Heights on Algebraic Varieties over Function Fields

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To all my family

Abstract

In this thesis we shall explore a number of aspects of algebraic varieties defined over function fields. The first two chapters look at the topic of height pairings, while the third chapter solves a problem about abelian varieties involving local heights.

First constructed by Beilinson in [1] for curves defined over an algebraically closed field; Rössler and Szamuely in [38] generalized Beilinson's height pairing to the higher dimensional setting. In the first chapter we study this pairing and relate it to the intersection product by constructing a new height pairing. We also prove that it satisfies a projection formula matching the one satisfied by the intersection product.

The existence of an analogous pairing in the Arakelovian setting, where our algebraically closed field is now replaced by an arithmetic ring, is conjectured in [38]. For the second chapter, we make some modifications to this conjecture, and using results from of Brosnan and Pearlstein in [8], define the analytic part of the height pairing.

In his paper [26] on the Mordell-Lang conjecture, Hrushovski employed techniques from model theory to prove the function field version of the conjecture. In doing so he was able to answer a related question of Voloch, which we refer to henceforth as Hrushovski's theorem. In the third and final chapter we shall give an alternative proof of said theorem in the characteristic p setting, but using purely algebro-geometric ideas.

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Contents

Notation

Let us go over some basic notation and concepts in the theory of schemes. A good reference for this is [23], or alternatively one can go to the source material in [16–18]. One can find a comprehensive overview on intersection theory in [15].

Start with a scheme X. For any point x on X let $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ be the residue field of x. We shall use the notation $|X|$ to denote the set of closed points of X. We say that X is:

- *reduced* if for every open affine $\mathrm{Spec}(R) \subseteq X$, the ring R is reduced.
- *integral* if for every open affine $\mathrm{Spec}(R) \subseteq X$, the ring R is an integral domain.
- *locally noetherian* if for every open affine $\mathrm{Spec}(R) \subseteq X$, R is a noetherian ring.
- *noetherian* if it is locally noetherian and quasi-compact.
- *regular* if for each open affine $\mathrm{Spec}(R) \subseteq X$, R is a noetherian and regular ring. We sometimes also refer to such a scheme as *non-singular*.

We can associate to any X a reduced scheme X_{red} by putting the reduced induced scheme structure on X. Note that X is integral if and only if it is reduced and irreducible. If X is noetherian then its underlying topological space is noetherian, that is: every descending chain of closed subsets stabilizes.

Now take $f : X \to S$ to be a morphism of schemes. We say that f is:

- *affine* if for every open affine $U \subseteq S$, $f^{-1}(U)$ is affine.
- *quasi-compact* if for every open affine $U \subseteq S$, $f^{-1}(U)$ is quasi-compact.
- *quasi-separated* if the diagonal $\Delta : X \to X \times_S X$ is quasi-compact.
- *separated* if the diagonal $\Delta : X \to X \times_S X$ is a closed immersion.

- *finite* if it is affine and for each open affine $\mathrm{Spec}(R) \subseteq S$, with $\mathrm{Spec}(R') = f^{-1}(\mathrm{Spec}(R))$, R' is a finite R-module.

- *locally of finite type* if for each open affine $\mathrm{Spec}(R') \subseteq X$, and any open affine $\mathrm{Spec}(R) \subseteq S$ containing $f(\mathrm{Spec}(R'))$, R' is a finitely generated algebra over R.

- *of finite type* if it is locally of finite type and quasi-compact.

- *locally of finite presentation* if for each open affine $\mathrm{Spec}(R') \subseteq X$, and any open affine $\mathrm{Spec}(R) \subseteq S$ containing $f(\mathrm{Spec}(R'))$, R' is a finitely presented algebra over R.

- *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

- *universally closed* if any pullback of f is a topologically closed map.

- *proper* if it is separated, of finite type and universally closed.

- *projective* if X is isomorphic over S to a closed subscheme of a projective bundle $\mathbb{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.

- *flat* if for each $x \in X$ the local ring $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{S,f(x)}$-module.

- *smooth* if it is locally of finite presentation, flat, and for each geometric point $\bar{s} \to S$, the fibre $X_{\bar{s}} = X \times_S \bar{s}$ is regular.

- *of relative dimension* n if for each $s \in S$, the fibre $f^{-1}(s)$ is of dimension n.

- *étale* if it is smooth and of relative dimension 0.

- *birational* if both X and S have finitely many irreducible components, and for every generic point $\eta \in X$ of an irreducible component of X, the map $\mathcal{O}_{S,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

- *dominant* if the image of f is a dense subset of S.

Let us point out here that both finite and projective morphisms are examples of proper morphisms. We say f is *compactifiable* if S is noetherian and f is separated and of finite type. Nagata's compactification theorem (see Theorem 4.1. in [12]) says that such a morphism can be factored as $f = pj$ where $j : X \hookrightarrow \overline{X}$ is an open immersion and $p : \overline{X} \to S$ is proper. Nearly all morphisms we use will be compactifiable, so when we speak of a morphism one can assume it is compactifiable unless stated otherwise.

The *scheme theoretic image* of f is the smallest closed subscheme $Z \subseteq S$ through which f factors. If f is quasi-compact then Z is determined by the ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_S \to f_*\mathcal{O}_X)$.

We write $X(T)$ for either the set of morphisms $T \to X$, or the set of such morphisms over a given scheme, it should usually be clear from the context which is meant.

If $S \to S'$ is a morphism of schemes, we say that $X' \to S'$ is a *model* of X over S' if $X = X'_S = X' \times_{S'} S$.

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

Moreover, we say for example X' is a smooth model of X if the morphism $X' \to S'$ is smooth, and likewise for any other property in place of smooth.

Suppose now $S = \mathrm{Spec}(k)$ where k is a field, then we say X is a *variety* over k if X is integral and f is separated and of finite type. If the dimension of X is 1, then we refer to X as a *curve*. A *subvariety* of X is any integral closed subscheme of X. We let $k(X) := \mathcal{O}_{X,\eta}$ denote the function field of X, where η is the generic point of X.

By the theorem of resolution of singularities (see Main Theorem I. in Chapter 3 of [24]), for any variety X in characteristic 0 one can always find a non-singular variety X' together with a proper birational map $X' \to X$. In characteristic p this is still an open problem in general.

We define the group $Z^i(X)$ to be the free abelain group generated by the set of codimension i subvarieties of X. Given $z = \sum_Z n_Z Z \in Z^i(X)$, we set

$$|z| = \mathrm{supp}(z) := \bigcup_Z \mathrm{supp}(Z)$$

We say that $z \in Z^i(X)$ and $z' \in Z^j(X)$ are *disjoint* if $|z| \cap |z'| = \emptyset$.

For a codimension $i-1$ cycle W on X, and any rational function f on W which is not identically zero, we set the divisor associated to f to be

$$(f) = \sum_Z \mathrm{ord}_Z(f)Z$$

where the sum is over all subvarieties Z of W which have codimension i in X. Here $\mathrm{ord}_Z(f)$ denotes the order of vanishing of f along Z, which is defined first for $f \in \mathcal{O}_{W,Z}$ as the length of the $\mathcal{O}_{W,Z}$-module $\mathcal{O}_{W,Z}/(f)$, and then for all rational functions $f = a/b$ (with $a, b \in \mathcal{O}_{W,Z}$) by setting $\mathrm{ord}_Z(f) = \mathrm{ord}_Z(a) - \mathrm{ord}_Z(b)$.

The subgroup of $Z^i(X)$ generated by all the (f) as we range over W and f is called the the group of codimension i cycles rationally equivalent to zero. The quotient group $CH^i(X)$ we define as the Chow group of codimension i cycles on X.

If X is smooth over k, the Chow groups form a graded commutative ring $CH^*(X)$ under the intersection product

$$\begin{aligned} CH^i(X) \times CH^j(X) &\to CH^{i+j}(X) \\ ([Y],[Z]) &\mapsto [Y] \cap [Z] \end{aligned}$$

If Y and Z intersect transversally, then $[Y] \cap [Z]$ is given by the sum

$$[Y] \cap [Z] = \sum_W m_W(Y,Z)W$$

over the irreducible components W of the set theoretic intersection $Y \cap Z$, where $m_W(Y,Z) \in \mathbb{Z}$ is the intersection multiplicity of Y and Z along W. The moving lemma states that for X projective: given $Y \in Z^i(X)$ and $Z \in Z^j(X)$, there is $Y' \in Z^i(X)$ such that $[Y] = [Y'] \in CH^i(X)$, and Y' intersects transversally with Z (see Theorem 11. in [39]).

Given a morphism $f : X \to X'$ of smooth schemes over k, there is always a pullback map for each i

$$f^* : CH^i(X') \to CH^i(X)$$

which gives rise to a graded ring homomorphism $f^* : CH^*(X') \to CH^*(X)$.

If f is also proper, then there is a pushforward map for each i

$$f_* : CH^i(X) \to CH^{i-\delta}(X') \quad (\delta = \dim X - \dim X')$$

The pullback and pushforward maps are related in the following way, known as the projection formula (see Example 8.1.7. in [15]):

$$f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta$$

for any $\alpha \in CH^i(X)$, $\beta \in CH^j(X')$. Furthermore, let $i : Z' \hookrightarrow X'$ be a subvariety with Z' also smooth over k, then $i^* f_* = g_* j^*$ (Theorem 6.2.(a) in [15]), where g and j are the maps in the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i"'] & X' \end{tikzcd}$$

We define the Picard group $\mathrm{Pic}(X)$ to be the group of isomorphism classes of line bundles on X under tensor product. When X is smooth over k then there is an isomorphism (Example 2.1.1. in [15])

$$\mathrm{Pic}(X) \xrightarrow{\sim} CH^1(X)$$

For a given prime number ℓ, we let $\mathbb{Z}_\ell$ denote the ring of ℓ-adic integers, and $\mathbb{Q}_\ell$ its fraction field, the field of ℓ-adic numbers. Assuming ℓ is invertible on X, we can consider the abelian category of $\mathbb{Z}_\ell$-sheaves on X. A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ is an inverse system of $\mathbb{Z}/\ell^n$-modules $\mathcal{F}_n$ in the étale topology, such that for all n

$$\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \xrightarrow{\sim} \mathcal{F}_n$$

The category of $\mathbb{Q}_\ell$-sheaves has the same objects, but if $\mathcal{F}, \mathcal{G}$ are two such sheaves, then

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

For each integer $m \in \mathbb{Z}$ we have the Tate sheaves

$$\mathbb{Q}_\ell(m) := \begin{cases} (\mu_{\ell^n}^{\otimes m})_n & \text{for } m \geq 0 \\ (\mathbb{Z}/\ell^n)_n & \text{for } m = 0 \\ (\mathbb{Q}_\ell(-m))^\vee & \text{for } m \leq 0 \end{cases}$$

where $\mathbb{Z}/\ell^n$ denotes the constant sheaf, μ_{ℓ^n} is the sheaf of ℓ^n-power roots of unity, and $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$. Moreover, for any sheaf $\mathcal{F}$ we have the Tate twists defined by

$$\mathcal{F}(m) := \mathcal{F} \otimes \mathbb{Q}_\ell(m)$$

If k is algebraically closed then for any constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$, which is to say that all the $\mathcal{F}_n$ are constructible, we can consider the ℓ-adic cohomology groups:

$$H^i_{\text{ét}}(X, \mathcal{F}) = \varprojlim_n H^i_{\text{ét}}(X, \mathcal{F}_n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

We may choose to write $H^i(X, \mathcal{F})$ in place of $H^i_{\text{ét}}(X, \mathcal{F})$ if it is clear from the context that we are referring to the ℓ-adic cohomology groups. One also has the cohomology groups with compact support $H^i_c(X, \mathcal{F})$ that are defined analogously.

If X is also smooth, then there is a cycle class map (see Theorem II.11.2. in [30])

$$\text{cl}_X : CH^i(X) \to H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell(i))$$

Moreover, if $k = \mathbb{C}$ then there is a comparison map from the singular cohomology groups (see Theorem 3.12. in [33])

$$H^{2i}(X(\mathbb{C}), \mathbb{Q}(i)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell(i))$$

which is always an isomorphism..

Introduction

In this introduction we cover the structure of each chapter, presenting some of the basic notation as-well as highlighting the main results that are new. We shall omit certain details for the moment that would only obfuscate the key ideas.

The primary object of study in Chapter 1 is the Beilinson height pairing on the groups of homologically trivial cycles

$$\langle \cdot, \cdot \rangle_X : CH^p_{\text{hom}}(X)_{\mathbb{Q}} \times CH^q_{\text{hom}}(X)_{\mathbb{Q}} \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell)$$

where B is a smooth integral scheme of dimension b of finite type over an algebraically closed field, X is a smooth projective variety of dimension d over the function field of B, and $p + q = d + 1$. It is defined using an open subscheme $j : U \hookrightarrow B$, and so we also denote this pairing by $\langle \cdot, \cdot \rangle_U$, even though the pairing itself does not depend on the choice of U.

We seek to compare Beilinson's pairing to the geometric height pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{X}} : CH^p_{\text{hom}}(X)_{\mathbb{Q}} \times CH^q_{\text{hom}}(X)_{\mathbb{Q}} \to \text{Pic}(B)_{\mathbb{Q}}$$

which is defined by intersecting extensions of cycles on a regular model $\mathcal{X} \xrightarrow{\pi} B$ of X. We also let $\mathcal{X}_U \xrightarrow{\pi_U} U$ be the pullback of π to U.

In order to compare these two pairings we define a third height pairing $\langle \cdot, \cdot \rangle_B$. In the same way that Beilinson's height pairing arises from a pairing

$$h_X : H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \times H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b]) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))$$

our pairing $\langle \cdot, \cdot \rangle_B$ is also derived from a cohomological one

$$h_B : H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \times H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q))) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))$$

where ${}^{\mathfrak{p}}\mathcal{H}^i(-)$ denotes the perverse cohomology. Indeed, one sets $\langle \alpha^p, \alpha^q \rangle_B := h_B(\alpha^p_B, \alpha^q_B)$ where $\alpha^p_B \in H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ is defined via an extension of the cycle to $\mathcal{X}$, and likewise for α^q_B.

Theorem 1.2.5. *For any $\alpha^p \in CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$, $\alpha^q \in CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}}$ we have*

$$\langle \alpha^p, \alpha^q \rangle_B = \mathrm{cl}_B \langle \alpha^p, \alpha^q \rangle_{\mathcal{X}}$$

Take $i : Z = B \setminus U \hookrightarrow B$ to be the closed complement. Then by the decomposition theorem we have

$${}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_* \mathbb{Q}_\ell(p)) \simeq j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \oplus i_* \widehat{C}_{2p-1}$$

for some perverse sheaf $\widehat{C}_{2p-1}$ on Z. This allows us to split α^p_B as

$$\alpha^p_B = \widetilde{\alpha^p_B} + \widehat{\alpha^p_B} \in H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus H^{1-b}_{\text{ét}}(Z, \widehat{C}_{2p-1})$$

Theorem 1.2.9. *For any $\alpha^p \in CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$, $\alpha^q \in CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}}$ we have*

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U + h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B})$$

Moreover, if α^p_B is chosen from a conjectural class of extensions then $\widehat{\alpha^p_B} = 0$, and consequently

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U$$

Corollary 1.2.10. *For any $\alpha^p \in CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$, $\alpha^q \in CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}}$ we have*

$$\mathrm{cl}_B \langle \alpha^p, \alpha^q \rangle_{\mathcal{X}} = \langle \alpha^p, \alpha^q \rangle_U + h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B})$$

Finally, we also prove an analogue of the projection formula for the Beilinson height. Let $f : B' \to B$ be a dominant proper morphism of relative dimension 0 which is generically finite, and let $K' \to K$ be the induced map on function fields. Then take a finite map $g : X' \to X$ which fits into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ K' & \longrightarrow & K \end{array}$$

Theorem 1.3.1. *For any $\alpha^p \in CH^p_{\mathrm{hom}}(X')_{\mathbb{Q}}$, $\beta^q \in CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}}$ the projection formula holds*

$$\langle g_* \alpha^p, \beta^q \rangle_X = f_* \langle \alpha^p, g^* \beta^q \rangle_{X'}$$

This result follows from the analogous one on the cohomology groups.

Proposition 1.3.2. *For any* $\alpha \in H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell(p)[b])$ *and* $\beta \in H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b])$ *the projection formula holds*

$$h_X(g_*\alpha, \beta) = f_*h_{X'}(\alpha, g^*\beta)$$

One of the main challenges to be overcome is simply defining the pushforward and pullback maps g_*, g^* in the statement of the proposition.

In chapter 2 we switch focus from the geometric setting to the Arekelovian one. We now take a regular, integral scheme B_0 which is flat and of finite type over an arithmetic ring. Then as before, one considers a smooth projective variety X_0 of dimension d over the function field of B_0.

It would be reasonable to assume there is a natural height pairing on the groups of homologically trivial cycles of X_0 that has values in $\widehat{\text{Pic}}(B_0)_{\mathbb{Q}}$, the group of hermitian line bundles on B_0, reflecting the fact B_0 carries additional structure. By studying the analytic part of such a pairing though, we see that this cannot hold in general, leading us to make the following conjecture instead.

Conjecture 2.1.2. *Suppose that* X_0 *has a model* $\mathcal{X}$ *over* B_0*, with smooth locus* $V \subset B_0$*, such that the Hodge structure* $H^{2p-1}(\mathcal{X}_V(\mathbb{C}), \mathbb{Z}(p))$ *has unipotent monodromy. Assuming that* $B_0 \setminus V$ *is a normal crossings divisor, then for* $p+q = d+1$ *there exists a height pairing*

$$\langle \cdot, \cdot \rangle_{X_0} : CH^p_{\text{hom}}(X_0)_{\mathbb{Q}} \otimes CH^q_{\text{hom}}(X_0)_{\mathbb{Q}} \to \widehat{\text{Pic}}'(B_0)_{\mathbb{Q}}$$

The definition of $\widehat{\text{Pic}}'(B_0)$ is somewhat convoluted, but suffice it to say, it is simply a weakening of the condition of being a hermitian line bundle.

This conjecture is informed by the following theorem for a smooth projective variety X of dimension d over a smooth integral scheme B of finite type over $\mathbb{C}$. The proof of this theorem is based on the work of Brosnan and Pearlstein on height jumping.

Theorem 2.2.2. *Suppose that* X *has a model* $\mathcal{X}$ *over* B*, with smooth locus* $V \subset B$*, such that the Hodge structure* $H^{2p-1}(\mathcal{X}_V, \mathbb{Z}(p))$ *has unipotent monodromy. Assuming that* $B \setminus V$ *is a normal crossings divisor, then for* $p + q = d + 1$ *there exists a height pairing*

$$\langle \cdot, \cdot \rangle_X : CH^p_{\text{hom}}(X)_{\mathbb{Q}} \otimes CH^q_{\text{hom}}(X)_{\mathbb{Q}} \to \text{Pic}'(B)_{\mathbb{Q}}$$

Finally, we also compare this pairing with Beilinson's, as-well as with the geometric height.

In Chapter 3 we look at a problem first considered by Voloch and then solved by Hrushovski using a model theoretic approach. This is not the first problem in algebraic geometry that was first solved using model theory, the Mordell-Lang conjecture being another notable example, and in fact Hrushovski's theorem can be viewed as a strengthening of Mordell-Lang. It is nonetheless desirable to have proofs of these theorems which are self-contained within algebraic geometry.

Let K be a function field of one variable over $k = \overline{\mathbb{F}}_p$, and take an abelian variety A over K with K/k-trace equal to zero. Given a finitely generated subgroup $\Gamma \subseteq A(K)$ one has the following theorem.

Theorem 3.1.4. *Let $X \subset A$ be a subvariety defined over K. There exists a linear subvariety $Y \subset X$ which is also defined over K, such that for each discrete valuation v of K there is a constant C_v for which*

$$\lambda_v(X, P) \leq \lambda_v(Y, P) + C_v$$

for all points $P \in \Gamma$.

This result, which we still refer to as Hrushovski's theorem, is slightly stronger than the result obtained by Hrushovski. Here λ_v denotes the local height, and by a linear subvariety we mean a finite union of translates of abelian subvarieties.

Corollary 3.1.8. *We have an equality of sets*

$$X(K) \cap \Gamma = Y(K) \cap \Gamma$$

Our proof of Hrushovski's theorem is based on the idea of exceptional schemes. These are closed subschemes $\mathrm{Exc}^n(A, X) \subseteq X$. We expand on their definition and show that they satisfy the following properties.

Proposition 3.2.4. *Assume that X is not linear. Then there exists $m \in \mathbb{N}$ such that for all $Q \in \Gamma$ we have a strict inclusion*

$$\mathrm{Exc}^m(A, X^{+Q}) \subsetneq X^{+Q}$$

Proposition 3.2.1. *For any separable field extension L/K, not necessarily finite, we have an inclusion*

$$X(L) \cap p^n A(L) \subseteq \mathrm{Exc}^n(A, X)(L)$$

We can use these two propositions to give a proof of the original version of Hrushovski's theorem. To obtain the stronger version stated above requires the following proposition.

Proposition 3.2.2. *Let $P \in p^k\Gamma$ be such that $d_v(X, P) \leq 1/p^n$ with n sufficiently large, then*

$$d_v(\mathrm{Exc}^k(A, X), P) \leq 1/p^n$$

In the above proposition d_v denotes the local distance, which is defined in terms of the local height as $d_v = e^{-\lambda_v}$.

Beilinson's Height Pairing

1.1 Introduction

Let X be a smooth projective variety of dimension d over a function field K, i.e. $K = k(B)$ where B is a smooth integral scheme of dimension b, of finite type over an algebraically closed field k.

We let $CH^i(X)$ denote the Chow groups of codimension i cycles on X, and $CH^i_{\mathrm{hom}}(X)_{\mathbb{Q}}$ be the kernel of the cycle map

$$CH^i(X)_{\mathbb{Q}} \to H^{2i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(i))$$

where ℓ is a prime invertible in K, and $\overline{K}$ is the algebraic closure of K.

1.1.1 Motivation

Let A be an abelian variety over a global field K, that is: either a number field or the function field of a curve defined over a finite field. For any symmetric invertible sheaf $\mathcal{L}$ on A one has the Néron-Tate height $h_{\mathcal{L}}$, which is a real-valued function on the points $A(K)$.

In the case of an elliptic curve E, there is essentially a canonical choice of $\mathcal{L}$, and we denote the associated Néron-Tate height by h. One can then associate to h a bilinear pairing like so:

$$\langle \cdot, \cdot \rangle : CH^1_{\mathrm{hom}}(E) \times CH^1_{\mathrm{hom}}(E) \to \mathbb{R}$$
$$\langle P - \infty, Q - \infty \rangle := \frac{1}{2}(h(P+Q) - h(P) - h(Q))$$

This gives one of the first examples of a height pairing.

The utility of heights is well known, for example in proving the Mordell-Weil Theorem.

Theorem 1.1.1 (Mordell-Weil Theorem)**.** *Let A be an abelian variety over a global field K, then the group $A(K)$ is finitely generated.*

The determinant of the Néron-Tate height pairing also shows up in the statement of the Birch and Swinnerton-Dyer Conjecture, further illustrating its important role.

In general a height pairing on X is a bilinear pairing

$$\langle \cdot, \cdot \rangle : CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \times CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}} \to \mathcal{Q}$$

for each p, q such that $p + q = d + 1$, and where $\mathcal{Q}$ is some coefficient group. Of course, we also expect such pairings to satisfy various nice properties, such as being non-degenerate.

1.1.2 The Geometric Height Pairing

There is a natural way to construct a height pairing on X, but this comes with a caveat. Take a regular projective model $\pi : \mathcal{X} \to B$ of X. We remark that such a model can always be found provided resolution of singularities holds, which is at least always true in characteristic zero.

Now, extend the cycles $a_1 \in CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$, $a_2 \in CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}}$ to cycles $\widetilde{a}_1 \in CH^p(\mathcal{X})_{\mathbb{Q}}$, $\widetilde{a}_2 \in CH^q(\mathcal{X})_{\mathbb{Q}}$ on $\mathcal{X}$. On the Chow groups of $\mathcal{X}$ the intersection product defines a bilinear pairing

$$\cdot \cap \cdot : CH^p(\mathcal{X})_{\mathbb{Q}} \times CH^q(\mathcal{X})_{\mathbb{Q}} \to CH^{d+1}(\mathcal{X})_{\mathbb{Q}}$$

So we can define $\langle a_1, a_2 \rangle_{\mathcal{X}}$ to be the intersection product $\widetilde{a}_1 \cap \widetilde{a}_2$, or rather the image $\pi_*(\widetilde{a}_1 \cap \widetilde{a}_2)$ under the pushforward map $CH^{d+1}(\mathcal{X})_{\mathbb{Q}} \to CH^1(B)_{\mathbb{Q}}$.

If $\mathcal{X} \to B$ is smooth this gives a well-defined height pairing by the following lemma, which is Proposition 6.1. in [38].

Lemma 1.1.2. *Suppose that $\pi : \mathcal{X} \to B$ is a smooth model. Then for any extension $\widetilde{a}_1 \in CH^p(\mathcal{X})$ of $a_1 \in CH^p_{\mathrm{hom}}(X)$, and any extension $\widetilde{a}_2 \in CH^q(\mathcal{X})$ of $0 \in CH^q(X)$, we have*

$$\pi_*(\widetilde{a}_1 \cap \widetilde{a}_2) = 0$$

Proof. By assumption on $\widetilde{a}_2$ there is some dense open $U \subset B$ such that $\widetilde{a}_2$ restricts to 0 in $CH^q(\mathcal{X}_U)$. So let $Z = B \setminus U$ be the complement, and consider the pullback diagram

$$\begin{tikzcd} \mathcal{X}_Z \arrow[r, hook, "i_{\mathcal{X}}"] \arrow[d, "\pi_Z"'] & \mathcal{X} \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"'] & B \end{tikzcd}$$

By the localization sequence $\widetilde{a}_2$ comes from a class $b_2 \in CH^q(\mathcal{X}_Z)$. Since the singular locus $Z_{\text{sing}} \subset Z$ has codimension ≥ 2 in B, it follows that $\text{Pic}(B \setminus Z_{\text{sing}}) \simeq \text{Pic}(B)$ and so we can assume wlog that both Z and $\mathcal{X}_Z$ are non-singular.

