Embeddings of infinite groups into Banach spaces

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For family and friends: undeniable proof that you don't need to understand to care.

Abstract

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In this thesis we build on the theory concerning the metric geometry of relatively hyperbolic and mapping class groups, especially with respect to the difficulty of embedding such groups into Banach spaces.

In Chapter 3 (joint with Alessandro Sisto) we construct simple embeddings of closed graph manifold groups into a product of three metric trees, answering positively a conjecture of Smirnov concerning the Assouad-Nagata dimension of such spaces. Consequently, we obtain optimal embeddings of such spaces into ℓ^p spaces. The ideas here have been extended to other closed 3-manifolds and to higher dimensional analogues of graph manifolds.

In Chapter 4 we give an explicit method of embedding relatively hyperbolic groups into ℓ^p spaces, which yields optimal bounds on the compression exponent of such groups relative to their peripheral subgroups. From this we deduce that the fundamental group of every closed 3-manifold has Hilbert compression exponent 1.

In Chapter 5 we prove that relatively hyperbolic spaces with a tree-graded quasi-isometry representative can be characterised by a relative version of Manning's bottleneck property. This applies to the Bestvina-Bromberg-Fujiwara quasi-trees of spaces, yielding an embedding of each mapping class group of a closed surface into a finite product of simplicial trees. From this we obtain explicit embeddings of mapping class groups into ℓ^p spaces and deduce that these groups have finite Assouad-Nagata dimension. It also applies to relatively hyperbolic groups, proving that such groups have finite Assouad-Nagata dimension if and only if each peripheral subgroup does.

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Chapter 1

Introduction

This year our members have put more things on top of other things than ever before. But, I should warn you, this is no time for complacency. No, there are still many things, and I cannot emphasise this too strongly, not on top of other things. I myself, on my way here this evening, saw a thing that was not on top of another thing in any way.
Crowd: Shame!! Shame!!

President of the Royal Society for Putting Things on Top of Other Things

– Monty Python

Given two metric spaces (X_1, d_1) and (X_2, d_2) it is a natural question to ask whether X_1 is a metric subspace of X_2, specifically, is there an isometric embedding of X_1 into X_2? This has increased significance when we wish to study the metric geometry of X_1 and X_2 is a space (or more generally, a class of spaces) which are geometrically well understood, for example Hilbert spaces. Of course, it is highly unlikely that interesting spaces X_1 will actually isometrically embed so the next question would be, how 'close' is X_1 to being a metric subspace of X_2? This leads to what might be described as a metric representation theory.

Embeddings of discrete metric spaces into Banach spaces have been an important topic in computer science for many years [LLR95, HLW06, BDG⁺05] and more recently became so in geometric group theory, combinatorics and K-theory.

To apply this to questions in geometric group theory, we will consider X_1 as the Cayley graph of a finitely generated group and X_2 as a class of ℓ^p spaces or finite products of metric trees. In the first case, knowing that a group admits even a rather weak metric representation inside some ℓ^p space - a coarse embedding - has strong consequences for rigidity conjectures in topology, K and L-theory [Yu00, KY06, GTY12].

Gromov outlined a construction of a group which fails to satisfy this property, leading to generalisations of small cancellation techniques and to some extent, to the study of lacunary hyperbolic groups [Gro00, Oll06, AD08]. Better metric representations - those with sufficiently large compression exponent - lead to information about amenability properties of the group and the nature of random walks on the Cayley graph [GK04, NP08, ANP09].

At the other extreme, admitting quasi-isometric embeddings into Hilbert spaces is conjecturally a highly restrictive condition, satisfied only by virtually abelian groups [dCTV07].

Moreover, obtaining quasi-isometric embeddings into finite products of trees - or more general finite dimensionality conditions - have a strong connection to the stable rigidity of manifolds, as well as links to embeddability of such groups into ℓ^p spaces, topological dimension controls on asymptotic cones and Lipschitz extension properties [GTY12, Gal08, LS05, BH09].

The purpose of the thesis is to construct embeddings of two well-studied, important classes of groups into ℓ^p spaces and finite products of trees, which are optimal in a strong sense and to show such groups are metrically 'finite dimensional'.

The first of the two classes of groups we consider are mapping class groups. Due to their close connections with geometry, topology and group theory and their similarities with lattices in higher rank semi-simple Lie groups and $\text{Out}(F_n)$, mapping class groups are one of the most interesting classes of finitely generated groups to study. Much more detail on these links can be found in [Iva02, FM12] and references therein.

Secondly, we study relatively hyperbolic groups; introduced by Gromov [Gro87], as a generalisation of hyperbolic groups which includes geometrically finite Kleinian groups. This wide class of groups includes: hyperbolic groups, amalgamated products and HNN-extensions over finite subgroups, fully residually free (limit) groups - which are key objects in solving the Tarski conjecture [Sel01, KM10] - and fundamental groups of non-geometric closed 3-manifolds with at least one hyperbolic component [Dah03a].

The understanding of such groups is a hugely active area of modern research and in this thesis we complete the solutions of the aforementioned questions for these groups: how well they embed into ℓ^p spaces and finite product of trees and the common notions of finite dimensionality they satisfy.

1.1 Main results

This section is split according to where the results appear. Chapter 2 is concerned only with background, so we begin with the results of Chapter 3.

1.1.1 Embeddings of graph manifolds

As mentioned previously, many compact 3-manifolds have relatively hyperbolic fundamental group. More specifically, they are hyperbolic relative to a collection of virtually polycyclic groups and to fundamental groups of graph manifolds. In order to better control embeddings of graph manifold groups into ℓ^p spaces and to give bounds on their dimensions, I proved the following with Alessandro Sisto:

Theorem 1. (Theorem 3.1.1 and Corollary 3.1.2)

Let M be a graph manifold which does not have the Nil geometry. Then the universal cover of M (denoted $\widetilde{M}$) quasi-isometrically embeds in the product of three metric trees. In particular, the universal cover of any graph manifold has Assouad-Nagata dimension at most 3*.*

This answers a conjecture of Smirnov, who was able to prove such groups have asymptotic Assouad-Nagata dimension at most 7 [Smi10].

Since this work was completed, Smirnov has produced a generalisation to higher dimensional manifolds satisfying a notion analogous to that of a non-geometric flip graph manifold [Smi12].
This result was also used by Mackay-Sisto to calculate dimension bounds on fundamental groups of 3-manifolds.

The results of this chapter appear in the paper [HS11].

1.1.2 Embeddings of relatively hyperbolic groups

These results are contained in Chapter 4. We develop the techniques required to construct explicit embeddings of relatively hyperbolic groups from given embeddings of the peripheral subgroups. The statements given here are weaker than what is actually proved in the thesis, but these statements are approachable without too much prior knowledge of the area. First we give explicit embeddings of hyperbolic spaces with very little compression.

Theorem 2. (Theorem 4.3.2)

Let X be a countable uniformly discrete Gromov hyperbolic metric space with bounded geometry. Then given any $p \geq 1$ there exists a map $\phi : X \to \bigoplus_{n\in\mathbb{N}} \ell^p(X)$ such that for all $x, y \in X$ and all $\alpha \in (0,1)$,

$$(d_X(x,y))^\alpha \leq \|\phi(x) - \phi(y)\|_p \leq d_X(x,y).$$

In particular, for every p, the ℓ^p compression exponent of X is 1.

The conclusion of this theorem is not new, but all previous proofs rely in a key way on major theorems of Bonk-Schramm or Buyalo-Schroeder [BS00, BS05]. It is the fact that this embedding is direct and easily constructed which is the main interest here as it allows the possibility of extending such a result to relatively hyperbolic spaces. The proof is short and has various generalisations which are discussed in Section 4.6. In particular, the same result is shown to hold for hyperbolic spaces satisfying a tight geodesic property defined by Bowditch, which is modelled on the Masur-Minsky notion of tight geodesics in curve complexes [MM00, Bow08]. This also applies, importantly in Chapter 5, to coned-off graphs of relatively hyperbolic groups.

The second step is to provide optimal embeddings of tree-graded spaces from given embeddings of the pieces.

Theorem 3. (Theorem 4.4.2 and Corollary 4.4.3)

Let Γ be the 0-skeleton of a connected countable simplicial graph which admits a tree-grading $\mathcal{P} = \{\Gamma_i\}_{i\in I}$. Fix some $p \in [1,\infty)$. If there exists a collection of coarse embeddings $\psi_i : \Gamma_i \to \ell^p(X_i)$ and a constant $c > 0$ such that for all $x, y \in \Gamma_i$,

$$c(d_\Gamma(x,y))^\alpha \leq \|\psi_i(x) - \psi_i(y)\|_p \leq d_\Gamma(x,y),$$

then $\alpha_p^*(\Gamma) \geq \alpha$. In particular, if G, H are finitely generated groups and F is a finite subgroup of G and H then

$$\alpha_p^*(G *_F H) = \min\left\{\alpha_p^*(G), \alpha_p^*(H)\right\} \quad \text{and} \quad \alpha_p^*(\mathrm{HNN}(G,F)) = \alpha_p^*(G).$$

This is optimal as peripheral subgroups are undistorted, and improves previous results of Dreesen [Dre10]. Together with the previous result for hyperbolic spaces, this will motivate the method for the final result in this chapter.

Theorem 4. (Theorem 4.5.4 and Corollary 4.5.10)

Let X be the 0-skeleton of a connected simplicial graph with bounded geometry which is asymptotically tree-graded with respect to a collection of pieces $\mathcal{P} = \{X_i\}_{i\in I}$. Fix some $p \in (1,\infty)$. If there exists a collection of coarse embeddings $\psi_i : X_i \to \ell^p(Y_i)$ and a constant $c > 0$ such that for all $x, y \in X_i$,

$$c(d_\Gamma(x,y))^\alpha \le \|\psi_i(x) - \psi_i(y)\|_p \le d_\Gamma(x,y),$$

then $\alpha_p^*(X) \ge \alpha$. In particular, if G is relatively hyperbolic with respect to a collection of subgroups $\{H_1, \dots, H_n\}$ then

$$\alpha_p^*(G) = \min\left\{\alpha_p^*(H_i)\right\}.$$

From this we deduce the following corollary for fundamental groups of closed 3-manifolds.

Corollary 5. (Corollary 4.5.11)

Let M be a closed 3-manifold. For all $p > 1$ and all $\alpha \in (0,1)$, there exists a map ϕ from $\pi_1(M)$ into some ℓ^p space, such that for all $x, y \in \pi_1(M)$,

$$(d(x,y))^\alpha \le \|\phi(x) - \phi(y)\|_p \le d(x,y).$$

In particular $\pi_1(M)$ has compression exponent 1 for all $p > 1$.

These results answer the question of how well relatively hyperbolic groups can be embedded into ℓ^p spaces but only when $p > 1$. This can be remedied if we instead consider embeddings into $L^1([0,1])$.

Corollary 6. (Corollary 4.5.12)

Let X be the 0-skeleton of a connected simplicial graph with bounded geometry which is asymptotically tree-graded with respect to a collection of pieces $\mathcal{P} = \{X_i\}_{i\in I}$. If there exists a collection of coarse embeddings $\psi_i : X_i \to \ell^2(Y_i)$ and a constant $c > 0$ such that for all $x, y \in X_i$,

$$c(d_\Gamma(x,y))^\alpha \le \|\psi_i(x) - \psi_i(y)\|_2 \le d_\Gamma(x,y),$$

then the L^1 compression exponent of X is at least α.

The reliance of L^1 compression on embeddings of pieces into Hilbert spaces is unsatisfactory as the situation for L^1 spaces should be better than for other values of p, since all hyperbolic groups quasi-isometrically embed into ℓ^1 spaces [BS00]. This is addressed in the next chapter, though the techniques there are principally motivated by the study of mapping class groups.

The results of this chapter appear in the paper [Hum13].

1.1.3 Quasi tree-graded spaces

A major breakthrough in the study of the metric geometry of mapping class groups of compact surfaces was obtained by Bestvina-Bromberg-Fujiwara in their construction of a finite collection of hyperbolic spaces (quasi-trees of spaces) which contain a quasi-isometric image of a given mapping class group. However, this construction has ramifications for other groups, for instance to relatively hyperbolic groups, where the quasi-trees of spaces constructed are certainly not hyperbolic. Chapter 5 is motivated by the desire to understand embeddings and dimension controls on these spaces in full generality.

At the same time, Theorems 3 and 4 show that local finiteness is an important distinction between trying to embed tree-graded and asymptotically tree-graded spaces. In general, quasi-trees of spaces are locally infinite; for instance, in the mapping class group case, they contain curve complexes.

This motivates one particular question: under what circumstances is a (geodesic) metric space quasi-isometric to a tree-graded space. We prove that this question is answered by a relative version of Manning's bottleneck property.

Theorem 7. (Theorem 5.1.1)

A geodesic metric space X satisfies the relative bottleneck property *with respect to a collection of sets $\{X_i \mid i \in I\}$ if and only if it is quasi-isometric to some tree-graded space $\mathcal{T}(X)$ with pieces $\mathcal{T}_i$ uniformly quasi-isometric to X_i. In particular, all quasi-trees of spaces are* quasi tree-graded spaces*.*

The conclusion that the pieces of $\mathcal{T}(X)$ are uniformly quasi-isometric to the pieces X_i is a crucial point for the corollaries we obtain for mapping class groups and relatively hyperbolic groups.

Using this we can obtain embeddings of the mapping class group into ℓ^p spaces by applying Theorem 3. Moreover, we show the following.

Theorem 8. (Theorem 5.1.2)

The mapping class group of any compact surface quasi-isometrically embeds into a finite product of trees.

Consequently, mapping class groups have finite Assouad-Nagata dimension, and hence they have ℓ^p compression exponent 1 for all $p \geq 1$. Applying Theorem 3 and a generalisation of Theorem 2 given in Section 4.6.1, we obtain explicit embeddings of mapping class groups into ℓ^p spaces exhibiting compression exponent 1.

This answers many of the questions concerning embeddings and dimension controls raised in the introduction for mapping class groups.

Moving to relatively hyperbolic groups, we use Theorem 7 to extend Theorem 4 in the case $p = 1$ and deduce bounds on dimension.

Corollary 9. (Corollary 5.1.3)

Let G be a finitely generated group, which is hyperbolic relative to a collection of subgroups $\{H_i \mid i \in I\}$.

- *G has finite Assouad-Nagata dimension if and only if each H_i does.*
- *G can be quasi-isometrically embedded into $\ell^1(\mathbb{N})$ if and only if each H_i can.*
- *For each $p \in [1, \infty)$, G admits an explicit embedding into some ℓ^p space which exhibits the optimal compression exponent $\alpha_p^*(G) = \min_i \left\{\alpha_p^*(H_i)\right\}$.*

For the final one of these results we require constructive embeddings of the coned-off graph of G into ℓ^p spaces. This is done in Section 4.6.1 using the fact that such graphs admit systems of tight geodesics.

The results of this chapter appear in [Hum12].

1.2 Plan of the Thesis

In Chapter 2, we give an overview of the background required to approach the remaining chapters. Section 2.1 presents the groups and metric spaces which appear throughout the document. The next section (2.2) gives a motivation for studying coarse embeddings and dimension controls on finitely generated groups via the Borel conjecture. The presentation here is mathematically clean and entirely suitable for our purposes, but is not historically justifiable, as notions of dimension for finitely generated groups have been studied for much longer than their connections with these topological conjectures have been known. Having motivated the two topics, we recall the fundamentals of metric embeddings in Section 2.3, including links to amenability and random walks, and we present a variety of notions of asymptotic dimension in Section 2.4, providing links with embeddings and a range of examples to distinguish the different dimension estimates.

Chapter 3 presents an efficient proof of Theorem 1 and discusses the optimality of the obtained result as well as subsequent work which uses this.

The main theorem of Chapter 4 (Theorem 4) is technically demanding, so is approached via Theorems 2 and 3. The presentation of these proofs will motivate the final result, where we also explain the limitation which prevents the proof applying in the case $p = 1$. The extensions to Theorem 2 presented in Section 4.6.1 do not appear in the paper [Hum13], but are needed in the following chapter, while the study of strong fellow-traveller properties in Section 4.6.2 is linked intricately with the gap in the isoperimetric spectrum.

Chapter 5 introduces the relative bottleneck property and gives an explicit construction of a tree-graded space (Section 5.3) which is proved to be quasi-isometric to a given space satisfying the relative bottleneck property in Section 5.4. The converse of Theorem 7 and the link with quasi-trees of spaces is explored in Section 5.2.

Chapter 2

Preliminaries

This book was written using 100% recycled words.

– Sir Terry Pratchett (Wyrd Sisters)

The first section of this chapter introduces the principal objects of study. In the second we motivate the study of metric embeddings via the Borel conjecture and results relating it to the coarse geometry of metric spaces. Section 2.3 introduces notions of metric embeddings, while section 2.4 is primarily concerned with various concepts of large-scale dimension.

It is not envisaged that the reader should attempt to read this chapter from beginning to end, rather for it to function as a detailed overview of the necessary definitions and results which are required in future chapters but which may disrupt the flow of the reader familiar with such topics.

2.1 Groups and spaces

In this section we introduce the key objects of the thesis and summarise preliminary results needed in later chapters.

2.1.1 Graphs and groups as metric spaces

This brief section recalls the notions of shortest path metric for a graph and the Cayley graph of a finitely generated group.

Definition 2.1.1. The shortest path metric

Let Γ be a simplicial graph. A path $P = \{p_1, p_2, \ldots, p_n\}$ in Γ is a finite collection of directed edges such that, for each i, the terminal vertex of p_i is the initial vertex of

p_{i+1}. *We define the* length *of* P, $|P|$ *to be its cardinality as a set.*

The initial vertex of P, $\iota(P)$ *is the initial vertex of* p_1 *and the terminal vertex of* P, $\tau(P)$ *is the terminal vertex of* p_n.

The shortest path metric $d : \Gamma \times \Gamma \to \mathbb{R}$ *is given by*

$$d(x,y) := \min \{|P| \mid \iota(P) = x,\ \tau(P) = y\}.$$

A path P *is called a* geodesic *if* $|P| = d(\iota(P), \tau(P))$.

In the remainder of the document we will implicitly assume that any graph is a metric space equipped with the shortest path metric. We will denote the set of geodesics between any two points x and y by $[\![x,y]\!]$.

To consider finitely generated groups as metric spaces, we introduce Cayley graphs.

Definition 2.1.2. Cayley graphs of finitely generated groups

Let G *be a finitely generated group with finite generating set* S, *with the properties that* S *is symmetric with respect to inversion and does not contain the identity element. The* Cayley graph $\mathrm{Cay}(G,S)$ *of* G *with respect to* S *is the simplicial graph with vertex set* G *and undirected edge set*

$$E := \{gh \mid g,h \in G,\ h = gs \text{ for some } s \in S\}.$$

We consider Cayley graphs as metric spaces by assigning the shortest path metric, which coincides with the word metric d_S *on* S. *The* length *of an element* $g \in G$ *is given by* $l_S(g) := d_S(\mathrm{id}_G, g)$.

A metric space X *is called a* Cayley graph *of* G *if there is some finite generating set* S *of* G *such that* X *is isometric to* $\mathrm{Cay}(G,S)$.

As our results concern the large-scale geometry of metric spaces, we consider spaces to be equivalent if they are quasi-isometric.

Definition 2.1.3. Quasi-isometries

Let (X,d) *and* (Y,d') *be metric spaces. If there exists a map* $\phi : X \to Y$ *and constants* $K \geq 1$, $C \geq 0$ *such that for all* $x_1, x_2 \in X$,

$$K^{-1} d(x_1,x_2) - C \leq d'(\phi(x_1), \phi(x_2)) \leq K d(x_1,x_2) + C,$$

and for every $y \in Y$ *there is some* $x \in X$ *such that* $d'(\phi(x), y) \leq C$, *then we say that* X *and* Y *are* quasi-isometric *and that* ϕ *is a* (K,C) quasi-isometry.

For any given finitely generated group G, all Cayley graphs of G are quasi-isometric, so it now makes sense to refer to the group G as a metric space, where we understand that the choice of Cayley graph does not matter up to quasi-isometry.

Two major facets of geometric group theory are to determine when two finitely generated groups are quasi-isometric, and to study the properties shared by quasi-isometric groups.

Finally in this section, we give the definitions of bounded geometry and uniformly discrete spaces.

Definition 2.1.4. Bounded geometry

Let (X,d) be a discrete metric space. X is said to have bounded geometry *if for every $k>0$ there is some constant $N(k)$ such that for every $x \in X$,*

$$|B(x;k)| \leq N(k),$$

where $B(x;k) := \{y \in X \mid d(x,y) < k\}$.

Definition 2.1.5. Uniformly discrete

A metric space (X,d) is uniformly discrete *if*

$$\inf\{d(x,y) \mid x \neq y\} > 0.$$

The 0-skeleton of every Cayley graph of a finitely generated group has bounded geometry and is uniformly discrete.

Now that we have introduced groups as metric spaces, we move onto the metric spaces we wish to consider embeddings into, starting with Banach spaces.

2.1.2 Banach spaces

Stefan Banach first introduced these spaces during the 1920's. Together with Hans Hahn and Eduard Helly, he made a systematic study of them which appears in [Ban93]. Since then they have become fundamental objects in functional analysis. We first recall the definition.

Definition 2.1.6. Banach spaces

Let X be a real vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm *on X if*

(i) $\|x\| \geq 0$ *for all $x \in X$, with equality if and only if $x = 0$,*

(ii) *for all* $\lambda \in \mathbb{R}$ *and* $x \in X$*,* $\|\lambda x\| = |\lambda| \, \|x\|$*,*

(iii) *for all* $x, y, z \in X$*,* $\|x - y\| + \|y - z\| \geq \|x - z\|$*.*

A pair $(X, \|\cdot\|)$ *where* X *is a vector space and* $\|\cdot\|$ *is a norm on* X *is called a* normed vector space*.*

A complete normed vector space $(X, \|\cdot\|)$ *is called a* Banach space*.*

Before proceeding, we introduce an important class of examples.

Example 2.1.7. **ℓ^p and L^p spaces**

Given a countable set X and some $p \in [1, \infty)$, $\ell^p(X)$ is defined to be the set of all functions $f : X \to \mathbb{R}$ such that

$$\|f\|_p := \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} < \infty.$$

Furthermore, we define $\ell^\infty(X)$ to be the set of all functions $f : X \to \mathbb{R}$ such that

$$\|f\|_\infty := \sup_{x \in X} |f(x)| < \infty.$$

The spaces $(\ell^p(X), \|\cdot\|_p)$ are Banach spaces. Notice that $\ell^p(X)$ is isomorphic to $\ell^q(Y)$ if and only if $|X| = |Y| \in \{0, 1\}$ or there exists a bijection $\psi : X \to Y$ and $p = q$.

Generalising this, we define $L^p([0,1])$ to be the set of all functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

is defined and finite. Again, we define $L^\infty([0,1])$ to be the collection of all functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)| < \infty.$$

Again $(L^p([0,1]), \|\cdot\|_p)$ is a Banach space.

These spaces are related by the following proposition.

Proposition 2.1.8. *Let* $X = \{x_0, x_1, \dots\}$ *be a countable set. For every* $p \in [1, \infty]$ *there is an isometric embedding of* $\ell^p(X)$ *into* $L^p([0,1])$*.*

Proof: We begin with the case $p \neq \infty$. Consider $\phi : \ell^p(X) \hookrightarrow L^p([0,1])$ as the linear extension of the mapping $x_n \mapsto f_n$ given by

$$f_n(x) = \begin{cases} 2^{\frac{n+1}{p}} & \text{if } x \in (2^{-(n+1)}, 2^{-n}) \\ 0 & \text{otherwise} \end{cases}$$

We now prove this is an isometric embedding. Firstly we notice that it is certainly injective.

Now consider $y = \sum_n a_n x_n \in \ell^p(X)$,

$$\begin{aligned} \|\phi(y)\|_p^p &= \left\|\sum_n a_n f_n\right\|_p^p \\ &= \int_0^1 \sum_n |a_n f_n(x)|^p \, dx = \sum_n |a_n|^p \int_0^1 |f_n(x)|^p \, dx \\ &= \sum_n |a_n|^p \, \|f_n\|_p^p = \|y\|_p^p, \end{aligned}$$

where the final step follows from the observation:

$$\|f_n\|_p := \left(\int_{2^{-(n+1)}}^{2^{-n}} \left(2^{\frac{n+1}{p}}\right)^p dx \right)^{\frac{1}{p}} = 1.$$

Moving on to the case $p = \infty$, we achieve the same result by defining

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (2^{-(n+1)}, 2^{-n}) \\ 0 & \text{otherwise.} \end{cases}$$

□

Since our eventual goal is to obtain useful consequences for metric spaces by considering embeddings into Banach spaces it is necessary to choose a target space carefully, as illustrated by the following proposition.

Proposition 2.1.9. *Every countable metric space X isometrically embeds into $\ell^\infty(X)$.*

Proof: Fix a basepoint $e \in X$ and consider the following embedding $\phi : X \to \ell^\infty(X)$:

$$(\phi(x))(y) = d(x,y) - d(y,e).$$

Notice that $\|\phi(x)\|_\infty := \sup_{y \in X} \{|d(x,y) - d(y,e)|\} = d(x,e) < \infty$, by the triangle inequality, so $\phi(x)$ is certainly an element of $\ell^\infty(X)$.

Moreover,

$$\|\phi(x_1) - \phi(x_2)\|_\infty := \sup_{y \in X} \{|d(x_1, y) - d(x_2, y)|\} = d(x_1, x_2).$$

The upper bound on $\|\phi(x_1) - \phi(x_2)\|_\infty$ is due to the triangle inequality, while the lower bound is obtained by setting y to be x_1 or x_2. □

One simple condition which can be imposed on Banach spaces to avoid the above situation is to ask that they are reflexive.

Definition 2.1.10. Reflexive Banach spaces

A Banach space $(X, \|\cdot\|)$ *is* reflexive *if and only if the natural inclusion of* X *into its double dual*

$$\iota : X \to X'' \text{ defined by } (\iota(x))(f) = f(x)$$

is surjective.

One subtle point about this definition is that it is not sufficient to ask merely that there exists a linear (isometric) isomorphism between X and X'', indeed James provides an example of a non-reflexive space with this property [Jam51].

For reasons which become apparent later in this chapter, (cf. Theorem 2.3.3), we wish to restrict our attention even further. With this in mind we will now define uniformly convex spaces - first studied by Clarkson [Cla36] - and super-reflexive spaces, introduced by James, though the definition we present is due to Enflo [Jam72, Enf72].

Definition 2.1.11. Uniformly convex and super-reflexive Banach spaces

A Banach space $(X, \|\cdot\|)$ *is* uniformly convex *if for each* $\epsilon > 0$ *there exists some* $\delta > 0$ *such that given* $x, y \in X$ *with* $\|x\| = \|y\| = 1$

$$\left\|\frac{x+y}{2}\right\| \le 1 - \delta \quad \text{whenever} \quad \|x - y\| \ge \epsilon.$$

A Banach space $(X, \|\cdot\|)$ *is* super-reflexive *if it admits a biLipschitz equivalent norm* $\|\cdot\|_0$ *such that* $(X, \|\cdot\|_0)$ *is uniformly convex.*

The Milman-Pettis Theorem [Mil38, Pet39] ensures that every uniformly convex Banach space is reflexive. It follows more readily from the original definition of James that super-reflexive Banach spaces are reflexive.

Returning to Example 2.1.7, Hanner's inequalities

$$2^p \left(\|f\|_p^p + \|g\|_p^p\right) \ge \left(\|f+g\|_p + \|f-g\|_p\right)^p + \left|\|f+g\|_p - \|f-g\|_p\right|^p, \text{ for } p \in [1,2] \text{ and}$$

$$\|f+g\|_p^p + \|f-g\|_p^p \leq \left(\|f\|_p + \|g\|_p\right)^p + \left|\|f\|_p - \|g\|_p\right|^p, \text{ for } p \in [2,\infty)$$

prove that $L^p([0,1])$ is uniformly convex when $p \in (1,\infty)$ and give optimal values of δ in terms of ϵ, simplifying the earlier proof of Clarke [Cla36, Han56]. Uniform convexity passes to subspaces, so all ℓ^p sequence spaces with $p \in (1,\infty)$ are also uniformly convex, and hence super-reflexive.

The extremal cases $L^1([0,1])$ and $L^\infty([0,1])$ are not reflexive, while the spaces $\ell^1(X)$ and $\ell^\infty(X)$ are reflexive if and only if X is finite, in which case they are super-reflexive, as all norms on finite dimensional vector spaces are biLipschitz equivalent, but they are not uniformly convex.

In the following figure, we present counter-examples to uniform convexity - x_1, y_1 and x_∞, y_∞ - in the cases of ℓ^1 and ℓ^∞ spaces, while x_2, y_2 gives the the corresponding situation in ℓ^2.

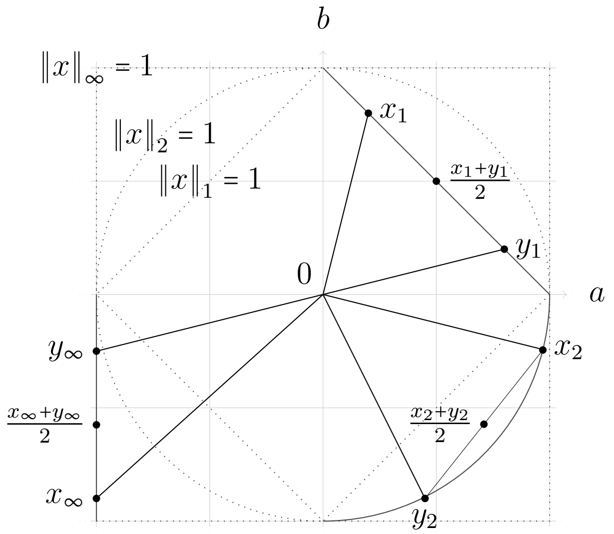

Figure 2.1: Uniform convexity for $\ell^p(\{a,b\})$ with $p \in \{1,2,\infty\}$

One collection of examples of infinite dimensional reflexive Banach spaces which are not super-reflexive, is the Hilbertian sum

$$\bigoplus_{i=1}^{\infty} \ell^{p_i}(N_i)$$

where $\{p_i\}_{i=1}^\infty$ is an increasing real sequence, with $p_1 \geq 1$ and $p_i \to \infty$ as $i \to \infty$ and each N_i is a non-empty finite set.

The optimality of the constants in Hanner's inequalities is sufficient to prove that such spaces are not uniformly convex, since although each $\ell^{p_i}(N_i)$ is uniformly convex, for fixed ϵ the constants $\delta(p,\epsilon)$ tend to 0 as $p \to \infty$.

The other spaces we will wish to consider embeddings into are finite products of metric trees; such spaces are also the first step towards a definition of asymptotically tree-graded spaces.

2.1.3 Trees and quasi-trees

The first class of metric spaces we consider are trees and those metric spaces quasi-isometric to a tree.

Definition 2.1.12. Trees and quasi-trees

A tree (T,d) *is a geodesic metric space satisfying the four point property:*

$$d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t),\ d(x,t) + d(y,z)\} \tag{2.1}$$

for all $x,y,z,t \in T$.

A geodesic metric space (Q,d) *is called a* quasi-tree *if it is quasi-isometric to a tree in the sense of* Definition 2.1.3.

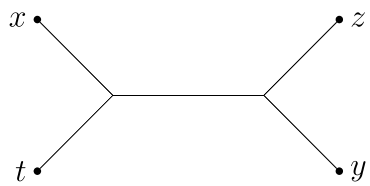

Figure 2.2: The four point property

One worthwhile point to note here is that the four point property clearly forbids the existence of geodesic loops, as shown by the next figure.

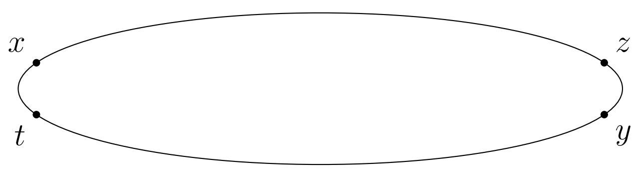

Figure 2.3: Loops fail the four point property

The main examples of trees considered here are simplicial trees, for instance the binary tree of depth k, which has as its vertex set the collection of all binary strings of length at most $k-1$ and edges ab if and only if $b = a0$ or $b = a1$.

Another example is the Cayley graph of a free group with respect to a minimal generating set. To ensure we consider this as a property of the group rather than just a single Cayley graph it can be preferable to work with quasi-trees.

Before continuing, we give one non-trivial example of a quasi-tree.

Example 2.1.13. The Farey graph

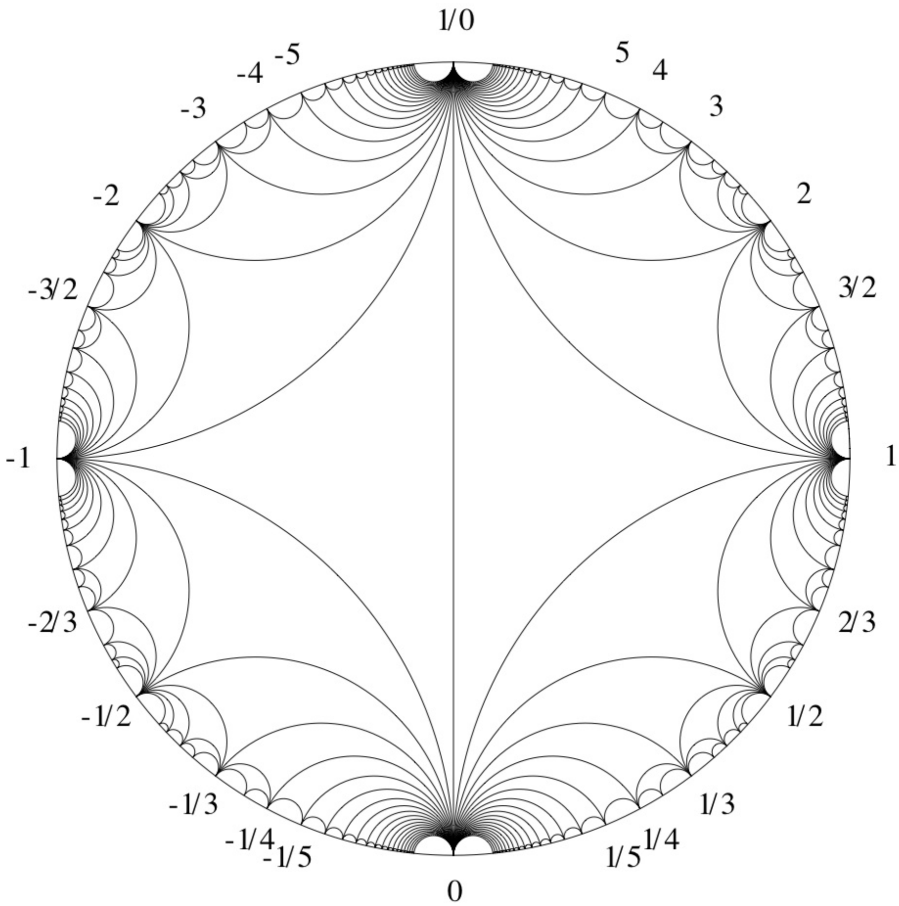

Figure 2.4: The Farey graph

The Farey graph $\mathcal{F}$ is a simplicial graph with vertex set

$$V \coloneqq \{(z,n) \mid z \in \mathbb{Z} \setminus \{0\},\ n \in \mathbb{N} \setminus \{0\}\} \cup \{(1,0),(0,1)\}$$

and (undirected) edge set $E = \bigcup_{n\in\mathbb{N}} E_n$ constructed as follows.

$E_0 \coloneqq \{(1,0)(0,1)\}$ and E_n is the union of E_{n-1} with the set of all $(z_0,n_0)(z_1,n_1)$ and $(z_1,n_1)(z_2,n_2)$ such that

- (z_0,n_0), (z_1,n_1) and (z_2,n_2) are distinct,

- $(z_0, n_0) \neq (1,0)$,
- $(z_0, n_0)(z_2, n_2) \in E_{n-1}$ and either

$$\begin{aligned} &(z_2,n_2) = (1,0) \quad \text{and} \quad n_1 = n_0,\ z_1 = z_0 \pm 1 \qquad\qquad \text{or} \\ &(z_2,n_2) \neq (1,0) \quad \text{and} \quad n_1 = n_0 + n_2,\ z_1 = z_0 + z_2. \end{aligned}$$

The first explicit occurrence of this definition in the literature seems to be in a paper of Hurwitz, on the reduction of quadratic binary forms [Hur94]. Figure 2.4 - obtained from [Ago] - illustrates this definition; it should be mentioned that the outer circle is not part of the Farey graph.

We sketch a proof that the Farey graph is a quasi-tree. Consider the tree $\mathbb{T}$ obtained from $\mathcal{F}$ by carrying out the following procedure.

- Replace each vertex $(z,n) \in V$ by two vertices $(z,n)^{\pm}$, (we will ignore $(0,1)^-$ and $(1,0)^-$).
- Replace each edge $(z_1,n_1)(z_2,n_2) \in E \smallsetminus E_0$ by

$$\begin{aligned} &(z_1,n_1)^+(1,0)^+ \qquad \text{if} \quad (z_2,n_2) = (1,0), \text{ or} \\ &(z_1,n_1)^+(z_2,n_2)^- \quad \text{if} \quad n_1, n_2 \neq 0 \text{ and } z_1/n_1 < z_2/n_2. \end{aligned}$$

The tree obtained from this procedure is simplicial and every vertex has countably infinite degree.

We define d and d' to be the shortest path metrics on $\mathcal{F}$ and $\mathbb{T}$ respectively, notice that $d'((z,n)^+,(z,n)^-) \leq 3$ for all $z \in \mathbb{Z} \smallsetminus \{0\}$ and $n \in \mathbb{N} \smallsetminus \{0\}$. This can be seen by traversing the unique triangle in $\mathcal{F}$ where (z,n) is the middle vertex by vertical co-ordinate in Figure 2.4.

The map $\phi : (\mathcal{F}, d) \to (\mathbb{T}, d')$ defined by $(z,n) \to (z,n)^+$ satisfies the inequalities

$$d(t_1,t_2) \leq d'(\phi(t_1),\phi(t_2)) \leq 4d(t_1,t_2).$$

The first of these is clear from the construction, for the second given any edge $(z,n)(z',n')$ in $\mathcal{F}$ with $|z/n| > |z'/n'|$ there is a path of length at most 3 from $(z,n)^+$ to $(z,n)^-$ and an edge connecting $(z,n)^-$ to $(z',n')^+$ in $\mathbb{T}$,

One more important collection of quasi-trees - projection complexes - are introduced in Section 2.1.8.

We finish this section with the following theorem of Manning, which classifies quasi-trees and provides a key motivation for the results obtained in Chapter 5.

Theorem 2.1.14. Manning's Bottleneck Property [Man05]

A geodesic metric space X is a quasi-tree if and only if there exists some constant Δ such that given

- *any two points x and $y \in X$*
- *any geodesic $\underline{g}$ from x to y with midpoint m*

every path from x to y in X intersects $B(m;\Delta) \coloneqq \{z \in X \mid d(z,m) < \Delta\}$.

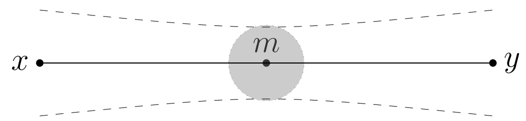

Figure 2.5: Manning's Bottleneck Property

In particular, this says that every tree in the sense of Definition 2.1.12 is quasi-isometric to a simplicial tree.

The constant Δ is called the *bottleneck constant* of X. We sketch the construction of a simplicial tree $\mathbb{T}$ which is quasi-isometric to X.

The figure below demonstrates the first few steps of this construction.

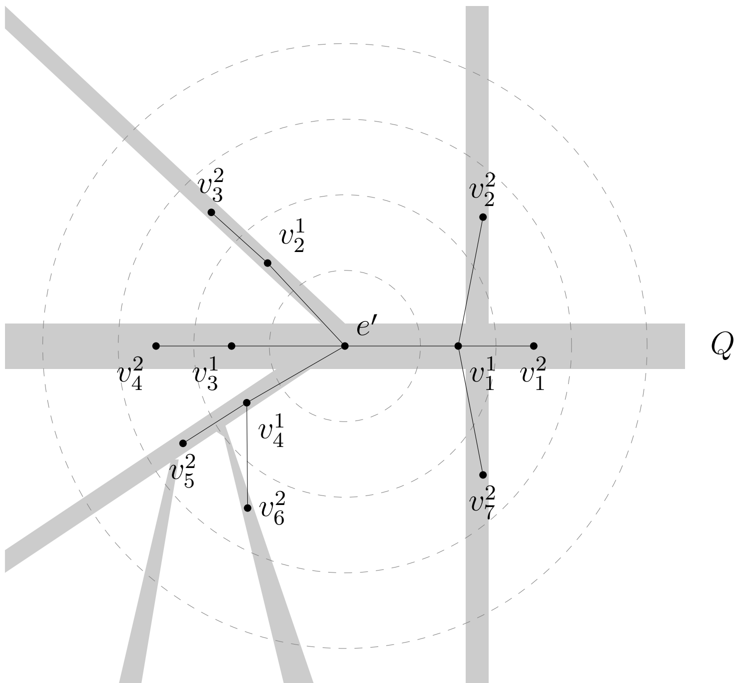

Figure 2.6: The construction of a tree $\mathbb{T}$ quasi-isometric to X

Proof: Choose $M \in \mathbb{N}$ with $M \geq 20\Delta$ and fix some basepoint $e \in X$. Denote the set of connected components of $X \setminus B(e; kM)$ by $\mathcal{C}_k$.

The tree $\mathbb{T} = \bigcup_{k \in \mathbb{N}} \mathbb{T}_k$ is constructed as follows

- $\mathbb{T}_0 := \{e\}$,
- $\mathbb{T}_1$ is obtained from $\mathbb{T}_0$ by adding a single vertex v_C for each $C \in \mathcal{C}_1$ and connecting it to e by a simplicial path of length M,
- finally, we build $\mathbb{T}_k$ from $\mathbb{T}_{k-1}$ by adding a vertex $v_{C'}$ for each $C' \in \mathcal{C}_k$ and connecting it to v_C by a simplicial path of length M, where C is the unique element of $\mathcal{C}_{k-1}$ such that $C' \subset C$.

2.1.4 Hyperbolic groups and spaces

The metric definition of hyperbolicity, introduced by Gromov [Gro87] as a generalisation of negative curvature for Riemannian surfaces, is now a key concept in geometric group theory, with a wealth of powerful consequences obtained for finitely generated groups whose Cayley graphs satisfy this metric property. The definition we give now is the 'slim triangles' condition, generally attributed to Rips.

Given a metric space (X, d) and two points $x, y \in X$ we denote the set of all geodesics from x to y in X by $[[x, y]]$.

Definition 2.1.15. Gromov hyperbolicity

A geodesic metric space (X, d) *is* δ-hyperbolic *if, given any three points* $x, y, z \in X$ *and any three geodesics* $\underline{g_1} \in [[x, y]]$, $\underline{g_2} \in [[y, z]]$ *and* $\underline{g_3} \in [[x, z]]$,

$$\underline{g_1} \subseteq N_\delta\left(\underline{g_2} \cup \underline{g_3}\right),$$

where $N_\delta(Y) := \{x \in X \mid d(x, Y) < \delta\}$.

A geodesic metric space (X, d) *is said to be* Gromov hyperbolic *(or just hyperbolic) if it is* δ*-hyperbolic for some* δ*. A finitely generated group* G *is* hyperbolic *if some (equivalently all) of its Cayley graphs are.*

Such a triple $(\underline{g_1}, \underline{g_2}, \underline{g_3})$ will be referred to as a *geodesic triangle.*

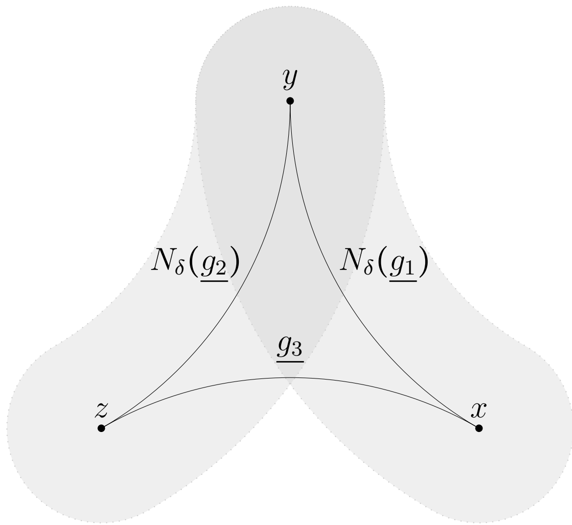

Figure 2.7: A δ-slim triangle

We now present some examples.

