SOME APPLICATIONS OF

THE THEORY OF GROUP REPRESENTATIONS

IN MATHEMATICAL PHYSICS

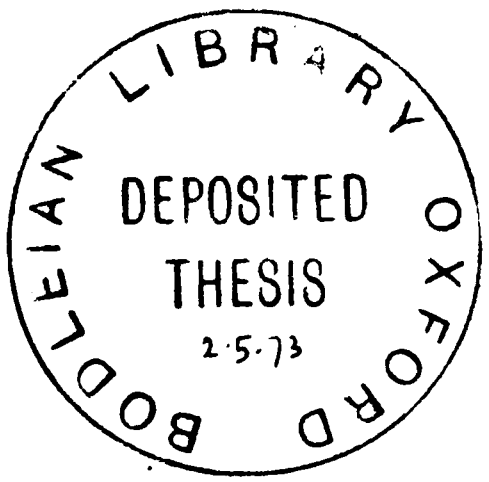
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ABSTRACT

This thesis is concerned mainly with the accidental degeneracy of the n-dimensional anisotropic harmonic oscillator.

The general concept of accidental degeneracy is defined, and a summary is given of some of the explanations for it.

A description is presented of Mackey's theory of group representations for general semi-direct product groups. This is followed by a method for the explicit application of the theory to the cases which arise in this thesis.

A group G associated with the n-dimensional anisotropic harmonic oscillator is constructed : G is essentially a group generated by the position and momentum observables, the identity operator and the Hamiltonian of the system. All the quantum mechanical irreducible representations of G are evaluated (up to unitary equivalence), using Mackey's theory.

G is shown to be embedded in the semi-direct product L of the Weyl group N (generated by the position and momentum observables and the identity operator) and the symplectic group $Sp(2n, \mathbb{R})$. It is then proved that the degeneracies of the energy levels of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1,\omega_2,\ldots,\omega_n)$ occur in the following way. A particular induced representation R^v of the group L, when restricted to G, is unitarily equivalent to :

$$\bigoplus_{s} d_{\omega,s} U_G^{v,-(\operatorname{sgn} v)s} ,$$

when $v \neq 0$, where $d_{\omega,s}$ is precisely the degeneracy of the energy level $E_{\omega,s}$ of the system, $U_G^{v,\ -(\operatorname{sgn} v)s}$ is an irreducible representation of G, and the summation may be regarded as over all distinct energy levels $E_{\omega,s}$ of the system.

The remainder of the thesis is devoted mostly to a study of the representation R^v of L.

Because the subgroups U(n) and SU(n) of L may be taken as degeneracy groups of the n-dimensional isotropic harmonic oscillator, the decompositions of the restrictions of R^v to U(n) and SU(n) are examined.

All the quantum mechanical irreducible representations of the subgroup $N \circledS SU(n)$ of L are found (up to unitary equivalence), using Mackey's theory. It is then shown that $N \circledS SU(n)$ is a non-invariance group for the n-dimensional isotropic harmonic oscillator, with a representation $U_N^v \ W_{SU(n)}$ $(v \neq 0)$ as the appropriate irreducible representation of $N \circledS SU(n)$. Moreover, U_N^v is the restriction to N of the representation $U_G^{v,-(\operatorname{sgn} v)s}$ of G.

The representation R^v $(v \neq 0)$ of L is proved to be unitarily equivalent to an inner Kronecker product representation of L : $U_N^v \ W_{Sp(2n,\ \mathbb{R})} \otimes \overline{W}_{Sp(2n,\ \mathbb{R})}$, where $U_N^v \ W_{Sp(2n,\ \mathbb{R})}$ is an irreducible projective representation of L, and $\overline{W}_{Sp(2n,\ \mathbb{R})}$ is

the complex conjugate of the projective representation $W_{Sp(2n, \mathbb{R})}$ of $Sp(2n, \mathbb{R})$. Furthermore, the restriction of $U_N^v W_{Sp(2n, \mathbb{R})}$ to $N \circledS SU(n)$ is the irreducible representation $U_N^v W_{SU(n)}$ of $N \circledS SU(n)$, which arises in proving that $N \circledS SU(n)$ is a non-invariance group in the isotropic case.

Finally, it is shown that R^v $(v \neq 0)$ splits into two irreducible representations of L.

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Certain equations are numbered, beginning with (1) in each new chapter. Equation (n) of chapter m is referred to as such in that chapter, and as equation m-(n) in later chapters.

References are indicated in the form : [r], and refer to the list on p.191.

CHAPTER 1

SYMMETRY AND DEGENERACY IN QUANTUM THEORY

In this introductory chapter, it is shown first how geometrical symmetry of the Hamiltonian of a quantum mechanical system leads to degeneracy of the energy levels. The concept of accidental degeneracy is then explained. Next, a summary is given of symmetry in classical mechanics, and the quantum mechanical analogue of this is briefly described. A short survey of the known connections between symmetry and degeneracy follows. Symmetry groups and non-invariance groups in quantum theory are then defined. Finally, there is a summary of the approach taken in the present work.

1.1 Degenerate eigenvalues

Suppose a quantum mechanical system has Hamiltonian H, and let E be an eigenvalue of H. If every solution ψ of

$$H\psi = E\psi$$

is a scalar multiple of one solution, ψ_o , then E is said to be a non-degenerate eigenvalue or a non-degenerate energy level In this case, the eigenspace of E is one-dimensional. If the eigenspace of E has dimension n, greater than one, then E is said to be an n-fold degenerate eigenvalue or an n-fold degenerate energy level.

The question then arises as to how the degeneracy which

occurs in certain quantum mechanical systems may be explained. Part of the answer lies in the geometrical symmetry possessed by many such systems.

1.2 Consequences of geometrical symmetry

Suppose the Hamiltonian of a system is an operator $H(x)$, depending on x (where x is a vector belonging to some space S) and acting on a function space $\mathcal{H}$. Let R be an invertible transformation of S which leaves the Hamiltonian invariant :

$$H(Rx) = H(x) \text{ , for all } x.$$

R is called a symmetry transformation of H.

Corresponding to R, define a linear operator O_R, which acts on $\mathcal{H}$, by :

$$O_R\psi(x) = \psi(R^{-1}x) \text{ , where } \psi \in \mathcal{H} .$$

Then $$O_R H(x)\psi(x) = H(R^{-1}x)\psi(R^{-1}x)$$

$$= H(x)O_R\psi(x) .$$

Therefore, if R is a symmetry transformation of H, O_R commutes with H.

Now suppose that ψ is an eigenfunction of H, corresponding to the eigenvalue E :

$$H\psi = E\psi .$$

If R is a symmetry transformation of H, then

$$H(O_R\psi) = O_R(H\psi)$$

$$= O_R(E\psi)$$

$$= E(O_R\psi) \ .$$

Hence $O_R\psi$ is also an eigenfunction of H with eigenvalue E. Therefore, unless $O_R\psi = c\psi$ (where c is a scalar, possibly dependent on R) for each symmetry transformation R of H, it follows that E is a degenerate eigenvalue of H.

Now suppose that E is n-fold degenerate, and that its eigenspace has a basis of n linearly independent eigenfunctions $\psi_1, \psi_2, \ldots, \psi_n$. Since $O_R\psi_\nu$ ($\nu = 1,2,\ldots,n$) belongs to this eigenspace,

$$O_R\psi_\nu = \sum_{\mu=1}^{n} \psi_\mu D_{\mu\nu}(R) \ ,$$

where $D(R) = (D_{\mu\nu}(R))$ is an n x n matrix of scalar functions of R. If S is another symmetry transformation of H, then

$$O_S\psi_\nu = \sum_{\mu=1}^{n} \psi_\mu D_{\mu\nu}(S) \ .$$

Hence $$O_S(O_R\psi_\nu) = O_S \sum_{\mu=1}^{n} \psi_\mu D_{\mu\nu}(R)$$

$$= \sum_{\mu,\lambda} \psi_\lambda D_{\lambda\mu}(S) D_{\mu\nu}(R)$$

$$= \sum_{\lambda=1}^{n} \psi_\lambda (D(S)D(R))_{\lambda\nu} \ .$$

By definition, $O_S O_R = O_{SR}$. Moreover,

$$O_{SR}\psi_\nu = \sum_{\lambda=1}^{n} \psi_\lambda D_{\lambda\nu}(SR) \ .$$

Hence $D(S)D(R) = D(SR)$.

Alternatively, D may be regarded as dependent on O_R, instead of R, in which case the above relation may be expressed as :

$$D(O_S)D(O_R) = D(O_S O_R) = D(O_{SR}) \ .$$

It follows that the eigenfunctions of each degenerate energy level form the basis for a representation of a group of operators which commute with the Hamiltonian.

1.3 Accidental degeneracy

Geometrical symmetry of the Hamiltonian thus leads to degenerate eigenvalues, and to a representation, on each eigenspace, of a group of operators which commute with the Hamiltonian. Very often, such a representation is irreducible. However, sometimes, it is reducible, and so sets of eigenfunctions belonging to different representations coincide in energy. In this case, the extra degeneracy is said to be accidental - since it was considered unlikely that it would occur.

For example, the Hamiltonian of a particle moving in a spherically symmetric potential is invariant under rotations,

and this leads to a (2l + 1)-fold degeneracy in the quantum number l, where l(l + 1) is an eigenvalue of the square of angular momentum. This, in turn, implies a (2l + 1)-fold degeneracy in the energy E. However, some spherically symmetric systems, such as the isotropic harmonic oscillator and the hydrogen atom, have energy levels which are more degenerate than this.

The suggestion has been made that, for accidentally degenerate systems, there is more symmetry than just the obvious geometrical symmetry, and that a group formed by these 'hidden' or 'higher' symmetries and the geometrical symmetries completely accounts for all the degeneracy present. Two properties of such a group would be that all its elements commute with the Hamiltonian, and that the eigenfunctions of any energy level form a basis for an irreducible representation of it.

1.4 Symmetry in classical mechanics

Before discussing symmetry groups in quantum theory any further, it is helpful to consider the concept of symmetry in classical mechanics, partly because results which are valid there often have a transcription into quantum theory. Classical results may thus throw light on the question of symmetry in quantum theory, and so illuminate the problem of accidental degeneracy.

The usual concept of invariance in classical mechanics is

that of a canonical transformation, that is, a transformation of the phase space variables which leaves the Hamiltonian form of the equations of motion invariant. Suppose such a transformation of the variables $q_1,\ldots,q_n,p_1,\ldots,p_n$ into the variables $q'_1,\ldots,q'_n,p'_1,\ldots,p'_n$ is generated by a function $F(q_1,\ldots,q_n,p'_1,\ldots,p'_n,t)$ (where t denotes time) ; then the variables are related by :

$$p_j = \frac{\partial F}{\partial q_j} , \quad q'_j = \frac{\partial F}{\partial p'_j} \qquad (j = 1,2,\ldots,n) ;$$

the Hamiltonian H of a system is transformed into a new Hamiltonian H' given by :

$$H' = H + \frac{\partial F}{\partial t} .$$

Now consider just those canonical transformations which are generated by functions F which are not explicitly time-dependent : for all such transformations, H' = H.

The identity transformation is generated by $F = \sum_{j=1}^{n} q_j p'_j$. Hence if ϵ is an infinitesimal parameter independent of q_j and p_j, an infinitesimal change in the variables is generated by :

$$F = \sum_{j=1}^{n} q_j p'_j + \epsilon G ,$$

where $G = G(q_1,\ldots,q_n,p'_1,\ldots,p'_n)$ is an arbitrary differentiable function, which is not explicitly time-dependent. The new

variables are given by :

$$\left.\begin{aligned} p_j &= \frac{\partial F}{\partial q_j} = p'_j + \epsilon \frac{\partial G}{\partial q_j} \quad , \\ q'_j &= \frac{\partial F}{\partial p'_j} = q_j + \epsilon \frac{\partial G}{\partial p'_j} \quad , \end{aligned}\right\} \quad (j = 1,2,\ldots,n) \ .$$

So $$\delta q_j = \epsilon \frac{\partial G}{\partial p'_j} \ , \qquad \delta p_j = - \epsilon \frac{\partial G}{\partial q_j} \qquad (j = 1,2,\ldots,n).$$

Since $(p'_j - p_j)$ is infinitesimal, p'_j may be replaced by p_j, giving :

$$\delta q_j = \epsilon \frac{\partial}{\partial p_j} G(q_1,\ldots,q_n,p_1,\ldots,p_n) \ ,$$

$$\delta p_j = - \epsilon \frac{\partial}{\partial q_j} G(q_1,\ldots,q_n,p_1,\ldots,p_n)$$

$(j = 1,2,\ldots,n)$.

Corresponding to the changes in q_j and p_j, generated by the function G, there is a change in the value of the Hamiltonian $H = H(q_1,\ldots,q_n,p_1,\ldots,p_n,t)$ given by :

$$\begin{aligned} \delta H &= \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} \delta q_j + \frac{\partial H}{\partial p_j} \delta p_j \right) \\ &= \epsilon \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial H}{\partial p_j} \frac{\partial G}{\partial q_j} \right) \\ &= \epsilon \{H, G\} \end{aligned}$$

(where $\{ \ , \ \}$ denotes the Poisson bracket).

Hence H is invariant under the infinitesimal canonical transformation generated by G if and only if $\{H, G\} = 0$.

Now, since G is not explicitly time-dependent,

$$\frac{dG}{dt} = \sum_{j=1}^{n} \left(\frac{\partial G}{\partial q_j} \dot{q}_j + \frac{\partial G}{\partial p_j} \dot{p}_j \right)$$

$$= \sum_{j=1}^{n} \left(\frac{\partial G}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial G}{\partial p_j} \frac{\partial H}{\partial q_j} \right)$$

(using Hamilton's equations)

$$= \{G, H\} \ . \qquad (1)$$

Hence G is a constant of the motion if and only if $\{G, H\} = 0$.

Therefore constants of the motion generate infinitesimal canonical transformations with respect to which H is invariant, and conversely.

1.5 The quantum mechanical analogue

It was noticed a long time ago, by Pauli [34], that some quantum mechanical systems with accidental degeneracy have classical analogues for which the constants of the motion generate a group which is larger than the group of obvious geometrical symmetries of the configuration space of the system.

Now, as shown above, if the Hamiltonian H of a quantum mechanical system is invariant under a symmetry transformation R, then the corresponding operator O_R commutes with H.

Moreover, the quantum mechanical analogue of equation (1), for an operator A which is not explicitly time-dependent, is :

$$i\hbar\frac{dA}{dt} = [A, H]$$

(where [,] denotes the commutator bracket). Hence the operator A is a constant of the motion if and only if A commutes with H. If A and B are constants of the motion, [A, B] is also a constant of the motion. Therefore the constants of the motion generate a group of operators which commute with H.

This connection between constants of the motion and invariance of the Hamiltonian, in both the classical and quantum mechanical cases, has been used in attempts to obtain more information about symmetry and degeneracy in quantum theory.

1.6 Historical survey

Bertrand's theorem

In the past, the study of degeneracy and symmetry has been concentrated mainly on the hydrogen atom and the isotropic harmonic oscillator. This arises from a theorem due to Bertrand, which states that, of classical systems for which the kinetic energy is the sum of the squares of the velocities, and the potential energy is spherically symmetric, the only potentials with bounded closed orbits are those

associated either with the hydrogen atom or with the isotropic harmonic oscillator. (A modern account of Bertrand's theorem is given by Greenberg [12]). The assumption was then made that quantum theoretical systems with accidental degeneracy have classical analogues for which all bounded orbits are closed.

The hydrogen atom

Pauli [34], in 1926, was the first to examine the hydrogen atom from a quantum theoretical viewpoint : he exhibited the angular momentum vector and the so-called Runge-Lenz vector as constants of the motion, and worked out their commutation rules Klein (as reported by Hulthen [18]) was apparently the first to recognise these as the commutation rules of the orthogonal group in four dimensions, O(4). In 1935, Fock [10] demonstrated that the Schrödinger equation for the hydrogen atom in momentum space could be transformed by stereographic projection into Laplace's equation for a hypersphere, in the case of bound states, or a suitable hyperboloid in the case of free states. This showed clearly that an appropriate symmetry group is the orthogonal group in four dimensions. Then, in 1936, Bargmann [3] showed explicitly that this group is generated by precisely the components of the angular momentum and of the Runge-Lenz vector. At about the same time, Laporte and Rainich [24] reached similar conclusions. The corresponding representations of O(4) defined on the eigenspaces of the Hamiltonian are all

irreducible. However, these representations do not include all the irreducible representations of O(4).

The isotropic harmonic oscillator

The Hamiltonian of the n-dimensional isotropic harmonic oscillator may be taken as

$$H = \tfrac{1}{2}\sum_{j=1}^{n}(P_j^2 + Q_j^2) \ ,$$

where Q_j and P_j are conjugate position and momentum variables in the classical case, and are position and momentum operators in the quantum mechanical case.

Its symmetry was first analysed in 1940, by Jauch and Hill [23]. Their approach was to look for classical constants of the motion and then to attempt to quantise these. They showed that

$$A_{jk} = \tfrac{1}{2}(Q_jP_k - P_jQ_k) \qquad (1 \leqslant j < k \leqslant n) \ ,$$

$$B_{1j} = \tfrac{1}{4}([P_1^2 - P_j^2] + [Q_1^2 - Q_j^2]) \qquad (2 \leqslant j \leqslant n) \ ,$$

$$C_{jk} = \tfrac{1}{2}(P_jP_k + Q_jQ_k) \qquad (1 \leqslant j < k \leqslant n)$$

are $(n^2 - 1)$ independent classical constants of the motion. The corresponding quantised quantities have the same form, and all commute with the quantum mechanical Hamiltonian. Moreover, they form the Lie algebra of the group SU(n), and the

corresponding representations of SU(n) on the eigenspaces of the Hamiltonian are all irreducible. However, these representations do not include all the irreducible representations of SU(n).

The constants A_{jk} are the components of angular momentum (which are conserved because of the spherical symmetry of the Hamiltonian); for each j, B_{1j} is the difference in energy between the first and the jth oscillators. However, Jauch and Hill were unable to find any physical significance for the constants C_{jk}.

It was not until 1949 that Säenz [36], developing the work of Laporte and Rainich, found a geometrically significant canonical transformation, by which the phase space of the classical n-dimensional isotropic harmonic oscillator could be regarded as a complex vector space, and the constants of the motion interpreted as generating unitary unimodular transformations of the phase space.

The form of the Hamiltonian of the n-dimensional isotropic harmonic oscillator suggests writing it in complex form. This approach was used by Baker [2], in 1956.

In the classical case, the position variables Q_j and the momenta P_k satisfy the Poisson bracket relation :

$$\{Q_j, P_k\} = \delta_{jk} .$$

Define $a_j = \frac{1}{\sqrt{2}} (Q_j + iP_j)$,

$$a_j^* = \frac{1}{\sqrt{2}} (Q_j - iP_j) ,$$

and introduce the bracket $(A, B) = i\{A, B\}$. Then

$$H = \sum_{j=1}^{n} a_j^* a_j \ ,$$

$$(a_j, a_k^*) = \delta_{jk} \ .$$

In the quantum theoretical case, commutator brackets are used instead of Poisson brackets :

$$[Q_j, P_k] = i\delta_{jk} \ .$$

With the above definitions of a_j and a_j^*,

$$H = \sum_{j=1}^{n} a_j^* a_j + \frac{n}{2} \ ,$$

$$[a_j, a_k^*] = \delta_{jk} \ .$$

From the form of the Hamiltonian, and the Poisson bracket and commutation relations for the a_j's, it is clear that they are all invariant under the transformation

$$a_j \longrightarrow \sum_{k=1}^{n} U_{jk} a_k \ ,$$

where (U_{jk}) is a unitary n x n matrix; hence they are all invariant under the group U(n) of unitary transformations in n dimensions.

Baker's construction of irreducible representations of

U(n) is not directly related to these symmetry considerations. He considers the unitary group acting on the vector space $\mathbb{C}^n$. Then it can be shown that, under these unitary transformations, the eigenfunctions of the Hamiltonian (namely, products of Hermite functions), corresponding to the same energy level, transform among themselves according to an irreducible unitary representation of U(n).

In 1958, Alliluev [1] demonstrated explicitly that the eigenfunctions of the two-dimensional isotropic harmonic oscillator transform according to irreducible representations of a group isomorphic to SU(2), whose generators commute with the Hamiltonian. Each irreducible representation of SU(2) occurs exactly once in this way. Moreover, no subgroup of SU(2) has these properties.

The anisotropic harmonic oscillator

In order to try to obtain a clearer understanding of the relationship between symmetry and degeneracy in quantum theory, more general systems than the hydrogen atom and the isotropic harmonic oscillator have been investigated. In particular, much attention has been given to the n-dimensional anisotropic harmonic oscillator.

The Hamiltonian of this system may be taken as

$$H = \tfrac{1}{2} \sum_{j=1}^{n} \left(P_j^2 + \omega_j^2 Q_j^2 \right) ,$$

where the ω_j are positive constants (the frequencies), and Q_j and P_j are conjugate position and momentum variables in the classical case, and are position and momentum operators in the quantum mechanical case.

Now

$$H = \sum_{j=1}^{n} H_j \ , \text{ where } H_j = \tfrac{1}{2}\,(P_j^2 + \omega_j^2 Q_j^2) \ .$$

In the quantum mechanical case, the eigenvalues of H_j are $\omega_j(m_j + \frac{1}{2})$, where $m_j \in \mathbb{Z}_+$ [11]. Hence each eigenvalue, or energy level, E of H is of the form

$$E = \sum_{j=1}^{n} \omega_j(m_j + \tfrac{1}{2}) \ .$$

Suppose there exists a positive number μ such that each $\mu\omega_j$ is integral; let λ be the smallest such number μ. Then, given $\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n$,

$$\sum_{j=1}^{n} \omega_j m_j = \frac{s}{\lambda} \ , \quad \text{for some } s \in \mathbb{Z}_+ \ .$$

It follows that the possible energy levels of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$ are :

$$E_{\omega,s} = \frac{s}{\lambda} + \tfrac{1}{2} \sum_{j=1}^{n} \omega_j \ ,$$

where s is an element of $\mathbb{Z}_+$ which is expressible in the form $\lambda\sum_{j=1}^{n}\omega_j m_j$, for some $\underline{m} = (m_1,m_2,\ldots,m_n) \in \mathbb{Z}_+^n$. The degeneracy $d_{\omega,s}$ of the energy level $E_{\omega,s}$ is the number of distinct ways of choosing $\underline{m} = (m_1,m_2,\ldots,m_n)\in \mathbb{Z}_+^n$ so that the relation $s = \lambda\sum_{j=1}^{n}\omega_j m_j$ is satisfied.

When no positive number λ exists such that each $\lambda\omega_j$ is integral, the energy levels are all non-degenerate.

In 1940, Jauch and Hill [23] found classical constants of the motion for the two-dimensional case with rational frequency ratio, but when these constants were quantised in a reasonable way, they did not form a Lie algebra.

More recently, Dulock [9] managed to extend the classical constants to the n-dimensional case with arbitrary frequency ratios, but because of the transcendental nature and multivaluedness of the constants, it was not possible to form the quantum theoretical analogues. He showed that his classical constants generated a group isomorphic to SU(n).

Demkov [8], in 1963, investigated the particular two-dimensional case with frequency ratio 2 : 1. The degeneracy $d_{(2,1),s}$ of the energy level $E_{(2,1),s}$ $\left(= s + \frac{3}{2} \right)$, namely the number of distinct ordered pairs of non-negative integers (m_1,m_2) such that $2m_1 + m_2 = s$, is equal to $\left[\frac{s}{2} \right] + 1$,

where $\left[\frac{s}{2}\right]$ denotes the largest integer not exceeding $\frac{s}{2}$. It follows that $d_{(2,1),2m} = d_{(2,1),2m+1}$ for integral m. The eigenfunctions may then be divided into two sets, of even and odd values of $\tilde{E}_s = E_{(2,1),s} - \frac{3}{2}$, respectively. To each even value of $\tilde{E}_s$, there corresponds an irreducible representation of SU(2), and each irreducible representation of SU(2) occurs in this way; this applies also to the odd values of $\tilde{E}_s$. Hence it may be said that the eigenfunctions are grouped into two systems of irreducible representations of SU(2). Il'kaeva [20], a student of Demkov, treated a somewhat more general case.

Cisneros and McIntosh [7] have recently examined the case of a two-dimensional anisotropic harmonic oscillator with rational frequency ratio a : b, where a and b are mutually prime integers. They have shown that the eigenfunctions are grouped into k systems of irreducible representations of SU(2), where k is the least common multiple of a and b.

However, in 1968, Vendramin [40] refuted Demkov's claim that, for $n > 2$, SU(n) is a possible type of symmetry group for an n-dimensional anisotropic harmonic oscillator with rational frequency ratios. He showed that, in general, the degeneracies of the energy levels are not equal to the dimensions of any irreducible representations of SU(n).

Symmetry in classical mechanics

Another recent approach to symmetry and degeneracy in quantum theory has been to re-examine symmetry in classical mechanics.

Mukunda [29], in 1967, showed that for all classical dynamical systems with n degrees of freedom, whatever the functional form of the Hamiltonian, there exists a set of constants of the motion whose Poisson bracket algebra coincides with the Lie algebra of O(n + 1), and there also exists a set of constants of the motion whose Poisson bracket algebra coincides with the Lie algebra of SU(n). O(n + 1) and SU(n) may therefore be regarded as 'universal symmetry groups' in classical mechanics.

This raised the question of whether there might be 'universal symmetry groups' in quantum theory. However, the results obtained for the anisotropic harmonic oscillator appear to suggest that this is not the case. Mukunda points out that it seems likely that the existence of classical 'universal symmetry groups' is purely a consequence of rather detailed properties of Poisson brackets, and so it is unlikely that these symmetries survive the transition to quantum theory, except for some special systems.

1.7 Definition of symmetry groups in quantum theory

The above survey illustrates some of the different ways in which groups arise in attempts to explain accidental

degeneracy. It is helpful to distinguish these different groups carefully.

The first requirement for an operator to be a symmetry operator for a system is that it should commute with the Hamiltonian of that system. If the symmetry operator is not explicitly time-dependent, then it is a constant of the motion. Furthermore, it is reasonable to require that a symmetry operator should leave all the probabilities of a system invariant; it follows, from Wigner's theorem [11], that such an operator must be a unitary or an antiunitary operator. If, in addition, the symmetry operators form a continuous group, then each symmetry operator must be unitary.

Therefore, the maximal symmetry group S of a quantum mechanical system is defined to be the group of all unitary operators commuting with the Hamiltonian of that system.

However, this group is usually too general, and so is not of much interest. If the energy levels are degenerate, with degree of degeneracy d_s for the energy level E_s, the matrix of a unitary operator U which commutes with the Hamiltonian H is quasi-diagonal in the energy representation; there are unitary matrices of dimension d_s along the diagonal, and so the maximal symmetry group is isomorphic to the direct product of the unitary groups of dimension d_s, which give independent unitary transformations of the eigenspaces of H :

$$S = U(d_1) \times U(d_2) \times \ldots \times U(d_s) \times \ldots .$$

Hence the group S is trivial in nature, and does not explain the degeneracy, but rather just describes it. If an irreducible representation of S corresponds to an energy level, then it is of a very special form, where the identity representation is chosen for all except one of the group factors, while, for that one, the representation by the unitary matrices themselves is chosen. All other irreducible representations of S do not correspond to any energy level of the system. Such groups which have superfluous representations may be called excessive; in particular, the maximal symmetry group is extremely excessive.

In order to eliminate this excessiveness, an ideal symmetry group I may be considered instead of the maximal symmetry group S. I is defined to be a subgroup of S for which :

(i) to each energy level of the system, there corresponds an irreducible representation of I;

(ii) to each irreducible representation of I, there corresponds exactly one energy level of the system;

(iii) no proper subgroup of I completely explains the degree of degeneracy of all the energy levels; that is, for any subgroup of I, there exists at least one energy level to which there corresponds a reducible representation of that subgroup.

The conditions defining an ideal symmetry group are obviously sufficient to explain the degeneracy. However, it

is not known whether such a group exists for every system; moreover, even if it does exist for a particular system, it may not be unique.

As shown by Alliluev [1], SU(2) is an ideal symmetry group for the two-dimensional isotropic harmonic oscillator, but ideal symmetry groups have not yet been found for the general n-dimensional isotropic or anisotropic harmonic oscillator, or the n-dimensional hydrogen atom.

One of the main difficulties in finding an ideal symmetry group is in obtaining a group which just has irreducible representations with dimensions equal to the degree of degeneracy of the energy levels, and no others.

Therefore, in practice, a more easily found subgroup of the maximal symmetry group S is a group which will be called a degeneracy group D for the system. D is defined to be a subgroup of S for which :

(i) to each energy level of the system, there corresponds an irreducible representation of D;

(ii) no irreducible representation of D corresponds to more than one energy level of the system, but some irreducible representations of D may not correspond to any energy level.

In the literature, the term 'symmetry group' or 'invariance group' is often used with little explanation. However, most authors, in talking about such a group, mean a degeneracy group. In general, once such a group has been

found for a system, the degeneracy is regarded as being explained, even though the group may be excessive.

O(n + 1) is a degeneracy group for the n-dimensional hydrogen atom, but, since all its representations do not correspond to energy levels, it is excessive, and so is not an ideal symmetry group [1]. Similarly, for the n-dimensional isotropic harmonic oscillator, U(n) and SU(n) are degeneracy groups, but not ideal symmetry groups. However, Vendramin's result implies that neither U(n) nor SU(n) can, in general, be a degeneracy group for the n-dimensional anisotropic harmonic oscillator.

An alternative group proposed by Demkov [8], mainly to overcome the excessiveness of the maximal symmetry group S, but again slightly more general than an ideal symmetry group is a <u>minimal symmetry group</u> Σ. Σ is defined to be a subgroup of S for which :

(i) to each energy level of the system, there corresponds an irreducible representation of Σ;

(ii) to each irreducible representation of Σ there corresponds at least one energy level of the system;

(iii) no proper subgroup of Σ completely explains the degree of degeneracy of all the energy levels.

In the case of a minimal symmetry group, an irreducible representation is allowed to correspond to more than one energy level, whereas, before Demkov's suggestion, it had been presumed that a symmetry group would be such that each

irreducible representation would correspond to at most one energy level.

It follows from the results obtained by Cisneros and McIntosh that SU(2) is a minimal symmetry group for the two-dimensional anisotropic harmonic oscillator with rational frequency ratio. However, from Vendramin's work again, when $n > 2$, SU(n) cannot be a minimal symmetry group for the n-dimensional anisotropic harmonic oscillator with rational frequency ratio.

1.8 Non-invariance groups

For some systems in which the accidental degeneracy can be explained by means of a degeneracy group, larger 'approximate' symmetry groups have recently been studied. Those irreducible representations of a degeneracy group of a system which occur as representations on the eigenspaces of the Hamiltonian are collected into one irreducible representation of a larger group. Such a larger group is known as a non-invariance group (distinguishing it from an invariance group - a term which usually refers to a degeneracy group). Precisely, a non-invariance group for a system is defined to be a Lie group G_{NI} with the following properties :

(i) G_{NI} contains a degeneracy group D of the system as a subgroup;

(ii) the Lie algebra of G_{NI} includes certain 'non-invariant' generators which do not commute with the

Hamiltonian;

(iii) G_{NI} has an irreducible representation whose restriction to the group D is unitarily equivalent to $\bigoplus_m \sigma_m$, where σ_m is an irreducible representation of the mth energy level of the system, and the summation may be regarded as over all energy levels of the system.

The concept of such a group was apparently first suggested by Barut [4], in 1964. In 1965, Mukunda, O'Raifeartaigh and Sudarshan [28] reported that they had found non-invariance groups for some fully solvable quantum mechanical systems : e.g. SU(n,1) for the n-dimensional isotropic harmonic oscillator, and O(n + 1,1) for the n-dimensional hydrogen atom.

One of the reasons for the interest in this approach is the hope that the Hamiltonian is connected in some natural way with the non-invariant generators of a non-invariance group, so that the energy spectrum is a consequence of the group property of the noncompact group. By examining the connection between the Hamiltonian of a system, when this is known, and the generators of a corresponding non-invariance group, it is hoped that, if a non-invariance group for the strongly interacting particles is found, then this may shed some light on the possible Hamiltonians for the system.

1.9 Approach taken in thesis

As shown above, the accidental degeneracy of the hydrogen atom and the isotropic harmonic oscillator may be explained in terms of symmetry. However, when $n > 2$, this approach fails to account for the accidental degeneracy of the n-dimensional anisotropic harmonic oscillator.

A rather different approach is taken here to the anisotropic case. The basic group G considered is essentially that generated by the position and momentum observables, the identity operator and the Hamiltonian of the system. The group G is shown to be embedded in a group L which describes systems with quadratic Hamiltonians : L is the semi-direct product of the Weyl group N (generated by the position and momentum observables and the identity operator) and the symplectic group $Sp(2n, \mathbb{R})$.

The degeneracies of the energy levels of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$ then occur in the following way. A particular induced representation R^v of the group L, when restricted to G, is unitarily equivalent to

$$\bigoplus_s d_{\omega,s} U_G^{v,-(\mathrm{sgn}\ v)s} ,$$

when $v \neq 0$, where $d_{\omega,s}$ is precisely the degeneracy of the energy level $E_{\omega,s}$ of the system, $U_G^{v,-(\mathrm{sgn}\ v)s}$ is an

irreducible representation of G, and the summation may be regarded as over all distinct energy levels $E_{\omega,s}$ of the system.

The remainder of the thesis is devoted mostly to a study of the representation R^v of L.

Firstly, because the groups U(n) and SU(n) may be taken as degeneracy groups of the n-dimensional isotropic harmonic oscillator, and are subgroups of L, the decompositions of the restrictions of R^v to U(n) and SU(n) are examined. Although the results do not throw much light on the representation R^v, they raise the possibility that a subgroup of L could be a non-invariance group in the isotropic case. It is shown that $N \circledS SU(n)$ is such a subgroup, with a representation $U_N^v \, W_{SU(n)}$ ($v \neq 0$), as the appropriate irreducible representation of $N \circledS SU(n)$. Moreover, U_N^v is the restriction to N of the representation $U_G^{v,-(\mathrm{sgn}\ v)s}$ of G.

Secondly, the representation R^v ($v \neq 0$) of L is proved to be unitarily equivalent to an inner Kronecker product representation of L : $U_N^v \, W_{Sp(2n,\,\mathbb{R})} \otimes \overline{W}_{Sp(2n,\,\mathbb{R})}$, where $U_N^v \, W_{Sp(2n,\,\mathbb{R})}$ is an irreducible projective representation of L, and $\overline{W}_{Sp(2n,\,\mathbb{R})}$ is the complex conjugate of the projective representation $W_{Sp(2n,\,\mathbb{R})}$ of $Sp(2n, \mathbb{R})$. Furthermore, the restriction of $U_N^v \, W_{Sp(2n,\,\mathbb{R})}$ to $N \circledS SU(n)$ is the irreducible representation $U_N^v \, W_{SU(n)}$ of $N \circledS SU(n)$, which arises in proving that $N \circledS SU(n)$ is a non-invariance group in the isotropic case.

Finally, it is shown that R^v ($v \neq 0$) splits into two irreducible representations of L.

CHAPTER 2

MACKEY'S THEORY OF GROUP REPRESENTATIONS

Mackey's general theory enables all irreducible unitary representations of a separable locally compact group G to be found from those of a proper normal subgroup N, when N is a so-called 'type 1' subgroup, a condition which is frequently satisfied. Just the particular application of Mackey's method which is needed in later chapters, namely that to semi-direct product groups, is discussed here. However, before describing the method for that case, it is necessary to discuss first the concepts of induced representation and projective representation.

Note

In what follows, a representation means a unitary representation acting on a separable Hilbert space.

2.1 Definition of induced representation

Let G be a separable locally compact group, and H a closed subgroup of G. Let σ be a unitary representation of H on a separable Hilbert space $\mathcal{H}_\sigma$, which has inner product (,). Suppose there exists an invariant measure μ on G/H.

Denote by $B_H(G, \mathcal{H}_\sigma, \mu)$ the set of all functions ϕ from G to $\mathcal{H}_\sigma$ which satisfy :

(i) $(\phi(g), v)$ is a Borel function on G for each $v \in \mathcal{H}_\sigma$;

(ii) $\phi(gh) = \sigma(h^{-1})\phi(g)$ for all $g \in G$, $h \in H$;

(iii) $\int_{G/H} (\phi(g), \phi(g)) d\mu(x) < \infty$, where $x = gH$.

(From (ii), $(\phi(g), \phi(g))$ is constant on left cosets of H in G, and so is independent of the choice of coset representative $g \in x$.)

Then, if functions which are equal almost everywhere are identified, $B_H(G, \mathcal{H}_\sigma, \mu)$ becomes a separable Hilbert space, with respect to the inner product defined by

$$\langle \phi_1, \phi_2 \rangle = \int_{G/H} (\phi_1(g), \phi_2(g)) d\mu(x), \text{ where } x = gH.$$

Now define an operator U(g) acting on $B_H(G, \mathcal{H}_\sigma, \mu)$ by

$$(U(g)\phi)(g') = \phi(g^{-1}g') , \qquad (1)$$

where $g, g' \in G$, $\phi \in B_H(G, \mathcal{H}_\sigma, \mu)$. It may be verified that U is a unitary representation of G on the space $B_H(G, \mathcal{H}_\sigma, \mu)$. U is called <u>the representation of G induced by the representation σ of H</u>. It will be denoted by $\sigma(H)\uparrow G$, or, when there is no risk of ambiguity, by $\sigma\uparrow G$. The notation $\rho(G)\downarrow H$ or $\rho\downarrow H$ will be used to denote the restriction of a representation ρ of G to a subgroup H.

This definition of induced representation is essentially that given by Mackey [25]. In practice, it is helpful to have

a more explicit expression for $\sigma\uparrow G$ than the one stated above.

From each coset $x \in G/H$, choose a coset representative $\wedge(x) \in G$ so that $\wedge : G/H \longrightarrow G$ is a Borel function; then $\wedge$ satisfies

$$\wedge(x)H = x.$$

Given a function $\phi \in B_H(G, \mathcal{H}_\sigma, \mu)$, define a corresponding function ψ from G/H to $\mathcal{H}_\sigma$ by

$$\psi(x) = \phi(\wedge(x)) \text{ , where } x \in G/H .$$

Then ψ belongs to the space $L^2(G/H, \mathcal{H}_\sigma, \mu)$ of functions from G/H to $\mathcal{H}_\sigma$ which are square-integrable with respect to the measure μ.

Conversely, a function $\psi \in L^2(G/H, \mathcal{H}_\sigma, \mu)$ determines a function $\phi \in B_H(G, \mathcal{H}_\sigma, \mu)$ given by

$$\phi(\wedge(x)) = \psi(x) \text{ for coset representatives } \wedge(x),$$

and $$\phi(\wedge(x)h) = \sigma(h^{-1})\phi(\wedge(x)) \quad (\text{where } h \in H)$$

$$= \sigma(h^{-1})\psi(x) \text{ otherwise.}$$

The action of the operator $U(g)$ on $B_H(G, \mathcal{H}_\sigma, \mu)$ may be transferred naturally to $L^2(G/H, \mathcal{H}_\sigma, \mu)$ by defining

$$(U(g)\psi)(x) = (U(g)\phi)(\wedge(x)) \text{ , where } x \in G/H.$$

By the definition, (1), of $U(g)$,

$$(U(g)\phi)(\wedge(x)) = \phi(g^{-1}\wedge(x)).$$

Now $g^{-1}\Lambda(x)$ is not necessarily of the form $\Lambda(y)$, for any $y \in G/H$. However,

$$g^{-1}\Lambda(x) \in g^{-1}\Lambda(x)H = g^{-1}x = \Lambda(g^{-1}x)H .$$

So $$\Lambda(g^{-1}x)^{-1}g^{-1}\Lambda(x) \in H .$$

Hence $$(U(g)\psi)(x) = \phi(g^{-1}\Lambda(x))$$

$$= \phi(\Lambda(g^{-1}x)[\Lambda(g^{-1}x)^{-1}g^{-1}\Lambda(x)])$$

$$= \sigma(\Lambda(x)^{-1}g\Lambda(g^{-1}x))\phi(\Lambda(g^{-1}x)),$$

since $\phi(gh) = \sigma(h^{-1})\phi(g)$ for any $g \in G$, $h \in H$.

