldsymbol{u}_m^\beta\|_2^2$ to both sides of (5.22), we deduce that

$$\begin{aligned}&\|\boldsymbol{u}_m^\beta\|_2^2+\|\delta\boldsymbol{u}_m^\beta\|_2^2+\|\boldsymbol{u}_m^\beta\|_{1,p'}^{p'}+\|v_m^\beta\|_{1,2}^2\\&\leq C\Bigg[1+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_2^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|l(0)\|_{-1,p}^p+\|l_m^\beta\|_{-1,p}^p+h\sum_{j=1}^m\|\delta l_j^\beta\|_{-1,p'}^{p'}\\&\quad+h\sum_{j=1}^m\|\boldsymbol{u}_{j-1}^\beta\|_{1,p'}^{p'}+h\sum_{j=1}^m\|\boldsymbol{u}_{j-1}^\beta\|_2^2\Bigg].\end{aligned}$$

We apply the discrete Gronwall inequality to get that

$$\begin{aligned}&\max_{1\leq m\leq M}\|\boldsymbol{u}_m^\beta\|_2^2+\max_{1\leq m\leq M}\|\delta\boldsymbol{u}_m^\beta\|_2^2+\max_{1\leq m\leq M}\|\boldsymbol{u}_m^\beta\|_{1,p'}^{p'}+\max_{1\leq m\leq M}\|v_m^\beta\|_{1,2}^2\\&\leq C\left[1+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_2^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|l\|_{W^{1,\infty}(W^{-1,p})}^p+\|l_t\|_{L^p(W^{-1,p})}^p\right].\end{aligned}$$

Using the equivalence of norms, the second bound follows immediately. □

In the elastodynamic equation, the term $b(v_m^\beta)$ is present, which is nonlinear in v_m^β. Hence, to obtain a pointwise convergence result for interpolants of the sequence $(b(v_m^\beta))_{m=1}^M$, we look for bounds on δv_m^β. The eventual aim is to use the Aubin–Lions lemma and deduce a strong convergence result. Due to the nonlinearity of F, when looking for bounds on the sequence $(\delta v_m^\beta)_{m=1}^M$, we cannot argue as in the proof of Lemma 3.4 from [73]. In particular, we cannot 'transfer' the effects of the nonlinearity between $\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta)$ and $\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta)$ in the sense that, for symmetric matrices $\mathbb{A}$ such that $\mathbb{A}=\mathbb{A}^{\frac12}\mathbb{A}^{\frac12}$, we have that

$$\mathbb{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta)\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta)=\mathbb{A}^{\frac12}\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta)\cdot\mathbb{A}^{\frac12}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta)=\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\beta)\cdot\mathbb{A}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\beta).$$

Instead, we make use of the embedding $V_N\subset W_D^{k,2}(\Omega)^d\subset C^1(\overline{\Omega})^d$ and Corollary 5.6. The idea is to 'differentiate' the terms in the square brackets in (5.17) in some 'discrete' sense in order to obtain $\|\delta v_m^\beta\|_{1,2}$.

Lemma 5.9. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}_m^\beta,v_m^\beta)_{m=1}^M$ be the solution sequence constructed there. There exists a constant $C=C(N)$, independent of M, such that, for every $1\leq m\leq M$,*

$$\|\delta v_m^\beta\|_{1,2}^2\leq C\left[1+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|v_0\|_{1,2}^2+\|l\|_{L^\infty(W^{-1,p})}^p+\|l_t\|_{L^p(W^{-1,p})}^p\right]^{p'}$$

Proof. For every $1\leq m\leq M$, by Proposition 5.5 and Corollary 5.6, respectively, we have that $0\leq\left[\partial_v\mathcal{E}(\boldsymbol{u}_{m-1}^\beta,v_{m-1}^\beta)+\mathcal{H}'(v_{m-1}^\beta)\right](\delta v_m^\beta)$ and $0=\left[\partial_v\mathcal{E}(\boldsymbol{u}_m^\beta,v_m^\beta)+\mathcal{H}'(v_m^\beta)\right](\delta v_m^\beta)$. Subtracting the second equation from the first inequality, we obtain

$$\frac{1}{2\epsilon}\|\delta v_m^\beta\|_2^2+2\epsilon\|\nabla\delta v_m^\beta\|_2^2\leq\frac1h\int_\Omega\left[\frac{b'(v_{m-1}^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_{m-1}^\beta))-\frac{b'(v_m^\beta)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\beta))\right]\cdot\delta v_m^\beta\,\mathrm{d}x.\quad(5.23)$$

We add and subtract $\alpha^{-1}b'(v^\beta_{m-1})\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^\beta_m))$ to the right-hand side of (5.23) and use that $b'(v) = 2v$ to yield

$$
\begin{aligned}
&\frac{1}{2\epsilon}\|\delta v^\beta_m\|_2^2 + 2\epsilon\|\nabla\delta v^\beta_m\|_2^2 + \frac{2}{\alpha}\int_\Omega |\delta v^\beta_m|^2\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^\beta_m))\,\mathrm{d}x\\
&\le -h\int_\Omega\int_0^1 b'(v^\beta_{m-1})\delta v^\beta_m F^{-1}(\alpha\boldsymbol{\varepsilon}(s\boldsymbol{u}^\beta_{m-1} + (1-s)\boldsymbol{u}^\beta_m))\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^\beta_m)\,\mathrm{d}s\,\mathrm{d}x\\
&\le \|b'(v^\beta_{m-1})\|_\infty\|\delta v^\beta_{m-1}\|_2\left(\|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^\beta_{m-1})\|_\infty^{p'-1} + \|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^\beta_m)\|_\infty^{p'-1}\right)\left(\|\boldsymbol{\varepsilon}(\boldsymbol{u}^\beta_m)\|_2 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}^\beta_{m-1})\|_2\right)\\
&\le \frac{\|\delta v^\beta_m\|_2^2}{4\epsilon} + C\left(\max_{0\le m\le M}\|\boldsymbol{u}^\beta_m\|_{V_N}^{2p'}\right).
\end{aligned}
$$

Absorbing the first term on the right-hand side into the left, using the second bound from Lemma 5.8 and maximising over m on the resulting left-hand side, the result follows. □

Lemma 5.10. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}^\beta_m, v^\beta_m)_{m=1}^M$ be the solution sequence constructed there. There exists a constant $C = C(N)$, independent of M, such that, for every $1 \le m \le M$,*

$$
\|\delta^2\boldsymbol{u}^\beta_m\|_{V_N}^2 \le C\big[1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|v_0\|_{1,2} + \|l\|^p_{L^\infty(W^{-1,p})} + \|l_t\|^p_{L^p(W^{-1,p})}\big].
$$

Proof. Recalling that the sequence $(v^\beta_m)_{m=0}^M$ is uniformly bounded above by 1, for any test function $\boldsymbol{w} \in V_N$, it follows that

$$
\begin{aligned}
\langle\delta^2\boldsymbol{u}^\beta_m, \boldsymbol{w}\rangle &= -\int_\Omega b(v^\beta_m)F^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^\beta_m + \alpha\boldsymbol{u}^\beta_m))\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x + \langle l^\beta_m, \boldsymbol{w}\rangle\\
&\le C\left[\|\delta\boldsymbol{u}^\beta_m\|_{V_N} + \|\boldsymbol{u}^\beta_m\|_{V_N} + \|l^\beta_m\|_{-1,p}\right]\|\boldsymbol{w}\|_{V_N}.
\end{aligned}
$$

Thus, we have that

$$
\|\delta^2\boldsymbol{u}^\beta_m\|_{V_N^*}^2 \le C\big[1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|v_0\|_{1,2}^2 + \|l\|^p_{L^\infty(W^{-1,p})} + \|l_t\|^p_{L^p(W^{-1,p})}\big],
$$

where $C = C(N)$ is independent of M. The norm in V_N^* is equivalent to the norm in V_N. Using this and maximising over the left-hand side with respect to m, the required result follows. □

5.2 The limit as $M \to \infty$ and the time-continuous Galerkin problem

For the solution sequence $(\boldsymbol{u}^\beta_m)_{m=0}^M$, we define the affine, backwards and forward interpolants, respectively, by

$$
\begin{aligned}
\overline{\boldsymbol{u}}^\beta(t) &= \frac{t - t^\beta_{m-1}}{h}\boldsymbol{u}^\beta_m + \frac{t^\beta_m - t}{h}\boldsymbol{u}^\beta_{m-1}, && t \in [t^\beta_{m-1}, t^\beta_m],\ m \in \{1,\dots,M\},\\
\boldsymbol{u}^{\beta,+}(t) &= \boldsymbol{u}^\beta_m, && t \in (t^\beta_{m-1}, t^\beta_m],\ m \in \{1,\dots,M\},\\
\boldsymbol{u}^{\beta,-}(t) &= \boldsymbol{u}^\beta_{m-1}, && t \in [t^\beta_{m-1}, t^\beta_m),\ m \in \{1,\dots,M\},
\end{aligned}
$$

where $\boldsymbol{u}^{\beta,+}$, $\boldsymbol{u}^{\beta,-}$ are extended continuously to $t = 0$ and $t = T$, respectively. Similarly, we denote by $\overline{\boldsymbol{u}}^{\beta,\prime}$, $\boldsymbol{u}^{\beta,+,\prime}$, $\boldsymbol{u}^{\beta,-,\prime}$ the interpolants of $(\delta\boldsymbol{u}^\beta_m)_{m=0}^M$, and $\boldsymbol{u}^{\beta,+,\prime\prime}$ the backwards interpolant of $(\delta^2 \boldsymbol{u}_m^\beta)_{m=1}^M$. We let $\overline{v}^\beta$, $v^{\beta,+}$, $v^{\beta,-}$ be the interpolants of $(v_m^\beta)_{m=0}^M$ and $v^{\beta,+,\prime}$ the backwards interpolant of $(\delta v_m^\beta)_{m=1}^M$. Clearly we have that

$$\overline{\boldsymbol{u}}_t^\beta = \boldsymbol{u}^{\beta,+,\prime}, \quad \overline{\boldsymbol{u}}_t^{\beta,\prime} = \boldsymbol{u}^{\beta,+,\prime\prime}, \quad \overline{v}_t^\beta = v^{\beta,+,\prime}.$$

Putting together the results of Lemma 5.8, Lemma 5.9 and Lemma 5.10, we immediately obtain the following.

Lemma 5.11. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}_m^\beta, v_m^\beta)_{m=1}^M$ be the solution sequence constructed there. There exists a constant $C = C(N)$, independent of M, such that*

$$\begin{aligned}
&\|\overline{\boldsymbol{u}}^\beta\|_{L^\infty(V_N)}^2 + \|\boldsymbol{u}^{\beta,\pm}\|_{L^\infty(V_N)}^2 + \|\overline{\boldsymbol{u}}^{\beta,\prime}\|_{L^\infty(V_N)}^2 + \|\boldsymbol{u}^{\beta,\pm,\prime}\|_{L^\infty(V_N)}^2 + \|\overline{v}^\beta\|_{L^\infty(H^1)}^2 \\
&\quad + \|v^{\beta,\pm}\|_{L^\infty(H^1)}^2 + \|v^{\beta,+,\prime}\|_{L^\infty(H^1)}^2 + \|\boldsymbol{u}^{\beta,+,\prime\prime}\|_{L^\infty(V_N)}^2 \\
&\leq C(N)\big[1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|v_0\|_{1,2} + \|l\|_{L^\infty(W^{-1,p})}^p + \|l_t\|_{L^p(W^{-1,p})}^p\big]^{p'}.
\end{aligned}$$

Lemma 5.12. *Let the assumptions of Theorem 5.4 hold and let $(\boldsymbol{u}_m^\beta, v_m^\beta)_{m=1}^M$ be the solution sequence constructed there. There exists a subsequence in M, not relabelled and independent of N, and a limiting couple $(\boldsymbol{u}^N, v^N)$ such that $\boldsymbol{u}^N \in W^{2,\infty}(0,T;V_N)$, $v^N \in W^{1,\infty}(0,T;H^1(\Omega))$ with $v^N(t) \in H^1_{D+1}(\Omega)$ for every $t \in [0,T]$, such that the following convergence results hold as $M \to \infty$:*

- $\overline{\boldsymbol{u}}^{(N,M)}$, $\overline{\boldsymbol{u}}^{(N,M),\prime} \overset{*}{\rightharpoonup} \boldsymbol{u}^N$, $\boldsymbol{u}_t^N$ *weakly-* in* $W^{1,\infty}(0,T;V_N)$, *respectively;*
- $\overline{\boldsymbol{u}}^{(N,M)}$, $\overline{\boldsymbol{u}}^{(N,M),\prime} \to \boldsymbol{u}^N$, $\boldsymbol{u}_t^N$ *strongly in* $C([0,T];C^1(\overline{\Omega})^d)$, *respectively;*
- $\boldsymbol{u}^{(N,M),\pm}$, $\boldsymbol{u}^{(N,M),\pm,\prime} \overset{*}{\rightharpoonup} \boldsymbol{u}^N$, $\boldsymbol{u}_t^N$ *weakly-* in* $L^\infty(0,T;V_N)$, *respectively;*
- $\overline{v}^{(N,M)} \overset{*}{\rightharpoonup} v^N$ *weakly-* in* $W^{1,\infty}(0,T;H^1(\Omega))$;
- $\overline{v}^{(N,M)} \to v^N$ *strongly in* $C([0,T];L^2(\Omega))$;
- $v^{(N,M),\pm} \overset{*}{\rightharpoonup} v^N$ *weakly-* in* $L^\infty(0,T;H^1(\Omega))$.

Furthermore, $v^N \in [0,1]$ and $v_t^N \leq 0$ a.e. in Q.

Proposition 5.13. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there. Then $(\boldsymbol{u}^N, v^N)$ is a weak solution of a Galerkin approximation from V_N from (5.1) in the following sense. For a.e. $t \in (0,T)$ and every $\boldsymbol{w} \in V_N$,*

$$\int_\Omega \boldsymbol{u}_{tt}^N(t) \cdot \boldsymbol{w} + b(v^N(t))\mathbf{T}^N(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \, \mathrm{d}x = \langle l(t), \boldsymbol{w} \rangle, \tag{5.24}$$

where the approximate stress $\mathbf{T}^N$ is defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N) = F(\mathbf{T}^N)$ pointwise a.e. in Q. The initial conditions hold in the sense that

$$\lim_{t \to 0+} \big[\|\boldsymbol{u}^N(t) - \boldsymbol{u}_{0,N}\|_{V_N} + \|\boldsymbol{u}_t^N(t) - \boldsymbol{u}_{1,N}\|_{V_N} + \|v^N(t) - v_0^N\|_{1,2}\big] = 0.$$

Proof. Rewriting (5.13) in terms of interpolant functions, for a.e. $t \in (0,T)$ and every test function $\boldsymbol{w} \in V_N$, we see that

$$\int_\Omega \overline{\boldsymbol{u}}_t^{\beta,+,\prime} \cdot \boldsymbol{w} + b(v^{\beta,-}(t))F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}^{\beta,+,\prime} + \alpha \boldsymbol{u}^{\beta,+})(t)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x = \langle l^{\beta,+}(t), \boldsymbol{w}\rangle.$$

Let $\psi \in C([0,T])$ be fixed but arbitrary. By the strong convergence results of Lemma 5.12, we see that, for every $t \in (0,T)$,

$$\begin{aligned}&\lim_{M\to\infty}\left[\int_Q \psi\left[b(v^{\beta,-}(t))F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}^{\beta,+,\prime} + \alpha\boldsymbol{u}^{\beta,+})(t)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x - \langle l^{\beta,+}(t), \boldsymbol{w}\rangle\right]\,\mathrm{d}t\right]\\ &= \int_Q \psi\left[b(v^N(t))F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha\boldsymbol{u}^N)(t)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x - \langle l(t), \boldsymbol{w}\rangle\right]\,\mathrm{d}t,\end{aligned} \tag{5.25}$$

and by the weak convergence result, we have that

$$\lim_{M\to\infty}\int_Q \overline{\boldsymbol{u}}_t^{\beta,+,\prime} \cdot \boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{u}_{tt}^N \cdot \boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{5.26}$$

Putting together (5.25) and (5.26) with a standard density argument, we obtain (5.24), defining $\mathbf{T}^N$ on Q by $F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha\boldsymbol{u}^N))$. Satisfaction of the initial conditions follows from the strong convergence results for $\boldsymbol{u}^N$ and $\boldsymbol{u}_t^N$ in $C([0,T]; C^1(\overline{\Omega})^d)$, and for v^N in $C([0,T]; L^2(\Omega))$, combined with the fact that $\boldsymbol{u}_{0,N}$, $\boldsymbol{u}_{1,N}$ and v_0^N are initialisations for the time discrete problem. □

Proposition 5.14. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there. For every $t \in [0,T]$, the approximate phase-field function $v^N(t)$ solves the minimisation problem*

$$\mathcal{E}(\boldsymbol{u}^N(t), v^N(t)) + \mathcal{H}(v^N(t)) = \inf\left\{\mathcal{E}(\boldsymbol{u}^N(t), v) + \mathcal{H}(v) \;:\; v \in H^1_{D+1}(\Omega), v \le v^N(t)\right\}. \tag{5.27}$$

Proof. By the choice of the initial data $\boldsymbol{u}_{0,N}$ and v_0^N, the result holds trivially at time $t = 0$. For $t \in (0,T]$, it suffices to show that

$$0 \le \int_\Omega \frac{b'(v^N(t))}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N(t))) + \frac{1}{2\epsilon}(v^N(t) - 1)\chi + 2\epsilon\nabla v^N(t) \cdot \nabla\chi\,\mathrm{d}x, \tag{5.28}$$

for every $\chi \in H^1_D(\Omega)$ with $\chi \le 0$. By Proposition 5.5, for such a χ and an arbitrary $\psi \in C([0,T])$ with $\psi \ge 0$, we have that

$$0 \le \int_Q \frac{b'(v^{\beta,+})}{\alpha}\chi\psi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^{\beta,+})) + \frac{1}{2\epsilon}(v^{\beta,+} - 1)\chi\psi + 2\epsilon\nabla v^{\beta,+} \cdot \nabla(\chi\psi)\,\mathrm{d}x\,\mathrm{d}t.$$

Letting $N \to \infty$ in the above, we use the weak convergence results and Lebesgue's differentiation theorem to deduce that for a.e. $t \in (0,T]$, (5.28) holds. By the regularity of $\boldsymbol{u}^N$ and v^N in the time variable, the right-hand side of (5.28) is continuous in the time variable. Hence (5.28) holds for every $t \in [0,T]$, as required. □

Proposition 5.15. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there. For every $t \in [0,T]$, we have the energy-dissipation equality*

$$\begin{aligned}\mathcal{F}(0; \boldsymbol{u}_{0,N}, \boldsymbol{u}_{1,N}, v_0^N) &= \mathcal{F}(t; \boldsymbol{u}^N(t), \boldsymbol{u}_t^N(t), v^N(t)) + \int_0^t \langle l_t, \boldsymbol{u}^N\rangle\,\mathrm{d}s\\ &+ \int_0^t\int_\Omega b(v^N)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha\boldsymbol{u}^N)) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N))\right] \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.29}$$

Proof. Trivially the result holds at the initial time $t = 0$. Furthermore, each term is continuous in t so it is enough to show that equality holds at a.e. $t \in (0,T)$. Considering the energy-dissipation inequality from Proposition 5.7 and taking the limit as $M \to \infty$ there, we deduce that (5.29) holds at a.e. $t \in (0,T)$ with the equality sign replaced by "$\leq$". For the opposite direction, first notice that

$$\begin{aligned}
&\mathcal{K}(\boldsymbol{u}_t^N(t)) - \mathcal{K}(\boldsymbol{u}_{1,N}) + \left[\langle l(t), \boldsymbol{u}^N(t)\rangle - \langle l(0), \boldsymbol{u}^N(0)\rangle\right] + \int_0^t \langle l_t, \boldsymbol{u}^N\rangle \, \mathrm{d}s \\
&\quad + \int_0^t \int_\Omega b(v^N) F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^N) \, \mathrm{d}x \, \mathrm{d}s \\
&= \int_0^t \int_\Omega \boldsymbol{u}_{tt}^N \cdot \boldsymbol{u}_t^N + b(v^N) F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^N) \, \mathrm{d}x \, \mathrm{d}s - \int_0^t \langle l, \boldsymbol{u}_t^N\rangle \, \mathrm{d}s \\
&= 0,
\end{aligned}$$

taking $\boldsymbol{u}_t^N$ as a test function in (5.24). Subtracting this from (5.29), for (5.29) to hold it suffices to show that

$$\left[\mathcal{E}(\boldsymbol{u}^N(s), v^N(s)) + \mathcal{H}(v^N(s))\right]_{s=0}^{s=t} - \int_0^t \int_\Omega b(v^N) F^{-1}(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^N) \, \mathrm{d}x \, \mathrm{d}s \geq 0. \tag{5.30}$$

To prove (5.30), we mimic an argument from [73]. We fix $t \in (0,T]$. For each $K \in \mathbb{N}_{\geq 2}$, define a time step $h_K = K^{-1} t$ and consider the corresponding time-discrete approximate solution sequence given by $\boldsymbol{u}_k^{N,K} := \boldsymbol{u}^N(hk)$, $v_k^{N,K} := v^N(hk)$ for $k \in \{0, \ldots, K\}$. Then,

$$\mathcal{E}(\boldsymbol{u}^N(t), v^N(t)) - \mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N) = \sum_{k=1}^K \left[\mathcal{E}(\boldsymbol{u}_k^{N,K}, v_k^{N,K}) - \mathcal{E}(\boldsymbol{u}_{k-1}^{N,K}, v_{k-1}^{N,K})\right].$$

By the variational problem (5.27), we have that

$$\begin{aligned}
&\mathcal{E}(\boldsymbol{u}_k^{N,K}, v_k^{N,K}) - \mathcal{E}(\boldsymbol{u}_{k-1}^{N,K}, v_{k-1}^{N,K}) + \mathcal{H}(v_k^{N,K}) - \mathcal{H}(v_{k-1}^{N,K}) \\
&\geq \int_\Omega \frac{b(v_k^{N,K})}{\alpha} \left(\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_k^{N,K})) - \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_{k-1}^{N,K}))\right) \mathrm{d}x \\
&= \frac{t}{K} \int_\Omega \int_0^1 \frac{b(v_k^{N,K})}{\alpha} F^{-1}(\alpha \boldsymbol{\varepsilon}(s \boldsymbol{u}_k^{N,K} + (1-s) \boldsymbol{u}_{k-1}^{N,K})) \cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_k^{N,K}) \, \mathrm{d}s \, \mathrm{d}x.
\end{aligned}$$

Using the previous notation for interpolants of sequences, we deduce that

$$\begin{aligned}
&\mathcal{E}(\boldsymbol{u}^N(t), v^N(t)) - \mathcal{E}(\boldsymbol{u}_{0,N}, v_0^N) - \mathcal{H}(v_0^N) + \mathcal{H}(v^N(t)) \\
&\geq \int_0^t \int_\Omega \int_0^1 \frac{b(v^{N,K,+})}{\alpha} F^{-1}(\alpha \boldsymbol{\varepsilon}(s \boldsymbol{u}^{N,K,+} + (1-s) \boldsymbol{u}^{N,K,-})) \cdot \boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^{N,K}) \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned} \tag{5.31}$$

Since $\boldsymbol{u}^N \in W^{2,\infty}(0,T;V_N)$ and $v^N \in W^{1,\infty}(0,T;H^1(\Omega)) \cap L^\infty(Q)$, we have that

- $\boldsymbol{u}^{N,K,\pm} \to \boldsymbol{u}^N$ strongly in $C([0,T];V_N)$,
- $\overline{\boldsymbol{u}}^{N,K} \to \boldsymbol{u}^N$ strongly in $W^{1,\infty}(0,T;V_N)$,
- $v^{N,K,+} \to v^N$ strongly in $L^\infty(0,T;L^2(\Omega)^d)$.

Letting $K \to \infty$ in (5.31) and, using the dominated convergence theorem, it follows that (5.30) holds. □

5.3 The limit as $N \to \infty$

5.3.1 Uniform estimates

Now we focus on constructing estimates on the couple $(\boldsymbol{u}^N, v^N)$ that are uniform with respect to N. The couple $(\boldsymbol{u}^N, v^N)$ constructed in Lemma 5.12 is not necessarily a unique solution of the Galerkin problem but we note that throughout this section we work with the solution couple constructed from the time discrete problem.

Lemma 5.16. *Let the assumptions of Lemma 5.12 hold, let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there, and denote $\mathbf{T}^N := F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N))$. There exists a constant C, independent of N, such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^N(t)\|_2^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}^N(t)\|_{1,p'}^{p'} + \sup_{t\in[0,T]} \|\boldsymbol{u}_t^N(t)\|_2^2 + \int_0^T \|\mathbf{T}^N(t)\|_p^p + \|\boldsymbol{u}_t^N(t)\|_{1,p'}^{p'} \,\mathrm{d}t$$
$$\leq C\left[1 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^p(W^{-1,p})}^p\right].$$

Proof. For the first, third and fourth terms on the left-hand side, we test against $\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N$ in the elastodynamic equation (5.24) and argue as in Theorem 2.4, for example, recalling the positivity of b to obtain the bound on $\mathbf{T}^N$. Using the constitutive relation, we notice that we have the memory kernel property

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^N(t)) = \mathrm{e}^{-\alpha t}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0,N}) + \int_0^t \mathrm{e}^{\alpha(s-t)} F(\mathbf{T}^N(s)) \,\mathrm{d}s,$$

and so we deduce that

$$\sup_{t\in[0,T]} \|\boldsymbol{\varepsilon}(\boldsymbol{u}^N(t))\|_{p'}^{p'} \leq C\left[1 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_{p'}^{p'} + \int_Q |\mathbf{T}^N|^p \,\mathrm{d}x \,\mathrm{d}t\right].$$

Applying the Korn–Poincaré inequality (Theorem 1.9), we get the $L^\infty(0,T;W_D^{1,p'}(\Omega)^d)$ bound on $\boldsymbol{u}^N$. With this and the bound on $\mathbf{T}^N$, it follows that

$$\int_0^T \|\boldsymbol{u}_t^N(t)\|_{1,p'}^{p'} \,\mathrm{d}t \leq C\left[1 + \int_0^T \|\boldsymbol{\varepsilon}(\boldsymbol{u}^N(t))\|_{p'}^{p'} \,\mathrm{d}t + \int_Q |\mathbf{T}^N|^p \,\mathrm{d}x \,\mathrm{d}t\right].$$

Putting the above together, we conclude the desired result. □

Lemma 5.17. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there. There exists a constant C, independent of N, such that*

$$\int_Q |\boldsymbol{u}_{tt}^N|^2 \,\mathrm{d}x \,\mathrm{d}t + \sup_{t\in[0,T]} \left[\|\mathbf{T}^N(t)\|_p^p + \|\boldsymbol{u}_t^N(t)\|_{1,p'}^{p'}\right] \leq C\left[1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|\boldsymbol{u}_1\|_{1,p'}^{p'}\right.$$
$$\left. + \|l\|_{L^\infty(W^{-1,p})}^p + \int_0^T \|l_t\|_{-1,p}^p \,\mathrm{d}t\right].$$

Proof. It suffices to prove the claim for the first two terms. In particular, if $\mathbf{T}^N$ is uniformly bounded in $L^\infty(0,T;L^p(\Omega)^{d\times d})$, and using the $L^\infty(0,T;W^{1,p'}(\Omega)^d)$ bound on $\boldsymbol{u}^N$ from Lemma 5.16, then

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N) = -\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N) + F(\mathbf{T}^N) = -\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N) + |\mathbf{T}^N|^{p-2}\mathbf{T}^N.$$

Thus we have

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\|_{p'}^{p'} \le \alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u}^N)\|_{p'}^{p'} + \|\mathbf{T}^N\|_p^p.$$

. Apply the Korn–Poincaré inequality, it follows that

$$\|\boldsymbol{u}_t^N\|_{1,p'}^{p'} \le C \left[\|\boldsymbol{\varepsilon}(\boldsymbol{u}^N)\|_{p'}^{p'} + \|\mathbf{T}^N\|_p^p \right].$$

Taking the supremum with respect to the time variable, the required bound on $\boldsymbol{u}_t^N$ follows.

Now we focus on the remaining estimates. We test against $\boldsymbol{u}_{tt}^N + \alpha \boldsymbol{u}_t^N$ in the elastodynamic equation (5.24) to see that

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_{tt}^N|^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha}{2}\|\boldsymbol{u}_t^N\|_2^2\right) + \int_\Omega b(v^N)\mathbf{T}^N \cdot F(\mathbf{T}^N)_t \,\mathrm{d}x = \langle l, \boldsymbol{u}_{tt}^N + \alpha \boldsymbol{u}_t^N \rangle \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle l, \boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N\rangle\right) - \langle l_t, \boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N\rangle.
\end{aligned}$$

For the term involving the stress tensor, we write

$$\begin{aligned}
\int_\Omega b(v^N)\mathbf{T}^N \cdot F(\mathbf{T}^N)_t \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega b(v^N)\frac{1}{p'}|\mathbf{T}^N|^p \,\mathrm{d}x\right) - \int_\Omega b(v^N) v_t^N \frac{1}{p'}|\mathbf{T}^N|^p \,\mathrm{d}x \\
&\ge \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega b(v^N)\frac{1}{p'}|\mathbf{T}^N|^p \,\mathrm{d}x\right),
\end{aligned}$$

where the inequality follows from the non-positivity of v_t^N. We integrate over $(0,t)$ for an arbitrary $t \in (0,T]$ to deduce that

$$\begin{aligned}
&\int_0^t \int_\Omega |\boldsymbol{u}_{tt}^N|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{\alpha}{2}\|\boldsymbol{u}_t^N(t)\|_2^2 + \int_\Omega b(v^N(t))\frac{1}{p'}|\mathbf{T}^N(t)|^p \,\mathrm{d}x \\
&\le C\Bigg[1 + \|\boldsymbol{u}_1\|_2^2 + \int_\Omega |F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{1,N} + \alpha \boldsymbol{u}_{0,N}))|^p \,\mathrm{d}x + \|l(t)\|_{-1,p}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N)(t)\|_{p'} \\
&\quad + \|l(0)\|_{-1,p}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_{1,N} + \alpha \boldsymbol{u}_{0,N})\|_{p'} + \int_0^t \|l_t\|_{-1,p}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N)(t)\|_{p'} \,\mathrm{d}s\Bigg] \\
&\le C\Bigg[1 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|\boldsymbol{u}_1\|_{1,p'}^{p'} + \|l\|_{L^\infty(W^{-1,p})}^p \\
&\quad + \int_0^t \|l_t\|_{-1,p}^p + \|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N)\|_{p'}^{p'} \,\mathrm{d}s\Bigg] + \frac{1}{2}\int_\Omega b(v^N(t))\frac{1}{p'}|\mathbf{T}^N(t)|^p \,\mathrm{d}x.
\end{aligned}$$

We absorb the final term on the right-hand side into the left and apply Gronwall's inequality to see that the required estimate is satisfied. □

Lemma 5.18. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N, v^N)$ be the limiting couple constructed there. There exists a constant C, independent of N, such that*

$$\sup_{t\in[0,T]} \|v^N\|_{1,2}^2 \le C\Big[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,p'}^{p'} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|l\|_{L^\infty(W^{-1,p})}^p + \int_0^T \|l_t\|_{-1,p}^p\Big].$$

Proof. From the energy-dissipation equality (5.29), we have that

$$\begin{aligned}
&\mathcal{H}(v^N(t)) \leq \mathcal{F}(t;\boldsymbol{u}^N(t),\boldsymbol{u}_t^N(t),v^N(t)) + \langle l(t),\boldsymbol{u}^N(t)\rangle \\
&\quad + \int_0^t\int_\Omega b(v^N)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N+\alpha\boldsymbol{u}^N)) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\,\mathrm{d}x\,\mathrm{d}s \\
&= \langle l(t),\boldsymbol{u}^N(t)\rangle + \mathcal{F}(0;\boldsymbol{u}_{0,N},\boldsymbol{u}_{1,N},v_0^N) - \int_0^t\langle l_t,\boldsymbol{u}^N\rangle\,\mathrm{d}s \\
&\leq C\Big[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|l\|^p_{L^\infty(W^{-1,p})}+\int_0^T\|l_t(s)\|^p_{-1,p}\,\mathrm{d}s\Big],
\end{aligned}$$

using Lemma 5.16 for the final line. By the definition of $\mathcal{H}$, we maximise the left-hand side with respect to $t\in[0,T]$ to yield the required result. □

Lemma 5.19. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N,v^N)$ be the limiting couple constructed there. For a.e. $t\in(0,T)$, we have that*

$$\left[\partial_v\mathcal{E}(\boldsymbol{u}^N(t),v^N(t)) + \mathcal{H}'(v^N(t))\right](v_t^N(t)) = 0. \tag{5.32}$$

Proof. We differentiate the energy-dissipation equality (5.29) with respect to the time variable to obtain the following:

$$\begin{aligned}
0 &= \int_\Omega \boldsymbol{u}_{tt}^N\cdot\boldsymbol{u}_t^N\,\mathrm{d}x + \int_\Omega\frac{1}{2\epsilon}(v^N-1)v_t^N + 2\epsilon\nabla v^N\cdot\nabla v_t^N\,\mathrm{d}x \\
&\quad + \int_\Omega\frac{b'(v^N)}{\alpha}v_t^N\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N)) + b(v^N)F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\,\mathrm{d}x - \langle l_t,\boldsymbol{u}^N\rangle - \langle l,\boldsymbol{u}_t^N\rangle \\
&\quad + \int_\Omega b(v^N)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N+\alpha\boldsymbol{u}^N)) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\,\mathrm{d}x + \langle l_t,\boldsymbol{u}^N\rangle \\
&= \left[\int_\Omega\boldsymbol{u}_{tt}^N\cdot\boldsymbol{u}_t^N + b(v^N)F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N+\alpha\boldsymbol{u}^N))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\,\mathrm{d}x - \langle l,\boldsymbol{u}_t^N\rangle\right] \\
&\quad + \left[\int_\Omega\frac{b'(v^N)}{\alpha}v_t^N\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N)) + \frac{1}{2\epsilon}(v^N-1)v_t^N + 2\epsilon\nabla v^N\cdot\nabla v_t^N\,\mathrm{d}x\right].
\end{aligned}$$

The sum of terms in the first set of square brackets vanishes because it is the elastodynamic equation (5.24) with test function $\boldsymbol{u}_t^N$. Hence the sum of terms in the second set of brackets must also vanish, but this is exactly the left-hand side of (5.32) and so the required result follows. □

Lemma 5.20. *Let the assumptions of Lemma 5.12 hold and let $(\boldsymbol{u}^N,v^N)$ be the limiting couple constructed there. Suppose additionally that $p\in(1,2]$. There exists a constant C, independent of N, such that*

$$\sup_{t\in[0,T]}\|v_t^N\|_{1,2}^2 \leq C\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|\boldsymbol{u}_0\|_{1,p'}^{p'}+\|\boldsymbol{u}_1\|_{1,p'}^{p'}+\|l\|^p_{L^\infty(W^{-1,p})}+\int_0^T\|l_t\|^p_{-1,p}\right].$$

Proof. Fix $t\in(0,T]$ such that (5.32) holds and let $h>0$ be sufficiently small such that $t-h>0$. We know that

$$0 \leq \left[\partial_v\mathcal{E}(\boldsymbol{u}^N(t-h),v^N(t-h)) + \mathcal{H}'(v^N(t-h))\right](v_t^N(t)), \tag{5.33}$$

because $v_t^N(t)$ is a suitable test function in (5.28). Subtracting (5.32) at time t from (5.33) yields

$$0 \le \int_\Omega \left[\frac{b'(v^N(t-h))}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t-h))) - \frac{b'(v^N(t))}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t)))\right] v_t^N(t)\,\mathrm{d}x$$
$$+ \frac{1}{2\epsilon}\int_\Omega \left[v^N(t-h) - v^N(t)\right] v_t^N(t)\,\mathrm{d}x + 2\epsilon \int_\Omega \nabla\left(v^N(t-h) - v^N(t)\right)\cdot \nabla v_t^N(t)\,\mathrm{d}x.$$

We use the fundamental theorem of calculus and divide the resulting inequality through by h to see that

$$\frac{1}{2\epsilon}\int_\Omega \frac{v^N(t) - v^N(t-h)}{h}\cdot v_t^N(t) + 2\epsilon \int_\Omega \nabla\left(\frac{v^N(t) - v^N(t-h)}{h}\right)\cdot \nabla v_t^N(t)$$
$$+ \frac{2}{\alpha}\int_\Omega \frac{v^N(t) - v^N(t-h)}{h}\cdot v_t^N(t)\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t-h)))$$
$$\le \int_\Omega\int_0^1 \frac{b'(v^N(t))}{\alpha} v_t^N(t) F^{-1}(\alpha \boldsymbol{\varepsilon}(s\boldsymbol{u}^N(t-h) + (1-s)\boldsymbol{u}^N(t)))\cdot \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}^N(t-h) - \boldsymbol{u}^N(t)}{h}\right).$$

Because $\boldsymbol{u}^N \in C^1([0,T];V_N)$ and $v^N \in W^{1,2}(0,T;H^1(\Omega))$, we can take the limit as $h \to 0+$ to deduce that, for a.e. $t \in (0,T)$,

$$\frac{1}{2\epsilon}\|v_t^N(t)\|_2^2 + 2\epsilon\|\nabla v_t^N(t)\|_2^2 + \frac{2}{\alpha}\int_\Omega |v_t^N(t)|^2\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t)))\,\mathrm{d}x$$
$$\le \int_\Omega \frac{b'(1)}{\alpha}|v_t^N(t)||F^{-1}(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t)))||\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N(t))|\,\mathrm{d}x$$
$$= \int_\Omega \frac{b'(1)}{\alpha}\left(|v_t^N(t)||\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t))|^{\frac{p'}{2}}\right)\cdot\left(|\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t))|^{\frac{p'-2}{2}}|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N(t))|\right)\,\mathrm{d}x.$$

We apply Hölder's inequality, followed by Young's inequality and absorb the term involving v_t^N into the left-hand side to deduce that

$$\frac{1}{2\epsilon}\|v^N(t)\|_2^2 + 2\epsilon\|\nabla v_t^N(t)\|_2^2 + \frac{1}{\alpha}\int_\Omega |v_t^N(t)|^2\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N(t)))\,\mathrm{d}x$$
$$\le C\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u}^N(t))|^{p'-2}|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N(t))|^2\,\mathrm{d}x.$$

If $p = 2$, the first factor on the right-hand side is 1 so we can apply Lemma 5.17. If $p \in (1,2)$, we apply Hölder's inequality with parameters $q = \frac{p'}{p'-2}$, $q' = \frac{p'}{2}$, valid because $p' > 2$. It follows that

$$\frac{1}{2\epsilon}\|v^N\|_2^2 + 2\epsilon\|\nabla v_t^N\|_2^2 + \frac{1}{\alpha}\int_\Omega |v_t^N|^2\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^N))\,\mathrm{d}x \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}^N)\|_{p'}^{p'-2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N)\|_{p'}^2.$$

Applying Lemma 5.17, we obtain the desired bound. □

Theorem 5.21. *Suppose that* $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$ *and* $v_0 \in H^1_{D+1}(\Omega)$ *such that* $v_0 \in [0,1]$ *a.e. and the compatibility condition* (5.7) *holds. Let* $\boldsymbol{f} \in C^1([0,T];W_D^{-1,p}(\Omega)^d)$ *and* $\boldsymbol{g} \in C^1([0,T];W_D^{-1,p}(\Omega)^d)$, *and define* l *as in* (5.4). *There exists a triple* $(\boldsymbol{u}, \mathbf{T}, v)$ *that is a weak energy solution of* (5.1)–(5.3) *in the sense of Theorem 5.1.*

Furthermore, for the sequence $(\boldsymbol{u}^N, v^N)_N$ *of approximate solutions constructed in Lemma 5.12 with stress tensor* $\mathbf{T}^N = F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha \boldsymbol{u}^N))$, *there exists a subsequence in* N, *not relabelled, such that the following convergence results hold:*

- $\boldsymbol{u}^N \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0,T;L^2(\Omega)^d)$ *and weakly-* in* $W^{1,\infty}(0,T;W_D^{1,p'}(\Omega)^d)$*;*
- $\mathbf{T}^N \overset{*}{\rightharpoonup} \mathbf{T}$ *weakly-* in* $L^\infty(0,T;L^p(\Omega)^{d\times d})$*;*
- $v^N \overset{*}{\rightharpoonup} v$ *weakly-* in* $W^{1,\infty}(0,T;H^1(\Omega))$*.*

Proof. The convergence results follow easily from the uniform bounds in Lemma 5.16, Lemma 5.17, Lemma 5.18 and Lemma 5.20. By the Aubin–Lions lemma, $\boldsymbol{u}^N \to \boldsymbol{u}$ strongly in $C^1([0,T];L^{p'}(\Omega)^d)$, and $v^N \to v$ strongly in $C([0,T];L^2(\Omega))$. The limiting phase-field function v clearly satisfies $v \in [0,1]$ and $v_t \leq 0$ a.e. in Q with $v(t) \in H^1_{D+1}(\Omega)$ for every $t \in [0,T]$. From (5.24), for every $\boldsymbol{w} \in C^\infty_D(\overline{\Omega})^d$ and $\psi \in C([0,T])$, we have that

$$\int_Q \boldsymbol{u}^N_{tt} \cdot (\psi P_N \boldsymbol{w}) + b(v^N)\mathbf{T}^N \cdot \boldsymbol{\varepsilon}(\psi P_N \boldsymbol{w}) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle l, \psi P_N \boldsymbol{w}\rangle \,\mathrm{d}t. \tag{5.34}$$

Using the weak convergence results and the fact that $P_N\boldsymbol{w} \to \boldsymbol{w}$ strongly in $C^1(\overline{\Omega})^d$, we let $N \to \infty$ in (5.34) to obtain

$$\int_Q \boldsymbol{u}_{tt} \cdot (\psi \boldsymbol{w}) + b(v)\mathbf{T} \cdot \boldsymbol{\varepsilon}(\psi \boldsymbol{w}) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle l, \psi \boldsymbol{w}\rangle \,\mathrm{d}t.$$

Using Lebesgue's differentation theorem and a standard density argument, it follows that the elastodynamic equation is satisfied at a.e. $t \in (0,T)$, for every test function $\boldsymbol{w} \in W_D^{1,p'}(\Omega)^d$. The satisfaction of the initial data follows from the usual arguments and the approximation properties of the sequences $(\boldsymbol{u}_{0,N})_N$, $(\boldsymbol{u}_{1,N})_N$ and $(v_0^N)_N$ (cf. Lemma 5.3).

