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Graphs with Nonnegative Resistance Curvature

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Abstract. This article introduces and studies a new class of graphs motivated by discrete curvature. We call a graph *resistance nonnegative* if there exists a distribution on its spanning trees such that every vertex has expected degree at most two in a random spanning tree; these are precisely the graphs that admit a metric with nonnegative resistance curvature, a discrete curvature introduced by Devriendt and Lambiotte. We show that this class of graphs lies between Hamiltonian and 1-tough graphs and, surprisingly, that a graph is resistance nonnegative if and only if its twice-dilated matching polytope intersects the interior of its spanning tree polytope. We study further characterizations and basic properties of resistance nonnegative graphs and pose several questions for future research.

1. Introduction

1.1. Motivation

In [9], the author and Lambiotte introduced a new notion of discrete scalar curvature on the vertices of a graph, based on the geometry of effective resistances. For a simple graph $G = (V, E)$ with edge weights $c > 0$, the *effective resistance* ω between two adjacent vertices is proportional to the c-weighted fraction of all spanning trees that contain the edge between them:

$$\omega_{uv} = c_{uv}^{-1} \frac{\sum_{T \ni uv} \prod_{t \in T} c_t}{\sum_T \prod_{t \in T} c_t} \qquad \text{for } uv \in E, \tag{1}$$

where T runs over all spanning trees in G. The effective resistance originates from electrical circuit analysis and potential theory and it has a rich combinatorial, algebraic and geometric theory; see for instance [3,7,14]. The *resistance curvature* p is a discrete scalar curvature on the vertices of a graph, defined in

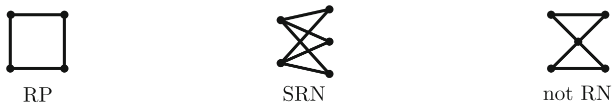

FIGURE 1. An RP, SRN and not RN graph. See Examples 3.3–3.6 for details

[9] as:

$$p_v(c) = 1 - \frac{1}{2} \sum_{uv \in E} \omega_{uv} c_{uv} \quad \text{for } v \in V.$$

As in differential geometry, graphs with nonnegative resistance curvature have a lot of nice properties [7,9,10] and we note in particular a strong structural theorem for positively curved graphs; a graph G is called *t-tough* if for every set of vertices $U \subset V$ whose removal disconnects the graph, the graph $G - U$ has at most $|U|/t$ components. The following is implied by a result of Fiedler in simplex geometry [14, Thm. 3.4.18] and another proof is given in [7, Thm. 6.31]:

Theorem 1.1. ([14]) *Connected weighted graphs with positive resistance curvature are* 1*-tough.*

In this result, we point out that positive resistance curvature depends on a choice of edge weights c, while 1-tough is a purely combinatorial property that only depends on the graph data $G = (V, E)$. Thus, if one can find some edge weights for which $p > 0$, then Theorem 1.1 applies and the graph is 1-tough (Fig. 1). This observation naturally motives the following question:

Question 1.2. For which graphs G does there exist a choice of edge weights c such that $p \geq 0$, respectively $p > 0$? We call these graphs *resistance nonnegative* (RN) and *resistance positive* (RP). A graph which is RN but not RP is called *strictly resistance nonnegative* (SRN).

Geometrically, by thinking of edge weights as inverse edge lengths, this question can be reformulated as "which graphs admit a metric with nonnegative/positive scalar curvature". Replacing "graph" by "manifold", this parallels a classical question in differential geometry [23]. With the definitions in Question 1.2, we can turn Theorem 1.1 into a purely structural result.

Theorem 1.3. *Connected resistance positive graphs are 1-tough.*

In this article, we will answer Question 1.2 and study characterizations and properties of RN, RP and SRN graphs. We formulate several questions for future research.

1.2. Main Results

As a first main result, we show that RN, RP and SRN graphs can be characterized in terms of the existence of certain distributions on their spanning trees; a distribution is *positive* if every spanning tree is assigned a non-zero probability.

Theorem 3.8. *A graph G is RN (resp. RP) if and only if there exists a positive distribution on the set of spanning trees of G, such that the expected degree of every vertex in a random spanning tree under this distribution is at most 2 (resp. strictly smaller than 2).*

The actual statement of Theorem 3.8 in the main text is stronger and allows for other types of distributions. Our proof makes use of the fact that $(c_{uv}\omega_{uv})_{uv\in E}$ are marginal probabilities of a so-called log-linear distribution on spanning trees induced by c. By classical results in statistics on log-linear distributions, as one ranges over all possible weights $0 < c < \infty$ these marginal probabilities trace out the (interior of the) set of marginals of all possible distributions on spanning trees. This set is a polytope $P(G) \subset \mathbb{R}^E$ and is called the *spanning tree polytope* of G. Since the expected degree constraints are affine, a practical consequence of Theorem 3.8 is that computing the RN property of a graph is a linear program on $P(G)$.
As a second main result, we give a geometric characterization of RN graphs via the intersection of two polytopes associated to a graph; recall that a matching is a subset of edges $M \subseteq E$ which covers every vertex at most once. The spanning tree polytope and *matching polytope* of G are

$$P(G) := \mathrm{conv}(e_T \mid T \text{ is a spanning tree}) \subset \mathbb{R}^E,$$
$$M(G) := \mathrm{conv}(e_M \mid M \text{ is a matching}) \subset \mathbb{R}^E,$$

where $e_T = \sum_{i\in T} e_i$ with e_i basis vectors of $\mathbb{R}^E$. We use the notation $2M(G) = \{2x \mid x \in M(G)\}$, and $P(G)^\circ$ for the relative interior of the spanning tree polytope (see Remark 2.5).

Theorem 4.1. *A graph G is RN if and only if $P(G)^\circ \cap 2M(G)$ is non-empty.*

Following this theorem, we initiate a study of the polytope $\Theta(G) = P(G) \cap 2M(G)$.

Proposition 4.6. *The integer points of $\Theta(G)$ are indicator vectors of Hamiltonian paths in G.*

As a direction for future research, we propose to study Ehrhart-type questions for this polytope: what can be said about the integer points of $k\Theta(G)$ for $k \in \mathbb{N}$? Combinatorially, this requires counting arrangements of double matchings that give arrangements of spanning trees.
As a third line of results, we study some operations on graphs and their effect on resistance nonnegativity. We show that adding edges preserves RP, that 'Kron reduction' (Definition 6.10) preserves both RP and RN but not SRN and that 'circle inversion' (Definition 6.15) preserves RN. One application of these closure results is the following theorem:

Theorem 6.2. *Hamiltonian graphs are resistance positive, but the converse is not always true.*

Graph toughness was introduced by Chvátal to study Hamiltonian graphs and 1-tough does not imply Hamiltonicity; see [2]. We ask whether the class of

RP graphs lies strictly between Hamiltonian and 1-tough graphs and propose a relaxation of a conjecture of Chvátal.

Finally, we discuss two further characterizations of RN and RP graphs. Similar to encoding the RN conditions for a graph G by affine inequalities on the space $P(G) \subset \mathbb{R}^E$ that parametrizes all effective resistances in a graph, we introduce two other parametrization spaces $\mathcal{K}(G) \subset \mathbb{R}^{V\times V}$ and $\mathcal{S}(G) \subset [0,1]^{2^V}$ on which the RN conditions are affine inequalities. Here, $\mathcal{K}(G)$ is the set of all inverse resistance matrices of a graph and $\mathcal{S}(G)$ the space of all resistance capacities. The latter are certain set function on 2^V induced by effective resistances. The conjectured inequalities on $\mathcal{S}(G)$ are the submodularity inequalities, which are known to be necessary for RN.