Therefore,

$$(U(g)\psi)(x) = \sigma(\Lambda(x)^{-1}g\Lambda(g^{-1}x))\psi(g^{-1}x) . \qquad (2)$$

2.2 Some properties of induced representations

Proofs of the following results may be found in Mackey [25].

Inducing in stages

If H_1 and H_2 are two closed subgroups of a separable locally compact group G, with $H_1 \subset H_2$, and if σ is a representation of H_1, then

$$\sigma(H_1)\uparrow G \simeq (\sigma(H_1)\uparrow H_2)\uparrow G$$

(where $\simeq$ denotes unitary equivalence).

Inducing from a reducible representation

If a representation σ of a closed subgroup H of a separable locally compact group G is unitarily equivalent to a direct integral decomposition of the form $\int_{\epsilon} \sigma^{y} d\mu(y)$, then

$$\sigma(H)\uparrow G \simeq \int_{\epsilon} (\sigma^{y}\uparrow G) d\mu(y) .$$

(A discussion of direct integrals may be found in Mackey [27] or in Naimark and Fomin [30].)

In particular, if

$$\sigma(H) \simeq \bigoplus_{r=1}^{n} \sigma^{r} ,$$

where n is finite or infinite, then

$$\sigma(H)\uparrow G \simeq \bigoplus_{r=1}^{n} (\sigma^{r}\uparrow G) .$$

Mackey's subgroup theorem

Let H_1 and H_2 be two closed subgroups of a separable locally compact group G, which are such that there exists a Borel set in G which intersects each H_1, H_2 double coset in exactly one point. Let K be the set of all H_1, H_2 double cosets, and let γ be the map which takes each element of G into the double coset to which it belongs. Give K the Borel structure induced by γ from that of G. Now take any finite measure μ on the Borel sets of G which is equivalent to Haar measure, and define

$\bar{\mu}$ on the Borel sets of K by the relation $\bar{\mu}(E) = \mu(\gamma^{-1}E)$. Let σ be a representation of the subgroup H_2. For each $g \in G$, define a representation σ_g of gH_2g^{-1} by

$$\sigma_g(gh_2g^{-1}) = \sigma(h_2) \quad \text{for each } h_2 \in H_2 .$$

Let $$\rho_g = \sigma_g(H_1 \cap gH_2g^{-1}) \uparrow H_1 .$$

Then ρ_g is determined up to unitary equivalence by the double coset κ to which g belongs; so ρ_g may be written ρ_κ. Then

$$(\sigma(H_2) \uparrow G) \downarrow H_1 \simeq \int_K \rho_\kappa \, d\bar{\mu}(\kappa) . \tag{3}$$

In particular, if there are only a countable number of double cosets in K, then

$$(\sigma(H_2) \uparrow G) \downarrow H_1 \simeq \bigoplus_K \rho_\kappa . \tag{4}$$

The direct integral also reduces to a direct sum if $\rho_\kappa = \rho$ is independent of κ. In this case,

$$(\sigma(H_2) \uparrow G) \downarrow H_1 \simeq n\rho , \tag{5}$$

where n is the number (possibly infinite) of double cosets in K.

A weak Frobenius reciprocity theorem

Let σ be an irreducible representation of a closed subgroup H of a separable locally compact group G, and let ρ be a finite-dimensional irreducible representation of G. Let $<\pi_0, \pi_1>_L$ denote the number of times that the irreducible representation π_1 of the group L is contained in the representation π_0 of L. Then, if G/H has a finite invariant measure,

$$<\sigma(H)\uparrow G, \rho>_G = <\rho\downarrow H, \sigma>_H .$$

Corollary : the Frobenius reciprocity theorem for compact groups

Let H be a closed subgroup of a separable compact group G, and suppose σ and ρ are irreducible representations of H and G respectively. Then

$$<\sigma(H)\uparrow G, \rho>_G = <\rho\downarrow H, \sigma>_H . \qquad (6)$$

2.3 Definition of projective representation and of properties of multipliers

Let G be a separable locally compact group. A projective representation of G is defined to be a mapping $g \longrightarrow U(g)$ of G into the group of all unitary operators on a separable Hilbert space $\mathcal{H}_U$ such that :

(i) $U(e) = I$ (where e is the identity element of G, and I is the identity operator on $\mathcal{H}_U$);

(ii) $U(g_1g_2) = \alpha(g_1,g_2)U(g_1)U(g_2)$ for each g_1, $g_2 \in G$, where $\alpha(g_1,g_2)$ is a complex number of unit modulus, dependent on g_1 and g_2;

(iii) the map $g \longrightarrow (U(g)v_1,v_2)$ is a Borel function on G, for each v_1, $v_2 \in \mathcal{H}_U$.

A number of properties follow immediately from this definition :

since $U(e) = I$,

$$\alpha(g,e) = \alpha(e,g) = 1 \text{ for all } g \in G \; ; \tag{7}$$

from the associativity of the operators U,

$$U(g_1)\left[U(g_2)U(g_3)\right] = \left[U(g_1)U(g_2)\right]U(g_3) \; ,$$

so $$\alpha(g_1g_2,g_3)\alpha(g_1,g_2) = \alpha(g_1,g_2g_3)\alpha(g_2,g_3) \; ; \tag{8}$$

from (iii),

$$\alpha \text{ is a Borel function on } G \times G. \tag{9}$$

Any function α defined on G x G, taking values in the multiplicative group, T, of all complex numbers of unit modulus and satisfying (7), (8) and (9) is called a multiplier on G.

If α is the multiplier appearing in (ii), the corresponding projective representation U is called an α-representation of G. A representation in the sense used previously is an α-representation with $\alpha(g_1,g_2) = 1$ for all g_1, $g_2 \in G$; it will sometimes be referred to as an ordinary representation.

However, unless otherwise stated, a 'representation' will mean an 'ordinary representation', not a 'projective representation'

If α_1 and α_2 are multipliers on G, then $\alpha_1\alpha_2$ is also a multiplier; if α is a multiplier, so is α^{-1}. Hence all the multipliers on G form an abelian group under multiplication.

A multiplier α_a is said to be <u>trivial</u> if there exists a Borel function a : G $\longrightarrow$ T such that

$$\alpha_a(g_1,g_2) = \frac{a(g_1g_2)}{a(g_1)a(g_2)} \quad \text{for all } g_1,\ g_2 \in G\ .$$

If the mapping $g \longrightarrow U(g)$ is a projective representation of G with trivial multiplier α_a, then the mapping $g \longrightarrow \frac{1}{a(g)} U(g)$ is an ordinary representation of G.

Two multipliers α_1 and α_2 are said to be <u>equivalent</u> (denoted by $\alpha_1 \sim \alpha_2$) if $\alpha_1\alpha_2^{-1}$ is trivial. $\sim$ is clearly an equivalence relation.

If $\alpha_1 \sim \alpha_2$, and U_1 is an α_1-representation, then an α_2-representation U_2 is given by putting

$$U_2(g) = a(g)U_1(g)\ , \text{ where a satisfies } \alpha_2 = \alpha_a\alpha_1.$$

U_1 and U_2 are said to be <u>projectively equivalent</u> representations. In particular, any projective representation with a trivial multiplier is projectively equivalent to an ordinary representation.

A multiplier α is said to be <u>locally trivial</u> if there

exists a neighbourhood V of e in G, and a Borel function $a : V \longrightarrow T$ such that

$$\alpha(g_1,g_2) = \frac{a(g_1g_2)}{a(g_1)a(g_2)} \quad \text{for all } g_1,\ g_2 \in V.$$

2.4 Some theorems about multipliers

Proofs of these results may be found in Parthasarathy [33].

(a) Let G be a separable, locally compact, connected and simply connected group. Then every locally trivial multiplier on G is globally trivial.

(b) Let G be a separable compact group, and α be a multiplier on G. Then there exists an integer k such that α^k is trivial. Furthermore, α is locally trivial.

(c) If G is a separable, compact, connected and simply connected group, then every multiplier on G is trivial. (This follows directly from (a) and (b).)

(d) Every multiplier on the real line (considered as an additive group) is trivial.

(e) Every multiplier on a connected semisimple Lie group is locally trivial.

2.5 Mackey's theory for general semi-direct product groups

Let G be a separable locally compact group containing a closed normal subgroup N and a closed subgroup H which satisfy $G = NH$ and $N \cap H = \{e\}$. Then G is called a semi-direct product

of N and H, denoted by $G = N \circledS H$.

Every element $g \in G$ can be expressed uniquely in the form nh, where $n \in N$, $h \in H$. The multiplication operation between two elements n_1h_1 and n_2h_2 can be written

$$n_1h_1n_2h_2 = n_1(h_1n_2h_1^{-1})h_1h_2 ,$$

for all $n_1, n_2 \in N$, $h_1, h_2 \in H$.

For each $g \in G$ and each irreducible representation σ of N, define

$$(g\sigma)(n') = \sigma(g^{-1}n'g) \text{ , where } n' \in N ; \qquad (10)$$

then $g\sigma$ is also an irreducible representation of N.

If $g \in N$, then

$$(g\sigma)(n') = \sigma(g^{-1}n'g) = \sigma(g)^{-1}\sigma(n')\sigma(g) ,$$

since σ is a representation of N. So, if $g \in N$, $g\sigma \simeq \sigma$.

Now let $\hat{N}$ denote the set of equivalence classes of irreducible representations of the subgroup N of $N \circledS H$.

Then G acts as a group of 1 - 1 transformations of $\hat{N}$ into itself in such a way that every member of N is the identity transformation. Hence G/N (which is isomorphic to H) acts as G does, and $\hat{N}$ is a G/N- or H-space.

The set $O_\sigma = \{[h\sigma] : h \in H\}$, where $[h\sigma]$ denotes the equivalence class of $h\sigma$ in $\hat{N}$, is called the <u>orbit</u> of σ in $\hat{N}$ under the action of H (or G). In this way, $\hat{N}$ is divided into disjoint orbits.

The isotropy group of σ under the action of G is defined to be $G_\sigma = \{g \in G : g\sigma \simeq \sigma\}$.

Since $n\sigma \simeq \sigma$ for all $n \in N$, $N \subset G_\sigma$.

Let $H_\sigma = H \cap G_\sigma$. H_σ is called the little group of σ : it is the isotropy group of σ under the action of H. Then $G_\sigma = N \circledS H_\sigma$.

If $[\sigma_1]$ and $[\sigma_2]$ are in the same orbit of $\hat{N}$ under the action of H, then their little groups H_{σ_1}, H_{σ_2} are isomorphic, and hence their isotropy groups $G_{\sigma_1} = N \circledS H_{\sigma_1}$ and $G_{\sigma_2} = N \circledS H_{\sigma_2}$ are also isomorphic.

The representation σ of N may be extended to a representation of $G_\sigma = N \circledS H_\sigma$ in the following way.

For each $g \in G_\sigma$, $g\sigma \simeq \sigma$. Therefore, there exists a unitary operator W(g) on $\mathcal{H}_\sigma$ such that

$$(g\sigma)(n') = W(g)^{-1}\sigma(n')W(g) ,$$

for each $g \in G_\sigma$, $n' \in N$.

Suppose $g = nh$, where $n \in N$, $h \in H_\sigma$. Then

$$\begin{aligned}(g\sigma)(n') &= \sigma(g^{-1}n'g) \\ &= \sigma(h^{-1}n^{-1}n'nh) \\ &= (h\sigma)(n^{-1}n'n) \\ &= W(h)^{-1}\sigma(n^{-1}n'n)W(h)\end{aligned}$$

$$= W(h)^{-1}\sigma(n)^{-1}\sigma(n')\,\sigma(n)W(h) \ ,$$

since σ is an ordinary representation of N. Hence the operator W(g) may be taken to be $\sigma(n)W(h)$, where

$$W(h)^{-1}\sigma(n')W(h) = (h\sigma)(n') = \sigma(h^{-1}n'h) \ . \qquad (11)$$

Since σ is an irreducible representation, it follows, from Schur's lemma, that W(h) (or, indeed, more generally, W(g)) is unique up to a multiplicative constant; this constant must be of unit modulus, as the operators W are unitary. There is no canonical way of choosing this arbitrary constant in W(h); however, for convenience, W(e) is chosen to be the identity operator.

Now, from (11),

$$(h_1h_2)\sigma(n') = W(h_1h_2)^{-1}\sigma(n')W(h_1h_2) \ .$$

Also
$$\begin{aligned}(h_1h_2)\,\sigma(n') &= \sigma(h_2^{-1}h_1^{-1}n'h_1h_2)\\ &= (h_2\sigma)(h_1^{-1}n'h_1)\\ &= W(h_2)^{-1}\sigma(h_1^{-1}n'h_1)W(h_2)\\ &= W(h_2)^{-1}W(h_1)^{-1}\sigma(n')W(h_1)W(h_2) \ .\end{aligned}$$

Since W is unique to within a multiplicative constant of unit modulus,

$$W(h_1h_2) = \alpha(h_1,h_2)W(h_1)W(h_2) \ ,$$

where $\alpha(h_1,h_2)$ is a complex number of unit modulus. The arbitrariness in W may be used to make α into a Borel function on G x G. Hence the operators W form a projective representation of H_σ, with multiplier α. Given N, H and σ, it may be shown that the multiplier α is uniquely determined up to trivial multipliers. α may be a non-trivial multiplier, and so W need not be an ordinary representation of H_σ. In general, W is a reducible projective representation of H_σ.

The map $W : nh \longrightarrow W(nh) = \sigma(n)W(h)$ of $N \circledS H_\sigma$ into the set of unitary operators on $\mathcal{H}_\sigma$ satisfies, for each $n_1, n_2 \in N$, $h_1, h_2 \in H_\sigma$,

$$W(n_1h_1n_2h_2) = W([n_1h_1n_2h_1^{-1}]h_1h_2)$$

$$= \sigma(n_1h_1n_2h_1^{-1})W(h_1h_2)$$

$$= \sigma(n_1)\sigma(h_1n_2h_1^{-1})\alpha(h_1,h_2)W(h_1)W(h_2)$$

(since σ is an ordinary representation of the normal subgroup N of G_σ, and W is an α-representation of H_σ)

$$= \sigma(n_1)W(h_1)\sigma(n_2)W(h_1)^{-1}\alpha(h_1,h_2)W(h_1)W(h_2$$

(since, using (11), $W(h_1)^{-1}\sigma(h_1n_2h_1^{-1})W(h_1) = \sigma(n_2)$)

$$= \alpha(h_1,h_2)W(n_1h_1)W(n_2h_2) .$$

Hence W is a projective representation of $G_\sigma = N \circledS H_\sigma$, with multiplier α, which is determined by its value on H_σ. The restriction of W to the subgroup N of G_σ is the irreducible

representation σ of N, and so W must be an irreducible projective representation of G_σ.

Now suppose that ρ is an irreducible projective representation of H_σ with multiplier $1/\alpha$. Then the map $\sigma W \otimes \rho$: nh $\longrightarrow$ $\sigma(n)W(h) \otimes \rho(h)$ (where $\otimes$ denotes the inner Kronecker product) is an ordinary representation of $G_\sigma = N \circledS H_\sigma$, and it may be shown to be irreducible.

An ordinary representation of G can now be obtained by inducing from this representation of G_σ to G.

Questions then arise as to whether this representation of G is irreducible and which representations of G may be obtained by using the above procedure. The answers are contained in a theorem due to Mackey [26].

Theorem

Suppose $G = N \circledS H$ is a semi-direct product group and that there exists a Borel subset of $\hat{N}$ which meets each orbit of $\hat{N}$ (under the action of H) in just one point.

Let σ be an irreducible ordinary representation of N, $H_\sigma = \{ h \in H : h\sigma \simeq \sigma \}$ and $G_\sigma = N \circledS H_\sigma$. Then there exists a projective representation W of H_σ satisfying $(h\sigma)(n') = W(h)^{-1}\sigma(n')W(h)$, for each $h \in H_\sigma$, $n' \in N$; W and the multiplier α of W are unique up to multiplication by trivial multipliers. For each irreducible $1/\alpha$-representation ρ of H_σ, the representation of G induced from the ordinary representation $\sigma W \otimes \rho$: nh $\longrightarrow$ $\sigma(n)W(h) \otimes \rho(h)$ of G_σ is an

irreducible representation of G.

Suppose $\tilde{\sigma}$ and σ belong to the same orbit of $\hat{N}$, so that $\tilde{\sigma} = \tilde{h}\sigma$ for some $\tilde{h} \in H$. Then : $H_{\tilde{\sigma}} = \tilde{h}H_{\sigma}\tilde{h}^{-1}$; corresponding to the extension $nh \longrightarrow \sigma(n)W(h)$ $(n \in N, h \in H_{\sigma})$ of the representation σ of N to G_{σ}, the extension of the representation $\tilde{\sigma}$ of N to $G_{\tilde{\sigma}}$ may be taken to be $nh \longrightarrow \tilde{\sigma}(n)W(\tilde{h}^{-1}h\tilde{h})$ $(n \in N, h \in H_{\tilde{\sigma}})$; the representations of G induced from the representation $nh \longrightarrow \sigma(n)W(h) \otimes \rho(h)$ $(n \in N, h \in H_{\sigma})$ of G_{σ}, and from the representation $nh \longrightarrow \tilde{\sigma}(n)W(\tilde{h}^{-1}h\tilde{h}) \otimes \rho(\tilde{h}^{-1}h\tilde{h})$ $(n \in N, h \in H_{\tilde{\sigma}})$ of $G_{\tilde{\sigma}}$ are unitarily equivalent.

Every irreducible ordinary representation of G is unitarily equivalent to one induced from a representation of the form $\sigma W \otimes \rho$: $nh \longrightarrow \sigma(n)W(h) \otimes \rho(h)$, where the orbit of the irreducible representation σ of N is uniquely determined, the projective representation W of H_{σ} is determined up to trivial multipliers by σ, and the irreducible projective representation ρ of H_{σ} is determined up to unitary equivalence.

2.6 A method of finding the operator W when the subgroup N and the little group H_{σ} are connected Lie groups, and H_{σ} has no non-trivial multipliers

Let $G = N \circledS H$ and let σ be an irreducible representation of N. Suppose that (with the notation of § 2.5) $G_{\sigma} = N \circledS H_{\sigma}$, where N and H_{σ} are connected Lie groups, and H_{σ} has no non-trivial multipliers.

Then, for $h \in H_\sigma$, $n' \in N$, the operator W satisfies

$$(h\sigma)(n') = W(h)^{-1}\sigma(n')W(h) \ . \qquad (11)$$

From Mackey's theory, W is a projective representation of H_σ and is determined up to trivial multipliers by σ. By hypothesis, H_σ has no non-trivial multipliers; hence W can always be chosen to be an ordinary representation W_o of H_σ. Any other ordinary representation W of H_σ satisfying (11) must be of the form $a(h)W_o(h)$, where the trivial multiplier

(§2.3) $\alpha(h_1,h_2) = \dfrac{a(h_1h_2)}{a(h_1)a(h_2)}$ is equal to one; so an ordinary representation W of H_σ satisfying (11) is uniquely determined up to a one-dimensional ordinary representation of H_σ.

The following method of finding the operator W was developed in dealing with a number of similar cases which arose in the course of the work to be described later. It depends on the use of one-parameter subgroups of H_σ and N.

Let L be an r-dimensional connected Lie group. A set $l_1(t), l_2(t),\ldots, l_m(t)$ of one-parameter subgroups of L is said to be independent if the corresponding left-invariant vector fields are linearly independent. Then :

Lemma (Pontryagin [35], p.296)

Let $l_1(t), l_2(t),\ldots, l_r(t)$ be r independent one-parameter subgroups of L. Then there is a non-negative real number c

such that the set

$$l(t_1,t_2,\ldots,t_r) = l_1(t_1)l_2(t_2)\ldots l_r(t_r) ,$$

where $|t_i| < c$, $i = 1,2,\ldots,r$, is a neighbourhood of the identity in L.

Lemma (Pontryagin [35], p.129)

Let V be a neighbourhood of the identity in L. Then every member of L can be expressed as a finite product $v_1v_2\ldots v_s$ of elements $v_i \in V$.

These lemmas lead to the following result.

Theorem

Every member of a connected r-dimensional Lie group L can be expressed as a finite product of the form

$$l = l_1(t_{11})l_2(t_{21})\ldots l_r(t_{r1})l_1(t_{12})l_2(t_{22})\ldots l_r(t_{r2})$$

$$\ldots l_1(t_{1s})l_2(t_{2s})\ldots l_r(t_{rs}) ,$$

where $l_1(t)$, $l_2(t)$,..., $l_r(t)$ are independent one-parameter subgroups of L.

By hypothesis, H_σ is a connected Lie group. Hence, by the above theorem, every member h of H_σ can be expressed as a finite product of the form

$$h = h_1(t_{11})h_2(t_{21})\ldots h_r(t_{r1})h_1(t_{12})h_2(t_{22})\ldots h_r(t_{r2})$$

$$\ldots h_1(t_{1s})h_2(t_{2s})\ldots h_r(t_{rs}) \ ,$$

where $h_1(t)$, $h_2(t)$,..., $h_r(t)$ (with r = dimension of H_σ) are independent one-parameter subgroups of H_σ.

For convenience of notation, this will be written as

$$h = \prod_j \prod_i h_i(t_{ij})$$

(where the order of the terms is important).

Since W_o is an ordinary representation of H_σ,

$$W_o(h) = \prod_j \prod_i W_o(h_i(t_{ij})) \ . \tag{12}$$

For each i, the map $h_i(t) \longrightarrow W_o(h_i(t))$ is an ordinary unitary representation of the one-parameter subgroup $h_i(t)$ of H_σ. Hence, by Stone's theorem [41], there exists a unique skew-adjoint operator J_i such that

$$W_o(h_i(t)) = \exp tJ_i \ . \tag{13}$$

J_i is given by

$$J_i\phi = \frac{d}{dt} \left(W_o(h_i(t))\phi\right)\Big|_{t=0} \ , \tag{14}$$

and its dense domain is the set of $\phi \in \mathcal{H}_{W_o}$ for which this derivative exists. The domain of J_i depends on i, but contains a dense linear subspace, called the Gårding subspace, which depends only on the representation W_o [15]. In what

follows, the domain of operators such as J_i will not be explicitly mentioned, it being assumed that their dense domain is the appropriate Gårding subspace.

It follows from (12) and (13) that

$$W_o(h) = \prod_j \prod_i \exp t_{ij} J_i \ .$$

For each one-parameter subgroup $h(t)$ of H_σ, the unique skew-adjoint operator J determined by $W_o(h(t)) = \exp tJ$ satisfies, from (11),

$$(h(t)\sigma)(n') = \exp(-tJ)\,\sigma(n')\exp(tJ) \ . \qquad (15)$$

As N is a connected Lie group, each $n' \in N$ can be expressed as a finite product of the form

$$n' = \prod_l \prod_k n_k(\mu_{kl}) \ ,$$

where k runs between 1 and the dimension of N, and the $n_k(\mu)$ are independent one-parameter subgroups of N.

Since $h(t)\sigma$ is an ordinary representation of N,

$$(h(t)\sigma)(n') = \prod_l \prod_k (h(t)\sigma)(n_k(\mu_{kl})) \ . \qquad (16)$$

For each k, the map $n_k(\mu) \longrightarrow (h(t)\sigma)(n_k(\mu))$ is an ordinary unitary representation of the one-parameter subgroup $n_k(\mu)$ of N. Hence, by Stone's theorem, there exists a unique skew-adjoint operator $Z_k(t)$ such that

$$(h(t)\sigma)(n_k(\mu)) = \exp \mu Z_k(t) \ . \qquad (17)$$

So, for each k, from (15) and (17),

$$\exp(\mu Z_k(t)) = \exp(-tJ)\exp(\mu Z_k(0))\exp(tJ) \ . \qquad (18)$$

Differentiating with respect to μ and putting $\mu = 0$ gives

$$Z_k(t) = \exp(-tJ)\ Z_k(0)\ \exp(tJ) \ . \qquad (19)$$

Then, differentiating this expression with respect to t and putting t = 0 gives

$$\left.\frac{dZ_k(t)}{dt}\right|_{t=0} = [Z_k(0), J] \text{ for all k} \ . \qquad (20)$$

The existence of a skew-adjoint operator J satisfying (20) follows from the existence of $W_o(h(t)) = \exp tJ$.

Conversely, suppose $\tilde{J}$ is a skew-adjoint operator satisfying (20). Obviously, for any imaginary constant η , $\tilde{J} + \eta I$ is also a skew-adjoint operator satisfying (20). To investigate to what further extent $\tilde{J}$ can differ from J, let $J' = J - \tilde{J}$. Then

$$[Z_k(0), J'] = 0 \text{ for all k} \ .$$

Hence J' commutes with all the operators $Z_k(0)$ of the representation of the Lie algebra of N corresponding to the representation σ of the Lie group N. So J' commutes with the irreducible representation σ of N. By Schur's lemma, it follows that J' is a scalar multiple of the identity operator, and, since J' is a skew-adjoint operator, J' must equal η I,

where η is imaginary. So

$$J = \tilde{J} + \eta I \ . \tag{21}$$

Hence every skew-adjoint operator $\tilde{J}$ satisfying (20) also satisfies (19). It also follows that (20) determines the skew-adjoint operator J satisfying (19) up to an imaginary constant. From the form of (19), this is as good as could be hoped for.

It follows that, for any $n' \in N$,

$$\begin{aligned}
(h(t)\sigma)(n') &= \prod_l \prod_k (h(t)\sigma)(n_k(\mu_{kl})) && \text{(from (16))} \\
&= \prod_l \prod_k \exp(\mu_{kl} Z_k(t)) && \text{(from (17))} \\
&= \prod_l \prod_k [\exp(-t(\tilde{J} + \eta I))\exp(\mu_{kl} Z_k(0)) \ x \\
&\quad \exp(t(\tilde{J} + \eta I))]. && \text{(from (18) and (21))} \\
&= \exp(-t(\tilde{J} + \eta I)) \prod_l \prod_k \exp(\mu_{kl} Z_k(0)) \ x \\
&\quad \exp(t(\tilde{J} + \eta I)) \\
&= \exp(-t(\tilde{J} + \eta I))\sigma(n')\exp(t(\tilde{J} + \eta I)),
\end{aligned}$$

where η is an arbitrary imaginary constant.

Therefore, for each i, the ordinary representation $\tilde{W} : h_i(t) \longrightarrow \exp t(\tilde{J}_i + \eta_i I)$ of the one-parameter subgroup $h_i(t)$ of H_σ satisfies the restriction of (11) to $h_i(t)$. However, because of the arbitrary constants η_i, the map

$$\tilde{W} : \prod_j \prod_i h_i(t_{ij}) \longrightarrow \prod_j \prod_i \exp t_{ij}(\tilde{J}_i + \eta_i I)$$

(cf.(12)) may not be an ordinary representation of H_σ. To ensure that $\tilde{W}$ is an ordinary representation of H_σ, the constants η_i cannot be allowed to be arbitrary, but must be appropriately chosen. If $\tilde{W}$ is an ordinary representation of the Lie group H_σ, it must give an ordinary representation of the Lie algebra of H_σ. This leads to constraints on the values of the constants η_i.

This method of finding the operator W is summarised in :

Theorem

Let $G = N \circledS H$ and let σ be an irreducible representation of N. Suppose $G_\sigma = N \circledS H_\sigma$, where N and H_σ are connected Lie groups, and H_σ has no non-trivial multipliers. Let $\left.\begin{matrix} h_i(t) \\ n_k(\mu) \end{matrix}\right\}$ be sets of independent one-parameter subgroups of $\left\{\begin{matrix} H_\sigma \\ N \end{matrix}\right.$ which generate $\left\{\begin{matrix} H_\sigma \\ N \end{matrix}\right.$. Suppose that $(h_i(t)\sigma)(n_k(\mu)) = \exp \mu Z_{ik}(t)$. Let $\{\tilde{J}_i\}$ be a set of skew-adjoint operators satisfying

$$\left.\frac{dZ_{ik}}{dt}\right|_{t=0} = [Z_{ik}(0), \tilde{J}_i] \text{ for each } k . \qquad (22)$$

Then there exists a choice of the imaginary constants η_i such that

$$W_o : \prod_j \prod_i h_i(t_{ij}) \longrightarrow \prod_j \prod_i \exp t_{ij}(\tilde{J}_i + \eta_i I) \qquad (23)$$

is an ordinary representation of H_σ satisfying

$$(h\sigma)(n') = W_o(h)^{-1}\sigma(n')W_o(h) ,$$

for all $h \in H_\sigma$, $n' \in N$.

2.7 Mackey's theory for semi-direct product groups N H with N abelian

If N is an abelian group, all its irreducible ordinary representations are one-dimensional. Hence, if

$$(g\sigma)(n') = W(g)^{-1}\sigma(n')W(g)$$

for all $g \in G_\sigma$, $n' \in N$, then $(g\sigma)(n') = \sigma(n')$ for all $g \in G_\sigma$, $n' \in N$. Therefore, in this case, the isotropy group of σ under the action of G is defined as

$$G_\sigma = \{g \in G : g\sigma = \sigma\} ,$$

and the little group of σ as

$$H = \{h \in H : h\sigma = \sigma\} .$$

The operator $W(h)$ $(h \in H_\sigma)$ can be taken to be the identity, ρ can be taken to be an ordinary representation of H_σ, and, since σ is a one-dimensional representation, the inner Kronecker product in $\sigma W \otimes \rho$ can be replaced by scalar multiplication. Mackey's results then reduce to the following.

Theorem

Suppose $G = N \circledS H$ is a semi-direct product group, with N abelian, and that there exists a Borel subset of $\hat{N}$ which meets each orbit of $\hat{N}$ (under the action of H) in just one point.

Let σ be an irreducible ordinary representation of N, $H_\sigma = \{ h \in H : h\sigma = \sigma \}$ and $G_\sigma = N \circledS H_\sigma$. Then, for each irreducible ordinary representation ρ of H_σ, the representation of G induced from the ordinary representation $\sigma\rho : nh \longrightarrow \sigma(n)\rho(h)$ $(n \in N, h \in H_\sigma)$ of G_σ is an irreducible representation of G.

Suppose $\tilde{\sigma}$ and σ belong to the same orbit of $\hat{N}$, so that $\tilde{\sigma} = \tilde{h}\sigma$ for some $\tilde{h} \in H$. Then : $H_{\tilde{\sigma}} = \tilde{h} H_\sigma \tilde{h}^{-1}$; the representations of G induced from the representation $nh \longrightarrow \sigma(n)\rho(h)$ $(n \in N, h \in H_\sigma)$ of G_σ, and from the representation $nh \longrightarrow \tilde{\sigma}(n)\rho(\tilde{h}^{-1}h\tilde{h})$ $(n \in N, h \in H_{\tilde{\sigma}})$ of $G_{\tilde{\sigma}}$ are unitarily equivalent.

Every irreducible ordinary representation of G is unitarily equivalent to one induced from a representation of the form $\sigma\rho : nh \longrightarrow \sigma(n)\rho(h)$, where the orbit of the irreducible representation σ of N is uniquely determined, and the irreducible representation ρ of H_σ is determined up to unitary equivalence.

CHAPTER 3

THE IRREDUCIBLE REPRESENTATIONS OF THE ANISOTROPIC HARMONIC OSCILLATOR GROUP G

The anisotropic harmonic oscillator group G is essentially that generated by the position and momentum observables, the identity operator, and the Hamiltonian of the system. In this chapter, it is shown how the explicit expressions for the set G and the composition law of G are suggested by a heuristic construction, which is based on the definition of the Lie algebra $\mathcal{L}$ of G. The simply connected covering group $\tilde{G}$ of G is also defined.

The irreducible representations of the groups $\tilde{G}$ and G are then found, using Mackey's theory.

3.1 Definition of the Lie algebra $\mathcal{L}$ of G

The quantum mechanical n-dimensional anisotropic harmonic oscillator is a particle of mass m with Hamiltonian

$$H_{osc} = \sum_{j=1}^{n} \frac{1}{2m} (P_j^2 + m^2\omega_j^2 Q_j^2) ,$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are positive constants (the frequencies), and $\{ Q_j : j = 1,2,\ldots,n \}$ and $\{ P_j : j = 1,2,\ldots,n \}$ are sets of self-adjoint operators corresponding to position and momentum observables respectively, and satisfying the commutation relations :

$$[Q_j, P_k] = i\hbar\delta_{jk}I \qquad (1 \leqslant j,k \leqslant n).$$

To avoid unimportant constants, define new self-adjoint operators :

$$\hat{Q}_j = \left(\frac{m\omega_j}{\hbar}\right)^{\frac{1}{2}} Q_j ,$$

$$\hat{P}_j = \left(\frac{1}{m\hbar\omega_j}\right)^{\frac{1}{2}} P_j ,$$

$$\hat{H} = \frac{H_{osc}}{\hbar} .$$

These satisfy the following commutation relations :

$$[\hat{Q}_j, \hat{P}_k] = i\delta_{jk}I ,$$

$$[\hat{H}, \hat{Q}_k] = -i\omega_k\hat{P}_k ,$$

$$[\hat{H}, \hat{P}_k] = i\omega_k\hat{Q}_k .$$

In order to construct a Lie algebra associated with these operators, define the skew-adjoint operators :

$$E = iI ,$$

$$B_j = i\hat{Q}_j ,$$

$$A_j = i\hat{P}_j ,$$

$$T = i\hat{H} .$$

Then :

$$
\left.
\begin{aligned}
\left[A_j,\ B_k\right] &= \delta_{jk}E\ , \\
\left[T,\ B_k\right] &= \omega_k A_k\ , \\
\left[T,\ A_k\right] &= -\omega_k B_k\ , \\
\left[B_j,\ E\right] &= 0\ , \\
\left[A_j,\ E\right] &= 0\ , \\
\left[T,\ E\right] &= 0\ .
\end{aligned}
\right\} \quad (1)
$$

Therefore, E, B_1, B_2,..., B_n, A_1, A_2,..., A_n and T form a basis of the Lie algebra defined by the above commutation relations. This Lie algebra will be denoted by $\mathcal{L}$.

3.2 Heuristic construction of the groups G and $\tilde{G}$

Every real Lie algebra is isomorphic to a subalgebra of some $gl(m, \mathbb{R})$ (where $gl(m, \mathbb{R})$, $m \in \mathbb{Z}_+$, denotes the Lie algebra of all real m x m matrices); therefore, there is a matrix representation of the Lie algebra $\mathcal{L}$ by elements of some $gl(m, \mathbb{R})$ ([22], p.202).

One straightforward representation to use is the adjoint representation. (For $X \in \mathcal{L}$, the adjoint of X is the linear transformation $\mathrm{Ad}\ X : Y \longrightarrow \left[X, Y\right]$ of $\mathcal{L}$, which is represented by a $(2n + 2) \times (2n + 2)$ matrix relative to the basis E, B_1, B_2,..., B_n, A_1, A_2,..., A_n, T of $\mathcal{L}$.) However, since E commutes with E, B_1, B_2,..., B_n, A_1, A_2,..., A_n and T, it follows that $\mathrm{Ad}\ E$ is the $(2n + 2) \times (2n + 2)$ zero matrix.

A significant property of the Lie algebra $\mathcal{L}$ is that $[A_j, B_k] = \delta_{jk}E$, where E is possibly not the zero operator. Hence many important features of the Lie algebra $\mathcal{L}$ would be lost if the adjoint representation were used. It is therefore necessary to look for some other matrix representation of $\mathcal{L}$.

Let E_{jk} denote the $(2n + 2) \times (2n + 2)$ matrix with 1 at the intersection of the jth row and kth column, and zeros elsewhere. Define a mapping $\sim$ of the chosen basis elements of $\mathcal{L}$ into $gl(2n + 2, \mathbb{R})$ by :

$$E \longrightarrow \tilde{E} = 2E_{2n+1,2n+2} ,$$

$$B_j \longrightarrow \tilde{B}_j = - E_{2n+1,j} + E_{n+j,2n+2} \qquad (1 \leqslant j \leqslant n) ,$$

$$A_j \longrightarrow \tilde{A}_j = E_{2n+1,n+j} + E_{j,2n+2} \qquad (1 \leqslant j \leqslant n) ,$$

$$T \longrightarrow \tilde{T} = \sum_{j=1}^{n} \omega_j (E_{j,n+j} - E_{n+j,j}) .$$

The mapping $\sim$ may be extended by linearity to the whole of $\mathcal{L}$.

It is easily verified that $\tilde{E}$, $\tilde{B}_1$, $\tilde{B}_2$,..., $\tilde{B}_n$, $\tilde{A}_1$, $\tilde{A}_2$,..., $\tilde{A}_n$, $\tilde{T}$ satisfy the same commutation relations as E, B_1, B_2,..., B_n, A_1, A_2,..., A_n, T respectively. Therefore the mapping $\sim$ gives a matrix representation of the Lie algebra $\mathcal{L}$.

Now consider the set of matrices formed by taking products of exponentials of the representative basis elements. A typical element of this set is

$$\exp(\sigma\tilde{E})\exp(\beta_1\tilde{B}_1)\exp(\beta_2\tilde{B}_2)\ldots\exp(\beta_n\tilde{B}_n)\ x$$

$$\exp(\alpha_1\tilde{A}_1)\exp(\alpha_2\tilde{A}_2)\ldots\exp(\alpha_n\tilde{A}_n)\exp(\tau\tilde{T})\ ,$$

where $\sigma \in \mathbb{R}$, $\underline{\beta} = (\beta_1,\beta_2,\ldots,\beta_n) \in \mathbb{R}^n$, $\underline{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. This element will be denoted by $(\sigma,\underline{\beta},\underline{\alpha},\tau)$. (The reason for this choice of the order of the exponentials will become apparent in § 4.2.)

A straightforward calculation gives that

$$(\sigma,\underline{\beta},\underline{\alpha},\tau) =$$

$$\left(\begin{array}{c|c|c|c}
\begin{matrix} \cos\omega_1\tau & & & \\ & \cos\omega_2\tau & & \\ & & \ddots & \\ & & & \cos\omega_n\tau \end{matrix} & \begin{matrix} \sin\omega_1\tau & & & \\ & \sin\omega_2\tau & & \\ & & \ddots & \\ & & & \sin\omega_n\tau \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{matrix} \\
\hline
\begin{matrix} -\sin\omega_1\tau & & & \\ & -\sin\omega_2\tau & & \\ & & \ddots & \\ & & & -\sin\omega_n\tau \end{matrix} & \begin{matrix} \cos\omega_1\tau & & & \\ & \cos\omega_2\tau & & \\ & & \ddots & \\ & & & \cos\omega_n\tau \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix} \\
\hline
\ldots,-\beta_j\cos\omega_j\tau-\alpha_j\sin\omega_j\tau,\ldots & \ldots,-\beta_j\sin\omega_j\tau+\alpha_j\cos\omega_j\tau,\ldots & 1 & 2\sigma-\underline{\alpha}.\underline{\beta} \\
\hline
0 & 0 & 0 & 1
\end{array}\right) \qquad (2)$$

(This construction for the anisotropic case is a generalisation of ideas used by Streater [37] in the one-dimensional case; however, his actual calculations contain several misprints.)