To see that the constitutive relationship holds in the limit, we use a variant of Minty's method as in the previous chapters so do not include the details. We also see that

$$\lim_{N\to\infty} \int_Q b(v^N)|\mathbf{T}^N|^p \,\mathrm{d}x\,\mathrm{d}t = \int_Q b(v)|\mathbf{T}|^p \,\mathrm{d}x\,\mathrm{d}t. \tag{5.35}$$

However, $(b(v^N)^{\frac{1}{p}}\mathbf{T}^N)_N$ converges weakly in $L^p(Q)^{d\times d}$ to $b(v)^{\frac{1}{p}}\mathbf{T}$. It follows that $\mathbf{T}^N \to \mathbf{T}$ strongly in $L^p(Q)^{d\times d}$ and so $(\boldsymbol{\varepsilon}(\boldsymbol{u}^N_t + \alpha\boldsymbol{u}^N))_N$ converges strongly in $L^{p'}(Q)^{d\times d}$ to $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$. Reasoning as before and using the memory kernel property, we have that $\boldsymbol{\varepsilon}(\boldsymbol{u}^N) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ strongly in $L^\infty(0,T;L^{p'}(\Omega)^{d\times d})$. In particular, $\boldsymbol{\varepsilon}(\boldsymbol{u}^N) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ pointwise a.e. in Q as $N \to \infty$. We also have that $\boldsymbol{\varepsilon}(\boldsymbol{u}^N_t) \to \boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ strongly in $L^{p'}(Q)^{d\times d}$ and pointwise a.e. on Q. We use Fatou's lemma with weak lower semi-continuity to take $N \to \infty$ in the energy-dissipation equality (5.29) and obtain

$$\begin{aligned}
&\mathcal{F}(t;\boldsymbol{u}(t),\boldsymbol{u}_t(t),v(t)) + \int_0^t \int_\Omega b(v)\left[\mathbf{T} - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\right] \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle l_t(s), \boldsymbol{u}(s)\rangle \,\mathrm{d}s \\
&\leq \liminf_{N\to\infty} \Bigg[\mathcal{F}(t;\boldsymbol{u}^N(t),\boldsymbol{u}^N_t(t),v^N(t)) + \int_0^t \int_\Omega b(v^N)\left[\mathbf{T}^N - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N))\right] \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}^N_t) \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \langle l_t(s), \boldsymbol{u}^N(s)\rangle \,\mathrm{d}s\Bigg] = \liminf_{N\to\infty} \mathcal{F}(0;\boldsymbol{u}_{0,N},\boldsymbol{u}_{1,N},v_0^N) = \mathcal{F}(0;\boldsymbol{u}_0,\boldsymbol{u}_1,v_0),
\end{aligned} \tag{5.36}$$

for a.e. $t \in (0,T)$. For the opposite inequality, we argue as in the proof of Proposition 5.15, provided that the limiting triple satisfies the minimisation problem (5.2). Thus it remains to show that

$$0 \leq \left[\partial_v \mathcal{E}(\boldsymbol{u}(t), v(t)) + \mathcal{H}'(v(t))\right](\chi),$$

for every $\chi \in H^1_D(\Omega)$ such that $\chi \leq 0$. Using Proposition 5.14, we recall that

$$0 \leq \lim_{N\to\infty} \int_Q \psi \left[\frac{b'(v^N)}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N)) + \frac{1}{2\epsilon}(v^N - 1)\chi + 2\epsilon\nabla v^N \cdot \nabla\chi\right] \mathrm{d}x\,\mathrm{d}t, \tag{5.37}$$

for every $\psi \in C([0,T])$ such that $\psi \geq 0$, with χ as above. By the strong convergence of $(\boldsymbol{\varepsilon}(\boldsymbol{u}^N))_N$ in $L^{p'}(Q)^{d\times d}$ and the definition of φ^*, we see that $\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N)) \to \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))$ strongly in $L^1(Q)$. Using the dominated convergence theorem, it follows that $(b'(v^N)\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N)))_N$ converges strongly in $L^1(Q)$ to $b'(v)\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))$. With this and the weak convergence of $(v^N)_N$ in $L^\infty(0,T;H^1(\Omega))$, it follows that

$$0 = \int_Q \psi \left[\frac{b'(v)}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) + \frac{1}{2\epsilon}(v - 1)\chi + 2\epsilon\nabla v \cdot \nabla\chi\right] \mathrm{d}x\,\mathrm{d}t.$$

Hence, the limiting triple satisfies both the energy-dissipation equality and the minimisation problem. We conclude that it is a solution of the dynamic fracture problem (5.1)–(5.3) in the required sense. □

Next, we consider when $p \in (2,\infty)$. The proof of Lemma 5.20 cannot be adapted to large values of p. Instead, we use the $L^\infty(0,T;H^1(\Omega))$ bound on $(v^N)_N$ and the monotonicity of $(v^N)_N$ as functions on $[0,T]$. Indeed $v^N(t) \leq v^N(s)$ in Ω, for every $0 \leq s \leq t \leq T$. Thus we can apply the following theorem from [26]. We combine this (Theorem 5.22) with the Rellich–Kondrachov compactness theorem to obtain a strong convergence result in $H^1(\Omega)$ (Corollary 5.23). For the limiting function constructed in this result, it satisfies a certain continuity property which is needed to prove one direction of the energy-dissipation equality.

Theorem 5.22. *Let $v^k : [a,b] \to L^2(\Omega)$ be a sequence of functions on a bounded interval $[a,b] \subset \mathbb{R}$ such that $v^k(t) \leq v^k(s)$ in Ω for every $a \leq s \leq t \leq b$ and $k \in \mathbb{N}$. Suppose that there exists a $C > 0$ such that $\|v^k(t)\|_{L^2(\Omega)} \leq C$ for every $k \in \mathbb{N}$ and $t \in [a,b]$.*

There exists a subsequence in k, not relabelled, and a function $v : [a,b] \to L^2(\Omega)$ such that, for every $t \in [a,b]$, we have that $v^k(t) \rightharpoonup v(t)$ weakly in $L^2(\Omega)$ as $k \to \infty$ where $\|v(t)\|_{L^2(\Omega)} \leq C$ for every $t \in [a,b]$ and $v(t) \leq v(s)$ for every $a \leq s \leq t \leq b$.

Corollary 5.23. *Let $(v^k)_{k\geq 1}$ be a sequence of functions on $[a,b]$ taking values in $H^1(\Omega)$ such that we have $v^k(t) \leq v^k(s)$ in Ω for every $a \leq s \leq t \leq b$ and there exists a $C > 0$ such that $\|v^k(t)\|_{H^1(\Omega)} \leq C$ for every $k \in \mathbb{N}$ and $t \in [a,b]$.*

There exists a subsequence in k, not relabelled, and a function $v : [a,b] \to H^1(\Omega)$ such that, for every $t \in [a,b]$, the subsequence converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to $v(t)$. Furthermore, $\|v(t)\|_{H^1(\Omega)} \leq C$ for every $t \in [a,b]$ and $v(t) \leq v(s)$ for every $a \leq s \leq t \leq b$.

Proposition 5.24. *Let the assumptions of Corollary 5.23 hold with corresponding sequence* $(v^k)_{k\geq 1}$ *and limiting function* v*. Then* v *is almost everywhere continuous on* $[a,b]$ *with respect to strong convergence in* $L^2(\Omega)$ *and weak convergence in* $H^1(\Omega)$*.*

This is proven using monotonicity and the fact that such a function can only have countably many discontinuities. We are now ready to prove the existence of a weak energy solution to the fracture problem (5.1)–(5.3) in the sense of Theorem 5.2.

Theorem 5.25. *Suppose that* $\boldsymbol{u}_0$*,* $\boldsymbol{u}_1 \in W_D^{1,p'}(\Omega)^d \cap L^2(\Omega)^d$ *and* $v_0 \in H^1_{D+1}(\Omega)$ *such that* $v_0 \in [0,1]$ *a.e. and the compatibility condition* (5.7) *holds. Let* $\boldsymbol{f} \in C^1([0,T]; W_D^{-1,p}(\Omega)^d)$ *and* $\boldsymbol{g} \in C^1([0,T]; L^{p'}(\partial\Omega_N)^d)$*, with* l *as in* (5.4)*. There exists a triple* $(\boldsymbol{u}, \mathbf{T}, v)$ *that is a weak energy solution of* (5.1)–(5.3) *in the sense of Theorem 5.2.*

Furthermore, for the sequence $((\boldsymbol{u}^N, v^N))_N$ *constructed in Lemma 5.12 with stress tensor* $\mathbf{T}^N = F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N + \alpha\boldsymbol{u}^N))$*, there exists a subsequence in* N*, not relabelled, such that the following convergence results hold:*

- $\boldsymbol{u}^N \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0,T; L^2(\Omega)^d)$ *and weakly-* in* $W^{1,\infty}(0,T; W^{1,p'}(\Omega)^d)$*;*
- $\mathbf{T}^N \overset{*}{\rightharpoonup} \mathbf{T}$ *weakly-* in* $L^\infty(0,T; L^p(\Omega)^{d\times d})$*;*
- $v^N(t) \to v(t)$ *strongly in* $L^2(\Omega)$ *and weakly in* $H^1(\Omega)$ *for every* $t \in [0,T]$*.*

Proof. The convergence results follow immediately from the uniform bounds in Lemma 5.16, Lemma 5.17 and Lemma 5.18, with Corollary 5.23. By the Aubin–Lions lemma, we have that $\boldsymbol{u}^N \to \boldsymbol{u}$ strongly in $C^1([0,T]; L^{p'}(\Omega)^d)$. Furthermore, the limiting function v satisfies $v \in [0,1]$ a.e. in Q with $v(t) \leq v(s)$ for every $0 \leq s \leq t \leq T$ and $v(t) \in H^1_{D+1}(\Omega)$. Furthermore, $v(0) = v_0$ since $v^N(0) = v_0^N$ and $(v_0^N)_N$ converges strongly in $H^1(\Omega)$ to v_0 by Lemma 5.3. Using the pointwise convergence of $(v^N)_N$, we argue as in the proof of Theorem 5.22 to see that the required elastodynamic equation is satisfied by the limiting triple. Similarly, we can use the same argument to deduce that $F(\mathbf{T}) = \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u})$ pointwise a.e. in Q. As before, we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{u}^N) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ in $L^\infty(0,T; L^{p'}(\Omega)^{d\times d})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^N) \to \boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ in $L^{p'}(Q)^{d\times d}$. As a result, we can take the limit in the energy-dissipation equality (5.29) to obtain (5.36).

For the minimisation problem, we recall that, for every $\chi \in H^1_D(\Omega)$ such that $\chi \leq 0$,

$$0 \leq \int_\Omega \frac{b'(v^N(t))}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^N(t))) + \frac{1}{2\epsilon}(v^N(t)-1)\chi + 2\epsilon\nabla v^N(t)\cdot\nabla\chi \,\mathrm{d}x.$$

Multiplying by an arbitrary $\psi \in C([0,T])$ such that $\psi \geq 0$ and integrating over $(0,T)$, we take the limit in the resulting inequality and apply a density argument in the time variable to deduce that, for a.e. $t \in (0,T)$, for every $\chi \in H^1_D(\Omega)$ such that $\chi \leq 0$,

$$0 \leq \int_\Omega \frac{b'(v(t))}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}(t))) + \frac{1}{2\epsilon}(v(t)-1)\chi + 2\epsilon\nabla v(t)\cdot\nabla\chi \,\mathrm{d}x.$$

This is exactly equivalent to having that, for a.e. $t \in (0,T)$,

$$\mathcal{E}(\boldsymbol{u}(t), v(t)) + \mathcal{H}(v(t)) = \inf\left\{\mathcal{E}(\boldsymbol{u}(t), v) + \mathcal{H}(v) \,:\, v \le v(t),\, v \in H^1_{D+1}(\Omega)\right\}.$$

Hence, the required minimisation problem (5.2) holds at a.e. $t \in (0,T)$. Due to a lack of continuity in the time variable of v, we cannot improve this to equality holding at every point in the time interval $[0,T]$.

To show that the opposite inequality to (5.36) holds (and thus prove satisfaction of the energy-dissipation equality), we reason as in Proposition 5.15 and use the continuity result from Proposition 5.24. Let $\mathcal{N} \subset [0,T]$ be a null set such that v is continuous on $[0,T] \setminus \mathcal{N}$ and the minimisation problem (5.2) is satisfied at every point in $[0,T] \setminus \mathcal{N}$. Fix $t \in (0,T] \setminus \mathcal{N}$ and $K \in \mathbb{N}_{\ge 2}$. We define a time step $h_K = \frac{t}{K}$. For each $1 \le k \le K-1$, pick $t_k^K \in (hk - \frac{h_K}{4}, hk + \frac{h_K}{4}) \cap ((0,T) \setminus \mathcal{N})$ and let $t_K^K = t$, $t_0^K = 0$. Then v is continuous at each time point t_k^K for $1 \le k \le K$. We define discrete sequences

$$\boldsymbol{u}_k^K = \boldsymbol{u}(t_k^K), \quad \mathbf{T}_k^K = \mathbf{T}(t_k^K), \quad v_k^K = v(t_k^K) \quad \text{for } 0 \le k \le K,$$

and write

$$\mathcal{E}(\boldsymbol{u}(t), v(t)) - \mathcal{E}(\boldsymbol{u}_0, v_0) = \sum_{k=1}^{K} \left[\mathcal{E}(\boldsymbol{u}_k^K, v_k^K) - \mathcal{E}(\boldsymbol{u}_{k-1}^K, v_{k-1}^K)\right]. \tag{5.38}$$

For the terms in the square brackets in the sum on the right-hand side, since the minimisation problem is satisfied at the given time points, we see that

$$\begin{aligned}
&\mathcal{E}(\boldsymbol{u}_k^K, v_k^K) - \mathcal{E}(\boldsymbol{u}_{k-1}^K, v_{k-1}^K) \\
&= \mathcal{E}(\boldsymbol{u}_k^K, v_k^K) - \mathcal{E}(\boldsymbol{u}_{k-1}^K, v_k^K) + \mathcal{E}(\boldsymbol{u}_{k-1}^K, v_k^K) - \mathcal{E}(\boldsymbol{u}_{k-1}^K, v_{k-1}^K) \\
&\ge \int_\Omega \frac{b(v_k^K)}{\alpha}\left(\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_k^K)) - \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_{k-1}^K))\right) \mathrm{d}x + \mathcal{H}(v_{k-1}^K) - \mathcal{H}(v_k^K) \\
&= h_K \int_\Omega \int_0^1 \frac{b(v_k^K)}{\alpha} F^{-1}(\alpha \boldsymbol{\varepsilon}(s\boldsymbol{u}_k^K + (1-s)\boldsymbol{u}_{k-1}^K)) \cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_k^K)\,\mathrm{d}s\,\mathrm{d}x + \mathcal{H}(v_{k-1}^K) - \mathcal{H}(v_k^K).
\end{aligned}$$

We substitute this into (5.38) and use the previous notation for interpolants of sequences to obtain

$$\begin{aligned}
\mathcal{E}(\boldsymbol{u}(t), v(t)) - \mathcal{E}(\boldsymbol{u}_0, v_0) &\ge \mathcal{H}(v_0) - \mathcal{H}(v(t)) \\
&\quad + \int_0^t \int_\Omega \int_0^1 \frac{b(v^{K,+})}{\alpha} F^{-1}(\alpha \boldsymbol{\varepsilon}(s\boldsymbol{u}^{K,+} + (1-s)\boldsymbol{u}^{K,-})) \cdot \boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

In the limit as $K \to \infty$, we have $b(v_+^K) \to b(v)$ pointwise a.e. on $(0,t) \times \Omega$ and

$$\int_0^1 F^{-1}(\alpha \boldsymbol{\varepsilon}(s\boldsymbol{u}^{K,+} + (1-s)\boldsymbol{u}^{K,-}))\,\mathrm{d}s \to F^{-1}(\alpha \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{strongly in } L^p((0,t) \times \Omega)^{d \times d}.$$

By the uniform bound on v, it follows that $b(v_+^K) \int_0^1 F^{-1}(\alpha \boldsymbol{\varepsilon}(s\boldsymbol{u}^{K,+} + (1-s)\boldsymbol{u}^{K,-}))\,\mathrm{d}s$ converges strongly to $b(v)F^{-1}(\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}))$ in $L^p((0,t) \times \Omega)^{d \times d}$. Suppose that $\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K) \rightharpoonup \boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ weakly in

$L^{p'}((0,t)\times\Omega)^{d\times d}$. We conclude that

$$\lim_{K\to\infty}\int_0^t\int_\Omega\int_0^1 \frac{b(v^{K,+})}{\alpha}F^{-1}(\alpha\boldsymbol{\varepsilon}(s\boldsymbol{u}^{K,+}+(1-s)\boldsymbol{u}^{K,-}))\cdot\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau$$
$$=\int_0^t\int_\Omega \frac{b(v)}{\alpha}F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\,\mathrm{d}x\,\mathrm{d}\tau,$$

and, in particular, we get that

$$\mathcal{E}(\boldsymbol{u}(t),v(t))+\mathcal{H}(v(t))\geq\mathcal{E}(\boldsymbol{u}_0,v_0)+\mathcal{H}(v_0)+\int_0^t\int_\Omega\frac{b(v)}{\alpha}F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Reasoning as in Proposition 5.15, it follows from this that the energy-dissipation equality is satisfied. It remains to show that $\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K)\rightharpoonup\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ weakly in $L^{p'}(Q)^{d\times d}$. In fact, we prove the stronger result of weak-* convergence in in $L^\infty(0,t;L^{p'}(\Omega)^{d\times d})$. By definition, we have that

$$\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_k^K)=\frac{1}{h_K}\boldsymbol{\varepsilon}(\boldsymbol{u}_k^K-\boldsymbol{u}_{k-1}^K)=\frac{t_k^K-t_{k-1}^K}{h_K}\fint_{t_{k-1}^K}^{t_k^K}\boldsymbol{\varepsilon}(\boldsymbol{u}_t(s))\,\mathrm{d}s,$$

where $\fint$ denotes the average integral operator. The difference $\frac{t_k^K-t_{k-1}^K}{h_K}$ is uniformly bounded with respect to K and k by construction of the time points t_k^K. Hence, we deduce that $\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_k^K)\|_{p'}\leq C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|_{L^\infty(L^{p'})}$ which implies that

$$\|\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K)\|_{L^\infty(0,t;L^{p'}(\Omega))}\leq C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\|_{L^\infty(L^{p'})}.$$

It follows that $(\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K))_{K\geq1}$ converges weakly-* in $L^\infty(0,t;L^{p'}(\Omega)^{d\times d})$. Since $\overline{\boldsymbol{u}}^K\to\boldsymbol{u}$ strongly in $L^{p'}((0,t)\times\Omega)^d$, it follows that the weak-* limit of $(\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}_t^K))_{K\geq1}$ is $\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ as required. □

This concludes our study of dynamic fracture problems in the case that the strain is not *a priori* bounded. We have developed a full existence theory that can be extended to a more general class of constitutive relations under certain growth and regularity conditions. In Chapter 6, we combine the techniques developed here with those of the strain-limiting problem from Chapter 4 in order to investigate strain-limiting dynamic fracture problems.

Chapter 6

Nonlinear fracture problem with strain-limiting constitutive relation

In this chapter, we combine all of the work of the previous chapters to study a strain-limiting dynamic fracture problem with mixed Dirichlet–Neumann boundary conditions. This is the key contribution of this thesis. The work is submitted and contained in [84]. If we let $p \to 1+$ in the constitutive relation from Chapter 5, we obtain

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = \frac{\mathbf{T}}{|\mathbf{T}|} =: F_1(\mathbf{T}).$$

However, the function F_1 is clearly not injective, nor is it continuous at the origin. Instead, we focus on a 'smoothed' version of this function. Namely, we consider F as in Chapters 2, 3 and 4, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} =: F(\mathbf{T}),$$

where $a > 0$ is a fixed problem parameter. We prove an existence result for a dynamic fracture problem with a phase-field approximation that satisfies a strain-limiting property defined by the above.

As in previous chapters, the main issue that needs to be overcome in the analysis is that the stress tensor $\mathbf{T}$ and its approximations are bounded in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ at best, which has very poor compactness properties. We are only able to deduce from this a weak-* convergence result in the space of Radon measures on $\overline{\Omega}$. However, for the constitutive relation to make sense pointwise, the stress tensor must be an element of some Lebesgue function space, or at least be a measurable function. Using higher regularity results, we improve the known information to deduce that the approximations in fact converge strongly in the space $L^1(0,T;L^1_{loc}(\Omega)^{d\times d})$, irrespective of the value of a. This leads to some issues near the boundary due to the lack of global convergence but it is possible with this convergence to show satisfaction of the constitutive relation pointwise on Q. In the case that the boundary conditions are fully Dirichlet, as in the problem without fracture from Chapter 3, a full existence result analogous to that of Chapter 5 is possible.

To prove the higher regularity estimates, the presence of the phase-field function provides an additional challenge. If we apply similar reasoning to that of Chapter 4, spatial derivatives of v will appear alongside the stress tensor. We require high regularity of the phase-field approximations due to the low integrability of the stress, namely, the gradient ∇v must be uniformly bounded on the space-time domain. Considering the construction of the problem, we would expect the gradient to have absolute value of order ϵ^{-1} because the parameter ϵ gives a sense of the 'thickness' of the approximation of the crack set. However, proving such a bound in practice is not possible at present. Hence, in the spirit of [26], we introduce a so-called rate-dependent term in the minimisation problem. By rate-dependent, we mean that we have a functional that has v_t as an argument. The presence of this term imposes a dependence on the speed at which the crack may grow. The physical implication of including such a term is discussed in detail in [26], particularly in the case $k = 0$. However, the presence of the term is necessary for the analysis of the strain-limiting problem.

If $(\mathbf{T}^n)_n$ is a sequence of approximations to the stress tensor $\mathbf{T}$, we prove that the sequence converges strongly in $L^1(0,T;L^1_{loc}(\Omega)^{d\times d})$ to $\mathbf{T}$. However, from the uniform bound in $L^\infty(0,T;L^1(\Omega)^{d\times d})$, we also see that the sequence converges weakly-* in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ to a limit $\overline{\mathbf{T}}$, say. As in Chapter 4, the difference between $\mathbf{T}$ and $\overline{\mathbf{T}}$ is confined to the boundary and in the weak form of the elastodynamic equation causes the presence of a penalty term on the Neumann part of the boundary. A further consequence is that we are unable to show that the energy-dissipation equality is satisfied when the Neumann part of the boundary is non-empty. At best, we have an energy-dissipation inequality. In fact, even if we *a priori* assume that the error term vanishes, we are still unable to prove that an energy-dissipation equality is satisfied.

The exact formulation of the problem of interest is as follows. We fix problem parameters $k \in \mathbb{N}$, $a > 0$ and $\alpha > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary such that $\partial\Omega$ has a Dirichlet part $\partial\Omega_D$ and a Neumann part $\partial\Omega_N$. Let $T > 0$ be a fixed finite final time and denote by $Q := (0,T)\times\Omega$ the space-time domain. We look for a triple $(\boldsymbol{u},\mathbf{T},v) : Q \to \mathbb{R}^d \times \mathbb{R}^{d\times d}\times\mathbb{R}$ that satisfies, in some weak sense,

$$\boldsymbol{u}_{tt} = \operatorname{div}(b(v)\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{6.1a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{in } Q, \tag{6.1b}$$

$$\boldsymbol{u} = \mathbf{0},\ v = 0 \qquad \text{on } (0,T]\times\partial\Omega_D, \tag{6.1c}$$

$$b(v)\mathbf{T}\mathbf{n} = \boldsymbol{g} \qquad \text{on } (0,T]\times\partial\Omega_N, \tag{6.1d}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1,\ v(0,\cdot) = v_0 \qquad \text{in } \Omega, \tag{6.1e}$$

with crack non-healing property $v_t \leq 0$, satisfaction of the minimisation problem

$$\begin{aligned}&\mathcal{E}(\boldsymbol{u}(t),v(t)) + \mathcal{H}(v(t)) + \mathcal{G}_k(v(t),v_t(t)) \\ &\quad = \inf\left\{\mathcal{E}(\boldsymbol{u}(t),v) + \mathcal{H}(v) + \mathcal{G}_k(v,v_t(t)) \,:\, v \in H^{\max\{k,1\}}_{D+1}(\Omega),\ v \leq v(t)\right\},\end{aligned} \tag{6.2}$$

and the energy-dissipation equality

$$\begin{aligned}\mathcal{F}(t;\boldsymbol{u}(t),\boldsymbol{u}_t(t),v(t)) &+ \int_0^t\int_\Omega b(v)\left(\mathbf{T}-F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\right)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\langle l_t,\boldsymbol{u}\rangle\,\mathrm{d}s\\ &+\int_0^t\|v_t(s)\|_{k,2}^2\,\mathrm{d}s = \mathcal{F}(0;\boldsymbol{u}_0,\boldsymbol{u}_1,v_0),\end{aligned}\tag{6.3}$$

for every $t\in[0,T]$. We recall that $H^k_{D+1}(\Omega)$ is the set of functions $v\in H^k(\Omega)$ such that $v=1$ on $\partial\Omega_D$ in the sense of traces.

We define the total energy functional $\mathcal{F}$, elastic energy $\mathcal{E}$ and surface energy $\mathcal{H}$ by

$$\begin{aligned}\mathcal{F}(t;\boldsymbol{u},\boldsymbol{w},v) &= \mathcal{K}(\boldsymbol{w})+\mathcal{E}(\boldsymbol{u},v)+\mathcal{H}(v)-\langle l(t),\boldsymbol{u}\rangle,\\ \mathcal{E}(\boldsymbol{u},v) &= \int_\Omega\frac{b(v)}{\alpha}\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\,\mathrm{d}x,\quad \mathcal{H}(v)=\frac{1}{4\epsilon}\|1-v\|_2^2+\epsilon\|\nabla v\|_2^2,\end{aligned}$$

with kinetic energy $\mathcal{K}$ and external force l given by

$$\mathcal{K}(\boldsymbol{w})=\frac{1}{2}\|\boldsymbol{w}\|_2^2,\quad \langle l(t),\boldsymbol{u}\rangle=\int_\Omega\boldsymbol{f}(t)\cdot\boldsymbol{u}\,\mathrm{d}x+\int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{u}\,\mathrm{d}S.$$

The functional $\mathcal{G}_k$ is defined on $H^{\max\{k,1\}}(\Omega)\times H^{\max\{k,1\}}(\Omega)$ by

$$\mathcal{G}_k(v,z)=(v,z)_{k,2}:=\sum_{|\alpha|\le k}\int_\Omega\partial^\alpha v\cdot\partial^\alpha w\,\mathrm{d}x.$$

In the elastic energy functional, φ^* is the convex conjugate of φ, where φ is defined on $\mathbb{R}^{d\times d}$ by

$$\varphi(\mathbf{T})=\int_0^{|\mathbf{T}|}\frac{t}{(1+t^a)^{\frac{1}{a}}}\,\mathrm{d}t.$$

We notice that $F(\mathbf{T})=\frac{\partial\varphi}{\partial\mathbf{T}}(\mathbf{T})$. On the open unit ball in $\mathbb{R}^{d\times d}$, we have $F^{-1}(\mathbf{T})=\frac{\partial\varphi^*}{\partial\mathbf{T}}$. Furthermore, for every $\mathbf{T}\in\mathbb{R}^{d\times d}$ (with $|\mathbf{T}|<1$ for the second equality) we have that

$$\mathbf{T}\cdot F(\mathbf{T})=\varphi(\mathbf{T})+\varphi^*(F(\mathbf{T})),\quad \mathbf{T}\cdot F^{-1}(\mathbf{T})=\varphi(F^{-1}(\mathbf{T}))+\varphi^*(\mathbf{T}).\tag{6.4}$$

This fact is used throughout this chapter and is vital in the derivation of certain estimates. The function φ^* is finite on the open unit ball in $\mathbb{R}^{d\times d}$ and infinite outside of the closed unit ball. Hence for the elastic energy to be finite in the strain-limiting setting, the strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ must be contained in the closed ball of radius α^{-1} around the origin. We make the following observation concerning a hypothetical solution of (6.1)–(6.3). Suppose that the initial data is chosen so that

$$\max\{\|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_0)\|_\infty,\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)\|_\infty\}:=C_*<1.\tag{6.5}$$

Using the memory kernel property, we have that

$$\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}(t))=\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_0)\mathrm{e}^{-\alpha t}+\int_0^t\alpha\mathrm{e}^{\alpha(s-t)}F(\mathbf{T}(s))\,\mathrm{d}s.$$

Taking the supremum norm, it follows that

$$\|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}(t))\|_\infty\le\mathrm{e}^{-\alpha t}\|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_0)\|_\infty+\int_0^t\alpha\mathrm{e}^{\alpha(s-t)}\,\mathrm{d}s\le 1-\mathrm{e}^{-\alpha T}(1-C_*)=:C_{**}$$

Since $C_* < 1$, the right-hand side C_{**} is strictly less than 1. Hence, $\varepsilon(\alpha\boldsymbol{u})$ is uniformly bounded away from 1, provided that (6.5) holds. This is a safety strain condition as in the previous chapters, except we now have the additional bound on the initial strain $\varepsilon(\boldsymbol{u}_0)$. Also, this is clearly sufficient in order to guarantee that the initial energy (particularly the initial elastic energy) is finite.

Due to the presence of the functional $\mathcal{G}_k$ in the minimisation problem (6.2), we can no longer guarantee that $v \geq 0$ on Q. For this reason, we must modify the choice of b from the previous chapter to ensure that b is uniformly bounded below and achieves its maximum over $(-\infty, 1]$ at 1. We later ask that $k > \frac{d}{2}$ so, by the Sobolev embedding theorem, $v \in L^\infty(Q)$. Approximations of the phase field function will also be bounded in this space. Hence such a boundedness restriction is essentially obsolete in this case. However, for simplicity, we assume that b is non-decreasing on $\mathbb{R}$ and bounded below. A standard choice of b to consider would be

$$b(v) = \max\{0, v\}^2 + \eta, \tag{6.6}$$

where $\eta > 0$ is a fixed constant. However, we can consider a more general class of functions satisfying the following requirements:

- $b \in C^1(\mathbb{R})$;
- there exists a positive parameter η such that $b(v) \geq \eta$ on $\mathbb{R}$;
- b is monotonically non-decreasing on $\mathbb{R}$.

Examples include $b(v) = \mathrm{e}^v + \eta$ where $\eta > 0$ is fixed, or a smoothing of (6.6).

For strain limiting problems of the type (6.1)–(6.3), we do not require a compatibility condition of the type (5.7). Such a condition does not make sense because we do not prescribe initial data for the rate dependent term $v_t(0)$. We only need the minimisation problem to hold at a.e. $t \in (0, T)$. However, as in Chapter 4, we require a compatibility condition between the Neumann boundary data and the initial data. Due to the presence of $b(v)$ in the elastodynamic equation, we modify the condition slightly to

$$\boldsymbol{g}(0) = b(v_0)F^{-1}(\varepsilon(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} \quad \text{ on } \partial\Omega_N. \tag{6.7}$$

This is vital in order to bound approximations of $\boldsymbol{u}_{tt}(0)$, used in obtaining higher regularity estimates of the displacement and the stress tensor with respect the time variable.

As with the analysis of the strain-limiting problem without fracture, the first step in proving the existence of a solution is to consider an approximation that replaces the bounded function F in the constitutive relation with one that is unbounded. Namely, we consider the problem of finding a triple $(\boldsymbol{u}, \mathbf{T}, v)$ such that

$$\boldsymbol{u}_{tt} = \operatorname{div}(b(v)\mathbf{T}) + \boldsymbol{f} \qquad \text{in } Q, \tag{6.8a}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = F_n(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{\mathbf{T}}{n} \qquad \text{in } Q, \tag{6.8b}$$

$$\boldsymbol{u} = \mathbf{0},\ v = 0 \qquad \text{on } (0,T]\times\partial\Omega_D, \tag{6.8c}$$

$$b(v)\mathbf{T}\mathbf{n} = \boldsymbol{g}^n \qquad \text{on } (0,T]\times\partial\Omega_N, \tag{6.8d}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{u}_t(0,\cdot) = \boldsymbol{u}_1,\ v(0,\cdot) = v_0 \qquad \text{in } \Omega, \tag{6.8e}$$

subject to the crack non-healing property $v_t \leq 0$, the minimisation problem

$$\begin{aligned}&\mathcal{E}_n(\boldsymbol{u}(t), v(t)) + \mathcal{H}(v(t)) + \mathcal{G}_k(v(t), v_t(t))\\ &\quad = \inf\left\{\mathcal{E}_n(\boldsymbol{u}(t), v) + \mathcal{H}(v) + \mathcal{G}_k(v, v_t(t))\ :\ v \in H^{\max\{k,1\}}_{D+1}(\Omega),\ v \leq v(t)\right\},\end{aligned} \tag{6.9}$$

and the energy-dissipation equality

$$\begin{aligned}&\mathcal{F}_n(t;\boldsymbol{u}(t),\boldsymbol{u}_t(t),v(t)) + \int_0^t\int_\Omega b(v)\left(\mathbf{T} - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\right)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle l_t^n, \boldsymbol{u}\rangle\,\mathrm{d}s\\ &\quad + \int_0^t \|v_t(s)\|_{k,2}^2\,\mathrm{d}s = \mathcal{F}_n(0;\boldsymbol{u}_0,\boldsymbol{u}_1,v_0),\end{aligned} \tag{6.10}$$

for every $t \in [0,T]$. The functional $\mathcal{F}_n$ is defined by replacing $\mathcal{E}$ and l in $\mathcal{F}$ with $\mathcal{E}_n$ and l^n which are defined as follows. The body force approximation l^n is defined by replacing $\boldsymbol{g}$ with $\boldsymbol{g}^n$ (see below for the definition) and the approximate elastic energy function $\mathcal{E}_n$ is defined as with $\mathcal{E}$ but φ^* is replaced by φ_n^*. The function φ_n^* is the convex conjugate of φ_n where φ_n is an anti-derivative for F_n. In particular, we define

$$\varphi_n(\mathbf{T}) = \int_0^{|\mathbf{T}|}\frac{t}{(1+t^a)^{\frac{1}{a}}}\,\mathrm{d}t + \frac{|\mathbf{T}|^2}{2n}, \quad \varphi_n^*(\mathbf{T}) = \sup_{\mathbf{S}\in\mathbb{R}^{d\times d}}\left\{\mathbf{T}\cdot\mathbf{S} - F_n(\mathbf{S})\right\}.$$

Expressions analogous to those in (6.4) hold but on the whole of $\mathbb{R}^{d\times d}$ because F_n is a C^1-diffeomorphism on $\mathbb{R}^{d\times d}$ to itself (cf. Lemma 2.3).