Example 2.1.16. Hyperbolic spaces

(i) A geodesic metric space is a tree if and only if it is 0-hyperbolic. This is easily proved by showing the four-point condition (cf. Definition 2.1.12) is equivalent to having $\underline{g_1} \subseteq \left(\underline{g_2} \cup \underline{g_3}\right)$ for all geodesic triangles $\left(\underline{g_1}, \underline{g_2}, \underline{g_3}\right)$. Moreover, all quasi-trees are δ-hyperbolic, where δ depends only on the bottleneck constant Δ from Theorem 2.1.14.

(ii) There are many hyperbolic groups whose Cayley graphs are not quasi-trees, indeed this is true of all one-ended hyperbolic groups, for instance fundamental groups of Riemannian manifolds with strictly negative curvature and hence of fundamental groups of surfaces with non-negative complexity [Gro87]. The second of these examples is explored in more detail in Section 2.1.7.

(iii) Three key classes of hyperbolic spaces: coned-off graphs of relatively hyperbolic groups, curve complexes and projection complexes - which do not (necessarily) have bounded geometry - will be introduced in Sections 2.1.6, 2.1.7 and 2.1.8 respectively.

To finish this section we give a different definition of hyperbolicity in terms of Dehn functions, which we now define.

Let $G = \langle S \,|\, R \rangle$ be a finitely presented group, then there is a natural surjection

$$\pi : \mathbb{F}(S) \to G \quad \text{with kernel} \quad \langle\!\langle R \rangle\!\rangle \,.$$

Therefore, every $w \in \mathbb{F}(S)$ which represents the identity in G can be written as a finite product of G-conjugates of elements of R:

$$w \underset{\mathbb{F}(S)}{=} \prod_{i=1}^{n} g_i^{-1} r_i g_i, \quad \text{where each} \quad g_i \in G \quad \text{and} \quad r_i \in R.$$

The *area* of w, $A(w)$ is the minimum value of n such that w can be written in the above form.

We then define the *Dehn function* of $G = \langle S \,|\, R \rangle$ as

$$D(n) := \max \left\{ A(w) \;\middle|\; l_{\mathbb{F}(S)}(w) \leq n, \; w \underset{G}{=} 1 \right\}.$$

To regard the Dehn function as an object dependent on a group rather than a given presentation we appeal to the following proposition.

Proposition 2.1.17. ([Alo90], a full proof in English can be found in [BH99])

Let $G = \langle S \,|\, R \rangle$ and $H = \langle S' \,|\, R' \rangle$ be finite presentations of quasi-isometric groups G and H. Define the Dehn functions of $\langle S \,|\, R \rangle$ and $\langle S' \,|\, R' \rangle$ as D and D' respectively.

There exist constants $a \geq 1$ and $b \geq 0$ such that

$$D(n) \leq aD'(an + b) + an + b \quad \text{and} \quad D'(n) \leq aD(an + b) + an + b.$$

An equivalent definition of hyperbolicity for finitely generated groups can be given in terms of Dehn functions, before this we require the following theorem.

Theorem 2.1.18. *Hyperbolic groups are finitely presented.*

This is attributed to Rips. We present a method of proof here, omitting some details. For a complete proof see [CDP90, Théorème 2.3]. We give a full proof of this in Proposition 4.6.6.

Let G be a finitely generated group and let S be a fixed finite generating set for G. For each natural number n we set

$$X_n := \{ g \in G \;|\; d_S(1, g) \leq n \} \,,$$

so X_n is the closed ball of radius n around the identity in the Cayley graph of G with respect to S.

For each collection X_n we define a collection of relations

$$R_n := \{xyz \in F(S) \mid x, y, z \in X_n \text{ and } xyz =_G 1\} \cup \left\{xx^{-1} \mid x \in X_n\right\}.$$

Despite the fact that the second collection of relations hold automatically, it is necessary to include them here.

If we now define $G_n = \langle X_n | R_n \rangle$ we obtain a sequence of finitely presented groups

$$G_1 \to G_2 \to \cdots \to G_\infty = G.$$

where each of these maps is a surjective homomorphism. It remains to show that under additional assumptions - hyperbolicity is more than sufficient [Bri93] - the homomorphism $G_N \to G_{N+1}$ is injective for all N sufficiently large, so $G = G_N$ and is finitely presented. □

Hyperbolicity for a finitely presented group is completely determined by its Dehn function.

Theorem 2.1.19. *All hyperbolic groups have linear Dehn function. In addition, a finitely presented group G is Gromov hyperbolic if and only if it has a subquadratic Dehn function with respect to some (equivalently all) finite presentations.*

Proof: See [Gro87, Ol′91, Bow95]. □

Moreover, this is the only gap in the isoperimetric spectrum [BB00].

We now move on to a different generalisation of trees.

2.1.5 Tree-graded spaces and Stallings' theorem

Heuristically, a tree-graded space is a metric space made up of a collection of pieces arranged in a tree-like manner. More formally, the notion is defined as follows:

Definition 2.1.20. Tree-graded spaces

Let X be a geodesic metric space and let $\mathcal{X} = \{X_i \mid i \in I\}$ be a collection of subsets of X. The space X is said to be tree-graded *with respect to $\mathcal{X}$ if and only if*

(i) *$\mathcal{X}$ covers X: that is, $\bigcup_{i \in I} X_i = X$,*

(ii) *pieces meet in at most one point: given $i, j \in I$ with $i \neq j$, $|X_i \cap X_j| \leq 1$,*

(iii) *the arrangement of pieces is 'tree-like': every simple geodesic triangle is contained in a single piece.*

The following figure illustrates this definition.

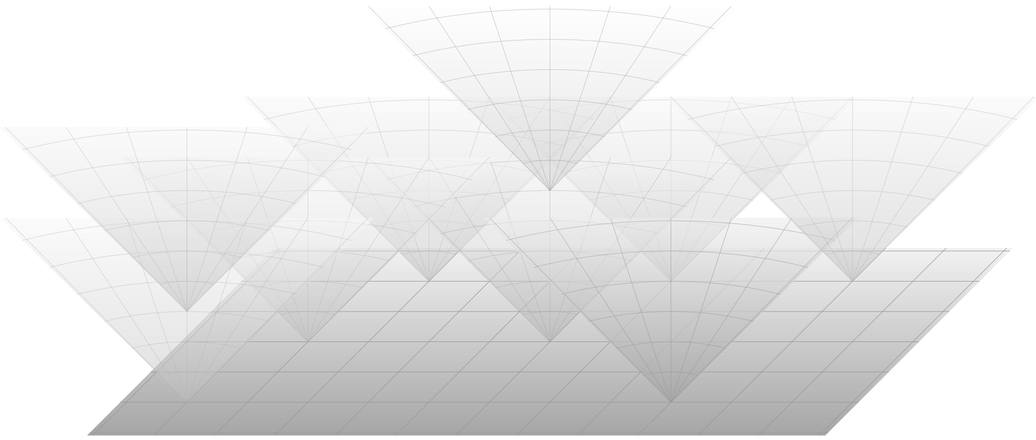

Figure 2.8: A tree-graded space

One important example of this is a free product of finitely generated groups. Let G and H be finitely generated groups with finite generating sets S_G and S_H respectively. Then the Cayley graph of $G * H$ with respect to the generating set $S_G \sqcup S_H$ is tree-graded with respect to the collection of $G * H$ translates of the Cayley graphs of G and H with respect to S_G and S_H respectively.

Stallings' Theorem is a strong converse of the above fact.

Theorem 2.1.21. Stallings' Theorem [Sta68, Sta71]

Let G be an infinite finitely generated group. If a Cayley graph X of G admits a bottleneck*, that is, there exists some metric ball B in X such that $X \setminus B$ is disconnected then one of the following holds:*

(i) *G splits as an amalgamated product $A *_C B$ where C is a finite subgroup of A and B, $A \neq C$ and $B \neq C$,*

(ii) *G is a HNN extension $G = \mathrm{HNN}(A, C, \theta) := \langle A, t \mid t^{-1}Ct = \theta(C) \rangle$, where C is a finite subgroups of A and $\theta : C \to A$ is an isomorphism onto its image.*

The original notion of Stallings was that of *relative ends*. Phrasing this theorem in terms of bottlenecks will link it closely with the results of Chapter 5, which proves a metric analogue of Stallings' Theorem.

It follows from results of Gromov and Papazoglou-Whyte that any Cayley graph of a group satisfying the hypotheses of Stallings' Theorem is quasi-isometric to a Cayley graph of a free product [Gro87, PW02].

Another example comes from a tree-grading of $\mathbb{R}^3$ equipped with the metric

$$d_Z((x,y,z),(x',y',z')) = \begin{cases} |x-x'| + |y-y'| & \text{if} \quad z = z' \\ |z-z'| + x + x' + y + y' & \text{if} \quad z \neq z' \end{cases}$$

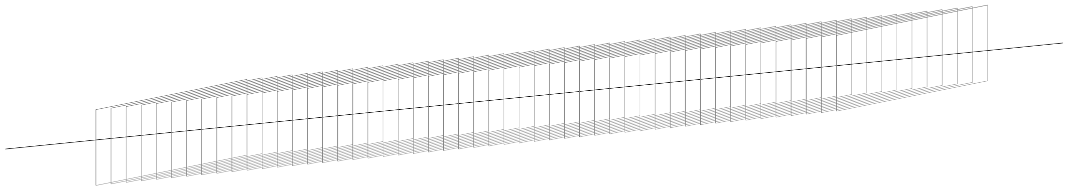

Figure 2.9: A tree-grading of $\mathbb{R}^3$

With respect to the metric d_Z, $\mathbb{R}^3$ is tree-graded with respect to the collection of hyperplanes

$$\{X_z := \{(x,y,z) \mid x,y \in X\} \mid z \in \mathbb{R}\},$$

each one of which is equipped with the usual ℓ^1 metric.

A more detailed study of the shortest-path metric on tree-graded graphs is carried out in Section 4.4, while Theorem 7 gives a characterisation of spaces quasi-isometric to tree-graded graphs.

Extending the previous two sections, we continue with a natural class of spaces which includes both hyperbolic and tree-graded spaces.

2.1.6 Relatively hyperbolic groups and asymptotically tree-graded spaces

Relatively hyperbolic groups were introduced by Gromov as a generalisation of hyperbolic and geometrically finite Kleinian groups (cf. Example 2.1.28(ii)) [Gro87].

Relative hyperbolicity has many different characterisations: in terms of group actions [Bow12], group-theoretic structure [Far98] [Dah03b] [Osi06], dynamics on the boundary [Yam04] and metric geometry [DS05].

We will present approaches of Bowditch and Druţu-Sapir, both of which will be necessary at different points in future chapters. For the first of these, we require the definitions of the coned-off graph and fine graphs.

Definition 2.1.22. Coned-off graphs

Let Γ be a simplicial graph and let $\{\Gamma_i \mid i \in I\}$ be a collection of (not necessarily *disjoint) subgraphs of* Γ. *The* coned-off graph *of* Γ *with respect to* $\{\Gamma_i \mid i \in I\}$ *is the graph* $\widehat{\Gamma}$ *with vertex set* $V(\Gamma) \sqcup I$ *and edge set*

$$\{xy \mid x, y \in X \text{ and } xy \in E(\Gamma)\} \cup \{xi \mid x \in X,\ i \in I \text{ and } x \in \Gamma_i\}$$

When the collection of subgraphs $\{\Gamma_i\}$ is clear from the context we will refer simply to the coned-off graph of Γ.

Definition 2.1.23. Fine graphs

Let Γ *be a simplicial graph.* Γ *is called* fine *if, for each* $n \in \mathbb{N}$*, every edge of* Γ *is contained in only finitely many simple loops (paths* P *such that no two edges in* P *have the same initial vertex and* $\iota(P) = \tau(P)$*) of length at most* n.

All locally finite graphs are fine. The coned-off graph of $\mathrm{Cay}(\mathbb{Z}^2, \{a, b\})$ with respect to the subgraphs corresponding to the cosets of $\langle b \rangle$ is a quasi-tree but is not fine, since for every $n \geq 1$ there is a simple loop $\{id \to a, a \to ab^n, ab^n \to b^n, b^n \to id\}$ of length 6.

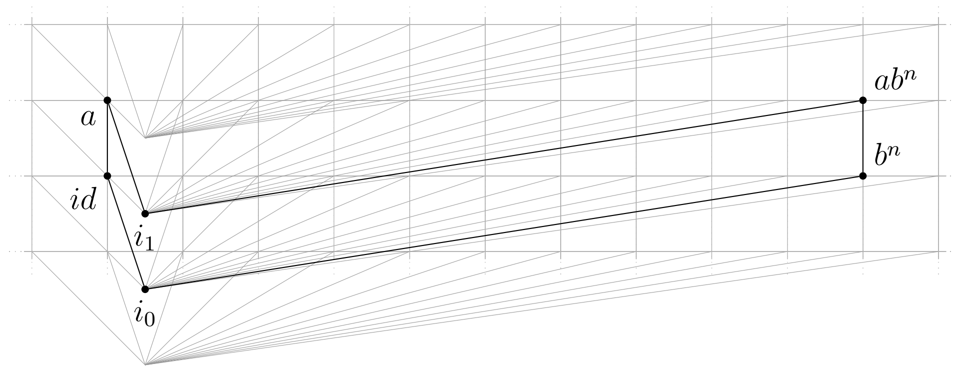

Figure 2.10: A quasi-tree which is not fine

It is important to note that the requirement that such loops are simple is necessary. For instance, taking the coned-off graph of the Cayley graph of $\mathbb{Z} \cong \langle a \rangle$ with respect to itself is fine, but the collection of loops

$$0 \to 1, 1 \to i, i \to n, n \to n+1, n+1 \to i, i \to 0$$

all have length 6.

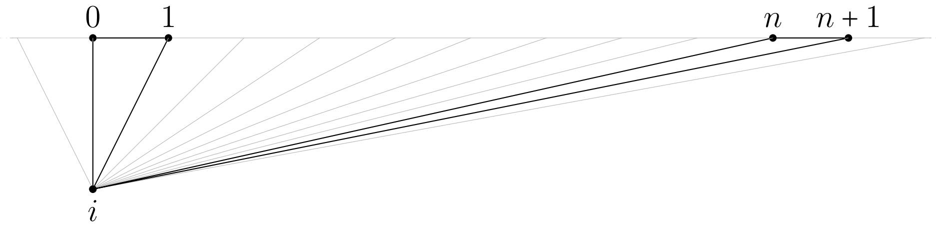

Figure 2.11: A coned-off copy of $\mathbb{Z}$

We now have all the terminology necessary to introduce the first definition of relative hyperbolicity for groups.

Definition 2.1.24. Relatively hyperbolic groups [Bow12]

A pair $(G, \{H_1, \ldots, H_n\})$ *is* relatively hyperbolic*, and* G *is called a* relatively hyperbolic group*, if* G *is a finitely generated group, each* H_i *is a subgroup of* G *and the coned-off graph* $\widehat{G}$ *of a Cayley graph of* G *with respect to the collection of all* H_i *cosets in* G *is hyperbolic and fine.*

With this definition it is easy to see that any hyperbolic group is hyperbolic relative to the trivial subgroup. $\mathbb{Z}^2$ is not hyperbolic relative to $\mathbb{Z}$ as mentioned previously: although the coned-off graph is a quasi-tree, and hence hyperbolic, it is not fine.

Example 2.1.25. Free products

In this example we prove that a free product of finitely generated groups $G = A * B$ is hyperbolic relative to $\{A, B\}$ in the sense of Definition 2.1.24.

We equip G with a finite generating set $S_A \sqcup S_B$, where S_A and S_B are finite generating sets of A and B respectively. The coned-off graph $\widehat{G}$ of G is a quasi-tree - bottleneck constant $\Delta = 1$ in Theorem 2.1.14 will suffice - so it just remains to prove $\widehat{G}$ is fine.

Fix some $n \in \mathbb{N}$ and let e be any edge in $\widehat{G}$. We show there are only finitely many simple loops in $\widehat{G}$ of length n containing e.

Firstly, we assume both end vertices of e, labelled x and y, lie in G. If x and y do not lie in a common coset of A or B in G then there are no simple loops in $\widehat{G}$ containing e. As every simple loop in G is contained in a piece, any simple loop containing e only meets the coset, gA say, containing e and its cone-point v_{gA}. A simple loop P meets v_{gA} at most once so if we write $B_n^G(x) \coloneqq \{zz' \in E(G) \mid d(z,x), d(z',x) \leq n-2\}$, we see that

$$P \subseteq B_n^G(x) \cup \left\{ v_{gA}z \;\middle|\; z \in B_n^G(x) \right\}$$

and hence there are only finitely many possible choices of P, as A and B have Cayley graphs with bounded geometry. The case where $e = v_{gA}z$ for some coset gA can be handled in a similar manner. □

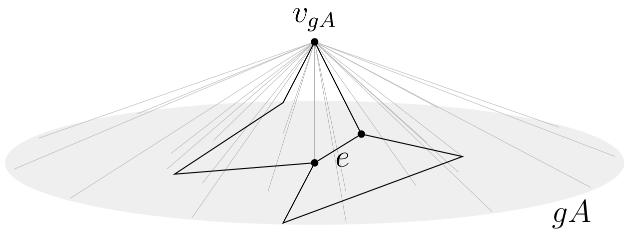

Figure 2.12: Simple loops in $\widehat{G}$ containing e

When introducing hyperbolicity (Definition 2.1.15), we first presented a metric definition and then applied it to Cayley graphs of groups. The analogous approach for relatively hyperbolic groups is given by the notion of asymptotically tree-graded spaces. One of the major conditions is to ensure that any non-hyperbolic behaviour (like 'fat' triangles) is contained in the collection of pieces in some way.

Definition 2.1.26. Asymptotically tree-graded spaces [DS05]

Let (X,d) be a geodesic metric space and let $\mathcal{A} = \{A_i \mid i \in I\}$ be a collection of subsets of X. We say X is asymptotically tree-graded *with respect to $\mathcal{A}$ if and only if the following three conditions are satisfied.*

(i) *For every $\delta > 0$ the diameters of the intersections of δ-neighbourhoods $N_\delta(A_i) \cap N_\delta(A_j)$ are uniformly bounded for all $i,j \in I$ with $i \neq j$.*

(ii) *For every $\theta \in \left[0, \frac{1}{2}\right)$ there exists some constant $M = M(\theta) > 0$ such that for every geodesic $\underline{q}$ of length l and every $A \in \mathcal{A}$ with $\underline{q}(0), \underline{q}(l) \in N_{\theta l}(A)$ we have $\underline{q}([0,l]) \cap N_M(A) \neq \varnothing$.*

(iii) *For every $k \geq 2$ there exists a constant $\chi > 0$ such that every 'sufficiently fat' k-gon P in X with geodesic edges satisfies $P \subset N_\chi(A)$ for some $A \in \mathcal{A}$.*

We will not give the full definition of a fat polygon here, the intuition is that it should be the negation of the slim triangle property (cf. Definition 2.1.15) and its natural extension to polygons. In what follows we will use the terminologies "asymptotically tree-graded space" and "relatively hyperbolic space" synonymously, where by either we are referring to a space satisfying Definition 2.1.26. The main result concerning such spaces is the following:

Theorem 2.1.27. [DS05, Theorem 8.5 and Appendix A]

A finitely generated group G is asymptotically tree-graded with respect to the set of cosets of subgroups $\{gH_i \mid g \in G,\ i = 1, \ldots, m\}$ if and only if G is hyperbolic relative to $\{H_1, \ldots, H_m\}$ in the sense of Definition 2.1.24 and each H_i is finitely generated.

As we have already seen, the class of relatively hyperbolic groups includes hyperbolic groups, and a small extension to Example 2.1.25 proves that amalgamated products $A *_C B$ and HNN-extensions $\mathrm{HNN}(A, C, \theta)$ over finite subgroups are hyperbolic relative to $\{A, B\}$ and $\{A\}$ respectively.

The full class of relatively hyperbolic groups is much richer than this.

Example 2.1.28. More relatively hyperbolic groups

(i) A finitely generated group G is said to be *fully residually free*, if given any finite set $F \subset G \smallsetminus \{id_G\}$ there is a surjection ϕ_F from G onto a free group $\mathbb{F}$, which is injective on F and $id_{\mathbb{F}} \notin \phi_F(F)$. Such groups are also called *limit* groups.

For each n, $\mathbb{Z}^n$ is fully residually free, so these groups are not hyperbolic in general, however, they are hyperbolic relative to a finite collection of maximal free abelian subgroups [Dah03a, Ali05].

These groups are key objects in the solution of the Tarski conjecture [Sel01, KM10].

(ii) A Kleinian group - a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$ - is *geometrically finite* if it admits a fundamental polyhedron (in hyperbolic 3-space) with finitely many sides. The origins of this definition stem from Greenberg, who proved that this is not an automatic consequence of being a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$ [Gre66].

Geometrically finite Kleinian groups are hyperbolic relative to their maximal parabolic (in the sense of linear groups) subgroups. This was a key motivation for the original definition of relative hyperbolicity.

(iii) Fundamental groups of non-geometric closed 3-manifolds with at least one hyperbolic component are hyperbolic relative to the fundamental groups of a finite collection of graph manifolds, tori and Klein bottles [Dah03a]. This will be discussed in greater detail in Section 2.1.9.

We now move on to the other key collection of groups considered in this thesis, mapping class groups.

2.1.7 Surfaces and mapping class groups

Before progressing to the definition of mapping class groups we recall the classification of oriented compact surfaces. Although this was known by both Jordan and Möbius, the first rigorous proof was given by Brahana, building on earlier work of Von Dyck and Dehn-Heegard.

Theorem 2.1.29. Classification of surfaces [Bra21]

Let S be a compact orientable 2-manifold, then there exist unique integers $g, n \geq 0$ such that S is homeomorphic to

$$S_{g,n} \coloneqq \underbrace{(S^1 \times S^1)\# \ldots \#(S^1 \times S^1)}_{g \text{ copies}} \smallsetminus \bigsqcup_{i=1}^{n} D_i$$

where $\#$ denotes the connect sum and $\{D_i \mid i = 1, \ldots, n\}$ is a disjoint collection of open 2-discs.

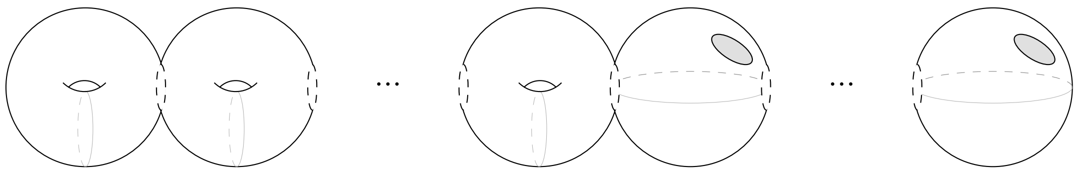

Figure 2.13: The surface $S_{g,n}$

With the classification theorem, we can now define the *Euler characteristic* of a surface S as $\chi(S) \coloneqq 2g + n - 2$ and the *complexity* of S as $\zeta(S) \coloneqq 3g + n - 4$. In what follows we will just identify a surface with its unique homeomorphism class representative $S_{g,n}$.

There are many possible choices of metrics which can be given to the surface $S_{g,n}$. We now give a construction of a very particular metric which is useful in Chapter 3.

Proposition 2.1.30. *Let S be a compact (but not closed) surface with $\zeta(S) \geq -1$ and boundary components $\partial_1, \partial_2, \ldots, \partial_n$, equipped with a Riemannian metric. There exists a biLipschitz equivalent metric d' such that the lengths of boundary components are maintained and there is a metric retraction from the universal cover of (S, d') to a tree T which is an isometry when restricted to any boundary component and such that the distance between each point and its retraction is at most 1.*

The restriction on complexity is necessary as it is clear this cannot possibly hold for any S homeomorphic to $S_{0,2}$ for which the boundary components are of different lengths. The insistence on there being a boundary component ensures that $\pi_1(S)$ is free. Without loss of generality we will assume that the boundary components are ordered so that the length are non-decreasing, so $|\partial_1| \leq |\partial_2| \leq \cdots \leq |\partial_n|$.

We first deal with the easier case of positive genus.

Proof of Proposition 2.1.30 in positive genus cases

We start by considering genus 1. Take a rectangle R of height $h = |\partial_1|/4$ and length $l = \frac{1}{2}\left(\sum_{i=1}^{n} |\partial_i| - 2nh\right) > 0$ and divide it into n rectangles R_i of height h and length $l_i = \frac{1}{2}\left(|\partial_i| - 2h\right)$, so R_i has perimeter $|\partial_i|$.

For each i we let D_i be a closed disc contained in the interior of R_i and give each $R_i \smallsetminus (D_i)^o$ the metric coming from a cylinder of length 1 and boundary perimeter $|\partial_i|$. We then identify the edges of R in the usual way to obtain the required metric surface

$$S = R \smallsetminus \bigsqcup_{i=1}^{n} (D_i)^o \;/ \sim .$$

We set Γ to be image of the union of the boundaries of the R_i in S. This is a metric graph.

This is illustrated by Figure 2.14 below in the specific case of two boundary components.

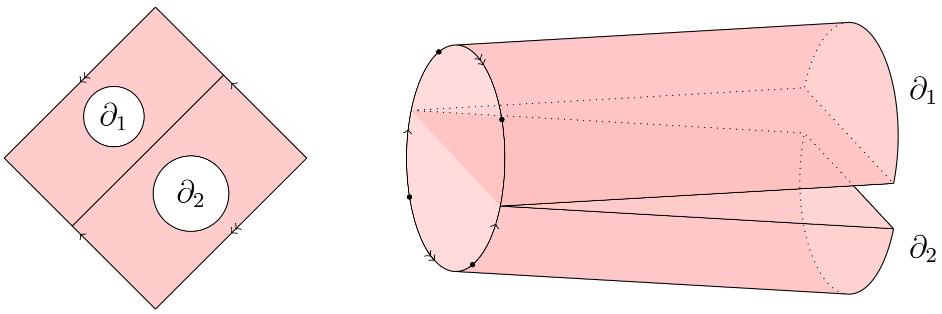

Figure 2.14: A metric surface of genus 1 with 2 boundary components

This process yields an oriented manifold S of genus 1 with n boundary components and an embedded graph Γ.

We now extend to genus $g \geq 2$, dealing initially with the case of 1 boundary component. Take a regular $4g$-gon P of circumference $|\partial_1|$ and let D be a closed disc contained in the interior of P. We give $P \smallsetminus D^o$ the metric of a cylinder of length 1 and boundary perimeter $|\partial_1|$.

Identifying the edges of $P \smallsetminus D^o$ in the usual way, we obtain a metric surface (S, d') of genus g with one boundary component. We set Γ to be the image of the boundary of P in S. In this case, Γ is a metric rose, with $2g$ edges of length $|\partial_1|/4g$.

The case of genus 2 is pictured below in Figure 2.15.

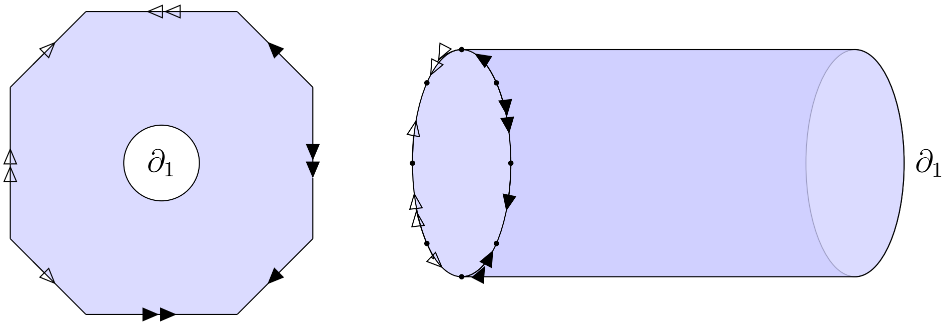

Figure 2.15: Metric surfaces of positive genus with 1 boundary component

The final case of positive genus $(g, n \geq 2)$ is a union of the above two. Start with a regular $4g - 2$-gon P_1 of perimeter $|\partial_1|$ with one marked edge e (of length $|e|$) and a rectangle R of height $h = |e|$ and length $l = \frac{1}{2}\left(\sum_{i=2}^{n} |\partial_i| - 2(n-1)h\right)$, identifying one edge of P_1 with one of the appropriate length in R, to obtain a $4g$-gon P. We divide R into $(n-1)$ rectangles P_i of height h and length $\frac{1}{2}(|\partial_i| - 2h)$.

Let $D_1, \ldots, D_n$ be closed discs contained in the interiors of $P_1, P_2, \ldots, P_n$ respectively. Each region $P_i \smallsetminus (D_i)^o$ is given the metric of a cylinder of height 1 and boundary perimeter $|\partial_i|$. We then identify the edges of $P \smallsetminus \bigsqcup_{i=1}^{n} (D_i)^o$ in the usual way - ensuring the edges of R of length l are identified - to obtain a metric surface (S, d') of genus g with n boundary components.

We set Γ to be the image of the union of the boundaries of the P_i in S. Γ is a metric graph with $2g + (n-1)$ edges.

We must now prove that this new metric, d', on S is biLipschitz equivalent to a Riemannian metric d. Since S (equipped with either metric) is compact it suffices to prove that (S, d') is locally biLipschitz equivalent to open balls in $\mathbb{R}^2$. This is obvious except at a single point where we have cone-angle $k\pi$, for some $k \geq 3$. Neighbourhoods of such points are shown in Figure 2.1.7.

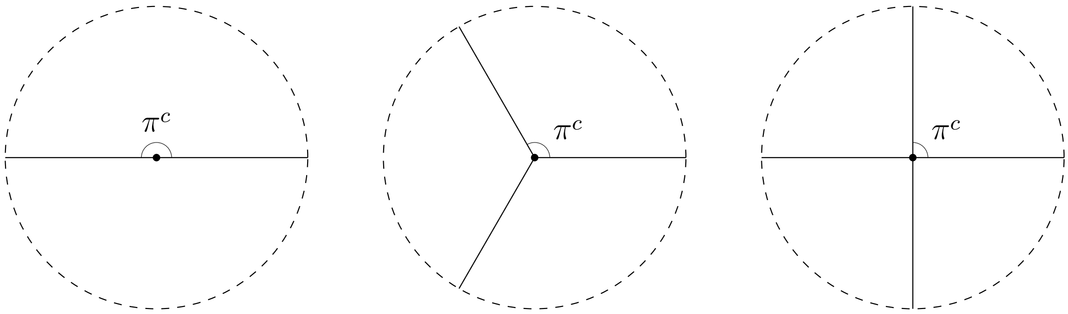

Figure 2.16: Small open neighbourhoods $D_{k\pi}$ of points with cone angle $k\pi$, $k \geq 2$

It remains to prove that such neighbourhoods are locally biLipschitz to the open unit disc (D, d) in $\mathbb{R}^2$. As a set one can identify $D_{k\pi}$ with $\{re^{i\theta} \mid 0 \leq r \leq 1,\ \theta \in [0, k\pi)\}$. We prove that the map

$$\psi_k : (D, d) \to (D_{k\pi}, d') \ \text{ given by } \ \psi_k(re^{i\theta}) = re^{i\frac{k}{2}\theta}$$

is biLipschitz.

Lemma 2.1.31. *The map ψ_k is biLipschitz.*

Proof: It is clear that ψ_k does not decrease distances. By homogeneity it suffices to prove an upper bound for the images of pairs of points $r = r^{i0}$ and $(r')^{i\theta}$ where $r' \geq r$ and $\theta \in [0, \pi]$.

If $r' \geq \frac{3}{2}r$, then $d(r, re^{i\theta}) \geq r' - r \geq r'/3$ while $d'(r, re^{i\frac{k}{2}\theta}) \leq r + r' \leq 2r'$, hence

$$d'(r, re^{i\frac{k}{2}\theta}) \leq 6d(r, re^{i\theta}).$$

Now suppose $r' \in [r, 3r/2]$ and $\theta \in [2\pi/k, \pi]$. This implies that $k\theta/2 \geq \pi$, and therefore $d'(r, r'e^{i\frac{k}{2}\theta}) = r + r' \leq 5r/2$.

Moreover, $d(r, r'e^{i\theta}) \geq d(r, re^{i\theta})$ which is equal to $2r\sin(\theta/2)$. Combining these we see that

$$d'(r, r'e^{i\frac{k}{2}\theta}) \leq \frac{5}{4\sin(\pi/k)} d(r, r'e^{i\theta}).$$

Finally, suppose $r' \in [r, 3r/2]$ and $\theta \in [0, 2\pi/k]$. Then $d'(r, r'e^{i\frac{k}{2}\theta}) \leq d'(r', r'e^{i\frac{k}{2}\theta}) = 2r'\sin(\theta/2)$. As above, $d(r, r'e^{i\theta}) \geq 2r\sin(\theta/2)$. Within the range $\theta \in [0, 2\pi/k]$,

$$\frac{k}{2\pi}\sin(2\pi/k)\theta \leq \sin(\theta) \leq \theta.$$

Combining these gives

$$d'(r, r'e^{i\frac{k}{2}\theta}) \leq \frac{3k}{\pi}\sin(2\pi/k) d(r, r'e^{i\theta}).$$

□

In the universal cover, Γ lifts to a tree T and the extension of the map which collapses all cylinders to circles defines a metric retraction of $\widetilde{S}$ onto T which is isometric when restricted to any boundary component, completing the positive genus case. □

We now move to the genus 0 case. This requires a different approach. We construct an n-holed sphere S from n metric cylinders $C_i = S^1 \times [0,1]$ of height 1 and boundary circumferences $|\partial_i|$. We identify each $S^1 \times \{1\}$ with the boundary component ∂_i. Each $S^1 \times \{0\}$ is glued to the boundary of a face in a planar metric graph Γ such that the union of these cylinders is homeomorphic to S. We then prove the new metric is biLipschitz equivalent to the original metric.

Lemma 2.1.32. *Let $n \geq 3$ and let $r_1, r_2, \ldots, r_n$ be a collection of positive real numbers with $r_1 \leq r_2 \leq \cdots \leq r_n$. There exists a metric planar graph Γ with the following properties:*

- *There are $n-1$ internal faces $F_1, \ldots F_{n-1}$ in Γ whose boundaries ∂F_i have length r_i.*
- *The external boundary of Γ has length r_n.*

Proof: We proceed by induction on n. In the case $n = 3$ the figure below deals with the situations $r_2 \leq r_3 < r_1 + r_2$, $r_3 = r_1 + r_2$ and $r_3 > r_1 + r_2$ respectively.

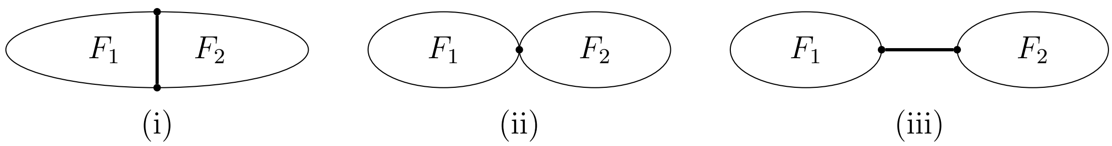

Figure 2.17: Three metric planar graphs with two interior faces

The thick lines in the first and third pictures have length $\frac{1}{2}|r_3 - (r_1 + r_2)|$.

Now we attack the general case. Let $r_1, r_2, \ldots, r_n$ be a collection of positive real numbers satisfying the above hypotheses and let Γ' be a metric graph satisfying the conclusion for the sequence $r_2, r_3, \ldots, r_n$. To obtain Γ we simply choose an arc α on the external boundary of Γ' of length $r_1/2$ and connect an additional edge β of length $r_1/2 \leq r_n$ with the same endpoints as α in the external face of Γ'. □

Proof of Proposition 2.1.30 for genus 0 surfaces

Consider a genus 0 surface with $n \geq 3$ boundary components ∂_i of lengths $|\partial_1| \leq |\partial_2| \leq \cdots \leq |\partial_n|$.

Let Γ be a graph obtained from Lemma 2.1.32 with $r_i = |\partial_i|$ for each i. To each internal face F_i we glue one end of a metric cylinder of height 1 and circumference $|\partial_i|$, and we glue a metric cylinder of height 1 and circumference $|\partial_n|$ to the external boundary. The resulting space S is homeomorphic to the surface $S_{0,n}$ presented in Figure 2.13. To see this, notice that S is homeomorphic to a thickening of Γ in the plane.

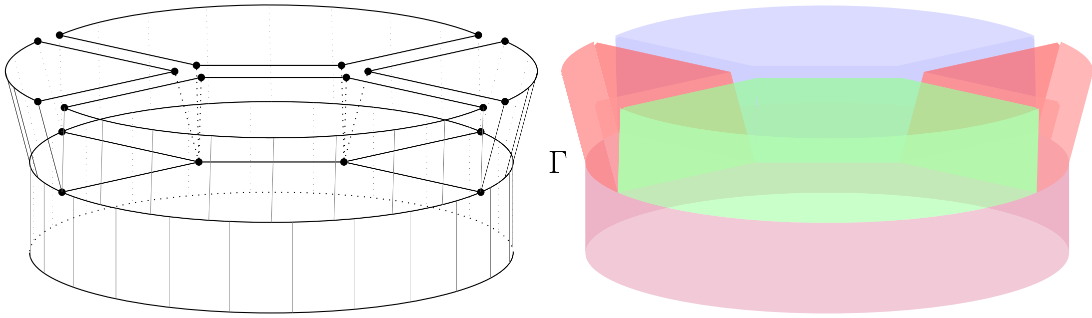

Figure 2.18: A metric surface of genus 0

With respect to the new metric d', S is a compact surface which, by Lemma 2.1.31, is locally biLipschitz equivalent to $\mathbb{R}^2$.

To complete the proof of Proposition 2.1.30, we notice that in the universal cover, as in the positive genus case, Γ lifts to a tree T and the extension of the map which collapses all cylinders onto cycles in Γ defines a metric retraction of $\widetilde{S}$ onto T which is isometric when restricted to any boundary component. □

Definition 2.1.33. Mapping class groups of surfaces

Let $S = S_{g,n}$ be a surface. The mapping class group *of S,* MCG(S) *is the quotient of the group of orientation-preserving diffeomorphisms of S which fix each component of the boundary ∂S by the normal subgroup of all such diffeomorphisms which are isotopic to the identity map.*

$$\mathrm{MCG}(S) := \mathrm{Diff}^{+}(S, \partial S)/\mathrm{Diff}_0^{+}(S, \partial S).$$

The highly non-trivial Dehn-Lickorish Theorem states that such groups are finitely generated by a collection of *Dehn twists* around a finite collection of curves [Lic64]. Any curve γ on a surface admits a neighbourhood homeomorphic to the product $S^1 \times [-1, 1]$ where γ is identified with $S^1 \times \{0\}$. The Dehn twist around γ is defined as the following diffeomorphism on $S^1 \times [-1, 1]$ extended smoothly to S by the identity:

$$D_\gamma(\theta, y) = (\theta + y\pi, y), \quad \text{where} \quad \theta \in \mathbb{R}/2\pi\mathbb{Z} \quad \text{and} \quad y \in [1-, 1].$$

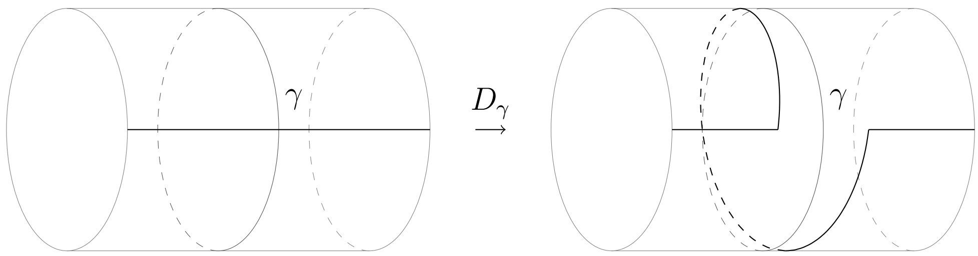

Figure 2.19: Applying a Dehn twist D_γ about the curve γ

Elements of the mapping class group are characterised by the following classification.

Theorem 2.1.34. Nielsen–Thurston classification [Thu88]

Let $[g] \in MCG(S)$, there is a map $f \in [g]$ such that at least one of the following holds:

(i) *f is periodic,*

(ii) *f is reducible, i.e. it preserves some finite union of disjoint simple closed curves on S,*

(iii) *f is a pseudo-Anosov; there exists a transverse pair of measured foliations on S, F_s and F_u, and a real number $\lambda > 1$ such that the foliations are preserved (up to Whitehead equivalence) by f and their transverse measures are multiplied by $1/\lambda$ and λ respectively.*

The results we obtain for relatively hyperbolic groups will not pass to mapping class groups immediately by the following theorem.

Theorem 2.1.35. [BDM09]

Let S be a surface with positive complexity. The mapping class group of S is not relatively hyperbolic with respect to any collection of infinite index subgroups.

By contrast, surfaces with non-positive complexity have virtually free mapping class groups, (cf. [Beh04]). Mapping class groups are often studied through their actions on *curve complexes*.

Definition 2.1.36. Curve complexes

Let $S = S_{g,n}$. A simple curve *γ on S is the image of any smooth embedding of S^1 into S. γ is said to be* essential *if it does not bound a disc in S and* non-peripheral *if it is not isotopic to a boundary component of S.*

The minimal intersection number *of a surface S, $\iota(S)$ is defined to be*

$$\iota(S) := \min\{|\gamma \cap \gamma'| \mid \gamma, \gamma' \text{ essential, non-peripheral, non-isotopic, simple curves}\}.$$

The curve complex *of S, $\mathcal{C}(S)$ is the simplicial complex whose vertex set is the set of isotopy classes of essential, non-peripheral curves on S where a finite set of such classes of curves $\{[\gamma_1], [\gamma_2], \ldots, [\gamma_n]\}$ spans a simplex if and only if, for each $i \neq j$ there are representatives*

$$\gamma_i' \in [\gamma_i] \text{ and } \gamma_j' \in [\gamma_j] \text{ such that } \left|\gamma_i' \cap \gamma_j'\right| = \iota(S).$$

The curve complex is flag, that is, whenever the 1-skeleton of an n-simplex lies in $\mathcal{C}(S)$, so does the entire n-simplex.

We begin by looking at low complexity cases. When $g = 0$ and $n \leq 3$ the curve complex is empty, while the curve complexes of $S_{1,0}$, $S_{1,1}$ and $S_{0,4}$ are all isomorphic to the Farey graph (cf. Example 2.1.13 and Figure 2.4). Here $\iota(S_{1,0}) = \iota(S_{1,1}) = 1$, while $\iota(S_{0,4}) = 2$. In all situations of positive complexity $\iota(S) = 0$, so simplexes correspond to collections of disjointly realisable curves.

In general, curve complexes are not locally finite, as the following example shows.

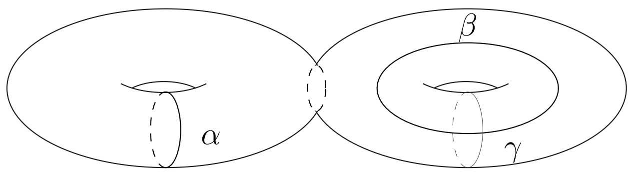

Figure 2.20: Neighbours of $[\alpha]$ in $\mathcal{C}(S_{2,0})$

For each $n \in \mathbb{N}$, $D_\gamma^n(\beta)$ and α are disjoint, so are at distance 1 in $\mathcal{C}(S)$. It remains to see that $D_\gamma^n(\beta)$ is isotopic to $D_\gamma^m(\beta)$ if and only if $m = n$. This is done by checking that the minimal intersection number of the isotopy classes $D_\gamma^m(\beta)$ and $D_\gamma^n(\beta)$ is $|m - n|$.

We require two key theorems concerning curve complexes, the Masur-Minsky Theorem and the existence of a collection of tight geodesics.

Theorem 2.1.37. The Masur-Minsky Theorem [MM99]

Let S be a surface. $\mathcal{C}(S)$ is Gromov hyperbolic.

A very recent proof of this theorem gives an upper bound of 17 on the hyperbolicity constant of the curve complex, independent of S [HPW13].