We now compute the intersection

$$\begin{aligned} \pi_*(\widetilde{a}_1 \cap \widetilde{a}_2) &= \pi_*(\widetilde{a}_1 \cap i_{\mathcal{X}*} b_2) \\ &= \pi_*(i_{\mathcal{X}*}(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2)) \\ &= i_*(\pi_{Z*}(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2)) \end{aligned}$$

where in the second line we have used the projection formula. To show that this class is zero then, it is enough to show that $\pi_{Z*}(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2) = 0 \in CH^0(Z)$. Since $CH^0(Z)$ is a finite direct sum of copies of $\mathbb{Z}$ indexed by the irreducible components of Z, in fact it suffices to show that $x^*(\pi_{Z*}(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2)) = 0$ for any closed point $x : \text{Spec}(k) \to Z$.

Let $\pi_x : \mathcal{X}_x \to \text{Spec}(k)$ be the fibre of π over x, and $i_x : \mathcal{X}_x \hookrightarrow \mathcal{X}_Z$ be the inclusion. Then we have another pullback diagram

$$\begin{tikzcd} \mathcal{X}_x \arrow[r, hook, "i_x"] \arrow[d, "\pi_x"'] & \mathcal{X}_Z \arrow[d, "\pi_Z"] \\ \text{Spec}(k) \arrow[r, hook, "x"'] & Z \end{tikzcd}$$

By using the base change compatibility $x^* \pi_{Z*} = \pi_{x*} i_x^*$, we have that

$$x^*(\pi_{Z*}(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2)) = \pi_{x*}(i_x^*(i_{\mathcal{X}}^* \widetilde{a}_1 \cap b_2)) = \pi_{x*}(i_x^* i_{\mathcal{X}}^* \widetilde{a}_1 \cap i_x^* b_2)$$

Since the restriction of $\widetilde{a}_1$ to the generic fibre X is homologically trivial, it follows by the smooth proper base change theorem that the cycle class of $i_x^* i_{\mathcal{X}}^* \widetilde{a}_1$ in $H^{2p}_{\text{ét}}(\mathcal{X}_x, \mathbb{Q}_\ell(p))$ is zero. Then because the cycle class map is compatible with pushforward and products, the cycle class of $\pi_{x*}(i_x^* i_{\mathcal{X}}^* \widetilde{a}_1 \cap i_x^* b_2)$ in $H^0_{\text{ét}}(k(x), \mathbb{Q}_\ell)$ is zero, and we are done.

□

If the model is not smooth however, the pairing may depend on how we extend the a_i to $\widetilde{a}_i$. This leads us on to the following conjecture.

Conjecture 1.1.3. *For each $a_1 \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$ there exists a so-called Beilinson extension $\widetilde{a}_1 \in CH^p(\mathcal{X})_{\mathbb{Q}}$ such that for any extension $\widetilde{a}_2 \in CH^q(\mathcal{X})_{\mathbb{Q}}$ of $0 \in CH^q(X)$ we have*

$$\pi_*(\widetilde{a}_1 \cap \widetilde{a}_2) = 0$$

Remark 1.1.4. *If we insist only that the degree of $\pi_*(\widetilde{a}_1 \cap \widetilde{a}_2)$ is zero, then this conjecture is known for example in the case where $\pi : \mathcal{X} \to B$ is a minimal elliptic surface (for $p = q = 1$). So if in addition $B = \mathbb{P}^1$, then the full conjecture holds. To derive this result we can use Propostion III.8.3. in* [42]*, noting that the homologically trivial divisors on an elliptic curve are of the form $(P) - (O)$ for P a rational point on said elliptic curve, and the canonical liftings of these divisors to $\mathcal{X}$ all intersect trivially with each fibre of π (see the discussion prior to Theorem III.9.3.). The proof of Propostion III.8.3. also makes clear why we must tensor the Chow groups with $\mathbb{Q}$.*

Assuming the truth of Conjecture 1.1.3 we could promote the geometric height pairing to a genuine height pairing by defining $\langle a_1, a_2 \rangle_{\mathcal{X}}$ to be $\pi_*(\widetilde{a}_1 \cap \widetilde{a}_2)$, where $\widetilde{a}_1$ is a Beilinson extension. For the meantime though, we must keep in mind the dependence on the choice of extension.

Work by Kahn in [29] has shown that one can define a genuine height pairing on a certain subgroup $CH^p(X)^{(0)} \subseteq CH^p(X)$:

$$CH^p(X)^{(0)}_{\mathbb{Q}} \times CH^q(X)^{(0)}_{\mathbb{Q}} \to \operatorname{Pic}(B)_{\mathbb{Q}}$$

with values in $\operatorname{Pic}(B)_{\mathbb{Q}}$, but without using Beilinson cycles. Moreover, it is equal to the Beilinson height pairing in a sense made precise in Proposition 2.11. of [29].

1.1.3 Cohomological Pairings

In the case where B is also a projective curve, Beilinson in his paper [1] constructs a height pairing using cohomological means, with $\mathcal{Q} = \mathbb{Q}_\ell$ where again ℓ is a prime which is invertible in k.

This is done by taking a smooth projective model $\mathcal{X} \to U$ of X, where U is an affine open subset of B, and then using Poincaré duality on $\mathcal{X}$ to get a perfect pairing

$$\langle \cdot, \cdot \rangle : H^{2p}_{!*}(X, \mathbb{Q}_\ell(p)) \times H^{2q}_{!*}(X, \mathbb{Q}_\ell(q)) \to \mathbb{Q}_\ell$$

where $H^{2p}_{!*}(X,\mathbb{Q}_\ell(p)) := \mathrm{Im}(H^{2p}_c(\mathcal{X},\mathbb{Q}_\ell(p)) \to H^{2p}_{\text{ét}}(\mathcal{X},\mathbb{Q}_\ell(p)))$. It is noted by Beilinson that

$$H^{2p}_{!*}(X,\mathbb{Q}_\ell(p)) = H^1_{\text{ét}}(B, j_*\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p))$$

where $j : U \hookrightarrow B$ is the open immersion, and moreover the above pairing is equal to the one induced by Poincaré duality on the fibres of π:

$$j_*\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p) \otimes j_*\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q) \to \mathbb{Q}_\ell(1)$$

It is this observation that allows the Beilinson pairing to generalized to higher dimensional B, as we shall see in the next section.

If the geometric pairing of the last section can be defined, that is to say that we can pick Beilinson extensions of our cycles, then it is equal to this pairing.

In addition to the global pairing, Beilinson also constructs a local pairing for each place v of the curve, which is a partially defined pairing

$$\langle \cdot, \cdot \rangle_v : Z^p(X_v) \times Z^q(X_v) \dashrightarrow \mathbb{Q}_\ell$$

on the cycles of $X_v = X \times_K K_v$ (K_v the completion of K at v) which are homologous to zero and which intersect trivially, where $p + q = d + 1$ as usual.

The global height pairing is connected to these local pairings in the following way by Lemma 1.1.2. and Lemma 2.1.5. in [1].

Theorem 1.1.5. *Let $a_1 \in Z^p(X)$, $a_2 \in Z^q(X)$ be cycles that do not intersect, and suppose that $a_1 \in CH^p_{\text{hom}}(X), a_2 \in CH^q_{\text{hom}}(X)$. Then for each place v of the curve B, $\mathrm{cl}(a_1)$ is zero in $H^{2p}(X_v,\mathbb{Q}_\ell(p))$ and likewise for a_2, so that $\langle a_1, a_2\rangle_v$ is well defined. Moreover, global height pairing is equal to the sum of the local pairings*

$$\langle a_1, a_2\rangle = \sum_v \langle a_1, a_2\rangle_v$$

where the sum is over the places v of B.

In their paper [38], Rössler and Szamuely generalize Beilinson's global height pairing to the more general case where B no longer has to be a curve, or even projective. In this case we have $\mathcal{Q} = H^2_{\text{ét}}(B,\mathbb{Q}_\ell(1))$. It coincides with Beilinson's construction via the trace isomorphism $H^2_{\text{ét}}(B,\mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$ when B is a projective curve.

1.1.4 Rössler and Szamuely's Pairing

Let us now outline the construction of Rössler and Szamuely in [38].

Let $\mathcal{X} \xrightarrow{\pi} U$ be a smooth projective model of X over an open affine subset $j : U \hookrightarrow B$. Then take the duality pairing on U:

$$\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b] \otimes^{\mathbf{L}} \mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b] \to \mathbb{Q}_\ell(1)[2b]$$

which under the Tensor-Hom adjunction corresponds to a mapping

$$\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b] \to D(\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b])$$

where $D(-) = \mathbf{R}\mathcal{H}om(-, \mathbb{Q}_\ell(b)[2b])$ is the duality functor. Now applying the intermediate extension functor $j_{!*}$, using the fact that $j_{!*}$ commutes with duality we get

$$j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b] \to D(j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b])$$

which corresponds to a pairing

$$j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b] \otimes^{\mathbf{L}} j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b] \to \mathbb{Q}_\ell(1)[2b]$$

in $D^b_c(B, \mathbb{Q}_\ell)$ which we call h_U.

Finally, by taking cohomology this descends to a pairing

$$H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b]) \times H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b]) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))$$

which we shall also denote by $h_U(-,-)$.

In order then to define a height pairing on the Chow groups, they construct a class $\alpha^p_U \in H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b])$ from an element $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$ (and likewise for q), and define

$$\langle \alpha^p, \alpha^q \rangle_U := h_U(\alpha^p_U, \alpha^q_U)$$

In more detail, α^p_U is defined in the following way. One takes a representative z of α^p and extends it to a cycle z_U on the model $\mathcal{X}$ (we can always shrink U if necessary). Since z is homologically trivial, by the smooth proper base change theorem $\text{cl}(z_U) \in H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p))$ lies in the kernel of the map

$$H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \to H^0_{\text{ét}}(U, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p))$$

and therefore produces an element $\alpha_U^p \in H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p))$, which they in turn prove always lies inside the subspace (see Proposition 2.3. in [38])

$$H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b]) \subset H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p))$$

Finally, let us briefly discuss how one could use this approach to build a conjectural motivic height pairing, i.e. a pairing

$$\langle \cdot, \cdot \rangle : CH^p_{\text{hom}}(X)_{\mathbb{Q}} \times CH^q_{\text{hom}}(X)_{\mathbb{Q}} \to \text{Pic}(B)_{\mathbb{Q}}$$

This construction hinges on the existence of a category $M(B, \mathbb{Q})$ of motivic $\mathbb{Q}$-sheaves analogous to the category of $\mathbb{Q}_\ell$-sheaves, as described in Chapter 5.10. of [1]. One also expects a perverse t-structure on the derived category of $M(B, \mathbb{Q})$, which would allow us to define a category $\text{Perv}(B, \mathbb{Q})$ of perverse motives, and so also a corresponding intermediate extension functor.

One could then, by the same argument, define a pairing

$$H^{1-b}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}(p)[b]) \times H^{1-b}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}(q)[b]) \to H^2(B, \mathbb{Q}(1)) = \text{Pic}(B)_{\mathbb{Q}}$$

where of course $H^i(B, -) = \text{Ext}^i_{M(B,\mathbb{Q})}(\mathbb{Q}, -)$. This pairing then induces the one on the Chow groups.

A candidate for the category $\text{Perv}(B, \mathbb{Q})$ in the case when k is of characteristic zero, with an embedding $k \hookrightarrow \mathbb{C}$, is given by Ivorra and Morel in [27]. See Definition 2.1. in their paper for a definition of the category of perverse motives $\mathcal{M}(B)$. They also prove in Theorem 5.1. that on the associated bounded derived category $D^b(\mathcal{M}(B))$ exists a pullback, pushforward, exceptional pullback and exceptional pushforward functor.

1.2 Comparing Height Pairings

In order to compare the pairing $\langle -, - \rangle_U$ (which we shall refer to simply as Beilinson's pairing) to the geometric one, we define a new height pairing as follows.

For this section, let $\mathcal{X} \xrightarrow{\pi} B$ be a regular projective model (which may not be smooth), and let U be an open affine subset of B such that $\mathcal{X}_U \xrightarrow{\pi_U} U$ is a smooth model of X. Also, let $j : U \hookrightarrow B$ the open embedding, and $i : Z = B \setminus U \hookrightarrow B$ the closed immersion.

We shall make use of the decomposition theorem for π (see Theorem A.2.5), which gives us

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \simeq \bigoplus_{k\geq 0} \big(j_{!*}(\mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus C_k\big)[-k-b] \tag{1.2.1}$$

where each $C_k = i_*\widehat{C}_k$ is a perverse sheaf supported on Z. Indeed, we have that

$$\begin{aligned}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) &\simeq \bigoplus_{k\geq 0} {}^{\mathfrak{p}}\mathcal{H}^k(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))[-k] \\ &\simeq \bigoplus_{k\geq 0} \big(j_{!*}A_k \oplus i_*\widehat{C}_{k-b}\big)[-k]\end{aligned}$$

using the fact that each ${}^{\mathfrak{p}}\mathcal{H}^k(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$ is semisimple, as-well as Proposition A.1.6 to deduce the form of each component. One can find the A_k by restricting to U and noting that the decomposition theorem for π_U (see Theorem A.2.6) tells us

$$\mathbf{R}\pi_{U*}\mathbb{Q}_\ell(p) \simeq \bigoplus_{k\geq 0} \mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[-k]$$

1.2.1 A New Cohomological Pairing

First of all, consider the perverse spectral sequence

$$H^s_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^t(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \Rightarrow H^{s+t}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p))$$

Given $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$, extending a representative z^p of α^p to a cycle z^p_B on the model $\mathcal{X} \to B$, we get a class $\text{cl}(z^p_B) \in H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p))$.

Lemma 1.2.1. *We have that*

$$H^c_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-c}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) = 0$$

for all $c < -b$.

Proof. Applying Proposition A.0.6 to $\mathcal{G} = {}^{\mathfrak{p}}\mathcal{H}^{2p-c}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$ we get that

$$\mathcal{H}^c(\mathcal{G}) = 0$$

for all $c < -b$ since $\dim(B) \leq b$. So using the hypercohomolgy spectral sequence

$$E_2^{s,t} = H^s_{\text{ét}}(B, \mathcal{H}^t(\mathcal{G})) \Rightarrow H^{s+t}_{\text{ét}}(B, \mathcal{G})$$

since $H^s_{\text{ét}}(B, \mathcal{H}^t(\mathcal{G})) = 0$ for $s < 0$ or $t < -b$, the lemma follows.

□

Hence the first non-zero term with grading $2p$ is $H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$.

Lemma 1.2.2. *The image of* $\mathrm{cl}(z^p_B)$ *in* $H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ *is zero.*

Proof. Using the decomposition theorem (1.2.1) we have that

$$\begin{aligned} {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) &\simeq \bigoplus_{k\geq 0} {}^{\mathfrak{p}}\mathcal{H}^{2p-k}(j_{!*}(\mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus C_k) \\ &= j_{!*}\mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p)[b] \oplus C_{2p} \end{aligned}$$

where the second equality uses the fact that the $j_{!*}(\mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[b])$ and C_k are perverse sheaves. Note that since C_{2p} is supported on Z, by the same argument as Lemma 1.2.1, using that $\dim(Z) \leq b-1$, we have

$$H^{-b}_{\text{ét}}(B, C_{2p}) = 0$$

which gives us that

$$H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) = H^{-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p)[b])$$

Using the distinguished triangle in Proposition A.1.4 for $\mathcal{F} = \mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p)[b]$, we get an exact sequence of cohomology groups

$$H^{-1-b}_{\text{ét}}(B, i_*{}^{\mathfrak{p}}\tau_{\geq 0}i^*\mathbf{R}j_*\mathcal{F}) \to H^{-b}_{\text{ét}}(B, j_{!*}\mathcal{F}) \to H^{-b}_{\text{ét}}(B, \mathbf{R}j_*\mathcal{F})$$

where the left hand group vanishes $H^{-1-b}_{\text{ét}}(B, i_*{}^{\mathfrak{p}}\tau_{\geq 0}i^*\mathbf{R}j_*\mathcal{F}) = 0$, again by the argument in Lemma 1.2.1.

Hence $H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ injects into $H^0_{\text{ét}}(U, \mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p))$. So now, using the map of spectral sequences

$$\begin{array}{ccc} H^{i-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-i+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) & \Longrightarrow & H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \\ \downarrow & & \downarrow \\ H^i_{\text{ét}}(U, \mathbf{R}^{2p-i}\pi_*\mathbb{Q}_\ell(p)) & \Longrightarrow & H^{2p}_{\text{ét}}(\mathcal{X}_U, \mathbb{Q}_\ell(p)) \end{array} \tag{1.2.2}$$

we get a commutative diagram

$$\begin{array}{ccc} H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) & \longrightarrow & H^{2p}_{\text{ét}}(\mathcal{X}_U, \mathbb{Q}_\ell(p)) \\ \downarrow & & \downarrow \\ H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) & \hookrightarrow & H^0_{\text{ét}}(U, \mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p)) \end{array}$$

where because the image of $\mathrm{cl}(z_U^p)$ in $H^0_{\text{ét}}(U, \mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p))$ is zero (this is implied by the smooth proper base change theorem since z^p is homologically trivial), the image of $\mathrm{cl}(z_B^p)$ inside $H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ is also zero, and the lemma follows.

$\square$

Let $P_p = \mathrm{Ker}(H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \to H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ and likewise for P_q. By the lemma then we have that $\mathrm{cl}(z_B^p) \in P_p$, $\mathrm{cl}(z_B^q) \in P_q$.

Definition 1.2.3. We define the class

$$\alpha_B^p \in H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$$

to be the image of $\mathrm{cl}(z_B^p) \in P_p$ under the map

$$P_p \to H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$$

and similarly for q.

Now, starting with the tensor product pairing

$$\mathbb{Q}_\ell(p) \otimes \mathbb{Q}_\ell(q) \to \mathbb{Q}_\ell(d+1)$$

using the fact $\mathbf{R}\pi_* D = D\mathbf{R}\pi_! = D\mathbf{R}\pi_*$ this induces the derived pairing

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \otimes^{\mathbf{L}} \mathbf{R}\pi_*\mathbb{Q}_\ell(q) \to \mathbf{R}\pi_*\mathbb{Q}_\ell(d+1)$$

We have the trace map $\mathbf{R}\pi_*\mathbb{Q}_\ell(d+b)[2(d+b)] = \mathbf{R}\pi_!\pi^!\mathbb{Q}_\ell(b)[2b] \to \mathbb{Q}_\ell(b)[2b]$, where we have used that $\mathcal{X}$ is regular to deduce

$$\begin{aligned}
\pi^!\mathbb{Q}_\ell(b)[2b] &= \pi^! K_B \\
&= \pi^! g^! K_k \\
&= (g \circ \pi)^! K_k = K_{\mathcal{X}} = \mathbb{Q}_\ell(d+b)[2(d+b)]
\end{aligned}$$

Here $g : B \to k$ is the structure morphism and K_Y denotes the dualizing complex on a scheme Y over k. Indeed, $\mathcal{X}$ regular means that $g \circ \pi$ is smooth, yielding the final equality.

So we get a map

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(d+1) \to \mathbb{Q}_\ell(1)[-2d]$$

which if we compose with the previous pairing gives us

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(p)[2p-1+b] \otimes^{\mathbf{L}} \mathbf{R}\pi_*\mathbb{Q}_\ell(q-1+b)[2q-1+b] \to \mathbb{Q}_\ell(b)[2b] \tag{1.2.3}$$

Since $\mathbb{Q}_\ell(b)[2b]$ is the dualizing sheaf on B, this corresponds to a map

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(p)[2p-1+b] \to D(\mathbf{R}\pi_*\mathbb{Q}_\ell(q-1+b)[2q-1+b]) \tag{1.2.4}$$

Noting that ${}^{\mathfrak{p}}\tau_{\leq 0}D = D{}^{\mathfrak{p}}\tau_{\geq 0}$ and ${}^{\mathfrak{p}}\tau_{\geq 0}D = D{}^{\mathfrak{p}}\tau_{\leq 0}$ because of the self dual nature of the perverse t-structure, we see that ${}^{\mathfrak{p}}\mathcal{H}^0 D = D{}^{\mathfrak{p}}\mathcal{H}^0$ and thus by taking perverse cohomology we have a map

$${}^{\mathfrak{p}}\mathcal{H}^0(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)[2p-1+b]) \to D({}^{\mathfrak{p}}\mathcal{H}^0(\mathbf{R}\pi_*\mathbb{Q}_\ell(q-1+b)[2q-1+b]))$$

which then corresponds to a pairing h_B

$${}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \otimes^{\mathbf{L}} {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q)) \to \mathbb{Q}_\ell(1)[2b]$$

Finally, taking cohomology gives us the cohomological pairing $h_B(-,-)$

$$H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \times H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q))) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))$$

We would like then to define the height pairing $\langle -,- \rangle_B$ by setting

$$\langle \alpha^p, \alpha^q \rangle_B := h_B(\alpha^p_B, \alpha^q_B)$$

however one must be aware that this may depend on the choice of z^p_B extending z^p, and likewise for z^q_B.

Finally, let us briefly describe how to proceed should one not have a regular projective model available, as could theoretically happen in the positive characteristic case. We can at least still take a projective model $\mathcal{X} \to B$ of X.

With the same notation as before, we consider $\mathrm{cl}(z^p_U) \in H^{2p}_{\text{ét}}(\mathcal{X}_U, \mathbb{Q}_\ell(p))$, where z^p_U is some extension of z^p on $\mathcal{X}_U$. Now, using de Jong's theorem on alterations [28] to find a proper and generically étale morphism $\mathcal{W} \to \mathcal{X}$ over k such that $\mathcal{W}$ is regular, we can prove the following.

Lemma 1.2.4. *The class* $\mathrm{cl}(z^p_U)$ *lies in the image of the restriction map*

$$H^{2p-b-d}_{\text{ét}}(\mathcal{X}, \mathrm{IC}_{\mathcal{X}}(p)) \to H^{2p}_{\text{ét}}(\mathcal{X}_U, \mathbb{Q}_\ell(p))$$

Proof. See Proposition 5.1. in [38]. $\square$

Here $\mathrm{IC}_{\mathcal{X}} = j_{\mathcal{X}!*}(\mathbb{Q}_\ell[b+d])$ is the intersection complex on $\mathcal{X}$, where $j_{\mathcal{X}} : \mathcal{X}_U \hookrightarrow \mathcal{X}$ is the inclusion. So using the lemma, we get a class in $H^{2p-b-d}_{\text{ét}}(\mathcal{X}, \mathrm{IC}_{\mathcal{X}}(p))$, and by the same argument as before this gives rise to a class

$$\alpha^p_B \in H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1-d}(\mathbf{R}\pi_* \mathrm{IC}_{\mathcal{X}}(p)))$$

Next, noting that $D(\mathrm{IC}_{\mathcal{X}}) = \mathrm{IC}_{\mathcal{X}}(b+d)$, we have the derived pairing (where $K_{\mathcal{X}}$ is the dualizing complex on $\mathcal{X}$)

$$\mathrm{IC}_{\mathcal{X}}(p) \otimes^{\mathbf{L}} \mathrm{IC}_{\mathcal{X}}(q) \to K_{\mathcal{X}}(1-b)$$

which we can use to define a cohomological pairing h_B in the same way as before

$$H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1-d}(\mathbf{R}\pi_* \mathrm{IC}_{\mathcal{X}}(p))) \times H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2q-1-d}(\mathbf{R}\pi_* \mathrm{IC}_{\mathcal{X}}(q))) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))$$

and then we can finally set $\langle \alpha^p, \alpha^q \rangle_B := h_B(\alpha^p_B, \alpha^q_B)$ just as before. All that follows applies equally to this pairing, but one must keep in mind that the intersection product is not well defined on $\mathcal{X}$ since it is not regular, so only the 'cohomological realization' of the geometric pairing (that we define in the next section) is well defined.

1.2.2 Cohomological vs Geometric

The geometric pairing has a cohomological realization $h_{\mathcal{X}}$ which fits into the commutative diagram

$$\begin{array}{ccc} CH^p(\mathcal{X}) \times CH^q(\mathcal{X}) & \xrightarrow{\quad \cap \quad} & CH^1(B) \\ \downarrow & & \downarrow \\ H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \times H^{2q}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(q)) & \xrightarrow[h_{\mathcal{X}}]{} & H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1)) \end{array}$$

It is constructed from the derived pairing (1.2.3) by taking cohomology directly, and so one might hope that we can relate it to the cohomological pairing in the previous section.

Theorem 1.2.5. *The cohomological pairing is equal to the geometric pairing*

$$h_B(\alpha^p_B, \alpha^q_B) = h_{\mathcal{X}}(\mathrm{cl}(z^p_B), \mathrm{cl}(z^q_B))$$

where α^p_B is the image of $\mathrm{cl}(z^p_B)$ under $P_p \to H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_* \mathbb{Q}_\ell(p)))$, and likewise for q.