Organization: In Sect. 2 we discuss effective resistances, their relation to distributions on spanning trees, and the spanning tree polytope $P(G)$. In Sect. 3, we define RN, RP and SRN graphs, give some examples and prove a first characterization result, Theorem 3.8. Section 4 deals with the second characterization result, Theorem 4.1, which relates resistance nonnegativity and matchings. We initiate the study of the polytope $\Theta(G)$. In Sect. 5 we prove a third characterization result, Corollary 5.4, in terms of the space of inverse resistance matrices $\mathcal{K}(G)$. Section 6 deals with three graph operations and their influence on resistance nonnegativity. Lastly, Sect. 7 conjectures a fourth characterization, Conjecture 7.7, which relates resistance nonnegativity to submodularity of the resistance capacity.

2. Relative Resistance and the Spanning Tree Polytope

Throughout the article, we will consider finite graphs $G = (V, E)$ which are simple (no multi-edges), loopless (no vv edges) and connected, unless stated otherwise. A graph is *biconnected* if $G - v$ is connected for every $v \in V$. Many of the presented results will hold in a more general setting, but this is left for later development. The following definition is central in this article:

Definition 2.1. (*Spanning tree*) A *spanning tree* of G is a maximal acyclic subset of edges.

In other words, a spanning tree is a set of edges $T \subseteq E$, such that T contains no cycles in G but adding any further edge from $E\setminus T$ creates a cycle. Every spanning tree in a connected graph has $|V| - 1$ edges. We will use the fact that spanning trees form a matroid [21, Ch. 13].

Definition 2.2. (*Spanning tree polytope*) The *spanning tree polytope* of a graph G is

$$P(G) = \mathrm{conv}(e_T \mid T \text{ is a spanning tree}) \subset \mathbb{R}^E.$$

Example 2.3. The 3-cycle has vertices u, v, w and edges uv, vw, uw. It has 3 spanning trees $\{uv, vw\}, \{vw, uw\}, \{uv, uw\}$ and $P(G)$ is a triangle in $\mathbb{R}^3$ with vertices $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.

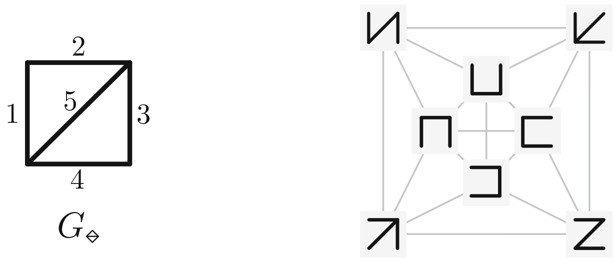

FIGURE 2. The diamond graph $G_{\Diamond}$ with edge labels and the 1-skeleton of its spanning tree polytope $P(G_{\Diamond})$. The 8 vertices of $P(G_{\Diamond})$ are indicator vectors of the spanning trees of $G_{\Diamond}$ and the edges of $P(G_{\Diamond})$ correspond to pairs of spanning trees that differ by two edges

Example 2.4. (Diamond graph) Figure 2 shows the 8 spanning trees of the diamond graph $G_{\Diamond} = K_4 - e$, arranged according to the 1-skeleton of the spanning tree polytope $P(G_{\Diamond}) \subset \mathbb{R}^5$.

Remark 2.5. (Open spanning tree polytope) The spanning tree polytope is not full-dimensional in $\mathbb{R}^E$ by Foster's theorems (see below). For this reason, when referring to the *open spanning tree polytope* $P(G)^\circ$, we mean its relative interior in the affine subspace in which it lives. In particular, the spanning tree polytope of a tree T is a point e_T, and so is its relative interior. This is important when G is a path, for instance in Theorem 4.1.

In a weighted graph, one associates finite positive weights $(c_e)_{e\in E}$ to the edges of G. Rather than working with the effective resistance ω_{uv} as introduced in (1), we consider the *relative resistance* $r_e = \omega_e c_e$ defined on the edges of a graph. We recall its definition.

Definition 2.6. *(Relative resistance)* The *relative resistance* of an edge $e \in E$ is

$$r_e(c) = \frac{\sum_{T\ni e}\prod_{t\in T} c_t}{\sum_T \prod_{t\in T} c_t}. \tag{2}$$

The homogeneous polynomial $Z_G(c) = \sum_T \prod_{t\in T} c_t$ in the denominator of (2) is called the Kirchhoff polynomial of G, and we note that $r_e \cdot Z_G = c_e \cdot \partial Z_G/\partial c_e$. From Euler's homogeneous function theorem, one recovers the following classical result in electrical circuit analysis:

Theorem 2.7. (Foster's theorem [15]) $\sum_{e\in E} r_e(c) = |V|-1$ *for any* $0 < c < \infty$.

We strengthen Foster's theorem to a localized version on the biconnected components of G:

Theorem 2.8. (Local Foster's theorem) $\sum_{e\in E(U)} r_e(c) = |U| - 1$ *for any* $0 < c < \infty$*, whenever* $U \subseteq V$ *and* $E(U) \subseteq E$ *are the vertices and edges of a biconnected component of* G.

Proof. This follows from (2) and the fact that every spanning tree has the same number of edges in every biconnected component, namely, the size of the component minus one. The biconnected components of a graph are the connected components of the corresponding cycle matroid. □

Next, we consider the subset of $\mathbb{R}^E$ traced out by $(r_e(c))_{e\in E}$ as we range over all possible weights c. This will turn out to be precisely the interior of the polytope $P(G)$. This generalizes Foster's theorem, since it characterizes all possible tuples of relative resistances by a collection of affine equalities and inequalities in r. Our discussion will follow a probabilistic approach.

Let $\mathcal{T} = \mathcal{T}(G)$ be the collection of all spanning trees of G. Then $\mu_c : \mathcal{T} \to \mathbb{R}$ given by

$$\mu_c(T) = \frac{\prod_{t\in T} c_t}{Z_G(c)} > 0 \quad \text{for } T \in \mathcal{T},$$

is a distribution on spanning trees, and we call $\mu_c(T)$ the probability of T. Since this distribution is of the form $\mu_c(T) \propto \exp(\sum_{t\in T} y_t)$ under the transformation $y_t = \log c_t$, this is also called a *log-linear distribution*; see for instance [25]. We consider two further types of distributions on $\mathcal{T}$, with increasing generality. First, a *positive distribution* assigns a positive probability to each spanning tree. For the second type, we need the following class of subsets of spanning trees:

Proposition 2.9. *The following are equivalent for a subset of spanning trees $\mathcal{F} \subseteq \mathcal{T}(G)$:*

1. $\mathrm{conv}(e_T \mid T \in \mathcal{F})$ *intersects* $P(G)^\circ$*;*
2. $\mathrm{conv}(e_T \mid T \in \mathcal{F})$ *is not contained in a proper face of* $P(G)$*;*
3. $\min(w(T) \mid T \in \mathcal{F}) < \max(w(T) \mid T \in \mathcal{T})$ *for all* $w \in \mathbb{R}^E$*,*

where $w(T) = \sum_{t\in T} w_t$ and w is not piecewise constant on the biconnected components of G. We call these sets non-separable*; this is not standard terminology.*

Proof. Equivalence of the first two statements follows immediately from the fact that $(e_T)_{T\in\mathcal{F}}$ is a subset of the vertices of $P(G)$ and that proper faces of a polytope have positive codimension. The third statement says that there is no linear function $\ell_w : x \mapsto \langle w, x\rangle$ on $P(G)$ which is maximized on the whole set $(e_T)_{T\in\mathcal{F}}$. This is equivalent to saying that these points are not contained in a face of $P(G)$, which is statement 2. By Foster's local theorem, it follows that ℓ_w is constant on the vertices of $P(G)$ when w is piecewise constant on biconnected components. □

A distribution on spanning trees is called *non-separable* if it is supported on a non-separable subset of spanning trees. We note that non-separable $\supset$ positive $\supset$ log-linear. Finally, the tuple $(\sum_{T\ni e} \mu(T))_{e\in E} \in \mathbb{R}^E$ are called the *edge-marginals* of the distribution μ. These are the probabilities that a random spanning tree following μ contains each of the respective edges.