Theorem 2.1.38. Tight geodesics [MM00, Bow08]

For every pair of vertices $x, y \in \mathcal{C}(S)$ *there exists a finite non-empty set of* tight geodesics $[[x,y]]_T$ *such that for each* k_0 *sufficiently large the following properties hold:*

(i) *There is some* K_0 *depending only on* $\zeta(S)$ *such that if* $a, b \in \mathcal{C}(S)$ *and* $c \in T(a,b) := \bigcup_{\underline{g} \in [[x,y]]_T} \underline{g}$*, then*

$$|T(a,b) \cap B(c; k_0)| \leq K_0.$$

(ii) *There are constants* k_1 *and* K_1*, which depend only on* $\zeta(S)$ *such that if* $a, b \in \mathcal{C}(S)$*,* $r \in \mathbb{N}$ *and* $c \in T(a,b)$ *with* $d(c, \{a,b\}) \geq r + k_1$*, then*

$$T(a,b;r) := \bigcup_{d(x,a), d(y,b) \leq r} T(x,y),$$

contains at most K_1 *elements of* $B(c; k_0)$*.*

Any hyperbolic space satisfying Theorem 2.1.38 will be said to have *Bowditch's tight geodesic* property. In particular, coned-off graphs of relatively hyperbolic groups have this property, but it is not true in general for fine hyperbolic graphs [Bow08].

The hyperbolicity of $\mathcal{C}(S)$ is sufficient to ensure that for sufficiently large k_0, every geodesic in $[[a,b]]_T$ meets $B(c; k_0)$ under the hypotheses of Theorem 2.1.38. This theorem is therefore a strong local finiteness property for these tight geodesics, which holds automatically in the case of hyperbolic groups. We will exploit this facet of curve complexes to construct good embeddings into ℓ^p spaces in section 4.6.

Another important notion is that of subsurface projections, before we can approach this however, we will need to introduce the arc complex.

Definition 2.1.39. Arc and curve complex

Let S *be a surface. An* arc α *is the image of any smooth embedding* $\alpha : [0,1] \to S$ *such that boundaries are mapped to boundaries, i.e.* $\alpha^{-1}(\partial S) = \{0,1\}$*.* α *is non-peripheral if it is not isotopic (relative to its boundary) to a subarc of a boundary component of* S*.*

The arc and curve complex *of* S*,* $\mathcal{AC}(S)$*, is the simplicial complex with vertex set given by isotopy classes of non-peripheral arcs and essential, non-peripheral curves, where a finite set of arcs and curves spans a simplex if they admit representatives with minimal possible number of crossings in* S*.*

Importantly, the 1-skeleton of the arc and curve complex, often known as the *curve graph* is hyperbolic and admits a collection of tight geodesics [MM00, Bow08].

Now we move on to projections onto subsurfaces, this notion, originally defined by Masur-Minsky inspires the definition of projection complexes which will appear in Section 2.1.8 [MM00].

Definition 2.1.40. Subsurface projections

Suppose that X *is an* essential non-simple *subsurface of* S*, so* $\pi_1(X)$ *is non-trivial and injects into* $\pi_1(S)$*. Let* $[\alpha] \in \mathcal{AC}(S)$ *and choose an isotopy representative* $\gamma \in [\alpha]$ *such that* $|\gamma \cap \partial X|$ *is minimised.*

We define the subsurface projection π_X *which maps elements of* $\mathcal{AC}(S)$ *to subsets of* $\mathcal{AC}(X)$ *according to the following trichotomy.*

- *If* $\gamma \in X$ *we define* $\pi_X([\alpha]) = \{[\alpha]\}$.
- *If* $\gamma \cap X = \varnothing$*, we define* $\pi_X([\alpha]) = \varnothing$.
- *Now suppose* $\gamma \cap \partial X \neq \varnothing$*. Define* X_γ *to be the set of all connected components of* γ *contained in* X*. For each* $\gamma' \in X_\gamma$ *set* $N_{\gamma'}$ *to be a closed neighbourhood of* $\gamma' \cup \partial X$ *in* S*.*
  $\pi_X([\alpha])$ *is the set of all isotopy classes in* $\mathcal{AC}(X)$ *with a representative* $\gamma' \in X_\gamma$ *(to find arcs), or coming from a component of* $\partial(N_{\gamma'})$ *for some* $\gamma' \in X_\gamma$ *(to find curves).*

The projection of one essential non-simple subsurface Y *onto another* X*, is given by*

$$\pi_X(Y) \coloneqq \bigcup_{[\alpha]\in\partial Y} \pi_X([\alpha]) \subseteq \mathcal{AC}(S).$$

The remarkable feature of this definition, is that despite there being many cases where $\pi_X(Y)$ is infinite, it always has bounded diameter in $\mathcal{AC}(S)$. A stronger statement is true in restricted situations and this will be crucial in the set-up of Section 2.1.8.

2.1.8 Quasi-trees of spaces

There are many possible definitions of a quasi-tree of spaces, perhaps as a space quasi-isometric to a tree-graded space; as a relatively hyperbolic graph whose coned-off graph is a quasi-tree; or we may just consider a simplicial graph which admits a coned-off graph which is a quasi-tree.

The first of these we will call a *quasi tree-graded space* in analogy with Definition 2.1.12. We have seen in Figures 2.10 and 2.11 that the second and third of these definitions lead to vastly different collections of spaces.

The connection between the first two suggestions is explored more fully in Section 5.2.3.

In their recent paper Bestvina-Bromberg-Fujiwara introduce the following list of axioms from which they define a projection complex.

Let $\mathbf{Y}$ be a set and suppose that for every $Y \in \mathbf{Y}$ we have a function

$$d_Y : (\mathbf{Y} \setminus \{Y\})^2 \to [0, \infty)$$

such that the following conditions are satisfied:

(i) for all $Y \in \mathbf{Y}$ and all $X, Z \in \mathbf{Y} \setminus \{Y\}$, $d_Y(X,Z) = d_Y(Z,X)$,

(ii) for all $Y \in \mathbf{Y}$ and all $W, X, Z \in \mathbf{Y} \setminus \{Y\}$, $d_Y(X,W) \leq d_Y(X,Z) + d_Y(Z,W)$,

(iii) there exists a positive constant η such that for any three distinct elements $X, Y, Z \in \mathbf{Y}$ at most one of $d_X(Y,Z)$, $d_Y(X,Z)$, $d_Z(X,Y)$ is greater than η,

(iv) there exists some K_0 such that for any $X, Z \in \mathbf{Y}$,

$$|\{Y \in \mathbf{Y} \setminus \{X, Z\} \mid d_Y(X,Z) > K_0\}| < \infty.$$

The example which is of most interest here is that of projections on subspaces. Given a metric space M and a collection of subspaces of M

$$\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\},$$

and notions of projection between spaces $\mathcal{C}(Y)$, π_Y which, on input $X \in \mathbf{Y} \setminus \{Y\}$, outputs a subset $\pi_Y(X) \subseteq M$, such that the function

$$d_Y(X,Z) \coloneqq \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$$

satisfies the axioms (i)-(iv) above.

We now give two examples of collections of spaces satisfying the above axioms with respect to projections.

Example 2.1.41. Curve complexes

Let S be a surface, and let $\mathbf{Y}$ be a collection of isotopic subsurfaces which pairwise intersect (considered up to equivalence by maps in $\mathrm{Diff}_0^+(S)$) and let $\mathcal{C}(Y)$ be the curve complex of Y. Projections between such subsurfaces are defined as in Definition 2.1.40.

Example 2.1.42. Relatively hyperbolic groups

Let $(G, \{H_i\})$ be a relatively hyperbolic group. Define $\mathbf{Y}$ to be the set of all H_i cosets in G. The projection $\pi_Y(X)$ is defined as a closest point projection of the coset $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$. This does not - in general - determine a unique point on the Cayley graph, but a set of uniformly bounded diameter.

This is a consequence of the coned-off graph being fine in Definition 2.1.24, or of condition (i) in Definition 2.1.26.

Definition 2.1.43. Projection complex

Given projections $\{d_Y \mid Y \in \mathbf{Y}\}$ satisfying axioms $(i)-(iv)$ above and a constant $K > 0$ we define the projection complex $\mathcal{P}_K(\mathbf{Y})$ to be the simplicial graph $(\mathbf{Y}, E)$, where two elements $X, Y \in \mathbf{Y}$ span an edge if and only if

$$d_Y(X,Z) < K \quad \text{for all} \quad Y \in \mathbf{Y} \smallsetminus \{X, Z\}\,.$$

It is not at all obvious from the construction, but there is some $K' > 0$ such that for all $K \geq K'$ the graph $\mathcal{P}_K(\mathbf{Y})$ is connected, and moreover, it is a quasi-tree with bottleneck constant independent of K [BBF10, Lemma 2.4 and Theorem D].

To do this, it is shown that given $X, Z \in \mathbf{Y} \smallsetminus \{Y\}$, with $X \neq Z$, the sets

$$Y_K(X,Z) \coloneqq \{Y \in \mathbf{Y} \mid d_Y(X,Z) \geq K\}\,,$$

which are certainly finite, in view of axiom (iv) above, come with a natural order $\{X, Y_1, \ldots, Y_n, Z\}$ which defines a path in $\mathcal{P}_K(\mathbf{Y})$ between X and Z.

Notice that every $\mathcal{P}_K(\mathbf{Y})$ contains $\mathcal{P}_{K_0}(\mathbf{Y})$ as a subgraph and for each X and Z, $Y_K(X,Z) \subseteq Y_{K_0}(X,Z)$.

Moreover, such paths are minimal in the sense that given any other path P from X to Z in $\mathcal{P}_K(\mathbf{Y})$, and any $Y_i \in Y_K(X,Z)$,

$$\mathrm{dist}_{\mathcal{P}(\mathbf{Y})}(Y_i, P) \leq 2,$$

where $\mathrm{dist}_{\mathcal{P}(\mathbf{Y})}$ is the shortest path metric on $\mathcal{P}_{K_0}(\mathbf{Y})$, for some fixed, but suitably large K_0.

Whenever a projection complex is defined as a collection of subspaces with notions of projection satisfying axioms (i)-(iv), we can "blow up" vertices of $\mathcal{P}_K(\mathbf{Y})$ to obtain a *quasi-tree of spaces* $\mathcal{C}(\mathbf{Y})$ in the following way.

Definition 2.1.44. Quasi-trees of spaces

Given a fixed constant $L \geq K$, $\mathcal{C}_L(\mathbf{Y})$ *is built from the disjoint union of the spaces* $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$, *where for every edge* $XZ \in E$, *we add an edge of length* L *from every vertex in* $\pi_X(Z) \in \mathcal{C}(X)$ *to every vertex in* $\pi_Z(X) \in \mathcal{C}(Z)$.

If L is sufficiently large in comparison to K, then each $\mathcal{C}(Y)$ is totally geodesically embedded in $\mathcal{C}_L(\mathbf{Y})$. In what follows, we will assume that this is the case and drop the L from the notation.

A space $\mathcal{C}(\mathbf{Y})$, constructed in this way (with suitably large choices of K and L) is called a *quasi-tree of spaces* $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$, in the terminology of [BBF10].

The importance of such spaces in the geometric understanding of mapping class groups is underlined by the following result.

Theorem 2.1.45. [BBF10, Theorem B]

Let S *be a compact orientable surface. There exist a finite number of collections of subsurfaces of* S, $\mathbf{Y}_1,\ \mathbf{Y}_2, \dots, \mathbf{Y}_n$ *and an equivariant quasi-isometric embedding*

$$\phi : \mathrm{MCG}(S) \hookrightarrow \prod_{i=1}^{n} \mathcal{C}(\mathbf{Y}_i),$$

where each $\mathcal{C}(\mathbf{Y}_i)$ *is a quasi-tree of curve complexes of isotopic subsurfaces* $\mathcal{C}(Y)$ *with projections given by* Definition 2.1.40.

The proof that there is a coarse embedding of the above form is a major part of [BBF10]. This extension is given using the Masur-Minsky formula for word length in the mapping class group.

Theorem 2.1.46. The Masur-Minsky formula [MM00]

Let $g \in \mathrm{MCG}(S)$ *then for all sufficiently large* M,

$$l(g) \asymp \sum_{Y \subseteq S} \{\{d_Y(\alpha, g(\alpha))\}\}_M$$

where the sum is taken over the set of all subsurfaces Y *of* S, *and*

$$\{\{x\}\}_M = \begin{cases} x & \text{if } x > M \\ 0 & \text{otherwise.} \end{cases}$$

The authors remark that this extension may be of independent interest, this additional restriction is necessary for the results we prove.

The study of such spaces in motivated by obtaining consequences for the geometry of the mapping class group. In particular, it inspires study of quasi-trees of spaces with curve complex pieces, which are hyperbolic and of finite asymptotic dimension [BBF10].

A more general study of these spaces is required in light of the following result.

Theorem 2.1.47. [MS12, Theorem 4.1]

Let G be a finitely generated group which is hyperbolic relative to a collection of subgroups $\{H_1, \ldots, H_n\}$. Then there is a quasi-isometric embedding

$$\phi : G \to \mathcal{C}(\mathbf{H}) \times \widehat{G},$$

where $\widehat{G}$ is the coned-off graph of G and $\mathcal{C}(\mathbf{H})$ is a quasi-tree of spaces $\{\mathcal{C}(H)\}$ which are Cayley graphs of H_i-cosets in G.

In this situation, the quasi-tree of spaces is almost never hyperbolic. A geometrically more enlightening feature is given by the following lemma [BBF10]:

Lemma 2.1.48. *Let $\mathcal{C}(\mathbf{Y})$ be a quasi-tree of spaces $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$. If there exists some $\Delta > 0$ such that every $\mathcal{C}(Y)$ is a quasi-tree with bottleneck constant Δ, then $\mathcal{C}(\mathbf{Y})$ is also a quasi-tree.*

The results of chapter 5 are principally inspired by this lemma. In that chapter we prove that given any quasi-tree of spaces $\mathcal{C}(\mathbf{Y})$, we may construct a quasi-isometric tree-graded space $\mathcal{T}(\mathbf{Y})$ with pieces $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$. This places the definition of Bestvina-Bromberg-Fujiwara as being at least as restrictive as any suggestion made at the start of the section.

2.1.9 3-manifolds and the Geometrisation Theorem

The study of 3-manifolds up to homotopy is a major area in topology, which has recently seen many long-standing conjectures resolved. We concentrate in this section on a classification (to some extent) of fundamental groups of compact 3-manifolds. This requires the Thurston-Perelman Geometrisation Theorem and Dahmani's combination theorem in essential ways. Before embarking on this, we introduce Seifert fibre surfaces - first introduced in [Sei33] - via fibred solid tori.

Definition 2.1.49. Fibred solid torus

Let (a,b) be a pair of coprime integers with $a > b \geq 0$. Denote by D the closed unit disc in the complex plane $\mathbb{C}$ and define

$$\psi_{a,b} : D \to D \text{ by } \psi_{a,b}(z) = ze^{2\pi i(b/a)}.$$

The fibred solid torus $T_{a,b}$ of type (a,b) is the quotient of the product space $D \times [0,1]$ by the relation $(z,0) \sim (\psi_{a,b}(z),1)$.

This is called a fibred solid torus because it admits a decomposition into a disjoint union of circles $\mathcal{O}_\psi(z) \times I$ where $\mathcal{O}_\psi(z)$ is the orbit of z under the action of the finite cyclic group $\langle \psi_{a,b} \rangle$ on D. These circles are called fibres.

Definition 2.1.50. Seifert fibre space

An orientable compact 3-manifold M is a Seifert fibre space *if it is can be written as a disjoint union of circles (called fibres) such that each fibre S has a closed neighbourhood T which is itself a union of fibres and admits a fibre-preserving homeomorphism to a standard fibred torus.*

The set of fibres B of a Seifert fibre space M can be viewed with an orbifold structure in such a way that the universal cover of M decomposes as a product

$$\widetilde{M} = \widetilde{B} \times \mathbb{R}.$$

We continue with the definition of graph manifolds.

Definition 2.1.51. (Non-geometric) Graph manifolds

An orientable 3-manifold M is a graph manifold *if it admits a decomposition into Seifert fibre spaces when cut along essential (π_1-injective) embedded tori. A graph manifold is* non-geometric *if it is not a Seifert fibre space, that is, any decomposition must be non-trivial.*

In particular, a graph manifold is a Seifert fibre space or it decomposes into a collection of Seifert fibre spaces with boundary. As such, it has a graph of groups decomposition, so comes equipped with a Bass-Serre tree which encodes the way the component surfaces are attached. This is exploited in Chapter 3.

Lemma 2.1.52. *Let M be a non-geometric graph manifold with Bass-Serre tree $T = (V(T), E(T))$. The universal cover of M is obtained by gluing a collection of metric spaces $\{X_v = F_v \times \mathbb{R} \mid v \in V(T)\}$ along embedded copies of $\mathbb{R}^2$. Moreover, each F_v is a quasi-tree.*

Each F_v is the universal cover of an orbifold B_v as mentioned above. Full details of this can be obtained from, for example [BN08].

We are interested in graph manifolds where copies of X_v are glued in the following way.

Definition 2.1.53. Flip graph manifolds

Let M be a non-geometric graph manifold M and let $T = (V(T), E(T))$ be the Bass-Serre tree corresponding to a decomposition of M into Seifert fibre spaces.

M is called a flip *graph manifold if for every $vv' \in E(T)$ there exist parametrisations $\gamma_v : \mathbb{R} \to F_v, \gamma_{v'} : \mathbb{R} \to F_{v'}$ of boundary components of F_v, $F_{v'}$ so that $(\gamma_v(t), u) \in F_v \times \mathbb{R}$ is identified with $(\gamma_{v'}(u), t) \in F_{v'} \times \mathbb{R}$ for each $t, u \in \mathbb{R}$.*

Flip graph manifolds are in some ways representative of the entire collection via the following result [KL98].

Theorem 2.1.54. *Let M be a non-geometric graph manifold, equipped with a Riemannian metric. There exists a non-geometric flip graph manifold N and a Riemannian metric on N such that $\widetilde{M}$ is quasi-isometric to $\widetilde{N}$.*

Using Proposition 2.1.30 we ensure that, up to biLipschitz equivalence, F_v admits a metric retraction onto a tree T_v

$$r_v : F_v \to T_v \text{ such that for each } x \in F_v,\ d_{F_v}(x, r_v(x)) \leq 1.$$

In addition, each r_v is an isometry when restricted to any boundary component of F_v. It is important to note that applying this proposition metrically preserves boundary components, so preserves the flip structure.

Finally in this section we give a description of the class of fundamental groups of compact 3-manifolds, which uses the Thurston-Perelman Geometrisation Theorem and a Combination Theorem of Dahmani.

Theorem 2.1.55. *Let M be a compact 3-manifold. $\pi_1(M)$ can be expressed as a free product of*

(i) *virtually polycyclic groups,*

(ii) *fundamental groups of graph manifolds,*

(iii) *groups hyperbolic relative to a collection of the above groups.*

Taking a prime decomposition of M equates, at the level of fundamental groups, to taking free factors by Van Kampen's Theorem. The result then follows from the Geometrisation Theorem [Thu82, Per02, Per03a, Per03b, CZ06a, CZ06b, KL08, MT07, MT08] and the Combination Theorem [Dah03a].

Much more detailed and specific versions of this theorem hold, see for instance [MS12], but this is more than sufficient for our purposes.

Now that we have introduced the spaces and groups we wish to study, we move onto a section which motivates the study of metric embeddings and dimension controls.

2.2 The Borel Conjecture and related questions

The Borel conjecture and various related questions have done much to motivate the study of embeddings and notions of dimension for finitely generated groups. We will present a few of these questions and explain how they can be approached in many cases through techniques of geometric group theory. This will inspire the following two sections on coarse embeddings and large-scale dimension. The Borel conjecture itself focuses on rigidity results, so we start with the definition of a rigid manifold.

Definition 2.2.1. Topologically rigid manifolds

A closed manifold M is said to be topologically rigid *if given any closed manifold N such that M is homotopy equivalent to N, then M and N are homeomorphic.*

It is, in general, very difficult to prove that even the simplest manifolds are topologically rigid, for instance the rigidity of the sphere S^n is equivalent to the Generalised Poincaré Conjecture. To find an example of a non-rigid manifold we consider Lens spaces,

$$L(p,q) \coloneqq S^3/\sim_{p,q},$$

where $(z_1,z_2) \sim_{p,q} (z_3,z_4)$ for two elements $(z_1,z_2),(z_3,z_4) \in S^3 \subset \mathbb{C}^2$ if they have the same image under the map $h_{p,q}$ defined as

$$h_{p,q}(z,z') \coloneqq \left(\exp\left(\frac{2\pi i}{p}\right)z, \exp\left(\frac{2\pi iq}{p}\right)z'\right).$$

The Lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent but not homeomorphic, so obviously neither is a rigid manifold. Notice that both these manifolds have S^3 as a universal cover, and both have finite fundamental group $\mathbb{Z}/7\mathbb{Z}$.

Such situations are avoided by restricting to *aspherical manifolds*, those whose universal covers are contractible. The Borel conjecture states that this is a sufficient restriction for closed manifolds.

Conjecture 2.2.2. The Borel Conjecture

All aspherical closed manifolds are topologically rigid.

The origins of the conjecture, as formulated in letters between Borel and Serre are explained in a text by Ranicki [Ran09].

There are now many variants of this conjecture, based on finding homotopy invariants of manifolds and developing links with K and L-theory, the first of these we present is the stable Borel conjecture. More information on the Borel Conjecture can be found in [Far02].

Conjecture 2.2.3. The stable Borel Conjecture

Let M and N be homotopy equivalent aspherical closed manifold. There exists some n such that $M \times \mathbb{R}^n$ and $N \times \mathbb{R}^n$ are homeomorphic.

Any closed manifold M which satisfies the conclusion of the stable Borel conjecture is said to be *stably rigid*. It is clear that the stable Borel Conjecture is implied by the original.

The first powerful link with geometric group theory is given by the following theorem.

Theorem 2.2.4. [GTY12] *Let M be an aspherical closed manifold whose fundamental group has* finite decomposition complexity. *Then M is stably rigid. Moreover, in almost all cases it is sufficient to set $n = 3$.*

We will formally introduce the notion of *finite decomposition complexity* later (cf. Definition 2.4.2), however, for the moment it suffices to think of it as a relatively weak notion of finite dimensional.

The Novikov conjecture, often regarded as one of the most important unsolved problems in topology, can be considered as a "linearisation" of the Borel conjecture. To state this conjecture we will first present some notation.

Let M be a smooth closed oriented n manifold and let G be a discrete group with classifying space BG.

Given any smooth map $f : M \to BG$, and some rational cocycle $x \in H^{n-4i}(BG, \mathbb{Q})$ we obtain an element of $H^n(M, \mathbb{Q})$ by considering $f^*(x) \cup L_i(M)$ where $\cup$ is the cup product in cohomology and L_i is the ith Hirzebruch polynomial, which can be expressed in terms of the rational Pontryagin classes of the tangent bundle of M. For more details on these polynomials, we refer the reader to [Hir54].

Conjecture 2.2.5. The Novikov conjecture

Let M and N be smooth closed oriented n-manifolds, let G be a discrete group with classifying space BG and let $f : M \to BG$ be a smooth map. Given any orientation preserving homotopy equivalence $h : N \to M$, any i with $4i \leq n$ and any cocycle $x \in H^{n-4i}(BG, \mathbb{Q})$, we have the following equality of higher signatures:

$$\left\langle f^*(x) \cup L^i(M), [M] \right\rangle = \left\langle (f \circ h)^*(x) \cup L^i(N), [N] \right\rangle,$$

where $\langle \cdot, [M] \rangle$ denotes taking the cap product with the standard class $[M] \in H_n(M, \mathbb{Q})$, yielding an element of $H_0(M, \mathbb{Q}) \simeq \mathbb{Q}$, since M is connected.

We give another statement of this conjecture using assembly maps shortly.

The Hirzebruch (or Hirzebruch-Riemann-Roch) Theorem, which was a key motivation for this conjecture, proves Novikov's conjecture in the case $i = 0$ [Hir54].

Over the course of many papers between 1965 and 1970, Novikov proves that these higher signatures are topological invariants, so if M and N are homeomorphic, their rational Pontryagin classes are equivalent. The major advances appear in [Nov66, Nov70].

We will say that a finitely generated group G satisfies the Novikov conjecture if the conjecture holds whenever the discrete group is chosen as G. Gromov recognised that the existence of a *coarse embedding* of G into a Hilbert space should provide an insight into the Novikov conjecture, this result was proved by Yu and then generalised by Kasparov-Yu [Gro93, Yu00, KY06].

Theorem 2.2.6. *Let G be a group which admits a* coarse embedding *into some uniformly convex Banach space. Then G satisfies the Novikov conjecture.*

Yu's original proof assumed that G could be coarsely embedded into a Hilbert space. The statement can be seen as a strengthening of this in light of results of Johnson-Randrianarivony and Randrianarivony [JR06, Ran06] which prove that there are uniformly convex Banach spaces which cannot be coarsely embedded into any Hilbert space.

As with finite decomposition complexity, coarse embeddings will be introduced formally in the next section.

This theorem actually proves the stronger coarse Baum-Connes Conjecture for such groups; to phrase this we will require *assembly maps.* Generically, assembly maps are considered (from the point of view of homotopy invariance) as a universal approximation of a homotopy invariant by a suitable notion of homology, or (from a more geometric point of view) as a 'local-to-global' tool (assembling local information to build a global picture). We give some specific cases below (cf. [FRR95]).

(i) The Novikov Conjecture can be expressed as requiring that the assembly map in integral L-theory is a rational split injection, while, within this terminology, the Borel conjecture states this map is an isomorphism.

(ii) The Baum-Connes Conjecture states that an assembly map for C^* algebras (acted on by a groupoid G) in terms of the topological K-theory of the C^* algebra of G is an isomorphism. There are known to be exceptions for groupoids, but no group counterexample is yet known [HLS02]. There is a variant of this called the coarse assembly map, and the conjecture requiring that this is an isomorphism is called the coarse Baum-Connes Conjecture.

(iii) There are also equivariant versions of these questions which make up the Farrell-Jones conjectures (cf. [Lüc10]).

More information on the Novikov Conjecture can be found in [FRR95, KL05].

The above results - particularly Theorem 2.2.6 - motivated the study of coarse embeddings of groups into Hilbert spaces and then into uniformly convex Banach spaces, which is the subject of the next section.

2.3 Embeddings

This section builds on the motivation provided by the previous section, introducing the notion of coarse embeddings and stronger variants, compression exponents and finally quasi-isometric embeddings. Later in the section we mention equivariant embeddings and the link between embeddings, amenability and dynamics on Cayley graphs.

2.3.1 Coarse embeddings

Coarse (sometimes called uniform) embeddings provide a weak way of asking for one metric space to be representable within another.

Definition 2.3.1. Coarse embeddings

Let (X_1, d_1) *and* (X_2, d_2) *be metric spaces. A map* $\phi : X_1 \to X_2$ *is a* coarse embedding *if there exist two functions* $\rho_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that* $\rho_\pm(r) \to \infty$ *as* $r \to \infty$ *and for all* $x, y \in X_1$,

$$\rho_-(d_1(x,y)) \leq d_2(\phi(x), \phi(y)) \leq \rho_+(d_1(x,y)).$$

We have seen that while this may seem like a very weak requirement it has powerful consequences (cf. Theorem 2.2.6). There are, however, examples of finitely generated groups which do not coarsely embed into any Hilbert space. This is because they coarsely contain a family of expander graphs.

Definition 2.3.2. Expander graphs

Let $\{\Gamma_n = V(\Gamma_n), E(\Gamma_n)) \mid n \in \mathbb{N}\}$ *be a collection of finite d-regular graphs such that* $|V(\Gamma_n)| \to \infty$ *as* $n \to \infty$*.* $\{\Gamma_n\}$ *is called a* family of expander graphs *if*

$$\inf_{n \in \mathbb{N}} (\mathrm{ch}(\Gamma_n)) > 0,$$

where the Cheeger constant of a finite graph Γ *is given by*

$$\mathrm{ch}(\Gamma) := \inf \left\{ \frac{|\partial(A)|}{|A|} \;\middle|\; |A| \leq \left| \frac{V(\Gamma)}{2} \right| \right\},$$

and the boundary of A *is given by* $\partial(A) := \{e \in E(\Gamma) \mid \iota(e) \in A,\ \tau(e) \notin A\}$.

Gromov demonstrated the existence of a finitely generated (and even finitely presented) group which coarsely contains a family of expanders (*Gromov's monster*), using a collection of expanders which cannot coarsely embed into any ℓ^p space with $p \in [1, \infty)$ [Gro00, Gro03, AD08]. The method is to use graphical small cancellation techniques which rely on the girth of the expander graphs (length of the shortest cycle) growing at the same rate as the diameter [Gro03, Oll06]. It is not known whether this family of expanders can be coarsely embedded into some other uniformly convex Banach space. There are, however, other families of expanders which cannot [Laf08], though there is no proof as yet that they can be coarsely embedded into any finitely generated group as they have uniformly bounded girth.

However, if we drop the assumption that the Banach spaces we are embedding into are super-reflexive, the following theorem shows that such examples above are not an obstruction.

Theorem 2.3.3. [BG05] - with extensions in [Kal07]

Every metric space with bounded geometry admits a coarse embedding into the Hilbertian sum $\bigoplus_{p\in\mathbb{N}} \ell^p(\mathbb{N})$.

Finally in this section, we give two conditions which guarantee the existence of a coarse embedding of a given group into a Hilbert space.

Example 2.3.4. Spaces admitting coarse embeddings

(i) Yu introduced property (A) as a non-equivariant form of amenability satisfied by both amenable and free groups [Yu00]. In the same paper he proves that any group with property (A) admits a coarse embedding into a Hilbert space.

(ii) Spaces with finite decomposition complexity have property (A), so coarsely embed into Hilbert spaces [GTY12].

To distinguish between groups admitting coarse embeddings, we can ask for additional conditions on the maps $\rho_\pm$ given by the definition of a coarse embedding. The first such work on this was that of Guentner and Kaminker in defining *compression exponents* [GK04].

2.3.2 Compression functions

Before giving any general definitions, we stop for a key example, non-abelian free groups.

Example 2.3.5. Embeddings of free groups into ℓ^p spaces

The Cayley graphs of free groups are infinite regular trees (up to quasi-isometry). In this situation we have a very good understanding of exactly which functions can appear as ρ_- in coarse embeddings (cf. Definition 2.3.1). Namely, given a function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and some $p \geq 2$ there exists a coarse embedding $\phi : \mathbb{F}_n \to \ell^p(\mathbb{N})$ with

$$f(d(x,y)) \leq \|\phi(x) - \phi(y)\|_p \leq d(x,y),$$

if and only if f satisfies Tessera's property (C_p),

$$\sum_n \frac{1}{n}\left(\frac{f(n)}{n}\right)^p < \infty. \tag{2.2}$$

For $p \in (1,2)$ any function satisfying property (C_p) appears as a compression function of some coarse embedding, while trees admit biLipschitz embeddings into $\ell^1(\mathbb{N})$ [Tes11, Gal08, Bou86, Bau07].

Definition 2.3.6. Compression exponents

Given two metric spaces (X, d_1) *and* (Y, d_2) *the* Y compression exponent *of* X, $\alpha^*_Y(X)$ *is given by the supremum of* $\alpha \in [0,1]$ *such that there is a map* $\phi: X \to Y$ *and constants* $K \geq 1$, $C \geq 0$ *such that*

$$K^{-1} d_1(x,y)^\alpha - C \leq d_2(\phi(x), \phi(y)) \leq K d_1(x,y) + C,$$

for all $x, y \in X$.

The affine upper bound given here is no restriction in the situation where X is a geodesic metric space. To see this, suppose that we are given a coarse embedding $\psi : (X, d_1) \to (Y, d_2)$, such that for every $x, y \in X$,

$$\rho_-(d_1(x,y)) \leq d_2(\psi(x), \psi(y)) \leq \rho_+(d_1(x,y).$$

We let $x, y \in X$ with $d_1(x,y) \in [n, n+1)$ and consider a geodesic $\underline{g} \in [[x,y]]$ with $x_1, x_2, \ldots, x_n \in \underline{g}$ arranged so that $d_1(x, x_m) = m$, then

$$\begin{aligned} d_2(\psi(x), \psi(y)) &\leq d_2(\psi(x), \psi(x_1)) + d_2(\psi(x_1), \psi(x_2)) \\ &\quad + \cdots + d_2(\psi(x_n), \psi(y)) \\ &\leq \rho_+(1) d_1(x,y) + \rho_+(1). \end{aligned}$$

Definition 2.3.6 can be generalised to embeddings of X into a family of spaces $\mathcal{Y}$ by setting $\alpha^*_{\mathcal{Y}}(X) \coloneqq \sup\{\alpha^*_Y(X) \mid Y \in \mathcal{Y}\}$. Most work on this constant focuses on the situation where X is the Cayley graph of a finitely generated group and $\mathcal{Y}$ is the family of all ℓ^p spaces for a fixed $p \in [1, \infty)$, where the values are denoted by $\alpha^*_p(X)$. Recall that for ℓ^∞ spaces there is no variety (Proposition 2.1.9).

It is known that every value in the interval $[0,1]$ occurs as the ℓ^p compression exponent of a finitely generated group for all $p \in (1, \infty)$ [ADS09]. The case for amenable groups is very different, as we shall see in Example 2.3.7.

A principal motivation for the work in Chapters 4 and 5 was to calculate the ℓ^p compression exponents of relatively hyperbolic and mapping class groups. We now provide some examples.

Example 2.3.7. Groups with known values of α^*_p

(i) Example 2.3.5 proves that every free group has compression exponent 1 for all $p \in [1, \infty)$.

(ii) Finitely generated virtually abelian groups A have $\alpha_p^*(A) = 1$ for all $p \in [1, \infty)$, this can obviously be obtained from the standard facts that A is virtually $\mathbb{Z}^n$ for some n, so the groups A and $\mathbb{Z}^n$ are quasi-isometric, and the standard inclusion of $\mathbb{Z}^n$ into $\ell^p(\{e_1, \dots, e_n\})$ is a quasi-isometry for each p.

(iii) Much less obviously, all polycyclic groups G - solvable groups with no infinitely generated subgroups - have $\alpha_p^*(G) = 1$ for all $p \in [1, \infty)$ [Tes11].

(iv) There are solvable groups G with $\alpha_2^*(G) = 0$, though the free solvable group $S_{r,d}$ of rank r and derived length d has $\alpha_p^*(S_{r,d}) \geq \frac{1}{d-1} \max\left\{\frac{1}{2}, \frac{1}{p}\right\}$ [Aus11, Sal12].

(v) A *wreath product* $G \wr H$ is defined as

$$\bigoplus_{h \in H} G_h \rtimes H,$$

where H acts on the normal subgroup by translation of the copies of G.
The wreath product $(\mathbb{Z})_1^{\wr} := \mathbb{Z} \wr \mathbb{Z}$ has Hilbert compression exponent $\frac{2}{3}$ [ANP09]. By iterating this wreath product, i.e. defining $(\mathbb{Z})_n^{\wr} = (\mathbb{Z})_{n-1}^{\wr} \wr \mathbb{Z}$, groups of ℓ^2 compression exponent $(2 - 2^{1-n})^{-1}$ for each n are obtained [Èrs03, NP08].

(vi) The wreath product $\mathbb{Z}^2 \wr \mathbb{Z}$ has ℓ^2 compression exponent $\frac{1}{2}$ [NP08].

The above list is exhaustive for currently known values the Hilbert compression exponent can take for amenable groups.

When specifically considering embeddings of groups, we can also consider the *equivariant compression exponent*, $\alpha_Y^{\#}(G)$, which is the same supremum as in Definition 2.3.6, but where we only consider G-equivariant embeddings, $\phi : G \to (Y, d_2)$ coming from some action of G on Y, i.e.

$$\phi(g) = g(y) \quad \text{for some fixed} \quad y \in Y.$$

It is clear that $\alpha_Y^{\#}(G) \leq \alpha_Y^*(G)$ for all G and Y but in the particular case of amenable groups G embedded into Hilbert spaces they are equal [AMM85, dCTV07].

These values are closely linked to the *speed of random walks* and amenability.

Definition 2.3.8. Speed of random walks

Let G be a finitely generated group and let S be a finite generating set. The speed of random walks $\beta_S^*(G)$ on G (with respect to S) is the supremum over all $\beta \in [0, 1]$ such that there exists a constant c_β with

$$\mathbb{E}(|W_t|) \geq c_\beta t^\beta,$$

where $\mathbb{E}(|W_t|)$ is the expected length of a random walk of length t, i.e. the average distance from the identity in Cay(G,S) of a concatenation of t independently chosen generators with repeats.

The speed of random walks $\beta^*(G)$ on G, is the supremum of $\beta^*_S(G)$ over all finite generating sets S.

There is no known example where the choice of finite generating set yields different values of $\beta^*_S(G)$. The link between speed of random walks and compression exponents is given by the following theorem.

Theorem 2.3.9. *Let G be a finitely generated group*

(i) *If G is amenable, then $\alpha^*_p(G) \leq \frac{1}{q\beta^*(G)}$, where $q = \max\{p, 2\}$,* [ANP09].

(ii) *$\alpha^{\#}_p(G) \leq \frac{1}{q\beta^*(G)}$, where $q = \max\{p, 2\}$,* [NP08].

Both of these inequalities are sharp, but neither is an equality, as the examples of Austin prove [Aus11].

This yields the following powerful link between amenability and compression exponents, coming from the fact that $\beta^*(G) < 1$ ensures G is amenable, though the converse does not hold.

Theorem 2.3.10. [GK04, Yu00] *Let G be a finitely generated group.*

(i) *If $\alpha^{\#}_p(G) > \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ then G is amenable.*

(ii) *If $\alpha^*_p(G) > \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ then G has Yu's property (A).*

Many groups have compression exponent 1 for all $p \in [1, \infty)$, as displayed in Example 2.3.7(i)-(iii), so we may ask which groups *attain* compression exponent 1. This is the subject of the next section.

2.3.3 Quasi-isometric embeddings

The following conjecture is due to de Cornulier-Tessera-Valette,

Conjecture 2.3.11. [dCTV07, Conjecture 1.1]

Let G be a finitely generated group which quasi-isometrically embeds into a Hilbert space. Then G is virtually abelian.

This conjecture states that although Hilbert compression exponent 1 is achieved by many groups, admitting a quasi-isometric embedding (i.e. attaining compression exponent 1) is highly restrictive. Proposition 2.1.9 states that no such restriction occurs for ℓ^∞ spaces, while all hyperbolic groups quasi-isometrically embed into ℓ^1 spaces [BS00, BDS07], for instance, free groups $\mathbb{F}_n := \langle s_1, s_2, \ldots, s_n \rangle$ admit the isometric embedding

$$\phi : \mathbb{F}_n \to \ell^1(\mathbb{F}_n) \text{ given by } \phi(x) = \sum_{y \in V(\underline{g_x})} y,$$

where $\underline{g_x}$ is the unique geodesic from x to e in the Cayley graph of $\mathbb{F}_n := \langle s_1, s_2, \ldots, s_n \rangle$.

One of the major tools used to approach this conjecture is the following metric interpretation of superreflexivity in terms of embeddings of rescaled - so every edge has length λ_n - finite depth binary trees $\lambda_n T_n^b$, due to Bourgain [Bou86].

Theorem 2.3.12. *A Banach space $(X, \|\cdot\|)$ is not super-reflexive if and only if there exist constants $K \geq 1$ and $C \geq 0$ and maps $\phi_n : \lambda_n T_n^b \to X$ - from (rescaled) finite binary trees of depth n, $(\lambda_n T_n^b, d_n)$, into G - such that for all $v, w \in V(T_n^b)$,*

$$K^{-1} d_n(v,w) - C \leq \|\phi_n(v) - \phi_n(w)\|_X \leq K d_n(v,w) + C.$$

Therefore any group admitting such a collection of well-embedded binary trees fails to have a quasi-isometric embedding into any uniformly convex Banach space.

Probabilistic combinatorial techniques of Benjamini-Schramm prove that every non-amenable group admits a quasi-isometrically embedded infinite binary tree, while results of de Cornulier-Tessera prove that every solvable group of exponential growth contains a quasi-isometrically embedded subsemigroup, so again such groups admit no quasi-isometric embedding into any uniformly convex Banach space [BS97, dCTV07, dCT08].

The case of nilpotent groups can currently only be resolved for Hilbert spaces, using results of [dCTV08]. We give a few details here. Any finitely generated nilpotent group G admits a torsion-free finite index subgroup. Either this subgroup is abelian, in which case, G is virtually abelian, or there is some element

$$z \in Z(G) \cap [G, G] \text{ such that } \langle z \rangle \cong \mathbb{Z}.$$

Using the fact that ℓ^2 compression is the same as ℓ^2 equivariant compression for finitely generated amenable groups - expressed in terms of functions, rather than just compression exponents [AMM85, Gro00, dCTV07] - we can instead consider 1-cocycles $b \in Z^1(G, \pi)$ where π is a unitary representation of G in a Hilbert space $\mathcal{H}$.

We split $\mathcal{H}$ into the subspace preserved by $\pi(Z(G))$ and its orthogonal complement

$$\mathcal{H} = \mathcal{H}_Z \oplus \mathcal{H}_Z^{\perp}$$

and obtain corresponding decompositions of $\pi = \pi_1 + \pi_2$ and $b = b_1 + b_2$.

Proving that G admits no quasi-isometric embedding is then equivalent to proving that $\|b(g)\|$ has sublinear growth. To do this, it is proved that $b_1(z) = 0$ and that $b_2(z)$ is approximated by bounded cocycles, and hence has sublinear growth. The first of these requires $z \in [G, G]$ and the second that it is central and of infinite order. □

There is no complementary result for other values of p, however, it is known that the discrete Heisenberg group does not embed into any ℓ^1 or ℓ^2 space [Pau01, CK10b, CK10a].

2.4 Coarse notions of dimension

In [GTY12] (cf. Theorem 2.2.4), it was proved that any group with *finite decomposition complexity* satisfies the stable Borel conjecture. This notion can be thought of as a "constructive" coarse embedding, linking it with the previous section, or as a weakening of finite asymptotic dimension. This correlation with large-scale notions of dimension is the focus of the current section.

Over the course of the section we provide many definitions, gradually becoming more restrictive. We then give a collection of examples (2.4.10), some of which will be required later in proceedings and others which highlight distinctions between the various methods of defining dimension.

2.4.1 Finite decomposition complexity

Here, we give the definition of this property and explain why it can be thought of as a "constructive" coarse embedding condition. In the following section (2.4.2) we show that it can also be considered as a notion of "finite dimensional".

Definition 2.4.1. Separated decompositions

*Let X be a metric space. For a fixed constant $r > 0$, a collection of subspaces $\{X_{ij} \mid i = 0, 1, \ldots, n-1\}$ of X is called an n-*piece, r-separated decomposition *of X if and only if*

- *the subspaces cover, so* $X = \bigcup_{i,j} X_{ij}$*,*
- *subspaces with the same* i*-index are* r*-separated, i.e. given* $i \in \{0, 1, \ldots, n-1\}$*,* $d(X_{ij}, X_{ij'}) \geq r$*, whenever* $j \neq j'$*.*

This can be extended by saying that a collection of subspaces $\mathcal{X}^m$ of X is an n-piece, r-separated decomposition of another collection of subspaces $\mathcal{X}^{m-1}$ of X if and only if every $X^{m-1} \in \mathcal{X}^{m-1}$ admits an n-piece, r-separated decomposition as above, and $\mathcal{X}^m = \left\{X_{ij}^{m-1}\right\}$.

We denote this $\mathcal{X}^{m-1} \xrightarrow[n]{r} \mathcal{X}^m$.

Definition 2.4.2. Finite decomposition complexity

A metric space X *has* finite decomposition complexity *if, given any finite sequence* $(r_i)_{i=0}^{n-1}$ *of natural numbers there is some set* $\mathcal{X}^{(r_0,\ldots,r_{n-1})}$ *such that* $\mathcal{X}^{()} = X$ *and*

$$\mathcal{X}^{(r_0,\ldots,r_{n-1})} \xrightarrow[2]{r_n} \mathcal{X}^{(r_0,\ldots,r_n)},$$

such that given any infinite sequence $(r_i)_{i=1}^{\infty}$ *there is some* m *satisfying*

$$\sup\left\{\operatorname{diam}(Y) \;\middle|\; Y \in \mathcal{X}^{(r_0,\ldots,r_{m-1})}\right\} < \infty.$$

The next theorem places this new definition in the context of the previous section [GTY11].