Proposition 1.2.6. *The following diagram commutes*

$$\begin{tikzcd}
P_p \times P_q \arrow[d] \arrow[r, "h_{\mathcal{X}}"] & H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1)) \\
H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \times H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q))) \arrow[ru, "h_B"'] &
\end{tikzcd}$$

Proof. First note that

$$H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \simeq H^{2p}_{\text{ét}}(B, \mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \simeq H^{2p}_{\text{ét}}(B, {}^{\mathfrak{p}}\tau_{\leq 2p+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$$

where the second isomorphism is from the long exact sequence associated to the exact triangle

$${}^{\mathfrak{p}}\tau_{\leq 2p+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to \mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to {}^{\mathfrak{p}}\tau_{\geq 2p+1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to {}^{\mathfrak{p}}\tau_{\leq 2p+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p)[1]$$

Indeed, the following groups vanish

Lemma 1.2.7. *We have*

$$H^c_{\text{ét}}(B, {}^{\mathfrak{p}}\tau_{\geq 2p+1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) = 0$$

for all $c < 2p + 1$.

Proof. We use the same argument as in Lemma 1.2.1. Let $\mathcal{G} = {}^{\mathfrak{p}}\tau_{\geq 2p+1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p)$, then by applying Proposition A.0.6 to $\mathcal{G}[2p + 1 + b]$ we have that

$$\mathcal{H}^c(\mathcal{G}) = 0$$

for all $c < 2p + 1$. Using the spectral sequence

$$H^s_{\text{ét}}(B, \mathcal{H}^t(\mathcal{G})) \Rightarrow H^{s+t}_{\text{ét}}(B, \mathcal{G})$$

since $H^s_{\text{ét}}(B, \mathcal{H}^t(\mathcal{G})) = 0$ for $s < 0$ or $t < 2p + 1$, the lemma now follows.

□

Next, using the exact triangle

$${}^{\mathfrak{p}}\tau_{\leq 2p-1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to {}^{\mathfrak{p}}\tau_{\leq 2p+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))[-2p-b] \xrightarrow{[1]}$$

and the long exact cohomology sequence associated to it, we have that the kernel of

$$H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \simeq H^{2p}_{\text{ét}}(B, {}^{\mathfrak{p}}\tau_{\leq 2p+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \to H^{-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$$

is equal to $H^{2p}_{\text{ét}}(B, {}^{\mathfrak{p}}\tau_{\leq 2p-1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$.

So the map $P_p \to H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$ is induced from

$${}^{\mathfrak{p}}\tau_{\leq 2p-1+b}\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$$

Let us momentarily denote $\mathcal{F}_p = \mathbf{R}\pi_*\mathbb{Q}_\ell(p)$ and likewise for $\mathcal{F}_q$.

Lemma 1.2.8. *We have the following commutative diagram involving the two derived pairings*

$$\begin{array}{ccc} {}^{\mathfrak{p}}\tau_{\leq 2p-1+b}\mathcal{F}_p \otimes^{\mathbf{L}} {}^{\mathfrak{p}}\tau_{\leq 2q-1+b}\mathcal{F}_q & \longrightarrow & \mathcal{F}_p \otimes^{\mathbf{L}} \mathcal{F}_q \\ \downarrow & & \downarrow \\ {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathcal{F}_p)[-2p+1-b] \otimes^{\mathbf{L}} {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathcal{F}_q)[-2q+1-b] & \longrightarrow & \mathbb{Q}_\ell(1)[-2d] \end{array}$$

Proof. We want to show two pairings

$${}^{\mathfrak{p}}\tau_{\leq 2p-1+b}\mathcal{F}_p \otimes^{\mathbf{L}} {}^{\mathfrak{p}}\tau_{\leq 2q-1+b}\mathcal{F}_q \to \mathbb{Q}_\ell(1)[-2d]$$

are equal. So let us show the maps

$${}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_p} \to D({}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_q})$$

which they correspond to are equal, where $\widetilde{\mathcal{F}_p} = \mathcal{F}_p[2p-1+b]$ and likewise for $\widetilde{\mathcal{F}_q}$.

This means showing the following diagram commutes

$$\begin{array}{ccc} {}^{\mathfrak{p}}\mathcal{H}^0\widetilde{\mathcal{F}_p} & \longrightarrow & D({}^{\mathfrak{p}}\mathcal{H}^0\widetilde{\mathcal{F}_q}) \\ \uparrow & & \downarrow \\ {}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_p} & \longrightarrow & D({}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_q}) \\ \downarrow & & \uparrow \\ \widetilde{\mathcal{F}_p} & \longrightarrow & D(\widetilde{\mathcal{F}_q}) \end{array}$$

Note that ${}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_p} \in {}^{\mathfrak{p}}D^{\leq 0}$ and $D({}^{\mathfrak{p}}\tau_{\leq 0}\widetilde{\mathcal{F}_q}) = {}^{\mathfrak{p}}\tau_{\geq 0}D(\widetilde{\mathcal{F}_q}) \in {}^{\mathfrak{p}}D^{\geq 0}$, so the commutativity follows from the standard result for t-structures that

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(H^0(A), H^0(B))$$

for $A \in D^{\leq 0}$ and $B \in D^{\geq 0}$.

$\square$

Taking cohomology of the commutative square in the statement of the lemma, the proposition now follows.

□

Proof. (of Theorem 1.2.5). Using Proposition 1.2.6 we just need to observe that the image of $\mathrm{cl}(z_B^p)$ under

$$P_p \to H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)))$$

is equal to α_B^p by definition, and likewise for q.

□

1.2.3 Comparing Cohomological Pairings

Let us now investigate how the two cohomological pairings h_B and h_U compare.

Theorem 1.2.9. *For any $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$, $\alpha^q \in CH^q_{\text{hom}}(X)_{\mathbb{Q}}$ we have*

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U + h_B(\widehat{\alpha_B^p}, \widehat{\alpha_B^q})$$

Moreover, if we can find z_B^p such that the image of $\mathrm{cl}(z_B^p)$ inside $H^0_{\text{ét}}(B, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p)))$ is zero, then for the corresponding α_B^p we have $\widehat{\alpha_B^p} = 0$, and consequently

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U$$

Combining with Theorem 1.2.5 gives the following corollary.

Corollary 1.2.10. *For any $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$, $\alpha^q \in CH^q_{\text{hom}}(X)_{\mathbb{Q}}$ we have*

$$\mathrm{cl}_B \langle \alpha^p, \alpha^q \rangle_{\mathcal{X}} = \langle \alpha^p, \alpha^q \rangle_U + h_B(\widehat{\alpha_B^p}, \widehat{\alpha_B^q})$$

Now to prove Theorem 1.2.9, we start with the decomposition theorem (1.2.1), which gives us a map

$$j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))$$

Proposition 1.2.11. *The following triangle commutes*

$$\begin{tikzcd}
j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \otimes^{\mathbf{L}} j_{!*}\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b] \arrow[d] \arrow[dr, "h_U"] & \\
{}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \otimes^{\mathbf{L}} {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q)) \arrow[r, "h_B"'] & \mathbb{Q}_\ell(1)[2b]
\end{tikzcd}$$

Proof. We simply go through the construction of h_B, using the decomposition theorem to decompose

$$\mathbf{R}\pi_*\mathbb{Q}_\ell(p) = j_{!*}(\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b])[-2p+1-b] \oplus C$$

and likewise for q, where C denotes the spare terms in the decomposition theorem.

Indeed, to construct h_B one starts with the map (1.2.4)

$$\begin{aligned} j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \oplus C = \mathbf{R}\pi_*\mathbb{Q}_\ell(p)[2p-1+b] &\to D(\mathbf{R}\pi_*\mathbb{Q}_\ell(q-1+b)[2q-1+b]) \\ &= D(j_{!*}\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b] \oplus C') \\ &= j_{!*}D(\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b]) \oplus D(C') \end{aligned}$$

which contains a map $j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \to j_{!*}D(\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b])$ that we claim is the one corresponding to h_U.

To see that this is true, note that if we pullback this map to U we get the duality map $\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \to D(\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b])$, from which h_U is defined by applying $j_{!*}$, so now we just need to appeal to the continuation principle Proposition A.1.5.

So when we apply ${}^{\mathfrak{p}}\mathcal{H}^0$ to (1.2.4) to get (the map corresponding to) the pairing h_B, as ${}^{\mathfrak{p}}\mathcal{H}^0$ does nothing on $j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]$ and $j_{!*}\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b]$, we see that the pairing h_B also contains h_U, and the proposition follows.

$\square$

When we pass to cohomology we get the commutative triangle

$$\begin{tikzcd} H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \times H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b]) \arrow[d] \arrow[dr, "h_U"] & \\ & H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1)) \\ H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \times H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q))) \arrow[ur, "h_B"'] & \end{tikzcd}$$

and what remains is to understand the relation between α^p_U and α^p_B.

To do this, first note that we have the commutative triangle

$$\begin{tikzcd}
j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \arrow[r] \arrow[d] & \mathbf{R}j_*\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \\
{}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \arrow[ur] &
\end{tikzcd}$$

which when we pass to cohomology gives

$$\begin{tikzcd}
H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \arrow[r, hook] \arrow[d, "\iota"'] & H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)) \\
H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \arrow[ur] &
\end{tikzcd}$$

Note that the top map is an inclusion by Proposition 2.1. in [38].

The kernel of the map

$$\begin{aligned}
H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus H^{1-b}_{\text{ét}}(B, C_{2p-1}) \simeq H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \\
\to H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p))
\end{aligned}$$

is exactly $H^{1-b}_{\text{ét}}(B, C_{2p-1})$, because C_{2p-1} is supported on Z.

Now consider the commutative diagram

$$\begin{tikzcd}
P_p \arrow[r] \arrow[d] & H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \arrow[d] \\
P'_p \arrow[r] & H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p))
\end{tikzcd}$$

where $P'_p = \mathrm{Ker}(H^{2p}_{\text{ét}}(\mathcal{X}_U, \mathbb{Q}_\ell(p)) \to H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)))$. The commutativity of the diagram follows from the map (1.2.2) of spectral sequences

If we trace the image of $\mathrm{cl}(z^p_B)$ anticlockwise we get $\mathrm{cl}(z^p_U)$ and then α^p_U. On the other hand, if we go clockwise we get α^p_B and then whatever it maps to, which therefore must be α^p_U.

So if we write

$$\alpha^p_B = \widetilde{\alpha^p_B} + \widehat{\alpha^p_B} \in H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus H^{1-b}_{\text{ét}}(B, C_{2p-1})$$

then we must have $\widetilde{\alpha^p_B} = \iota(\alpha^p_U)$.

We have then that

$$\begin{aligned}\langle \alpha^p, \alpha^q \rangle_B &= h_B(\widetilde{\alpha^p_B}, \widetilde{\alpha^q_B}) + h_B(\widetilde{\alpha^p_B}, \widehat{\alpha^q_B}) + h_B(\widehat{\alpha^p_B}, \widetilde{\alpha^q_B}) + h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B}) \\ &= \langle \alpha^p, \alpha^q \rangle_U + h_B(\widetilde{\alpha^p_B}, \widehat{\alpha^q_B}) + h_B(\widehat{\alpha^p_B}, \widetilde{\alpha^q_B}) + h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B})\end{aligned}$$

Lemma 1.2.12. *Both of the cross-terms are zero*

$$h_B(\widetilde{\alpha^p_B}, \widehat{\alpha^q_B}) = 0 = h_B(\widehat{\alpha^p_B}, \widetilde{\alpha^q_B})$$

Proof. Notice that the pairing h_B corresponds to the map

$${}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p)) \to D({}^{\mathfrak{p}}\mathcal{H}^{2q-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(q)))$$

Using the decomposition theorem this splits as a direct sum of maps, and the cross-terms come from the following respective parts of this decomposition

$$j_{!*}\mathbf{R}^{2p-1}\pi_{U*}\mathbb{Q}_\ell(p)[b] \to i_*D(\widehat{C}'_{2q-1})$$

$$i_*\widehat{C}_{2p-1} \to j_{!*}D(\mathbf{R}^{2q-1}\pi_{U*}\mathbb{Q}_\ell(q)[b])$$

where $i_*\widehat{C}_{2p-1} = C_{2p-1}$, $i_*\widehat{C}'_{2q-1} = C'_{2q-1}$ are the terms supported on Z.

We claim that in general for $A \in \mathrm{Perv}(U)$, $B \in \mathrm{Perv}(Z)$

$$\mathrm{Hom}(j_{!*}A, i_*B) = 0 = \mathrm{Hom}(i_*B, j_{!*}A)$$

from which the lemma follows. The two statements are dual, so we just prove the first. Indeed, we have

$$\mathrm{Hom}(j_{!*}A, i_*B) \simeq \mathrm{Hom}(i^*j_{!*}A, B)$$

where $i^*j_{!*}A \in {}^{\mathfrak{p}}D^{\leq -1}(Z)$ by Proposition A.1.3, and $B \in {}^{\mathfrak{p}}D^{\geq 0}(Z)$ since B is perverse. Hence,

$$\mathrm{Hom}(i^*j_{!*}A, B) = 0$$

and so we are done.

$\square$

It follows from the lemma that

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U + h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B})$$

In general the spare term $h_B(\widehat{\alpha^p_B}, \widehat{\alpha^q_B})$ will be non-zero, but if we choose our extension of α^p from a conjectural class of extensions, then we claim that the class $\widehat{\alpha^p_B}$ will be zero.

Theorem 1.2.13. *Suppose that the cycle z_B^p on $\mathcal{X}$ is such that the image of* $\mathrm{cl}(z_B^p)$ *inside $H^0_{\text{ét}}(B, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p)))$ is zero, then $\widehat{\alpha_B^p} = 0$.*

In the above situation then, we have

$$\langle \alpha^p, \alpha^q \rangle_B = \langle \alpha^p, \alpha^q \rangle_U$$

and this completes the proof of Theorem 1.2.9. We conjecture that such a cycle like this can always be found.

Conjecture 1.2.14. *For any $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$, we can find an extension z_B^p on $\mathcal{X}$ such that the image of* $\mathrm{cl}(z_B^p)$ *inside $H^0_{\text{ét}}(B, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p)))$ is zero.*

Remark 1.2.15. *Conjecture 1.1.3 and Conjecture 1.2.14 are not unrelated, however neither one implies the other a priori. In principle though, we should be able to find extensions that satisfy both conjectures.*

Let us now prove Theorem 1.2.13.

Proof. (of Theorem 1.2.13). We start by noting that if we pull the decomposition (1.2.1) back to Z we get

$$\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p) \simeq \bigoplus_{k\geq 0} \left(i^* j_{!*}(\mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[b]) \oplus \widehat{C}_k\right)[-k-b] \tag{1.2.5}$$

Since j is affine, we have that $i^*j_{!*}(\mathbf{R}^k\pi_{U*}\mathbb{Q}_\ell(p)[b])[-1]$ is a perverse sheaf on Z (see Proposition A.1.3), and therefore

$${}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)) = i^*j_{!*}(\mathbf{R}^{2p}\pi_{U*}\mathbb{Q}_\ell(p)[b])[-1] \oplus \widehat{C}_{2p-1}$$

Now, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{R}\pi_*\mathbb{Q}_\ell(p) \arrow[r, two heads] \arrow[d] & {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))[-2p+1-b] \arrow[r, two heads] & C_{2p-1}[-2p+1-b] \arrow[d, "\sim"] \\
i_*\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p) \arrow[r, two heads] & i_*{}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))[-2p+1-b] \arrow[r, two heads] & i_*\widehat{C}_{2p-1}[-2p+1-b]
\end{tikzcd}$$

where the horizontal arrows are the projection maps from the respective decompositions (1.2.1) and (1.2.5).

When we take cohomology of the above diagram we get

$$\begin{tikzcd}
H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \arrow[r] \arrow[d] & H^{1-b}_{\text{ét}}(B, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_*\mathbb{Q}_\ell(p))) \arrow[r, two heads] & H^{1-b}_{\text{ét}}(B, C_{2p-1}) \arrow[d, "\sim"] \\
H^{2p}_{\text{ét}}(\mathcal{X}_Z, \mathbb{Q}_\ell(p)) \arrow[r] & H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))) \arrow[r, two heads] & H^{1-b}_{\text{ét}}(Z, \widehat{C}_{2p-1})
\end{tikzcd}$$

which tells us that $\widehat{\alpha^p_B}$ is equal to the image of $\mathrm{cl}(z^p_B)$ as we go round the diagram anticlockwise.

So to prove the theorem, it suffices to prove that the image of $\mathrm{cl}(z^p_B)$ in $H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)))$ is zero. By assumption on z^p_B we know that the image in $H^0_{\text{ét}}(B, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p)))$ is zero. We claim that there is a map fitting into the following commutative diagram

$$\begin{tikzcd}
H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(p)) \arrow[r] \arrow[d] & H^0_{\text{ét}}(B, \mathbf{R}^{2p}\pi_*\mathbb{Q}_\ell(p))) \arrow[d] \\
H^{2p}_{\text{ét}}(\mathcal{X}_Z, \mathbb{Q}_\ell(p)) \arrow[r] \arrow[dr] & H^0_{\text{ét}}(Z, \mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p))) \arrow[d, dashed] \\
& H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)))
\end{tikzcd} \tag{1.2.6}$$

from which the theorem would then follow.

To construct said map, first recall that for any K we have an exact triangle

$$\tau_{\leq 2p}K \to K \to \tau_{\geq 2p+1}K \to \tau_{\leq 2p}K[1]$$

Applying this to $\tau_{\geq 2p}\mathcal{F}$ where $\mathcal{F} = \mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)$ gives us the exact triangle

$$\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[-2p] \to \tau_{\geq 2p}\mathcal{F} \to \tau_{\geq 2p+1}\mathcal{F} \to \mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[-2p+1]$$

The perverse long exact sequence induces a map

$${}^{\mathfrak{p}}\mathcal{H}^0(\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1]) \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\tau_{\geq 2p}\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)) \tag{1.2.7}$$

Next, we use the exact triangle

$$\tau_{\leq 2p-1}\mathcal{F} \to \mathcal{F} \to \tau_{\geq 2p}\mathcal{F} \to \tau_{\leq 2p-1}\mathcal{F}[1]$$

and the perverse long exact sequence gives

$${}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\tau_{\leq 2p-1}\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\tau_{\geq 2p}\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^{2p+b}(\tau_{\leq 2p-1}\mathcal{F})$$

Lemma 1.2.16. *We have that*

$$^{\mathfrak{p}}\mathcal{H}^c(\tau_{\leq 2p-1}\mathcal{F}) = 0$$

for all $c > 2p - 2 + b$.

Proof. We use the exact triangle

$$\tau_{\leq 2p-2}\mathcal{F} \to \tau_{\leq 2p-1}\mathcal{F} \to \mathbf{R}^{2p-1}\pi_{Z*}\mathbb{Q}_\ell(p)[-2p+1] \to \tau_{\leq 2p-2}\mathcal{F}[1]$$

The perverse long exact sequence gives us the exact sequence

$$^{\mathfrak{p}}\mathcal{H}^c(\tau_{\leq 2p-2}\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^c(\tau_{\leq 2p-1}\mathcal{F}) \to {}^{\mathfrak{p}}\mathcal{H}^{c-2p+1}(\mathbf{R}^{2p-1}\pi_{Z*}\mathbb{Q}_\ell(p)) \tag{1.2.8}$$

Note that

$$^{\mathfrak{p}}\mathcal{H}^{c-2p+1}(\mathbf{R}^{2p-1}\pi_{Z*}\mathbb{Q}_\ell(p)) = {}^{\mathfrak{p}}\mathcal{H}^{c-2p+2-b}(\mathbf{R}^{2p-1}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1])$$

and we have the following sub-lemma

Lemma 1.2.17. *For any sheaf* $\mathcal{G}$ *on* Z *we have*

$$^{\mathfrak{p}}\mathcal{H}^c(\mathcal{G}[b-1]) = 0$$

for all $c > 0$.

Proof. We claim that $\tau_{\geq 0}\mathcal{G}[b] = 0$, which is enough to prove the lemma.

Indeed, for any $A \in {}^{\mathfrak{p}}D^{\geq 0}(Z)$ we know that $A \in D^{\geq 1-b}(Z)$ by Proposition A.0.6, and since obviously $\mathcal{G}[b] \in D^{\leq -b}(Z)$, it follows that

$$\mathrm{Hom}(\mathcal{G}[b], A) = 0$$

which implies that $\tau_{\geq 0}\mathcal{G}[b] = 0$ as desired.

□

So now, to prove Lemma 1.2.16 using (1.2.8); by Lemma 1.2.17 the RHS group is zero for all $c > 2p - 2 + b$ and thus it just remains to show that the LHS group ${}^{\mathfrak{p}}\mathcal{H}^c(\tau_{\leq 2p-2}\mathcal{F})$ is also zero for such c.

To do this one simply uses the same argument, replacing $2p - 1$ with $2p - 2$, and so on until we find n large enough such that $\tau_{\leq 2p-n}\mathcal{F} = 0$, using the fact that $\mathcal{F}$ is bounded.

□

By Lemma 1.2.16 then, we have an isomorphism

$$ {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathcal{F}) \xrightarrow{\sim} {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\tau_{\geq 2p}\mathcal{F}) $$

which if we compose with (1.2.7) gives us a map

$$ {}^{\mathfrak{p}}\mathcal{H}^{0}(\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1]) \to {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)) $$

Taking cohomology gives

$$ H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{0}(\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1])) \to H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))) $$

which if we compose with the spectral sequence map

$$ H^{0}_{\text{ét}}(Z, \mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)) \to H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{0}(\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1])) $$

gives us the desired map

$$ H^{0}_{\text{ét}}(Z, \mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)) \to H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))) \tag{1.2.9} $$

Finally, the commutativity of (1.2.6) results from the following commutative diagram which fits all the spectral sequence maps together

$$ \begin{tikzcd}
H^{0}_{\text{ét}}(Z, \mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)) \arrow[r] \arrow[d] & H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{0}(\mathbf{R}^{2p}\pi_{Z*}\mathbb{Q}_\ell(p)[b-1])) \arrow[d] \\
H^{2p}_{\text{ét}}(Z, \tau_{\geq 2p}\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p)) \arrow[r] & H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\tau_{\geq 2p}\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))) \\
H^{2p}_{\text{ét}}(\mathcal{X}_Z, \mathbb{Q}_\ell(p)) \arrow[r] \arrow[u] \arrow[uu, bend left] & H^{1-b}_{\text{ét}}(Z, {}^{\mathfrak{p}}\mathcal{H}^{2p-1+b}(\mathbf{R}\pi_{Z*}\mathbb{Q}_\ell(p))) \arrow[u, "\sim"']
\end{tikzcd} $$

Observe that the map (1.2.9) is obtained by going clockwise around the diagram. This completes the proof of Theorem 1.2.13.

□

1.3 The Projection Formula

From this point on, let us denote Beilinson's pairing as $\langle -, - \rangle_X$ (recall that we previously denoted it by $\langle -, - \rangle_U$), and likewise the pairing h_U as h_X.

Suppose $f : B' \to B$ is a dominant proper morphism of relative dimension 0 which is generically finite, and let $K' \to K$ be the induced map on function fields. Then take a finite map $g : X' \to X$ fitting into the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ K' & \longrightarrow & K \end{array}$$

Theorem 1.3.1 (Projection Formula)**.** *For any $\alpha^p \in CH^p_{\text{hom}}(X')_{\mathbb{Q}}$ and $\beta^q \in CH^q_{\text{hom}}(X)_{\mathbb{Q}}$ the projection formula holds*

$$\langle g_*\alpha^p, \beta^q\rangle_X = f_*\langle \alpha^p, g^*\beta^q\rangle_{X'}$$

This is analogous to the projection formula for the intersection product. Indeed, if we could spread g out to a finite morphism $g_{\mathcal{X}}$ over B

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g_{\mathcal{X}}} & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \xrightarrow[f]{} & B \end{array}$$

as-well as find Beilinson cycles $\widetilde{\alpha^p}$, $\widetilde{\beta^q}$ extending α^p, β^q, we could immediately deduce the above theorem:

$$\langle g_*\alpha^p, \beta^q\rangle_X = g_{\mathcal{X}*}\widetilde{\alpha^p} \cap \widetilde{\beta^q} = f_*(\widetilde{\alpha^p} \cap g^*_{\mathcal{X}}\widetilde{\beta^q}) = f_*\langle \alpha^p, g^*\beta^q\rangle_{X'}$$

In lieu of such extensions, we shall prove the proposition from first principles using the analogous cohomological result.

We can find some dense open affine subset $j : U \hookrightarrow B$ such that g spreads out to a finite morphism $g_{\mathcal{X}}$ over U

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ f^{-1}(U) & \xrightarrow[f_U]{} & U \end{array}$$

and for which: f_U is finite and $\mathcal{X}' \to f^{-1}(U)$, $\mathcal{X} \to U$ are smooth projective models. Let us also denote $j' : U' = f^{-1}(U) \hookrightarrow B'$.

Proposition 1.3.2. *For any $\alpha \in H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell(p)[b])$ and $\beta \in H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell(q)[b])$ the projection formula holds*

$$h_X(g_*\alpha, \beta) = f_* h_{X'}(\alpha, g^*\beta)$$

Theorem 1.3.1 then follows as a corollary, simply by definition of the height pairing. We have not yet specified how to define $g_*\alpha$ and $g^*\beta$; we shall define these pushforward and pullback maps in the next section.

1.3.1 Constructing the Pushforward and Pullback

Consider the derived pushforward

$$f_* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell(p)[b]$$

In what follows we drop the Tate twist for simplicity. Now, let us restrict this complex to U:

$$\begin{aligned} j^* f_* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] &\simeq f_{U*} j'^* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \\ &\simeq f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \\ &\simeq \mathbf{R}^{2p-1}\pi_* g_{\mathcal{X}*}\mathbb{Q}_\ell[b] \end{aligned}$$

where the last line follows from the fact that $g_{\mathcal{X}*}$ and f_{U*} are exact (since $g_{\mathcal{X}}$ and f_U are both finite).