Theorem 2.10. *Let G be a graph, then the following subsets of $\mathbb{R}^E$ coincide:*

1. *the open spanning tree polytope* $P(G)^\circ$*;*

2. *edge marginals of non-separable distributions on $\mathcal{T}(G)$;*
3. *edge marginals of positive distributions on $\mathcal{T}(G)$;*
4. *edge marginals of log-linear distributions on $\mathcal{T}(G)$;*
5. *relative resistances of the edges in G, over all possible edge weights.*

Every point in this subset corresponds to a unique choice of edge weights, up to common rescaling in the biconnected components of G.

Proof. The translation between probability and geometry goes via interpreting the probabilities $\mu(T)$ as convex coefficients of the vertices e_T. The edge marginals of a distribution are then precisely the coordinates of the point in $P(G)$ determined by these convex coefficients:

$$\Big(\sum_T \mu(T)e_T\Big)_i = \sum_T \mu(T)(e_T)_i = \sum_{T\ni i} \mu(T) \quad \text{for } i \in E.$$

With this translation, statements 1., 2. and 3. are equivalent by definition of non-separable and positive distributions. By the inclusion positive $\supset$ log-linear we have 4. $\Rightarrow$ 3. and by definition of relative resistances, we have 5. $\Leftrightarrow$ 4. It remains to prove 3. $\Rightarrow$ 4. and the correspondence of points in $P(G)^\circ$ with unique weights.

Birch's theorem in statistics [21, Cor. 8.25] says that there is a unique log-linear distribution μ_c among all positive distributions with the same edge marginals. This is precisely 3. $\Rightarrow$ 4.

Finally, the only way in which different choices of edge weights $c, \tilde{c}$ lead to the same log-linear distribution $\mu_c = \mu_{\tilde{c}}$ is if they differ by a multiplicative factor in each biconnected component. This follows from the fact that all spanning trees T contain equally many edges from each of the biconnected components, and that these are the only edge subsets with this property. Thus, by uniqueness from Birch's theorem, every point $r \in P(G)^\circ$ corresponds to a unique choice of finite positive edge weights c, up to rescaling in the biconnected components of G. $\square$

We illustrate the equivalences in Theorem 2.10 for the diamond graph (Example 2.4).

Example 2.11. The open spanning tree polytope $P(G_{\diamond})^\circ$ contains the point $(\frac{3}{5}, \dots, \frac{3}{5})$. We give three distributions μ_1, μ_2, μ_3 on $\mathcal{T}(G_{\diamond})$ that have this point as edge marginal. First, let

$$\begin{aligned} \mu_1(|/|) = \mu_1(\mathsf{Z}) = \tfrac{3}{10} \text{ and } \mu_1(\sqcap) = \mu_1(\sqcup) = \mu_1(\sqsubset) \\ = \mu_1(\sqsupset) = \tfrac{1}{10} \text{ and } \mu_1 = 0 \text{ otherwise.} \end{aligned}$$

The distribution μ_1 is non-separable but not positive (e.g. $\mu_1(\overline{/|}) = 0$). Second, let

$$\begin{aligned} \mu_2(|/|) = \mu_2(\mathsf{Z}) = \mu_2(\overline{/|}) = \mu_2(\underline{|/}) = \tfrac{3}{20} \text{ and } \mu_2(\sqcap) = \mu_2(\sqcup) \\ = \mu_2(\sqsubset) = \mu_2(\sqsupset) = \tfrac{2}{20}. \end{aligned}$$

The distribution μ_2 is a positive, non-separable and log-linear distribution induced by edge weights $c = (2, 2, 2, 2, 3)$, following the edge labeling in Fig. 2.

Third, the distribution $\mu_3 = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ is positive and non-separable but not log-linear. Finally, we note that the relative resistances in $G_{\diamondsuit}$ with edge weights $c = (2, 2, 2, 2, 3)$ are equal to $r(c) = (\frac{3}{5}, \dots, \frac{3}{5})$.

Remark 2.12. Following Theorem 2.10, relative resistances $r \in P(G)^\circ$ can be used as an alternative way to parametrize edge weights on graphs. The theory of moment maps in symplectic geometry [20] and toric geometry [21, Thm. 8.24] furthermore implies that the relation between these two parametrizations is a diffeomorphism. More precisely, the interior of $P(G)$ is diffeomorphic, under the inverse moment map, to the space of all c-weighted cut spaces of the graph, i.e., the c-scaled images of the coboundary operator $d_G^* : \mathbb{R}^V \to \mathbb{R}^E$. These vector subspaces form the real open part of a toric subvariety of the Grassmannian. This is a well-known perspective for matroid theorists since [18], but the interpretation of effective and relative resistances as images of the moment map seems to have gone unnoticed until recently in [8].

Remark 2.13. (Computation) In practice, the relation between edge weights and relative resistances is efficiently computable. Given c, one can compute the relative resistances r as the diagonal of the projection matrix onto the c-weighted cut space [8]. The hardest step in this procedure is a $|V|$-sized matrix inversion. Alternatively, there are highly optimized approximation algorithms [24]. Given r, one can compute the corresponding edge weights iteratively: pick a random point $\hat{c} \in \mathbb{R}^E_{>0}$ and compute $\hat{r}(\hat{c})$. Then repeat the steps below until $\hat{r} \approx r$ as desired:

1. Let $\hat{c}_e \leftarrow \hat{c}_e \cdot (r_e/\hat{r}_e)^{1/(|V|-1)}$ for all $e \in E$
2. Compute $\hat{r}(\hat{c})$

The theory of operator scaling guarantees convergence of the iterates in polynomial time [16].

We conclude this section by giving an explicit characterization of relative resistances based on the hyperplane description of $P(G)$. This greatly generalizes the bounds in [7, Prop. 2.16-17].

Corollary 2.14. *Let G be a graph. Then $r \in \mathbb{R}^E$ are relative resistances in G if and only if*

$$0 \leq \sum_{e \in A} r_e \leq |V| - \#\textit{components of } (G - E \backslash A) \qquad \textit{for all } A \subseteq E, \tag{3}$$

with equality in the upper bound if and only if A are the edges of an induced subgraph on some (possibly none or all) biconnected components of G, and equality in the lower bound if and only if A is empty.

Proof. Expression (3) are the hyperplane inequalities that cut out the spanning tree polytope $P(G)$ in $\mathbb{R}^E$; this follows for instance from the hyperplane description of matroid base polytopes in [13, Prop. 2.3]. By Foster's local theorem, the upper bound in (3) is an equality when A are the edges of an induced subgraph on biconnected components of G. This includes $A = E$, and $A = \emptyset$ where also the lower bound is attained and equal to zero.

Since the codimension of $P(G)$ is equal to the number of biconnected components of G—the corresponding matroid polytope has codimension equal to the number of connected components of the matroid [13, Prop. 2.4]—no further independent equalities can hold on the relative interior of $P(G)$. Hence all other inequalities in (3) must be strict, which completes the proof. $\square$

Finally, we note that the results above extend to weights which can be zero or infinite. This closure is well-understood from the perspective of toric geometry, and it completes the space of possible relative resistances to the full spanning tree polytope $P(G)$.

3. Resistance Nonnegative Graphs

We return to the main topic of this article. Recall the definition of resistance curvature.

Definition 3.1. (*Resistance curvature*) The *resistance curvature* of a vertex $v \in V$ is

$$p_v(c) = 1 - \frac{1}{2} \sum_{uv \in E} r_{uv}(c).$$

We make the definitions introduced in Question 1.2 explicit.