The composition law on this set of matrices is taken to be matrix multiplication. It may be shown that

$$(\sigma', \underline{\beta}', \underline{\alpha}', \tau')(\sigma, \underline{\beta}, \underline{\alpha}, \tau) = (\sigma'', \underline{\beta}'', \underline{\alpha}'', \tau''),$$

where
$$\sigma'' = \sigma' + \sigma + \sum_{j=1}^{n} \left[\frac{\beta_j^2 - \alpha_j^2}{2} \cos \omega_j \tau' \sin \omega_j \tau' \right.$$
$$\left. - \alpha_j \beta_j \sin^2 \omega_j \tau' + \alpha_j' (\beta_j \cos \omega_j \tau' - \alpha_j \sin \omega_j \tau') \right],$$

$$\beta_j'' = \beta_j \cos \omega_j \tau' - \alpha_j \sin \omega_j \tau' + \beta_j' ,$$

$$\alpha_j'' = \beta_j \sin \omega_j \tau' + \alpha_j \cos \omega_j \tau' + \alpha_j' ,$$

and τ'' satisfies :

$$\left. \begin{aligned} \cos \omega_j \tau'' &= \cos \omega_j (\tau' + \tau) , \\ \sin \omega_j \tau'' &= \sin \omega_j (\tau' + \tau) , \end{aligned} \right\} (1 \leqslant j \leqslant n) .$$

The ambiguity in the definition of τ'' arises from the fact that a Lie algebra determines a Lie group only up to local isomorphism. If τ, τ' and τ'' are all restricted to be small, then it follows that $\tau'' = \tau' + \tau$. However, in general, all that can be said is that τ'' must be expressible in the form

$$\tau'' = \tau' + \tau + \frac{2\pi r_j}{\omega_j} ,$$

for each j, where $r_1, r_2, \ldots, r_n$ are appropriate integers. So, if τ'' is to be uniquely defined, the r_j's must satisfy

$$\frac{r_k}{\omega_k} = \frac{r_l}{\omega_l} , \qquad r_k, r_l \in \mathbb{Z} \ (1 \leqslant k, l \leqslant n) .$$

If the ratio of two numbers is rational, they are said to be rationally related. When each pair of ω_j's is rationally related, $\frac{r_k}{\omega_k}$ $(1 \leqslant k \leqslant n)$ can take any value in the set $\{\lambda s : s \in \mathbb{Z}\}$, where λ is the smallest positive number which is integral on division by any $\frac{1}{\omega_j}$. Hence τ'' is determined modulo $2\pi\lambda$. Also, in the matrix representation of $(\sigma, \underline{\beta}, \underline{\alpha}, \tau)$, values of τ which differ by integral multiples of $2\pi\lambda$ give the same matrix.

When each pair of ω_j's is rationally related, this suggests regarding τ and τ' as elements of $(-\pi\lambda, \pi\lambda]$, and defining $\tau'' \in (-\pi\lambda, \pi\lambda]$ by $\tau'' \equiv \tau' + \tau \pmod{2\pi\lambda}$. $(\sigma, \underline{\beta}, \underline{\alpha}, \tau)$ is then regarded as an element of the set $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times (-\pi\lambda, \pi\lambda]$, rather than as a matrix. This leads to the definition of the set G as

$$\{(\sigma, \underline{\beta}, \underline{\alpha}, \tau) : \sigma \in \mathbb{R}, \ \underline{\beta}, \underline{\alpha} \in \mathbb{R}^n, \ \tau \in (-\pi\lambda, \pi\lambda]\} ,$$

and to an unambiguous composition law for G.

An alternative procedure for removing the ambiguity in τ'' is to define the set $\tilde{G}$ to be

$$\{(\sigma,\underline{\beta},\underline{\alpha},\tau) : \sigma\in\mathbb{R}, \underline{\beta},\underline{\alpha}\in\mathbb{R}^n, \tau\in\mathbb{R}\}$$

and to take $\tau'' = \tau' + \tau$ in the composition law.

If any two of the ω_j's are not rationally related, each r_j can take only the value zero. τ'' is then uniquely determined as $\tau' + \tau$. In this case, each $\tau\in\mathbb{R}$ defines a different matrix. This suggests defining G (and $\tilde{G}$) to be the set

$$\{(\sigma,\underline{\beta},\underline{\alpha},\tau) : \sigma\in\mathbb{R}, \underline{\beta},\underline{\alpha}\in\mathbb{R}^n, \tau\in\mathbb{R}\},$$

and indicates the appropriate form of the composition law.

The case of non-rationally related ω_j's can be considered as the limiting case of rationally related ω_j's as $\lambda\longrightarrow\infty$.

3.3 Definition and properties of G and $\tilde{G}$

The above heuristic considerations suggest the following definitions of the sets G and $\tilde{G}$, and of the composition laws for these sets.

If each pair of ω_j's is rationally related, the set G is defined to be

$$G = \{(\sigma,\underline{\beta},\underline{\alpha},\tau) : \sigma\in\mathbb{R}, \underline{\beta},\underline{\alpha}\in\mathbb{R}^n, \tau\in(-\pi\lambda,\pi\lambda]\}, \qquad (3)$$

with λ the smallest positive number such that each $\lambda\omega_j$ is integral.

If any two of the ω_j's are not rationally related, the

set G is taken to be the set $\tilde{G}$ defined below.

Whatever the values of the ω_j's, the set $\tilde{G}$ is defined to be

$$\tilde{G} = \{(\sigma,\underline{\beta},\underline{\alpha},\tau) : \sigma\in \mathbf{R},\ \underline{\beta},\underline{\alpha}\in \mathbf{R}^n,\ \tau\in \mathbf{R}\} . \tag{4}$$

Define the composition laws for G and $\tilde{G}$ by

$$(\sigma',\underline{\beta}',\underline{\alpha}',\tau)(\sigma,\underline{\beta},\underline{\alpha},\tau) = (\sigma'',\underline{\beta}'',\underline{\alpha}'',\tau'') , \tag{5}$$

where, for G and $\tilde{G}$,

$$\sigma'' = \sigma' + \sigma + \sum_{j=1}^{n}\left[\frac{\beta_j^2-\alpha_j^2}{2}\cos\omega_j\tau'\sin\omega_j\tau' - \alpha_j\beta_j\sin^2\omega_j\tau' + \alpha_j'(\beta_j\cos\omega_j\tau' - \alpha_j\sin\omega_j\tau')\right] ,$$

$$\beta_j'' = \beta_j\cos\omega_j\tau' - \alpha_j\sin\omega_j\tau' + \beta_j' ,$$

$$\alpha_j'' = \beta_j\sin\omega_j\tau' + \alpha_j\cos\omega_j\tau' + \alpha_j' ,$$

and, for G,

$$\tau'' \equiv \tau' + \tau \pmod{2\pi\lambda} ,$$

whereas, for $\tilde{G}$,

$$\tau'' = \tau' + \tau .$$

It is readily verified that, with these composition laws, the sets G and $\tilde{G}$ are groups.

Furthermore, G is a homomorphic image of $\tilde{G}$ under the map

$$\nu : (\sigma,\underline{\beta},\underline{\alpha},\tau)(\in \tilde{G}) \longrightarrow (\sigma,\underline{\beta},\underline{\alpha},\hat{\tau})(\in G) ,$$

where $\hat{\tau} \in (-\pi\lambda,\pi\lambda]$, and is given by $\hat{\tau} \equiv \tau \bmod 2\pi\lambda$. Hence

$$G \approx \frac{\tilde{G}}{Z} , \tag{6}$$

where $Z = \{(o,\underline{o},\underline{o},2\pi\lambda s) : s \in \mathbb{Z}\} \approx \mathbb{Z}$,

with $\approx$ denoting isomorphism.

Each element $(\sigma,\underline{\beta},\underline{\alpha},\tau)$ of $\tilde{G}$ may be regarded as an element of $\mathbb{R}^{2n+2}$, and conversely. $\tilde{G}$ may therefore be given the usual topology of $\mathbb{R}^{2n+2}$. The topology on G is then chosen to be the topology inherited from $\tilde{G}$ under the above map $\nu : \tilde{G} \longrightarrow G$.

It follows that $\tilde{G}$ and G are connected Lie groups.

It remains to check that the Lie algebras of $\tilde{G}$ and G are isomorphic to the Lie algebra $\mathcal{L}$ (defined by (1)).

The Lie algebra of a general Lie group L can be defined as the set of left-invariant vector fields on the Lie group [15]. The left-invariant vector field X_μ, say, corresponding to the one-dimensional Lie subgroup $\mu(t)$ (where t is a parameter) is given by

$$(X_\mu \phi)(1) = \frac{\partial}{\partial t} \phi(1\mu(t))\Big|_{t = 0} ,$$

where $1 \in L$, $\phi \in C^\infty(L)$.

Now every element of the Lie groups $\tilde{G}$ and G can be specified by assigning values to the (2n + 2) parameters :

$\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \tau$. For each $\mu \in \{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \tau\}$, let $\mu(t)$ be the one-dimensional Lie subgroup formed by those elements for which the value of every parameter except μ is zero. Then, for $(\sigma, \underline{\beta}, \underline{\alpha}, \tau) \in \tilde{G}$ (or G), and $\phi \in C^\infty(\tilde{G})$ (or $C^\infty(G)$),

$$(X_\sigma \phi)(\sigma, \underline{\beta}, \underline{\alpha}, \tau) = \frac{\partial}{\partial t} \phi((\sigma, \underline{\beta}, \underline{\alpha}, \tau)(t, \underline{0}, \underline{0}, 0))\Big|_{t=0}$$

$$= \frac{\partial}{\partial t} \phi(\sigma + t, \underline{\beta}, \underline{\alpha}, \tau)\Big|_{t=0} \quad \text{from (5)}$$

$$= \frac{\partial}{\partial \sigma} \phi(\sigma, \underline{\beta}, \underline{\alpha}, \tau) .$$

Therefore, for both G and $\tilde{G}$,

$$X_\sigma = \frac{\partial}{\partial \sigma} .$$

Similarly, for both $\tilde{G}$ and G,

$$X_{\beta_j} = \alpha_j \cos \omega_j \tau \frac{\partial}{\partial \sigma} + \cos \omega_j \tau \frac{\partial}{\partial \beta_j} + \sin \omega_j \tau \frac{\partial}{\partial \alpha_j} ,$$

$$X_{\alpha_j} = -\alpha_j \sin \omega_j \tau \frac{\partial}{\partial \sigma} - \sin \omega_j \tau \frac{\partial}{\partial \beta_j} + \cos \omega_j \tau \frac{\partial}{\partial \alpha_j} ,$$

$$X_\tau = \frac{\partial}{\partial \tau} .$$

Hence :

$$[X_{\alpha_j}, X_{\beta_k}] = \Big[- \alpha_j \sin\omega_j\tau \frac{\partial}{\partial\sigma} - \sin\omega_j\tau \frac{\partial}{\partial\beta_j} + \cos\omega_j\tau \frac{\partial}{\partial\alpha_j} ,$$

$$\alpha_k \cos\omega_k\tau \frac{\partial}{\partial\sigma} + \cos\omega_k\tau \frac{\partial}{\partial\beta_k} + \sin\omega_k\tau \frac{\partial}{\partial\alpha_k} \Big]$$

$$= \delta_{jk} \Big(\sin^2\omega_j\tau \Big[- \alpha_j, \frac{\partial}{\partial\alpha_j} \Big] \frac{\partial}{\partial\sigma} + \cos^2\omega_j\tau \Big[\frac{\partial}{\partial\alpha_j} , \alpha_j \Big] \frac{\partial}{\partial\sigma}$$

$$= \delta_{jk} \frac{\partial}{\partial\sigma} \qquad \Big(\text{since } \Big[\frac{\partial}{\partial\alpha_j} , \alpha_j \Big] = 1 \Big)$$

$$= \delta_{jk} X_\sigma ;$$

$$[X_\tau , X_{\beta_k}] = \Big[\frac{\partial}{\partial\tau} , \alpha_k \cos\omega_k\tau \frac{\partial}{\partial\sigma} + \cos\omega_k\tau \frac{\partial}{\partial\beta_k} + \sin\omega_k\tau \frac{\partial}{\partial\alpha_k} \Big]$$

$$= \Big[\frac{\partial}{\partial\tau} , \cos\omega_k\tau \Big] \Big(\alpha_k \frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\beta_k} \Big) + \Big[\frac{\partial}{\partial\tau} , \sin\omega_k\tau \Big] \frac{\partial}{\partial\alpha_k}$$

$$= - \omega_k \sin\omega_k\tau \Big(\alpha_k \frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\beta_k} \Big) + \omega_k \cos\omega_k\tau \frac{\partial}{\partial\alpha_k}$$

$$= \omega_k X_{\alpha_k} .$$

Similarly,

$$[X_\tau , X_{\alpha_k}] = - \omega_k X_{\beta_k} ,$$

$$[X_{\beta_j} , X_\sigma] = 0 ,$$

$$[X_{\alpha_j} , X_\sigma] = 0 ,$$

$$[X_\tau , X_\sigma] = 0 .$$

Therefore the Lie algebra with basis X_σ, X_{β_1}, X_{β_2},..., X_{β_n}, X_{α_1}, X_{α_2},..., X_{α_n}, X_τ is isomorphic to the Lie algebra $\mathcal{L}$ with basis E, B_1, B_2,..., B_n, A_1, A_2,..., A_n, T (defined by (1)).

Therefore the Lie algebras of $\tilde{G}$ and G are each isomorphic to $\mathcal{L}$.

Moreover, since $\tilde{G}$ has underlying topological space homeomorphic to $\mathbb{R}^{2n+2}$, and $\mathbb{R}$ is simply connected, $\tilde{G}$ is also simply connected. Therefore, as $\tilde{G}$ and G are connected Lie groups with isomorphic Lie algebras, it follows that $\tilde{G}$ is the simply connected covering group of G.

Now, from (5), an arbitrary element of $\tilde{G}$ (or G) may be expressed in the form

$$(\sigma,\underline{\beta},\underline{\alpha},\tau) = (\sigma,\underline{\beta},\underline{0},0)(0,\underline{0},\underline{\alpha},0)(0,\underline{0},\underline{0},\tau) \ . \qquad (7)$$

It is then easily verified, using the group laws, (5), that

$$\left.\begin{aligned} \tilde{G} &= N \circledS M = (Y \circledS X) \circledS \tilde{M} \ , \\ G &= N \circledS M = (Y \circledS X) \circledS M \ , \end{aligned}\right\} \qquad (8)$$

where $N = Y \circledS X = \{(\sigma,\underline{\beta},\underline{\alpha},0) : \sigma \in \mathbb{R}, \underline{\beta},\underline{\alpha} \in \mathbb{R}^n\}$

is a normal subgroup of $\tilde{G}$ and G;

$$Y = \{(\sigma,\underline{\beta},\underline{0},0) : \sigma \in \mathbb{R}, \underline{\beta} \in \mathbb{R}^n\}$$

is a normal subgroup of N, and is isomorphic to $\mathbb{R}^{n+1}$;

$$X = \{(0,\underline{0},\underline{\alpha},0) \ : \ \underline{\alpha} \in \mathbb{R}^n\}$$

is a subgroup of $\tilde{G}$ and G, and is isomorphic to $\mathbb{R}^n$;

$$\tilde{M} = \{(0,\underline{0},\underline{0},\tau) \ : \ \tau \in \mathbb{R}\}$$

is a subgroup of $\tilde{G}$ and is isomorphic to $\mathbb{R}$;

$$M = \{(0,\underline{0},\underline{0},\tau) \ : \ \tau \in (-\pi\lambda, \pi\lambda]\}$$

is a subgroup of G and is isomorphic to $\mathbb{R}/\mathbb{Z}$.

3.4 The irreducible representations of the group N

All the irreducible representations of $\tilde{G} = N \circledS \tilde{M}$ may be found by using Mackey's theory for general semi-direct product groups (described in § 2.5). In order to apply this method to $\tilde{G}$, all the irreducible representations of N must first be evaluated. Since $N = Y \circledS X$, this also may be done by applying Mackey's theory for semi-direct product groups; Y is abelian, so, in this case, the method is simplified (see § 2.7).

The group law of N is, from (5),

$$(\sigma',\underline{\beta}',\underline{\alpha}',0)(\sigma,\underline{\beta},\underline{\alpha},0) = (\sigma' + \sigma + \underline{\alpha}'.\underline{\beta} \ , \ \underline{\beta}' + \underline{\beta} \ , \ \underline{\alpha}' + \underline{\alpha} \ , \ 0) \quad (9)$$

Since Y is isomorphic to $\mathbb{R}^{n+1}$, each of its irreducible representations is of the form

$$\chi_{v,\underline{u}}(\sigma,\underline{\beta},\underline{0},0) = \exp(iv\sigma + i\underline{u}.\underline{\beta}) \ ,$$

where $(\sigma,\underline{\beta},\underline{0},0) \in Y$, $v \in \mathbb{R}$ and $\underline{u} \in \mathbb{R}^n$.

The action of $(0,\underline{0},\underline{\alpha},0) \in X$ on $\chi_{v,\underline{u}}$ is, by definition 2-(10),

$$\begin{aligned}\left[(0,\underline{0},\underline{\alpha},0)\chi_{v,\underline{u}}\right](\sigma,\underline{\beta},\underline{0},0) &= \chi_{v,\underline{u}}\left[(0,\underline{0},\underline{\alpha},0)^{-1}(\sigma,\underline{\beta},\underline{0},0)(0,\underline{0},\underline{\alpha},0)\right] \\ &= \chi_{v,\underline{u}}(\sigma - \underline{\alpha}.\underline{\beta}\ ,\underline{\beta},\underline{0},0) \qquad \text{from (9)} \\ &= \exp(iv(\sigma - \underline{\alpha}.\underline{\beta}) + i\underline{u}.\underline{\beta}) \\ &= \chi_{v,-v\underline{\alpha}+\underline{u}}\ .\end{aligned}$$

The orbits of $\hat{Y}$ under the action of X are therefore

$$\left.\begin{aligned} O_v &= \{\chi_{v,\underline{u}} : \underline{u} \in \mathbb{R}^n\} \quad (v \neq 0,\ v \in \mathbb{R})\ , \\ O_{\underline{u}} &= \{\chi_{0,\underline{u}}\} \quad (\underline{u} \in \mathbb{R}^n)\ . \end{aligned}\right\} \quad (10)$$

Firstly, consider the representations of N arising from orbits of the form O_v.

As a representative point of the orbit O_v, choose $\chi_{v,\underline{0}}$. The little group of $\chi_{v,\underline{0}}$ is just the identity $(0,\underline{0},\underline{0},0)$, whose only irreducible ordinary representation is, of course, the one-dimensional identity representation. Hence the corresponding representation of N is given by inducing the representation $\chi_{v,\underline{0}} : (\sigma,\underline{\beta},\underline{0},0) \longrightarrow \exp iv\sigma$ of Y $(= Y \circledS (0,\underline{0},\underline{0},0))$ to N.

The representation $\chi_{v,\underline{0}}(Y)\uparrow N$ may be defined on square-integrable functions from N/Y ($\approx X \approx \mathbb{R}^n$ qua N-spaces) to $\mathbb{C}$. An explicit expression can be given for it once coset representatives of N/Y in N have been chosen (see § 2.1). Now

$(\sigma,\underline{\beta},\underline{\alpha},0)$ and $(\sigma',\underline{\beta}',\underline{\alpha}',0)$ belong to the same coset of Y in N if and only if $\underline{\alpha} = \underline{\alpha}'$; denote the coset to which $(\sigma,\underline{\beta},\underline{\alpha},0)$ belongs by $\underline{\alpha}$ $(\in \mathbb{R}^n)$. Then the natural action of N on the coset $\underline{\alpha} \in N/Y$ is given by

$$(\sigma',\underline{\beta}',\underline{\alpha}',0)\underline{\alpha} = \underline{\alpha} + \underline{\alpha}' \ .$$

The coset representatives $\Lambda : N/Y \longrightarrow N$ must satisfy

$$\Lambda(\underline{\alpha})\underline{0} = \underline{\alpha} \ .$$

Hence a possible choice for Λ is $\Lambda(\underline{\alpha}) = (0,\underline{0},\underline{\alpha},0)$ $(\in N)$.

It follows, from 2-(2), that the representation of N induced by $\chi_{v,\underline{0}}$ is

$$\begin{aligned}
&\left[U_N^v(\sigma,\underline{\beta},\underline{\alpha},0)\psi\right](\underline{x}) \\
&\quad = \chi_{v,\underline{0}}\left[\Lambda(\underline{x})^{-1}(\sigma,\underline{\beta},\underline{\alpha},0)\Lambda((\sigma,\underline{\beta},\underline{\alpha},0)^{-1}\underline{x})\right]\psi((\sigma,\underline{\beta},\underline{\alpha},0)^{-1}\underline{x}) \\
&\quad = \chi_{v,\underline{0}}(\sigma - \underline{x}.\underline{\beta}\ ,\underline{\beta},\underline{0},0)\psi(\underline{x} - \underline{\alpha}) \\
&\quad = \exp iv(\sigma - \underline{x}.\underline{\beta})\ \psi(\underline{x} - \underline{\alpha}) \ , \qquad (11)
\end{aligned}$$

where $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$.

Secondly, consider the representations of N arising from orbits of $\hat{Y}$ of the form $O_{\underline{u}}$, (10).

$O_{\underline{u}}$ is an orbit consisting of the single point $\chi_{0,\underline{u}}$; the little group of $\chi_{0,\underline{u}}$ is the whole of the subgroup X. As $X \approx \mathbb{R}^n$, each of its irreducible representations is of the form

$$\chi_{\underline{t}} : (0,\underline{0},\underline{\alpha},0) \longrightarrow \exp i\underline{t}.\underline{\alpha} ,$$

where $\underline{t} \in \mathbb{R}^n$.

Since the isotropy group of $\chi_{0,\underline{u}}$ is the whole of N, the representation of N obtained from the representations $\chi_{0,\underline{u}}$ of Y and $\chi_{\underline{t}}$ of X is just their product :

$$U_N^{\underline{u},\underline{t}}(\sigma,\underline{\beta},\underline{\alpha},0) = \exp i(\underline{u}.\underline{\beta} + \underline{t}.\underline{\alpha}) . \qquad (12)$$

Hence, up to unitary equivalence, each irreducible representation of N is of the form

$$\left[U_N^{v}(\sigma,\underline{\beta},\underline{\alpha},0)\psi\right](\underline{x}) = \exp iv(\sigma - \underline{x}.\underline{\beta})\ \psi(\underline{x} - \underline{\alpha}) , \qquad (11)$$

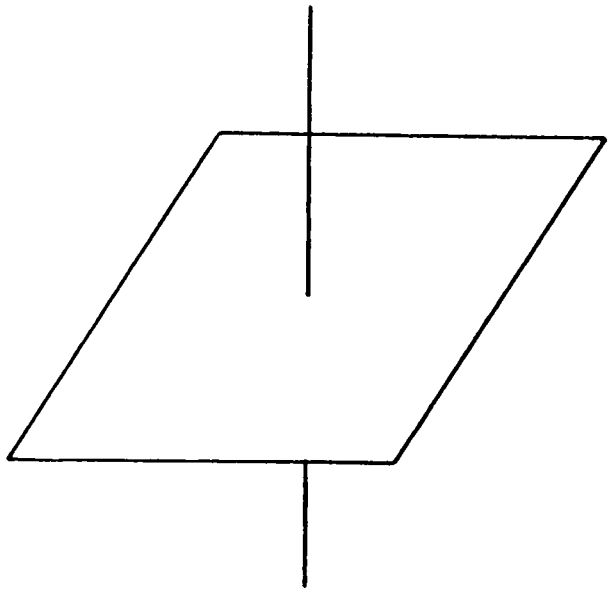
where $v\ (\neq 0) \in \mathbb{R}$, and $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$;

or $$U_N^{\underline{u},\underline{t}}(\sigma,\underline{\beta},\underline{\alpha},0) = \exp i(\underline{u}.\underline{\beta} + \underline{t}.\underline{\alpha}) , \qquad (12)$$

where $\underline{u}, \underline{t} \in \mathbb{R}^n$.

When n = 1, the set $\hat{N}$ may be represented pictorially as a plane (containing points corresponding to $U_N^{u,t}$), together with a line (containing points corresponding to U_N^{v}, $v \neq 0$) perpendicular to the plane; the point of intersection of the plane and the line is regarded as a point of the plane only, since there is no representation of the form U_N^{v} for v = 0.

The space $\hat{\Pi}$ for n = 1



3.5 The irreducible representations of the group $\tilde{G}$

As the next step towards obtaining all the irreducible representations of $\tilde{G} = N \circledS \tilde{M}$, (8), consider the action of $\tilde{M}$ on $\hat{N}$ (see §2.5).

The action of $(0,\underline{0},\underline{0},\tau) \in \tilde{M}$ on an irreducible representation U_N of N is, by definition 2-(10),

$$\left[(0,\underline{0},\underline{0},\tau)U_N\right](\sigma,\underline{\beta},\underline{\alpha},0) = U_N\left[(0,\underline{0},\underline{0},\tau)^{-1}(\sigma,\underline{\beta},\underline{\alpha},0)(0,\underline{0},\underline{0},\tau)\right]$$

$$= U_N\Big(\sigma + \sum_{j=1}^{n}\left[\frac{\alpha_j^2 - \beta_j^2}{2}\cos\omega_j\tau\sin\omega_j\tau - \beta_j\alpha_j\sin^2\omega_j\tau\right],\ldots,$$

$$\beta_j\cos\omega_j\tau + \alpha_j\sin\omega_j\tau,\ldots,\ -\beta_j\sin\omega_j\tau + \alpha_j\cos\omega_j\tau,\ldots,0\Big)$$

Therefore , in particular, for U_N^V, from (11),

$$\left(\left[(0,\underline{0},\underline{0},\tau)U_N^v\right](\sigma,\underline{\beta},\underline{\alpha},0)\psi\right)(\underline{x}) \qquad (13)$$

$$= \exp iv\Big(\sigma + \sum_{j=1}^{n}\Big[\frac{\alpha_j^2 - \beta_j^2}{2}\cos\omega_j\tau\,\sin\omega_j\tau - \beta_j\alpha_j\sin^2\omega_j\tau$$

$$- x_j(\beta_j\cos\omega_j\tau + \alpha_j\sin\omega_j\tau)\Big]\Big)\psi(\ldots,\ x_j + \beta_j\sin\omega_j\tau - \alpha_j\cos\omega_j\tau,\ldots).$$

$(0,\underline{0},\underline{0},\tau)U_N^v$ is an irreducible representation of N. Restricted to the subgroup $\{(\sigma,\underline{0},\underline{0},0)\}$, it is just the phase $\exp iv\sigma$. It follows that $(0,\underline{0},\underline{0},\tau)U_N^v$ must be unitarily equivalent to U_N^v, since U_N^v is the only irreducible representation of N which has the required form on restriction to the subgroup $\{(\sigma,\underline{0},\underline{0},0)\}$.

Hence there exists a unitary operator $W(\tau)$, dependent on τ, such that

$$(0,\underline{0},\underline{0},\tau)U_N^v = W(\tau)^{-1}U_N^v\,W(\tau)\ ,$$

for each $(0,\underline{0},\underline{0},\tau)\in\tilde{M}$.

Each orbit of $\{[U_N^v] : v\ (\neq 0)\in\mathbb{R}\}$ $(\subset\hat{N})$ under the action of $\tilde{M}$ thus consists of a single point U_N^v; the little group of U_N^v is the whole of $\tilde{M}$, and so the isotropy group of U_N^v is $N\circledS\tilde{M} = \tilde{G}$.

In order to obtain all the irreducible representations of $\tilde{G}$ explicitly, the operator $W(\tau)$ must be found. Now N and $\tilde{M}$ are connected Lie groups. Also $\tilde{M}$ has no non-trivial multipliers, since it is isomorphic to $\mathbb{R}$ (considered as an additive group), which has no non-trivial multipliers (§2.4(d)).

The method for finding $W(\tau)$ described in §2.6 is therefore applicable here.

Each element $(\sigma,\underline{\beta},\underline{\alpha},0) \in N$ can be expressed as a product of elements of one-parameter subgroups, using (7), by

$$(\sigma,\underline{\beta},\underline{\alpha},0) = (\sigma,\underline{0},\underline{0},0)\prod_{j=1}^{n}(0,\beta_j\underline{e}_j,\underline{0},0)\prod_{j=1}^{n}(0,\underline{0},\alpha_j\underline{e}_j,0), \quad (14)$$

where $\underline{e}_j$ $(\in \mathbb{R}^n)$ is a unit vector with 1 in the jth position and zeros elsewhere.

By Stone's theorem, for each $j = 1,2,\ldots,n$,

$$\left[(0,\underline{0},\underline{0},\tau)U_N^V\right](0,\beta_j\underline{e}_j,\underline{0},0) = \exp \beta_j Z_{\beta_j}(\tau),$$

where (cf. 2-(14)), using (13),

$$\begin{aligned} Z_{\beta_j}(\tau)\psi(\underline{x}) &= \frac{\partial}{\partial\beta_j}\left(\left[(0,\underline{0},\underline{0},\tau)U_N^V\right](0,\beta_j\underline{e}_j,\underline{0},0)\psi\right)(\underline{x})\Big|_{\beta_j = 0} \\ &= \frac{\partial}{\partial\beta_j}\Big[\exp iv\Big(-\frac{\beta_j^2}{2}\cos\omega_j\tau\sin\omega_j\tau - x_j\beta_j\cos\omega_j\tau, \\ &\qquad \psi(\ldots,x_{j-1},\ x_j + \beta_j\sin\omega_j\tau, x_{j+1},\ldots)\Big]\Big|_{\beta_j = 0} \\ &= \Big(-ivx_j\cos\omega_j\tau + \sin\omega_j\tau\,\frac{\partial}{\partial x_j}\Big)\psi(\underline{x}) \ . \end{aligned}$$

In the same way, and with a similar notation,

$$Z_{\alpha_j}(\tau) = \Big(-ivx_j\sin\omega_j\tau - \cos\omega_j\tau\,\frac{\partial}{\partial x_j}\Big),$$

$$Z_\sigma(\tau) = iv \ .$$

Hence, from §2.6, a possible form of the operator $W(\tau)$ is exp tJ, where J is a skew-adjoint operator satisfying

$$\left.\frac{dZ_\mu(\tau)}{d\tau}\right|_{\tau = 0} = [Z_\mu(0), J] \qquad \text{(cf. 2-(22)} \ ,$$

for all $\mu \in \{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Therefore :

$$0 = [iv, J] \ ,$$

$$\left.\begin{aligned} \omega_j \frac{\partial}{\partial x_j} &= [-ivx_j, J] \ , \\ -iv\omega_j x_j &= \left[-\frac{\partial}{\partial x_j}, J\right] \ , \end{aligned}\right\} \quad \text{for each } j = 1,2,\ldots,n.$$

A solution of these is

$$\tilde{J} = \frac{i}{2}\sum_{k=1}^{n} \omega_k\left(-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + vx_k^2\right) \ .$$

With this $\tilde{J}$, and any imaginary constant η, $\tilde{W}(\tau) = \exp \tau(\tilde{J} + \eta I)$ is an ordinary representation of $\tilde{M}$ satisfying

$$(0,\underline{0},\underline{0},\tau)U_N^v = \tilde{W}(\tau)^{-1} U_N^v \tilde{W}(\tau) \ .$$

It follows from Mackey's theory (§2.5) that $U_N^v \tilde{W} : (\sigma,\underline{\beta},\underline{\alpha},\tau) \longrightarrow U_N^v(\sigma,\underline{\beta},\underline{\alpha},0)W(\tau)$ is an irreducible representation of $N \circledS \tilde{M} = \tilde{G}$.

Now every irreducible representation of $\tilde{M}$ ($\approx \mathbb{R}$) is of the form

$$(0,\underline{0},\underline{0},\tau) \longrightarrow \exp il\tau \text{ , where } l \in \mathbf{R} \text{ .}$$

Hence, from Mackey's theory, every irreducible representation of $\tilde{G}$ obtained from U_N^V is unitarily equivalent to one of the form

$$(U_{\tilde{G}}^{V,h}(\sigma,\underline{\beta},\underline{\alpha},\tau)\psi)(\underline{x}) = U_N^V(\sigma,\underline{\beta},\underline{\alpha},0)\tilde{W}(\tau)\exp il\tau\,\psi(\underline{x}) \qquad (15)$$

$$= \exp iv\sigma \prod_{j=1}^{n} \exp(-iv\beta_j x_j) \prod_{j=1}^{n} \exp(-\alpha_j \frac{\partial}{\partial x_j})$$

$$\exp i\tau\left[h + \tfrac{1}{2}\sum_{j=1}^{n} \omega_j(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + v x_j^2)\right]\psi(\underline{x}),$$

where $h = l - i\eta$ can take any value in $\mathbf{R}$.

In the irreducible representation $U_N^{\underline{u},\underline{t}}$ of N given by (12) :

$$U_N^{\underline{u},\underline{t}}(\sigma,\underline{\beta},\underline{\alpha},0) = \exp i(\underline{u}.\underline{\beta} + \underline{t}.\underline{\alpha}) \text{ ,}$$

the generator of the subgroup $\{(\sigma,\underline{0},\underline{0},0) : \sigma \in \mathbf{R}\}$ is represented by the zero operator, and, from the general theory, it is easily seen that this is also true for the irreducible representations of $\tilde{G}$ obtained from $U_N^{\underline{u},\underline{t}}$ by Mackey's method.

Among the commutation relations of $\mathcal{L}$ are, from (1),

$$[A_j, B_k] = \delta_{jk}E \qquad (1 \leq j,k \leq n) \text{ .}$$

Hence, when E is represented by the zero operator, the representatives of A_j and B_k must commute, for all j,k = 1,2,..

..,n; this implies that the representatives of all the momentum and position observables commute. This situation corresponds to the classical case. In subsequent chapters, it is the quantum mechanical case which is of interest, and so the representations arising from $U_N^{\underline{u},\underline{t}}$ need not be considered here.

3.6 The irreducible representations of the group G

The quantum mechanical irreducible ordinary representations of G may be obtained from those of its simply connected covering group $\tilde{G}$ in the following way.

From (6),

$$G \approx \frac{\tilde{G}}{Z} ,$$

where $Z = \{(0,\underline{0},\underline{0},2\pi\lambda s) : s \in \mathbb{Z}\}$.

Hence $U_{\tilde{G}}^{v,h}$ defines a single-valued representation of G if $U_{\tilde{G}}^{v,h}$ has the same value on each element of the subgroup Z.

Now, for $s \in \mathbb{Z}$,

$$\left[U_{\tilde{G}}^{v,h}(0,\underline{0},\underline{0},2\pi\lambda s)\psi\right](\underline{x})$$

$$= \exp i2\pi\lambda s\left[h + \tfrac{1}{2}\sum_{j=1}^{n}\omega_j\left(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + vx_j^2\right)\right]\psi(\underline{x}) .$$

As a basis for the representation space $L^2(\mathbb{R}^n, \mathbb{C})$ of $U_{\tilde{G}}^{v,h}$,

take the set $\{\psi_{\underline{m}} : \underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n\}$, where

$$\psi_{\underline{m}}(\underline{x}) = \prod_{j=1}^{n} u_{m_j}(|v|^{\frac{1}{2}} x_j) \ , \tag{16}$$

with u_{m_j} a Hermite function of order m_j.

Then, with $\operatorname{sgn} v = \begin{cases} +1 \text{ if } v \text{ is positive} \\ -1 \text{ if } v \text{ is negative} \end{cases}$,

$$\left[U_{\tilde{G}}^{v,h}(0,\underline{0},\underline{0},2\pi\lambda s)\psi_{\underline{m}}\right](\underline{x})$$

$$= \exp i2\pi\lambda s\left[h + \operatorname{sgn} v \sum_{j=1}^{n} \omega_j(m_j + \tfrac{1}{2})\right]\psi_{\underline{m}}(\underline{x})$$

$$= \exp i2\pi\lambda s\left[h + \frac{\operatorname{sgn} v}{2} \sum_{j=1}^{n} \omega_j\right]\psi_{\underline{m}}(\underline{x})$$

since, by the definition, (3), of λ, $\lambda\omega_j$ is integral for each j.

Hence $U_{\tilde{G}}^{v,h}$ defines a single-valued representation of G only if

$$h = \frac{r}{\lambda} - \frac{\operatorname{sgn} v}{2} \sum_{j=1}^{n} \omega_j \ , \text{ for some } r \in \mathbb{Z} .$$

Therefore every quantum mechanical irreducible ordinary representation of G is unitarily equivalent to one of the form

$$\left[U_G^{v,r}(\sigma,\underline{\beta},\underline{\alpha},\tau)\psi\right](\underline{x}) = \exp iv\sigma \prod_{j=1}^{n} \exp(-iv\beta_j x_j) \prod_{j=1}^{n} \exp(-\alpha_j \frac{\partial}{\partial x_j})$$

$$\exp i\tau\left[\frac{r}{\lambda} + \tfrac{1}{2} \sum_{j=1}^{n} \omega_j(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + vx_j^2 - \operatorname{sgn} v)\right] \psi(\underline{x}) \ , \tag{17}$$

where $r \in \mathbb{Z}$.

CHAPTER 4

VAN HOVE'S GROUP Γ, SOME SUBGROUPS, AND TWO REPRESENTATIONS R AND R^v OF Γ

In a paper concerned mainly with a possible method of quantisation, van Hove [38] defines a group Γ of transformations which leave a certain differential form invariant. Among the subgroups of Γ is a group L which van Hove shows is associated with systems having quadratic Hamiltonians. In this chapter, a description of the group Γ and some of its subgroups is given; in particular, it is proved that the oscillator group G is isomorphic to a subgroup of L.

Van Hove defines a representation R of Γ, which he shows is unitarily equivalent to a direct integral of irreducible representations R^v of Γ. It is proved that the restriction of R^v to L is actually an induced representation.

4.1 Definition of van Hove's group Γ

Define a differential form $\bar{\theta}$ on the vector space of real $(2n + 1)$-tuples : $\{(s,q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n) \in \mathbb{R}^{2n+1}\}$ by

$$\bar{\theta} = ds - \sum_{j=1}^{n} p_j dq_j$$

$$= ds - \underline{p}.d\underline{q} \ ,$$

where $\underline{q} = (q_1,q_2,\ldots,q_n)$, $\underline{p} = (p_1,p_2,\ldots,p_n)$.

Let Γ be the family of invertible C^∞ transformations of this space, $\{(s,\underline{q},\underline{p})\}$, into itself which leave the differential form $\bar{\theta}$ invariant. It can be shown that Γ is a group, and that every element of Γ is a transformation of the type ([38] p.17) :

$$\left.\begin{aligned} \underline{q}' &= \underline{q}'(\underline{q},\underline{p}) \ , \\ \underline{p}' &= \underline{p}'(\underline{q},\underline{p}) \ , \\ s' &= s + \pi(\underline{q},\underline{p}) \ , \end{aligned}\right\} \quad (1)$$

where $\underline{p}'.d\underline{q}' - \underline{p}.d\underline{q} = d\pi$.

In classical mechanics, the different possible motions of a dynamical system with Lagrangian L can be described by the extremals of the integral $\int L dt$, where t denotes time. The Hamiltonian of the system

$$H = \sum_{j=1}^{n} p_j \frac{dq_j}{dt} - L,$$

where q_j, p_j $(j = 1,2,\ldots,n)$ are pairs of conjugate canonical variables. Hence each integral $\int L dt$ has an associated differential form which may be written as

$$\theta = \sum_{j=1}^{n} p_j dq_j - H dt \ .$$

Using θ, a group $\mathcal{B}$ can be defined which is intrinsically associated with the particular system under consideration :

$\mathcal{B}$ is the group of all invertible C^∞ transformations of the vector space, $\{(t,\underline{q},\underline{p})\}$, of real $(2n+1)$-tuples : $\{(t,q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n)\in \mathbb{R}^{2n+1}\}$, which leave the differential form Θ invariant.

In classical mechanics, it is usually the quotient of $\mathcal{B}$ by its centre which is studied, rather than $\mathcal{B}$ itself; this quotient group is essentially the group of canonical transformations.

In general, for each point of the space $\{(t,\underline{q},\underline{p})\}$, there is a local change of variables which transforms Θ into $\bar{\Theta}$. In the special case when there is a single change of variables which transforms Θ into $\bar{\Theta}$ for the whole space $\{(t,\underline{q},\underline{p})\}$, the group $\mathcal{B}$ can be taken to be the group Γ defined above.

Van Hove shows (p.24) that there is a 1 - 1 correspondence between the infinitesimal generators of Γ and the first integrals of the equations of motion, in which commutator brackets and Poisson brackets correspond to one another. This property of Γ is important in quantisation.