The approximation $\boldsymbol{g}^n$ is defined as in Chapter 4. Let $\chi \in C_c^\infty([0,\infty))$ be a smooth cut-off function such that $\chi = 1$ on $[0,\frac{1}{2}]$, $\chi = 0$ on $[1,\infty)$ and there exists a constant C such that $|\chi'(t)| \leq Ct^{-1}$ on $[0,\infty)$. Define $\boldsymbol{g}^n$ on $[0,T]\times\partial\Omega_N$ by

$$\boldsymbol{g}^n(t,x) = \chi(nt)b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} + (1-\chi(nt))\boldsymbol{g}(t,x).$$

We choose this approximation so that we have a compatibility condition like (6.7). Indeed, we have that

$$\boldsymbol{g}^n(0) = b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} \tag{6.11}$$

Reasoning as in Chapter 4, if $\boldsymbol{g} \in W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$, we have $\boldsymbol{g}^n \to \boldsymbol{g}$ strongly in $C([0,T];L^2(\partial\Omega_N)^d)$ and weakly-* in $W^{1,\infty}(0,T;L^2(\partial\Omega_N)^d)$. Furthermore, we have that

$$\begin{aligned}\|\boldsymbol{g}^n\|_{L^\infty(L^2(\partial\Omega_N))} &\leq C\left(1 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))} + \|\boldsymbol{g}_t\|_{L^2(L^2(\partial\Omega_N))}\right),\\ \|\boldsymbol{g}_t^n\|_{L^\infty(L^2(\partial\Omega_N))} &\leq C\left(1 + \|\boldsymbol{g}_t\|_{L^\infty(L^2(\partial\Omega_N))}\right),\end{aligned}$$

where C is a constant that is independent of n. If $\boldsymbol{g} \in W^{2,1}(0,T;L^2(\partial\Omega_N)^d)$, then $(\boldsymbol{g}^n)_n$ is bounded in $W^{2,1}(0,T;L^2(\partial\Omega_N)^d)$ and converges in this space to $\boldsymbol{g}$. It follows that, provided $\boldsymbol{f} \in W^{2,1}(0,T;L^2(\Omega)^d)$, then $(l^n)_n$ is bounded in $W^{2,1}(0,T;W_D^{-1,2}(\Omega)^d)$ and converges weakly in this space to l.

The proof of the existence of a solution to (6.8)-(6.10) follows similar lines to the proof in Chapter 5. In fact, the results of that chapter already tell us that a solution to the problem exists in a weak energy sense. However, to obtain the necessary higher regularity estimates, we use the exact structure of the time discrete problem. Furthermore, due to the presence of the linear term in the constitutive relation, we do not need to use a Galerkin approximation in space. We are essentially working in the $p = 2$ setting, so appropriate bounds are available in the time discrete setting that allow us to use the Browder–Minty Theorem (Theorem 2.2) immediately without a discretisation in space.[1]

Now we introduce a time discrete approximation of the regularised problem. We prove that the solution satisfies various properties such as a discrete energy-dissipation inequality, as well as uniform estimates. These allow us to take the limit in the time step to obtain a solution of the time continuous regularised problem (6.8)–(6.10). We start with assuming $k \geq 0$ and gradually introduce restrictions on k only when they are needed.

Theorem 6.1. *Fix a regularisation parameter $n \in \mathbb{N}$ and a time discretisation parameter $M \in \mathbb{N}$. Denote $\gamma = (n, M)$. Suppose that $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W_D^{1,2}(\Omega)^d$, $\boldsymbol{f} \in C^1([0,T];W_D^{-1,2}(\Omega)^d)$ and $\boldsymbol{g} \in C^1([0,T];L^2(\partial\Omega_N)^d)$. Let l^n be as defined above. Suppose that $v_0 \in H_{D+1}^{\max\{k,1\}}(\Omega)$ with $v_0 \in L^\infty(\Omega)$. Furthermore, suppose that $b \in C^1(\mathbb{R})$, b is uniformly bounded below by some $\eta > 0$, and b is non-decreasing. Denote the time step $h = \frac{T}{M}$ and define initialisations for the problem by*

$$\boldsymbol{u}_0^\gamma = \boldsymbol{u}_0, \quad \boldsymbol{u}_{-1}^\gamma = \boldsymbol{u}_0 - h\boldsymbol{u}_1, \quad v_0^\gamma = v_0.$$

We denote $l_m^\gamma = l^n(t_m^\gamma)$ where $t_m^\gamma = mh$ for $0 \leq m \leq M$.

Considering the statements recursively, for every $1 \leq m \leq M$ there exists a unique $\boldsymbol{u}_m^\gamma \in W_D^{1,2}(\Omega)^d$ such that

$$\int_\Omega \delta^2 \boldsymbol{u}_m^\gamma \cdot \boldsymbol{w} + b(v_{m-1}^\gamma) F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \,\mathrm{d}x = \langle l_m^\gamma, \boldsymbol{w} \rangle, \tag{6.12}$$

for every $\boldsymbol{w} \in W_D^{1,2}(\Omega)^d$, and there exists a unique $v_m^\gamma \in H_{D+1}^{\max\{k,1\}}(\Omega)$ such that

$$\begin{aligned} &\mathcal{E}_n(\boldsymbol{u}_m^\gamma, v_m^\gamma) + \mathcal{H}(v_m^\gamma) + \frac{1}{2h}\mathcal{G}_k(v_m^\gamma - v_{m-1}^\gamma, v_m^\gamma - v_{m-1}^\gamma) \\ &\quad = \inf\left\{\mathcal{E}_n(\boldsymbol{u}_m^\gamma, v) + \mathcal{H}(v) + \frac{1}{2h}\mathcal{G}_k(v - v_{m-1}^\gamma, v - v_{m-1}^\gamma) \,:\, v \in H_{D+1}^{\max\{k,1\}}(\Omega),\, v \leq v_{m-1}^\gamma\right\}. \end{aligned} \tag{6.13}$$

[1] We could prove the results in Chapter 5 without using a Galerkin approximation in space but the uniform estimates become much more technical to deal with. It is simpler to introduce the approximation in space and obtain a solution to a discrete in space, continuous in time problem from the fully discrete problem rather than going directly from a discrete in time problem to the original problem.

Proof. First, we note that by construction $v_m^\gamma \le v_0$ for every $0 \le m \le M$. It follows that $\eta \le b(v_m^\gamma) \le b(v_0)$. The right-hand side is uniformly bounded above by $b(\|v_0\|_\infty)$. Hence, if the solution sequence exists, $(b(v_m^\gamma))_m$ is uniformly bounded on Ω. Fix $m \in \{1,\dots,M\}$ and suppose that the result holds at the $(m-1)$-st time level. Consider the solution map $S : W_D^{1,2}(\Omega)^d \to W_D^{-1,2}(\Omega)^d$ defined by

$$\langle S(\boldsymbol{u}), \boldsymbol{w}\rangle = \int_\Omega \frac{\boldsymbol{u} - 2\boldsymbol{u}_{m-1}^\gamma + \boldsymbol{u}_{m-2}^\gamma}{h^2} \cdot \boldsymbol{w} + b(v_{m-1}^\gamma) F_n^{-1}\left(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u} - \boldsymbol{u}_{m-1}^\gamma}{h} + \alpha \boldsymbol{u}\right)\right) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \,\mathrm{d}x.$$

It is easy to check that the hypotheses of the Browder–Minty Theorem (Theorem 2.2) are satisfied and, in particular, that S is a bijection. We deduce that there exists a unique $\boldsymbol{u}_m^\gamma \in W_D^{1,2}(\Omega)^d$ such that $S(\boldsymbol{u}_m^\gamma) = l_m^\gamma$ in $W_D^{-1,2}(\Omega)^d$. Equivalently, $\boldsymbol{u}_m^\gamma$ is the unique solution of (6.12).

For the existence of a function $v_m^\gamma \in H_{D+1}^{\max\{k,1\}}(\Omega)$ solving the minimisation problem (6.13), define a functional $\mathcal{A}$ on $H_{D+1}^{\max\{k,1\}}(\Omega)$ by

$$\begin{aligned}\mathcal{A}(v) &= \mathcal{E}_n(\boldsymbol{u}_m^\gamma, v) + \mathcal{H}(v) + \frac{1}{2h}\mathcal{G}_k(v - v_{m-1}^\gamma, v - v_{m-1}^\gamma)\\ &= \int_\Omega \frac{b(v)}{\alpha}\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_m^\gamma)) + \frac{1}{4\epsilon}(1-v)^2 + \epsilon|\nabla v|^2 \,\mathrm{d}x + \frac{1}{2h}\|v - v_{m-1}^\gamma\|_{k,2}^2.\end{aligned}$$

Clearly $\mathcal{A}(v) \ge 0$ and $\mathcal{A}(0) < \infty$ so the infimum exists. If $(v_l)_{l\ge1}$ is a minimising sequence for $\mathcal{A}$, it follows that $(v_l)_{l\ge1}$ is uniformly bounded in $H^{\max\{k,1\}}(\Omega)$. Hence, up to a subsequence, not relabelled, it converges weakly in $H^{\max\{k,1\}}(\Omega)$, strongly in $L^2(\Omega)$ and pointwise a.e. on Ω as $l \to \infty$. By weak lower semi-continuity, the continuity of b and Fatou's lemma,

$$\inf_{\tilde v \in H_{D+1}^{\max\{k,1\}}(\Omega), \tilde v \le v_{m-1}^\gamma} \mathcal{A}(\tilde v) = \liminf_{l\to\infty} \mathcal{A}(v_l) \ge \mathcal{A}(v) \ge \inf_{\tilde v \in H_{D+1}^{\max\{k,1\}}(\Omega), \tilde v \le v_{m-1}^\gamma} \mathcal{A}(\tilde v).$$

To see that the choice of minimiser is unique, we note that $\tilde v \mapsto \mathcal{A}(\tilde v)$ is strictly convex and apply a standard argument. □

We state Lemma 6.2 and Corollary 6.3 without proof, as they can be proven as in Proposition 5.5 and Corollary 5.6, taking account of the extra rate-dependent term. Similarly, we can obtain an energy-dissipation inequality (Proposition 6.4), following the proof of Proposition 5.7.

Lemma 6.2. *Let the assumptions of Theorem 6.1 hold and let $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ be the sequence of solutions to the time discrete, regularised problem. For every $1 \le m \le M$ and every $\tilde\chi \in H_{D+1}^{\max\{k,1\}}(\Omega)$ such that $\tilde\chi \le v_{m-1}^\gamma$, we have*

$$\begin{aligned}0 &\le \left[\partial_v \mathcal{E}_n(\boldsymbol{u}_m^\gamma, v_m^\gamma) + \mathcal{H}'(v_m^\gamma) + \frac{1}{2h}\partial_v \mathcal{G}_k(v_m^\gamma - v_{m-1}^\gamma, v_m^\gamma - v_{m-1}^\gamma)\right](\tilde\chi - v_{m-1}^\gamma)\\ &= \int_\Omega \frac{b'(v_m^\gamma)}{\alpha}(\tilde\chi - v_{m-1}^\gamma)\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}_m^\gamma)) + \frac{1}{2\epsilon}(v_m^\gamma - 1)(\tilde\chi - v_{m-1}^\gamma)\\ &\quad + 2\epsilon\nabla v_m^\gamma \cdot \nabla(\tilde\chi - v_{m-1}^\gamma)\,\mathrm{d}x + (\delta v_m^\gamma, \tilde\chi - v_{m-1}^\gamma)_{k,2},\end{aligned}$$

where $\partial_v \mathcal{G}_k(v,w)[z] = (w,z)_{k,2}$. In particular, for every $\chi \in H_D^{\max\{k,1\}}(\Omega)$ with $\chi \le 0$,

$$0 \le \left[\partial_v \mathcal{E}_n(\boldsymbol{u}_m^\gamma, v_m^\gamma) + \mathcal{H}'(v_m^\gamma) + \frac{1}{2h}\partial_v \mathcal{G}_k(v_m^\gamma - v_{m-1}^\gamma, v_m^\gamma - v_{m-1}^\gamma)\right](\chi).$$

Corollary 6.3. *Let the assumptions of Theorem 6.1 hold and let* $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ *be the sequence of solutions to the time discrete, regularised problem. For every* $1 \le m \le M$*, we have that*

$$0 = \left[\partial_v \mathcal{E}_n(\boldsymbol{u}_m^\gamma, v_m^\gamma) + \mathcal{H}'(v_m^\gamma) + \frac{1}{2h}\partial_v \mathcal{G}_k(v_m^\gamma - v_{m-1}^\gamma, v_m^\gamma - v_{m-1}^\gamma)\right](\delta v_m^\gamma).$$

Proposition 6.4. *Let the assumptions of Theorem 6.1 hold and let* $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ *be the sequence of solutions to the time discrete, regularised problem. For every* $1 \le m \le M$*, the following energy-dissipation inequality holds:*

$$\begin{aligned}
&\mathcal{F}_n(t_m^\gamma; \boldsymbol{u}_m^\gamma, \delta \boldsymbol{u}_m^\gamma, v_m^\gamma) + h\sum_{j=1}^m \|\delta v_j^\gamma\|_{k,2}^2 + h\sum_{j=1}^m \langle \delta l_j^\gamma, \boldsymbol{u}_{j-1}^\gamma\rangle \\
&\quad + h\sum_{j=1}^m \int_\Omega b(v_{j-1}^\gamma)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma + \alpha \boldsymbol{u}_j^\gamma)) - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_j^\gamma))\right]\cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma)\,\mathrm{d}x \le \mathcal{F}_n(0; \boldsymbol{u}_0, \boldsymbol{u}_1, v_0).
\end{aligned}$$

Now we investigate M-independent bounds on the solution sequence so that we can take the limit as $M \to \infty$ and obtain a weak energy solution of the regularised problem (6.8)–(6.10).

Lemma 6.5. *Let the assumptions of Theorem 6.1 hold and let* $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ *be the sequence of solutions to the time discrete, regularised problem. There exists a constant* $C = C(n)$*, independent of* M*, such that*

$$\begin{aligned}
&\max_{1\le m\le M} \|\boldsymbol{u}_m^\gamma\|_{1,2}^2 + \max_{1\le m\le M} \|\delta \boldsymbol{u}_m^\gamma\|_2^2 + \max_{1\le k\le M} \|v_m^\gamma\|_{k,2}^2 + h\sum_{m=1}^M \left(\|\delta \boldsymbol{u}_m^\gamma\|_{1,2}^2 + \|\delta v_m^\gamma\|_{k,2}^2\right) \\
&\le C(n)\left[1 + (\|b(v_0)\|_\infty + 1)\|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}^2\right].
\end{aligned}$$

Proof. Expanding out the energy-dissipation inequality, we use the Korn–Poincaré inequality to see that

$$\begin{aligned}
&\|\delta \boldsymbol{u}_m^\gamma\|_2^2 + \|\boldsymbol{u}_m^\gamma\|_{1,2}^2 + \|v_m^\gamma\|_{1,2}^2 + h\sum_{j=1}^m \|\delta v_j^\gamma\|_{k,2}^2 + h\sum_{j=1}^m \|\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma)\|_2^2 \\
&\le \|\delta \boldsymbol{u}_m^\gamma\|_2^2 + C(n)\int_\Omega b(v_{m-1}^\gamma)\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_m^\gamma))\,\mathrm{d}x + \|v_m^\gamma\|_{1,2}^2 + h\sum_{j=1}^m \|\delta v_j^\gamma\|_{k,2}^2 \\
&\quad + C(n)h\sum_{j=1}^m \int_\Omega b(v_{j-1}^\gamma)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma + \alpha \boldsymbol{u}_j^\gamma)) - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_j^\gamma))\right]\cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma)\,\mathrm{d}x \\
&\le C\left[1 + \mathcal{F}_n(0; \boldsymbol{u}_0^\gamma, \boldsymbol{u}_1^\gamma, v_0^\gamma) + \langle l_m^\gamma, \boldsymbol{u}_m^\gamma\rangle - h\sum_{j=1}^m \langle \delta l_j^\gamma, \boldsymbol{u}_{j-1}^\gamma\rangle\right] \\
&\le C\Bigg[1 + (\|b(v_0)\|_\infty + 1)\|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|l^n(0)\|_{-1,2}^2 + \|l_m^\gamma\|_{-1,2}^2 \\
&\quad + h\sum_{j=1}^m \|\delta l_j^\gamma\|_{-1,2}^2 + h\sum_{j=1}^{m-1} \|\boldsymbol{u}_j^\gamma\|_{1,2}^2\Bigg] + \frac{\|\boldsymbol{u}_m^\gamma\|_{1,2}^2}{2}.
\end{aligned}$$

We absorb the final term on the right-hand side into the left and apply the discrete Gronwall inequality to yield the following:

$$\max_{1\le m\le M}\|\delta \boldsymbol{u}_m^\gamma\|_2^2 + \max_{1\le m\le M}\|\boldsymbol{u}_m^\gamma\|_{1,2} + \max_{1\le m\le M}\|v_m^\gamma\|_{1,2}^2 + h\sum_{m=1}^M \left(\|\delta \boldsymbol{u}_m^\gamma\|_{1,2}^2 + \|\delta v_m^\gamma\|_{k,2}^2\right)$$
$$\le C\left[1 + (\|b(v_0)\|_\infty + 1)\|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}\right].$$

We can write $v_m^\gamma = v_0^\gamma + h\sum_{j=1}^m \delta v_j^\gamma$ and so we get that

$$\max_{1\le m\le M}\|v_m^\gamma\|_{k,2}^2 \le C\left(\|v_0\|_{k,2}^2 + h\sum_{m=1}^M \|\delta v_m^\gamma\|_{k,2}^2\right).$$

Combining this with the above, we conclude the required result. □

Lemma 6.6. *Let the assumptions of Theorem 6.1 hold. Let $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ be the sequence of solutions to the time discrete, regularised problem. There exists a constant $C = C(n)$, independent of M, such that*

$$h\sum_{m=1}^M \|\delta^2\boldsymbol{u}_m^\gamma\|_2^2 + \max_{1\le m\le M}\|\delta\boldsymbol{u}_m^\gamma\|_{1,2}^2 \le C(n)\Big[1 + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}^2$$
$$+ (1+\|b(v_0)\|_\infty)\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)\|_2^2 + \|\boldsymbol{u}_1\|_2^2\Big].$$

Proof. We use an argument similar to one from Chapter 4, adapted to the time discrete setting. Let $1 \le m \le M$. We test against $h(\delta^2\boldsymbol{u}_m^\gamma + \alpha\delta_m^\gamma)$ in the elastodynamic equation (6.12) and now consider each term in turn. We rewrite the term involving the nonlinear function F_n in the following way:

$$\begin{aligned}
&\int_\Omega b(v_{m-1}^\gamma)F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma))\cdot\boldsymbol{\varepsilon}(h(\delta^2\boldsymbol{u}_m^\gamma + \alpha\delta\boldsymbol{u}_m^\gamma))\,\mathrm{d}x\\
&= \int_\Omega b(v_{m-1}^\gamma)F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma))\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma)\,\mathrm{d}x\\
&\quad - \int_\Omega b(v_{m-1}^\gamma)F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_{m-1}^\gamma + \alpha\boldsymbol{u}_{m-1}^\gamma))\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_{m-1}^\gamma + \alpha\boldsymbol{u}_{m-1}^\gamma)\,\mathrm{d}x\\
&\quad - \int_\Omega b(v_{m-1}^\gamma)\Big[F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma))\\
&\qquad - F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_{m-1}^\gamma + \alpha\boldsymbol{u}_{m-1}^\gamma))\Big]\cdot\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_{m-1}^\gamma + \alpha\boldsymbol{u}_{m-1}^\gamma)\,\mathrm{d}x.
\end{aligned}\tag{6.14}$$

Define a fourth-order tensor $\mathcal{B}_n(\mathbf{T})$ for each $\mathbf{T}\in\mathbb{R}^{d\times d}$ by $\mathcal{B}_n^{ijkl}(\mathbf{T}) = \frac{\partial(F_n^{-1})_{ij}}{\partial\mathbf{T}_{kl}}(\mathbf{T})$ with corresponding inner product given by

$$(\mathbf{S},\mathbf{U})_{\mathcal{B}_n(\mathbf{T})} = \mathcal{B}_n^{ijkl}(\mathbf{T})\mathbf{S}_{ij}\mathbf{U}_{kl}.$$

By definition and the fundamental theorem of calculus, we have that

$$(F_n^{-1}(\mathbf{T}) - F_n^{-1}(\mathbf{S}))\cdot\mathbf{T} = \int_0^1(\mathbf{T},\mathbf{T}-\mathbf{S})_{\mathcal{B}_n(s\mathbf{T}+(1-s)\mathbf{S})}\,\mathrm{d}s.$$

On the other hand, we see that

$$\begin{aligned}
&\varphi_n(F_n^{-1}(\mathbf{T})) - \varphi_n(F_n^{-1}(\mathbf{S})) \\
&\quad = -\int_0^1 (1-s)(\mathbf{T}-\mathbf{S}, \mathbf{T}-\mathbf{S})_{\mathcal{B}_n(s\mathbf{T}+(1-s)\mathbf{S})} \,\mathrm{d}s + \int_0^1 (\mathbf{T}, \mathbf{T}-\mathbf{S})_{\mathcal{B}_n(s\mathbf{T}+(1-s)\mathbf{S})} \,\mathrm{d}s
\end{aligned}$$

Rearranging and using the inner product property yields

$$\begin{aligned}
&(F_n^{-1}(\mathbf{T}) - F_n^{-1}(\mathbf{S})) \cdot \mathbf{T} = \varphi_n(F_n^{-1}(\mathbf{T})) - \varphi_n(F_n^{-1}(\mathbf{S})) \\
&\quad + \int_0^1 (1-s)(\mathbf{T}-\mathbf{S}, \mathbf{T}-\mathbf{S})_{\mathcal{B}_n(s\mathbf{T}+(1-s)\mathbf{S})} \,\mathrm{d}s \geq \varphi_n(F_n^{-1}(\mathbf{T})) - \varphi_n(F_n^{-1}(\mathbf{S})).
\end{aligned}$$

Returning to (6.14), it follows that

$$\begin{aligned}
&\int_\Omega b(v_{m-1}^\gamma) F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) \cdot \boldsymbol{\varepsilon}(h(\delta^2 \boldsymbol{u}_m^\gamma + \alpha \delta \boldsymbol{u}_m^\gamma)) \,\mathrm{d}x \\
&\geq \int_\Omega b(v_{m-1}^\gamma) F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) \cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma) \,\mathrm{d}x \\
&\quad - \int_\Omega b(v_{m-1}^\gamma) F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma)) \cdot \boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma) \,\mathrm{d}x \\
&\quad + \int_\Omega b(v_{m-1}^\gamma) \left[\varphi_n(F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma))) - \varphi_n(F_n^{-1}(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)))\right] \,\mathrm{d}x \\
&= \int_\Omega b(v_{m-1}^\gamma) \left[\varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) - \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma))\right] \,\mathrm{d}x \\
&\geq \int_\Omega b(v_m^\gamma) \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) - b(v_{m-1}^\gamma) \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma)) \,\mathrm{d}x,
\end{aligned} \tag{6.15}$$

using the relation $\varphi_n(F_n^{-1}(\mathbf{T})) - F_n^{-1}(\mathbf{T}) \cdot \mathbf{T} = \varphi_n^*(\mathbf{T})$ and the non-decreasing property of b. We insert (6.15) into (6.12) with $\boldsymbol{w} = h(\delta^2 \boldsymbol{u}_m^\gamma + \alpha \delta \boldsymbol{u}_m^\gamma)$ to deduce that

$$\begin{aligned}
&h \|\delta^2 \boldsymbol{u}_m^\gamma\|_2^2 + \frac{\alpha}{2}\left(\|\delta \boldsymbol{u}_m^\gamma\|_2^2 - \|\delta \boldsymbol{u}_{m-1}^\gamma\|_2^2 + \|\delta \boldsymbol{u}_m^\gamma - \delta \boldsymbol{u}_{m-1}^\gamma\|_2^2\right) \\
&\quad + \int_\Omega b(v_m^\gamma) \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) - b(v_{m-1}^\gamma) \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma)) \,\mathrm{d}x \\
&\leq \langle l_m^\gamma, \delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma \rangle - \langle l_{m-1}^\gamma, \delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma \rangle + h \langle \delta l_m^\gamma, \delta \boldsymbol{u}_{m-1}^\gamma + \alpha \boldsymbol{u}_{m-1}^\gamma \rangle.
\end{aligned}$$

Using this recursively, it follows that

$$\begin{aligned}
&h \sum_{j=1}^m \|\delta^2 \boldsymbol{u}_j^\gamma\|_2^2 + \|\delta \boldsymbol{u}_m^\gamma\|_2^2 + \int_\Omega b(v_m^\gamma) \varphi_n^*(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)) \,\mathrm{d}x \\
&\leq C \Bigg[\|l_m^\gamma\|_{-1,2} \|\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)\|_2 + \|l_0^\gamma\|_{-1,2} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)\|_2 + h \sum_{j=1}^m \|\delta l_j^\beta\|_{-1,2}^2 \\
&\quad + h \sum_{j=0}^{m-1} \|\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_j^\gamma + \alpha \boldsymbol{u}_j^\gamma)\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \int_\Omega b(v_0) \varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)) \,\mathrm{d}x \Bigg].
\end{aligned}$$

Noting that $\varphi_n^*(\mathbf{T}) \geq C(n)|\mathbf{T}|^2$, we apply Young's inequality to absorb the $\boldsymbol{\varepsilon}(\delta \boldsymbol{u}_m^\gamma + \alpha \boldsymbol{u}_m^\gamma)$ term on the right-hand side into the left. After an application of the discrete Gronwall inequality, the required result holds. □

The following estimate demonstrates the extra regularity due to the presence of the functional $\mathcal{G}_k$. However, it is only possible under a restriction on the size of k.

Lemma 6.7. *Let the assumptions of Theorem 6.1 hold. Let $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ be the sequence of solutions to the time discrete, regularised problem. Suppose additionally that $k > \frac{d}{2}$. There exists a constant $C = C(n)$, independent of M, such that*

$$\max_{1\le m\le M} \|\delta v_m^\gamma\|_{k,2}^2 \le C(n)\left[1 + \|b'\|_{L^\infty(I_{\|v^\gamma\|_{\infty,*}})}\right]\Big[1 + (\|b(v_0)\|_\infty + 1)\|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}^2\Big],$$

where $I_{\|v^\gamma\|_{\infty,*}}$ denotes the interval $[-\|v^\gamma\|_{\infty,*}, \|v^\gamma\|_{\infty,*}]$ and $\|v^\gamma\|_{\infty,*} := \max_{1\le m\le M}\|v_m^\gamma\|_\infty$.

Proof. First, we note that $\|v^\gamma\|_{\infty,*} < \infty$ because of the estimate from Lemma 6.5 and the Sobolev embedding $H^k(\Omega) \subset L^\infty(\Omega)$. From Corollary 6.3, we see that

$$\begin{aligned}\|\delta v_m^\gamma\|_{k,2}^2 &= -\int_\Omega \frac{b'(v_m^\gamma)}{\alpha}\delta v_m^\gamma \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_m^\gamma)) + \frac{1}{2\epsilon}(v_m^\gamma - 1)\delta v_m^\gamma + 2\epsilon \nabla v_m^\gamma \cdot \nabla \delta v_m^\gamma \,\mathrm{d}x \\ &\le C(n)\left[\|b'(v_m^\gamma)\|_\infty \|\delta v_m^\gamma\|_\infty \|\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\gamma)\|_2^2 + \|v_m^\gamma - 1\|_2\|\delta v_m^\gamma\|_2 + \|\nabla v_m^\gamma\|_2\|\nabla\delta v_m^\gamma\|_2\right] \\ &\le C(n)\left[\|b'(v_m^\gamma)\|_\infty\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\gamma)\|_2^2 + \|v_m^\gamma - 1\|_2 + \|\nabla v_m^\gamma\|_2\right]\|\delta v_m^\gamma\|_{k,2},\end{aligned}$$

using the Sobolev embedding theorem to get $\|\delta v_m^\gamma\|_\infty \le C\|\delta v_m^\gamma\|_{k,2}$. We notice that b' is continuous and so

$$\|b'(v_m^\gamma)\|_\infty \le \|b'\|_{L^\infty(I_{\|v^\gamma\|_{\infty,*}})} < \infty,$$

Using Lemma 6.5, we have that

$$\begin{aligned}&\max_{1\le m\le M}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m^\gamma)\|_2^2 + \max_{1\le m\le M}\|v_m^\gamma - 1\|_2 + \max_{1\le m\le M}\|\nabla v_m^\gamma\|_2 \\ &\le C(n)\left[1 + \|b'\|_{L^\infty(I_{\|v^\gamma\|_{\infty,*}})}\right]\Big[1 + (\|b(v_0)\|_\infty + 1)\|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 \\ &\quad + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}^2\Big],\end{aligned}$$

from which the asserted bound follows. □

Using the notation of Chapter 5 for interpolants, we immediately have the following.

Lemma 6.8. *Let the assumptions of Theorem 6.1 hold. Let $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ be the sequence of solutions to the time discrete, regularised problem. Suppose additionally that $k > \frac{d}{2}$. There exists a constant $C = C(n)$, independent of M, such that*

$$\begin{aligned}&\max_{t\in[0,T]}\|\overline{\boldsymbol{u}}^\gamma(t)\|_{1,2} + \max_{t\in[0,T]}\|\boldsymbol{u}^{\gamma,\pm}\|_{1,2} + \max_{t\in[0,T}\|\overline{\boldsymbol{u}}^{\gamma,\prime}(t)\|_{1,2} + \max_{t\in[0,T]}\|\boldsymbol{u}^{\gamma,\pm,\prime}(t)\|_{1,2} \\ &\quad + \int_0^T \|\boldsymbol{u}^{\gamma,+,\prime\prime}(t)\|_2^2\,\mathrm{d}t + \max_{t\in[0,T]}\|\overline{v}^\gamma(t)\|_{k,2} + \max_{t\in[0,T]}\|v^{\gamma,\pm}(t)\|_{k,2} + \max_{t\in[0,T]}\|v^{\gamma,+,\prime}(t)\|_{k,2} \\ &\le C(n)\left[1 + \|b(v_0)\|_\infty + \|b'\|_{L^\infty(I_{\|v^\gamma\|_{\infty,*}})}\right]\Big[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_{1,2}^2 + \|v_0\|_{1,2}^2 \\ &\quad + \|l\|_{L^\infty(W^{-1,2})}^2 + \|l_t\|_{L^2(W^{-1,2})}^2\Big].\end{aligned}$$

6.1 A solution to the regularised problem

Theorem 6.9. *Let the assumptions of Theorem 6.1 hold and suppose additionally that $k > \frac{d}{2}$. There exists a weak energy solution $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ of the regularised problem (6.8)-(6.10) in the following sense. For every test function $\boldsymbol{w} \in W^{1,2}_D(\Omega)^d$ and a.e. $t \in (0,T)$, we have the weak elastodynamic equation*

$$\int_\Omega \boldsymbol{u}^n_{tt}(t) \cdot \boldsymbol{w} + b(v^n(t))\mathbf{T}^n(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \,\mathrm{d}x = \langle l^n(t), \boldsymbol{w} \rangle, \tag{6.16}$$

where the stress tensor is identified by (6.8b) pointwise a.e. in Q. For a.e. $t \in [0,T]$, the minimisation problem (6.9) is satisfied and for every $t \in [0,T]$, the energy-dissipation balance holds. We have the crack non-healing property $v^n_t \leq 0$ and the initial conditions are satisfied in the sense that

$$\lim_{t\to 0+} [\|\boldsymbol{u}^n(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}^n_t(t) - \boldsymbol{u}_1\|_2 + \|v^n(t) - v_0\|_{k,2}] = 0. \tag{6.17}$$

Proof. Using the bounds in Lemma 6.8 and standard compactness results, there exists a subsequence in M, not relabelled and independent of n, such that the following convergence results hold:

- $\overline{\boldsymbol{u}}^\gamma \overset{*}{\rightharpoonup} \boldsymbol{u}^n$ weakly in $W^{1,\infty}(0,T; W^{1,2}_D(\Omega)^d)$;
- $\boldsymbol{u}^{\gamma,\pm}$, $\boldsymbol{u}^{\gamma,\pm,\prime} \overset{*}{\rightharpoonup} \boldsymbol{u}^n$, $\boldsymbol{u}^n_t$ weakly-* in $L^\infty(0,T; W^{1,2}_D(\Omega)^d)$, respectively;
- $\overline{\boldsymbol{u}}^{\gamma,\prime} \overset{*}{\rightharpoonup} \boldsymbol{u}^n_t$ weakly-* in $L^\infty(0,T; W^{1,2}_D(\Omega)^d)$ and weakly in $W^{1,2}(0,T; L^2(\Omega)^d)$;
- $\overline{v}^\gamma \overset{*}{\rightharpoonup} v^n$ weakly-* in $W^{1,\infty}(0,T; H^k(\Omega))$;
- $v^{\gamma,\pm} \overset{*}{\rightharpoonup} v^n$ weakly-* in $L^\infty(0,T; H^k_{D+1}(\Omega))$.

We have $\overline{v}^\gamma_t \leq 0$ on Q by construction and weak convergence preserves ordering so $v^n_t \leq 0$ a.e. on Q. We apply the Aubin–Lions lemma and the Sobolev embedding theorem to see that $(\overline{v}^\gamma)_M$ converges strongly in $C([0,T]; H^1(\Omega) \cap C(\overline{\Omega}))$. Consequently, $(v^{\gamma,\pm})_M$ converges strongly in $L^\infty(0,T; H^1(\Omega) \cap C(\overline{\Omega}))$. In particular, for $1 \leq m \leq M$ and $t \in (t^\gamma_{m-1}, t^\gamma_m)$, we get

$$v^{\gamma,+}(t) - \overline{v}^\gamma(t) = v^\gamma_m - \frac{t - t^\gamma_{m-1}}{h} v^\gamma_m - \frac{t^\gamma_m - t}{h} v^\gamma_{m-1} = (t^\gamma_m - t)\delta v^\gamma_m,$$

which yields

$$\|v^{\gamma,+} - \overline{v}^\gamma\|_{L^\infty(H^k)} \leq h \max_{1\leq m\leq M} \|\delta v^\gamma_m\|_{k,2} \leq Ch,$$

for a constant C, independent of M. Thus $v^{\gamma,+} - \overline{v}^\gamma \to 0$ strongly in $L^\infty(0,T; H^k(\Omega))$. By the strong convergence of $\overline{v}^\gamma$ and the Sobolev embedding theorem, the strong convergence of $(v^{\gamma,+})_M$ follows. Applying the Aubin–Lions lemma to $\overline{\boldsymbol{u}}^\gamma$ and $\overline{\boldsymbol{u}}^{\gamma,\prime}$, we see that these sequences converge strongly in $C([0,T]; L^2(\Omega)^d)$ to their respective limits. Recalling the initialisation for the discrete problem, $\boldsymbol{u}^n(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}^n_t(0) = \boldsymbol{u}_1$ in $L^2(\Omega)^d$, and $v^n(0) = v_0$ in $H^1(\Omega)$ by standard arguments, so (6.17) holds.

We define the stress tensor by $\mathbf{T}^\gamma = F_n^{-1}(\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}^\gamma_t + \alpha \boldsymbol{u}^{\gamma,+}))$. By Lemma 6.8 and the bound $|F_n^{-1}(\mathbf{T})| \leq n|\mathbf{T}|$, it follows that $\mathbf{T}^\gamma$ is bounded in $L^\infty(0,T;L^2(\Omega)^{d\times d})$, independent of M. In particular, we have that $\mathbf{T}^\gamma \overset{*}{\rightharpoonup} \mathbf{T}^n$ weakly-* in $L^\infty(0,T;L^2(\Omega)^{d\times d})$ as $M\to\infty$ for some limit $\mathbf{T}^n$.

We rewrite the elastodynamic equation (6.12) in terms of the interpolant functions, test against a fixed $\boldsymbol{w} \in W^{1,2}_D(\Omega)^d$, multiply by an fixed but arbitrary $\chi \in C([0,T])$, and integrate over $[0,T]$ to deduce that

$$\int_Q \overline{\boldsymbol{u}}^{\gamma,\prime}_t \cdot (\chi \boldsymbol{w}) + b(v^{\gamma,-})\mathbf{T}^\gamma \cdot \boldsymbol{\varepsilon}(\chi\boldsymbol{w}) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle l^{\gamma,+}, \chi\boldsymbol{w}\rangle \,\mathrm{d}t.$$

Letting $M\to\infty$ yields

$$\int_0^T \chi \cdot \left(\int_\Omega \boldsymbol{u}^n_{tt}\cdot \boldsymbol{w} + b(v^n(t))\mathbf{T}^n(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x - \langle l^n, \boldsymbol{w}\rangle \right) \mathrm{d}t = 0.$$

By Lebesgue's differentiation theorem, the weak elastodynamic equation (6.16) holds for a.e. $t\in(0,T)$ and every test function $\boldsymbol{w}\in W^{1,2}_D(\Omega)^d$ as required.

Applying the Aubin–Lions lemma to $(\overline{\boldsymbol{u}}^{\gamma,\prime}_t + \alpha\boldsymbol{u}^{\gamma,+})_M$, we see that the sequence converges strongly in $L^2(Q)^d$ to $\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n$ as $M\to\infty$. Hence we have that

$$\begin{aligned}
&\lim_{M\to\infty}\int_Q b(v^{\gamma,-})\mathbf{T}^\gamma\cdot\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}^\gamma_t + \alpha\boldsymbol{u}^{\gamma,+})\,\mathrm{d}x\,\mathrm{d}t\\
&= \lim_{M\to\infty}\left[-\int_Q \overline{\boldsymbol{u}}^{\gamma,\prime}_t\cdot(\overline{\boldsymbol{u}}^\gamma_t + \alpha\boldsymbol{u}^{\gamma,+})\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle l^{\gamma,+}, \overline{\boldsymbol{u}}^\gamma_t + \alpha\boldsymbol{u}^{\gamma,+}\rangle\,\mathrm{d}t\right]\\
&= -\int_Q \boldsymbol{u}^n_{tt}\cdot(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\langle l^n, \boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n\rangle\,\mathrm{d}t\\
&= \int_Q b(v^n)\mathbf{T}^n\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Applying Minty's method (cf. the proof of Theorem 2.7), we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n) = F_n(\mathbf{T}^n)$ pointwise a.e. in Q.

