

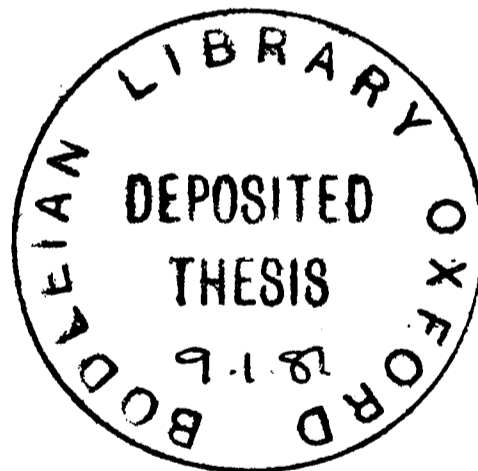
SPACES OF COMPLEX GEODESICS

& RELATED STRUCTURES

by

C.R. LeBrun, Jr.

Hertford College, Oxford



A thesis submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
at the University of Oxford

1980

SPACES OF COMPLEX GEODESICS AND RELATED STRUCTURES

C.R. LeBrun, Jr., Hertford College, Oxford

A thesis submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy

Trinity Term, 1980

ABSTRACT

On a complex n -manifold with holomorphic projective connexion, any point has a neighbourhood U of which the space of geodesics has naturally the structure of a (Hausdorff) complex $(2n-2)$ -manifold; it is shown that the complex structure of this auxiliary space encodes, in a sense, the original projective connexion by means of a complete analytic family of \mathbb{P}_{n-1} 's. Rather strikingly, small deformations of the space of geodesics correspond precisely to small deformations of the projective connexion of the primary space.

Similar results are obtained concerning the space of null geodesics of a complex n -fold, $n \geq 4$, with *conformal connexion*, a geometrical structure amounting to a holomorphic conformal structure plus a torsion tensor. For $n=3$ the analysis is complicated by the fact that the analytic family of submanifolds (quadrics; in this case \mathbb{P}_1 's) representing the points of the primary space fails to be complete; but it can be completed to give a 4-dimensional family, effecting a unique embedding of the original 3-fold in a 4-fold with conformal structure, of which the conformal curvature is self-dual, in such a way that the induced conformal structure is the original one and such that the conformal torsion is related to the second conformal fundamental form of the hypersurface in a canonical linear fashion. In any case, the small deformations of the complex structure of the space of null geodesics correspond precisely to the small deformations of the conformal

connexion. It is shown that a space of torsion-free null geodesics admits a holomorphic contact structure, and that conversely, for $n \geq 4$, the admission of a contact structure forces the conformal torsion to vanish; for $n=3$, the contact form constructs automatically a unique metric on the ambient 4-fold in the previously constructed self-dual conformal class which solves *Einstein's equations with cosmological constant 1* and blows up on the 3-fold, which is a general umbilic hypersurface. These results are in turn used to show that a real-analytic 3-fold with real-analytic positive definite conformal structure and a real-analytic symmetric form of conformal weight 1 can be embedded (in a locally unique fashion) in a real-analytic 4-fold with positive-definite conformal structure for which the conformal curvature is self-dual in such a way as to realize the given structures as the first and second conformal fundamental forms of the hypersurface; and it is shown that a real analytic 3-fold with positive-definite conformal bounds a locally unique positive-definite solution of Einstein's equations with cosmological constant -1 as its umbilic conformal infinity. By contrast, these results fail when "real-analytic" is replaced by "smooth".

ACKNOWLEDGEMENTS

My years in Oxford have been interesting, illuminating and quite enjoyable; for this I must thank a number of people. Roger Penrose has been an excellent supervisor, always encouraging, inquisitive, and helpful, yet always allowing his students to find their own ways; his insight and enthusiasm have been a great inspiration. Nigel Hitchin was the source of a great deal of enlightenment on a whole array of issues; his knowledge, patience, and approachability made conversing with him a dependable source of enjoyment. Then, too, the entire twistor group is to be thanked for providing a stimulating atmosphere in which to work; among those contributing to the line of thought represented by this thesis in one way or another, through their questions or remarks, were Mike Eastwood, Lane Hughston, George Burnett-Stuart, Paul Tod, Nick Woodhouse, and Paul Green.

As my student days draw to a close, I should especially like to thank my family - in particular, my parents, grandmother, and uncle - for their constant support.

CONTENTS

Page

PREFACE

(i)

0. INTRODUCTION

§0.1	HOLOMORPHIC FOLIATIONS	1
§0.2	SOME VANISHING THEOREMS	3
§0.3	RIGIDITY THEORY	4
§0.4	COMPLETE ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS	7
§0.5	CR MANIFOLDS	9

1. PROJECTIVE CONNEXIONS

§1.1	HOLOMORPHIC AFFINE CONNEXIONS	11
§1.2	PROJECTIVE CONNEXIONS	15
§1.3	COMPLEX GEODESICALLY CONVEX NEIGHBOURHOODS	19
§1.4	SPACES OF COMPLEX GEODESICS	22
§1.5	THE NORMAL BUNDLE OF P_x	25
§1.6	RELEVANT COHOMOLOGY OF $\mathcal{O}(N)$ AND $\mathcal{O}(\text{HOM}(N,N))$	29
§1.7	PROJECTIVE CONNEXIONS: THE MAIN THEOREM	31
§1.8	EXPLICIT DEFORMATIONS	38

2. CONFORMAL CONNEXIONS IN DIMENSION ≥ 4

§2.1	CONFORMAL STRUCTURES, CONFORMAL CONNEXIONS	40
§2.2	SPACES OF NULL GEODESICS	50
§2.3	COHOMOLOGICAL CALCULATIONS: THE NORMAL BUNDLE TO Q_x	51
§2.4	DEFORMATIONS OF SPACES OF NULL GEODESICS	52
§2.5	SPACES OF TORSION-FREE NULL GEODESICS	54

3. CONFORMAL CONNEXIONS IN DIMENSION 3

§3.1	SPACES OF NULL GEODESICS IN DIMENSION THREE	64
------	---	----

	<u>Page</u>
§3.2 THE HYPERSURFACE TWISTOR CONSTRUCTION	75
§3.3 DEFORMATIONS OF SPACES OF NULL GEODESICS: THE IMPROVED RESULT	78
§3.4 SPACES OF TORSION-FREE NULL GEODESICS IN THREE DIMENSIONS AND SOLUTIONS OF EINSTEIN'S EQUATIONS	79
§3.5 HYPERSURFACE TWISTOR CR MANIFOLDS	88
§3.6 REAL SLICES OF "HEAVEN-ON-EARTH"	100
§3.7 POSITIVE-DEFINITE HEAVEN-WITH-A-COSMOLOGICAL- CONSTANT	103
4. PROBLEMS AND PROSPECTS	105
BIBLIOGRAPHY	109

PREFACE

Complex analytic structures are comparatively rigid ones, so much so that the analogy of geometrical structures often proves useful in thinking about them. But there is more than an analogy here. It was first noticed by Penrose [15] that certain types of (holomorphic) local geometrics (specifically, self-dual conformal structures or self-dual metrics in four dimensions) can be encoded into the global structure of auxiliary complex manifolds, and that small deformations of the complex structures of these auxiliary manifolds correspond precisely to deformations of the original geometrical structure; the crucial ingredient here is a theorem of Kodaira [9] on holomorphic motions of compact submanifolds. In this thesis we consider a much wider spectrum of geometrical structures, namely those that are characterized by the specification of geodesic curves, which are amenable to an analogous treatment.

After presenting, in Chapter 0, a loose assortment of more or less widely known results which will prove valuable in the course of our investigations, we proceed, in Chapter 1, to examine holomorphic projective connexions and their spaces of geodesics. We show that any complex n -manifold with holomorphic projective connexion admits a neighbourhood basis such that the space of geodesics of any basis neighbourhood is a complex $(2n-2)$ -fold, and that the \mathbb{P}_{n-1} 's embedded in such a space of geodesics, which consist of those geodesics through some fixed point of the original n -manifold, form a complete analytic family - i.e. any small holomorphic motion of such a submanifold gives another one. Small deformations of the complex structure of such a space of geodesics are shown to all be other spaces of geodesics.

In Chapter 2, the theory of holomorphic *conformal connexions* on n -manifolds, $n \geq 4$, is seen to be much the same. This type of geometrical structure amounts locally to the specification of a holomorphic conformal

metric along with an equivalence class of holomorphic affine connexions preserving it, where two connexions are equivalent if they have the same null geodesics, considered as immersed submanifolds. Further, it is demonstrated that the null geodesics are those of torsion-free metric connexions iff the space of null geodesics admits a contact structure; we can thus restrict our attention, if we so desire, to the more familiar class of *conformal structures*.

In Chapter 3, we investigate the rather remarkable breakdown of the above results when $n = 3$. The submanifolds (\mathcal{P}_1 's) which represent the points of a three-dimensional conformal connexion in its three-dimensional space of null geodesics do *not* form a complete analytic family, but rather sit naturally in a *four*-dimensional complete family of \mathcal{P}_1 's; this embeds the given 3-fold in a 4-fold with conformal structure for which the conformal curvature is self-dual, and in such a way as to realize (in a locally unique fashion) the original conformal metric as the induced one and the "conformal torsion" that characterizes the null geodesics as, essentially, the conformal part of the second fundamental form of the hypersurface. When the conformal torsion vanishes (i.e. if the corresponding hypersurface is umbilic), then the space of null geodesics admits a contact structure, and this structure constructs for us a unique solution of Einstein's equations with cosmological constant λ which lies in the previously constructed conformal class and which has a simple pole on the umbilic hypersurface. If the original three-dimensional conformal structure admits a positive-definite slice, so does the Einstein metric (if we renormalize the cosmological constant to be negative). This construction is, in effect, a cosmological constant analogue of Newman's H-space [27], but as it is considerably better behaved one may hope that this new construction may shed some light on that of Newman.

Chapter 3 also includes an analysis of the smooth analogue of the

analytic embedding problem mentioned above; it is shown that such embeddings give rise to embeddings of certain CR 5-folds which do not generally exist, and so such geometrical embeddings are generally equally impossible. In the process, an interesting correspondence between *smooth* geometry and certain CR manifolds is brought out.

Finally, we point out in Chapter 4 a few of the interesting problems related to our investigations which remain unsolved.

Throughout, indices may (and often must) be interpreted as numerical indices, except in equations involving both spinor and tensor indices. The summation convention always applies.

The author has avoided the use of spinors whenever possible within the bounds of sane behaviour; although spinor techniques should, in principle, be mastered by anyone vaguely interested in these topics, it is sometimes illuminating to see spinorial results and constructions from a less exotic perspective. It is hoped that perhaps a reader or two may be enticed into learning about spinors in order to understand a few points of the argument more fully.

As for notation:

- $T^{\mathbb{C}}X$ Denotes the holomorphic tangent bundle of a complex manifold X , as distinguished from its smooth tangent bundle TX . We usually think of $T^{\mathbb{C}}X$ as a sub-bundle of the complexified smooth tangent bundle $\mathbb{C}TX$.
- $\mathbb{C}E$ Denotes $E \otimes_{\mathbb{R}} \mathbb{C}$ if $E \rightarrow X$ is a real vector bundle.
- $\mathbb{P}E$ Denotes the projectivization of a complex vector bundle $E \rightarrow X$: in other words, $\mathbb{P}E$ is obtained from E by excising the zero section and modding out by the multiplicative action of the non-zero complex numbers \mathbb{C}^* .
- \mathcal{O} Denotes the sheaf of germs of holomorphic functions on some (understood) complex manifold.

- $\mathcal{O}(E)$ Denotes the sheaf of germs of holomorphic sections of a holomorphic vector bundle E .
- $\mathcal{E}(E)$ Denotes the sheaf of germs of smooth (C^∞) sections of a smooth vector bundle E .
- Θ_X Denotes $\mathcal{O}(T^*X)$.
- $\mathcal{O}(m)$ Denotes $\mathcal{O}(H^m)$, where H is the hyperplane section line bundle (Chern class + 1) over \mathbb{P}^k , for some k understood; or the restriction of this sheaf to a projective variety.

§0.1 HOLOMORPHIC FOLIATIONS

Let M be a complex n -manifold. A *holomorphic k -distribution* D on M is a k -dimensional holomorphic sub-bundle of the holomorphic tangent bundle T^*M . We say that D is *involutive* if the Lie bracket of two sections of D (considered as vector fields) is again a section of D : $[\mathcal{O}(D), \mathcal{O}(D)] = \mathcal{O}(D)$. Given an involutive holomorphic k -distribution D on M , there are holomorphic coordinates $\varphi: U \rightarrow \mathbb{C}^n$ on some neighbourhood \hat{U} of any point $x \in M$ with the property that D is everywhere spanned by $\{\partial/\partial z^1, \dots, \partial/\partial z^k\}$. This follows from the usual smooth analogue of this statement, the Frobenius theorem [7], since, by induction, we can find holomorphic vector fields X_1, \dots, X_k spanning D in some neighbourhood of x such that $[X_j, X_\ell] = 0$ for all $j, \ell = 1, \dots, k$, and then notice that $[X_j, \bar{X}_\ell] = 0$ also for all j and ℓ , since the fields are holomorphic; the real, smooth fields

$$\left. \begin{aligned} Y_j &= X_j + \bar{X}_j \\ Y_{j+k} &= -i(X_j - \bar{X}_j) \end{aligned} \right\} \quad j = 1, \dots, k$$

now satisfy $Y_\ell, Y_m = 0$ for $\ell, m = 1, \dots, 2n$. Take some complex $(n-k)$ -fold N transverse to D at x , with coordinates $\tilde{\varphi}: N \rightarrow \mathbb{C}^{n-k}$; the usual proof of the Frobenius theorem now shows that there exist local coordinates $\psi: U \rightarrow \mathbb{R}^{2n}$ for some neighbourhood U of x such that $\partial/\partial x^\ell = Y_\ell$ for $\ell = 1, \dots, 2k$, and $\psi|_N = 0 \times \tilde{\varphi}$, modulo the identification of $\mathbb{R}^{2(n-k)}$ and \mathbb{C}^{n-k} . It then turns out that the coordinates $z^j, j = 1, \dots, n$ defined by

$$z^j = x^j + ix^{k+j}, \quad j = 1, \dots, k$$

$$z^j = x^{2j-1} + ix^{2j}, \quad j = k+1, \dots, n$$

are holomorphic, since the pseudo-group of diffeomorphisms generated by the real part of a holomorphic vector field consists of biholomorphisms.

The above result is the only one that we shall need regarding

holomorphic differential equations ("ODE's").

Given an involutive holomorphic k -distribution D on a complex manifold M , a maximal connected holomorphic immersed k -fold L tangent to D is called a *leaf*. The system of all such leaves is called the *foliation* of M tangent to D , and constitutes a *partition* of M . There is thus a quotient map $q: M \rightarrow F$ from M to the space of leaves of this foliation; F is given the quotient topology. The projection q is an open map, since our special "Frobenius" local coordinates show that the set of points on a given leaf having neighbourhoods meeting no leaf which does not meet some fixed open set $V \subset M$ is closed, and hence is the entire leaf. In particular, the topology of F admits a countable basis.

F need not be a topological manifold, but if it is one, and if q carries any submanifold transverse to the leaves locally homeomorphically into F , then F has a natural structure as a complex manifold. Namely, we give F the atlas consisting of those local homeomorphisms from F to \mathbb{C}^{ℓ} , where ℓ is the codimension of a typical leaf, induced by the last ^{of} coordinates of a sufficiently small Frobenius chart. The transition functions between such charts on F will be biholomorphisms; for if $x, y \in M$ are such that $q(x) = q(y)$, we can find a sequence of Frobenius charts about portions of the leaf $q^{-1}(q(x))$ with domains of definition $U_1, \dots, U_N \subset M$ such that $x \in U_1, y \in U_N$, and $U_j \cap U_{j+1} \cap q^{-1}(q(x)) \neq \emptyset$ for all $j = 1, \dots, N-1$, and since the transition functions between the pair of charts for F induced by those defined on U_j and U_{j+1} will necessarily be biholomorphic near $q(x)$, it follows by composition that the given transition function is biholomorphic near any arbitrary point $q(x)$.

§0.2 SOME VANISHING THEOREMS

We shall need to know something about the cohomology of complex projective n -space \mathbb{P}_n with coefficients in the sheaf $\mathcal{O}(m)$ of germs of holomorphic functions of homogeneity m , as well as something about the cohomology of quadrics with the restrictions of these sheaves as coefficients. The main result is that of Bott [2].

Theorem. ("Bott's Rule"). $H^p(\mathbb{P}_n, \mathcal{O}(m)) = 0$ if $0 < p < n$, or if $p = n$ and $m \geq -n$.

This can be derived from the more general Kodaira-Nagano vanishing theorem using Serre duality; see [3], [17].

Corollary. If $Q \subset \mathbb{P}_{n+1}$ is a (non-degenerate) quadric, then $H^p(Q, \mathcal{O}(m)) = 0$ if $0 < p < n$, or if $p = n$ and $m \geq -n+1$.

Proof. This follows directly from the long exact sequence induced by the short exact sequence

$$0 \rightarrow \mathcal{O}(m-2) \xrightarrow{g} \mathcal{O}(m) \rightarrow \mathcal{O}_Q(m) \rightarrow 0$$

of sheaves on \mathbb{P}_{n+1} , where $\mathcal{O}_Q(m)$ has stalk 0 at points not on Q , and consists of germs of holomorphic functions on Q homogeneous of degree m , and where g is multiplication by the defining function of the quadric; $H^p(Q, \mathcal{O}(m))$ is a subspace of $H^{p+1}(\mathbb{P}_{n+1}, \mathcal{O}(m-2))$ for $p > 0$. \square

§0.3 RIGIDITY THEORY [11], [12]

By a deformation of a compact complex manifold M we mean a proper regular holomorphic map $\alpha: \tilde{M} \rightarrow U$ between complex manifolds such that M is biholomorphically equivalent to $\alpha^{-1}(x)$ for some specified $x \in U$. (By *proper* we mean that the inverse image of a compact set is compact; by *regular* we mean that the rank of the Jacobian is everywhere maximal. Elementary differential topology shows that such a projection is differentiably trivial).

Let Θ denote the sheaf $\mathcal{O}(T^*M)$ of germs of holomorphic vector fields on M .

Theorem. If $H^1(\Theta) = 0$, any small deformation of M is trivial: if $\alpha: \tilde{M} \rightarrow U$ is a deformation of $M \simeq \alpha^{-1}(x)$, then, for some neighbourhood $V \subset U$ of x , $\alpha^{-1}(V) \simeq V \times M$ biholomorphically.

For the proof, see either of the given references. One can think of a Čech cochain with coefficients in Θ as an infinitesimal change in the transition functions which define M as a complex manifold, and factoring out by coboundaries just removes the ambiguity of infinitesimal changes in holomorphic coordinates; thus the theorem can be thought of as saying that there are no small deformations of M if there are no infinitesimal ones. (This picture of $H^1(\Theta)$ as the infinitesimal deformation of the complex structure is also rigorous when $H^1(\Theta) \neq 0$ but $H^2(\Theta) = 0$, in which case there is a *moduli space* of complex dimension $\dim H^1(\Theta)$ which classifies small deformations of M ; in any case, a 1-parameter deformation has a "derivative" in $H^1(\Theta)$).

When $H^1(\Theta) = 0$, we say that M is rigid.

There is an analogous theory of deformations of holomorphic fibre bundles over a fixed base, as developed in [12]. This specialises to give the following theorem, which is rather easier to prove than the first.

Theorem. If $E \rightarrow M$ is a holomorphic vector bundle over a compact complex manifold such that $H^1(E \otimes E^*) = 0$, then any small deformation of E is trivial.

This should be interpreted as saying that if \tilde{E} is a vector bundle over $M \times B$, where B is a neighbourhood of $0 \in \mathbb{C}^k$, such that $\tilde{E}/_{M \times \{0\}} \cong E$, then $\tilde{E} \cong \pi^* E$, where $\pi: M \times B \rightarrow M$ is projection to the first factor.

When $H^1(E \otimes E^*) = 0$, we say that E is rigid.

We will be interested in only two examples of the first theorem, namely when $M = \mathbb{P}_n$ or M is a quadric in \mathbb{P}_n , for any n . We now quickly demonstrate that the necessary hypothesis holds in each case.

Example 1. $M = \mathbb{P}_n$. The tangent bundle of \mathbb{P}_n is described by the exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda} (n+1)\mathcal{O}(1) \rightarrow \mathcal{O} \rightarrow 0$$

where λ multiplies a holomorphic function by the $n+1$ homogeneous coordinates on \mathbb{P}_n . Bott's Rule tells us that $H^1(\mathbb{P}_n, \mathcal{O}(1)) = H^2(\mathbb{P}_n, \mathcal{O}) = 0$, so $H^1(\mathbb{P}_n, \mathcal{O}) = 0$.

Example 2. $M = Q$, a quadric hypersurface in \mathbb{P}_n . Let $T := \mathcal{O}(T\mathbb{P}_n|_Q)$, let $N := T/\mathcal{O} \cong \mathcal{O}(2)$ be the sheaf of sections of the normal bundle of Q in \mathbb{P}_n .

We have the long exact sequence

$$0 \rightarrow H^0(\mathcal{O}) \rightarrow H^0(T) \xrightarrow{\mu} H^0(N) \rightarrow H^1(\mathcal{O}) \rightarrow H^1(T) \rightarrow \dots$$

Now T is given (cf. example 1) by the exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda} (n+1)\mathcal{O}(1) \rightarrow T \rightarrow 0$$

and so our corollary to Bott's Rule demonstrates that $H^1(T) = 0$, while the $(n+1)^2 - 1$ dimensional space of global sections of T is filled out by the Lie algebra of $SL(n+1, \mathbb{C})$, acting on \mathbb{P}_n in the usual way; we can thus identify $H^0(T)$ with trace-free $(n+1) \times (n+1)$ matrices. The subgroup of

$SL(n+1, \mathbb{C})$ taking Q to itself is just $SO(n+1, \mathbb{C})$, and so the image of $H^0(\Theta)$ in $H^0(T)$ can be identified with the anti-symmetric matrices; so the image of μ has dimension $(n+1)^2 - 1 - \binom{n+1}{2} = \binom{n+2}{2} - 1$, while $H^0(N) \cong H^0(\mathcal{O}(2))$ can be identified (via the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(2) \rightarrow \mathcal{O}_Q(2) \rightarrow 0$$

of sheaves on \mathbb{P}_n) with symmetric $(n+1) \times (n+1)$ matrices modulo the symmetric form defining Q , and so has dimension $\binom{n+2}{2} - 1$. Hence μ is onto, and $H^1(\Theta) = 0$.

§ 0.4 COMPLETE ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS

By an *analytic family of compact submanifolds* of a complex manifold M we mean a complex submanifold S of $M \times X$, where X is some complex manifold, such that the projection map $\pi_2: S \rightarrow X$ to the second factor is a regular proper surjection. For each $x \in X$ we say that the compact complex submanifold $\pi_1[\pi_2^{-1}(x)] \subset M$ belongs to the family X .

If $S \subset M \times X$ is an analytic family of compact submanifolds, there is a natural linear map from $T_x X$ to holomorphic global sections of the normal bundle of $S_x := \pi_1[\pi_2^{-1}(x)] \subset M$, where the *normal bundle* is $(T^*M|_{S_x})/T^*S_x$. Namely, the projection π_2 sets up an isomorphism between the global sections of the (trivial) normal bundle of $\pi_2^{-1}(x)$ relative to S and the tangent space $T_x X$, and global sections of this normal bundle give rise to global sections of the normal bundle to $S_x \subset M$ via the Jacobian of π_1 . An analytic family of compact submanifolds is said to be *complete* if this map is everywhere an isomorphism.

An analytic family $S \subset M \times X$ of compact submanifolds is said to be *maximal* if for any other such family $T \subset M \times Y$ such that $S_x = T_y$ as point sets for some $x \in X$, $y \in Y$, there are neighbourhoods $U \subset X$ of x and $V \subset Y$ of y such that for every $y' \in V$ there exists $x' \in U$ for which $S_{x'} = T_{y'}$, as point sets.

Kodaira [9] proved the following beautiful result, for which we shall find many uses:

Theorem. Let $K \subset M$ be a compact complex submanifold whose normal bundle N satisfies $H^1(K, \mathcal{O}(N)) = 0$. Then there exists a complete analytic family of compact submanifolds of M containing K . This family is maximal and is of complex dimension $\dim_{\mathbb{C}} H^0(K, \mathcal{O}(N))$; moreover, it is locally unique.

This result is proved by working in local coordinates, expanding the defining functions of nearby compact submanifolds in terms of power series on the parameter space, and using the vanishing of $H^1(\mathcal{O}(N))$ to insure

agreement on the overlaps one order at a time.

As a consequence, we have the

Corollary [10]. If $K \subset M$ is a compact complex submanifold with normal bundle N satisfying $H^1(K, \mathcal{O}(N)) = 0$, then small deformations of M do not destroy the complete analytic family to which K belongs, i.e. if $\omega: \tilde{M} \rightarrow \Pi$ is a regular holomorphic surjection of complex manifolds such that $M \cong \omega^{-1}(p)$ for some $p \in \Pi$, then there is a complete analytic family of compact submanifolds containing $K \subset M$ such that each fibre of ω contains precisely an m -dimensional subfamily, where $m = \dim H^0(K, \mathcal{O}(N))$.