By the decomposition theorem we have a (non-canonical) isomorphism

$$f_* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \simeq j_{!*}\mathbf{R}^{2p-1}\pi_* g_{\mathcal{X}*}\mathbb{Q}_\ell[b] \oplus i_* C \tag{1.3.1}$$

where $i_* C$ is supported on $Z = B \setminus U$.

We want to define pushforward and pullback maps between $f_* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell(p)[b]$ and $j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell(p)[b]$ which make the following diagrams commute

$$\begin{tikzcd} CH^p_{\mathrm{hom}}(\mathcal{X}')_{\mathbb{Q}} \arrow[r, bend left, "g_{\mathcal{X}*}"] \arrow[d] & CH^p_{\mathrm{hom}}(\mathcal{X})_{\mathbb{Q}} \arrow[l, bend left, "g^*_{\mathcal{X}}"] \arrow[d] \\ H^{2p}_{\text{ét}}(\mathcal{X}', \mathbb{Q}_\ell) \arrow[r, bend left] \arrow[d] & H^{2p}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell) \arrow[l, bend left] \arrow[d] \\ H^1_{\text{ét}}(U', \mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell) \arrow[r, bend left] & H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell) \arrow[l, bend left] \\ H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \arrow[u] \arrow[r, bend left, "g_*"] & H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell[b]) \arrow[u] \arrow[l, bend left, "g^*"] \end{tikzcd}$$

The pushforward and pullback maps on the middle two rows are defined easily using the adjunction maps $g_{\mathcal{X}*}\mathbb{Q}_\ell = g_{\mathcal{X}*}g^!_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ and $\mathbb{Q}_\ell \to g_{\mathcal{X}*}g^*_{\mathcal{X}}\mathbb{Q}_\ell = g_{\mathcal{X}*}\mathbb{Q}_\ell$ respectively, where to get $g^!_{\mathcal{X}}\mathbb{Q}_\ell = \mathbb{Q}_\ell$ we use that $g^!_{\mathcal{X}}K_{\mathcal{X}} = K_{\mathcal{X}'}$.

Since the bottom vertical maps are inclusions, this determines the pushforward and pullback maps on the bottom row, however we wish to understand these maps in the derived category. We seek maps that fit into the following commutative diagram

$$\begin{tikzcd}
f_*j'_*\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \arrow[r, shift left, "\sim"] & j_*f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \arrow[l, shift left, "\sim"] \arrow[r, shift left] & j_*\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell[b] \arrow[l, shift left] \\
f_*j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \arrow[u] \arrow[r, shift left, dashed] & j_{!*}f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \arrow[u] \arrow[l, shift left, dashed] \arrow[r, shift left] & j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell[b] \arrow[u] \arrow[l, shift left]
\end{tikzcd} \tag{1.3.2}$$

where the vertical maps are induced by $j_{!*} \to j_*$ and $j'_{!*} \to j'_*$, while the rightmost horizontal arrows are again determined by the adjunction maps mentioned above.

To see that these dashed maps exist; applying the isomorphism (1.3.1) we have

$$\begin{aligned}
&\mathrm{Hom}(f_*j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b], j_*f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \\
\simeq\ &\mathrm{Hom}(j_{!*}f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \oplus i_*C, j_*f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \\
\simeq\ &\mathrm{Hom}(j_{!*}f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b], j_*f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \oplus \mathrm{Hom}(i_*C, j_*f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \\
\simeq\ &\mathrm{Hom}(j^*j_{!*}f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b], f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \oplus \mathrm{Hom}(j^*i_*C, f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \\
\simeq\ &\mathrm{Hom}(f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b], f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b])
\end{aligned}$$

using the adjunction $j^* \dashv j_*$ and the fact $j^*j_{!*} = \mathrm{id}$, $j^*i_* = 0$.

Call $A = f_{U*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]$, then we see that the map $f_*j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \to f_*j'_*\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \xrightarrow{\sim} j_*A$ in (1.3.2) is determined by some $h \in \mathrm{End}(A)$, and in fact h is an automorphism since h is obtained by simply restricting to U (on which the above map is an isomorphism).

Obviously the map $j_{!*}A \to j_*A$ is also determined by an element of $\mathrm{End}(A)$, namely id_A. So we can define the map

$$f_*j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \simeq j_{!*}A \oplus i_*C \to j_{!*}A$$

by $(j_{!*}h, 0)$ and then the required diagram commutes. Furthermore, the map in the opposite direction

$$j_{!*}A \to j_{!*}A \oplus i_*C \simeq f_*j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]$$

we take to be $(j_{!*}h^{-1}, 0)$, and again the required diagram in (1.3.2) commutes.

In order to prove the projection formula, we will also want to know that the following diagram commutes

$$\begin{tikzcd} f_* j'_{!*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[r, dashed, shift left] & j_{!*} f_{U*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[l, dashed, shift left] \\ f_* j'_! \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[r, shift left, "\sim"] & j_! f_{U*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[l, shift left, "\sim"] \end{tikzcd} \tag{1.3.3}$$

where the vertical maps are now determined by $j_! \to j_{!*}$, $j'_! \to j'_{!*}$, and the dashed arrows are the ones just constructed.

By an analogous argument, one sees that maps making this diagram commute are determined by $h' \in \mathrm{End}(A)$, so we just need to prove that $h = h'$.

For this, consider the full diagram

$$\begin{tikzcd} f_* j'_* \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[r, shift left, "\sim"] & j_* f_{U*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[l, shift left, "\sim"] \\ f_* j'_{!*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[r, dashed, shift left] & j_{!*} f_{U*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[l, dashed, shift left] \\ f_* j'_! \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[r, shift left, "\sim"] & j_! f_{U*} \mathbf{R}^{2p-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[l, shift left, "\sim"] \end{tikzcd}$$

which if we pullback to U we get

$$\begin{tikzcd} A \arrow[r, shift left] & A \arrow[l, shift left] \\ A \arrow[u, "\sim"] \arrow[ur, "h"] \arrow[r, dashed, shift left] \arrow[dr, "h'"'] & A \arrow[u, "\mathrm{id}"'] \arrow[l, dashed, shift left] \\ A \arrow[u, "\sim"] \arrow[r, shift left] & A \arrow[u, "\mathrm{id}"'] \arrow[l, shift left] \end{tikzcd}$$

and therefore $h = h'$, which concludes the proof that the diagram (1.3.3) commutes.

If we now dualize this diagram, we get

$$\begin{tikzcd} f_* j'_* \mathbf{R}^{2q-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[r, shift left, "\sim"] & j_* f_{U*} \mathbf{R}^{2q-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[l, shift left, "\sim"] \\ f_* j'_{!*} \mathbf{R}^{2q-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[r, dashed, shift left] & j_{!*} f_{U*} \mathbf{R}^{2q-1} \pi'_* \mathbb{Q}_\ell[b] \arrow[u] \arrow[l, dashed, shift left] \end{tikzcd}$$

which tells us that the dual of the pushforward map is the pullback map and vice versa.

1.3.2 Proving the Projection Formula

For any map $L \to M$ in the derived category we have a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & D(D(L)) \arrow[d] \\ M \arrow[r] & D(D(M)) \end{tikzcd}$$

where the horizontal maps are the duality morphisms. Applying this to the pushforward map just constructed we obtain

$$\begin{tikzcd} f_* j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b] \arrow[r] \arrow[d] & D(f_* j'_{!*}\mathbf{R}^{2q-1}\pi'_*\mathbb{Q}_\ell[b]) \arrow[d] \\ j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell[b] \arrow[r] & D(j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b]) \end{tikzcd}$$

where by the above remarks, $j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b] \to f_* j'_{!*}\mathbf{R}^{2q-1}\pi'_*\mathbb{Q}_\ell[b]$ is the pullback map.

Let us apply $H^{1-b}_{\text{ét}}(B, -)$ to this diagram, we get:

$$\begin{tikzcd} H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b]) \arrow[r, "\widehat{\cdot}"] \arrow[d, "g_*"'] & \mathrm{Hom}(j'_{!*}\mathbf{R}^{2q-1}\pi'_*\mathbb{Q}_\ell[b], \mathbb{Q}_\ell[1+b]) \arrow[d, "g^*"] \\ H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbb{Q}_\ell[b]) \arrow[r, "\widehat{\cdot}"'] & \mathrm{Hom}(j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b], \mathbb{Q}_\ell[1+b]) \end{tikzcd}$$

So if we take $\alpha \in H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2p-1}\pi'_*\mathbb{Q}_\ell[b])$, we have that

$$\widehat{g_*\alpha} = g^*\widehat{\alpha} \tag{1.3.4}$$

where by definition $g^*\widehat{\alpha}$ is the composition

$$j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b] \xrightarrow{g^*} f_* j'_{!*}\mathbf{R}^{2q-1}\pi'_*\mathbb{Q}_\ell[b] \xrightarrow{f_*(\widehat{\alpha})} f_*\mathbb{Q}_\ell[1+b] \xrightarrow{f_*} \mathbb{Q}_\ell[1+b]$$

Let us apply $H^{1-b}_{\text{ét}}(B, -)$ again, but now to the equality (1.3.4), we compute that

$$H^{1-b}_{\text{ét}}(B, -)(\widehat{g_*\alpha}) : H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b]) \to H^2_{\text{ét}}(B, \mathbb{Q}_\ell)$$

sends $\beta \mapsto h_X(g_*\alpha, \beta)$.

On the other hand, $H^{1-b}_{\text{ét}}(B, -)(g^*\widehat{\alpha})$ is the composite map

$$\begin{tikzcd}
H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbb{Q}_\ell[b]) \arrow[r] \arrow[d, "g^*"'] & H^2_{\text{ét}}(B, \mathbb{Q}_\ell) \\
H^{1-b}_{\text{ét}}(B', j'_{!*}\mathbf{R}^{2q-1}\pi'_*\mathbb{Q}_\ell[b]) \arrow[r] & H^2_{\text{ét}}(B', \mathbb{Q}_\ell) \arrow[u, "f_*"']
\end{tikzcd}$$

which sends $\beta \mapsto g^*\beta \mapsto h_{X'}(\alpha, g^*\beta) \mapsto f_*h_{X'}(\alpha, g^*\beta)$.

Putting everything together then, we have that

$$h_X(g_*\alpha, \beta) = f_*h_{X'}(\alpha, g^*\beta)$$

completing the proof of Proposition 1.3.2.

Heights in Arakelov Geometry

2.1 Introduction

Let R be a regular arithmetic ring, i.e. a regular, excellent, noetherian domain, together with a finite set $\mathcal{S}$ of injective ring homomorphisms $R \to \mathbb{C}$ which is invariant under complex conjugation.

Take a regular, integral scheme B_0, flat and of finite type over R. Write $B_0(\mathbb{C})$ for the points of the complex variety given by the disjoint union of the $B_0 \times_{R,\sigma} \mathbb{C}$ for all $\sigma \in \mathcal{S}$. This naturally has the structure of a complex manifold.

A hermitian line bundle $\mathcal{L}$ on B_0 is a line bundle together with a hermitian metric $||\cdot||$ on the associated holomorphic line bundle on $B_0(\mathbb{C})$, which is invariant under complex conjugation and which varies smoothly over the points of $B_0(\mathbb{C})$. Write $\widehat{\mathrm{Pic}}(B_0)$ for the group of isomorphism classes of hermitian line bundles on B_0, with the group structure given by the tensor product.

Let X_0 be a smooth projective variety of dimension d over the function field K_0 of B_0. For $p+q=d+1$, one might conjecture, as is done in Conjecture 7.1. of [38], the existence of a height pairing

$$\langle \cdot, \cdot \rangle_{X_0} : CH^p_{\mathrm{hom}}(X_0)_{\mathbb{Q}} \otimes CH^q_{\mathrm{hom}}(X_0)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(B_0)_{\mathbb{Q}}$$

with $\widehat{\mathrm{Pic}}(B_0)$ being the natural choice for the value group in the Arakelov setting. As it turns out though, this is too optimistic and it is generally not possible to define such a pairing in a natural way. The reason being is that the metric on the holomorphic line bundle is usually only smooth on a dense open subset, and usually only continuous over a dense open subset whose complement is of codimension at least 2. We shall discuss an example of this later in the chapter. This leads us to make the following definition.

Definition 2.1.1. We define $\widehat{\mathrm{Pic}}'(B_0)$ to be the group of (isomorphism classes of) line bundles on B_0, with a continuous hermitian metric on the associated holomorphic line bundle on $B_0(\mathbb{C})$ which is: defined an open subset of $B_0(\mathbb{C})$ whose complement is of codimension at least 2, that is invariant under complex conjugation, and which is smooth over a dense open subset of $B_0(\mathbb{C})$.

Furthermore, we identify any two metrics that are equal on the set on which they are both defined. The group structure, as before, is given by the tensor product. Of course then, there are natural restriction maps

$$\widehat{\mathrm{Pic}}(B_0) \to \widehat{\mathrm{Pic}}'(B_0) \to \mathrm{Pic}(B_0)$$

Conjecture 2.1.2. *Suppose that X_0 has a model $\mathcal{X}$ over B_0, with smooth locus $V \subset B_0$, such that the Hodge structure $H^{2p-1}(\mathcal{X}_V(\mathbb{C}), \mathbb{Z}(p))$ has unipotent monodromy. Assuming that $B_0 \setminus V$ is a normal crossings divisor, then for $p+q = d+1$ there exists a height pairing*

$$\langle \cdot, \cdot \rangle_{X_0} : CH^p_{\mathrm{hom}}(X_0)_{\mathbb{Q}} \otimes CH^q_{\mathrm{hom}}(X_0)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}'(B_0)_{\mathbb{Q}}$$

The rest of this chapter will be devoted to constructing the analytic half of this height pairing; indeed this is given by the height pairing in Theorem 2.2.2. It remains open on how to define the appropriate line bundle on B_0 without using Beilinson cycles, however we know at least what the associated holomorphic line bundle on $B_0(\mathbb{C})$ must be.

2.1.1 The Biextension Line Bundle

Let $\mathcal{H}$ be a weight -1 polarizable variation of pure Hodge structure over a complex manifold S, and assume that the local system $\mathcal{H}_{\mathbb{Z}}$ is torsion free.

Definition 2.1.3. The group of normal functions into $\mathcal{H}$ is defined as

$$\mathbf{NF}(S, \mathcal{H}) := \mathrm{Ext}^1_{\mathbf{VMHS}(S)}(\mathbb{Z}, \mathcal{H})$$

where $\mathbf{VMHS}(S)$ is the category of variations of mixed hodge structures on S. We also define

$$\mathbf{NF}(S, \mathcal{H})^\vee := \mathrm{Ext}^1_{\mathbf{VMHS}(S)}(\mathcal{H}, \mathbb{Z}(1))$$

Remark 2.1.4. *Note that by duality* $\mathbf{NF}(S, \mathcal{H})^\vee \simeq \mathbf{NF}(S, \mathcal{H}^\vee)$*, where* $\mathcal{H}^\vee = \mathrm{Hom}(\mathcal{H}, \mathbb{Z}(1))$.

Suppose that we have an inclusion $S \hookrightarrow \overline{S}$ of S as a Zariski open subset of $\overline{S}$. Then we also have the notion of an admissible normal function.

Definition 2.1.5. The group of admissible normal functions is

$$\mathbf{ANF}(S, \mathcal{H})_{\overline{S}} := \mathrm{Exc}^1_{\mathbf{VMHS}^{\mathrm{ad}}_{\overline{S}}}(\mathbb{Z}, \mathcal{H})$$

where $\mathbf{VMHS}^{\mathrm{ad}}_{\overline{S}}$ is the category of admissible variations of mixed hodge structures (see [6] for a definition of this category).

Definition 2.1.6. A biextension variation of type $\mathcal{H}$ is a variation of mixed hodge structure $\mathcal{V}$ over S equipped with isomorphisms

$$\mathrm{Gr}^W_0(\mathcal{V}) \simeq \mathbb{Z} \quad \mathrm{Gr}^W_{-1}(\mathcal{V}) \simeq \mathcal{H} \quad \mathrm{Gr}^W_{-2}(\mathcal{V}) \simeq \mathbb{Z}(1)$$

where W denotes the weight filtration on $\mathcal{V}$ and $\mathrm{Gr}^W_i(\mathcal{V}) = (W_i/W_{i-1})\mathcal{V}$.

To a biextension $\mathcal{V}$, or indeed any variation of mixed hodge structure, there is an associated monodromy representation of the fundamental group of S

$$\rho_s : \pi_1(S, s) \to \mathrm{Aut}_{\mathbb{Z}}(\mathcal{V}_s)$$

for any point $s \in S$. We say that $\mathcal{V}$ has unipotent monodromy if this representation is unipotent. By Proposition 1.3. in [22] a biextension $\mathcal{V}$ of type $\mathcal{H}$ is unipotent iff $\mathcal{H}$ is unipotent.

For any such $\mathcal{V}$ there is an associated C^∞ function (see [19] and also Proposition 2.2. in [10])

$$h_{\mathcal{V}} : S \to \mathbb{R} \tag{2.1.1}$$

Given $\nu \in \mathbf{NF}(S, \mathcal{H})$ and $\omega \in \mathbf{NF}(S, \mathcal{H})^\vee$, the set of biextension variations of type (ν, ω) consists of all biextension variations over S such that

$$(W_0/W_{-2})\mathcal{V} \simeq \nu$$
$$(W_{-1}/W_{-3})\mathcal{V} \simeq \omega$$

For any open subset $U \subset S$, we can consider the set $\mathcal{L}(\nu, \omega)$ of isomorphism classes of biextension variations of type (ν, ω) over U. $\mathcal{L}(\nu, \omega)$ is naturally a torsor over $\mathcal{O}_S^\times$, therefore its sections can be identified with the non vanishing sections of an analytic line bundle $\mathcal{L}(\nu, \omega)$ over S, which Hain and Reed in [21] refer to as the biextension line bundle.

Furthermore, they noted that $\mathcal{L}(\nu, \omega)$ comes with a natural metric defined by

$$||\mathcal{V}|| = \exp(-h_{\mathcal{V}})$$

where $h_{\mathcal{V}}$ is the function in (2.1.1).

2.2 Analytic Height Pairing

Let B be a smooth integral scheme of finite type over $\mathbb{C}$ with function field $K = k(B)$, and take a smooth projective variety X over K of dimension d.

Definition 2.2.1. We define $\widehat{\mathrm{Pic}}'(B)$ to be the group of (isomorphism classes of) line bundles on B, with a continuous hermitian metric on the associated holomorphic line bundle on B which is: defined an open subset of B whose complement is of codimension at least 2, that is invariant under complex conjugation, and which is smooth over a dense open subset of B. We identify any two metrics that are equal on the set on which they are both defined. The group structure is given by the tensor product.

Now, suppose we have a smooth projective model $\mathcal{X}_U$ of X over an open subscheme $U \subset B$. Let $\tilde{z}^p$, $\tilde{z}^q$ be a pair of algebraic cycles on $\mathcal{X}_U$ (of codimension p and q respectively, where $p + q = d + 1$) which are homologically trivial on each fibre.

If we take $\mathcal{H}$ to be the Hodge structure $H^{2p-1}(\mathcal{X}_U, \mathbb{Z}(p))$ over U, then there is a natural way to associate normal functions

$$\tilde{z}^p \rightsquigarrow \nu \in \mathbf{NF}(U, \mathcal{H})$$
$$\tilde{z}^q \rightsquigarrow \omega \in \mathbf{NF}(U, \mathcal{H})^\vee$$

See Section 4.3. in [31] for an explanation of this. In fact, these normal functions turn out to be admissible.

We can then consider the biextension line bundle $\mathcal{L}(\nu, \omega)$. If additionally the cycles $\tilde{z}^p$, $\tilde{z}^q$ are disjoint, then there is a canonical biextension $\mathcal{V}$ (which is a subquotient of $H^{2p-1}(\mathcal{X}_U \backslash |\tilde{z}^p|, |\tilde{z}^q|, \mathbb{Z}(p))$), for which $||\mathcal{V}||$ is naturally identified with the archimedean height pairing $\langle \tilde{z}^p, \tilde{z}^p \rangle : U \to \mathbb{R}$, see [19] and [34]. It follows from this that $\mathcal{L}(\nu, \omega) = \tilde{z}^p \cap \tilde{z}^q$.

2.2.1 Constructing the Analytic Height Pairing

Theorem 2.2.2. *Suppose that X has a model $\mathcal{X}$ over B, with smooth locus $V \subset B$, such that the Hodge structure $H^{2p-1}(\mathcal{X}_V, \mathbb{Z}(p))$ has unipotent monodromy. Assuming that $B \setminus V$ is a normal crossings divisor, then for $p + q = d + 1$ there exists a height pairing*

$$\langle \cdot, \cdot \rangle_X : CH^p_{\text{hom}}(X)_{\mathbb{Q}} \otimes CH^q_{\text{hom}}(X)_{\mathbb{Q}} \to \text{Pic}'(B)_{\mathbb{Q}}$$

Proof. We can assume wlog that B is proper by embedding B into a compactification $B \hookrightarrow \overline{B}$, and then using the restriction map $\text{Pic}'(\overline{B}) \to \text{Pic}'(B)$.

Given $\alpha^p \in CH^p_{\text{hom}}(X)_{\mathbb{Q}}$ and $\beta^q \in CH^q_{\text{hom}}(X)_{\mathbb{Q}}$, take representatives z^p and z^q. Choose an open subset $U \subset B$ such that we can find a smooth projective model $\mathcal{X}_U$ of X over U, and we can find extensions $\tilde{z}^p$ and $\tilde{z}^q$ on $\mathcal{X}_U$ with disjoint supports, and which are homologous to zero on each fibre.

Let ν be the normal function associated to $\tilde{z}^p$, and ω be the normal function associated to $\tilde{z}^q$. Then consider the biextension line bundle $\mathcal{L}(\nu, \omega)$ on U, which has a canonical metric $|| \cdot ||$ on it. This line bundle has an algebraic structure given by the divisor associated to $\tilde{z}^p \cap \tilde{z}^q$, i.e. it is the trivial bundle.

By Theorem 233. in [8] $\mathcal{L}(\nu, \omega)$ has an extension $\overline{\mathcal{L}} \in \text{Pic}(B)_{\mathbb{Q}}$, the exact choice is as in Remark 234. Now, since $B \setminus V$ is a normal crossings divisor, it is given locally by equations of the form $z_1 z_2 \cdots z_r = 0$ for $1 \leq r \leq \dim(B)$. If we throw out the points where $z_i = 0$ for more than one i, which we can do because the complement of these points is of codimension at least 2, then what we are left with looks locally like $z_1 = 0$. This means that we can apply Theorem 24. in [8] to extend the metric so that $(\overline{\mathcal{L}}, || \cdot ||) \in \text{Pic}'(B)_{\mathbb{Q}}$. Furthermore, by the following sub-lemma our extension $(\overline{\mathcal{L}}, || \cdot ||)$ must be unique.

Lemma 2.2.3. *Let $U \subset B$ be a dense open subscheme. The natural restriction map is injective*

$$\text{Pic}'(B) \hookrightarrow \text{Pic}'(U)$$

Proof. Take $(\mathcal{L}, || \cdot ||) \in \text{Pic}'(B)$. If s is a meromorphic section on $\mathcal{L}$, then we can compute $\text{div}(s)$ by looking at the zeroes and poles (and their orders) of $||s||$. Explicitly,

$$\text{div}(s) = \sum_Z \text{ord}_Z(||s||) \cdot Z$$

As the metric is continuous on the complement of a codimension 2 subset, $\mathrm{ord}_Z(||s||)$ can be determined by looking solely at the restriction of $||s||$ to U. Hence, $\mathcal{L} = \mathcal{O}(\mathrm{div}(s))$ is uniquely determined by the restriction of $(\mathcal{L}, ||\cdot||)$ to U. By continuity the metric is also uniquely determined by its restriction to U, and this completes the proof of the lemma.

□

So we can define the height pairing as follows:

$$\begin{aligned} \langle \cdot, \cdot \rangle_X : CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \otimes CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}} &\to \mathrm{Pic}'(B)_{\mathbb{Q}} \\ (\alpha^p, \beta^q) &\mapsto (\overline{\mathcal{L}}, ||\cdot||) \end{aligned}$$

To see that this height pairing does not depend on any choices, consider that for a different extension $\tilde{z}^{p'}$ on another model $\mathcal{X}'_{U'}$ over $U' \subset B$, there is some $U'' \subset U \cap U'$ such that $\tilde{z}^p|_{U''} = \tilde{z}^{p''} = \tilde{z}^{p'}|_{U''}$.

Clearly $\mathcal{L}(\nu, \omega)|_{U''} = \mathcal{L}(\nu'', \omega) = \mathcal{L}(\nu', \omega)|_{U''}$ (where ν' and ν'' are the normal functions associated to $\tilde{z}^{p'}$ and $\tilde{z}^{p''}$ respectively), and likewise for the metric. Of course, the analogous statement for the second variable is true also, and so the fact that the pairing does not depend on any choices made follows from the uniqueness of the extension $(\overline{\mathcal{L}}, ||\cdot||)$ on B. Lemma 2.2.3 also implies that $\langle \cdot, \cdot \rangle_X$ is bilinear, etc. Indeed, by the lemma we just have to check this on an open subset, over which the height pairing is given by the intersection pairing together with the archimedean height, both of which are bilinear.