To prove that the minimisation problem (6.9) is satisfied, we note that $\boldsymbol{u}^{\gamma,+}\to\boldsymbol{u}^n$ and $\boldsymbol{u}^{\gamma,\prime,+}\to\boldsymbol{u}^n_t$ strongly in $L^2(0,T;W^{1,2}(\Omega)^d)$. This is proven by adapting the reasoning in [73] (namely, Lemmas 3.9, 3.10 and 3.11) to the nonlinear problem. The sequence $(v^{\gamma,+})_M$ is bounded in $H^k(\Omega)$ and thus in $L^\infty(\Omega)$. By the previous reasoning, the sequence also converges pointwise a.e. on Q. The same is then also true for $(b'(v^{\gamma,+}))_M$ with limit $b'(v^n)$ with respect to pointwise convergence on Q and weak-* convergence in $L^\infty(Q)$. Using the dominated convergence theorem, we see that $(b'(v^{\gamma,+})\varphi^*_n(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^{\gamma,+})))_M$ converges strongly in $L^1(Q)$ to $b'(v^n)\varphi^*_n(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))$. Furthermore, by the strong convergence of $\boldsymbol{u}^{\gamma,+}$ and $\boldsymbol{u}^{\gamma,\prime,+}$ in $L^2(0,T;W^{1,2}(\Omega)^d)$, it follows that $\boldsymbol{\varepsilon}(\overline{\boldsymbol{u}}^\gamma_t + \alpha\boldsymbol{u}^{\gamma,+})$ converges strongly in $L^2(Q)^{d\times d}$. Thus, $\mathbf{T}^\gamma\to\mathbf{T}^n$ strongly in $L^2(Q)^{d\times d}$.

From Lemma 6.2, for every $\chi \in H^k_D(\Omega)$ such that $\chi \le 0$ and every $\psi \in C([0,T])$ with $\psi \ge 0$, we have

$$0 \le \int_Q \left(\frac{b'(v^{\gamma,+})}{\alpha} \chi\psi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^{\gamma,+})) + \frac{1}{2\epsilon}\psi(v^{\gamma,+}-1)\chi + 2\epsilon \nabla v^{\gamma,+}\cdot\nabla\chi \right) + \int_0^T (\overline{v}_t^{\gamma}, \psi\chi)_{k,2},$$

which, taking the limit as $M \to \infty$, leads to

$$0 \le \int_Q \left(\frac{b'(v^n)}{\alpha} \chi\psi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u})) + \frac{1}{2\epsilon}\psi(v^n-1)\chi + 2\epsilon \nabla v^n\cdot\nabla\chi \right) + \int_0^T (v_t^n, \psi\chi)_{k,2}.$$

It follows that, for a.e. $t \in (0,T)$ and every $\chi \in H^k_D(\Omega)$ such that $\chi \le 0$,

$$0 \le \left[\partial_v \mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t)) + \partial_v \mathcal{G}_k(v^n(t), v_t^n(t))\right](\chi).$$

Thus the limiting couple satisfies the minimisation problem (6.9).

It remains to show that energy-dissipation equality holds. Using the energy-dissipation inequality for the discrete solution sequence, we immediately deduce that

$$\begin{aligned} &\mathcal{F}_n(t; \boldsymbol{u}^n(t), \boldsymbol{u}_t^n(t), v^n(t)) + \int_0^t \int_\Omega b(v^n)\left(\mathbf{T}^n - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\right)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t \langle l_t^n, \boldsymbol{u}^n\rangle\,\mathrm{d}s + \int_0^t \|v_t^n(s)\|_{k,2}^2\,\mathrm{d}s \le \mathcal{F}_n(0; \boldsymbol{u}_0, \boldsymbol{u}_1, v_0). \end{aligned} \tag{6.18}$$

For the opposite inequality, we cannot argue as in Proposition 5.15, due to a lack of continuity of the rate-dependent term v_t^n. Instead, suppose that, for a.e. $t \in (0,T)$,

$$\begin{aligned} 0 \le &\int_\Omega \boldsymbol{u}_{tt}^n(t)\cdot\boldsymbol{u}_t^n(t) + b(v^n(t))\mathbf{T}^n(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n(t))\,\mathrm{d}x - \langle l^n(t), \boldsymbol{u}_t^n(t)\rangle \\ &+ \left[\partial_v\mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t))\right](v_t^n(t)) + \|v_t^n(t)\|_{k,2}^2. \end{aligned} \tag{6.19}$$

Integration with respect to the time variable yields

$$\begin{aligned} 0 &\le \int_0^t \int_\Omega \boldsymbol{u}_{tt}^n\cdot\boldsymbol{u}_t^n + b(v^n)\mathbf{T}^n\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \langle l^n, \boldsymbol{u}_t^n\rangle\,\mathrm{d}s \\ &\quad + \int_0^t \int_\Omega \frac{b'(v^n)}{\alpha} v_t^n \varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) + \frac{1}{2\epsilon}(v^n-1)v_t^n + 2\epsilon\nabla v^n\cdot\nabla v_t^n\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \|v_t^n\|_{k,2}^2\,\mathrm{d}s \\ &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\boldsymbol{u}_t^n\|_2^2}{2} + \int_\Omega \frac{b(v^n)}{\alpha}\varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n))\,\mathrm{d}x - \langle l^n, \boldsymbol{u}^n\rangle\right) + \langle l_t^n, \boldsymbol{u}^n\rangle\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega b(v^n)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)) - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \mathcal{H}(v^n(t)) - \mathcal{H}(v^n(0)) + \int_0^t \|v_t^n\|_{k,2}^2\,\mathrm{d}s \\ &= \left[\mathcal{K}(\boldsymbol{u}^n(s)) + \mathcal{E}_n(\boldsymbol{u}^n(s), v^n(s)) + \mathcal{H}(v^n(s)) - \langle l^n(s), \boldsymbol{u}^n(s)\rangle\right]_{s=0}^{s=t} + \int_0^t \langle l_t^n, \boldsymbol{u}^n\rangle\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega b(v^n)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)) - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \|v_t^n\|_{k,2}^2\,\mathrm{d}s. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \mathcal{F}_n(0; \boldsymbol{u}_0, \boldsymbol{u}_1, v_0) &\le \mathcal{F}_n(t; \boldsymbol{u}^n(t), \boldsymbol{u}_t^n(t), v^n(t)) + \int_0^t \langle l_t^n, \boldsymbol{u}^n\rangle\,\mathrm{d}s + \int_0^t \|v_t^n\|_{k,2}^2\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega b(v^n)\left[F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)) - F_n^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

which, together with (6.18), shows that the energy-dissipation equality is satisfied at a.e. $t \in (0,T)$. Each term in the energy-dissipation equality is continuous as a function of the time variable. Hence the energy-dissipation equality is satisfied at every point in $[0,T]$.

It remains to show that (6.19) is satisfied. First, we test in the elastodynamic equation (6.16) against $\boldsymbol{u}_t^n$ to see that

$$0 = \int_\Omega \boldsymbol{u}_{tt}^n \cdot \boldsymbol{u}_t^n + b(v^n)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)\,\mathrm{d}x - \langle l^n, \boldsymbol{u}_t^n\rangle.$$

Thus it remains to show that

$$0 \leq \left[\partial_v \mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t))\right](v_t^n(t)) + \|v_t^n(t)\|_{k,2}^2. \tag{6.20}$$

However, we know that the minimisation problem is satisfied and in particular we have that

$$0 \leq \left[\partial_v \mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t)) + \partial_v \mathcal{G}_k(v^n(t), v_t^n(t))\right](\chi),$$

for every $\chi \in H_D^k(\Omega)$ such that $\chi \leq 0$. Since $v_t^n(t)$ satisfies these hypotheses for a.e. $t \in (0,T)$, (6.20) holds as required. □

Corollary 6.10. *Let the assumptions of Theorem 6.9 hold and let $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ be the solution triple constructed from the limit of the time discrete problem. For a.e. $t \in [0,T]$, the following equality holds:*

$$\left[\partial_v \mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t)) + \partial_v \mathcal{G}_k(v^n(t), v_t^n(t))\right](v_t^n(t)) = 0.$$

Lemma 6.11. *Let the assumptions of Theorem 6.9 hold and let $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ be the solution triple constructed there. There exists a constant C, independent of n, such that*

$$\begin{aligned}&\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{1,2}^2 + \sup_{t\in[0,T]} \|\boldsymbol{u}_t^n(t)\|_2^2 + \int_Q |\mathbf{T}^n| + \frac{|\mathbf{T}^n|^2}{n} + |\nabla \boldsymbol{u}_t^n|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_1\|_2^2 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^T \|\boldsymbol{f}\|_2^2 + \|\boldsymbol{g}_t\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}t\right].\end{aligned}$$

Proof. The reasoning follows the same argument as that of Theorem 2.4. However, we need to take care of is the term resulting from the presence of the boundary traction term $\boldsymbol{g}^n$. First, we note that

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)| \leq 1 + \frac{|\mathbf{T}^n|}{n}.$$

We apply the trace theorem and the Korn–Poincaré inequality to get

$$\begin{aligned}&\int_0^t \int_{\partial\Omega_N} \boldsymbol{g}^n \cdot \boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n\,\mathrm{d}S\,\mathrm{d}s \leq C\int_0^t \|l^n\|_{-1,2}\|\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n\|_{1,2}\,\mathrm{d}s \\ &\leq C(\delta)\left[1 + \|\boldsymbol{g}\|_{L^\infty(L^2(\partial\Omega_N))}^2 + \int_0^t \|\boldsymbol{g}\|_{L^2(\partial\Omega_N)}^2\,\mathrm{d}s\right] + \delta\int_0^t \frac{\|\mathbf{T}^n\|_2^2}{n^2}\,\mathrm{d}s.\end{aligned}$$

The final term on the right-hand side can be absorbed into the left of the elastodynamic equation, testing with $\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n$. □

With Lemma 6.11, we can use the energy-dissipation equality to obtain bounds on the phase-field function. We do not include the details because it is similar to Lemma 5.18. It is at this stage that we need the safety strain condition (6.5) to hold. This ensures that the initial elastic energy $\mathcal{E}_n(\boldsymbol{u}_0, v_0)$ can be bounded above independent of n. In particular, we claim that the sequence $(\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)))_n$ is uniformly bounded in $L^1(\Omega)$. Convex conjugation is an order reversing process and $\varphi \leq \varphi_n$ so we have that $\varphi_n^*(\mathbf{T}) \leq \varphi^*(\mathbf{T})$ for every $\mathbf{T} \in \mathbb{R}^{d\times d}$. Recalling that $\frac{\partial \varphi^*}{\partial \mathbf{T}} = F^{-1}(\mathbf{T})$ and $\varphi^*(\mathbf{0}) = 0$, we get

$$\int_\Omega \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0))\,\mathrm{d}x \leq \int_\Omega \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0))\,\mathrm{d}x = \int_\Omega \int_0^{|\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)|} \frac{s}{(1-s^a)^{\frac{1}{a}}}\,\mathrm{d}s\,\mathrm{d}x \leq \frac{|\Omega| C_*^2}{(1-C_*^a)^{\frac{1}{a}}},$$

where C_* is the constant from the safety strain condition (6.5).

Lemma 6.12. *Let the assumptions of Theorem 6.9 hold and let* $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ *be the solution triple constructed there. Suppose additionally that the safety strain condition* (6.5) *holds. There exists a constant C, independent of n, such that*

$$\sup_{t\in[0,T]} \int_\Omega \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x + \sup_{t\in[0,T]} \|v^n(t)\|_{k,2}^2 + \int_0^T \|v_t^n(t)\|_{k,2}^2\,\mathrm{d}t \leq C\Big[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|b(v_0)\|_\infty + \|l\|_{L^\infty(W^{-1,2})}^2 + \int_0^T \|l_t(t)\|_{-1,2}^2\,\mathrm{d}t\Big].$$

Lemma 6.13. *Let the assumptions of Theorem 6.9 hold and let* $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ *be the solution triple constructed there. Suppose additionally that the safety strain condition* (6.5) *holds. There exists a constant C, independent of n, such that*

$$\sup_{t\in[0,T]} \|v_t^n(t)\|_{k,2}^2 \leq C\left[1 + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}^2\right]\left[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|b(v_0)\|_\infty + \|l\|_{L^\infty(W^{-1,2})}^2 + \int_0^T \|l_t(t)\|_{-1,2}^2\,\mathrm{d}t\right]^2.$$

Proof. From Corollary 6.10, we have that, for a.e. $t \in (0,T)$,

$$\begin{aligned}
\|v_t^n(t)\|_{k,2}^2 &= -\left[\partial_v \mathcal{E}_n(\boldsymbol{u}^n(t), v^n(t)) + \mathcal{H}'(v^n(t))\right](v_t^n(t)) \\
&= -\int_\Omega \frac{b'(v^n(t))}{\alpha} v_t^n(t) \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t))) + \frac{1}{2\epsilon}(v^n(t)-1)v_t^n(t) + 2\epsilon \nabla v^n(t)\cdot \nabla v_t^n(t)\,\mathrm{d}x \\
&\leq C\Big[\|b'\|_{L^\infty(I_{\|v^n(t)\|_\infty})} \|v_t^n(t)\|_{k,2} \int_\Omega \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x + \|v^n(t)-1\|_2 \|v_t^n(t)\|_2 \\
&\qquad + \|\nabla v^n(t)\|_2 \|\nabla v_t^n(t)\|_2\Big] \\
&\leq C\left[1 + \|b'\|_{L^\infty(I_{\|v^n(t)\|_\infty})}^2 \left(\int_\Omega \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x\right)^2 + \|v^n(t)\|_2^2 + \|\nabla v^n(t)\|_2^2\right] \\
&\qquad + \frac{\|v_t^n(t)\|_{k,2}^2}{2},
\end{aligned}$$

using the Sobolev embedding theorem for the second inequality. Applying Lemma 6.12, the result follows. □

Now we focus on improving the regularity estimates on the approximations of the stress tensor $(\mathbf{T}^n)_n$. As in Chapter 4, we look for weighted estimates on the time and spatial derivatives. These allow us to deduce a pointwise convergence result for $(\mathbf{T}^n)_n$, mimicking the ideas from the chapters on strain-limiting problems without fracture. Now we have the added complication of the phase-field function. Lemma 6.13 is essential in the derivation of higher regularity estimates for $(\mathbf{T}^n)_n$. For time regularity, the assumption $k > \frac{d}{2}$ is sufficient because it guarantees that $(v_t^n)_n$ is a bounded sequence in $L^\infty(Q)$. However, for the spatial regularity estimates, we need $(\nabla v^n)_n$ to be uniformly bounded in $L^\infty(Q)^d$. Thus, we ask for $k > \frac{d}{2} + 1$ in this case.

We start with the regularity in the time variable. We work directly with the time discrete approximation rather than the time continuous solution of the regularised problem to ensure a fully rigorous argument. We construct a bound that is uniform in n and M, and let $M \to \infty$. By weak lower semi-continuity and the pointwise convergence of $(\mathbf{T}^\gamma)_M$, we deduce an n-independent bound on $\mathbf{T}_t^n$. The issue that occurs if we try to work with the time continuous solution of the regularised problem is that $\boldsymbol{u}_{tt}^n$ is not continuous at $t = 0$. Thus it cannot be identified via the initial data. However, for the time discrete problem, $\delta^2 \boldsymbol{u}_0^\gamma$ is not currently defined. We need to define an initialisation term $\boldsymbol{u}_{-2}^\gamma$ and use it to define $\delta^2 \boldsymbol{u}_0^\gamma$, an approximation of $\boldsymbol{u}_{tt}^n(0)$. We choose $\boldsymbol{u}_{-2}^\gamma$ so that $\delta^2 \boldsymbol{u}_0^\gamma$ is bounded in $L^2(\Omega)^d$, independent of M and n, which is used to show that $(\delta^2 \boldsymbol{u}_m^\gamma)_{m=1}^M$ is uniformly bounded in $L^2(\Omega)^d$ with respect to M and n. Taking $M \to \infty$, we get a uniform bound on $(\boldsymbol{u}_{tt}^n)_n$ in $L^\infty(0,T; L^2(\Omega)^d)$. Hence $\mathbf{T}^n \in W^{1,2}(0,T; L^2(\Omega)^{d\times d})$, but with bound depending on n. However, this higher regularity can be used to find a weighted estimate on $\mathbf{T}_t^n$ that is uniform in n.

The compatibility condition is essential here. In particular, we make us of the fact that the approximation $\boldsymbol{g}^n$ is chosen so that (6.11) holds. Suppose hypothetically that $\boldsymbol{u}_{tt}^n$ is continuous from $[0,T]$ into $L^2(\Omega)^d$. Testing in the elastodynamic equation (6.16) at time $t = 0$ against $\boldsymbol{u}_{tt}^n(0)$, we have that

$$
\begin{aligned}
&\|\boldsymbol{u}_{tt}^n(0)\|_2^2 = \int_\Omega -b(v^n(0))\mathbf{T}^n(0) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_{tt}^n(0))\,\mathrm{d}x + \langle l^n(0), \boldsymbol{u}_{tt}^n(0)\rangle \\
&= \int_\Omega -b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_{tt}^n(0)) + \boldsymbol{f}(0) \cdot \boldsymbol{u}_{tt}^n(0)\,\mathrm{d}x + \int_{\partial\Omega_N} F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\mathbf{n} \cdot \boldsymbol{u}_{tt}^n(0)\,\mathrm{d}S \\
&= \int_\Omega \left[\operatorname{div}\left(b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))\right) + \boldsymbol{f}(0)\right] \cdot \boldsymbol{u}_{tt}^n(0)\,\mathrm{d}x,
\end{aligned}
$$

where the boundary integral from the integration by parts vanishes due to (6.11). It remains to show that $\operatorname{div}(b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))$ is bounded in $L^2(\Omega)^d$ independent of n. This is shown in the proof of Proposition 6.14 below. Without such a compatibility condition, we would need a bound on $\boldsymbol{u}_{tt}^n(0)$ on the Neumann part of the boundary $\partial\Omega_N$. We use analogous reasoning to show that $\delta^2 \boldsymbol{u}_0^\gamma$ is bounded in $L^2(\Omega)^d$, independent of n and M.

Proposition 6.14. *Let the assumptions of Theorem 6.9 hold and let* $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ *be the solution triple constructed there. Suppose additionally that the safety strain condition* (6.5) *holds. Furthermore, assume that* $\boldsymbol{f} \in W^{2,1}(0,T; W_D^{-1,2}(\Omega)^d)$*,* $\boldsymbol{g} \in W^{2,1}(0,T; L^2(\partial\Omega_N)^d)$*, and*

$\boldsymbol{u}_1+\alpha\boldsymbol{u}_0 \in W_D^{2,2}(\Omega)^d$. *Then* $\mathbf{T}^n \in W^{2,1}(0,T;L^2(\Omega)^d)$ *and there exists a constant* C*, independent of* n*, such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_{tt}^n(t)\|_2^2 + \int_Q \frac{|\mathbf{T}_t^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} + \frac{|\mathbf{T}_t^n|^2}{n} \,\mathrm{d}x\,\mathrm{d}t \le C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2$$
$$\cdot\left[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|b(v_0)\|_\infty+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|_{L^2(W^{-1,2})}^2\right.$$
$$\left.+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}+\|l_{tt}\|_{L^1(W^{-1,2})}\right]^2.$$

Proof. Fix $\gamma=(n,M)$ and let $(\boldsymbol{u}_m^\gamma, v_m^\gamma)_{m=1}^M$ be the discrete solution sequence from Theorem 6.1. Denote the discrete stress $\mathbf{T}_m^\gamma = F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma+\alpha\boldsymbol{u}_m^\gamma))$ for $0\le m\le M$. We fix $m\in\{2,\dots,M\}$ and test in the elastodynamic equation (6.12) at levels m and $m-1$ against $\boldsymbol{v}=\delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma$. We subtract the resulting inequalities to obtain

$$\begin{aligned}\int_\Omega &\left(b(v_{m-1}^\gamma)\mathbf{T}_m^\gamma - b(v_{m-2}^\gamma)\mathbf{T}_{m-1}^\gamma\right)\cdot\boldsymbol{\varepsilon}\left(\delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma\right)\\ &+\left(\delta^2\boldsymbol{u}_m^\gamma-\delta^2\boldsymbol{u}_{m-1}^\gamma\right)\cdot\left(\delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma\right)\,\mathrm{d}x = \langle l_m^\gamma - l_{m-1}^\gamma, \delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma\rangle.\end{aligned} \tag{6.21}$$

We can rewrite the first set of brackets using (5.18) to get that

$$\begin{aligned}&\int_\Omega \left(\delta^2\boldsymbol{u}_m^\gamma-\delta^2\boldsymbol{u}_{m-1}^\gamma\right)\cdot\left(\delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma\right)\,\mathrm{d}x\\ &=\frac{1}{2}\left(\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2-\|\delta^2\boldsymbol{u}_{m-1}^\gamma\|_2^2+\|\delta^2\boldsymbol{u}_m^\gamma-\delta^2\boldsymbol{u}_{m-1}^\gamma\|_2^2\right)\\ &\quad+\int_\Omega \delta^2\boldsymbol{u}_m^\gamma\cdot\delta\boldsymbol{u}_m^\gamma - \delta^2\boldsymbol{u}_{m-1}^\gamma\cdot\delta\boldsymbol{u}_{m-1}^\gamma\,\mathrm{d}x - \int_\Omega h\delta^2\boldsymbol{u}_m^\gamma\cdot\delta^2\boldsymbol{u}_{m-1}^\gamma\,\mathrm{d}x.\end{aligned}$$

For the terms involving the body forces, we perform a discrete integration by parts with respect to the time variable, moving the derivative in time from the displacement terms to the body force term so that the trace of second order derivatives $\delta^2\boldsymbol{u}_m^\gamma$ are not present. We obtain

$$\begin{aligned}&\langle l_m^\gamma - l_{m-1}^\gamma, \delta^2\boldsymbol{u}_m^\gamma+\alpha\delta\boldsymbol{u}_m^\gamma\rangle\\ &=\frac{1}{h}\langle l_m^\gamma-l_{m-1}^\gamma, (\delta\boldsymbol{u}_m^\gamma+\alpha\boldsymbol{u}_m^\gamma)-(\delta\boldsymbol{u}_{m-1}^\gamma+\alpha\boldsymbol{u}_{m-1}^\gamma)\rangle\\ &=\langle\delta l_m^\gamma, \delta\boldsymbol{u}_m^\gamma+\alpha\boldsymbol{u}_m^\gamma\rangle - \langle\delta l_{m-1}^\gamma, \delta\boldsymbol{u}_{m-1}^\gamma+\alpha\boldsymbol{u}_{m-1}^\gamma\rangle - h\langle\delta^2 l_m^\gamma, \delta\boldsymbol{u}_{m-1}^\gamma+\alpha\boldsymbol{u}_{m-1}^\gamma\rangle.\end{aligned}$$

For the nonlinear term, first define the fourth-order tensor $\mathcal{A}_n(\mathbf{T})$ by

$$\mathcal{A}_n(\mathbf{T})_{ijkl} = \frac{\partial F_n(\mathbf{T})_{ij}}{\partial \mathbf{T}_{kl}}, \tag{6.22}$$

with corresponding inner product on $\mathbb{R}^{d\times d}$ defined in the usual way. Then we get

$$\begin{aligned}
&\int_\Omega \left[b(v_{m-1}^\gamma)\mathbf{T}_m^\gamma - b(v_{m-2}^\gamma)\mathbf{T}_{m-1}^\gamma\right]\cdot \boldsymbol{\varepsilon}(\delta^2 \boldsymbol{u}_m^\gamma + \alpha\delta\boldsymbol{u}_m^\gamma)\,\mathrm{d}x\\
&= \int_\Omega \left[b(v_{m-1}^\gamma)\mathbf{T}_m^\gamma - b(v_{m-2}^\gamma)\mathbf{T}_{m-1}^\gamma\right]\cdot \frac{F_n(\mathbf{T}_m^\gamma)-F_n(\mathbf{T}_{m-1}^\gamma)}{h}\,\mathrm{d}x\\
&= \int_\Omega b(v_{m-1}^\gamma)\left(\mathbf{T}_m^\gamma - \mathbf{T}_{m-1}^\gamma\right)\cdot \frac{F_n(\mathbf{T}_m^\gamma)-F_n(\mathbf{T}_{m-1}^\gamma)}{h}\,\mathrm{d}x\\
&\quad + \int_\Omega \left(b(v_{m-1}^\gamma)-b(v_{m-2}^\gamma)\right)\mathbf{T}_{m-1}^\gamma\cdot \frac{F_n(\mathbf{T}_m^\gamma)-F_n(\mathbf{T}_{m-1}^\gamma)}{h}\,\mathrm{d}x\\
&= h\int_\Omega\int_0^1 b(v_{m-1}^\gamma)\left(\frac{\mathbf{T}_m^\gamma-\mathbf{T}_{m-1}^\gamma}{h},\frac{\mathbf{T}_m^\gamma-\mathbf{T}_{m-1}^\gamma}{h}\right)_{\mathcal{A}_n(s\mathbf{T}_m^\gamma+(1-s)\mathbf{T}_{m-1}^\gamma)}\mathrm{d}s\,\mathrm{d}x\\
&\quad + h\int_\Omega\int_0^1\left[\frac{b(v_{m-1}^\gamma)-b(v_{m-2}^\gamma)}{h}\right]\left(\mathbf{T}_{m-1}^\gamma,\frac{\mathbf{T}_m^\gamma-\mathbf{T}_{m-1}^\gamma}{h}\right)_{\mathcal{A}_n(s\mathbf{T}_m^\gamma+(1-s)\mathbf{T}_{m-1}^\gamma)}\mathrm{d}s\,\mathrm{d}x\\
&\geq \frac{h}{2}\int_\Omega\int_0^1 b(v_{m-1}^\gamma)\left(\frac{\mathbf{T}_m^\gamma-\mathbf{T}_{m-1}^\gamma}{h},\frac{\mathbf{T}_m^\gamma-\mathbf{T}_{m-1}^\gamma}{h}\right)_{\mathcal{A}_n(s\mathbf{T}_m^\gamma+(1-s)\mathbf{T}_{m-1}^\gamma)}\mathrm{d}s\,\mathrm{d}x\\
&\quad - \frac{h}{2\eta}\int_\Omega\int_0^1\left(\frac{b(v_{m-1}^\gamma)-b(v_{m-2}^\gamma)}{h}\right)^2(\mathbf{T}_{m-1}^\gamma,\mathbf{T}_{m-1}^\gamma)_{\mathcal{A}_n(s\mathbf{T}_m^\gamma+(1-s)\mathbf{T}_{m-1}^\gamma)}\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}$$

using the Cauchy–Schwarz inequality in the transition to the last line. Summing over the indices between 2 and m, it follows that

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2 - \|\delta^2\boldsymbol{u}_1^\gamma\|_2^2\right) + \alpha(\delta^2\boldsymbol{u}_m^\gamma,\delta\boldsymbol{u}_m^\gamma) - \alpha(\delta^2\boldsymbol{u}_1^\gamma,\delta\boldsymbol{u}_1^\gamma)\\
&\quad + \frac{h}{2}\sum_{j=2}^m\int_\Omega\int_0^1 b(v_{j-1}^\gamma)\left(\delta\mathbf{T}_j^\gamma,\delta\mathbf{T}_j^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_j^\gamma+(1-s)\mathbf{T}_{j-1}^\gamma)}\,\mathrm{d}s\,\mathrm{d}x - \alpha h\sum_{j=2}^m(\delta^2\boldsymbol{u}_j^\gamma,\delta^2\boldsymbol{u}_{j-1}^\gamma)\\
&\leq \langle\delta l_m^\gamma,\delta\boldsymbol{u}_m^\gamma+\alpha\boldsymbol{u}_m^\gamma\rangle - \langle\delta l_1^\gamma,\delta\boldsymbol{u}_1^\gamma+\alpha\boldsymbol{u}_1^\gamma\rangle + h\sum_{j=2}^m\langle\delta^2 l_j^\gamma,\delta\boldsymbol{u}_{j-1}^\gamma+\alpha\boldsymbol{u}_{j-1}^\gamma\rangle\\
&\quad + \frac{h}{2\eta}\sum_{j=1}^{m-1}\int_\Omega\int_0^1\left(\frac{b(v_j^\gamma)-b(v_{j-1}^\gamma)}{h}\right)^2(\mathbf{T}_j^\gamma,\mathbf{T}_j^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{j+1}^\gamma+(1-s)\mathbf{T}_j^\gamma)}\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}\tag{6.23}$$

To remove the term $\delta^2\boldsymbol{u}_1^\gamma$, we define $\boldsymbol{u}_{-2}^\gamma$ and thus $\delta^2\boldsymbol{u}_0^\gamma$, which we will bound in terms of only the problem data. We look for $\boldsymbol{u}_{-2}^\gamma \in L^2(\Omega)^d$ such that, for every $\boldsymbol{w}\in W_D^{1,2}(\Omega)^d$,

$$\begin{aligned}
\int_\Omega \frac{\boldsymbol{u}_0^\gamma - 2\boldsymbol{u}_{-1}^\gamma + \boldsymbol{u}_{-2}^\gamma}{h^2}\cdot\boldsymbol{w}\,\mathrm{d}x &= -\int_\Omega b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_0^\gamma+\alpha\boldsymbol{u}_0^\gamma))\cdot\boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x + \langle l^n(0),\boldsymbol{w}\rangle\\
&= \int_\Omega\left[\mathrm{div}\left(b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\right)+\boldsymbol{f}(0)\right]\cdot\boldsymbol{w}\,\mathrm{d}x,
\end{aligned}$$

using the compatibility condition (6.11). Thus define $\boldsymbol{u}_{-2}^\gamma$ by

$$\boldsymbol{u}_{-2}^\gamma := 2\boldsymbol{u}_{-1}^\gamma - \boldsymbol{u}_0^\gamma + h^2\left[\mathrm{div}\left(b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\right)+\boldsymbol{f}(0)\right].$$

The right-hand side is an element of $L^2(\Omega)^d$ and so $\boldsymbol{u}_{-2}^\gamma\in L^2(\Omega)^d$. Furthermore, it satisfies

$$\int_\Omega \frac{\boldsymbol{u}_0^\gamma - 2\boldsymbol{u}_{-1}^\gamma + \boldsymbol{u}_{-2}^\gamma}{h^2}\cdot\boldsymbol{w}\,\mathrm{d}x = \int_\Omega\left[\mathrm{div}\left(b(v_0)F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))\right)+\boldsymbol{f}(0)\right]\cdot\boldsymbol{w}\,\mathrm{d}x, \tag{6.24}$$

for every test function $\boldsymbol{w} \in L^2(\Omega)^d$.

We claim that, under the assumption $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W^{2,2}_D(\Omega)^d$ and the safety strain condition (6.5), there exists a constant C, independent of n, such that

$$\left\| \operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right) + \boldsymbol{f}(0) \right\|_2 \leq C. \tag{6.25}$$

A bound on $\operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right)$ in $L^2(\Omega)^d$ that is uniform with respect to n can be deduced in the following way. Expanding the derivative, justified by the regularity assumption, we have that

$$\begin{aligned} &\operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right)_i = \frac{\partial}{\partial x_j}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))_{ij}\right) \\ &= b'(v_0) \frac{\partial v_0}{\partial x_j} F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))_{ij} + b(v_0) \mathcal{A}_n(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))_{ijkl} \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)_{kl}}{\partial x_j}. \end{aligned}$$

Using the safety strain condition and the fact that $|F(\mathbf{T})| \leq |F_n(\mathbf{T})|$ for every $\mathbf{T} \in \mathbb{R}^{d\times d}$, we notice that

$$|F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))| \leq |F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))| \leq \frac{C_*}{(1 - C_*^a)^{\frac{1}{a}}}.$$

By direct computation of the derivative of F_n, we also see that

$$|\mathcal{A}_n(\mathbf{T})_{ijkl}| = \left| \delta_{ik}\delta_{jl} \left(\frac{1}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} + \frac{1}{n} \right) - \frac{|\mathbf{T}|^{a-2} \mathbf{T}_{ij} \mathbf{T}_{kl}}{(1 + |\mathbf{T}|^a)^{1+\frac{1}{a}}} \right| \leq 3,$$

and so

$$\left| \operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right) \right| \leq C \left(\frac{C_*}{(1 - C_*^a)^{\frac{1}{a}}} |\nabla v_0| + |\nabla \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0)| \right).$$

The right-hand side is an element of $L^2(\Omega)$ so $\operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right) \in L^2(\Omega)^d$ with bound in this space independent of n since $v_0 \in L^2(\Omega)$ and $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W^{2,2}_D(\Omega)^d$. Hence (6.25) holds.

Returning to (6.24), we choose $\boldsymbol{w} = \delta^2 \boldsymbol{u}_0^\gamma$ as a test function to yield

$$\begin{aligned} \|\delta^2 \boldsymbol{u}_0^\gamma\|_2^2 &= \int_\Omega \left[\operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right) + \boldsymbol{f}(0)\right] \cdot \delta^2 \boldsymbol{u}_0^\gamma \, \mathrm{d}x \\ &\leq \frac{\|\operatorname{div}\left(b(v_0) F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0))\right) + \boldsymbol{f}(0)\|_2^2}{2} + \frac{\|\delta^2 \boldsymbol{u}_0^\gamma\|_2^2}{2}. \end{aligned}$$

Applying (6.25), there exists a constant C, independent of n and M, such that

$$\|\delta^2 \boldsymbol{u}_0^\gamma\|_2 \leq C \left(\|v_0\|_{1,2} + \|\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0\|_{2,2} \right).$$

Now, we compare

$$\int_\Omega \delta^2 \boldsymbol{u}_0^\gamma \cdot \boldsymbol{w} + b(v_0^\gamma) \mathbf{T}_0^\gamma \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \, \mathrm{d}x = \langle l_0^\gamma, \boldsymbol{w} \rangle,$$

and

$$\int_\Omega \delta^2 \boldsymbol{u}_1^\gamma \cdot \boldsymbol{w} + b(v_0^\gamma)\mathbf{T}_1^\gamma \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x = \langle l_1^\gamma, \boldsymbol{w}\rangle.$$

Subtracting these equalities from each other and choosing $\boldsymbol{w} = \delta^2\boldsymbol{u}_1^\gamma + \alpha\delta\boldsymbol{u}_1^\gamma$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta^2\boldsymbol{u}_1^\gamma\|_2^2 - \|\delta^2\boldsymbol{u}_0^\gamma\|_2^2\right) + \alpha(\delta^2\boldsymbol{u}_1^\gamma, \delta\boldsymbol{u}_1^\gamma) - \alpha(\delta^2\boldsymbol{u}_0^\gamma, \delta\boldsymbol{u}_0^\gamma) + \alpha(\delta^2\boldsymbol{u}_0^\gamma, \delta\boldsymbol{u}_0^\gamma - \delta\boldsymbol{u}_1^\gamma)\\
&\quad + \frac{h}{2}\int_\Omega\int_0^1 b(v_0^\gamma)\left(\delta\mathbf{T}_1^\gamma, \delta\mathbf{T}_1^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_1^\gamma+(1-s)\mathbf{T}_0^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\
&\le \langle \delta l_1^\gamma, \delta\boldsymbol{u}_1^\gamma + \alpha\boldsymbol{u}_1^\gamma\rangle - \langle \delta l_1^\gamma, \delta\boldsymbol{u}_0^\gamma + \alpha\boldsymbol{u}_0^\gamma\rangle.
\end{aligned}$$

Adding this to (6.23), we deduce that, for every $m \in \{2, \dots, M\}$,

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2 - \|\delta^2\boldsymbol{u}_0^\gamma\|_2^2\right) + \alpha(\delta^2\boldsymbol{u}_m^\gamma, \delta\boldsymbol{u}_m^\gamma) - \alpha(\delta^2\boldsymbol{u}_0^\gamma, \delta\boldsymbol{u}_0^\gamma)\\
&\quad + \frac{h}{2}\sum_{j=1}^m\int_\Omega\int_0^1 b(v_{j-1}^\gamma)\left(\delta\mathbf{T}_j^\gamma, \delta\mathbf{T}_j^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_j^\gamma+(1-s)\mathbf{T}_{j-1}^\gamma)}\,\mathrm{d}s\,\mathrm{d}x - \alpha h\sum_{j=1}^m(\delta^2\boldsymbol{u}_j^\gamma, \delta^2\boldsymbol{u}_{j-1}^\gamma)\\
&\le \langle \delta l_m^\gamma, \delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma\rangle - \langle \delta l_1^\gamma, \delta\boldsymbol{u}_0^\gamma + \alpha\boldsymbol{u}_0^\gamma\rangle + h\sum_{j=2}^m\langle \delta^2 l_j^\gamma, \delta\boldsymbol{u}_{j-1}^\gamma + \alpha\boldsymbol{u}_{j-1}^\gamma\rangle\\
&\quad + \frac{h}{2\eta}\sum_{j=1}^{m-1}\int_\Omega\int_0^1\left(\frac{b(v_j^\gamma) - b(v_{j-1}^\gamma)}{h}\right)^2(\mathbf{T}_j^\gamma, \mathbf{T}_j^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{j+1}^\gamma+(1-s)\mathbf{T}_j^\gamma)}\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

A corresponding inequality holds in the case $m = 1$ so, from now on, we assume that $1 \le m \le M$. Young's inequality yields

$$\alpha h\sum_{j=1}^m(\delta^2\boldsymbol{u}_j^\gamma, \delta^2\boldsymbol{u}_{j-1}^\gamma) \le C(\alpha)h\sum_{j=0}^{m-1}\|\delta^2\boldsymbol{u}_j^\gamma\|_2^2 + \frac{\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2}{8},$$

and

$$\alpha(\delta^2\boldsymbol{u}_m^\gamma, \delta\boldsymbol{u}_m^\gamma) \le 2\alpha^2\|\delta\boldsymbol{u}_m^\gamma\|_2^2 + \frac{\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2}{8}.$$

It follows that

$$\begin{aligned}
&\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2 + h\sum_{j=1}^m\int_\Omega\int_0^1 b(v_{j-1}^\gamma)\left(\delta\mathbf{T}_j^\gamma, \delta\mathbf{T}_j^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_j^\gamma+(1-s)\mathbf{T}_{j-1}^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\
&\le C\Bigg[\|\delta^2\boldsymbol{u}_0^\gamma\|_2^2 + h\sum_{j=0}^{m-1}\|\delta^2\boldsymbol{u}_j^\gamma\|_2^2 + \|\delta\boldsymbol{u}_m^\gamma\|_2^2 + \|\delta^2\boldsymbol{u}_0^\gamma\|_2^2 + \|\delta l_m^\gamma\|_{-1,2}\|\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma\|_{1,2}\\
&\quad + \|\delta\boldsymbol{u}_0^\gamma\|_2^2 + \|\delta l_1^\gamma\|_{-1,2}\|\delta\boldsymbol{u}_0^\gamma + \alpha\boldsymbol{u}_0^\gamma\|_{1,2} + h\sum_{j=2}^m\|\delta^2 l_j^\gamma\|_{-1,2}\|\delta\boldsymbol{u}_{j-1}^\gamma + \alpha\boldsymbol{u}_{j-1}^\gamma\|_{1,2}\\
&\quad + h\sum_{j=1}^{m-1}\int_\Omega\int_0^1\left(\frac{b(v_j^\gamma) - b(v_{j-1}^\gamma)}{h}\right)^2(\mathbf{T}_j^\gamma, \mathbf{T}_j^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{j+1}^\gamma+(1-s)\mathbf{T}_j^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\Bigg].
\end{aligned}$$