Definition 3.2. (*RN, RP, SRN*) A graph is *resistance nonnegative* (RN) if there exist weights $0 < c < \infty$ such that $p_v(c) \geq 0$ for all vertices v. A graph is *resistance positive* (RP) if there exists weights such that $p_v(c) > 0$ for all vertices v. A graph is *strictly resistance nonnegative* (SRN) if it is RN but not RP.

Applying the definition, we find that the diamond graph in Example 2.4 is RP, as certified by the relative resistances $(\frac{3}{5}, \ldots, \frac{3}{5}) \in P(G_{\diamond})^{\circ}$. We further consider the examples in Fig. 1.

Example 3.3. (*RP graph*) If G is vertex-transitive, then for constant edge weights $c = 1$ we find $p_v(1) = |V|^{-1}$ for all $v \in V$. This is shown for instance in [9, Prop. 7]. As a result, every vertex-transitive graph is RP. In particular, cycle graphs and the Petersen graph are RP.

Example 3.4. (*SRN graph*) For $n \in \mathbb{N}$, let $G = K_{n,n+1}$ be the near-balanced complete bipartite graph on vertex sets V_n, V_{n+1}, and pick constant edge weights $c = 1$. Since G is edge-transitive, $r_e(1)$ is constant and by Foster's theorem we find $p_v(1) = 0$ for $v \in V_n$ and $p_u(1) = 1/(n+1)$ for $u \in V_{n+1}$. Thus $K_{n,n+1}$ is RN.

Then, for any choice of edge weights c, we use Foster's theorem to find that

$$\sum_{v \in V_n} p_v(c) = n - \tfrac{1}{2} \sum_{v \in V_n} \sum_{u \in V_{n+1}} r_{uv}(c) = n - \tfrac{1}{2} \sum_{e \in E} r_e(c) \overset{\text{(Foster)}}{=} n - \tfrac{1}{2}(2n+1-1) = 0.$$

So in particular we cannot have $p_v(c) > 0$ for all $v \in V_n$ and thus $K_{n,n+1}$ must be SRN. The same approach works in any bipartite graph, which leads to the following result:

Proposition 3.5. *A bipartite graph G is not RN if its bipartition sizes differ by more than one and G is not RP if its bipartition sizes differ.*

Example 3.6. (*Not RN graph*) Let G be a graph with cut vertex x, i.e., $G - x$ is disconnected. Following [9, Prop. 2], we then have that $p_x(c) \leq 0$ with equality if and only if x is an interior vertex in a path graph G. Thus the only non-biconnected RN graphs are paths, which are SRN.

Proposition 3.7. *If G is RN and not biconnected then G is a path, and if $|V| \geq 3$ it is SRN.*

We continue with a first characterization theorem that answers Question 1.2.

Theorem 3.8. *A graph G is RN (resp. RP) if and only if there exists a {non-separable, log-linear, positive} distribution on $\mathcal{T}(G)$, such that the expected degree of every vertex in a random spanning tree under this distribution is at most 2 (resp. strictly smaller than 2). Any type of distribution is sufficient and all three types are necessary.*

Proof. The expected degree of a vertex v in a random spanning tree with distribution μ is

$$\sum_{T} \mu(T) \sum_{uv \in T} 1 = \sum_{uv \in E} \Big(\sum_{T \ni uv} \mu(T) \Big).$$

In other words, the expected degree is given in terms of the edge marginals of μ. Let μ^* be a non-separable distribution for which the expected degree of each vertex is at most 2. By Theorem 2.10, the edge marginals of μ^* are equal to relative resistances for some choice of edge weights c^* and thus we find for every $v \in V$ that

$$2 \geq \sum_{uv \in E} \Big(\sum_{T \ni uv} \mu^*(T) \Big) = \sum_{uv \in E} r_{uv}(c^*) = 2 - 2p_v(c^*),$$

which shows that $p(c^*) \geq 0$ and thus that G is RN. Since non-separable distributions are the most general type, sufficiency of all three types follows. The converse statement and the strict version for RP graphs are proven in the same way. □

We now turn to a geometric version of this characterization. As the proof above illustrates, the expected degree constraints are affine inequalities on $\mathbb{R}^E$. We define the (open) halfspaces

$$\mathcal{H}_v = \Big\{ x \in \mathbb{R}^E \mid \sum_{uv \in E} x_{uv} \leq 2 \Big\},\ \mathcal{H}_v{}^\circ = \Big\{ x \in \mathbb{R}^E \mid \sum_{uv \in E} x_{uv} < 2 \Big\} \qquad \text{for } v \in V.$$

Corollary 3.9. *A graph G is RN (resp. RP) if and only $P(G)^\circ$ and the halfspaces $\mathcal{H}_v$ (resp. open halfspaces $\mathcal{H}_v^\circ$) for all $v \in V$ have a common intersection point.*

Proof. As in the proof of Theorem 3.8, a graph is RN if and only if it has relative resistances that satisfy $\sum_{uv\in E} r_{uv} \leq 2$ for all $v \in V$, equivalently, if $r \in \mathcal{H}_v$ for all v. Since $P(G)^\circ$ is the set of all possible relative resistances, this completes the proof; the same proof holds for RP. □

Remark 3.10. (Computation) Corollary 3.9 shows that determining if a graph is RN can be done using linear programming. In practice, this requires some tolerance ε to turn strict inequalities into non-strict ones. While spanning tree polytopes can have exponentially many (in $|E|$) defining inequalities, they admit a polynomial-sized extended formulation [1] using which these linear programs can be computed efficiently.

Question 3.11. What is the computational complexity of the decision problems:

For a given graph G, decide if G is RN/RP/SRN.

4. Trees and Double Matchings

We continue with the geometric characterization of RN and RP graphs in Corollary 3.9, and show that the intersection of $P(G)$ with the halfspaces $(\mathcal{H}_v)_{v\in V}$ has a combinatorial interpretation in terms of double matchings. Recall that a matching is a (possibly empty) subset of edges $M \subseteq E$ which covers every vertex at most once, and the matching polytope $M(G)$ is the convex hull of indicator vectors of matchings. Edmonds derived the hyperplane description of the matching polytope in [12]; scaled by a factor of two, this becomes:

$$\begin{aligned} &2M(G) \\ &\quad = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} \sum_{uv\in E} x_{uv} \leq 2 & \text{for all } v \in V, \\ \sum_{uv\in E(U)} x_{uv} \leq 2\lfloor \frac{1}{2}|U| \rfloor & \text{for all odd-sized } U \subseteq V \end{array} \right\} \subset \mathbb{R}^E, \end{aligned} \tag{4}$$

where $E(U)$ are the edges in the induced subgraph on U. Note that the v-based inequalities correspond to the expected degree condition. We show that the U-based inequalities are implied by the hyperplanes of $P(G)$, which leads to the following characterization of RN graphs:

Theorem 4.1. *A graph G is RN if and only if $P(G)^\circ \cap 2M(G)$ is non-empty.*

Proof. If a point $r \in P(G)^\circ$ lies in $2M(G)$, then by the v-based inequalities of $2M(G)$ we know that r lies in the intersection of $(\mathcal{H}_v)_{v\in V}$ and thus that G is RN by Corollary 3.9.

If G is RN then there is a point $r \in P(G)^\circ \subset \mathbb{R}^E_{\geq 0}$ which satisfies the v-based inequalities of $2M(G)$. It remains to show that r also satisfies the U-based inequalities. Let $U \subseteq V$ have odd size and $G[U]$ be the induced subgraph on U with edges $E(U)$. We have

$$\sum_{e\in E(U)} r_e \leq |V| - \#\text{components of } \big(G - E \setminus E(U)\big) \quad \text{(Corollary 2.14)}$$

$$= |U| - \#\text{components of } \big(G[U]\big) \le |U| - 1 = 2\lfloor \tfrac{1}{2}|U| \rfloor.$$

This confirms that $r \in 2M(G)$ and completes the proof. □

Remark 4.2. Similar to how $P(G)^\circ$ parametrizes all possible edge weights, the set $P(G)^\circ \cap 2M(G)$ parametrizes all edge weights for which G has nonnegative resistance curvature. Geometrically, thinking of edge weights as inverse edge lengths, this is the "space of metrics with nonnegative curvature on G". The fact that the closure of this set is a polytope is in stark contrast with the complexity of the "space of metrics with positive/nonnegative curvature on a manifold" as studied in differential geometry; see for instance [22, Sec. 2].