□

Let us finally comment on why the metric may not extend smoothly in general. Theorem 22. of [8] gives an explicit condition necessary for the metric to extend, and this condition is not in general met. We can give an illustrative example in the case B is the compactified stack $\overline{\mathcal{M}_3}$ of smooth projective curves of genus 3 (so not actually a scheme). As is explained in Section 1.1. of [8], if C is a smooth projective curve of genus 3 then the Ceresa cycle $C - C^-$ is a homologically trivial 1-cycle on $\mathrm{Jac}(C)$. So we can consider the height of this cycle $\langle C - C^-, C - C^- \rangle_{\mathrm{Jac}(C)}$, which is equal to the extension of the line bundle $\mathcal{L}(\nu, \nu^\vee)$, where ν is the normal function associated to $C - C^-$. Calculations of Brosnan and Pearlstein show the condition mentioned above is not satisfied in this case, and hence the metric does not extend smoothly. One can also find the relevant calculations in Section 14.3. of [19].

2.2.2 Analytic vs Geometric Height Pairing

Conjecture 2.2.4. *Let $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ denote the conjectural geometric height pairing formed by intersecting Beilinson extensions on a model $\mathcal{X}$ of X over B (see Section 1.1.2). Then there is a commutative diagram*

$$\begin{tikzcd} CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \otimes CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}} \arrow[r, "\langle\cdot,\cdot\rangle_X"] \arrow[dr, "\langle\cdot,\cdot\rangle_{\mathcal{X}}"'] & \mathrm{Pic}'(B)_{\mathbb{Q}} \arrow[d] \\ & \mathrm{Pic}(B)_{\mathbb{Q}} \end{tikzcd}$$

Let us give some informal justification why this conjecture should be true. As we have explained before, the geometric pairing $\langle\alpha^p,\beta^q\rangle_{\mathcal{X}}$ should be computed by taking the intersection product $\tilde{z}^p \cap \tilde{z}^q$ of Beilinson extensions of the two cycles on a model $\mathcal{X}$ over B. We can scale these by positive integers if necessary so to ensure they are integral cycles.

As we have seen, over some open subscheme $U \subset B$, this line bundle coincides with the biextension line bundle $\mathcal{L}(\nu,\omega)$, where ν and ω are the two normal functions associated to $\tilde{z}^p|_U$ and $\tilde{z}^q|_U$ respectively. Moreover, this line bundle on U has a canonical C^∞-metric.

Intuitively, because $\tilde{z}^p$, $\tilde{z}^q$ are Beilinson extensions, this should in some sense allow us to associate normal functions $\tilde{\nu}$, $\tilde{\omega}$ extending ν and ω. In turn, this should allow us to construct the unique extension $\overline{\mathcal{L}}(\tilde{\nu},\tilde{\omega}) \in \mathrm{Pic}'(B)_{\mathbb{Q}}$ of $\mathcal{L}(\nu,\omega) = \tilde{z}^p|_U \cap \tilde{z}^q|_U$, whose underlying line bundle is $\tilde{z}^p \cap \tilde{z}^q$.

Of course, by construction the Arakelovian pairing $\langle\alpha^p,\beta^q\rangle_X$ is equal to this unique extension $\overline{\mathcal{L}}(\tilde{\nu},\tilde{\omega})$, and therefore the diagram should commute.

In the case where B is a curve and so are all the fibres of $\mathcal{X} \to B$, Holmes and de Jong give what amounts to a full proof of the conjecture via this method, see Theorem 2.2. in [25].

Finally, we compare the analytic pairing to the cohomological Beilinson height pairing.

Conjecture 2.2.5. *Let $\langle\cdot,\cdot\rangle_U$ denote the Beilinson height paring (see Section 1.1.4). Then there is a commutative diagram*

$$\begin{tikzcd}
CH^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \otimes CH^q_{\mathrm{hom}}(X)_{\mathbb{Q}} \arrow[r, "\langle\cdot,\cdot\rangle_X"] \arrow[dr, "\langle\cdot,\cdot\rangle_U"'] & \mathrm{Pic}'(B)_{\mathbb{Q}} \arrow[d] \\
& H^2_{\text{ét}}(B, \mathbb{Q}_\ell(1))
\end{tikzcd}$$

This conjecture would obviously follow from the previous one, but work in progress by Brosnan, Pearlstein and Szamuely would give a direct proof. Roughly, this is done by constructing a map

$$\mathrm{Ext}^2_{\mathrm{MHM}(B)}(\mathbb{Q}, \mathbb{Q}(1)) \xrightarrow{\sim} \mathrm{Pic}(B)_{\mathbb{Q}}$$

which is an isomorphism. Here $\mathrm{MHM}(B)$ is the category of mixed Hodge modules on B. Moreover, this map takes $j_{!*}\nu \cup j_{!*}\omega$ to the line bundle $\overline{\mathcal{L}}$ (which is the extension of $\mathcal{L}(\nu, \omega)$ described previously). The intermediate extension functor $j_{!*}$ in this context is a map

$$j_{!*} : \mathbf{ANF}(U, \mathcal{H})_B \to \mathrm{Exc}^1_{\mathrm{MHM}(B)}(\mathbb{Q}, j_{!*}\mathcal{H})$$

see Lemma 2.18. in [7].

Hrushovski's Theorem

3.1 Introduction

Let U be a smooth curve over $k = \overline{\mathbb{F}}_p$, and let K be its function field. Then take an abelian variety A over K with K/k-trace equal to zero, and a finitely generated subgroup $\Gamma \subseteq A(K)$. Given a discrete valuation v on K, recall the notion of v-adic distance $d_v(Y, P)$ between a subvariety Y of A, and a point P on A. We can also define the local height $\lambda_v(Y, P) := -\log d_v(Y, P)$.

Remark 3.1.1. *For this chapter, a subvariety of A can be any reduced closed subscheme of $A_{K'} = A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$, where K' is any algebraic field extension of K. So in particular, it may not be irreducible.*

Definition 3.1.2. A subvariety $Y \subset A$ is said to be linear if it is a finite union of translates of abelian subvarieties of A.

Theorem 3.1.3 (Hrushovski's Theorem)**.** *Let $X \subset A$ be a subvariety defined over K. There exists a linear subvariety $Y \subset X$ which is also defined over K, such that for each discrete valuation v of K there is a constant C_v for which*

$$\lambda_v(X, P) \leq C_v \cdot \lambda_v(Y, P)$$

for all points $P \in \Gamma$.

Originally conjectured by Voloch, Hrushovski first proved this theorem using deep results from model theory, see Theorem 6.4. in [26]. Our goal then, is to give a proof without this input. In fact, we shall give two proofs, the second of which implies a slightly stronger result.

Theorem 3.1.4. *Let $X \subset A$ be a subvariety defined over K. There exists a linear subvariety $Y \subset X$ which is also defined over K, such that for each discrete valuation v of K there is a constant C_v for which*

$$\lambda_v(X, P) \leq \lambda_v(Y, P) + C_v$$

for all points $P \in \Gamma$.

Remark 3.1.5. *In terms of the v-adic distance, the above inequality translates into*

$$d_v(Y,P) \leq e^{C_v} \cdot d_v(X,P)$$

Of course, the v-adic distance is only defined up to a multiplicative constant, and so the constant in Hrushovski's theorem is unavoidable.

Remark 3.1.6. *If we remove the condition on the K/k-trace being zero we still can define a subvariety $Y \subset X$ satisfying the same inequality, however Y may no longer be linear. Instead, Y will be a finite union of "special" subvarieties, see Definition D.0.4.*

Remark 3.1.7. *In Hrushovski's original proof Y is defined only over some extension field of K. Poonen and Voloch showed in* [36] *that one can always find such a Y defined over K by using an argument from model theory. Our method then reproves this fact, but without the model theoretic input.*

Corollary 3.1.8. *We have an equality of sets*

$$X(K) \cap \Gamma = Y(K) \cap \Gamma$$

Proof. Given $P \in X(K) \cap \Gamma$, for any valuation v we have $\lambda_v(X,P) = \infty$ and therefore by Hrushovski's theorem $\lambda_v(Y,P) = \infty$ too. Hence $P \in Y(K) \cap \Gamma$.

□

Remark 3.1.9. *The corollary above implies the Mordell-Lang conjecture in the sense that if $X(K) \cap \Gamma$ is dense in X, then so must $Y(K) \cap \Gamma$ be, and hence $X = Y$ is linear. Thus demonstrating that Hrushovski's theorem can be thought of as a continuous version of Mordell-Lang.*

We can and shall assume throughout that X is irreducible. Our approach to proving Hrushovski's theorem is to consider the closed subschemes

$$\mathrm{Exc}^n(A, X^{+Q}) \subseteq X^{+Q}$$

which we refer to generally as exceptional schemes. Here $n \in \mathbb{N}$ and X^{+Q} denotes the translation of X by a point $Q \in \Gamma$.

Take the Néron model $\mathcal{A}$ of A over U, and assume U is sufficiently small so that the closed immersion $X \hookrightarrow A$ extends to a closed immersion $\mathcal{X} \hookrightarrow \mathcal{A}$, with $\mathcal{X}$ flat over U. We do not however consider U to be fixed, as we will need to vary it throughout the chapter.

Note that for the Néron model there is a bijective correspondence between $A(K)$ and $\mathcal{A}(U)$, therefore we shall seldom distinguish between a point $Q \in A(K)$ and its lifting $Q \in \mathcal{A}(U)$.

In [37] Rössler defines the exceptional schemes $\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})$ over U for the purposes of proving the Mordell-Lang conjecture. We shall go over the construction and define the analogous objects $\mathrm{Exc}^n(A, X)$ over K, as-well as of course proving they are compatible, i.e.

$$\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})_K = \mathrm{Exc}^n(A, X)$$

3.1.1 Defining the Exceptional Schemes

Let us first introduce the notion of the Weil restriction functor. Start with a scheme T and a morphism $T' \to T$. Then for each T'-scheme Z we can consider the functor

$$W/T \mapsto \mathrm{Hom}_{T'}(W \times_T T', Z)$$

If T' is finite, flat and locally of finite presentation over T then by 7.6. in [5] this functor is representable by a T-scheme which we denote $\mathfrak{R}_{T'/T}(Z)$.

Now, consider the diagonal immersion $\Delta : U \to U \times_k U$. Let $\mathcal{I}_\Delta \subseteq \mathcal{O}_{U\times U}$ be the ideal sheaf of $\Delta_* U$. For each $n \geq 0$ define U_n, the n-th infinitesimal neighbourhood of the diagonal inside $U \times_k U$, as the closed subscheme associated to $\mathcal{O}_{U\times U}/\mathcal{I}_\Delta^{n+1}$.

From the two projection maps $\pi_1, \pi_2 : U \times_k U \to U$, we obtain the induced maps $\pi_1^{U_n}, \pi_2^{U_n} : U_n \to U$. We view U_n as a U-scheme via $\pi_1^{U_n}$.

Lemma 3.1.10. *U_n is finite and flat as a U-scheme.*

Proof. To see that U_n is finite over U, note that it is proper and also quasi-finite because $(U_n)_{\mathrm{red}} = \Delta_*(U)$, and $\Delta_*(U) \to U$ is obviously bijective as a map of the underlying spaces.

For flatness, consider the exact sequence of $\mathcal{O}_U$-modules

$$0 \to \mathcal{I}_\Delta^{n+1}/\mathcal{I}_\Delta^{n+2} \to \mathcal{O}_{U_{n+1}} \to \mathcal{O}_{U_n} \to 0$$

We observe that $\mathcal{I}_\Delta^{n+1}/\mathcal{I}_\Delta^{n+2}$ is naturally isomorphic to $\mathrm{Sym}_{U_1}^{n+1}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$ as a $\mathcal{O}_{U_0}$-module since $\mathcal{I}_\Delta$ is locally generated by a regular sequence in $U \times_k U$.

Hence, $\mathcal{I}_\Delta^{n+1}/\mathcal{I}_\Delta^{n+2}$ is a locally free $\mathcal{O}_U$-module. So because $U_0 = \Delta_*(U)$ is locally free, by inductively using the above exact sequence we see that $\mathcal{O}_{U_n}$ is locally free as a $\mathcal{O}_U$-module too.

□

This allows us to make the following definition for a scheme W over U.

Definition 3.1.11. The n-th jet scheme of W over U is defined as

$$J^n(W/U) := \mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W)$$

For each $m \leq n$ there are morphisms $U_m \to U_n$, and these subsequently induce morphisms $\Lambda_{n,m}^W : J^n(W/U) \to J^m(W/U)$ of the jet schemes. Furthermore, there is a map of sets

$$\lambda_n^W : W(U) \to J^n(W/U)(U)$$

which sends $f : U \to W$ to $J^n(f) : U = J^n(U/U) \to J^n(W/U)$.

Lemma 3.1.12. *We have the following identity for all $m \leq n$*

$$\Lambda_{n,m}^W \circ \lambda_n^W = \lambda_m^W$$

Proof. This follows from the commutative diagram for any $f : U \to W$

$$\begin{array}{ccc} J^n(U/U) & \xrightarrow{\Lambda_{n,m}^U = \mathrm{id}_U} & J^m(U/U) \\ {\scriptstyle J^n(f)}\downarrow & & \downarrow{\scriptstyle J^m(f)} \\ J^n(W/U) & \xrightarrow[\Lambda_{n,m}^W]{} & J^m(W/U) \end{array}$$

□

Lemma 3.1.13. *For any U-morphism $g : W \to W'$ we have*

$$J^n(g) \circ \lambda_n^W = \lambda_n^{W'} \circ g$$

Proof. This simply follows from the fact that J^n is a functor and so

$$J^n(g \circ f) = J^n(g) \circ J^n(f)$$

for any $f : U \to W$.

□

Lemma 3.1.14. *For any closed point $u \in U$, we have the following description of the special fibre of $J^n(W/U)$ over u:*

$$J^n(W/U)_u(k) = W(u_n)$$

where $u_n = u \times_U U_n$ is the n-th infinitesimal neighbourhood of u.

Proof. We simply compute

$$\begin{aligned} J^n(W/U)_u(k) &= \mathrm{Hom}_{U_n}(u \times_U U_n, \pi_2^{U_n,*}W) \\ &= \mathrm{Hom}_{U_n}(u_n, W_{u_n}) \\ &= \mathrm{Hom}_{u_n}(u_n, W_{u_n}) = W(u_n) \end{aligned}$$

□

Next, let $K \to U$ be the generic point of U. We can consider the analogous diagonal immersion $\Delta_K : K \to K \times_k K$, and the associated ideal $I_\Delta \subseteq \mathcal{O}_{K\times K}$, in order to define $K_n := \mathcal{O}_{K\times K}/I_\Delta^{n+1}$.

As before, we also have maps $\pi_1^{K_n}, \pi_2^{K_n} : K_n \to K$, and we view K_n as a K-scheme via $\pi_1^{K_n}$.

Lemma 3.1.15. *We have that*

$$K_n = U_n \times_U K$$

Proof. We can assume wlog that $U = \mathrm{Spec}(R)$, in which case we need to show that

$$(R \otimes_k R/\mathcal{I}_\Delta^{n+1}) \otimes_R K = K \otimes_k K/I_\Delta^{n+1}$$

where

$$\begin{aligned} \mathcal{I}_\Delta &= \langle r \otimes 1 - 1 \otimes r : r \in R \rangle \\ I_\Delta &= \langle x \otimes 1 - 1 \otimes x : x \in K \rangle \end{aligned}$$

First of all, we have a natural inclusion

$$\begin{aligned} \phi : (R \otimes_k R/\mathcal{I}_\Delta^{n+1}) \otimes_R K &\hookrightarrow K \otimes_k K/I_\Delta^{n+1} \\ r \otimes r' \otimes x &\mapsto rx \otimes r' \end{aligned}$$

where we have used the fact that $I_\Delta^{n+1} \cap R \otimes_k R = \mathcal{I}_\Delta^{n+1}$.

To get surjectivity, take any $r \in R^\times$ and note that

$$(r \otimes 1 \otimes 1) \cdot (1 \otimes 1 \otimes r^{-1}) = r \otimes 1 \otimes r^{-1} = 1 \otimes 1 \otimes rr^{-1} = 1$$

and thus $r \otimes 1 \otimes 1$ is invertible. Furthermore, since $r \otimes 1 - 1 \otimes r \in \mathcal{I}_\Delta$, the element $r \otimes 1 \otimes 1 - 1 \otimes r \otimes 1$ is nilpotent.

Since the sum of an invertible element and a nilpotent one is also invertible:

$$\frac{1}{i+n} = \frac{i^{-1}}{1+i^{-1}n} = i^{-1}(1 - i^{-1}n + \cdots + (-i^{-1}n)^k)$$

it follows that $1 \otimes r \otimes 1 = r \otimes 1 \otimes 1 - (r \otimes 1 \otimes 1 - 1 \otimes r \otimes 1)$ is also invertible.

The image under ϕ of these inverses must be $r^{-1} \otimes 1$ and $1 \otimes r^{-1}$ respectively. Hence, ϕ is surjective and so an isomorphism.

□

By the above lemma, K_n is also finite and flat as a K-scheme, so we can also define the jet schemes for a scheme W now over K.

Definition 3.1.16. The n-th jet scheme of W over K is

$$J^n(W/K) := \mathfrak{R}_{K_n/K}(\pi_2^{K_n,*}W)$$

Let us check that these two definitions are compatible.

Lemma 3.1.17. *We have that*

$$J^n(W/U)_K = J^n(W_K/K)$$

Proof. Since the Weil restriction functor is compatible with base change (see 7.6. in [5]), we have that

$$\begin{aligned} J^n(W/U)_K &= \mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W)_K \\ &= \mathfrak{R}_{K_n/K}((\pi_2^{U_n,*}W)_{K_n}) \\ &= \mathfrak{R}_{K_n/K}(\pi_2^{K_n,*}W_K) = J^n(W_K/K) \end{aligned}$$

where we have used that $U_n \times_U K = K_n$, as-well as the following Cartesian diagram

$$\begin{tikzcd}
& \pi_2^{K_n,*}W_K \arrow[rr] \arrow[dl] \arrow[dd] & & W_K \arrow[dl] \arrow[dd] \\
\pi_2^{U_n,*}W \arrow[rr] \arrow[dd] & & W \arrow[dd] & \\
& K_n \arrow[rr] \arrow[dl] & & K \arrow[dl] \\
U_n \arrow[rr] & & U &
\end{tikzcd}$$

□

Lemma 3.1.18. *Let V be étale over U. Then we have*

$$J^n(V/U) = V$$

Proof. Since V is smooth over U, by Lemma 2.3. in [37] $J^n(V/U)$ is a $J^{n-1}(V/U)$-torsor under the vector bundle $\Lambda_{n,0}^{V,*}(\Omega_{V/U}^{\vee}) \otimes \mathrm{Sym}^n(\Omega_{U/k})$. As V is in fact étale over U, this means $\Omega_{V/U}^{\vee} = 0$ and thus

$$\Lambda_{n,0}^{V,*}(\Omega_{V/U}^{\vee}) \otimes \mathrm{Sym}^n(\Omega_{U/k}) = 0$$

Hence we have an isomorphism $J^n(V/U) \simeq J^{n-1}(V/U)$. By the same reasoning $J^{n-1}(V/U) \simeq J^{n-2}(V/U)$, and so on, so we eventually get that

$$J^n(V/U) \simeq V$$

which completes the proof. □

We must now impose the additional condition that U is sufficiently small so that $\mathcal{A}$ is an abelian scheme over U. Then we can make the following definition.

Definition 3.1.19. The n-th critical scheme of $\mathcal{X}$ over U is

$$\mathrm{Crit}^n(\mathcal{A}, \mathcal{X}) := [p^n]_*(J^n(\mathcal{A}/U)) \cap J^n(\mathcal{X}/U)$$

where $[p^n]_*(J^n(\mathcal{A}/U))$ is the scheme theoretic image of $J^n(\mathcal{A}/U)$ by $[p^n]_{J^n(\mathcal{A}/U)}$.

Remark 3.1.20. *Note that because $\mathcal{A}$ is proper over U, it follows that $[p^n]_*(J^n(\mathcal{A}/U))$ is closed inside $J^n(\mathcal{A}/U)$, and also that $[p^n]_*(J^n(\mathcal{A}/U)) \to \mathcal{A}$ is finite.*

The $\Lambda_{n,n-1}^{\mathcal{A}} : J^n(\mathcal{A}/U) \to J^{n-1}(\mathcal{A}/U)$ induce a system of maps

$$\cdots \to \mathrm{Crit}^2(\mathcal{A}, \mathcal{X}) \to \mathrm{Crit}^1(\mathcal{A}, \mathcal{X}) \to \mathcal{X}$$

with each morphism finite. Hence we can finally define the exceptional schemes.

Definition 3.1.21. The n-th exceptional scheme $\mathrm{Exc}^n(\mathcal{A}, \mathcal{X}) \subseteq \mathcal{X}$ over U is the scheme theoretic image of the the morphism $\mathrm{Crit}^n(\mathcal{A}, \mathcal{X}) \to \mathcal{X}$.

Remark 3.1.22. *Likewise, we can carry out this construction generically over K in order to define* $\mathrm{Crit}^n(A, X)$ *and* $\mathrm{Exc}^n(A, X)$.

Proposition 3.1.23. *We have that*

$$\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})_K = \mathrm{Exc}^n(A, X)$$

Proof. Noting that taking scheme theoretic image commutes with flat base change, the pullback $\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})_K$ is equal to the scheme theoretic image of the morphism $\mathrm{Crit}^n(\mathcal{A}, \mathcal{X})_K \to X$, and so it suffices to compute

$$\begin{aligned}\mathrm{Crit}^n(\mathcal{A}, \mathcal{X})_K &= [p^n]_*(J^n(\mathcal{A}/U))_K \cap J^n(\mathcal{X}/U)_K \\ &= [p^n]_*(J^n(A/K)) \cap J^n(X/K) \\ &= \mathrm{Crit}^n(A, X)\end{aligned}$$

where we have used Lemma 3.1.17 in the second line. □

Note that the exceptional schemes give us a descending chain of closed subschemes

$$\cdots \subseteq \mathrm{Exc}^n(\mathcal{A}, \mathcal{X}) \subseteq \cdots \subseteq \mathrm{Exc}^1(\mathcal{A}, \mathcal{X}) \subseteq \mathcal{X}$$

By noetherianity this chain must stabilize at some $\mathrm{Exc}^N(\mathcal{A}, \mathcal{X})$. We set

$$\mathrm{Exc}(\mathcal{A}, \mathcal{X}) := \cap_n \mathrm{Exc}^n(\mathcal{A}, \mathcal{X}) = \mathrm{Exc}^N(\mathcal{A}, \mathcal{X})$$

to be the intersection of all the exceptional schemes.

3.1.2 Basic Properties of the Exceptional Schemes

Proposition 3.1.24. *Suppose that $Q, Q' \in \mathcal{A}(U)$ are such that $Q - Q' \in p^n\mathcal{A}(U)$, then*

$$\mathrm{Exc}^n(\mathcal{A}, \mathcal{X}^{+Q})^{-Q} = \mathrm{Exc}^n(\mathcal{A}, \mathcal{X}^{+Q'})^{-Q'}$$

Proof. Consider first the diagram

$$\begin{tikzcd} J^n(\mathcal{A}/U) \arrow[r] \arrow[d] & J^n(\mathcal{A}/U) \arrow[d] \\ \mathcal{A} \arrow[r, "+Q'-Q"] & \mathcal{A} \end{tikzcd}$$

where the top map is

$$\begin{aligned}
+\lambda_n^{\mathcal{A}}(Q-Q') &= J^n(-/U)(+Q-Q') \\
&= J^n(-/U)(+Q) - J^n(-/U)(Q') = +\lambda_n^{\mathcal{A}}(Q) - \lambda_n^{\mathcal{A}}(Q')
\end{aligned}$$

using the fact that J^n is functorial.

From the following commutative square

$$\begin{tikzcd}
J^n(\mathcal{A}/U) \arrow[r, "+P"] \arrow[d, "p^n"'] & J^n(\mathcal{A}/U) \arrow[d, "p^n"] \\
J^n(\mathcal{A}/U) \arrow[r, "+p^nP"'] & J^n(\mathcal{A}/U)
\end{tikzcd}$$

we deduce that $[p^n]_* J^n(\mathcal{A}/U) + p^n P \subseteq [p^n]_* J^n(\mathcal{A}/U)$.

Since $Q - Q' \in p^n \mathcal{A}(U)$, it follows that $J^n(-/U)(+Q-Q') \in p^n J^n(\mathcal{A}/U)(U)$, and hence we get the induced diagram

$$\begin{tikzcd}
{[p^n]_* J^n(\mathcal{A}/U)} \arrow[r] \arrow[d] & {[p^n]_* J^n(\mathcal{A}/U)} \arrow[d] \\
\mathcal{A} \arrow[r, "+Q'-Q"] & \mathcal{A}
\end{tikzcd}$$

We also have the analogous diagram on the subschemes

$$\begin{tikzcd}
J^n(\mathcal{X}^{+Q}/U) \arrow[r] \arrow[d] & J^n(\mathcal{X}^{+Q'}/U) \arrow[d] \\
\mathcal{X}^{+Q} \arrow[r, "+Q'-Q"] & \mathcal{X}^{+Q'}
\end{tikzcd}$$

Combining them together, by definition of the critical scheme we get

$$\begin{tikzcd}
\mathrm{Crit}^n(\mathcal{A}, \mathcal{X}^{+Q}) \arrow[r] \arrow[d] & \mathrm{Crit}^n(\mathcal{A}, \mathcal{X}^{+Q'}) \arrow[d] \\
\mathcal{X}^{+Q} \arrow[r, "+Q'-Q"] & \mathcal{X}^{+Q'}
\end{tikzcd}$$

This means that $\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})$ maps into $\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q'})$ under translation by $+Q-Q'$. By the same token, $\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q'})$ maps into $\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})$ under translation by $+Q'-Q$, and so the lemma follows.