The proof simply notes that the normal bundle of K *relative to* \tilde{M} is just $N + E$, where E is a trivial bundle whose dimension is the dimension of Π . Every compact submanifold of the complete family lies in a fibre of ω because holomorphic functions on a compact manifold are constant.

§ 0.5 CR MANIFOLDS [13], [14]

The holomorphic tangent space $T^{\mathbb{C}}M$ to a complex n -manifold M is a complex n -distribution (n -dimensional complex vector sub-bundle of the complexified smooth tangent bundle $\mathbb{C}TM$) which is

(a) totally complex: $T^{\mathbb{C}}M \cap \overline{T^{\mathbb{C}}M} = 0$, where the complex conjugation is the usual one in $\mathbb{C}TM$;

(b) involutive: the Lie bracket of two smooth sections of $T^{\mathbb{C}}M$ is again a smooth section of $T^{\mathbb{C}}M$: $[\mathcal{E}(T^{\mathbb{C}}M), \mathcal{E}(T^{\mathbb{C}}M)] = \mathcal{E}(T^{\mathbb{C}}M)$.

Conversely, the Newlander-Nirenberg theorem [14] states that any complex n -distribution on a smooth $2n$ -manifold M satisfying (a) and (b) is the holomorphic tangent bundle for a (unique) complex structure on M ; i.e. there exist local complex coordinates z^i , $i = 1, \dots, n$, such that the $\partial/\partial z^i \in \mathbb{C}TM$ span the given distribution.

If $X \subset M$ is a smooth hypersurface (real codimension 1) in a complex $(n+1)$ -manifold, then $T^{\mathbb{C}}M \cap \mathbb{C}TX$ is a smooth totally complex, involutive complex n -distribution on the smooth $(2n+1)$ -fold X . More generally, we define a *CR structure* on a smooth $(2n+1)$ -fold to be a smooth totally complex, involutive complex n -distribution H . As it turns out, not every CR manifold (i.e. odd-dimensional smooth manifold with CR structure) can be realised as a smooth hypersurface of a complex manifold [13].

A crude classification of CR manifolds is given by the *Levi-form*; if X is a CR manifold with CR distribution H , this is the line-bundle-valued Hermitian form

$$L: H \times H \rightarrow \mathbb{C}TX/(H + \bar{H})$$

$$(V_x, W_x) \rightarrow [\tilde{V}, \tilde{W}]_x + (H + \bar{H})_x$$

where \tilde{V} and \tilde{W} are any smooth sections of H taking on the values V and W , respectively, at a given point $x \in X$; this is well defined (i.e. independent of the choice of extensions) since if f is a smooth function vanishing at

x one has

$$[f\tilde{V}, \tilde{W}]_x = f(x) \cdot [\tilde{V}, \tilde{W}]_x - (\tilde{W}f)(x) \cdot \tilde{V}_x \in H$$

for any sections \tilde{V} and \tilde{W} of H .

A differentiable complex valued function f on a CR manifold X with CR distribution H is said to be a *CR function* if $df|_{\bar{H}} \equiv 0$. Clearly, if X arises as a real hypersurface in a complex manifold M , then the restriction of a holomorphic function on a neighbourhood of X in M to X is a CR function on X ; conversely, provided that the Levi-form of a smooth hypersurface $X \subset M$ in a complex manifold has, at each point, both positive and negative eigenvalues, then every CR function on X is the restriction of a (locally unique) holomorphic function on a neighbourhood of X [18]. (If the Levi-form is merely non-vanishing, then a CR function will be the one-sided boundary value of a holomorphic function). Notice that this extension property holds more generally if the CR function takes values in a complex manifold, since locally this is simply a generalisation to an m -tuple of CR functions; we may define a smooth map $f: X \rightarrow N$ from a CR manifold to a complex manifold to be a *CR map* if its Jacobian carries the CR distribution of X into the holomorphic tangent space of N .

§1.1 HOLOMORPHIC AFFINE CONNEXIONS

For the sake of completeness, we give a brief exposition of holomorphic affine connexions, which are related to the analogous objects in the smooth category in the most obvious possible manner. For holomorphic affine connexions see [1], [5].

By a *holomorphic affine connexion* on a complex n -manifold M we mean a sheaf morphism $\nabla: \Theta \rightarrow \Omega^1 \otimes_{\mathcal{O}} \Theta$ (where $\Theta := \mathcal{O}(T^*M)$ is the sheaf of holomorphic vector fields on M and $\Omega^1 := \mathcal{O}(T^*M)$ is the sheaf of holomorphic 1-forms) satisfying the *Leibnitz Rule*,

$$\nabla(f \cdot X) = \partial f \otimes X + f \cdot \nabla X \quad f \in \mathcal{O}, X \in \Theta.$$

Though we have defined ∇ on the level of germs, we shall be completely cavalier in using the same symbol to denote the induced morphism on sections over an open set.

Let $|M|$ denote the underlying smooth manifold of M ; we define an underlying smooth affine connexion $|\nabla|$ on $|M|$ for an arbitrary holomorphic affine connexion ∇ on M . If X is a holomorphic vector field in some region of M , we define

$$|\nabla| (X + \bar{X}) := \nabla X + \bar{\nabla} \bar{X}$$

which is a *real* smooth $(1, 1)$ tensor; by linearity and the Leibnitz Rule we extend this to all smooth vector fields. One must now check that this is well defined (i.e. entails no contradictions). To do so, we must simply certify that if

$$\sum_{j=1}^N f^j (X_j + \bar{X}_j) = 0$$

(i.e. if $\sum f^j X_j = 0$) for X holomorphic vector fields and f^j real smooth functions, then

$$\sum df^j \otimes (X_j + \bar{X}_j) + \sum f^j (\nabla X_j + \bar{\nabla} \bar{X}_j) = 0,$$

which is seen by using an arbitrary holomorphic basis y_j , $j = 1, \dots, n$, for the holomorphic vector fields on some neighbourhood (e.g. a coordinate frame); writing $X_j = \sum_{k=1}^n A_j^k y_k$ for some holomorphic $N \times n$ matrix A , and noticing that $\sum f^j A_j^k = 0$, $k = 1, \dots, n$, we see that

$$\begin{aligned} \sum f^j (\nabla X_j + \bar{\nabla} \bar{X}_j) + \sum df^j (X_j + \bar{X}_j) = \\ \sum f^j (A_j^k \nabla y_k + dA_j^k \otimes y_k + \text{c.c.}) + df^j (A_j^k y_k + \text{c.c.}) = \\ \sum d(f^j A_j^k) \otimes y_k + \text{c.c.} = 0. \end{aligned}$$

We will now define the geodesics of a holomorphic connexion and presently see their intimate relationship with the geodesics of the underlying smooth connexion. By a *geodesic* for the holomorphic connexion ∇ on the complex n -manifold M we mean a holomorphic curve with parameter ζ such that for some, and hence for any, local holomorphic extension X of $d/d\zeta$ about any point of the curve, one has

$$X \lrcorner \nabla X = 0 \tag{1.1.1}$$

on the curve. (If f is a holomorphic function vanishing on the curve, then $(X + f \cdot W) \lrcorner \nabla(X + f \cdot W) = X \lrcorner (f \cdot W + \partial f \otimes W) = 0$ on the curve if (1.1.1) is satisfied; since any holomorphic vector field vanishing on the curve can be written as a finite sum $\sum f^i W_i$ where the f^i vanish on the curve, (the requisite number of terms being the codimension of the curve), we've shown by induction that (1.1.1) does not depend on the extension. The fact that there exist extensions X at all is easily seen by the holomorphic implicit function theorem - see [3]).

Given any holomorphic tangent vector V on a complex n -manifold M with holomorphic connexion ∇ there is precisely one geodesic curve with parameter ζ , defined in any suitably small neighbourhood of the origin of the complex line, for which $\frac{d}{d\zeta} \Big|_{\zeta=0} = V$.

This is true because of the theory of \mathcal{O}_{DE} 's previously expounded; in local holomorphic coordinates $(z^j): U \rightarrow \mathbb{C}^n$, $U \stackrel{\text{open}}{\subseteq} M$, defining holomorphic Christoffel symbols $\chi_{k\ell}^j$, $j, k, \ell = 1, \dots, n$, by

$$\nabla \frac{\partial}{\partial z^\ell} = \chi_{k\ell}^j dz^k \otimes \frac{\partial}{\partial z^j},$$

The geodesics are just the solutions to the equations

$$\frac{d^2 z^j}{d\zeta^2} + \chi_{k\ell}^j \frac{dz^k}{d\zeta} \frac{dz^\ell}{d\zeta} = 0.$$

Now we come to the relation with the geodesics underlying smooth connexion. If one restricts the parameter ζ of a complex geodesic τ to the real axis, one gets a smooth (in fact, a real analytic) curve satisfying

$$\frac{d}{d\xi} \lrcorner |\nabla| \frac{d}{d\xi} = 0$$

where $\xi = \text{Re } \zeta$; i.e. one gets a geodesic of the underlying smooth (actually real analytic) connexion. Conversely, one may use the exponential map of the smooth connexion (see [7] or any book on differential geometry) to find the complex geodesics; this fact is crucial to our proof of the existence of geodesically convex neighbourhoods in the holomorphic category.

To finish off this section, we notice the form of the Christoffel symbols of the underlying smooth connexion in the underlying real coordinates of a complex chart. Let z^j , $j = 1, \dots, n$, be local holomorphic coordinates, let $x^j = \text{Re } z^j$, $x^{j'} := \text{Im } z^j$, $j = 1, \dots, n$. Then, letting hatted χ 's denote the Christoffel symbols for the underlying smooth connexion of a holomorphic connexion with Christoffel symbols χ , we have

$$\hat{\chi}_{k\ell}^j = \text{Re } \chi_{k\ell}^j, \hat{\chi}_{k\ell}^{j'} = \text{Im } \chi_{k\ell}^j, \hat{\chi}_{k'-\ell}^{j'} = \text{Re } \chi_{k\ell}^j$$

$$\hat{\chi}_{k'-\ell}^j = -\text{Im } \chi_{k\ell}^j, \hat{\chi}_{k\ell}^{j'} = -\text{Im } \chi_{k\ell}^j, \hat{\chi}_{k'-\ell'}^{j'} = -\text{Im } \chi_{k\ell}^j$$

$$\hat{\chi}_{k\ell}^j = -\text{Re } \chi_{k\ell}^j, \quad \hat{\chi}_{k\ell}^{j'} = \text{Re } \chi_{k\ell}^j.$$

Henceforth, "affine connexion" shall mean "holomorphic affine connexion" unless contraindication is given.

§1.2 PROJECTIVE CONNEXIONS

Let M be a complex n -manifold. By a *holomorphic projective connexion* on M we mean a system \mathcal{L} of complex curves (*inextendible* immersed connected complex 1-manifolds) such that there is a unique curve of the system tangent to any holomorphic direction (i.e. to any element of the projectivized holomorphic tangent bundle $\mathbb{P}T^*M$) and such that the curve varies holomorphically with the "initial" direction, which is to say the "lifted" curves

$$\hat{\mathcal{L}} := \{(x, T_x^* \ell) \mid x \in \ell\}, \ell \in \mathcal{L}$$

foliate $\mathbb{P}T^*M$ holomorphically. We call the curves of \mathcal{L} *geodesics*.

Proposition (1.2.1). Let M be a complex n -manifold with holomorphic projective structure \mathcal{L} , and let $(z^1, \dots, z^n): U \rightarrow \mathbb{C}^n$ be a local chart. n open M

Then there exists a unique set of *trace-free Christoffel symbols*

$$\Gamma_{jk}^i \in \mathcal{O}_U, \quad i, j, k = 1, \dots, n$$

such that

$$\Gamma_{jk}^i = \Gamma_{kj}^i, \quad \Gamma_{ij}^i = 0$$

and such that the images of the solutions of the equations

$$\frac{d^2 z^i}{d\zeta^2} + \Gamma_{jk}^i \frac{dz^j}{d\zeta} \frac{dz^k}{d\zeta} = 0 \quad (1.2.1)$$

are the intersections of the curves of \mathcal{L} with U .

Proof. We let $\mathcal{O}(m)$ denote the sheaf on $\mathbb{P}T^*M$ of germs of holomorphic functions homogeneous of degree m in the fibres; i.e. $\mathcal{O}(m)$ has the complete presheaf which assigns to a region $V \subset \mathbb{P}T^*M$ the holomorphic functions f on the corresponding region \tilde{V} in $T^*M - 0_M$, 0_M the image of the zero section, for which $f(zw) = z^m f(w)$ for any vector $w \in \tilde{V}$ and $z \in \mathbb{C}^*$. Then we define a sheaf S on $\mathbb{P}T^*U$ by the exact sheaf sequence

$$0 \rightarrow \mathcal{O}(1) \xrightarrow{\alpha} n \mathcal{O}(2) \xrightarrow{\beta} S \rightarrow 0 \quad (1.2.2)$$

where $\alpha(f) \left(w^a \frac{d}{dz^a} \Big|_p \right) = \left(w^1 f \left(w^a \frac{d}{dz^a} \Big|_p \right), \dots, w^n f \left(w^a \frac{d}{dz^a} \Big|_p \right) \right)$

for any $f \in \mathcal{O}_V(1)$, $V \subset \mathbb{P}T^*U$. We now define a global section σ of S by requiring that

$$\sigma \left(\frac{dz^a}{d\zeta} \Big|_p \frac{d}{dz^a} \Big|_p \right) = \beta \left(\frac{d^2 z^a}{d\zeta^2} \Big|_p \right)$$

for any geodesic through any $p \in U$ parametered by ζ . This is well defined because a change from parameter ζ to parameter $\hat{\zeta} = \zeta + \zeta^2 g(\zeta)$ (for g a holomorphic function in \mathcal{U} near 0) simply forces

$$\frac{d^2 z^a}{d\hat{\zeta}^2} \Big|_{\hat{\zeta}=0} = \frac{d^2 z^a}{d\zeta^2} \Big|_{\zeta=0} - 2g(0) \frac{dz^a}{d\zeta} \Big|_{\zeta=0}$$

i.e. $\frac{d^2 z^a}{d\hat{\zeta}^2} - \frac{d^2 z^a}{d\zeta^2} \in \text{IM } \alpha = \text{KER } \beta$. Moreover, this is indeed a holomorphic object, and not some other sort of section of the vector bundle whose holomorphic sections are S ; for since lifts of the curves of \mathcal{L} foliate $\mathbb{P}T^*M$ holomorphically, we can find local coordinates on $\mathbb{P}T^*M$ such that varying the first coordinate while holding the others fixed parameterizes a lifted geodesic, and so we can choose $d^2 z^a / d\zeta^2$ to be holomorphic functions (of homogeneity 2) on $\mathbb{P}T^*U$, at least locally. But in fact (and this is the whole point of the proposition) we can choose these functions over the entire coordinate chart; this is true because, assuming for the moment that U is stein,

$$H^1(\mathbb{P}T^*U, \mathcal{O}(1)) = 0$$

since $H^p(U \times \mathbb{P}_{n-1}, \mathcal{O}(m)) \cong H^p(\mathbb{P}_{n-1}, \mathcal{O}(m)) \otimes_{\mathbb{C}} \mathcal{O}_U$

(as observed by inspecting the Leray cover $\{W_j \times U, j=1, \dots, n\}$ where $\{W_j\}$ is a covering of \mathbb{P}_{n-1} by affine charts) and since $H^1(\mathbb{P}_{n-1}, \mathcal{O}(1)) = 0$

(see "Bott's Rule" in the Chapter of preliminary material); thus over any Stein sub-region of the coordinate patch, the exactness of the sequence

$$0 \rightarrow \Gamma^{\mathcal{O}}(1) \xrightarrow{\alpha_*} n\Gamma^{\mathcal{O}}(2) \xrightarrow{\beta_*} \Gamma^{\mathcal{S}} \rightarrow 0 \quad (1.2.3)$$

means there is a unique set of *trace-free* Christoffel symbols such that the solutions of (1.2.1) parameterize curves of \mathcal{L} (notice that something in the image of α_* - that is, a pure trace term - does not change the image curves of the solution, but only the parameterization); and we therefore have a unique choice of such symbols over the entire patch dictated by the requirement that it agrees with the unique choices on Stein sub-regions. \square

Remark: If $(\tilde{z}^j): \tilde{U} \rightarrow \mathbb{C}^n$ is another coordinate system such that $U \cap \tilde{U} \neq \emptyset$, the corresponding symbols $\tilde{\Gamma}_{jk}^i$ are related to the Γ_{jk}^i on $U \cap \tilde{U}$ by

$$\begin{aligned} \tilde{\Gamma}_{jk}^i &= \frac{\partial \tilde{z}^i}{\partial z^\ell} \left(\frac{\partial^2 z^\ell}{\partial \tilde{z}^j \partial \tilde{z}^k} + \frac{\partial z^p}{\partial \tilde{z}^j} \frac{\partial z^q}{\partial \tilde{z}^k} \Gamma_{pq}^\ell \right) \\ &\quad - \frac{1}{2n} \left[\delta_j^i \frac{\partial z^q}{\partial \tilde{z}^k} + \delta_k^i \frac{\partial z^q}{\partial \tilde{z}^j} \right] \Gamma_{\ell q}^\ell \end{aligned} \quad (1.2.4)$$

Conversely, an assignment of symbols to each coordinate system of a covering satisfying (1.2.4) on the overlaps defines a projective connexion by taking the images of solutions of (1.2.2) to be the geodesics - noting that this does not introduce inconsistencies on overlaps. This equivalent definition is perhaps the most common (cf. [5]). (Notice, incidentally, that the foliation of T^*M by the lifted curves is automatically holomorphic by the theory of $\mathcal{O}DE$'s).

Given an affine connexion ∇ on a manifold M , the *projective class* of that connexion will mean the projective connexion whose geodesics are the geodesics of L . Locally, every projective connexion arises this way—

for instance from the connexion defined on a coordinate patch by

$$\nabla \frac{\partial}{\partial z^j} = \Gamma_{jk}^i \frac{\partial}{\partial z^i} \otimes dz^k -$$

but of course one projective connexion locally represents many affine connexions — namely, those whose Christoffel symbols have the Γ 's as their trace-free symmetric parts.

Finally, note that our approach to projective connexions is impossible in the smooth category; nothing analogous to the above proposition would be true of a general system of smooth curves depending smoothly upon their tangent directions since a smooth homogeneous function on $\mathbb{R}P_n$ need not be algebraic by any stretch of the imagination. Rather, one simply imposes the existence of local representative affine connexions as an extra hypothesis in one form or another. (For the theory of smooth projective connexions see [8], which shows that they are indeed Cartan connexions — on suitable sub-bundles of the second order frame bundle).

By "projective connexion", we shall henceforth always mean "holomorphic projective connexion".

§1.3 COMPLEX GEODESICALLY CONVEX NEIGHBOURHOODS

Let M be a complex manifold with holomorphic projective connexion. We will show that the topology of M has a basis of open sets of the following type: If U is a basis set, any two points of U are joined by precisely one geodesic, which is a simply-connected closed one-dimensional submanifold of U . Open sets with this property are said to be *geodesically convex*.

Our proof exploits the well-known analogous result of Whitehead [20] for the smooth category as applied to the underlying smooth connexion of a local representative holomorphic connexion for the projective connexion. We shall therefore spend some time recalling a few details about the exponential map and the proof of Whitehead's theorem that will be of use to us.

On the (real) smooth tangent bundle $\pi:TN \rightarrow N$ of a smooth n -manifold with smooth affine connexion there is a smooth *canonical horizontal vector field* W characterised by the fact that if $\gamma:(-\epsilon,\epsilon) \rightarrow N$ is a geodesic, then $\gamma':(-\epsilon,\epsilon) \rightarrow TN:t \mapsto (d\gamma/dt)(t)$ is tangent to W ; if (x^1, \dots, x^n) are local coordinates on M , then in the local coordinates $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ where $\xi^i := dx^i$ as a local function on TM for $i = 1, \dots, n$, one has

$$W(x, \xi) = \xi^i \frac{\partial}{\partial x^i} - \chi_{jk}^i \xi^j \xi^k \frac{\partial}{\partial \xi^i}$$

where the χ 's are the Christoffel symbols for the connexion. Let $\{\phi_t | t \in \mathbb{R}\}$ be the one parameter pseudo-group of maximal diffeomorphisms of regions of TN to TN integrating W ; since W vanishes on the zero section of TN , ϕ_t is defined on a neighbourhood of the zero section for all t . The exponential map $\exp: \mathcal{W} \rightarrow N$, where $\mathcal{W} \subset TN$ is the domain of ϕ_1 , is given by $\exp := \pi \circ \phi_1$; it takes a vector V to the point having unit parameter value on the geodesic with tangent V at parameter value nought.

Now let M be a complex n -manifold with holomorphic affine connexion ∇ ,

$\tilde{\pi}: T^*M \rightarrow M$ the holomorphic tangent bundle, and let $\text{Re}: T^*M \rightarrow TM$ be the real-linear isomorphism $X \rightarrow \frac{1}{2}(X + \bar{X})$. We define the complex exponential map $\text{EXP}_{\nabla}: \text{Re}^{-1}(W_M) \rightarrow M$ to be $\exp_{|\nabla|} \circ \text{Re}$; then $\text{EXP}_{\nabla}(x)$ is the point of parameter value units on the unique geodesic with holomorphic tangent vector X at parameter value nought. The important point to note, however, is that EXP is a *holomorphic* map, something perhaps not evident from the way we have defined it; this is true because $(\text{Re})_*^{-1}(W)$ is the real part of a holomorphic field \tilde{W} . The *canonical holomorphic horizontal field* associated to ∇ , given in local holomorphic coordinates $(z^1, \dots, z^n, \zeta^1, \dots, \zeta^n)$ on T^*M , where (z^1, \dots, z^n) are local holomorphic coordinates for M and $\zeta^i := dz^i$ as local functions on T^*M , by

$$\tilde{W} = 2\zeta^j \frac{\partial}{\partial z^j} - 2\chi_{kl}^j \zeta^k \zeta^l \frac{\partial}{\partial \zeta^j}$$

where the χ 's are the holomorphic Christoffel symbols of ∇ . (The diffeomorphisms integrating the real part of a holomorphic vector field are always biholomorphisms. One sees this in local coordinates by letting X^j , $j=1, \dots, n$, be holomorphic functions of the coordinates and then noticing that

$$\begin{aligned} \mathbb{E} \text{Re}\left(X^j \frac{\partial}{\partial z^j}\right) \left(\frac{\partial}{\partial z^k}\right) &= \frac{1}{2} \left[X^j \frac{\partial}{\partial z^j} + \bar{X}^j \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k} \right] \\ &= -\frac{1}{2} \frac{\partial X^j}{\partial z^k} \frac{\partial}{\partial z^j} \end{aligned}$$

and hence the flow preserves the complex structure).

What Whitehead actually showed for the smooth category was that the exponential image of a sufficiently small ball about zero in any tangent plane (fibre of π) is geodesically convex. Let U be such a geodesically convex neighbourhood for $|\nabla|$, and let $V \subset T^*U$ be the domain of EXP_{∇} . Since EXP_{∇} takes every real line through zero in any fibre of $\tilde{\pi}$ to a $|\nabla|$ geodesic, and any two points of U are joined by one such geodesic, it

follows that \mathcal{W} is fibrewisely star-shaped about the zero section, and so, in particular, every complex line through the origin of any fibre meets \mathcal{W} in a bounded star-shaped set (so, by the Riemann mapping theorem, in a biholomorphic copy of the disc in \mathbb{C}), which in turn is mapped biholomorphically onto a complex geodesic in U . We therefore see that every two points in U are joined by precisely one complex geodesic.

We've actually proven something a bit stronger than our original claim, and this stronger statement will prove useful. We define a complex manifold with projective connexion to be *strongly geodesically convex* if it admits a compatible smooth affine connexion (i.e. the underlying smooth connexion for a global holomorphic affine connexion representing the projective connexion) for which it is geodesically convex. We've demonstrated the

Fact. Every complex manifold with projective connexion admits a basis of strongly geodesically convex neighbourhoods. \square

The constructed strongly geodesically convex neighbourhoods are biholomorphically balls in \mathbb{C}^n . In general, strongly geodesically convex manifolds are biholomorphic to domains of holomorphy. Firstly, since any two points can be joined by a unique complex geodesic, EXP restricted to a fibre of $\tilde{\pi}$ is a biholomorphism that shows us that the manifold can be embedded openly in \mathbb{C}^n . Secondly, the manifold is holomorphically convex, since if K is a compact subset, the geodesic convex hull \hat{K} defined in terms of the compatible smooth connexion is compact and has a domain of holomorphy as interior - for if $y \in \text{Bd}[\hat{K}]$, \hat{K} geodesically convex, then $\text{EXP}_y^{-1} \hat{K}$ must avoid a hyperplane through 0_y , and so we can define a holomorphic function on $\text{INT}[\hat{K}]$ blowing up at y . (Here $\text{EXP}_y := \text{EXP}|_{\tilde{\pi}^{-1}(y)}$). Hence the manifold is biholomorphically a domain of holomorphy (see [6]).

§1.4 SPACES OF COMPLEX GEODESICS

Let M be a strongly geodesically convex n -fold with projective connexion. (By our previous work, this does not introduce any *local* loss of generality). We shall show that the space $L(M)$ of complex geodesics in M is naturally a (Hausdorff) complex $(2n-2)$ -fold.