Theorem 2.4.3. *Let* X *be a metric space with bounded geometry and finite decomposition complexity. Then* X *has property* (A)*, in particular, for each* $p \in [1, \infty)$ *it admits a coarse embedding into some* ℓ^p *space.*

This gives no bound on compression exponent, even for groups, as every solvable - and indeed every elementary amenable - group has finite dimensional complexity, but there are solvable groups with compression exponent 0 [GTY11, Aus11]. As with coarse embeddings, our main focus will be to prove stronger constraints on groups, so we now move on to asymptotic dimension.

2.4.2 Asymptotic dimension

Originally introduced by Gromov as a coarse invariant of finitely generated groups, the notion of asymptotic dimension has since been extended to general metric spaces [Gro93].

Definition 2.4.4. Asymptotic dimension

Let (X, d) *be a metric space. We say* X *has* asymptotic dimension *at most* n*,* $\operatorname{asdim}(X) \leq n$ *if for every* $r > 0$ *there is an* $(n+1)$*-piece,* r*-separated decomposition*

$$X \xrightarrow[n+1]{r} \mathcal{X},$$

such that all subsets of $\mathcal{X}$ *are uniformly bounded.*

There is a a suprising degree of subtlety in the proof that any space with asymptotic dimension at most n has finite decomposition complexity [GTY12].

Although finite asymptotic dimension is more restrictive than finite decomposition complexity (cf. Example 2.4.10) it still gives no bounds on compression exponent, given the family of groups $(G_\alpha)_{\alpha\in[0,1]}$ with compression exponent α constructed in [ADS09] all have asymptotic dimension 2.

This is to be expected, as asymptotic dimension is a coarse invariant, and compression exponents were specifically designed to distinguish between behaviours of coarse embeddings [Gro93]. As we will use this fact later in the thesis, we now enumerate it as a point of reference.

Proposition 2.4.5. *Let* $\phi : X \to Y$ *be a coarse embedding, then* $\operatorname{asdim}(X) \leq \operatorname{asdim}(Y)$*.*

2.4.3 (Asymptotic) Assouad-Nagata dimension

Assouad-Nagata dimension is a linearly controlled version of asymptotic dimension, so is more restrictive. The asymptotic version (2.4.6) is a quasi-isometry invariant and allows some wild localised behaviour, the same is not true of Assouad-Nagata dimension (2.4.7) which observes the dimension on all scales. First we give the definitions.

Definition 2.4.6. Asymptotic Assouad-Nagata dimension

Let (X, d) *be a metric space. We say* X *has* asymptotic Assouad-Nagata dimension *at most* n *- denoted* $\operatorname{asdim}_{AN}(X) \leq n$ *- if there exists a constant* C*, such that for every* $r > 0$ *there is an* $(n+1)$*-piece,* r*-separated decomposition*

$$X \xrightarrow[n+1]{r} \mathcal{X},$$

such that all subsets of $\mathcal{X}$ *have diameter at most* $Cr + C$*.*

Definition 2.4.7. Assouad-Nagata dimension

Let (X,d) be a metric space. We say X has Assouad-Nagata dimension *at most n - denoted* $\dim_{AN}(X) \leq n$ *- if there exists a constant C, such that for every $r > 0$ there is an $(n+1)$-piece, r-separated decomposition*

$$X \xrightarrow[n+1]{r} \mathcal{X},$$

such that all subsets of $\mathcal{X}$ have diameter at most Cr.

It is clear from this that, given any metric space X,

$$\text{asdim}(X) \leq \text{asdim}_{AN}(X) \leq \dim_{AN}(X),$$

but in the case of discrete metric spaces the second of these inequalities is actually an equality [BDHM09].

Finite Assouad-Nagata dimension implies many useful properties for a group, namely, it bounds the topological dimension of asymptotic cones [DH08], it provides certain Lipschitz extension properties [BDHM09, LS05] and it guarantees ℓ^p compression exponent 1 for all $p \geq 1$ [Gal08].

While Assouad-Nagata dimension is a more restrictive condition than asymptotic Assouad-Nagata dimension, it is not a quasi-isometry invariant of a space, for instance, the universal cover of $S^1 \times S^k$ has asymptotic Assouad-Nagata dimension 1, but Assouad-Nagata dimension $k+1$. We will see a proof of this in Lemma 2.4.11. It is, however, a quasi-isometry invariant of finitely generated groups by the above note.

We progress now to an even more restrictive condition.

2.4.4 Embeddings into products of trees

Although, this is not usually considered as a dimension, any space admitting a quasi-isometric embedding into a finite product of simplicial trees is finite dimensional in a strong sense. This becomes more apparent if we think of trees as a natural generalisation of the typical 1-dimensional space, the real line.

Any group admitting a quasi-isometric embedding into a product of n metric trees has asymptotic Assouad-Nagata dimension at most n, while any group with finite asymptotic Assouad-Nagata dimension quasi-symmetrically embeds into a finite product of trees, in particular, it coarsely embeds into a finite product of trees [LS05].

A useful example, which will reoccur later, is that of right-angled Artin groups.

Example 2.4.8. Right-angled Artin groups

First, we recall the definition: let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simplicial graph. The *right-angled Artin group* of Γ is given by the finite presentation

$$A_\Gamma := \langle V(\Gamma) \,|\, R \rangle \quad \text{where} \quad R = \{[x,y] \mid xy \in E(\Gamma)\}\,.$$

Every virtually special group and every limit group (cf. Example 2.1.28(i)) can be quasi-isometrically embedded into a right-angled Artin group [HW08, Wis11].

The *right-angled Coxeter group* of Γ, is the quotient of A_Γ by $\langle\!\langle \{x^2 \mid x \in V(\Gamma)\} \rangle\!\rangle$, so has presentation

$$R_\Gamma := \langle V(\Gamma) \,|\, R \rangle \quad \text{where} \quad R = \left\{x^2 \mid x \in V(\Gamma)\right\} \cup \{[x,y] \mid xy \in E(\Gamma)\}\,.$$

Every right-angled Artin group A_Γ is a finite index subgroup of some right-angled Coxeter group $R_{\Gamma'}$ [DJ00]. Γ' is obtained by taking two disjoint copies of Γ, Γ^1 and Γ^2 with their vertex sets enumerated coherently - so the map $v_i^1 \mapsto v_i^2$ is a graph isomorphism - then adding the edges $v_i^1 v_j^2$ whenever $i \neq j$. The map $\phi : A_\Gamma \to R_{\Gamma'}$ given by $\phi(v_i) = v_i^1 v_i^2$ is a monomorphism onto a finite index subgroup of $R_{\Gamma'}$.

The figure below gives two examples of this construction in simple cases.

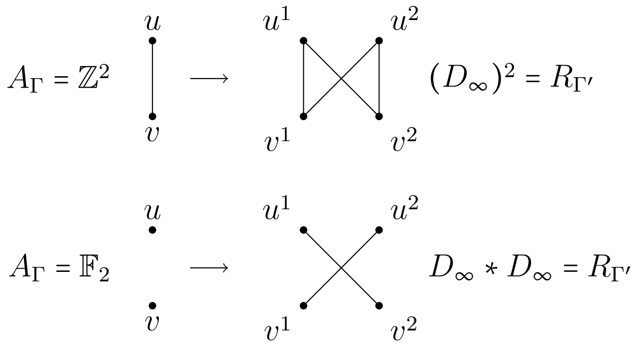

Figure 2.21: Constructing a commensurable Coxeter group

Here, $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group.

Then, we use the fact that right-angled Coxeter groups quasi-isometrically embed into a finite product of trees to deduce that the same is true of right-angled Artin groups [DJ99].

It is possible to take this one step further and ask which groups quasi-isometrically embed into $\mathbb{R}^n$ for some n. This is a (very) restricted case of Conjecture 2.3.11 where the conclusion is known to hold.

Proposition 2.4.9. *Let G be a finitely generated group which quasi-isometrically embeds into $\mathbb{R}^n$ equipped with some norm $\|\cdot\|$. Then G has a finite index abelian subgroup A of rank at most n.*

Proof: As every norm on $\mathbb{R}^n$ is biLipschitz equivalent, we may assume that we have a quasi-isometric embedding

$$\phi : G \to (\mathbb{R}^n, \|\cdot\|_2).$$

By considering the growth of balls in $\mathbb{R}^n$ and applying Gromov's polynomial growth theorem, we deduce that G is virtually nilpotent (with growth bounded by d^n for some d) [Gro81].

On the level of asymptotic cones this descends to a biLipschitz embedding of a nilpotent Lie group into $\mathbb{R}^n$, which is only possible if the Lie group is abelian [Gro81, Pau01]. If the asymptotic cone of G is abelian, then G is virtually abelian, so the result follows [Pan83]. □

The same method can be used to prove that any virtually nilpotent group which quasi-isometrically embeds into a Hilbert space is virtually abelian, except that we require the characterisation of inner product spaces given by Jordan and Von Neumann, to ensure that the asymptotic cone of a Hilbert space is itself a Hilbert space [JVN35].

2.4.5 Relationship between different notions of dimension

The primary purpose of this section is to provide illuminating examples to distinguish between the various notions of dimension previously introduced.

Example 2.4.10. Groups with various dimensional behaviour

(i) Any bounded metric space X has finite decomposition complexity (in 0 steps), asymptotic and asymptotic Assouad-Nagata dimension equal to 0 but can take any value of Assouad-Nagata dimension, for instance a closed ball in $\mathbb{R}^n$ has Assouad-Nagata dimension n. Naturally, X also quasi-isometrically embeds into a product of 0 trees.

(ii) A metric tree (one satisfying the 4-point property (cf. Definition 2.1.12) has Assouad-Nagata dimension 1 (unless it is of bounded diameter), it also quasi-isometrically embeds into a product of 1 simplicial tree (cf. Theorem 2.1.14). Quasi-trees have asymptotic Assouad-Nagata dimension 1 but can take any value of Assouad-Nagata dimension at least 1.

(iii) As $\operatorname{asdim}_{AN}(X \times Y) \leq \operatorname{asdim}_{AN}(X) + \operatorname{asdim}_{AN}(Y)$ any metric space X which quasi-isometrically embeds into a product of n trees has $\operatorname{asdim}_{AN}(X) \leq n$ [LS05].

(iv) $\operatorname{asdim}(\mathbb{R}^n) = \dim_{AN}(\mathbb{R}^n) = n$ and $\mathbb{R}^n$ quasi-isometrically embeds into a product of n trees, but no fewer.

(v) The discrete Heisenberg group H^3 has $\operatorname{asdim}(H^3) = \operatorname{asdim}_{AN}(H^3) = 3$, but fails to quasi-isometrically embed into any finite product of trees, or indeed any other CAT(0) space [DH08, Pau01].

(vi) If G is finite and non-trivial, then the wreath product $G \wr H$ (cf. Example 2.3.7(v)) has finite Assouad-Nagata dimension if and only if H has at most linear growth, while $\operatorname{asdim}(G \wr H) = \operatorname{asdim}(H)$ [Now07, BDL06].

(vii) The wreath product $\mathbb{Z} \wr \mathbb{Z}$ has infinite asymptotic dimension (as it coarsely contains $\mathbb{Z}^n$ for each n), but it does have finite decomposition complexity [GTY11].

We give one more instructive lemma here, to help place the difference between Assouad-Nagata and asymptotic Assouad-Nagata dimensions. This is required in a subtle way during Chapter 3.

Lemma 2.4.11. *Let $\widetilde{M}$ be the universal cover of a smooth compact n-manifold, then*

$$\dim_{AN}(\widetilde{M}) = \max\left\{\operatorname{asdim}_{AN}(\widetilde{M}), n\right\}.$$

Proof: The lower bound on $\dim_{AN}(\widetilde{M})$ is clear. As any discrete ϵ-net X of $\widetilde{M}$ has $\dim_{AN}(X) = \operatorname{asdim}_{AN}(\widetilde{M})$ and any sufficiently small ball in $\widetilde{M}$ is biLipschitz equivalent to a closed subset of $\mathbb{R}^n$, combining the definitions of asymptotic Assouad-Nagata dimension for X and Assouad-Nagata dimension for $\mathbb{R}^n$ yields the upper bound. □

2.4.6 Linking dimensions and embeddability

We present two results here which show that controls on the dimension of a space ensure the existence of coarse embeddings. The first concerns finite decomposition complexity.

Theorem 2.4.12. *Let X be a metric space with bounded geometry and finite decomposition complexity. Then, for every $p \in [1, \infty)$, X admits a coarse embedding into $\ell^p(\mathbb{N})$.*

Proof: Every space with bounded geometry and finite decomposition complexity has property (A) [GTY12] and every space with property (A) admits a coarse embedding into $\ell^p(\mathbb{N})$ [Yu00]. □

In particular, this holds for any metric space with bounded geometry and finite asymptotic dimension, as such spaces have finite decomposition complexity. However, there are interesting spaces with finite asymptotic dimension which do not have bounded geometry, for instance curve complexes. We can bypass bounded geometry in the this case using the following theorem [HR00].

Theorem 2.4.13. *Let X be a metric space with finite asymptotic dimension. Then, for every $p \in [1, \infty)$, X admits a coarse embedding into $\ell^p(\mathbb{N})$.*

There are examples of spaces which admit coarse embeddings into Hilbert spaces but do not have finite dimensional complexity [AGŠ12]. In the other direction, there are solvable groups of compression exponent 0 which have finite dimensional complexity (as they are elementary amenable), and non-amenable groups of asymptotic dimension 2 and any given compression exponent $\alpha \in [0, 1]$ [Aus11, GTY11, ADS09].

There is a strengthening of this result in the specific case of spaces with finite asymptotic Assouad-Nagata dimension [Gal08].

Theorem 2.4.14. *Let X be a metric space with finite asymptotic Assouad-Nagata dimension. Then, for all $p \geq 1$, $\alpha_p^*(X) = 1$.*

The link between finite Assouad-Nagata dimension and ℓ^p compression exponent 1 is less clear. A natural question given a space or construction which has or preserves one of the above properties is to ask how it behaves with respect to the other. Free products preserve finite Assouad-Nagata dimension [BH09], in fact,

$$\operatorname{asdim}_{AN}(G * H) = \max\{\operatorname{asdim}_{AN}(G), \operatorname{asdim}_{AN}(H), 1\},$$

while understanding the behaviour of ℓ^p compression under free products is the main aim of Section 4.4. In the same spirit, it is known that all polycyclic groups have ℓ^p compression exponent 1 [Tes11], but the corresponding result for Assouad-Nagata dimension remains, at present, a folklore theorem.

Chapter 3

Assouad-Nagata dimension of graph manifolds

It's people like that who make you realize how little you've accomplished. It is a sobering thought, for example, that when Mozart was my age, he had been dead for two years.

– Tom Lehrer

This brief chapter concerns joint work completed with Alessandro Sisto [HS11].

An introduction to the necessary details concerning graph manifolds can be found in Section 2.1.9, while the various definitions of dimension and their basic properties can be found in Section 2.4. We will recall the essential features of this discussion as we progress.

3.1 History and results

The main result of this chapter is the following:

Theorem 3.1.1. (cf. Theorem 1)

Let M be a graph manifold which does not have the Nil geometry. Then the universal cover of M quasi-isometrically embeds in a product of three metric trees.

This result is optimal for the following reasons. The asymptotic dimension of universal covers of closed non-geometric graph manifolds is known to be 3. This follows from bounds obtained in [BD08], the details of which are presented in [Smi10], so three metric trees are indeed necessary. Also, the Heisenberg group does not quasi-isometrically embed into any finite product of trees, so this exclusion is also needed [Pau01].

In addition, it is known that the Sol geometry quasi-isometrically embeds into $\mathbb{H}^2 \times \mathbb{H}^2$ [Gro93], and hence into a product of four trees but it is not known whether this can be improved to three [dC08, HP13].

As an application of this result, we determine the Assouad-Nagata dimension of the universal cover of closed graph manifolds.

Corollary 3.1.2. *Let M be a closed graph manifold. Exactly one of the following holds:*

(i) *M is geometric of type S^3, $\dim_{AN}(\widetilde{M}) = 3$ and $\operatorname{asdim}_{AN}(\widetilde{M}) = 0$,*

(ii) *M is geometric of type $S^2 \times \mathbb{R}$, $\dim_{AN}(\widetilde{M}) = 3$ and $\operatorname{asdim}_{AN}(\widetilde{M}) = 1$,*

(iii) $\dim_{AN}(\widetilde{M}) = \operatorname{asdim}_{AN}(\widetilde{M}) = 3$.

Smirnov [Smi10], showed that non-geometric graph manifolds have Assouad-Nagata dimension at most 7 and conjectured that it actually equals 3. Corollary 3.1.2 answers his conjecture positively.

Proof: The first two cases are self-explanatory, given Lemma 2.4.11. In all remaining cases, we prove that $\operatorname{asdim}_{AN}(\widetilde{M}) = 3$ and apply Lemma 2.4.11. If $\widetilde{M}$ has the Nil geometry, then $\operatorname{asdim}_{AN}(\widetilde{M}) = 3$ by [DH08]. In other geometric situations, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL_2}$, the asymptotic dimension is at least 3 and this provides the lower bound on asymptotic Assouad-Nagata dimension.

In the non-geometric situation, asymptotic dimension provides a lower bound on asymptotic Assouad-Nagata dimension, so this value is at least 3 [BD08]. Results in [LS05] prove $\operatorname{asdim}_{AN}(X) \leq n$ when X is an $n-$fold product of trees and $\operatorname{asdim}_{AN} A \leq \operatorname{asdim}_{AN} B$ whenever A admits a quasi-isometric embedding into B, so we get the upper bound on asymptotic Assouad-Nagata dimension via Theorem 3.1.1. This is deduced from the fact that any unbounded tree has asymptotic Assouad-Nagata dimension 1, and this value is sub-additive under taking Cartesian products of spaces [LS05]. □

We now move on to the proof of the main theorem of this chapter.

3.2 Proof of Theorem 3.1.1

We begin with the case of geometric graph manifolds. The universal covers of these are quasi-isometric to either S^3, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$. The first three of these obviously quasi-isometrically embed into a product of 0, 1 and 3 trees respectively, but no fewer. $\mathbb{H}^2$ quasi-isometrically embeds into a product of two trees by [BS05], so $\mathbb{H}^2 \times \mathbb{R}$ quasi-isometrically embeds into a product of three trees.

In what follows, we only have to consider non-geometric flip graph manifolds (cf. Definition 2.1.53), by Theorem 2.1.54. We recall the necessary consequences of this definition here.

Let M be a non-geometric flip graph manifold and let T be its Bass-Serre tree.

The universal cover $\widetilde{M}$ of M is constructed by suitably gluing certain metric spaces $\{X_v \coloneqq F_v \times \mathbb{R} \mid v \in T\}$. Each F_v is the universal cover of a compact surface with non-empty boundary. Applying a bi-Lipschitz transformation to $\widetilde{M}$ if necessary, we may assume - using Proposition 2.1.30 - that each F_v admits a metric retraction $r_v : F_v \to T_v$, where $T_v \subseteq F_v$ is a tree, with the further properties that r_v is an isometry when restricted to any boundary component of F_v and such that for each $x \in F_v$ we have $d_{F_v}(x, r_v(x)) \leq 1$. In particular, this implies that each F_v is a quasi-tree with bottleneck constant 2, (cf. Theorem 2.1.14).

Finally, the spaces are glued together in the following manner. Let v, w be adjacent vertices. There exist parametrisations $\gamma_v : \mathbb{R} \to F_v$ and $\gamma_w : \mathbb{R} \to F_w$ of boundary components of F_v, F_w so that $(\gamma_v(t), u) \in F_v \times \mathbb{R}$ is identified with $(\gamma_w(u), t) \in F_w \times \mathbb{R}$ for each $t, u \in \mathbb{R}$. (cf. Definition 2.1.53)

Step 1: Defining the three trees

The first tree will just be the Bass-Serre tree $T_0 = T$. We define the other two trees T_1 and T_2 as follows.

Subdivide the vertices of T into disjoint families V_1, V_2 such that if $v, v' \in V_i$ then $d_T(v, v')$ is even and set $T_i' = \bigsqcup_{v \in V_i} T_v$.

We now define an equivalence relation $\sim$ on T_i'. Suppose that $v, v' \in V_i$, $v \neq v'$ and there exists w such that $d_T(v, w) = d_T(v', w) = 1$. Given $x \in T_v$ and $x' \in T_{v'}$ we write $x \sim_d x'$, if there exist points $y \in F_v$ and $y' \in F_{v'}$ with the following properties

(i) $r_v(y) = x$ and $r_{v'}(y') = x'$,

(ii) the points in $F_w \times \mathbb{R}$ identified with $(y, 0) \in F_v \times \mathbb{R}$ and $(y', 0) \in F_{v'} \times \mathbb{R}$ have the same $\mathbb{R}$−coordinate.

Such a relation is clearly reflexive and symmetric. To ensure an equivalence relation, we set $\sim$ to be the transitive closure of $\sim_d$. The fact that these manifolds are flip means that the $\mathbb{R}$ factors of $F_v \times \mathbb{R}$ and $F_{v'} \times \mathbb{R}$ are both mapped into $F_w \times \mathbb{R}$ with a fixed second co-ordinate. The equivalence relation just identifies lines in T_i' for which this value is the same.

We define $T_i := T_i' / \sim$, this is a metric tree with countably many branching points. In fact, it can be described as the increasing union of metric spaces $\{Y_k\}_{k \in \mathbb{N}}$ such that Y_0 is a tree and Y_{k+1} is obtained from Y_k by identifying a line in Y_k with a line in some tree.

We denote the ℓ^1 product metric on $T_0 \times T_1 \times T_2$ by d.

Step 2: The components of the embedding

Define $f_0 : \widetilde{M} \to T_0$ to be any map such that for all $x \in \widetilde{M}$, $x \in F_{f_0(x)} \times \mathbb{R}$. Now we define $f_i : \widetilde{M} \to T_i$.

For each v, we let $\pi_v : F_v \times \mathbb{R} \to F_v$ be the projection on the first factor, and as usual denote the equivalence classes of $\sim$ with square brackets.

If $x \in F_v \times \mathbb{R}$ for some $v \in V_i$, then set $f_i(x) = [r_v(\pi_v(x))]$.

Otherwise we have $x \in F_w \times \mathbb{R}$ for $w \notin V_i$. Let $v \in V_i$ be any vertex such that $d_T(v, w) = 1$. Set $f_i(x) = [p]$ where $p \in T_v$ is such that $(p, 0)$ has, as a point in $F_w \times \mathbb{R}$, the same $\mathbb{R}$–coordinate as x. The equivalence relation states that this does not depend on the choice of v.

We now prove that the map

$$f : \widetilde{M} \to \prod_{i=0}^{2} T_i \quad \text{given by} \quad f(x) = (f_0(x), f_1(x), f_2(x))$$

is a quasi-isometric embedding.

Step 3: Metric comparisons

The easier inequality is $d(f(x), f(y)) \leq K d_{\widetilde{M}}(x, y) + C$: the maps π_v and r_v are 1-Lipschitz, so the same is true of f_1 and f_2, while f_0 satisfies $d_{T_0}(f_0(x), f_0(y)) \leq d_{\widetilde{M}}(x, y)/\rho + 1$ where

$$0 < \rho := \inf \{ d_{\widetilde{M}}(x, x') \mid x \in X_v,\ x' \in X_{v'},\ d_{T_0}(v, v') = 2 \}.$$

For the other inequality we start with a geodesic δ in $\prod T_i$ connecting $f(x)$ to $f(y)$ and construct a path γ in $\widetilde{M}$ connecting x to y such that $l(\gamma) \leq K l(\delta) + C$. Let δ_1, δ_2 be the projections of δ on the factors T_1 and T_2 respectively.

Suppose that $x \in X_{v_0}$, $y \in X_{v_n}$ and let $v_0, \dots, v_n$ be the vertices of T in the geodesic connecting v_0 to v_n.

For $j = 0, \dots, n$ let $i(j) \in \{1, 2\}$ be such that $v_j \in V_{i(j)}$ and choose $\alpha_j \subseteq \delta_{i(j)}$ so that $\alpha_j \subseteq \left[r_{v_j}\left(F_{v_j}\right)\right]$.

We require that the final point of α_j is the starting point of α_{j+2}, that the starting point of α_0 is $f_{i(0)}(x)$ and that the final point of α_n is $f_{i(n)}(y)$. This is easily arranged using the fact that each $[r_v(F_v)]$ is convex in the corresponding T_i.

For $j = 0, \dots, n-1$, let t_j be the $\mathbb{R}$−coordinate as a point in $F_{v_j} \times \mathbb{R}$ of $(p_j, 0) \in F_{v_{j+1}} \times \mathbb{R}$, where p_j is the starting point of α_{j+1}. Let t_n be the $\mathbb{R}$−coordinate of $y \in F_{v_n} \times \mathbb{R}$.

For $j = 0, \dots, n$ let γ_j be the path $\alpha_j \times t_j$ in X_{v_j}, i.e. the path with fixed $\mathbb{R}$ coordinate t_j. Notice that the distance between the final point of γ_j and the starting point of γ_{j+1} is at most 2μ, so we can concatenate the paths γ_j with n geodesics of length at most 2μ to obtain a path γ from x to y. Clearly $l(\gamma_j) = l(\alpha_j)$ so

$$\begin{aligned} l(\gamma) &\leq \sum l(\gamma_j) + 2n\mu \\ &= l(\delta_1) + l(\delta_2) + 2n\mu \\ &= d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2n\mu. \end{aligned}$$

Finally, as $d(f_0(x), f_0(y)) \geq n - 2$ we have

$$l(\gamma) \leq d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2\mu d(f_0(x), f_0(y)) + 4\mu,$$

and we are done. □

3.3 Subsequent results

Since this work was carried out, two extensions have been made to it by Smirnov and Mackay-Sisto. The first of these defines a class of higher dimensional manifolds called *orthogonal graph manifolds* for which the above method can be generalised. The orthogonality condition is to avoid requiring a higher dimensional analogue of the reduction to flip graph manifolds ensured by [KL98] (cf. Theorem 2.1.54). Smirnov's main result is to prove that n-dimensional orthogonal graph manifolds quasi-isometrically embed into a product of n metric trees [Smi12].

The controls obtained in Theorem 1 and Corollary 3.1.2 are used in a key way by Mackay-Sisto when calculating bounds on the Assouad-Nagata dimension and eco-dimension (eco-dim(X): the minimal number of trees such that X quasi-isometrically embeds into a product of n metric trees) for 3-manifold groups [MS12].

Chapter 4

Embeddings of relatively hyperbolic spaces with optimal ℓ^p compression exponent

When I use a word, it means just what I choose it to mean — neither more, nor less.

Through the Looking-Glass, and What Alice Found There

– Lewis Carroll

4.1 Introduction and statement of results

Recall that a *coarse (or uniform) embedding* is a map $\phi : (X, d) \to (Y, d')$ for which there are functions $\rho_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\rho_\pm(r) \to \infty$ as $r \to \infty$ and $\rho_- (d(x_1, x_2)) \leq d'(\phi(x_1), \phi(x_2)) \leq \rho_+(d(x_1, x_2))$.

More specifically we measure how 'faithfully' a group embeds into a space using compression exponents, $\alpha^*_Y(G)$ which is defined to be the supremum of those values $\alpha \in [0, 1]$ for which there exists some Lipschitz coarse embedding $\phi : G \to Y$ with $\rho_-(n) \geq C^{-1}n^\alpha - C$ for some constant C.

A more thorough overview of this can be found in Sections 2.3.1 and 2.3.2.

In [Gro87], Gromov introduced relatively hyperbolic groups as a generalisation of hyperbolic groups. The class of relatively hyperbolic groups includes: hyperbolic groups, amalgamated products and HNN-extensions over finite subgroups, fully residually free (limit) groups [Dah03a, Ali05] - which are key objects in solving the Tarski conjecture [Sel01, KM10], geometrically finite Kleinian groups and fundamental groups of non-geometric closed 3-manifolds with at least one hyperbolic component [Dah03a].

Relative hyperbolicity has many different characterisations: in terms of group actions [Bow12], group-theoretic structure [Far98] [Dah03b] [Osi06], topology [Yam04] and metric geometry [DS05]. Two definitions of relative hyperbolicity are given in Section 2.1.6.

Our main result in this chapter is the following:

Theorem 4.1.1. (cf. Theorem 4)

Let G be a finitely generated group which is hyperbolic relative to a collection of subgroups $\{H_i\}$. For all $p > 1$, $\alpha_p^*(G) = \min\{\alpha_p^*(H_i)\}$.

In fact, we obtain this result in greater generality (Theorem 4.5.4), and use it to calculate the ℓ^p compression of closed 3-manifolds.

Corollary 4.1.2. (Corollary 5)

Let M be a closed 3-manifold, then for all $p > 1$ and all $\alpha \in (0,1)$, there exists a map ϕ from $\pi_1(M)$ into some ℓ^p space, such that for all $x, y \in \pi_1(M)$,

$$(d(x,y))^\alpha \leq \|\phi(x) - \phi(y)\|_p \leq d(x,y).$$

So $\alpha_p^*(\pi_1(M)) = 1$ for all $p > 1$.

Some estimates on compression exponents are already known for particular kinds of relatively hyperbolic groups. Indeed, Sela proved in [Sel92] the coarse embeddability of hyperbolic groups into Hilbert spaces, his direct methods yielding a lower bound of $\frac{1}{2}$ on compression exponent. For free groups, compression exponent 1 was determined by Guentner and Kaminker, with refinements by Brodskiy and Higes and optimal results on compression obtained independently by Gal and Tessera [GK04, BS08, Gal08, Tes11]. A proof that general hyperbolic groups have ℓ^p compression exponent 1 for all $p \in [1, \infty)$ can be obtained using major theorems from [BS00] and [BS05, BDS07] or from [Tes11].

All relatively hyperbolic groups admit coarse embeddings into Hilbert spaces (provided that their maximal peripheral subgroups do) [DG07], (previous results can be found in [CDGY03, DG03]), but their methods do not generalise to ℓ^p spaces and provide no bound on compression exponent so a completely different technique is required to progress here. However, bounds are available in restricted cases. Dreesen [Dre10], proves that given finitely generated groups A, B and C, where C is a finite subgroup of A and B, $\min\{\alpha_2^*(A), \alpha_2^*(B), \frac{1}{2}\} \leq \alpha_2^*(A *_C B)$ and $\min\{\alpha_2^*(A), \frac{1}{2}\} \leq \alpha_2^*(\mathrm{HNN}(A, C, \theta))$.

For limit groups Theorem 4.1.1 also follows from work of Wise, who proves that limit groups embed quasi-isometrically into right-angled Artin groups, so these specific relatively hyperbolic groups have ℓ^p compression exponent 1 for all $p \geq 1$ [Wis11, DJ00, DJ99, GK04] (cf. Examples 2.1.28(i) and 2.4.8).

In order to make the intricate proof of Theorem 4.1.1 more transparent, we approach it via new results for two key sub-collections. The methods presented aid the intuition and clarify the strategy behind the final proof. In Section 4.3 we provide a new, direct, self-contained proof that the ℓ^p compression of any hyperbolic group is 1. This uses ideas from [Tu01] to construct weighted cylinders around geodesics which have good cancellation conditions imposed by the hyperbolic structure. Then we find embeddings of amalgamated products and HNN-extensions over finite groups displaying the exact compression exponent (Section 4.4), by applying a careful weighting procedure to Dreesen's method. A naïve implementation of these two methods will never yield a Lipschitz embedding in the general situation, so for the final proof we are required to refine these two arguments and ensure that a suitable combination of them also provides an optimal lower bound on compression exponent.

4.2 Preliminaries

Here we define collections of functions that will be used throughout this paper as lower bounds to the coarse embeddings we construct, following this we present some initial facts about them. Definitions and results are modelled on similar ideas found in [Tes11], (cf. Example 2.3.5).

Definition 4.2.1. Concave functions and property (C_p^c)

We will call a function $f:\mathbb{N}\to[0,\infty)$ concave *if f is non-decreasing and for all $m,n\in\mathbb{N}$ with $n\geq m$:*

$$f(n+m)-f(n)\leq f(n)-f(n-m).$$

This is based on the usual concavity condition $f''\leq 0$ given for smooth functions.

Let $f:\mathbb{N}\to[0,\infty)$ be a concave function satisfying Tessera's property (C_p),

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f(n)}{n}\right)^p<\infty.$$

f is said to satisfy (C_p^c) if, in addition, $\frac{f(n)^p}{n}$ is non-decreasing for all n sufficiently large.

We observe here that the class of functions satisfying (C_1^c) is empty, but for all $\epsilon > 0$ and all $p > 1$,

$$f(n) = \frac{n}{(\log_2(n+2)(\log_2\log_2)^{1+\epsilon}(n+2))^{\frac{1}{p}}}$$

has property (C_p^c).

We now present two technical lemmas which will later explain the necessity of the preceding definition. Lemma 4.2.2 will provide information about the lower bounding function ρ_- in the coarse embeddings we construct, while Lemma 4.2.3 is used to ensure the defined embeddings are Lipschitz.

Lemma 4.2.2. *Let M be a finite subset of $\mathbb{N}$ such that $M = \{m_1, m_2, \ldots, m_{2k}\}$ with $m_i < m_{i+1}$ and $m_1 \geq 1$.*

Let $p \geq 1$ and let $f : \mathbb{N} \to [0, \infty)$ be a concave function such that $\frac{f(n)^p}{n}$ is non-decreasing. Then

$$\sum_i \frac{f(m_{2i})^p}{m_{2i}}(m_{2i} - m_{2i-1}) \geq \left(\frac{1}{2}\right)^{3+p} f\left(\sum_i m_{2i} - m_{2i-1}\right)^p.$$

Proof: For ease of notation we set $m = \sum_{i=1}^{k} m_{2i} - m_{2i-1}$.

As $\frac{f(n)^p}{n}$ is non-decreasing,

$$\sum_{i=1}^{k} \frac{f(m_{2i})^p}{m_{2i}}(m_{2i} - m_{2i-1}) \geq \sum_{n=1}^{m} \frac{f(n)^p}{n}.$$

We then rewrite $\sum_{n=1}^{m} \frac{f(n)^p}{n}$ as follows

$$\begin{aligned} \sum_{n=1}^{m} \frac{f(n)^p}{n} &\geq \sum_{n=m/2}^{m} \frac{1}{n} f([m/2])^p \\ &\geq \frac{1}{4} f([m/2])^p \geq \left(\frac{1}{2}\right)^{3+p} f(m). \end{aligned}$$

□

Lemma 4.2.3. *Let M be a finite subset of $\mathbb{N}$ such that $M = \{m_1, m_2, \ldots, m_{2k}\}$ with $m_i < m_{i+1}$ and $m_1 \geq 1$.*

Let $p \geq 1$ and let $f : \mathbb{N} \to [0, \infty)$ be a concave function with property (C_p). Then there exists some uniform constant C such that

$$\sum_i \left(\frac{f(m_{2i})}{m_{2i}}\right)^p \frac{m_{2i} - m_{2i-1}}{m_{2i}} \leq C.$$

Moreover, if for each i, $m_{2i} \leq 2m_{2i-1}$, then

$$\sum_i \left(\frac{f(m_{2i-1})}{m_{2i-1}}\right)^p \frac{m_{2i} - m_{2i-1}}{m_{2i-1}} \leq 2^{p+1}C.$$

Proof: As f is concave $\frac{f(n)}{n}$ is non-increasing. Hence, for each i,

$$\left(\frac{f(m_{2i})}{m_{2i}}\right)^p (m_{2i} - m_{2i-1}) \leq \sum_{n=m_{2i-1}+1}^{m_{2i}} \left(\frac{f(n)}{n}\right)^p.$$

Therefore,

$$\sum_i \left(\frac{f(m_{2i})}{m_{2i}}\right)^p \frac{m_{2i} - m_{2i-1}}{m_{2i}} \leq \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{f(n)}{n}\right)^p$$

which is uniformly bounded as f has property (C_p).

For the second part just notice that $\frac{m_{2i}-m_{2i-1}}{m_{2i-1}} \leq 2\frac{m_{2i}-m_{2i-1}}{m_{2i}}$ and as f is non-decreasing,

$$\left(\frac{f(m_{2i-1})}{m_{2i-1}}\right)^p \leq 2^p \left(\frac{f(m_{2i})}{m_{2i}}\right)^p.$$

□

With this complete, we now move to the first step towards proving Theorem 4.1.1: building an explicit embedding of hyperbolic groups with optimal compression exponent.

4.3 Hyperbolic metric spaces

In this section we provide a short, self-contained and explicit method of embedding uniformly discrete hyperbolic metric spaces with bounded geometry into ℓ^p spaces with optimal compression exponent.

We first require the following basic lemma in hyperbolic geometry.

Lemma 4.3.1. *Let X be a δ-hyperbolic metric space and let $e \in X$. Let $n \geq 3\delta$ and let $x, y \in X$ with $d(x,e) \geq n$ and $d(x,y) \leq \frac{n}{4}$. For all geodesics $\underline{g_0} \in [[x,e]]$, $\underline{g} \in [[y,e]]$ and points $p \in \underline{g}([n,2n])$,*

$$d\left(p, \underline{g_0}([n/2, 5n/2])\right) \leq 3\delta.$$

Proof: We use the Rips definition of hyperbolicity, so in a geodesic triangle any edge is contained in the union of the δ-neighbourhoods of the other two. Select $p \in \underline{g}([n,2n])$, if p lies within the δ-neighbourhood of $\underline{g_0}$ then we are done as a sufficiently close point must lie within the required range.

Alternatively, p must lie within the δ-neighbourhood of any geodesic in $[[x,y]]$.

Let z be a point on some geodesic in $[[x,y]]$ with $d(p,z) < \delta$, then

$$d(x,p) \leq d(x,z) + d(z,p) \leq \frac{n}{4} + \delta.$$

However, $d(x,p) \geq d(y,p) - d(x,y) \geq \frac{3n}{4}$, which is a contradiction as $n \geq 3\delta$. □

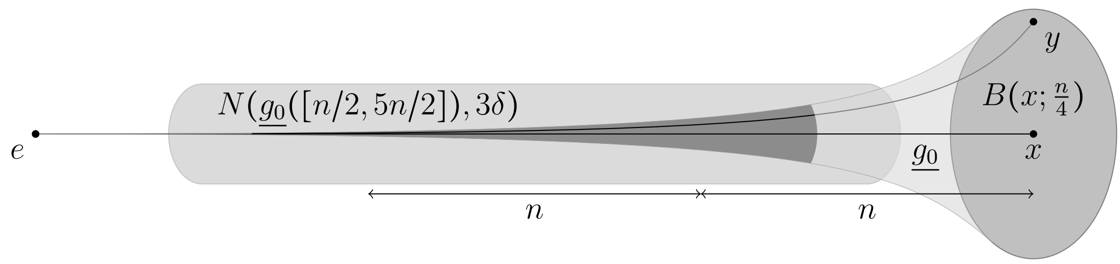

Figure 4.1: The conclusion of Lemma 4.3.1

Theorem 4.3.2. (cf. Theorem 2)

Let X be a countable uniformly discrete δ-hyperbolic metric space with bounded geometry. Then given any $p \geq 1$ and any concave function f with property (C_p) there exists a map $\phi : X \to \bigoplus_n \ell^p(X)$ such that for all $x, y \in X$,

$$f(d_X(x,y)) \leq \|\phi(x) - \phi(y)\|_p \leq d_X(x,y).$$

In particular, for all $p \geq 1$, $\alpha_p^*(X) = 1$.

Throughout this chapter, any direct sum of ℓ^p spaces is equipped with the ℓ^p norm, so all such spaces are isometric to $\ell^p(\mathbb{N})$. We include the additional detail to more clearly define how each embedding is constructed.

Proof: We can reduce our problem to the case where X is the 0-skeleton of a connected simplicial graph, using [Tu01, Lemmas 4.1 and 7.3]. As X has bounded geometry (cf. Definition 2.1.4) we can define $N(k)$ to be a bound on the cardinality of any ball of radius k.

Fix a basepoint $e \in X$. Given the following collection of restricted geodesics

$$\underline{G_{x,k,n}} := \left\{ \underline{g}([n, 2n]) \;\middle|\; \underline{g} \in [[y, e]] \text{ for some } y \text{ with } d(x,y) \leq k \right\},$$

we define $F_{x,k,n}$ to be the set of all points in X lying on some $\underline{g} \in \underline{G_{x,k,n}}$ but not in $B(e; 3\delta)$ and set $F(x,k,n)$ to be the characteristic function of $F_{x,k,n}$. We use the bounded geometry of X to ensure that $F_{x,k,n}$ is a finite set, thus $F(x,k,n) \in \ell^p(X)$.

We then average these functions over all suitable values of k,

$$H(x,n) = \frac{1}{n} \sum_{k \le \frac{n}{4}} F(x,k,n).$$

For increasing values of k, the collection of all points on a geodesic in the set $G_{x,k} := \left\{ \underline{g} \in [\![y,e]\!] \mid d_X(x,y) \le k \right\}$ forms a sequence of 'trumpets'.

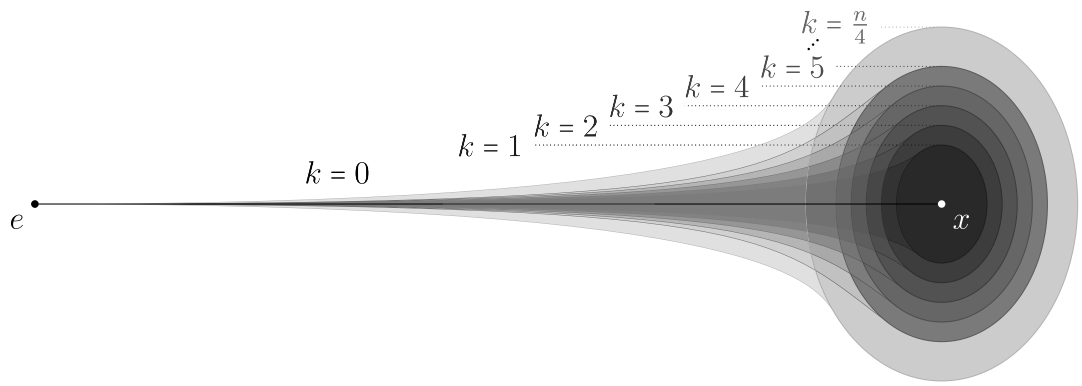

Figure 4.2: A weighted sum of hyperbolic 'trumpets'

The restriction of these to the desired interval (after rescaling) gives the function $H(x,n)$:

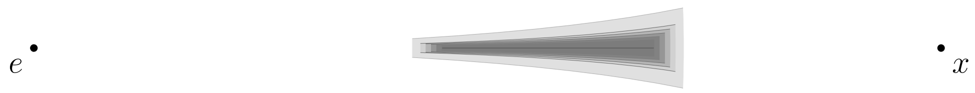

Figure 4.3: Restriction to the function $H(x,n)$

The following three lemmas provide bounds on the p-norms of these functions.

Lemma 4.3.3. *There exists some constant C such that for all $x \in X$, $k \le \frac{n}{4}$ and $n \in \mathbb{N} \smallsetminus \{0\}$, $\|F(x,k,n)\|_p^p \le Cn$.*

If, in addition, $d(x,e) \ge 2n$, then $n - 3\delta \le \|F(x,k,n)\|_p^p$

Proof: The first inequality is obvious as $|F_{x,k,n}| \ge n - 3\delta$. For the second we use Lemma 4.3.1 and the bounded geometry of X,

$$\|F(x,k,n)\|_p^p \; = \; \|F(x,k,n)\|_1 \; \le \; (2n+1)N(3\delta) \; \le \; 3N(3\delta)n \le Cn$$

completing the proof. □

Lemma 4.3.4. *If $d(x,y) \le R$ then $\|H(x,n) - H(y,n)\|_p \le 2C(R+1)n^{-\frac{p-1}{p}}$.*

Proof: Choose $x, y \in X$ with $d(x,y) \le R$. Then as $F_{x,k,n} \subseteq F_{y,k+R,n}$,

$$\begin{aligned}\sum_{0\le k\le \frac{n}{4}} F(x,k,n) &\le \sum_{\frac{n}{4}-R\le k\le \frac{n}{4}} F(x,k,n) + \sum_{0\le k<\frac{n}{4}-R} F(y,k+R,n) \\ &\le \sum_{\frac{n}{4}-R\le k\le \frac{n}{4}} F(x,k,n) + \sum_{0\le k\le \frac{n}{4}} F(y,k,n).\end{aligned}$$

Switching x and y in the above argument we conclude that

$$\begin{aligned}\|H(x,n)-H(y,n)\|_p^p &\le \frac{1}{n^p}\sum_{\frac{n}{4}-R\le k\le \frac{n}{4}} \|F(x,k,n)\|_p^p + \|F(y,k,n)\|_p^p \\ &\le \frac{1}{n^p}(2C(R+1))n \le (2C(R+1))n^{-(p-1)}.\end{aligned}$$

Notice we have made no assumption that $H(x,n)$, $H(y,n) \ne 0$. □

Lemma 4.3.5. *$\|H(x,n)\|_p^p \le n$, and whenever $d(x,e) \ge 2n$, $\|H(x,n)\|_p^p \asymp n$.*

Proof: For the lower bound on $\|H(x,n)\|_p^p$ we notice that given any fixed geodesic $\underline{g} \in [[x,e]]$, $\underline{g}([n,2n]) \subseteq F(x,k,n)$ for all k.