$\square$

Remark 3.1.25. *The lemma of course also implies that the exceptional scheme is invariant on the generic fibre, i.e. that*

$$\mathrm{Exc}^n(A, X^{+Q})^{-Q} = \mathrm{Exc}^n(A, X^{+Q'})^{-Q'}$$

for any $Q, Q' \in A(K)$ *such that* $Q - Q' \in p^n A(K)$.

Proposition 3.1.26. *Let* u *be a closed point of* U*, then*

$$\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})_u(k) = \{P \in \mathcal{X}_u(k) \mid P \text{ lifts to an element of } \mathcal{X}(u_n) \cap p^n \mathcal{A}(u_n)\}$$

where u_n *is the* n*-th infinitesimal neighbourhood of* u.

Proof. We first compute

$$\begin{aligned}\mathrm{Crit}^n(\mathcal{A}, \mathcal{X})_u(k) &= ([p^n]_*(J^n(\mathcal{A}/U)))_u(k) \cap J^n(\mathcal{X}/U)_u(k) \\ &= \{P \in J^n(\mathcal{X}/U)_u(k) \mid \exists \widetilde{P} \in J^n(\mathcal{A}/U)_u(k) : p^n \widetilde{P} = P\} \\ &= \{P \in \mathcal{X}(u_n) \mid \exists \widetilde{P} \in \mathcal{A}(u_n) : p^n \widetilde{P} = P\}\end{aligned}$$

where the last line follows from Lemma 3.1.14.

By definition of the exceptional scheme, $\mathrm{Exc}^n(\mathcal{A}, \mathcal{X})_u(k)$ is the set of points on $\mathcal{X}(u)$ which lift to a point on $\mathrm{Crit}^n(\mathcal{A}, \mathcal{X})_u(k)$. So by the above identification, the proposition follows.

□

3.2 Further Properties of the Exceptional Schemes

In this section we prove a number of important results that will be used in the proof of Hrushovski's theorem.

3.2.1 Separable Points

Proposition 3.2.1. *For any separable field extension* L/K*, not necessarily finite, we have an inclusion*

$$X(L) \cap p^n A(L) \subseteq \mathrm{Exc}^n(A, X)(L)$$

Proof. We first prove the case when L is algebraic over K. So suppose we have a point $P \in X(L) \cap p^n A(L)$, which gives us a diagram like so

$$\begin{tikzcd}
L \arrow[r] \arrow[drr] & A \arrow[r, "p^n"] & A \\
& & X \arrow[u, hook]
\end{tikzcd}$$

Since $J^n(-/K)$ is functorial, and because $J^n([p^n]_A) = [p^n]_{J^n(A/K)}$, this gives us a point

$$\widetilde{P} \in J^n(X/K)(L) \cap p^n J^n(A/K)(L) \subseteq \mathrm{Crit}^n(A, X)(L)$$

where we have used (the generic version of) Lemma 3.1.18 to get that $J^n(L/K) = L$ since $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is étale.

By Lemma 3.1.12 and Lemma 3.1.13 we have

$$\Lambda^A_{n,0} \circ J^n(f) \circ \lambda^L_n = \Lambda^A_{n,0} \circ \lambda^A_n \circ f = f$$

So if we take $f : L \to A$ to be given by the point P, we get a commutative diagram

$$\begin{tikzcd}
J^n(L) \arrow[r, "J^n(P)"] & J^n(A) \arrow[d, "\Lambda^A_{n,0}"] \\
L \arrow[u, "\sim"] \arrow[ur, "\widetilde{P}"] \arrow[r, "P"'] & A
\end{tikzcd}$$

and therefore it follows that $\Lambda^A_{n,0}(\widetilde{P}) = P$. This gives us $P \in \mathrm{Exc}^n(A, X)(L)$, and so we have

$$X(L) \cap p^n A(L)) \subseteq \mathrm{Exc}^n(A, X)(L)$$

Now in general, if we have a point $L \to W$ on a scheme W of finite type, then it must factor through a point $L' \to W$ where L' is finitely generated over K. For such L' we can find a smooth variety U' such that the function field of U' is L'.

By restricting U' to an open subset if necessary, we can assume that $L' \to W$ factors through a point $U' \to W$. Then, for any separable point $K^s \to U'$ we get a point $K^s \to W$.

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & L' \arrow[r] \arrow[dl] & U' \arrow[dll] \\
W & & K^s \arrow[ll] \arrow[u]
\end{tikzcd}$$

So if we have a point $P \in X(L) \cap p^n A(L)$, it factors through a point $P \in X(U') \cap p^n A(U')$ which we also call P. For each separable point $x : K^s \to U'$ we get a point

$$P_x \in X(K^s) \cap p^n A(K^s) \subseteq \mathrm{Exc}^n(A, X)(K^s)$$

where we have used what we showed above.

$$\begin{tikzcd}
K^s \arrow[r] \arrow[ddrrr, bend right] & U' \arrow[r] \arrow[drr] \arrow[ddrr] & A \arrow[r, "p^n"] & A \\
 & & & X \arrow[u, hook] \\
 & & & \mathrm{Exc}^n(A, X) \arrow[u, hook]
\end{tikzcd}$$

Since the separable points x are dense in U', it follows that

$$P \in X(U') \cap p^n A(U') \subseteq \mathrm{Exc}^n(A, X)(U')$$

Indeed, consider the pullback diagram

$$\begin{tikzcd}
V' \arrow[r] \arrow[d, hook] & A \setminus \mathrm{Exc}^n(A, X) \arrow[d, hook] \\
U' \arrow[r, "P"] & A
\end{tikzcd}$$

If V' is non-empty then it must contain a separable point, which would give us a contradiction. So V' must be empty and thus P factors through $\mathrm{Exc}^n(A, X)$ as required.

Hence we have

$$P \in X(L) \cap p^n A(L) \subseteq \mathrm{Exc}^n(A, X)(L)$$

which completes the proof.

□

3.2.2 Nearby Points

For the duration of this proposition only we suppose that U contains the closed point u corresponding to the discrete valuation v. This means that $\mathcal{A}$ may no longer be an abelian scheme, and hence we only have access to the the jet schemes over U, not the critical or exceptional schemes.

Proposition 3.2.2. *Let $P \in p^k\Gamma$ be such that $d_v(X, P) \leq 1/p^n$ with n sufficiently large, then*

$$d_v(\mathrm{Exc}^k(A, X), P) \leq 1/p^n$$

Proof. Since $d_v(X, P) \leq 1/p^n$, we have that $P \in \mathcal{X}(u_n)$, where u_n is the n-th infinitesimal neighbourhood of u. That is to say that there is a map $u_n \to \mathcal{X}$ which completes the following diagram

$$\begin{tikzcd} u_n \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, hook] \\ U \arrow[r, "P"'] & \mathcal{A} \end{tikzcd}$$

Now consider $\widetilde{P} = \lambda_k^A(P) \in p^k J^k(A/K)(K) = [p^k]_* J^k(A/K)(K)$. If we apply $J^k(-/U)$ to the commutative diagram above we get

$$\begin{tikzcd} u_n \arrow[r] \arrow[d] & J^k(u_n/U) \arrow[r] \arrow[d] & J^k(\mathcal{X}/U) \arrow[d, hook] \\ U \arrow[r, equal] \arrow[rr, bend right, "\widetilde{P}"'] & J^k(U/U) \arrow[r] & J^k(\mathcal{A}/U) \end{tikzcd}$$

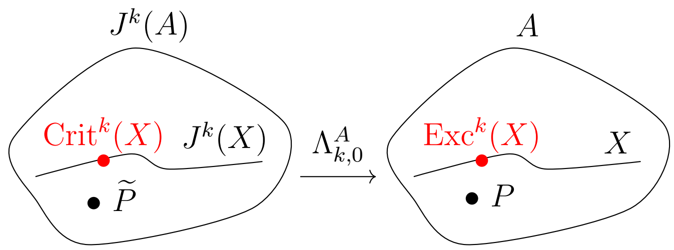
where we have used the fact that

$$\begin{aligned} J^k(W/U)(u_n) &= \mathrm{Hom}_{U_k}(u_n \times_U U_k, \pi_2^{U_k,*}W) \\ &= \mathrm{Hom}_{U_k}(u_n \times_u (u \times_U U_k), \pi_2^{U_k,*}W) \\ &= \mathrm{Hom}_{U_k}(u_n \times_u u_k, \pi_2^{U_k,*}W) \\ &= \mathrm{Hom}_{u_k}(u_n \times_u u_k, W_{u_k}) \\ &= W(u_n \times_u u_k) \end{aligned}$$

to define the map $u_n \to J^k(u_n/U)$ given by the projection $u_n \times_u u_k \to u_n$. Note also that the commutativity of the left hand square follows because $J^k(U/U) = U$ is terminal. Hence we have that $\widetilde{P} \in J^k(\mathcal{X}/U)(u_n)$.

Thus this tells us, at least for n sufficiently large, that

$$\begin{aligned} d_v([p^n]_* J^k(A/K), \widetilde{P}) = 0 \leq 1/p^n \qquad d_v(J^k(X/K), \widetilde{P}) \leq 1/p^n \\ \implies d_v(\mathrm{Crit}^k(A, X), \widetilde{P}) \leq 1/p^n \end{aligned}$$



since by definition $\mathrm{Crit}^k(A, X) = [p^k]_* J^k(A/K) \cap J^k(X/K)$. So we can conclude that, since $\mathrm{Crit}^k(A, X) \subseteq \Lambda^{A*}_{k,0}\mathrm{Exc}^k(A, X)$, we have

$$\begin{aligned} d_v(\mathrm{Exc}^k(A, X), P) &= d_v(\Lambda^{A*}_{k,0}\mathrm{Exc}^k(A, X), \widetilde{P}) \\ &\leq d_v(\mathrm{Crit}^k(A, X), \widetilde{P}) \leq 1/p^n \end{aligned}$$

□

3.2.3 Density Results

We have the following result which is essentially Theorem 3.1. in [37]. Indeed, note that since X is assumed not to be linear, by Theorem 1.2. in [37] this implies that for any field extension L/K and any $Q \in A(L)$ the set $X_L^{+Q} \cap \mathrm{Tor}(A(L))$ is not Zariski dense in X_L^{+Q}, where here we use the assumption that A has K/k-trace zero.

Proposition 3.2.3. *Assume that X is not linear. Then there exists $m \in \mathbb{N}$ such that for all $Q \in \Gamma$ we have a strict inclusion*

$$\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q}) \subsetneq \mathcal{X}^{+Q}$$

Proof. It suffices to show that

$$\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})_u \subsetneq \mathcal{X}_u^{+Q}$$

for any closed point $u \in U$. By Proposition 3.1.26 though, we have

$$\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})_u(k) = \{P \in \mathcal{X}_u^{+Q}(k) \mid P \text{ lifts to an element of } \mathcal{X}^{+Q}(u_m) \cap p^m\mathcal{A}(u_m)\}$$

therefore it is enough to show that the set

$$\{P \in \mathcal{X}_u^{+Q}(k) \mid P \text{ lifts to an element of } \mathcal{X}^{+Q}(u_m) \cap p^m\mathcal{A}(u_m)\}$$

is not Zariski dense in $\mathcal{X}_u^{+Q}$ for all $Q \in \Gamma$.

Note that we can take an abelian scheme $\widetilde{\mathcal{A}}$ which is a model of $\mathcal{A}$ over a curve $\widetilde{U}$, which is smooth over a finite field. Furthermore, since Γ is finitely generated, we can assume it is the image of some $\widetilde{\Gamma} \subseteq \widetilde{\mathcal{A}}(\widetilde{U})$. Finally, we can also suppose that $\mathcal{X} \hookrightarrow \mathcal{A}$ has a model $\widetilde{\mathcal{X}} \hookrightarrow \widetilde{\mathcal{A}}$ over $\widetilde{U}$.

Hence, we can apply Corollary 3.2.8 to the base change of $\mathcal{X} \hookrightarrow \mathcal{A}$ to the completion of U at u to complete the proof.

□

Proposition 3.2.4. *Assume that X is not linear. Then there exists $m \in \mathbb{N}$ such that for all $Q \in \Gamma$ we have a strict inclusion*

$$\mathrm{Exc}^m(A, X^{+Q}) \subsetneq X^{+Q}$$

Proof. By Proposition 3.2.3 we have a strict containment over U:

$$\mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q}) \subsetneq \mathcal{X}^{+Q}$$

and so if we set $V := \mathcal{X}^{+Q} \setminus \mathrm{Exc}^m(\mathcal{A}, \mathcal{X}^{+Q})$, this is a non-empty open subscheme over U. Since $\mathcal{X}^{+Q}$ is flat over U, it follows that V is too.

Now, if the generic fibre V_K were empty, we could find some dense open $U' \subset U$ such that $V_{U'}$ was also empty. Then, as $f : V \to U$ is flat, we have that $f^{-1}(U') = V_{U'} = \emptyset$ is dense in V by Theorem 2.3.10. in [16], which gives us a contradiction. Hence, V_K must be non-empty, and so we have that

$$\mathrm{Exc}^m(A, X^{+Q}) \subsetneq X^{+Q}$$

This completes the proof.

□

So it just remains to explain and prove Corollary 3.2.8. This result is due to Rössler and is Corollary 4.5. in [37], but we shall cover the argument in full for completeness.

Let S be the spectrum of a complete discrete local ring with fraction field K and residue field k, a finite field of characteristic p. For any $n \in \mathbb{N}$ we have S_n, the n-th infinitesimal neighbourhood of the closed point in S.

Take S^{sh} be the strict henselization of S, with fraction field L and whose residue field we can identify with $\bar{k}$. Likewise, we also have S^{sh}_n, the n-th infinitesimal neighbourhood of the closed point in S^{sh}

Consider an abelian scheme $\mathcal{A}$ over S together with an integral subscheme $\mathcal{X} \hookrightarrow \mathcal{A}$, with generic fibre $X \hookrightarrow A$ and special fibre $X_0 \hookrightarrow A_0$.

Proposition 3.2.5. *Suppose that* $\mathrm{Tor}(A(\overline{K})) \cap X_{\overline{K}}$ *is not dense in* $X_{\overline{K}}$. *Then for some* $m \in \mathbb{N}$, *the set*

$$\{P \in X_0(\bar{k}) \mid P \textit{ lifts to an element of } \mathcal{X}(S_m^{\mathrm{sh}}) \cap p^m \mathcal{A}(S_m^{\mathrm{sh}})\}$$

is not Zariski dense in X_0.

Proof. The Galois group $\mathrm{Gal}(\bar{k}/k)$ has topological generator ϕ. The Weil conjectures inform us that there is an monic integer polynomial $Q(x) \in \mathbb{Z}[x]$ such that $Q(\phi)(P) = 0$ for all $P \in A_0(\bar{k})$, and which has no roots of unity among its complex roots. We can write Q as

$$Q(x) = x^{2g} - (a_{2g-1}x^{2g-1} + \cdots + a_0)$$

We then define the following matrix according to the coefficients of Q

$$M = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{2g-2} & a_{2g-1} \end{bmatrix}$$

Let $\tau \in \mathrm{Aut}_S(S^{\mathrm{sh}})$ be a lifting of ϕ. Given $x \in \mathcal{A}(S_n^{\mathrm{sh}})$, set

$$u(P) := (P, \tau(P), \tau^2(P) \cdots \tau^{2g-1}(P)) \in (\prod_{s=0}^{2g-1} \mathcal{A})(S_n^{\mathrm{sh}})$$

and note that if $Q(\tau)(P) = 0$, then $M(u(P)) = u(\tau(P))$.

Let us define the following closed subscheme of $\mathcal{A}^{2g}$

$$\mathcal{Z} := \cap_{t \geq 0} M_*^t(\cap_{r \geq 0} M^{r,*}(\prod_{s=0}^{2g-1} \mathcal{X}))$$

where M^t is simply the composition of M with itself t times, and M_*^t is the scheme theoretic image of said map.

$\mathcal{Z}$ has two key properties that we state as sub-lemmas.

Lemma 3.2.6. *If $P \in \mathcal{X}(S_n^{\mathrm{sh}})$ satisfies $Q(\tau)(P) = 0$, then $u(P) \in \mathcal{Z}(S_n^{\mathrm{sh}})$.*

Proof. If we have P as in the statement of the lemma, then $M^r(M^k(u(P))) = u(\tau^{r+k}(P))$ for all r and k, with $u(\tau^{r+k}(P)) \in (\prod_{s=0}^{2g-1} \mathcal{X})(S_n^{\mathrm{sh}})$. Hence, for any k

$$M^k(u(P)) \in (\cap_{r\geq 0} M^{r,*}(\prod_{s=0}^{2g-1} \mathcal{X}))(S_n^{\mathrm{sh}})$$

Furthermore, we note that there is t' such that $\tau^{t'}(P) = P$, and so for all t we have

$$M^t(M^k(u(P))) = M^{t+k}(u(P)) = u(\tau^{t+k}(P)) = u(P)$$

where k is chosen such that $t + k$ is a multiple of t'. It follows that

$$u(P) \in (\cap_{t\geq 0} M_*^t(\cap_{r\geq 0} M^{r,*}(\prod_{s=0}^{2g-1} \mathcal{X})))(S_n^{\mathrm{sh}}) = \mathcal{Z}(S_n^{\mathrm{sh}})$$

□

Lemma 3.2.7. *The set $\mathcal{Z}$ is invariant under M, i.e. $M(\mathcal{Z}) = \mathcal{Z}$.*

Proof. As M is proper, we have that as sets

$$\mathcal{Z} = \cap_{t\geq 0} M^t(\cap_{r\geq 0} M^{r,-1}(\prod_{s=0}^{2g-1} \mathcal{X}))$$

Now since $M(M^{r,-1}(-)) \subseteq M^{r-1,-1}(-)$, by construction then

$$M(\cap_{r\geq 0} M^{r,-1}(\prod_{s=0}^{2g-1} \mathcal{X})) \subseteq \cap_{r\geq 0} M^{r,-1}(\prod_{s=0}^{2g-1} \mathcal{X})$$

So we get a decreasing chain of subsets

$$\cdots \subseteq M^2(\cap_{r\geq 0} M^{r,-1}(-)) \subseteq M(\cap_{r\geq 0} M^{r,-1}(-)) \subseteq \cap_{r\geq 0} M^{r,-1}(-)$$

By noetherianity, the chain must stabilize at some t', and thus

$$\begin{aligned} M(\mathcal{Z}) &= M(M^{t'}(\cap_{r\geq 0} M^{r,-1}(\prod_{s=0}^{2g-1} \mathcal{X})) \\ &= M^{t'+1}(\cap_{r\geq 0} M^{r,-1}(\prod_{s=0}^{2g-1} \mathcal{X}) = \mathcal{Z} \end{aligned}$$

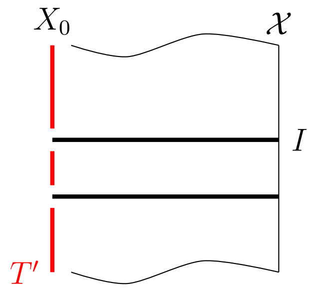
□

By Theorem D.0.7 then the set $\mathcal{Z}_{\overline{K},\mathrm{red}} \cap \mathrm{Tor}(\prod_{s=0}^{2g-1} A(\overline{K}))$ is dense in $\mathcal{Z}_{\overline{K},\mathrm{red}}$, and therefore the projection map $\mathcal{Z} \to \mathcal{X}$ onto the first factor cannot be surjective on the generic fibre, else it would contradict the hypothesis of the proposition.

Let T be the scheme theoretic image of $\mathcal{Z} \to \mathcal{X}$. Then because every $P \in X_0(\bar{k})$ is a root of $Q(\phi)$, it follows that X_0 is a closed subscheme of T.

Let I be a reduced irreducible component of T which surjects onto S. Then I must be flat over S, and since $I \neq \mathcal{X}$ (because $T \neq \mathcal{X}$) we must have that the dimension of I over the special fibre is strictly less than the dimension of X_0.

So if we set H to the union of such I, then the same is true for the dimension of H over the special fibre. Defining $T' = T \setminus H$, we see that T' must be a non-empty open subset of X_0.



Now, suppose for a contradiction that the conclusion of the proposition was false. Then for all ℓ there is some $P \in T'(\bar{k})$ which lifts to a point on $\mathcal{X}(S_\ell^{\mathrm{sh}}) \cap p^\ell \mathcal{A}(S_\ell^{\mathrm{sh}})$.

So let $\widetilde{P} \in \mathcal{A}(S_\ell^{\mathrm{sh}})$ be such that $p^\ell \widetilde{P} \in \mathcal{X}(S_\ell^{\mathrm{sh}})$ and $p^\ell \widetilde{P}_0 = P$. Since $Q(\phi)(P) = 0$ it follows that $Q(\tau)(p^\ell \widetilde{P}) = 0$ too. Hence

$$u(p^\ell \widetilde{P}) \in \mathcal{Z}(S_\ell^{\mathrm{sh}}) \implies p^\ell \widetilde{P} \in T'(S_\ell^{\mathrm{sh}})$$

but since the generic fibre of T' is empty though, this is a contradiction for ℓ large enough.

□

Corollary 3.2.8. *Suppose that* $\mathrm{Tor}(A(\overline{K})) \cap X_{\overline{K}}^{+Q}$ *is not dense in* $X_{\overline{K}}^{+Q}$ *for all* $Q \in \mathcal{A}(S)$. *Then for some* $m \in \mathbb{N}$, *the set*

$$\{P \in X_0(\bar{k}) \mid P \textit{ lifts to an element of } \mathcal{X}^{+Q}(S_m^{\mathrm{sh}}) \cap p^m \mathcal{A}(S_m^{\mathrm{sh}})\}$$

is not Zariski dense in X_0^{+Q} *for all* $Q \in \mathcal{A}(S)$.

Proof. By Proposition 3.2.5 we can choose, for each $Q \in \mathcal{A}(S)$, the minimal $m_Q \in \mathbb{N}$ such that

$$\{P \in X_0(\bar{k}) \mid P \text{ lifts to an element of } \mathcal{X}^{+Q}(S^{\mathrm{sh}}_{m_Q}) \cap p^{m_Q}\mathcal{A}(S^{\mathrm{sh}}_{m_Q})\}$$

is not Zariski dense in X_0^{+Q}. Suppose for a contradiction, that we can find a sequence $(Q_n)_{n\in\mathbb{N}}$ such that m_{Q_n} is strictly increasing. Replacing (Q_n) by a subsequence if necessary, we can assume that it converges to some $Q \in \mathcal{A}(S)$, and that $Q_n = Q$ over S_n.

By construction $m_{Q_n} \geq n$, therefore $m_Q \geq n$ for all n, which contradicts Proposition 3.2.5.

□

3.3 Proving Hrushovski's Theorem

Having compiled a number of results on the exceptional schemes, we are now in a position to prove Hrushovski's theorem. Our first proof demonstrates Theorem 3.1.3, utilizing Proposition 3.2.1 and Corollary 3.2.8 together with a small trick from model theory used in Hrushovski's original proof.

Our second, perhaps more succinct proof, uses Proposition 3.2.2 and Corollary 3.2.8 to show Theorem 3.1.4, which is the strong form of Hrushovski's theorem.

3.3.1 First Proof

If X is linear then we are done, so assume otherwise, then by Proposition 3.2.4 we can choose $m_1 \in \mathbb{N}$ so that $\mathrm{Exc}^{m_1}(A, X^{+Q}) \subsetneq X^{+Q}$ for all $Q \in \Gamma$. Then define

$$Y_1 := \bigcup_{j \in J} \mathrm{Exc}^{m_1}(A, X^{-\widetilde{Q_j}})^{+\widetilde{Q_j}} \subsetneq X$$

where the J is the set of equivalence classes of $\Gamma/p^{m_1}\Gamma$, and each $\widetilde{Q_j}$ is a representative of the j-th class.

Lemma 3.3.1. *Let $m \geq m_1$, and suppose that $\{Q_i\}$ are representatives of the equivalence classes of $\Gamma/p^m\Gamma$. Then for any separable field extension L/K and for each Q_i, we have the inclusion*

$$X(L) \cap (Q_i + p^m A(L)) \subseteq Y_1(L) \cap (Q_i + p^m A(L))$$

Proof. Take a point P in the left hand side, then by Proposition 3.2.1

$$P - Q_i \in X^{-Q_i}(L) \cap p^m A(L) \subseteq \mathrm{Exc}^m(A, X^{-Q_i})(L) \subseteq \mathrm{Exc}^{m_1}(A, X^{-Q_i})(L)$$

and therefore since $Q_i - \widetilde{Q_j} \in p^{m_1}\Gamma$ for some j, using Proposition 3.1.24 we have

$$P \in \mathrm{Exc}^{m_1}(A, X^{-Q_i})^{+Q_i}(L) = \mathrm{Exc}^{m_1}(A, X^{-\widetilde{Q_j}})^{+\widetilde{Q_j}}(L) \subseteq Y_1(L)$$

which proves the lemma.

□

Although Y_1 may not be linear, if we repeat this construction with (each irreducible component of) Y_1 in place of X, we get another subvariety Y_2. We also keep track of the corresponding integer m_2. Doing this inductively, we get a decreasing chain of subvarieties

$$\cdots \subseteq Y_n \subseteq \cdots \subseteq Y_2 \subseteq Y_1 \subseteq X$$

together with an integer $m_n \in \mathbb{N}$ associated to each Y_n.

By noetherianity, this chain must stabilize at some point, say $Y_{N+1} = Y_N$. This means that by Proposition 3.2.4 the subvariety $Y := Y_N$ is linear.