By the definition of a projective connexion, the lifts

$$\tilde{\ell} := \{(x, T_x^{-1}\ell) \mid x \in \ell\}$$

of geodesics $\ell \in \mathcal{L}$ foliate $\mathbb{P}T^*M$ holomorphically. The space $L(M)$ of geodesics in M is, as a topological space, the space of leaves of this foliation with the quotient topology. We shall see that this space is Hausdorff, and that the Frobenius charts of the foliation induce charts on $L(M)$, giving it the structure of a complex manifold.

First let us see that the space of leaves is Hausdorff. Whitehead's theorem [20], with our previously discussed additions, sets up a biholomorphism ψ between $W - 0_M$, where $W \subset T^*M$ is a neighbourhood of the zero section 0_M , and $M \times M - \Delta$, where Δ is the diagonal such that the following diagram commutes:

$$\begin{array}{ccc} W - 0_M & \xrightarrow[\psi]{\cong} & M \times M - \Delta \\ \text{EXP} \searrow & & \swarrow \text{pr}_1 \\ & M & \end{array}$$

where pr_1 is projection to the first factor. Now suppose that $\{\ell_i\}$ is a sequence of leaves of the foliation of $\mathbb{P}T^*M$ by lifted geodesics which converges to two distinct leaves ℓ and ℓ' . Then we can find distinct points $\tilde{x}, \tilde{y} \in \ell$, $\tilde{z}, \tilde{w} \in \ell'$, and $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i, \tilde{w}_i \in \ell_i$ such that $\tilde{x}_i \rightarrow \tilde{x}$, $\tilde{y}_i \rightarrow \tilde{y}$, $\tilde{z}_i \rightarrow \tilde{z}$, $\tilde{w}_i \rightarrow \tilde{w}$; projecting this into M we have geodesics $\gamma_i, \gamma, \gamma'$ and distinct points $x_i, y_i, w_i, z_i \in \gamma_i$, $x, y \in \gamma$, and $w, z \in \gamma'$ such that $x_i \rightarrow x$, $y_i \rightarrow y$, $z_i \rightarrow z$, and $w_i \rightarrow w$ in M . Thus $\psi^{-1}(x_i, y_i) \rightarrow \psi^{-1}(x, y)$ and

$\psi^{-1}(x_i, z_i) \rightarrow \psi^{-1}(x, z)$ in $W - 0_M$. Letting $p: (T^*M - 0_M) \rightarrow \mathbb{P}T^*M$ be the canonical projection, we see that $p\psi^{-1}(x_i, y_i) = p\psi^{-1}(x_i, z_i)$ for all i because $x_i, y_i,$ and z_i all lie on the geodesic γ_i ; since $\mathbb{P}T^*M$ is Hausdorff and $p\psi^{-1}(x_i, y_i)$ converges to both $p\psi^{-1}(x, y)$ and $p\psi^{-1}(x, z)$, it follows that $p\psi^{-1}(x, y) = p\psi^{-1}(x, z)$, and so z lies on the geodesic γ joining x to y . Similarly, w lies on γ . So $\gamma' = \gamma$ by geodesic convexity. Hence $\ell' = \ell$. This shows that $L(M)$ is Hausdorff, since it has a countable basis.

Let $\Phi: U \rightarrow \mathbb{C}^{2n-1}$, where $U \subset \mathbb{P}T^*M$ is open, be a Frobenius chart for the foliation of $\mathbb{P}T^*M$ by lifted geodesics; by a Frobenius chart we mean that $(r)^{-1}(z)$ lies in a single leaf for each $z \in \mathbb{C}^{2n-2}$, where $r: \mathbb{C}^{2n-1} \rightarrow \mathbb{C}^{2n-2}$ is the projection to the last factors; suppose, without loss of generality, that $\Phi[U]$ is the unit polydisc. For some $x \in U$, find a neighbourhood V of $\pi(x)$, where $\pi: \mathbb{P}T^*M \rightarrow M$ is the canonical projection, which is geodesically convex and so small that $[\pi^{-1}(\bar{V})] \cap r^{-1}(r(x))$ is compact, where \bar{V} denotes the topological closure of V in M . Then there is some neighbourhood B of $r(x)$ in \mathbb{C}^{2n-2} such that $[\pi^{-1}(\bar{V})] \cap r^{-1}(z)$ is compact for all $z \in B$, since $[\pi^{-1}(\bar{V})] \cap r^{-1}(z)$ is connected for all z (otherwise there would exist a geodesic leaving and re-entering V , contrary to our hypotheses, which state that both V and M are geodesically convex; the unique geodesic joining two points of V must also be the unique geodesic joining those points in M) and so the fact that $[\pi^{-1}(\bar{V})] \cap \Omega$ is compact, where

$$\Omega := \{(z_1, \dots, z_{2n-1}) \mid |z_1| < 1-\epsilon, |z_j| < \frac{1}{2} \forall j > 1\}$$

and ϵ is chosen so that $\Omega \cap r^{-1}(0) = \emptyset$, forces the compactness of

$[\pi^{-1}(\bar{V})] \cap r^{-1}(z)$ for all z sufficiently small. Since any geodesic intersects V in a connected set, it follows that each $(r)^{-1}(z)$, $z \in B$, is part of a *different* leaf of the foliation of $\mathbb{P}T^*M$. Letting $q: \mathbb{P}T^*M \rightarrow L(M)$

denote the quotient map, we see that $q \circ (r)^{-1}$ defines an open one-to-one map $B \rightarrow L(M)$. In particular, $L(M)$ is a topological manifold. But, as we saw in §0.1, this means that charts of the constructed kind give $L(M)$ a complex structure automatically. We've proven the

Theorem (1.4.1). If M is a strongly geodesically convex complex manifold with projective connexion, then its space $L(M)$ of geodesics is naturally a (Hausdorff) paracompact complex manifold. (If $\dim_{\mathbb{C}} M = n$, then $\dim_{\mathbb{C}} L(M) = 2(n-1)$). \square

If $x \in M$, let $P_x \subset L(M)$ be the set of geodesics passing through x . Then P_x is the image under the (holomorphic) quotient map $q: \mathbb{P}T^*M \rightarrow L(M)$ of the fibre over x ; since every leaf is transverse to the fibres of the projection $\pi: \mathbb{P}T^*M \rightarrow M$, this $\mathbb{P}P_{n-1}$ is immersed in $L(M)$; but geodesic convexity guarantees that there are no geodesic loops and hence P_x is in fact an embedded $\mathbb{P}P_{n-1}$.

The family of P_x 's completely encodes the projective structure of M . Namely, three points $x, y, z \in M$ are geodesically colinear iff $P_x \cap P_y \cap P_z \neq \emptyset$ (it will then consist of just one point; $\#(P_x \cap P_y) = 1$ for any x and y since each pair of points are joined by just one geodesic). What we shall soon see is that the family of P_x 's is in a certain sense defined by the complex structure of $L(M)$, and that, even more strikingly, it survives any slight deformation of the complex structure of $L(M)$ - i.e. deforming the complex structure of $L(M)$ corresponds just to deforming the projective connexion on M .

§1.5 THE NORMAL BUNDLE OF P_x

Let M be a strongly geodesically convex complex n -manifold with projective connexion. We have the following diagram

$$\begin{array}{ccc} & \mathbb{P}T M & \\ \pi \swarrow & & \searrow q \\ M & & L(M) \end{array}$$

where π is the canonical projection and where q is the quotient map; both π and q are holomorphic subjections with nowhere singular Jacobians (i.e. they are *regular*). We have defined $P_x := q[\pi^{-1}(x)]$ for all $x \in M$, and have noted that each P_x is an embedded \mathbb{P}^{n-1} ; we now proceed to compute the isomorphism type of the normal bundle of P_x , which turns out to be not only independent of x , but also independent of the projective connexion.

Let \tilde{T} denote the holomorphic tangent bundle of $\mathbb{P}T M$ restricted to the fibre over x , and let T denote the holomorphic tangent bundle of $L(M)$ restricted to P_x : $\tilde{T} := T(\mathbb{P}T M)|_{\pi^{-1}(x)}$, $T := T(L(M))|_{P_x}$. The Jacobian q_* of q , restricted to $\pi^{-1}(x)$, defines a morphism of holomorphic vector bundles over P_x (we shall not make the pedantic distinction between $\pi^{-1}(x)$ and P_x since the identification of the two via q is clear) which has as its kernel a one dimensional vector bundle K :

$$0 \rightarrow K \rightarrow \tilde{T} \rightarrow T \rightarrow 0 \quad (\text{exact}) \quad (1.5.1)$$

$\alpha \quad q^*$

K consists of those vectors at $\pi^{-1}(x)$ which are tangent to the foliation by lifted geodesics. Now let T_0 be tangent bundle to $\pi^{-1}(x)$ (or P_x ; i.e. $q_* T_0$), and let N and \tilde{N} be the normal bundles to P_x and $\pi^{-1}(x)$ respectively:

$$0 \rightarrow T_0 \rightarrow \tilde{T} \rightarrow \tilde{N} \rightarrow 0$$

$\tilde{\gamma} \quad \tilde{\delta}$

(exact) (1.5.2)

$$0 \rightarrow T_0 \rightarrow T \rightarrow N \rightarrow 0$$

$\gamma \quad \delta$

(again, $\gamma = q_*\tilde{\gamma}$). We thus have a unique morphism $\varepsilon: \tilde{N} \rightarrow N$ defined by the commutativity of the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & K & & \\
 & & & \swarrow \alpha & & \searrow \beta & \\
 & & & \tilde{T} & \longrightarrow & \tilde{N} & \rightarrow 0 \\
 0 \rightarrow T_0 & \xrightarrow{\tilde{\gamma}} & & \downarrow q_* & \xrightarrow{\tilde{\delta}} & \downarrow \varepsilon & \\
 & \searrow \gamma & & T & \longrightarrow & N & \rightarrow 0 \\
 & & & \downarrow & \xrightarrow{\delta} & \downarrow & \\
 & & & 0 & & 0 &
 \end{array} \tag{1.5.3}$$

which has exact columns as well as exact rows; the important fact is that

$$0 \rightarrow K \xrightarrow{\beta} \tilde{N} \rightarrow N \rightarrow 0 \tag{1.5.4}$$

is exact, where $\beta := \tilde{\delta}\alpha$, as follows directly from the exactness of (1.5.1) and the fact that $T_0 \cap K$ is the zero section of \tilde{T} (the leaves of the foliation are transverse to the π -fibres).

Now the Jacobian of π sets up a natural isomorphism between \tilde{N} and the trivial bundle $\pi^*(T_x)$ where T_x is the holomorphic tangent plane to M at x , thought of as a vector bundle over $\{x\}$; this is true because π is a regular map, and so π_* is onto, with kernel the tangent bundle of the fibres.

Recalling that K consists of vectors tangent to curves of the form

$$\{(y, T'_y \ell) \mid y \in \ell\}, \quad \ell \text{ a geodesic in } M$$

it follows that $\pi_*(K_z)$, $z \in \pi^{-1}(x)$ consists of those vectors in T_x that lie on the line $z \in T_x$; it follows that K is the Hopf bundle (Chern class -1) and that the exact sequence (1.5.4) is equivalent to the sequence

$$0 \rightarrow \mathcal{O}(-1) \xrightarrow{\lambda} n\mathcal{O} \rightarrow \mathcal{O}(N) \rightarrow 0 \tag{1.5.5}$$

of sheaves on \mathbb{P}_{n-1} ; here $\mathcal{O}(N)$ is the sheaf of germs of holomorphic sections of N , and $(\lambda(f))([z^1, \dots, z^n]) := (z^1 f([z^1, \dots, z^n]), \dots, z^n f([z^1, \dots, z^n]))$ for f a function of the homogeneous coordinates of

where (a) is derived from the original sequence by tensoring with $\mathcal{O}(N^*)$, while (b) is derived from the dual (c) of the original sequence by tensoring with $\mathcal{O}(-1)$. The map λ^* is the adjoint of λ and given explicitly by

$$[\lambda^* (f_1, \dots, f_n)] ([z^1, \dots, z^n]) = \left(\sum_{j=1}^n \right) z^j f_j([z^1, \dots, z^n])$$

where the f_j are local functions of the homogeneous coordinates of homogeneity zero. Now notice that on the level of global sections, λ^* induces a *surjection* $nH^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(1))$, since every global section of $H^0(\mathcal{O}(1))$ is a linear functional $(z^1, \dots, z^n) \rightarrow \sum \alpha_j z^j$; so the long exact sequence

$$\dots \rightarrow H^0(n\mathcal{O}) \rightarrow H^0(\mathcal{O}(1)) \xrightarrow{(0)} H^1(\mathcal{O}(N^*)) \rightarrow H^1(n\mathcal{O}) \rightarrow \dots$$

derived from (c) tells us that $H^1(\mathcal{O}(N^*)) = 0$. Meanwhile, the long exact sequence derived from (b)

$$\dots \rightarrow H^1(\mathcal{O}) \rightarrow H^2(\mathcal{O}(N^*) \otimes_{\mathcal{O}} \mathcal{O}(-1)) \rightarrow nH^2(\mathcal{O}(-1)) \rightarrow \dots$$

shows that $H^2(\mathcal{O}(N^*) \otimes_{\mathcal{O}} \mathcal{O}(-1))$ vanishes. So now the induced long exact sequence of (a)

$$\dots \rightarrow nH^1(\mathcal{O}(N^*)) \rightarrow H^1(\mathcal{O}(N \otimes N^*)) \rightarrow H^2(\mathcal{O}(N^*) \otimes_{\mathcal{O}} \mathcal{O}(-1)) \rightarrow \dots$$

proves the claim. □

Remarks: 1. The fact that the tensor product of a short exact sequence of sheaves of germs of holomorphic sections of vector bundles with another such sheaf is again exact is true because exactness is a local criterion, and because such sheaves are locally free over \mathcal{O} ; *locally*, such a sequence can be split as a direct sum (though in no preferred way), and then the fact that "tensoring up" by another (locally) free sheaf causes no difficulty is obvious. For the general problem in homological algebra of

homogeneity -1. However, we may notice that the sheaf of holomorphic vector fields θ on \mathbb{P}_{n-1} is described by the exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda \text{ id}} n\mathcal{O}(1) \rightarrow \theta \rightarrow 0 \quad (1.5.6)$$

and so we see the following

Theorem (1.5.I) The normal bundle of $P_x \subset L(M)$ is isomorphic to the tensor product of the holomorphic tangent bundle $T\mathbb{P}_{n-1}$ and the Hopf bundle:

$$\mathcal{O}(N) \cong \theta \otimes \mathcal{O}(-1). \quad \square$$

Notice that this isomorphism is really canonical, up to a constant overall factor, once the identification of P_x with \mathbb{P}_{n-1} is chosen.

§ 1.6 RELEVANT COHOMOLOGY OF $\mathcal{O}(N)$ AND $\mathcal{O}(\text{HOM}(N, N))$

We now use the exact sequence we derived in order to calculate $H^0(\mathcal{O}(N))$, $H^1(\mathcal{O}(N))$, and $H^1(\mathcal{O}(N \otimes N^*))$, as these are the groups needed to apply the stability theorems which we will use.

Proposition (1.6.I) Let M be a complex n -manifold with projective connexion for which M is strongly geodesically convex. Then

1. $H^1(\mathcal{O}(N)) = 0$
2. $\dim H^0(\mathcal{O}(N)) = n$
3. $H^1(\mathcal{O}(\text{HOM}(N, N))) = 0$

where $N \rightarrow P_x$ is the normal bundle in $L(M)$ to the \mathbb{P}_{n-1} of geodesics through any arbitrary point $x \in M$.

Proof

1, 2. The exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(-1) \xrightarrow{\lambda} n\mathcal{O} \rightarrow \mathcal{O}(N) \rightarrow 0 \quad (1.4.5)$$

gives rise to the long exact sequence

$$0 \rightarrow H^0(\mathcal{O}(-1)) \rightarrow nH^0(\mathcal{O}) \xrightarrow{\cong} H^0(\mathcal{O}(N)) \rightarrow H^1(\mathcal{O}(-1))$$

$\begin{array}{c} \text{0} \\ \parallel \\ \mathbb{C}^n \end{array}$

$$nH^1(\mathcal{O}) \rightarrow H^1(\mathcal{O}(N)) \rightarrow H^2(\mathcal{O}(-1)) \rightarrow \dots$$

$\begin{array}{c} \text{0} \\ \parallel \\ \text{0} \end{array}$

where all the cohomology groups are on P_x (i.e. on \mathbb{P}_{n-1}); we've used *Bott's Rule* in noting that certain groups vanish; see Chapter 0.

3. This is slightly more subtle, but again essentially straightforward.

One has the three exact sequences

$$(a) \quad 0 \rightarrow \mathcal{O}(N^*) \otimes_{\mathcal{O}} \mathcal{O}(-1) \rightarrow n\mathcal{O}(N^*) \rightarrow \mathcal{O}(N \otimes N^*) \rightarrow 0$$

$$(b) \quad 0 \rightarrow \mathcal{O}(N^*) \otimes_{\mathcal{O}} \mathcal{O}(-1) \rightarrow n\mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0 \quad (1.6.1)$$

$$(c) \quad 0 \rightarrow \mathcal{O}(N^*) \xrightarrow{\lambda^*} n\mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow 0$$

deciding to what degree the tensor ruins (left) exactness, see [16] or any other reference book on algebraic topology.

2. We re-emphasise that the only vanishing result used in the above proof is "Bott's Rule":

$$H^p(\mathbb{P}^k, \mathcal{O}(q)) = 0 \quad \text{for } p > 0 \quad \underline{\text{if}} \quad q \neq k \text{ or } q \geq -k.$$

§1.7 PROJECTIVE CONNEXIONS: THE MAIN THEOREM

Proposition (1.7.I). Let L be a complex $(2n-2)$ -manifold endowed with an n -dimensional family of submanifolds, each an embedded \mathbb{P}_{n-1} with normal bundle $T\mathbb{P}_{n-1} \otimes H^{-1}$. Letting \mathcal{M} denote the parameter space of the family, there is a projective connexion induced on \mathcal{M} by the rule that a curve is geodesic iff the intersection of all the corresponding submanifolds is non-void.

Proof. Let \mathcal{O}_y be the submanifold of L corresponding to $y \in \mathcal{M}$, and let N denote its normal bundle; by hypothesis, $\mathcal{O}(N) \cong \Theta \otimes \mathcal{O}(-1)$. The exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(-1) \xrightarrow{\lambda} n\mathcal{O} \xrightarrow{p} \mathcal{O}(N) \rightarrow 0 \quad (1.4.5)$$

induces an isomorphism of \mathbb{C}^n with $\Gamma(\mathcal{O}(N))$, since $\Gamma(\mathcal{O}(-1)) = \underline{H}^1(\mathcal{O}(-1)) = 0$, and one sees that the image of a non-zero (constant) section of $n\mathcal{O}$ vanishes precisely when evaluated on the line in \mathbb{C}^n (point in \mathbb{P}_{n-1}) on which that vector lies; notice also that the map from $\Gamma(\mathcal{O}(N))$ to N_z , $z \in \mathcal{O}_y$, by *evaluation* is *onto* - indeed, it is spanned by the image under p of $(n-1)$ vectors spanning a subspace of \mathbb{C}^n complementary to the line in \mathbb{C}^n corresponding to z . Now for any $z \in \mathcal{O}_y$ pick an $(n-1)$ manifold B through z transverse to \mathcal{O}_y ; let us say it's biholomorphically a ball in \mathbb{C}^{n-1} . Then define a holomorphic function φ on a neighbourhood U of y in \mathcal{M} with values in B by taking $x \in \mathcal{M}$ to the point $\mathcal{O}_x \cap B$; this is well defined near y if $\bar{B} \cap \mathcal{O}_y$ is the single point z because of compactness of the \mathcal{O}_x 's and holomorphic by the definition of an analytic family. This function is of maximal rank at y because, by Kodaira's theorem, normal sections on \mathcal{O}_y are the derivatives of variations corresponding to curves in \mathcal{M} , and the evaluation of a normal section at z (which is represented, so far as φ is concerned, by the tangent vector to B in that equivalence class) is a surjective linear map. Hence $\varphi^{-1}(\varphi(y))$ is a curve through y , at least

near y , and we call this curve a segment of a geodesic, as is indeed in accordance with our rule; applying this to all $z \in \mathcal{O}_y$ for all y then generates a family of curves which are *closed* as immersed submanifolds of \mathcal{M} ; and there is precisely one geodesic tangent to each vector, since the corresponding normal field on the appropriate \mathcal{O}_y vanishes at some point. It remains to see that the geodesics depend holomorphically on the initial direction.

To do this, it is sufficient to check that the holomorphicity of the map $\alpha: \mathbb{P}T^*\mathcal{M} \rightarrow L$ which takes the projective class of a nonzero holomorphic vector V at $x \in \mathcal{M}$ to the place where corresponding normal field on \mathcal{O}_y vanishes. But, restricted to the submanifold of $\mathbb{P}T^*\mathcal{M}$ which is the image of the local section consisting of tangents to the fibres of φ , the map α is simply φ composed with the projection $\pi: \mathbb{P}T^*\mathcal{M} \rightarrow \mathcal{M}$, and so holomorphic; and, restricted to the fibres of π , α is holomorphic because the map $\mathbb{P}(H^0(\mathcal{O}(N))) \xrightarrow{\cong} \mathcal{O}_x$ is. Since we can find two transverse submanifolds through any point of $\mathbb{P}T^*\mathcal{M}$ such that the sum of their tangent spaces is the tangent space at the point, and such that α is holomorphic on each, it follows that the real Jacobian α_* is everywhere complex linear (with respect to the complex structure tensor) and so α is indeed holomorphic. This proves the claim. \square

Corollary. (1.7.II) If $\{P_x | x \in \mathcal{M}\}$ is a complete analytic family of \mathbb{P}^n 's in L with normal bundle $T^*P_x \otimes H^{-1}$, then

$$V := \bigcup_{x \in \mathcal{M}} P_x$$

is open and there is a discrete holomorphic surjection $L(\mathcal{M}) \rightarrow V$ which is biholomorphic on a neighbourhood of each P_x .

Proof. Consider the map $\hat{\alpha}: L(\mathcal{M}) \rightarrow L$ which takes a geodesic $\gamma \subset \mathcal{M}$ to the single point which comprises

$$\bigcap_{x \in \gamma} P_x$$

then we've already shown that $\hat{\alpha}$ is holomorphic; in fact it's the unique holomorphic map such that

$$\begin{array}{ccc} \mathbb{P}T\mathcal{M} & \xrightarrow{\hat{\alpha}} & L \\ q \downarrow & \nearrow \hat{\alpha} & \\ L(\mathcal{M}) & & \end{array}$$

commutes. The Jacobian of $\hat{\alpha}$ is everywhere of maximal rank because the normal bundle of P_x is taken isomorphically to that of P_x by $\hat{\alpha}_*$ for any $x \in \mathcal{M}$, and we can think of $T^*L(M)$ at γ as canonically $N_x|_\gamma \oplus N_y|_\gamma$, where $N_x|_\gamma$ and $N_y|_\gamma$ are the fibres at γ of the normal bundles of P_x and P_y , respectively, where x and y are on the geodesic γ - any two such submanifolds of $L(\mathcal{M})$ intersect transversely. Thus $\hat{\alpha}$ is at worst a discrete mapping, and is a biholomorphism on a neighbourhood of any P_x . □

Lemma (1.7.III). ("UNIFORM INVERSE FUNCTION THEOREM")

Let $M \xrightarrow{\pi} \mathbb{C}^k$ and $\tilde{M} \xrightarrow{\tilde{\pi}} \mathbb{C}^k$ be smooth fibre-n-manifolds and let $\varphi: M \rightarrow \tilde{M}$ be a fibre map (i.e. $\tilde{\pi} \circ \varphi = \pi$) such that $M_0 = \pi^{-1}(0)$ is mapped diffeomorphically onto $\tilde{M}_0 = \tilde{\pi}^{-1}(0)$. Then there is a neighbourhood $V \supset M_0$ in M such that $\varphi|_V$ is a diffeomorphism.

Proof. The Jacobian of φ is non-singular at each point of M_0 , so there's some neighbourhood V_1 on which φ is a local diffeomorphism, by the inverse function theorem. Without loss of generality we may take $\tilde{M} \xrightarrow{\tilde{\pi}} \mathbb{C}^k$ to be $U \rightarrow \mathbb{C}^k$ where $U \subset \tilde{M}_0 \times \mathbb{C}^k$ is a neighbourhood of $M_0 \times \{0\}$; this is true because pr_2 there always exists a fibre manifold map $\tilde{M} \rightarrow \tilde{M}_0 \times \mathbb{C}^k$ satisfying the hypotheses of the theorem, as is most easily seen by induction on k using the existence of non-vanishing vector fields everywhere transverse to M_0 ; for an arbitrary Riemannian metric on M_0 , put the product metric on U , where "product" is to be interpreted as stemming from j . Call this metric g , and call the associated topological metric d ; on $\varphi^{-1}(U) \cap V_1 =: V_2$, we put the metric $\tilde{g} := \varphi^*g$ and call the associated topological metric ρ . Obviously, $\rho(x,y) > d(\varphi(x), \varphi(y))$.