Since the transformations which leave $\bar{\Theta}$ invariant depend on arbitrary C^∞ functions, Γ is an infinite continuous group. The problem of giving such groups a suitable topological structure is very difficult. However, the topological structure of finitely-generated continuous groups is known : they are all compact or locally compact [6].

4.2 Some subgroups of Γ

The following finitely-generated subgroups of Γ are of special importance. (In each case, the topology on a subgroup with m generators is taken to be that inherited from the usual topology on $\mathbb{R}^m$.)

The centre C

This is the subgroup given by transformations of the form :

$$\left.\begin{aligned} s' &= s + \sigma \,, \\ \underline{q}' &= \underline{q} \,, \\ \underline{p}' &= \underline{p} \,, \end{aligned}\right\} \quad (2)$$

where $\sigma \in \mathbb{R}$.

The subgroup N

A simple transformation of the space $\{(\underline{q},\underline{p})\}$ is that of translation :

$$\left.\begin{aligned} \underline{q}' &= \underline{q} + \underline{\beta} \,, \\ \underline{p}' &= \underline{p} + \underline{\alpha} \,, \end{aligned}\right\} \quad (3a)$$

where $\underline{\beta}$, $\underline{\alpha} \in \mathbb{R}^n$ (retaining van Hove's choice of notation, both here and for the other subgroups he mentions).

This gives an element of Γ in which, from (1),

$$s' = s + \underline{\alpha}.\underline{q} + \sigma , \tag{3b}$$

where $\sigma \in \mathbb{R}$.

This set of transformations is a subgroup of Γ of which a typical element will be written $(\sigma,\underline{\beta},\underline{\alpha},I)$ (for reasons which will soon become apparent). The group law is

$$(\sigma',\underline{\beta}',\underline{\alpha}',I)(\sigma,\underline{\beta},\underline{\alpha},I) = (\sigma'',\underline{\beta}'',\underline{\alpha}'',I) , \tag{4}$$

where $\sigma'' = \sigma' + \sigma + \underline{\alpha}'.\underline{\beta}$,

$$\underline{\beta}'' = \underline{\beta}' + \underline{\beta} ,$$

$$\underline{\alpha}'' = \underline{\alpha}' + \underline{\alpha} .$$

It follows, from 3-(9), that this subgroup is isomorphic (as a topological group) to the subgroup N of the oscillator group G. N can therefore be identified with a subgroup of Γ.

The symplectic group $Sp(2n, \mathbb{R})$

Another simple transformation of the space $\{(\underline{q},\underline{p})\}$ is that of linear transformation :

$$q'_j = \sum_{k=1}^{n} (d_{jk}q_k + c_{jk}p_k) ,$$

$$p'_j = \sum_{k=1}^{n} (b_{jk}q_k + a_{jk}p_k) ,$$

or

$$\left.\begin{aligned} \underline{q}' &= D\underline{q} + C\underline{p} , \\ \underline{p}' &= B\underline{q} + A\underline{p} , \end{aligned}\right\} \tag{5a}$$

with $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix} \in GL(2n, \mathbb{R})$ (where $GL(2n, \mathbb{R})$ is the group of all 2n x 2n real non-singular matrices). This leads to a transformation of $\{(\underline{q},\underline{p})\}$ belonging to Γ only if $\sum_{j=1}^{n} (p'_j dq'_j - p_j dq_j)$ is a perfect differential. This condition implies :

$$\sum_{i=1}^{n} (a_{ij}c_{ik} - c_{ij}a_{ik}) = 0 \ ,$$

$$\sum_{i=1}^{n} (b_{ij}d_{ik} - d_{ij}b_{ik}) = 0 \ ,$$

$$\sum_{i=1}^{n} (a_{ij}d_{ik} - c_{ij}b_{ik}) = \delta_{jk} \ .$$

Equivalently,

$$S^T J S = J \ ,$$

where T denotes transpose and $J = \sum_{j=1}^{n} (E_{j,j+n} - E_{j+n,j})$ (with E_{jk} a 2n x 2n matrix having 1 in the (j,k) position and zeros elsewhere). Hence S belongs to the group $Sp(2n, \mathbb{R})$ of 2n x 2n real symplectic matrices. Conversely, this condition is sufficient to give a corresponding transformation of $\{(s,\underline{q},\underline{p})\}$ belonging to Γ.

When $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$,

$$s' = s + \tfrac{1}{2}A\underline{p}.C\underline{p} + \tfrac{1}{2}B\underline{q}.D\underline{q} + B\underline{q}.C\underline{p} + \sigma \,, \qquad (5b)$$

where $\sigma \in \mathbb{R}$. This set of transformations of $\{(s,\underline{q},\underline{p})\}$ is a subgroup of Γ whose composition law is :

$$\sigma'' = \sigma' + \sigma \,,$$

$$S'' = S'S \,.$$

Hence this subgroup is isomorphic to the direct product of C and Sp(2n, $\mathbb{R}$). In particular, Sp(2n, $\mathbb{R}$) can be identified with the subgroup of Γ to which it is isomorphic.

The subgroup L

The subgroups N and Sp(2n, $\mathbb{R}$) may be combined by considering transformations of the space $\{(\underline{q},\underline{p})\}$ of the form

$$\left.\begin{aligned} \underline{q}' &= D\underline{q} + C\underline{p} + \underline{\beta} \,, \\ \underline{p}' &= B\underline{q} + A\underline{p} + \underline{\alpha} \,, \end{aligned}\right\} (6a)$$

where $\underline{\beta}$, $\underline{\alpha} \in \mathbb{R}^n$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in$ Sp(2n, $\mathbb{R}$). (As before, only the subgroup Sp(2n, $\mathbb{R}$) of GL(2n, $\mathbb{R}$) leads to transformations belonging to Γ.)

It follows that

$$s' = s + \tfrac{1}{2}A\underline{p}.C\underline{p} + \tfrac{1}{2}B\underline{q}.D\underline{q} + B\underline{q}.C\underline{p} + \underline{\alpha}.(D\underline{q} + C\underline{p}) + \sigma \,, \qquad (6b)$$

where $\sigma \in \mathbb{R}$. (The expression given in van Hove's paper is

incorrect, because the term $\sum_{j,k} \alpha_j c_{jk} p_k$ has been omitted.)

This set of transformations, L, is a subgroup of Γ of which a typical element may be written $(\sigma,\underline{\beta},\underline{\alpha},S)$. The group law is

$$(\sigma',\underline{\beta}',\underline{\alpha}',S')(\sigma,\underline{\beta},\underline{\alpha},S) = (\sigma'',\underline{\beta}'',\underline{\alpha}'',S'') \ , \tag{7}$$

where $\sigma'' = \sigma' + \sigma + \frac{1}{2}A'\underline{\alpha}.C'\underline{\alpha} + \frac{1}{2}B'\underline{\beta}.D'\underline{\beta} + B'\underline{\beta}.C'\underline{\alpha}$

$$+ \underline{\alpha}'.(D'\underline{\beta} + C'\underline{\alpha}) \ ,$$

$$\underline{\beta}'' = D'\underline{\beta} + C'\underline{\alpha} + \underline{\beta}' \ ,$$

$$\underline{\alpha}'' = B'\underline{\beta} + A'\underline{\alpha} + \underline{\alpha}' \ ,$$

$$S'' = S'S \ .$$

An arbitrary element of L may be expressed in the form

$$(\sigma,\underline{\beta},\underline{\alpha},S) = (\sigma,\underline{\beta},\underline{\alpha},I)(0,\underline{0},\underline{0},S) \ . \tag{8}$$

It is then easily verified, using (7), that

$$L = N \circledS Sp(2n, \mathbb{R}) \ , \tag{9}$$

where $N = \{(\sigma,\underline{\beta},\underline{\alpha},I) : \sigma \in \mathbb{R}, \underline{\beta},\underline{\alpha} \in \mathbb{R}^n\}$

is a normal subgroup of L, and

$$Sp(2n, \mathbb{R}) = \{(0,\underline{0},\underline{0},S) \in L\} \ .$$

(The previously mentioned identifications of $N(\subset G)$ and of $Sp(2n, \mathbb{R})$ with the corresponding isomorphic subgroups of L

have been used here.)

The subgroup G

The isomorphism of the subgroup N ($\subset$ G) with a subgroup of L raises the possibility that G (or $\tilde{G}$) is also isomorphic to a subgroup of L. The heuristic construction of G suggests (cf. 3-(2)) consideration of the subset of L given by $\{(\sigma,\underline{\beta},\underline{\alpha},S(\tau)\}$, where $S(\tau)$ is a symplectic matrix with

$$\left.\begin{aligned} a_{jk} &= \delta_{jk}\cos\omega_j\tau\,, & b_{jk} &= \delta_{jk}\sin\omega_j\tau\,, \\ c_{jk} &= -\delta_{jk}\sin\omega_j\tau\,, & d_{jk} &= \delta_{jk}\cos\omega_j\tau\,. \end{aligned}\right\} \quad (10)$$

Suppose each pair of ω_j's is rationally related. To obtain all possible matrices of the form $S(\tau)$, it is sufficient to take $\tau\in(-\pi\lambda,\pi\lambda]$, where, as in 3-(3), λ is the smallest positive number such that $\lambda\omega_j$ is integral for each j. If τ and τ' are distinct elements of $(-\pi\lambda,\pi\lambda]$, then $S(\tau)\neq S(\tau')$. Moreover, $S(\tau')S(\tau) = S(\tau'')$, where τ'' is uniquely determined by $\tau''\in(-\pi\lambda,\pi\lambda]$ and $\tau''\equiv\tau'+\tau(\mathrm{mod}\ 2\pi\lambda)$. Hence the subgroup M (of G) $= \{(0,\underline{0},\underline{0},\tau) : \tau\in(-\pi\lambda,\pi\lambda]\}$ is isomorphic to the subgroup (of L) $\{(0,\underline{0},\underline{0},S(\tau)) : \tau\in(-\pi\lambda,\pi\lambda]\}$ under the map $\tau \longleftrightarrow S(\tau)$. It then follows that $\{(\sigma,\underline{\beta},\underline{\alpha},S(\tau)) : \sigma\in\mathbb{R}, \underline{\beta},\underline{\alpha}\in\mathbb{R}^n, \tau\in(-\pi\lambda,\pi\lambda]\}$ is a subgroup of L and is isomorphic to G.

If any two of the ω_j's are not rationally related, then $\{(\sigma,\underline{\beta},\underline{\alpha},S(\tau)) : \sigma\in\mathbb{R}, \underline{\beta},\underline{\alpha}\in\mathbb{R}^n, \tau\in\mathbb{R}\}$ is a subgroup of L isomorphic to the group G (= $\tilde{G}$, in this case).

Hence G is isomorphic to a subgroup of L, and it will often be identified with this subgroup.

The unitary group U(n)

Suppose $A + iB = (A_{jk} + iB_{jk})$, where $A_{jk}, B_{jk} \in \mathbb{R}$, is an n x n unitary matrix. When it acts on the space $\mathbb{C}^n = \{(q_1 + ip_1, q_2 + ip_2, \ldots, q_n + ip_n) : q_j, p_j \in \mathbb{R}\}$ in the usual way, the coordinates q_j, p_j are transformed according to

$$\left.\begin{aligned} q'_j &= \sum_{k=1}^{n} (A_{jk}q_k - B_{jk}p_k) \ , \\ p'_j &= \sum_{k=1}^{n} (B_{jk}q_k + A_{jk}p_k) \end{aligned}\right\} \quad \text{for each } j = 1,2,\ldots,n.$$

Comparison with the definition, (7), of L suggests a map of U(n) to GL(2n, $\mathbb{R}$) given by

$$A + iB \longrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix} .$$

When A + iB is unitary, $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ is symplectic, and conversely. It is easily shown that this map is, in fact, an isomorphism of U(n) with the group $\left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(2n, \mathbb{R})\right\}$.

Therefore U(n) too may be identified with a subgroup of L, under the map

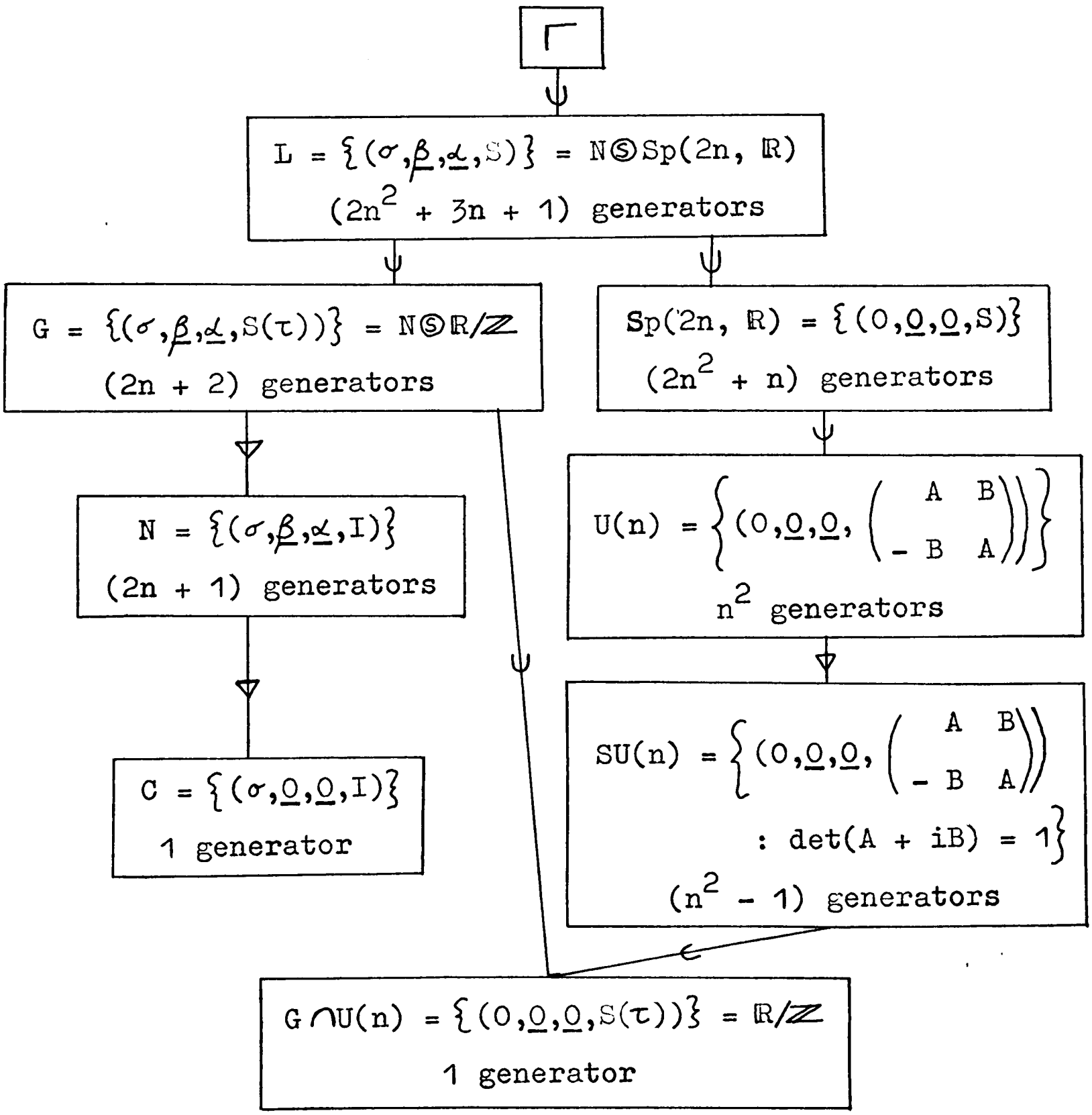
$$A + iB \longleftrightarrow (0, \underline{0}, \underline{0}, \begin{pmatrix} A & B \\ -B & A \end{pmatrix}) . \qquad (11)$$

The unimodular unitary group SU(n)

The subgroup SU(n) of U(n) consisting of matrices with determinant one may also be identified with a subgroup of L :

$$SU(n) \approx \left\{ (0, \underline{0}, \underline{0}, \begin{pmatrix} A & B \\ -B & A \end{pmatrix}) \in L : \det(A + iB) = 1 \right\} .$$

Diagrammatic summary (12)



4.3 Definition of the representations R and R^v of Γ

Van Hove introduces two types of representation of Γ :

(i) To simplify the notation, let z denote a point of the space $\{(s,\underline{q},\underline{p})\}$ and dz a volume element $dsdq_1\ldots dq_n dp_1\ldots dp_n$. The fact that transformations belonging to Γ leave the volume element dz invariant is used to construct a representation of Γ. The complex-valued, measurable, square-integrable functions on $\{(s,\underline{q},\underline{p})\}$ form a separable Hilbert space $\mathcal{H}_{2n+1}$, with scalar product defined by

$$(\Phi,\Psi) = \int \overline{\Phi(z)}\Psi(z)dz \ ,$$

where the integral is taken over the whole space $\{(s,\underline{q},\underline{p})\}$, and $\bar{}$ denotes complex conjugation.

Define

$$(R(\gamma)\Phi)(z) = \Phi(\gamma^{-1}z) \ ,$$

where $\gamma\in\Gamma$, $\Phi\in\mathcal{H}_{2n+1}$, and $\gamma^{-1}z$ denotes the image of z under γ^{-1}.

Since $d(\gamma z) = dz$, R is a faithful, unitary representation of Γ on the space $\mathcal{H}_{2n+1}$.

(ii) Let w denote a point of the space $\{(\underline{q},\underline{p})\}$, and dw a volume element $dq_1\ldots dq_n dp_1\ldots dp_n$. For convenience, write the transformation $\gamma\in\Gamma$ in the form

$$\left.\begin{aligned} s' &= s + \sigma_\gamma(w) \ , \\ w' &= \gamma w \ . \end{aligned}\right\} \quad (13)$$

The volume element dw is invariant under transformations of Γ : $d(\gamma w) = dw$, for all $\gamma \in \Gamma$. Let $\mathcal{H}_{2n}$ be the separable Hilbert space of complex-valued, measurable, square-integrable functions on $\{(\underline{q},\underline{p})\}$, with scalar product defined by

$$(\phi, \zeta) = \int \overline{\phi(w)}\zeta(w)dw \ ,$$

where the integral is taken over the whole space $\{(\underline{q},\underline{p})\}$.

Define

$$(R^{v}(\gamma)\phi)(w) = \exp\left[iv\pi_{\gamma}(\gamma^{-1}w)\right]\phi(\gamma^{-1}w) \ , \qquad (14)$$

where $\gamma \in \Gamma$, $v \in R$ and $\phi \in \mathcal{H}_{2n}$.

Since $d(\gamma w) = dw$, $R^{v}(\gamma)$ is a unitary operator.

Suppose the transformation $\gamma_2 \in \Gamma$ is followed by the transformation $\gamma_1 \in \Gamma$. If

$$\left.\begin{aligned} s' &= s + \pi_{\gamma_2}(w) \\ w' &= \gamma_2 w \end{aligned}\right\} \text{ and } \left.\begin{aligned} s'' &= s' + \pi_{\gamma_1}(w') \\ w'' &= \gamma_1 w' \end{aligned}\right\} ,$$

then $\quad s'' = s + \pi_{\gamma_2}(w) + \pi_{\gamma_1}(\gamma_2 w)$;

hence, since the product transformation $\gamma_1\gamma_2$ satisfies

$$s'' = s + \pi_{\gamma_1\gamma_2}(w) \ ,$$

it follows that

$$\pi_{\gamma_1\gamma_2}(w) = \pi_{\gamma_2}(w) + \pi_{\gamma_1}(\gamma_2 w) \ . \qquad (15)$$

Therefore

$$(R^v(\gamma_1)[R^v(\gamma_2)\phi])(w) = \exp iv\pi_{\gamma_1}(\gamma_1^{-1}w)\,[R^v(\gamma_2)\phi](\gamma_1^{-1}w)$$

$$= \exp iv\pi_{\gamma_1}(\gamma_1^{-1}w)\exp iv\pi_{\gamma_2}(\gamma_2^{-1}\gamma_1^{-1}w)\,\phi(\gamma_2^{-1}\gamma_1^{-1}w)$$

$$= \exp iv[\pi_{\gamma_2}(\gamma_2^{-1}\gamma_1^{-1}w) + \pi_{\gamma_1}(\gamma_2\gamma_2^{-1}\gamma_1^{-1}w)]\,\phi(\gamma_2^{-1}\gamma_1^{-1}w)$$

$$= \exp iv\pi_{\gamma_1\gamma_2}((\gamma_1\gamma_2)^{-1}w)\,\phi((\gamma_1\gamma_2)^{-1}w)$$

(using (15), with w replaced by $(\gamma_1\gamma_2)^{-1}w$)

$$= (R^v(\gamma_1\gamma_2)\phi)(w) .$$

Hence R^v is a faithful, unitary representation of Γ, when $v \neq 0$, and a faithful, unitary representation of Γ/C when $v = 0$ (where C is the centre of Γ, (2)).

The reason for introducing the representations R^v is that the representation R is unitarily equivalent to a direct integral of them, as follows.

For each $\Phi \in \mathcal{h}_{2n+1}$, $v \in \mathbb{R}$, $w \in \{(\underline{q},\underline{p})\}$, define

$$(F\Phi)(v,w) = \Phi'(v,w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp ivs_z\, \Phi(z)ds_z$$

(with $z = (s_z,w)$). F is a Fourier transform of $\mathcal{h}_{2n+1}$ onto a function space $L^2(\mathbb{R}^{2n+1}, \mathbb{C})$, and so, by Plancherel's theorem [41], F is unitary.

Since $\Phi' \in L^2(R^{2n+1}, \mathbb{C})$, it follows, by Fubini's theorem [13], that, for almost all values of v, $\Phi'(v,w)$ is a measurable, square-integrable function of w on the space $\{(\underline{q},\underline{p})\}$, and hence defines an element of a space $L^2_v(R^{2n}, \mathbb{C})$; this element, the so-called component of Φ' in $L^2_v(R^{2n}, \mathbb{C})$, is denoted by ϕ^v, and Φ' is a direct integral : $\Phi' = \int \phi^v$.

Van Hove shows (p.31) that

$$FR(\gamma)F^{-1}\Phi' = FR(\gamma)F^{-1}\int \phi^v = \int R^v(\gamma)\phi^v \ .$$

4.4 Expression of $R^v \downarrow L$ as an induced representation

The representation R^v of the group Γ can be expressed as

$$(R^v(\gamma)\phi)(w) = m(w,\gamma)\phi(\gamma^{-1}w) \ ,$$

where $$m(w,\gamma) = \exp iv\pi_\gamma(\gamma^{-1}w) \ . \qquad (16)$$

This is similar in form to the explicit expression 2-(2) for an induced representation. However, induced representations are usually defined only for separable locally compact groups; the group Γ is not locally compact, but its subgroup L is locally compact, and separable. It is for this reason that attention is restricted in this section to the subgroup L of Γ.

The space $\{(\underline{q},\underline{p})\}$ is transitive under the action of the subgroup N (of L), which acts as a group of translations on

the space $\{(\underline{q},\underline{p})\}$ (from (3)). Hence, a priori, $\{(\underline{q},\underline{p})\}$ is transitive under the action of L.

Let w_0 be an arbitrary point of the space $\{(\underline{q},\underline{p})\}$, and let H be the closed subgroup of L which leaves w_0 fixed. Then the space $\{(\underline{q},\underline{p})\}$ is L-isomorphic to the space L/H. Since R^v is a representation of L, m satisfies the so-called cocycle condition

$$m(w,\gamma_1\gamma_2) = m(w,\gamma_1)m(\gamma_1^{-1}w,\gamma_2) , \tag{17}$$

for each $w \in \{(\underline{q},\underline{p})\}$, $\gamma_1,\gamma_2 \in L$.

In particular, for $w = w_0$ and $h \in H$,

$$m(w_0,h_1h_2) = m(w_0,h_1)m(w_0,h_2) .$$

Hence, since $m(w_0,h)$ is a complex number of unit modulus, $h \longrightarrow m(w_0,h)$ is a unitary representation of H.

The representation of L induced from this representation of H is, from 2-(2),

$$(\rho(\gamma)\phi)(w) = m(w_0,\Lambda(w)^{-1}\gamma\Lambda(\gamma^{-1}w))\phi(\gamma^{-1}w) ,$$

where $\gamma \in L$, $\phi \in L^2(L/H, \mathbb{C})$, $w \in \{(\underline{q},\underline{p})\}$ and $\Lambda : L/H \longrightarrow L$ satisfies $\Lambda(w)w_0 = w$.

From (17),

$$\begin{aligned} m(w_0,\Lambda(w)^{-1}\gamma\Lambda(\gamma^{-1}w)) &= m(w_0,\Lambda(w)^{-1})m(w,\gamma\Lambda(\gamma^{-1}w)) \\ &= m(w_0,\Lambda(w)^{-1})m(w,\gamma)m(\gamma^{-1}w,\Lambda(\gamma^{-1}w)) . \end{aligned}$$

Since $m(w,\gamma)$ is a scalar, $R^v \downarrow L$ is the induced representation ρ if w_0 and Λ can be chosen such that

$$m(w_0,\Lambda(w)^{-1})m(\gamma^{-1}w,\Lambda(\gamma^{-1}w)) = 1 .$$

From (16), this condition is equivalent to

$$\exp iv\left[\pi_{\Lambda(w)^{-1}}(\Lambda(w)w_0) + \pi_{\Lambda(\gamma^{-1}w)}(w_0)\right] = 1 .$$

If $v = 0$, this is the case for any choice of w_0 and Λ.

If $v \neq 0$, it holds if

$$\pi_{\Lambda(w)}(w_0) = \text{constant} , \text{ for all } w \in \{(\underline{q},\underline{p})\} ,$$

since, from (13), $\pi_{\Lambda(w)}(w_0) + \pi_{\Lambda(w)^{-1}}(\Lambda(w)w_0) = \pi_I(w_0) = 0$.

A simple choice of Λ (satisfying $\Lambda(w)w_0 = w$) is

$$\Lambda(w)\hat{w} = \hat{w} + w - w_0 ,$$

$$\pi_{\Lambda(w)}\hat{w} = \sigma_w + (\underline{p} - \underline{p}_0)\cdot\hat{\underline{q}} ,$$

(using (3)), where $\sigma_w \in \mathbb{R}$, $w = (\underline{q},\underline{p})$, $\hat{w} = (\hat{\underline{q}},\hat{\underline{p}})$ and $w_0 = (\underline{q}_0,\underline{p}_0)$

In particular,

$$\pi_{\Lambda(w)}(w_0) = \sigma_w + (\underline{p} - \underline{p}_0)\cdot\underline{q}_0 .$$

This is constant for all w if σ_w is constant for all w and $\underline{q}_0 = \underline{0}$. For simplicity, take $\sigma_w = 0$ and $\underline{p}_0 = \underline{0}$. Then $w_0 = (\underline{0},\underline{0})$, $H = C \times Sp(2n, \mathbb{R})$, and $m(w_0,h) = \exp iv\pi_h(\underline{0},\underline{0})$ $= \exp iv\sigma$, where $h = (\sigma,\underline{0},\underline{0},S) \in H$ (from (6), (13) and (16)).

This proves the following result.

Theorem

The representation $R^v \downarrow L$ is a representation of L induced from the representation

$$D : (\sigma, \underline{0}, \underline{0}, S) \longrightarrow \exp iv\sigma \qquad (18)$$

of the subgroup $H = C \times Sp(2n, \mathbb{R})$, with choice of coset representatives

$$\Lambda(w)\hat{w} = \hat{w} + w ,$$

$$\pi_{\Lambda(w)}\hat{w} = \underline{p}.\hat{\underline{q}} .$$

Note

Although induced representations are usually defined only for locally compact groups, the explicit expression 2-(2) for an induced representation makes sense even for the non-locally compact group Γ. With this formal idea of an induced representation for a non-locally compact group, the following result holds :

The representation R^v of Γ is (formally) the representation induced from the representation $D : k \longrightarrow \exp iv\pi_k(\underline{0},\underline{0})$ of the subgroup $K = \{k \in \Gamma : k(\underline{0},\underline{0}) = (\underline{0},\underline{0})\}$, with the same choice of coset representatives as in the above theorem.

The proof is similar to that of the above theorem.

Now $K \cap L = C \times Sp(2n, \mathbb{R})$. Hence, from the above two results, in a formal sense,

$$(D(K) \uparrow \Gamma) \downarrow L = (D(K) \downarrow D(K \cap L)) \uparrow L .$$

CHAPTER 5

THE DECOMPOSITION OF THE REPRESENTATION $R^v \downarrow G$ $(v \neq 0)$

When $v = 0$, $R^v \downarrow G$ is given by, from 4-(14),

$$\left[R^v(\sigma,\underline{\beta},\underline{\alpha},S(\tau))\phi\right](w) = \phi((\sigma,\underline{\beta},\underline{\alpha},S(\tau))^{-1}w) \ ,$$

where $(\sigma,\underline{\beta},\underline{\alpha},S(\tau)) \in G$.

Since $(\sigma,\underline{\beta},\underline{\alpha},S(\tau))^{-1}w$ is independent of σ (from 4-(6)), the generator of the subgroup $\{(\sigma,\underline{0},\underline{0},I) : \sigma \in \mathbf{R}\}$ of G is represented by the zero operator in the representation $R^v \downarrow G$. This corresponds to the classical case, and so will be ignored.

In this chapter, the representation $R^v \downarrow G$ $(v \neq 0)$ is decomposed into a sum of irreducible representations. It is shown that

$$R^v \downarrow G \simeq \bigoplus_S d_{\omega,s} U_G^{v,-(\text{sgn } v)s},$$

where $d_{\omega,s}$ is the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1,\omega_2,\ldots,\omega_n)$, $U_G^{v,-(\text{sgn } v)s}$ is an irreducible representation of G (defined by 3-(17)), and the summation may be regarded as over all distinct energy levels of the system.

To obtain the decomposition, $R^v \downarrow G$ is first expressed as an induced representation, using Mackey's subgroup theorem. If G were a compact group, the usual form of the Frobenius reciprocity theorem (2-(6)) could then be used to find the decomposition of this representation. However, G is only

locally compact. There are various extensions of the Frobenius reciprocity theorem, but none of them appear to apply to this particular case, partly because all the quantum mechanical representations of G are infinite-dimensional. Nevertheless, a formal application of the theorem is still of some help, since it suggests a result which can be proved, by other methods, to be correct.

5.1 Expression of $R^V\downarrow G$ as an induced representation

$$R^V\downarrow G = (R^V\downarrow L)\downarrow G = (D(H)\uparrow L)\downarrow G \text{ (from § 4.4)} .$$

Hence $R^V\downarrow G$ is the restriction of an induced representation of L, and so, since G and H are closed subgroups of the separable locally compact group L, Mackey's subgroup theorem (2-(3)) may be used as a first step in splitting $R^V\downarrow G$, provided there exists a Borel set in L which intersects each G, H double coset in exactly one point.

An arbitrary element $(\sigma,\underline{\beta},\underline{\alpha},S) \in L$ can be written as :

$$(\sigma,\underline{\beta},\underline{\alpha},S) = (0,\underline{\beta},\underline{\alpha},I)(\sigma,\underline{0},\underline{0},S) \text{ , using 4-(7),}$$

where $(0,\underline{\beta},\underline{\alpha},I) \in G$ and $(\sigma,\underline{0},\underline{0},S) \in H$. Therefore L can be expressed as a single double coset $G(0,\underline{0},\underline{0},I)H$. Moreover, $\{(0,\underline{0},\underline{0},I)\}$ is, trivially, a Borel set in L which intersects the double coset $G(0,\underline{0},\underline{0},I)H$ exactly once. Hence Mackey's subgroup theorem may be applied :

$$(D(H)\uparrow L)\downarrow G = (D(H)\downarrow(G\cap H))\uparrow G.$$

Therefore,

$$R^{v}\downarrow G = D(G\cap H)\uparrow G \ , \qquad (1)$$

where $G\cap H = \{(\sigma,\underline{0},\underline{0},S(\tau))\}$

and $D : (\sigma,\underline{0},\underline{0},S(\tau)) \longrightarrow \exp iv\sigma$

is an irreducible representation of $G\cap H$.

5.2 Formal decomposition of $R^{v}\downarrow G$

Suppose ρ is an arbitrary irreducible representation of G. A formal application of the Frobenius reciprocity theorem (2-(6)) yields

$$\langle R^{v}\downarrow G, \rho\rangle_{G} = \langle D(G\cap H)\uparrow G, \rho\rangle_{G} = \langle \rho\downarrow(G\cap H), D(G\cap H)\rangle_{G\cap H} . \qquad (2)$$

This suggests that ρ occurs in $R^{v}\downarrow G$ the same number of times that $D(G\cap H)$ occurs in $\rho\downarrow(G\cap H)$.

As explained in §3.5, for a classical irreducible representation ρ_{cl} of G, $\rho_{cl}(\sigma,\underline{\beta},\underline{\alpha},S(\tau))$ is independent of the parameter σ, and so $\rho_{cl}\downarrow(G\cap H)$ cannot contain the representation $D : (\sigma,\underline{0},\underline{0},S(\tau)) \longrightarrow \exp iv\sigma$ of $G\cap H$, when $v \neq 0$. This suggests that no classical irreducible representation ρ_{cl} can occur in $R^{v}\downarrow G$.

From 3-(17), every quantum mechanical irreducible representation of G is unitarily equivalent to one of the form

$$\left[U_G^{u,r}(\sigma,\underline{\beta},\underline{\alpha},S(\tau))\psi\right](\underline{x})$$

$$= \exp iu\sigma \prod_{j=1}^{n} \exp(-iu\beta_j x_j) \prod_{j=1}^{n} \exp(-\alpha_j \frac{\partial}{\partial x_j})$$

$$\exp i\tau\left[\frac{r}{\lambda} + \tfrac{1}{2}\sum_{j=1}^{n} \omega_j(-\frac{1}{u}\frac{\partial^2}{\partial x_j^2} + ux_j^2 - \operatorname{sgn} u)\right] \psi(\underline{x}) ,$$

where $u\ (\neq 0) \in \mathbb{R}$, $r \in \mathbb{Z}$, $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$.

On restriction to the subgroup $G \cap H$, $U_G^{u,r}$ becomes

$$\left[U_G^{u,r}(\sigma,\underline{0},\underline{0},S(\tau))\psi\right](\underline{x}) \qquad (3)$$

$$= \exp iu\sigma \exp i\tau\left[\frac{r}{\lambda} + \tfrac{1}{2}\sum_{j=1}^{n} \omega_j(-\frac{1}{u}\frac{\partial^2}{\partial x_j^2} + ux_j^2 - \operatorname{sgn} u)\right] \psi(\underline{x}) .$$

It now remains to find how often the representation $D(G \cap H)$ occurs in $U_G^{u,r}\downarrow(G \cap H)$.

As a basis for the representation space $L^2(\mathbb{R}^n, \mathbb{C})$ of $U_G^{u,r}$, take the set $\{\psi_{\underline{m}} : \underline{m} \in \mathbb{Z}_+^n\}$ (defined in 3-(16)).

Then, from (3),

$$\left[U_G^{u,r}(\sigma,\underline{0},\underline{0},S(\tau))\psi_{\underline{m}}\right](\underline{x})$$

$$= \exp iu\sigma \exp i\tau\left[\frac{r}{\lambda} + (\operatorname{sgn} u)\sum_{j=1}^{n} \omega_j m_j\right] \psi_{\underline{m}}(\underline{x}) .$$

Therefore,

$$U_G^{u,r}\downarrow(G \cap H) = \bigoplus_{\underline{m}} \chi^{u,r+\lambda(\operatorname{sgn} u)\sum_{j=1}^{n} \omega_j m_j} ,$$

where $\chi^{\tilde{u},\tilde{r}}$ is the irreducible representation of $G \cap H$ given by

$$\chi^{\tilde{u},\tilde{r}} : (\sigma,\underline{0},\underline{0},S(\tau)) \longrightarrow \exp i\tilde{u}\sigma \exp i\frac{\tilde{r}}{\lambda}\tau \quad (\tilde{u}\in\mathbb{R},\ \tilde{r}\in\mathbb{Z}).$$

Hence the representation $D : (\sigma,\underline{0},\underline{0},S(\tau)) \longrightarrow \exp iv\sigma$ of $G\cap H$ occurs in $U_G^{u,r}\downarrow(G\cap H)$ if and only if

$$u = v \ , \text{ and}$$

$$r + \lambda(\operatorname{sgn} u)\sum_{j=1}^{n}\omega_j m_j = 0 \quad \text{for some } \underline{m}\in\mathbb{Z}_+^n \ . \tag{4}$$

Suppose that, for fixed $r\in\mathbb{Z}$, there are $f_{\omega,r}$ distinct ways of choosing $\underline{m}$ so that (4) is satisfied. Then $D(G\cap H)$ occurs $f_{\omega,r}$ times in $U_G^{u,r}\downarrow(G\cap H)$ provided $u = v$. Hence (2) suggests that $U_G^{u,r}$ occurs in $R^v\downarrow G$ if and only if $u = v$, and that, in this case, it occurs $f_{\omega,r}$ times. Therefore,

$$R^v\downarrow G \simeq \bigoplus_r f_{\omega,r}U_G^{v,r} \quad \text{(formally)} \ , \tag{5}$$

where the summation is over all distinct values of r which are expressible in the form

$$r = -\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j \quad (m_j\in\mathbb{Z}_+) \ , \tag{6}$$

and, for fixed r, $f_{\omega,r}$ is the number of distinct ways of choosing $\underline{m} = (m_1,m_2,\ldots,m_n)\in\mathbb{Z}_+^n$ so that (6) is satisfied.

5.3 Proof of the decomposition of $R^V \downarrow G$

Alternative expression for $R^V \downarrow G$

Since every element of G can be written in the form

$$(\sigma, \underline{\beta}, \underline{\alpha}, S(\tau)) \tag{7}$$

$$= (\sigma, \underline{0}, \underline{0}, I) \prod_{j=1}^{n} (0, \beta_j \underline{e}_j, \underline{0}, I) \prod_{j=1}^{n} (0, \underline{0}, \alpha_j \underline{e}_j, I)(0, \underline{0}, \underline{0}, S(\tau))$$

(from 3-(14) and 4-(8)), the representation $R^V \downarrow G$ may be expressed as a product of representations of one-dimensional Lie subgroups of G. These representations may be calculated explicitly, by using 4-(11) :

$$[R^V(g)\phi](\underline{q},\underline{p}) = \exp iv\pi_g(g^{-1}(\underline{q},\underline{p}))\, \phi(g^{-1}(\underline{q},\underline{p})) ,$$

where $g \in G$, $(\underline{q},\underline{p}) \in \mathbb{R}^{2n}$, $\phi \in L^2(\mathbb{R}^{2n}, \mathbb{C})$, together with 4-(6), -(10) and -(13).

$$[R^V(\sigma, \underline{0}, \underline{0}, I)\phi](\underline{q},\underline{p}) = \exp iv\sigma\, \phi(\underline{q},\underline{p}) , \tag{8a}$$

$$[R^V(0, \beta_j \underline{e}_j, \underline{0}, I)\phi](\underline{q},\underline{p}) = \phi(\underline{q} - \beta_j \underline{e}_j, \underline{p})$$

$$= \exp\left(-\beta_j \frac{\partial}{\partial q_j}\right) \phi(\underline{q},\underline{p}) , \tag{8b}$$

$$[R^V(0, \underline{0}, \alpha_j \underline{e}_j, I)\phi](\underline{q},\underline{p}) = \exp(iv\alpha_j q_j)\, \phi(\underline{q}, \underline{p} - \alpha_j \underline{e}_j)$$

$$= \exp \alpha_j \left(ivq_j - \frac{\partial}{\partial p_j}\right) \phi(\underline{q},\underline{p}) \tag{8c}$$

(using Stone's theorem to obtain (8b) and (8c)).

$$\left[R^{V}(0,\underline{0},\underline{0},S(\tau))\phi\right](\underline{q},\underline{p})$$

$$= \exp iv\pi_{S(\tau)}(S(\tau)^{-1}(\underline{q},\underline{p}))\ \phi(S(\tau)^{-1}(\underline{q},\underline{p}))$$

$$= \exp\left[-iv\pi_{S(\tau)^{-1}}(\underline{q},\underline{p})\right]\phi(S(\tau)^{-1}(\underline{q},\underline{p}))$$

(since, from 4–(15), $\pi_{\gamma^{-1}}(\underline{q},\underline{p}) + \pi_{\gamma}(\gamma^{-1}(\underline{q},\underline{p})) = \pi_{I}(\underline{q},\underline{p}) = 0$)

$$= \exp\left(iv\sum_{k=1}^{n}\left[\sin\omega_k\tau\cos\omega_k\tau\,\frac{q_k^2 - p_k^2}{2} + \sin^2\omega_k\tau\, p_k q_k\right]\right)$$

$$\phi(\ldots,\cos\omega_j\tau\, q_j + \sin\omega_j\tau\, p_j,\ldots,-\sin\omega_j\tau\, q_j + \cos\omega_j\tau\, p_j,\ldots$$

(since $S(\tau)^{-1} = S(\tau)$ if $\tau = \pi\lambda$, and $S(\tau)^{-1} = S(-\tau)$ otherwise).