A matching in a graph can have size at most $\lfloor \frac{1}{2}|V| \rfloor$. Such a matching is called *perfect* if the graph is even, and *near-perfect* if the graph is odd. Perfect matchings in graphs and their statistics are very actively studied; they are also called *dimers* [19]. It follows from 1-tough that RP graphs have a (near-)perfect matching, but we can strengthen this result to all RN graphs.

Corollary 4.3. *Every resistance nonnegative graph has a (near-)perfect matching.*

Proof. By Foster's theorem, the spanning tree polytope lies in the hyperplane of points in $\mathbb{R}^E$ whose entries sum to $|V| - 1$. If a graph G is RN, then by Theorem 4.1 there must be a point in $2M(G)$ that lies in this hyperplane. Since this point is a convex combination of indicator vectors of double matchings, G must have a matching of size at least $\lceil \frac{1}{2}(|V| - 1) \rceil = \lfloor \frac{1}{2}|V| \rfloor$. □

We note that Corollary 4.3 is useful as a necessary condition to check if a class of graphs could be RN. It can be used for instance to show that imbalanced bipartite graphs are not RN, as in Proposition 3.5. The converse to Corollary 4.3 does not hold. For instance, a 3-cycle glued to a 4-cycle at a single vertex has a near-perfect matching, but it is not RN (see Example 3.6).

Motivated by Theorem 4.1 and Remark 4.2, we take a better look at the following polytope:

Definition 4.4. The *tree double matching* (TDM) polytope of a graph is $\Theta(G) = P(G) \cap 2M(G)$.

Example 4.5. (*Even cycle graphs*) For $n = 2k \ge 3$ even, the cycle graph C_n has n spanning trees, 2 perfect matchings of size k and n matchings of size $k - 1$. Each of the spanning trees can be composed by combining a $(k-1)$-sized matching and one of the two k-sized matching. The TDM polytope is the whole spanning tree polytope $\Theta(C_n) = P(C_n)$.

Integer points of polytopes are important in many areas of combinatorics and its applications; a polytope is called a *lattice polytope* if its vertices are integer points. For the TDM polytope, we find that the integer points $\Theta(G) \cap \mathbb{Z}^E$ have a simple characterization; recall that a *Hamiltonian path* in a graph is a spanning tree which is also a path.

Proposition 4.6. *The integer points of $\Theta(G)$ are indicator vectors of Hamiltonian paths.*

Proof. Since the vertices of $P(G)$ lie in $\{0,1\}^E$ the only integer points in $P(G)$ are its vertices, which are indicator vectors of spanning trees. A vertex e_T of $P(G)$ lies in $\Theta(G)$ if and only if it can be written as a convex combination of double matchings in G. By Foster's theorem as in the proof of Corollary 4.3 and integrality of the point e_T, this point must be a combination of (i) two matchings with $\frac{1}{2}(|V|-1)$ edges if G is odd, or (ii) one matching with $\frac{1}{2}|V|$ edges and another one with $\frac{1}{2}|V|-1$ edges if G is even. Combining these matchings gives a subgraph with degree at most 2, which is connected if and only if it is a path. This completes the proof. □

We can use this result to give a combinatorial condition for the RN property; a set of Hamiltonian paths $\mathcal{F}$ is called *independent* if the indicator vectors $(e_T)_{T\in\mathcal{F}}$ are affinely independent.

Corollary 4.7. *If G contains ($|E|-\#$biconnected components of G) independent Hamiltonian paths, then G is RN.*

Proof. The result is true if G is a path graph so we may assume that G is biconnected. If G contains $|E|-1$ independent Hamiltonian paths, then $\operatorname{conv}(e_T \mid T \text{ is Hamiltonian path}) \subseteq \Theta(G)$ has dimension $|E|-1$. Since this is equal to the dimension of $P(G)$, this convex hull and thus also $\Theta(G)$ must intersect $P(G)^\circ$. By Theorem 4.1 this completes the proof. □

Proposition 4.6 and its proof illustrate how the integer points of the TDM polytope correspond combinatorially to trees that can be constructed as the union of two matchings. A similar combinatorial interpretation holds for rational points of $\Theta(G)$ or, equivalently, integer points of its dilations. We propose the following Ehrhart-type question for TDM polytopes:

Question 4.8. Count the number of integer points $n_k = \#(k\Theta(G)\cap\mathbb{Z}^E)$ for $k\in\mathbb{N}$.

Let us return to Example 4.5, where we mentioned that $\Theta(G)$ coincides with $P(G)$ for cycle graphs. It turns out that this is the only class of graphs where this occurs.

Corollary 4.9. *The following are equivalent for a biconnected graph G:*

1. $\Theta(G) = P(G)$*;*
2. $p(c) > 0$ *for all weights* $0 < c < \infty$*;*
3. G *is a cycle or a single edge* $G = K_2$.

Proof. Note that $\Theta(G) = P(G)$ is equivalent to $P(G)^\circ \cap 2M(G) = P(G)^\circ$, which says that every point $r \in P(G)^\circ$ lies in the halfspaces $(\mathcal{H}_v)_{v\in V}$. This establishes 1. ⇔ 2. The implication 3. ⇒ 1. follows from Proposition 4.6. Conversely, if $\Theta(G) = P(G)$ then all vertices of $P(G)$ must be Hamiltonian paths, so all spanning trees of G are paths. For a biconnected graph, this occurs if and only if G is a cycle or a single edge, which proves 1. ⇒ 3. □

We note that paths with more than 3 vertices satisfy similar properties: $\Theta(G) = P(G) = e_G$ and $p(c) \geq 0$ for all weights. We propose to study the vertices of general TDM polytopes:

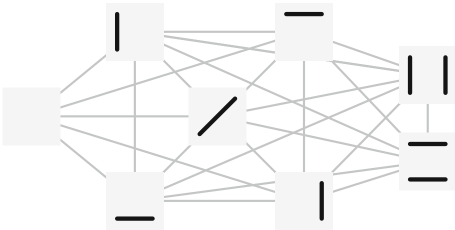

Figure 3. The 1-skeleton of the matching polytope $M(G_{\diamond})$

Question 4.10. What are the vertices of $\Theta(G)$? What is the smallest k such that the k-dilated TDM polytope is a lattice polytope, i.e., determine $k_G = \min\{k \in \mathbb{N} \mid k\Theta(G)$ is lattice polytope$\}$.

More generally, the points in $\Theta(G)$ are edge marginals of distributions μ on spanning trees in G and distributions λ on matchings in G, at the same time. In both cases, these distributions can be taken log-linear μ_c and $\lambda_{\tilde{c}}$ induced by some weights c and $\tilde{c}$ on the edges.

Question 4.11. What is the relation between the edge weights c and $\tilde{c}$, for which the distributions μ_c on spanning trees and $\lambda_{\tilde{c}}$ on matchings have the same edge marginals $x \in \Theta(G)$?

To conclude the section, we revisit Example 2.4 and illustrate the new definitions and results.