For each point $x \in M_0$, there is an $\varepsilon(x) > 0$ such that $B_{\varepsilon(x)}(x) := \{y \in V_2 \mid \rho(x,y) < \varepsilon(x)\}$ is a fundamental region for φ ; i.e. $\varphi|_{B_{\varepsilon(x)}(x)}$ is a diffeomorphism. Then, letting $V := \bigcup_{x \in M_0} B_{\frac{1}{5}\varepsilon(x)}(x)$, $\varphi|_V$ is a diffeomorphism. For if $\rho(x,y) < \frac{1}{5}\varepsilon(x)$, $\rho(x',y') < \frac{1}{5}\varepsilon(x')$, and if $\varphi(y) = \varphi(y')$, then

$$\rho(y,y') < \frac{1}{5}\varepsilon(x) + \frac{1}{5}\varepsilon(x') + \rho(x,x')$$

$$= \frac{1}{5}\varepsilon(x) + \frac{1}{5}\varepsilon(x') + d(x,x')$$

$$< \frac{2}{5}\varepsilon(x) + \frac{2}{5}\varepsilon(x')$$

$$< \text{MAX}(\varepsilon(x), \varepsilon(x'))$$

Hence $\text{MAX}(\rho(x,y'), \rho(x',y))$

$$< \text{MAX}(\varepsilon(x), \varepsilon(x'))$$

and so $\rho(x, y') < \varepsilon(x)$ or $\rho(x', y) < \varepsilon(x')$. So both y and y' lie within a fundamental region (e.g. $B_{\varepsilon(x)}(x)$), and so $y = y'$. Thus $\varphi|_V$ is indeed a diffeomorphism. \square

Remark: All the mucking about with the product metric was to insure that $\rho(x, x') = d(x, x')$.

Theorem (1.7.IV). Let $\mu: \mathcal{E} \rightarrow U \subseteq \mathbb{C}^k, 0 \in U$, be a deformation of $\mathcal{E}_0 := \mu^{-1}(0) \approx L(M)$ the space of geodesics of a strongly geodesically convex complex n -manifold M with projective structure. Then there exists a neighbourhood $V \subset \mathcal{E}$ of \mathcal{E}_0 such that, for every t , $\mu^{-1}(t) \cap V$ is biholomorphically the space of geodesics of some n -manifold M_t with projective structure.

Proof. By Kodaira's theorem, each $P_x \subset \mathcal{E}_0$ is a member of a maximal complete $(n+k)$ -dimensional analytic family of compact complex submanifolds of \mathcal{E} , since the normal bundle \tilde{N} of the P_x in \mathcal{E} is just the sum of N with a k -dimensional trivial bundle; each submanifold of the family lies in some $\mu^{-1}(t)$ because a holomorphic function on a compact manifold is constant; and the map so defined from the parameter space of such a family to \mathbb{C}^k is everywhere of maximal rank since μ_* induces a surjection from $H^0(\mathcal{O}(\tilde{N}))$ to \mathbb{C}^k . Moreover, we can take the union of all these families to produce a single connected $(n+k)$ -dimensional family containing each $P_x \subset \mathcal{E}_0$ since each family is maximal, and the induced "transition functions" between the various parameter spaces are biholomorphisms due to the fact that each tangent space to a parameter space is identified complex linearly with $H^0(\mathcal{O}(\tilde{N}))$ for the appropriate normal bundle. Since \mathbb{P}_{n-1} is rigid, we can take each of the submanifolds of \mathcal{E} to be a \mathbb{P}_{n-1} by reducing the parameter space M of \mathcal{E} . (Actually this step is unnecessary - see [24] for the "absolute" rigidity of \mathbb{P}_m); and by a further reduction of the parameter space, we can take the normal bundle relative to the fibres of μ of each \mathbb{P}_{n-1} of the family to be $T\mathbb{P}_{n-1} \otimes H^{-1}$, since this bundle is rigid. Now let V be defined as the union of all the submanifolds of the resulting family, then V is an open set, just as in the corollary, the global sections of the normal bundle of a submanifold of the family take on all possible values at any point of the submanifold. Let $\tilde{\mathcal{E}} \rightarrow \mathbb{C}^k$ be the space of geodesics in the fibres of the projection from the parameter space of

the family to \mathbb{C}^k ; these fibres are indeed endowed with projective connexions by the proposition, and the space of geodesics is again clearly a complex manifold just as before. We get a discrete holomorphic surjection $\beta: \tilde{\mathfrak{F}} \rightarrow V$ defined as in the corollary, but now done uniformly with respect to \mathbb{C}^k .

This mapping is a biholomorphism on the fibre over zero, and also a biholomorphism on a neighbourhood of any submanifold of the family. Hence, by the "uniform inverse function theorem" there is a neighbourhood \tilde{V} of the fibre over zero in \mathfrak{F} which is mapped biholomorphically into V , and which has the property that there is a P_x through each point \tilde{V} . □

§1.8 EXPLICIT DEFORMATIONS

We have shown that a small deformation of a space of complex geodesics is again a space of geodesics. We will now show that, if we work sufficiently locally in the primary space, an arbitrary space of geodesics can be deformed to its flat analogue (a neighbourhood of an $(m-1)$ plane in $G(2, n+1)$ if n is the primary space dimension) through spaces of geodesics.

Proposition (1.8.I). Let M be a complex n -manifold with projective structure. Each point $x \in M$ has a neighbourhood U such that $L(U)$ may be deformed into the space $L(B)$ of lines in the Euclidean ball $B \subset \mathbb{C}^n$ through spaces of geodesics; i.e. there exists a fibre-manifold $\omega: L \rightarrow W$, where $W \subset^{\text{open}} \mathbb{C}$ is a neighbourhood of the unit interval, such that $L_t := \omega^{-1}(t)$ is a space of geodesics for each t , and such that $L_1 \cong L(U)$, $L_0 \cong L(B)$.

Proof. In a region on M admitting a connexion, use the exponential map to identify a neighbourhood of the origin in \mathbb{C}^n with an open set in M in such a way that the straight lines through the origin correspond to geodesics; if $\hat{\Gamma}_{jk}^i$ are the trace-free symmetric Christoffel symbols in this chart and Γ_{jk}^i are the Christoffel of the underlying smooth connexion in the underlying smooth chart, then these vanish at the origin by Jacobi's equation of geodesic deviation [7]. Thus we can find a small ball B on which, for some $\epsilon > 0$,

$$\left| \sum_{i,j,k=1}^{2n} \hat{\Gamma}_{jk}^i(x) x^i \xi^j \xi^k \right| < (\frac{1}{2} - \epsilon) |\xi|^2 \quad (1.8.1)$$

for all ξ , and for $x \in B$; Whitehead [20] showed that such a ball is geodesically convex. (Notice that, unlike in §1.1, we are letting i, j, k range over both "primed" and "unprimed" values, so that x^i can represent *either* the real part or the imaginary part of a holomorphic coordinate).

Now defining Christoffel symbols $\Gamma_{jk}^i(x,t)$ depending upon an extra complex variable t by $\Gamma_{jk}^i(x,t) := t \Gamma_{jk}^i(x)$, it follows that (1.8.1) holds on $B \times W$ for some W a neighbourhood of $[0, 1]$ in \mathbb{C} and possibly for a smaller $\epsilon > 0$; for (Cf. §1.1)

$$|\Gamma_{jk}^i(x,t) - |t|\Gamma_{jk}^i(x)| < \sqrt{2}|Imt| |\hat{\Gamma}_{jk}^i(x)| \quad i, j, k = 1, \dots, 2n,$$

and so

$$\left| \sum_{i,j,k=1}^{2n} \hat{\Gamma}_{jk}^i(x,t) x^i \xi^j \xi^k \right| < (|t| + \sqrt{2}|Imt|) \left| \sum_{i,j,k=1}^{2n} \hat{\Gamma}_{jk}^i(x) x^i \xi^j \xi^k \right|$$

and this quantity is less than $(\frac{1}{2} - \frac{\epsilon}{2})|\xi|^2$ if $|t| < 1 + \frac{\epsilon}{4}$ and $|Imt| < \epsilon/4\sqrt{2}$, where $\epsilon \in (0, \frac{1}{2})$ is the number appearing in (1.8.1). Thus, defining L_t to be the space of geodesics of $\Gamma_{jk}^i(x,t)$ for t in this range, we have constructed a deformation of the required type.

§2.1 CONFORMAL STRUCTURES, CONFORMAL CONNEXIONS

A *conformal structure* on a complex manifold is a holomorphic assignment of quadric null cones in the holomorphic tangent bundle, which is the same as specifying, locally, a *conformal class* of metrics under the relation of multiplication by a non-vanishing holomorphic function. By a metric we mean here a holomorphic symmetric covariant (i.e. contrafunctorial) 2-tensor field which is everywhere non-degenerate as a quadratic form on the holomorphic tangent space; if g is such a metric, we say that a non-zero holomorphic vector X is *null* if $g(X,X) = 0$, and clearly this is independent of the choice of g within a conformal class. The neatest but perhaps least intuitive definition of a conformal structure on a complex n -fold is a holomorphic line sub-bundle of $(T^*M)^{\otimes 2}$, \otimes the symmetric tensor product, which is everywhere non-degenerate in the sense that, for any local section g the induced local morphism of TM to T^*M by $X \rightarrow X \lrcorner g$, where $(X \lrcorner g)(Y) := g(X, Y)$ for any holomorphic vector Y , is a local isomorphism. (Notice that for a manifold to admit a global metric one must have $TM \cong T^*M$, which is rare indeed). For conformal structures in the smooth category see [23].

A definition of a conformal structure which is simultaneously reasonably intuitive and elegant is as follows: a conformal structure on a complex n -fold is a holomorphic quadric bundle $\subset \mathbb{P}T^*M$. By such a quadric bundle we mean a complex submanifold of $\mathbb{P}T^*M$ which intersects each fibre in a non-degenerate quadric. It is immediate that the projective image of the null cones of a conformal structure in our original sense is a quadric sub-bundle. The converse is rather more subtle, and calls for a short diversion.

Let $Q \subset \mathbb{P}T^*M$ be a holomorphic hypersurface which intersects each fibre in a quadric. Then Q is biholomorphically an analytic fibre bundle of quadrics over M by the Kodaira-Spencer theory espoused in Section §0.3

and by the rigidity of quadrics. Over a Stein region $U \subset M$ on which it is trivial, consider the exact sequence of sheaves on $\mathbb{P}T'U$

$$0 \rightarrow \mathcal{I}_Q(2) \rightarrow \mathcal{O}(2) \xrightarrow{\rho} \mathcal{O}_Q(2) \rightarrow 0$$

defining the ideal sheaf of Q of homogeneity 2; ρ is the restriction map to Q . Since $H^1(Q, \mathcal{O}) = 0$, the restriction of the Hopf bundle to Q is just the same as the Hopf bundle on $Q \times U$, so the fact that $H^0(\mathcal{O}_Q(2)) = H^0(Q \times U, \mathcal{O}(2)) = H^0(U, \mathcal{O}) \otimes_{\mathbb{C}} H^0(Q, \mathcal{O}(2))$

$$= \left(\frac{n(n+1)}{2} - 1 \right) \mathcal{O}_U$$

while $H^0(\mathcal{O}(2))$

$$= \frac{n(n+1)}{2} \mathcal{O}_U$$

demonstrates that the kernel of ρ , acting on global sections is non zero, so in fact there is a global defining function $g \in H^0(\mathcal{I}_Q(2))$; this function is a local metric representing the conformal structure.

Armed with this equivalence, we can now hope to construct a theory of spaces of null geodesics with some aesthetic virtues. In analogy to our definition of a projective connexion we now define a *conformal connexion* on M as consisting of a holomorphic conformal structure on M together with a family \mathcal{N} of null curves, one tangent to each null direction and varying holomorphically in the sense that their canonical lifts to $\mathbb{P}T'M$, defined as before, foliate the quadric sub-bundle of the conformal structure.

Proposition. Let M be a complex manifold with conformal connexion \mathcal{N} . If g is a local representative metric for the conformal structure underpinning \mathcal{N} , then there exists a local affine connexion ∇ near any point such that

$$\nabla g = 0$$

and such that the null geodesics of ∇ are just the curves of \mathcal{N} .

(Remarks: The expression " $\nabla g = 0$ " means that $Z \lrcorner \partial(g(X, Y)) = g(Z \lrcorner \nabla X, Y) + g(Z, X \lrcorner \nabla Y)$ for any holomorphic vector fields X, Y, Z ; "parallel transport is an isometry". Under this condition, an initially null geodesic remains null, for $X \lrcorner \nabla X = 0 \Rightarrow X \lrcorner \nabla(g(X, X)) = 0$).

Proof. First we construct a local connexion $\hat{\nabla}$ near any point of M with the property that the curves of η are $\hat{\nabla}$ geodesics. This is done in essentially the same fashion as that occurring in our treatment of projective connexions.

Let $(z^i): U \rightarrow \mathbb{C}^n$, U a Stein region in M , be a coordinate chart; on the quadric bundle $Q \subset \mathbb{P}T^*M$ which defines the conformal structure, we define a section f of the sheaf $n\mathcal{O}(z)/\mathcal{O}(1)$ (the sheaf S of section 1.2, but now restricted to Q) over the region $\pi^{-1}(U)$, where $\pi: Q \rightarrow M$ is the canonical projection, by

$$f\left(\frac{d}{d\zeta}\Big|_{\zeta=0}\right) = \left(\frac{d^2 z^1}{d\zeta^2}\Big|_{\zeta=0}, \dots, \frac{d^2 z^n}{d\zeta^2}\Big|_{\zeta=0}\right) \text{ MOD } \left(\frac{dz^1}{d\zeta}\Big|_{\zeta=0}, \dots, \frac{dz^n}{d\zeta}\Big|_{\zeta=0}\right)$$

for any curve of η with non-singular parameter ζ ; as before this is well defined and holomorphic.

Since $H^1(\pi^{-1}(U), \mathcal{O}(1)) \cong H^1(Q, \mathcal{O}(1)) \otimes_{\mathbb{C}} \mathcal{O}_U = 0$, by the same sort of argument as before (see the quadric vanishing result in section 02), it follows that f represents a global section of $n\mathcal{O}_Q(2)$, where $\mathcal{O}_Q(n)$ is just the restriction of $\mathcal{O}(n)$ to Q from $\mathbb{P}T^*M$. We have the exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\alpha_g} \mathcal{O}(2) \rightarrow \mathcal{O}_Q(2) \rightarrow 0 \quad (2.1.1)$$

of sheaves on $\mathbb{P}T^*V$, $V \subset M$ the domain of definition of a metric g representing the conformal structure, where α_g is multiplication by g ; we may assume that U lies within some such region V . Since $H^1(\pi^{-1}(U), \mathcal{O}) \cong H^1(Q, \mathcal{O}) \otimes_{\mathbb{C}} \mathcal{O}_U = 0$, it follows that f can be represented as a set of n holomorphic symmetric forms; i.e. there exist $\Gamma_{k\ell}^j = \Gamma_{\ell k}^j \in \mathcal{O}_U$, $j, k, \ell = 1, \dots, n$, such that the solutions of

$$\frac{d^2 z^j}{d\zeta^2} + \Gamma_{k\ell}^j \frac{dz^k}{d\zeta} \frac{dz^\ell}{d\zeta} = 0 \quad (2.1.2)$$

include all the curves of η ; the ambiguity of the Γ 's, widening our scope to include the possibility of unsymmetrical Γ 's, is precisely

$$\Delta \Gamma_{k\ell}^j = A_{k\ell}^j + \delta_k^j \psi_\ell + \delta_\ell^j \psi_k + \varphi^j g_{k\ell} \quad (2.1.3)$$

where $A_{k\ell}^j = -A_{\ell k}^j$, and where ψ_j , φ^j , and $A_{k\ell}^j$ are holomorphic.

Tangent to any null vector x there is a solution of (2.1.2) satisfying

$$\frac{dz^i}{d\zeta} \frac{dz^j}{d\zeta} g_{ij} = 0 \quad \forall \zeta.$$

Thus

$$\frac{dz^i}{d\zeta} \frac{dz^j}{d\zeta} \frac{dz^k}{d\zeta} \left(\frac{\partial g_{ij}}{\partial z^k} - 2\Gamma_{ij}^\ell g_{\ell k} \right) = 0$$

and so the function α

$$y^i \frac{\partial}{\partial z^i} \rightarrow y^i y^j y^k \left(\frac{\partial g_{ij}}{\partial z^k} - 2\Gamma_{ij}^\ell g_{\ell k} \right) = 0$$

of homogeneity 3 on the projectivized tangent space at an arbitrary point of the chart vanishes on the quadric $y^i y^j g_{ij} = 0$; it follows (cf. [3]) that α is a multiple of the defining function of the quadric; i.e. that for some η_i one has

$$\partial_{(i} g_{jk)} - 2\Gamma_{(ij}^\ell g_{k)\ell} = \eta_{(i} g_{jk)}$$

where round brackets denote symmetrization and where $\partial_i = \partial/\partial z^i$. So

$$\partial_i g_{jk} - \Gamma_{ij}^\ell g_{k\ell} - \Gamma_{ik}^\ell g_{j\ell} = \eta_i g_{jk} + L_{ijk} + L_{ikj}$$

for some $L_{ijk} = -L_{jik}$, since all 3-tensors which are symmetric in two indices but have no totally the symmetric part can be expressed as

$L_{ijk} + L_{ikj}$ where $L_{ijk} = -L_{jik}$; indeed if $B_{ijk} = B_{ikj}$ and $B_{ijk} + B_{jik} + B_{kij} = 0$, then setting $L_{ijk} = \frac{1}{3}(B_{ijk} - B_{jik})$ one has

$$B_{ijk} - (L_{ijk} + L_{ikj}) = B_{ijk} - \frac{2}{3} B_{ijk} + \frac{1}{3} B_{jik} + \frac{1}{3} B_{kij} = 0.$$

Now setting $\hat{\Gamma}_{jk}^i := \Gamma_{jk}^i - g^{i\ell} L_{\ell jk} + \frac{1}{2} \eta_j \delta_k^i$, where $g^{i\ell} g_{\ell j} = \delta_j^i$, we have

$$\begin{aligned} \frac{\partial g_{jk}}{\partial z^i} - \hat{\Gamma}_{ij}^{\ell} g_{k\ell} - \hat{\Gamma}_{ik}^{\ell} g_{j\ell} &= \eta_i g_{ik} + L_{ijk} + L_{ikj} \\ - \frac{1}{2} \eta_i (\delta_j^{\ell} g_{k\ell} + \delta_k^{\ell} g_{j\ell}) + \delta_k^{\ell} L_{\ell ij} + \delta_j^{\ell} L_{\ell ik} &= 0 \end{aligned}$$

and so defining a connexion on U by

$$\nabla(y^i \frac{\partial}{\partial z^i}) = \frac{\partial y^i}{\partial z^j} dz^j \otimes \frac{\partial}{\partial z^i} + \hat{\Gamma}_{jk}^i y^k dz^j \otimes \frac{\partial}{\partial z^i}$$

we see that ∇g , as defined by the Leibnitz Rule, vanishes, and that the null geodesics of ∇ are the curves of η in U .

We can now give a tensorial description of the extra information that differentiates between the conformal connexion and its conformal structure. Associated with an affine connexion ∇ there is a torsion tensor (cf. [7] for the smooth version)

$$T(X, Y) = X \lrcorner \nabla Y - Y \lrcorner \nabla X - [X, Y] \quad (2.1.4)$$

defined for two holomorphic vector fields X, Y , but depending only on their values at the point of evaluation; T is a holomorphic 2-form with values in the tangent bundle. In terms of the Christoffel symbols χ_{jk}^i of ∇ , T is just the anti-symmetric part of the Christoffel symbols:

$$T = (\chi_{jk}^i - \chi_{kj}^i) dz^j \otimes dz^k \otimes \frac{\partial}{\partial z^i}$$

On a manifold M with conformal connexion, let g be a metric and let ∇ be an affine connexion, both defined in some region $U \subset M$, which represent the conformal connexion in the sense of the above proposition:

$\nabla g = 0$, and the null geodesics of ∇ are the geodesics of the conformal connexion. The *conformal torsion* τ of the conformal connexion is the holomorphic vector valued 2-form $\tau_{jk}^i = -\tau_{kj}^i$ satisfying

$$g_{i\ell} \tau_{jk}^\ell + g_{j\ell} \tau_{ki}^\ell + g_{k\ell} \tau_{ij}^\ell \equiv 0 \quad (2.1.5)$$

(i.e. the totally skew part vanishes) and

$$\tau_{ji}^i \equiv 0 \quad (2.1.6)$$

which is defined by the equation

$$\begin{aligned} \tau_{jk}^i := & \frac{2}{3} T_{jk}^i - \frac{1}{3} g^{i\ell} (g_{jm} T_{k\ell}^m + g_{km} T_{\ell j}^m) - \\ & \frac{1}{(n-1)} [\delta_j^i T_{\ell k}^\ell + \delta_k^i T_{j\ell}^\ell]. \end{aligned} \quad (2.1.7)$$

(We are uninterested in $n=1$ as there is only one conformal structure and not a single null geodesics in this case; but one may simply define $\tau := 0$ if $n = 1$. Notice that τ always vanishes if $n = 2$, as it should, since in this dimension, the only null curves are the torsion-free geodesics).

Fact. The conformal torsion τ defined by (2.1.7) depends only on the conformal connexion, and not on the representatives g and ∇ . Moreover, τ (together with the conformal structure) completely determines the conformal connexion. Finally, any holomorphic vector valued 2-form $\tau_{jk}^i = -\tau_{kj}^i$ satisfying (2.1.5) and (2.1.6) with respect to some fixed conformal structure is the conformal torsion of some conformal connexion with the given conformal structure underlying it.

Proof. Fix for the moment some g , and consider an arbitrary connexion $\hat{\nabla}$ satisfying $\hat{\nabla} g \equiv 0$. Let ∇ denote the unique *torsion-free* connexion preserving g , with Christoffel^{symb.l} given by the classical Levi-Civita formula

$$\chi_{jk}^i = \frac{1}{2} g^{i\ell} \left[\frac{\partial g_{\ell j}}{\partial z^k} + \frac{\partial g_{\ell k}}{\partial z^j} - \frac{\partial g_{jk}}{\partial z^\ell} \right];$$

then the difference tensor Δ_{jk}^i which distinguishes $\hat{\nabla}$ from ∇ (i.e. which is the difference between the respective Christoffel symbols) satisfies

$$\Delta_{jk}^{\ell} g_{\ell i} + \Delta_{ji}^{\ell} g_{\ell k} \equiv 0$$

because $\nabla g - \hat{\nabla} g \equiv 0$. (The uniqueness of ∇ follows because if $\Delta_{jk}^{\ell} = \Delta_{kj}^{\ell}$, then $\Delta_{jk}^{\ell} g_{\ell i}$ is symmetric in j and k but skew in k and i , and so vanishes).

Now let

$$E_{jk}^{\ell} = \Delta_{jk}^{\ell} - A_{jk}^{\ell}$$

where

$$A_{jk}^{\ell} = \frac{1}{3}(\Delta_{jk}^{\ell} + g^{\ell i} g_{jm} \Delta_{ki}^m + g^{\ell i} g_{km} \Delta_{ij}^m)$$

so that $g_{i\ell} A_{jk}^{\ell}$ is totally skew and the totally skew part of E vanishes; then the torsion of $\hat{\nabla}$ is given by

$$\begin{aligned} T_{jk}^i &= E_{jk}^i - E_{kj}^i + A_{jk}^i - A_{kj}^i \\ &= g^{i\ell} (g_{\ell m} E_{jk}^m + g_{km} E_{j\ell}^m + g_{jm} E_{\ell k}^m) + 2A_{jk}^i \\ &= g^{i\ell} g_{km} E_{j\ell}^m + 2A_{jk}^i \end{aligned}$$

which shows that τ is indifferent to the totally skew part of Δ - as are the geodesics of $\hat{\nabla}$, null or otherwise. Next we define

$$F_{jk}^i = E_{jk}^i - \frac{1}{n-1} (E_{j\ell}^{\ell} \Delta_k^i - g^{im} g_{jk} E_{m\ell}^{\ell})$$

so that F retains the antisymmetry $g_{\ell i} F_{jk}^i = -g_{ji} F_{\ell k}^i$, and so still is the difference tensor distinguishing the torsion-free metric connexion from some metric connexion, but now $F_{ji}^i = 0$; notice that

$$(F_{jk}^i - E_{jk}^i) x^j x^k = -\frac{1}{n-1} (E_{jk}^{\ell} x^j) x^i$$

if X is null, and so the null geodesics of the metric connexion corresponding to F are the same as those of $\hat{\nabla}$. Now the conformal torsion of $\hat{\nabla}$ is

$$\tau_{jk}^i = g^{i\ell} g_{km} F_{j\ell}^m$$

and we must check that if F_{jk}^i is a holomorphic tensor field such that $g_{\ell i} F_{jk}^i = -g_{ji} F_{\ell k}^i$, $F_{ji}^i = 0$, and $g_{\ell i} F_{jk}^i + g_{ki} F_{\ell j}^i + g_{ji} F_{k\ell}^i = 0$, and such that

$$F_{jk}^i X^j X^k \propto X^i \quad \forall X \text{ null}$$

then $F_{jk}^i = 0$.