As τ occurs only in $\sin\omega_j\tau$ and $\cos\omega_j\tau$, R^{V} can be extended from a representation of M $(= \{(0,\underline{0},\underline{0},S(\tau)) : \tau\in(-\pi\lambda,\pi\lambda]\})$ to a representation of $\tilde{M}$ $(= \{(0,\underline{0},\underline{0},\tau) : \tau\in\mathbb{R}\})$ by defining $R^{V}(0,\underline{0},\underline{0},\tau) = R^{V}(0,\underline{0},\underline{0},S(\hat{\tau}))$, where $\hat{\tau}\in(-\pi\lambda,\pi\lambda]$, and $\hat{\tau} \equiv \tau \bmod 2\pi\lambda$. Since R^{V} is then regarded as a unitary representation of R, it follows, by Stone's theorem, that there exists a unique skew-adjoint operator J such that

$$R^{V}(0,\underline{0},\underline{0},\tau) = \exp\tau J\ .$$

$$J\phi(\underline{q},\underline{p}) = \frac{\partial}{\partial\tau}\left[R^{V}(0,\underline{0},\underline{0},\tau)\phi\right](\underline{q},\underline{p})\Big|_{\tau=0}$$

$$= \sum_{k=1}^{n}\omega_k\left[\frac{iv}{2}(q_k^2 - p_k^2) + p_k\frac{\partial}{\partial q_k} - q_k\frac{\partial}{\partial p_k}\right]\phi(\underline{q},\underline{p}).$$

Hence, as a representation of $\tilde{M}$ or M,

$$\left[R^{v}(0,\underline{0},\underline{0},S(\tau))\phi\right](\underline{q},\underline{p}) \tag{8d}$$

$$= \exp \tau\left(\sum_{k=1}^{n}\omega_k\left[\frac{iv}{2}(q_k^2 - p_k^2) + p_k\frac{\partial}{\partial q_k} - q_k\frac{\partial}{\partial p_k}\right]\right)\phi(\underline{q},\underline{p}) .$$

Therefore, from (7) and (8),

$$\left[R^{v}(\sigma,\underline{\beta},\underline{\alpha},S(\tau))\phi\right](\underline{q},\underline{p}) \tag{9}$$

$$= \exp iv\sigma\prod_{j=1}^{n}\exp\left(-\beta_j\frac{\partial}{\partial q_j}\right)\prod_{j=1}^{n}\exp\alpha_j\left(ivq_j - \frac{\partial}{\partial p_j}\right)$$

$$\exp\tau\left(\sum_{k=1}^{n}\omega_k\left[\frac{iv}{2}(q_k^2 - p_k^2) + p_k\frac{\partial}{\partial q_k} - q_k\frac{\partial}{\partial p_k}\right]\right)\phi(\underline{q},\underline{p}) .$$

Alternative expression for $\bigoplus_r f_{\omega,r}U_G^{v,r}$

It is helpful to express the representation $\bigoplus_r f_{\omega,r}U_G^{v,r}$ in terms of a single representation, rather than a direct sum.

For fixed r, the $f_{\omega,r}$ copies of $U_G^{v,r}$ arise from the $f_{\omega,r}$ distinct ways of choosing $\underline{m} = (m_1,m_2,\ldots,m_n) \in \mathbb{Z}_+^n$ so that (6) is satisfied. Hence

$$f_{\omega,r}U_G^{v,r} = \bigoplus_{\substack{\underline{m} \text{ such that} \\ r=-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j}} U_G^{v,-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j} .$$

The summation in $\bigoplus_r f_{\omega,r}U_G^{v,r}$ is over all distinct values

of r which are expressible in the form $-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j$ ($m_j \in \mathbb{Z}_+$). Therefore

$$\bigoplus_r f_{\omega,r} U_G^{v,r} = \bigoplus_{\underline{m}\in\mathbb{Z}_+^n} U_G^{v,-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j} . \tag{10}$$

Now, from 3-(17),

$$\left[U_G^{v,-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j}(\sigma,\underline{\beta},\underline{\alpha},S(\tau))\psi\right](\underline{x}) \tag{11}$$

$$= \exp iv\sigma \prod_{j=1}^{n}\exp(-iv\beta_j x_j)\prod_{j=1}^{n}\exp\left(-\alpha_j\frac{\partial}{\partial x_j}\right)$$

$$\exp i\tau\left[-(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j(m_j+\tfrac{1}{2}) + \tfrac{1}{2}\sum_{j=1}^{n}\omega_j\left(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + vx_j^2\right)\right]\psi(\underline{x}).$$

The summation over $\underline{m}$ of such representations can be simplified by taking $U_G^{v,-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j}$ to act on the space $L^2(\mathbb{R}^n, \mathbb{C})\psi_{\underline{m}}$ (where $\psi_{\underline{m}}$ is defined by 3-(16)). For each $\underline{m}\in\mathbb{Z}_+^n$,

$$\exp\left[-i\tau(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j(m_j+\tfrac{1}{2})\right]\psi_{\underline{m}}(\underline{y})$$

$$= \exp\left[-\frac{i\tau}{2}\sum_{j=1}^{n}\omega_j\left(-\frac{1}{v}\frac{\partial^2}{\partial y_j^2} + vy_j^2\right)\right]\psi_{\underline{m}}(\underline{y}) . \tag{12}$$

Therefore, from (11) and (12), for each $\underline{m}\in\mathbb{Z}_+^n$,

$$\left[U_G^{v,-\lambda(\operatorname{sgn} v)\sum_{j=1}^{n}\omega_j m_j(\sigma,\underline{\beta},\underline{\alpha},S(\tau))}\psi\right](\underline{x})\psi_{\underline{m}}(\underline{y}) \tag{13}$$

$$= \exp iv\sigma \prod_{j=1}^{n} \exp(-iv\beta_j x_j) \prod_{j=1}^{n} \exp(-\alpha_j \frac{\partial}{\partial x_j})$$

$$\exp \frac{i\tau}{2}\left(\sum_{j=1}^{n}\omega_j\left[\left(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + vx_j^2\right) - \left(-\frac{1}{v}\frac{\partial^2}{\partial y_j^2} + vy_j^2\right)\right]\right)\psi(\underline{x})\psi_{\underline{m}}(\underline{y}) .$$

Now the set $\{\psi_{\underline{m}} : \underline{m}\in\mathbb{Z}_+^n\}$ forms a basis of the space $L^2(\mathbb{R}^n, \mathbb{C})$ of functions from $\{\underline{y}\in\mathbb{R}^n\}$ to $\mathbb{C}$; so the set $\{\psi_{\underline{m}(1)}(\underline{x})\psi_{\underline{m}(2)}(\underline{y}) : \underline{m}(1), \underline{m}(2)\in\mathbb{Z}_+^n\}$ is a basis of the space $L^2(\mathbb{R}^{2n}, \mathbb{C})$ of functions from $\{(\underline{x},\underline{y})\in\mathbb{R}^{2n}\}$ to $\mathbb{C}$. Hence, from linearity, (10) and (13), it follows that, with $\zeta\in L^2(\mathbb{R}^{2n}, \mathbb{C})$,

$$\left(\left[\bigoplus_r f_{\omega,r}U_G^{v,r}\right](\sigma,\underline{\beta},\underline{\alpha},S(\tau))\zeta\right)(\underline{x},\underline{y}) \tag{14}$$

$$= \exp iv\sigma \prod_{j=1}^{n} \exp(-iv\beta_j x_j) \prod_{j=1}^{n} \exp(-\alpha_j \frac{\partial}{\partial x_j})$$

$$\exp \frac{i\tau}{2}\left(\sum_{j=1}^{n}\omega_j\left[\left(-\frac{1}{v}\frac{\partial^2}{\partial x_j^2} + vx_j^2\right) - \left(-\frac{1}{v}\frac{\partial^2}{\partial y_j^2} + vy_j^2\right)\right]\right)\zeta(\underline{x},\underline{y}) .$$

The unitary equivalence of $R^v\downarrow G$ and $\bigoplus_r f_{\omega,r}U_G^{v,r}$

If the representations $R^v\downarrow G$ and $\bigoplus_r f_{\omega,r}U_G^{v,r}$ are unitarily equivalent, there exists a unitary operator V such that

$$V^{-1}(R^v\downarrow G)V = \bigoplus_r f_{\omega,r}U_G^{v,r} .$$

For each $\mu\in\{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \tau\}$, let $\mu(t)$ be the one-dimensional Lie subgroup of G formed by those

elements for which the value of every parameter except μ is zero. Let Y_μ, Y'_μ be the representatives of the generators of $\mu(t)$ in the representations $R^V \downarrow G$ and $\bigoplus_r f_{\omega,r} U_G^{V,r}$ respectively. Then

$$\left.\begin{aligned} R^V \downarrow G &= \prod_\mu \exp(\mu Y_\mu) \ , \\ \text{and} \qquad \bigoplus_r f_{\omega,r} U_G^{V,r} &= \prod_\mu \exp(\mu Y'_\mu) \end{aligned}\right\} \quad (15)$$

(where the parameters in the product occur from left to right in the order $\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \tau$).

Hence, if V exists, it satisfies

$$\begin{aligned} \prod_\mu \exp(\mu Y'_\mu) &= V^{-1}\left[\prod_\mu \exp(\mu Y_\mu)\right] V \\ &= \prod_\mu V^{-1} \exp(\mu Y_\mu) V \\ &= \prod_\mu \exp \mu(V^{-1} Y_\mu V) \ . \end{aligned}$$

So $Y'_\mu = V^{-1} Y_\mu V$ for each μ. Conversely, if V is a unitary operator such that $Y'_\mu = V^{-1} Y_\mu V$ for each μ, then $\bigoplus_r f_{\omega,r} U_G^{V,r} = V^{-1}(R^V \downarrow G)V$.

Therefore, to demonstrate the unitary equivalence of $R^V \downarrow G$ and $\bigoplus_r f_{\omega,r} U_G^{V,r}$, it is sufficient to show that there exists a unitary operator V satisfying

$$Y'_\mu = V^{-1} Y_\mu V \ , \tag{16}$$

for each $\mu \in \{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \tau\}$.

(Superficially, this problem perhaps appears similar to that of finding the operator W, defined on a little group, in Mackey's theory (§ 2.5 and § 2.6). However, there are some fundamental differences between the two situations. Firstly, in Mackey's theory, W is known to be a projective representation of a little group, and so results such as Stone's theorem are sometimes of help in evaluating W. In contrast, the operator V (if it exists) is independent of group elements of G. Secondly, the existence of W can sometimes be proved easily, and the problem is then to find W, rather than to show it exists. However, the existence of V has still to be proved.)

Explicitly, V must satisfy, from (9), (14), (15) and (16):

when $\mu = \sigma$,
$$iv = V^{-1}(iv)V \ ; \tag{17}$$

when $\mu = \beta_j$,
$$ivx_j = V^{-1} \frac{\partial}{\partial q_j} V \ ; \tag{18}$$

when $\mu = \alpha_j$,
$$\frac{\partial}{\partial x_j} = V^{-1}(- ivq_j + \frac{\partial}{\partial p_j})V \ ; \tag{19}$$

when $\mu = \tau$,

$$\sum_{j=1}^{n} \omega_j \left[\frac{i}{2}(- \frac{1}{v} \frac{\partial^2}{\partial x_j^2} + v x_j^2) - \frac{i}{2}(- \frac{1}{v} \frac{\partial^2}{\partial y_j^2} + v y_j^2) \right]$$

$$= V^{-1}\Big(\sum_{j=1}^{n} \omega_j \left[\frac{iv}{2}(q_j^2 - p_j^2) + p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j} \right] \Big) V \ . \tag{20}$$

From the form of these relations, it should be possible to express V, if it exists, as a product of operators $V_1V_2\ldots V_n$, where V_j transforms operators dependent on one or both of x_j and y_j into operators dependent on one or both of q_j and p_j, and the action of V_j on operators which are independent of x_j and y_j is simply to take them into the operators obtained by replacing x_k, y_k by q_k, p_k respectively ($k = 1,\ldots, j-1, j+1,\ldots, n$). Moreover, if $\zeta \in L^2(\mathbb{R}^{2n}, \mathbb{C})$ is a function defined on the space $\{(x_1,\ldots,x_n,y_1,\ldots,y_n) \in \mathbb{R}^{2n}\}$, then ζ is measurable and square-integrable when considered as a function of x_j and y_j only, and hence, in this way, ζ can be regarded as an element of $L^2(\mathbb{R}^2, \mathbb{C})$. It follows that V_j can be considered as an operator on a space $L^2(\mathbb{R}^2, \mathbb{C})$. Furthermore, if V_j is unitary when regarded as acting on $L^2(\mathbb{R}^2, \mathbb{C})$, then V_j is unitary as an operator on $L^2(\mathbb{R}^{2n}, \mathbb{C})$.

This reduces the problem to showing the existence of a unitary operator $\tilde{V}$, defined on $L^2(\mathbb{R}^2, \mathbb{C})$, which satisfies (since, trivially, (17) is satisfied by any V) :

$$ivx = \tilde{V}^{-1} \frac{\partial}{\partial q} \tilde{V} \; ; \tag{21}$$

$$\frac{\partial}{\partial x} = \tilde{V}^{-1}(-ivq + \frac{\partial}{\partial p})\tilde{V} \; ; \tag{22}$$

$$\frac{i}{2}\left[(-\frac{1}{v}\frac{\partial^2}{\partial x^2} + vx^2) - (-\frac{1}{v}\frac{\partial^2}{\partial y^2} + vy^2)\right]$$

$$= \tilde{V}^{-1}\left[\frac{iv}{2}(q^2 - p^2) + p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}\right]\tilde{V} \; . \tag{23}$$

The existence of the operator $\tilde{V}$ will be shown by explicit

construction, in two stages.

First, let $\zeta \in L^2(\mathbb{R}^2, \mathbb{C})$ be a function from the space $\{(x,y) \in \mathbb{R}^2\}$ to $\mathbb{C}$; define a Fourier transform F_1 of ζ into the space $L^2(\mathbb{R}^2, \mathbb{C})$ of functions from the space $\{(q,p) \in \mathbb{R}^2\}$ to $\mathbb{C}$ by

$$(F_1\zeta)(q,p) = \sqrt{\frac{v}{2\pi}} \int_{-\infty}^{\infty} \exp(ivxq)\zeta(x,p)dx \ . \qquad (24)$$

By Plancherel's theorem [41], F_1 is a unitary operator.

Let $\phi \in L^2(\mathbb{R}^2, \mathbb{C})$ be a function from the space $\{(q,p) \in \mathbb{R}^2\}$ to $\mathbb{C}$; then F_1^{-1} maps ϕ into the space $L^2(\mathbb{R}^2, \mathbb{C})$ of functions from the space $\{(x,y) \in R^2\}$ to $\mathbb{C}$, and is given by

$$(F_1^{-1}\phi)(x,y) = \sqrt{\frac{v}{2\pi}} \int_{-\infty}^{\infty} \exp(-\ ivxq)\phi(q,y)dq \ .$$

Then (with slight abuse of notation) :

$$(F_1^{-1} \frac{\partial}{\partial q} F_1\zeta)(x,y) = F_1^{-1}\left(\sqrt{\frac{v}{2\pi}} \int_{-\infty}^{\infty} ivx \exp(ivxq)\zeta(x,p)dx \right)$$

$$= ivx\zeta(x,y) \ ; \qquad (25a)$$

$$(F_1^{-1}(ivq)F_1\zeta)(x,y) = F_1^{-1}\left(\sqrt{\frac{v}{2\pi}} \int_{-\infty}^{\infty} ivq \exp(ivxq)\zeta(x,p)dx \right)$$

(integrating by parts :) $$= F_1^{-1}\left(\sqrt{\frac{v}{2\pi}} \int_{-\infty}^{\infty} \exp(ivxq) .- \frac{\partial}{\partial x}\zeta(x,p)dx \right)$$

$$= - \frac{\partial}{\partial x}\zeta(x,y) \ ; \qquad (25b)$$

$$F_1^{-1} \frac{\partial}{\partial p} F_1 = \frac{\partial}{\partial y} \ , \quad \text{and} \quad F_1^{-1}(ivp)F_1 = ivy \ . \quad (25c \ \& \ d)$$

So, for the operators occurring in (21) and (22),

$$F_1^{-1} \frac{\partial}{\partial q} F_1 = ivx \ ,$$

and $$F_1^{-1}(- ivq + \frac{\partial}{\partial p})F_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \ .$$

Hence the introduction of the operator F_1 reduces the problem of finding a unitary operator $\tilde{V}$ satisfying (21) and (22) to that of finding a unitary operator U satisfying :

$$ivx = U^{-1}(ivx)U \ ; \tag{26}$$

$$\frac{\partial}{\partial x} = U^{-1}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})U \ . \tag{27}$$

If such an operator U exists, then $\tilde{V} = F_1 U$ satisfies the conditions (21) and (22).

Define an operator $\tilde{U}$ on $L^2(\mathbb{R}^2, \mathbb{C})$ by

$$(\tilde{U}\zeta)(x,y) = \zeta(u_{11}x + u_{12}y, u_{21}x + u_{22}y) \ , \tag{28a}$$

where $u_{11}, u_{12}, u_{21}, u_{22} \in \mathbb{R}$.

$\tilde{U}$ is unitary if $|\Delta| = 1$, where $\Delta = u_{11}u_{22} - u_{12}u_{21}$.

$\tilde{U}^{-1}$ is given by

$$(\tilde{U}^{-1}\zeta)(x,y) = \zeta(\frac{u_{22}}{\Delta} x - \frac{u_{12}}{\Delta} y, - \frac{u_{21}}{\Delta} x - \frac{u_{11}}{\Delta} y). \tag{28b}$$

It follows that

$$\tilde{U}^{-1}(ivx)\tilde{U} = \frac{iv}{\Delta}(u_{22}x - u_{12}y) \ ,$$

$$\tilde{U}^{-1}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\tilde{U} = u_{11}\frac{\partial}{\partial x} + u_{21}\frac{\partial}{\partial y} + u_{12}\frac{\partial}{\partial x} + u_{22}\frac{\partial}{\partial y} \ .$$

Hence, when $u_{11} = 1$, $u_{12} = 0$, $u_{21} = -\Delta$ and $u_{22} = \Delta$, $\tilde{U}$ satisfies conditions (26) and (27). Also, with these values, $u_{11}u_{22} - u_{12}u_{21} = \Delta$, as required; to make $\tilde{U}$ unitary, Δ may be taken as 1.

Therefore, when $\tilde{V} = F_1 U$, with

$$(U\zeta)(x,y) = \zeta(x, -x + y) \ , \tag{29}$$

conditions (21) and (22) are satisfied. It remains to check that, with this choice of $\tilde{V}$, (23) is also satisfied.

Now, from (25), (28) and (29),

$$\tilde{V}^{-1} q \tilde{V} = U^{-1}\left(\frac{i}{v}\frac{\partial}{\partial x}\right)U = \frac{i}{v}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) ; \tag{30}$$

$$\tilde{V}^{-1} p \tilde{V} = U^{-1} y U = x + y ; \tag{31}$$

$$\tilde{V}^{-1}\frac{\partial}{\partial p}\tilde{V} = U^{-1}\frac{\partial}{\partial y}U = \frac{\partial}{\partial y} . \tag{32}$$

It follows that

$$\begin{aligned}
&\tilde{V}^{-1}\left[\frac{iv}{2}(q^2 - p^2) + p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}\right]\tilde{V} \\
&= \frac{iv}{2}\left[(\tilde{V}^{-1}q\tilde{V})^2 - (\tilde{V}^{-1}p\tilde{V})^2\right] + (\tilde{V}^{-1}p\tilde{V})(\tilde{V}^{-1}\frac{\partial}{\partial q}\tilde{V}) \\
&\quad - (\tilde{V}^{-1}q\tilde{V})(\tilde{V}^{-1}\frac{\partial}{\partial p}\tilde{V}) \\
&= \frac{iv}{2}\left[-\frac{1}{v^2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 - (x + y)^2\right] \\
&\quad + (x + y)(ivx) - \frac{i}{v}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) \\
&= \frac{i}{2}\left[\left(-\frac{1}{v}\frac{\partial^2}{\partial x^2} + vx^2\right) - \left(-\frac{1}{v}\frac{\partial^2}{\partial y^2} + vy^2\right)\right] .
\end{aligned}$$

Therefore $\tilde{V} = F_1 U$ is a unitary operator satisfying conditions (21), (22) and (23).

A unitary operator V satisfying (17) - (20) can now be constructed. Let $\zeta \in L^2(\mathbb{R}^{2n}, \mathbb{C})$ be a function from the space $\{(\underline{x},\underline{y}) \in \mathbb{R}^{2n}\}$ to $\mathbb{C}$, and define the operator V mapping ζ into the space $L^2(\mathbb{R}^{2n}, \mathbb{C})$ of functions from the space $\{(\underline{q},\underline{p}) \in \mathbb{R}^{2n}\}$ to $\mathbb{C}$ by

$$(V\zeta)(q_1,\ldots,q_n,p_1,\ldots,p_n) \tag{33}$$

$$= \left(\sqrt{\frac{v}{2\pi}}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\, iv \sum_{j=1}^{n} x_j q_j$$

$$\zeta(x_1,\ldots,x_n, -x_1 + p_1,\ldots, -x_n + p_n)dx_1\ldots dx_n \ .$$

From the properties of $\tilde{V}$, it follows that V is a unitary operator on $L^2(\mathbb{R}^{2n}, \mathbb{C})$ satisfying conditions (17) - (20).

Hence $Y'_\mu = V^{-1} Y_\mu V$, for each $\mu \in \{\sigma, \beta_1, \beta_2,\ldots,\beta_n, \alpha_1, \alpha_2,\ldots,\alpha_n, \tau\}$; this implies that

$$\prod_\mu \exp(\mu Y'_\mu) = V^{-1}\left[\prod_\mu \exp(\mu Y_\mu)\right] V \ .$$

Therefore

$$V^{-1}(R^V \downarrow G)V = \bigoplus_r f_{\omega,r} U_G^{V,r} \ , \tag{34}$$

where V is the unitary operator defined by (33).

This proves that the decomposition of $R^V \downarrow G$ suggested by a formal application of the Frobenius reciprocity theorem is

correct. It also shows explicitly the unitary equivalence of $R^{\nu}\downarrow G$ with its decomposition into irreducible representations.

5.4 The significance of the decomposition of $R^{\nu}\downarrow G$

$$R^{\nu}\downarrow G \simeq \bigoplus_{r} f_{\omega,r} U_G^{\nu,r} ,$$

where the summation is over all distinct values of r which are expressible in the form $-\lambda(\operatorname{sgn}\nu)\sum_{j=1}^{n}\omega_j m_j$ $(m_j \in \mathbb{Z}_+)$, and, for fixed r, $f_{\omega,r}$ is the number of distinct ways of choosing $\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n$ so that $r = -\lambda(\operatorname{sgn}\nu)\sum_{j=1}^{n}\omega_j m_j$ is satisfied.

For fixed r occurring in the summation, let $s = -\frac{r}{\operatorname{sgn}\nu}$. Then $s \in \mathbb{Z}_+$, and $f_{\omega,r}$ is the number of distinct ways of choosing $\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n$ so that $s = \lambda\sum_{j=1}^{n}\omega_j m_j$ is satisfied. Hence $f_{\omega,r}$ is precisely the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$ (§1.6); moreover, all energy levels of the system occur in this way.

Therefore,

$$R^{\nu}\downarrow G \simeq \bigoplus_{s} d_{\omega,s} U_G^{\nu,-(\operatorname{sgn}\nu)s} , \qquad (35)$$

where $d_{\omega,s}$ is the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$, and the summation may be regarded as over all distinct energy levels $E_{\omega.s}$ of the system.

CHAPTER 6

THE DECOMPOSITION OF THE REPRESENTATIONS $R^V \downarrow U(n)$ AND $R^V \downarrow SU(n)$

U(n) may be considered as a subgroup of L, under the map :

$$A + iB \longleftrightarrow (0, \underline{0}, \underline{0}, \begin{pmatrix} A & B \\ -B & A \end{pmatrix}) \ . \qquad 4\text{-}(11)$$

Since U(n) and SU(n) are degeneracy groups of the n-dimensional isotropic harmonic oscillator, it is interesting to evaluate the decompositions of $R^V \downarrow U(n)$ and of $R^V \downarrow SU(n)$. To do this, $R^V \downarrow U(n)$ is first expressed as a sum of induced representations, using Mackey's subgroup theorem. Since U(n) is a separable compact group, the Frobenius reciprocity theorem theorem can then be applied to find the decomposition of each of these induced representations into irreducible representations. The decomposition of $R^V \downarrow SU(n)$ then follows from that of $R^V \downarrow U(n)$.

6.1 Expression of $R^V \downarrow U(n)$ as a sum of induced representations

$$R^V \downarrow U(n) = (R^V \downarrow L) \downarrow U(n) = (D(H) \uparrow L) \downarrow U(n) \text{ (from §4.4).(1)}$$

Hence $R^V \downarrow U(n)$ is the restriction of an induced representation of L, and so, since U(n) and H are closed subgroups of the separable locally compact group L, Mackey's subgroup theorem (2-(3)) may be used as a first step in splitting $R^V \downarrow U(n)$, provided there exists a Borel set in L which intersects each U(n), H double coset in exactly one point.

First, L must be factorized into double cosets of U(n) and H. This is achieved in the following lemma.

Lemma

(a) The map $\upsilon : L \longrightarrow \mathbb{R}_+$ defined by

$$\upsilon : (\sigma', \underline{\beta}', \underline{\alpha}', S') \longrightarrow |\underline{\beta}'|^2 + |\underline{\alpha}'|^2$$

is constant on the double coset $U(n)(\sigma, \underline{\beta}, \underline{\alpha}, S)H$, for each $(\sigma, \underline{\beta}, \underline{\alpha}, S) \in L$.

(b) If $\upsilon(\sigma', \underline{\beta}', \underline{\alpha}', S') \neq \upsilon(\sigma, \underline{\beta}, \underline{\alpha}, S)$, then $(\sigma', \underline{\beta}', \underline{\alpha}', S')$ and $(\sigma, \underline{\beta}, \underline{\alpha}, S)$ belong to different double cosets of U(n) and H in L.

(c) If $\upsilon(\sigma', \underline{\beta}', \underline{\alpha}', S') = \upsilon(\sigma, \underline{\beta}, \underline{\alpha}, S)$, then $(\sigma', \underline{\beta}', \underline{\alpha}', S')$ and $(\sigma, \underline{\beta}, \underline{\alpha}, S)$ belong to the same double coset of U(n) and H in L.

(d) A possible choice of double coset representatives of U(n) and H in L is the set

$$K = \{(0, \kappa \underline{e}_1, \underline{0}, I) : \kappa \in \mathbb{R}_+\} . \qquad (2)$$

Proof

(a) Choose arbitrary elements $(0, \underline{0}, \underline{0}, \begin{pmatrix} A & B \\ -B & A \end{pmatrix}) \in U(n)$, $(\sigma, \underline{\beta}, \underline{\alpha}, S) \in L$, $(\hat{\sigma}, \underline{0}, \underline{0}, \hat{S}) \in H$ $(= C \times Sp(2n, \mathbb{R}))$.

Then, from 4-(7),

$$\upsilon\left[(0,\underline{0},\underline{0},\begin{pmatrix} A & B \\ -B & A \end{pmatrix})(\sigma,\underline{\beta},\underline{\alpha},S)(\hat{\sigma},\underline{0},\underline{0},\hat{S})\right] \qquad (3)$$

$$= \upsilon(\sigma + \hat{\sigma} + \sigma^*(A,B,\underline{\alpha},\underline{\beta}),\ A\underline{\beta} - B\underline{\alpha},\ B\underline{\beta} + A\underline{\alpha},\ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} S\hat{S})$$

(where σ^* is determined, but not required in detail)

$$= |A\underline{\beta} - B\underline{\alpha}|^2 + |B\underline{\beta} + A\underline{\alpha}|^2$$

$$= |\underline{\beta}|^2 + |\underline{\alpha}|^2 \quad \text{since } A + iB \text{ is unitary.}$$

Hence υ is constant on the double coset $U(n)(\sigma,\underline{\beta},\underline{\alpha},S)H$, for each $(\sigma,\underline{\beta},\underline{\alpha},S) \in L$.

(b) This follows at once, from (a).

(c) Suppose $\upsilon(\sigma',\underline{\beta}',\underline{\alpha}',S') = \upsilon(\sigma,\underline{\beta},\underline{\alpha},S) = \kappa \neq 0$.

In order to prove that $(\sigma',\underline{\beta}',\underline{\alpha}',S')$ and $(\sigma,\underline{\beta},\underline{\alpha},S)$ belong to the same double coset of U(n) and H in L, it is sufficient, from the form of the elements of $U(n)(\sigma,\underline{\beta},\underline{\alpha},S)H$ (see (3)), to show that there exists $(0,\underline{0},\underline{0},\begin{pmatrix} A & B \\ -B & A \end{pmatrix}) \in U(n)$ such that

$$\underline{\beta}' = A\underline{\beta} - B\underline{\alpha}\ ,$$

$$\underline{\alpha}' = B\underline{\beta} + A\underline{\alpha}\ .$$

Since $\underline{f}_1 = \dfrac{\underline{\beta} + i\underline{\alpha}}{\sqrt{\kappa}}$ is a unit vector of $\mathbb{C}^n$, it may be taken as the first vector of an orthonormal basis of $\mathbb{C}^n$: $\underline{f}_1$, $\underline{f}_2$,..

., $\underline{f}_n$ say; similarly, $\underline{f}'_1 = \dfrac{\underline{\beta}' + i\underline{\alpha}'}{\sqrt{\kappa}}$ may be taken as the first vector of another orthonormal basis : $\underline{f}'_1, \underline{f}'_2, \ldots, \underline{f}'_n$; hence there exists a unitary matrix $\tilde{A} + i\tilde{B}$ such that

$$(\tilde{A} + i\tilde{B})\underline{f}_j = \underline{f}'_j \qquad \text{(for all } j = 1,2,\ldots,n) \ .$$

In particular, for $j = 1$,

$$(\tilde{A} + i\tilde{B}) \frac{\underline{\beta} + i\underline{\alpha}}{\sqrt{\kappa}} = \frac{\underline{\beta}' + i\underline{\alpha}'}{\sqrt{\kappa}} \ .$$

So $\underline{\beta}'$ and $\underline{\alpha}'$ may be expressed as

$$\underline{\beta}' = \tilde{A}\underline{\beta} - \tilde{B}\underline{\alpha} \ ,$$

$$\underline{\alpha}' = \tilde{B}\underline{\beta} + \tilde{A}\underline{\alpha} \ ,$$

where $\tilde{A} + i\tilde{B}$ is a unitary matrix.

Therefore, when $\upsilon(\sigma',\underline{\beta}',\underline{\alpha}',S') = \upsilon(\sigma,\underline{\beta},\underline{\alpha},S) \neq 0$, $(\sigma',\underline{\beta}',\underline{\alpha}',S')$ belongs to the double coset $U(n)(\sigma,\underline{\beta},\underline{\alpha},S)H$.

When $\upsilon(\sigma',\underline{\beta}',\underline{\alpha}',S') = \upsilon(\sigma,\underline{\beta},\underline{\alpha},S) = 0$, then $\underline{\beta} = \underline{0} = \underline{\alpha}$, $\underline{\beta}' = \underline{0} = \underline{\alpha}'$, and so $(\sigma',\underline{\beta}',\underline{\alpha}',S')$ and $(\sigma,\underline{\beta},\underline{\alpha},S)$ both belong to the double coset $U(n)(0,\underline{0},\underline{0},I)H$.

(d) From (a), (b) and (c), there is a 1 - 1 correspondence between double cosets of $U(n)$ and H in L, and values (in $\mathbb{R}_+$) taken by the map υ. Therefore, the double coset representative of $U(n)(\sigma,\underline{\beta},\underline{\alpha},S)H$ in L may be taken to be $(0,\kappa\underline{e}_1,\underline{0},I)$, where $\kappa = \upsilon(\sigma,\underline{\beta},\underline{\alpha},S)$, and $\underline{e}_1$ $(\in \mathbb{R}^n)$ is a vector

with 1 in the first position and zeros elsewhere.

Hence a possible choice of double coset representatives of U(n) and H in L is the set

$$K = \{(0, \kappa \underline{e}_1, \underline{0}, I) : \kappa \in \mathbb{R}_+\} .$$

This completes the proof of the lemma.

Now L is isomorphic to $(\mathbb{R}^{n+1} \circledS \mathbb{R}^n) \circledS Sp(2n, \mathbb{R})$, and its topological structure is that inherited from $\mathbb{R}^{2n^2+3n+1}$ (from 4-(9) and -(12)). In this topology, the complement of K in L is an open set, and therefore a Borel set; it follows that K is also a Borel set, since the complement of a Borel set is Borel. Therefore all the conditions of Mackey's subgroup theorem are satisfied.

To simplify the notation, put $\tilde{\kappa} = (0, \kappa \underline{e}_1, \underline{0}, I)$. Then, using (1) and (2), Mackey's subgroup theorem (2-(3)) gives :

$$R^V \downarrow U(n) = \int_K (D_\kappa \uparrow U(n)) d\bar{\mu}(\kappa) , \qquad (4)$$

where D_κ is the representation of $U(n) \cap \tilde{\kappa} H \tilde{\kappa}^{-1}$ defined by $D_\kappa(\tilde{\kappa} h \tilde{\kappa}^{-1}) = D(h)$, with $h \in H$.

It is easily verified that :

$$\text{when } \kappa \neq 0, \quad U(n) \cap \tilde{\kappa} H \tilde{\kappa}^{-1} = U(n-1) \text{ and } D_\kappa = I_{U(n-1)} \qquad (5a)$$

(where $I_{U(n-1)}$ denotes the one-dimensional identity representation of U(n - 1));

when $\kappa = 0$, $\quad U(n) \cap \tilde{\kappa} H \tilde{\kappa}^{-1} = U(n)$ and $D_\kappa = I_{U(n)}$. (5b)

Since $D_\kappa \uparrow U(n)$ is the same representation for each $\kappa \neq 0$, the direct integral over $\mathbb{R}_+ - \{0\}$ is equivalent to a direct sum of countably many copies of $D_\kappa \uparrow U(n)$ (cf. 2-(5)).

Therefore, using (5), (4) becomes :

$$R^V \downarrow U(n) = \int_{\{0\}} (D_\kappa \uparrow U(n)) d\bar{\mu}(\kappa) \oplus \int_{\mathbb{R}_+ - \{0\}} (D_\kappa \uparrow U(n)) d\bar{\mu}(\kappa)$$

$$= I_{U(n)} \oplus \infty (I_{U(n-1)} \uparrow U(n)) \, . \qquad (6)$$

6.2 The decomposition of $I_{U(n-1)} \uparrow U(n)$

Since U(n) is a separable compact group, the decomposition of $I_{U(n-1)} \uparrow U(n)$ into irreducible representations may be found by applying the Frobenius reciprocity theorem (2-(6)).

Suppose ρ is an arbitrary irreducible representation of U(n). Then

$$\langle I_{U(n-1)} \uparrow U(n), \rho \rangle_{U(n)} = \langle \rho \downarrow U(n-1), I_{U(n-1)} \rangle_{U(n-1)} \, . (7)$$

Hence, to decompose $I_{U(n-1)} \uparrow U(n)$, it is sufficient to determine, for each ρ, how many times $I_{U(n-1)}$ occurs in the restriction of ρ to $U(n-1)$.

The irreducible representations of U(n) may be obtained as operators on tensor spaces with particular symmetries, and are most easily described in terms of Young tableaux ([5], [14] and [21]).

Every irreducible representation of U(n) is of the form $U\ (\in U(n)) \longrightarrow (\det U)^{s} C_{\xi}(U)$, where s is an integer and C_{ξ} is an irreducible representation of U(n), corresponding to a Young tableau with ξ_j boxes in the jth row $(1 \leqslant j \leqslant n)$, and $\xi_n = 0$; this representation will be denoted by $\det^{s}(\xi_1,\ldots,\xi_{n-1},0)$. Two such representations of U(n) with different s or different Young pattern $[\xi_1,\ldots,\xi_{n-1},0]$ are inequivalent.

If $k \in \mathbb{Z}_+$, then $\det^{k}$ $(= \det^{k}(0,0,\ldots,0,0))$ can be represented by a tableau with Young pattern $[k,k,\ldots,k]$ (n entries); more generally, if $k \in \mathbb{Z}_+$, $\det^{s}(\xi_1,\ldots,\xi_{n-1},0)$ is equivalent to $\det^{s-k}(\xi_1 + k,\ \xi_2 + k,\ldots,\ \xi_{n-1} + k,\ k)$.

When the irreducible representation $(\xi_1,\ldots,\xi_{n-1},0)$ of U(n) is restricted to U(n - 1), its decomposition into irreducible representations is given by :

$$(\xi_1,\ldots,\xi_{n-1},0)\downarrow U(n-1) = \bigoplus (\xi'_1,\xi'_2,\ldots,\xi'_{n-1}),$$

where the direct sum contains exactly once each irreducible representation $(\xi'_1,\xi'_2,\ldots,\xi'_{n-1})$ of U(n - 1) satisfying

$$\xi_1 \geqslant \xi'_1 \geqslant \xi_2 \geqslant \xi'_2 \geqslant \ldots \geqslant \xi'_{n-1} \geqslant \xi_n = 0\ .$$

Hence, $I_{U(n-1)}$ occurs in $\det^{s}(\xi_1,\ldots,\xi_{n-1},0)\downarrow U(n-1)$ if and only if $\det^{-s}\downarrow U(n-1)$ occurs in $(\xi_1,\ldots,\xi_{n-1},0)\downarrow U(n-1)$. This happens if and only if $s \leqslant 0$. For $-s \geqslant 0$, $\det^{-s}\downarrow U(n-1)$ has Young pattern $[-s, -s,\ldots, -s]$ $(n-1$ entries), that is, $\xi'_1 = \xi'_2 = \ldots = \xi'_{n-1} = -s$. It follows that

$\det^s(\xi_1,\ldots,\xi_{n-1},0)\downarrow U(n-1)$ contains $I_{U(n-1)}$ if and only if $s \leqslant 0$, and $[\xi_1,\ldots,\xi_{n-1},0] = [\,j - s, - s, - s,\ldots, - s, 0]$, for some $j \geqslant 0$.

Therefore, from (7),

$$I_{U(n-1)}\uparrow U(n) = \bigoplus_{j,k=0}^{\infty} \det^{-k}(j + k,k,k,\ldots,k,0) \ . \quad (8)$$

6.3 The decompositions of $R^v\downarrow U(n)$ and of $R^v\downarrow SU(n)$

It follows from (5) and (7) that the decomposition of $R^v\downarrow U(n)$ into irreducible representations is

$$R^v\downarrow U(n) = \infty \bigoplus_{j,k=0}^{\infty} \det^{-k}(j + k,k,k,\ldots,k,0)_{U(n)}, \quad (9)$$

where $(j + k,k,k,\ldots,k,0)_{U(n)}$ is the irreducible representation of $U(n)$ with Young pattern $[j + k,k,k,\ldots,k,0]$.