Example 4.12. (Diamond graph, continued) Figure 3 shows the 8 matchings of the diamond graph arranged according to the 1-skeleton of the matching polytope $M(G_{\diamond}) \subset \mathbb{R}^5$. Since the diamond graph is RP, the intersection $P(G_{\diamond})^{\circ} \cap 2M(G_{\diamond})$ is non-empty. The corresponding TDM polytope $\Theta(G_{\diamond})$ is a lattice polytope whose 6 vertices are indicator vectors of the Hamiltonian paths of $G_{\diamond}$. The number of rational points in the TDM polytope can be found from its Ehrhart polynomial; we compute this polynomial using `Polymake` [17] as

$$\#(k\Theta(G_{\diamond}) \cap \mathbb{Z}^5) = 1 + \frac{9}{4}k + \frac{15}{8}k^2 + \frac{3}{4}k^3 + \frac{1}{8}k^4 \text{ for } k \in \mathbb{N}.$$

5. Algebraic Characterization

Until here, we have used the relation between effective resistances and spanning trees to characterize RN and RP graphs. However, the effective resistance has a rich algebraic theory independent of the combinatorics of spanning trees and matroids; in this section, we use this theory to derive one further characterization. More precisely, we start from an alternative parametrization for the space of all possible effective resistances in a graph—a space $\mathcal{K}(G)$ with a similar role to $P(G)^{\circ}$—and derive an analogue of Corollary 3.9 to characterize

RN and RP graphs. Interestingly, the condition $p \geq 0$ again translates to affine inequalities on $\mathcal{K}(G)$.

We give a minimal treatment of Laplacian and resistance matrices and refer to [7, Ch. 3] for more details. Let G be a connected graph with $n = |V|$ vertices and edge weights $0 < c < \infty$. The Laplacian matrix of G is the $n \times n$ symmetric matrix L with entries

$$L_{uv} = 0 \text{ for } uv \notin E, \quad L_{uv} = -c_{uv} \text{ for } uv \in E, \quad L_{vv} = -\sum_{u \in V} L_{uv} \text{ for } v \in V.$$

This matrix is real, symmetric, positive semidefinite and has $\ker(L) = \operatorname{span}(\mathbf{1})$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones. Together with off-diagonal nonpositivity, these four properties characterize Laplacian matrices [7, Prop. 2.5]. Let $L^\dagger = (L + \frac{1}{n}\mathbf{1}\mathbf{1}^T)^{-1} - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ be the Moore–Penrose pseudoinverse of L, then the effective resistance between two vertices is defined as

$$\omega_{uv} = L^\dagger_{uu} + L^\dagger_{vv} - 2L^\dagger_{uv}, \qquad \text{for } u, v \in V. \tag{5}$$

For adjacent vertices, this agrees with Definition (1) in the introduction and we recall that $r_{uv} = \omega_{uv} c_{uv}$ for $uv \in E$. The *resistance matrix* of G is the $n \times n$ matrix $\Omega = (\omega_{uv})_{u,v \in V}$. It is a real, symmetric, invertible matrix with zero diagonal [7, Prop. 3.2 & 3.5]. We will further work with the inverse resistance matrix $K = \Omega^{-1}$ and consider a special class of edge weights:

Definition 5.1. The edge weights $0 < c < \infty$ are called *normalized* if $\sum_{u,v \in V} K_{uv} = 1$.

We call a graph with normalized weights a *normalized graph*. Assuming normalized edge weights is not restrictive, since any choice of edge weights can be normalized by the common rescaling $c_e \mapsto c_e / (\sum_{u,v \in V} K_{uv})$ for all $e \in E$. The resistance curvature of a normalized graph has a very simple expression in terms of K.

Proposition 5.2. ([7, Cor. 3.10]) *Let G be a normalized graph, then $p_v(c) = \sum_{u \in V} K_{uv}$.*

We now rephrase [7, Thm. 3.9] and characterize all possible inverse resistance matrices associated to a given graph as a semialgebraic set inside $\mathrm{GL}_n^{\mathrm{sym}}(\mathbb{R})$, the set of real symmetric invertible $n \times n$ matrices. We think of this as an alternative parametrization to $P(G)^\circ$.

Theorem 5.3. [7] *The set of inverse resistance matrices of G with normalized weights is*

$$\mathcal{K}(G) = \left\{ K \in \mathrm{GL}_n^{sym}(\mathbb{R}) \;\middle|\; \begin{array}{l} \sum_{u,v \in V} K_{uv} = 1 \\ K_{uv} = \left(\sum_{w \in V} K_{wv}\right)\left(\sum_{w \in V} K_{uw}\right) \textit{ for all } uv \notin E \\ K_{uv} > \left(\sum_{w \in V} K_{wv}\right)\left(\sum_{w \in V} K_{uw}\right) \textit{ for all } uv \in E \end{array} \right\}.$$

Proof. We summarize the proof of [7, Thm. 3.9]. Given $K \in \mathcal{K}(G)$, we construct the matrix

$$L = -2K + 2(K\mathbf{1})(K\mathbf{1})^T. \tag{6}$$

It can be checked that this matrix is real, symmetric, positive semidefinite with $\ker(L) = \operatorname{span}(\mathbf{1})$ and that it has nonpositive off-diagonal entries: $L_{uv} < 0$ if $uv \in E$ and zeros otherwise. Thus, L is a Laplacian matrix of G with weights $c^*_{uv} = -L_{uv}$, and K is the inverse resistance matrix of this graph by [7, Cor. 3.8]. Here, $K \in \mathcal{K}(G)$ confirms that the weights c^* are normalized.

Conversely, it can be checked that every inverse resistance matrix of G lies in the set $\mathcal{K}(G)$; this follows for instance by relation (6) between the Laplacian and inverse resistance matrix of a normalized graph, and then using the properties of Laplacian matrices. $\square$

We now formulate an analogue of Corollary 3.9 based on the parametrization $\mathcal{K}(G)$. As before, we define (open) halfspaces to keep track of signs of the resistance curvature

$$\widetilde{\mathcal{H}}_v = \Big\{K \in \mathrm{GL}_n^{\mathrm{sym}}(\mathbb{R}) \mid \sum_{u \in V} K_{uv} \geq 0\Big\},$$

$$\widetilde{\mathcal{H}}_v^{\circ} = \Big\{K \in \mathrm{GL}_n^{\mathrm{sym}}(\mathbb{R}) \mid \sum_{u \in V} K_{uv} > 0\Big\} \text{ for } v \in V.$$

Corollary 5.4. *A graph G is RN (resp. RP) if and only if $\mathcal{K}(G)$ and the halfspaces $\widetilde{\mathcal{H}}_v$ (resp. open halfspaces $\widetilde{\mathcal{H}}_v^{\circ}$) for all $v \in V$ have a common intersection point.*

Proof. We may assume that all weights are normalized, without loss of generality. The corollary then immediately follows from the expression of p in terms of K in Proposition 5.2. $\square$

Example 5.5. The Laplacian, resistance and inverse resistance matrix of $G_{\diamond}$ with $c = 1$ are

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \quad \Omega = \frac{1}{8}\begin{pmatrix} 0 & 5 & 8 & 5 \\ 5 & 0 & 5 & 4 \\ 8 & 5 & 0 & 5 \\ 5 & 4 & 5 & 0 \end{pmatrix} \quad K = \frac{1}{1.36}\begin{pmatrix} -1 & 0.8 & 0.36 & 0.8 \\ 0.8 & -2 & 0.8 & 0.72 \\ 0.36 & 0.8 & -1 & 0.8 \\ 0.8 & 0.72 & 0.8 & -1 \end{pmatrix}$$

6. Graph Operations

We consider three graph operations and their effect on resistance nonnegativity and positivity. Figure 4 at the end of the section illustrates the three operations.