Next, we notice that

$$\mathbf{T}_m^\gamma = h\sum_{j=1}^m\delta\mathbf{T}_j^\gamma + \mathbf{T}_0^\gamma = h\sum_{j=1}^m\delta\mathbf{T}_j^\gamma + F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)),$$

and so

$$\max_{1\le j\le m}\frac{\|\mathbf{T}^\gamma_j\|_2^2}{n} \le C\left[h\sum_{j=1}^m \frac{\|\delta\mathbf{T}^\gamma_j\|_2^2}{n} + \frac{F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))}{n}\right]$$
$$\le C\left[h\sum_{j=1}^m\int_\Omega\int_0^1 b(v^\gamma_{j-1})\left(\delta\mathbf{T}^\gamma_j,\delta\mathbf{T}^\gamma_j\right)_{\mathcal{A}_n(s\mathbf{T}^\gamma_j+(1-s)\mathbf{T}^\gamma_{j-1})}\,\mathrm{d}s\,\mathrm{d}x + 1\right],$$

where C depends only on the data of the original problem. From the definition of $\mathbf{T}^\gamma_m$, we have $|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^\gamma_m+\alpha\boldsymbol{u}^\gamma_m)| \le 1 + n^{-1}|\mathbf{T}^\gamma_m|$ and so

$$\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^\gamma_j+\alpha\boldsymbol{u}^\gamma_j)\|_2^2 \le C\Bigg(1+h\sum_{j=1}^m\int_\Omega\int_0^1 b(v^\gamma_{j-1})\left(\delta\mathbf{T}^\gamma_j,\delta\mathbf{T}^\gamma_j\right)_{\mathcal{A}_n(s\mathbf{T}^\gamma_j+(1-s)\mathbf{T}^\gamma_{j-1})}\,\mathrm{d}s\,\mathrm{d}x\Bigg),$$

for every $1\le m\le M$. Applying the Korn–Poincaré inequality, it follows that

$$\|\delta^2\boldsymbol{u}^\gamma_m\|_2^2 + \max_{1\le j\le m}\|\delta\boldsymbol{u}^\gamma_j+\alpha\boldsymbol{u}^\gamma_j\|_{1,2}^2 + \max_{1\le j\le m}\frac{\|\mathbf{T}^\gamma_j\|_2^2}{n}$$
$$+h\sum_{j=1}^m\int_\Omega\int_0^1 b(v^\gamma_{j-1})\left(\delta\mathbf{T}^\gamma_j,\delta\mathbf{T}^\gamma_j\right)_{\mathcal{A}_n(s\mathbf{T}^\gamma_j+(1-s)\mathbf{T}^\gamma_{j-1})}\,\mathrm{d}s\,\mathrm{d}x$$
$$\le C\Bigg[1+\|\delta^2\boldsymbol{u}^\gamma_0\|_2^2 + h\sum_{j=0}^{m-1}\|\delta^2\boldsymbol{u}^\gamma_j\|_2^2 + \|\delta\boldsymbol{u}^\gamma_m\|_2^2 + \|\delta^2\boldsymbol{u}^\gamma_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|\delta l^\gamma_m\|_{-1,2}^2$$
$$+\|\delta l^\gamma_1\|_{-1,2}^2 + h\sum_{j=1}^{m-1}\int_\Omega\int_0^1\left(\frac{b(v^\gamma_j)-b(v^\gamma_{j-1})}{h}\right)^2(\mathbf{T}^\gamma_j,\mathbf{T}^\gamma_j)_{\mathcal{A}_n(s\mathbf{T}^\gamma_{j+1}+(1-s)\mathbf{T}^\gamma_j)}\,\mathrm{d}s\,\mathrm{d}x$$
$$+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2 + \left(h\sum_{j=2}^m\|\delta^2 l^\gamma_j\|_{-1,2}\right)^2\Bigg] + \frac{1}{2}\max_{1\le j\le m}\|\delta\boldsymbol{u}^\gamma_j+\alpha\boldsymbol{u}^\gamma_j\|_{1,2}^2.$$

Applying the discrete Gronwall inequality, we see that

$$\max_{1\le m\le M}\left(\|\delta^2\boldsymbol{u}^\gamma_m\|_2^2 + \|\delta\boldsymbol{u}^\gamma_m++\alpha\boldsymbol{u}^\gamma_m\|_{1,2}^2 + \frac{\|\mathbf{T}^\gamma_m\|_2^2}{n}\right)$$
$$+h\sum_{j=1}^M\int_\Omega\int_0^1 b(v^\gamma_{j-1})\left(\delta\mathbf{T}^\gamma_j,\delta\mathbf{T}^\gamma_j\right)_{\mathcal{A}_n(s\mathbf{T}^\gamma_j+(1-s)\mathbf{T}^\gamma_{j-1})}\,\mathrm{d}s\,\mathrm{d}x$$
$$\le C\Bigg[\|\delta^2\boldsymbol{u}^\gamma_0\|_2^2 + \max_{1\le m\le M}\left(\|\delta\boldsymbol{u}^\gamma_m\|_2^2 + \|\delta l^\gamma_m\|_{-1,2}^2\right) + \|\boldsymbol{u}_1\|_2^2 + \left(h\sum_{j=2}^M\|\delta^2 l^\gamma_j\|_{-1,2}\right)^2$$
$$+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{1,2}^2 + h\sum_{j=1}^{M-1}\int_\Omega\int_0^1\left(\frac{b(v^\gamma_j)-b(v^\gamma_{j-1})}{h}\right)^2(\mathbf{T}^\gamma_j,\mathbf{T}^\gamma_j)_{\mathcal{A}_n(s\mathbf{T}^\gamma_{j+1}+(1-s)\mathbf{T}^\gamma_j)}\,\mathrm{d}s\,\mathrm{d}x\Bigg]. \tag{6.26}$$

To bound $\max_{1\le m\le M}\|\delta\boldsymbol{u}^\gamma_m\|_2^2$ from above, independent of n and M, we test in the elastodynamic equation (6.12) against $\boldsymbol{w} = h(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma)$ to get

$$
\begin{aligned}
&\langle l_m^\gamma, h(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma)\rangle \\
&= \int_\Omega \delta^2\boldsymbol{u}_m^\gamma \cdot h(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma) + b(v_{m-1}^\gamma)F_n^{-1}(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma)) \cdot \boldsymbol{\varepsilon}(h(\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma))\,\mathrm{d}x \\
&= \frac{1}{2}\left(\|\delta\boldsymbol{u}_m^\gamma\|_2^2 - \|\delta\boldsymbol{u}_{m-1}^\gamma\|_2^2 + \|\delta\boldsymbol{u}_m^\gamma - \delta\boldsymbol{u}_{m-1}^\gamma\|_2^2\right) + \alpha(\delta\boldsymbol{u}_m^\gamma, \boldsymbol{u}_m^\gamma) \\
&\quad - \alpha(\delta\boldsymbol{u}_{m-1}^\gamma, \boldsymbol{u}_{m-1}^\gamma) - \alpha h(\delta\boldsymbol{u}_m^\gamma, \delta\boldsymbol{u}_{m-1}^\gamma) + h\int_\Omega b(v_{m-1}^\gamma)\mathbf{T}_m^\gamma \cdot F_n(\mathbf{T}_m^\gamma)\,\mathrm{d}x.
\end{aligned}
$$

Using this recursively, we deduce that

$$
\begin{aligned}
&\frac{\|\delta\boldsymbol{u}_m^\gamma\|_2^2}{2} + h\sum_{j=1}^m \int_\Omega b(v_{j-1}^\gamma)\mathbf{T}_j^\gamma \cdot F_n^{-1}(\mathbf{T}_j^\gamma)\,\mathrm{d}x \\
&\le \frac{\|\delta\boldsymbol{u}_0^\gamma\|_2^2}{2} + \alpha(\delta\boldsymbol{u}_0^\gamma, \boldsymbol{u}_0^\gamma) - \alpha(\delta\boldsymbol{u}_m^\gamma, \boldsymbol{u}_m^\gamma) + \alpha h\sum_{j=1}^m(\delta\boldsymbol{u}_j^\gamma, \delta\boldsymbol{u}_{j-1}^\gamma) + h\sum_{j=1}^m\langle l_j^\gamma, \delta\boldsymbol{u}_j^\gamma + \alpha\boldsymbol{u}_j^\gamma\rangle \\
&\le C\left[\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2\right] - \alpha\left(\delta\boldsymbol{u}_m^\gamma, \boldsymbol{u}_0^\gamma + h\sum_{j=1}^m \delta\boldsymbol{u}_j^\gamma\right) + Ch\sum_{j=1}^m \|l_j^\gamma\|_{-1,2}\|\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_j^\gamma + \alpha\boldsymbol{u}_j^\gamma)\|_2 \\
&\le C\left[\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2\right] - \alpha\|\delta\boldsymbol{u}_m^\gamma\|_2^2 - \alpha\left(\delta\boldsymbol{u}_m^\gamma, h\sum_{j=1}^{m-1}\delta\boldsymbol{u}_j^\gamma\right) + Ch\sum_{j=1}^m\|l_j^\gamma\|_{-1,2}\left(1 + \frac{\|\mathbf{T}_j^\gamma\|_2}{n}\right) \\
&\le C\left[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^2(W^{-1,2})} + h\sum_{j=1}^{m-1}\|\delta\boldsymbol{u}_j^\gamma\|_2^2\right] + \frac{\|\delta\boldsymbol{u}_m^\gamma\|_2^2}{4} + \frac{h\eta}{2}\sum_{j=1}^m\frac{\|\mathbf{T}_j^\gamma\|_2^2}{n}.
\end{aligned}
$$

Absorbing the final two terms on the right into the left-hand side, we apply the discrete Gronwall inequality to get

$$
\max_{1\le m\le M}\|\delta\boldsymbol{u}_m^\gamma\|_2^2 + h\sum_{m=1}^M\int_\Omega |\mathbf{T}_m^\gamma| + \frac{|\mathbf{T}_m^\gamma|^2}{n}\,\mathrm{d}x \le C\left[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^2(W^{-1,2})}\right].
$$

In fact, we further have that

$$
\max_{1\le m\le M}\|\boldsymbol{u}_m^\gamma\|_2^2 + h\sum_{j=1}^M\|\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma\|_{1,2}^2 \le C\left[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^2(W^{-1,2})}^2\right].
$$

Returning to (6.26), we see that, for every $1 \le m \le M$,

$$
\begin{aligned}
&\left(\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2 + \|\delta\boldsymbol{u}_m^\gamma + \alpha\boldsymbol{u}_m^\gamma\|_{1,2}^2\right) + h\sum_{j=1}^M\int_\Omega\int_0^1 b(v_{j-1}^\gamma)\left(\delta\mathbf{T}_j^\gamma, \delta\mathbf{T}_j^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_j^\gamma + (1-s)\mathbf{T}_{j-1}^\gamma)} \\
&\le C\Bigg[1 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2}^2 + \|v_0\|_{1,2}^2 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^2(W^{-1,2})}^2 + \max_{1\le m\le M}\|\delta l_m^\gamma\|_{-1,2}^2 \qquad (6.27) \\
&\quad + \left(h\sum_{m=2}^M\|\delta^2 l_m^\gamma\|_{-1,2}^2\right)^2 + h\sum_{m=1}^{M-1}\int_\Omega\int_0^1(\delta b(v_m^\gamma))^2(\mathbf{T}_m^\gamma, \mathbf{T}_m^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{m+1}^\gamma + (1-s)\mathbf{T}_m^\gamma)}\Bigg].
\end{aligned}
$$

Since $l \in W^{2,1}(0,T;W_D^{-1,2}(\Omega)^d)$, we must have $l \in W^{1,\infty}(0,T;W_D^{-1,2}(\Omega)^d)$. There exists a constant C, independent of n, such that

$$
\begin{aligned}
&\|l_t^n\|_{L^\infty(W^{-1,2})} \le C\left(1 + \|l_t\|_{L^\infty(W^{-1,2})}\right), \quad \text{and} \\
&\|l_{tt}^n\|_{L^1(W^{-1,2})} \le C\left(1 + \|l_t\|_{L^\infty(W^{-1,2})} + \|l_{tt}\|_{L^1(W^{-1,2})}\right).
\end{aligned}
$$

It follows that

$$\max_{1\le m\le M}\|\delta l_m^\gamma\|_{-1,2}^2+\left(h\sum_{m=2}^{M}\|\delta^2 l_m^\gamma\|_{-1,2}\right)^2\le C\left(1+\|l_t\|_{L^\infty(W^{-1,2})}+\|l_{tt}\|_{L^1(W^{-1,2})}\right).$$

Next, we consider the final term on the right-hand of (6.27). By the fundamental theorem of calculus for weak derivatives, we have

$$\begin{aligned}|\delta b(v_m^\gamma)|=\left|\frac{b(v_m^\gamma)-b(v_{m-1}^\gamma)}{h}\right|&=\left|\frac{1}{h}\int_0^1 b'(sv_m^\gamma+(1-s)v_{m-1}^\gamma)(v_m^\gamma-v_{m-1}^\gamma)\,\mathrm{d}s\right|\\&\le\|b'\|_{L^\infty(I_{\|v^\gamma\|_\infty,*})}|\delta v_m^\gamma|,\end{aligned}$$

using the notation of Lemma 6.7. Applying Lemma 6.7, there exists a positive constant $C=C(n)$, possibly depending on n but independent of M, such that

$$\begin{aligned}&\max_{1\le m\le M}\|\delta v_m^\gamma\|_\infty\le C\max_{1\le m\le M}\|\delta v_m^\gamma\|_{k,2}\\&\le C\left[1+\|b'\|_{L^\infty(I_{\|v^\gamma\|_\infty,*})}\right]\Big[1+(1+\|b(v_0)\|_\infty)\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2\\&\quad+\|l\|_{L^\infty(W^{-1,2})}^2+\|l_t\|_{L^2(W^{-1,2})}^2\Big]=:C_1(n).\end{aligned}$$

It follows that

$$\begin{aligned}&h\sum_{m=1}^{M-1}\int_\Omega\int_0^1\left(\frac{b(v_m^\gamma)-b(v_{m-1}^\gamma)}{h}\right)^2(\mathbf{T}_m^\gamma,\mathbf{T}_m^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{m+1}^\gamma+(1-s)\mathbf{T}_m^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\&\le C_1(n)^2h\sum_{m=1}^{M}\int_\Omega\int_0^1(\mathbf{T}_m^\gamma,\mathbf{T}_m^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{m+1}^\gamma+(1-s)\mathbf{T}_m^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\&\le C_1(n)^2hn\sum_{m=1}^{M}\int_\Omega\frac{|\mathbf{T}_m^\gamma|^2}{n}\,\mathrm{d}x\\&\le C_1(n)^2Cn\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|_{L^2(W^{-1,2})}^2\right],\end{aligned}$$

using Lemma 6.5 in the transition to the final line. It follows that

$$\begin{aligned}&\max_{1\le m\le M}\|\delta^2\boldsymbol{u}_m^\gamma\|_2^2+\max_{1\le m\le M}\|\delta\boldsymbol{u}_m^\gamma+\alpha\boldsymbol{u}_m^\gamma\|_{1,2}^2+\max_{1\le m\le M}\|\mathbf{T}_m^\gamma\|_2^2\\&\quad+h\sum_{j=1}^{M}\int_\Omega\int_0^1 b(v_{j-1}^\gamma)\left(\delta\mathbf{T}_j^\gamma,\delta\mathbf{T}_j^\gamma\right)_{\mathcal{A}_n(s\mathbf{T}_j^\gamma+(1-s)\mathbf{T}_{j-1}^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\&\le C(n)\left[C_1(n)^2n+1\right]\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|_{L^2(W^{-1,2})}^2\right],\end{aligned}$$

where C depends on n and the problem parameters but is independent of M. Rewriting this in terms of the interpolant functions, we get

$$\begin{aligned}&\sup_{t\in[0,T]}\|\overline{\boldsymbol{u}}_t^{\gamma,\prime}(t)\|_2^2+\sup_{t\in[0,T]}\|\overline{\boldsymbol{u}}_t^\gamma+\alpha\boldsymbol{u}^{\gamma,+}\|_{1,2}^2+\int_Q\frac{|\overline{\mathbf{T}}_t^\gamma|^2}{n}\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in[0,T]}\|\overline{\mathbf{T}}^\gamma(t)\|_2^2\\&\le C\left[C_1(n)^2n+1\right]\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|_{L^2(W^{-1,2})}^2\right],\end{aligned}$$

using the fact that $n^{-1}|\mathbf{T}|^2\le(\mathbf{T},\mathbf{T})_{\mathcal{A}_n(\mathbf{S})}$ for every $\mathbf{T}$, $\mathbf{S}\in\mathbb{R}^{d\times d}$. By standard compactness arguments, there exists a subsequence in M, independent of n, such that

- $\overline{\boldsymbol{u}}^{\gamma,\prime} \overset{*}{\rightharpoonup} \boldsymbol{u}_t^n$ weakly-* in $W^{1,\infty}(0,T;L^2(\Omega)^d)$,
- $\overline{\boldsymbol{u}}_t^\gamma + \alpha \boldsymbol{u}^{\gamma,+} \overset{*}{\rightharpoonup} \boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n$ weakly-* in $L^\infty(0,T;W_D^{1,2}(\Omega)^d)$,
- $\overline{\mathbf{T}}^\gamma \rightharpoonup \mathbf{T}$ weakly in $W^{1,2}(0,T;L^2(\Omega)^{d\times d})$ and weakly-* in $L^\infty(0,T;L^2(\Omega)^{d\times d})$.

Recalling the strong convergence results from Theorem 6.9, $(\overline{\boldsymbol{u}}_t^\gamma + \alpha \boldsymbol{u}^{\gamma,+})_M$ converges strongly in $L^2(0,T;W_D^{1,2}(\Omega)^d)$ to $\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n$. It follows that $\mathbf{T}^{\gamma,+} \to \mathbf{T}^n$ strongly in $L^2(Q)^{d\times d}$. Similarly, we have that $\mathbf{T}^{\gamma,-} \to \mathbf{T}^n$ strongly in $L^2(Q)^{d\times d}$. Furthermore, we can write

$$(\mathbf{T}^{\gamma,+}, \mathbf{T}^{\gamma,+})_{\mathcal{A}_n(\mathbf{S})} = \frac{|\mathbf{T}^{\gamma,+}|^2}{n} + (\mathbf{T}^{\gamma,+}, \mathbf{T}^{\gamma,+})_{\mathcal{A}(\mathbf{S})}, \tag{6.28}$$

where $(\cdot,\cdot)_{\mathcal{A}(\mathbf{S})}$ is the inner product corresponding to the fourth-order tensor $\mathcal{A}(\mathbf{S})$ defined by $\mathcal{A}(\mathbf{S})_{ijkl} = \frac{\partial F(\mathbf{S})_{ij}}{\partial \mathbf{S}_{kl}}$, and recalling $\mathcal{A}_n$ defined by (6.22). We claim that if $(\mathbf{T}^M)_M$, $(\mathbf{S}^M)_M$ converge strongly to limits $\mathbf{T}$, $\mathbf{S}$, respectively, in $L^2(Q)^{d\times d}$, then

$$(\mathbf{T}^M, \mathbf{T}^M)_{\mathcal{A}_n(\mathbf{S}^M)} \to (\mathbf{T}, \mathbf{T})_{\mathcal{A}_n(\mathbf{S})} \quad \text{strongly in } L^1(Q)^{d\times d}. \tag{6.29}$$

By (6.28), it is sufficient to show that the result holds for $\mathcal{A}$ rather than $\mathcal{A}_n$. We have that

$$\begin{aligned}
&(\mathbf{T}^M, \mathbf{T}^M)_{\mathcal{A}(\mathbf{S}^M)} - (\mathbf{T}, \mathbf{T})_{\mathcal{A}(\mathbf{S})} \\
&= (\mathbf{T}^M, \mathbf{T}^M)_{\mathcal{A}(\mathbf{S}^M)} - (\mathbf{T}, \mathbf{T})_{\mathcal{A}(\mathbf{S}^M)} + (\mathbf{T}, \mathbf{T})_{\mathcal{A}(\mathbf{S}^M)} - (\mathbf{T}, \mathbf{T})_{\mathcal{A}(\mathbf{S})} \\
&= \frac{|\mathbf{T}^M|^2 - |\mathbf{T}|^2}{(1+|\mathbf{S}^M|^a)^{\frac{1}{a}}} + \frac{|\mathbf{S}^M|^{a-2}}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}} \left[(\mathbf{S}^M \cdot \mathbf{T}^M)^2 - (\mathbf{S}^M \cdot \mathbf{T})^2\right] \\
&\quad + |\mathbf{T}|^2 \left[\frac{1}{(1+|\mathbf{S}^M|^a)^{\frac{1}{a}}} - \frac{1}{(1+|\mathbf{S}|^a)^{\frac{1}{a}}}\right] + \frac{|\mathbf{S}|^{a-2}(\mathbf{S}\cdot\mathbf{T})^2}{(1+|\mathbf{S}|^a)^{1+\frac{1}{a}}} - \frac{|\mathbf{S}^M|^{a-2}(\mathbf{S}^M\cdot\mathbf{T})^2}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}}.
\end{aligned}$$

Using the dominated convergence theorem and the fact that $\mathbf{T} \in L^2(\Omega)^{d\times d}$, we see that

$$|\mathbf{T}|^2 \left[\frac{1}{(1+|\mathbf{S}^M|^a)^{\frac{1}{a}}} - \frac{1}{(1+|\mathbf{S}|^a)^{\frac{1}{a}}}\right] + \frac{|\mathbf{S}|^{a-2}(\mathbf{S}\cdot\mathbf{T})^2}{(1+|\mathbf{S}|^a)^{1+\frac{1}{a}}} - \frac{|\mathbf{S}^M|^{a-2}(\mathbf{S}^M\cdot\mathbf{T})^2}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}} \to 0,$$

strongly in $L^1(Q)$. Similarly, we have that

$$\frac{|\mathbf{T}^M|^2 - |\mathbf{T}|^2}{(1+|\mathbf{S}^M|^a)^{\frac{1}{a}}} \to 0 \quad \text{strongly in } L^1(Q).$$

For the remaining term, write

$$\begin{aligned}
&\frac{|\mathbf{S}^M|^{a-2}}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}} \left[(\mathbf{S}^M \cdot \mathbf{T}^M)^2 - (\mathbf{S}^M \cdot \mathbf{T})^2\right] \\
&= \frac{|\mathbf{S}^M|^a}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}} \left[\left(\frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}^M\right)^2 - \left(\frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}\right)^2\right] \\
&= \frac{|\mathbf{S}^M|^a}{(1+|\mathbf{S}^M|^a)^{1+\frac{1}{a}}} \left[\frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}^M - \frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}\right] \left[\frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}^M + \frac{\mathbf{S}^M}{|\mathbf{S}^M|} \cdot \mathbf{T}\right] \\
&\leq |\mathbf{T}^M - \mathbf{T}||\mathbf{T}^M + \mathbf{T}|.
\end{aligned}$$

The factors on the right-hand side converge strongly in $L^2(Q)$ to 0 and $2|\mathbf{T}|$ respectively. Hence the product converges in $L^1(Q)$ to 0. Thus (6.29) follows and we conclude that

$$\chi_{(0,T-h)}(t)\int_0^1 (\mathbf{T}^{\gamma,+},\mathbf{T}^{\gamma,+})_{\mathcal{A}_n(s\mathbf{T}^{\gamma,+}(\cdot+h)+(1-s)\mathbf{T}^{\gamma,+})}\,\mathrm{d}s \to (\mathbf{T}^n,\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}$$

strongly in $L^1(Q)$ as $M\to\infty$.

On the other hand, recalling the uniform bound on $(v_m^\gamma)_{m=0}^M$ with respect to M and the pointwise convergence of $(v^{\gamma,\pm})_M$, we have that

$$\int_0^1 b'(sv^{\gamma,+}+(1-s)v^{\gamma,-})\overline{v}_t^\gamma\,\mathrm{d}s \overset{*}{\rightharpoonup} b'(v^n)v_t^n,$$

weakly-* in $L^\infty(Q)$. It follows that

$$\begin{aligned}
&h\sum_{m=1}^{M-1}\int_\Omega\int_0^1\left(\frac{b(v_m^\gamma)-b(v_{m-1}^\gamma)}{h}\right)^2(\mathbf{T}_m^\gamma,\mathbf{T}_m^\gamma)_{\mathcal{A}_n(s\mathbf{T}_{m+1}^\gamma+(1-s)\mathbf{T}_m^\gamma)}\,\mathrm{d}s\,\mathrm{d}x\\
&=\int_0^{T-h}\int_\Omega\left(\int_0^1 b'(sv^{\gamma,+}+(1-s)v^{\gamma,-})\overline{v}_t^\gamma\right)\left(\int_0^1(\mathbf{T}^{\gamma,+},\mathbf{T}^{\gamma,+})_{\mathcal{A}_n(s\mathbf{T}^{\gamma,+}(\cdot+h)+(1-s)\mathbf{T}^{\gamma,+}}\right)\\
&\to\int_Q b'(v^n)v_t^n(\mathbf{T}^n,\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

as $M\to\infty$. Taking the limit in (6.26), we apply weak lower semi-continuity and the weak convergence results to deduce that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^n(t)\|_2^2+\sup_{t\in[0,T]}\|\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n\|_{1,2}^2+\int_Q\frac{|\mathbf{T}_t^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in[0,T]}\frac{\|\mathbf{T}^n(t)\|_2^2}{n}\\
&\le C\Big[1+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}^2+\|v_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|_{L^\infty(W^{-1,2})}\\
&\quad+\|l_{tt}\|_{L^1(W^{-1,2})}^2+\int_Q b'(v^n)v_t^n(\mathbf{T}^n,\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t\Big],
\end{aligned}$$

where the constant C is independent of n. Using the n-independent bounds of Lemma 6.11 with Lemma 6.13, we get

$$\begin{aligned}
&\int_Q b'(v^n)v_t^n(\mathbf{T}^n,\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\le\|b'(v_n)\|_{L^\infty(Q)}\|v_t^n\|_{L^\infty(Q)}\int_Q\frac{|\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}+\frac{|\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|v_t^n\|_{L^\infty(H^k)}\left[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_1\|_2^2+\|l\|_{L^2(W^{-1,2})}^2\right]\\
&\le C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\big[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2\\
&\quad+\|b(v_0)\|_\infty+\|l\|_{L^\infty(W^{-1,2})}^2+\|l_t\|_{L^2(W^{-1,2})}^2\big]^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^n(t)\|_2^2+\sup_{t\in[0,T]}\|\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n\|_{1,2}+\int_Q\frac{|\mathbf{T}_t^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in[0,T]}\frac{\|\mathbf{T}^n(t)\|_2^2}{n}\\
&\le C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\big[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|b(v_0)\|_\infty\\
&\quad+\|l\|_{L^\infty(W^{-1,2})}^2+\|l_t\|_{L^2(W^{-1,2})}^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}+\|l_{tt}\|_{L^1(W^{-1,2})}\big]^2.
\end{aligned}$$

The sequence $(v^n)_n$ is uniformly bounded in $L^\infty(Q)$ with respect to n and b is bounded on bounded sets. Thus $\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}$ is bounded above independent of n and the right-hand side can be bounded above by an n-independent constant C_2, say.

Using the extra regularity of $\mathbf{T}^n$, we now look for an n-uniform estimate on $\mathbf{T}^n_t$. Since we have $\mathbf{T}^n \in W^{1,2}(0,T;L^2(\Omega)^{d\times d})$, then also $F_n(\mathbf{T}^n) \in W^{1.2}(0,T;L^2(\Omega)^{d\times d})$ with weak derivative $\mathcal{A}_n(\mathbf{T}^n)\mathbf{T}^n_t$. For a.e. $t \in (0,T)$, we note that

$$\lim_{h\to 0+} \frac{\mathbf{T}^n(t+h) - \mathbf{T}^n(t)}{h} = \mathbf{T}^n_t, \quad \lim_{h\to 0+} \frac{F_n(\mathbf{T}^n(t+h)) - F_n(\mathbf{T}^n(t))}{h} = \mathcal{A}_n(\mathbf{T}^n)\mathbf{T}^n_t, \tag{6.30}$$

where the limits are with respect to strong convergence in $L^2(\Omega)^{d\times d}$. Let Δ^h_t be the undivided difference quotient of length h in the time variable. Let $t \in (0,T)$. Assume that $h > 0$ is sufficiently small so that $t + h < T$. Considering the elastodynamic equation (6.16) at t and $t+h$ with test function $\Delta^h_t(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n)(t)$, the difference of the two yields

$$\begin{aligned}
&\langle \Delta^h_t l^n(t), \Delta^h_t(\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n_t)(t)\rangle \\
&= \int_\Omega \Delta^h_t \boldsymbol{u}^n_{tt}(t)\cdot\Delta^h_t(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)(t) + \Delta^h_t\left(b(v^n)\mathbf{T}^n\right)(t)\cdot\boldsymbol{\varepsilon}(\Delta^h_t(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n))(t)\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\Delta^h_t\boldsymbol{u}^n_t(t)\|_2^2}{2} + \alpha\int_\Omega \Delta^h_t\boldsymbol{u}^n_t\cdot\Delta^h_t\boldsymbol{u}^n\,\mathrm{d}x\right) - \alpha\int_\Omega |\Delta^h_t\boldsymbol{u}^n_t|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \Delta^h_t b(v^n)\tau^h_t\mathbf{T}^n\cdot\Delta^h_t F_n(\mathbf{T}^n) + b(v^n)\Delta^h_t\mathbf{T}^n\cdot\Delta^h_t F_n(\mathbf{T}^n)\,\mathrm{d}x,
\end{aligned} \tag{6.31}$$

where τ^h_t denotes a translation of distance h in the positive time direction. From the uniform $L^\infty(Q)$ bounds with respect to n on $(v^n)_n$ and $(v^n_t)_n$, there exists a constant C, independent of h and n, such that

$$\begin{aligned}
|\Delta^h_t b(v^n)| &= \left|\int_0^h b'(v^n(t+s))v^n_t(t+s)\,\mathrm{d}s\right| \le h\|b'(v^n)\|_{L^\infty(Q)}\|v^n_t\|_{L^\infty(Q)} \\
&\le Ch\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_\infty})}\right]^2\Big[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2 \\
&\qquad + \|b(v_0)\|_\infty + \|l\|^2_{L^\infty(W^{-1,2})} + \|l_t\|^2_{L^2(W^{-1,2})}\Big] \le C_2.
\end{aligned}$$

Recalling that $(\cdot,\cdot)_{\mathcal{A}_n(\mathbf{T})}$ is an inner product for every $\mathbf{T}\in\mathbb{R}^{d\times d}$, it follows that

$$\begin{aligned}
\left|\int_\Omega \Delta^h_t b(v^n)\tau^h_t\mathbf{T}^n\cdot\Delta^h_t F_n(\mathbf{T}^n)\,\mathrm{d}x\right| &\le CC_2h\int_\Omega\int_0^1\left|(\tau^h_t\mathbf{T}^n,\Delta^h_t\mathbf{T}^n)_{\mathcal{A}_n(s\tau^h_t\mathbf{T}^n+(1-s)\mathbf{T}^n)}\right|\,\mathrm{d}s\,\mathrm{d}x \\
&\le CC_2^2h^2\int_\Omega\int_0^1(\tau^h_t\mathbf{T}^n,\tau^h_t\mathbf{T}^n)_{\mathcal{A}_n(s\tau^h_t\mathbf{T}^n+(1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \frac{\eta}{2}\int_\Omega\int_0^1(\Delta^h_t\mathbf{T}^n,\Delta^h_t\mathbf{T}^n)_{\mathcal{A}_n(s\tau^h_t\mathbf{T}^n+(1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

There exists a constant C, independent of n, such that

$$|F_n(\mathbf{T}) - F_n(\mathbf{S})|^2 \le C(\mathbf{T}-\mathbf{S})\cdot(F_n(\mathbf{T}) - F_n(\mathbf{S})).$$

As a result, $|\Delta_t^h F_n(\mathbf{T}^n)|^2 \le C\Delta_t^h \mathbf{T}^n \cdot \Delta_t^h F_n(\mathbf{T}^n)$. Using this and the constitutive relation (6.8b), the external force term can be bound in the following way:

$$\begin{aligned}|\langle \Delta_t^h l^n, \Delta_t^h(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)\rangle| &\le C\|\Delta_t^h l^n\|_{-1,2}\|\Delta_t^h \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\|_2 \\ &\le C\|\Delta_t^h l^n\|_{-1,2}^2 + \frac{\eta}{4}\int_\Omega \Delta_t^h \mathbf{T}^n \cdot \Delta_t^h F_n(\mathbf{T}^n)\,\mathrm{d}x,\end{aligned}$$

where C is independent of n. Returning to (6.31), we integrate over $(t_1, t_2) \subset [0, T]$ to deduce that

$$\begin{aligned}&\|\Delta_t^h \boldsymbol{u}_t^n(t_2)\|_2^2 + \int_{t_1}^{t_2}\int_\Omega b(v^n)\Delta_t^h\mathbf{T}^n \cdot \Delta_t^h F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\bigg[\int_{t_1}^{t_2} \|\Delta_t^h l^n\|_{-1,2}^2\,\mathrm{d}t + h^2\int_{t_1}^{t_2}\int_\Omega\int_0^1 (\tau_t^h\mathbf{T}^n, \tau_t^h\mathbf{T}^n)_{\mathcal{A}_n(s\tau_t^h\mathbf{T}^n + (1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2}\|\Delta_t^h \boldsymbol{u}_t^n(t)\|_2^2\,\mathrm{d}t + \|\Delta_t^h\boldsymbol{u}_t^n(t_1)\|_2^2 + \|\Delta_t^n \boldsymbol{u}^n(t_1)\|_2^2\bigg].\end{aligned}$$

We divide through by h^2 and use standard properties of difference quotients to get

$$\begin{aligned}&\frac{\|\Delta_t^h\boldsymbol{u}_t^n(t_2)\|_2^2}{h^2} + \int_{t_1}^{t_2}\int_\Omega b(v^n)\frac{\Delta_t^h\mathbf{T}^n}{h}\cdot\frac{\Delta_t^h F_n(\mathbf{T}^n)}{h}\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\bigg[\int_0^T \|l_t^n\|_{-1,2}^2\,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega\int_0^1(\tau_t^h\mathbf{T}^n, \tau_t^h\mathbf{T}^n)_{\mathcal{A}_n(s\tau_t^h\mathbf{T}^n+(1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sup_{t\in[0,T]}\|\boldsymbol{u}_{tt}^n(t)\|_2^2 + \sup_{t\in[0,T]}\|\boldsymbol{u}_t^n(t)\|_2^2\bigg] \\ &\le C\left[1 + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\big[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|b(v_0)\|_\infty \\ &\quad + \|l\|_{L^\infty(W^{-1,2})} + \|l_t\|_{L^2(W^{-1,2})}^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2} + \|l_{tt}\|_{L^1(W^{-1,2})}\big]^2 \\ &\quad + C\int_{t_1}^{t_2}\int_\Omega\int_0^1 (\tau_t^h\mathbf{T}^n, \tau_t^h\mathbf{T}^n)_{\mathcal{A}_n(s\tau_t^h\mathbf{T}^n + (1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where C is independent of n. In the limit as $h \to 0+$, we have that $\tau_t^h\mathbf{T}^n \to \mathbf{T}^n$ strongly in $L^2((t_1, t_2)\times\Omega)^{d\times d}$. Using this and the dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{h\to 0+}\int_{t_1}^{t_2}\int_\Omega\int_0^1 (\tau_t^h\mathbf{T}^n, \tau_t^h\mathbf{T}^n)_{\mathcal{A}_n(s\tau_t^h\mathbf{T}^n+(1-s)\mathbf{T}^n)}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t &= \int_{t_1}^{t_2}\int_\Omega (\mathbf{T}^n, \mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t \\ \le C\int_{t_1}^{t_2}\int_\Omega |\mathbf{T}^n| + \frac{|\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t &\le C\left[1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{u}_1\|_2^2 + \|l\|_{L^2(W^{-1,2})}^2\right].\end{aligned}$$

It follows that

$$\begin{aligned}&\limsup_{h\to 0+}\frac{\|\Delta_t^h\boldsymbol{u}_t^n(t_2)\|_2^2}{h^2} + \limsup_{h\to 0+}\int_{t_1}^{t_2}\int_\Omega b(v^n)\frac{\Delta_t^h\mathbf{T}^n}{h}\cdot\frac{\Delta_t^h F_n(\mathbf{T}^n)}{h}\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\left[1 + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\big[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|b(v_0)\|_\infty \\ &\quad + \|l\|_{L^\infty(W^{-1,2})} + \|l_t\|_{L^2(W^{-1,2})}^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2} + \|l_{tt}\|_{L^1(W^{-1,2})}\big]^2,\end{aligned}$$

where C depends only on the parameters of the original problem. Using (6.30) with Fatou's lemma and the monotonicity of F_n, we have that

$$\limsup_{h\to 0+}\int_{t_1}^{t_2}\int_\Omega b(v^n)\frac{\Delta_t^h \mathbf{T}^n}{h}\cdot\frac{\Delta_t^h F_n(\mathbf{T}^n)}{h}\,\mathrm{d}x\,\mathrm{d}t \geq \int_{t_1}^{t_2}\int_\Omega b(v^n)\mathbf{T}_t^n\cdot\mathcal{A}_n(\mathbf{T}^n)\mathbf{T}_t^n\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2}\int_\Omega b(v^n)(\mathbf{T}_t^n,\mathbf{T}_t^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t.$$

As with (6.30), for a.e. $t\in(0,T)$, we have that

$$\frac{\boldsymbol{u}_t^n(t+h)-\boldsymbol{u}_t^n(t)}{h}\to \boldsymbol{u}_{tt}^n(t)\quad \text{strongly in } L^2(\Omega)^d.$$

Taking $t_1\to 0+$ and setting $t_2=t$, we deduce that, for a.e. $t\in(0,T)$,

$$\begin{aligned}&\|\boldsymbol{u}_{tt}^n(t)\|_2^2+\int_0^t\int_\Omega b(v^n)(\mathbf{T}_t^n,\mathbf{T}_t^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}s\\ &\leq C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\left[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|b(v_0)\|_\infty\right.\\ &\quad\left.+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|_{L^2(W^{-1,2})}^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}+\|l_{tt}\|_{L^1(W^{-1,2})}\right]^2.\end{aligned}$$

Taking the essential supremum over the left-hand side with respect to t and using the definition of $\mathcal{A}_n$, the result follows. □

Corollary 6.15. *Let the hypotheses of Proposition 6.14 hold. There exists a constant C, independent of n, such that*

$$\begin{aligned}\int_0^T\|\boldsymbol{u}_{tt}^n(t)\|_{1,2}^2\,\mathrm{d}t &\leq C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\left[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2\right.\\ &\left.+\|b(v_0)\|_\infty+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|_{L^2(W^{-1,2})}^2+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}+\|l_{tt}\|_{L^1(W^{-1,2})}\right]^2.\end{aligned}$$

Proof. From the constitutive relation, we have

$$|\Delta_t^h\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)|^2=|\Delta_t^h F_n(\mathbf{T}^n)|^2\leq C\Delta_t^h\mathbf{T}^n\cdot\Delta_t^h F_n(\mathbf{T}^n).$$

It follows that $h^{-1}\Delta_t^h\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)$ is uniformly bounded in $L^2((t_1,t_2)\times\Omega)^d$ with respect to h for any fixed interval (t_1,t_2) that is compactly contained in $(0,T)$. We deduce that $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\in W^{1,2}(0,T;L^2(\Omega)^{d\times d})$. By the Korn–Poincaré inequality (Theorem 1.9), in fact $\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n\in W^{1,2}(0,T;W_D^{1,2}(\Omega)^d)$. However, $\boldsymbol{u}_t^n\in L^2(0,T;W_D^{1,2}(\Omega)^{d\times d})$. Thus $\boldsymbol{u}_{tt}^n\in L^2(0,T;W_D^{1,2}(\Omega)^d)$. The stated bound can be obtained from applying the estimates from the previous proposition with Lemma 6.11. □

With this extra time regularity, we can improve the $L^1(Q)^{d\times d}$ uniform bound on $(\mathbf{T}^n)_n$ to a bound in $L^\infty(0,T;L^1(\Omega)^{d\times d})$. The space $L^\infty(0,T;L^1(\Omega)^{d\times d})$ embeds continuously into $L^\infty_{w*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$, which is the dual of $L^1(0,T;C(\overline{\Omega})^{d\times d})$. By the sequential form of the Banach–Alaoglu Theorem, a bounded sequence in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ has a weakly-* convergent subsequence in $L^\infty_{w*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$. The measure is only seen on the spatial domain. If we only had an $L^1(Q)^{d\times d}$ bound, the best weakly-* convergent subsequence would be in $\mathcal{M}(\overline{Q})^{d\times d}$, which provides significantly less information.