Every irreducible representation of $U(n)$ remains irreducible on restriction to the subgroup $SU(n)$; also $\det^{-k}\downarrow SU(n) = I_{SU(n)}$, for any k. Hence the decomposition of $R^v\downarrow SU(n)$ into irreducible representations is

$$R^v\downarrow SU(n) = \infty \bigoplus_{j,k=0}^{\infty} (j + k,k,k,\ldots,k,0)_{SU(n)} \ , \quad (10)$$

where $(j + k,k,k,\ldots,k,0)_{SU(n)}$ is the irreducible representation $(j + k,k,k,\ldots,k,0)_{U(n)}\downarrow SU(n)$. Moreover, $(j_1 + k_1,k_1,k_1,\ldots,k_1,0)_{SU(n)}$ and $(j_2 + k_2,k_2,k_2,\ldots,k_2,0)_{SU(n)}$ are inequivalent if (and only if) $j_1 \neq j_2$ and $k_1 \neq k_2$.

6.4 The significance of a subrepresentation of $I_{U(n-1)} \uparrow U(n)$ and of $I_{SU(n-1)} \uparrow SU(n)$

From (8),

$$I_{U(n-1)} \uparrow U(n) = \tilde{\bigoplus_{j,k=0}} \det{}^{-k}(j + k,k,k,\ldots,k,0)_{U(n)} \cdot$$

Applying Mackey's subgroup theorem (2-(4)) to the representation $(I_{U(n-1)} \uparrow U(n)) \downarrow SU(n)$ gives

$$(I_{U(n-1)} \uparrow U(n)) \downarrow SU(n) = I_{SU(n-1)} \uparrow SU(n) ,$$

since U(n) factorized into double cosets of SU(n) and U(n - 1) is just U(n) = SU(n)IU(n - 1), where I is the identity of U(n)

Hence, from (8),

$$I_{SU(n-1)} \uparrow SU(n) = \tilde{\bigoplus_{j,k=0}} (j + k,k,k,\ldots,k,0)_{SU(n)} \cdot \quad (11)$$

Now the dimension of the representation of U(n) or of SU(n) with Young pattern $[j + k,k,k,\ldots,k,0]$ is

$$\frac{(j + n - 2)!\ (k + n - 2)!\ (j + k + n - 1)}{j!\ k!\ (n - 1)!\ (n - 2)!} .$$

In particular, if j = 0, this is $\dfrac{(j + n - 1)!}{j!\ (n - 1)!}$; if, instead, k = 0, it becomes $\dfrac{(k + n - 1)!}{k!\ (n - 1)!}$. Now $\dfrac{(j + n - 1)!}{j!\ (n - 1)!}$ is

precisely the degeneracy of the (j + 1)th energy level of the n-dimensional isotropic harmonic oscillator.

It follows that the representations $I_{U(n-1)}\uparrow U(n)$ of U(n) and $I_{SU(n-1)}\uparrow SU(n)$ of SU(n) may be regarded as splitting a subspace of the representation space into two complete sets of eigenspaces of the n-dimensional isotropic harmonic oscillator.

CHAPTER 7

A NON-INVARIANCE GROUP FOR THE N-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

The results of the previous chapter, particularly those of §6.4, suggest the possibility that there exists a subgroup of L which is a non-invariance group for the n-dimensional isotropic harmonic oscillator (see §1.8). Such a subgroup of L must contain a degeneracy group for the n-dimensional isotropic harmonic oscillator, and be locally compact, but not compact. Two reasonable subgroups to test as non-invariance groups are thus $N \circledS U(n)$ and $N \circledS SU(n)$. Because SU(n) is a separable, compact, connected and simply connected group, it has no non-trivial multipliers (§2.4(c)), and this simplifies the problem of finding all the irreducible representations of $N \circledS SU(n)$. For this reason, attention is given to $N \circledS SU(n)$, rather than to $N \circledS U(n)$.

First, all the quantum mechanical irreducible representations of $N \circledS SU(n)$ are found, by using Mackey's theory for semi-direct product groups, together with the method of §2.6 for evaluating the representation W. (In the irreducible representations of N of the form $U_N^{\underline{u},\underline{t}}$ (3-(12)), the generator of the subgroup $\{(\sigma,\underline{0},\underline{0},I) : \sigma \in \mathbb{R}\}$ of N is represented by the zero operator, and it is easily seen that this is also true for the irreducible representations of $N \circledS SU(n)$ obtained from

$U_N^{\underline{u},\underline{t}}$ by Mackey's method. Such representations correspond to the classical case and so will be ignored here.)

Some of the simplest of the quantum mechanical irreducible representations of N⊛SU(n), when restricted to SU(n), become just the representation W of SU(n). It is proved that $W = \bigoplus_{a=0}^{\infty} W_a$, where W_a is an irreducible representation of SU(n) of dimension equal to the degeneracy of the (a + 1)th energy level of the n-dimensional isotropic harmonic oscillator. These results show that N⊛SU(n) is a possible non-invariance group for the n-dimensional isotropic harmonic oscillator.

7.1 The little group of $U_N^{\underline{v}}$

From 4-(7) and -(11), the group law of N⊛SU(n) (isomorphic to a subgroup of L) may be taken as :

$$(\sigma',\underline{\beta}',\underline{\alpha}',U')(\sigma,\underline{\beta},\underline{\alpha},U) = (\sigma'',\underline{\beta}'',\underline{\alpha}'',U'') \qquad (1)$$

where $\sigma'' = \sigma' + \sigma + \frac{1}{2}A'\underline{\beta}.B'\underline{\beta} - \frac{1}{2}A'\underline{\alpha}.B'\underline{\alpha} - B'\underline{\beta}.B'\underline{\alpha}$

$$+ \underline{\alpha}'.(A'\underline{\beta} - B'\underline{\alpha}) ,$$

$$\underline{\beta}'' = A'\underline{\beta} - B'\underline{\alpha} + \underline{\beta}' ,$$

$$\underline{\alpha}'' = B'\underline{\beta} + A'\underline{\alpha} + \underline{\alpha}' ,$$

$$U'' = U'U ,$$

with $U' = A' + iB' \in SU(n)$, and $U \in SU(n)$.

Every quantum mechanical irreducible representation of N is unitarily equivalent to one of the form 3-(11) :

$$\left[U_N^v(\sigma,\underline{\beta},\underline{\alpha},I)\psi\right](\underline{x}) = \exp iv(\sigma - \underline{x}.\underline{\beta})\,\psi(\underline{x} - \underline{\alpha}) \ ,$$

where $v\ (\neq 0) \in \mathbb{R}$, $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$.

The action of $A + iB \in SU(n)$ on U_N^v is, by definition 2-(10),

$$\left[((A + iB)U_N^v)(\sigma,\underline{\beta},\underline{\alpha},I)\psi\right](\underline{x})$$

$$= \left[U_N^v((0,\underline{0},\underline{0},\ (A + iB)^{-1})(\sigma,\underline{\beta},\underline{\alpha},I)(0,\underline{0},\underline{0},\ A + iB))\psi\right](\underline{x})$$

$$= \left[U_N^v((0,\underline{0},\underline{0},\ A^T - iB^T)(\sigma,\underline{\beta},\underline{\alpha},\ A + iB))\psi\right](\underline{x})$$

$$= \left[U_N^v(\sigma + \tfrac{1}{2}(A^T\underline{\alpha}.B^T\underline{\alpha} - A^T\underline{\beta}.B^T\underline{\beta}) - B^T\underline{\beta}.B^T\underline{\alpha}\ ,\ A^T\underline{\beta} + B^T\underline{\alpha},\right.$$

$$\left. - B^T\underline{\beta} + A^T\underline{\alpha},\ I)\psi\right](\underline{x}) \qquad \text{from (1)}$$

$$= \exp iv(\sigma + \tfrac{1}{2}(A^T\underline{\alpha}.B^T\underline{\alpha} - A^T\underline{\beta}.B^T\underline{\beta}) - B^T\underline{\beta}.B^T\underline{\alpha}$$

$$- \underline{x}.(A^T\underline{\beta} + B^T\underline{\alpha}))\ \psi(\underline{x} + B^T\underline{\beta} - A^T\underline{\alpha}) \ . \qquad (2)$$

$(A + iB)U_N^v$ is an irreducible representation of N. Restricted to the subgroup $\{(\sigma,\underline{0},\underline{0},I)\}$, it is just the phase $\exp iv\sigma$. Hence $(A + iB)U_N^v$ must be unitarily equivalent to U_N^v, since U_N^v is the only irreducible representation of N which has the required form on restriction to the subgroup $\{(\sigma,\underline{0},\underline{0},I)\}$.

Therefore, there exists a unitary operator $W(A + iB)$, dependent on $A + iB$, such that

$$(A + iB)U_N^v = W(A + iB)^{-1}U_N^v\, W(A + iB) \ , \qquad (3)$$

for each $A + iB \in SU(n)$.

The orbits of the quantum mechanical part of $\hat{N}$ under the action of SU(n) thus consist of single points U_N^v. The little group of U_N^v under the action of SU(n) is the whole of SU(n), and so the isotropy group of U_N^v is $N \circledS SU(n)$.

In order to obtain all the irreducible representations of $N \circledS SU(n)$ explicitly, the operator W must be found. Now N and SU(n) are connected Lie groups, and SU(n) has no non-trivial multipliers (§ 2.4(c)). The method for finding W described in §2.6 is therefore applicable here.

7.2 A parametrisation of SU(n)

As a first step towards evaluating W, a set of independent one-parameter subgroups of SU(n) which generate SU(n) will be found. In order to do this, a basis for the Lie algebra of SU(n) is first obtained.

Let U(t) be an analytic curve in SU(n), and suppose U(0) = I. Then, differentiating :

$$U(t)\overline{U(t)}^T = I \ , \qquad \det U(t) = 1 \ ,$$

with respect to t, and putting t = 0, gives

$$U'(0) + \overline{U'(0)}^T = 0 \ , \quad \text{trace } U'(0) = 0 \ ,$$

where ' denotes differentiation with respect to t. Hence

the Lie algebra of SU(n) is the algebra of all n x n complex skew-adjoint matrices of trace zero, denoted by su(n).

A basis for su(n) is the set of $(n^2 - 1)$ elements :

$$\left.\begin{aligned} R^{(1)}_{jk} &= i(E_{jk} + E_{kj}) && (1 \leqslant j < k \leqslant n)\ , \\ R^{(2)}_{jk} &= E_{jk} - E_{kj} && (1 \leqslant j < k \leqslant n)\ , \\ H_k &= i(E_{kk} - \tfrac{1}{n} I_n) && (2 \leqslant k \leqslant n) \end{aligned}\right\} \quad (4)$$

(where E_{lm} is the n x n matrix with 1 in the (l,m) position and zeros elsewhere, and I_n is the n x n identity matrix).

Since SU(n) is an $(n^2 - 1)$-dimensional connected Lie group, it is generated by the $(n^2 - 1)$ independent one-parameter subgroups generated by $R^{(1)}_{jk}$, $R^{(2)}_{jk}$ and H_k respectively. However, there is a somewhat simpler expression for elements of SU(n), in terms of $R^{(1)}_{jk}$, $R^{(2)}_{jk}$ and H_k : in general, the exponential map from a Lie algebra of matrices to a corresponding Lie group is into the Lie group, but not onto the group; for SU(n), however, the exponential map is, in fact, onto, and so SU(n) is said to be an exponential group. For, any matrix of SU(n) can be diagonalised by means of a unitary matrix, which can be chosen to be of determinant 1; that is, given $U \in SU(n)$, there exists $U^* \in SU(n)$ such that

$$U^{*-1}UU^* = \begin{pmatrix} \epsilon_1 & & & 0 \\ & \epsilon_2 & & \\ & & \ddots & \\ 0 & & & \epsilon_n \end{pmatrix},$$

where the ϵ_j are the eigenvalues of U, of modulus one, and $\epsilon_1\epsilon_2\ldots\epsilon_n = 1$ [17]. Hence

$$U = \exp Q \ ,$$

where Q is a skew-adjoint matrix, which can be chosen to be of trace zero.

Therefore, every element of SU(n) is expressible in the form

$$\exp\left[\sum_{1\leqslant j<k\leqslant n}(t_{jk}^{(1)}R_{jk}^{(1)} + t_{jk}^{(2)}R_{jk}^{(2)}) + \sum_{k=2}^{n} t_k H_k\right] , \qquad (5)$$

where $t_{jk}^{(1)}$, $t_{jk}^{(2)}$, $t_k \in \mathbb{R}$ $(1 \leqslant j < k \leqslant n)$.

7.3 The operator W

General conditions to be satisfied

The operator W(A + iB) satisfies

$$(A + iB)U_N^V = W(A + iB)^{-1}U_N^V\, W(A + iB) \ . \qquad (3)$$

Using 3-(14) :

$$(\sigma,\underline{\beta},\underline{\alpha},I) = (\sigma,\underline{0},\underline{0},I)\prod_{j=1}^{n}(0,\beta_j\underline{e}_j,\underline{0},I)\prod_{j=1}^{n}(0,\underline{0},\alpha_j\underline{e}_j,I) \ ,$$

together with Stone's theorem for unitary representations of one-parameter subgroups of N, it follows, from (2), that the representation $(A + iB)U_N^V$ of N may be expressed as

$$\left[((A + iB)U_N^v)(\sigma,\underline{\beta},\underline{\alpha},I)\psi\right](\underline{x})$$

$$= \exp iv\sigma \prod_{j=1}^{n} \exp \beta_j \left[- iv \sum_{k=1}^{n} a_{jk}x_k + \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k} \right]$$

$$\prod_{j=1}^{n} \exp \alpha_j \left[- iv \sum_{k=1}^{n} b_{jk}x_k - \sum_{k=1}^{n} a_{jk} \frac{\partial}{\partial x_k} \right] \psi(\underline{x}) ,$$

where $A + iB = (a_{jk} + ib_{jk})$.

Hence, if $h(t) = A(t) + iB(t) = (a_{jk}(t) + ib_{jk}(t))$ is a one-parameter subgroup of the little group SU(n), the representatives of the generators of N in the representation $h(t)U_N^v$ of N are, with the previous notation (cf. §3.5) :

$$\left.\begin{aligned} Z_\sigma(t) &= iv , \\ Z_{\beta_j}(t) &= - iv \sum_{k=1}^{n} a_{jk}(t)x_k + \sum_{k=1}^{n} b_{jk}(t) \frac{\partial}{\partial x_k} , \\ Z_{\alpha_j}(t) &= - iv \sum_{k=1}^{n} b_{jk}(t)x_k - \sum_{k=1}^{n} a_{jk}(t) \frac{\partial}{\partial x_k} . \end{aligned}\right\} \quad (6)$$

From §2.6, a possible form of the operator $W(h(t))$ is $\exp tJ$, where J is a skew-adjoint operator satisfying

$$\left.\frac{dZ_\mu(t)}{dt}\right|_{t = 0} = \left[Z_\mu(0), J\right] \quad \text{(cf. 2-(22))} , \qquad (7)$$

for all $\mu \in \{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Explicitly, from (6) and (7), J must satisfy :

when $\mu = \sigma$, $\qquad 0 = \left[iv, J\right]$,

which is satisfied by any J;

when $\mu = \beta_j$,

$$- iv \sum_{k=1}^{n} a'_{jk}(0) x_k + \sum_{k=1}^{n} b'_{jk}(0) \frac{\partial}{\partial x_k} = \left[- ivx_j, J \right] ; \qquad (8)$$

when $\mu = \alpha_j$,

$$- iv \sum_{k=1}^{n} b'_{jk}(0) x_k - \sum_{k=1}^{n} a'_{jk}(0) \frac{\partial}{\partial x_k} = \left[- \frac{\partial}{\partial x_j} , J \right] \qquad (9)$$

(where ' denotes differentiation with respect to t).

The operator W for the subgroup generated by $R^{(1)}_{lm}$

$$R^{(1)}_{lm} = i(E_{lm} + E_{ml}) \quad (1 \leqslant l < m \leqslant n) \quad \text{(from (4))} .$$

Suppose, with the above notation, that $R^{(1)}_{lm}$ is the generator of the one-parameter subgroup $h(t) = A(t) + iB(t) = (a_{jk}(t) + ib_{jk}(t))$. Then,

$$R^{(1)}_{lm} = A'(0) + iB'(0) = (a'_{jk}(0) + ib'_{jk}(0)) .$$

In this case, the only non-zero elements of A'(0) and B'(0) are :

$$b'_{lm}(0) = 1 = b'_{ml}(0) .$$

Hence, the conditions (8) and (9) become :

for $j \neq 1$ or m,

$$0 = \left[- ivx_j, J \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j}, J \right] ;$$

for $j = 1 \ (\neq m)$,

$$\frac{\partial}{\partial x_m} = \left[- ivx_1, J \right] \quad \text{and} \quad - ivx_m = \left[- \frac{\partial}{\partial x_1}, J \right] ;$$

for $j = m \ (\neq 1)$,

$$\frac{\partial}{\partial x_1} = \left[- ivx_m, J \right] \quad \text{and} \quad - ivx_1 = \left[- \frac{\partial}{\partial x_m}, J \right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{J} = \frac{1}{iv} \frac{\partial^2}{\partial x_1 \partial x_m} + ivx_1 x_m .$$

Therefore, from §2.6,

$$\tilde{W}(\exp tR_{lm}^{(1)}) = \exp t\left(\frac{1}{iv} \frac{\partial^2}{\partial x_1 \partial x_m} + ivx_1 x_m + \eta_{lm}^{(1)} \right) , \quad (10a)$$

where $\eta_{lm}^{(1)}$ is any imaginary constant, satisfies

$$(\exp tR_{lm}^{(1)}) U_N^v = \tilde{W}(\exp tR_{lm}^{(1)})^{-1} U_N^v \tilde{W}(\exp tR_{lm}^{(1)}) .$$

The operator W for the subgroup generated by $R_{lm}^{(2)}$

$$R_{lm}^{(2)} = E_{lm} - E_{ml} \qquad (1 \leqslant l < m \leqslant n) .$$

In this case, the only non-zero elements of A'(0) and B'(0) are :

$$a'_{lm}(0) = 1 = - a'_{ml}(0) \ .$$

Hence, (8) and (9) become :

for $j \neq 1$ or m,

$$0 = \left[- ivx_j, J \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} , J \right] ;$$

for $j = 1 \ (\neq m)$,

$$- ivx_m = \left[- ivx_1, J \right] \quad \text{and} \quad - \frac{\partial}{\partial x_m} = \left[- \frac{\partial}{\partial x_1} , J \right] ;$$

for $j = m \ (\neq 1)$,

$$ivx_1 = \left[- ivx_m, J \right] \quad \text{and} \quad \frac{\partial}{\partial x_1} = \left[- \frac{\partial}{\partial x_m} , J \right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{J} = x_1 \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_1} \ .$$

Therefore,

$$\tilde{W}(\exp tR^{(2)}_{lm}) = \exp t\left(x_1 \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_1} + \eta^{(2)}_{lm} \right) , \quad (10b)$$

where $\eta^{(2)}_{lm}$ is any imaginary constant, satisfies

$$(\exp tR^{(2)}_{lm}) U^v_N = \tilde{W}(\exp tR^{(2)}_{lm})^{-1} U^v_N \tilde{W}(\exp tR^{(2)}_{lm}) \ .$$

The operator W for the subgroup generated by H_m

$$H_m = i(E_{mm} - \frac{1}{n} I_n) \qquad (2 \leqslant m \leqslant n) \ .$$

In this case, the only non-zero elements of $A'(0)$ and $B'(0)$ are :

$$b'_{mm}(0) = 1 - \frac{1}{n} , \qquad b'_{kk}(0) = -\frac{1}{n} \quad (k \neq m) .$$

Hence, (8) and (9) become :

for $j \neq m$,

$$-\frac{1}{n}\frac{\partial}{\partial x_j} = \left[-ivx_j, J\right] \quad \text{and} \quad \frac{iv}{n}x_j = \left[-\frac{\partial}{\partial x_j}, J\right] ;$$

for $j = m$,

$$(1 - \frac{1}{n})\frac{\partial}{\partial x_m} = \left[-ivx_m, J\right] \quad \text{and} \quad -iv(1 - \frac{1}{n})x_m = \left[-\frac{\partial}{\partial x_m}, J\right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{J} = \frac{i}{2}(-\frac{1}{v}\frac{\partial^2}{\partial x_m^2} + vx_m^2) - \frac{i}{2n}\sum_{k=1}^{n}(-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + vx_k^2) .$$

Therefore,

$$\tilde{W}(\exp tH_m) \tag{10c}$$

$$= \exp it\left(\frac{1}{2}(-\frac{1}{v}\frac{\partial^2}{\partial x_m^2} + vx_m^2) - \frac{1}{2n}\sum_{k=1}^{n}(-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + vx_k^2) - i\eta_m\right) ,$$

where η_m is any imaginary constant, satisfies

$$(\exp tH_m)U_N^v = \tilde{W}(\exp tH_m)^{-1}U_N^v\,\tilde{W}(\exp tH_m) .$$

Calculation of the values of the constants $\eta^{(1)}_{lm}$, $\eta^{(2)}_{lm}$ and η_m

If $\tilde{W}$ is to be an ordinary representation of SU(n), it must give an ordinary representation $\tilde{w}$ of su(n) (the Lie algebra of SU(n)). This leads to constraints on the values of the constants $\eta^{(1)}_{lm}$, $\eta^{(2)}_{lm}$ and η_m, as follows.

It may be shown that the commutation relations of su(n), in terms of the basis $R^{(1)}_{jk}$, $R^{(2)}_{jk}$, H_k (see (4)), are :

$$\left.\begin{aligned}
\left[R^{(1)}_{jk}, R^{(1)}_{lm}\right] &= -(\delta_{km}R^{(2)}_{jl} + \delta_{kl}R^{(2)}_{jm} + \delta_{jm}R^{(2)}_{kl} + \delta_{jl}R^{(2)}_{km}),\\
\left[R^{(1)}_{jk}, R^{(2)}_{lm}\right] &= -\delta_{km}R^{(1)}_{jl} + \delta_{kl}R^{(1)}_{jm} - \delta_{jm}R^{(1)}_{kl} + \delta_{jl}R^{(1)}_{km},\\
\left[R^{(2)}_{jk}, R^{(2)}_{lm}\right] &= -\delta_{km}R^{(2)}_{jl} + \delta_{kl}R^{(2)}_{jm} + \delta_{jm}R^{(2)}_{kl} - \delta_{jl}R^{(2)}_{km},\\
\left[R^{(1)}_{jk}, H_l\right] &= (\delta_{jl} - \delta_{kl})R^{(2)}_{jk},\\
\left[R^{(2)}_{jk}, H_l\right] &= (\delta_{kl} - \delta_{jl})R^{(1)}_{jk},\\
\left[H_l, H_m\right] &= 0,
\end{aligned}\right\} \qquad (11)$$

where, for convenience of notation, define

$$R^{(1)}_{jk} = R^{(1)}_{kj}, \quad R^{(2)}_{jk} = -R^{(2)}_{kj}, \quad \text{for } j>k;$$

$$H_1 = -\sum_{j=2}^{n} H_j;$$

$$R^{(1)}_{jj} = 2H_j, \quad R^{(2)}_{jj} = 0, \quad \text{for } j = 1,2,\ldots,n.$$

When the restrictions of $\tilde{W}$ to the one-parameter subgroups generated by $R^{(1)}_{jk}$, $R^{(2)}_{jk}$, H_k are those given by (10), then, in

the corresponding representation $\tilde{w}$ of su(n), the matrices $R^{(1)}_{jk}$, $R^{(2)}_{jk}$, H_k are represented respectively by

$$\left.\begin{aligned}
\tilde{w}(R^{(1)}_{jk}) &= i(-\frac{1}{v}\frac{\partial^2}{\partial x_j \partial x_k} + v x_j x_k) + \eta^{(1)}_{jk}, \\
\tilde{w}(R^{(2)}_{jk}) &= x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} + \eta^{(2)}_{jk}, \\
\tilde{w}(H_k) &= \frac{i}{2}((-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + v x_k^2) - \frac{1}{n}\sum_{m=1}^{n}(-\frac{1}{v}\frac{\partial^2}{\partial x_m^2} + v x_m^2)) + \eta_k,
\end{aligned}\right\} \quad (12)$$

where $\eta^{(1)}_{jk}$, $\eta^{(2)}_{jk}$ and η_k are imaginary constants.

Since $\tilde{w}$ is to be an ordinary representation of su(n), $\tilde{w}(R^{(1)}_{jk})$, $\tilde{w}(R^{(2)}_{jk})$, $\tilde{w}(H_k)$ must satisfy the commutation relations, (11), of su(n). Explicit calculation shows that this implies :

$$\delta_{km}\eta^{(2)}_{jl} + \delta_{kl}\eta^{(2)}_{jm} + \delta_{jm}\eta^{(2)}_{kl} + \delta_{jl}\eta^{(2)}_{km} = 0 , \quad (13a)$$

$$\delta_{km}\eta^{(1)}_{jl} - \delta_{kl}\eta^{(1)}_{jm} + \delta_{jm}\eta^{(1)}_{kl} - \delta_{jl}\eta^{(1)}_{km} = 0 , \quad (13b)$$

$$\delta_{km}\eta^{(2)}_{jl} - \delta_{kl}\eta^{(2)}_{jm} - \delta_{jm}\eta^{(2)}_{kl} + \delta_{jl}\eta^{(2)}_{km} = 0 , \quad (13c)$$

$$(-\delta_{jl} + \delta_{kl})\eta^{(2)}_{jk} = 0 , \quad (13d)$$

$$(-\delta_{kl} + \delta_{jl})\eta^{(1)}_{jk} = 0 , \quad (13e)$$

where, for convenience of notation, define

$$\eta^{(1)}_{jk} = \eta^{(1)}_{kj}, \quad \eta^{(2)}_{jk} = -\eta^{(2)}_{kj}, \quad \text{for } j > k ;$$

$$\eta_1 = -\sum_{j=2}^{n} \eta_j \ ;$$

$$\eta_{jj}^{(1)} = 2\eta_j \ , \quad \eta_{jj}^{(2)} = 0 \ , \qquad \text{for } j = 1,2,\ldots,n.$$

Conditions (13d) and (13e) give that $\eta_{jk}^{(1)} = 0 = \eta_{jk}^{(2)}$ for $j < k$ (and hence for $j > k$ aswell). Condition (13b) gives that, when $j = 1$, $k = m$ $(j < k)$, $\eta_{jj}^{(1)} - \eta_{kk}^{(1)} = 0$, so $\eta_j = \eta_k$; but $\sum_{j=1}^{n} \eta_j = 0$, so $\eta_j = 0$ for all $j = 1,2,\ldots,n$.

Hence the condition that $\tilde{w}$ is an ordinary representation of su(n) is sufficient to determine the η's uniquely : they must all be zero.

Therefore the ordinary representation W of SU(n) satisfying :

$$(A + iB)U_N^V = W(A + iB)^{-1} U_N^V \, W(A + iB) \ , \qquad (3)$$

for all $A + iB \in SU(n)$, is uniquely determined.

$W(A + iB)$ can be written down explicitly, once $A + iB$ has been expressed as a product of elements of the one-parameter subgroups of SU(n) generated by $R_{jk}^{(1)}$, $R_{jk}^{(2)}$ and H_k.

However, $W(A + iB)$ can be expressed more simply using the parametrisation (5) of SU(n). For, since SU(n) is a connected, simply connected, exponential Lie group, it follows that the ordinary representation of su(n) determined by $\tilde{w}$ (12), with all the η's zero, exponentiates to an ordinary representation of SU(n) [32]; moreover, this representation is

precisely the representation $\tilde{W}$ determined by (10), with all the η's zero.

Therefore, if $A + iB \in SU(n)$ is parametrised by $t^{(1)}_{jk}$, $t^{(2)}_{jk}$, $t_k \in \mathbb{R}$ $(1 \leqslant j < k \leqslant n)$, so that (5)

$$A + iB = \exp\left[\sum_{1\leqslant j<k\leqslant n}(t^{(1)}_{jk}R^{(1)}_{jk} + t^{(2)}_{jk}R^{(2)}_{jk}) + \sum_{k=2}^{n} t_k H_k\right] ,$$

then

$$W(A + iB) \tag{14}$$

$$= \exp\Bigg[\sum_{1\leqslant j<k\leqslant n}\Big(it^{(1)}_{jk}\Big(-\frac{1}{v}\frac{\partial^2}{\partial x_j \partial x_k} + v x_j x_k\Big) + t^{(2)}_{jk}\Big(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\Big)\Big) + \sum_{k=2}^{n}\frac{i}{2} t_k\Big(-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + v x_k^2 - \frac{1}{n}\sum_{m=1}^{n}\Big(-\frac{1}{v}\frac{\partial^2}{\partial x_m^2} + v x_m^2\Big)\Big)\Bigg]$$

is the unique ordinary representation of SU(n) satisfying

$$(A + iB)U^v_N = W(A + iB)^{-1} U^v_N W(A + iB) , \tag{3}$$

for all $A + iB \in SU(n)$.

7.4 The irreducible representations of N⊙SU(n)

It now follows from Mackey's theory for semi-direct product groups (§2.5) that every quantum mechanical irreducible (ordinary) representation of N⊙SU(n) is unitarily equivalent to one of the form :

$$U_N^v\, W\otimes\rho\ :\ (\sigma,\underline{\beta},\underline{\alpha},\ A+iB)\longrightarrow \tag{15}$$

$$\exp\, iv(\sigma - \underline{x}.\underline{\beta})\ \exp\Big(-\sum_{j=1}^{n}\alpha_j\,\frac{\partial}{\partial x_j}\Big)\ W(A+iB)\otimes\rho(A+iB)\ ,$$

where $(\sigma,\underline{\beta},\underline{\alpha},\ A+iB)\in N\circledS SU(n)$, W is the representation of $SU(n)$ defined by (14), ρ is an irreducible representation of $SU(n)$, and the operators in the first part of the inner Kronecker product are defined on a dense subspace of $L^2(\mathbb{R}^n,\ \mathbb{C})$.

7.5 The decomposition of the representation W of SU(n)

Some of the simplest of the quantum mechanical irreducible representations of $N\circledS SU(n)$ (15) are those for which ρ is the one-dimensional identity representation of $SU(n)$: $\rho(A+iB) = 1$ for all $A+iB\in SU(n)$. The restriction of $U_N^v\ W\otimes\rho$ to $SU(n)$ is then just the representation W of $SU(n)$.

In order to decompose W into irreducible representations, the closed subspaces of $L^2(\mathbb{R}^n,\ \mathbb{C})$ which are invariant and irreducible under W will be found. Suppose w is the representation of $su(n)$ corresponding to the representation W of $SU(n)$. Then a closed subspace Ω is invariant under W if and only if the dense subspace of Ω on which w is defined is invariant under w [31]. Hence it is sufficient to find the subspaces of $L^2(\mathbb{R}^n,\ \mathbb{C})$ which are invariant and irreducible under the representation w.

As a basis for the representation space $L^2(\mathbb{R}^n,\ \mathbb{C})$ of W, take the set $\{\psi_{\underline{m}} : \underline{m} = (m_1,m_2,\ldots,m_n)\in\mathbb{Z}_+^n\}$, where

$$\psi_{\underline{m}}(\underline{x}) = \prod_{j=1}^{n} u_{m_j}(|v|^{\frac{1}{2}}x_j) \ , \qquad \text{(as in 3-(16))}$$

with u_{m_j} a Hermite function of order m_j.

A Hermite function $u_m(x)$ of order m satisfies [11] :

$$\tfrac{1}{2}(-\frac{d^2}{dx^2} + x^2)u_m(x) = (m + \tfrac{1}{2})u_m(x) \ ,$$

$$xu_m(x) = \sqrt{\frac{m}{2}}\, u_{m-1}(x) + \sqrt{\frac{m+1}{2}}\, u_{m+1}(x) \ ,$$

$$\frac{d}{dx}u_m(x) = \sqrt{\frac{m}{2}}\, u_{m-1}(x) - \sqrt{\frac{m+1}{2}}\, u_{m+1}(x) \ .$$

Hence the function $\psi_m(x) = u_m(|v|^{\frac{1}{2}}x)$ satisfies :

$$\tfrac{1}{2}(-\frac{1}{v}\frac{d^2}{dx^2} + vx^2)\psi_m(x) = (\text{sgn } v)(m + \tfrac{1}{2})\psi_m(x) \ , \qquad (16a)$$

$$x\psi_m(x) = \frac{1}{|v|^{\frac{1}{2}}}(\sqrt{\frac{m}{2}}\,\psi_{m-1}(x) + \sqrt{\frac{m+1}{2}}\,\psi_{m+1}(x)) \ , \qquad (16b)$$

$$\frac{d}{dx}\psi_m(x) = |v|^{\frac{1}{2}}(\sqrt{\frac{m}{2}}\,\psi_{m-1}(x) - \sqrt{\frac{m+1}{2}}\,\psi_{m+1}(x)) \ . \qquad (16c)$$

Therefore, from (14) and (16),

$$w(R^{(1)}_{jk})\psi_{\underline{m}}(\underline{x}) = i(-\frac{1}{v}\frac{\partial^2}{\partial x_j \partial x_k} + vx_jx_k)\psi_{m_1}(x_1)\psi_{m_2}(x_2)\ldots\psi_{m_n}(x_n)$$

$$= i(\text{sgn } v)\Big[-\Big(\sqrt{\frac{m_j}{2}}\,\psi_{m_j-1}(x_j) - \sqrt{\frac{m_j+1}{2}}\,\psi_{m_j+1}(x_j)\Big)\,x$$

$$\Big(\sqrt{\frac{m_k}{2}}\,\psi_{m_k-1}(x_k) - \sqrt{\frac{m_k+1}{2}}\,\psi_{m_k+1}(x_k)\Big)$$

$$+\Big(\sqrt{\frac{m_j}{2}}\,\psi_{m_j-1}(x_j) + \sqrt{\frac{m_j+1}{2}}\,\psi_{m_j+1}(x_j)\Big)\,x$$

$$\Big(\sqrt{\frac{m_k}{2}}\,\psi_{m_k-1}(x_k) + \sqrt{\frac{m_k+1}{2}}\,\psi_{m_k+1}(x_k)\Big)\Big]\prod_{l\neq j,k}^{n}\psi_{m_l}(x_l)$$

$$= i(\text{sgn } v)\Big[\sqrt{m_j(m_k+1)}\,\psi_{m_j-1}(x_j)\psi_{m_k+1}(x_k) +$$

$$\sqrt{(m_j+1)m_k}\,\psi_{m_j+1}(x_j)\psi_{m_k-1}(x_k)\Big]\prod_{l\neq j,k}^{n}\psi_{m_l}(x_l)\ . \qquad (17a)$$

Similarly,

$$w(R^{(2)}_{jk})\psi_{\underline{m}}(\underline{x}) \qquad (17b)$$

$$= (\text{sgn } v)\Big[-\sqrt{m_j(m_k+1)}\,\psi_{m_j-1}(x_j)\psi_{m_k+1}(x_k)$$

$$+\sqrt{(m_j+1)m_k}\,\psi_{m_j+1}(x_j)\psi_{m_k-1}(x_k)\Big]\prod_{l\neq j,k}^{n}\psi_{m_l}(x_l)\ ,$$

$$w(H_k)\psi_{\underline{m}}(\underline{x}) = i(\text{sgn } v)\Big(m_k - \frac{1}{n}\sum_{l=1}^{n} m_l\Big)\psi_{\underline{m}}(\underline{x})\ . \qquad (17c)$$

Hence the representatives of each of the basis elements of su(n) transform any $L^2(\mathbb{R}^n, \mathbb{C})$ basis vector $\psi_{m_1}(x_1)\ldots\psi_{m_n}(x_n)$ with $\sum_{j=1}^{n} m_j$ = a into a linear combination of $L^2(\mathbb{R}^n, \mathbb{C})$ basis vectors, each of which has the sum of the subscripts of the

component functions also equal to a.

Therefore the subspace Ω_a of $L^2(\mathbb{R}^n, \mathbb{C})$ spanned by the set

$$\{\psi_{\underline{m}} : \sum_{j=1}^{n} m_j = a\} \tag{18}$$

contains a dense subspace on which the representation w of su(n) is defined, and which is invariant under w.

It follows that, for each $a \in \mathbb{Z}_+$, Ω_a is invariant under the representation W of SU(n). Hence

$$W = \bigoplus_{a=0}^{\infty} W_a \ , \tag{19}$$

where W_a is the restriction of W to the subspace Ω_a.

The representations W_a must now be decomposed into irreducible representations. This will be done with the aid of two preliminary lemmas.

Lemma A

Any one-dimensional subspace of Ω_a which is invariant under the subalgebra generated by $w(H_k)$ $(k = 2,\ldots,n)$ consists of scalar multiples of some $\psi_{\underline{m}}(\underline{x}) = \psi_{m_1}(x_1)\ldots\psi_{m_n}(x_n)$, where $\sum_{j=1}^{n} m_j = a$.

Proof

From (14) and (16), for each $\psi_{\underline{m}} \in \Omega_a$,

$$w(H_k)\psi_{\underline{m}} = i(\text{sgn } v)(m_k - \tfrac{a}{n})\psi_{\underline{m}} \qquad (k = 2,\ldots,n) \ . \tag{20}$$

Hence each $\psi_{\underline{m}}$ spans a one-dimensional subspace which is invariant under the subalgebra generated by $w(H_k)$ $(k = 2,\ldots,n)$.

Suppose $\phi_a \in \Omega_a$ also spans a one-dimensional subspace of Ω_a which is invariant under the subalgebra generated by $w(H_k)$ $(k = 2,\ldots,n)$. Then, for each $k = 2,\ldots,n$:

$$w(H_k)\phi_a = h_k\phi_a \ , \quad \text{for some } h_k \in \mathbb{C} \ . \tag{21}$$

Now Ω_a is spanned by the functions $\psi_{\underline{m}}$ for which $\sum_{j=1}^{n} m_j = a$. Therefore,

$$\phi_a = \sum_{\underline{m} \text{ such that } \sum_{j=1}^{n} m_j = a} c_{\underline{m}}\psi_{\underline{m}} \ , \quad \text{for some } c_{\underline{m}} \in \mathbb{C} \ . \tag{22}$$

Hence, using (20),

$$w(H_k)\phi_a = \sum c_{\underline{m}} w(H_k)\psi_{\underline{m}} = i(\text{sgn } v) \sum c_{\underline{m}}(m_k - \tfrac{a}{n})\psi_{\underline{m}} \ .$$

Also, from (21) and (22),

$$w(H_k)\phi_a = h_k\phi_a = h_k \sum c_{\underline{m}}\psi_{\underline{m}} \ .$$

Since the $\psi_{\underline{m}}$ are linearly independent, it follows that :

$$i(\text{sgn } v)c_{\underline{m}}(m_k - \tfrac{a}{n}) = h_k c_{\underline{m}} \ ,$$

for all $\underline{m} = (m_1,\ldots,m_n) \in \mathbb{Z}_+^n$ such that $\sum_{j=1}^{n} m_j = a$. So either $c_{\underline{m}} = 0$, or $m_k = \frac{a}{n} - ih_k$. Hence all non-zero terms in the sum

$\sum c_{\underline{m}}\psi_{\underline{m}}$ must have the same value of m_k, for each $k = 2,\ldots,n$; moreover, since $\sum_{j=1}^{n} m_j = a$ for each term in the sum $\sum c_{\underline{m}}\psi_{\underline{m}}$, it follows that each term must have the same value of m_1 aswell.

Hence $\phi_a = c_{\underline{m}}\psi_{\underline{m}}$ for some $\psi_{\underline{m}}$ satisfying $\sum_{j=1}^{n} m_j = a$.

Therefore any one-dimensional subspace of Ω_a which is invariant under the subalgebra generated by $w(H_k)$ $(k = 2,\ldots, n)$ consists of scalar multiples of some $\psi_{\underline{m}}$, with $\sum_{j=1}^{n} m_j = a$.