Hence, the function $H(x,n)$ has at least $n-3\delta$ points on which it takes value at least $\frac{1}{4}$, so the lower bound is justified.

As an upper bound,

$$\|H(x,n)\|_p \le n^{-1}\sum_{k\le \frac{n}{4}} \|F(x,k,n)\|_p \le n^{-1}\left(\frac{n}{4}+1\right)(Cn)^{\frac{1}{p}} \le n^{\frac{1}{p}}.$$

□

With these three lemmas we are now in a position to define our embedding $\phi : X \to \bigoplus_n \ell^p(X)$.

$$\phi(x) := \sum_{n\ge 1} \frac{f(2^n)}{2^{\frac{n}{p}}} H(x,2^n).$$

To show ϕ is Lipschitz, consider $x, y \in X$ with $d(x,y) \le R$. Then, using Lemma 4.3.4:

$$\begin{aligned}\|\phi(x)-\phi(y)\|_p^p &\le \sum_{n=1}^{\infty} \frac{f(2^n)^p}{2^n} \|H(x,2^n)-H(y,2^n)\|_p^p \\ &\le \sum_{n=1}^{\infty}\left(\frac{f(2^n)}{2^n}\right)^p.\end{aligned}$$

As f is concave and has property (C_p), $f(2^n)^p \le 2^{p+1} \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i} f(i)^p$. Thus

$$\|\phi(x)-\phi(y)\|_p^p \le \sum_{n=1}^{\infty}\left(\frac{f(2^n)}{2^n}\right)^p \le \sum_{n=1}^{\infty} 2^{-np} \sum_{i=2^n+1}^{2^{n+1}} \frac{1}{i} f(i)^p \le \sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{f(i)}{i}\right)^p \le 1.$$

For the lower bound on ϕ, consider two points $x, y \in X$, with $d(x,y) > 12\delta$. We assume, without loss of generality, that $d(x,e) \geq d(y,e)$.

We wish to find a value k_x such that $2^{k_x} \asymp d(x,y)$ and for all $n \in \{1, 2, \ldots, k_x\}$, the functions $H(x, 2^n)$ and $H(y, 2^n)$ have disjoint support. Lemma 4.3.1 implies that setting

$$k_x := \lfloor \log_2 ((x.y)_e - 5\delta) \rfloor$$

suffices, where $(x.y)_e = \frac{1}{2}(d(x,e) + d(x,y) - d(y,e))$ is the Gromov product.

Then, by Lemma 4.3.5,

$$\|\phi(x) - \phi(y)\|_p^p \geq \sum_{n=1}^{k_x} \frac{f(2^n)^p}{2^n} \|H(x,2^n)\|_p^p \geq \sum_{n=1}^{k_x} f(2^n)^p \geq f(d(x,y))^p.$$

The final step is due to the fact that f is concave and has property (C_p). □

4.4 Tree-graded spaces

During this section, we prove that the compression exponent of amalgamated products and HNN extensions over finite groups depend only on the compression of the initial groups. Specifically we prove that any tree-graded graph (cf. Definition 2.1.20) can be metrically 'decomposed' into a collection of pieces and an underlying tree. Embeddings of tree-graded spaces are found by embedding the two 'components' separately, in such a way as to preserve the metric of the original space.

We begin with the definition of a tree-graded graph.

Definition 4.4.1. *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a connected simplicial graph. We say Γ is* tree-graded *(in the sense of Druţu and Sapir,* [DS05]*), with respect to a collection of non-empty connected subgraphs $\mathcal{P} := \{\Gamma_i\}_{i \in I}$ if the following properties the satisfied.*

(i) *Every vertex and every simple loop of Γ is contained in some Γ_i.*

(ii) *If $i \neq j$, then $\Gamma_i \notin \Gamma_j$ and $|V(\Gamma_i) \cap V(\Gamma_j)| \leq 1$.*

In particular, given two finitely generated groups A and B with finite generating sets S_A and S_B respectively, the Cayley graph $\text{Cay}(A * B, S_A \sqcup S_B)$ is tree-graded with pieces given by the cosets of A and B in G.

The main result of this section is

Theorem 4.4.2. *Let Γ be a simplicial graph which is tree-graded with respect to the collection of pieces $\mathcal{P} = \{\Gamma_i\}_{i\in I}$. Suppose we are given a concave function $\rho' : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and a collection of coarse embeddings of pieces $\psi_i : \Gamma_i \to \ell^p(X_i)$ such that for all $x, y \in \Gamma_i$,*

$$\rho'(d_\Gamma(x,y)) \leq \|\psi_i(x) - \psi_i(y)\|_p \leq d_\Gamma(x,y).$$

If $p = 1$, then there is a coarse embedding ϕ of Γ into an ℓ^1 space with

$$\rho'(d_\Gamma(x,y)) \leq \|\phi_i(x) - \phi_i(y)\|_p \leq d_\Gamma(x,y).$$

For $p > 1$, given any function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfying (C_p) there is a coarse embedding ϕ of Γ into an ℓ^p space with

$$\rho(d_\Gamma(x,y)) \leq \|\phi_i(x) - \phi_i(y)\|_p \leq d_\Gamma(x,y),$$

where $\rho(n) = \min\{\rho'(n), f(n)\}$.

We will provide a detailed set-up of the problem which makes the final proof rather short.

Notice that we have made no assumption that Γ has any finiteness property, in particular, we have not assumed I is countable. This means that to be absolutely technically accurate we are viewing $\ell^p(X)$ as the space of all countably supported functions $f : X \to \mathbb{R}$ with $\|f\|_p^p := \sum_{i\in I} |f(i)|^p < \infty$. For our purposes, finitely supported will suffice.

Theorem 4.4.2 allows us to calculate compression exponents of certain groups. The list of immediate consequences below is exhaustive by Stallings' Theorem (cf. Theorem 2.1.21 and [Sta68, Sta71]).

Corollary 4.4.3. (cf. Theorem 3)

Let G and H be finitely generated groups and let F be a finite subgroup of G and H. For all $p \geq 1$,

(i) $\alpha_p^*(G *_F H) = \min\{\alpha_p^*(G),\ \alpha_p^*(H)\}$ *and*

(ii) $\alpha_p^*(\mathrm{HNN}(G, F, \theta)) = \alpha_p^*(G)$.

Proof: It is an obvious consequence of Theorem 4.4.2 that (i) holds whenever F is the trivial group.

If G and H are both finite, then $G *_F H$ and $\mathrm{HNN}(G, F)$ are both hyperbolic by

[Gro87], so the result holds. If G is infinite then $\mathrm{HNN}(G, F)$ is quasi-isometric to $G * \mathbb{Z}$, similarly, if at least one of G or H is infinite then $G *_F H$ is quasi-isometric to $G * H$ [PW02]. $\mathbb{Z}$ isometrically embeds into any non-trivial ℓ^p space, so we may choose $\rho'(n) = n$, completing the result. □

The following lemma is a key tool in this section.

Lemma 4.4.4. *Let Γ be a tree-graded graph with respect to the collection of subgraphs $\mathcal{P} := \{\Gamma_i\}_{i \in I}$ and fix a basepoint $e \in V(\Gamma)$. For each $x \in V(\Gamma)$ there are finite sets $I_x = \{i_0, i_1, \dots, i_k\}$ and $\{x_0, \dots, x_k = x\}$ such that any geodesic $\underline{g} \in [\![e, x]\!]$ can be decomposed into a concatenation of subgeodesics*

$$\underline{g} = \underline{g_0}\,\overline{g_0}\,\underline{g_1} \cdots \overline{g_{k-1}}\,\underline{g_k},$$

such that for each j,

(i) *$\underline{g_j} \subseteq \Gamma_j$, where $\iota(\underline{g_j}) = e_j$ is the unique vertex of Γ_j at minimal distance from $e_0 = e$ and $\tau(\underline{g_j}) = x_j$,*

(ii) $\left|\overline{g_j}\right| \leq 1$.

Proof: $\underline{g}$ intersects a finite number of pieces, $\{\Gamma_i \mid i \in I_x\}$ and $\underline{g_i} := \underline{g} \cap \Gamma_i$ is connected, so is a sub-geodesic. $\underline{g_i}$ and $\underline{g_{i+1}}$ have at most one vertex in common, by definition 4.4.1(ii). If they are disjoint, then there is a path of length 1 between them (otherwise Γ must meet another piece, or I_x has been ordered incorrectly). This then gives a decomposition satisfying (i) and (ii).

Now take the above decomposition of any other geodesic in $[\![e, x]\!]$. Any counterexample to this lemma forces there to be a simple loop which is not contained in a single piece. □

In what follows we will assume, by applying a translation if necessary, that $\psi_i(e_i) = 0$ for all $i \in I$.

Returning to the free product example $G = A * B$, if we are given some word $x = a_0 b_1 a_2 \dots a_{n-1} b_n$, where $a_i \in A \smallsetminus \{1_A\}$ and $b_i \in B \smallsetminus \{1_B\}$, each $\overline{g_j}$ is just the vertex $x_j = a_0 b_1 \dots c_j$, where $c \in \{a, b\}$ depends only on the parity of j. Let us assume $c = a$, the other case is similar. Each $\underline{g_j}$ is a geodesic in the coset $x_{j-1}A$ from x_{j-1} to x_j, although there may be many choices of such geodesic, they have the same end points.

Definition 4.4.5. *Let* Γ *be a tree-graded graph with pieces* $\{\Gamma_i \mid i \in I\}$ *and let* $e \in V(\Gamma)$. *The* e-distance tree *of* Γ *is the simplicial tree* T^e_Γ*, obtained from* Γ *by identifying vertices under the relation* $x \sim y$ *if and only if* $x, y \in \Gamma_i$ *for some* $i \in I$ *and* $d(e_i, x) = d(e_i, y)$.

This is not analogous to the coned-off graph of a relatively hyperbolic group, where we imagine collapsing each Γ_i to a point, instead we are projecting them to rays.

This leads to a slightly non-trivial tree: even in simple situations like the free product of two groups it yields a tree which does not have bounded geometry, though it is locally finite. What matters here is that geodesics with e as one end vertex inject isometrically into T^e_Γ under the obvious graph quotient. This motivates the name e-distance tree, as that is precisely what it preserves. We will drop the e notation as we have already prescribed a fixed basepoint.

We now introduce a new metric d' on $V(\Gamma)$ which splits distances into a "tree part" and a "pieces part". d' is the sum of two pseudo-metrics σ_T and σ_I where $\sigma_T(x,y)$ is the distance between the projections of x and y on the tree T_Γ and $\sigma_I(x,y) = \sum_i d(x'_i, y'_i)$ where $z'_i = z_i$ if $i \in I_z$ and e_i otherwise, so $d(x'_i, y'_i) = 0$ for all but finitely many values of i.

Checking that d' is a metric only requires showing that $d'(x,y) = 0$ implies $x = y$. Assume $d'(x,y) = 0$, which ensures $\phi_T(x,y) = 0$ and $d(x'_i, y'_i) = 0$ for all i. The first of these implies that x and y lie in a common piece Γ_j, and therefore, by definition, $x'_j = x$ and $y'_j = y$. Thus $d(x,y) = d(x'_j, y'_j) = 0$ and, as d is a metric, $x = y$.

The following lemma greatly reduces the workload of proving Theorem 4.4.2.

Lemma 4.4.6. *Let* d *be the shortest path metric on* Γ. d *and* d' *are* 2 *bi-Lipschitz.*

Proof: It is clear that $\sigma_I(x,y)$ and $\sigma_T(x,y)$ are bounded from above by $d(x,y)$, so $d'(x,y) \leq 2d(x,y)$.

For the other bound suppose that for all i, $d(x'_i, y'_i) < \frac{1}{2}d(x,y)$, if this is not the case then the result is clear. We show $\sigma_T(x,y) \geq \frac{1}{2}d(x,y)$.

Fix $j \in I_x \cap I_y = \{i_0, \ldots, j = i_l\}$ with $d(e, e_j)$ maximal. As $0 < d(x'_j, y'_j) < \frac{1}{2}d(x,y)$, in the tree T_Γ the images of geodesics in $[[x_j, x]]$ and $[[y_j, y,]]$ have at most one vertex in common.
Moreover, every geodesic in $[[e, x]]$ (resp. $[[e, y]]$) meets x_j (resp. y_j) by Lemma 4.4.4, so $\sigma_T(x,y) \geq d(x, x_j) + d(y, y_j) \geq \frac{1}{2}d(x,y)$. □

Proof of Theorem 4.4.2: Results of Tessera and Gal [Tes11, Gal08] imply that for every $p > 1$ and every function f with property (C_p) there is a map $\phi_T : V(\Gamma) \to \ell^p(V(T_\Gamma))$ such that

$$f(\sigma_T(x,y)) \leq \|\phi_T(x) - \phi_T(y)\|_p \leq \sigma_T(x,y). \tag{4.1}$$

While for $p = 1$, the above works with $f(n) = n$.

By Lemma 4.4.6, this embedding satisfies the conclusion of Theorem 4.4.2 whenever $d(x'_i, y'_i) < \frac{1}{2}d(x,y)$ for all i. To deal with the other situation, we consider the map $\phi_I : V(\Gamma) \to \bigoplus_{i\in I} \ell^p(V(\Gamma_i))$ given by

$$\phi_I(x) = \sum_{i\in I} \psi_i(x'_i).$$

By definition, $\|\phi_I(x) - \phi_I(y)\|_p \leq \sigma_I(x,y)$, so this is also Lipschitz. Moreover, given $x, y \in V(\Gamma)$ such that there exists some i with $d(x'_i, y'_i) < \frac{1}{2}d(x,y)$, then

$$\|\phi_I(x) - \phi_I(y)\|_p \geq \rho(d(x'_i, y'_i)) \geq \rho(d(x,y)).$$

Hence the theorem follows for the embedding $\phi = \phi_I + \phi_T$. □

However, the embedding ϕ_T does not have a good analogue in the relatively hyperbolic case, and as the role of this section is to motivate the methods employed there, we now present another map $\phi_T : V(\Gamma) \to \ell^p(\{e_i \mid i \in I\})$.

$$(\phi_T(x))(e_k) = \begin{cases} f(d(e_k,x))\left(\dfrac{d(e_k,e_{k+1})}{d(e_k,x)}\right)^{\frac{1}{p}} & \text{if } k \in I_x \text{ and } e_k \neq x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

This only works for functions f satisfying property (C_p^c), and hence only when $p > 1$, as it requires Lemma 4.2.3 to prove that it is Lipschitz and Lemma 4.2.2 to provide a lower bound as in Equation (4.1). We will not give full details here as they are presented in greater generality later.

We now adapt the methods presented so far in this chapter to the relatively hyperbolic situation.

4.5 Relatively hyperbolic spaces

In this section we mimic the ideas of the previous section, splitting our embedding into two pieces ϕ^s and ϕ^l which perform the same function as ϕ_T and ϕ_I in Section

4.4 respectively. We also require the averaging techniques from Section 4.3. In most of what follows, trying to directly embed an amalgamated product $A *_C B$ where C is finite and non-trivial is sufficient to see why the proof of Theorem 4.5.4 is so much more intricate than anything previously undertaken in this chapter.

We begin with an outline of various properties which are necessary for the method of proof we give. This is a highly notation-heavy process, something we attempt to offset using various informative figures. Once this is complete we formally describe (Definition 4.5.2) the collection of simplicial graphs we are considering and prove that asymptotically tree-graded simplicial graphs with bounded geometry satisfy this definition (Proposition 4.5.3).

We assume in what follows that X is the 0-skeleton of a simplicial graph with bounded geometry equipped with the shortest path metric d, $e \in X$ is a fixed basepoint and

$$\mathcal{A} = \{A_i \mid i \in I\}$$

is a countable collection of subsets - which we will refer to as *pieces* - of X which cover, i.e. $\bigcup_{i \in I} A_i = X$.

In the case of a relatively hyperbolic group, we add the trivial subgroup to the list of peripheral subgroups and take $\mathcal{A}$ to be the set of K-neighbourhoods of cosets of these peripheral subgroups, for some fixed $K \geq 1$.

Firstly, we impose some restrictions on geodesics in X, in what follows we assume that $\underline{g} \in [[x, e]]$, where e is the fixed basepoint of X and $x \in X \setminus \{e\}$:

Interactions between geodesics and pieces:

We define the *i-domain* of $\underline{g}$, denoted $\underline{g}|_i$, to be the convex hull (in $\underline{g}$) of $\underline{g} \cap A_i$ and considering $\underline{g}|_i$ as a directed path, we denote its initial vertex $\iota(\underline{g}|_i)$ by $\underline{g}|_i^+$ and its terminal vertex $\tau(\underline{g}|_i)$ by $\underline{g}|_i^-$.

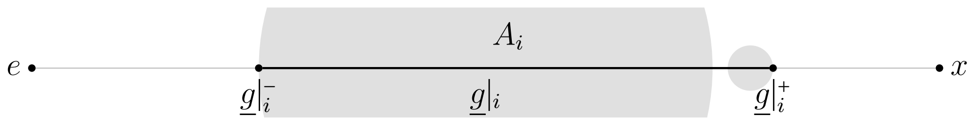

Figure 4.4: The i-domain of a geodesic

The *i-length* of $\underline{g}$ is defined to be $l_i(\underline{g}) := d\left(\underline{g}|_i^-, \underline{g}|_i^+\right) + 1$.

We require a result analogous to Lemma 4.3.1 to ensure an averaging technique like the one in Section 4.3 can be applied. Hence we impose the following two conditions on X and $\mathcal{A}$.

(C1) There exists a constant $K > 0$ such that for every $i \in I$ and any two geodesics $\underline{g_1}$ and $\underline{g_2}$ which intersect A_i,

$$d\left(\underline{g_1}|_i^-, \underline{g_2}|_i^-\right) \leq K.$$

(C2) There exists a constant $K > 0$ such that for every $i \in I$, every pair of points x, y with $d(x,y) \leq \max\left\{\frac{d(x,A_i)}{4}, 1\right\}$ and any two geodesics $\underline{g_x} \in [[x,e]]$ and $\underline{g_y} \in [[y,e]]$, which intersect A_i,

$$d(\underline{g_x}|_i^+, \underline{g_y}|_i^+) \leq K.$$

Moreover, if $\underline{g_y} \cap A_i = \varnothing$, then $l_i(\underline{g_x}) \leq K$.

It is a simple consequence of these two conditions that $\left|l_i(\underline{g_x}) - l_i(\underline{g_y})\right| \leq 2K$ under the hypotheses of condition (C2).

We illustrate the above conditions with the following figure.

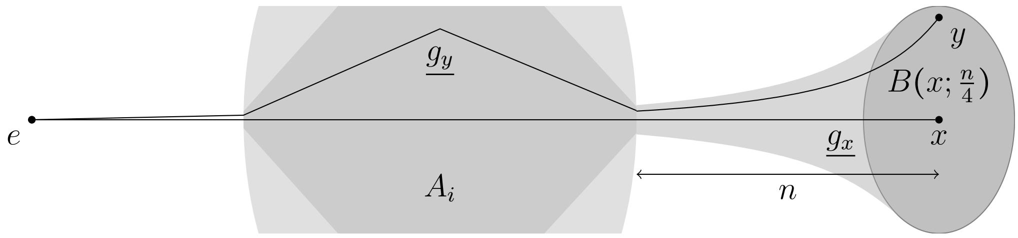

Figure 4.5: A relative version of Lemma 4.3.1

For relatively hyperbolic groups, both these conditions follow from the bounded coset penetration property, originally defined by Farb [Far98].

We will also need various finiteness conditions to ensure Lipschitz upper bounds for the averaging procedures.

Finiteness conditions:

Figure 4.5 illustrates a key finiteness issue we will have to address, namely, if we consider the collection of all geodesics from a ball of radius k around x to the origin

$$\underline{G_{x,k}} := \bigcup_{d(x,y)\leq k} [[y,e]],$$

then, within pieces where such geodesics have large i-domains, this collection of geodesics may meet a large number of other pieces. As a result, any embedding we make taking the form of those in Sections 4.3 and 4.4 will fail to be Lipschitz in general. We remedy this by discounting pieces whose i-domains lie 'deep inside' some large j-domain

For a given $x \in X \setminus \{e\}$ we restrict the collection of considered pieces in the following way:

Given a collection of geodesics $\underline{G}$, and a constant $K \geq 1$ we define the *i-boundary of $\underline{G}$*, $\partial_i^K(\underline{G})$ to be the set of $\underline{g}|_i^+$ which satisfy the following condition.

For all $\underline{g'} \in \underline{G}$ and any $j \in I$ with $l_j(\underline{g'}) \geq 5K$,

$$d(e, \underline{g}|_i^+) \notin \left[d(e, \underline{g'}|_j^-) + 2K, d(e, \underline{g'}|_j^+) - 2K\right]. \tag{4.3}$$

This unpleasant-looking restriction discounts those i-domains which lie 'deep' inside some j-domain. In Figure 6 one can imagine that by taking A_i to be a copy of $\mathbb{Z}^2$ in some free product, the collection of all geodesics from a fixed x to e with i-domain of length n meet around n^2 different pieces while inside A_i. This restriction - equation (4.3) - throws out all but a uniformly bounded number of such domains and is crucial in showing that the two maps we will construct are Lipschitz, which we do in Lemmas 4.5.8 and 4.5.9.

Then when we consider the analogue of the "hyperbolic trumpets" considered in Section 4.3, we collect all suitable boundary points of relevant i-domains into the set

$$\partial_i^K(x) := \bigcup_{k \leq d(x, A_i)/4} \partial_i^K\left(\underline{G_{x,k}}\right).$$

We define the set of *x-relevant i-domains* - $I_x(K)$ - to be the set of $i \in I$ which are crossed by some geodesic in a "trumpet" around x, but not too close to the basepoint. More formally, we require that

$$\partial_i^K(x) \neq \emptyset \text{ and } d(e, \partial_i^K(x)) \geq 3K.$$

It is far simpler - and thus very tempting - to consider only pieces with a sufficiently large i-domain. However, simply considering the situation of a hyperbolic (but not free) group as hyperbolic relative to the trivial group, we obtain an empty collection of pieces.

As in the hyperbolic case, we dismiss any piece which is too close to the basepoint.

The subset $I'_x(K) \subseteq I_x(K)$ consists of those $i \in I_x(K)$ such that $x \notin A_i$.

Technical point: Unlike the hyperbolic situation (cf. Lemma 4.3.4), it is not in general true that $\partial_i^K(\underline{G_{x,k}}) \subseteq \partial_i^K(\underline{G_{y,k+R}})$ whenever $d(x,y) \leq R$. However, we do have the following result.

Lemma 4.5.1. *For each* $a \in X$ *define* $n_{x,i}(a) := \left|\left\{k \;\middle|\; a \in \partial_i^K(\underline{G_{x,k}})\right\}\right|$. *Then, for any two points* $x, y \in X$ *with* $d(x,y) \leq R$,

$$|n_{x,i}(a) - n_{y,i}(a)| \leq \left|\left\{k \;\middle|\; a \in \partial_i^K(\underline{G_{x,k}}) \triangle \partial_i^K(\underline{G_{y,k}})\right\}\right| \leq 4R,$$

where $\triangle$ *denotes the symmetric difference* $A \triangle B := (A \setminus B) \sqcup (B \setminus A)$.

Proof: We bound the number of possible values of k such that

$$a \in \partial_i^K(\underline{G_{x,k}}) \setminus \partial_i^K(\underline{G_{y,k}})$$

For this to occur, either a is not an end point of any i-domain of a geodesic in $\underline{G_{y,k}}$, or it is, but there is some $\underline{g'} \in \underline{G_{y,k}}$ and some $j \in I$ such that Equation (4.3) fails to hold.

In the second situation, $a \notin \partial_i^K(\underline{G_{x,m}}) \cup \partial_i^K(\underline{G_{y,m}})$ for any $m \geq k + R$.

In the first situation we are only interested in the case where $a \notin \partial_i^K(\underline{G_{y,k+R}})$, which only occurs when the second situation holds for $k + R$. Thus,

$$\left|\left\{k \;\middle|\; a \in \partial_i^K(\underline{G_{x,k}}) \setminus \partial_i^K(\underline{G_{y,k}})\right\}\right| \leq 2R,$$

and the bound on $|n_{x,i}(a) - n_{y,i}(a)|$ follows. □

This lemma will be sufficient to develop an analogue of Lemma 4.3.4, once we have the following finiteness properties.

(C3) There exists a constant K such that $1 \leq |\{i \mid x \in A_i\}| \leq K$ for all $x \in X$ and $\operatorname{diam}(A_i \cap A_j) \leq K$ for all $i, j \in I$ with $i \neq j$.

(C4) There exists a constant K such that $|\{i \in I_x(K) \mid d(x, A_i) = t\}| \leq K$, for each $t \in \mathbb{N}$.

We are now ready to define the collection of spaces considered.

Definition 4.5.2. SPQR relatively hyperbolic graphs

Let X *be the* 0*-skeleton of a simplicial graph with bounded geometry and a fixed basepoint* $e \in X$. *Let* $\mathcal{A} = \{A_i \mid i \in I\}$ *be a collection of subsets of* X. *The triple* $(X, \mathcal{A}, e)$ *is* SPQR relatively hyperbolic *if it satisfies conditions* (C1)-(C4) *for a fixed constant* K.

Intuitively, what this says is that X appears to be a relatively hyperbolic graph if all we can see is geodesics to the basepoint e, i.e. all those roads that lead to Rome.

The main examples of SPQR relatively hyperbolic spaces are given by the following proposition.

Proposition 4.5.3. *Let X be a simplicial graph of uniformly bounded degree which is asymptotically tree-graded in the sense of* [DS05], *with set of pieces $\mathcal{A}'$. Then $(X, \mathcal{A}, x)$ is SPQR relatively hyperbolic where $\mathcal{A}$ is a set of M-neighbourhoods of elements of $\mathcal{A}'$ and M-balls around points and $x \in X$ is arbitrary.*

Proof: Suppose X is asymptotically tree-graded with respect to a collection of subsets, which we will label $\{A_i \mid i \in I\}$. Then we set $\mathcal{A}$ to be the collection of all M-neighbourhoods of these pieces and all M balls centred at points not lying in some A_i, where M is the constant obtained in the proof of the Rips' hyperbolicity of saturations [DS05, Corollary 4.27].

Property (C1) is the conclusion of [DS05, Corollary 8.14]. Suppose that (C2) fails, and choose a collection of counterexample triples (x_n, y_n, A_n) such that $d(x_n, y_n) \leq d(x_n, A_n)/4$ and there are geodesics $\underline{g_n^x} \in [[x_n, e]]$ and $\underline{g_n^y} \in [[y_n, e]]$ such that

$$d(\underline{g_n^x}|_i^+, \underline{g_n^y}|_i^+) \geq n.$$

Quadrilaterals with vertex set $(x_n, \underline{g_n^x}|_i^+, \underline{g_n^y}|_i^+, y_n, x_n)$ are either uniformly pinched - in the sense of Figure 4.6 below - or increasingly fat - in the sense of Definition 2.1.26.

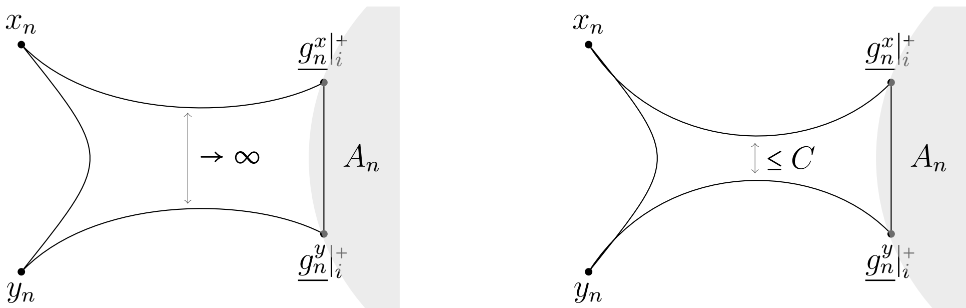

Figure 4.6: Fat and 'pinched' quadrilaterals

If they are increasingly fat, then we obtain a contradiction to [DS05, Theorem 4.1(α_3)] as such polygons are certainly not contained in a uniform neighbourhood of a single piece. However, if they are pinched we obtain a contradiction to [DS05, Lemma 4.11], which can be thought of as a quasi-geodesic version of property (C1). Here we are using the quasi-convexity of pieces in an asymptotically tree-graded space [DS05, Lemma 4.3].

Property (C3) follows verbatim from [DS05, Theorem 4.1(α_1)]. Finally, property (C4) follows from the other three properties and the bounded geometry of X. Properties (C1) and (C2) ensure that we only need consider points in uniformly bounded neighbourhood of any fixed geodesic $\underline{g_x} \in [[x, e]]$ - here the restriction to $i \in I_x(K)$ is crucial. This contains a uniformly bounded number of points at any fixed distance from x, each of which lies in only finitely many pieces, by (C3). □

In particular, Cayley graphs of relatively hyperbolic groups with finitely generated peripheral subgroups are SPQR relatively hyperbolic [DS05, Appendix A].

Now we present the main theorem of the paper in its most general form.

Theorem 4.5.4. (cf. Theorem 4.5.10)

Let $(X, \mathcal{A}, e)$ be SPQR relatively hyperbolic, where $\mathcal{A} = \{A_i \mid i \in I\}$. Suppose we are provided with maps $\psi_i : A_i \to \ell^p(X_i)$ and a concave function $\rho' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in A_i$,

$$\rho'(d(x,y)) \leq \|\psi_i(x) - \psi_i(y)\|_p \leq d(x,y).$$

Then, for all $p > 1$ and all functions $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ with property (C_p^c) there exists a coarse embedding ϕ of X into some ℓ^p space such that for all $x, y \in X$,

$$\rho(d(x,y)) \preceq \|\phi(x) - \phi(y)\|_p \preceq d(x,y),$$

where $\rho(n) = \min\{\rho'(n), f(n)\}$.

We define e_i to be some closest point of A_i to e, by condition (C3), the diameter of the set of possible choices for e_i is at most K. Without loss of generality we may assume $\psi_i(e_i) = 0$ for each $i \in I$. As the constant K is now fixed we will write I_x and I'_x in place of $I_x(K)$ and $I'_x(K)$ respectively. Similarly, we drop the K in the notation ∂_i^K.

The proof is now split into three parts, in the first two we introduce two maps from X into ℓ^p spaces with the equivalent roles of ϕ_T and ϕ_I in Section 4.4 and prove they are Lipschitz. In the third (4.5.3) we prove their sum satisfies the conclusion of Theorem 4.5.4.

4.5.1 Embedding small pieces

We construct an embedding with the role of identifying and separating points whose geodesics spend most of their time in pieces they see very little of.

It is important to note that this uses the technique of replacing each piece by a ray like in the construction of the e-distance tree (Definition 4.4.5). We then use the averaging method from Section 4.3 to ensure a Lipschitz map. Crucially, in what follows, we can only average a length proportional to the distance between a point x and a piece A_i - in the hyperbolic case we take geodesics of length n at distance approximately n from x. For this reason, we define the following capped version of the hyperbolic trumpets $F_{x,k}$ seen in Section 4.3.

For each $i \in I'_x$, we define functions $F_i(x,k) \in \ell^p(X)$ as follows:

$$F_i(x,k)(y) = \begin{cases} \min\{d(x,A_i), d_X(y,e_i)+1\}^{\frac{1}{p}} & \text{if } y \in \partial_i(\underline{G_{x,k}}) \\ 0 & \text{otherwise.} \end{cases}$$

As a necessary shorthand we set $d_{x,i}(y) := \min\{d(x,A_i), d_X(y,e_i)+1\}$. We then define

$$H_i(x) = \frac{1}{d(x,A_i)} \sum_{k \le \frac{d(x,A_i)}{4}} F_i(x,k).$$

The following three lemmas (mirroring Lemmas 4.3.3, 4.3.4 and 4.3.5) provide useful information on these new objects.

Lemma 4.5.5. *For all $x \in X$, $\underline{g} \in [[x,e]]$, $i \in I'_x$ and $k \le \frac{d(x,A_i)}{4}$,*

$$d_{x,i}(\underline{g}|_i^+) \le \|F_i(x,k)\|_p^p \le \left|\partial_i(\underline{G_{x,k}})\right| \left(d_{x,i}(\underline{g}|_i^+) + K\right) \le d_{x,i}(\underline{g}|_i^+).$$

Moreover, if $\underline{g}|_i^+ \in \bigcap_{k \le d(x,A_i)/4} \partial_i(\underline{G_{x,k}})$, then

$$\|F_i(x,k)\|_p^p \ge d_{x,i}(\underline{g}|_i^+).$$

Proof: By assumption, $\underline{g}|_i^+ \in \partial_i(\underline{G_{x,k}})$ for all $k \le \frac{d(x,A_i)}{4}$, so the first bound is satisfied.

For the second bound,

$$\|F_i(x,k)\|_p^p \le \sum_{y \in \partial_i(\underline{G_{x,k}})} d_{x,i}(y) \le \left|\partial_i(\underline{G_{x,k}})\right| (d_{x,i}(\underline{g}|_i^+) + K).$$

The final inequality holds as X has uniformly bounded valency and the diameter of $\partial_i(\underline{G_{x,k}})$ is at most K, by condition (C2).

Lemma 4.5.6. *For all $x \in X$, $i \in I'_x$ and $\underline{g} \in [\![x, e]\!]$,*

$$\|H_i(x)\|_p^p \leq |\partial_i(x)| \left(d_{x,i}(\underline{g}|_i^+) + K\right).$$

Moreover, if $\underline{g}|_i^+ \in \bigcap_{k \leq d(x,A_i)/4} \partial_i(\underline{G_{x,k}})$, then

$$\|H_i(x)\|_p^p \geq \frac{1}{4} d_{x,i}(\underline{g}|_i^+).$$

Proof: The upper bound follows from Lemma 4.5.5, as $\underline{g}|_i^+ \in \partial_i(\underline{G_{x,k}})$ for all $k \leq \frac{d(x,A_i)}{4}$, so

$$\|H_i(x)\|_p \leq \frac{1}{d(x,A_i)} \frac{d(x,A_i)+1}{4} \left\| F_i\left(x, \frac{d(x,A_i)}{4}\right) \right\|_p.$$

For the lower bound, we evaluate the contribution to $\|H_i(x)\|_p$ coming from the point $\underline{g}|_i^+$:

$$\|H_i(x)\|_p \geq \frac{1}{d(x,A_i)} \frac{d(x,A_i)+1}{4} d_{x,i}\left(\underline{g}|_i^+\right)^{\frac{1}{p}}.$$

Lemma 4.5.7. *There exists some constant $C > 0$ such that for all $x, y \in X$ with $d(x,y) \leq 1$, all $\underline{g} \in [\![x, e]\!]$ and all $i \in I'_x \cup I'_y$,*

$$\|H_i(x) - H_i(y)\|_p^p \leq C \frac{d_{x,i}(\underline{g}|_i^+)}{d(x,A_i)^p}.$$

Proof: We first bound the absolute value of $H_i(x) - H_i(y)$ at some point $a \in \partial_i(x) \cup \partial_i(y)$.

Recall that $n_{z,i}(a) := \left|\left\{n \leq \frac{d(z,A_i)}{4} \;\middle|\; a \in \partial_i(\underline{G_{z,k}})\right\}\right|$. By Lemma 4.5.1, we know that whenever $d(x,y) \leq 1$, $|n_{x,i}(a) - n_{y,i}(a)| \leq 4$, for all a. Therefore,

$$|\left(H_i(x) - H_i(y)\right)(a)| = \left| \frac{n_{x,i}(a)}{d(x,A_i)} d_{x,i}(a)^{\frac{1}{p}} - \frac{n_{x,i}(a)}{d(x,A_i)} d_{x,i}(a)^{\frac{1}{p}} \right|,$$

If $i \in I'_x \smallsetminus I'_y$ then $H_i(y) = 0$, and $n_{x,i}(a) \leq 2$. Again, we use the fact that $|\partial_i(x)|$ is uniformly bounded by condition (C2) and the uniformly bounded valency of X, so we are done. The case $i \in I'_y \smallsetminus I'_x$ is treated in the same way.

Suppose now that $i \in I'_x \cap I'_y$, so $d(x,A_i), d(y,A_i) \geq 1$. Notice that $d_{x,i}(a) = d_{y,i}(a)$ unless one (or both) are equal to $d(x,A_i)$ (respectively $d(y,A_i)$). Therefore,

$$\left| d_{x,i}(a)^{\frac{1}{p}} - d_{y,i}(a)^{\frac{1}{p}} \right| \leq \min\left\{d(x,A_i), d(y,A_i)\right\}^{-\frac{p-1}{p}} \leq 2d(x,A_i)^{-\frac{p-1}{p}}.$$

At this point it is crucial that we capped the lengths considered before defining the embedding.

Finally, combining these observations we have

$$|(H_i(x) - H_i(y))(a)| = \frac{d(y, A_i) n_{x,i}(a) d_{x,i}(a)^{\frac{1}{p}} - d(x, A_i) n_{y,i}(a) d_{y,i}(a)^{\frac{1}{p}}}{d(x, A_i) d(y, A_i)}.$$

By the triangle inequality, we can bound this from above by

$$\frac{n_{x,i} d_{x,i}(a)^{\frac{1}{p}} |d(x, A_i) - d(y, A_i)|}{d(x, A_i) d(y, A_i)} + \frac{d(x, A_i) d_{x,i}(a)^{\frac{1}{p}} |n_{x,i}(a) - n_{y,i}(a)|}{d(x, A_i) d(y, A_i)}$$
$$+ \frac{d(x, A_i) n_{y,i}(a) \left| d_{y,i}(a)^{\frac{1}{p}} - d_{x,i}(a)^{\frac{1}{p}} \right|}{d(x, A_i) d(y, A_i)}.$$

Applying all the previous deductions and noticing that $n_{x,i}(a) \le d(x, A_i)$ we obtain a uniform constant C' such that

$$|(H_i(x) - H_i(y))(a)| \le C' \frac{d_{x,i}(a)^{\frac{1}{p}}}{d(x, A_i)}.$$

Finally, we use condition (C2) again to deduce that the set $\partial_i(x) \cup \partial_i(y)$ has uniformly bounded cardinality and the lemma follows. □

We are now ready to define the first part of our embedding:

$$\phi^{s}(x) := \sum_{i \in I'_x} \frac{f(d(x, A_i))}{d(x, A_i)^{\frac{1}{p}}} H_i(x).$$

Lemma 4.5.8. *$\phi^s : X \to \ell^p(X)$ is Lipschitz for all $p > 1$.*

Proof: Consider two points $x, y \in X$ with $d(x, y) \le 1$.

Firstly, suppose $i \in I'_x \setminus I'_y$. Then by Lemma 4.5.7,

$$\|H_i(x)\|_p^p \le C \frac{d_{x,i}(\underline{g}|_i^+)}{d(x, A_i)^p},$$

for any geodesic $\underline{g} \in [\![x, e]\!]$, but by condition (C2), $l_i(\underline{g})$ is uniformly bounded, so

$$\|H_i(x)\|_p^p \le \frac{1}{(d(x, A_i) + 1)^p}.$$

The case $i \in I'_y \setminus I'_x$ is treated similarly and as $|d(x, A_i) - d(y, A_i)| \le 1$,

$$\|H_i(y)\|_p^p \le \frac{1}{(d(x, A_i) + 1)^p}.$$

By the triangle inequality, the contribution made to $\|\phi^s(x) - \phi^s(y)\|$ by those pieces A_i with $i \in I'_x \cap I'_y$ is at most

$$\sum_{i \in I'_x \cap I'_y} \frac{f(d(x, A_i))^p}{d(x, A_i)} \|H_i(x) - H_i(y)\|_p^p \tag{4.4}$$

$$+\sum_{i\in I'_x\cap I'_y}\left(\frac{f(d(x,A_i))}{d(x,A_i)^{\frac{1}{p}}}-\frac{f(d(y,A_i))}{d(y,A_i)^{\frac{1}{p}}}\right)^p\|H_i(x)\|_p^p. \tag{4.5}$$

As $f(n)(n^{\frac{-1}{p}})$ is non-decreasing (cf. Definition 4.2.1) we may use the same argument as in the tree-graded case to deduce that (4.5) is bounded from above (up to some uniform multiplicative constant) by

$$\sum_{i\in I'_x\cap I'_y}\frac{\min\{\operatorname{diam}(\partial_i(x))+1,d(x,A_i)\}}{d(x,A_i)}\left(\frac{f(d(x,A_i))}{d(x,A_i)}\right)^p.$$

Also, by Lemma 4.5.6 and the fact that f is concave, (4.4) is bounded from above (up to some uniform multiplicative constant) by

$$\sum_{i\in I'_x\cap I'_y}\frac{\min\{\operatorname{diam}(\partial_i(x))+1,d(x,A_i)\}}{d(x,A_i)}\left(\frac{f(d(x,A_i))}{d(x,A_i)}\right)^p.$$

Hence,

$$\|\phi^s(x)-\phi^s(y)\|_p^p\preceq\sum_{i\in I'_x\cup I'_y}\frac{\min\{\operatorname{diam}(\partial_i(x))+1,d(x,A_i)\}}{d(x,A_i)}\left(\frac{f(d(x,A_i))}{d(x,A_i)}\right)^p$$

which is uniformly bounded. To see this notice that by conditions (C3) and (C4) we can partition $I'_x\cup I'_y$ into a uniformly bounded number of subsets in such a way that the above sum restricted to any such subset satisfies the hypotheses of Lemma 4.2.3. □

4.5.2 Embedding large pieces

For the second part of the embedding we make a complementary construction, set the task of identifying long pieces using the existing embeddings of pieces $(\psi_i)_{i\in I}$. The difficulty here is in ensuring the map is Lipschitz. To do this we combine geodesics using the averaging methods from Section 4.3, normalised so that each 'thick geodesic' has suitable weight.

We define $a_{x,i}:=\sum_{k=0}^{\frac{d(x,A_i)}{4}}\left|\partial_i(\underline{G_{x,k}})\right|$ and define $k_{x,i}=\min\left\{a_{x,i},1+\frac{d(x,A_i)}{4}\right\}$. This will be the normaliser of our thick geodesic.

Recall that we made the convention $\psi_i(e_i)=0$ for each $i\in I$.

We then proceed towards the definition of the second part of the embedding, by defining

$$F'_i(x,k):=\sum_{a\in\partial_i(\underline{G_{x,k}})}\psi_i(a).$$

We then normalise using $k_{x,i}$,

$$H'_i(x) = \frac{1}{k_{x,i}} \sum_{k \le \frac{d(x,A_i)}{4}} F'_i(x,k).$$

The second part of the embedding $\phi^l : X \to \bigoplus_{i \in I} \ell^p(X_i)$ is defined as

$$\phi^l(x) = \sum_{i \in I_x} H'_i(x).$$

Lemma 4.5.9. *For all $p > 1$, ϕ^l is Lipschitz.*

Proof: Let $x, y \in X \setminus B(e; 3K)$ with $d(x,y) \le 1$. We show that for each $i \in I_x \cup I_y$,

$$\|H'_i(x) - H'_i(y)\|_p \le \frac{C}{d(x,A_i) + 1},$$

for some $C > 0$ not depending on i. This suffices by the finiteness conditions (C3) and (C4).

Initially, suppose $k_{x,i} = a_{x,i}$ and $k_{y,i} = a_{y,i}$. Then notice that the function

$$\frac{1}{k_{x,i}} \sum_{k \le \frac{d(x,A_i)}{4}} \chi(\partial_i(\underline{G_{x,k}}))$$

where $\chi(S)$ is the characteristic function of the set S, is non-negative and has ℓ^1 norm exactly 1, as $k_{x,i} = \sum_{k \le \frac{d(x,A_i)}{4}} \left|\partial_i(\underline{G_{x,k}})\right|$.