To do this, we notice that

$$A_{ijk} \rightarrow A_{ijk} + A_{ikj}$$

is an isomorphism between tensors skew in i and j with vanishing totally skew part, and tensors symmetric in j and k with vanishing totally symmetric part; in fact, the inverse function is just

$$B_{ijk} \rightarrow \frac{2}{3} (B_{ijk} - B_{jik}).$$

Now suppose that $G_{jk}^i = G_{kj}^i$ satisfies

$$G_{jk}^i X^j X^k \propto X^i \quad X \text{ null};$$

then G represents $0 \in H^0(Q, (n\mathcal{O}(2))/\mathcal{O}(1))$, where Q is the quadric of null directions at any point, and so, as an element of $H^0(Q, n\mathcal{O}(2))$ is in the image of $H^0(Q, \mathcal{O}(1))$ as represented by

$$v_i \rightarrow v_i \delta_k^j + v_k \delta_i^j;$$

on the other hand, the kernel of the restriction $H^0(\mathbb{P}_{n-1}, \mathcal{O}(2)) \rightarrow H^0(Q, \mathcal{O}(2))$ is spanned by the metric, while the restriction $H^0(\mathbb{P}_{n-1}, \mathcal{O}(1)) \rightarrow H^0(Q, \mathcal{O}(1))$ is

zero, as we've previously argued from exact sequences; so, in short, G is of the form

$$G_{jk}^i = \mu^i g_{jk} + \nu_j \delta_k^i + \nu_k \delta_j^i$$

for some μ and ν . If we require that G have vanishing totally symmetric part we get

$$G_{jk}^i = -2\mu^i g_{jk} + \mu_j \delta_k^i + \mu_k \delta_j^i$$

where $\mu_j = g_{ji} \mu^i$. Skewing over i and j , we get

$$G_{jk}^i - g^{il} g_{jm} G_{lk}^m = 3\mu_j \delta_k^i - 3\mu^i g_{jk}$$

which has trace $3(n-1)\mu_j$ over i and k ; this proves the claim that F_{jk}^i , skew in i and j , trace-free in i and k , with vanishing totally skew part, and satisfying

$$F_{jk}^i \chi^j \chi^j \propto \chi^i \quad \forall \chi \text{ null}$$

must in fact vanish.

Thus we've demonstrated that, for fixed g , the conformal torsion τ is independent of the representative torsion metric connexion.

Making a conformal change in the metric $g \rightarrow \Omega^2 g$ just changes the torsion free metric connexion ∇ to $\nabla + S$, where S_{jk}^i is the tensor

$$S_{jk}^i = \delta_j^i \eta_k + \delta_k^i \eta_j - \eta^i g_{jk}$$

where $\eta = \Omega^{-1} d\Omega$ and where $\eta^i := \eta_j g^{ij}$, if $\hat{\nabla}$ is a torsion connexion preserving g and representing a given conformal connexion, then $\hat{\nabla} + S$ preserves $\Omega^2 g$ and represents the same conformal connexion; but the torsion of $\hat{\nabla} + S$ is just that of $\hat{\nabla}$. So the conformal torsion is indeed independent of our choice of representatives.

To see that every holomorphic vector-valued 2-form τ satisfying

(2.1.5) and (2.1.6) with respect to a given conformal structure is the conformal connexion with the given underlying conformal structure, it suffices to note that

$$\hat{\nabla} = \nabla + F + A + B$$

where ∇ is the torsion free connexion preserving an element g of the given conformal class of metrics, where

$$F_{jk}^i = g^{i\ell} g_{km} \tau_{ji}^m$$

$$B_{jk}^i = u_j \delta_k^i - g^{im} g_{jk} u_m \text{ for some } u$$

$$g_{i\ell} A_{jk}^\ell + g_{j\ell} A_{ki}^\ell + g_{k\ell} A_{ij}^\ell = 0$$

preserves g and has conformal torsion τ for any choice of A and B , is the most general connexion (as A and B vary) preserving g and possessing conformal torsion τ , and has null geodesics independent of A and B .

§2.2 SPACES OF NULL GEODESICS

We've previously seen that a complex n -manifold with affine connexion has a neighbourhood basis consisting of sets whose spaces of geodesics are $(2n-2)$ -manifolds in which the points of the primary space are represented by embedded complex projective $(n-1)$ -spaces with normal bundle $T^*\mathbb{P}_{n-1} \otimes H^{-1}$. It follows that a complex n -manifold with conformal connexion has a neighbourhood basis consisting of sets whose spaces of *null* geodesics are (Hausdorff) $(2n-3)$ -manifolds in which the points of the primary space are represented by embedded copies of the non-degenerate quadric $Q \subset \mathbb{P}_{n-1}$, each copy of which has normal bundle $(T^*\mathbb{P}_{n-1} \otimes H^{-1})|_Q$. We call a neighbourhood meeting these requirements *civilized*.

The deduction is as follows: let M be a complex n -manifold with metric g , connexion ∇ such that $\nabla g = 0$, and Hausdorff space of ∇ geodesics $L(M)$. Let $QM \subset T^*M$ be the bundle of null directions as defined by g . Then the quotient map $q: T^*M \rightarrow L(M)$ has the property that $y \in QM \Rightarrow q^{-1}(q(y)) \subset QM$, since an initially null geodesic is everywhere null; so $q(QM)$ is a hypersurface $N(M) \subset L(M)$, and the quotient map $q|_{QM}: QM \rightarrow N(M)$ induces the exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow n\mathcal{O} \rightarrow \mathcal{O}(N^{\vee}) \rightarrow 0$$

where N^{\vee} is the normal bundle to the image of a fibre of $\pi: QM \rightarrow M$, by precisely the same argument used to calculate the normal bundle of P_x . We label the submanifold of $N(M)$ consisting of those null geodesics through $x \in M$ by Q_x .

§2.3 COHOMOLOGICAL CALCULATIONS: THE NORMAL BUNDLE TO Q_x

We carry out the calculations for the normal bundle to Q_x analogous to those done in Section 1.5 for that of P_x .

Proposition. Let M be a complex n -manifold with civilized conformal connexion, and let N' be the normal bundle to $Q_x \subset N(M)$. Then

1. $H^1(\mathcal{O}(N')) = 0$
2. $\dim H^0(\mathcal{O}(N')) = n$ if $n \geq 4$; $\dim H^0(\mathcal{O}(N')) = 4$ if $n = 3$.
3. $H^1(\mathcal{O}(\text{HOM}(N', N'))) = 0$.

Proof. 1, 2. The exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(-1) \rightarrow n\mathcal{O} \rightarrow \mathcal{O}(N') \rightarrow 0$$

gives rise to a long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}(-1)) \rightarrow n \underset{\mathbb{C}^n}{H^0(\mathcal{O})} \rightarrow H^0(\mathcal{O}(N')) \rightarrow H^1(\mathcal{O}(-1)) \rightarrow n \underset{0}{H^1(\mathcal{O})} \\ \rightarrow H^1(\mathcal{O}(N')) \rightarrow \underset{0}{H^2(\mathcal{O}(-1))} \rightarrow 0 \end{aligned}$$

which proves the claim, provided that we remember that $H^1(\mathcal{O}(-1))$ vanishes on a quadric in \mathbb{P}_{n-1} if $n \geq 4$, but that on a conic curve $H^1(\mathcal{O}(-1)) = H^1(\mathbb{P}_1, \mathcal{O}(-2)) \simeq \mathbb{C}$. (The map $[\xi, \eta] \rightarrow [\xi^2 + \eta^2, \xi^2 - \eta^2, 2\xi\eta]$ which identifies the conic $x^2 + y^2 + z^2 = 0$ with \mathbb{P}_1 is of homogeneity two!). The vanishing of all cohomology groups is by the corollary to Bott's Rule given in the preliminary chapter.

3. The proof is identical with that in Section 1.5 as the same relevant cohomology groups vanish.

§2.4 DEFORMATIONS OF SPACES OF NULL GEODESICS

Theorem. Let M be a complex n -manifold, $n \geq 4$, with civilized conformal connexion; let $N(M)$ be its manifold of null geodesics. Then any small deformation of $L(M)$ is a space of null geodesics; more specifically, if $p: \mathcal{X} \rightarrow \mathbb{C}^k$ is a regular holomorphic map such that $p^{-1}(0) \cong N(M)$ biholomorphically, then there is a neighbourhood V of $\mathcal{X}_0 := p^{-1}(0)$ such that, for every $t \in \mathbb{C}^k$, $V_t := V \cap p^{-1}(t)$ is biholomorphically the space of null geodesics for some complex n -manifold M_t with civilized conformal connexion.

Remark. We shall see that this result holds water *even if* $n < 4$; but the proof is rather different for $n = 3$, where the Q_x 's don't form a complete analytic family. For $n = 2$ the result is trivial as the correspondence between M and $N(M)$ amounts to finding null coordinates (i.e. coordinates such that the coordinate axes are null).

Proof. The proof is essentially the same as that given in Section 1.7 for the analogous theorem concerning projective structures. We merely catalogue the crucial facts that allow it to work. They are:

- a) $H^1(\mathcal{O}(N')) = 0$, $\dim H^0(\mathcal{O}(N')) = n$; this allows us to find a complete analytic family of submanifolds of \mathcal{X} containing the Q_x 's as well as an n -parameter of submanifolds of each fibre of p .
- b) $H^1(Q, \mathcal{O}(T^*Q)) = 0$ (see Section 0.3); quadrics are *rigid* so we can take the submanifolds of the family to all be quadrics, simply by restricting our attention to an open subfamily.
- c) $H^1(Q, \mathcal{O}(\text{END}(N'))) = 0$, so we can also take the normal bundles relative to the fibres of p of each submanifold of the family to be $N' := (T^*\mathbb{P}_{n-1} \otimes H^{-1})|_Q$.
- d) N has the property that every vector is the value of some global section, which means that the subsets of the space M_t of submanifolds of the family which lie in $p^{-1}(t)$ defined as the set of submanifolds through

a given point $y \in p^{-1}(t)$ which lies on *some* manifold of the family constitutes a holomorphic curve, since the codimension of a quadric of the family relative to $p^{-1}(t)$ is $n-1$. Moreover, since every section of N' is a restriction to $Q \subset \mathbb{P}_{n-1}$ of a section of $T'\mathbb{P}_{n-1} \otimes H^{-1}$, and since there is a biholomorphic identification of $\mathbb{P}(\Gamma(T'\mathbb{P}_{n-1} \otimes H^{-1}))$ with \mathbb{P}_{n-1} via "[section] \rightarrow zero set", it follows that precisely a quadric cone's worth of sections of N' vanish at some point of Q , and thus the above defined family of curves in M_t have the property that through every point x of M_t there is precisely a quadric's worth of these curves; we call the corresponding quadric cone in the holomorphic tangent space to M_t at x the *null cone* at x and call the curves of the above defined family *null geodesics*. To see this is all holomorphic, and thus defines a conformal connexion on M_t , we proceed as follows: firstly, there are, locally, holomorphic foliations of M_t by "null geodesics" which include any null geodesic desired; and so the space $QM_t \subset \mathbb{P}T'M_t$ of "null directions" contains two transverse complex submanifolds of $\mathbb{P}T'M_t$ through any point $y \in QM_t$ - namely, the fibre of the canonical projection $QM_t \rightarrow M_t$, which we've already shown to be a quadric, and the tangent field to the mentioned foliation; so the tangent space of QM_t (which is, in particular, a smooth submanifold) is everywhere the real part of a complex subspace of the holomorphic tangent bundle, and so QM_t is a complex submanifold (e.g. by applying the Newlander-Nirenberg Theorem [14]); but the Jacobian of the quotient map $q: QM_t \rightarrow p^{-1}(t)$ is complex linear, since it is when restricted to either of the spanning submanifolds through any point; and so we do in fact have "holomorphically varying null geodesics" in the desired sense.

e) Finally, we apply the "uniform inverse function theorem" (Lemma 1.7.III), just as before, to show that the induced map from the space of null geodesics in all the M_t to \mathcal{N} is a biholomorphism on some neighbourhood of $N(M)$.

§2.5 SPACES OF TORSION-FREE NULL GEODESICS

When does a space of null geodesics correspond to a conformal connexion without conformal torsion? This is a natural question to ask, since the torsion-free null geodesics arise naturally out of a conformal structure - they are, for instance, critical points of such functionals as

$$E(\gamma) := \int_a^b g\left(\frac{d\gamma}{d\zeta}, \frac{d\gamma}{d\zeta}\right) d\zeta$$

defined on the space of holomorphic maps $\gamma: D \rightarrow M$, D the unit disc in \mathbb{C} , which take $a, b \in D$ to two given points $p, q \in M$ where g is a conformal metric. (The integral is to be evaluated on any path in D from a to b ; since the integrand is holomorphic, the integral does not depend on the path. The torsion-free null geodesics are the *only* curves which can be parameterized in such a way as to be critical for *any* choice of g in the conformal class). In fact, one has the following:

Theorem (2.5.I) Let $N(M)$ be a space of null geodesics for a civilized conformal connexion on the complex n -manifold M . The conformal connexion has vanishing conformal torsion iff there is a holomorphic $2n-4$ distribution D on $N(M)$ to which every $Q_x, x \in M$, is tangent:

$$T^*Q_x \subset D|_{Q_x} \quad \forall x.$$

(Remark. The existence of such a distribution just says that the tangents to the Q_x 's nowhere span $T^*N(M)$).

Proof. Pick a local metric g representing the conformal structure and a connexion ∇ preserving g and representing the conformal connexion. If $z \in N(M)$ corresponds to the null geodesic $\gamma \subset M$, then a vector $v \in T_z^*N(M)$ corresponds naturally to an equivalence class of holomorphic (*Jacobi*) vector fields J along γ satisfying

$$X \lrcorner \nabla(X \lrcorner \nabla J) = R(X,J) X + X \lrcorner \nabla(T(X,J)) \quad (2.5.1a)$$

and

$$\begin{aligned} g(X \lrcorner \nabla J, X) &= g(T(X,J), X) \\ & (= g(\tau(X,J), X)) \end{aligned} \quad (2.5.1b)$$

where X is a tangent field for γ satisfying $X \lrcorner \nabla X = 0$ and where the equivalence relation on the fields J is just $J \sim J'$ if $J - J' \propto X$; the tensors T and τ are the earlier defined torsion and conformal torsion and R is the *Riemann tensor* which is defined by

$$R(X,Y) Z = \tilde{X} \lrcorner \nabla(\tilde{Y} \lrcorner \nabla \tilde{Z}) - \tilde{Y} (\tilde{X} \lrcorner \nabla \tilde{Z}) - [\tilde{X}, \tilde{Y}] \lrcorner \nabla \tilde{Z}$$

for any holomorphic fields taking on the desired values X, Y, Z ; it is a simple calculation to see that this is indeed well-defined (cf. [7]).

The correspondence is seen as follows: if X is a holomorphic vector field defined on some region of M satisfying $X \lrcorner \nabla X = 0$ and if J is another vector field satisfying $[X,J] = 0$ (which can be arranged by simply defining J tangent to a single surface transverse to the (geodesic) integral curves of X , and then lie propagating J along by $\text{Re}(e^{i\theta} X)$ for all $\theta \in [0, 2\pi]$) then J satisfies

$$\begin{aligned} X \lrcorner \nabla(X \lrcorner \nabla J) &= X \lrcorner \nabla(J \lrcorner \nabla X + T(X,J)) \\ &= X \lrcorner \nabla(T(X,J)) + \\ &\quad X \lrcorner \nabla(J \lrcorner \nabla X) - J \lrcorner \nabla(X \lrcorner \nabla X) \\ &= R(X,J) X + X \lrcorner \nabla(T(X,J)) \end{aligned}$$

(*Jacobi's equation with torsion*); if one additionally has $g(X,X) \equiv 0$, then also

$$\begin{aligned} 0 &= Jg(X,X) = 2g(J \lrcorner \nabla X, X) = \\ &= 2(g(X \lrcorner \nabla J + T(J,X), X)); \end{aligned}$$

now J and J' are tangent to the same one parameter family of geodesics iff

$J - J'$ is a multiple of X ; and such fields satisfy the desired equations even at their zeroes by continuity; finally, a dimension count certifies that every solution to this pair of equations arises from a tangent vector to $N(M)$ because the solution space of the equations, modulo multiples of X , is $(2n-3)$ -dimensional since we can freely specify the initial value of J and the initial value of $X \lrcorner \nabla J$ subject only to the constraint that $g(X \lrcorner \nabla J + \tau(J, X), X)$ is initially zero.

Now we can get down to business. Let W be the $(2n-1)$ dimensional space of solutions of the equations (2.5.1a), (2.5.1b) and let $V \subset W$ be the smallest linear subspace which contains every J which vanishes somewhere along γ . (J vanishes somewhere just if it's tangent to a one-parameter family of null geodesics through the point at which it vanishes, also by a dimension count). Thus $V \neq W$ precisely if the desired distribution exists. The crucial observation is that, for any point $p \in \gamma$, and for any vectors $Y, Z \in T_p M$ such that $\langle Y, X \rangle = \langle Z, X \rangle = 0$, the unique J satisfying (2.5.1) with $J_p = Y$ and $(X \lrcorner \nabla J)_p = Z$ is an element of V . To see this, first notice that it is obvious if $Y = 0$, so we must merely check that it is also true for $Z = 0$, and Y general. Parameterizing γ by ζ such that $X = d/d\zeta$ and p corresponds to $\zeta = 0$, let J_ϵ be the solution of (2.5.1) such that

$$J_\epsilon|_{\zeta=-\epsilon} = 0, (X \lrcorner \nabla J_\epsilon)|_{\zeta=-\epsilon} = \tilde{Y}|_{\zeta=-\epsilon}$$

where $X \lrcorner \nabla \tilde{Y} = 0$ and $\tilde{Y}_p = Y$; then

$$(J_\epsilon)_p = \epsilon Y + \frac{\epsilon^2}{2} (T(X, \tilde{Y}))|_{\zeta=-\epsilon} + O(\epsilon^3)$$

$$(X \lrcorner \nabla J_\epsilon)_p = \epsilon (T(X, \tilde{Y}))|_{\zeta=-\epsilon} + O(\epsilon^2)$$

where $O(\epsilon^k)$ denotes a vector whose components in any frame are dominated by $K\epsilon^k$, for $\epsilon < \delta$ for some $\delta > 0$, where K and δ may depend on the frame, but not on ϵ ; here $T(X, \tilde{Y})|_{\zeta=-\epsilon}$ is parallel-propagated along γ to p . Now

since $g(T(X, \tilde{Y}), X) = g(X \lrcorner \nabla J_\epsilon, X)|_{\zeta=-\epsilon} = g(\tilde{Y}, X)|_{\zeta=-\epsilon} = 0$, and since v is linear, it follows that V contains the unique solution \hat{J}_ϵ of (2.5.1) for which

$$(\hat{J}_\epsilon)_p = \frac{1}{\epsilon} (J_\epsilon)_p = Y + O(\epsilon),$$

$$(X \lrcorner \nabla \hat{J}_\epsilon)_p = \frac{1}{\epsilon} (X \lrcorner \nabla J_\epsilon)_p - T(X, \tilde{Y})|_{\zeta=-\epsilon} = O(\epsilon)$$

and so, since V is closed, it follows that V contains the unique solution J of (2.5.1) such that $J_p = Y$, $(X \lrcorner \nabla J)_p = 0$.

Thus it is that $V \neq W$ iff a solution J of these equations satisfies

$$X g(J, X) \equiv 0,$$

which is to say iff $g(\tau(X, J), X) \equiv 0$. Thus the existence of the desired distribution on $N(M)$ occurs just in the case that

$$g(X, X) = g(x, y) = 0 \Rightarrow g(\tau(X, Y), X) = 0.$$

We shall now see that this is equivalent to the *vanishing* of the conformal torsion τ .

Indeed, if \hat{v} is torsion-free and $\hat{v}g = 0$, then we've shown that the null integral curves a vector field X satisfying

$$(X \lrcorner \hat{v} X) + (X^b \lrcorner \tau(X))^\# = 0$$

are null geodesics (cf. Eq. (2.1.6)), where

$$g((X^b \lrcorner \tau(X))^\#, Y) := g(\tau(X, Y), X)$$

for any y (i.e. $\#$, b denote "raising and lowering of indices" by the metric).

Moreover, we also showed that the null geodesics determine τ - it vanishes precisely when $(X^b \lrcorner \tau(X))^\# \propto X$ for every null X . But since

$$g((X^b \lrcorner \tau(X))^\#, X) = g(\tau(X, X), X) = 0$$

by the anti-symmetry of τ , and because we know that the non-vanishing of τ forces the existence of a null X such that $(X^b \lrcorner \tau(X))^\#$ is not a multiple

of X , it follows that, for such an X , there exists J such that $g(X,J) = 0$ and $g((X \lrcorner \tau(X)), J) \neq 0$, since the degeneracy of g restricted to the null hyperplane $\{Y | g(X,Y) = 0\}$ is just the span of X . Hence τ vanishes precisely when $g(\tau(X,J), X) = 0$ for all X null and J orthogonal to X , from whence follows the theorem. \square

Notice that the proof of this theorem does *not* rely on analyticity in any way; the major reason for stating it as a theorem concerning *complex* geodesics is that in the smooth category the Q_x 's are not rigidly determined by the structure of $N(M)$, but rather must be filled in by hand. Nonetheless, the smooth analogue will be of interest to us in our discussion of hypersurface-twistor CR structures (Chapter 4).

There is something rather unsatisfactory about this criterion because of the peculiar fashion it mixes local and global requirements. However, when the dimension exceeds 3 one has a much nicer formulation.

Theorem (2.5.II). Let M be a complex n -manifold, $n \geq 4$, with civilized conformal connexion. The conformal torsion vanishes iff there exists a holomorphic line sub-bundle $E \subset T^*N(M)$ of the holomorphic cotangent bundle to the space of null geodesics whose restriction to some (and hence to any) Q_x is isomorphic to the Hopf bundle H^{-1} .

Proof. First suppose the conformal torsion vanishes. By the proposition there exists a distribution D on $N(M)$ of codimension 1 tangent to every Q_x , which is given by the set of Jacobi fields J along the corresponding null geodesic in M such that $g(X,J) \equiv 0$, for X a tangent field to the geodesic. Let E be the annihilator of D ; i.e. let $\omega \in E \subset T^*N(M)$ iff $v \lrcorner \omega = 0$ for every $v \in D$. Letting g be a metric on some region of M in the given conformal class, then for every autoparallel tangent field X , $X \lrcorner \nabla X = 0$, to a null geodesic, $X \lrcorner$ (the 1-form $Y \rightarrow g(X,Y)$) defines an element of E , since, as we saw in the proof of the proposition, $g(X,J) =$

constant if J is a Jacobi field corresponding to a tangent vector on $N(M)$ (i.e. if J "connects null geodesics"), and this constant is unchanged by adding a multiple of X to J . Thus, if $x \in M$ is an element of the region on which we picked a specific metric, then $E|_{Q_x}$ is simply described as the bundle over the quadric of null directions at x whose fibre over a null direction consists of null vectors pointing in that direction; i.e. $E|_{Q_x}$ is isomorphic to H^{-1} .

Conversely, suppose that we're given a holomorphic line bundle $E \subset T^*N(M)$ which restricts to every Q_x as H^{-1} , and where $\dim M \geq 4$. Then $E \otimes E^* \subset T^*N(M) \otimes E^*$ is trivial, and so admits a non-vanishing global section ω . We want to show that $\omega|_{T^*Q_x} \equiv 0$ for all x ; this is true because $H^0(Q, \Omega^1 \otimes \mathcal{O}(1)) = 0$, where $Q \subset \mathbb{P}_{n-1}$ is a quadric, as we shall now demonstrate. Indeed, one has the exact sequences

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \Omega^1_{\mathbb{P}_{n-1}} \otimes \mathcal{O}(1) \rightarrow \Omega^1_Q \otimes \mathcal{O}(1) \rightarrow 0$$

$$0 \rightarrow \Omega^1_{\mathbb{P}_{n-1}} \otimes \mathcal{O}(1) \xrightarrow{\lambda^*} n\mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow 0$$

of sheaves on $Q \subset \mathbb{P}_{n-1}$, where $\Omega^1_Q = \mathcal{O}(T^*Q)$, $\Omega^1_{\mathbb{P}_{n-1}} = \mathcal{O}(T^*\mathbb{P}_{n-1})$; the first comes from dualising

$$0 \rightarrow \theta_Q \rightarrow \theta_{\mathbb{P}_{n-1}} \rightarrow \mathcal{O}(2) \rightarrow 0$$

and tensoring by $\mathcal{O}(1)$, while the second results from dualising

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda} n\mathcal{O} \rightarrow \theta_{\mathbb{P}_{n-1}} \rightarrow 0$$

and tensoring by $\mathcal{O}(1)$; $[\lambda(f)]([z_1, \dots, z_n]) = (z_1 f([z_1, \dots, z_n]), \dots, z_n f([z_1, \dots, z_n]))$, and so the adjoint λ^* is given by $[\lambda^*(f_1, \dots, f_n)]([z_1, \dots, z_n]) = (\sum_{i=1}^n f_i z_i) ([z_1, \dots, z_n])$. On the level of global sections, λ^* is an isomorphism, so $H^0(Q, \Omega^1_{\mathbb{P}_{n-1}} \otimes \mathcal{O}(1)) = 0$, and, since $H^1(Q, \mathcal{O}(-1)) = 0$ for $n \geq 4$, it follows that $H^0(Q, \Omega^1_Q \otimes \mathcal{O}(1)) = 0$. Hence the distribution annihilated by E is tangent to Q_x .