Lemma B

Each $\psi_{\underline{m}}$ is contained in an irreducible invariant subspace under the representation W.

Proof

Choose $\underline{m} = (m_1,\ldots,m_n) \in \mathbb{Z}_+^n$ arbitrarily.

Then $\psi_{\underline{m}} \in \Omega_a$, where $a = \sum_{j=1}^{n} m_j$.

Suppose Ω_a splits into $s(a)$ irreducible invariant subspaces under W :

$$\Omega_a = \bigoplus_{r=1}^{s(a)} \Omega_{ar} ,$$

where Ω_{ar} is an irreducible invariant subspace under W. Then

$$\psi_{\underline{m}} = \sum_{r=1}^{s(a)} \phi_{ar} \ , \quad \text{where } \phi_{ar} \in \Omega_{ar} \ . \tag{23}$$

So $w(H_k)\psi_{\underline{m}} = \sum_{r=1}^{s(a)} w(H_k)\phi_{ar} = \sum_{r=1}^{s(a)} \phi'_{ar}$,

where $\phi'_{ar} = w(H_k)\phi_{ar} \in \Omega_{ar}$, since Ω_{ar} is invariant under W, and therefore under $w(H_k)$.

But also, from (20) and (23),

$$w(H_k)\psi_{\underline{m}} = i(\text{sgn } v)(m_k - \tfrac{a}{n})\psi_{\underline{m}} = i(\text{sgn } v)(m_k - \tfrac{a}{n}) \sum_{r=1}^{s(a)} \phi_{ar} \ .$$

Since ϕ_{ar} and $\phi'_{ar} \in \Omega_{ar}$, and the Ω_{ar} (r = 1,...,s(a)) are disjoint subspaces, it follows that

$$\phi'_{ar} = w(H_k)\phi_{ar} = i(\text{sgn } v)(m_k - \tfrac{a}{n})\phi_{ar} \ .$$

Hence ϕ_{ar} spans a one-dimensional subspace of Ω_a, which is invariant under the subalgebra generated by $w(H_k)$ (k = 2,..., n). So, by lemma A,

$$\phi_{ar} = c_{\underline{m}_r} \psi_{\underline{m}_r} \ ,$$

for some $c_{\underline{m}_r} \in \mathbb{C}$ and $\psi_{\underline{m}_r}$ satisfying $\sum_{j=1}^{n} (m_r)_j = a$. Since the ψ's are linearly independent functions, it follows, from (23), that the chosen $\psi_{\underline{m}}$ is itself of the form ϕ_{ar} (for some r), and so $\psi_{\underline{m}} \in \Omega_{ar}$.

Therefore each $\psi_{\underline{m}}$ belongs to an irreducible invariant subspace under the representation W.

Proposition

For each $a \in \mathbb{Z}_+$, Ω_a is an irreducible invariant subspace under the representation W.

Proof

Choose $a \in \mathbb{Z}_+$ arbitrarily.

From lemma B, $\psi_{(a,0,\ldots,0)}(\underline{x}) = \psi_a(x_1)\psi_0(x_2)\ldots\psi_0(x_n)$ belongs to an irreducible invariant subspace, $\mathcal{H}$ say, of Ω_a.

Now, from (17b),

$$w(R_{12}^{(2)})\psi_{(a,0,\ldots,0)}(\underline{x})$$

$$= -(\operatorname{sgn} v)\sqrt{a}\,\psi_{a-1}(x_1)\psi_1(x_2)\psi_0(x_3)\ldots\psi_0(x_n) ,$$

unless $a = 0$.

If $a = 0$, then $\psi_{(a,0,\ldots,0)} = \psi_{\underline{0}}$ spans a one-dimensional subspace, Ω_0, invariant under W, since it is the only vector of the form $\psi_{\underline{m}}$ for which $\sum_{j=1}^{n} m_j = 0$.

If $a \neq 0$, then $\psi_{(a-1,1,0,\ldots,0)} \in \mathcal{H}$, since $\mathcal{H}$ is invariant under W and therefore under $w(R_{12}^{(2)})$.

Using (17b) again,

$$w(R_{12}^{(2)})\psi_{(a-1,1,0,\ldots,0)}(\underline{x})$$

$$= (\operatorname{sgn} v)(-\sqrt{2(a-1)}\,\psi_{a-2}(x_1)\psi_2(x_2) + \sqrt{a}\,\psi_a(x_1)\psi_0(x_2))$$

$$\psi_0(x_3)\ldots\psi_0(x_n) .$$

Since $\psi_{(a,0,\ldots,0)} \in \Theta$, it follows that, unless a = 1, $\psi_{(a-2,2,0,\ldots,0)} \in \Theta$ aswell.

By induction, it may be shown that the invariant subspace Θ containing $\psi_{(a,0,\ldots,0)}$ also contains $\psi_{(a-j,j,0,\ldots,0)}$, for j = 1,2,...,a.

Choose arbitrarily $\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n$ satisfying $\sum_{j=1}^{n} m_j = a$.

Since $\psi_{(a,0,\ldots,0)} \in \Theta$, it follows, from above, that $\psi_{(a-m_2,m_2,0,\ldots,0)} \in \Theta$.

A similar argument to that given above, using $R_{13}^{(2)}$ instead of $R_{12}^{(2)}$, gives that $\psi_{(a-m_2-m_3,m_2,m_3,0,\ldots,0)} \in \Theta$.

Similarly, using $R_{1k}^{(2)}$ (k = 4,...,n), it follows that $\psi_{(a-\sum_{j=2}^{n} m_j, m_2, m_3, \ldots, m_n)} \in \Theta$.

So $\psi_{(m_1,m_2,\ldots,m_n)} \in \Theta$, since $\sum_{j=1}^{n} m_j = a$.

Hence Θ contains all the basis functions $\psi_{\underline{m}}$ for which $\sum_{j=1}^{n} m_j = a$, and so $\Theta = \Omega_a$.

Therefore Ω_a is an irreducible invariant subspace under W.

(This completes the proof of the proposition.)

It follows, from (19), that the decomposition of the representation W of SU(n) is :

$$W = \bigoplus_{a=0}^{\infty} W_a \ , \tag{24}$$

where each W_a is an irreducible representation (namely the restriction of W to the subspace Ω_a).

7.6 N⊛SU(n) as a non-invariance group

The dimension of Ω_a is the number of distinct ways of choosing $\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n$ such that $\sum_{j=1}^{n} m_j = a$. This is precisely the degeneracy of the $(a + 1)$th energy level of the n-dimensional isotropic harmonic oscillator.

The results of §7.4 and §7.5 then lead to the following.

Theorem

For each $v\ (\neq 0) \in \mathbb{R}$, the ordinary representation $U_N^v W$ of N⊛SU(n) given by

$$U_N^v W : (\sigma, \underline{\beta}, \underline{\alpha}, A + iB) \longrightarrow \tag{25}$$

$$\exp iv(\sigma - \underline{x}.\underline{\beta}) \exp\left(- \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j}\right) W(A + iB)$$

(where W is the representation of SU(n) defined by (14) and the operators are defined on a dense subspace of $L^2(\mathbb{R}^n, \mathbb{C})$), is irreducible as a representation of N⊛SU(n).

The restriction of $U_N^v W$ to SU(n), which is a degeneracy group for the n-dimensional isotropic harmonic oscillator, is

$$(U_N^v W)\downarrow SU(n) = W = \bigoplus_{a=0}^{\infty} W_a , \tag{26}$$

where W_a is an irreducible representation of SU(n), of dimension equal to the degeneracy of the $(a + 1)$th energy level of the n-dimensional isotropic harmonic oscillator.

Therefore, N⊛SU(n) is a non-invariance group for the n-dimensional isotropic harmonic oscillator.

7.7 Expression of the irreducible representation W_a of SU(n) as a Young tableau

For comparison with §6.4 and for later use, the irreducible representation W_a of SU(n) will be expressed as a Young tableau. To do this, some general results will be quoted (from a review article by Itzykson and Nauenberg [21]), and then applied to the representation W_a.

The generators H_k (k = 2,...,n) of SU(n) (4), together with the operator H_1, defined by

$$H_1 = - \sum_{k=2}^{n} H_k \ , \tag{27}$$

commute among themselves, and hence they can be simultaneously diagonalised.

In any particular representation $\tilde{H}_j$ of H_j (j = 1,2,...,n), a set of n real eigenvalues μ_j (j = 1,2,...,n) satisfying

$$- i\tilde{H}_j \phi = \mu_j \phi \tag{28}$$

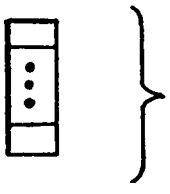
is called a weight of the representation.

It can be proved that the irreducible representations of SU(n) are uniquely characterised by their highest weight. The adjective highest refers to an ordering of the weights of a representation in which $(\mu_1, \mu_2, \ldots, \mu_n)$ is defined to be higher

than $(\mu'_1,\mu'_2,\ldots,\mu'_n)$ if the first non-vanishing difference $\mu_j - \mu'_j$ is greater than zero. It can be shown that the μ_j's of a particular weight of a representation, aswell as satisfying the condition $\sum_{j=1}^{n} \mu_j = 0$, are at most fractions with denominator n, which differ by integers. Furthermore, the highest weight is a linear combination with non-negative integral coefficients of (n - 1) fundamental ones :

$$\left.\begin{aligned}
M^{(1)} &= \left(\frac{n-1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}\right), \\
M^{(2)} &= \left(\frac{n-2}{n}, \frac{n-2}{n}, -\frac{2}{n}, \ldots, -\frac{2}{n}\right), \\
M^{(3)} &= \left(\frac{n-3}{n}, \frac{n-3}{n}, \frac{n-3}{n}, -\frac{3}{n}, \ldots, -\frac{3}{n}\right), \\
&\vdots \\
M^{(n-1)} &= \left(\frac{1}{n}, \ldots, \frac{1}{n}, -\frac{n-1}{n}\right).
\end{aligned}\right\} \quad (29)$$

$M^{(j)}$ (j = 1,2,..., n - 1) corresponds to the irreducible representation in terms of antisymmetric tensors of rank j :



Now, with each Young tableau, or irreducible representation of SU(n), two sets of (n - 1) numbers may be associated :

(i) $\xi_1, \xi_2, \ldots, \xi_{n-1}$ ($\xi_n = 0$) giving the number of boxes in each row;

(ii) $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, where λ_j is the number of columns of length j (j = 1,2,..., n - 1).

The highest weight M of the representation $(\xi_1, \ldots, \xi_{n-1}, 0)$ is directly related to the second set by :

$$M = \lambda_1 M^{(1)} + \lambda_2 M^{(2)} + \ldots + \lambda_{n-1} M^{(n-1)} \ . \qquad (30)$$

Now suppose, in particular, that the representation $\tilde{H}_j$ of H_j (j = 1,2,...,n) is that corresponding to the representation W_a of SU(n).

Then, from lemma A of §7.5, it follows that any simultaneous eigenfunction of $\tilde{H}_2, \ldots, \tilde{H}_n$ (and hence, by (27), also of $\tilde{H}_1$) is a scalar multiple $c_{\underline{m}} \psi_{\underline{m}}$ of some $\psi_{\underline{m}}$; the eigenvalues of the $\tilde{H}_j$ are, of course, independent of $c_{\underline{m}}$.

For $\psi_{\underline{m}} \in \Omega_a$ and for each j = 1,2,...,n, (20) gives that :

$$- i\tilde{H}_j \psi_{\underline{m}} = (\text{sgn } v)(m_j - \tfrac{a}{n}) \psi_{\underline{m}} \ .$$

Hence the weights of the irreducible representation W_a of SU(n) are (cf. (28)) :

$$\left\{ (\text{sgn } v)(m_1 - \tfrac{a}{n} \ , \ m_2 - \tfrac{a}{n} \ , \ldots, \ m_n - \tfrac{a}{n} \ ) : \sum_{j=1}^{n} m_j = a \right\}. \qquad (31)$$

Firstly, suppose $v > 0$; then sgn v = 1. In this case, the highest weight is that given by $m_1 = a$, $m_2 = \ldots = m_n = 0$, namely, using (29),

$$\left(a - \tfrac{a}{n} \ , \ - \tfrac{a}{n} \ , \ldots, \ - \tfrac{a}{n}\right) = a\left(\tfrac{n-1}{n} \ , \ - \tfrac{1}{n} \ , \ldots, \ - \tfrac{1}{n}\right) = aM^{(1)} \ .$$

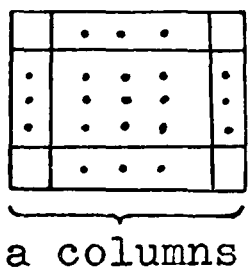
For $v>0$, W_a is therefore, from (30), the irreducible representation of SU(n) whose Young tableau has a columns of length 1 and no columns of length greater than 1 :

$$W_a = \underbrace{\square\cdots\square}_{a \text{ boxes}} = (a,0,\ldots,0) \quad (v>0) \,. \tag{32a}$$

Secondly, suppose $v<0$; then sgn $v = -1$. In this case, from (31), the highest weight is that given by $m_1 = \ldots = m_{n-1} = 0$, $m_n = a$, namely, using (29),

$$\left(\frac{a}{n}, \ldots, \frac{a}{n}, -a + \frac{a}{n}\right) = a\left(\frac{1}{n}, \ldots, \frac{1}{n}, -\frac{n-1}{n}\right) = aM^{(n-1)} \,.$$

For $v<0$, W_a is therefore, from (30), the irreducible representation of SU(n) whose Young tableau has a columns of length n - 1 :

$W_a =$  $= (a,\ldots,a,0) \quad (v<0)$. (32b)

It follows, from (24) and (32), that the representation W of SU(n) can be expressed as :

$$W = \bigoplus_{a=0}^{\infty} W_a = \bigoplus_{a=0}^{\infty} (a,0,\ldots,0) \quad \text{when } v>0 \,, \tag{33a}$$

$$W = \bigoplus_{a=0}^{\infty} W_a = \bigoplus_{a=0}^{\infty} (a,\ldots,a,0) \quad \text{when } v<0 \,. \tag{33b}$$

CHAPTER 8

THE QUANTUM MECHANICAL IRREDUCIBLE REPRESENTATIONS OF L AND SOME OF THEIR PROPERTIES

This chapter is mainly a preliminary to chapter 9, in which van Hove's representation $R^V \downarrow L$ of L is decomposed into irreducible representations of L.

First, all the quantum mechanical irreducible representations of $L = N \circledS Sp(2n, \mathbb{R})$ are found, by using Mackey's theory for semi-direct product groups, together with an extension of the method of §2.6 for evaluating the projective representation W. Next, it is shown that L can be regarded as a local non-invariance group for the n-dimensional isotropic harmonic oscillator. (The meaning of the description 'local' is explained later.) Then, to obtain more information about the structure of the representations of L, the projective representation W of $Sp(2n, \mathbb{R})$ is decomposed into irreducible projective representations of $Sp(2n, \mathbb{R})$.

8.1 The little group of U_N^V

From 4-(7), the group law of L is given by :

$$(\sigma', \underline{\beta}', \underline{\alpha}', S')(\sigma, \underline{\beta}, \underline{\alpha}, S) = (\sigma'', \underline{\beta}'', \underline{\alpha}'', S'') \ , \qquad (1)$$

where $$\sigma'' = \sigma' + \sigma + \tfrac{1}{2}(A'\underline{\alpha}.C'\underline{\alpha} + B'\underline{\beta}.D'\underline{\beta}) + B'\underline{\beta}.C'\underline{\alpha}$$

$$+ \underline{\alpha}'.(D'\underline{\beta} + C'\underline{\alpha}) \ ,$$

$$\underline{\beta}'' = D'\underline{\beta} + C'\underline{\alpha} + \underline{\beta}' \ ,$$

$$\underline{\alpha}'' = B'\underline{\beta} + A'\underline{\alpha} + \underline{\alpha}' \ ,$$

$$S'' = S'S \ ,$$

with $S' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(2n, \mathbb{R})$, and $S \in Sp(2n, \mathbb{R})$.

Every quantum mechanical irreducible representation of N is unitarily equivalent to one of the form 3-(11) :

$$\left[U_N^{\nu}(\sigma, \underline{\beta}, \underline{\alpha}, I)\psi\right](\underline{x}) = \exp i\nu(\sigma - \underline{x}.\underline{\beta})\ \psi(\underline{x} - \underline{\alpha}) \ ,$$

where $\nu\ (\neq 0) \in \mathbb{R}, \psi \in L^2(\mathbb{R}^n, \mathbb{C})$.

The action of $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$ on U_N^{ν} is, by definition 2-(10),

$$\left[(SU_N^{\nu})(\sigma, \underline{\beta}, \underline{\alpha}, I)\psi\right](\underline{x})$$

$$= \left[U_N^{\nu}((0, \underline{0}, \underline{0}, S)^{-1}(\sigma, \underline{\beta}, \underline{\alpha}, I)(0, \underline{0}, \underline{0}, S))\psi\right](\underline{x})$$

$$= \left[U_N^{\nu}((0, \underline{0}, \underline{0}, \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix})(\sigma, \underline{\beta}, \underline{\alpha}, S))\psi\right](\underline{x})$$

$$= \left[U_N^{\nu}(\sigma + \tfrac{1}{2}(-D^T\underline{\alpha}.C^T\underline{\alpha} - B^T\underline{\beta}.A^T\underline{\beta}) + B^T\underline{\beta}.C^T\underline{\alpha}\ ,\ A^T\underline{\beta} - C^T\underline{\alpha},\right.$$

$$\left. - B^T\underline{\beta} + D^T\underline{\alpha},\ I)\psi\right](\underline{x}) \qquad \text{from (1)}$$

$$= \exp i\nu(\sigma + \tfrac{1}{2}(-D^T\underline{\alpha}.C^T\underline{\alpha} - B^T\underline{\beta}.A^T\underline{\beta}) + B^T\underline{\beta}.C^T\underline{\alpha}$$

$$- \underline{x}.(A^T\underline{\beta} - C^T\underline{\alpha}))\ \psi(\underline{x} + B^T\underline{\beta} - D^T\underline{\alpha}) \ . \qquad (2)$$

SU_N^v is an irreducible representation of N. Restricted to the subgroup $\{(\sigma,\underline{0},\underline{0},I)\}$, it is just the phase exp ivσ. Hence SU_N^v must be unitarily equivalent to U_N^v, since U_N^v is the only irreducible representation of N which has the required form on restriction to the subgroup $\{(\sigma,\underline{0},\underline{0},I)\}$.

Therefore, there exists a unitary operator W(S), dependent on S, such that

$$SU_N^v = W(S)^{-1} U_N^v W(S) \quad \text{for each } S \in Sp(2n, \mathbb{R}) . \qquad (3)$$

The orbits of the quantum mechanical part of $\hat{N}$ under the action of $Sp(2n, \mathbb{R})$ thus consist of single points U_N^v. The little group of U_N^v under the action of $Sp(2n, \mathbb{R})$ is the whole of $Sp(2n, \mathbb{R})$, and so the isotropy group of U_N^v is $N \circledS Sp(2n, \mathbb{R})$.

8.2 A method of finding the operator W for the group Sp(2n, $\mathbb{R}$)

The group $Sp(2n, \mathbb{R})$ possesses non-trivial multipliers, and so the previous method of finding W (§2.6) (where it was assumed that the little group had no non-trivial multipliers) cannot be applied directly in this case. However, since $Sp(2n, \mathbb{R})$ is a connected semi-simple Lie group, every multiplier is locally trivial (§2.4(e)). It follows that the (possibly) projective representation W of $Sp(2n, \mathbb{R})$ can be chosen in such a way that the corresponding representation w of the Lie algebra of $Sp(2n, \mathbb{R})$ is ordinary.

Suppose $h_i(t)$ is a one-dimensional Lie subgroup of $Sp(2n, \mathbb{R})$. For each one-parameter subgroup $n_k(\mu)$ of N, let

$Z_{ik}(t)$ be the unique skew-adjoint operator defined by

$$(h_i(t)U_N^v)(n_k(\mu)) = \exp \mu Z_{ik}(t) \quad \text{(cf. 2-(17))} .$$

Then (cf. 2-(22)) the skew-adjoint operator $\left.\frac{dW(h_i(t))}{dt}\right|_{t=0}$ is determined up to an arbitrary imaginary constant η_i by

$$\left.\frac{dZ_{ik}(t)}{dt}\right|_{t=0} = \left[Z_{ik}(0), \left.\frac{dW(h_i(t))}{dt}\right|_{t=0}\right] \quad \text{for each } k . \quad (4)$$

(The proof of this result is similar to the corresponding one of §2.6, with $\exp tJ$ replaced by $W(h_i(t))$.)

Since the operators $\left.\frac{dW(h_i(t))}{dt}\right|_{t=0}$ can be chosen to give an ordinary representation w of the Lie algebra of $Sp(2n, \mathbb{R})$, there are constraints on the constants η_i; as will be shown, these are sufficient to determine the constants η_i uniquely.

This ordinary representation w of the Lie algebra of $Sp(2n, \mathbb{R})$ may be extended uniquely to an ordinary representation of the connected, simply connected, covering group $\widetilde{Sp(2n, \mathbb{R})}$ of $Sp(2n, \mathbb{R})$ [32]. This ordinary representation of $\widetilde{Sp(2n, \mathbb{R})}$ may in turn be used to determine the required projective representation W of $Sp(2n, \mathbb{R})$.

8.3 The Lie algebra of $Sp(2n, \mathbb{R})$

As a first step towards evaluating the representation w, a basis for the Lie algebra of $Sp(2n, \mathbb{R})$ will be obtained.

The symplectic group $Sp(2n, \mathbb{R})$ is the set of matrices $S \in GL(2n, \mathbb{R})$ for which

$$S^T J S = J \ ,$$

where $J = \sum_{j=1}^{n} (E_{j,j+n} - E_{j+n,j})$ (with E_{jk} a $2n \times 2n$ matrix having 1 in the (j,k) position and zeros elsewhere).

Let $S(t)$ be an analytic curve in $Sp(2n, \mathbb{R})$, and suppose $S(0) = I$. Then, differentiating :

$$S(t)^T J S(t) = J \ ,$$

with respect to t, and putting $t = 0$, gives

$$S'(0)^T J + J S'(0) = 0 \ ,$$

where ' denotes differentiation with respect to t. Hence the Lie algebra of $Sp(2n, \mathbb{R})$ is the algebra of all matrices of the form $\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, where A, B and C are $n \times n$ real matrices, with A arbitrary, $B = B^T$ and $C = C^T$; this algebra will be denoted by $sp(2n, \mathbb{R})$.

From 4-(11), the group $U(n)$ is isomorphic to the subgroup of $Sp(2n, \mathbb{R})$ of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Hence the Lie algebra of $U(n)$, denoted by $u(n)$, can be regarded as a subalgebra of $sp(2n, \mathbb{R})$; this subalgebra consists of

matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A and B are n x n real matrices, with $A = -A^T$, $B = B^T$.

su(n) can in turn be regarded as the subalgebra of u(n) consisting of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A and B are n x n real matrices, with $A = -A^T$, $B = B^T$, trace A = 0, trace B = 0 (from §7.2).

However, for ease of calculation, instead of choosing a basis of $sp(2n, \mathbb{R})$ which extends one of su(n), choose the basis :

$$\left.\begin{aligned} X_{jk} &= \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix} && (1 \leqslant j, k \leqslant n), \\ Y_{jk} &= \begin{pmatrix} 0 & E_{jk} + E_{kj} \\ 0 & 0 \end{pmatrix} && (1 \leqslant j \leqslant k \leqslant n), \\ Z_{jk} &= \begin{pmatrix} 0 & 0 \\ E_{jk} + E_{kj} & 0 \end{pmatrix} && (1 \leqslant j \leqslant k \leqslant n) \end{aligned}\right\} \quad (5)$$

(where E_{lm} is an n x n matrix having 1 in the (l,m) position and zeros elsewhere).

The basis elements of the subalgebra su(n) which correspond to the basis elements : $R^{(1)}_{jk}$, $R^{(2)}_{jk}$, H_k, used previously (7-(4)) are then, respectively :

$$\hat{R}^{(1)}_{jk} = Y_{jk} - Z_{jk} \qquad (1 \leqslant j < k \leqslant n), \qquad (6a)$$

$$\hat{R}^{(2)}_{jk} = X_{jk} - X_{kj} \qquad (1 \leqslant j < k \leqslant n)\,, \quad (6b)$$

$$\hat{H}_k = \tfrac{1}{2}(Y_{kk} - Z_{kk} - \frac{1}{n}\sum_{l=1}^{n}(Y_{ll} - Z_{ll})) \qquad (2 \leqslant k \leqslant n)\,. \quad (6c)$$

8.4 The projective representation W of Sp(2n, $\mathbb{R}$)

General conditions to be satisfied

The operator W(S) satisfies

$$SU^V_N = W(S)^{-1} U^V_N W(S) \quad \text{for each } S \in Sp(2n, \mathbb{R})\,. \qquad (3)$$

Using 3-(14) :

$$(\sigma, \underline{\beta}, \underline{\alpha}, I) = (\sigma, \underline{0}, \underline{0}, I)\prod_{j=1}^{n}(0, \beta_j \underline{e}_j, \underline{0}, I)\prod_{j=1}^{n}(0, \underline{0}, \alpha_j \underline{e}_j, I)\,,$$

together with Stone's theorem for unitary representations of one-parameter subgroups of N, it follows, from (2), that the ordinary representation SU^V_N of N may be expressed as

$$\left[(SU^V_N)(\sigma, \underline{\beta}, \underline{\alpha}, I)\psi\right](\underline{x})$$

$$= \exp\, iv\sigma \prod_{j=1}^{n} \exp \beta_j \left[- iv \sum_{k=1}^{n} a_{jk} x_k + \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k}\right]$$

$$\prod_{j=1}^{n} \exp \alpha_j \left[iv \sum_{k=1}^{n} c_{jk} x_k - \sum_{k=1}^{n} d_{jk} \frac{\partial}{\partial x_k}\right] \psi(\underline{x})\,,$$

where $S = \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix}$.

Hence, if $h(t) = \begin{pmatrix} a_{jk}(t) & b_{jk}(t) \\ c_{jk}(t) & d_{jk}(t) \end{pmatrix}$ is a one-dimensional Lie subgroup of the little group $Sp(2n, \mathbb{R})$, the representatives of the generators of N in the ordinary representation $h(t)U_N^v$ of N are, with the previous notation (cf. §3.5) :

$$Z_\sigma(t) = iv \ ,$$

$$Z_{\beta_j}(t) = - iv \sum_{k=1}^{n} a_{jk}(t)x_k + \sum_{k=1}^{n} b_{jk}(t) \frac{\partial}{\partial x_k} \ , \qquad (7)$$

$$Z_{\alpha_j}(t) = iv \sum_{k=1}^{n} c_{jk}(t)x_k - \sum_{k=1}^{n} d_{jk}(t) \frac{\partial}{\partial x_k} \ .$$

From (4), it follows that the skew-adjoint operator $\left.\frac{dW(h(t))}{dt}\right|_{t=0}$ satisfies

$$\left.\frac{dZ_\mu(t)}{dt}\right|_{t=0} = \left[Z_\mu(0), \left.\frac{dW(h(t))}{dt}\right|_{t=0}\right] , \qquad (8)$$

for all $\mu \in \{\sigma, \beta_1, \beta_2, \ldots, \beta_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Explicitly, from (7) and (8), $\left.\frac{dW(h(t))}{dt}\right|_{t=0} = w(h'(0))$ must satisfy :

when $\mu = \sigma$, $\qquad 0 = [iv, w(h'(0))]$,

which is satisfied by any $w(h'(0))$;

when $\mu = \beta_j$,

$$- iv \sum_{k=1}^{n} a'_{jk}(0) x_k + \sum_{k=1}^{n} b'_{jk}(0) \frac{\partial}{\partial x_k} = \left[- ivx_j, w(h'(0)) \right] \quad ;(9)$$

when $\mu = \alpha_j$,

$$iv \sum_{k=1}^{n} c'_{jk}(0) x_k - \sum_{k=1}^{n} d'_{jk}(0) \frac{\partial}{\partial x_k} = \left[- \frac{\partial}{\partial x_j} , w(h'(0)) \right] \quad (10)$$

(where ' denotes differentiation with respect to t).

The operator $w(X_{lm})$

$$X_{lm} = \begin{pmatrix} E_{lm} & 0 \\ 0 & - E_{ml} \end{pmatrix} \qquad (1 \leqslant l , m \leqslant n) \qquad \text{(from (5))} .$$

Suppose, with the above notation, that X_{lm} is the tangent at the identity of a one-dimensional Lie subgroup $h(t) = \begin{pmatrix} a_{jk}(t) & b_{jk}(t) \\ c_{jk}(t) & d_{jk}(t) \end{pmatrix}$ in $Sp(2n, \mathbb{R})$. Then

$$X_{lm} = h'(0) = \begin{pmatrix} a'_{jk}(0) & b'_{jk}(0) \\ c'_{jk}(0) & d'_{jk}(0) \end{pmatrix} .$$

In this case, the only non-zero elements of $h'(0)$ are :

$$a'_{lm}(0) = 1 , \qquad d'_{ml}(0) = - 1 .$$

Hence, the conditions (9) and (10) become :

for $j \neq 1$ or m,

$$0 = \left[- ivx_j,\ w(X_{lm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} ,\ w(X_{lm}) \right];$$

for $j = 1$,

$$- ivx_m = \left[- ivx_l,\ w(X_{lm}) \right] \quad \text{and} \quad \delta_{lm} \frac{\partial}{\partial x_l} = \left[- \frac{\partial}{\partial x_l} ,\ w(X_{lm}) \right];$$

for $j = m$,

$$- iv\delta_{lm} x_m = \left[- ivx_m,\ w(X_{lm}) \right] \quad \text{and} \quad \frac{\partial}{\partial x_l} = \left[- \frac{\partial}{\partial x_m} ,\ w(X_{lm}) \right].$$

If $1 \neq m$, these relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(X_{lm}) = - x_m \frac{\partial}{\partial x_l} + \alpha_{lm} ,$$

where α_{lm} is an arbitrary imaginary constant.

If $1 = m$, these relations are all satisfied by

$$\tilde{w}(X_{mm}) = - x_m \frac{\partial}{\partial x_m} + \epsilon_m ,$$

where ϵ_m is an arbitrary constant. $\tilde{w}(X_{mm})$ is a skew-adjoint operator provided $\tilde{w}(X_{mm})^* = - \tilde{w}(X_{mm})$ (where * denotes adjoint). Now

$$\tilde{w}(X_{mm})^* = (- \frac{\partial}{\partial x_m})^* x_m^* + \overline{\epsilon_m} = \frac{\partial}{\partial x_m} x_m + \overline{\epsilon_m} = x_m \frac{\partial}{\partial x_m} + 1 + \overline{\epsilon_m} .$$

So $\tilde{w}(X_{mm})^* = - \tilde{w}(X_{mm})$ if $1 + \overline{\epsilon_m} = - \epsilon_m$; thus $\epsilon_m = - \frac{1}{2} + \alpha_{mm}$, where α_{mm} is an arbitrary imaginary constant.

Therefore, for $1 \leqslant l , m \leqslant n$,

$$\tilde{w}(X_{lm}) = - x_m \frac{\partial}{\partial x_l} - \tfrac{1}{2}\delta_{lm} + \alpha_{lm} . \qquad (11a)$$

The operator $w(Y_{lm})$ $(l < m)$

$$Y_{lm} = \begin{pmatrix} 0 & E_{lm} + E_{ml} \\ 0 & 0 \end{pmatrix} \qquad (1 \leqslant l < m \leqslant n) .$$

In this case, the only non-zero elements of $h'(0)$ are :

$$b'_{lm}(0) = 1 = b'_{ml}(0) .$$

Hence, (9) and (10) become :

for $j \neq l$ or m,

$$0 = \left[- ivx_j, w(Y_{lm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} , w(Y_{lm}) \right] ;$$

for $j = l$ $(\neq m)$,

$$\frac{\partial}{\partial x_m} = \left[- ivx_l, w(Y_{lm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_l} , w(Y_{lm}) \right] ;$$

for $j = m$ $(\neq l)$,

$$\frac{\partial}{\partial x_l} = \left[- ivx_m, w(Y_{lm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_m} , w(Y_{lm}) \right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Y_{lm}) = \frac{1}{iv} \frac{\partial^2}{\partial x_l \partial x_m} + \beta_{lm} , \qquad (11b)$$

where β_{lm} is an arbitrary imaginary constant.

The operator $w(Y_{mm})$

$$Y_{mm} = \begin{pmatrix} 0 & 2E_{mm} \\ 0 & 0 \end{pmatrix} \qquad (1 \leqslant m \leqslant n) \ .$$

In this case, the only non-zero element of $h'(0)$ is :

$$b'_{mm}(0) = 2 \ .$$

Hence, (9) and (10) become :

for $j \neq m$,

$$0 = \left[- ivx_j, \ w(Y_{mm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} \ , \ w(Y_{mm}) \right] \ ;$$

for $j = m$,

$$2 \frac{\partial}{\partial x_m} = \left[- ivx_m, \ w(Y_{mm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_m} \ , \ w(Y_{mm}) \right] \ .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Y_{mm}) = \frac{1}{iv} \frac{\partial^2}{\partial x_m^2} + \beta_{mm} \ , \qquad (11c)$$

where β_{mm} is an arbitrary imaginary constant.

The operator $w(Z_{lm})$ $(l < m)$

$$Z_{lm} = \begin{pmatrix} 0 & 0 \\ E_{lm} + E_{ml} & 0 \end{pmatrix} \qquad (1 \leqslant l < m \leqslant n) \ .$$

In this case, the only non-zero elements of h'(0) are :

$$c'_{lm}(0) = 1 = c'_{ml}(0) \ .$$

Hence, (9) and (10) become :

for $j \neq l$ or m,

$$0 = \left[- ivx_j, \ w(Z_{lm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} , \ w(Z_{lm}) \right] ;$$

for $j = l \ (\neq m)$,

$$0 = \left[- ivx_l, \ w(Z_{lm}) \right] \quad \text{and} \quad ivx_m = \left[- \frac{\partial}{\partial x_l} , \ w(Z_{lm}) \right] ;$$

for $j = m \ (\neq l)$,

$$0 = \left[- ivx_m, \ w(Z_{lm}) \right] \quad \text{and} \quad ivx_l = \left[- \frac{\partial}{\partial x_m} , \ w(Z_{lm}) \right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Z_{lm}) = - ivx_l x_m + \gamma_{lm} \ , \qquad (11d)$$

where γ_{lm} is an arbitrary imaginary constant.

The operator $w(Z_{mm})$

$$Z_{mm} = \begin{pmatrix} 0 & 0 \\ 2E_{mm} & 0 \end{pmatrix} \qquad (1 \leqslant m \leqslant n) \ .$$

In this case, the only non-zero element of h'(0) is :

$$c'_{mm}(0) = 2 \ .$$

Hence, (9) and (10) become :

for $j \neq m$,

$$0 = \left[- ivx_j, \; w(Z_{mm}) \right] \quad \text{and} \quad 0 = \left[- \frac{\partial}{\partial x_j} , \; w(Z_{mm}) \right] ;$$

for $j = m$,

$$0 = \left[- ivx_m, \; w(Z_{mm}) \right] \quad \text{and} \quad 2ivx_m = \left[- \frac{\partial}{\partial x_m} , \; w(Z_{mm}) \right] .$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Z_{mm}) = - ivx_m^2 + \gamma_{mm} , \tag{11e}$$

where γ_{mm} is an arbitrary imaginary constant.

Calculation of the values of the constants α_{lm}, β_{lm}, γ_{lm}

Since $\tilde{w}$ is to be an ordinary representation of $sp(2n, \mathbb{R})$, there are constraints on the values of the constants α_{lm}, β_{lm}, γ_{lm}, as follows.

It may be shown that the commutation relations of $sp(2n, \mathbb{R})$, in terms of the basis X_{jk}, Y_{jk}, Z_{jk} (see (5)), are :

$$\left[X_{jk}, \; X_{lm} \right] = \delta_{kl} X_{jm} - \delta_{jm} X_{lk} , \tag{12a}$$

$$\left[X_{jk}, \; Y_{lm} \right] = \delta_{kl} Y_{jm} + \delta_{km} Y_{jl} , \tag{12b}$$

$$\left[X_{jk}, \; Z_{lm} \right] = - \delta_{jl} Z_{km} - \delta_{jm} Z_{kl} , \tag{12c}$$

$$\left[Y_{jk}, \; Y_{lm} \right] = 0 , \tag{12d}$$

$$\left[Y_{jk}, Z_{lm} \right] = \delta_{km} X_{jl} + \delta_{kl} X_{jm} + \delta_{jm} X_{kl} + \delta_{jl} X_{km} , \qquad (12e)$$

$$\left[Z_{jk}, Z_{lm} \right] = 0 , \qquad (12f)$$

where, for convenience of notation, define

$$Y_{jk} = Y_{kj}, \qquad Z_{jk} = Z_{kj} , \qquad \text{for } j > k .$$

Since $\tilde{w}$ is to be an ordinary representation of $sp(2n, \mathbb{R})$, $\tilde{w}(X_{jk})$, $\tilde{w}(Y_{jk})$, $\tilde{w}(Z_{jk})$ must satisfy the commutation relations of $sp(2n, \mathbb{R})$. Explicit calculation, using (11) and (12), shows that this implies :

$$- \delta_{kl} \alpha_{jm} + \delta_{jm} \alpha_{lk} = 0 , \qquad (13a)$$

$$- \delta_{kl} \beta_{jm} - \delta_{km} \beta_{jl} = 0 , \qquad (13b)$$

$$\delta_{jl} \gamma_{km} + \delta_{jm} \gamma_{kl} = 0 , \qquad (13c)$$

$$- \delta_{km} \alpha_{jl} - \delta_{kl} \alpha_{jm} - \delta_{jm} \alpha_{kl} - \delta_{jl} \alpha_{km} = 0 , \qquad (13d)$$

where, for convenience of notation, define

$$\beta_{jk} = \beta_{kj} , \qquad \gamma_{jk} = \gamma_{kj} , \qquad \text{for } j > k .$$

Condition (13a) gives that $\alpha_{jm} = 0$ for $j \neq m$ (putting $k = l$). Condition (13d) gives that $\alpha_{jj} = 0$ for $1 \leqslant j \leqslant n$ (putting $j = k = l = m$). From condition (13b), it follows that $\beta_{jm} = 0$ for $1 \leqslant j , m \leqslant n$ (putting $k = l \neq m$, if $n > 1$, and $j = k = l = m = 1$, if $n = 1$). Similarly, from condition (13c), $\gamma_{km} = 0$ for $1 \leqslant k , m \leqslant n$.

Hence, $\tilde{w}$ is an ordinary representation w of $sp(2n, \mathbb{R})$ if and only if the α's, β's and γ's are all zero.

Therefore, from (11), the ordinary representation w of $sp(2n, \mathbb{R})$ is determined by :

$$\left.\begin{aligned} w(X_{jk}) &= -x_k \frac{\partial}{\partial x_j} - \tfrac{1}{2}\delta_{jk} && (1 \leqslant j, k \leqslant n), \\ w(Y_{jk}) &= -\frac{i}{v} \frac{\partial^2}{\partial x_j \partial x_k} && (1 \leqslant j \leqslant k \leqslant n), \\ w(Z_{jk}) &= -ivx_j x_k && (1 \leqslant j \leqslant k \leqslant n) \end{aligned}\right\} \quad (14)$$

(where X_{jk}, Y_{jk}, Z_{jk} are defined by (5)).

In this representation w of $sp(2n, \mathbb{R})$, the basis elements $\hat{R}^{(1)}_{jk}$, $\hat{R}^{(2)}_{jk}$, $\hat{H}_k$ of su(n) (defined by (6)) are represented by :

$$\left.\begin{aligned} w(\hat{R}^{(1)}_{jk}) &= i\left(-\frac{1}{v}\frac{\partial^2}{\partial x_j \partial x_k} + vx_j x_k\right), \\ w(\hat{R}^{(2)}_{jk}) &= x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}, \\ w(\hat{H}_k) &= \frac{i}{2}\left(-\frac{1}{v}\frac{\partial^2}{\partial x_k^2} + vx_k^2 - \frac{1}{n}\sum_{l=1}^{n}\left(-\frac{1}{v}\frac{\partial^2}{\partial x_l^2} + vx_l^2\right)\right) \end{aligned}\right\} \quad (15)$$

$(1 \leqslant j < k \leqslant n)$ (from (6) and (14)). This determines a representation of su(n), which is, of course, identical to the representation w of su(n) obtained before (cf. 7-(14)).