Setting $m = \max\{m_n\}_{n=1}^N$ and applying the previous lemma repeatedly, we obtain as a corollary:

Corollary 3.3.2. *There is a linear subvariety $Y \subseteq X$ that satisfies the following. For some $m \in \mathbb{N}$, if $\{Q_i\}$ are representatives of the equivalence classes of $\Gamma/p^m\Gamma$, then for any separable field extension L/K and for each Q_i, we have the inclusion*

$$X(L) \cap (Q_i + p^m A(L)) \subseteq Y(L) \cap (Q_i + p^m A(L))$$

Remark 3.3.3. *This implies in particular that $X(K) \cap \Gamma = Y(K) \cap \Gamma$.*

Before we derive the main theorem, let us prove a weakened version.

Proposition 3.3.4. *Let $P_n \in \Gamma$ be a sequence of points such that $P_n \to X$, then $P_n \to Y$.*

Proof. Suppose for a contradiction that there is a subsequence P_{n_k} for which $d(P_{n_k}, Y) \geq \delta$, where $\delta > 0$ is a fixed constant. By taking another subsequence if necessary we can assume that $P_{n_k} \to P \in X(\widehat{K})$, and also that for some Q_i, $P_{n_k} - Q_i \in p^m\Gamma$ for all k. Here $\widehat{K}$ denotes the completion of K. Using the corollary, this means that

$$P \in X(\widehat{K}) \cap (Q_i + p^m A(\widehat{K})) \subseteq Y(\widehat{K})$$

which contradicts the assumption that $d(P_{n_k}, Y) \geq \delta$ for all k.

□

To obtain the full version of the theorem, we appeal to the compactness theorem from model theory, which we state before proving the lemma. See Chapter 13 of [4] for a proof.

Theorem 3.3.5 (Compactness Theorem)**.** *A set of first order sentences has a model if and only if every finite subset has a model.*

Lemma 3.3.6. *Let $Y \subseteq X$ be a subvariety which satisfies the following: there exists $m \in \mathbb{N}$, such that if $\{Q_i\}$ are representatives of the equivalence classes of $\Gamma/p^m\Gamma$, then for any separable field extension L/K and for each Q_i, we have the inclusion*

$$X(L) \cap (Q_i + p^m A(L)) \subseteq Y(L) \cap (Q_i + p^m A(L))$$

Then there exists a constant $C_v > 0$ such that for any point $P \in \Gamma$ we have

$$\lambda_v(P, X) \leq C_v \cdot \lambda_v(P, Y)$$

Proof. We adapt the proof of Lemma 6.6. in [26]. Suppose for a contradiction that there is no such C_v. Then for all $n \in \mathbb{N}$ we can find $P_n \in \Gamma$ such that $\lambda_v(P_n, X) > n\lambda_v(P_n, Y)$, and wlog we can assume $P_n - Q_i \in p^m\Gamma$ for all n.

Consider the language describing a field extension $\widetilde{K}$ of K, a valuation $\tilde{v}$ of $\widetilde{K}$ extending v with value group $\widetilde{Z}$, and an element $\widetilde{P}$ of $A(\widetilde{K})$. Let $\tilde{\lambda}$ be defined according to $\tilde{v}$. Now consider the following first order statements

- $\widetilde{P} \in Q_i + p^m A(\widetilde{K})$
- $\tilde{v}$ extends v, and $\widetilde{Z}$ is an ordered group with least positive element
- $\tilde{\lambda}(\widetilde{P}, X) > n\tilde{\lambda}(\widetilde{P}, Y)$ $(n = 1, 2, \cdots)$

By assumption, any finite number of these axioms can be satisfied, and so by the compactness theorem there exist $\widetilde{K}$, $\tilde{v}$, $\widetilde{P}$ satisfying all of them.

Let $\widetilde{R}$ be the valuation ring of $\tilde{v}$ and define

$$M = \{x \in \widetilde{R} : \tilde{v}(x) > n\tilde{\lambda}(\widetilde{P}, Y) \text{ for all } n\}$$

Note that if $x, y \notin M$, so that $\tilde{v}(x) \leq n_1\tilde{\lambda}(\widetilde{P}, Y)$ and $\tilde{v}(y) \leq n_2\tilde{\lambda}(\widetilde{P}, Y)$, then

$$\begin{aligned} \tilde{v}(xy) &= \tilde{v}(x) + \tilde{v}(y) \\ &\leq (n_1 + n_2)\tilde{\lambda}(\widetilde{P}, Y) \end{aligned}$$

and therefore $xy \notin M$. Hence M is a prime and so maximal ideal of $\widetilde{R}$. Moreover, since $\tilde{v}(x) = v(x) \in \mathbb{Z}$ for all $x \in \widetilde{R} \cap K$, M intersects $\widetilde{R} \cap K$ trivially, and thus K embeds into the field $L = \widetilde{R}/M$.

To see that L is a separable extension of K we use the following sub-lemma. See Chapter 10 of [32] for proofs of any of the standard facts on differentials used in the proof of the lemma.

Lemma 3.3.7. *If L/K is an inseparable extension, with K a function field over a perfect field k, then any $t \in K$ has a p-th root $t^{1/p}$ in L.*

Proof. We can assume L/K is finitely generated, as any inseparable extension L/K contains a finitely generated inseparable extension L^{fin}/K.

Then, we use the exact sequence of differentials

$$\Omega_{K/k} \otimes_K L \to \Omega_{L/k} \to \Omega_{L/K} \to 0$$

Since L/k is a finitely generated extension over a perfect field, it is separable, and therefore $\dim_L \Omega_{L/k} = \text{tr.deg}(L/k)$. On the other hand, as L/K is assumed to be inseparable, we must have that

$$\dim_L \Omega_{L/K} \geq \text{tr.deg}(L/K) + 1 = \text{tr.deg}(L/k) = \dim_L \Omega_{L/k}$$

This implies that $\Omega_{L/k} \to \Omega_{L/K}$ is an isomorphism, and so in particular $dt = 0$ in $\Omega_{L/k}$ for any $t \in K$.

Consider the extension $L' = L(t^{1/p})$. If $t^{1/p} \notin L$ then as $dt = 0$, it follows that

$$\text{tr.deg}(L'/k) = \dim_{L'}\Omega_{L'/k} = \dim_L \Omega_{L/k} + 1 = \text{tr.deg}(L/k) + 1$$

using the fact that L'/k and L/k are both separable extensions. This contradicts the fact that $\text{tr.deg}(L'/L) = 0$ though, since $t^{1/p}$ is algebraic over L.

$\square$

So now suppose L/K were inseparable. If we pick $t \in K$ with $v(t) = 1$, then by the lemma we can find a p-th root of t. This means we have $s \in \widetilde{R}$ such that $s^p - t \in M$, which tells us in particular that $\tilde{v}(s^p - t) \notin \mathbb{Z}$. If $\tilde{v}(s) = 0$ then $\tilde{v}(s^p - t) = 0$, and if $\tilde{v}(s) \geq 1$ then $\tilde{v}(s) \geq p$ and so $\tilde{v}(s^p - t) = 1$, in either case we have a contradiction, thereby implying that L/K is separable.

Let $P \in A(L)$ be the point induced by $\widetilde{P}$. By construction P is in $X(L)$ but not in $Y(L)$. To see this, note the set of equivalences

$$\begin{aligned} P \in Z(L) &\iff f(\widetilde{P}) \in M, \ \forall f \in I_Z \\ &\iff v(f(\widetilde{P})) > n\tilde{\lambda}(\widetilde{P}, Y), \ \forall f \in I_Z, \forall n \in \mathbb{N} \\ &\iff \tilde{\lambda}(\widetilde{P}, Z) > n\tilde{\lambda}(\widetilde{P}, Y), \ \forall n \in \mathbb{N} \end{aligned}$$

and clearly the last set of inequalities is true for $Z = X$, but not for $Z = Y$.

Finally, $P \in Q_i + p^m A(L)$ because $\widetilde{P} \in Q_i + p^m A(\widetilde{K})$. Hence, by the assumption made in the statement of the lemma, we must have

$$P \in X(L) \cap (Q_i + p^m A(L)) \subseteq Y(L)$$

which gives the required contradiction.

$\square$

Combining Corollary 3.3.2 with Lemma 3.3.6 then, Theorem 3.1.3 follows immediately.

3.3.2 Second Proof

Assume X is not linear, then by Proposition 3.2.4 we can choose $m \in \mathbb{N}$ such that

$$\text{Exc}^m(A, X^{+Q}) \subsetneq X^{+Q}$$

for all $Q \in \Gamma$. In particular, since Γ is finitely generated, we can take $\{Q_i\}$ to be a set of representatives of the equivalence classes of the finite group $\Gamma/p^m\Gamma$, and define

$$Y_1 := \cup_i \mathrm{Exc}^m(A, X^{-Q_i})^{+Q_i} \subsetneq X$$

Although Y_1 may still not be linear, we can repeat this construction with each irreducible component of Y_1 in place of X to get $Y_2 \subseteq Y_1$. Inductively, this gives a descending chain of closed subschemes

$$\cdots \subseteq Y_k \subseteq \cdots \subseteq Y_2 \subseteq Y_1 \subseteq X$$

which by noetherianity must stabilize at some $Y_{N+1} = Y_N$. By Proposition 3.2.4 $Y := Y_N$ then has to be linear.

Now, if $P \in \Gamma$ is such that $d_v(X, P) \leq 1/p^n$, then we can choose Q_i such that $P - Q_i \in p^m\Gamma$, and we will have

$$d_v(X^{-Q_i}, P - Q_i) \leq 1/p^n$$

therefore by Proposition 3.2.2 we get that

$$\begin{aligned} d_v(\mathrm{Exc}^m(A, X^{-Q_i}), P - Q_i) \leq 1/p^n &\implies d_v(\mathrm{Exc}^m(A, X^{-Q_i})^{+Q_i}, P) \leq 1/p^n \\ &\implies d_v(Y_1, P) \leq 1/p^n \end{aligned}$$

Repeating this argument inductively to get that $d_v(Y_k, P) \leq 1/p^n$ for all k, which of course implies $d_v(Y, P) \leq 1/p^n$, this completes the proof of Theorem 3.1.4.

Appendix A

Perverse Sheaves

Let $X \xrightarrow{g} \mathrm{Spec}(k)$ be a scheme of finite type over an algebraically closed field k. Then one can construct the triangulated category $D^b_c(X, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-sheaves, where ℓ is invertible on X. This can be thought of as essentially the derived category of $\mathbb{Q}_\ell$-sheaves, although the precise construction is more technical.

In fact, for any finite extension E of $\mathbb{Q}_\ell$ one can construct $D^b_c(X, E)$, and then define $D^b_c(X, \overline{\mathbb{Q}_\ell})$ as a direct limit of the $D^b_c(X, E)$ over all finite extensions E of $\mathbb{Q}_\ell$. See Chapters II.5. and II.6. in [30] for the details.

For a morphism $f : X \to X'$ over k, one has the standard six functor formalism

$$f_*,\ f^*,\ f_!,\ f^!,\ -\otimes^{\mathbf{L}} -,\ \mathbf{R}Hom(-,-)$$

with the adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$. Note that we shall often write $\mathbf{R}f_*$ instead of f_* and $\mathbf{R}f_!$ instead of $f_!$. Given a pullback diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ S \arrow[r, "f'"'] & S' \end{tikzcd}$$

one has a quasi-isomorphism $f'^* \circ \mathbf{R}h'_! \xrightarrow{\sim} \mathbf{R}h_! \circ f^*$ given by base change.

Now $D^b_c(X, \mathbb{Q}_\ell)$ has a standard t-structure with truncation functors $\tau_{\leq i}, \tau_{\geq i}$ and cohomology functors $\mathcal{H}^i = \tau_{\leq i}\tau_{\geq i}$, but that does not mean it is the only one of interest.

Let $K_X = g^!\mathbb{Q}_\ell$ be the dualizing complex of X. If X is smooth of dimension d, then $K_X = \mathbb{Q}_\ell(d)[2d]$. Set $D(-) = \mathbf{R}Hom(-, K_X)$ to be the dualizing functor on X, which satisfies $D^2 = id$, as-well as:

- $D \circ f_* = f_! \circ D$ and $D \circ f_! = f_* \circ D$
- $D \circ f^* = f^! \circ D$ and $D \circ f^! = f^* \circ D$
- $\mathbf{R}Hom(A, B) = D(A \otimes^{\mathbf{L}} D(B))$

Definition A.0.1. The perverse t-structure on $D_c^b(X, \mathbb{Q}_\ell)$ is defined by

$$K \in {}^{\mathfrak{p}}D^{\leq 0} \Leftrightarrow \dim \operatorname{supp}(\mathcal{H}^{-i}(K)) \leq -i, \ \forall i \in \mathbb{Z}$$
$$K \in {}^{\mathfrak{p}}D^{\geq 0} \Leftrightarrow \dim \operatorname{supp}(\mathcal{H}^{-i}(DK)) \leq -i, \ \forall i \in \mathbb{Z}$$

Remark A.0.2. *Note that $K \in {}^{\mathfrak{p}}D^{\leq 0} \Leftrightarrow DK \in {}^{\mathfrak{p}}D^{\geq 0}$; we say the perverse t-structure is self-dual.*

As usual when we have a t-structure, we can define the core of the t-structure, which is always an abelain category.

Definition A.0.3. The abelian category of perverse sheaves on X is the core

$$\operatorname{Perv}(X) := {}^{\mathfrak{p}}D^{\leq 0} \cap {}^{\mathfrak{p}}D^{\geq 0}$$

We also have the associated truncation functors ${}^{\mathfrak{p}}\tau_{\leq i}, {}^{\mathfrak{p}}\tau_{\geq i}$, as-well as the cohomological functors ${}^{\mathfrak{p}}\mathcal{H}^i$. For any exact triangle (sometimes called distinguished) in $D_c^b(X, \mathbb{Q}_\ell)$

$$A \to B \to C \to A[1]$$

there is an associated long exact sequence in $\operatorname{Perv}(X)$

$$\cdots \to {}^{\mathfrak{p}}\mathcal{H}^{i-1}(C) \to {}^{\mathfrak{p}}\mathcal{H}^i(A) \to {}^{\mathfrak{p}}\mathcal{H}^i(B) \to {}^{\mathfrak{p}}\mathcal{H}^i(C) \to {}^{\mathfrak{p}}\mathcal{H}^{i+1}(A) \to \cdots$$

A good introduction to perverse sheaves can be found in Chapter III of [30], but we shall summarise some important results here. To get an intuition first of all for what constitutes a perverse sheaf, suppose for this proposition only that X is smooth.

Proposition A.0.4. *Let K be a smooth complex in $D_c^b(X, \mathbb{Q}_\ell)$, then*

$$K \in \operatorname{Perv}(X) \Leftrightarrow K = \mathcal{G}[\dim X]$$

for a smooth sheaf $\mathcal{G}$.

Proof. See Remark III.2.2 in [30].

□

Remark A.0.5. *For any* $K \in D^b_c(X, \mathbb{Q}_\ell)$ *and* $n \in \mathbb{N}$, $K[n]$ *denotes the shift of* K. *Also, for any smooth sheaf* $\mathcal{G}$, *we can think of* $\mathcal{G} \in D^b_c(X, \mathbb{Q}_\ell)$ *as the complex concentrated in degree* 0.

In general, we can relate the perverse t-structure to the standard t-structure with the next proposition.

Proposition A.0.6. *Assume that* $\dim(X) \leq d$. *Let* $K \in {}^{\mathfrak{p}}D^{\geq 0}$ *then we have that*

$$\mathcal{H}^i(K) = 0 \quad \text{for all } i < -d$$

which is to say that ${}^{\mathfrak{p}}D^{\geq 0}$ *is contained in* $D^{\geq -d}$.

Proof. See Lemma III.5.13. in [30]. □

One may consider the derived category $D^b(\mathrm{Perv}(X))$. There is a natural exact functor $D^b(\mathrm{Perv}(X)) \to D^b_c(X, \mathbb{Q}_\ell)$, and it turns out due to a theorem of Beilinson (see Theorem 1.3. in [2]) that this is an equivalence of categories. In particular, this implies the following proposition.

Proposition A.0.7. *For any perverse sheaves* $K, K' \in \mathrm{Perv}(X)$ *we have*

$$\mathrm{Ext}^n_{\mathrm{Perv}(X)}(K, K') = \mathrm{Hom}_{D^b_c(X,\mathbb{Q}_\ell)}(K, K'[n])$$

and so if $\mathcal{G}, \mathcal{G}'$ *are smooth sheaves on* X *then*

$$\mathrm{Ext}^n_{\mathrm{Perv}(X)}(\mathcal{G}[b], \mathcal{G}'[b]) = \mathrm{Ext}^n_{\mathbb{Q}_\ell}(\mathcal{G}, \mathcal{G}')$$

A.1 The Intermediate Extension Functor

Now let $j : U \hookrightarrow X$ be an open subscheme, then we have the following useful functor.

Definition A.1.1. The intermediate extension functor $j_{!*} : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$ is defined by

$$j_{!*}K := \mathrm{image}({}^{\mathfrak{p}}\mathcal{H}^0(\mathbf{R}j_!K) \to {}^{\mathfrak{p}}\mathcal{H}^0(\mathbf{R}j_*K))$$

Given $K \in \mathrm{Perv}(U)$, the intermediate extension $j_{!*}K \in \mathrm{Perv}(X)$ is unique with the property that: $j^*j_{!*}K = K$ and it is has neither quotients or subobjects of the type i_*A for $A \in \mathrm{Perv}(Z)$, where $i : Z = X \setminus U \hookrightarrow X$ is the closed immersion.

The intermediate extension functor has a plethora of nice properties. We state some of them here for reference.

Proposition A.1.2. *The intermediate extension functor commutes with duality*

$$D(j_{!*}K) = j_{!*}D(K)$$

Proof. See Corollary III.5.3. in [30]. $\square$

Proposition A.1.3. *For any perverse sheaf $K \in \mathrm{Perv}(U)$ we have*

$$i^* j_{!*}K \in {}^{\mathfrak{p}}D_{\leq -1} \quad \textit{and} \quad i^! j_{!*}K \in {}^{\mathfrak{p}}D_{\geq 1}$$

Moreover, if $j : U \hookrightarrow X$ is affine, then $(i^* j_! K)[-1] \in \mathrm{Perv}(Z)$.

Proof. See Lemma III.5.1. and the discussion after Corollary III.6.2. in [30]. $\square$

Proposition A.1.4. *For any perverse sheaf $K \in \mathrm{Perv}(U)$ there is a distinguished triangle*

$$j_{!*}K \to \mathbf{R}j_*K \to i_*{}^{\mathfrak{p}}\tau_{\geq 0} i^*\mathbf{R}j_*K \to j_{!*}K[1]$$

Proof. See Lemma III.5.1. in [30]. $\square$

Proposition A.1.5 (Continuation Principle)**.** *Take any $A, B \in \mathrm{Perv}(U)$, then*

$$\mathrm{Hom}_{\mathrm{Perv}(U)}(A, B) = \mathrm{Hom}_{\mathrm{Perv}(X)}(j_{!*}A, j_{!*}B)$$

Proof. See Corollary III.5.11. in [30]. $\square$

Proposition A.1.6. *Any simple object $K \in \mathrm{Perv}(X)$ is either of the form i_*A for some simple object $A \in \mathrm{Perv}(Z)$ or $j_{!*}B$ for some simple $B \in \mathrm{Perv}(U)$.*

Proof. See Corollary III.5.4. in [30]. $\square$

Let X be an irreducible k-variety. Using the intermediate extension functor, one can define the intersection complex $\mathrm{IC}_X \in D^b_c(X, \mathbb{Q}_\ell)$ as $j_{!*}(\mathbb{Q}_\ell[\dim(X)])$ for any dense open immersion $j : U \hookrightarrow X$ with U smooth over k.

A.2 The Decomposition Theorem

Given a finite field $\mathbb{F}_q$, one has the geometric Frobenius $\mathrm{Fr} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ which is inverse to the arithmetic Frobenius that sends $a \mapsto a^q$.

Let X_0 be scheme of finite type over $\mathbb{F}_q$ and $\mathcal{G}_0$ a $\mathbb{Q}_\ell$-sheaf on X_0. Take a closed point $x \in |X_0|$ with $\kappa(x) = \mathbb{F}_{q^n}$. For each geometric point $\bar{x} : \mathrm{Spec}(\overline{\mathbb{F}}_q) \to X_0$ lying over x, the n-th power of the geometric Frobenius acts on the stalk of $\mathcal{G}_0$:

$$\mathrm{Fr}_x^n : \mathcal{G}_{0,\bar{x}} \to \mathcal{G}_{0,\bar{x}}$$

Up to isomorphism, this action depends only on the closed point x and not the particular choice of geometric point $\bar{x}$ over x.

Definition A.2.1. A $\mathbb{Q}_\ell$-sheaf $\mathcal{G}_0$ is said to be pure of weight w if for all isomorphisms $\tau : \mathbb{Q}_\ell \simeq \mathbb{C}$ and all $x \in |X_0|$, the eigenvalues α of Fr_x^n on $\mathcal{G}_{0,\bar{x}}$ satisfy:

$$|\tau(\alpha)| = q^{nw/2}$$

Definition A.2.2. $K \in D_c^b(X_0, \mathbb{Q}_\ell)$ is said to be pure of weight w if for all i the cohomology sheaves $\mathcal{H}^i(K_0)$ are pure of weight $\leq w + i$, and the same holds true for $D(K_0)$.

Theorem A.2.3. *Let $f_0 : X_0 \to Y_0$ be a proper morphism of $\mathbb{F}_q$-varieties, $K_0 \in D_c^b(X_0, \mathbb{Q}_\ell)$ be pure of weight w, with $f : X \to Y$ and K the pullback over $\overline{\mathbb{F}}_q$. Then $\mathbf{R}f_{0*}K_0$ is pure of weight w and there is an isomorphism*

$$\mathbf{R}f_*K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(\mathbf{R}f_*K)[-i]$$

Furthermore, each ${}^{\mathfrak{p}}\mathcal{H}^i(\mathbf{R}f_*K)$ is semisimple.

Proof. See Theorem III.10.6. in [30].

□

Proposition A.2.4 (Gabber Purity Theorem)**.** *The intersection complex IC_{X_0} of an $\mathbb{F}_q$-variety X_0 is pure of weight $\dim(X_0)$.*

Proof. See the discussion after Proposition III.9.1. in [30].

□

Using Theorem A.2.3 one can deduce the following theorem for an algebraically closed field k of any characteristic using the standard specialization argument, see Chapter 6 in [3].

Theorem A.2.5 (The Decomposition Theorem)**.** *Let $f : X \to Y$ be a proper morphism of k-varieties. Then there is an isomorphism*

$$\mathbf{R}f_*\mathrm{IC}_X \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(\mathbf{R}f_*\mathrm{IC}_X)[-i]$$

Furthermore, each ${}^{\mathfrak{p}}\mathcal{H}^i(\mathbf{R}f_*\mathrm{IC}_X)$ is semisimple.

If in addition f is smooth and X, Y are non-singular, then all the $\mathbf{R}^i f_*\mathbb{Q}_\ell$ are smooth sheaves and so $\tau_{\geq i-\dim(Y)}\mathbf{R}f_*\mathbb{Q}_\ell \in {}^{\mathfrak{p}}D^{\geq i}$. Furthermore, by Proposition A.0.6 ${}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell \in D^{\geq i-\dim(Y)}$, therefore ${}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell = \tau_{\geq i-\dim(Y)}\mathbf{R}f_*\mathbb{Q}_\ell$ is a smooth complex.

Using the exact triangle

$${}^{\mathfrak{p}}\tau_{\leq i}{}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell \to {}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell \to {}^{\mathfrak{p}}\tau_{\geq i+1}\mathbf{R}f_*\mathbb{Q}_\ell \to {}^{\mathfrak{p}}\tau_{\leq i}{}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell[1]$$

and the long exact sequence for $\mathcal{H}^i(-)$, since ${}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell$ and ${}^{\mathfrak{p}}\tau_{\geq i+1}\mathbf{R}f_*\mathbb{Q}_\ell$ are both smooth, the same is true of ${}^{\mathfrak{p}}\tau_{\leq i}{}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell$. This means that ${}^{\mathfrak{p}}\tau_{\leq i}{}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell = \tau_{\leq i-\dim(Y)}\tau_{\geq i-\dim(Y)}\mathbf{R}f_*\mathbb{Q}_\ell$, and so

$$\begin{aligned}
{}^{\mathfrak{p}}\mathcal{H}^i(\mathbf{R}f_*\mathbb{Q}_\ell) &= {}^{\mathfrak{p}}\tau_{\leq i}{}^{\mathfrak{p}}\tau_{\geq i}\mathbf{R}f_*\mathbb{Q}_\ell[i] \\
&= \tau_{\leq i-\dim(Y)}\tau_{\geq i-\dim(Y)}\mathbf{R}f_*\mathbb{Q}_\ell[i] \\
&= \mathbf{R}^{i-\dim(Y)}f_*\mathbb{Q}_\ell[\dim(Y)]
\end{aligned}$$

Hence, we recover the original decomposition theorem due to Deligne, see Proposition 2.4. in [14].

Theorem A.2.6 (The Decomposition Theorem)**.** *Let $f : X \to Y$ be a smooth proper morphism of non-singular k-varieties. Then there is an isomorphism*

$$\mathbf{R}f_*\mathbb{Q}_\ell \simeq \bigoplus_i \mathbf{R}^i f_*\mathbb{Q}_\ell[-i]$$

Furthermore, each $\mathbf{R}^i f_*\mathbb{Q}_\ell$ is semisimple.

Appendix B

Spectral Sequences

Like most things in homological algebra, spectral sequences have a complicated construction that can be treated as a black box. The aim of this appendix is instead to give the basic intuition for how spectral sequences work, which is usually sufficient to prove things using them.