All that remains to prove is that the type of $E|_{Q_x}$ is independent of x . Line bundles, of course, are just elements of $H^1(Q, \mathcal{O}^*)$, and this group can be calculated from the famous exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0;$$

Since $H^1(Q, \mathcal{O}) = H^2(Q, \mathcal{O}) = 0$, it follows that the boundary (Chern Class) operator $H^1(Q, \mathcal{O}^*) \rightarrow H^2(Q, \mathbb{Z})$ is an isomorphism, showing that the holomorphic type of line bundle on a quadric is determined by its topological type. \square

It might seem that the line bundle E in the above discussion need bear no particular holomorphic relation to any other structure of the manifold $N(M)$; but in fact E is automatically the $(n-1)$ -st root of the canonical bundle of $N(M)$ - a local section of E^{n-1} is, up to a constant overall scale, a $(2n-3)$ -form. In fact, if $\theta \in \Gamma \Omega^1(E^*)$ is a non-zero section of $E \otimes E^*$ (there are just \mathbb{C} 's worth of such sections; any holomorphic function on $N(M)$ is constant), then we shall show that the twisted $(2n-3)$ -form $\theta \wedge (d\theta)^{\wedge(n-2)} \in \Gamma \Omega^{2n-3}(E^{*(n-1)})$ vanishes nowhere, and so identifies E with the $(n-1)$ -st root of the canonical bundle. Thus the structure we've been discussing is just a *contact form* for $N(M)$; cf. [25] for a discussion of the smooth analogue. (Notice, however, that the "twist" in our definition of a contact form on a complex $(2p+1)$ -manifold X as a section ω of $\Omega^1(\underline{k}^{-1}/(p+1))$, where \underline{k} is the canonical bundle $\wedge^{2p+1} T^*X$, such that $\omega \wedge (d\omega)^{\wedge p}$ vanishes nowhere, is completely necessary, since otherwise we would be limited to the rare and boring cases in which \underline{k} is trivial; by contrast, in the smooth category the triviality of the canonical bundle is just the requirement of orientability, which is hardly severe).

Let $N(M)$ be the space of null geodesics for a civilized torsion-free conformal connexion on a complex n -manifold M , let $\tilde{Q} \subset \mathbb{P} T^*M$ be the bundle of projectivized holomorphic null *cotangent* vectors, and let $q: \tilde{Q} \rightarrow N(M)$ be the quotient map as before, where \tilde{Q} is identified with the projectivized null tangent vectors by the use of the conformal structure. If $\tilde{E} \rightarrow \tilde{Q}$ is

the "Hopf" line bundle whose fibre over a codirection consists of covectors "pointing" in that codirection, then there is a canonical form $\tilde{\theta} \in \Omega^1(\tilde{E}^*)$ given on \tilde{Q} by

$$(V \lrcorner \tilde{\theta})|_{\omega} = (\pi_* V) \lrcorner \omega$$

where $\pi: \tilde{Q} \rightarrow M$ is the canonical projection, $\omega \in \tilde{E}$ (i.e. a null covector) and where $V \in T'_{[\omega]} \tilde{Q}$ (i.e. a holomorphic vector on \tilde{Q} at the direction represented by ω). Then $\tilde{\theta}$ is of the form $q^*\theta$ for a twisted form θ on $N(M)$, since the vanishing of torsion implies, as noted in our earlier discussion, that if X is an auto-parallel tangent field to a null geodesic with respect to the torsion-free metric connexion of the conformal metric g , and if J is a Jacobi field along the null geodesic which "points to another null geodesic" then $g(X, J)$ is constant; but thinking of $X\flat = X \lrcorner g$ as an element of \tilde{E} , and J as $\pi_* V$, this just says that $\tilde{\theta}$ is the pull-back of a twisted form θ on $N(M)$. (Notice, incidentally, that X is a conformally invariant object; if ∇ is the torsion-free metric connexion for the conformal metric g , and if $X \lrcorner \nabla X \flat = 0$, then replacing g by $\Omega^2 g$ results in a change

$$\Delta_{bc}^a = \delta_b^a v_c + \delta_c^a v_b - g^{ad} v_d g_{bc}$$

in the connexion, where $v = \Omega^{-1} d\Omega$, and so, letting $\hat{\nabla}$ be new torsion-free metric connexion one has

$$X \lrcorner \hat{\nabla} X = (X \lrcorner X\flat)v + (X \lrcorner v) X\flat - g(X, v\sharp) X \lrcorner g = 0.$$

Thus we can, in good faith, define a line bundle E over $N(M)$ whose fibre over a given point consists of autoparallel "tangent" covectors along the corresponding null geodesic; the 1-form θ such that $\tilde{\theta} = q^*\theta$ takes its values in E^*).

Now given a 1-form ψ with values in a line bundle L , let V_ψ denote the annihilator of ψ - the set of vectors w such that $w \lrcorner \psi = 0$; if ψ is non-

vanishing then V_ψ is a vector sub-bundle of the tangent bundle of codimension 1. Then $d\psi$ is a well defined section of $\wedge^2 V_\psi^* \otimes L$; for though by changing the local trivialization of L by multiplication by a nowhere vanishing function f we can change $d\psi$, interpreted as taking values in L , to

$$f^{-1} d(f\psi) = d\psi + f^{-1} df \wedge \psi$$

the difference is unseen on V_ψ . Moreover, since $d\psi$, interpreted as an honest-to-God 2-form taking values in L , is only ambiguous by a multiple of ψ , it follows that

$$\psi \wedge (d\psi)^{\wedge p}$$

is a well defined $(2p+1)$ -form with values in L^{p+1} for any non-negative integer p .

I now claim that the twisted volume element

$$\theta \wedge (d\theta)^{\wedge (n-2)}$$

vanishes nowhere on $N(M)$. This would be true if $d\theta: V_\theta \rightarrow V_\theta^*$ were of maximal rank; on the other hand, $q^* d\theta = d(q^*\theta)$, so it suffices to check that $d\theta: V_{\tilde{\theta}} \rightarrow V_{\tilde{\theta}}^*$ has rank $n-2$. Now $\tilde{\theta}$ is obtained from the familiar contact form on T^*M (cf. 25), by restricting to the null convectors and projectivization; this contact form of T^*M is given in local coordinates by

$$\hat{\theta} = p_a dz^a$$

where z^a are coordinates on M , pulled back to T^*M by the canonical projection, and where the functions p_a are just the 1-forms dz^a considered as functions on T^*M ; thus we can compute $d\tilde{\theta}|_{V_{\tilde{\theta}}}$ by restricting $d\hat{\theta}$ to a $(2n-3)$ -dimensional distribution tangent to $g^{ab} p_a p_b = 0$, annihilated by $\hat{\theta}$ and transverse to $p_a \frac{\partial}{\partial p_a}$. Now

$$d\hat{\theta} = dp_a \wedge dz^a$$

is clearly non-degenerate; restricting to $V_{\hat{\theta}}$ will introduce a degeneracy, namely in the direction $p_a \frac{\partial}{\partial p_a}$, since

$$-p_a \frac{\partial}{\partial p_a} \lrcorner dp_a \wedge dz^a = p_a dz^a = \hat{\theta},$$

but restricting to a subspace transverse to $p_a \frac{\partial}{\partial p_a}$ eliminates this. Now restricting to tangents to $g^{ab} p_a p_b = 0$ introduces another degeneracy, but this one is just the span of the horizontal lift of $g^{ab} p_a \frac{\partial}{\partial z^b}$ - that is, the tangent space to the fibre of q ; this we can see explicitly by taking all the derivatives of the metric at the origin of the coordinates z^a on M to vanish (by taking, for instance, the coordinates to be obtained by exponentiating euclidean coordinates on a tangent space), so that the said horizontal vector over the origin is just $g^{ab} p_a \frac{\partial}{\partial z^b}$, while the normal form over the origin of $g^{ab} p_a p_b = 0$ is just

$$g^{ab} p_a dp_b = -g^{ab} p_a \frac{\partial}{\partial z^b} \lrcorner dp_c \wedge dz^c.$$

Thus the rank of $d\hat{\theta}|_{V_{\hat{\theta}}}$, and hence that of $d\theta|_{V_{\theta}}$, is $2n-4$, as claimed.

We've demonstrated the

Theorem (2.5.III). A space of torsion-free null geodesics admits a contact structure. Conversely, if M is a complex n -manifold, $n \geq 4$, with civilized conformal^{connexion}, the space $N(M)$ of null geodesics admits a holomorphic contact structure only if the torsion vanishes.

Notice that we have also shown that the total space of the line bundle $\kappa^{1/(n-1)}$ admits a symplectic structure if the conformal torsion vanishes. The implications of this result remain uninvestigated; we shall make no use of it.

In Chapter 3 we shall see that for $n=3$ the existence of a contact structure on a space of torsion-free null geodesics has remarkable consequences.

§3.1. SPACES OF NULL GEODESICS IN DIMENSION THREE

Up until now we have been considering spaces of geodesics in which the submanifolds consisting of all geodesics through a point of the primary space form a complete analytic family. For null geodesics in three dimensions, this fails to be the case, but because the normal bundle N' to a Q_x still satisfies $H^1(N') = 0$ (see §2.3), one can use Kodaira's theorem to complete the family to yield a family of dimension $4 = \dim H^0(N')$. This fact catapults us into the realm of Penrose's twistor theory [22], as shall be explained presently.

Q_x is a conic curve, and so [3] is a biholomorphically \mathbb{P}_1 ; this can be seen from the fact that the map $\rho: \mathbb{P}_1 \rightarrow \mathbb{P}_2: [a, b] \rightarrow [(a^2 + b^2), (a^2 - b^2), 2ab]$ is a biholomorphism onto the quadric $z_1^2 + z_2^2 + z_3^2 = 0$. (Notice that, because this map is of homogeneity 2, $\mathcal{O}_Q(m) \cong \mathcal{O}_{\mathbb{P}_1}(2m)$ for all $m \in \mathbb{Z}$). But every vector bundle over \mathbb{P}_1 is a sum of line bundles [4]. Thus, $\mathcal{O}(N') \cong \mathcal{O}(m_1) + \mathcal{O}(m_2)$ as sheaves on \mathbb{P}_1 for some integers m_1, m_2 . Then

$$\begin{aligned} H^1(\mathcal{O}(N' \otimes N'^*)) &= 2H^1(\mathcal{O}) + H^1(\mathcal{O}(m_1 - m_2)) \\ &\quad + H^1(\mathcal{O}(m_2 - m_1)) \\ &= H^1(\mathcal{O}(-|m_1 - m_2|)); \end{aligned}$$

but we demonstrated in §2.3 that $H^1(\mathcal{O}(\text{END}(N'))) = 0$, and so $|m_1 - m_2| \leq 1$.

Yet we also saw that $\dim H^0(N') = 4$, so $4 = \text{Max}(m_1 + 1, 0) + \text{Max}(m_2 + 1, 0) =$

$$\begin{cases} 2 \text{Max}(m_1 + 1, 0), & m_1 = m_2 \\ 2 \text{Max}(m_1 + 1, m_2 + 1, 0) - 1, & |m_1 - m_2| = 1 \end{cases}$$

and thus $m_1 = m_2 = 1$; and

$$\mathcal{O}(N') \cong 2\mathcal{O}(1).$$

Now this is precisely the case familiar from the non-linear graviton

construction [15]; the four dimensional complete analytic family of \mathbb{P}_1 's with normal bundle $H + H$ containing a given curve of this type has a natural conformal structure with self-dual conformal curvature; and every complex 4-manifold with self-dual conformal structure arises (locally) in this manner. We shall briefly sketch why this is the case; cf. [15], [21].

We define the conformal structure on the space of curves of the above type by requiring that a vector be null iff the corresponding section of the normal bundle (to the appropriate curve) vanishes somewhere; this is, in fact, a quadric cone, because global sections of $2\mathcal{O}(1)$ are pairs of linear functionals on \mathbb{C}^2 , and vanish somewhere on \mathbb{P}_1 iff these two functionals are proportional; thinking of this pair of linear forms as a linear map $\lambda: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, the section vanishes somewhere precisely when $\det \lambda = 0$, which is indeed a non-degenerate quadratic condition on the rows of a 2×2 matrix. Now the set of curves of the family through some fixed point constitutes a complex 2-manifold by an argument precisely analogous to that used in the proof of proposition (1.7.1), the point being that every point in the total space of $H + H$ is the value of some global holomorphic section, and so the codimension of the manifold of curves through a point is the same as the codimension of a curve - that is, two. Now these 2-folds are totally null - meaning that each vector tangent to one is null - as follows directly from our definition of the conformal structure; there is, moreover, precisely one such totally null 2-fold, or *twistor surface*, tangent to each direction. (The set of tangent vectors to a twistor surface at some point x corresponds, projectively, to a generator in one of the two possible families of \mathbb{P}_1 's foliating the null quadric, which, as a quadric in \mathbb{P}_3 , is biholomorphically $\mathbb{P}_1 \times \mathbb{P}_1$; this is the complex version of the familiar two family of straight lines on a hyperboloid of one sheet, and is tied up with the structure of spinors in

four dimensions [21], [23]. Conversely, it will become clear presently that if a conformal structure admits such a family of twistor surfaces, one tangent to each null direction, then, at least if we work in a suitable region, the space of twistor surfaces forms a complex 3-manifold, called the *twistor space* of the conformal structure, in which the points of the original space correspond to \mathbb{P}_1 's with normal sheaf $2\mathcal{O}(1)$. (As it turns out, the family of twistor surfaces is unique except in the case of the *flat* conformal structure, where there are two families of such surfaces corresponding to the two families of \mathbb{P}_1 's on a quadric in \mathbb{P}_3 , both with twistor spaces biholomorphic to some neighbourhood of a line in \mathbb{P}_3).

We now remark that the existence of a family of twistor surfaces, one surface tangent to each null direction, is precisely that the conformal (Weyl) curvature be self-dual or anti-self-dual. By *self-dual* we mean that the Weyl curvature C_{jkl}^i , which is the conformally invariant part of the Riemann Tensor for torsion-free metric connexions with the given conformal structure

$$C_{jkl}^i := R_{jkl}^i - \delta_{[l}^i R_{k]j} - g_{j[k} R_{l]}^i + \frac{1}{3} R \delta_{[k}^i g_{l]j};$$

$$R_{jk}^i := R_{jik}^i, \quad R_j^k := R_{j\ell} g^{\ell k}, \quad R := R_i^i$$

(see [23] for the independence of C from the representative connexion) satisfies

$$C = * C; \text{ i.e. } C_{jkl}^i = \frac{1}{2} \epsilon_{kl}^{mn} C_{jmn}^i$$

where ϵ is a preferred holomorphic 4-form such that $\epsilon(X_1, X_2, X_3, X_4) = 1$ if $g(X_i, X_j) = \delta_{ij}$. (There are just two such forms, one the opposite of the other. The *Hodge star* $*$ so defined on vector-bundle-valued 2-forms satisfies $*^2 = 1$ and is conformally invariant; it suffers, though, the

infirmity of depending upon an "orientation", for although $GL(n, \mathbb{C})$ is connected, $O(n, \mathbb{C})$ is not, and so "complex orientations" come into being once a metric is introduced. All our discussion is equally valid for *anti*-self dual curvatures, where $C = - *C$, but by switching the arbitrary orientation we eliminate the need to include this other possibility). This fact is most easily seen by spinor algebra [23], [21], but in deference to those unfamiliar with these important techniques we will sketch a proof using only tensor algebra.

We mentioned earlier that, in the holomorphic tangent plane to a point on a 4-manifold with conformal structure, there are two kinds of totally null 2-planes; these are those with self-dual and anti-self-dual volume elements, respectively. (And any 2-plane with self-dual or anti-self-dual volume is totally null since the Hodge star takes a volume element for a two plane onto a volume element of an orthogonal two plane). The self-duality of the Weyl curvature is equivalent to the vanishing of $g(Y, R(X, Y)X)$ for every X and Y such that $X \wedge Y = - * X \wedge Y$; again, R is here the Riemann tensor of the torsion free connexion preserving g .

Suppose the Weyl curvature is self-dual. Then consider the surface swept out by the null geodesics whose tangents at some point lie in an anti-self dual plane. A Jacobi field J along such a null geodesic with tangent X satisfies

$$\begin{aligned} X(X(g(X, J))) &= g(X, X \lrcorner \nabla(X \lrcorner \nabla J)) \\ &= g(X, R(X, J)X) = 0; \end{aligned}$$

and if Y is parallel propagated along X and $X \wedge Y = - * X \wedge Y$ (initially, and hence always), then

$$X(X(g(Y, J))) = g(Y, R(X, J)X) = 0;$$

but since the Weyl curvature is self-dual, $g(X, J) = g(Y, J) = 0$

$g(Y, R(X, J)X) = 0$. So a Jacobi field along X vanishing at some point and such that $g(X, X \lrcorner \nabla J) = g(y, X \lrcorner \nabla J) = 0$ at that point satisfies $g(X, J) \equiv g(Y, J) \equiv 0$. So the surface we've described is *everywhere* totally null, and so is a twistor surface.

Conversely, suppose that there are vector fields X, y satisfying $[X, Y] = 0$, $X \wedge y = - * X \wedge Y$; that is, suppose there exists a foliation by twistor surfaces. Then $g(X, X \lrcorner \nabla Y) = g(X, Y \lrcorner \nabla X) = \frac{1}{2} Y(g(X, X)) = 0$, and, similarly, $g(Y, Y \lrcorner \nabla X) = g(Y, X \lrcorner \nabla Y) = 0$; also $g(X \lrcorner \nabla X, X) = \frac{1}{2} X(g(X, X)) = 0$, and similarly $g(Y \lrcorner \nabla Y, Y) = 0$; and $g(X \lrcorner \nabla X, Y) = -g(X, X \lrcorner \nabla Y) = 0$, and similarly $g(Y \lrcorner \nabla Y, X) = 0$. Thus, for some functions $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ one has

$$X \lrcorner \nabla X = \alpha X + \beta Y$$

$$y \lrcorner \nabla X = X \lrcorner \nabla Y = \gamma X + \delta Y$$

$$y \lrcorner \nabla y = \epsilon X + \eta Y$$

which means that the twistor surface is totally geodesic, and so, by the inversion of the earlier Jacobi field argument, $\langle R(X, Y) X, Y \rangle \equiv 0$; so the existence of twistor surfaces tangent to each infinitesimal anti-self dual plane is equivalent to the self-duality of the conformal curvature.

We will henceforth say that a holomorphic conformal structure on a complex 4-manifold is *half-flat* iff the conformal curvature is self-dual or anti-self-dual (depending upon the chosen orientation).

We will say that a 4-manifold with half-flat conformal structure is *polished* if the *twistor space* of its twistor surfaces (of the "handedness" corresponding to anti-self-duality if the conformal curvature is self-dual) is a Hausdorff (paracompact three-) manifold.

A region which is geodesically convex for some conformal factor is clearly polished, by an argument precisely analogous to that which shows that its space of geodesics is Hausdorff.

What our calculation of the normal bundle of $Q_x \subset N(M)$ for $\dim M = 3$ shows is that any three-dimensional conformal connexion can be realised, at least locally, as that induced on a nowhere null hypersurface in a complex 4-manifold with half-flat conformal structure by defining the null geodesics to be the intersections of twistor surfaces with the hypersurface. The underlying conformal structure of this three-dimensional conformal connexion is just the "conformal first fundamental form" - a conformal metric restricted to the surface is a conformal metric for this structure. Rather beautifully, it turns out that the torsion of this conformal structure is, in essence, the conformally invariant part of the second form.

The reader familiar with the differential geometry of hypersurfaces [7] will recall that the second fundamental form of a hypersurface is given by

$$II(X, Y) = g(X, Y \lrcorner \nabla V)$$

where X and Y are tangent to the hypersurface and where V is a chosen unit normal vector field (locally, one of two possible fields). This is a symmetric form because

$$\begin{aligned} g(X, Y \lrcorner \nabla V) &= -g(Y \lrcorner \nabla X, V) \\ &= -g(X \lrcorner \nabla Y, V) + g([X, Y], V) \\ &= -g(X \lrcorner \nabla Y, V) \\ &= g(Y, X \lrcorner \nabla V). \end{aligned}$$

We let \hat{II} denote the trace-free part of this form; for a hypersurface of three-dimensions $\hat{II} = II - \frac{1}{3}(\text{Tr} II)g$.

Now consider the behaviour of II under a conformal transformation. If we replace g by $\tilde{g} := \Omega^2 g$, then it becomes necessary to replace V by $\tilde{V} := \Omega^{-1} V$. The new symmetric connexion $\tilde{\nabla}$ preserving the metric differs from ∇ by

$$\Delta_{jk}^i = \delta_j^i \omega_k + \delta_k^i \omega_j - g^{il} \omega_l g_{jk}$$

where $\omega = d(\log \Omega)$, and so

$$\begin{aligned} \tilde{I}\tilde{I}(X,y) &= \tilde{g}(X,Y \lrcorner \tilde{\nabla} \tilde{V}) = \Omega^2 g(X,Y \lrcorner \tilde{\nabla} \Omega^{-1} V) \\ &= \Omega g(X,Y \lrcorner \tilde{\nabla} V) - g(X,V) Y(\Omega) \\ &= \Omega II(X,Y) + \Omega g(X,(V \lrcorner \omega)Y) \\ &\quad + \Omega g(X,(Y \lrcorner \omega)V - g(V,Y)\omega) \\ &= \Omega II(X,Y) + V(\Omega) g(X,Y). \end{aligned}$$

By monkeying with the normal derivative of Ω we can thus make $\text{Tr} \tilde{I}\tilde{I}$ whatever we like; but the trace-free part $\hat{I}\hat{I}$ simply transforms as a tensor of conformal weight 1:

$$\widetilde{\hat{I}\hat{I}}(X,y) = \Omega \hat{I}\hat{I}(X,y).$$

We call this conformally weighted tensor the *conformal second fundamental form* of the hypersurface; it can be thought of as a section of the bundle $\Lambda^2 T^*S \otimes (K_S)^{1/3}$, where S is the hypersurface and K_S is the canonical bundle whose sections are holomorphic 3-forms. (The cube-root may only exist locally. Notice that thinking about $\hat{I}\hat{I}$ in this fashion eliminates the sign ambiguity, since the sign of the volume 3-form e associated with a metric can be chosen by convention such $V \wedge e$ is a standard 4-form on the ambient space).

We now can state the beguiling

Proposition (3.1.I). Let $S \subset M$ be a nowhere null complex hypersurface in a complex 4-manifold with polished half-flat conformal structures. Define a conformal connexion on S by taking the null geodesics to be the intersections of twistor surfaces with S . Then the conformal torsion of this conformal connexion is

$$\tau_{jk}^i = e_{jk\ell} \hat{\Pi}^{\ell i} \quad (3.1.1)$$

for any choice of the conformal factor; here e is the metric volume element on the hypersurface, and $\hat{\Pi}^{ij} := g^{i\ell} g^{jk} \hat{\Pi}_{\ell k}$.

Remark. This expression for τ is of conformal weight 0, since e has weight 3 and $\hat{\Pi}$ with *raised* indices has weight $-4 + 1 = -3$. There is still a sign of ambiguity in this expression, which will be set straight in a moment.

Proof. First let's notice that τ is anti-symmetric in its downstairs indices, as it should be, and that traces vanish, as does its totally skew part, since

$$e^{jki} e_{jk\ell} \hat{\Pi}^{\ell}_i = 2\delta^i_{\ell} \hat{\Pi}^{\ell}_i = 2\text{Tr}\hat{\Pi} = 0.$$

Now we must merely show that the null geodesics defined by this torsion are just the intersections of twistor surfaces with S .

As noted earlier, the twistor surfaces are totally geodesic (for any conformal factor). So if ∇ is, for some choice of metric, representing the conformal structure on M , the symmetric metric connexion, and if Π is a twistor surface in M , it follows that a tangent field X to $\gamma = \Pi \cap S$ satisfies

$$X \lrcorner \nabla X \in T\Pi.$$

Also, the part of the covariant acceleration out of the surface is given by

$$\begin{aligned} g(V, X \lrcorner \nabla X) &= -g(X \lrcorner \nabla V, X) = -II(X, X) \\ &= -\hat{II}(X, X) \end{aligned}$$

and so if $\hat{\nabla}$ is the symmetric Riemannian connexion on S - that is, if $\hat{\nabla}$ is the connexion on S induced by ∇ - then

$$X \lrcorner \hat{\nabla} X = X \lrcorner \nabla X + \hat{\mathbb{I}}(X, X) V$$

where V is, as before, the preselected unit normal field to S . We choose the orientation of S by requiring

$$e = V \lrcorner \varepsilon \quad (\text{i.e. } e_{ijk} = V^\ell \varepsilon_{\ell ijk})$$

where ε is the metric volume element for which the conformal curvature is *anti-self dual* and the twistor surfaces are *self dual*. Now since X and $X \lrcorner \nabla X$ are tangent to the same twistor surface we have

$$\begin{aligned} e_{ijk} X^i (X \lrcorner \hat{\nabla} X)^j &= \varepsilon_{\ell ijk} V^\ell X^i (X \lrcorner \nabla X)^j \\ &= 2V^\ell \cdot \frac{1}{2} \varepsilon_{ij\ell k} X^{[i} (X \lrcorner \nabla X)^{j]} \\ &= 2V^\ell X_{[\ell} (X \lrcorner \nabla X)_{k]} \\ &= -g(V, X \lrcorner \nabla X) X_k \\ &= \hat{\mathbb{I}}(X, X) X_k \end{aligned}$$

where indices are lowered using the metric, and where square brackets indicate projection to the anti-symmetric part. But if we define σ_{jk}^i by

$$\sigma_{jk}^i = e_j^{i\ell} \hat{\mathbb{I}}_{\ell k}$$

then

$$\begin{aligned} e_{ijk} X^i \sigma_{\ell m}^j X^\ell X^m &= 2g_{\ell i} \delta_k^n \hat{\mathbb{I}}_{nm} X^i X^\ell X^m \\ &= -X_k \hat{\mathbb{I}}_{im} X^i X^m \end{aligned}$$

for all null X , and so

$$X \lrcorner \hat{\nabla} X + \sigma(X, X) \propto X$$

when X is tangent to the intersection of a twistor surface with S .