Lemma 6.16. *Let the hypotheses of Proposition 6.14 hold and let $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ be the triple constructed in the proof of Theorem 6.9. There exists a constant C, independent of n, such that*

$$\sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)|\,\mathrm{d}x \le C\left[1 + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2 \left[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 \right.$$
$$\left. + \|b(v_0)\|_\infty + \|l\|_{L^\infty(W^{-1,2})} + \|l_t\|_{L^2(W^{-1,2})}^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2} + \|l_{tt}\|_{L^1(W^{-1,2})}\right]^2.$$

Proof. Fix $t \in (0,T)$ and $h > 0$ such that $t + h < T$. Test in the elastodynamic equation (6.16) at times t and $t+h$ against $\boldsymbol{v} = (\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)(t)$. The difference of the two equalities yields

$$\langle \Delta_t^h l(t), (\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)(t)\rangle = \int_\Omega \frac{\partial}{\partial t}\left(\Delta_t^h \boldsymbol{u}_t^n \cdot (\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\right)$$
$$- \Delta_t^h \boldsymbol{u}_t^n(t)\cdot(\boldsymbol{u}_{tt}^n + \alpha\boldsymbol{u}_t^n)(t) + \Delta_t^h\left(b(v^n)\mathbf{T}^n\right)(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)(t)\,\mathrm{d}x.$$

We integrate over (t_1, t_2), where $0 < t_1 < t_2 < T$, divide through by h and let $h \to 0+$ to get

$$\begin{aligned}\int_{t_1}^{t_2} \langle l_t^n, \boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n\rangle\,\mathrm{d}t &= \left[\int_\Omega \boldsymbol{u}_{tt}^n\cdot(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\,\mathrm{d}x\right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_\Omega \boldsymbol{u}_{tt}^n\cdot(\boldsymbol{u}_{tt}^n + \alpha\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_\Omega b'(v^n)v_t^n\mathbf{T}^n\cdot F_n(\mathbf{T}^n) + b(v^n)\mathbf{T}_t^n\cdot F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.32}$$

Recalling that F_n is the derivative of φ_n, using the chain rule for weak derivatives, we have that $\varphi_n(\mathbf{T}^n) \in W^{1,1}(0,T;L^1(\Omega))$ with weak time derivative $\mathbf{T}_t^n \cdot F_n(\mathbf{T}^n)$. Since $\boldsymbol{u}_{tt}^n + \alpha\boldsymbol{u}_t^n \in L^2(0,T;W_D^{1,2}(\Omega)^d)$, we have $\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n \in C([0,T];W_D^{1,2}(\Omega)^d)$ and so by the constitutive relation, $F_n(\mathbf{T}^n) \in C([0,T];L^2(\Omega)^{d\times d})$ with $F_n(\mathbf{T}^n(0)) = \boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)$. Since $\mathbf{T}^n \in W^{1,2}(0,T;L^2(\Omega)^{d\times d})$, then $\mathbf{T}^n \in C([0,T];L^2(\Omega)^{d\times d})$. Combining these facts, it follows that $\mathbf{T}^n(0) = F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0))$ and the following holds:

$$\begin{aligned}&\lim_{t_1\to 0+}\int_{t_1}^{t_2}\int_\Omega b(v^n)\mathbf{T}_t^n\cdot F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{t_1\to 0+}\left\{\left[\int_\Omega b(v^n(t))\varphi_n(\mathbf{T}^n(t))\,\mathrm{d}x\right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_\Omega b'(v^n)v_t^n\varphi_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t\right\} \\ &= \int_\Omega b(v^n(t_2))\varphi_n(\mathbf{T}^n(t_2)) - b(v_0)\varphi_n(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0)))\,\mathrm{d}x - \int_0^{t_2}\int_\Omega b'(v^n)v_t^n\varphi_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From (6.32), we also have that

$$\begin{aligned}&\left[\int_\Omega b(v^n(t))\varphi_n(\mathbf{T}^n(t))\,\mathrm{d}x\right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\int_\Omega b'(v^n)v_t^n\varphi_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\langle l_t^n, \boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n\rangle\,\mathrm{d}t - \int_\Omega \boldsymbol{u}_{tt}^n(t_2)\cdot(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)(t_2) - \boldsymbol{u}_{tt}^n(t_1)\cdot(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)(t_1)\,\mathrm{d}x \\ &\quad - \int_{t_1}^{t_2}\int_\Omega b'(v^n)v_t^n\mathbf{T}^n\cdot F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \boldsymbol{u}_{tt}^n\cdot(\boldsymbol{u}_{tt}^n + \alpha\boldsymbol{u}_t^n)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\Bigg[1 + \|l_t\|_{L^2(W^{-1,2})}^2 + \|\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n\|_{L^2(W^{1,2})}^2 + \|\boldsymbol{u}_{tt}^n\|_{L^\infty(L^2)}^2 + \|\boldsymbol{u}_t^n\|_{L^\infty(L^2)}^2 + \|\boldsymbol{u}^n\|_{L^\infty(L^2)}^2 \\ &\quad + \|b'(v^n)\|_{L^\infty(Q)}\|v_t^n\|_{L^\infty(Q)}\int_Q |\mathbf{T}^n| + \frac{|\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\Bigg] \\ &\le C\left[1 + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\left[1 + \|\boldsymbol{u}_0\|_{1,2}^2 + \|\boldsymbol{u}_0\|_{1,\infty} + \|\boldsymbol{u}_1\|_2^2 + \|v_0\|_{1,2}^2 + \|b(v_0)\|_\infty\right. \\ &\quad \left. + \|l\|_{L^\infty(W^{-1,2})} + \|l_t\|_{L^2(W^{-1,2})}^2 + \|\boldsymbol{u}_1 + \alpha\boldsymbol{u}_0\|_{2,2} + \|l_{tt}\|_{L^1(W^{-1,2})}\right]^2.\end{aligned}$$

We deduce that

$$\sup_{t\in[0,T]}\int_\Omega b(v^n(t))\varphi_n(\mathbf{T}^n(t))\,\mathrm{d}x \le C\left[1+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]^2\Big[1+\|\boldsymbol{u}_0\|_{1,2}^2+\|\boldsymbol{u}_0\|_{1,\infty}$$
$$+\|\boldsymbol{u}_1\|_2^2+\|v_0\|_{1,2}^2+\|b(v_0)\|_\infty+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|^2_{L^2(W^{-1,2})}+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2} \tag{6.33}$$
$$+\|l_{tt}\|_{L^1(W^{-1,2})}\Big]^2+\int_\Omega b(v_0)\varphi_n(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\,\mathrm{d}x.$$

There exist constants c_1, c_2, independent of n, such that $c_1|\mathbf{T}|-c_2\le\varphi_n(\mathbf{T})$. We apply this to the left-hand side of (6.33). For the final term on the right-hand side, we have that $\varphi_n(F_n^{-1}(\mathbf{T}))\le F_n^{-1}(\mathbf{T})\cdot\mathbf{T}$. Using the safety strain condition (6.5), it follows that

$$\int_\Omega b(v_0)\varphi_n(F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)))\,\mathrm{d}x \le \|b(v_0)\|_\infty\int_\Omega |F_n^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0))||\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)|\,\mathrm{d}x$$
$$\le C_*|\Omega|\|b(v_0)\|_\infty\|F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)\|_\infty\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1+\alpha\boldsymbol{u}_0)\|_\infty\le\frac{C_*^2|\Omega|}{(1-C_*^a)^{\frac{1}{a}}}\|b(v_0)\|_\infty.$$

□

Next, we focus on improving the spatial regularity estimates on the solution. We prove a weighted estimate on $\nabla\mathbf{T}^n$, analogous to the bound from Proposition 6.14. We use arguments that mimic those in [5] and those in Chapter 4. The presence of the phase-field function causes additional difficulty. Indeed, we will see that to obtain a suitable bound on $\nabla\mathbf{T}^n$, we require $(\nabla v^n)_n$ to be uniformly bounded in $L^\infty(Q)$. Hence we now impose that $k>\frac{d}{2}+1$ and so, by the Sobolev embedding theorem, we have that $H^k(\Omega)$ embeds compactly into $C^1(\overline{\Omega})$.

Proposition 6.17. *Let the assumptions of Theorem 6.9 hold and let $(\boldsymbol{u}^n,\mathbf{T}^n,v^n)$ be the solution triple constructed in the proof. Suppose that the safety strain condition (6.5) holds and that $\boldsymbol{g}\in W^{2,1}(0,T;L^2(\partial\Omega_N)^d)$, $\boldsymbol{f}\in L^2(0,T;W^{1,2}_{loc}(\Omega)^d)\cap W^{2,1}(0,T;W^{-1,2}_D(\Omega)^d)$, $\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\in W^{2,2}_D(\Omega)^d$ and $\boldsymbol{u}_0\in W^{2,2}_{loc}(\Omega)^d$. Furthermore, assume that $k>\frac{d}{2}+1$. For every pair of open sets Ω_0, Ω_1 such that $\overline{\Omega_0}\subset\Omega_1$ and $\overline{\Omega_1}\subset\Omega$, there exists a constant C, depending on Ω_0 and Ω_1 but independent of n, such that*

$$\int_0^T\int_{\Omega_0}\frac{|\nabla\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}}+\frac{|\nabla\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\le C\Big[\Big[1+\sup_n\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}$$
$$+C_1\sup_n\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\Big]^2C_1^4+\|\boldsymbol{f}\|^2_{L^2(L^2)}+\|\nabla\boldsymbol{f}\|^2_{L^2(Q_1)}\Big],$$

where $Q_1=(0,T)\times\Omega_1$ and the constant C_1 is defined by

$$C_1=\left[1+\sup_n\|b'\|_{I_{\|v^n\|_{L^\infty(Q)}}}\right]\Big[1+\|\boldsymbol{u}_0\|_2+\|\boldsymbol{u}_0\|_{1,\infty}^{\frac{1}{2}}+\|\boldsymbol{u}_1\|_2+\|v_0\|_{1,2}+\|b(v_0)\|_\infty^{\frac{1}{2}}$$
$$+\|l\|_{L^\infty(W^{-1,2})}+\|l_t\|_{L^2(W^{-1,2})}\Big].$$

Proof. For sets Ω_0, Ω_1 as above, let $\zeta\in C_c^\infty(\Omega)$ be a cut-off function such that $\zeta=1$ on Ω_0 and $\mathrm{supp}(\zeta)$ is compactly contained in Ω_1. There exists a $h_0=h_0(\Omega_0,\Omega_1)>0$ such that for every $h\in(0,h_0)$, the function $\zeta\Delta_i^h\boldsymbol{v}$ is well-defined on Ω for arbitrary $\boldsymbol{v}\in L^1(\Omega)^d$. We denote by Q_i the set $(0,T)\times\Omega_i$ for $i\in\{0,1\}$.

First we show that the solution triple has higher regularity in space than is presently known. Then we use this to prove an estimate that is uniform with respect to n. Taking $\zeta^2\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)$ as a test function in the elastodynamic equation (6.16) and noticing that the Neumann boundary term vanishes, we get that

$$\begin{aligned}
&\int_\Omega \zeta^2\Delta_i^h\boldsymbol{u}_{tt}^n\cdot\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)+\Delta_i^h\left(b(v^n)\mathbf{T}^n\right)\cdot\boldsymbol{\varepsilon}\left(\zeta^2\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\right)\,\mathrm{d}x\\
&=\int_\Omega \zeta^2\Delta_i^h\boldsymbol{f}(t)\cdot\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\,\mathrm{d}x\\
&\le C(\eta)h^2\|\nabla\boldsymbol{f}(t)\|_{L^2(\Omega_1)}\|\nabla(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)(t)\|_{L^2(\Omega_1)}.
\end{aligned}$$

Using that $\operatorname{supp}(\zeta\Delta_i^h\boldsymbol{v})\subset\Omega_1$ for $\boldsymbol{v}\in L^2(\Omega)^d$, we see that

$$\int_\Omega \zeta^2\Delta_i^h\boldsymbol{u}_{tt}^n\cdot\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\,\mathrm{d}x\le C(\zeta)h^2\|\nabla\boldsymbol{u}_{tt}^n\|_{L^2(\Omega_1)}\|\nabla(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\|_{L^2(\Omega_1)},$$

which is valid by Corollary 6.15. For the terms that contain the nonlinearities, that is $F_n(\mathbf{T}^n)$ and $b(v^n)$, we have

$$\begin{aligned}
&\int_\Omega \Delta_i^h\left(b(v^n)\mathbf{T}^n\right)\cdot\boldsymbol{\varepsilon}\left(\zeta^2\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\right)\,\mathrm{d}x\\
&=\int_\Omega\left(\Delta_i^h b(v^n)\tau_i^h\mathbf{T}^n+b(v^n)\Delta_i^h\mathbf{T}^n\right)\cdot\left(2\zeta\nabla\zeta\otimes\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)+\zeta^2\boldsymbol{\varepsilon}(\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right)\,\mathrm{d}x,
\end{aligned}$$

where τ_i^h is a translation of length h in the i-th coordinate direction. Integration over $(0,T)$ yields

$$\begin{aligned}
&\int_Q \zeta^2 b(v^n)\Delta_i^h\mathbf{T}^n\cdot\Delta_i^h F_n(\mathbf{T}^n)\\
&\le C\left[\int_0^T\left(h^2\|\nabla\boldsymbol{f}(t)\|_{L^2(\Omega_1)}\|\nabla(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\|_{L^2(\Omega_1)}+h^2\|\nabla\boldsymbol{u}_{tt}^n\|_{L^2(\Omega_1)}\|\nabla(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\|_{L^2(\Omega_1)}\right)\right]\\
&\quad+\int_Q\left|\zeta^2\Delta_i^h b(v^n)\tau_i^h\mathbf{T}^n\cdot\boldsymbol{\varepsilon}(\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right|+\int_Q\left|2\zeta b(v^n)\Delta_i^h\mathbf{T}^n\cdot(\nabla\zeta\otimes\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right|\\
&\quad+\int_Q\left|2\zeta\Delta_i^h b(v^n)\tau_i^h\mathbf{T}^n\cdot(\nabla\zeta\otimes\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right|.
\end{aligned}$$

Recalling that $|F_n(\mathbf{T})-F_n(\mathbf{S})|\le 3|\mathbf{T}-\mathbf{S}|$ for every n and every $\mathbf{T}$, $\mathbf{S}\in\mathbb{R}^{d\times d}$, we can write

$$\begin{aligned}
&\int_Q\left|\zeta^2\Delta_i^h b(v^n)\tau_i^h\mathbf{T}^n\cdot\boldsymbol{\varepsilon}(\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right|\,\mathrm{d}x\,\mathrm{d}t\\
&\le Ch\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|\nabla v^n\|_{L^\infty(Q)}\int_Q\zeta^2|\tau_i^h\mathbf{T}^n||\Delta_i^h\mathbf{T}^n|\,\mathrm{d}x\,\mathrm{d}t\\
&\le Ch\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|\nabla v^n\|_{L^\infty(Q)}\|\mathbf{T}^n\|_{L^2(Q_1)}\|\zeta\Delta_i^h\mathbf{T}^n\|_{L^2(Q)}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\int_Q\left|2\zeta b(v^n)\Delta_i^h\mathbf{T}^n\cdot(\nabla\zeta\otimes\Delta_i^h(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n))\right|\,\mathrm{d}x\,\mathrm{d}t\\
&\le C(\zeta)h\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|\zeta\Delta_i^h\mathbf{T}^n\|_{L^2(Q)}\|\nabla(\boldsymbol{u}_t^n+\alpha\boldsymbol{u}^n)\|_{L^2(Q_1)},
\end{aligned}$$

and

$$\int_Q \left|2\zeta\Delta_i^h b(v^n)\tau_i^h \mathbf{T}^n \cdot (\nabla\zeta \otimes \Delta_i^h(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n))\right| \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C(\zeta)h\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|\nabla v^n\|_{L^\infty(Q)}\|\mathbf{T}^n\|_{L^2(Q_1)}\|\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\|_{L^2(Q_1)}.$$

It follows that

$$\int_{Q_0} \frac{|\Delta_i^h \mathbf{T}^n|^2}{n} \,\mathrm{d}x\,\mathrm{d}t \leq Ch^2[1+n]\left[1 + \|b'\|^2_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\|\nabla v^n\|^2_{L^\infty(Q)} + \|b\|^2_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}\right]$$
$$\cdot\left[\|\nabla \boldsymbol{f}(t)\|^2_{L^2(Q_1)} + \|\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)\|^2_{L^2(Q_1)} + \|\nabla\boldsymbol{u}_{tt}^n\|^2_{L^2(Q_1)} + \|\mathbf{T}^n\|^2_{L^2(Q_1)}\right] \leq C(n)h^2,$$

where C is a positive constant that depends on n, ζ, the problem parameters and the problem data. Dividing through by h^2 and letting $h \to 0+$ yields $\mathbf{T}^n \in L^2(0,T;W^{1,2}(\Omega_0)^{d\times d})$. Since Ω_0 is arbitrary, it follows that $\mathbf{T}^n \in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$. We also have the extra spatial regularity $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n) \in L^2(0,T;W^{1,2}_{loc}(\Omega)^{d\times d})$. Using Korn's inequality and a standard density argument as in Chapter 4, we see that $\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n \in L^2(0,T;W^{2,2}_{loc}(\Omega)^{d\times d})$. By the memory kernel property and the assumption that $\boldsymbol{u}_0 \in W^{2,2}_{loc}(\Omega)^d$, we further deduce that $\boldsymbol{u}^n \in L^\infty(0,T;W^{2,2}_{loc}(\Omega)^d)$. As a result, $\boldsymbol{u}_t^n \in L^2(0,T;W^{2,2}_{loc}(\Omega)^d)$. We now use this improved regularity to deduce estimates that are uniform in n.

Considering an arbitrary test function $\boldsymbol{v} \in C_c^\infty(\Omega)^d$ in the elastodynamic equation (6.16), using the improved regularity of $\mathbf{T}^n$, we deduce that

$$\boldsymbol{u}_{tt}^n = \mathrm{div}(\mathbf{T}^n) + \boldsymbol{f} \quad \text{pointwise a.e. in } Q. \tag{6.34}$$

Each term in (6.34) is an element of $L^2(0,T;L^2_{loc}(\Omega)^d)$. We take the dot product of (6.34) with $\zeta^2\nabla\cdot\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)$ and integrate over Ω to give the following:

$$\begin{aligned}&\int_\Omega \zeta^2\boldsymbol{u}_{tt}^n \cdot (\nabla\cdot\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n))\,\mathrm{d}x\\ &= \int_\Omega \zeta^2\mathrm{div}(b(v^n)\mathbf{T}^n)\cdot(\nabla\cdot\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)) + \zeta^2\boldsymbol{f}\cdot(\nabla\cdot\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n))\,\mathrm{d}x.\end{aligned} \tag{6.35}$$

For the first term on the right-hand side, expanding out the derivatives, we get that

$$\int_\Omega \zeta^2\mathrm{div}(b(v^n)\mathbf{T}^n)\cdot(\nabla\cdot\nabla(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n))\,\mathrm{d}x$$
$$= \int_\Omega \zeta^2 b(v^n)\frac{\partial\mathbf{T}^n_{ij}}{\partial x_j}\frac{\partial^2}{\partial x_k^2}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_i + \zeta^2 b'(v^n)\mathbf{T}^n_{ij}\frac{\partial v^n}{\partial x_j}\frac{\partial^2}{\partial x_k^2}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_i\,\mathrm{d}x.$$

Using an integration by parts argument, justified by considering a smooth approximation of $\mathbf{T}^n$ on Ω_1 with compact support in Ω, we see that

$$\int_\Omega \zeta^2 b(v^n)\frac{\partial\mathbf{T}^n_{ij}}{\partial x_j}\frac{\partial^2}{\partial x_k^2}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_i\,\mathrm{d}x$$
$$= \int_\Omega \zeta^2 b(v^n)\frac{\partial\mathbf{T}^n_{ij}}{\partial x_k}\frac{\partial}{\partial x_k}\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_{ij} - \mathbf{T}^n_{ij}\frac{\partial^2}{\partial x_k^2}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_i\left[b'(v^n)\frac{\partial v^n}{\partial x_j}\zeta^2 + 2\zeta\frac{\partial\zeta}{\partial x_j}b(v^n)\right]$$
$$+ \mathbf{T}^n_{ij}\frac{\partial}{\partial x_k}\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha\boldsymbol{u}^n)_{ij}\left[b'(v^n)\frac{\partial v^n}{\partial x_k}\zeta^2 + 2\zeta\frac{\partial\zeta}{\partial x_k}b(v^n)\right]\,\mathrm{d}x.$$

The first term on the right-hand side can be identified as

$$\int_\Omega \zeta^2 b(v^n) \frac{\partial \mathbf{T}^n_{ij}}{\partial x_k} \frac{\partial}{\partial x_k} \boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)_{ij} \,\mathrm{d}x = \int_\Omega \zeta^2 b(v^n) \nabla \mathbf{T}^n \cdot \nabla F_n(\mathbf{T}^n) \,\mathrm{d}x,$$

where the gradient operator is defined by $\nabla \mathbf{T} = (\partial_k \mathbf{T}_{ij})^d_{i,j,k=1}$. This yields

$$\begin{aligned}
&\int_\Omega \zeta^2 \mathrm{div}\,(b(v^n)\mathbf{T}^n) \cdot (\nabla \cdot \nabla(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)) \,\mathrm{d}x \\
&= \int_\Omega \zeta^2 b(v^n) \nabla \mathbf{T}^n \cdot \nabla F_n(\mathbf{T}^n) + \zeta^2 b'(v^n) \mathbf{T}^n_{ij} \frac{\partial v^n}{\partial x_j} \frac{\partial^2}{\partial x_k^2} (\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)_i \\
&\quad - \zeta \mathbf{T}^n_{ij} \frac{\partial^2}{\partial x_k^2} (\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)_i \left[b'(v^n) \frac{\partial v^n}{\partial x_j} \zeta + 2 \frac{\partial \zeta}{\partial x_j} b(v^n) \right] \\
&\quad + \zeta \mathbf{T}^n_{ij} \frac{\partial}{\partial x_k} \boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)_{ij} \left[b'(v^n) \frac{\partial v^n}{\partial x_k} \zeta + 2 \frac{\partial \zeta}{\partial x_k} b(v^n) \right] \mathrm{d}x \\
&= \int_\Omega \zeta^2 b(v^n) \nabla \mathbf{T}^n \cdot \nabla F_n(\mathbf{T}^n) + \zeta \frac{\partial}{\partial x_k} \boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n)_{ij} \mathbf{B}^{n,k}_{ij} \,\mathrm{d}x,
\end{aligned}$$

where, for each $k \in \{1, \ldots, d\}$, the second-order tensor $\mathbf{B}^{n,k}$ is defined by

$$\begin{aligned}
\mathbf{B}^{n,k}_{ij} &= \mathbf{T}^n_{ij} \left[2b(v^n)\partial_k \zeta + \zeta b'(v^n) \partial_k v^n \right] - 2\delta_{jk} \mathbf{T}^n_{im} \left[2b(v^n) \partial_m \zeta + \zeta b'(v^n) \partial_m v^n \right] \\
&\quad + \delta_{ij} \mathbf{T}^n_{km} \left[2b(v^n) \partial_m \zeta + \zeta b'(v^n) \partial_m v^n \right] + 2\delta_{jk} \zeta \mathbf{T}^n_{im} b'(v^n) \partial_m v^n - \delta_{ij} \zeta \mathbf{T}^n_{km} b'(v^n) \partial_m v^n,
\end{aligned}$$

using the identity

$$\frac{\partial^2 \boldsymbol{v}_i}{\partial x_k^2} = 2 \frac{\partial}{\partial x_k} \boldsymbol{\varepsilon}(\boldsymbol{v})_{ik} - \frac{\partial}{\partial x_i} \boldsymbol{\varepsilon}(\boldsymbol{w})_{kk}.$$

For a constant $C = C(\zeta)$, independent of n, we have that

$$\begin{aligned}
|\mathbf{B}^{n,k}| &\le C \left[\|b(v^n)\|_{L^\infty(Q_1)} + \|b'(v^n) v^n_t\|_{L^\infty(Q_1)} \right] |\mathbf{T}^n| \\
&\le C \left[\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} C_1 \right] |\mathbf{T}^n|,
\end{aligned}$$

where we recall that C_1 is the constant given by

$$\begin{aligned}
C_1 = &\left[1 + \sup_n \|b'\|_{I_{\|v^n\|_{L^\infty(Q)}}} \right] \left[1 + \|\boldsymbol{u}_0\|_2 + \|\boldsymbol{u}_0\|^{\frac{1}{2}}_{1,\infty} + \|\boldsymbol{u}_1\|_2 + \|v_0\|_{1,2} + \|b(v_0)\|^{\frac{1}{2}}_\infty \right. \\
&\left. + \|l\|_{L^\infty(W^{-1,2})} + \|l_t\|_{L^2(W^{-1,2})} \right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_\Omega \zeta \frac{\partial}{\partial x_k} \boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n) \cdot \mathbf{B}^{n,k} \,\mathrm{d}x \right| = \left| \int_\Omega \zeta \frac{\partial}{\partial x_k} F_n(\mathbf{T}^n) \cdot \mathbf{B}^{n,k} \,\mathrm{d}x \right| \\
&\le \int_\Omega \frac{\zeta^2 \eta}{2} (\partial_k \mathbf{T}^n, \partial_k \mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)} + C(\eta) \chi_{\mathrm{supp}(\zeta)} (\mathbf{B}^{n,k}, \mathbf{B}^{n,k})_{\mathcal{A}_n(\mathbf{T}^n)} \,\mathrm{d}x,
\end{aligned}$$

where $\chi_{\text{supp}(\zeta)}$ denotes the indicator function of $\text{supp}(\zeta)$ and $\mathcal{A}_n$ is defined as in (6.22). Using properties of the inner product $(\cdot,\cdot)_{\mathcal{A}_n(\mathbf{T})}$, it follows that

$$
\begin{aligned}
&\int_Q \chi_{\text{supp}(\zeta)}(\mathbf{B}^{n,k},\mathbf{B}^{n,k})_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\int_Q \frac{|\mathbf{B}^{n,k}|^2}{n}+\frac{|\mathbf{B}^{n,k}|^2}{1+|\mathbf{T}^n|}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\left[\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2\int_Q \frac{|\mathbf{T}^n|^2}{n}+\frac{|\mathbf{T}^n|^2}{1+|\mathbf{T}^n|}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\left[\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|^2_{L^2(W^{-1,2})}\right].
\end{aligned}
$$

We return to (6.35) and see that

$$
\begin{aligned}
&\int_{Q_0}\frac{|\nabla\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}}+\frac{|\nabla\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\int_Q \zeta^2 b(v^n)(\partial_k\mathbf{T}^n,\partial_k\mathbf{T}^n)_{\mathcal{A}_n(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\int_Q \zeta^2(\boldsymbol{u}^n_{tt}-\boldsymbol{f})\cdot(\nabla\cdot\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n))\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C\left[\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|^2_{L^2(W^{-1,2})}\right].
\end{aligned}
$$

Recalling that $\boldsymbol{u}^n_{tt}\in L^2(0,T;W^{1,2}(\Omega)^d)$, we use integration by parts and note that boundary terms disappear due to the compact support of ζ to see that

$$
\begin{aligned}
&\int_\Omega \zeta^2(\boldsymbol{u}^n_{tt}-\boldsymbol{f})\cdot(\nabla\cdot\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n))\,\mathrm{d}x\\
&=\int_\Omega \zeta^2\nabla(\boldsymbol{f}-\boldsymbol{u}^n_{tt})\cdot\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)+2\zeta\,(\nabla\zeta\otimes(\boldsymbol{f}-\boldsymbol{u}^n_{tt}))\cdot\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)\,\mathrm{d}x\\
&\leq-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\zeta\nabla\boldsymbol{u}^n_t\|_2^2}{2}+\alpha\int_\Omega\zeta\nabla\boldsymbol{u}^n_t\cdot\nabla\boldsymbol{u}^n\,\mathrm{d}x\right)+\|\zeta\nabla\boldsymbol{f}\|_2\|\zeta\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)\|_2\\
&\quad+\alpha\int_\Omega\zeta^2|\nabla\boldsymbol{u}^n_t|^2\,\mathrm{d}x+C(\zeta)\|\boldsymbol{f}\|_2\|\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)\|_2+C(\zeta)\|\boldsymbol{u}^n_{tt}\|_2\|\nabla(\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n)\|_2.
\end{aligned}
$$

We apply Gronwall's inequality to deduce that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\zeta\nabla\boldsymbol{u}^n(t)\|_2^2+\sup_{t\in[0,T]}\|\zeta\nabla\boldsymbol{u}^n_t(t)\|_2^2+\int_{Q_1}\frac{|\nabla\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}}+\frac{|\nabla\mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\Bigg[\left[\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2\left[1+\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{u}_1\|_2^2+\|l\|^2_{L^2(W^{-1,2})}\right]\\
&\quad+\|\boldsymbol{u}^n_t+\alpha\boldsymbol{u}^n\|_{L^2(W^{1,2})}+\|\boldsymbol{u}^n_{tt}\|^2_{L^2(L^2)}+\|\boldsymbol{f}\|^2_{L^2(L^2)}+\|\nabla\boldsymbol{f}\|^2_{L^2(Q_1))}\Bigg]\\
&\leq C\Bigg[\left[1+\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2\Big[1+\|\boldsymbol{u}_0\|^2_{1,2}+\|\boldsymbol{u}_0\|_{1,\infty}+\|\boldsymbol{u}_1\|_2^2\\
&\quad+\|v_0\|^2_{1,2}+\|b(v_0)\|_\infty+\|l\|^2_{L^\infty(W^{-1,2})}+\|l_t\|_{L^2(W^{-1,2})}+\|\boldsymbol{u}_1+\alpha\boldsymbol{u}_0\|_{2,2}+\|l_{tt}\|_{L^1(W^{-1,2})}\Big]^2\\
&\quad+\|\boldsymbol{f}\|^2_{L^2(L^2)}+\|\nabla\boldsymbol{f}\|^2_{L^2(Q_1))}\Bigg]\\
&\leq C\Bigg[\left[1+\|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}+\|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})}C_1\right]^2C_1^4+\|\boldsymbol{f}\|^2_{L^2(L^2)}+\|\nabla\boldsymbol{f}\|^2_{L^2(Q_1))}\Bigg].
\end{aligned}
$$

□

Corollary 6.18. *Let the assumptions of Proposition 6.17 hold and let* $(\boldsymbol{u}^n, \mathbf{T}^n, v^n)$ *be the solution of* (6.8) *constructed in Theorem 6.9. With* Ω_0*,* Ω_1 *and* C_1 *as in Proposition 6.17, there exists a constant* C*, independent of* n *but depending on* Ω_0 *and* Ω_1*, such that*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\nabla^2 \boldsymbol{u}^n(t)\|^2_{L^2(\Omega_0)} + \int_0^T \|\nabla^2 \boldsymbol{u}^n_t(t)\|^2_{L^2(\Omega_0)}\,\mathrm{d}t \\
&\le C\Bigg[\Big[1 + \sup_n \|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} + \sup_n \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} C_1\Big]^2 C_1^4 + \|\boldsymbol{f}\|^2_{L^2(L^2)} \\
&\quad + \|\nabla \boldsymbol{f}\|^2_{L^2(Q_1))} + \|\nabla^2 \boldsymbol{u}_0\|^2_{L^2(\Omega_1)}\Bigg].
\end{aligned}$$

Proof. From Proposition 6.17, $\boldsymbol{u}^n \in L^\infty(0,T;W^{2,2}_{loc}(\Omega)^d)$ and $\boldsymbol{u}^n_t \in L^2(0,T;W^{2,2}_{loc}(\Omega)^d)$. For the n-uniform estimate, from the memory kernel property, we have that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}^n(t)) = \mathrm{e}^{-\alpha t}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \int_0^t \mathrm{e}^{-\alpha(t-s)} F_n(\mathbf{T}^n(s))\,\mathrm{d}s.$$

We recall that

$$|\Delta^h_i F_n(\mathbf{T}^n)|^2 \le C\Delta^h_i \mathbf{T}^n \cdot \Delta^h_i F_n(\mathbf{T}^n), \tag{6.36}$$

so, for h sufficiently small, we get

$$\|\Delta^h_i \boldsymbol{\varepsilon}(\boldsymbol{u}^n(t))\|^2_{L^2(\Omega_0)} \le C\left(\|\Delta^h_i \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|^2_{L^2(\Omega_0)} + \int_0^t\int_{\Omega_0} \Delta^h_i \mathbf{T}^n \cdot \Delta^h_i F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}s\right).$$

Dividing through by h^2 and letting $h \to 0+$, we deduce that

$$\begin{aligned}
&\limsup_{h\to 0+} \frac{\|\Delta^h_i \boldsymbol{\varepsilon}(\boldsymbol{u}^n(t))\|^2_{L^2(\Omega_0)}}{h^2} \le C\left(\|\nabla^2 \boldsymbol{u}_0\|^2_{L^2(\Omega_0)} + \int_{Q_0} \nabla \mathbf{T}^n \cdot \nabla F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\le C\Bigg[\Big[1 + \|b\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} + \|b'\|_{L^\infty(I_{\|v^n\|_{L^\infty(Q)}})} C_1\Big]^2 C_1^4 + \|\boldsymbol{f}\|^2_{L^2(L^2)} + \|\nabla \boldsymbol{f}\|^2_{L^2(Q_1))} \\
&\quad + \|\boldsymbol{u}_0\|^2_{L^2(\Omega_1)}\Bigg].
\end{aligned}$$

Using Korn's inequality, it follows that $\boldsymbol{u}^n \in L^\infty(0,T;W^{2,2}_{loc}(\Omega)^d)$ with bound as in the statement of the corollary. Applying (6.36) again, we have $\boldsymbol{u}^n_t + \alpha\boldsymbol{u}^n \in L^2(0,T;W^{2,2}_{loc}(\Omega)^d)$ with bound as in Proposition 6.17. Hence, $\boldsymbol{u}^n_t$ satisfies the required estimate. □

6.2 Limit in the regularisation parameter

We now have sufficient uniform estimates and are able to take the limit as $n \to \infty$ to obtain a weak solution of the strain-limiting fracture problem (6.1)–(6.3), up to the Neumann boundary condition. First, we focus solely on the elastodynamic equation. Following this, we consider the minimisation problem (6.2) and then the energy-dissipation equality (6.3). The issues caused by the presence of a Neumann boundary condition means that we are not able to obtain a full existence result as in Chapter 5. However, in the case that we assume a fully Dirichlet boundary condition, we prove a strong convergence result that is not possible when we have mixed boundary conditions. We use this to show the existence of a weak energy solution.

Theorem 6.19. *Suppose that $k > \frac{d}{2}+1$ with data $v_0 \in H^k_{D+1}(\Omega)$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in W^{1,2}_D(\Omega)^d$ such that $\boldsymbol{u}_0 \in W^{2,2}_{loc}(\Omega)^d$, $\boldsymbol{u}_1 + \alpha \boldsymbol{u}_0 \in W^{2,2}_D(\Omega)^d$ and the safety strain condition* (6.5) *holds. Furthermore, suppose that we have $\boldsymbol{f} \in W^{2,1}(0,T; W^{-1,2}_D(\Omega)^d) \cap L^2(0,T; W^{1,2}_{loc}(\Omega)^d)$, $\boldsymbol{g} \in W^{2,1}(0,T; L^2(\partial\Omega_N)^d)$ and the compatibility condition* (6.7). *Let $((\boldsymbol{u}^n, \mathbf{T}^n, v^n))_n$ be the sequence of solutions to the regularised problem* (6.8)–(6.10) *constructed in Theorem 6.9 from the limit of the time discrete problem. There exists a subsequence in n, not relabelled, and a limiting triple $(\boldsymbol{u}, \mathbf{T}, v)$ such that*

- $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ *weakly-* in* $W^{2,\infty}(0,T; L^2(\Omega)^d) \cap W^{1,\infty}(0,T; W^{1,2}_D(\Omega)^d)$,
- $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{2,2}(0,T; W^{1,2}_D(\Omega)^d) \cap W^{1,2}(0,T; W^{2,2}_{loc}(\Omega)^d)$,
- $\boldsymbol{\varepsilon}(\boldsymbol{u}^n_t + \alpha \boldsymbol{u}^n) \to \boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})$ *pointwise a.e. in* Q,
- $v^n \overset{*}{\rightharpoonup} v$ *weakly-* in* $W^{1,\infty}(0,T; H^k(\Omega))$ *with* $v(t) \in H^k_{D+1}(\Omega)$ *for every* $t \in [0,T]$,
- $\mathbf{T}^n \to \mathbf{T}$ *pointwise a.e. in* Q *where* $\mathbf{T} \in L^\infty(0,T; L^1(\Omega)^{d\times d})$.