We start with a bi-graded object $E_0^{\bullet,\bullet}$, where the $E_0^{p,q}$ are objects in some abelian category, together with a map $d_0 : E_0^{\bullet,\bullet} \to E_0^{\bullet,\bullet}$ such that $d_0(E_0^{p,q}) \subseteq E_0^{p,q+1}$ and $d_0 \circ d_0 = 0$.

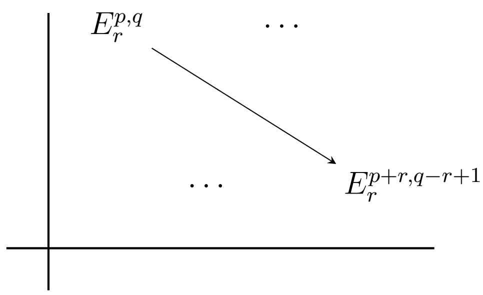
$$\begin{tikzcd}
E_0^{1,3} & E_0^{2,3} & E_0^{3,3} & E_0^{4,3} \\
E_0^{1,2} \arrow[u] & E_0^{2,2} \arrow[u] & E_0^{3,2} \arrow[u] & E_0^{4,2} \arrow[u] \\
E_0^{1,1} \arrow[u] & E_0^{2,1} \arrow[u] & E_0^{3,1} \arrow[u] & E_0^{4,1} \arrow[u]
\end{tikzcd}$$

We call $E_0^{\bullet,\bullet}$ the zeroth sheet. We shall assume $E_0^{p,q} = 0$ unless $p, q > 0$.
Now since $d_0^2 = 0$, we can consider the cohomology of d_0, and we set $E_1^{\bullet,\bullet}$ to be this cohomology. We call this the first sheet. Part of the data of a spectral sequence is that we also have then a map $d_1 : E_1^{\bullet,\bullet} \to E_1^{\bullet,\bullet}$ such that $d_1(E_1^{p,q}) \subseteq E_1^{p+1,q}$ and $d_1 \circ d_1 = 0$.

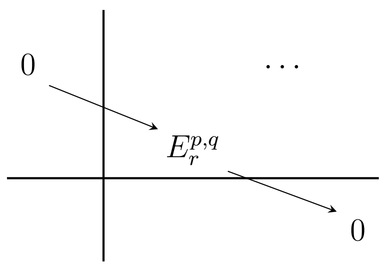
$$
\begin{array}{l}
E_1^{1,3} \longrightarrow E_1^{2,3} \longrightarrow E_1^{3,3} \longrightarrow E_1^{4,3} \\[1em]
E_1^{1,2} \longrightarrow E_1^{2,2} \longrightarrow E_1^{3,2} \longrightarrow E_1^{4,2} \\[1em]
E_1^{1,1} \longrightarrow E_1^{2,1} \longrightarrow E_1^{3,1} \longrightarrow E_1^{4,1}
\end{array}
$$

Again we can take the cohomology (of d_1), and this will form the second sheet $E_2^{\bullet,\bullet}$, which comes with a map $d_2 : E_2^{\bullet,\bullet} \to E_2^{\bullet,\bullet}$ such that $d_2(E_2^{p,q}) \subseteq E_2^{p+2,q-1}$ and $d_2 \circ d_2 = 0$.

This continues on as expected. The r-th sheet $E_r^{\bullet,\bullet}$ is formed from the cohomology of the $(r-1)$-th sheet, and comes with a map $d_r : E_r^{\bullet,\bullet} \to E_r^{\bullet,\bullet}$ such that $d_r(E_r^{p,q}) \subseteq E_r^{p+r,q-r+1}$ and $d_r \circ d_r = 0$.



Note that for each (p, q), for r sufficiently large, the arrow into $E_r^{p,q}$ will come from outside the first quadrant, and the arrow out of $E_r^{p,q}$ will also land outside the first quadrant.



Outside the first quadrant is all zeros, and thus when we calculate the cohomology at (p, q), it just returns $E_r^{p,q}$, i.e. $E_{r+1}^{p,q} = E_r^{p,q}$. Moreover, this will continue so that

$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots =: E_\infty^{p,q}$. Hence the spectral sequence eventually terminates at each point.

This means that the spectral sequence "converges", and we write

$$E_1^{p,q} \Rightarrow E_\infty^*$$

This is mainly just pure notation, and sometimes one writes $E_2^{p,q}$ in place of $E_1^{p,q}$, or whichever page is of interest to us. E_∞^* denotes a collection of objects E_∞^n, each with a filtration

$$E_\infty^n = F_0^n \supseteq F_1^n \supseteq \cdots \supseteq F_n^n \supseteq 0$$

such that $F_j^n / F_{j+1}^n \cong E_\infty^{j,n-j}$ for each j.

When our objects are finite dimensional vector spaces then we have

$$\dim E_\infty^n = \sum_{j=0}^{n} \dim E_\infty^{j,n-j}$$

We say that the spectral sequence "degenerates" at sheet r if for all $r' \geq r$ the map d_r is zero. In most useful cases degeneration occurs (often at the first or second sheet), and it tells in particular that $E_r^{p,q} = E_\infty^{p,q}$ (note that in general $E_\infty^{p,q}$ is a subquotient of $E_r^{p,q}$).

One of the main classes of spectral sequences are those associated to a bi-complex $C^{\bullet,\bullet}$. The zeroth sheet $E_0^{\bullet,\bullet}$ has the same entries as $C^{\bullet,\bullet}$, and d_0 is given by the vertical maps in $C^{\bullet,\bullet}$. Moreover, the horizontal maps induce maps on the cohomology of $E_0^{\bullet,\bullet} = C^{\bullet,\bullet}$, which gives d_1.

We have the complex $\operatorname{Tot}(C^{\bullet,\bullet})$, and the infinity page is such that $E_\infty^n = \mathcal{H}^n \operatorname{Tot}(C^{\bullet,\bullet})$. Hence we have a spectral sequence

$$E_1^{p,q} = H^q(C^{p,\bullet}) \Rightarrow \mathcal{H}^* \operatorname{Tot}(C^{\bullet,\bullet}) \tag{B.0.1}$$

By essentially switching the columns and rows, we also get another spectral sequence with the same infinity page, with $E_1^{p,q} = H^q(C^{\bullet,p})$ and

$$E_2^{p,q} = H^p H^q(C^{\bullet,\bullet}) \Rightarrow \mathcal{H}^* \operatorname{Tot}(C^{\bullet,\bullet}) \tag{B.0.2}$$

Now consider the general set-up where $F : A \to B$ is an additive functor between abelian categories. In what follows we shall freely use the language of derived categories, one can consult [46] for a reference.

Assume A has enough injectives, then we have the right derived functor between the (bounded-below) derived categories of A and B

$$\mathbf{R}F : \mathcal{D}^+(A) \longrightarrow \mathcal{D}^+(B)$$

which is calculated by $\mathbf{R}F(K) \cong F(K')$, where $K \to K'$ is a quasi-isomorphism and K' is bounded below with injective components. We denote $H^i \circ \mathbf{R}F : \mathcal{D}^+(A) \to B$ by $\mathbf{R}^i F$.

Proposition B.0.1. *Let K be a complex, bounded below, with components in A. Then we have a spectral sequence*

$$E_1^{p,q} = \mathbf{R}^q F(K^p) \Rightarrow \mathbf{R}^* F(K)$$

called the first spectral sequence of hypercohomology of F.

Proposition B.0.2. *Let K be a complex, bounded below, with components in A. Then we have a spectral sequence*

$$E_2^{p,q} = \mathbf{R}^p F(H^q(K)) \Rightarrow \mathbf{R}^* F(K)$$

called the second spectral sequence of hypercohomology of F.

Both propositions can be proven by choosing a resolution $K \to L$ of K by a bi-complex L such that each column $L^{p,\bullet}$ is an injective resolution of K^p, and then applying the spectral sequences (B.0.1) and (B.0.2) respectively.

Although for simplicity we specified that A was an abelian category, the corresponding result holds for a triangulated category D with a given t-structure. Indeed, if every object of D is bounded, and $F : D \to B$ is a cohomology functor into an abelian category, then we have a spectral sequence for any $K \in D$

$$E_2^{p,q} = F^p(H^q(K)) \Rightarrow F^*(K)$$

See for example Theorem IV.5.1. in [30] for a proof. In particular, this implies the following corollary pertaining to perverse sheaves.

Corollary B.0.3. *Let $f : X \to Y$ be a morphism between finitely generated schemes over an algebraically closed field. Then we have a spectral sequence*

$$E_2^{p,q} = H^p_{\text{ét}}(Y, {}^{\mathfrak{p}}\mathcal{H}^q(\mathbf{R}f_* K)) \Rightarrow H^*_{\text{ét}}(X, K)$$

for any $K \in D^b_c(X, \mathbb{Q}_\ell)$.

Appendix C

Arakelov Theory

A good reference for the material covered in this appendix is [43]. Let us briefly mention Serre's GAGA theorems, the original reference for which can be found in [41]. It says essentially that any analytic coherent sheaf on a complex projective variety comes from an algebraic coherent sheaf. In particular, any analytic subvariety is algebraic. The upshot of this is that we shall seldom distinguish an analytic object from its algebraic counterpart.

We begin with X a smooth projective complex variety of dimension d. Let $A^{p,q}(X)$ denote the vector space of complex differential forms of type (p, q). If $z = (z_1, \ldots, z_d)$ are local coordinates, then for $\omega \in A^{p,q}(X)$ we have

$$\omega = \sum_{|I|=p,|J|=q} f_{I,J}(z, \overline{z}) dz_I \wedge d\overline{z}_J$$

where $f_{I,J}(z, \overline{z})$ are C^∞-functions. We also define the space of differential forms of degree n as

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$$

There are the standard differential operators $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$ and $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$, as-well as $d = \partial + \overline{\partial} : A^n(X) \to A^{n+1}(X)$. Furthermore, we can set $d^c = (4\pi i)^{-1}(\partial - \overline{\partial})$.

We define $D^{p,q}(X) := A^{d-p,d-q}(X)^*$, the space of continuous linear functionals on $A^{d-p,d-q}(X)$. The differential operators on $A^{d-p,d-q}(X)$ induce ones on $D^{p,q}(X)$. There is a dualization map

$$A^{p,q}(X) \to D^{p,q}(X)$$
$$\omega \mapsto [\omega]$$

which is defined by

$$[\omega](\alpha) = \int_X \omega \wedge \alpha$$

for any $\alpha \in A^{d-p,d-q}(X)$.

For an irreducible subvariety $i : Z \hookrightarrow X$ of codimension p, we can define a current $\delta_Z \in D^{p,p}(X)$ via the formula

$$\delta_Z(\alpha) := \int_{Z^{ns}} i^*\alpha$$

for $\alpha \in A^{d-p,d-p}(X)$. Here Z^{ns} denotes the non-singular locus of Z. In fact, we can extend this definition to any cycle $Z \in CH^p(X)$.

We define a Green current for $Z \in CH^p(X)$ to be a current $g_Z \in D^{p-1,p-1}(X)$ which satisfies the equation

$$dd^c g_Z + \delta_Z = [\omega]$$

for some (p,p)-form $\omega \in A^{p,p}(X)$. One has the following existence theorem, a proof of which can be found in [43], see Theorem II.3.

Theorem C.0.1. *For every irreducible subvariety $Z \subset X$ there exists a smooth form g_Z on $X \setminus Z$ of logarithmic type along Z such that $[g_Z]$ is a Green current for Z, i.e.*

$$dd^c[g_Z] + \delta_Z = [\omega]$$

where ω is smooth on X.

If $Y \in CH^q(X)$ is another cycle then we can form the product of the two Green currents g_Y and g_Z

$$g_Y * g_Z := g_Y \wedge \delta_Z + [\omega_Y] \wedge g_Z$$

which gives a well defined commutative, associative product modulo $\mathrm{im}\partial + \mathrm{im}\overline{\partial}$. Of course, one has to make sense of the product $g_Y \wedge \delta_Z$.

It turns out that $g_Y * g_Z$ is a green current for $Y \cap Z$, that is:

$$dd^c(g_Y * g_Z) + \delta_{Y \cap Z} = [\omega_Y \wedge \omega_Z]$$

C.1 Arithmetic Theory

For this next part we take R to be a regular arithmetic ring, i.e. a regular, excellent, noetherian domain, together with a finite set $\mathcal{S}$ of injective ring homomorphisms $R \to \mathbb{C}$ which is invariant under complex conjugation. From now on X is a regular scheme which is projective and flat over $\mathrm{Spec}(R)$. There is a continuous involution $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ given by complex conjugation.

For such X, we can define

$$A^{p,p}(X) := \{\omega \in A^{p,p}(X(\mathbb{C})) \mid F_\infty^* \omega = \omega\}$$
$$D^{p,p}(X) := \{T \in D^{p,p}(X(\mathbb{C})) \mid F_\infty^* T = (-1)^p T\}$$

All the definitions from the previous section then also carry over.

A hermitian line bundle $\mathcal{L}$ on X is a line bundle together with a hermitian metric $|| \cdot ||$ on the associated holomorphic line bundle on $X(\mathbb{C})$, which is invariant under F_∞, and which varies smoothly over the points of $X(\mathbb{C})$. Write $\widehat{\mathrm{Pic}}(X)$ for the group of isomorphism class of hermitian line bundles on X, with the group structure given by the tensor product.

We can also define the arithmetic Chow groups

$$\widehat{CH}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X)$$

Here $\widehat{Z}^p(X)$ is the group of arithmetic cycles (Z, g_Z), with $Z \in Z^p(X)$ and g_Z a Green current for Z, while $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ is the subgroup generated by pairs of the form $(\mathrm{div} f, -[\log|f|^2])$ and $(0, \partial u + \overline{\partial} v)$.

There is an isomorphism

$$\widehat{\mathrm{Pic}}(X) \to \widehat{CH}^1(X)$$

given by sending $(\mathcal{L}, || \cdot ||)$ to $(\mathrm{div} s, -[\log||s||^2])$ for any rational section s of $\mathcal{L}$.

Now, using the product structure on Green currents, we can define an arithmetic intersection pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$$
$$([(Y, g_Y)], [(Z, g_Z)]) \mapsto [(Y \cap Z, g_Y * g_Z)]$$

Given a map $f : X \to X'$ one has the pullback $f^* : \widehat{CH}^p(X') \to \widehat{CH}^p(X)_{\mathbb{Q}}$, which respects the product structure. Moreover, if f is proper, smooth on the generic fibre $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to X'_{\mathbb{Q}}$ and X, X' are both equidimensional, then there is also a pushforward map

$$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-\delta}(X') \quad (\delta = \mathrm{dim}X - \mathrm{dim}X')$$

As with the standard intersection pairing, the projection formula holds for the arithmetic pairing too

$$f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta$$

for any $\alpha \in \widehat{CH}^i(X)$, $\beta \in \widehat{CH}^j(X')$, see Theorem III.3. in [43].

Appendix D

Abelian Varieties

In this appendix we shall cover some aspects of abelian varieties that will be relevant for us. A good reference for this is [13].

We begin with the definition of an abelian scheme. For this, we must first define the notion of a group scheme. Let S be a general scheme.

Definition D.0.1. An S-group scheme is a scheme G over S together with an S-morphism $m : G \times_S G \to G$ which satisfies the following three axioms:

- (Associativity) There is a commutative square

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "id\times m"] \arrow[d, "m\times id"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

- (Identity) There exists a section $e : S \to G$ which fits into the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "id\times e"] \arrow[d, "e\times id"'] \arrow[dr, "id"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

- (Inverse) There is an S-morphism $i : G \to G$ making both diagrams below commute

$$\begin{tikzcd} G \times_S G \arrow[r, "i\times id"] & G \times_S G \arrow[d, "m"] \\ G \arrow[u, "\Delta"] \arrow[r, "e\circ\pi"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \times_S G \arrow[r, "id\times i"] & G \times_S G \arrow[d, "m"] \\ G \arrow[u, "\Delta"] \arrow[r, "e\circ\pi"'] & G \end{tikzcd}$$

where Δ is the diagonal and $\pi : G \to S$ is the structure morphism.

Definition D.0.2. An abelian scheme over S is an S-group scheme $\mathcal{A}$ such that $\pi : \mathcal{A} \to S$ is proper, smooth and has connected geometric fibres.

Remark D.0.3. *An abelian scheme is always commutative.*

If F is a field, then an abelian variety over F is simply an abelian scheme over F. One also has the notion of a semi-abelian variety, which is a group variety A that fits into an exact sequence

$$0 \to T \to A \to A_0 \to 0$$

where A_0 is an abelian variety and T is a torus, i.e. $T \otimes_F \overline{F} \simeq \mathbb{G}^n_{m,\overline{F}}$ for some $n \geq 0$, where $\mathbb{G}_{m,\overline{F}}$ is the multiplicative group of $\overline{F}$.

From now on we suppose F is algebraically closed and of characteristic p. If $X \hookrightarrow A$ is an irreducible closed subscheme, we can define its translation stabilizer $\mathrm{Stab}(X)$ as the closed subgroup of A whose points stabilize X in A. We let $C = \mathrm{Stab}(X)_{\mathrm{red}}$.

Definition D.0.4. We say that X is special if there exists:

- a semi-abelian variety B over F
- a homomorphism with a finite kernel $h : B \to A/C$
- a model $\mathbf{B}$ of B over a finite field $\mathbb{F}_{p^r} \subset F$
- an irreducible reduced closed subscheme $\mathbf{X} \subset \mathbf{B}$
- a point $a \in (A/C)(F)$ such that $X/C = a + h_*(\mathbf{X} \times_{\mathbb{F}_{p^r}} F)$

With the notation as above, we have the following two important conjectures, which are both now theorems.

Theorem D.0.5 (Manin-Mumford Conjecture)**.** *If $X \cap \mathrm{Tor}(A(F))$ is Zariski dense in X, then X is special.*

Theorem D.0.6 (Mordell-Lang Conjecture)**.** *Let $\Gamma \subseteq A(F)$ be a subgroup such that $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module. Then if $X \cap \Gamma$ is Zariski dense in X, it follows that X is special.*

Furthermore, we have the following theorem which is Proposition 6.1. in [35].

Theorem D.0.7. *Suppose $\psi : A \to A$ is an endomorphism such that $R(\psi) = 0$ where $R \in \mathbb{Z}[T]$ is an integer polynomial with no roots of unity among its complex roots. If $\psi(X) = X$ then $\mathrm{Tor}(A(F)) \cap X$ is dense in X.*

D.1 Abelian Varieties over Function Fields

Now, let U be a smooth curve over $k = \overline{\mathbb{F}}_p$, and let K be the function field of U. Take an abelian variety A over K.

Theorem D.1.1. *There exists a smooth separated scheme $\mathcal{A}$ over U such that $\mathcal{A}_K = A$ and which satisfies: if $\mathcal{B}$ is a smooth separated scheme over U, then any K-morphism $B_K \to A_K$ extends to a unique R-morphism $\mathcal{B} \to \mathcal{A}$.*

One can consult Chapter VIII in [13] for a proof of this fact.

Definition D.1.2. The Néron model of A over U is the unique scheme $\mathcal{A}$ guaranteed by the above lemma.

By the theorem, there is a natural bijection between $A(K)$ and $\mathcal{A}(U)$, and this is chiefly what makes the Néron model so useful to work with.

In addition, there is a dense open subset $U' \subset U$ such that the restriction $\mathcal{A}\mid_{U'}$ is an abelian scheme. If U' is maximal with this property, then the points of U' are exactly the points of good reduction.

Proposition D.1.3. *There exists an abelian variety B over k and a homomorphism $f : B_K \to A$ with finite kernel, such that: for any abelian variety B' over k and homomorphism $f' : B'_K \to A$ with finite kernel, there exists a unique homomorphism $g : B' \to B$ such that $f' = f \circ g$.*

See Theorem 20.5. in Chapter V of [13] for a proof. We remark that K need not be a function field, the proposition holds for any regular field extension K/k, which is to say that K is linearly disjoint from $\bar{k}$.

Definition D.1.4. The K/k-trace of A is the unique abelian variety $\mathrm{Tr}_{K/k}(A)$ over k determined by the previous theorem.

Remark D.1.5. *If* $\mathrm{Tr}_{K/k}(A)$ *is the trivial abelian variety over k, then we simply say A has K/k-trace zero.*

Dual to the concept of trace is the image $\mathrm{Im}_{K/k}(A)$ (see for example [11]). It is an abelian variety over k equipped with a homomorphism $A \to \mathrm{Im}_{K/k}(A)_K$, which satisfies the universal property that any homomorphism $A \to B_K$ (with B defined over k) factors through it.

$\mathrm{Tr}_{K/k}(A)$ and $\mathrm{Im}_{K/k}(A)$ are isogenous over k (Theorem 6.9. in [11]), therefore if the trace of A is zero then so too is the image. It is easy to see from the universal properties of the trace and image, that if we have an exact sequence of abelian varieties

$$0 \to A' \to A \to A'' \to 0$$

and A has K/k-trace zero, then the same is true for A' and A''.

Now suppose we had a homomorphism $B \to A$, where B is a semi-abelian variety with a model over a finite field. Then we have an exact sequence

$$0 \to T \to B \to B_0 \to 0$$

with B_0 an abelian variety and T a torus, which also have a model over a finite field. The composite map $T \to A$ must be trivial, and so $B \to A$ factors through B_0. If the trace of A is zero though, $B_0 \to A$ must be trivial and hence so too is $B \to A$. This tells us then, that any homomorphism from a semi-abelian variety to an abelian variety of trace zero is trivial.

Lemma D.1.6. *Assume A has K/k-trace zero. For any irreducible subvariety X of A, if X is special then X is a translate of an abelian subvariety of A.*

Proof. Recall Definition D.0.4 of what it means for X to be special. Take the homomorphism $h : B \to A/C$ in the definition. Since A has trace zero, we have that $\mathrm{Tr}_{K/k}(A/C) = 0$, and so by the above discussion it follows that h is trivial. This means that X/C is a single point, and thus X is a translate of an abelian subvariety of A.

□

Appendix E

Local Height Functions

Take K to be a field which is complete with respect a given metric $|\cdot|_v$. Let $\mathcal{O}_K$ be the valuation ring of K, with maximal ideal $\mathfrak{m}$ and a finite residue field $k = \mathcal{O}_K/\mathfrak{m}$ of characteristic p.

Let A be an abelian variety over K. We can take the Néron model $\mathcal{A}$ of A over $\mathcal{O}_K$. Furthermore, for any closed subscheme $X \hookrightarrow A$ and any point $P \in A(K)$, we can take extensions $\mathcal{X} \hookrightarrow \mathcal{A}$ and $P \in \mathcal{A}(\mathcal{O}_K)$.

Define $\mathfrak{m}_\varepsilon = \{x \in K : |x| < \varepsilon\}$, and set $\mathcal{O}_\varepsilon = \mathcal{O}_K/\mathfrak{m}_\varepsilon$ for $0 < \varepsilon \leq 1$. Take the pullbacks $\mathcal{X}_\varepsilon = \mathcal{X} \times_{\mathcal{O}} \mathcal{O}_\varepsilon$ and $P_\varepsilon \in \mathcal{A}(\mathcal{O}_\varepsilon)$.

Definition E.0.1. The local distance associated to X is defined as

$$d_v(X,P) = \inf\{\varepsilon > 0 : P_\varepsilon \in \mathcal{X}_\varepsilon\}$$

if the set is non-empty, and $d_v(X,P) = 1$ otherwise. Note that the condition $P_\varepsilon \in \mathcal{X}_\varepsilon$ means that we have a point $\mathcal{O}_\varepsilon \to \mathcal{X}_\varepsilon$ that fits into a commutative diagram like so

$$\begin{tikzcd}
\mathcal{O}_\varepsilon \arrow[dr] \arrow[drr] \arrow[drrr, bend left, "P_\varepsilon"] \arrow[ddr, bend right] & & & \\
& \mathcal{X}_\varepsilon \arrow[r] \arrow[d] & \mathcal{X} \arrow[r, hook] \arrow[d] & \mathcal{A} \arrow[dl] \\
& \mathcal{O}_\varepsilon \arrow[r] & \mathcal{O} &
\end{tikzcd}$$

We can also define the local height in terms of the local distance as follows.

Definition E.0.2. The local height function associated to X is defined as

$$\lambda_v(X,P) = -\log\, d_v(X,P)$$

Let us record here a couple of useful results on the local distance. One can see Theorem 1. in [45] for proofs.

Proposition E.0.3. *Let Y be another closed subscheme of A. Then for any point P we have*

$$d_v(X \cap Y, P) = \max(d_v(X, P), d_v(Y, P))$$

Proposition E.0.4. *If $f : A \to B$ is a morphism of schemes, then for any closed subscheme $Y \hookrightarrow B$ and $P \in A(K)$, letting $X := f^*Y$, we have*

$$d_v(X, P) = d_v(Y, f(P))$$

Set $S = \mathrm{Spec}(\mathcal{O}_K)$ and $S_n = \mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^{n+1})$. Then for any point $P \in A(K)$ one has the criterion:

$$d_v(P, X) \leq p^{-1/n} \iff P \in \mathcal{X}(S_n)$$

at least for n for sufficiently large.

Moreover, if we set $S^{\mathrm{sh}} = \mathrm{Spec}(\mathcal{O}_K^{\mathrm{sh}})$, where $\mathcal{O}_K^{\mathrm{sh}}$ is the strict henselization of $\mathcal{O}_K$, then in the same way we have that for n sufficiently large

$$d_v(P, X) \leq p^{-1/n} \iff P \in \mathcal{X}(S_n^{\mathrm{sh}})$$

for any point $P \in A(K^{\mathrm{nr}})$, where K^{nr} is the maximal unramified extension of K.

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