Because $\hat{\nabla} + \sigma$ preserves the metric, due to the skewness of σ_{jk}^i in i and j , it follows that the conformal torsion is τ . \square

Remark: An alternate set of conventions for which the proposition is true without sign change would be to take the twistor surfaces to be anti-self dual with respect to ϵ and to take

$$e_{ijk} = \epsilon_{ijkl} v^l.$$

This leads immediately to the

Theorem(3.1.II). Every complex 3-manifold with holomorphic conformal first and second fundamental forms (i.e. with conformal structure and a trace-free symmetric two-tensor of conformal weight 1) can be embedded in a complex 4-manifold with half-flat conformal structure in such a way that these forms are realised. Moreover, this embedding is locally unique up to conformal biholomorphic mapping.

Proof. Define a conformal connexion on the given 3-manifold with underlying conformal structure the "first fundamental form" and with torsion related to the "conformal second fundamental form" by (3.1.1). Cover the manifold by civilized neighbourhoods. Then in the space of null geodesics for each such neighbourhood use Kodaira's theorem to find a four-parameter-family of \mathbb{P}_1 's with half-flat conformal structure as described before; the original neighbourhood sits in this family as a hypersurface. Then using the maximality clause of Kodaira's theorem, there is a canonical identification of curves sufficiently close to regions of such hypersurfaces corresponding to the overlaps of neighbourhoods in our cover, and we can thus glue all these manifolds together; moreover, the resulting ambient 4-manifold can be assumed to be Hausdorff by restricting one's attention to a neighbourhood of the original 3-manifold. Finally, we can again

use the maximality clause to assure that any other such embedding is conformally biholomorphic to the one we've constructed in some neighbourhood of the 3-manifold.

§3.2 THE HYPERSURFACE TWISTOR CONSTRUCTION

The hypersurface twistor construction [21], [22] is a rather natural way of inducing a conformal connexion on a hypersurface of a complex 4-manifold with conformal structure. (This construction arose from consideration of the case in which the 4-manifold and hypersurface are obtained by analytic continuation from an analytic Lorentzian ^{manifold} with Cauchy surface; and in fact, there's a bit of extra structure arising the real slice that will be of importance in motivating certain later sections). As it turns out, the conformal torsion of this conformal connexion is related to the second fundamental form of the hypersurface by (3.1.1), and so this construction actually corresponds to embedding the hypersurface in a complex 4-manifold with *half-flat* conformal structure in such a way that the original conformal first and second fundamental forms are realised.

Let $S \subset M$ be a complex hypersurface in a complex 4-manifold with holomorphic metric g and metric volume element ϵ , and suppose there is a global holomorphic unit normal vector field V on S . Let ∇ denote the torsion-free metric connexion of g . ∇ naturally extends to a connexion of holomorphic 2-forms, and, since ∇ preserves the volume element ϵ , it follows that ∇ defines a unique connexion on *self-dual* 2-forms, and this connexion (also denoted ∇) preserves the nondegenerate quadratic form.

$$q(F_{[ab]}, G_{[ab]}) := \epsilon^{abcd} F_{ab} G_{cd} = F_{ab} G^{cd}$$

on the self-dual 2-forms. Now V defines an isomorphism of the bundle of self-dual 2-forms, restricted to S , with holomorphic tangent bundle to S via

$$F_{ab} \rightarrow F^a_b V^b$$

("taking the electric field"); the anti-symmetry of F guarantees that the image is orthogonal to V , while the expression

$$F_{ab} = 2 F_{c[b} V_{a]} V^c + 2 \epsilon_{ba}{}^{cd} F_{ed} V_c V^e$$

for a self-dual 2-form at S shows that this isomorphism takes $\frac{1}{2}q$ onto g ; ∇ therefore defines a (torsion) metric connexion on S via this isomorphism. The conformal connexion represented by this connexion is the "twistorial" conformal connexion, and its null geodesics are called *hypersurface twistors*.

Let us see that our definition coincides with the usual definition of hypersurface twistors. The null vectors correspond to degenerate F 's that can be written in spinorial [23] notation as

$$F_{ab} = \epsilon_{AB} \pi_{A'} \pi_{B'}$$

for some spinor π ; the corresponding null vector tangent to S is

$$V^a F_a{}^b = V^{BA'} \pi_{A'} \pi^{B'}$$

and equation $(V \lrcorner F) \lrcorner \nabla F = 0$ for a geodesic becomes

$$V^{BA'} \pi_{A'} \pi^{B'} \nabla_{BB'} \pi^{C'} = 0$$

as in [21].

If $X = V \lrcorner F$, then $X \wedge (X \lrcorner \nabla X)$ is self-dual, as is most easily seen from the spinorial approach, since

$$V^{BA'} \pi_{A'} \pi^{B'} \nabla_{BB'} (V^{CD'} \pi_{D'} \pi^{C'}) =$$

$$V^{BA'} \pi_{A'} \pi^{B'} \pi_{D'} \pi^{C'} \nabla_{BB'} V^{CD'} \propto \pi^{C'}$$

while $X^{AA'} \propto \pi^{A'}$ also; from the strictly tensorial point of view one must check that

$$U^a V^b F_a{}^{[c} F_b{}^{d]}$$

is self-dual if F is. We also have that $g(X, X \lrcorner \nabla X) = -\hat{\mathbb{I}}(X, X)$, and so

the same algebra used in §3.1 demonstrates that the conformal torsion of the "twistorial" conformal connexion is $e_{ijk} \hat{\Gamma}^{kl}$, which shows, incidentally, that this conformal connexion is independent of the original conformal factor on M .

Thus, this rather natural conformal connexion on $S \subset M$ is the same one obtained by first embedding S in a 4-manifold \tilde{M} with *half-flat* conformal structure in such a way as to realise the conformal first and second fundamental forms of S , and then taking the null geodesics to be the intersections of twistor surfaces with S . The half-flat embedding space \tilde{M} is known in twistor folk-lore as "heaven-on-earth", a name remarkably appropriate in terms of these rather unexpected paradisiacal properties. (In fact, it was previously only recognised that "heaven-on-earth" existed provided that the hypersurface was sufficiently "flat").

§3.3 DEFORMATIONS OF SPACES OF NULL GEODESICS: THE
IMPROVED RESULT

Theorem (3.3.I). Any small deformation of the space of null geodesics of a complex manifold with civilised conformal connexion is again such a space.

Proof. We have proved this for all dimensions except 3. In this dimension, the result is essentially that of Penrose [15], which notices that a small deformation of a twistor space is (in a fashion not made precise there, but which can in fact be interpreted after the manner of our own results) again a twistor space; a rigorous version of the proof is precisely analogous to our proofs that small deformations of spaces of geodesics are again spaces of geodesics, the information required for the proof to work being that the normal bundle $N \cong 2H$ of a \mathbb{P}_1 of the initial family satisfies $H^1(\mathcal{O}(N)) = 0$, $H^0(\mathcal{O}(N)) \simeq \mathbb{C}^4$, $H^1(\mathcal{O}(N \otimes N^*)) = 0$, and every vector in N is the value of a global section, all of which we've noted previously. One now finds a hypersurface in any of the 4-folds with half-flat conformal structure which correspond to the deformed twistor spaces, and the deformed twistor space is the space of null geodesics for the induced "hypersurface twistor" conformal connexion.

§3.4 SPACES OF TORSION-FREE NULL GEODESICS IN
THREE DIMENSIONS AND SOLUTIONS OF EINSTEIN'S EQUATIONS

Let M be a complex 3-fold with civilized conformal structure (torsion-free conformal connexion), and let $N(M)$ be its space of torsion-free null geodesics. We have seen (§2.5) that $N(M)$ carries a contact structure such that the annihilator distribution of the contact form is tangent to each \mathbb{P}_1 corresponding to a point of M , and this contact structure removes (at least locally) the nagging ambiguity we formerly witnessed in reconstructing M from $N(M)$. But more importantly, this contact form is precisely the structure needed to construct a metric satisfying Einstein's equations with cosmological constant 1 on the complement of M in the previously fabricated ambient 4-fold with half-flat conformal structure, in which M sits as the general umbilic (i.e. conformally totally geodesic) hypersurface; M is the conformal infinity for this Einstein metric. Conversely, the twistor space of a half-conformally-flat 4-fold with Einstein metric and non-vanishing scalar curvature carries a contact structure, and tangency to the annihilator distribution of the contact form defines a (possibly empty) 3-fold with torsion-free conformal connexion that is the conformal infinity of the given Einstein metric. (The connexion between half-conformally-flat Einstein metrics and contact forms on the twistor spaces was noticed independently by Nigel Hitchin and by Richard Ward; in both cases this work remains unpublished).

First, let us see that if X is a complex 4-fold of \mathbb{P}_1 's in a complex 3-fold N with contact form $\theta \in H^0(N, \Omega^1(K^{-\frac{1}{2}}))$, and if each curve of the family X has normal bundle $2H$, then the set S of curves of X which lie tangent to the annihilator of θ constitutes an umbilic hypersurface in X . First notice that S is given by the vanishing of a holomorphic section of a line bundle over X ; for if $\pi: B \rightarrow X$ is the \mathbb{P}_1 bundle of self-dual 2-plane elements tangent to X , and if $q: B \rightarrow N$ takes a 2-plane element to the tangent

twistor surface, then $q^*\theta$ defines a global holomorphic section of the line bundle $\Omega_{\pi}^1(q^*K^{-\frac{1}{2}}) \rightarrow B$ of twisted 1-forms restricted to the fibres of π ; but the sheaf over X given by $U \rightarrow H^0(\pi^{-1}(U), \Omega_{\pi}^1(q^*K^{-\frac{1}{2}}))$ is locally free over \mathcal{O} , and in fact, for suitably small U one has $H^0(\pi^{-1}(U), \Omega_{\pi}^1(q^*K^{-\frac{1}{2}})) \cong \mathcal{O}_U \otimes_{\mathbb{C}} H^0(\mathbb{P}_1, \mathcal{O}) \cong \mathcal{O}_U$, so θ defines a section of a line bundle over X which vanishes precisely on S . (Notice that since each curve of the family X has normal bundle $2H$, it follows that $K^{-\frac{1}{2}}$ restricted to such a curve is given by $(T^*\mathbb{P}_1 \otimes \Lambda^2 2H)^{\frac{1}{2}} \cong H^2$. Since the cotangent bundle of \mathbb{P}_1 is isomorphic to H^{-2} , it follows that $\Omega_{\pi}^1(q^*K^{-\frac{1}{2}})$ is trivial on each fibre of π). Moreover, this section vanishes in a non-singular fashion and has non-null normal; for if γ is a \mathbb{P}_1 tangent to the annihilator of θ , θ defines an element of the dual of the normal bundle with values in $K^{-\frac{1}{2}}$ of the curve *which never vanishes*, and, assuming the normal bundle is isomorphic to $2H$, this is an element of $H^0(\mathbb{P}_1, 2\mathcal{O}(-1) \otimes \mathcal{O}(2)) = 2H^0(\mathbb{P}_1, \mathcal{O}(1))$, which we identify with a pair (λ_A, π_A) of spinors (just elements of \mathbb{C}^2 here), and the condition that the section never vanishes is just that λ and π are linearly independent, which is to say that $x_{AA'} = (\lambda_A, \pi_A)$ satisfies $\epsilon^{AB} \epsilon^{A'B'} x_{AA'} x_{BB'} \neq 0$ for any non-zero skew epsilons; now any section of the normal bundle $2H$ which vanishes somewhere, and so can be written as $y^{AA'} = (\xi^A, \xi^A) \in 2H^0(\mathbb{P}_1, \mathcal{O}(1))$, which satisfies $y^{AA'} x_{AA'} \neq 0$ represents the first derivative of a 1-parameter family of \mathbb{P}_1 's which is initially tangent to the annihilator distribution of θ but is transverse to it at the first order, showing that $S \subset X$ is indeed non-singular; and the non-degenerate cone of null vectors $y^{AA'} = (\eta^A, \eta^A) \in 2H^0(\mathbb{P}_1, \mathcal{O}(1))$ such that $y^{AA'} x_{AA'} = 0$ lies tangent to $S \subset X$, since 1-parameter families of \mathbb{P}_1 's with these initial derivatives are tangent, to first order, to the annihilator distribution of θ . It follows that S is a nowhere-null hypersurface, and since the annihilator distribution of θ , thought of as a distribution on the space of "hypersurface twistor" null geodesics of S , lies tangent to each \mathbb{P}_1 representing a point of S , it follows by (2.5.I) and the analysis in §3.1 that S is umbilic.

The form $\theta \in H^0(N(M), \Omega^1(K^{-\frac{1}{2}}))$ does a great deal more for us, however. We take θ to be normalized so that $\theta \wedge d\theta \in H^0(\mathcal{O}) = \mathbb{C}$ is 1, and then define a metric on $X - M$, where X is the half-flat ambient 4-fold of §3.1, as follows: on the $\mathbb{P}_1 \gamma_x$ corresponding to any point $x \in X - M$, we notice the fact that $\theta|_{T\gamma_x}$ is everywhere non-vanishing and view θ as an isomorphism between $T\gamma_x$ and $K^{-\frac{1}{2}}|_{\gamma_x}$, which in turn determines (up to sign) an isomorphism of $(T\gamma_x)^{\frac{1}{2}}$ and $K^{-\frac{1}{4}}|_{\gamma_x} \simeq H$ which is the "square root" of the isomorphism θ . Now the annihilator distribution D of θ is everywhere transverse to γ_x , and so $D|_{\gamma_x}$ can be identified with the normal bundle of γ_x ; now we define two 2-dimensional *spinor bundles* \mathcal{S} and \mathcal{S}' over $X - M$ which come equipped with an isomorphism $\psi: \mathcal{S} \otimes \mathcal{S}' \cong T^*X$ (having the property that simple products correspond to null vectors) by $\mathcal{S}_x := H^0(\gamma_x, \mathcal{O}(D \otimes K^{\frac{1}{4}}))$, $\mathcal{S}'_x := H^0(\gamma_x, \mathcal{O}(K^{-\frac{1}{4}}))$, ψ being the natural isomorphism: $H^0(\gamma_x, \mathcal{O}(D \otimes K^{\frac{1}{4}})) \otimes H^0(\gamma_x, \mathcal{O}(K^{-\frac{1}{4}})) = H^0(\gamma_x, \mathcal{O}(D)) = T^*_x X$, since $D \otimes K^{\frac{1}{4}}|_{\gamma_x}$ is trivial. Now $d\theta$ defines a non-degenerate volume element on D with values in $K^{-\frac{1}{2}}$, and so a non-degenerate volume element on $D \otimes K^{\frac{1}{4}}$ with values in \mathbb{C} ; since this element is holomorphic it defines a volume element on $\mathcal{S}_x := H^0(\gamma_x, \mathcal{O}(D \otimes K^{\frac{1}{4}}))$ for each x . Meanwhile, $\mathcal{S}'_x := H^0(\gamma_x, \mathcal{O}(K^{\frac{1}{4}}))$ carries a volume element, too - the *Wronskian*; for we've identified $K^{\frac{1}{4}}|_{\gamma_x}$ with $(T\gamma_x)^{\frac{1}{2}}$ via " $\theta^{\frac{1}{2}}$ ", and this converts the skew form

$$W: 2(K^{\frac{1}{4}}) \rightarrow \Omega^1(K^{\frac{1}{2}})$$

$$(\chi, \omega) \rightarrow \omega d\chi - \chi d\omega$$

(which is well defined, since a change of local trivialization of $K^{\frac{1}{2}}$ by multiplication by a non-vanishing holomorphic function f results in the new expression

$$\begin{aligned} W(\chi, \omega) &= f^{-2} [f\chi \, d(f\omega) - f\omega \, d(f\chi)] \\ &= \omega \, d\chi - \chi \, d\omega \end{aligned}$$

into a skew form with values in \mathbb{C} . The product of these non-zero skew forms on \mathcal{S} and \mathcal{S}' defines a non-degenerate metric on $X - M$ which lies in the conformal class with which X came equipped. I claim that this metric satisfies *Einstein's equations* (with cosmological constant):

$$R_{ab} = \frac{1}{4} R g_{ab}$$

where R_{ab} is the Ricci tensor R_{acb}^c and where R is the curvature scalar $g^{ab} R_{ab}$. (Bianchi's identities then imply that R is a constant, which can be made any non-zero ^{number} we like by multiplying g by a constant; this choice of "cosmological constant" corresponds to the choice in normalization of θ).

Because $D \otimes K^{\frac{1}{4}}$ is trivial on each \mathbb{P}_1 of the family $X - M$, there is a natural connexion on \mathcal{S} , having the property that the connexion is *flat on any twistor surface*, defined on any twistor surface α by taking a spinor $\lambda \in \mathcal{S}_x$ to be obtained by parallel transport within a twistor surface α from a spinor $\tilde{\lambda} \in \mathcal{S}_{\tilde{x}}$, where $x, \tilde{x} \in \alpha$, iff the corresponding sections of $D \otimes K^{\frac{1}{4}}|_{\gamma_x}$ and $D \otimes K^{\frac{1}{4}}|_{\gamma_{\tilde{x}}}$ agree at the point $\gamma_x \cap \gamma_{\tilde{x}}$ (i.e. at the twistor α); this does, indeed, define a unique connexion, since the "horizontal lifts" so defined of null vectors to some spinor $\pi \in \mathcal{S}$ must all lie within a (unique) 4-plane because a section of $m(\alpha, 1)$ on a quadric in \mathbb{P}_n extends uniquely to a section of $m(\alpha, 1)$ over the whole of \mathbb{P}_n via the usual exact sequence. (This construction of a self-dual connexion is a special case of the Ward construction [26] of self-dual Yang-Mills fields). Now this connexion preserves the volume element ϵ on \mathcal{S} induced by $d\theta$ because, by construction, $V \lrcorner \nabla \epsilon = 0$ for all null vectors V , where ∇ denotes the constructed connexion, and hence $\nabla \epsilon = 0$. The other spin bundle also has a natural connexion, since the realization of the normal bundle to γ_x as $D|_{\gamma_x}$ gives us a way, roughly speaking, of identifying curves of the family which are first-order separated, and thus identifying the square roots of their tangent bundles and hence, via θ , identifying the associated spin spaces; this connexion will have non-vanishing curvature everywhere, since the integrability obstruction $d\theta$ of

D is everywhere non-degenerate. This connexion also preserves the volume element constructed earlier, since the construction respects the Wronskian.

These connexions on the spin bundles define a metric-preserving affine connexion on M via the Leibnitz^{rule}. Remarkably enough, this connexion is torsion-free. To see this, notice that torsion resides on the first order neighbourhood of a point of M - it does not involve any derivatives of the connexion. Now we can identify the first-order neighbourhood of any two curves with isomorphic normal bundles; moreover, we can choose to do this in such a way as to preserve given holomorphic splittings of the ambient tangent bundles into transverse and tangent parts. Now the Griffiths obstruction [31] to the uniqueness of the extension of a holomorphic vector bundle E specified on a submanifold S to the first-order neighbourhood of that submanifold lies in $H^1(S, \mathcal{O}(E \otimes E^* \otimes N^*))$, where N is the normal bundle to S ; in our case, this means that the extensions of the line bundle $K^{-\frac{1}{4}}$ and the rank-2 sub-bundle D of the ambient tangent bundle to the first order neighbourhood of a \mathbb{P}_1 with normal bundle $2H$ are unique, since

$$H^1(\mathbb{P}_1, \mathcal{O}(H \otimes H^{-1} \otimes 2H^{-1})) = 2H^1(\mathbb{P}_1, \mathcal{O}(-1)) = 0$$

and

$$H^1(\mathbb{P}_1, \mathcal{O}(2H \otimes 2H^{-1} \otimes 2H^{-1})) = 8H^1(\mathbb{P}_1, \mathcal{O}(-1)) = 0.$$

Now since the value at x of the connexion on $\$$ depends only on the first-order extension of $D \otimes K^{\frac{1}{4}}$ about γ_x , while the value at x of the connexion on $\tilde{\$}$ depends only on the first extension of $K^{-\frac{1}{4}}$ about γ_x and upon the splitting of the ambient tangent restricted to γ_x , as determined by D , it follows that all the affine connexions defined by our rules are torsion-free provided that at least one is; we shall see before long that this is indeed the case, but simply for the moment assume that it is true.

Now it follows immediately from the spinor calculus that the metrics

that arise through the outlined construction satisfy Einstein's equations with cosmological constant, since the unprimed spin bundle is flat on every twistor surface with respect to the torsion-free connexion, which has curvature

$$\Phi_{BC'D'}^A \epsilon_{CD}$$

where [23] Φ_{ab} is one-fourth the trace-free part of the Ricci curvature. (The interested reader may wish instead to deduce the pure-trace nature of the Ricci curvature from the fact that

$$X \wedge Y = - * X \wedge Y \Rightarrow R(X, Y) X = 0$$

because there also exist solutions of the equation

$$\pi^{A'} \nabla_{AA'} \pi^{B'} = 0$$

on any 4-fold with self-dual conformal curvature; see below). And since the connexion on the projectivized primed spin-bundle is the pull-back of the contact form via the quotient map to the twistor space, it follows that the curvature (i.e. Frobenius integrability obstruction) of the connexion on the primed spin-bundle vanishes nowhere, since $\theta \wedge d\theta \neq 0$, and so the (constant) scalar curvature of the constructed metric is always non-zero. Moreover, it is not difficult to see that there exists a universal constant by which our definition of the Wronskian could be multiplied to insure that the cosmological constant is 1.

Now on any 4-fold with self-dual conformal structure every twistor surface comes naturally equipped with an autoparallel "tangent" spinor field $\pi^{A'}$, determined up to an overall constant factor, so that the tangent space to the twistor surface consists of vectors of the form

$$V^{AA'} = \lambda^A \pi^{A'}$$

and such that

$$\pi^{A'} \nabla_{AA'} \pi_{B'} = 0.$$

(We have written $\pi_{B'}$ with a lower index because the equation is conformally invariant in this form). This follows immediately from the expression [23]

$$\Phi^{A'} B' CD \epsilon_{C'D'} + \Lambda \epsilon_{(C'}^{A'} \epsilon_{D')} B' \epsilon_{CD}$$

for the curvature of the primed spin-bundle when the conformal curvature is self-dual. This defines a standard line bundle E over the twistor space, with restriction to any "celestial sphere" isomorphic to $H^{-1} \rightarrow \mathbb{P}^1$, whose fibre over any point consists of covariantly auto-parallel "tangent" spinor fields on the corresponding twistor surface. (The Λ in the above expression is $R/24$, where R is the scalar curvature). If our 4-fold has an Einstein metric, this gives the twistor space the additional structure of a non-vanishing holomorphic 1-form θ with values in E^* . To see this, we need to find an analogue of Jacobi's equation, so as to be able to realize the tangent space to the twistor space as a space of fields defined on a twistor surface.

Since every (torsion-free) null geodesic lies in a unique twistor surface, a Jacobi field $J^{AA'}$ defined along a null geodesic with tangent $X^{AA'} = \mu^A \pi^{A'}$ and satisfying

$$X^a X_b \nabla_a J^b = 0$$

defines a unique tangent vector on the twistor space; two such fields define the same vector iff their difference is tangent to the twistor surface. Thus the unprimed spinor field

$$\omega^A = J^{AA'} \pi_{A'}$$

which satisfies

$$\mu_A \mu_B \pi^{B'} \nabla_{B'} \omega^A = 0$$

completely determines the tangent vector on the twistor space; moreover, the value of ω^A is independent of our original choice of the null geodesic within the given twistor surface, and so we in fact have a field ω^A defined over the entirety of the twistor surface which satisfies the twistor equation (cf. [22])

$$\pi^{B'} \nabla_{B'} (\omega^A) = 0.$$

We may identify a twistor surface with (a neighbourhood of the origin in) the unprimed spin-space at some point upon it via $\lambda^A \rightarrow \exp(\lambda^A \pi^{A'})$, and when the Ricci curvature is of pure-trace type this identifies vector fields on the spin-space, with their canonical flat connexion, with unprimed spinors on the twistor surface, with the metric connexion; the solutions of the twistor equation are then explicitly given by

$$\{\omega^A|_{\lambda^A} := \omega^A|_0 + \alpha \lambda^A\}, \alpha \in \mathbb{C}$$

and one sees that $\pi^{A'} \nabla_{AA'} \omega^A = \pi_{A'} \pi_{B'} \nabla_A^{A'} J^{AB'}$ is constant, and thus defines a holomorphic 1-form on the twistor space with values in E^{*2} .