Moreover,

$$\frac{1}{k_{x,i}} \sum_{k \le \frac{d(x,A_i)}{4}} \chi(\partial_i(\underline{G_{x,k}})) - \frac{1}{k_{y,i}} \sum_{k \le \frac{d(y,A_i)}{4}} \chi(\partial_i(\underline{G_{y,k}})) \tag{4.6}$$

has ℓ^1 norm at most $\frac{C'}{d(x,A_i)}$ for some uniform constant C', and the sum of its entries is 0.

The second of these claims follows from the fact that this is a difference of non-negative functions of ℓ^1 norm 1. Recall that $|n_{x,i}(a) - n_{y_i}(a)| \le 4$ for all a, by Lemma 4.5.1 and the set $\partial_i(\underline{G_{x,k}}) \cup \partial_i(\underline{G_{y,k}})$ has uniformly bounded cardinality (independent of k). Hence, $|k_{x,i} - k_{y,i}|$ is uniformly bounded by some constant C''.

Next, fix any point $a \in X$. The contribution to (4.6) coming from a is at most

$$\left|\frac{n_{x,i}(a)}{k_{x,i}} - \frac{n_{y,i}(a)}{k_{y,i}}\right|.$$

As $n_{y,i}(a) \leq k_{y,i}$,

$$\begin{aligned}\left|\frac{n_{x,i}(a)}{k_{x,i}} - \frac{n_{y,i}(a)}{k_{y,i}}\right| &\leq \left|\frac{n_{x,i}(a)}{k_{x,i}} - \frac{n_{y,i}(a)}{k_{x,i}}\right| + \left|\frac{n_{y,i}(a)}{k_{x,i}} - \frac{n_{y,i}(a)}{k_{y,i}}\right| \\ &\leq \frac{|n_{x,i}(a) - n_{y,i}(a)|}{k_{x,i}} + \frac{n_{y,i}(a)\,|k_{y_i} - k_{x,i}|}{k_{y,i}k_{x,i}} \\ &\leq \frac{1}{k_{x,i}} + \frac{C''}{k_{x,i}} \leq \frac{C'}{d(x,A_i)+1},\end{aligned}$$

with the final step coming from the fact that $k_{x,i} \geq 1 + \frac{d(x,A_i)}{4}$. Now we return our attention to $H_i'(x) - H_i'(y)$, which we deduce from our previous arguments can be written in the following way:

$$H_i'(x) - H_i'(y) = \sum \mu_n \psi_i(b_n),$$

where each $b_n \in \partial_i(x) \cup \partial_i(y)$ and μ_n is the value of the function (4.6) at b_n. From the above argument we know that $\sum \mu_n = 0$ and $\sum |\mu_n| \leq \frac{C'}{d(x,A_i)+1}$.

But for any two points $a, b \in \partial_i(x) \cup \partial_i(y)$, $\|\psi_i(a) - \psi_i(b)\| \leq 2K$, by conditions (C1) and (C2) and the fact that each ψ_i is 1-Lipschitz. Therefore,

$$\|H_i'(x) - H_i'(y)\|_p \leq \frac{2KC'}{d(x,A_i)+1}.$$

Instead, assume without loss of generality that $k_{x,i} > a_{x,i}$, then $\partial_i(\underline{G_{x,k}}) = \varnothing$ for some k and by condition (C2) the length of any i-domain of any considered geodesic is bounded from above by K.

Hence, using the fact that $|k_{x,i} - k_{y,i}| \leq K$, we deduce in the same way as above that (4.6) has ℓ^1 norm bounded by $\frac{C'}{d(x,A_i)+1}$ for some uniform constant C'.

Again writing

$$H_i'(x) - H_i'(y) = \sum \mu_n \psi_i(b_n),$$

we see that as each ψ_i is 1-Lipschitz,

$$\|H_i'(x) - H_i'(y)\|_p \leq \sum |\mu_n| \, \|\psi_i(b_n)\|_p \leq \frac{2KC'}{d(x,A_i)+1},$$

completing the lemma. □

4.5.3 The proof of Theorem 4.5.4

Now we are ready to prove Theorem 4.5.4 using the embedding

$$\phi : X \to \ell^p(X) \oplus \bigoplus_{i \in I} \ell^p(X_i) \quad \text{given by} \quad \phi(x) = \phi^s(x) + \phi^l(x).$$

This is Lipschitz by Lemmas 4.5.8 and 4.5.9.

Consider $x, y \in X$ with $d(x,y) \geq CK$ (C is chosen such that $\rho'(CK) \geq 35K$ and $C \geq 35$).

Fix geodesics $\underline{g_x} \in [\![x, e]\!]$ and $\underline{g_y} \in [\![y, e]\!]$.

Set x_y to be the closest point $p_{x,y}$ on $\underline{g_x}$ to e such that $d(p_{x,y}, \underline{g_y}) \geq 5K$ and define y_x similarly.

Notice that if $x_y, y_x \in A_i$ for some i, then that i is unique, as the intersection of any two pieces has diameter at most K, by condition (C3).

Let $J_x = \left\{ j \in I_x \;\middle|\; \underline{g_x}|_{[\![x,x_y]\!]} \cap A_j \neq \varnothing \right\}$ and $J'_x = J_x \cap I'_x$. We define J_y and J'_y similarly.

$J_x \cap J_y$ has cardinality at most 1, by condition (C1).

We now deal with the cases where there x and y are separated by a large piece. To detect this, we will be using the embedding ϕ^l.

Suppose $|J_x \cap J_y| = 1$, label that index i and suppose $d(\underline{g_x}|_i^+, \underline{g_y}|_i^+) \geq \frac{1}{7} d(x,y)$, then

$$\|\phi(x) - \phi(y)\|_p^p \geq \|H'_i(x) - H'_i(y)\|_p^p .$$

We notice that the sets $\partial_i(x)$ and $\partial_i(y)$ are disjoint as $\partial_i(x)$ has diameter at most K, so the function defined in the proof of Lemma 4.5.9, (4.6) has ℓ^1 norm 2 in this case. Therefore, we can write

$$H'_i(x) - H'_i(y) = \sum_n \mu_n H'_i(x).\chi(\{b_n\}) - \sum_m \mu_m H'_i(y) \chi(\{b_m\})$$

with μ_a again being the value of (4.6) evaluated at b_a. Notice that $\sum_n \mu_n = \sum_m \mu_m = 1$ and the sets $\{b_m\}$ and $\{b_n\}$ are disjoint. Pairing up the μ_n and μ_m and applying condition (C2) we see that

$$\|H'_i(x) - H'_i(y)\|_p \geq \rho'(d(\underline{g_x|_i^+}, \underline{g_y|_i^+})) - 4K \geq \frac{1}{35} \rho'(d(x,y)),$$

where the last step comes from the concavity of ρ' and the upper bound on $d(x,y)$.

If $J_x \cap J_y = \{i\}$ we now set $x_y = \underline{g_x}|_i^+$ and $y_x = \underline{g_y}|_i^+$. Otherwise we leave x_y and y_x as before. In particular, $d(x_y, y_x) \geq \frac{6}{7}d(x,y) - 2K$.

Without loss of generality, suppose that $d(x, x_y) \geq d(y, y_x)$. From this we deduce that $d(x, x_y) \geq \frac{3}{7}d(x,y) - K$. If there exists some $j \in J_x \smallsetminus J_y$ with $l_j(\underline{g_x}) \geq \frac{1}{7}d(x,y)$, then

$$\begin{aligned} \|\phi(x) - \phi(y)\|_p &\geq \left\|H'_j(x) - H'_j(y)\right\|_p = \left\|H'_j(x)\right\|_p \\ &\geq \rho'(l_j(\underline{g_x})) - 2K \geq \tfrac{1}{35}\rho'(d(x,y)). \end{aligned}$$

If this does not happen, then we use the embedding ϕ^s to detect the distance between x and y.

$$\|\phi(x) - \phi(y)\|_p^p \geq \sum_{j \in J'_x \smallsetminus J_y} \frac{f(d(x, A_i))^p}{d(x, A_i)} \|H_j(x)\|_p^p .$$

As every point p lying on $\underline{g_x}$ at distance between $\frac{2}{3}d(x, x_y)$ and $d(x, x_y)$ from x lies in some A_j with $j \in J'_x \smallsetminus J_y$, we use Lemma 4.2.2 and the lower bound of Lemma 4.5.6 to deduce

$$\|\phi(x) - \phi(y)\|_p^p \geq f\left(\frac{1}{3}d(x, x_y) - 1\right)^p .$$

Finally, f is concave and $d(x, x_y) \geq \frac{3}{7}d(x,y) - K$, so this gives

$$\|\phi(x) - \phi(y)\|_p \geq f\left(\frac{1}{7}d(x,y) - (K+1)\right) \geq f(d(x,y)).$$

□

Corollary 4.5.10. *Let G be a finitely generated group which is hyperbolic relative to a collection of subgroups $\{H_i\}$. Given any $p > 1$, any collection of coarse embeddings of H_i into ℓ^p spaces with associated concave lower bounding functions ρ^i_- and any function f with property (C^c_p) there is a map ϕ from G into an ℓ^p space such that for all $x, y \in G$,*

$$\rho(d(x,y)) \leq \|\phi(x) - \phi(y)\|_p \leq d(x,y),$$

where $\rho(n) = \min\{\rho^i_-(n), f(n)\}$.

Proof: This follows from Theorem 4.5.4, Proposition 4.5.3 and Appendix A of [DS05].

In particular, we obtain an embedding result for all closed 3-manifolds.

Corollary 4.5.11. (cf. Corollary 5)

Let M be a closed 3-manifold, then for all $p > 1$ and all f satisfying property (C^c_p), there exists a map ϕ from $\pi_1(M)$ into some ℓ^p space, such that for all $x, y \in \pi_1(M)$,

$$f(d(x,y)) \leq \|\phi(x) - \phi(y)\|_p \leq d(x,y).$$

So $\alpha^*_p(\pi_1(M)) = 1$ for all $p > 1$.

Proof: Consider first the geometric manifolds. The fundamental groups of these are quasi-isometric to one of eight Thurston geometries, which are either compact, Euclidean, hyperbolic or have a suitable embedding via [Tes11, Theorem 1].

In the non-geometric case, we decompose the manifold along tori using the Geometrisation Theorem [Per02, Per03b, CZ06a, CZ06b, KL08, MT07, MT08]. If M has no hyperbolic part then it is a graph manifold and Smirnov proves this has finite Assouad-Nagata dimension [Smi10]. A theorem of Gal then gives suitable bounds on compression [Gal08]. For an explicit embedding, one can use Theorems 1 and 4.4.2.

Finally, if it has a hyperbolic part, then $\pi_1(M)$ is hyperbolic relative to the fundamental groups of a finite collection of graph manifold groups and virtually polycyclic groups [Dah03a]. Using [Tes11, Theorem 1] and [Gal08] again and applying Corollary 4.5.10 completes the result. □

By way of complete contrast, Sapir [Sap11] proves the existence of a closed aspherical 4-manifold M where $\pi_1(M)$ coarsely contains expanders and hence admits no coarse embedding into any Hilbert space. This uses Gromov's proof [Gro00] of the existence of a group coarsely containing expanders.

Finally, we also obtain an estimate for L_p compression using proposition 2.1.8.

Corollary 4.5.12. (cf. Corollary 6)

Let X be an asymptotically tree-graded simplicial graph of uniformly bounded degree and let $\{A_i \mid i \in I\}$ be a suitable choice of pieces. Suppose we are given a collection of maps $\psi_i : A_i \to L_p([0,1])$ and a concave function $\rho' : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for all $i \in I$ and all $x, y \in A_i$,

$$\rho'(d(x,y)) \leq \|\psi_i(x) - \psi_i(y)\|_p \leq d(x,y).$$

For each function f satisfying property (C_q^c) where $q = \max\{p, 2\}$ there exists a map ϕ of X into $L_p([0,1])$ with

$$\min\{f(d(x,y)), \rho'(d(x,y))\} \leq \|\phi(x) - \phi(y)\| \leq d(x,y).$$

Proof: By Proposition 2.1.8, $\ell^p(X)$ isometrically embeds into $L^p([0,1])$. The remainder then follows by recalling the fact that $L^2([0,1])$ isometrically embeds into $L_p([0,1])$ when $p \in [1,2]$ [Woj91] and applying Theorem 4.5.4. □

4.6 Extensions

Here we review a few extensions which can be made to the above results, particularly those of Section 4.3. We construct explicit embeddings of hyperbolic metric spaces with Bowditch's tight geodesic property into ℓ^p spaces in Section 4.6.1, and give an extension away from hyperbolic metric spaces in situations where the space satisfies some weakened version of Lemma 4.3.1.

4.6.1 A step away from bounded geometry

Here we show how to extend Theorem 4.3.2 to curve complexes of surfaces and coned-off graphs of relatively hyperbolic groups, (cf. Sections 2.1.7 and 2.1.6).

Theorem 4.6.1. *Let X be a hyperbolic simplicial graph satisfying Bowditch's tight geodesic property (cf. Theorem 2.1.38). Then for every concave function $f : \mathbb{N} \to \mathbb{R}$ satisfying Tessera's property (C_p) there is an embedding ϕ of X into an ℓ^p space with*

$$f(d(x,y)) \leq \|\phi(x) - \phi(y)\|_p \leq d(x,y).$$

In what follows we only deal with geodesics which are tight in the sense of Bowditch [Bow08]. We recall here Bowditch's notation. $\mathcal{L}_T(a,b)$ denotes the set of all tight geodesics connecting a and b, of which there are finitely many (and at least one [MM00]). Moreover, we set

$$\mathcal{L}_T(A,b) = \bigcup_{a \in A} \mathcal{L}_T(a,b).$$

Fix a basepoint $e \in X$, we define the set of restricted geodesics

$$\underline{G_{x,k,n}} = \bigcup_{\underline{g} \in \mathcal{L}_T\left(\overline{B(x;k)},e\right)} \left\{\underline{g}([n,2n])\right\}.$$

We define $F_{x,k,n}$ to be the set of all points in $\mathcal{C}(X)$ lying on some $\underline{g} \in \underline{G_{x,k,n}}$ but not in $B(e;3\delta)$ and set $F(x,k,n)$ to be the characteristic function of $F_{x,k,n}$.

Lemma 4.6.2. *There exists some n_0 such that for all $n \geq n_0$, $F_{x,k,n}$ is a finite set, so $F(x,k,n) \in \ell^p(X)$.*

Proof: This follows from [Bow08, Theorem 1.1] and Lemma 4.3.1. □

Next define

$$H(x,n) = \frac{1}{n} \sum_{k \leq \frac{n}{4}} F(x,k,n).$$

The following lemma captures the necessary properties of these functions.

Lemma 4.6.3. *There exists some constant C such that for all $x, y \in X$, $k \le \frac{n}{4}$ and $n \ge n_0$,*

(i) $\|F(x,k,n)\|_p^p \le Cn$, *and if* $d(x,e) \ge 2n$, *then* $\|F(x,k,n)\|_p^p \ge n - 3\delta$,

(ii) $\|H(x,k,n)\|_p^p \le Cn$, *and if* $d(x,e) \ge 2n$, *then* $\|H(x,k,n)\|_p^p \ge \frac{n-3\delta}{4}$,

(iii) *if* $d(x,y) \le R$, *then* $\|H(x,n) - H(y,n)\|_p \le 2C(R+1)n^{-\frac{p-1}{p}}$.

The proofs are so similar to those of Lemmas 4.3.3, 4.3.4 and 4.3.5 respectively that we omit additional details here. □

We can now define our embedding $\phi : X \to \bigoplus_n \ell^p(X)$, by setting

$$\phi(x) = \sum_{n \ge \log_2(n_0+3)} \frac{f(2^n)}{2^{\frac{n}{p}}} H(x, 2^n).$$

The result then follows in exactly the same way as Theorem 4.3.2. □

4.6.2 Strong fellow-traveller properties

One may view the conclusion of Lemma 4.3.1 - we will give a full statement momentarily to avoid long-distance cross-checking - as a strengthening (albeit an enormous one) of the fellow-traveller property for automatic groups. We can grade these strong fellow-traveller properties as follows:

Definition 4.6.4. The g fellow-traveller property

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simplicial graph and let $g : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ be a function. We say Γ satisfies the g fellow-traveller property *if, for some point $e \in V(\Gamma)$ there exist a collection of geodesics (in other terminology, a normal form) $\underline{g_x} \in [[x,e]]$ for all $x \in V(\Gamma)$ such that for all n sufficiently large ($n \ge n_0$), there exists some constant $K \ge 0$ such that for all*

- *$x, y \in V(\Gamma)$ with $d_\Gamma(x,y) \le g(n)$,*
- *geodesics $\underline{g_0} \in [[x,e]]$ and $\underline{g} \in [[y,e]]$,*
- *points $p \in \underline{g}([n,2n])$,*

$$d_\Gamma\left(p, \underline{g_0}([0,3n])\right) \le K.$$

We note first that $g(n) \le n$. The case where g is a constant function defines the ordinary fellow-traveller property, while the conclusion of Lemma 4.3.1 is the situation where $g(n) \ge n$.

Theorem 4.3.2 can be extended to all simplicial graphs satisfying some version of this fellow-traveller property. Unfortunately, but also unsurprisingly, this says nothing about automatic groups in general.

Theorem 4.6.5. *Let Γ be a simplicial graph with bounded geometry satisfying the g fellow-traveller property for some function g. For every function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)\frac{n}{g(n)}$ is concave and has property (C_p) there exists an embedding $\phi: \Gamma \to \bigoplus_{n\in\mathbb{N}} \ell^p(\Gamma)$ such that for all $x, y \in V(\Gamma)$,*

$$f(d_\Gamma(x,y)) \le \|\phi(x) - \phi(y)\|_p \le d_\Gamma(x,y).$$

Proof: This follows the exact method of the proof of Theorem 4.3.2 with the following adjustments:

- the definition of $G_{x,k,n}$ is restricted to the collection of prescribed geodesics $\{\underline{g_x} \mid x \in V(\Gamma) \setminus \{e\}\}$,
- we define $H(x,n) \coloneqq \frac{1}{g(n)} \sum_{k \le g(n)} F(x,k,n)$,
- the conclusion of Lemma 4.3.4 is replaced with

$$\|H(x,n) - H(y,n)\|_p \le 2C(R+1)\frac{n}{g(n)^{-p}}.$$

□

However, for finitely generated groups nothing new is gained, as shown by the following proposition.

Proposition 4.6.6. *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be the Cayley graph of a finitely generated group G. If Γ satisfies the g fellow-traveller property for some function g where $g(n) \to \infty$ as $n \to \infty$, then G is hyperbolic.*

To prove this, we will show that G has subquadratic Dehn function and apply Theorem 2.1.19. First, we show that such a group is necessarily finitely presented. Groups satisfying weaker fellow travelling properties are explored in much greater detail in [Bri93]. Notice that all automatic groups have $g(n)$ uniformly bounded, so the result is optimal in that sense.

Proof of Proposition 4.6.6 First we prove that G is finitely presentable, the method here will also imply (once we know finite presentability) that the Dehn function of G is at most quadratic. Given any loop L in Γ of length at most m originating at the identity e, we enumerate vertices on that loop as $A \coloneqq \{e = a_0, a_1, \ldots, a_{m-1}\}$ and label the geodesics given by the definition as $\underline{g_i} \in [[a_i, e]]$ for all $i \geq 1$.

Applying the g fellow-traveller property to $n = n_0$, we see that the distance between $\underline{g_i}(k)$ and $\underline{g_{i+1}}(k)$ is uniformly bounded independent of k, so adding geodesics between all such pairs of points we fill L using at most m^2 loops of uniformly bounded length. Hence, every relation in G can be written as product of uniformly bounded length loops, so G is finitely presentable and the Dehn function of G is at most quadratic.

This proof is illustrated by Figure 4.7 below.

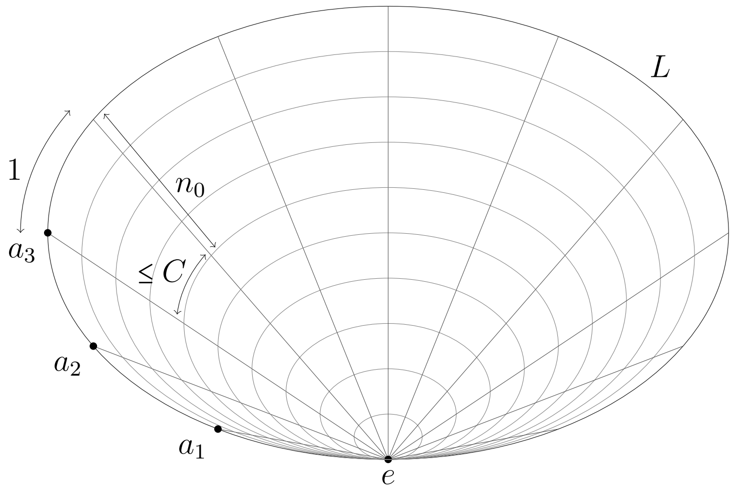

Figure 4.7: Finite presentability and quadratic Dehn function

We now use this to prove subquadratic Dehn function. This time we take a loop of length at most $m \geq (n_0)^2$ in Γ and consider a subset $\{e = a_{i_0}, a_{i_1}, \ldots, a_{i_j}\}$ of A where $j \leq 2m/g(\sqrt{m})$ and $1 \leq i_{k+1} - i_k \leq g(\sqrt{m})$. Applying the g fellow-traveller property to $\sqrt{m} \geq n_0$, we see that the distance between $\underline{g_i}(k)$ and $\underline{g_{i+1}}$ is uniformly bounded independent of k for all $k \geq \sqrt{m}$. Again, adding geodesics between all such pairs of points we fill L using at most $2m/g(\sqrt{m})$ loops of length at most $4\sqrt{m}$ and $(2m/g(\sqrt{m}))^2$ loops of uniformly bounded length. As the Dehn function of G is at most quadratic, $A(G) \leq m^2/g(\sqrt{m})$. □

We illustrate this proof by Figure 4.8 below.

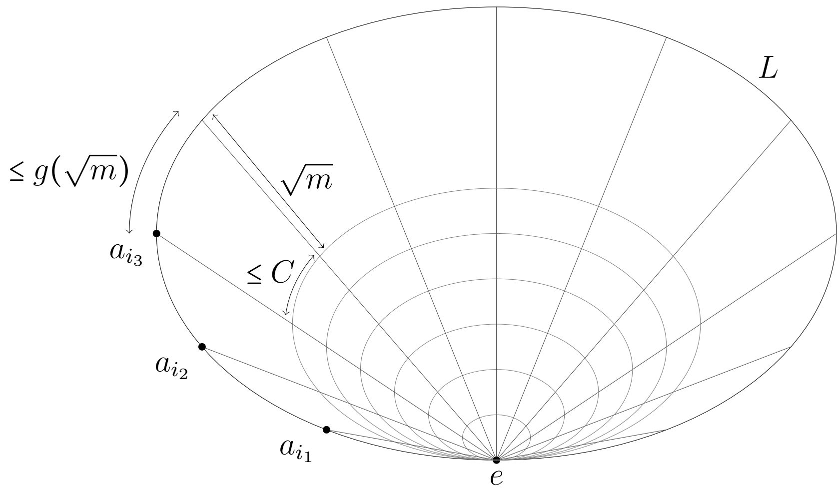

Figure 4.8: Sub-quadratic Dehn function

This yields the following definition.

Theorem 4.6.7. Gromov hyperbolicity

A finitely generated group G is Gromov hyperbolic if and only if it satisfies the g fellow-traveller property for some function g such that $g(n) \to \infty$ as $n \to \infty$.

Proof: Follows from Lemma 4.3.1 and Proposition 4.6.6. □

4.7 Subsequent work

In [MS12], (cf. Theorem 2.1.47) the authors prove that a relatively hyperbolic group quasi-isometrically embeds into a finite product of trees if and only if its peripheral subgroups have this property, this gives an optimal bound on compression exponent - including the seemingly elusive $p = 1$ case - in this restricted setting. Using this, they prove that every closed 3-manifold group has finite Assouad-Nagata dimension (cf. Definition 2.4.7) giving an optimal version of Corollary 4.5.11 in light of results of [Gal08].

Chapter 5

Embedding mapping class groups into finite products of trees

Let's think the unthinkable, let's do the undoable, let's prepare to grapple with the ineffable itself, and see if we may not eff it after all.

– Douglas Adams

5.1 Introduction

As mentioned previously, mapping class groups are one of the most interesting classes of finitely generated groups, due to their close connections with geometry, topology and group theory and their similarities with lattices in higher rank semisimple Lie groups and $\mathrm{Out}(F_n)$.

In this chapter, we study these groups from the viewpoint of their quasi-isometric embeddings into finite products of (locally infinite) simplicial trees and coarse embeddings into ℓ^p spaces.

Trees occur naturally as an important and well-studied subclass of Gromov hyperbolic metric spaces. Many finitely generated groups are already known to admit quasi-isometric embeddings into a finite product of trees: Hyperbolic, Coxeter, right-angled Artin and virtually special groups are all examples, (cf. Example 2.4.8 and the following references contained therein [BDS07, DJ99, DJ00, HW08]). By contrast: the discrete Heisenberg group, Thompson's group and wreath products of infinite finitely generated groups admit no such embedding, (cf. Example 2.4.10 and [Pau01]).

Quasi-isometric embeddability into a finite product of trees is an important metric constraint, which is in general not easy to verify. Within this setting it is of course sufficient to consider embeddings into products of spaces quasi-isometric to trees.

The quasi-isometry classes of hyperbolic spaces with tree representatives provides an important subclass characterised by Manning's bottleneck property, (cf. Theorem 2.1.14 and [Man05]):

A geodesic metric space X satisfies the *bottleneck property* (BP) if and only if there is some constant $\Delta > 0$ such that given any two distinct points $x, y \in X$ and some geodesic $\underline{g}$ from x to y with midpoint m, every path from x to y in X intersects $B(m; \Delta) = \{z \in X \mid d_X(z, m) < \Delta\}$.

Within the collection of relatively hyperbolic spaces, (cf. Definition 2.1.26 and [DS05]) the analogue of a tree is the notion of a tree-graded space (cf. Section 2.1.5. We recall (Definition 2.1.20) that a geodesic metric space X is tree-graded with respect to a collection of subsets $\{X_i \mid i \in I\}$ (called pieces) if and only if

- for all $i \neq j$, $|X_i \cap X_j| \leq 1$ and
- every simple geodesic triangle (a simple loop consisting of three geodesic edges) is contained in a single piece.

These occur as a subclass of relatively hyperbolic spaces, one of the simplest (and most natural) non-degenerate examples being a free product of groups. Moreover, every asymptotic cone of a relatively hyperbolic space is tree-graded [DS05].

Spaces contained in quasi-isometry classes of relatively hyperbolic spaces with tree-graded representatives have additional structural properties not present in the general class of relatively hyperbolic spaces. If a space X is quasi-isometric to a tree-graded space $\mathcal{T}(X)$ with pieces $\{\mathcal{T}_i \mid i \in I\}$:

- X has asymptotic dimension at most n/asymptotic Assouad-Nagata dimension at most n/quasi-isometrically embeds into a product of at most n trees if and only if the same is uniformly true for the collection of pieces $\{\mathcal{T}_i \mid i \in I\}$ [BH09].
- The ℓ^p compression exponent of X - the supremum over all $\alpha \in [0,1]$ with the property that there is some Lipschitz embedding ϕ of X into $\ell^p(\mathbb{N})$ with $\|\phi(x) - \phi(y)\|_p \geq K^{-1} d_X(x,y)^\alpha - C$ for all $x, y \in X$ - equals the uniform ℓ^p compression exponent of $\{\mathcal{T}_i \mid i \in I\}$, i.e. the supremum over α for which K, C can be chosen independent of i, by Theorem 3.
- X is a quasi-tree if and only if there is some Δ such that each $\mathcal{T}_i$ satisfies (BP) with constant Δ.

The first two of these conditions have some analogues for general relatively hyperbolic groups [Osi05, MS12] and Theorem 4, but almost nothing is known once we leave the realm of spaces with bounded geometry.

The recent paper of Bestvina-Bromberg-Fujiwara represents an important advancement in understanding the geometry of mapping class groups and its axiomatic approach has since been used to study embeddings of relatively hyperbolic groups into products of trees [MS12]. An overview of this construction is contained in Section 2.1.8, but we will recall the key features here.

In [BBF10], starting with a general list of axioms concerning a collection of metric spaces $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$ and notions of projection

$$\pi_Y : \mathbf{Y} \setminus \{Y\} \to \mathcal{C}(Y),$$

a quasi-tree Q with vertex set $\mathbf{Y}$ is produced. This can be enlarged to a *quasi-tree of spaces* $\mathcal{C}(\mathbf{Y})$ by "blowing up" each vertex Y of Q by the corresponding space $\mathcal{C}(Y)$. The main theorem of this paper is that mapping class groups coarsely embed into a finite product of such spaces, the authors then note that the Masur-Minsky formula (cf. Theorem 2.1.46) implies that this embedding is quasi-isometric.

We prove that this construction always yields a space quasi-isometric to a tree-graded space. Moreover, we prove that the quasi-isometry class of tree-graded spaces is characterised by a relative bottleneck property, (Definition 5.2.1) and all quasi-trees of spaces satisfy this property. Moreover, we do this in a constructive way, so that the collection of pieces of the tree-graded space are naturally twinned with selected 'pieces' of the original metric space.

Theorem 5.1.1. (cf. Theorem 7)

A geodesic metric space X has the relative bottleneck property with respect to a collection of sets $\{X_i \mid i \in I\}$ if and only if it is quasi-isometric to some tree-graded space $\mathcal{T}(X)$ with pieces $\mathcal{T}_i$ uniformly quasi-isometric to X_i.

From this we deduce several consequences for mapping class groups of compact surfaces and relatively hyperbolic groups.

Corollary 5.1.2. (cf. Corollary 8)

Mapping class groups quasi-isometrically embed into a finite product of simplicial (but locally infinite) trees. In particular, they

- *have finite Assouad-Nagata dimension,*
- *can be quasi-isometrically embedded into* $\ell^1(\mathbb{N})$*,*
- *admit explicit embeddings into* ℓ^p *spaces which exhibit compression exponent* 1.

The first two of these are consequences of the embedding into a product of trees but the third is more subtle and builds on the work in Chapter 4.

Previously there was little information concerning how mapping class groups may embed into Banach spaces. Finite asymptotic dimension does imply coarse embeddability into Hilbert spaces, so mapping class groups satisfy the Novikov and Coarse Baum-Connes conjectures - the Novikov conjecture had already been granted independently by work of Hamenstädt, Kida and Behrstock-Minsky. Kida, moreover, proves that mapping class groups are exact and hence have Yu's property (A). This follows from finite asymptotic dimension [BBF10, HR00, Yu00, Ham09, Kid08, BM11].

The conclusions of Corollary 5.1.2 were previously only known in low complexity cases, where the mapping class group is virtually free, see for instance [Beh04].

Corollary 5.1.3. (cf. Corollary 9)

If G *a finitely generated group, which is hyperbolic relative to a collection of subgroups* $\{H_i \mid i \in I\}$ *then*

- G *has finite Assouad-Nagata dimension if and only if each* H_i *does.*
- G *can be quasi-isometrically embedded into* $\ell^1(\mathbb{N})$ *if and only if each* H_i *can,*
- *for each* p, G *admits explicit embeddings into* ℓ^p *spaces which exhibit compression exponent* $\min\{\alpha_P^*(H_i) \mid i \in I\}$.

The first of these was previously known for asymptotic dimension [Osi05], the other two are generalisations of results contained in [MS12] and an extension of Theorem 4.

We obtain Corollaries 5.1.2 and 5.1.3 from Theorem 5.1.1 in the following way.

Using the results of [BBF10] (cf. Section 2.1.8) together with Theorem 5.1.1 we obtain quasi-isometric embeddings of mapping class groups into finite products of tree-graded spaces, each of which have pieces uniformly quasi-isometric to a particular curve graph of a subsurface.

$$MCG(S) \to \prod_{i=1}^{k} \mathcal{C}(\mathbf{Y}) \to \prod_{i=1}^{k} \mathcal{T}(\mathbf{Y}).$$

A version of the theorem of Mackay and Sisto (cf. Theorem 2.1.47 and [MS12]), together with Theorem 5.1.1 implies that: given a group G, which is hyperbolic relative to $\{H_i\}$ we can quasi-isometrically embed G into the product of a tree-graded space $\mathcal{T}(\mathbf{H})$ with pieces quasi-isometric to subgroups H_i with its coned-off graph $\hat{G}$.

$$G \to \mathcal{C}(\mathbf{H}) \times \hat{G} \to \mathcal{T}(\mathbf{H}) \times \hat{G}.$$

These two corollaries then descend from studying embeddings of quasi-trees of spaces to instead studying embeddings of curve graphs, coned-off graphs and subgroups H_i [BH09]. It follows from work of Buyalo [Buy05], that curve graphs and coned-off graphs can be quasi-isometrically embedded into a finite product of trees. This requires the Masur-Minsky Theorems that curve complexes are hyperbolic and admit a family of tight geodesics, Bowditch's results on tight geodesics in coned-off graphs and the Bell-Fujiwara bounds on asymptotic dimension of such spaces [MM99, MM00, Bow08, BF08]. (cf. Sections 2.1.7 and 2.1.6)

Layout of the Chapter: Section 5.2 gives the precise definition of the relative bottleneck property and proves that it is satisfied by all quasi-trees of spaces constructed from the axiomatisation in [BBF10]. We also prove that the property is a quasi-isometry invariant, which completes the reverse implication of Theorem 5.1.1. Section 5.3 gives the construction of a tree-graded space $\mathcal{T}(X)$ from a space X satisfying the relative bottleneck property and in Section 5.4 we prove that $\mathcal{T}(X)$ is quasi-isometric to X completing the forwards implication of Theorem 5.1.1. The final section (5.5) gives the full proof of Corollaries 5.1.2 and 5.1.3.

5.2 Relative Bottleneck Property

In this section we introduce the relative bottleneck property, prove it is a quasi-isometry invariant, deduce some immediate consequences of the definition and give one technical lemma which is essential for the proof of Theorem 5.1.1 in its most general guise. Following this we give the two key examples of spaces satisfying this property, tree-graded spaces and quasi-trees of spaces satisfying the axiomatic construction defined in [BBF10].

Formally, the relative bottleneck property is defined as follows:

Definition 5.2.1. Relative Bottleneck Property

Let X be a geodesic metric space. We say X has the relative bottleneck property (RBP) *if there exists a collection of pieces $\{X_i \mid i \in I\}$ with $X = \bigcup_{i\in I} X_i$ and* *a constant $M > 0$ such that given $i, j \in I$ with $i \neq j$ there is a finite ordered set $I_{i,j} = \{i = i_0, i_1, i_s = j\}$ and for all $r \in \{0, \ldots s-1\}$ there is some point $w_r \in X_{i_r} \cap X_{i_{r+1}}$ such that every path from X_i to X_j in X passes through*

$$B(w_r; M) := \{x \in X \mid d_X(w_r, x) < M\}.$$

The following figure presents this definition in a more intuitive format. The focus of Section 5.2.1 is to justify the extent to which this picture is a valid approximation.

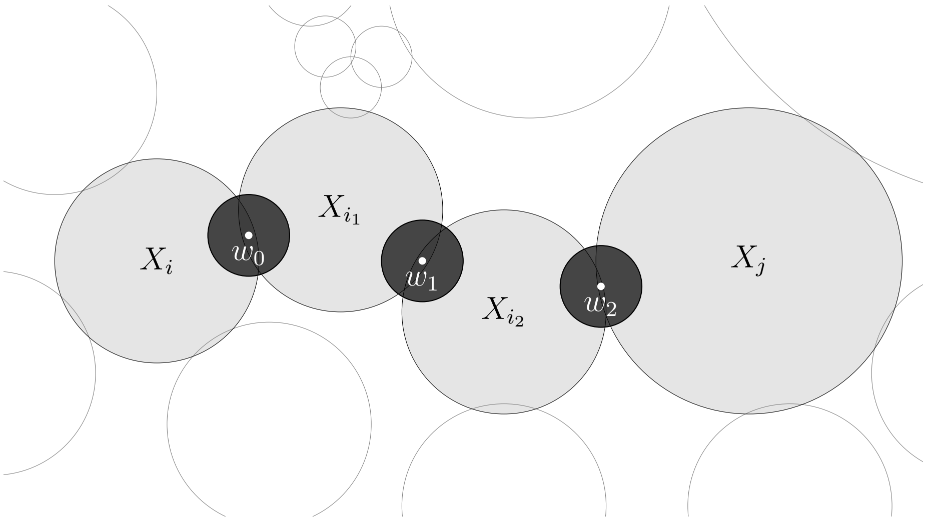

Figure 5.1: The relative bottleneck property

As a simplification to notation, given $I_{i,j} = \{i = i_0, i_1, \ldots, i_s = j\}$ we define the collection of bottlenecks between X_i and X_j to be

$$W_{i,j} := \{w_r \mid r = 0, \ldots, s-1\}.$$

Theorem 5.1.1 implies that (RBP) is a quasi-isometry invariant, however, this is a straightforward consequence of the definition given by the following proposition.

Proposition 5.2.2. *Let (X, d_X) and (Y, d_Y) be geodesic metric spaces. If X and Y are quasi-isometric and X has* (RBP) *then so does Y. Moreover, one can ensure pieces are uniformly quasi-isometric.*

Proof: We assume X has (RBP) with respect to a collection of pieces $\{X_i \mid i \in I\}$ and some constant $M > 0$. Let $q : X \to Y$ be a (K, C) quasi-isometry. We will show Y has (RBP) with respect to $\{Y_i := N_C(q(X_i)) \mid i \in I\}$ and constant $M' = M'(M, K, C)$.

It is clear that $\bigcup_{i\in I} Y_i = Y$ as q is C-onto. Notice also that each Y_i is uniformly quasi-isometric to X_i.

Let $i, j \in I$ with $i \neq j$ and let $w_k \in W_{i,j}$. We compute the distance between $q(w_k) \in Y_k \cap Y_{k+1}$ and some path P from Y_i to Y_j in Y.

The pre-image under q of P defines a subset of X whose C neighbourhood contains a path from $N_{KC+C}(X_i)$ to $N_{KC+C}(X_j)$. Hence, $N_{KC+2C}(q^{-1}(P)) \cap B(w_k; M) \neq \emptyset$. Applying q we see that $d_Y(P, q(w_k)) \leq K(KC + 2C + M) + C$. □

Another property which should be present in any sensible definition of a relative bottleneck property is a notion of convexity for pieces. This is inherent in our definition via the following lemma.

Lemma 5.2.3. *Suppose X has the bottleneck property relative to the collection of pieces $\{X_i \mid i \in I\}$ and some constant $M > 0$, then each X_i is $4M$ quasi-convex.*

Specifically, if $x, y \in N_C(X_i)$ and $\underline{g} \in [[x, y]]$, then $\underline{g}$ is contained in the $2(M + M')$ tubular neighbourhood of X_i, where $M' := \max\{M, C\}$.

Proof: We define $M' := \max\{M, C\}$. Let x', y' be the end points of any component of $\underline{g}$ outside $N_{M'}(X_i)$, so $d(x', X_i), d(y', X_i) = M'$ and let m be the mid-point of this component. As pieces cover X, $m \in X_k$ for some $k \in I$. Let $x'', y'' \in X_i$ be points at distance exactly M' from x', y' respectively. By (RBP) there is some point $w \in X_i$ such that every path from X_i to X_k meets $B(w; M)$, in particular this occurs for the paths from m to x'' and y'' via x' and y' respectively.

Hence $d(x'', y'') < 4M$. Therefore $\underline{g}$ is contained in the $2M + 2M'$ tubular neighbourhood of X_i. □

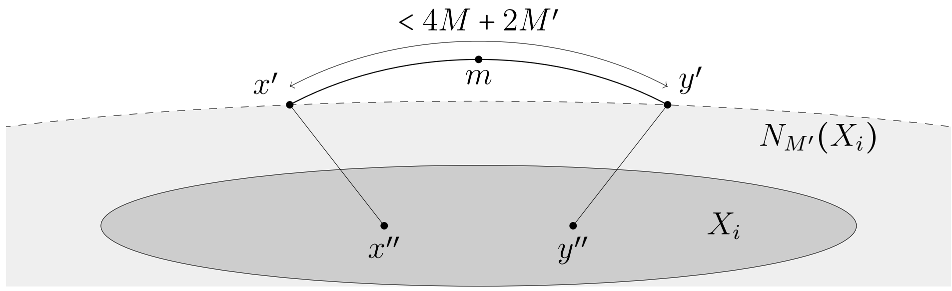

Figure 5.2: Quasi convexity of pieces

5.2.1 Unwanted bottlenecks

Most of the arguments presented in this paper revolve around setting up a contradiction to (RBP) by constructing pairs of paths between common pieces but at large Hausdorff distance. To make finding paths easy we want to be in a situation where pieces are connected in some strong sense: it is not even apparent from the definition that the pieces X_i are connected. This is easily dealt with by Lemma 5.2.3.

Moreover, we want no bottlenecks inside the pieces X_i on the same scale as those between different pieces. No such claim is made in the definition, but a simple quasi-isometric transformation of the space achieves this. The robustness of the resulting connectivity is parametrised by a constant b and - crucially - the bottleneck constant of the transformed space does not depend on b.

Proposition 5.2.4. *Let X' be a geodesic metric space satisfying* (RBP) *with respect to a collection of subsets $\{X_i' \mid i \in I\}$ and constant $\frac{M}{9}$. X' is quasi-isometric to a space X satisfying* (RBP) *with respect to subsets $\{X_i \mid i \in I\}$ and constant M such that X_i is uniformly quasi-isometric to X_i' and*

- *there is a point e (which will become the basepoint) contained in a unique piece X_e,*
- *given any metric ball B and any i such that $B \cap X_i$ has diameter bounded by $2b$, $X_i \smallsetminus B$ is (path-)connected.*

Proof: Each piece X_i' is $\frac{4M}{9}$ quasi-convex by Lemma 5.2.3, so the $\frac{4M}{9}$ tubular neighbourhood of X_i' (which we will label X_i'') is connected. Moreover, X' has the relative bottleneck property with respect to $\{X_i'' \mid i \in I\}$ with constant M (cf. Figure 5.3).

We then achieve the first additional claim by defining a new point e and attaching it to a unique piece X_e'' by a line of length 1 (this line is added to X_e''). The resulting space under this construction so far is $(1,1)$ quasi-isometric to the original with uniformly $(1, \frac{8M}{9}+1)$ quasi-isometric pieces and has (RBP) with constant M.

Now to achieve the second additional property we make the following construction.

We define $X_i = X_i'' \times [0, 2b+1]$ with the supremum product metric where the interval is given the standard Euclidean metric. Then we set

$$X = \bigsqcup_{i\in I} X_i \Big/ \sim \quad \text{where} \quad (x,a) \sim (y,b) \text{ iff } a = b = 0 \text{ and } x = y.$$

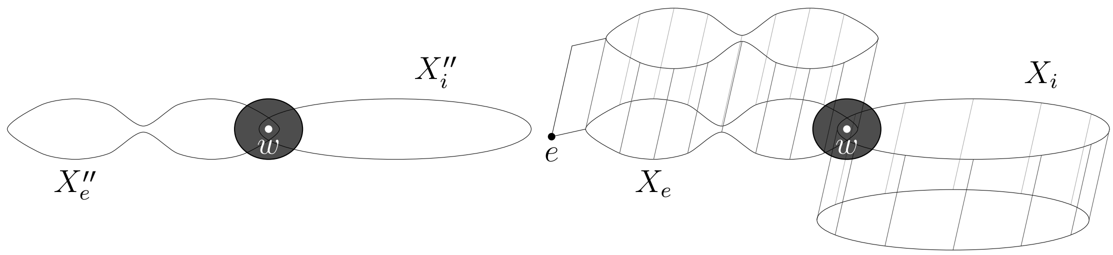

Figure 5.3: The process in Proposition 5.2.4

It is clear that X_i cannot be disconnected by a metric ball of diameter at most $2b$ with centre inside X_i. A ball centred outside X_i which intersects this piece in a set of diameter at most $2b$ completely misses $X_i'' \times \{2b+1\}$ so any two points (x_1'', r_1) and (x_2'', r_2) can be connected via $(x_1'', 2b+1)$ and $(x_2'', 2b+1)$ taking geodesics in the $[0, 2b+1]$ direction and using the fact that X_i'' is connected. Also, as pieces only meet when the component of $[0, 2b+1]$ is 0 we have not changed the constant M.