6.1. Subgraphs

Since adding edges to a graph increases its set of spanning trees, we obtain the following:

Lemma 6.1. *Let $H \subseteq G$ with $V(H) = V(G)$ and H connected. If H is RP then G is RP.*

Proof. Let H be a connected RP subgraph of G on the same vertex set. The spanning tree sets of these two graphs satisfy $\mathcal{T}(H) \subseteq \mathcal{T}(G)$. Since H is RP, there exists a positive distribution μ on $\mathcal{T}(H)$ whose edge marginals $r \in P(H)^{\circ}$

satisfy $\sum_{uv \in E(H)} r_{uv} < 2$ for all $v \in V$. We then define a distribution μ^* on $\mathcal{T}(G)$ as

$$\mu^* : T \mapsto \begin{cases} (1-\varepsilon)\cdot\mu(T) & \text{if } T \in \mathcal{T}(H) \\ \varepsilon \cdot \big(|\mathcal{T}(G)\backslash\mathcal{T}(H)|\big)^{-1} & \text{if } T \notin \mathcal{T}(H), \end{cases} \qquad \text{for } T \in \mathcal{T}(G).$$

This is a positive distribution on $\mathcal{T}(G)$ with edge marginals $r^* \in P(G)^\circ$. For sufficiently small $\varepsilon > 0$ these marginals satisfy $\sum_{uv\in E(G)} r^*_{uv} < 2$ for all v, by linearity of r, r^* in terms of μ, μ^*. □

A graph G is *Hamiltonian* if it contains a $|V(G)|$-length cycle graph as a subgraph.

Theorem 6.2. *Hamiltonian graphs are resistance positive, but the converse is not always true.*

Proof. If G is Hamiltonian, then it contains a cycle graph $H \subseteq G$ as a subgraph on the same vertex set. Since H is connected and RP, G is also RP by Lemma 6.1. The Petersen graph is RP (since it is vertex-transitive) but not Hamiltonian, so RP does not imply Hamiltonicity. □

The study of Hamiltonicity is closely related to graph toughness. Recall that a graph is t-tough if for every set of vertices $U \subset V$ whose removal disconnects the graph, the graph $G - U$ has at most $|U|/t$ components; cliques are ∞-tough. Chvátal conjectured the following:

Conjecture 6.3. [4, Conj. 2.3] *There exists t_0 such that every t_0-tough graph is Hamiltonian.*

We refer to the survey [2] for history and progress on this conjecture. Combining Theorem 1.3 and Theorem 6.2, we find that RP graphs lie between two graph classes: *Hamiltonian* $\subset$ *RP* $\subseteq$ *1-tough*. Following this relation, we propose a relaxation of Chvátal's conjecture:

Conjecture 6.4. *There exists t_{RP} such that every t_{RP}-tough graph is resistance positive.*

Proposition 6.5. *Conjecture 6.3 implies Conjecture 6.4.*

Proof. If Conjecture 6.3 holds for t_0, then t_0-tough implies Hamiltonicity which implies RP. □

The latter conjecture was also posed for $t_{RP} = 1$ by Fiedler in [14, p59]. We ask the following:

Question 6.6. Does 1-tough imply *RP*? Is every non-Hamiltonian 1-tough graph RP?

As a further application we consider grid graphs. The Cartesian product $G \times H$ of two graphs has vertex set $V(G) \times V(H)$ and edge set $\{(u,x)(v,y) \mid uv \in E(G) \text{ or } xy \in E(H)\}$, where "or" is exclusive. A *grid graph* is the Cartesian product of two path graphs $P_n \times P_m$. In [6] it was shown that for constant edge weights $c = 1$ and $m, n \geq 3$, all grid graphs have negative resistance curvature on interior vertices and nonnegative resistance curvature on boundary vertices and if $m > 3$ or $n > 3$, then the boundary vertices have positive curvature. Here we find that some of these grid graphs are RP. This illustrates the relevance of considering edge weights.

Proposition 6.7. *For $m \cdot n$ even and $m, n > 1$, the grid graph $P_n \times P_m$ is RP.*

Proof. For these parameters, $P_n \times P_m$ is Hamiltonian and thus RP. □

We conjecture that all grid graphs are resistance nonnegative, and ask about generalizations.

Conjecture 6.8. (Grid graphs are RN) *$P_n \times P_m$ is RN for all $m, n \in \mathbb{N}$.*

Question 6.9. Are all graphs $P_n \times \cdots \times P_m$ RN? Which biconnected planar graphs are RN?

6.2. Kron Reduction

Kron reduction is an important tool in electrical circuit analysis, which allows to reduce a large circuit to a smaller one while maintaining its electrical properties [11]. Combinatorially, this operation is very simple; we write $U^c = V \setminus U$ and $\partial U = \{v \in U^c \mid uv \in E \text{ for some } u \in U\}$.

Definition 6.10. (*Kron reduction*) Let G be a connected graph with vertex subset $U \subset V$. The *Kron reduction* of U in G is the graph obtained by removing U and adding all possible edges between ∂U; we write $\text{kron}_U(G)$.

Definition 6.11. (*Matched weights)*) Let $\widetilde{G}$ be the Kron reduction of $x \in V$ in G with edge weights c. The *matched weights* $\tilde{c}$ in $\widetilde{G}$ are defined as

$$\tilde{c}_{uv} = c_{uv} + \frac{c_{ux} \cdot c_{xv}}{\sum_{xy \in E(G)} c_{xy}} \text{ if } u, v \in \partial x,$$
$$\tilde{c}_{uv} = c_{uv} \text{ otherwise,} \qquad \text{for } uv \in E(\widetilde{G}).$$

We recall some relevant properties of Kron reductions; see [7, Ch. 3],[11] for more details.

Lemma 6.12. (Quotient property, [11]) *Let $U = \{u_1, \ldots, u_k\} \subset V(G)$ in arbitrary order. Then*

$$\text{kron}_U(G) = \text{kron}_{u_k} \circ \cdots \circ \text{kron}_{u_1}(G). \tag{7}$$

To understand Kron reduction it often suffices to understand Kron reduction of single vertices. For a general vertex subset $U \subset V$, if the weights are matched at every step in the decomposition of $\widetilde{G} = \text{kron}_U(G)$ along single vertices as in (7), then we say that the final weights $\tilde{c}$ on $E(\widetilde{G})$ are matched; these weights can also be obtained directly from the Schur complement of the associated Laplacian matrix. The following is relevant for RN graphs:

Lemma 6.13. [7, Prop. 3.25] *Let* $\widetilde{G} = \mathrm{kron}_x(G)$ *with matched edge weights* $\tilde{c}$. *Then*

$$\tilde{p}_v(\tilde{c}) = p_v(c) + \frac{c_{xv} \cdot p_x(c)}{\sum_{xy \in E(G)} c_{xy}} \text{ if } v \in \partial x,$$
$$\tilde{p}_v(\tilde{c}) = p_v(c) \text{ otherwise}, \quad \text{for } v \in V(\widetilde{G}).$$

In the above lemma, we note that if $p_v(c), p_x(c) \geq 0$ then also $\tilde{p}_v(\tilde{c}) \geq 0$, and similarly if all inequalities are strict. This leads to the following closure property:

Proposition 6.14. *If G is RN (resp. RP), then any Kron reduction of G is RN (resp. RP).*

Proof. Let G be an RN graph with edge weights c such that $p(c) \geq 0$ and take any $U \subset V$. Then by taking Kron reductions of single vertices as in (7) with matched edge weights, and using Lemma 6.13, we find that $\mathrm{kron}_U(G)$ is again RN. The same derivation holds for RP graphs. ☐

6.3. Circle Inversion

Different from the other two operations, the next operation is defined with respect to a choice of edge weights; it originates from Fiedler's matrix theory of simplices, where it corresponds to spherical inversion of a simplex around one of its vertices [14, Rem. 3.4.5].

Definition 6.15. (*Circle inversion,* [14]) Let G be an RN graph with vertex $x \in V$ and weights c such that $p(c) \geq 0$. The *circle inversion over* x is the graph obtained by removing x and adding a new vertex $\hat{x}$ with edges $\{\hat{x}y \mid p_y(c) > 0 \text{ in G}\}$; we write $\mathrm{cinv}_x(G, c)$.

Similar to Kron reduction, circle inversion comes with a distinguished choice of edge weights. We make use of the effective resistance ω defined as in (5) between any pair of vertices.