Furthermore, there exists an error term $\tilde{\boldsymbol{g}} \in L^\infty_{w^*}(0,T; (C^1_0(\partial\Omega_N)^d)^*)$ such that

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{v} + b(v(t))\mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \,\mathrm{d}x + \langle \boldsymbol{g}(t) - \tilde{\boldsymbol{g}}(t), \boldsymbol{v} \rangle_{C^1_0(\partial\Omega_N)}, \tag{6.37}$$

for a.e. $t \in (0,T)$ and every test function $\boldsymbol{v} \in C^1_D(\overline{\Omega})^d$. The constitutive relation (6.1b) *holds pointwise a.e. in Q and the initial conditions hold strongly in the sense that*

$$\lim_{t\to 0+} \big(\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_2^2 + \|v(t) - v_0\|_{k,2} \big) = 0. \tag{6.38}$$

Proof. Putting together the results of Lemma 6.11, Lemma 6.12, Lemma 6.13, Proposition 6.14, Corollary 6.15, Lemma 6.16, Proposition 6.17 and Corollary 6.18, there exists a constant C, independent of n but dependent on the problem data and problem parameters, such that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|^2_{1,2} + \sup_{t\in[0,T]} \|\boldsymbol{u}^n_t(t)\|^2_{1,2} + \sup_{t\in[0,T]} \|\boldsymbol{u}^n_{tt}(t)\|^2_2 + \int_0^T \|\boldsymbol{u}^n_{tt}(t)\|^2_{1,2} \,\mathrm{d}t \\
&\quad + \sup_{t\in[0,T]} \int_\Omega \varphi^*_n(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t))) \,\mathrm{d}x + \sup_{t\in[0,T]} \|v^n(t)\|^2_{k,2} + \sup_{t\in[0,T]} \|v^n_t(t)\|^2_{k,2} \\
&\quad + \sup_{t\in[0,T]} \int_\Omega |\mathbf{T}^n(t)| + \frac{|\mathbf{T}^n(t)|^2}{n} \,\mathrm{d}x + \int_Q \frac{|\mathbf{T}^n_t|^2}{(1+|\mathbf{T}^n|)^{1+a}} + \frac{|\mathbf{T}^n_t|^2}{n} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\tilde{C}} \Bigg[\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{W^{2,2}(\Omega_0)} + \int_0^T \|\boldsymbol{u}^n_t(t)\|^2_{W^{2,2}(\Omega_0)} \,\mathrm{d}t \\
&\quad + \int_{Q_0} \frac{|\nabla \mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} + \frac{|\nabla \mathbf{T}^n|^2}{n} \,\mathrm{d}x\,\mathrm{d}t \Bigg] \le C
\end{aligned} \tag{6.39}$$

where Ω_0, Ω_1 are open subsets of Ω such that $\overline{\Omega_0} \subset \Omega_1$ and $\overline{\Omega_1} \subset \Omega$ and $\tilde{C} = \tilde{C}(\Omega_0, \Omega_1, \Omega)$ is a constant.

Up to a subsequence in n that we do not relabel, the first, second and fourth asserted convergence results must hold for some limiting functions $\boldsymbol{u}$ and v. Using the initial conditions for the solution of the regularised problem, we use the Aubin–Lions lemma and standard arguments to deduce that (6.38) holds. For the pointwise convergence of $(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n))_n$, suppose that $(\mathbf{T}^n)_n$ converges pointwise a.e. on Q to a limit $\mathbf{T}$, say. Then $(F_n(\mathbf{T}^n))_n$ converges pointwise to $F(\mathbf{T})$ a.e. on Q, but $F_n(\mathbf{T}^n) = \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)$. Thus by the uniqueness of pointwise and weak limits in $L^2(Q)$, it follows that $\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)$ converges pointwise in Q to $\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})$. From (6.39) and Fatou's lemma, if pointwise convergence holds then the limiting function $\mathbf{T}$ must be an element of $L^\infty(0,T; L^1(\Omega)^{d\times d})$. Hence it is enough to prove the pointwise convergence of $(\mathbf{T}^n)_n$.

We prove the pointwise convergence result in an analogous way to the method used in Chapter 4. Define sequences $(\mathbf{S}^n)_n$, $(s^n)_n$, respectively, by

$$\mathbf{S}^n = \frac{\mathbf{T}^n}{(1+|\mathbf{T}^n|)^{1+a}}, \quad s^n = \frac{1}{(1+|\mathbf{T}^n|)^{1+a}}.$$

By the differentiability of $\mathbf{T}^n$ and the chain rule for weak derivatives, we see that $\mathbf{S}^n$ and s^n are weakly differentiable in both time and space. Furthermore, for sets Ω_0 and Ω_1 as above, we have

$$\begin{aligned}
&\|\mathbf{S}^n\|_{L^\infty(Q)} + \|s^n\|_{L^\infty(Q)} + \int_0^T \|\nabla \mathbf{S}^n\|_{L^2(\Omega_0)}^2 + \|s^n\|_{L^2(\Omega_0)}^2 + \|\mathbf{S}_t^n\|_{L^2(\Omega)}^2 + \|s_t^n\|_{L^2(\Omega)}^2 \,\mathrm{d}t \\
&\leq 2 + C\int_0^T \int_{\Omega_0} \frac{|\nabla \mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} \,\mathrm{d}x + \int_\Omega \frac{|\mathbf{T}_t^n|^2}{(1+|\mathbf{T}^n|)^{1+a}} \,\mathrm{d}x \,\mathrm{d}t \leq C,
\end{aligned}$$

where C is independent of n. Applying the Aubin–Lions lemma, we deduce that $(\mathbf{S}^n)_n$ and $(s^n)_n$ converge strongly in $L^2(0,T; L^2_{loc}(\Omega))$ to respective limits $\mathbf{S}$ and s. In particular, they converge pointwise a.e. on Q. Considering Fatou's lemma and the uniform $L^1(Q)^{d\times d}$ bound on $(\mathbf{T}^n)_n$, then $s^{-\frac{1}{1+a}}$ is an element of $L^1(\Omega)$. Thus $s > 0$ a.e. in Q and so $\mathbf{T}^n = (s^n)^{-1}\mathbf{S}^n \to s^{-1}\mathbf{S} =: \mathbf{T}$ pointwise a.e. in Q. As the pointwise limit of measurable functions, $\mathbf{T}$ is measurable and $\mathbf{T} \in L^\infty(0,T; L^1(\Omega)^{d\times d})$.

Next, we claim that, for every test function $\boldsymbol{v} \in W_0^{1,2}(\Omega)^d$ such that $\boldsymbol{\varepsilon}(\boldsymbol{v}) \in L^\infty(\Omega)^{d\times d}$, for a.e. $t \in (0,T)$, we have that

$$\int_\Omega \boldsymbol{u}_{tt}(t)\cdot \boldsymbol{v} + b(v(t))\mathbf{T}(t)\cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t)\cdot \boldsymbol{v} \,\mathrm{d}x. \tag{6.40}$$

By a density argument and Lemma 3.1, it is enough to show that (6.40) is satisfied for smooth test functions $\boldsymbol{v} \in C_c^\infty(\Omega)^d$. We fix an arbitrary $\boldsymbol{v} \in C_c^\infty(\Omega)^d$ and $\psi \in C([0,T])$. We consider $\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi \in W_0^{1,2}(\Omega)^d$ as a test function in the elastodynamic equation (6.16) where $\tau \in C_c^1([0,\infty))$ is a fixed smooth function. This choice is valid because $\tau(|\mathbf{T}^n|)$ is uniformly bounded on Q and, by the differentiability of $\mathbf{T}^n$ and the chain rule for weakly differentiable functions, $\tau(|\mathbf{T}^n|) \in L^2(0,T; W^{1,2}(\Omega))$. It follows that

$$\begin{aligned}
0 &= \int_Q \boldsymbol{u}_{tt}^n \cdot (\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) + b(v^n)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_Q \boldsymbol{u}_{tt}^n \cdot (\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) + b(v^n)\tau(|\mathbf{T}^n|)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau(|\mathbf{T}^n|)\boldsymbol{v}\psi) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_Q \psi b(v^n)\mathbf{T}^n \cdot (\nabla \tau(|\mathbf{T}^n|) \otimes \boldsymbol{v}) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

Since τ has compact support in $[0,\infty)$ and $\mathbf{T}^n \to \mathbf{T}$ pointwise on Q, $\tau(|\mathbf{T}^n|) \to \tau(|\mathbf{T}|)$ and $\tau(|\mathbf{T}^n|)\mathbf{T}^n \to \tau(|\mathbf{T}|)\mathbf{T}$ strongly in $L^p(Q)$ for every $p \in [1,\infty)$. Furthermore, $(v^n)_n$ is uniformly bounded in Q and converges pointwise to v by the Aubin–Lions lemma. Hence, by the dominated convergence theorem, $b(v^n)\tau(|\mathbf{T}^n|)\mathbf{T}^n \to b(v)\tau(|\mathbf{T}|)\mathbf{T}$ strongly in $L^p(Q)^{d\times d}$ for every $p \in [1,\infty)$. It follows that

$$\begin{aligned}&\lim_{n\to\infty}\left(-\int_Q \psi b(v^n)\mathbf{T}^n \cdot (\nabla\tau(|\mathbf{T}^n|)\otimes \boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\right)\\&= \int_Q \boldsymbol{u}_{tt}\cdot(\tau(|\mathbf{T}|)\boldsymbol{v}\psi) + b(v)\tau(|\mathbf{T}|)\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\tau(|\mathbf{T}|)\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.41}$$

Now replace τ by $\tau_k \in C^1_c([0,\infty))$ in (6.41) where τ_k is such that $0 \le \tau_k \le 1$ on $[0,\infty)$ with

$$\tau_k(s) = \begin{cases} 1 & \text{if } s \le k,\\ 0 & \text{if } s > 2k,\end{cases}$$

and there exists a constant C, independent of k, such that $|\tau_k'| \le Ck^{-1}$. Letting $k \to \infty$ in (6.41) and using that $\mathbf{T} \in L^1(Q)$, we apply the dominated convergence theorem to get

$$\begin{aligned}&\lim_{k\to\infty}\lim_{n\to\infty}\left(-\int_Q \psi b(v^n)\mathbf{T}^n\cdot(\nabla\tau_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\right)\,\mathrm{d}x\,\mathrm{d}t\\&= \int_Q \boldsymbol{u}_{tt}\cdot(\boldsymbol{v}\psi) + b(v)\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}\psi) - \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.42}$$

If we show that the limit on the left-hand side of (6.42) vanishes, since ψ is arbitrary, it will follow immediately that (6.40) holds. We can write

$$\begin{aligned}&\int_Q \psi b(v^n)\mathbf{T}^n\cdot(\nabla\tau_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\&= \int_Q \psi\tau_k'(|\mathbf{T}^n|)b(v^n)\mathbf{T}^n\cdot(\nabla|\mathbf{T}^n|\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\&= \int_Q \psi\tau_k'(|\mathbf{T}^n|)(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}b(v^n)F(\mathbf{T}^n)\cdot(\nabla|\mathbf{T}^n|\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\&= \int_Q \psi b(v^n)F(\mathbf{T}^n)\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where g_k is defined on $[0,\infty)$ by

$$g_k(s) = \int_0^s (1+u^a)^{\frac{1}{a}}\tau_k'(u)\,\mathrm{d}u.$$

Integration by parts yields

$$\begin{aligned}&\int_Q \psi b(v^n)F(\mathbf{T}^n)\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\&= \int_Q \psi b(v^n)F(\mathbf{T}^n)_{ij}\frac{\partial g_k(|\mathbf{T}^n|)}{\partial x_i}\boldsymbol{v}_j\,\mathrm{d}x\,\mathrm{d}t\\&= -\int_Q \psi g_k(|\mathbf{T}^n|)\frac{\partial F(\mathbf{T}^n)_{ij}}{\partial x_i}\boldsymbol{v}_j + \psi g_k(|\mathbf{T}^n|)F(\mathbf{T}^n)_{ij}\frac{\partial \boldsymbol{v}_j}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t\\&= -\int_Q \psi g_k(|\mathbf{T}^n|)\mathcal{A}_{ijlp}(\mathbf{T}^n)\frac{\partial \mathbf{T}^n_{lp}}{\partial x_i}\boldsymbol{v}_j + \psi g_k(\mathbf{T}^n|)F(\mathbf{T}^n)_{ij}\frac{\partial \boldsymbol{v}_j}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{6.43}$$

where $\mathcal{A}$ is the derivative of F. Considering the inner product induced by $\mathcal{A}(\mathbf{T}) = \frac{\partial F}{\partial \mathbf{T}}(\mathbf{T})$, we apply the Cauchy–Schwarz inequality followed by the Hölder inequality to get that

$$\begin{aligned}
&\left|\int_Q \psi g_k(|\mathbf{T}^n|)\mathcal{A}_{ijlp}(\mathbf{T}^n)\frac{\partial \mathbf{T}^n_{lp}}{\partial x_i}\boldsymbol{v}_j \,\mathrm{d}x\,\mathrm{d}t\right| \\
&= \left|\int_Q \left(\left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\psi|\boldsymbol{v}|\partial_l \mathbf{T}^n_{ij}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t\right| \\
&\le \left[\int_Q \left(\left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad \cdot \left[\int_Q \left(\left(\psi|\boldsymbol{v}|\partial_l \mathbf{T}^n_{ij}\right)_{i,j}, \left(\psi|\boldsymbol{v}|\partial_l \mathbf{T}^n_{ij}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}}.
\end{aligned}$$

For the second factor on the right-hand side, we have that

$$\begin{aligned}
\int_Q \left(\left(\psi|\boldsymbol{v}|\partial_l \mathbf{T}^n_{ij}\right)_{i,j}, \left(\psi|\boldsymbol{v}|\partial_l \mathbf{T}^n_{ij}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t &\le C(\boldsymbol{v},\psi)\int_Q (\partial_l \mathbf{T}^n, \partial_l \mathbf{T}^n)_{\mathcal{A}(\mathbf{T}^n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\le C(\boldsymbol{v},\psi)\int_Q \nabla \mathbf{T}^n \cdot \nabla F(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C(\boldsymbol{v},\psi),
\end{aligned}$$

where C is independent of n. For the other factor,

$$\int_Q \left(\left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t \le C\int_Q \frac{g_k(|\mathbf{T}^n|)^2}{1+|\mathbf{T}^n|}\,\mathrm{d}x\,\mathrm{d}t.$$

By definition, we have that $g_k(s) = 0$ for $s \le k$ and, for $s \ge k$,

$$g_k(s) = \int_k^s (1+u^a)^{\frac{1}{a}}\tau_k'(u)\,\mathrm{d}u \le \int_k^{2k} (1+u^a)^{\frac{1}{a}}\tau_k'(u)\,\mathrm{d}u \le Ck \le Cs.$$

Using that $g_k(s) \le Cs$ for every $s \ge 0$, it follows that

$$\int_Q \left(\left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}, \left(\delta_{il} g_k(|\mathbf{T}^n|)\frac{\boldsymbol{v}_j}{|\boldsymbol{v}|}\right)_{i,j}\right)_{\mathcal{A}(\mathbf{T}^n)} \mathrm{d}x\,\mathrm{d}t \le C\int_Q g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t.$$

For the second term on the right-hand side of (6.43), by the boundedness of F, we get that

$$\int_Q \psi g_k(|\mathbf{T}^n|)F(\mathbf{T}^n)_{ij}\frac{\partial \boldsymbol{v}_j}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \le C(\boldsymbol{v},\psi)\int_Q g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t.$$

Putting these together yields

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\left|\int_Q \psi b(v^n)F(\mathbf{T}^n)\cdot(\nabla g_k(|\mathbf{T}^n|)\otimes \boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\le C\lim_{k\to\infty}\lim_{n\to\infty}\left[\int_Q g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t + \left(\int_Q g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Using the pointwise convergence of $(\mathbf{T}^n)_n$, the bound on g_k for each fixed k and the fact that $\mathbf{T} \in L^1(Q)^{d\times d}$ with the dominated convergence theorem, we see that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_Q g_k(|\mathbf{T}^n|)\,\mathrm{d}x\,\mathrm{d}t = \lim_{k\to\infty}\int_Q g_k(|\mathbf{T}|)\,\mathrm{d}x\,\mathrm{d}t \le C\lim_{k\to\infty}\int_{\{|\mathbf{T}|>k\}} |\mathbf{T}|\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Thus (6.40) holds, as required.

The sequence $(\mathbf{T}^n)_n$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega)^{d\times d})$ and hence is uniformly bounded in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$. This is the dual space of $L^1(0,T;C(\overline{\Omega})^{d\times d})$ so by the Banach–Alaoglu theorem, there is a subsequence, not relabelled, that converges weakly-* in $L^\infty_{w^*}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d})$ to a limit $\overline{\mathbf{T}}$, say. Taking the limit in the elastodynamic equation (6.16) when testing against an arbitrary $\boldsymbol{v}\in C^1_D(\overline{\Omega})^d$, we deduce that

$$\int_\Omega \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \langle b(v(t))\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))} = \langle l(t),\boldsymbol{v}\rangle. \tag{6.44}$$

We note that $b(v^n)\to b(v)$ strongly in $L^1(0,T;C(\overline{\Omega}))$. Indeed, since $v^n \overset{*}{\rightharpoonup} v$ weakly-* in $W^{1,\infty}(0,T;H^k(\Omega))$ and $H^k(\Omega)$ is compactly embedded into $C^1(\overline{\Omega})$, we apply the Aubin–Lions lemma to see that $v^n\to v$ strongly in $C([0,T];C^1(\overline{\Omega}))$. By the continuity of b, we immediately deduce that $b(v^n)\to b(v)$ strongly in $C(\overline{Q})$ and so, for every $\boldsymbol{w}\in C(\overline{\Omega})^{d\times d}$ and $\psi\in C([0,T])$,

$$\lim_{n\to\infty}\int_0^T\int_\Omega b(v^n)\mathbf{T}^n\cdot\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T\langle\mathbf{T}^n, b(v^n)\boldsymbol{w}\rangle\psi\,\mathrm{d}t = \int_0^T\langle\overline{\mathbf{T}}, b(v)\boldsymbol{w}\rangle\psi\,\mathrm{d}t.$$

Using (6.40) and (6.44), and arguing as in Chapter 4, we can construct an error term $\tilde{\boldsymbol{g}}\in L^\infty_{w^*}(0,T;(C^1_0(\partial\Omega_N)^d)^*)$ such that (6.37) holds. Recall that $C^1_0(\partial\Omega_N)$ is the set of functions $\boldsymbol{w}\in C^1(\partial\Omega_N)$ such that there exists an extension $\tilde{\boldsymbol{w}}\in C^1_D(\overline{\Omega})^d$ of $\boldsymbol{w}$ to $\overline{\Omega}$ such that it vanishes on $\partial\Omega_D$ and there exists a continuous extension operator $E: C^1(\partial\Omega)^d\to C^1(\overline{\Omega})^d$ such that $E|_{C^1_0(\partial\Omega_N)}$, the restriction to functions in $C^1_0(\partial\Omega_N)^d$, has image contained in $C^1_D(\overline{\Omega})^d$. Furthermore, for every $\boldsymbol{v}\in C^1(\partial\Omega)^d$, we have that

$$\|E\boldsymbol{v}\|_{1,\infty}\le C\|\boldsymbol{v}\|_{C^1(\partial\Omega)},$$

where C is independent of $\boldsymbol{v}$. We define the normal component of $b(v(t))\mathbf{T}(t)$ and $b(v(t))\overline{\mathbf{T}}$ on the Neumann part of the boundary, respectively, by

$$\langle b(v(t))\mathbf{T}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} := \int_\Omega \boldsymbol{u}_{tt}(t)\cdot\tilde{\boldsymbol{v}} + b(v(t))\mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})\,\mathrm{d}x,$$

and

$$\langle b(v(t))\overline{\mathbf{T}}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} := \int_\Omega \boldsymbol{u}_{tt}(t)\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}x + \langle b(v(t))\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))},$$

where $\tilde{\boldsymbol{v}}$ denotes an extension of $\boldsymbol{v}\in C^1_0(\partial\Omega_N)^d$ to an element of $C^1_D(\overline{\Omega})^d$. Considering (6.40) and (6.44), these normal components are well-defined, with the right-hand sides independent of the choice of extension. Considering the extension operator E, we conclude that $b(v(t))\mathbf{T}(t)\mathbf{n}$ and $b(v(t))\overline{\mathbf{T}}(t)\mathbf{n}$ define elements of $(C^1_0(\partial\Omega_N)^d)^*$ for a.e. $t\in(0,T)$. By the regularity of $\boldsymbol{u}_{tt}$, $\mathbf{T}$ and $\overline{\mathbf{T}}$, it follows that $b(v)\mathbf{T}\mathbf{n}$, $b(v)\overline{\mathbf{T}}\mathbf{n}\in L^\infty_{w^*}(0,T;(C^1_0(\partial\Omega_N)^d)^*)$.

We define the penalisation $\tilde{\boldsymbol{g}}$ on the Neumann part of the boundary by

$$\langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} := \langle b(v(t))\overline{\mathbf{T}}(t)\mathbf{n} - b(v(t))\mathbf{T}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N},$$

for a.e. $t \in (0,T)$ and every $\boldsymbol{v} \in C_0^1(\partial\Omega_N)^d$. It follows that, for $\boldsymbol{v} \in C_D^1(\overline{\Omega})^d$ and a.e. $t \in (0,T)$, we have

$$
\begin{aligned}
\int_\Omega \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v} + b(v(t))\mathbf{T}(t)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x &= \langle b(v(t))\mathbf{T}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} \\
&= \langle b(v(t))\overline{\mathbf{T}}(t)\mathbf{n},\boldsymbol{v}\rangle|_{\partial\Omega_N} - \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} \\
&= \int_\Omega \boldsymbol{u}_{tt}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \langle\overline{\mathbf{T}}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{(\mathcal{M}(\overline{\Omega}),C(\overline{\Omega}))} - \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} \\
&= \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}x - \langle\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N} \\
&= \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \langle\boldsymbol{g}(t)-\tilde{\boldsymbol{g}}(t),\boldsymbol{v}\rangle|_{\partial\Omega_N}.
\end{aligned}
$$

Thus the elastodynamic equation (6.1a) is satisfied weakly up to a penalisation on the Neumann part of boundary in the required sense. □

Proposition 6.20. *Let the assumptions of Theorem 6.19 hold and let $(\boldsymbol{u},\mathbf{T},v,\tilde{\boldsymbol{g}})$ be the solution quadruple constructed there. For a.e. $t \in (0,T)$, $v(t)$ solves the minimisation problem*

$$
\begin{aligned}
&\mathcal{E}(\boldsymbol{u}(t),v(t)) + \mathcal{H}(v(t)) + \mathcal{G}_k(v(t),v_t(t)) \\
&\quad= \inf_{v\in H^k_{D+1}(\Omega),\, v\le v(t)} \left\{\mathcal{E}(\boldsymbol{u}(t),v) + \mathcal{H}(v) + \mathcal{G}_k(v,v_t(t))\right\}.
\end{aligned}
$$

Furthermore, $v_t \le 0$ a.e. in Q.

Proof. First, recall that $v_t^n \le 0$ in Q. Weak convergence preserves ordering so clearly $v_t \le 0$. To prove that the minimisation problem holds, recall that, for every $\chi \in H_D^k(\Omega)$ such that $\chi \le 0$, for a.e. $t \in (0,T)$,

$$
\begin{aligned}
0 &\le \left[\partial_v\mathcal{E}_n(\boldsymbol{u}^n(t),v^n(t)) + \mathcal{H}'(v^n(t)) + \partial_v\mathcal{G}_k(v^n(t),v_t^n(t))\right](\chi) \\
&= \int_\Omega \frac{b'(v^n)}{\alpha}\chi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) + \frac{1}{2\epsilon}(v^n(t)-1)\chi + 2\epsilon\nabla v^n(t)\cdot\nabla\chi\,\mathrm{d}x + (\chi,v_t^n(t))_{k,2}.
\end{aligned}
$$

Let $\psi \in C([0,T])$ such that $\psi \ge 0$. Multiply the above inequality by ψ and integrate over $(0,T)$. Using the weak convergence results, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T \psi\left[\int_\Omega \frac{1}{2\epsilon}(v^n(t)-1)\chi + 2\epsilon\nabla v^n(t)\cdot\nabla\chi\,\mathrm{d}x + (\chi,v_t^n(t))_{k,2}\right]\mathrm{d}t \\
&= \int_0^T \psi\left[\int_\Omega \frac{1}{2\epsilon}(v(t)-1)\chi + 2\epsilon\nabla v(t)\cdot\nabla\chi\,\mathrm{d}x + (\chi,v_t^n(t))_{k,2}\right]\mathrm{d}t.
\end{aligned}
$$

It remains to deal with the nonlinear term. It is sufficient to show that

$$
\liminf_{n\to\infty}\int_Q \psi\frac{b'(v^n)}{\alpha}\chi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \psi\frac{b'(v)}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\,\mathrm{d}x\,\mathrm{d}t
$$

First, we claim that $\boldsymbol{\varepsilon}(\boldsymbol{u}^n) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ pointwise a.e. in Q. We have that $(\boldsymbol{u}^n)_n$ converges weakly in the space $W^{1,2}(0,T;W^{2,2}_{loc}(\Omega)^d)$. By the Aubin–Lions lemma, $\boldsymbol{u}^n \to \boldsymbol{u}$ strongly in $L^2(0,T;W^{1,2}_{loc}(\Omega)^d)$ and so $\boldsymbol{\varepsilon}(\boldsymbol{u}^n) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ pointwise a.e. in Q. Next, we claim that $\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) \to$

$\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))$ pointwise a.e. in Q. We have $\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u}) = F(\mathbf{T})$ pointwise a.e. in Q. By the memory kernel property and safety strain condition (6.5), we have

$$\begin{aligned}
|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))| &= \left|\mathrm{e}^{-\alpha t}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \int_0^t \mathrm{e}^{\alpha(s-t)}F(\mathbf{T}(s))\,\mathrm{d}s\right| \\
&\leq \frac{C_*\mathrm{e}^{-\alpha t}}{\alpha} + \int_0^t \mathrm{e}^{\alpha(s-t)}\,\mathrm{d}s \\
&= \frac{1}{\alpha}\left[1+\mathrm{e}^{-\alpha T}(C_*-1)\right] \\
&= C_{**},
\end{aligned}$$

where $C_{**} \in (0,1)$ is a fixed constant depending only on C_*, T and α. In particular, we have that $\|\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})\|_{L^\infty(Q)} \leq C_{**}$. Thus, for a.e. $(t,x) \in Q$, $|\boldsymbol{\varepsilon}(\boldsymbol{u}^n(\alpha\boldsymbol{u}(t,x)))| < 1$ for n sufficiently large (depending on (t,x)). The function φ^* is well-defined on the open unit ball in $\mathbb{R}^{d\times d}$ and in this set $\varphi_n^*(\mathbf{T}) \to \varphi^*(\mathbf{T})$ for each fixed $\mathbf{T}$. It follows that

$$\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) \to \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \quad \text{pointwise a.e. in } Q.$$

Recalling the pointwise convergence of $(v^n)_n$ and continuity of b', it follows that

$$\psi\frac{b'(v^n)}{\alpha}\chi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) \to \psi\frac{b'(v)}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \quad \text{pointwise a.e. in } Q.$$

The function on the left-hand side is non-positive so by Fatou's lemma, it follows that

$$\begin{aligned}
\int_0^T \psi\partial_v\mathcal{E}(\boldsymbol{u})(t),v(t))(\chi)\,\mathrm{d}t &= \int_Q \psi\frac{b'(v)}{\alpha}\chi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \limsup_{n\to\infty}\int_Q \psi\frac{b'(v^n)}{\alpha}\chi\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since ψ is arbitrary, we conclude the desired result. □

Using these pointwise convergence results with Fatou's lemma, and the weak lower semi-continuity of norms, taking $n \to \infty$ in the energy-dissipation equality for the regularised solution yields the following result immediately.

Proposition 6.21. *Let the assumptions of Theorem 6.19 hold and let $(\boldsymbol{u}, \mathbf{T}, v, \tilde{\boldsymbol{g}})$ be the solution quadruple constructed there. For every $t \in [0,T]$, the following energy-dissipation inequality holds:*

$$\begin{aligned}
&\mathcal{F}(t;\boldsymbol{u}(t),\boldsymbol{u}_t(t),v(t)) + \int_0^t \|v_t(s)\|_{k,2}^2\,\mathrm{d}s + \int_0^t \langle l_t(s),\boldsymbol{u}(s)\rangle\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega b(v)\left[F^{-1}(\boldsymbol{\varepsilon}(\boldsymbol{u}_t+\alpha\boldsymbol{u})) - F^{-1}(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))\right]\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\,\mathrm{d}x\,\mathrm{d}s \leq \mathcal{F}(0;\boldsymbol{u}_0,\boldsymbol{u}_1,v_0).
\end{aligned}$$

To prove that equality holds, we must assume that the Neumann part of the boundary is empty, i.e., fully Dirichlet boundary conditions. The main problem is the presence of the error term on the Neumann part of the boundary. We discuss an equivalent formulation of the energy-dissipation equality to see why we must make this restriction. Suppose that we have some sufficiently smooth solution of the strain-limiting problem which satisfies the energy-dissipation equality. Differentiating with respect to the time variable and applying the chain rule for weak derivatives, we see that

$$0 = \left\{ \int_\Omega \boldsymbol{u}_{tt} \cdot \boldsymbol{u}_t + b(v)\mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \,\mathrm{d}x - \langle l, \boldsymbol{u}_t \rangle \right\}$$
$$+ \left[\int_\Omega \frac{b'(v)v_t}{\alpha} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u})) + \frac{1}{2\epsilon}(v-1)v_t + 2\epsilon \nabla v \cdot \nabla v_t \,\mathrm{d}x + \|v_t\|_{k,2}^2 \right],$$

for a.e. $t \in (0,T)$. The terms in the curly brackets correspond to the elastodynamic equation and the terms in the square brackets are exactly

$$\left[\partial_v \mathcal{E}(\boldsymbol{u}, v) + \mathcal{H}'(v) + \partial_v \mathcal{G}_k(v, v_t)\right](v_t). \tag{6.45}$$

From Propsition 6.17, the expression (6.45) is bounded below by 0. If we additionally have that the expression (6.45) vanishes, then we must have that

$$\int_\Omega \boldsymbol{u}_{tt} \cdot \boldsymbol{u}_t + b(v)\mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \,\mathrm{d}x - \langle l, \boldsymbol{u}_t \rangle = 0, \tag{6.46}$$

for a.e. $t \in (0,T)$. It is clear that (6.46) holds if there is no penalisation on the Neumann part of the boundary when testing with $\boldsymbol{u}_t$, which is trivially the case if $\partial\Omega_N = \emptyset$. In the mixed boundary condition case, we would need to show that

$$\lim_{n\to\infty} \int_Q b(v^n)\mathbf{T}^n \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n) \,\mathrm{d}x \,\mathrm{d}t = \int_Q b(v)\mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \,\mathrm{d}x \,\mathrm{d}t.$$

However, we do not have any global convergence results that are sufficient in order to prove such a claim. From now on, we assume that the Neumann part of the boundary is empty so $\partial\Omega = \partial\Omega_D$. Thus (6.46) holds and now we focus on showing that the expression (6.45) vanishes at a.e. $t \in (0,T)$. We have proven that

$$0 \le \left[\partial_v \mathcal{E}(\boldsymbol{u}, v) + \mathcal{H}'(v) + \partial_v \mathcal{G}_k(v, v_t)\right](v_t),$$

for a.e. $t \in (0,T)$. Now we will show that the opposite inequality holds. By weak lower semi-continuity, we see that, for every $\psi \in C([0,T])$ such that $\psi \ge 0$,

$$0 = \lim_{n\to\infty} \int_0^T \psi \left[\int_\Omega \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)) + \frac{1}{2\epsilon}(v^n-1)v_t^n + 2\epsilon \nabla v^n \cdot \nabla v_t^n \,\mathrm{d}x + \|v_t^n\|_{k,2}^2 \right] \mathrm{d}t$$
$$\ge \lim_{n\to\infty} \int_Q \psi \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \int_0^T \psi \left[\int_\Omega \frac{1}{2\epsilon}(v-1)v_t + 2\epsilon \nabla v \cdot \nabla v_t \,\mathrm{d}x + \|v_t\|_{k,2}^2 \right] \mathrm{d}t.$$

We need to prove that

$$\lim_{n\to\infty} \int_Q \psi \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)) \,\mathrm{d}x \,\mathrm{d}t \ge \int_Q \psi \frac{b'(v)v_t}{\alpha} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u})) \,\mathrm{d}x \,\mathrm{d}t. \tag{6.47}$$

At best, the convergence of $(v_t^n)_n$ is weak in $L^\infty(0,T;H^k(\Omega))$. Hence a strong convergence result for the sequence $(\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)))_n$ needs to be found. We will prove that strong convergence in $L^1(0,T;L^1_{loc}(\Omega))$ holds.

First, we show that $\mathbf{T}^n \to \mathbf{T}$ strongly in $L^1(0,T;L^1_{loc}(\Omega)^{d\times d})$. If $a \in (0,\frac{2}{d})$, in fact this can be improved to $L^{1+\delta}(0,T;L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for some $\delta = \delta(a) > 0$. Next, we show that $\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n))$ converges to $\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}))$ strongly in $L^1(0,T;L^1_{loc}(\Omega))$ and use the fact that $(v_t^n)_n$ and v_t vanish on the boundary to prove (6.47). The energy-dissipation equality can be concluded from the aforementioned arguments and we deduce the existence of a weak energy solution to the strain-limiting problem in the case of fully Dirichlet boundary conditions.

Lemma 6.22. *Let the assumptions of Theorem 6.19 hold and suppose additionally that $\partial\Omega_D = \partial\Omega$. Let $(\boldsymbol{u},\mathbf{T},v)$ be the solution triple constructed there and let $((\boldsymbol{u}^n,\mathbf{T}^n,v^n))_n$ be the sequence of solutions to the regularised problem from Theorem 6.9. Then,*

$$\mathbf{T}^n \to \mathbf{T} \quad \textit{strongly in } L^1(0,T;L^1_{loc}(\Omega)^{d\times d}).$$

Proof. By the pointwise convergence result for $(\mathbf{T}^n)_n$, it is enough to show that $|\mathbf{T}^n| \to |\mathbf{T}|$ strongly in $L^1(0,T;L^1_{loc}(\Omega))$. There exists a constant C, independent of n, such that, for every Ω_0 compactly contained in Ω,

$$\sup_{t\in[0,T]} \int_\Omega \frac{|\mathbf{T}^n(t)|^2}{n}\,\mathrm{d}x + \int_Q \frac{|\mathbf{T}^n_t|^2}{n}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{c(\Omega_0)}\int_0^T\int_{\Omega_0} \frac{|\nabla \mathbf{T}^n|^2}{n}\,\mathrm{d}x\,\mathrm{d}t \le C,$$

where $c(\Omega_0)$ depends only on Ω and Ω_0. By the Aubin–Lions lemma, it follows that $(n^{-\frac{1}{2}}\mathbf{T}^n)_n$ converges to $\mathbf{0}$ strongly in $L^2(0,T;L^2_{loc}(\Omega)^{d\times d})$. On the other hand, we have that

$$\lim_{n\to\infty}\int_Q b(v^n)\mathbf{T}^n\cdot F_n(\mathbf{T}^n)\,\mathrm{d}x\,\mathrm{d}t = \int_Q b(v)\mathbf{T}\cdot F(\mathbf{T})\,\mathrm{d}x\,\mathrm{d}t.$$

By the pointwise convergence and non-negativity of the integrands, we deduce that

$$b(v^n)\mathbf{T}^n\cdot F_n(\mathbf{T}^n) \to b(v)\mathbf{T}\cdot F(\mathbf{T}) \quad \text{strongly in } L^1(0,T;L^1(\Omega)),$$

and so $\mathbf{T}^n\cdot F_n(\mathbf{T}^n) \to \mathbf{T}\cdot F(\mathbf{T})$ strongly in $L^1(0,T;L^1(\Omega))$. Using the convergence result for $(n^{-\frac{1}{2}}\mathbf{T}^n)_n$, it follows that

$$\mathbf{T}^n\cdot F(\mathbf{T}^n) \to \mathbf{T}\cdot F(\mathbf{T}) \quad \text{strongly in } L^1(0,T;L^1_{loc}(\Omega)). \tag{6.48}$$

We can write

$$|\mathbf{T}^n| = \left(\frac{|\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}\right)\cdot\left(\frac{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}{|\mathbf{T}^n|}\right)\chi_{\{|\mathbf{T}^n|>1\}} + |\mathbf{T}^n|\chi_{\{|\mathbf{T}^n|\le 1\}},$$

where χ_A is the indicator function of a set. By the dominated convergence theorem,

$$|\mathbf{T}^n|\chi_{\{|\mathbf{T}^n|\le 1\}} \to |\mathbf{T}|\chi_{\{|\mathbf{T}|\le 1\}} \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

For the other term, the first factor converges strongly in $L^1(0,T;L^1_{loc}(\Omega))$. The second is uniformly bounded and converges pointwise in Q. It follows that

$$\left(\frac{|\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}\right)\cdot\left(\frac{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}{|\mathbf{T}^n|}\right)\chi_{\{|\mathbf{T}^n|>1\}}\to\left(\frac{|\mathbf{T}|^2}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}\right)\cdot\left(\frac{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}{|\mathbf{T}|}\right)\chi_{\{|\mathbf{T}|>1\}},$$

strongly in $L^1(0,T;L^1_{loc}(\Omega))$. The asserted convergence result follows immediately. □

Lemma 6.23. *Let the assumptions of Theorem 6.19 hold and suppose additionally that $\partial\Omega_D = \partial\Omega$. Let $(\boldsymbol{u},\mathbf{T},v)$ be the solution triple constructed there and let $((\boldsymbol{u}^n,\mathbf{T}^n,v^n))_n$ be the solutions to the regularised problem constructed in Theorem 6.9. Furthermore, assume that $a\in(0,\frac{2}{d})$. Let Ω_0 be an open set that is compactly contained in Ω. For every $\delta>0$ such that $a+\delta<\frac{2}{d}$, there exists a constant $C=C(\Omega_0)$ independent of n such that*

$$\int_0^T\int_{\Omega_0}|\mathbf{T}^n|^{1+\delta}+|\mathbf{T}|^{1+\delta}\,\mathrm{d}x\,\mathrm{d}t\le C.$$

Furthermore, $\mathbf{T}^n\to\mathbf{T}$ strongly in $L^{1+\delta}(0,T;L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for every such δ.