The natural injection of X' into X is a $(2b+2)$-onto isometric embedding. □

For completeness we note that $b = 15M$ suffices for all arguments in this paper.

5.2.2 Examples

The two key examples of spaces satisfying (RBP) are tree-graded spaces and quasi-trees of spaces satisfying the axioms of [BBF10].

Proposition 5.2.5. *Let X be tree-graded with respect to a collection of pieces $\{X_i \mid i \in I\}$. Then X has* (RBP) *with respect to $\{N_1(X_i) \mid i \in I\}$ and constant $M = 2$.*

This may seem a little unnecessary at first glance, but there is no reason to assume that pieces intersect in a tree-graded space. Also, we must be wary of pieces accumulating as we require the sets $I_{i,j}$ to be finite. This is exhibited by the tree-grading of $\mathbb{R}^3$ with respect to the set of hyperplanes $\{z = a \mid a \in \mathbb{R}\}$ given in Figure 2.9.

Proof: Let $i, j \in I$, $i \neq j$. Pick any geodesic $\underline{g}$ from X_i to X_j and set $I'_{i,j}$ to be the set of pieces met by $\underline{g}$ at integer distance points from the start taking repetitions wherever possible. If a point lies in multiple pieces (none of which the geodesic has previously met) we simply choose one. When required we suffix $I'_{i,j}$ by j and define this to be $I_{i,j} = \{i = i_0, i_1, \ldots, i_n = j\}$.

Given $k \in \{0, \ldots, n-1\}$ there is some minimal $t \in \mathbb{N}$ such that $\underline{g}(t) \in X_{i_k}$. But then any path from X_i to X_j must pass within distance 2 of $\underline{g}\left(t - \frac{1}{2}\right) \in N_1(X_{i_{k-1}}) \cap N_1(X_{i_k})$ completing the proof. □

Combined with Proposition 5.2.2 this proves the easier direction of Theorem 5.1.1.

Notice that there is no need for any form of 'continuity' - or, more accurately, consistency - in the choice of $I_{i,j}$ in terms of i and j.

The second class of examples are the quasi-trees of spaces defined axiomatically in [BBF10] (cf. Section 2.1.8). We recall the properties of such spaces required here for convenience:

- $\mathcal{C}(\mathbf{Y})$ is a geodesic metric spaces in which the subsets $\{\mathcal{C}(Y) \mid Y \in \mathbf{Y}\}$ are totally geodesically embedded.
- There are projections π_Y which map any $X \in \mathbf{Y} \setminus \{Y\}$ to a subset of $\mathcal{C}(Y)$ with diameter bounded by some uniform constant L.
- There exist standard paths between any two pieces, the internal pieces of which are written as a finite ordered set $Y_K(X,Z)$ where two spaces $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ are joined by a complete bipartite graph with edges of length L between $\pi_Z(X)$ and $\pi_X(Z)$ if and only if $Y_K(X,Z) = \varnothing$.

More importantly they also satisfy the relative bottleneck property:

Proposition 5.2.6. *Let $\mathcal{C}(\mathbf{Y})$ be a quasi-tree of spaces satisfying the axioms of* [BBF10]*. Then X satisfies* (RBP) *with respect to $\{N_L(\mathcal{C}(Y)) \mid Y \in \mathbf{Y}\}$ and constant $M = 10L$. Specifically, let $X, Z \in \mathbf{Y}$ with $X \neq Z$ and let $Y \in Y_K(X,Z) \cup \{Z\}$. There is some point $w_Y \in \pi_Y(X)$ such that all paths from $\mathcal{C}(X)$ to $\mathcal{C}(Z)$ pass within distance $9L$ of w_Y.*

Proof: For $Y \in Y_K(X,Z)$ the result follows directly from [BBF10, Lemma 3.9] with $w_Y \in \pi_Y(X)$. We now deal case $Y = Z$.

Suppose first that $Y_K(X,Z) = \varnothing$. Using precisely the same thickening technique as in Lemma 5.2.4 we may assume the space $\mathcal{C}(Z)$ is of sufficiently large diameter that we may choose a point $z \in \mathcal{C}(Z)$ such that $Y_K(X,z) = \{Z\}$, using the axiom $\operatorname{diam}(\pi_Z(X)) \leq K$. Take any path P from some $x \in \mathcal{C}(X)$ to z. By [BBF10, Lemma 3.9] there is a point w lying on P such that $d_{\mathcal{C}(\mathbf{Y})}(w, \pi_Z(X)) \leq 7L$. As any $\mathcal{C}(X)$ to $\mathcal{C}(Z)$ path not meeting $N_{7L}(\pi_Z(X))$ can be extended to a paths from $\mathcal{C}(X)$ to z which also misses this set and the Hausdorff distance between $\pi_Z(X)$ and $\pi_X(Z)$ is L, we are done in this case.

For the next part we require the fact that the set $Y_K(X,Z)$ admits a total order [BBF10, Theorem 2.3 (G)].

Suppose Y is the maximal element of $Y_K(X,Z) \smallsetminus \{X,Z\}$, so $Y_K(Y,Z) = \varnothing$. We apply [BBF10, Lemma 3.9] to Y and in doing so deduce that every path from $\mathcal{C}(X)$ to $\mathcal{C}(Z)$ meets $N_{7L}(\pi_Y(Z))$. As $\operatorname{diam}(\pi_Y(Z)) \leq L$ and the Hausdorff distance between $\pi_Z(Y)$ and $\pi_Y(Z)$ is L the result is complete.

To obtain a suitable point w_X we simply flip the roles of X and Z in the above argument.

Therefore, $\mathcal{C}(\mathbf{Y})$ satisfies (RBP) with pieces $\{N_L(\mathcal{C}(Y)) \mid Y \in \mathbf{Y}\}$ and constant $M = 10L$. □

5.2.3 Groups satisfying (RBP)

The relative bottleneck property is already well understood for finitely generated groups, via Stallings' Theorem (Theorem 2.1.21), which states that $\operatorname{Cay}(G,S)$ has (RBP) with respect to some subsets (in a non-trivial way) if and only if G splits as an amalgam or HNN extension $G = A *_C B$ or $G = \operatorname{HNN}(A,\theta)$ (in a non-trivial way) [Sta68, Sta71]. Moreover, the graph of groups decomposition induced by a collection of subsets which satisfy the relative bottleneck property for a fixed constant M is accessible via results of Linnell [Lin83], as the cardinality of subgroups over which we may amalgamate is uniformly bounded (cf. Section 2.1.5).

5.3 Construction of the tree-graded space

Here we will assume that X has (RBP) with respect to pieces $\{X_i \mid i \in I\}$ and a constant M with a basepoint e contained in a unique piece X_e such that no metric ball which intersects X_i in a set of diameter at most $2b$ disconnects X_i.

As M does not depend on b results from here on will just assume that b is sufficiently large, though $b = 15M$ will suffice.

Our goal is to construct a suitable tree-graded space $\mathcal{T}(X)$ which has the collection of pieces $\{N_{4M}(X_i) \mid i \in I\}$.

For each $i \in I \smallsetminus \{e\}$ we define $e_i \in X_i$ to be the point w_0 given by the bottleneck property such that all paths from X_i to X_e meet $B(e_i; M)$. Notice that $d(e,e_i) \leq d(e,X_i) + M$. We think of e_i as a basepoint of X_i.

Our construction relies on organising pieces into strata parametrised by a (large) constant R which will be determined later, ($R = 160M$ will suffice). To this end we define a collection of strata $I^n \coloneqq \{i \in I \mid d(e,e_i) \leq nR\}$ and set $I_n \coloneqq I^n \smallsetminus I^{n-1}$. The level of i, $\mathrm{lv}(i)$ is the unique n such that $i \in I_n$. By assumption $I^0 = \{e\}$.

At this point we fix for each X_i with $i \in I_{n+1}$, with $n \geq 0$, a geodesic $\underline{g_i} \in [\![e_i, e]\!]$ and define c_i to be the point on $\underline{g_i}$ at distance exactly nR from e.

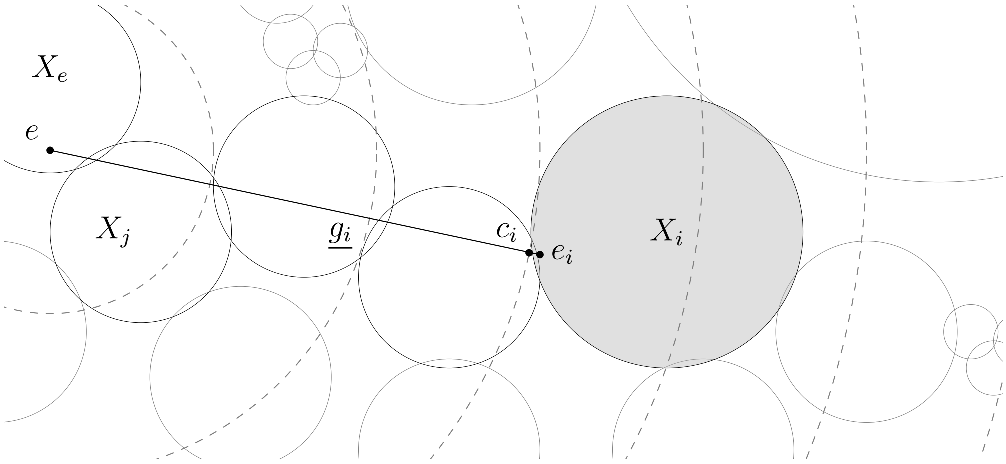

Figure 5.4: R-separated strata, in this example $i \in I_4$ and $j \in I_1$

The next two lemmas collect observations which will prove useful later. In what follows, we call a path P with endpoints x and y a *K-slack geodesic* if $|P| \leq d(x, y) + K$. We denote the reverse direction of a path P by $\overline{P}$ and denote concatenation of paths by $P_1 \circ P_2$, whenever the terminal point of P_1 agrees with the initial point of P_2.

Lemma 5.3.1. *For each $x \in N_{4M}(X_i)$ with $i \in I_{n+1}$ there is some $10M$-slack geodesic $\underline{q^i_x} \subseteq N_{4M}(X_i) \cup B(e; nR)$.*

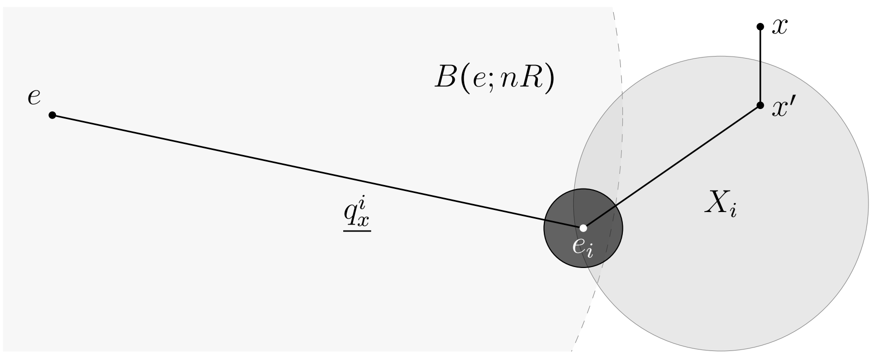

Figure 5.5: $10M$-slack geodesics

Proof: Say $x \in N_{4M}(X_i)$ with $i \in I^n$, then there is some $x' \in X_i$ with $d(x, x') \leq 4M$. We define the quasi-geodesic $\underline{q^i_x}$ as the concatenation of some $\underline{g_1} \in [\![x, x']\!]$, $\underline{g_2} \in [\![x', e_i]\!]$ and $\underline{g_i}$.

As X_i is $4M$ quasi-convex by Lemma 5.2.3 and $e_i \in B(e;nR)$, $\underline{q_x^i} \subseteq N_{4M}(X_i) \cup B(e;nR)$. Every geodesic from x to e passes within M of e_i by (RBP). Hence, $|q_x^i| \leq 4M + d(x', e) + 2M \leq d_X(x, e) + 10M$.

Lemma 5.3.2. *Let $i, j \in I$, $i \neq j$. If $d_X(e_i, e) \geq d_X(e_j, e)$ then every path from X_i to X_j in X passes through $B(e_i; 4M)$.*

Proof: Suppose there is a path P from $x \in X_i$ to $y \in X_j$ which avoids the ball $B(e_i; M)$. If $d(e_i, e_j) \geq 2M$ then any geodesic in $[[e_j, e]]$ avoids this ball, and as we may assume X_j has no small cut-sets there is a path from y to e also avoiding this ball (for instance extend a path from y to e_j by $\underline{g_j}$) contradicting (RBP).

Now consider a path P of length at most $2M$ from e_i to e_j, some point on this path lies on a bottleneck for paths between X_i and X_j and hence P contains a point within $B(e_i; 4M)$. □

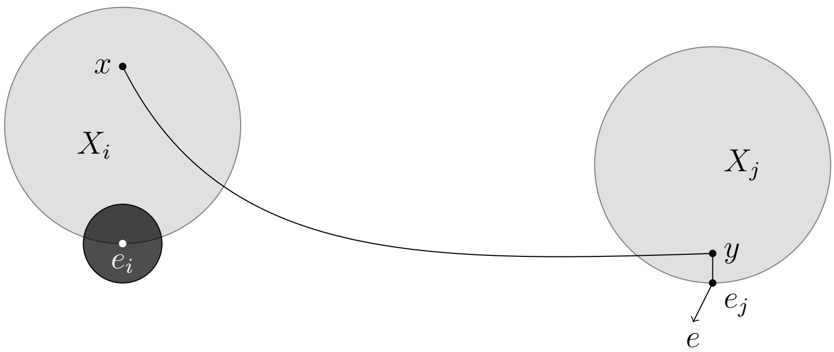

Figure 5.6: Passing to lower levels when $d(e_i, e_j) \geq 2M$

One key element of this paper is deciding when pieces in the same level should have an immediate common ancestor. We introduce the following equivalence relation on each level I_{n+1} to determine this:

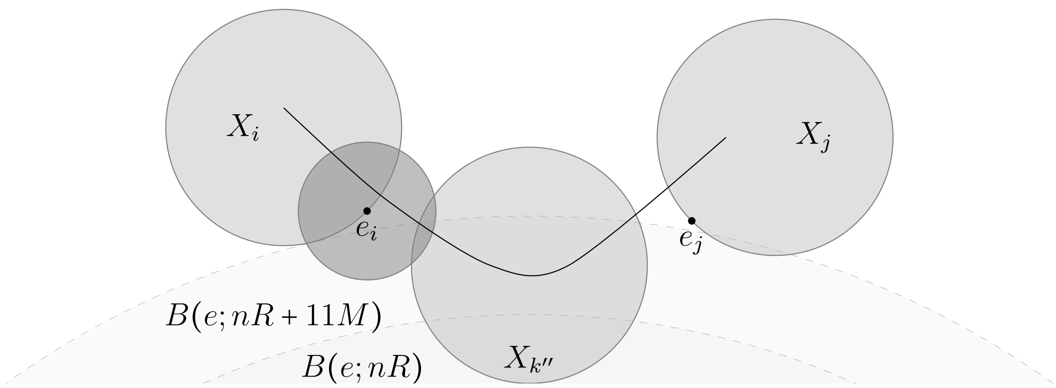

Figure 5.7: The equivalence $i \sim j$

Given $i, j \in I_{n+1}$ we write $i \sim j$ if and only if there exists some path from X_i to X_j in X such that $P \cap B(e; nR + 11M)$ is contained in some $N_{4M}(X_k)$ with $k \in I^n$.

Such a path intersects the $4M$ ball around e_i by Lemma 5.3.2. The fact that this does define an equivalence relation is not obvious so we provide a proof.

Lemma 5.3.3. *The relation $\sim$ is an equivalence relation.*

Proof: We need only check transitivity. Suppose $i \sim j \sim l$ with $|\{i, j, l\}| = 3$. There is nothing to prove unless the paths P_1 from X_i to X_j and P_2 from X_j to X_l both meet different pieces $N_{4M}(X_k), N_{4M}(X_{k'})$ with $k, k' \in I^n$. In this situation we look at two paths from $N_{4M}(X_k)$ to $N_{4M}(X_{k'})$.

- $\underline{g_k} \circ \overline{g_{k'}}$ (contained in $B(e; nR)$),
- P (avoids $B(e; nR + 11M)$): follow P_1 from $N_{4M}(X_k)$ to X_j then take any path in X_j from the terminal point of P_1 to the initial point of P_2 avoiding $B(e; nR + 11M)$ and follow P_2 to $N_{4M}(X_{k'})$.

These paths are at Hausdorff distance at least $11M \geq 6M$, which contradicts (RBP). We have tacitly used here the fact that $B(e; nR + 11M)$ intersects X_j in a set of diameter less than $30M$, thus Lemma 5.2.4 ensures such a path P exists. □

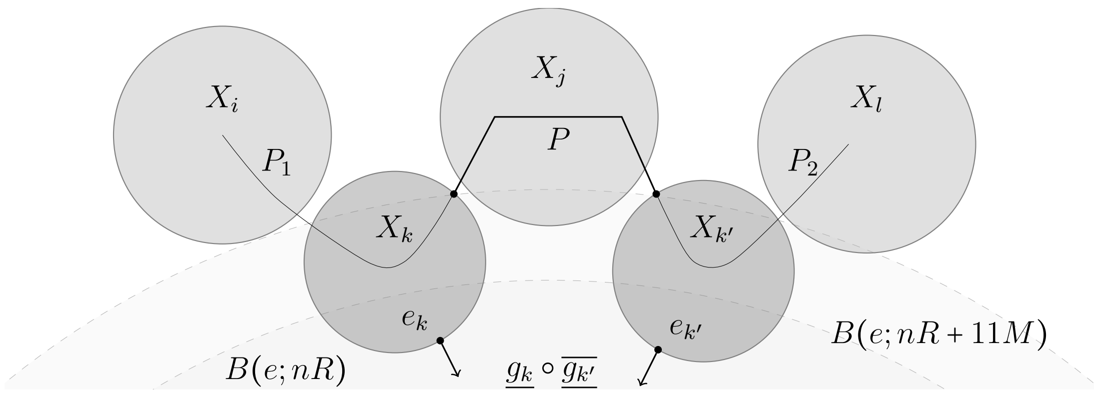

Figure 5.8: Transitivity of the relation $\sim$

The following lemma is another key step in the construction we will shortly make. It ensures that we have suitable candidate pieces in lower levels to glue each member of an equivalence class of pieces to.

Lemma 5.3.4. *Let $i \in I_{n+1}$ with $n \geq 0$. For all $k \in I_{i,e} \cap I^n$ such that one bottleneck $w \in X_k \cap W_{i,e}$ for paths from X_i to X_e satisfies $d_X(e, w) \geq nR - M$,*

$$\{c_j \mid j \in [i]\} \subseteq N_{4M}(X_k).$$

It will not necessarily be the case that $k \in I_{j,e}$ for all $j \in [i]$, however, the conclusion of this lemma is that it will still satisfy the same property. The condition governing the distance between the bottleneck point and e is purely to avoid looking at pieces which geodesics heading towards e have not yet interacted with in any way.

In some sense Lemma 5.2.3 states that this result is as much as could be hoped for.

Proof: We first prove $c_i \in N_{4M}(X_k)$. Set $B := B(w; M)$. By hypothesis and (RBP), $\underline{g_i} \cap B \neq \varnothing$ so let $m_i \in \underline{g_i} \cap B$.

If $d_X(e, w) \leq nR + 2M$ then $d_X(m_i, e) \in (nR - 2M, nR + 3M)$, which implies that

$$d_X(w, c_i) \leq d_X(w, m_i) + d_X(m_i, c_i) < M + 3M = 4M.$$

Otherwise, $d_X(e, w) > nR + 2M$. Then all paths from X_i to e meet $B(e_k; M)$, because there is a path from X_j to X_i avoiding this ball, moreover, $d_X(e_k, e) \leq nR$ since $k \in I^n$. Then, considering the position of c_i on the geodesic $\underline{g_i}$, we see that $c_i \in N_{4M}(X_k)$ by Lemma 5.2.3.

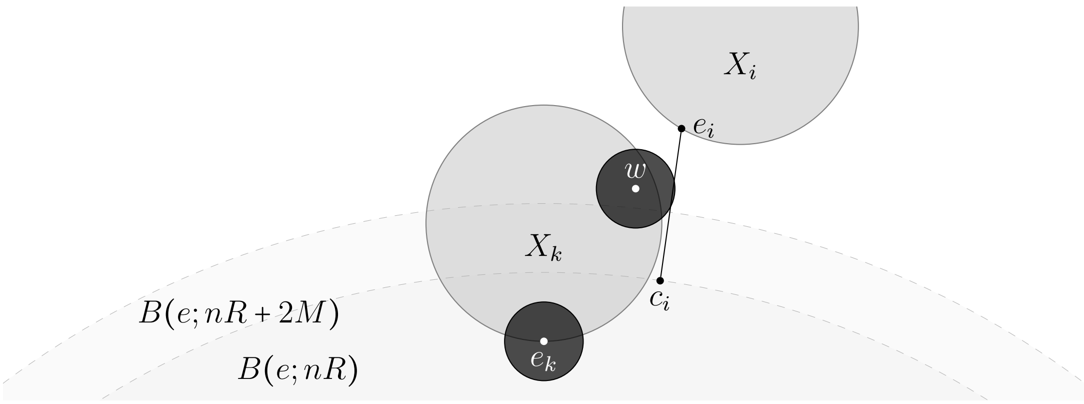

Figure 5.9: Proving $c_i \in N_{4M}(X_k)$

We now deal with the general case.

Suppose first that there is some path P inferring the relation $i \sim j$ which does not meet B, then $e_j \in B$ or P can be extended to a path P'' from X_i to e_j avoiding B.

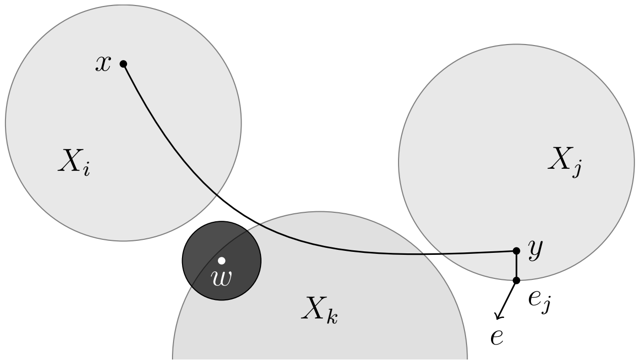

Figure 5.10: A contradiction of (RBP) when $\underline{g_j} \cap B = \varnothing$

In either case $\underline{g_j} \cap B \neq \varnothing$ as all paths from X_i to e intersect B. We are then in the same situation as the special case above and the same argument holds.

Otherwise, B meets every path P with this property. We now show $k \in I_{j,e} = \{j = j_0, \dots, j_t = e\}$. Consider the collection of paths from X_j to e defined below:

- start at the end of P contained in X_j and follow it until it meets $y \in N_{4M}(X_k)$,
- take a fixed path of length at most $4M$ to some $y' \in X_k \smallsetminus B(e; nR + 11M)$,
- follow some path in X_k from y' to e_k,
- follow $\underline{g_k}$ to e.

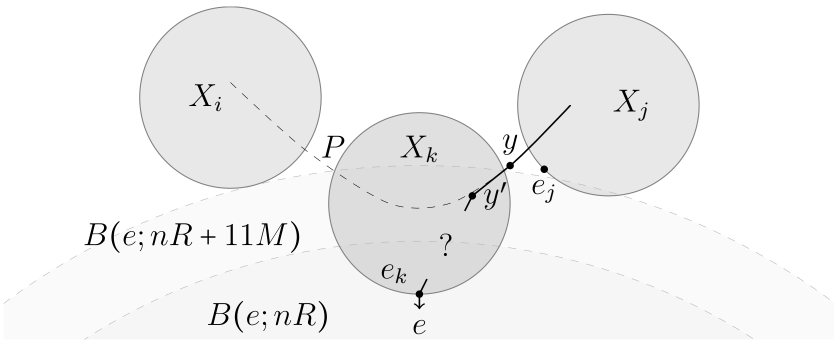

Figure 5.11: Paths satisfying these properties

As we cannot cut X_k by any ball of diameter at most $2M$ it follows from the above that $d_X(w, e) \in [0, nR + M] \cup [nR + 6M, (n+1)R + 2M]$ for all $w \in W_{j,e}$.

In particular, there is some $s \geq 1$ such that $d_X(w_{j_s}, e) \leq nR + M$ and $d_X(w_{j_{s-1}}, e) \geq nR + 6M$. To ease notation we set $w_1 := w_{j_{s-1}}$ and $w_2 := w_{j_s}$. Both points lie in a unique piece X_l by (RBP). This implies that $d_X(e_l, e) \leq d_X(e_l, w_2) + d_X(w_2, e) < nR + 3M$.

Suppose $l \neq k$ then there are two paths P_1 and P_2 (see below) from X_l to X_k at Hausdorff distance at least $2M$, which contradicts (RBP). Note here that $e_j = w_0$ so $d_X(e_j, e) \geq nR + 6M$.

- P_1 (avoids $B(e; nR{+}5M)$): follow any path from w_1 to e_j avoiding $B(e; nR{+}5M)$ (using the fact that $\underline{g_j}$ meets $B(w_1, M)$), then join this via a path in X_j to the end of P contained in X_j, follow P to $N_{4M}(X_k)$ and take any path of length at most $4M$ into X_k.
- P_2 (contained in $B(e; nR + 3M)$): $\underline{g_l} \circ \overline{\underline{g_k}}$.

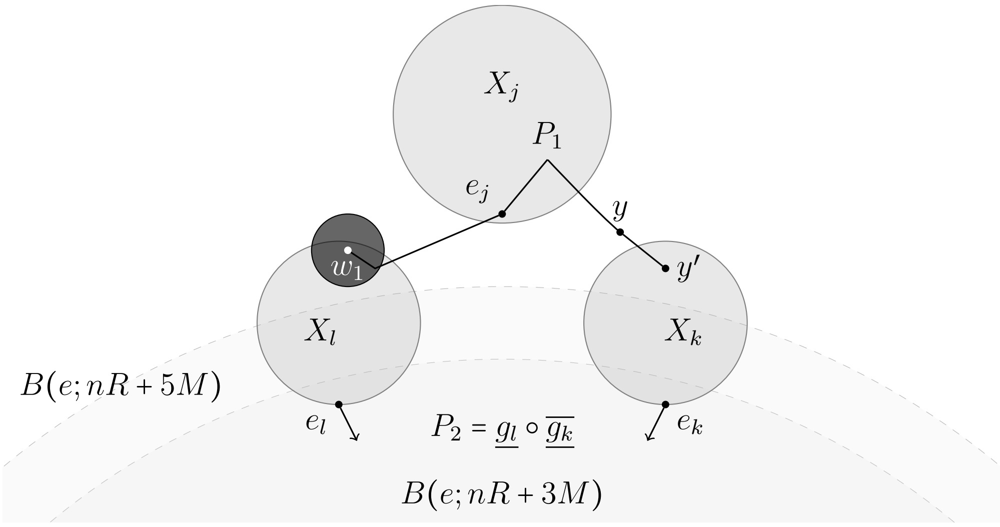

Figure 5.12: Paths P_1 and P_2

Hence $k = l \in I_{j,e}$. Using the same argument as in the special case we deduce that $c_j \in N_{4M}(X_k)$. □

We complete this section by giving the definition of the tree-graded space $\mathcal{T}(X)$ associated to X.

We define a level-decreasing function $c : I \smallsetminus \{e\} \to I$ with the following properties:

- if $i \sim j$ then $c(i) = c(j)$,
- if $c(i) = k$, then there exists some $i' \sim i$ and some bottleneck point $w \in X^k \cap W_{i',e}$ such that for all $w' \in \bigcup_{j \sim i} W_{j,e} \cap \bigcup_{\mathrm{lv}(k') < \mathrm{lv}(i)} X_{k'}$, $d_X(w', e) \leq d_X(w, e)$.

In particular, $c_j \in N_{4M}(X_{c(i)})$ for all $j \sim i$ by Lemma 5.3.4. Intuitively, $X_{c(i)}$ is the piece in a lower level which "works hardest" to approach $\{e_j \mid j \in [i]\}$.

This definition may seem awkward at first, but the following lemma shows it has merits.

Lemma 5.3.5. *If* $\mathrm{lv}(i) := n+1 > \mathrm{lv}(j)$ *and there exists some path* P *from some* $x_i \in X_i$ *to some* $x_j \in X_j$ *avoiding* $B(e; nR + 7M)$ *then* $c(i) = j$.

Proof: We are required to prove two things. Firstly we show $j \in I_{i,e}$, the method used here will also imply that $c_i \in N_{4M}(X_j)$. Following this, we prove that no other suitable piece has a bottleneck further from e.

Consider the collection $W = \{w_0, w_1, \ldots, w_s\}$ of bottlenecks defined by the set $I_{i,e}$. As $X_j \setminus B(w_r; M)$ is connected for each $r \in \{0, \ldots, s\}$, we deduce that

$$W \cap (B(e; nR + 6M) \setminus B(e; nR + M)) = \varnothing.$$

Therefore there is some r such that $w_{r+1} \in B(e; nR + M)$ and $w_r \notin B(e; nR + 6M)$, and $w_r, w_{r+1} \in X_k$ for some $k \in I_{i,e}$.

Then $k = j$, as otherwise there are two paths P_1, P_2 from X_k to X_j at Hausdorff distance at least $2M$ contradicting (RBP): (cf. figure 5.12)

- P_1 (contained in $B(e; nR + 3M)$): concatenate $\underline{g_k}$ with $\overline{\underline{g_j}}$,
- P_2 (avoids $B(e; nR + 5M)$): take a path of length at most M from w_r to some $m_i \in \underline{g_i}$, follow $\underline{g_i}$ to e_i, then take some path from e_i to x_i contained in X_i and finally follow P to $x_j \in X_j$.

Now suppose $c(i) = k' \neq j$, so there is some $i' \sim i$ and $k' \in I_{i',e} \cap I^n$ such that $c(i') = c(i) = k'$. By definition, $X_{k'}$ contains a bottleneck point $w \in W_{i',e}$ for all paths from $X_{i'}$ to X_e such that $d_X(w, e) \geq nR + 6M$.

Let P_0 be some path from $y_i \in X_i$ to $y_{i'} \in X_{i'}$ which induces the relation $i \sim i'$ and consider the paths P_3, P_4 from X_j to $X_{k'}$ given below:

- P_3: (contained in $B(e; nR)$) concatenate $\underline{g_j}$ with $\overline{\underline{g_{k'}}}$,
- P_4: start at w_r and take a path of length at most M to some $m_i \in \underline{g_i}$ then follow the reverse of $\underline{q^i_{y_i}}$ to y_i, take P_0 to $y_{i'}$, $\underline{q^{i'}_{y_{i'}}}$ to some $m_{i'} \in \underline{g_{i'}} \cap B(w; M)$ then take some path of length at most M to w.

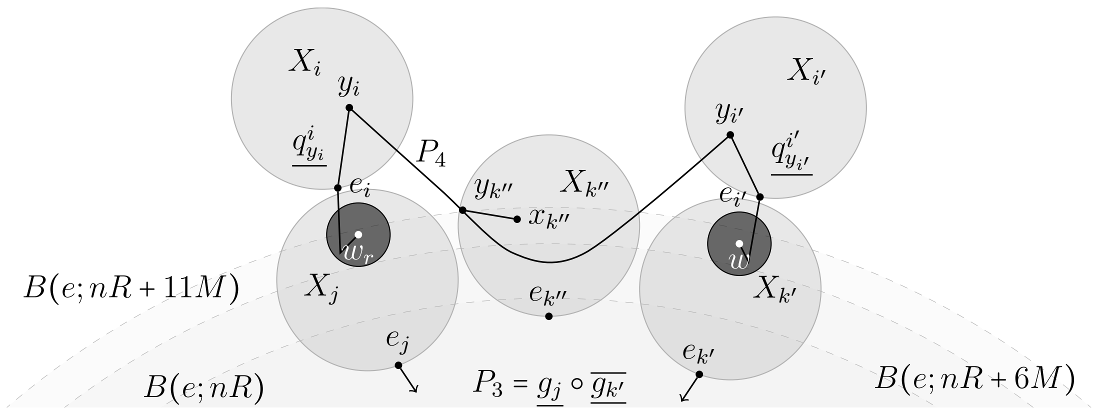

Figure 5.13: Paths P_3 and P_4

These paths are at Hausdorff distance at least $2M$ - contradicting (RBP) - unless P_0 passes within $2M$ of P_3. Such a meeting must occur inside $B(e; nR + 11M)$, and therefore within $N_{4M}(X_{k''})$ for some $k'' \in I^n$ by the definition of the equivalence relation $\sim$.

In this situation we prove $j = k'' = k'$, we present only the first of these, the second follows using the same method. To do this we give two paths P_5 and P_6 from X_j to $X_{k''}$ at Hausdorff distance at least $5M$ (cf. figure 5.13).

- P_5: (contained in $B(e; nR)$) concatenate $\underline{g_j}$ with $\overline{g_{k''}}$.
- P_6: (avoids $B(e; nR + 5M)$) follow P_4 from w_r to a point $y_{k''} \in P_0 \cap N_{4M}(X_{k''})$ but outside $B(e; nR + 10M)$ then take any path of length at most $4M$ to some point $x_{k''} \in X_{k''}$.

This completes the proof. □

$\mathcal{T}(X)$ is defined inductively with base space $\mathcal{T}(X)_0 = N_{4M}(X_e)$. We construct $\mathcal{T}(X)_k$ from $\mathcal{T}(X)_{k-1}$ by gluing on a copy of $N_{4M}(X_i)$ for each $i \in I_k$. To do this we attach $e_i \in N_{4M}(X_i)$ to $c_i \in N_{4M}(X_{c(i)})$ by a path of length $d_X(e_i, c_i)$.

Defining $\mathcal{T}(X) = \bigcup_{k \in \mathbb{N}} \mathcal{T}(X)_k$ gives rise to a tree-graded space, as each $\mathcal{T}(X)_k$ is tree-graded and the collection of such spaces is nested. The set of pieces of $\mathcal{T}(X)$ is $\{\mathcal{T}_i := N_{4M}(X_i) \mid i \in I\}$. We denote the natural metric on $\mathcal{T}(X)$ by $d_{\mathcal{T}(X)}$.

The underlying tree $\mathcal{T}$ for this construction is defined to have vertex set I and ij is an edge if and only if $c(i) = j$ or $c(j) = i$. The simplicial graph metric on $\mathcal{T}$ is denoted by $d_{\mathcal{T}}$.

We make one important observation at this point. If X is a simplicial graph, then it is easy to give $\mathcal{T}(X)$ the structure of a simplicial graph by dividing the (integer length) edges $e_i c_i$ into edges of length 1.

5.4 The proof of Theorem 5.1.1

Here we show that the natural collapse $\phi : \mathcal{T}(X) \to X$ which maps each $\mathcal{T}_i$ onto $N_{4M}(X_i)$ in the obvious way defines a quasi-isometry.

From the construction it follows immediately that ϕ is 1-Lipschitz.

We denote by e_i' and c_i' the unique points in $\mathcal{T}(X)$ contained in $\phi^{-1}(e_i) \cap \mathcal{T}_i$ and $\phi^{-1}(c_i) \cap \mathcal{T}_{c(i)}$ respectively.

To prove the other inequality we take any two points $x \in \mathcal{T}_i$ and $y \in \mathcal{T}_j$ and write the $\mathcal{T}$-geodesic between i and j as

$$i = i_0, i_1, \ldots, i_a = l = j_b, j_{b-1}, \ldots, j_0 = j,$$

where l is the piece along this geodesic of minimal level.

Without loss of generality we may assume $d_X(e_i, e) \geq d_X(e_j, e)$.

We firstly deal with the case where at least one of $a, b = 0$. By our above assumption, it must be the case that $b = 0$. To achieve this we present a base case (Lemma 5.4.1) and then apply an inductive process on a (Lemma 5.4.2).

Lemma 5.4.1. *Suppose in the above situation $a \leq 1$ and $b = 0$, then*

$$d_{\mathcal{T}(X)}(x, y) \leq d_X(\phi(x), \phi(y)) + 2R + 40M.$$

Proof: If $a = 0$ then $i = j$ and the result is obvious as X_i is $4M$ quasi-convex. For $a = 1$, $\mathrm{lv}(j) < \mathrm{lv}(i)$ so Lemma 5.3.2 yields

$$d_X(\phi(x), \phi(y)) \geq d_X(\phi(x), e_i) + d_X(e_i, \phi(y)) - 24M.$$

Hence,

$$\begin{aligned} d_{\mathcal{T}(X)}(x, y) &\leq (d_X(\phi(x), e_i) + 8M) + d_X(e_i, c_i) + (d_X(c_i, \phi(y)) + 8M) \\ &\leq d_X(\phi(x), e_i) + d_X(e_i, \phi(y)) + 16M + 2R. \end{aligned}$$

The result follows by combining the two inequalities. □

Our first inductive step completes the proof in the case $b = 0$.

Lemma 5.4.2. *Suppose $a \geq 2$ and $b = 0$. Then*

$$d_{\mathcal{T}(X)}(x,y) \leq d_X(\phi(x),\phi(y)) + 2R + 58Ma + 16M.$$

Proof: Note that by construction there is some $i' \sim i$ such that $c(i) = i'_{s+1} \in I_{i',e}$, with $s \geq 1$.

Our first step is to prove that every geodesic from $\phi(x)$ to $\phi(y)$ intersects $B := B(w_s; 5M)$.

Suppose some geodesic $\underline{g} \in [\![\phi(x),\phi(y)]\!]$ avoids B. If $d_X(e_j, w_s) > 10M$, then as $\mathrm{lv}(j) < \mathrm{lv}(i)$ $\underline{g}$ can be extended to a path from $\phi(x)$ to e also avoiding B, by taking some path within $N_{4M}(X_j)$ to e_j then following the geodesic from e_j to e.

Using the proof of Lemma 5.3.4 we see that either there is some path avoiding B from $X_{i'}$ to X_i provided by the relation $i \sim i'$ or $c(i) \in I_{i,e}$ which also yields the claim.

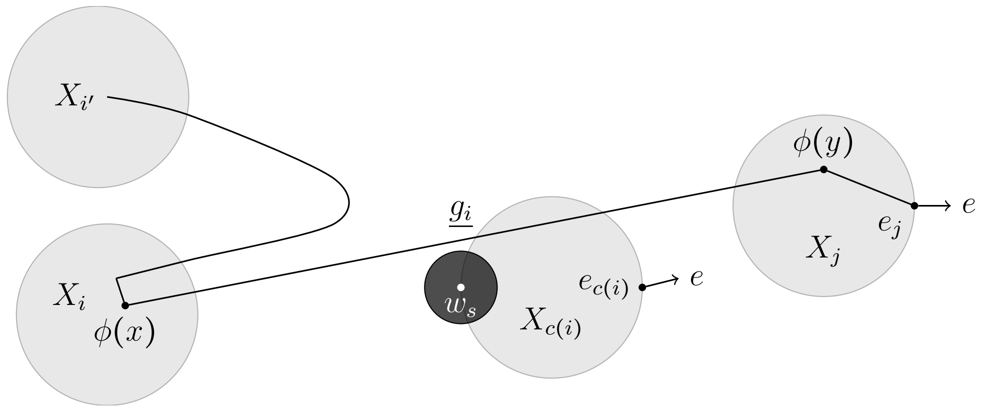

Figure 5.14: A path avoiding $B(w_s; 5M)$

However, if $d_X(e_j, w_s) < 10M$, then $\mathrm{lv}(c(i)) = \mathrm{lv}(j) + 1$. Moreover, it follows that $d_X(c_i, e_{c(i)}) \geq R - 12M$ so the geodesic from e_i to c_i extended by any path of length at most $4M$ from c_i into $X_{c(i)}$ still avoids B. This can be extended to a path from e_i to e avoiding B (and hence contradicting Lemma 5.2.3) unless $\underline{g_{c(i)}}$ meets B.

If $\underline{g_{c(i)}} \cap B \neq \varnothing$, then $d_X(e_j, X_{c(i)}) < 10M$. This ensures

$$d_X(e_{c(i)}, e) < (\mathrm{lv}(c(i)) - 1)R + 10M.$$

But then $W_{i,e}$ is contained within the $5M$-neighbourhood of $\underline{g_i}$ restricted to the sub-paths C_1 from e_i to c_i and C_2 from some point in B to e as any metric ball of radius M in between can be bypassed using paths inside $X_{c(i)}$.

In particular there is some t such that $w_t \in N_{5M}(C_1)$ and $w_{t+1} \in N_{5M}(C_2)$. Set $i'' = i_{j_t}$. We now construct two paths from $X_{c(i)}$ to $X_{i''}$:

- P_1: (contained in $B(e;(\mathrm{lv}(c(i))-1)R+15M))$ concatenate $\underline{g_{c(i)}}$ with a geodesic in $[\![e,w_{t+1}]\!]$.

- P_2: (avoids $B(e;\mathrm{lv}(c(i))R-6M))$ take a path of length $4M$ from $X_{c(i)}$ to c_i, then follow $\underline{g_i}$ to a point in $B(w_{t+1};M)$, then take a path of length at most M to $w_t \in X_{i''}$.

As these paths are at Hausdorff distance at least $2M$, we deduce via (RBP) that $i'' = c(i)$ and therefore $s = t$. But then $10M > d_X(w_s, e_j) \geq R - 7M$, which is a contradiction.

Hence, every geodesic from $\phi(x)$ to $\phi(y)$ meets B, so they also must meet $B' = B(e_{c(i)};9M)$ by Lemma 5.3.2. We then obtain a $68M$-slack geodesic $\underline{q}$ from $\phi(x)$ to $\phi(y)$ by following $q_i^{\phi(x)}$ from $\phi(x)$ to some point in B', if this sub-path meets c_i then we take a path of length at most $16M$ inside B' to meet up with some point on a geodesic $\underline{g} \in [\![\phi(x),\phi(y)]\!]$ and follow that to $\phi(y)$. This provides a $50M$-slack geodesic.

If this sub-path does not meet c_i then as $d_X(e_{c(i)},e) \leq d_X(c_i,e)$ we see that c_i lies in $B(e_{c(i)};18M)$, in this case we follow the path $q_i^{\phi(x)}$ from $\phi(x)$ to c_i, take a path of length at most $27M$ to some point in $\underline{g}$ and follow that to $\phi(y)$. In this situation we obtain a $68M$-slack geodesic.

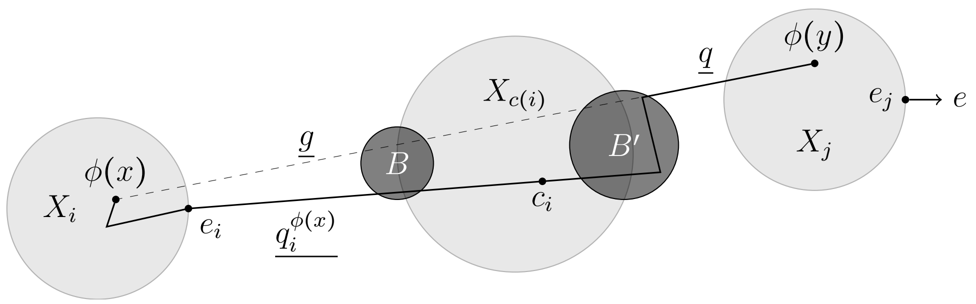

Figure 5.15: The $68M$-slack geodesic $\underline{q}$

Importantly, $\underline{q}$ meets c_i, so

$$d_X(\phi(x),\phi(y)) \geq d_X(\phi(x),c_i) + d_X(c_i,\phi(y)) - 68M.$$

We recall that by the inductive hypothesis,

$$d_{\mathcal{T}(X)}(c_i',y) \leq d_X(c_i,\phi(y)) + 2R + 76M(a-1) + 16M.$$

Finally, by Lemma 5.3.1, $d_{\mathcal{T}(X)}(x, c'_i) = d_X(\phi(x), e_i) + d_X(e_i, c_i) \leq d_X(\phi(x), c_i) + 8M$, so combining these we see that

$$\begin{aligned} d_{\mathcal{T}(X)}(x,y) &= d_{\mathcal{T}(X)}(x,c'_i) + d_{\mathcal{T}(X)}(c'_i,y) \\ &\leq d_X(\phi(x),\phi(y)) + 2R + 76M(a-1) + 68M + 8M + 16M \\ &= d_X(\phi(x),\phi(y)) + 2R + 76Ma + 16M. \end{aligned}$$

□

Now we come to the case $b \geq 1$. Again we start with a base case before progressing to the general result.