Procedure for obtaining W

Since w is an ordinary representation of the Lie algebra

of the connected group $Sp(2n, \mathbb{R})$, w exponentiates to an ordinary representation, $\hat{W}$ say, of the connected, simply connected, covering group $\widetilde{Sp(2n, \mathbb{R})}$ of $Sp(2n, \mathbb{R})$ [32].

$Sp(2n, \mathbb{R})$ is the image of $\widetilde{Sp(2n, \mathbb{R})}$ under a homomorphism δ, whose kernel is a discrete central subgroup of $\widetilde{Sp(2n, \mathbb{R})}$ [6] :

$$Sp(2n, \mathbb{R}) \approx \frac{\widetilde{Sp(2n, \mathbb{R})}}{\ker \delta} . \tag{16}$$

In order to construct a projective representation of $Sp(2n, \mathbb{R})$, using the ordinary representation $\hat{W}$ of $\widetilde{Sp(2n, \mathbb{R})}$, the action of $\hat{W}$ on $\ker \delta$ will first be investigated.

Now, in general, for a connected Lie group K with connected, simply connected, covering group $\tilde{K}$ and covering homomorphism $\delta_{\tilde{K}}$, the group $\ker \delta_{\tilde{K}}$ is known as the fundamental group (or Poincaré group) of K. The fundamental group of $Sp(2n, \mathbb{R})$ is just the fundamental group of the subgroup U(n) ([16], p.143).

Suppose $U \in U(n)$; then $|\det U| = 1$, and so there exists $t \in (-\pi, \pi]$ such that $\det U = \exp it$. With this t, define an element of U(n) ($\subset Sp(2n, \mathbb{R})$, see §8.3) by :

$$\rho(t) = \begin{pmatrix} \cos t\, E_{11} & \sin t\, E_{11} \\ -\sin t\, E_{11} & \cos t\, E_{11} \end{pmatrix} \tag{17}$$

(where E_{11} is an n x n matrix having 1 in the (1,1) position and zeros elsewhere). Then $U = \rho(t)S$, where $S \in SU(n)$. Since

$\{\rho(t) : t \in (-\pi, \pi]\} \cap SU(n)$ contains only the identity of $U(n)$, every $U \in U(n)$ can be written in one and only one way in the form $U = \rho(t)S$ (where $\rho(t)$ is defined by (17) and $S \in SU(n)$).

Now the set $\{\rho(t) : t \in (-\pi, \pi]\}$ is a subgroup of $U(n)$ isomorphic to the factor group of $\mathbb{R}$ by $\mathbb{Z}$; this factor group will be denoted by T. It then follows that the underlying space of $U(n)$ is homeomorphic to $T \times SU(n)$. Since $T = \mathbb{R}/\mathbb{Z}$, and $\mathbb{R}$ is simply connected, $\mathbb{R}$ is the simply connected covering group $\tilde{T}$ of T. Hence, since $SU(n)$ is simply connected, the simply connected covering group $\widetilde{U(n)}$ of $U(n)$ has underlying space homeomorphic to $\mathbb{R} \times SU(n)$. It follows that

$$\ker \delta_{\widetilde{Sp(2n, \mathbb{R})}} \approx \ker \delta_{\widetilde{U(n)}} \approx \ker \delta_{\tilde{T}} \approx \mathbb{Z} .$$

Suppose $\tilde{T}$ $(= \mathbb{R})$ is identified with the subgroup of $\widetilde{Sp(2n, \mathbb{R})}$ to which it is isomorphic. Restricted to $\tilde{T}$, the covering homomorphism $\delta_{\widetilde{Sp(2n, R)}} = \delta$ may then be taken as :

$$\delta(\tilde{t}) = \rho(t) ,$$

where $\tilde{t} \in \tilde{T}$, $t \in (-\pi, \pi]$ and $\tilde{t} \equiv t \bmod 2\pi$.

Hence $\ker \delta = \{2\pi s : s \in \mathbb{Z}\}$.

The tangent at the identity of the subgroup $\{\rho(t) : t \in (-\pi, \pi]\}$ is, from (17),

$$\left.\frac{d\rho(t)}{dt}\right|_{t = 0} = \begin{pmatrix} 0 & E_{11} \\ -E_{11} & 0 \end{pmatrix} = \tfrac{1}{2}(Y_{11} - Z_{11})$$

(with the notation of (5)).

Since $\tilde{T}$ is the simply connected covering group of $\{\rho(t) : t \in (-\pi, \pi]\}$, it follows, from the definition of the ordinary representation $\hat{W}$ of $\widetilde{Sp(2n, \mathbb{R})}$, and from (14), that, for $\tilde{t} \in \tilde{T}$,

$$\hat{W}(\tilde{t}) = \exp \tilde{t}w(\tfrac{1}{2}(Y_{11} - Z_{11})) = \exp \frac{i}{2}\tilde{t}(-\frac{1}{v}\frac{\partial^2}{\partial x_1^2} + vx_1^2) .$$

Hence, for $\tilde{t} \in \tilde{T}$, with $\psi_{\underline{m}}$ defined by 3-(16),

$$\hat{W}(\tilde{t})\psi_{\underline{m}}(\underline{x}) = \exp i\tilde{t}(\operatorname{sgn} v)(m_1 + \tfrac{1}{2})\,\psi_{\underline{m}}(\underline{x}) .$$

In particular, for $\tilde{t} = 2\pi s \in \ker \delta$ $(s \in \mathbb{Z})$,

$$\hat{W}(2\pi s)\psi_{\underline{m}}(\underline{x}) = (-1)^s\psi_{\underline{m}}(\underline{x}) .$$

Thus for any function ψ in the domain of W,

$$\hat{W}(2\pi s)\psi(\underline{x}) = (-1)^s\psi(\underline{x}) .$$

Therefore, for $\tilde{S} \in \widetilde{Sp(2n, \mathbb{R})}$, $\tilde{t} = 2\pi s \in \ker \delta$,

$$\hat{W}(\tilde{S}\tilde{t}) = \hat{W}(\tilde{S})\hat{W}(\tilde{t}) = (-1)^s\hat{W}(\tilde{S}) .$$

This implies that, for any $S \in Sp(2n, \mathbb{R})$, all the operators $\hat{W}(\tilde{S})$ $(\tilde{S} \in \widetilde{Sp(2n, \mathbb{R})})$ with $\delta(\tilde{S}) = S$ (cf. (16)) define the same element of the projective group $\mathcal{P}$, where $\mathcal{P}$ is defined to be the unitary group $\mathcal{U}$ of the representation space of $\hat{W}$, factored by scalar multiples of the identity operator.

It follows that there is a Borel homomorphism, τ say, of $Sp(2n, \mathbb{R})$ into the projective group $\mathcal{P}$. It can then be shown that there exists a projective representation W of $Sp(2n, \mathbb{R})$,

defined on the representation space of $\hat{W}$, such that $\Upsilon = \iota W$, where ι is the canonical homomorphism of $\mathcal{U}$ onto $\mathcal{P}$. In this way, W is determined up to trivial multipliers. (Details may be found in Varadarajan [39], chapter 10).

Moreover, from the properties of the ordinary representation w of sp(2n, $\mathbb{R}$), the projective representation W of Sp(2n, $\mathbb{R}$) satisfies (3).

Therefore, the projective representation W which satisfies (3) may be determined up to trivial multipliers by the above procedure, starting from the ordinary representation w of sp(2n, $\mathbb{R}$), defined by (14). However, for what follows, it is not necessary to know W explicitly; it is sufficient that the ordinary representation w of sp(2n, $\mathbb{R}$) is known in detail.

8.5 The irreducible ordinary representations of L

It now follows from Mackey's theory for semi-direct product groups (§2.5) that every quantum mechanical irreducible ordinary representation of $L = N \circledS Sp(2n, \mathbb{R})$ is unitarily equivalent to one of the form :

$$U_N^V \; W \otimes \rho \; : \; (\sigma, \underline{\beta}, \underline{\alpha}, S) \longrightarrow \tag{18}$$

$$\exp iv(\sigma - \underline{x}.\underline{\beta}) \exp\Big(-\sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j}\Big) \; W(S) \otimes \rho(S) \; ,$$

where $(\sigma, \underline{\beta}, \underline{\alpha}, S) \in L$, W is the projective representation of Sp(2n, $\mathbb{R}$) determined up to trivial multipliers by (14), ρ is an irreducible projective representation of Sp(2n, $\mathbb{R}$) with

multiplier inverse to that of W, and the operators in the first part of the inner Kronecker product are defined on a dense subspace of $L^2(\mathbb{R}^n, \mathbb{C})$.

8.6 L as a local non-invariance group for the n-dimensional isotropic harmonic oscillator

From Mackey's general theory (§2.5), $U_N^V W$ is an irreducible projective representation of $L = N \circledS Sp(2n, \mathbb{R})$. The corresponding representation of the Lie algebra of L is an ordinary representation.

The restriction of $U_N^V W$ to the subgroup SU(n) is precisely the ordinary representation W of SU(n) obtained previously (7-(14)). Hence, from 7-(24),

$$(U_N^V W)(L) \downarrow SU(n) = \bigoplus_{a=0}^{\infty} W_a \ , \tag{19}$$

where W_a is an irreducible ordinary representation of SU(n), of dimension equal to the degeneracy of the (a + 1)th energy level of the n-dimensional isotropic harmonic oscillator.

Therefore $N \circledS Sp(2n, \mathbb{R})$ is a local non-invariance group for the n-dimensional isotropic harmonic oscillator. (The adjective local refers to the fact that the projective representation $U_N^V W$ has a multiplier which is locally equal to 1, but which is globally non-trivial.)

8.7 The decomposition of the projective representation W of Sp(2n, ℝ)

$$W(Sp(2n, \mathbb{R}))\downarrow SU(n) = \bigoplus_{a=0}^{\infty} W_a \qquad \text{(from (19))},$$

where W_a is an irreducible ordinary representation of SU(n) defined on the subspace Ω_a (7-(18)) of $L^2(\mathbb{R}^n, \mathbb{C})$.

Suppose Ω is a subspace of $L^2(\mathbb{R}^n, \mathbb{C})$ which is invariant and irreducible under the projective representation W of Sp(2n, ℝ). Then, a priori, Ω is invariant, although not necessarily irreducible, under $W(Sp(2n, \mathbb{R}))\downarrow SU(n)$. Hence Ω is a direct sum of closed subspaces of $L^2(\mathbb{R}^n, \mathbb{C})$ which are invariant and irreducible under $W(Sp(2n, \mathbb{R}))\downarrow SU(n)$. Since the irreducible representations W_a of SU(n) have different dimensions, they are inequivalent, and hence, by the uniqueness (up to unitary equivalence) of the decomposition of $W(Sp(2n, \mathbb{R}))\downarrow SU(n)$, it follows that Ω must be a direct sum of subspaces of the form Ω_a.

Suppose Ω contains the subspace Ω_b spanned by the set $\{\psi_{\underline{m}} : \sum_{j=1}^{n} m_j = b\}$. Since Ω is invariant under the projective representation W of Sp(2n, ℝ), the space generated by Ω_b, $w(X_{jk})\Omega_b$ $(1 \leqslant j, k \leqslant n)$, $w(Y_{jk})\Omega_b$ and $w(Z_{jk})\Omega_b$ $(1 \leqslant j \leqslant k \leqslant n)$ must be contained in Ω. Now, from (14) and 7-(16), for $j \neq k$,

$$w(X_{jk})\psi_{\underline{m}}(\underline{x}) = -x_k \frac{\partial}{\partial x_j}\psi_{m_1}(x_1)\psi_{m_2}(x_2)\ldots\psi_{m_n}(x_n)$$

$$= - \left(\sqrt{\frac{m_k}{2}}\,\psi_{m_k-1}(x_k) + \sqrt{\frac{m_k+1}{2}}\,\psi_{m_k+1}(x_k)\right) x$$

$$\left(\sqrt{\frac{m_j}{2}}\,\psi_{m_j-1}(x_j) - \sqrt{\frac{m_j+1}{2}}\,\psi_{m_j+1}(x_j)\right) \prod_{l\neq j,k}^{n} \psi_{m_l}(x_l) \ .$$

Therefore Ω contains elements which are linear combinations of elements belonging to Ω_{b-2}, Ω_b and Ω_{b+2}. Hence, since Ω is the direct sum of subspaces of the form Ω_a, Ω must contain the subspaces Ω_{b-2}, Ω_{b+2} aswell as Ω_b. By induction, it follows that Ω contains all the subspaces Ω_a for which a has the same parity as b.

Hence, the projective representation W of $Sp(2n, \mathbb{R})$ splits into at most two irreducible projective representations.

From the form of the representation w of the remaining basis elements of $sp(2n, \mathbb{R})$ (14), it follows, again using 7-(16), that the subspaces :

$$\Omega_{even}, \text{ spanned by } \{\psi_{\underline{m}} : \sum_{j=1}^{n} m_j \text{ is even}\} \ , \qquad (20a)$$

and $$\Omega_{odd}, \text{ spanned by } \{\psi_{\underline{m}} : \sum_{j=1}^{n} m_j \text{ is odd}\} \ , \qquad (20b)$$

are each invariant under the representation w of $sp(2n, \mathbb{R})$.

Hence, the ordinary representation w of $sp(2n, \mathbb{R})$ is the direct sum of two irreducible ordinary representations. These representations may be regarded as acting on the subspaces $\Omega_{even}, \Omega_{odd}$ defined above.

Therefore, the decomposition of the projective representation W of $Sp(2n, \mathbb{R})$ is :

$$W = W_{even} \oplus W_{odd} \ , \tag{21}$$

where W_{even}, W_{odd} are irreducible projective representations of $Sp(2n, \mathbb{R})$ (namely the restriction of W to the subspaces Ω_{even}, Ω_{odd} respectively).

(This result is similar to one obtained by Hwa and Nuyts [19], in which they show that su(n) can be embedded in $sp(2n, \mathbb{R})$, with the spectrum of the n-dimensional isotropic harmonic oscillator contained in the adjoint representation of $sp(2n, \mathbb{R})$; this representation (which corresponds to an ordinary representation of $\widetilde{Sp(2n, \mathbb{R})}$, but not of $Sp(2n, \mathbb{R})$) decomposes into two irreducible representations, one of which contains all the even levels of the spectrum, the other the odd levels.)

CHAPTER 9

VAN HOVE'S REPRESENTATION $R^v \downarrow L$ $(v \neq 0)$

In this chapter, it is shown that van Hove's representation $R^v \downarrow L$ $(v \neq 0)$ of $L = N \circledS Sp(2n, \mathbb{R})$ is unitarily equivalent to the representation $U_N^v \, W \otimes \overline{W}$ of L, where $\overline{W}$ is the complex conjugate of the projective representation W of $Sp(2n, \mathbb{R})$ (§8.4). The operator giving the equivalence is the operator V which occurred in decomposing $R^v \downarrow G$ $(v \neq 0)$ (§5.3).

First, various informal arguments which suggest the above result are given. Next, the result is proved. Then, by using the fact that $R^v \downarrow L$ $(v \neq 0)$ is unitarily equivalent to $U_N^v \, W \otimes \overline{W}$, the representation $R^v \downarrow L$ $(v \neq 0)$ is decomposed. Finally, the significance of these results is briefly discussed.

(It will be assumed throughout this chapter that $v \neq 0$; also, to simplify the notation, let $R^v \downarrow L = R_L^v$.)

9.1 Heuristic arguments suggesting that $V^{-1}(R^v \downarrow L)V = U_N^v \, W \otimes \overline{W}$

The result stated above was found through an attempt to decompose R_L^v into irreducible representations of L, of the form 8-(18). It was hoped that the decomposition of R_L^v would be suggested by that of $R_L^v \downarrow (N \circledS SU(n))$. The decomposition of $R_L^v \downarrow (N \circledS SU(n))$ may be obtained as follows.

$$R_L^v \downarrow (N \circledS SU(n)) = (D(H) \uparrow L) \downarrow (N \circledS SU(n)) \qquad \text{(from § 4.4)} .$$

Hence $R_L^v \downarrow (N \circledS SU(n))$ is the restriction of an induced

representation of L. Since $N \circledS SU(n)$ and H are closed subgroups of the separable locally compact group L, and L can be expressed as a single double coset of $N \circledS SU(n)$ and H : $L = N \circledS SU(n)(0,\underline{0},\underline{0},I)H$ (from 4-(8)), Mackey's subgroup theorem (2-(4)) may be applied :

$$(D(H) \uparrow L) \downarrow (N \circledS SU(n)) = D((N \circledS SU(n)) \cap H) \uparrow (N \circledS SU(n)) \ .$$

Therefore,

$$R_L^V \downarrow (N \circledS SU(n)) = D(C \times SU(n)) \uparrow (N \circledS SU(n)) \ . \qquad (1)$$

The group $N \circledS SU(n)$ is locally compact, but not compact, so the Frobenius reciprocity theorem (2-(6)) does not necessarily hold for it. However, a formal application of the theorem may, nevertheless, give the correct decomposition of $R_L^V \downarrow (N \circledS SU(n))$.

Suppose $U_N^t \ W \otimes \rho$ is an arbitrary quantum mechanical irreducible representation of $N \circledS SU(n)$, where W is now regarded as an ordinary representation of SU(n) (7-(15)).

Then, formally,

$$\begin{aligned} &\langle R_L^V \downarrow (N \circledS SU(n)), \ U_N^t \ W \otimes \rho \rangle_{N \circledS SU(n)} \\ &= \langle D(C \times SU(n)) \uparrow (N \circledS SU(n)), \ U_N^t \ W \otimes \rho \rangle_{N \circledS SU(n)} \\ &= \langle (U_N^t \ W \otimes \rho) \downarrow (C \times SU(n)), \ D(C \times SU(n)) \rangle_{C \times SU(n)} \ . \qquad (2) \end{aligned}$$

The representation

$$D : (\sigma,\underline{0},\underline{0},U) \longrightarrow \exp iv\sigma \quad ((\sigma,\underline{0},\underline{0},U) \in C \times SU(n))$$

is contained in the representation

$$U_N^t \; W \otimes \rho \; : \; (\sigma, \underline{0}, \underline{0}, U) \longrightarrow \exp it\sigma \; W(U) \otimes \rho(U)$$

($(\sigma, \underline{0}, \underline{0}, U) \in C \times SU(n)$) only when $t = v$; in this case, the number of times that D is contained in $U_N^t \; W \otimes \rho$ (as representations of $C \times SU(n)$) equals the frequency of the one-dimensional identity representation $I_{SU(n)}$ in the representation $U \longrightarrow W(U) \otimes \rho(U)$ of $SU(n)$.

Now, from 7-(24),

$$W \otimes \rho = (\bigoplus_{a=0}^{\infty} W_a) \otimes \rho = \bigoplus_{a=0}^{\infty} (W_a \otimes \rho) \; . \tag{3}$$

It thus remains to determine, for each a, how often $I_{SU(n)}$ occurs in $W_a \otimes \rho$. A general method for decomposing inner Kronecker products of irreducible representations of $SU(n)$ is given in [21].

When $v > 0$, from 7-(32a),

$$W_a = \underbrace{\square\cdots\square}_{a \text{ boxes}} = (a, 0, \ldots, 0) \; .$$

It follows that, when $v > 0$, $W_a \otimes \rho$ contains $I_{SU(n)}$ if and only if

$$\rho = \left.\begin{array}{c} \square\cdots\square \\ \vdots \quad \vdots \\ \square\cdots\square \end{array}\right\} n-1 \text{ rows} \; = (a, \ldots, a, 0) \; .$$

$$\underbrace{\qquad\qquad}_{a \text{ columns}}$$

When $v<0$, from 7-(32b), $W_a = (a,\ldots,a,0)$. It follows that, when $v<0$, $W_a \otimes \rho$ contains $I_{SU(n)}$ if and only if $\rho = (a,0,\ldots,0)$.

Now the representations $(a,0,\ldots,0)$ and $(a,\ldots,a,0)$ are mutually contragredient [21] (where the representation contragredient to W_a is defined to be $(W_a^{-1})^T$). Since W_a is unitary, the representation contragredient to W_a is just the complex conjugate $\overline{W}_a$ of W_a.

So, for any v ($\neq 0$), $I_{SU(n)}$ occurs in $W_a \otimes \rho$ if and only if $\rho = \overline{W}_a$, and then it occurs exactly once.

Hence, using (3), D is contained in $U_N^t\ W \otimes \rho$ (as representations of $C \times SU(n)$) if and only if $t = v$ and $\rho = \overline{W}_a$ for some a; in this case, D occurs exactly once.

A formal application (2) of the Frobenius reciprocity theorem therefore suggests that the decomposition of $R_L^v\downarrow (N \circledS SU(n))$ into irreducible representations is :

$$R_L^v \downarrow (N \circledS SU(n)) \simeq \bigoplus_{a=0}^{\infty} (U_N^v\ W \otimes \overline{W}_a) \ .$$

Now

$$\bigoplus_{a=0}^{\infty} (U_N^v\ W \otimes \overline{W}_a) = U_N^v\ W \otimes (\bigoplus_{a=0}^{\infty} \overline{W}_a) = U_N^v\ W \otimes \overline{W} \ .$$

This indicates that, perhaps, van Hove's representation R_L^v of L is unitarily equivalent to $U_N^v\ W \otimes \overline{W}$, where W is now regarded as a projective representation of $Sp(2n, \mathbb{R})$. $U_N^v\ W \otimes \overline{W}$ is an ordinary representation of L, since $\overline{W}$ has multiplier inverse to that of W.

From 3-(17), 5-(14) and -(34), it follows that the restrictions of R_L^v and $U_N^v\ W\otimes\overline{W}$ to the subgroup G of L are certainly unitarily equivalent :

$$V^{-1}(R_L^v \downarrow G)V = (U_N^v\ W\otimes\overline{W})\downarrow G\ , \tag{4}$$

where V is the operator defined by 5-(33).

The operator V was chosen so that the relations 5-(18) and -(19) were satisfied, that is, so that the representations of the generators of the subgroup N of G transformed in the required manner. Once V had been chosen in this way, it was then shown that the representations of the remaining generator of G also transformed in the required manner. It might thus be hoped that in transforming $R_L^v\downarrow \mathrm{Sp}(2n, \mathbb{R})$ by the operator V some useful information might be obtained.

These heuristic arguments suggest that, perhaps,

$$V^{-1}R_L^v\ V = U_N^v\ W\otimes\overline{W}$$

as representations of L.

9.2 Statement and proof of result

Theorem

$$V^{-1}R_L^v\ V = U_N^v\ W\otimes\overline{W} \quad (v \neq 0)\ , \tag{5}$$

where $R_L^v = R^v\downarrow L$ is van Hove's representation of L (4-(14)), U_N^v is a quantum mechanical irreducible representation of N (3-(11)), V is a unitary operator defined on $L^2(\mathbb{R}^{2n}, \mathbb{C})$

(5-(33)) and W is a projective representation of Sp(2n, $\mathbb{R}$) (§8.4).

Proof

Method

Every element $l \in L = N \circledS Sp(2n, \mathbb{R})$ can be expressed uniquely in the form $l = nS$, where $n \in N$, $S \in Sp(2n, \mathbb{R})$. From (4), the restriction of the theorem to N is certainly true. Hence, since $V^{-1}R_L^V V$ is an ordinary representation of L, it is sufficient to show that the theorem holds for Sp(2n, $\mathbb{R}$).

Since Sp(2n, $\mathbb{R}$) is a connected Lie group, every element may be expressed as a product of elements of one-parameter subgroups (§2.6). As $(V^{-1}R_L^V V) \downarrow Sp(2n, \mathbb{R})$ is an ordinary representation of Sp(2n, $\mathbb{R}$), it is thus enough to prove the theorem for a set of independent one-parameter subgroups which generate Sp(2n, $\mathbb{R}$).

If h(t) is a one-parameter subgroup of Sp(2n, $\mathbb{R}$), then $(V^{-1}R_L^V V) \downarrow h(t)$ and $(W \otimes \overline{W}) \downarrow h(t)$ are unitary ordinary representations of the one-parameter subgroup h(t), to which Stone's theorem may be applied. Hence, it is sufficient to show that V transforms the representation r_L^V of sp(2n, $\mathbb{R}$) into the representation $w \otimes \overline{w}$ of sp(2n, $\mathbb{R}$) (where r_L^V, $w \otimes \overline{w}$ are the representations of sp(2n, $\mathbb{R}$) corresponding to the representations R_L^V, $W \otimes \overline{W}$ of Sp(2n, $\mathbb{R}$) respectively).

The representation $w \otimes \overline{w}$

Suppose W acts on functions ($\in L^2(\mathbb{R}^n, \mathbb{C})$) dependent on $\underline{x} \in \mathbb{R}^n$, and that $\overline{W}$ acts on functions ($\in L^2(\mathbb{R}^n, \mathbb{C})$) dependent on $\underline{y} \in \mathbb{R}^n$. Let h(t) be a one-parameter subgroup of $Sp(2n, \mathbb{R})$. Then :

$$(w \otimes \overline{w})(h'(0)) = w(h'(0)) + \overline{w}(h'(0)) \qquad (6)$$

(where ' denotes differentiation with respect to t).

$\overline{w}(h'(0))$ may be obtained from $w(h'(0))$ by replacing x_m by y_m ($m = 1,2,\ldots,n$), and taking the complex conjugate.

From (6) and 8-(14), it follows that :

$$\left.\begin{aligned}
(w \otimes \overline{w})(X_{jk}) &= - x_k \frac{\partial}{\partial x_j} - y_k \frac{\partial}{\partial y_j} - \delta_{jk} \quad (1 \leqslant j, k \leqslant n), \\
(w \otimes \overline{w})(Y_{jk}) &= \frac{i}{v}\left(- \frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k}\right) \quad (1 \leqslant j \leqslant k \leqslant n), \\
(w \otimes \overline{w})(Z_{jk}) &= iv(- x_j x_k + y_j y_k) \quad (1 \leqslant j \leqslant k \leqslant n),
\end{aligned}\right\} \qquad (7)$$

where the X_{jk}, Y_{jk}, Z_{jk} form a basis of $sp(2n, \mathbb{R})$ (8-(5)).

Explicit expression for $r_L^v(h'(0))$

By definition, 4-(5) and -(14), $R_L^v \downarrow Sp(2n, \mathbb{R})$ is given by

$$\left[R_L^v(\gamma)\phi\right](\underline{q},\underline{p}) = \exp iv\pi_\gamma(\gamma^{-1}(\underline{q},\underline{p}))\ \phi(\gamma^{-1}(\underline{q},\underline{p})) ,$$

where $\gamma = (0, \underline{0}, \underline{0}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in Sp(2n, \mathbb{R})$,

$$\pi_\gamma(\underline{q},\underline{p}) = \tfrac{1}{2}A\underline{p}.C\underline{p} + \tfrac{1}{2}B\underline{q}.D\underline{q} + B\underline{q}.C\underline{p} ,$$

and $\phi \in L^2(\mathbb{R}^{2n}, \mathbb{C})$.

Since $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$, it follows from 4-(5) that

$$\gamma^{-1}(\underline{q},\underline{p}) = (A^T\underline{q} - C^T\underline{p}, -B^T\underline{q} + D^T\underline{p}) .$$

Therefore,

$$\left[R_L^V(\gamma)\phi\right](\underline{q},\underline{p}) = \exp iv\Big[\tfrac{1}{2}(-AB^T\underline{q} + AD^T\underline{p}).(-CB^T\underline{q} + CD^T\underline{p})$$

$$+ \tfrac{1}{2}(BA^T\underline{q} - BC^T\underline{p}).(DA^T\underline{q} - DC^T\underline{p})$$

$$+ (BA^T\underline{q} - BC^T\underline{p}).(-CB^T\underline{q} + CD^T\underline{p})\Big]$$

$$\phi(A^T\underline{q} - C^T\underline{p}, -B^T\underline{q} + D^T\underline{p}) . \qquad (8)$$

Suppose $h(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$ is a one-parameter subgroup of $Sp(2n, \mathbb{R})$. Then, from (8), since $A(0) = I_n' = D(0)$ and $B(0) = 0 = C(0)$,

$$\left[r_L^V(h'(0))\phi\right](\underline{q},\underline{p}) = \frac{d}{dt}\left[R_L^V(h(t))\phi\right](\underline{q},\underline{p})\Big|_{t=0}$$

$$= \Big[\frac{iv}{2}(\underline{p}.C'(0)\underline{p} + B'(0)\underline{q}.\underline{q}) + \sum_{m=1}^{n}((A'(0)^T\underline{q} - C'(0)^T\underline{p})_m \frac{\partial}{\partial q_m}$$

$$+ (-B'(0)^T\underline{q} + D'(0)^T\underline{p})_m \frac{\partial}{\partial p_m}\Big)\Big]\phi(\underline{q},\underline{p}) . \qquad (9)$$

The action of V

From 5-(21), -(30), -(31) and -(32),

$$\left.\begin{aligned} V^{-1}q_j V &= \frac{i}{v}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j}\right), \\ V^{-1}p_j V &= x_j + y_j , \\ V^{-1}\frac{\partial}{\partial q_j} V &= ivx_j , \\ V^{-1}\frac{\partial}{\partial p_j} V &= \frac{\partial}{\partial y_j} . \end{aligned}\right\} \quad (10)$$

Evaluation of $V^{-1}r_L^V(X_{jk})V$

$$X_{jk} = \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix} \qquad (1 \leqslant j, k \leqslant n) \quad \text{(from 8-(5))}.$$

So, in this case,

$$A'(0)^T = E_{kj} , \qquad B'(0) = 0 ,$$

$$C'(0) = 0 , \qquad D'(0)^T = -E_{jk} .$$

Hence, from (9),

$$r_L^V(X_{jk}) = q_j \frac{\partial}{\partial q_k} - p_k \frac{\partial}{\partial p_j} .$$

Therefore, using (10),

$$V^{-1}r_L^V(X_{jk})V = (V^{-1}q_j V)(V^{-1}\frac{\partial}{\partial q_k} V) - (V^{-1}p_k V)(V^{-1}\frac{\partial}{\partial p_j} V)$$

$$= \frac{i}{v}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j} \right)(ivx_k) - (x_k + y_k)\left(\frac{\partial}{\partial y_j}\right)$$

$$= - x_k \frac{\partial}{\partial x_j} - y_k \frac{\partial}{\partial y_j} - \delta_{jk}$$

$$= (w \otimes \overline{w})(X_{jk}) \qquad \text{from (7)} .$$

Evaluation of $V^{-1} r_L^V(Y_{jk}) V$

$$Y_{jk} = \begin{pmatrix} 0 & E_{jk} + E_{kj} \\ 0 & 0 \end{pmatrix} \qquad (1 \leqslant j \leqslant k \leqslant n) .$$

So, in this case,

$$A'(0) = 0 , \qquad B'(0) = B'(0)^T = E_{jk} + E_{kj} ,$$

$$C'(0) = 0 , \qquad D'(0) = 0 .$$

Hence, from (9),

$$r_L^V(Y_{jk}) = ivq_j q_k - q_k \frac{\partial}{\partial p_j} - q_j \frac{\partial}{\partial p_k} .$$

Therefore, using (10),

$$V^{-1} r_L^V(Y_{jk}) V$$

$$= iv(V^{-1} q_j V)(V^{-1} q_k V) - (V^{-1} q_k V)(V^{-1} \frac{\partial}{\partial p_j} V) - (V^{-1} q_j V)(V^{-1} \frac{\partial}{\partial p_k} V)$$

$$= - \frac{i}{v}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j} \right)\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial y_k} \right) - \frac{i}{v}\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial y_k} \right)\left(\frac{\partial}{\partial y_j}\right)$$

$$- \frac{i}{v}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j} \right)\left(\frac{\partial}{\partial y_k}\right)$$

$$= \frac{i}{v}\left(- \frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k}\right)$$

$$= (w \otimes \overline{w})(Y_{jk}) \qquad \text{from (7)} .$$

Evaluation of $V^{-1} r_L^v(Z_{jk}) V$

$$Z_{jk} = \begin{pmatrix} 0 & 0 \\ E_{jk} + E_{kj} & 0 \end{pmatrix} \qquad (1 \leqslant j \leqslant k \leqslant n) .$$

So, in this case,

$$A'(0) = 0 , \qquad B'(0) = 0 ,$$

$$C'(0) = C'(0)^T = E_{jk} + E_{kj} , \qquad D'(0) = 0 .$$

Hence, from (9),

$$r_L^v(Z_{jk}) = ivp_j p_k - p_k \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial q_k} .$$

Therefore, using (10),

$$V^{-1} r_L^v(Z_{jk}) V$$

$$= iv(V^{-1} p_j V)(V^{-1} p_k V) - (V^{-1} p_k V)(V^{-1} \frac{\partial}{\partial q_j} V) - (V^{-1} p_j V)(V^{-1} \frac{\partial}{\partial q_k} V)$$

$$= iv(x_j + y_j)(x_k + y_k) - (x_k + y_k)(ivx_j) - (x_j + y_j)(ivx_k)$$

$$= iv(- x_j x_k + y_j y_k)$$

$$= (w \otimes \overline{w})(Z_{jk}) \qquad \text{from (7)} .$$

This completes the proof of the theorem.

9.3 The decomposition of van Hove's representation $R^v \downarrow L$

From 8-(21), the decomposition of the projective representation W of $Sp(2n, \mathbb{R})$ is :

$$W = W_{even} \oplus W_{odd} ,$$

where W_{even}, W_{odd} are irreducible projective representations, each of which has the same multiplier as W. Hence,

$$\overline{W} = \overline{W}_{even} \oplus \overline{W}_{odd} ,$$

where $\overline{W}_{even}$, $\overline{W}_{odd}$ are irreducible projective representations, each of which has multiplier inverse to that of W.

Therefore,

$$\begin{aligned} R^v \downarrow L &\simeq U_N^v \; W \otimes \overline{W} \\ &= U_N^v \; W \otimes (W_{even} \oplus W_{odd}) \\ &= (U_N^v \; W \otimes \overline{W}_{even}) \oplus (U_N^v \; W \otimes \overline{W}_{odd}) \; . \end{aligned} \tag{11}$$

From 8-(18), $U_N^v \; W \otimes \overline{W}_{even}$ and $U_N^v \; W \otimes \overline{W}_{odd}$ are irreducible ordinary representations of L.

It follows that van Hove's representation $R^v \downarrow L$ $(v \neq 0)$ splits into two irreducible ordinary representations of L.

9.4 Conclusion

For clarity, let $W_{Sp(2n, \mathbb{R})}$ denote the projective representation W of $Sp(2n, \mathbb{R})$ determined by 8-(14).

The structure of van Hove's representation $R^v \downarrow L$ is more

clearly seen by using the unitarily equivalent inner Kronecker product $U_N^V \, W_{Sp(2n, \mathbb{R})} \otimes \overline{W}_{Sp(2n, \mathbb{R})}$, rather than the explicit expression for $R^V \downarrow L$ given by 4-(14). In particular, the restriction of $U_N^V \, W_{Sp(2n, \mathbb{R})} \otimes \overline{W}_{Sp(2n, \mathbb{R})}$ to G is more easily decomposed than $R^V \downarrow G$: the decomposition follows in a straightforward way from the explicit form of $(U_N^V \, W_{Sp(2n, \mathbb{R})} \otimes \overline{W}_{Sp(2n, \mathbb{R})}) \downarrow G$, as is shown (in reverse) in §5.3.

Hence, as far as the anisotropic harmonic oscillator is concerned, almost all reference to van Hove [38] can now be removed. His representation $R^V \downarrow L$ can be replaced by $U_N^V \, W_{Sp(2n, \mathbb{R})} \otimes \overline{W}_{Sp(2n, \mathbb{R})}$; all that is then required from his paper is the action of $Sp(2n, \mathbb{R})$ on N in the definition of $N \circledS Sp(2n, \mathbb{R})$.

The main results can now be rewritten without explicit mention of R^V; this alternative approach emphasises the significant role played by the projective representation $W_{Sp(2n, \mathbb{R})}$.

Preliminary results

(i) It is possible to define a group G associated with the n-dimensional anisotropic harmonic oscillator (§3.3).

(ii) G is isomorphic to a subgroup of $L = N \circledS Sp(2n, \mathbb{R})$ (§4.2).

(iii) Every quantum mechanical irreducible representation of N is unitarily equivalent to one of the form U_N^v, where $v\ (\neq 0) \in \mathbb{R}$ (§3.4).

(iv) Every quantum mechanical irreducible representation of G is unitarily equivalent to one of the form $U_G^{v,r}$, where $v\ (\neq 0) \in \mathbb{R}$, $r \in \mathbb{Z}$ (§3.6).

(v) Every quantum mechanical irreducible (ordinary) representation of L is unitarily equivalent to one of the form $U_N^v\ W_{Sp(2n,\ \mathbb{R})} \otimes \rho_{Sp(2n,\ \mathbb{R})}$, where $W_{Sp(2n,\ \mathbb{R})}$ is a projective representation of $Sp(2n, \mathbb{R})$, determined up to trivial multipliers, and $\rho_{Sp(2n,\ \mathbb{R})}$ is an irreducible projective representation of $Sp(2n, \mathbb{R})$, with multiplier inverse to that of W (§8.4 and §8.5).

(vi) $U_N^v\ W_{Sp(2n,\ \mathbb{R})} \otimes \overline{W}_{Sp(2n,\ \mathbb{R})}$ is an ordinary representation of L (from (v)).

Basic result

$$\text{(vii)}\quad (U_N^v\ W_{Sp(2n,\ \mathbb{R})} \otimes \overline{W}_{Sp(2n,\ \mathbb{R})}) \downarrow G = \bigoplus_s d_{\omega,s} U_G^{v,-(\operatorname{sgn} v)s},$$

where $d_{\omega,s}$ is the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$, and the summation may be regarded as over all distinct energy levels $E_{\omega,s}$ of the system (§5.3 and §9.2).

Further results

(viii) $W_{Sp(2n, \mathbb{R})} \downarrow SU(n) = W_{SU(n)}$ is an ordinary representation of SU(n); its decomposition is :

$$W_{SU(n)} = \bigoplus_{a=0}^{\infty} W_a \ ,$$

where W_a is an irreducible (ordinary) representation of SU(n), of dimension equal to the degeneracy of the (a + 1)th energy level of the n-dimensional isotropic harmonic oscillator (§7.5).

(ix) The decomposition of the projective representation $W_{Sp(2n, \mathbb{R})}$ is :

$$W_{Sp(2n, \mathbb{R})} = W_{even} \oplus W_{odd} \ ,$$

where W_{even}, W_{odd} are irreducible projective representations of $Sp(2n, \mathbb{R})$ (§7.5 and §8.7).

(x) The (ordinary) representation $U_N^V \, W_{Sp(2n, \mathbb{R})} \otimes \overline{W}_{Sp(2n, \mathbb{R})}$ of L splits into two irreducible (ordinary) representations of L (from (v) and (ix)).

(xi) Every quantum mechanical irreducible (ordinary) representation of $N \circledS SU(n)$ is unitarily equivalent to one of the form $U_N^V \, W_{SU(n)} \otimes \rho_{SU(n)}$, where $\rho_{SU(n)}$ is an irreducible (ordinary representation of SU(n) (§7.4).

(<u>xii</u>) $U_N^V W_{SU(n)}$ is an irreducible (ordinary) representation of $N \circledS SU(n)$ (from (xi)); its decomposition is :

$$(U_N^V W_{SU(n)}) \downarrow SU(n) = \overset{\infty}{\underset{a=0}{\oplus}} W_a$$

(from (viii)). Hence, $N \circledS SU(n)$ is a non-invariance group for the n-dimensional isotropic harmonic oscillator.

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