Definition 6.16. Let $\widetilde{G}$ be the circle inversion over x with respect to weights c. The *matched weights* $\tilde{c}$ in $\widetilde{G}$ are defined as

$$\tilde{c}_{\hat{x}y} = 2 \cdot p_y(c) \cdot \omega_{xy} \text{ if } y \in \partial\hat{x}, \qquad \tilde{c}_{uv} = c_{uv} \cdot \omega_{ux} \cdot \omega_{xv} \text{ if } u, v \notin \partial\hat{x}.$$

These weights are only nonnegative if $p(c) \geq 0$. When fixing a choice of edge weights on a graph and keeping these weights matched under circle inversions, the map $\mathrm{cinv}_\bullet : (G, c) \mapsto (\widetilde{G}, \tilde{c})$ commutes and has an inverse, i.e., $\mathrm{cinv}_x \circ \mathrm{cinv}_y = \mathrm{cinv}_y \circ \mathrm{cinv}_x$ up to relabeling vertices, and $\mathrm{cinv}_{\hat{x}} \circ \mathrm{cinv}_x = \mathrm{Id}$. The relevance of circle inversion for RN graphs is due to the following result:

Lemma 6.17. [14, Thm. 3.4.4] *Let* $\widetilde{G} = \mathrm{cinv}_x(G, c)$ *with matched edge weights* $\tilde{c}$. *Then*

$$\tilde{p}_v(\tilde{c}) = \tfrac{1}{2} \cdot c_{xv} \cdot \omega_{xv} \text{ if } v \in \partial x, \qquad \tilde{p}_v(\tilde{c}) = p_x(c) \text{ if } v = \hat{x},$$
$$\tilde{p}_v(\tilde{c}) = 0 \text{ otherwise, for } v \in V(\widetilde{G}).$$

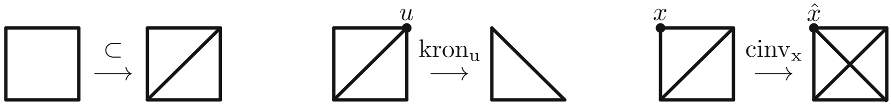

FIGURE 4. Example of a subgraph, Kron reduction and circle inversion (with $c = 1$)

This directly leads to another closure property for RN and RP graphs.

Proposition 6.18. *If G is RN, then any circle inversion of G is RN. Furthermore, if $\mathrm{cinv}_x(G, c)$ is over a vertex x with $p_x(c) > 0$ and which is connected to all other vertices, then it is RP.*

7. Nonnegativity from Submodularity

We conclude the article with a speculative third approach to characterizing resistance nonnegativity. We again start from a space $\mathcal{S}(G)$ that parametrizes all possible effective resistances in a graph and conjecture that the condition $p \geq 0$ translates to linear inequalities on $\mathcal{S}(G)$; more precisely, we conjecture that these inequalities are those of the submodular cone.

For $U \subseteq V$ nonempty, we write $\Omega[U]$ for the submatrix of the resistance matrix Ω with rows and columns in U. A variant of the following set function was introduced in [7, Ch. 3.4].

Definition 7.1. (*Resistance capacity*) For a normalized graph G, the *resistance capacity* is a set function $\tau_\bullet(c) : 2^V \to \mathbb{R}$ defined on the subsets of V as: $\tau_\emptyset(c) = 0$, and $\tau_v(c) = \frac{1}{2}$ for $v \in V$, and

$$\tau_U(c) = \frac{1}{2} + \frac{1}{2}\Big(\sum_{u,v\in U}(\Omega[U]^{-1})_{uv}\Big)^{-1} \qquad \text{for } U \subseteq V \text{with } |U| \geq 2.$$

Remark 7.2. The resistance capacity τ_V is a discrete analogue of a well-studied invariant in potential theory on metric graphs, the so-called tau constant of a metric graph [5, Thm. 1.27].

A set function f is called *monotone* if it satisfies $f(A) \leq f(B)$ for all $A \subseteq B$.

Proposition 7.3. *The resistance capacity is monotone and lies in* $[0, 1]$.

Proof. Monotonicity was proven in [7, Prop 3.38]. For $U = \emptyset$ and $U = V$ we directly compute $\tau_\emptyset = 0$ and $\tau_V = 1$, and by monotonicity this implies that $\tau_U \in [0, 1]$ for all $U \subseteq V$. □

Next, we observe that the resistance capacity parametrizes all data of a weighted graph:

Proposition 7.4. *The graph G and edge weights c can be recovered from* $(\tau_U(c))_{U\subseteq V}$.

Proof. For $U = uv$, the value of τ was computed in [7, p53] to be $\tau_{uv}(c) = \frac{1}{2} + \frac{1}{4}\omega_{uv}$. We can thus recover all effective resistances in the graph as $\omega_{uv} = 4\tau_{uv}(c) - 2$. From this data, the graph and edge weights can be found by constructing the Laplacian matrix via $K = \Omega^{-1}$ as in (6). □

We thus have a third parametrization in addition to $P(G)^\circ$ and $\mathcal{K}(G)$: the set of resistance capacities $(\tau_U(c))_{U\subseteq V}$ for all normalized edge weights c on a given graph G. We denote this set by $\mathcal{S}(G) \subseteq [0,1]^{2^V}$. We now relate certain inequalities on this space to resistance nonnegativity. A set function $f : 2^V \to \mathbb{R}$ is called *submodular* if it satisfies

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \qquad \text{for all } A, B \subseteq V. \tag{8}$$

The set of all submodular set functions on a given ground set V is called the submodular cone; it is a polyhedral cone in $\mathbb{R}^{2^V}$ cut out by the submodularity inequalities (8). We rephrase [7, Thm. 6.33], which implies submodularity for the resistance curvature when $p(c) \geq 0$:

Theorem 7.5. *Let G and c be normalized with $p(c) \geq 0$. Then $\tau_\bullet(c)$ is submodular.*

Proof. Let $\sigma_U^2 = \tau_U - \frac{1}{2}\delta_{U\neq\emptyset}$ be the resistance capacity shifted by $-\frac{1}{2}$ for all sets except for the empty set $U = \emptyset$. It was shown in [7, Thm. 6.33] that σ^2 satisfies the submodularity inequality (8) whenever the two sets A, B intersect. The main obstruction in proving full submodularity of σ^2 is that inequality (8) fails for distinct singletons $A = \{a\}, B = \{b\}$. Indeed, we have

$$\sigma_a^2(c) + \sigma_b^2(c) = 0 < \sigma_{ab}^2(c).$$

This obstruction is no longer present for τ due to the $-\frac{1}{2}$ shift and normalization; we find

$$\tau_a(c) + \tau_b(c) = 1 = \tau_E(c) \geq \tau_{ab}(c) \qquad \text{for distinct } a, b \in V.$$

All other cases are then dealt with inductively in the proof of [7, Thm. 6.33]. □

Conjecture 7.6. *Let G and c be normalized with $\tau_\bullet(c)$ submodular. Then $p(c) \geq 0$.*

The author proved the partial result "submodular σ^2 implies $p(c) \geq -1$" in [7, Prop. 6.34]. Conjecture 7.6 is equivalent to the following conjecture on resistance nonnegative graphs:

Conjecture 7.7. *A graph G is RN if and only if $\mathcal{S}(G)$ intersects the submodular cone in $[0,1]^{2^V}$.*

Example 7.8. We consider the diamond graph $G_{\diamond}$ with normalized constant weights $c \approx 0.531$. Figure 5 shows the resistance capacity $\tau_U(c)$ of the 6 distinct subsets $U \subseteq V(G_{\diamond})$ of size $|U| \geq 2$, up to automorphisms. The resistance capacity is both monotonous and submodular.

FIGURE 5. The resistance capacity of subsets of the vertices of $G_{\diamond}$ (rounded to 3-digit precision)

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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