Lemma 5.4.3. *Suppose $a = b = 1$. Then*

$$d_{\mathcal{T}(X)}(x,y) \leq d_X(\phi(x),\phi(y)) + 7R + 80M.$$

Proof: Recall that $l = c(i) = c(j)$. Without loss of generality we assume $d(e_i, e) \geq d(e_j, e)$, so in particular, $n := \mathrm{lv}(i) \geq m := \mathrm{lv}(j)$. By Lemma 5.3.2, every path from $\phi(x)$ to $N_{4M}(X_j)$ passes through $B(e_i; 8M)$. If some geodesic in $[[\phi(x), \phi(y)]]$ meets $B(e_j; 16M)$, then

$$\begin{aligned} d_X(\phi(x),\phi(y)) &\geq d_X(\phi(x),e_j) + d_X(e_j,\phi(y)) - 32M \\ &\geq d_X(\phi(x),e_i) + d_X(e_i,e_j) - 16M + d_X(e_j,\phi(y)) - 32M. \end{aligned}$$

Combining these bounds we see that

$$\begin{aligned} d_{\mathcal{T}(X)}(x,y) &= d_{\mathcal{T}(X)}(x,e'_i) + d_{\mathcal{T}(X)}(e'_i,c'_i) + d_{\mathcal{T}(X)}(c'_i,c'_j) + \\ &\quad\; d_{\mathcal{T}(X)}(c'_j,e'_j) + d_{\mathcal{T}(X)}(e'_j,y) \\ &\leq d_X(\phi(x),e_i) + d_X(c_i,c_j) + d_X(e_j,\phi(y)) + 2R + 32M \\ &\leq d_X(\phi(x),e_i) + d_X(e_i,e_j) + d_X(e_j,\phi(y)) + 4R + 32M \\ &\leq d_X(\phi(x),\phi(y)) + 4R + 80M. \end{aligned}$$

Now suppose all geodesics avoid $B(e_j; 16M)$. By Lemma 5.3.2 we know that geodesics must also avoid $\bigcup_{k \in I^m} N_{12M}(X_k)$, so, in particular they avoid the set $N_{8M}(\underline{g_i^c})$ where we define $\underline{g_i^c}$ to be the restriction of $\underline{g_i}$ to a geodesic in $[[c_i, e,]]$. Moreover, all geodesics must also avoid $N_{8M}(\underline{g_j})$ otherwise one can find a path from X_j to e avoiding $B(e_j; M)$.

Hence, the bottleneck $w_0 \in W_{j,i}$ lying in X_j must be within M of some point of $\underline{g_i} \setminus B(e; nR{+}8M)$. In particular there is a path from X_i to X_j avoiding $B(e; nR{+}7M)$.

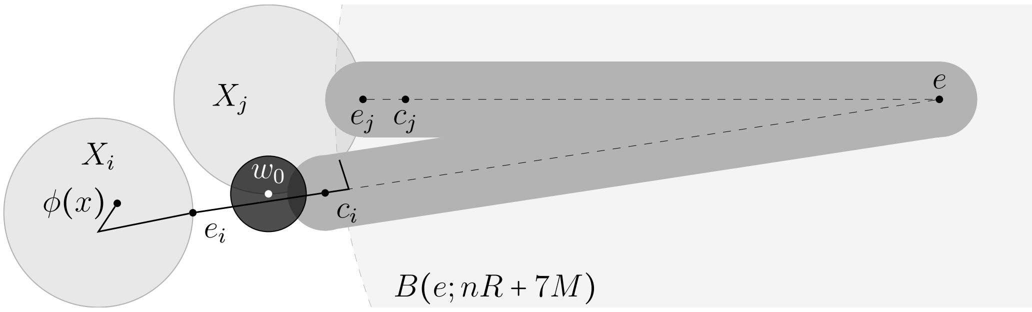

Figure 5.16: A path from X_i to X_j

If $n > m$ then $c(i) = j$, by Lemma 5.3.5, which contradicts the assumption that $a = b = 1$. However, if $n = m$ then $d_X(w, e_j) \le R + 2M$. Hence,

$$\begin{aligned} d_X(c_i, c_j) &\le d_X(c_i, w) + d_X(w, e_j) + d_X(e_j, c_j) \\ &\le (R + M) + (R + 2M) + R = 3R + 3M, \end{aligned}$$

while $d_X(e_i, e_j) \le d_X(e_i, w) + d_X(w, e_j) \le (R - 7M) + (R + 2M) = 2R - 5M$.

Therefore,

$$\begin{aligned} d_{\mathcal{T}(X)}(x, y) &\le d_X(\phi(x), e_i) + R + (3R + 3M) + R + d_X(e_j, \phi(y)) + 16M \\ &\le d_X(\phi(x), e_i) + d_X(e_i, \phi(y)) + (2R - 5M) + 5R + 19M \\ &\le d_X(\phi(x), \phi(y)) + 7R + 14M. \end{aligned}$$

The final step uses Lemma 5.3.2. □

This leads to the final lemma required for the proof of Theorem 5.1.1.

Lemma 5.4.4. *Suppose $a, b \ge 1$ and $d_{\mathcal{T}}(i, j) = a + b \ge 3$, then*

$$d_{\mathcal{T}(X)}(x, y) \le d_X(\phi(x), \phi(y)) + 9R + 80M(a + b).$$

Proof: We proceed by induction on $a + b$ using the previous three lemmas as base cases, we do not include the extra $+16M$ as we will not require the situation $a = b = 0$ in our inductive step. To ease notation we set $\mathrm{lv}(i) := n + 1$ and $\mathrm{lv}(j) := m + 1$, by assumption $\mathrm{lv}(i), \mathrm{lv}(j) \ge 1$.

If some $45M$-slack geodesic from $\phi(x)$ to $\phi(y)$ meets $\{c_i, c_j\}$, (we deal with the case of c_i, the other case is very similar) then $d_X(\phi(x), \phi(y)) \ge d_X(\phi(x), c_i) + d_X(c_i, \phi(y)) - 45M$.

Lemma 5.3.1 gives $d_X(\phi(x), c_i) \ge d_X(\phi(x), e_i) + d_X(e_i, c_i) - 10M$, while by the inductive hypothesis

$$d_{\mathcal{T}(X)}(y, c_i') \le d_X(\phi(y), c_i) + 9R + 80M(a + b - 1).$$

Combining these we see that

$$\begin{aligned} d_{\mathcal{T}(X)}(x,y) &= d_{\mathcal{T}(X)}(x,c_i') + d_{\mathcal{T}(X)}(c_i',y) \\ &\leq d_X(\phi(x),e_i) + d_X(e_i,c_i) + 8M + d_X(c_i,\phi(y)) \\ &\quad +9R + 80M(a+b-1) \\ &\leq d_X(\phi(x),\phi(y))9R + 80M(a+b). \end{aligned}$$

So far we have not supposed that $d_X(e,e_i) \geq d_X(e_j,e)$, but from this point on we will assume that this is the case.

Now suppose every $45M$-slack geodesic from $\phi(x)$ to $\phi(y)$ avoids $\{c_i,c_j\}$, then every geodesic in $[[\phi(x),\phi(y)]]$ misses $N_{15M}(\underline{g_i^c} \cup \underline{g_j^c})$, where we recall $\underline{g_k^c}$ is the restriction of $\underline{g_k}$ to a geodesic in $[[c_k,e]]$. If this is not the case then it is easy to find a suitable slack geodesic $\underline{q}$ which hits either c_i or c_j.

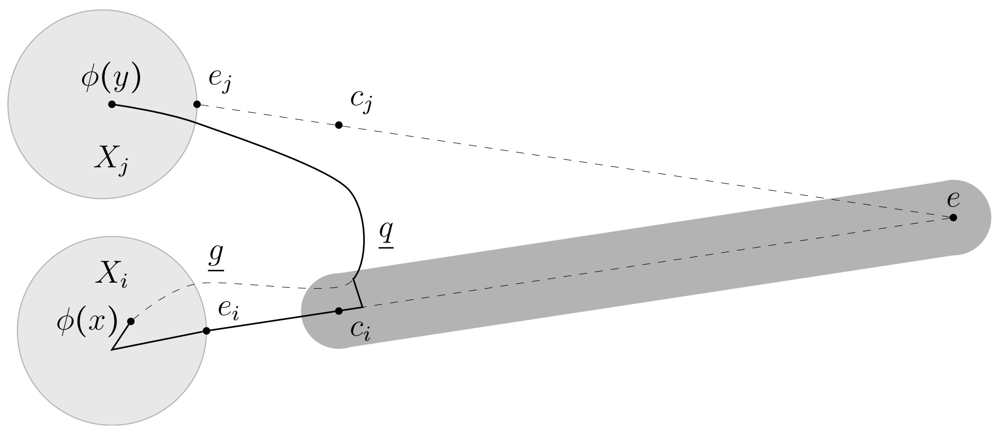

Figure 5.17: Finding slack geodesics meeting c_i

We now have two paths from $N_{4M}(X_i)$ to $N_{4M}(X_j)$ given by $\underline{g_i} \circ \overline{\underline{g_j}}$ and some $\underline{g} \in [[\phi(x),\phi(y)]]$.

As $\underline{g} \cap N_{15M}(\underline{g_i^c} \cup \underline{g_j^c}) = \varnothing$, we deduce that the collection of bottlenecks $W_{i,j}$ given by (RBP) is contained in

$$\left(N_M(\underline{g_i}) \smallsetminus B(e;nR+13M)\right) \cup \left(N_M\left(X_j \cup \underline{g_j}\right) \smallsetminus B(e;mR+13M)\right).$$

We label the first of these two sets A and the second one B. Here we are using Lemma 5.2.4 to ensure that X_j is (path-)connected.

If $A \cap B \neq \varnothing$ then it is clear that $i \sim j$ if $\mathrm{lv}(i) = \mathrm{lv}(j)$ or $c(i) = j$, by Lemma 5.3.5, if $\mathrm{lv}(i) > \mathrm{lv}(j)$, both of which contradict the assumption that $d_{\mathcal{T}}(i,j) \geq 3$. This situation is similar to that of figure 5.16.

Now we may assume that they are disjoint, then there is some piece X_k, with $k \in I_{i,j}$ containing two bottlenecks, one in each of A and B. We label the bottleneck point in A by w_1 and the one in B by w_2.

From here on we split into a number of cases depending on the relationship between $\mathrm{lv}(i)$, $\mathrm{lv}(j)$ and $\mathrm{lv}(k)$.

Case 1: $\mathrm{lv}(i) = \mathrm{lv}(j)$ It follows immediately from the above that $i \sim j$, regardless of $\mathrm{lv}(k)$, contradicting the assumption that $d_{\mathcal{T}}(i,j) \geq 3$.

From now on we assume $\mathrm{lv}(i) > \mathrm{lv}(j)$.

Case 2: $\mathrm{lv}(k) > \mathrm{lv}(i)$ In this situation we prove that $c(i) = j$.

$w_1, w_2 \in X_k$, so $d_X(w_1, e),\ d_X(w_2, e) \geq (n+1)R - M$. Hence there is a path from X_i to X_j avoiding $B(e; (n+1)R - 2M)$.

Thus, $c(i) = j$ by Lemma 5.3.5 as there is a path from X_i to X_j avoiding $B(e; nR + 6M)$.

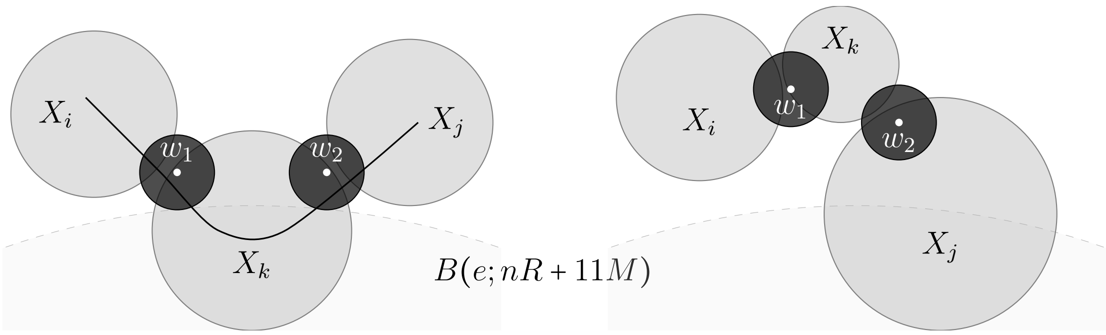

Figure 5.18: Cases 1 (left) and 2 (right)

Case 3: $\mathrm{lv}(k) = \mathrm{lv}(i)$ Here we prove that either $c(i) = j$ or contradict the assumption that no $45M$-slack geodesic from $\phi(x)$ to $\phi(y)$ meets c_i.

The fact that $i \sim k$ is immediate from the location of bottleneck w_1.

If $d_X(e_k, c_k) \geq 9M$ then there is a path from X_k to X_j (via w_2) avoiding $B(e; nR{+}6M)$, so $c(k) = j$ by Lemma 5.3.5. Hence, $c(i) = j$.

Now suppose $d_X(e_k, c_k) < 9M$, then $\underline{g_i} \cap B(e_k; M) \neq \varnothing$ as otherwise we would obtain (via w_1 and $\underline{g_i}$) a path from X_k to e avoiding $B(e_k; M)$, which contradicts (RBP). Notice that here we have used the fact that $d_X(w_1, e_k) \geq d_X(w_1, e) - d_X(e_k, e) \geq 11M - 9M \geq 2M$.

Let $m_i \in \underline{g_i} \cap B(e_k; M)$. Then,

$$d_X(e_k, c_i) \le d_X(e_k, m_i) + d_X(m_i, c_i) < M + 9M + M = 11M.$$

As every path from $\phi(x)$ to $\phi(y)$ meets $B(w_1; M)$ it also meets $B(e_k; 5M)$ by Lemma 5.3.1. Thus every such path meets $B(c_i; 16M)$. In particular, there is some $32M$-slack geodesic from $\phi(x)$ to $\phi(y)$ which meets c_i, contradicting the initial assumption.

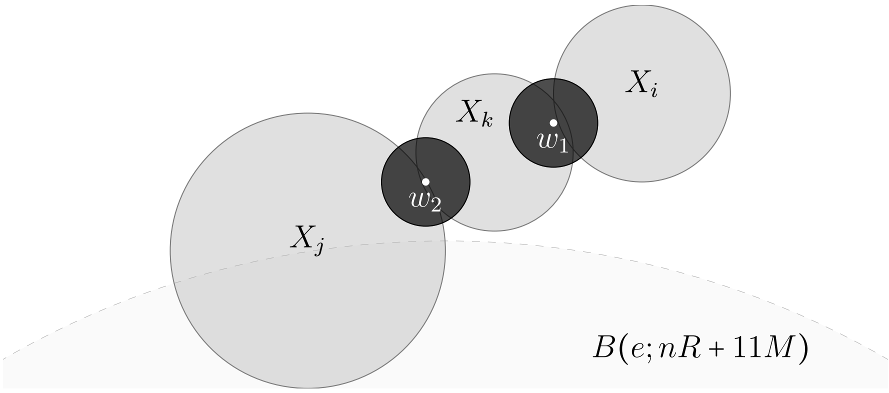

Figure 5.19: Case 3: $c(i) = j$

From here on we assume $\text{lv}(i) > \text{lv}(k)$, from this and the location of the bottleneck w_1 we know that $c(i) = k$.

Case 4: $\text{lv}(k) > \text{lv}(j)$ As in case 3 we find a $45M$-slack geodesic meeting c_i.

Immediately we see that $d_X(w_2, e_k) \le 2M$ as the bottleneck must cut the path $\underline{g_k} \circ \underline{\overline{g_j}}$. But as every path from $\phi(x)$ to $\phi(y)$ meets $N_{5M}(X_k)$ we see that such paths meet $B(e_k; 9M)$ by Lemma 5.3.1.

Fix some $\underline{g} \in [[\phi(x), \phi(y)]]$. We obtain a $45M$-slack geodesic $\underline{q}$ from $\phi(x)$ to $\phi(y)$ passing through c_i by following $\underline{q_i^{\phi(x)}}$ to a point $m_i \in B(e_k; 9M) \cap \underline{g}$ - if this restriction of $\underline{q_i^{\phi(x)}}$ does not include c_i we include a diversion of length at most $18M$ along $\underline{q_i^{\phi(x)}}$ to c_i and then back again - then follow $\underline{g}$ to $\phi(y)$.

As every path meets $B(e_k; 9M)$,

$$\begin{aligned} d_X(\phi(x), \phi(y)) &\ge d_X(\phi(x), e_k) + d_X(e_k, \phi(y)) - 18M \\ &\ge (d_X(\phi(x), m_i) - d_X(m_i, e_k)) + d_X(e_k, \phi(y)) - 18M \\ &\ge l(\underline{q}) - 18M - 9M - 18M. \end{aligned}$$

(The first $-18M$ comes from the possible detour to c_i.) This contradicts the assumption made at the start. (cf. Figure 5.17.)

Case 5: $\mathrm{lv}(j) > \mathrm{lv}(k)$ In this situation we prove that $c(i) = c(j) = k$ contradicting the assumption that $d_{\mathcal{T}}(i,j) \geq 3$.

We already know that $c(i) = k$. It is immediate from the location of w_2 that $d_X(e, w_2) \geq mR + 10M$, so there is a path from X_j to X_k avoiding $B(e; mR + 7M)$ and we apply Lemma 5.3.5 to deduce that $c(j) = k$.

Case 6: $\mathrm{lv}(i) > \mathrm{lv}(j) = \mathrm{lv}(k)$ Here $c(i) = k \sim j$, so $d_{\mathcal{T}}(i,j) = m + n = 3$. We deal with this case directly.

$j \sim k$, as the bottleneck between $X_j \cup \underline{g_j}$ and X_k yields a path from X_j to X_k avoiding $B(e; nR + 11M)$. To avoid contradicting RBP for paths between X_j and X_k it follows that $w_2 \in B(e_j; 3M) \cup B(e_k; 3M)$. If this is not the case then the path of length M from w_2 to X_j and $\underline{g_k} \circ \overline{\underline{g_j}}$ are at Hausdorff distance at least $2M$.

If $w_2 \in B(e_j; 3M)$, then

$$d_X(e_j, e_k) \leq d_X(e_j, w_2) + d_X(w_2, e_k) \leq 3M + (R + 5M)$$

and if $w_2 \in B(e_k; 3M)$, then

$$d_X(e_j, e_k) \leq d_X(e_j, w_2) + d_X(w_2, e_k) \leq (R + 7M) + 3M.$$

Here we are using the fact that any geodesic from w_2 to e meets $B(e_k; M)$ or $B(e_j; 2M)$. In either situation, $d_X(c_j, c_k) \leq 3R + 10M$.

Hence, as any path from $\phi(x)$ to $\phi(y)$ meets $B(e_k; 5M)$ or $B(e_j; 5M)$, by Lemma 5.3.2,

$$d_X(\phi(x), \phi(y)) \geq d_X(\phi(x), e_k) + d_X(e_j, \phi(y)) - 2(R + 10M) - 10M.$$

Using Lemma 5.4.1 we see that

$$d_{\mathcal{T}(X)}(x, e'_k) \leq d_X(\phi(x), e_k) + 2R + 40M$$

Then as $d_{\mathcal{T}(X)}(x,y) \leq d_{\mathcal{T}(X)}(x, e'_k) + d_{\mathcal{T}(X)}(c'_k, c'_j) + d_{\mathcal{T}(X)}(e'_j, y) + 2R$,

$$\begin{aligned} d_{\mathcal{T}(X)}(x,y) &\leq (d_X(\phi(x), e_k) + 2R + 40M) + (3R + 10M + 16M) + \\ &\quad (d_X(e_j, \phi(y)) + 8M) + 2R \\ &\leq d_X(\phi(x), \phi(y)) + 9R + 104M. \end{aligned}$$

□

Proof of Theorem 5.1.1: The easier implication follows from Lemma 5.2.2 and

Proposition 5.2.5. From Lemmas 5.4.1, 5.4.2, 5.4.3 and 5.4.4 we know that for all $x \in \mathcal{T}_i$ and all $y \in \mathcal{T}_j$

$$d_{\mathcal{T}(X)}(x,y) \le d_X(\phi(x),\phi(y)) + 9R + 80M d_{\mathcal{T}}(i,j) + 16M.$$

Now, $d_{\mathcal{T}(X)}(x,y) \ge R(\max\{\mathcal{T}(i,j) - 2, 0\})$, so setting $R = 2(80M) = 160M$ we see that

$$d_{\mathcal{T}(X)}(x,y) \le d_X(\phi(x),\phi(y)) + 9R + \frac{1}{2} d_{\mathcal{T}(X)}(x,y) + 320M + 16M.$$

Hence,

$$d_{\mathcal{T}(X)}(x,y) \le 2d_X(\phi(x),\phi(y)) + 18R + 672M = 2d_X(\phi(x),\phi(y)) + 3552M.$$

□

5.5 Consequences

In this Section we prove Corollaries 5.1.2 and 5.1.3.

Corollary 5.5.1. *Mapping class groups of compact surfaces quasi-isometrically embed into a finite product of trees.*

This construction is not equivariant, even for finite index subgroups. Such an equivariant construction would provide a proof that mapping class groups do not have Kazhdan's property (T).

Proof: Consider the surface $S = S_{g,n}$. If $3g + n - 4 \le 0$ then $MCG(S)$ is virtually free and the result follows [Beh04]. We now assume $3g + n > 4$, from [BBF10] and Theorem 5.1.1 we have a quasi-isometric embedding of $MCG(S)$ into a finite product of tree-graded spaces as follows:

$$MCG(S) \to \prod_{i=1}^{k} \mathcal{C}(\mathbf{Y})_i \to \prod_{i=1}^{k} \mathcal{T}(\mathcal{C}(\mathbf{Y})_i).$$

where in each $\mathcal{T}(\mathcal{C}(\mathbf{Y})_i)$ the pieces are uniformly quasi-isometric to the curve complex of a fixed subsurface U_i of S.

There are only finitely many subsurfaces of S up to homeomorphism, so their curve complexes can be uniformly (K,C) quasi-isometrically embedded into a product of l trees, for some K, C, l depending only on S [Buy05, MS12].

To complete this corollary we now outline a simple argument which states that if each piece in a tree-graded space (K,C) quasi-isometrically embeds (for some fixed

K and C) into a product of l trees then so does the whole tree-graded space. We simply replace each piece $\mathcal{T}_i$ in the tree-graded space $\mathcal{T}(X)$ by the appropriate ordered product of trees, $T_{i1} \times T_{i2} \times \cdots \times T_{il}$ to obtain the new tree-graded space $\mathcal{T}(X)'$. It is clear that $\mathcal{T}(X)$ quasi-isometrically embeds into $\mathcal{T}(X)'$. The natural mapping

$$\mathcal{T}(X)' \to \prod_{j=1}^{l} T_j$$

where each T_j is the tree obtained by collapsing each piece of $\mathcal{T}(X)'$ to the jth tree is a quasi-isometry. Certainly, distances are not decreased by this map, but also the map onto each tree is Lipschitz. Thus we obtain a quasi-isometric embedding

$$MCG(S) \to \prod_{i=1}^{k} \prod_{j=1}^{l} T_{jl},$$

completing the proof. □

Up to another quasi-isometry, each T_{jl} can be assumed to be a simplicial (but still locally infinite) tree [Man05].

We move now to consequences for relatively hyperbolic groups.

Corollary 5.5.2. *Relatively hyperbolic groups have finite Assouad-Nagata dimension or quasi-isometrically embed into an ℓ^1 space if and only if each maximal parabolic subgroup of G does.*

Proof: Given a group G which is hyperbolic relative to $\{H_i\}$, combining Theorem 5.1.1 with [MS12] we obtain a quasi-isometric embedding:

$$G \to \prod_{i=1}^{k} \mathcal{C}(\mathbf{H})_i \times \hat{G} \to \prod_{i=1}^{k} \mathcal{T}(\mathcal{C}(\mathbf{H})_i) \times \hat{G},$$

where $\mathcal{T}(\mathcal{C}(\mathbf{H}))$ is a tree-graded space whose pieces are uniformly quasi-isometric to parabolic subgroups of G, also $\hat{G}$ quasi-isometrically embeds into a finite product of trees [Buy05, MS12]. The result then follows from [BH09] and Theorem 3. □

We now come to the final result, displaying explicit embedding of mapping class groups and relatively hyperbolic groups into ℓ^p spaces exhibiting the optimal compression exponent. We have the following embeddings already

$$MCG(S) \to \prod \mathcal{C}(\mathbf{Y}) \to \prod \mathcal{T}(\mathcal{C}(\mathbf{Y})), \text{ and}$$

$$G \to \prod \mathcal{C}(\mathbf{H}) \times \hat{G} \to \prod \mathcal{T}(\mathcal{C}(\mathbf{H})) \times \hat{G}.$$

It follows from Theorem 3 that a tree-graded space (with countably many pieces) can be explicitly embedded into an ℓ^p space displaying compression exponent α whenever this is uniformly true for all the pieces. Therefore, the result for mapping class groups follows from Theorem 4.6.1. Notice that this theorem applies to both curve graphs and coned-off graphs [MM99, MM00, Bow08], so we can also apply that theorem in the relatively hyperbolic case to deduce the final part of Corollary 5.1.3.

Bibliography

[AD08] Goulnara Arzhantseva and Thomas Delzant. Examples of Random Groups. www.unige.ch/math/folks/arjantse/Abs/random.pdf, 2008.

[ADS09] Goulnara Arzhantseva, Cornelia Druţu and Mark Sapir. Compression functions of uniform embeddings of groups into Hilbert and Banach spaces. *J. Reine Angew. Math.*, 633:213–235, 2009.

[Ago] Ian Agol (mathoverflow.net/users/1345). Is there an elementary proof of a result about the parity of the period of the repeating block in the continued fraction expansion of square roots of integers. MathOverflow. `http://mathoverflow.net/questions/13449` (version: 2010-01-30).

[AGŠ12] Goulnara Arzhantseva, Erik Guentner and Ján Špakula. Coarse non-amenability and coarse embeddings. *Geom. Funct. Anal.*, 22(1):22–36, 2012.

[Ali05] Emina Alibegović. A combination theorem for relatively hyperbolic groups. *Bull. London Math. Soc.*, 37(3):459–466, 2005.

[Alo90] Juan M. Alonso. Inégalités isopérimétriques et quasi-isométries. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(12):761–764, 1990.

[AMM85] Israel Aharoni, Bernard Maurey and Boris S. Mityagin. Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces. *Israel J. Math.*, 52(3):251–265, 1985.

[ANP09] Tim Austin, Assaf Naor and Yuval Peres. The wreath product of $\mathbb{Z}$ with $\mathbb{Z}$ has Hilbert compression exponent $\frac{2}{3}$. *Proc. Amer. Math. Soc.*, 137(1):85–90, 2009.

[Aus11] Tim Austin. Amenable groups with very poor compression into Lebesgue spaces. *Duke Math. J.*, 159(2):187–222, 2011.

[Ban93] Stefan Banach. *Théorie des opérations linéaires.* Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original.

[Bau07] Florent Baudier. Metrical characterization of super-reflexivity and linear type of Banach spaces. *Arch. Math. (Basel)*, 89(5):419–429, 2007.

[BB00] N. Brady and M. R. Bridson. There is only one gap in the isoperimetric spectrum. *Geom. Funct. Anal.*, 10(5):1053–1070, 2000.

[BBF10] Mladen Bestvina, Ken Bromberg and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. Preprint available from arXiv:1006.1939, 2010.

[BD08] Gregory Bell and Alexander Dranishnikov. Asymptotic dimension. *Topology Appl.*, 155(12):1265–1296, 2008.

[BDG+05] Mihai Bădoiu, Kedar Dhamdhere, Anupam Gupta, Yuri Rabinovich, Harald Räcke, R. Ravi and Anastasios Sidiropoulos. Approximation algorithms for low-distortion embeddings into low-dimensional spaces. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 119–128 (electronic), New York, 2005. ACM.

[BDHM09] Nikolay Brodskiy, Jerzy Dydak, Jose Higes and Atish Mitra. Assouad-Nagata dimension via Lipschitz extensions. *Israel J. Math.*, 171:405–423, 2009.

[BDL06] Nikolay Brodskiy, Jerzy Dydak and Urs Lang. Assouad-Nagata dimension of wreath products of groups. Preprint available from arXiv: math/0611331, 2006.

[BDM09] Jason Behrstock, Cornelia Druţu and Lee Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Math. Ann.*, 344(3):543–595, 2009.

[BDS07] Sergei Buyalo, Alexander Dranishnikov and Viktor Schroeder. Embedding of hyperbolic groups into products of binary trees. *Invent. Math.*, 169(1):153–192, 2007.

[Beh04] Jason A. Behrstock. *Asymptotic Geometry of the Mapping Class Group and Teichmuller Space.* PhD thesis, SUNY Stony Brook, 2004.

[BF08] Gregory C. Bell and Koji Fujiwara. The asymptotic dimension of a curve graph is finite. *J. Lond. Math. Soc. (2)*, 77(1):33–50, 2008.

[BG05] Nathanial Brown and Erik Guentner. Uniform embeddings of bounded geometry spaces into reflexive Banach space. *Proc. Amer. Math. Soc.*, 133(7):2045–2050 (electronic), 2005.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BH09] Nikolay Brodskiy and Jose Higes. Assouad-Nagata dimension of tree-graded spaces. Preprint available from arXiv:0910.2378v1, 2009.

[BM11] Jason A. Behrstock and Yair N. Minsky. Centroids and the rapid decay property in mapping class groups. *J. Lond. Math. Soc. (2)*, 84(3):765–784, 2011.

[BN08] Jason A. Behrstock and Walter D. Neumann. Quasi-isometric classification of graph manifold groups. *Duke Math. J.*, 141(2):217–240, 2008.

[Bou86] Jean Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56(2):222–230, 1986.

[Bow95] Brian H. Bowditch. A short proof that a subquadratic isoperimetric inequality implies a linear one. *Michigan Math. J.*, 42(1):103–107, 1995.

[Bow08] Brian H. Bowditch. Tight geodesics in the curve complex. *Invent. Math.*, 171(2):281–300, 2008.

[Bow12] Brian H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[Bra21] Henry R. Brahana. Systems of circuits on two-dimensional manifolds. *Ann. of Math. (2)*, 23(2):144–168, 1921.

[Bri93] Martin R. Bridson. On the geometry of normal forms in discrete groups. *Proc. London Math. Soc. (3)*, 67(3):596–616, 1993.

[BS97] Itai Benjamini and Oded Schramm. Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *Geom. Funct. Anal.*, 7(3):403–419, 1997.

[BS00] Mario Bonk and Oded Schramm. Embeddings of Gromov hyperbolic spaces. *Geom. Funct. Anal.*, 10(2):266–306, 2000.

[BS05] Sergei Buyalo and Viktor Schroeder. Embedding of hyperbolic spaces in the product of trees. *Geom. Dedicata*, 113:75–93, 2005.

[BS08] Nikolay Brodskiy and Dmitry Sonkin. Compression of uniform embeddings into Hilbert space. *Topology Appl.*, 155(7):725–732, 2008.

[Buy05] Sergei V. Buyalo. Capacity dimension and embedding of hyperbolic spaces into the product of trees. *Algebra i Analiz*, 17(4):42–58, 2005.

[CDGY03] Xiaoman Chen, Marius Dadarlat, Erik Guentner and Guoliang Yu. Uniform embeddability and exactness of free products. *J. Funct. Anal.*, 205(1):168–179, 2003.

[CDP90] Michel Coornaert, Thomas Delzant and Athanase Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.

[CK10a] Jeff Cheeger and Bruce Kleiner. Differentiating maps into L^1, and the geometry of BV functions. *Ann. of Math. (2)*, 171(2):1347–1385, 2010.

[CK10b] Jeff Cheeger and Bruce Kleiner. Metric differentiation, monotonicity and maps to L^1. *Invent. Math.*, 182(2):335–370, 2010.

[Cla36] James A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.

[CZ06a] Huai-Dong Cao and Xi-Ping Zhu. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.*, 10(2):165–492, 2006.

[CZ06b] Huai-Dong Cao and Xi-Ping Zhu. Erratum to: "A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow" [Asian J. Math. **10** (2006), no. 2, 165–492]. *Asian J. Math.*, 10(4):663, 2006.

[Dah03a] François Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963 (electronic), 2003.

[Dah03b] François Dahmani. *Les groupes relativement hyperboliques et leurs bords*. Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 2003/13. Université Louis Pasteur Département de Mathématique Institut de Recherche Mathématique Avancée, Strasbourg, 2003. Thèse, l'Université Louis Pasteur (Strasbourg I), Strasbourg, 2003.

[dC08] Yves de Cornulier. Dimension of asymptotic cones of Lie groups. *J. Topol.*, 1(2):342–361, 2008.

[dCT08] Yves de Cornulier and Romain Tessera. Quasi-isometrically embedded free sub-semigroups. *Geom. Topol.*, 12(1):461–473, 2008.

[dCTV07] Yves de Cornulier, Romain Tessera and Alain Valette. Isometric group actions on Hilbert spaces: growth of cocycles. *Geom. Funct. Anal.*, 17(3):770–792, 2007.

[dCTV08] Yves de Cornulier, Romain Tessera and Alain Valette. Isometric group actions on Hilbert spaces: structure of orbits. *Canad. J. Math.*, 60(5):1001–1009, 2008.

[DG03] Marius Dadarlat and Erik Guentner. Constructions preserving Hilbert space uniform embeddability of discrete groups. *Trans. Amer. Math. Soc.*, 355(8):3253–3275 (electronic), 2003.

[DG07] Marius Dadarlat and Erik Guentner. Uniform embeddability of relatively hyperbolic groups. *J. Reine Angew. Math.*, 612:1–15, 2007.

[DH08] Jerzy Dydak and Jose Higes. Asymptotic cones and Assouad-Nagata dimension. *Proc. Amer. Math. Soc.*, 136(6):2225–2233, 2008.

[DJ99] Alexander Dranishnikov and Tadeusz Januszkiewicz. Every Coxeter group acts amenably on a compact space. *Topology Proc.*, 24(Spring):135–141, 1999.

[DJ00] Michael W. Davis and Tadeusz Januszkiewicz. Right-angled Artin groups are commensurable with right-angled Coxeter groups. *J. Pure Appl. Algebra*, 153(3):229–235, 2000.

[Dre10] Dennis Dreesen. Hilbert space compression for free products and HNN-extensions. Preprint available from arXiv:1002.3879v3, 2010.

[DS05] Cornelia Druţu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.

[Enf72] Per Enflo. Banach spaces which can be given an equivalent uniformly convex norm. In *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972)*, volume 13, pages 281–288 (1973), 1972.

[Èrs03] Anna Èrshler. On drift and entropy growth for random walks on groups. *Ann. Probab.*, 31(3):1193–1204, 2003.

[Far98] Benson Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.

[Far02] F. T. Farrell. The Borel conjecture. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 225–298. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 2012.

[FRR95] Steven C. Ferry, Andrew Ranicki and Jonathan Rosenberg. A history and survey of the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 7–66. Cambridge Univ. Press, Cambridge, 1995.

[Gal08] Światosław R. Gal. Asymptotic dimension and uniform embeddings. *Groups Geom. Dyn.*, 2(1):63–84, 2008.

[GK04] Erik Guentner and Jerome Kaminker. Exactness and uniform embeddability of discrete groups. *J. London Math. Soc. (2)*, 70(3):703–718, 2004.

[Gre66] Leon Greenberg. Fundamental polyhedra for kleinian groups. *Ann. of Math. (2)*, 84:433–441, 1966.

[Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[Gro87] Misha Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Gro93] Misha Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gro00] Misha Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).

[Gro03] M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[GTY11] Erik Guentner, Romain Tessera and Guoliang Yu. Discrete Groups with Finite Decomposition Complexity. Preprint available from http://www.math.hawaii.edu/~erik/papers/FDC.pdf, 2011.

[GTY12] Erik Guentner, Romain Tessera and Guoliang Yu. A notion of geometric complexity and its application to topological rigidity. *Invent. Math.*, 189(2):315–357, 2012.

[Ham09] Ursula Hamenstädt. Geometry of the mapping class groups. I. Boundary amenability. *Invent. Math.*, 175(3):545–609, 2009.

[Han56] Olof Hanner. On the uniform convexity of L^p and l^p. *Ark. Mat.*, 3:239–244, 1956.

[Hir54] Friedrich Hirzebruch. Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties. *Proc. Nat. Acad. Sci. U. S. A.*, 40:110–114, 1954.

[HLS02] Nigel Higson, Vincent Lafforgue and Georges Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[HLW06] Shlomo Hoory, Nathan Linial and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[HP13] J. Higes and I. Peng. Assouad–Nagata dimension of connected Lie groups. *Math. Z.*, 273(1-2):283–302, 2013.

[HPW13] Sebastian Hensel, Piotr Przytycki and Richard C. H. Webb. Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs. Preprint available from arXiv:1301.5577, 2013.

[HR00] Nigel Higson and John Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.

[HS11] David Hume and Alessandro Sisto. Embedding universal covers of graph manifolds in products of trees. *To appear in Proceedings of the American Mathematical Society*, 2011.

[Hum12] David Hume. Embedding mapping class groups into finite products of trees. Preprint available from arXiv:1207.2132., 2012.

[Hum13] David Hume. Direct embeddings of relatively hyperbolic groups with optimal ℓ^p compression exponent. *To appear in J. Reine Angew. Math.*, 2013.

[Hur94] Adolf Hurwitz. Über die Reduction der binären quadratischen Formen. *Math. Ann.*, 45(1):85–117, 1894.

[HW08] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

[Iva02] Nikolai V. Ivanov. Mapping class groups. In *Handbook of geometric topology*, pages 523–633. North-Holland, Amsterdam, 2002.

[Jam51] Robert C. James. A non-reflexive Banach space isometric with its second conjugate space. *Proc. Nat. Acad. Sci. U. S. A.*, 37:174–177, 1951.

[Jam72] Robert C. James. Super-reflexive Banach spaces. *Canad. J. Math.*, 24:896–904, 1972.

[JR06] William B. Johnson and N. Lovasoa Randrianarivony. l_p $(p > 2)$ does not coarsely embed into a Hilbert space. *Proc. Amer. Math. Soc.*, 134(4):1045–1050 (electronic), 2006.

[JVN35] Pascual Jordan and John Von Neumann. On inner products in linear, metric spaces. *Ann. of Math. (2)*, 36(3):719–723, 1935.

[Kal07] Nigel J. Kalton. Coarse and uniform embeddings into reflexive spaces. *Q. J. Math.*, 58(3):393–414, 2007.

[Kid08] Yoshikata Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916):viii+190, 2008.

[KL98] Michael Kapovich and Bernhard Leeb. 3-manifold groups and nonpositive curvature. *Geom. Funct. Anal.*, 8(5):841–852, 1998.

[KL05] Matthias Kreck and Wolfgang Lück. *The Novikov conjecture*, volume 33 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2005. Geometry and algebra.

[KL08] Bruce Kleiner and John Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008.

[KM10] Olga Kharlampovich and Alexei G. Myasnikov. Equations and fully residually free groups. In *Combinatorial and geometric group theory*, Trends Math., pages 203–242. Birkhäuser/Springer Basel AG, Basel, 2010.

[KY06] Gennadi Kasparov and Guoliang Yu. The coarse geometric Novikov conjecture and uniform convexity. *Adv. Math.*, 206(1):1–56, 2006.

[Laf08] Vincent Lafforgue. Un renforcement de la propriété (T). *Duke Math. J.*, 143(3):559–602, 2008.

[Lic64] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.*, 60:769–778, 1964.

[Lin83] P. A. Linnell. On accessibility of groups. *J. Pure Appl. Algebra*, 30(1):39–46, 1983.

[LLR95] Nathan Linial, Eran London and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[LS05] Urs Lang and Thilo Schlichenmaier. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.*, (58):3625–3655, 2005.

[Lüc10] Wolfgang Lück. On the Farrell-Jones and related conjectures. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 269–341. Int. Press, Somerville, MA, 2010.

[Man05] Jason Fox Manning. Geometry of pseudocharacters. *Geom. Topol.*, 9:1147–1185 (electronic), 2005.

[Mil38] D. Milman. On some criteria for the regularity of spaces of type (b). *C. R. (Doklady) Acad. Sci. U.R.S.S*, 20:243–246, 1938.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[MM00] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.

[MS12] John M. Mackay and Alessandro Sisto. Embedding relatively hyperbolic groups in products of trees. Preprint available from arXiv:1207.3008v1, 2012.

[MT07] John Morgan and Gang Tian. *Ricci flow and the Poincaré conjecture*, volume 3 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2007.

[MT08] John Morgan and Gang Tian. Completion of the proof of the geometrization conjecture. Preprint available from arXiv:0809.4040v1, 2008.

[Nov66] S. P. Novikov. On manifolds with free abelian fundamental group and their application. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:207–246, 1966.

[Nov70] S. P. Novikov. Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. I. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:253–288; ibid. 34 (1970), 475–500, 1970.

[Now07] Piotr W. Nowak. On exactness and isoperimetric profiles of discrete groups. *J. Funct. Anal.*, 243(1):323–344, 2007.

[NP08] Assaf Naor and Yuval Peres. Embeddings of discrete groups and the speed of random walks. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 076, 34, 2008.

[Ol′91] A. Yu. Ol′shanskiĭ. Hyperbolicity of groups with subquadratic isoperimetric inequality. *Internat. J. Algebra Comput.*, 1(3):281–289, 1991.

[Oll06] Yann Ollivier. On a small cancellation theorem of Gromov. *Bull. Belg. Math. Soc. Simon Stevin*, 13(1):75–89, 2006.

[Osi05] Denis Osin. Asymptotic dimension of relatively hyperbolic groups. *Int. Math. Res. Not.*, (35):2143–2161, 2005.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Pan83] Pierre Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.

[Pau01] Scott D. Pauls. The large scale geometry of nilpotent Lie groups. *Comm. Anal. Geom.*, 9(5):951–982, 2001.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. Preprint available from arXiv:math/0211159v1, 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. Preprint available from arXiv: math/0307245v1, 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds. Preprint available from arXiv:math/0303109v1, 2003.

[Pet39] B. J. Pettis. A proof that every uniformly convex space is reflexive. *Duke Math. J.*, 5(2):249–253, 1939.

[PW02] Panos Papasoglu and Kevin Whyte. Quasi-isometries between groups with infinitely many ends. *Comment. Math. Helv.*, 77(1):133–144, 2002.

[Ran06] N. Lovasoa Randrianarivony. Characterization of quasi-Banach spaces which coarsely embed into a Hilbert space. *Proc. Amer. Math. Soc.*, 134(5):1315–1317 (electronic), 2006.

[Ran09] Andrew Ranicki. The birth of the Borel conjecture, 7 October 2009. Available from `http://www.maths.ed.ac.uk/~aar/surgery/borel.pdf`.

[Sal12] Andrew W. Sale. On the Magnus Embedding and the Conjugacy Length Function of Wreath Products and Free Solvable Groups. Preprint available from arXiv:1202.5343, 2012.

[Sap11] Mark Sapir. Aspherical groups and manifolds with extreme properties. arXiv:1103.3873v3, 2011.

[Sei33] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

[Sel92] Zlil Sela. Uniform embeddings of hyperbolic groups in Hilbert spaces. *Israel J. Math.*, 80(1-2):171–181, 1992.

[Sel01] Zlil Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, (93):31–105, 2001.

[Smi10] Alexander Smirnov. The linearly controlled asymptotic dimension of the fundamental group of a graph manifold. *Algebra i Analiz*, 22(2):185–203, 2010.

[Smi12] Alexander Smirnov. Quasi-isometric embedding of the fundamental group of an orthogonal graph-manifold into a product of metric trees. Preprint available from arXiv:1204.5874., 2012.

[Sta68] John Stallings. Groups of dimension 1 are locally free. *Bull. Amer. Math. Soc.*, 74:361–364, 1968.

[Sta71] John Stallings. *Group theory and three-dimensional manifolds.* Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 4.

[Tes11] Romain Tessera. Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces. *Comment. Math. Helv.*, 86(3):499–535, 2011.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[Tu01] Jean-Louis Tu. Remarks on Yu's "property A" for discrete metric spaces and groups. *Bull. Soc. Math. France*, 129(1):115–139, 2001.

[Wis11] Daniel Wise. The structure of groups with a quasiconvex hierarchy. Available from www.math.mcgill.ca/wise/papers.html, 2011.

[Woj91] Przemysław Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

[Yam04] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[Yu00] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.