Proof. First, recall that there exists a constant $C=C(\Omega_0)$ such that

$$\begin{aligned}C(\Omega_0)&\ge\sup_{t\in[0,T]}\int_\Omega|\mathbf{T}^n(t)|\,\mathrm{d}x+\int_0^T\int_{\Omega_0}\frac{|\nabla\mathbf{T}^n|^2}{(1+|\mathbf{T}^n|^a)^{\frac{1}{a}}}\,\mathrm{d}x\,\mathrm{d}t\\&\ge c_a\left(c_p\int_0^T\|(1+|\mathbf{T}^n(t)|)^{\frac{1-a}{2}}\|_{L^p(\Omega_0)}\,\mathrm{d}t-1\right)\\&\ge c_ac_p\left(\int_0^T\|\mathbf{T}^n\|^{1-a}_{L^{\frac{p(1-a)}{2}}(\Omega_0)}\,\mathrm{d}t-1\right),\end{aligned}$$

where $p\in(2,\frac{2d}{d-2}]$ if $d>2$, $p\in(2,\infty)$ if $d=2$, applying the Sobolev embedding theorem applied to $(1+|\mathbf{T}^n(t)|)^{\frac{1-a}{2}}$. We argue in the case $d>2$ but the case $d=2$ is similar. Choosing $p=\frac{2d}{d-2}$, it follows that

$$\sup_{t\in[0,T]}\int_\Omega|\mathbf{T}^n(t)|\,\mathrm{d}x+\int_0^T\left(\int_{\Omega_0}|\mathbf{T}^n(t)|^{\frac{d(1-a)}{d-2}}\,\mathrm{d}x\right)^{\frac{d-2}{d}}\,\mathrm{d}t\le C(\Omega_0).\tag{6.49}$$

Take $\delta>0$ such that $a+\delta\in(0,\frac{2}{d})$. A standard manipulation shows that

$$\frac{d(1+\delta)-2}{d-2}<\frac{d(1-a)}{d-2}.$$

Applying Hölder's inequality with parameters $\frac{d}{2}$ and $\frac{d}{d-2}$, we get

$$\begin{aligned}\int_{\Omega_0}|\mathbf{T}^n(t)|^{1+\delta}\,\mathrm{d}x&\le\left(\int_{\Omega_0}|\mathbf{T}^n(t)|\,\mathrm{d}x\right)^{\frac{2}{d}}\left(\int_{\Omega_0}|\mathbf{T}^n(t)|^{1+\frac{\delta d}{d-2}}\,\mathrm{d}x\right)^{\frac{d-2}{d}}\\&\le C\left(\int_{\Omega_0}|\mathbf{T}^n(t)|^{\frac{d(1+\delta)-2}{d-2}}\,\mathrm{d}x\right)^{\frac{d-2}{d}}\\&\le C\left(1+\int_{\Omega_0}|\mathbf{T}^n(t)|^{\frac{d(1-a)}{d-2}}\,\mathrm{d}x\right)^{\frac{d-2}{d}}.\end{aligned}$$

Integration over $(0, T)$ and use of (6.49) yields

$$\int_0^T \int_{\Omega_0} |\mathbf{T}^n(t)|^{1+\delta} \, \mathrm{d}x \, \mathrm{d}t \leq C(\Omega_0).$$

Since $\delta > 0$, we deduce that $\mathbf{T}^n \rightharpoonup \mathbf{T}$ weakly in $L^{1+\delta}(0, T; L^{1+\delta}_{loc}(\Omega)^{d\times d})$. For strong convergence, let $\alpha \in (0, 1)$. Applying Hölder's inequality with parameter $p = \frac{1}{\alpha}$, we see that

$$\begin{aligned}
\int_0^T \int_{\Omega_0} |\mathbf{T}^n - \mathbf{T}|^{1+\alpha\delta} \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_{\Omega_0} |\mathbf{T}^n - \mathbf{T}|^{\alpha+\alpha\delta+(1-\alpha)} \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \left(\int_0^T \int_{\Omega_0} |\mathbf{T}^n - \mathbf{T}|^{1+\delta} \, \mathrm{d}x \, \mathrm{d}t \right)^{\alpha} \left(\int_0^T \int_{\Omega_0} |\mathbf{T}^n - \mathbf{T}| \, \mathrm{d}x \, \mathrm{d}t \right)^{1-\alpha} \\
&\leq C \left(\int_0^T \int_{\Omega_0} |\mathbf{T}^n - \mathbf{T}| \, \mathrm{d}x \, \mathrm{d}t \right)^{1-\alpha},
\end{aligned}$$

where the right-hand side vanishes in the limit. Hence the strong convergence result follows. □

Lemma 6.24. *Let the assumptions of Theorem 6.19 hold and suppose additionally that $\partial\Omega_D = \partial\Omega$. Let $(\boldsymbol{u}, \mathbf{T}, v)$ be the solution triple and $((\boldsymbol{u}^n, \mathbf{T}^n, v^n))_n$ the sequence of solutions to the regularised problem from Theorem 6.9. Then,*

$$\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)) \to \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u})) \quad \textit{strongly in } L^1(0, T; L^1_{loc}(\Omega)).$$

Proof. First, we show that $\varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)) \to \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}))$ strongly in $L^1(0, T; L^1_{loc}(\Omega))$. From Lemma 6.22, $(n^{-1}|\mathbf{T}^n|^2)_n$ converges strongly to 0 in $L^1(0, T; L^1_{loc}(\Omega))$. Using this with the dominated convergence theorem, $\varphi_n(\mathbf{T}^n) \to \varphi(\mathbf{T})$ strongly in $L^1(0, T; L^1_{loc}(\Omega))$. Using this with the strong convergence of $(\mathbf{T}^n \cdot F_n(\mathbf{T}^n))_n$, we get

$$\varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)) \to \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})) \quad \text{strongly in } L^1(0, T; L^1_{loc}(\Omega)).$$

To prove the asserted convergence result for $(\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)))_n$, by the memory kernel property and Jensen's inequality, we see that

$$\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t))) \leq \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) + \int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)) \, \mathrm{d}s. \tag{6.50}$$

By the pointwise convergence result for $(\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)))_n$ and the strong convergence of $(\varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)))_n$, it follows that

$$\begin{aligned}
0 &\leq \int_0^T \int_{\Omega_0} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) - \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}(t))) + \left[\int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})) \, \mathrm{d}s \right] \mathrm{d}x \, \mathrm{d}t \\
&\leq \liminf_{n\to\infty} \int_0^T \int_{\Omega_0} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) - \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t))) + \left[\int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi_n^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t^n + \alpha \boldsymbol{u}^n)) \, \mathrm{d}s \right] \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_{\Omega_0} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) + \left[\int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u})) \, \mathrm{d}s \right] \mathrm{d}x \, \mathrm{d}t \\
&\quad - \limsup_{n\to\infty} \int_0^T \int_{\Omega_0} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t))) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

In particular, we have

$$
\begin{aligned}
&\int_0^T \int_{\Omega_0} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) - \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}(t))) + \left[\int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}))\,\mathrm{d}s\right] \mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \int_{\Omega_0} \varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0)) + \left[\int_0^t \alpha \mathrm{e}^{\alpha(s-t)} \varphi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}))\,\mathrm{d}s\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad - \limsup_{n\to\infty} \int_0^T \int_{\Omega_0} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

and so

$$
\limsup_{n\to\infty} \int_0^T \int_{\Omega_0} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^T \int_{\Omega_0} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n(t)))\,\mathrm{d}x\,\mathrm{d}t,
$$

using Fatou's lemma to obtain the second inequality. Thus $(\varphi_n^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}^n)))_n$ converges to $\varphi^*(\boldsymbol{\varepsilon}(\alpha \boldsymbol{u}_0))$ both in the sense of norms and pointwise a.e. in Q. Thus we obtain strong convergence in $L^1(0,T;L^1_{loc}(\Omega))$. □

The final piece of information needed in order to deduce the desired energy equality concerns the sequence $(v_t^n)_n$. Namely, we must show that the functions vanish uniformly on the boundary, which is only possible due to the fully Dirichlet boundary conditions. This allows us to overcome the lack of global convergence in space. We require the following auxiliary result.

Lemma 6.25. *Let $w \in W^{1,\infty}(\Omega)$ be such that $w = 0$ on $\partial\Omega$ in the trace sense. There exist positive constants C and δ_0, depending only on Ω, such that*

$$
|w(x)| \leq C\|\nabla w\|_\infty d(x,\partial\Omega) \quad \textit{for a.e. } x \in \Omega \textit{ such that } d(x,\partial\Omega) < \delta_0.
$$

The idea of the proof is to cover the boundary by finitely many balls such that the boundary is given by a Lipschitz function within each ball. The fundamental theorem of calculus is used alongside comparing points near the boundary and those on the boundary to deduce the required result.

Corollary 6.26. *Let the assumptions of Theorem 6.19 hold and assume additionally that $\partial\Omega_D = \partial\Omega$. Let δ_0 be the constant from Lemma 6.25. For every $\delta \in (0,\delta_0)$, let Ω_δ be the set of points $x \in \Omega$ such that $d(x,\partial\Omega) < \delta$. There exists a constant C, independent of n, such that, for every $\delta \in (0,\delta_0)$,*

$$
\|b'(v^n)v_t^n\|_{L^\infty((0,T)\times\Omega_\delta)} \leq C\delta. \tag{6.51}
$$

Proof. For $x \in \Omega_\delta$ and a.e. $t \in (0,T)$, we have

$$
|b'(v^n(t,x))v_t^n(t,x)| \leq \|b'(v^n)\|_\infty |v_t^n(t,x)| \leq C\|v^n\|_\infty \|\nabla v_t^n\|_\infty d(x,\partial\Omega) \leq C\delta.
$$

Taking the essential supremum over the left-hand side, the bound (6.51) follows. □

Lemma 6.27. *Let the assumptions of Theorem 6.19 hold and assume additionally that $\partial\Omega_D = \partial\Omega$. Let $(\boldsymbol{u}, \mathbf{T}, v)$ be the solution triple from Theorem 6.19 and $((\boldsymbol{u}^n, \mathbf{T}^n, v^n))_n$ the solutions to the regularised problem constructed in Theorem 6.9. Let $\psi \in C([0,T])$ be fixed but arbitrary. Then,*

$$\lim_{n\to\infty} \int_Q \psi \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) \,\mathrm{d}x\,\mathrm{d}t = \int_Q \psi \frac{b'(v)v_t}{\alpha} \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $\delta \in (0, \delta_0)$ where δ_0 is the constant from Lemma 6.25. Let Ω_δ be as in Corollary 6.26. Then we have that

$$\begin{aligned}
&\left| \int_Q \psi \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) - \psi \frac{b'(v)v_t}{\alpha} \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq \left| \int_0^T \int_{\Omega\setminus\Omega_\delta} \psi \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \left(\frac{b'(v)v_t}{\alpha} - \frac{b'(v^n)v_t^n}{\alpha} \right) \mathrm{d}x\,\mathrm{d}t \right| \\
&\quad + C \int_0^T \int_{\Omega\setminus\Omega_\delta} |\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) - \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}))| \,\mathrm{d}x\,\mathrm{d}t + C\delta,
\end{aligned} \tag{6.52}$$

where C is a positive constant that is independent of n and δ.

Notice that $\Omega \setminus \Omega_\delta$ is a compact subset of Ω. Using this with the local convergence results for $\varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n))$, we see that the second term on the right-hand side of (6.52) vanishes as $n \to \infty$. For the first term, we use that $(b'(v^n)v_t^n)_n$ converges weakly-* in $L^\infty(Q)$ as $n \to \infty$, with the fact that $\psi\varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \in L^1(Q)$. It follows that the first term also vanishes in the limit. This yields

$$\lim_{n\to\infty} \left| \int_Q \psi \frac{b'(v^n)v_t^n}{\alpha} \varphi_n^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u}^n)) - \psi \frac{b'(v)v_t}{\alpha} \varphi^*(\boldsymbol{\varepsilon}(\alpha\boldsymbol{u})) \,\mathrm{d}x\,\mathrm{d}t \right| \leq C\delta,$$

for every $\delta \in (0, \delta_0)$. Letting $\delta \to 0$, we conclude the asserted result. □

Corollary 6.28. *Let the assumptions of Theorem 6.19 hold and suppose additionally that $\partial\Omega_D = \partial\Omega$. Let $(\boldsymbol{u}, \mathbf{T}, v)$ be the corresponding solution triple that is constructed in Theorem 6.19. Then $(\boldsymbol{u}, \mathbf{T}, v)$ is a weak energy solution of the strain-limiting dynamic fracture problem in the following sense. For every test function $\boldsymbol{w} \in W_0^{1,2}(\Omega)^d \cap W^{1,\infty}(\Omega)^d$, we have that*

$$\int_\Omega \boldsymbol{u}_{tt}(t) \cdot \boldsymbol{w} + b(v(t))\mathbf{T}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) \,\mathrm{d}x = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{w} \,\mathrm{d}x,$$

for a.e. $t \in (0,T)$, with constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha\boldsymbol{u}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} \quad \textit{pointwise a.e. in } Q.$$

The minimisation problem (6.2) holds for a.e. $t \in (0,T)$ and the energy-dissipation equality (6.3) is satisfied for every $t \in [0,T]$. The initial conditions hold in the sense that

$$\lim_{t\to 0+} [\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{1,2} + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_1\|_{1,2} + \|v(t) - v_0\|_{k,2}] = 0.$$

Chapter 7

Conclusions and open problems

To conclude this thesis, we will gather up the results that have been presented here and discuss the relevant open problems that are associated with this work.

In the work from Chapters 2, 3 and 4, we prove a full existence result to the strain-limiting viscoelastic problem in the setting of periodic and Dirichlet boundary conditions for an arbitrary value of the parameter $a > 0$ in the constitutive relation. We note that the magnitude of a essentially refers to the differentiability properties of the function F in the constitutive relation. The existence result is despite the best bound on the sequence of approximate stress tensors $(\mathbf{T}^n)_n$ being in $L^\infty(0,T;L^1(\Omega)^{d\times d})$. When we have that $a \in (0, \frac{2}{d})$, it is possible to show that $(\mathbf{T}^n)_n$ is a bounded sequence in $L^{1+\delta}(0,T;L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for every $\delta > 0$ such that $a + \delta < \frac{2}{d}$ (with a global bound in the periodic case). This improves the convergence of $(\mathbf{T}^n)_n$ from only pointwise in Q to strong convergence in $L^1(0,T;L^1_{loc}(\Omega)^{d\times d})$.

In the case of mixed Dirichlet–Neumann boundary conditions, we have an existence result up to satisfaction of the Neumann boundary condition. That is, there is an error term on $\partial\Omega_N$ that appears in the weak form of the elastodynamic equation. The reason for such a failure is that the convergence of the approximate stress tensors $(\mathbf{T}^n)_n$ is only pointwise on Q or, if a is small, weak in $L^{1+\delta}(0,T;L^{1+\delta}_{loc}(\Omega)^{d\times d})$ for some $\delta > 0$. In either case, we have no suitable convergence result near the boundary.

An immediate question that we must ask is whether or not a solution exists that solves the strain-limiting problem in some weak sense without this error. It is reassuring that if a solution does exist, the solution that we construct from the limit of the regularised solutions coincides with the true solution. The work in [5] shows that, even in the steady case where we have no time dependence, it is not clear if a solution exists without the penalisation on the Neumann part of the boundary. It would make sense to investigate this issue further in the steady case and then try to adapt any results to the dynamic problem studied here. A possible place to start might be to try to construct some type of higher integrability estimates near the boundary. However, comparing this problem to the minimal surface equation where such a singularity cannot be removed from the boundary [16], proving such a claim seems less promising.

A related problem that would be interesting to study is the balance of momentum coupled with the stress-rate constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = h(\mathbf{T}) - \gamma \mathbf{T}_t,$$

where h is a nonlinear function and γ is a constant. Of particular interest is when h is bounded. This relation is first mentioned in [43] where the authors impose that γ is positive in order for the model to be thermodynamically consistent. They study the one-dimensional problem, investigating the similarities between the stress-rate model and the strain-rate model. The results suggest that it would be interesting to study what might happen in higher dimensions. However, there are no obvious energy estimates resulting from such a system so it is not immediately clear how the existence of solutions might be proven. This suggests that perhaps some modification of the system must be made in order to perform any analysis.

We must also mention the strain-limiting dynamic elastic problem. This is the elastodynamic equation coupled with the constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}. \tag{7.1}$$

There is no strain-rate term $\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ present in (7.1), which provided a certain level of smoothing in the problems studied in this thesis. Little attention has been given to this problem despite many investigations in the steady case, for example, [5], [19], [18], to name a few. We note that some work has been done in the parabolic case [22] where $\boldsymbol{u}_{tt}$ is replaced with $\boldsymbol{u}_t$ in the elastodynamic equation. However, this greatly simplifies the problem compared to when $\boldsymbol{u}_{tt}$ is present. As with the stress-rate problem, the first issue that occurs is that there are no obvious energy estimates for the system. One possible way to tackle this might be to introduce a damping term into the elastodynamic equation, since we have removed it from the constitutive relation. For example, one might study (7.1) coupled with

$$\boldsymbol{u}_{tt} = \operatorname{div}(\mathbf{T}) + \beta \nabla \cdot \nabla \boldsymbol{u}_t, \tag{7.2}$$

where β is a fixed positive constant. More generally, it could be interesting to investigate (7.2) coupled with the strain-limiting viscoelastic relation that was studied throughout this thesis and consider the effects of an extra damping term.

In Chapter 5, we developed a general theory for nonlinear dynamic fracture problems with a phase-field approximation that mimics an *a priori* unknown crack set. The existence of a weak energy solution was proven for the specific constitutive relation

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t + \alpha \boldsymbol{u}) = |\mathbf{T}|^{p-2}\mathbf{T},$$

where $p \in (1, \infty)$. The solutions satisfied the elastodynamic equation in the usual weak sense, the constitutive relation held pointwise on the space-time domain, and both the minimisation problem (ensuring crack growth) and energy-dissipation equality held at every point in time.

In Chapter 6, we combined the techniques developed in Chapter 5 with those of Chapter 4 to investigate a strain-limiting dynamic fracture problem. For mixed Dirichlet–Neumann boundary conditions, a singularity appeared on the Neumann part of the boundary. For this reason, only an energy-dissipation inequality is proven and partial satisfaction of the elastodynamic equation. We do not obtain a weak energy solution in the complete sense. However, for fully Dirichlet boundary conditions, a full existence result can be proven.

The first question that arises from this work is the uniqueness of solutions. Even in the linear case [73], the uniqueness of solutions remains open. The other obvious question, at least in the strain-limiting case, is whether or not the existence can be proven without the presence of the rate-dependent term in the minimisation problem. It ensures good regularity of the phase-field approximation which is necessary for deriving higher regularity estimates on the stress tensor. Even removing the restriction on k would be a starting point in order to develop the theory further.

As discussed in the introduction, an issue if we were to take the limit in the approximation parameter ϵ is the presence of the strain rate term $\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ in the constitutive relation. It can cause the viscoelastic paradox, where only stationary solutions are allowed. Hence, of interest would be an investigation into nonlinear dynamic fracture problems without the strain-rate term. However, even in the linear case, it is not clear how to do this. It has been attempted when there is a rate-dependent term in the minimisation problem [26], but there has been no analysis achieved, to my knowledge, for the elastodynamic problem with an elastic constitutive relation and a minimisation problem involving only the elastic energy and surface energy. As discussed above, it is not clear how to study the strain-limiting elastic problem even without the complication of damage so there is much to be learnt before we can think of tackling a strain-limiting elastic problem with fracture. Following this, the ultimate goal would be an existence result for a dynamic fracture problem with an exact crack set, rather than a phase-field approximation, and a strain-limiting constitutive relation between the stress and the strain. Numerical simulations of such a model are currently out of reach due to the highly nonlinear nature of the problem. However, an interesting problem to study would be the numerical analysis and simulations of a dyanmic fracture problem with linear constitutive relation. The next step may be to investigate nonlinear fracture problems in the quasi-static case as this could simplify matters slightly. Such simulations may give an idea of how such a phase-field model relates to actual crack formation.

Bibliography

[1] L. Ambrosio, S. Mortola, and V. M. Tortorelli. Functionals with linear growth defined on vector valued BV functions. J. Math. Pures Appl. (9), 70(3):269–323, 1991.

[2] L. Ambrosio and V. M. Tortorelli. On the approximation of free discontinuity problems. Boll. Un. Mat. Ital. B (7), 6(1):105–123, 1992.

[3] T. M. Atanackovic. A modified Zener model of a viscoelastic body. Contin. Mech. Thermodyn., 14(2):137–148, 2002.

[4] T. Bárta. One-dimensional model describing the non-linear viscoelastic response of materials. Comment. Math. Univ. Carolin., 55(2):227–246, 2014.

[5] L. Beck, M. Bulíček, J. Málek, and E. Süli. On the existence of integrable solutions to non-linear elliptic systems and variational problems with linear growth. Arch. Ration. Mech. Anal., 225(2):717–769, 2017.

[6] B. Benešová, M. Kružík, and A. Schlömerkemper. A note on locking materials and gradient polyconvexity. Math. Models Methods Appl. Sci., 28(12):2367–2401, 2018.

[7] J. Bergh and J. Löfström. Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[8] A. Bonito, V. Girault, and E. Süli. Finite element approximation of a strain-limiting elastic model. IMA J. Numer. Anal., 40(1):29–86, 2020.

[9] M. J. Borden, T. J. Hughes, C. M. Landis, and C. V. Verhoosel. A higher-order phase-field model for brittle fracture: Formulation and analysis within the isogeometric analysis framework. Computer Methods in Applied Mechanics and Engineering, 273:100–118, 2014.

[10] B. Bourdin, C. Larsen, and C. Richardson. A time-discrete model for dynamic fracture based on crack regularization. International Journal of Fracture, 168(2):133–143, 2011.

[11] A. Braides. Γ-convergence for beginners, volume 22 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2002.

[12] F. E. Browder. Existence and perturbation theorems for nonlinear maximal monotone operators in Banach spaces. Bull. Amer. Math. Soc., 73:322–327, 1967.

[13] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska Gwiazda. On steady flows of incompressible fluids with implicit power-law-like rheology. Adv. Calc. Var., 2(2):109–136, 2009.

[14] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. SIAM J. Math. Anal., 44(4):2756–2801, 2012.

[15] M. Bulíček, P. Kaplický, and M. Steinhauer. On existence of a classical solution to a generalized Kelvin-Voigt model. Pacific J. Math., 262(1):11–33, 2013.

[16] M. Bulíček, J. Málek, K. Rajagopal, and E. Süli. On elastic solids with limiting small strain: modelling and analysis. EMS Surv. Math. Sci., 1(2):283–332, 2014.

[17] M. Bulíček, J. Málek, and K. R. Rajagopal. On Kelvin–Voigt model and its generalizations. Evol. Equ. Control Theory, 1(1):17–42, 2012.

[18] M. Bulíček, J. Málek, K. R. Rajagopal, and J. R. Walton. Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies. Calc. Var. Partial Differential Equations, 54(2):2115–2147, 2015.

[19] M. Bulíček, J. Málek, and E. Süli. Analysis and approximation of a strain-limiting nonlinear elastic model. Math. Mech. Solids, 20(1):92–118, 2015.

[20] M. Bulíček, V. Patel, Y. Şengül, and E. Süli. Existence and uniqueness of global weak solutions to strain-limiting viscoelasticity with Dirichlet boundary data. Submitted. *https://arxiv.org/pdf/2011.11683.pdf*, 2020.

[21] M. Bulíček, V. Patel, Y. Şengül, and E. Süli. Existence of large-data global weak solutions to a model of a strain-limiting viscoelastic body. Commun. Pure Appl. Anal., 20(5):1931–1960, 2021.

[22] M. Bulíček, D. Hruška, and J. Málek. On evolutionary problems with a-priori bounded gradients. Submitted. *https://arxiv.org/abs/2102.13447.pdf*, 2021.

[23] R. Bustamante and K. Rajagopal. Solutions of some simple boundary value problems within the context of a new class of elastic materials. International Journal of Non-Linear Mechanics, 46(2):376–386, 2011.

[24] C. Canuto and A. Quarteroni. Approximation results for orthogonal polynomials in Sobolev spaces. Math. Comp., 38(157):67–86, 1982.

[25] M. Caponi. Linear hyperbolic systems in domains with growing cracks. Milan J. Math., 85(1):149–185, 2017.

[26] M. Caponi. Existence of solutions to a phase-field model of dynamic fracture with a crack-dependent dissipation. NoDEA Nonlinear Differential Equations Appl., 27(2):Paper No. 14, 48, 2020.

[27] M. Caponi and F. Sapio. A dynamic model for viscoelastic materials with prescribed growing cracks. Ann. Mat. Pura Appl. (4), 199(4):1263–1292, 2020.

[28] M. Chirita and C. M. Ionescu. Models of biomimetic tissues for vascular grafts. In On biomimetics, pages 43–52. In-Tech, 2011.

[29] P. G. Ciarlet. On Korn's inequality. Chinese Ann. Math. Ser. B, 31(5):607–618, 2010.

[30] J. C. Criscione and K. R. Rajagopal. On the modeling of the non-linear response of soft elastic bodies. International Journal of Non-Linear Mechanics, 56:20–24, 2013.

[31] Y. Şengül. Nonlinear viscoelasticity of strain rate tye: an overview. Proc. R. Soc. A., 477(2245):20200715–20200715, 2021.

[32] Y. Şengül. Viscoelasticity with limiting strain. Discrete Contin. Dyn. Syst. Ser. S, 14(1):57–70, 2021.

[33] G. Dal Maso and C. J. Larsen. Existence for wave equations on domains with arbitrary growing cracks. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 22(3):387–408, 2011.

[34] G. Dal Maso, C. J. Larsen, and R. Toader. Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition. J. Mech. Phys. Solids, 95:697–707, 2016.

[35] G. Dal Maso and I. Lucardesi. The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on the data. Appl. Math. Res. Express. AMRX, (1):184–241, 2017.

[36] G. Dal Maso and R. Toader. A model for the quasi-static growth of brittle fractures: existence and approximation results. Arch. Ration. Mech. Anal., 162(2):101–135, 2002.

[37] F. Demengel and P. Suquet. On locking materials. Acta Appl. Math., 6(2):185–211, 1986.

[38] M. Destrade, P. M. Jordan, and G. Saccomandi. Compact travelling waves in viscoelastic solids. EPL (Europhysics Letters), 87(4):48001, aug 2009.

[39] R. G. Durán and M. A. Muschietti. The Korn inequality for Jones domains. Electron. J. Differential Equations, pages No. 127, 10, 2004.

[40] E. Emmrich and M. Thalhammer. Convergence of a time discretisation for doubly nonlinear evolution equations of second order. Found. Comput. Math., 10(2):171–190, 2010.

[41] E. Emmrich, D. Šiška, and A. Wróblewska-Kamińska. Equations of second order in time with quasilinear damping: existence in Orlicz spaces via convergence of a full discretisation. Math. Methods Appl. Sci., 39(10):2449–2460, 2016.

[42] H. A. Erbay and Y. Şengül. Traveling waves in one-dimensional non-linear models of strain-limiting viscoelasticity. International Journal of Non-Linear Mechanics, 77(C):61–68, 2015.

[43] H. A. Erbay and Y. Şengül. A thermodynamically consistent stress-rate type model of one-dimensional strain-limiting viscoelasticity. Z. Angew. Math. Phys., 71(3):Paper No. 94, 10, 2020.

[44] H. A. Erbay, A. Erkip, and Y. Şengül. Local existence of solutions to the initial-value problem for one-dimensional strain-limiting viscoelasticity. J. Differential Equations, 269(11):9720–9739, 2020.

[45] L. C. Evans. Partial differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1998.

[46] C. Fefferman. C^m extension by linear operators. Ann. of Math. (2), 166(3):779–835, 2007.

[47] L. E. Figueroa and E. Süli. Greedy approximation of high-dimensional Ornstein-Uhlenbeck operators. Found. Comput. Math., 12(5):573–623, 2012.

[48] G. A. Francfort and C. J. Larsen. Existence and convergence for quasi-static evolution in brittle fracture. Comm. Pure Appl. Math., 56(10):1465–1500, 2003.

[49] G. A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids, 46(8):1319–1342, 1998.

[50] A. D. Freed and K. R. Rajagopal. A viscoelastic model for describing the response of biological fibers. Acta Mech., 227(12):3367–3380, 2016.

[51] N. Gelmetti and E. Süli. Spectral approximation of a strain-limiting nonlinear elastic model. Mat. Vesnik, 71(1-2):63–89, 2019.

[52] A. Giacomini. Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures. Calc. Var. Partial Differential Equations, 22(2):129–172, 2005.

[53] D. M. Gordin, R. Ion, C. Vasilescu, S. I. Drob, A. Cimpean, and T. Gloriant. Potentiality of the "gum metal" titanium-based alloy for biomedical applications. Materials Science & Engineering C, 44:362–370, 2014.

[54] K. Gou, M. Mallikarjuna, K. R. Rajagopal, and J. R. Walton. Modeling fracture in the context of a strain-limiting theory of elasticity: a single plane-strain crack. Internat. J. Engrg. Sci., 88:73–82, 2015.

[55] A. A. Griffith. The phenomena of rupture and flow in solids. Philos. Trans. Roy. Soc. London, 221(582-593):163–198, 1921.

[56] P. Gwiazda and A. Świerczewska Gwiazda. On non-Newtonian fluids with a property of rapid thickening under different stimulus. Math. Models Methods Appl. Sci., 18(7):1073–1092, 2008.

[57] P. Gwiazda, A. Świerczewska Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. Math. Methods Appl. Sci., 33(2):125–137, 2010.

[58] C. O. Horgan and G. Saccomandi. Constitutive modelling of rubber-like and biological materials with limiting chain extensibility. Math. Mech. Solids, 7(4):353–371, 2002.

[59] C. O. Horgan and G. Saccomandi. A description of arterial wall mechanics using limiting chain extensibility constitutive models. Biomechanics and Modeling in Mechanobiology, 1(4):251–266, 2003.

[60] C. O. Horgan and G. Saccomandi. Constitutive models for compressible nonlinearly elastic materials with limiting chain extensibility. J. Elasticity, 77(2):123–138 (2005), 2004.

[61] H. Itou, V. A. Kovtunenko, and K. R. Rajagopal. Contacting crack faces within the context of bodies exhibiting limiting strains. JSIAM Lett., 9:61–64, 2017.

[62] H. Itou, V. A. Kovtunenko, and K. R. Rajagopal. Nonlinear elasticity with limiting small strain for cracks subject to non-penetration. Math. Mech. Solids, 22(6):1334–1346, 2017.

[63] H. Itou, V. A. Kovtunenko, and K. R. Rajagopal. On the states of stress and strain adjacent to a crack in a strain-limiting viscoelastic body. Math. Mech. Solids, 23(3):433–444, 2018.

[64] H. Itou, V. A. Kovtunenko, and K. R. Rajagopal. Crack problem within the context of implicitly constituted quasi-linear viscoelasticity. Math. Models Methods Appl. Sci., 29(2):355–372, 2019.

[65] P. A. Johnson and P. N. J. Rasolofosaon. Manifestation of nonlinear elasticity in rock: convincing evidence over large frequency and strain intervals from laboratory studies. Nonlinear Processes in Geophysics, 3(2):77–88, 1996.

[66] P. M. Jordan and A. Puri. A note on traveling wave solutions for a class of nonlinear viscoelastic media. Physics Letters A, 335(2):150–156, 2005.

[67] V. A. Kondrat'ev and O. A. Oleĭnik. Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities. Uspekhi Mat. Nauk, 43(5(263)):55–98, 239, 1988.

[68] C. Kuhn and R. Müller. A continuum phase field model for fracture. Engineering Fracture Mechanics, 77(18):3625–3634, 2010. Computational Mechanics in Fracture and Damage: A Special Issue in Honor of Prof. Gross.

[69] V. Kulvait, J. Málek, and K. R. Rajagopal. Modeling gum metal and other newly developed titanium alloys within a new class of constitutive relations for elastic bodies. Archives of Mechanics, 69(3), 2017.

[70] V. Kulvait, J. Málek, and K. R. Rajagopal. The state of stress and strain adjacent to notches in a new class of nonlinear elastic bodies. J. Elasticity, 135(1-2):375–397, 2019.

[71] V. Kulvait, J. Málek, and K. R. Rajagopal. Anti-plane stress state of a plate with a v-notch for a new class of elastic solids. International Journal of Fracture, 179(1-2):59–73, 2013.

[72] C. J. Larsen. Models for dynamic fracture based on Griffith's criterion. In IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials, pages 131–140. Springer, 2010.

[73] C. J. Larsen, C. Ortner, and E. Süli. Existence of solutions to a regularized model of dynamic fracture. Math. Models Methods Appl. Sci., 20(7):1021–1048, 2010.

[74] J. Merodio and K. R. Rajagopal. On constitutive equations for anisotropic nonlinearly viscoelastic solids. Math. Mech. Solids, 12(2):131–147, 2007.

[75] G. J. Minty. On a "monotonicity" method for the solution of nonlinear equations in banach spaces*. Proceedings of the National Academy of Sciences, 50(6):1038–1041, 1963.

[76] N. Mott. Brittle fracture in mild steel plates. Engineering, 165:16–18, 1948.

[77] A. Muliana, K. R. Rajagopal, and A. S. Wineman. A new class of quasi-linear models for describing the nonlinear viscoelastic response of materials. Acta Mech., 224(9):2169–2183, 2013.

[78] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. Comm. Pure Appl. Math., 42(5):577–685, 1989.

[79] M. Negri. Γ-convergence for high order phase field fracture: continuum and isogeometric formulations. Comput. Methods Appl. Mech. Engrg., 362:112858, 21, 2020.

[80] S. Nicaise and A.-M. Sändig. Dynamic crack propagation in a 2D elastic body: the out-of-plane case. J. Math. Anal. Appl., 329(1):1–30, 2007.

[81] A. Ortiz, R. Bustamante, and K. R. Rajagopal. A numerical study of a plate with a hole for a new class of elastic bodies. Acta Mech., 223(9):1971–1981, 2012.

[82] A. Ortiz-Bernardin, R. Bustamante, and K. R. Rajagopal. A numerical study of elastic bodies that are described by constitutive equations that exhibit limited strains. International Journal of Solids and Structures, 51(3-4):875–885, 2014.

[83] N. S. Papageorgiou and S. T. Kyritsi-Yiallourou. Handbook of applied analysis, volume 19 of Advances in Mechanics and Mathematics. Springer, New York, 2009.

[84] V. Patel. Nonlinear dynamic fracture problems with polynomial and strain-limiting constitutive relations. Submitted. *https://arxiv.org/pdf/2108.03896.pdf*, 2021.

[85] N. Payne and K. Pochiraju. Methodologies for constitutive model parameter identification for strain locking materials. Mechanics of Materials, 134:30–37, 2019.

[86] A. Phillips. The theory of locking materials. Trans. Soc. Rheol., 3:13–26, 1959.

[87] W. Prager. On ideal locking materials. Transactions of the Society of Rheology, 1(1):169–175, 1957.

[88] K. R. Rajagopal. On implicit constitutive theories. Appl. Math., 48(4):279–319, 2003.

[89] K. R. Rajagopal. On implicit constitutive theories for fluids. J. Fluid Mech., 550:243–249, 2006.

[90] K. R. Rajagopal. The elasticity of elasticity. Zeitschrift für angewandte Mathematik und Physik, 58(2):309–317, 2007.

[91] K. R. Rajagopal. A note on a reappraisal and generalization of the Kelvin—Voigt model. Mechanics Research Communications, 36(2):232–235, 2009.

[92] K. R. Rajagopal. On a new class of models in elasticity. Math. Comput. Appl., 15(4):506–528, 2010.

[93] K. R. Rajagopal. Conspectus of concepts of elasticity. Math. Mech. Solids, 16(5):536–562, 2011.

[94] K. R. Rajagopal. Non-linear elastic bodies exhibiting limiting small strain. Math. Mech. Solids, 16(1):122–139, 2011.

[95] K. R. Rajagopal. On the nonlinear elastic response of bodies in the small strain range. Acta Mech., 225(6):1545–1553, 2014.

[96] K. R. Rajagopal and G. Saccomandi. Shear waves in a class of nonlinear viscoelastic solids. Quart. J. Mech. Appl. Math., 56(2):311–326, 2003.

[97] K. R. Rajagopal and G. Saccomandi. Circularly polarized wave propagation in a class of bodies defined by a new class of implicit constitutive relations. Z. Angew. Math. Phys., 65(5):1003–1010, 2014.

[98] K. R. Rajagopal and A. R. Srinivasa. A Gibbs-potential-based formulation for obtaining the response functions for a class of viscoelastic materials. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467(2125):39–58, 2011.

[99] K. R. Rajagopal and J. R. Walton. Modeling fracture in the context of a strain-limiting theory of elasticity: a single anti-plane shear crack. International Journal of Fracture, 169(1):39–48, 2011.

[100] K. J. R. Rasmussen. Full-range stress–strain curves for stainless steel alloys. Journal of Constructional Steel Research, 59(1):47–61, 2003.

[101] F. Riesz and B. Sz.-Nagy. Functional analysis. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1990. Translated from the second French edition by Leo F. Boron, Reprint of the 1955 original.

[102] T. Roubíček. Nonlinear partial differential equations with applications, volume 153 of International Series of Numerical Mathematics. Birkhäuser/Springer Basel AG, Basel, second edition, 2013.

[103] M. Ruzhansky and M. Sugimoto. On global inversion of homogeneous maps. Bull. Math. Sci., 5(1):13–18, 2015.

[104] T. Saito, T. Furuta, J.-H. Hwang, S. Kuramoto, K. Nishino, N. Suzuki, R. Chen, A. Yamada, K. Ito, Y. Seno, T. Nonaka, H. Ikehata, N. Nagasako, C. Iwamoto, Y. Ikuhara, and T. Sakuma. Multifunctional alloys obtained via a dislocation-free plastic deformation mechanism. Science (New York, N.Y.), 300(5618):464–467, 2003.

[105] E. Süli and L. Oparnica. Well-posedness of the fractional Zener wave equation for heterogenous viscoelastic materials. Fractional Calculus and Applied Analysis, 23:126–166, 2020.

[106] E. Tasso. Weak formulation of elastodynamics in domains with growing cracks. Ann. Mat. Pura Appl. (4), 199(4):1571–1595, 2020.

[107] B. Tvedt. Quasilinear equations for viscoelasticity of strain-rate type. Arch. Ration. Mech. Anal., 189(2):237–281, 2008.

[108] J. Warga. Optimal control of differential and functional equations. Academic Press, New York; London, 1972.