(The reader may also deduce the constancy of $\pi_{A'} \pi_{B'} \nabla_A^{A'} J^{AB'}$ from Jacobi's equation given the vanishing of the trace-free Ricci curvature). This twisted 1-form never vanishes. The annihilator distribution of this 1-form at some point in the twistor space corresponds precisely to the covariantly constant spinor fields ω^A on the corresponding twistor surface.

Now if we consider the quotient map $q: \mathbb{P}\mathcal{S}' \rightarrow \mathcal{T}$ from the projectivized primed spin bundle to the twistor space which takes a spinor to the tangent twistor surface, we notice that the Jacobian of this map takes horizontal vectors to vectors in the annihilator distribution of the above 1-form. For if we consider a 1-parameter family of null geodesics with tangents $\mu^A \pi^{A'}$ connected by a vector field $J^{AA'}$ we have

$$J^{AA'} \nabla_{AA'} \mu^B \pi^{B'} - \mu^A \pi^{A'} \nabla_{AA'} J^{BB'} = 0$$

and so

$$J^{AA'} \nabla_{AA'} \pi^{B'} = 0$$

implies that

$$\mu^A \pi^{A'} \pi_{B'} \nabla_{AA'} J^{BB'} = 0.$$

Thus the integrability obstruction of the constructed 1-form fails to vanish precisely if the scalar curvature is non-zero. This reverses the original construction, and verifies that the connexion constructed in it was in fact torsion-free. Thus we have the

Theorem (3.4.I). Every holomorphic three-dimensional conformal structure is the umbilic conformal infinity of a unique holomorphic solution of Einstein's equations with cosmological constant 1 and self-dual conformal curvature, where by conformal infinity we indicate that the conformal structure is regular across the hypersurface but that the metric has a simple pole there. Conversely, every solution of Einstein's equations with non-zero cosmological constant and self-dual conformal curvature has a (possibly empty) non-null umbilic hypersurface as its conformal infinity.

This provides a cosmological constant analogue of Newman's "heaven" construction [27]. As some of the properties of our construction are rather nicer than in the case of vanishing scalar curvature (due to the non-null nature of the conformal infinity in the present case) it may be hoped that this analogue may shed some light on the original construction.

§3.5 HYPERSURFACE TWISTOR CR MANIFOLDS

We have now seen that a pair of holomorphic conformal fundamental forms on a complex 3-manifold gives rise, in a natural way, to the construction of an auxiliary 3-complex-dimensional *twistor space* by which one realises the fundamental forms as those of a hypersurface in a complex 4-manifold with half-flat conformal structure. It will now be shown that the specification of a *smooth* (i.e. C^∞ ; let's not quibble) pair of complex conformal fundamental forms on a *smooth* 3-manifold gives rise to a CR 5-manifold of "twistors", and this CR 5-manifold can be embedded if these fundamental forms can, in a suitable sense, be realised as those of a smooth submanifold of a complex 4-manifold with half-flat conformal structure. This latter fact will then be used to show that this geometric embedding problem has, in general, no solution, even though it admits one whenever the "data" are real analytic (as follows by analytic continuation and the application of our results from §3.1). Meanwhile, we shall meet a rather pleasing correspondence theorem that tells us precisely which abstract CR manifolds arise via this construction when the first fundamental form is required to have no real null vectors; and we can also identify the abstract CR manifolds corresponding to *real* data.

Let us first begin by making precise the meanings attached to the terms we've been bandying about. First of all, we wish to have a notion of first and second fundamental forms for an appropriate class of n -dimensional smooth submanifolds of a complex $(n+1)$ -manifold. The "appropriate class" is that of *totally real* submanifolds; a smooth submanifold $R \subset M$ of a complex manifold is said to be *totally real* if $\mathcal{C}TR \cap T^*M = 0_\Sigma$, the zero section. (This is the smooth generalisation of a "real slice" of a complex submanifold; the condition on the tangent bundle to the submanifold may also be written as $J[TR] \cap TR = 0$ - that is, "multiplication by i " takes the tangent plane to a transverse subspace). Associated with a *totally real*

submanifold $R \subset M$ of a complex manifold is a *hypertangent bundle* $\tilde{TR} \subset T^*M|_R$, defined to be the smallest complex sub-bundle containing every vector whose real part lies tangent to R , and a canonical isomorphism $\varphi: \mathbb{C}TR \rightarrow \tilde{TR}$ of complex vector bundles such that $\text{Re} \circ \varphi|_{TR} = \text{id}_{TR}$. (In terms of the almost complex structure J , $\tilde{TR} = \{V - iJV | V \in \mathbb{C}TR\}$, and φ is given by $V \rightarrow V - iJV$).

Now suppose that $R \subset M$ is a totally real submanifold of a complex manifold with holomorphic metric g . Then the *first fundamental form* of R is the symmetric form φ^*g on $\mathbb{C}TR$. We say that R is *non-null* if this form is non-degenerate, and *pseudo-spacelike* if no null vector (null with respect to φ^*g) is real. The *conformal first fundamental form* of R is just the equivalence class of the first fundamental form under complex multiplication and is characterised by the quadric sub-bundle of $\mathbb{P}\mathbb{C}TR$ consisting of null directions; if the fibres of this sub-bundle are all non-singular quadrics then R is non-null, and if this sub-bundle never meets $\mathbb{R}TR \subset \mathbb{P}\mathbb{C}TR$ then R is pseudo-spacelike. Finally, the first fundamental form is said to be real if its restriction to TR is a real form, and this conformally equivalent to the invariance of the null quadrics under complex conjugation; naturally enough, a real pseudo-spacelike submanifold is called *spacelike* or *definite*.

Now suppose further that $R \subset M$ is a totally real non-null submanifold, and that $\dim_{\mathbb{C}} M = 1 + \dim R$. Then we define the *second fundamental form* is the quadratic form on $\mathbb{C}TR$ given by

$$II(x,y) := \frac{i}{2} g(\varphi(x), Y \lrcorner \hat{\nabla} V)$$

where V is one of the two possible smooth sections of $T^*M|_R$ such that $g(V,V) \equiv 1$ and such that $g(V,Z) = 0$ for every $Z \in \tilde{TR}$, and where $\hat{\nabla}$ is the underlying smooth connexion of the holomorphic torsion-free metric connexion ∇ ; to see that this is well-defined, notice that $Y \lrcorner \hat{\nabla} V$ is a smooth section of $T^*M|_R$. II changes sign when V is replaced with the other unit normal

field, $-V$. The *conformal second fundamental form* of R is the trace-free part \hat{II} of II , with conformal weight 1. (If you like, the actual object is the corresponding totally symmetric 4-spinor $\hat{II}_{ABC}{}^D$ with one raised index; this is conformally invariant).

Notice that if $S \subset M$ is a complex hypersurface of a complex manifold with holomorphic metric, and if $R \subset S$ is a real slice (i.e. R is a real analytic submanifold of half the real dimension of S) then the second fundamental form (and more obviously, the first fundamental form) of S , restricted to R , is, via φ , just the second (respectively, first) fundamental form of $R \subset M$. This is true (in the case of the second fundamental form) because for $Y \in TR$ and $Z \in \tilde{TR}$ such that $y = \frac{1}{2}(Z + \bar{Z})$, one has $Y \lrcorner \hat{\nabla} V = \frac{1}{2}(Z + \bar{Z}) \lrcorner \nabla V = \frac{1}{2} Z \lrcorner \nabla V = \frac{1}{2} \varphi(y) \lrcorner \nabla V$. Therefore, any two analytic fundamental forms specified on \mathbb{R}^3 may be conformally realised, in a local unique way, as those of a real analytic totally real non-null half-flat conformal structure; this follows immediately by analytic continuation to a region of \mathbb{C}^3 and §3.1.

Now consider a smooth totally real non-null 3-manifold R in a complex 4-manifold M with half-flat conformal structure. The twistor surfaces meeting R transversely form a real hypersurface Σ in the twistor space of any suitably small region $U \subset M$ which meets R ; this is true because any local (holomorphic) foliation of M by twistor surfaces meeting R transversely and uniquely if at all has precisely a smooth 3-manifold of leaves meeting R , and such a foliation corresponds to a generic complex hypersurface in the twistor space. We shall now manufacture an abstract model of the CR 5-manifold we've given as a hypersurface Σ in the twistor space.

Let $Q \subset \mathbb{P}\mathbb{C}TR$ be the submanifold consisting of *non-real* complex null directions with respect to the (conformal) first fundamental form of R , and let D_1 be the 1-dimensional complex distribution of anti-holomorphic

tangents to the fibres of the map $Q \rightarrow R$ which is the restriction of the canonical projection $\mathcal{P}\mathcal{C}TR \rightarrow R$. Despite the fact that the metric (first fundamental form) is complex, the classical Levi-Civita formula defines the unique torsion-free connexion ∇ on the complex tangent bundle $\mathcal{C}TR$ preserving the metric, and we define a torsion connexion $\hat{\nabla}$ preserving the first fundamental form by adding, for some 1-form α , the defect

$$\Delta_{jk}^i = e_{j\ell}^i II_k^\ell + \alpha^i g_{jk} - \alpha_j \delta_k^i$$

where II is the second fundamental form and where e is the metric (complex) volume element; by letting II range over all smooth quadratic forms this gives the most general smooth torsion metric connexion. There is a one-dimensional distribution on $\mathcal{C}TR$ spanned at the complex vector V by the horizontal lift of V with respect to $\hat{\nabla}$, and this distribution is tangent to the null cone (when V is null); restricting this distribution to the null cone and projectivizing, we get a 1-dimensional complex distribution D_2 on Q , and since we've excised the real null directions (if there are any) it follows that D_2 is nowhere real ($D_2 \cap TQ = 0$), as is D_1 . We now define a 2-dimensional complex distribution H on Q by $H := D_1 + D_2$. Notice that this depends only on the conformal class of the fundamental forms. *I claim that H is a CR structure on Q , and that Q is CR-isomorphic to Σ in the case that the given fundamental forms can be realised for a totally real 3-manifold in a complex 4-manifold with half-flat conformal structure.*

First let's notice that H contains no real vectors, since $D_1 \cap D_2 = D_1 \cap \bar{D}_1 = D_2 \cap \bar{D}_2 = 0$. Now we must see that H is integrable. Take an orthonormal frame $\{V_1, V_2, V_3\}$ for $\mathcal{C}TR$, and define a complex coordinate ζ on Q by

$$\zeta \rightarrow [(1 + \zeta^2)V_1 + i(1 - \zeta^2)V_2 + 2i\zeta V_3]$$

which covers every null direction (including, actually, the real null

directions we've removed) except $V_1 - iV_2$ (which corresponds to $\zeta = \infty$); this ζ is painted on all fibres of $Q \rightarrow R$ and we thus identify Q with $\mathbb{P}_1 \times R$ (minus isolated points corresponding to real null directions).

The orthonormal frame is covariantly constant for some torsion metric connexion, say corresponding to Π_0 ; define, for convenience, $\tilde{\Pi} = \Pi - \Pi_0$. Now $\zeta = \text{constant}$ is canonically identified with R via the canonical projection, and we define a vector field w on Q by requiring that it be tangent to the foliation $\zeta = \text{constant}$ and be given for any ζ by

$$w = (1 + \zeta^2)V_1 + i(1 - \zeta^2)V_2 + 2i\zeta V_3$$

under the said identification. Then

$$H = \text{span} \left\{ \frac{\partial}{\partial \bar{\zeta}}, w + \frac{1}{2} \tilde{\Pi}(w, w) \frac{\partial}{\partial \zeta} \right\}$$

which we see as follows: clearly, all we need to check is that the vertical part of the horizontal lift of w_0 at w_0 is given, modulo $\partial/\partial \bar{\zeta}$, by $\tilde{\Pi}(w, w) \frac{\partial}{\partial \zeta}$; since we are uninterested in the antiholomorphic part we can work in $T \uparrow T_x R$, $x \in R$, instead of in $\mathbb{C}T \uparrow T_x R$ and then identify this former space with $\mathbb{C}T_x R$ in the obvious fashion, now being saved from the horror of having two kinds of "i's" floating around; now the horizontal lift of w_0 to $w_0 \in \mathbb{C}TR$ has vertical part, relative to trivialization induced by the chosen frame, given by $-w \times (\tilde{\Pi} \lrcorner w)$, where the " \times " is the familiar 3-vector product induced by the metric and volume element from the exterior product; we're looking for α such that $\frac{\partial}{\partial \zeta} \equiv -w_0 \times (\tilde{\Pi} \lrcorner w_0) \text{ mod } w_0$, which is to say that

$$\begin{aligned} \alpha \frac{\partial}{\partial \zeta} \times w &= w \times (w \times (\tilde{\Pi} \lrcorner w)) \\ &= \tilde{\Pi}(w, w) w - g(w, w) (\tilde{\Pi} \lrcorner w) \\ &= \tilde{\Pi}(w, w) w; \end{aligned}$$

but

$$\begin{aligned}
\frac{\partial}{\partial \zeta} \times w_0 &= 2[\zeta e_1 - i\zeta e_2 + ie_3] \times [(1 + \zeta^2)e_1 + i(1 - \zeta^2)e_2 + 2i\zeta e_3] \\
&= 2[(1 + \zeta^2)e_1 + i(1 - \zeta^2)e_2 + 2i\zeta e_3] \\
&= 2w.
\end{aligned}$$

And so $\alpha = \frac{1}{2} \tilde{I}\tilde{I}(w_0, w_0)w_0$. But now the integrability is easy; it follows immediately from the observation that

$$\left[\frac{\partial}{\partial \bar{\zeta}}, w + \frac{1}{2} \tilde{I}\tilde{I}(w, w) \frac{\partial}{\partial \zeta} \right] = 0.$$

While we're at it, we can see that the Levi form of this abstract CR structure is of type (1,1). Since $[\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}}] = 0$, it follows that the Levi form is not positive definite; on the other hand

$$\left[w + \frac{1}{2} \tilde{I}\tilde{I}(w, w) \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}} \right] = \left[w, \frac{\partial}{\partial \bar{\zeta}} \right] \text{ mod } H + \bar{H} = -\frac{\partial w}{\partial \bar{\zeta}} \notin H + \bar{H}$$

so the Levi form is not zero, and we can find a second null direction by adjusting the function α in

$$\begin{aligned}
&\left[w + \frac{1}{2} \tilde{I}\tilde{I}(w, w) \frac{\partial}{\partial \zeta} + \alpha \frac{\partial}{\partial \bar{\zeta}}, \bar{w} + \frac{1}{2} \tilde{I}\tilde{I}(w, w) \frac{\partial}{\partial \bar{\zeta}} + \alpha \frac{\partial}{\partial \zeta} \right] = \\
&\left[w, \bar{w} \right] + \alpha \frac{\partial \bar{w}}{\partial \bar{\zeta}} - \bar{\alpha} \frac{\partial w}{\partial \zeta} \text{ mod } H + \bar{H}.
\end{aligned}$$

Now let us see that H (or rather \bar{H} ; we chose to work with H because of the smaller number of complex conjugations) is the induced CR structure on Σ when the fundamental forms can be realised for a totally real 3-manifold R in a complex 4-fold with half-flat conformal structure. Recall that the diffeomorphism set up between $Q \subset \mathcal{P}\mathcal{C}TR$ and Σ took a complex null direction $[V]$ to the twistor surface to which $\varphi(V)$ is tangent; if we look at the set of twistor surfaces through a point of R , we get a \mathcal{T}_1 in the twistor space which is contained (except perhaps at isolated excised points, corresponding to twistor surfaces tangent to R) in Σ , and this curve

corresponds holomorphically to the conic curve of complex null directions tangent to R at that point, showing that our inclusion of the anti-holomorphic tangent spaces to the fibres of Q into the induced conjugate CR structure is in fact correct. If we let $\mathcal{B} \rightarrow M$ denote the \mathbb{P}_1 bundle over M whose fibre at a point $x \in M$ consists of the self-dual 2-planes in the holomorphic tangent space to x , we can identify \mathcal{B}/R with Q , since a self-dual 2-plane at a point x of R contains just one (null) direction in $\tilde{T}R$; there is a canonical projection $\mathcal{B} \rightarrow \mathcal{J}$ to the twistor space of M taking a self-dual 2-plane in a tangent space to the unique twistor surface to which it is tangent, and this projection takes Q onto Σ via the previously defined diffeomorphism; the kernel of the Jacobian of this map at a point y of \mathcal{B} consists of the horizontal lifts to \mathcal{B} of tangent vectors of M which lie in the tangent 2-plane represented by y . Letting \tilde{Q} denote the null-quadratic sub-bundle of $\mathbb{P}\tilde{T}R$ (i.e. Q carried over by φ), letting $\pi: \tilde{Q} \rightarrow \Sigma$ be the canonical projection induced by the projection $\mathcal{B} \rightarrow \mathcal{J}$, and letting J and \tilde{J} denote, respectively, the almost complex structure tensors of \mathcal{B} and \mathcal{J} , we see that for any $v, w \in T\tilde{Q}$, one has $\tilde{J}(\pi_*v) = \pi_*w$ iff $Jv - w \in \text{KER } \pi_*$. Now since the connexion on M is holomorphic, the horizontal subspaces of \mathcal{B} are J -invariant, so that the horizontal lift \tilde{u} of a complex null vector $u \in \mathbb{C}TR$ to $[\varphi(u)] \in \tilde{Q}$ satisfies $\pi_*(\tilde{u} - iJ\tilde{u}) = 0$ because $\tilde{u} - iJ\tilde{u}$ is the horizontal lift of $u - iJu = \varphi(u)$; so $\pi_*(\tilde{u}) = i\tilde{J}[\pi_*(u)]$, and so $\pi_*(\tilde{u})$ is in the anti-holomorphic tangent space to \mathcal{B} , as claimed. (Notice that the horizontal subspaces considered are just those of the hypersurface twistor connexion, since \mathcal{B} is the bundle of self-dual 2-planes, which we may consider as sitting in the projectivized self-dual 2-forms). Thus our abstract CR structure agrees with (the complex conjugate of) the concrete one.

In particular, if the conformal first fundamental form is pseudo-spacelike (i.e. if there are no real null vectors) then we've constructed

a corresponding CR 5-manifold with non-degenerate Levi-form and a foliation by \mathcal{P}_1 's. Conversely, we have the rather remarkable

Proposition (3.5.1). Every smooth CR 5-manifold with non-degenerate Levi-form and foliation by \mathcal{P}_1 's arises from this construction for a (unique) pair of conformal fundamental forms, of which the first is pseudo-spacelike.

Proof. Let \bar{X} be the CR 5-fold, and let $H \subset \mathbb{P} \mathbb{C} T \bar{X}$ be its CR tangent space; let S be the 3-fold of leaves of the foliation by \mathcal{P}_1 's; we assume that our conventions are such that $H \cap \mathbb{P} \mathbb{C} T \mathcal{P}_1$ is the *anti-holomorphic* tangent space for any leaf. Over *each* leaf there is a line bundle $E := H/T\mathcal{P}_1$, which, by the non-degeneracy of the Levi-form (making it dual to the holomorphic tangent bundle), has Chern class -2; this line bundle has a natural structure as a *holomorphic* line bundle over the leaf, which comes from defining a local section f of E in a region of \bar{X} to be holomorphic on each leaf if

$$[v, w] = 0$$

for all smooth sections v of the anti-holomorphic tangent bundles to the fibres and vector fields w representing $f \pmod{v}$, all of which is independent of representatives and satisfies the necessary and sufficient condition for the acceptability of such a definition that multiplication of a "local holomorphic section" by a holomorphic function gives another "local holomorphic section". Now the quotient map $\text{pr}: \bar{X} \rightarrow S$ defines a complex curve in the projectivized complexified tangent space $\mathbb{P} \mathbb{C} T_x S$ of any point $x \in S$ consisting of the image of H under pr_* , which is also the image of E ; but in fact, this curve is the *holomorphic* image of the \mathcal{P}_1 which is the leaf over x , because, for any representatives v and w as above, one has

$$v(\text{pr}_* w) = \text{pr}_* \mathcal{L}_v w = \text{pr}_* [v, w]$$

where the expression on the left is to be interpreted as the ordinary vector space derivative. Furthermore, pr_* carries E onto the restricted Hopf bundle of this holomorphic plane curve, and since E has Chern class -2 it follows that the curve is a conic. We define the *first conformal fundamental form* by taking its null quadric to be this curve; notice that this complex conformal metric is pseudo-spacelike, since H would otherwise include a real vector.

We now identify \bar{X} with the bundle of complex null directions and remark that we've now *two* CR structures on \bar{X} : H , and a second corresponding to our earlier construction, taking the second conformal fundamental form to be nought. The difference between the two defines a linear map from the Hopf bundle over the bundle of null directions to the holomorphic tangent bundles of the fibres of pr ; since both CR distributions include the anti-holomorphic fibre-tangents and are involutory, it follows that this map is holomorphic. Such holomorphic maps form a five-dimensional vector space:

$$H^0(\mathbb{P}_1, \mathcal{O}(T\mathbb{P}_1 \otimes (H^{-2})^*)) \cong H^0(\mathbb{P}_1, \mathcal{O}(4)).$$

But the trace-free symmetric forms sit in this space in the fashion used in our earlier construction, and a dimension count shows that they fill it. So the construction produces all such CR manifolds with foliation. \square

Let us emphasise the fact that, even for non-pseudo-spacelike first fundamental forms, the CR structure and the foliation corresponding to projection to the original 3-fold contain the full information of the conformal fundamental forms, as is clear from inspection.

We conclude this detour by characterising those CR manifolds corresponding to real (positive) definite first conformal fundamental form and real second fundamental form.

Proposition (3.5.II). A smooth CR 5-manifold \bar{X} with non-degenerate Levi-form and foliation by \mathcal{P}_1 's corresponds to a *real* pair of conformal fundamental forms (the first of which is (positive) definite) iff there is an anti-CR (cf. "anti-holomorphic") map $\alpha: \bar{X} \rightarrow \bar{X}$ preserving the foliation, such that $\alpha^2 = 1$.

Proof. Let us make clear that a map between CR manifolds is *anti-CR* if the Jacobian of the map takes the CR distribution onto the complex conjugate of the CR distribution.

First, suppose we carry out our construction of a CR manifold corresponding to real data on a 3-fold S with the conformal metric positive definite. Then the map $\rho: \mathbb{C}TS \rightarrow \mathbb{C}TS$ by complex conjugation in the fibres induces a map $\alpha: Q \rightarrow Q$ of the null quadric bundle over S which takes the anti-holomorphic tangents to the fibres to the holomorphic tangents to the fibres. As for the complementary direction, we can best see that its image is as desired by working in our previously constructed local coordinates, *taking the frame $\{e_i\}$ to be real*, and thus taking \tilde{I} to be real; complex conjugation ρ induces a map on the fibres of Q given by

$$z(\zeta) = -\frac{1}{\bar{\zeta}}$$

since one then has

$$\begin{aligned} \overline{w(\zeta)} &= \overline{[(1 + \zeta^2)e_1 + i(1 - \zeta^2)e_2 + 2i\zeta e_3]} = \\ &= \frac{1}{\bar{z}^2} [(1 + z^2)e_1 + i(1 - z^2)e_2 + 2ize_3] = \frac{w(z)}{\bar{z}^2} \end{aligned}$$

while the image of $\frac{\partial}{\partial \bar{\zeta}}$ is $z^2 \frac{\partial}{\partial z}$; so the image of $w(\zeta) + \frac{1}{2} \tilde{I}(w(\zeta)) \frac{\partial}{\partial \zeta}$ is

$$\frac{w(z)}{\bar{z}^2} + \frac{1}{2} \frac{1}{\bar{z}^4} \tilde{I}(\overline{w(z)}, \overline{w(z)}) \frac{\partial}{\partial \zeta} = \frac{1}{\bar{z}^2} \overline{[w(z) + \frac{1}{2} \tilde{I}(w(z), w(z)) \frac{\partial}{\partial z}]}$$

(the Jacobian of ρ on tangents to S is, of course, the identity, and so $w(\zeta)$ is indeed taken to itself, which can be rewritten as $w(z)/\bar{z}^2$) which

shows that our involution is anti-CR, as desired. (The real efficacy of working in special local coordinates will become apparent in the proof of the converse).

Conversely, suppose \bar{X} is a CR manifold by \mathbb{P}_1 's preserved by an anti-CR involution $\alpha: \bar{X} \rightarrow \bar{X}$; let H denote the CR distribution. If S is the space of leaves of the given foliation and $\pi: \bar{X} \rightarrow S$ is the quotient projection, then α is a map over π :

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\alpha} & \bar{X} \\ \pi \searrow & & \swarrow \pi \\ & S & \end{array} \quad \text{commutes.}$$

As in Theorem (3.5.I), \bar{X} can be identified with the quadric bundle Q of S for some pseudo-spacelike conformal metric, and H will then be given by our earlier construction for some (trace-free) conform second fundamental form $\hat{I}\hat{I}$; but since α is a map over π , it follows that α is just the map $Q \rightarrow Q$ induced by complex conjugation in the fibres of $\mathcal{C}TS$, because for any complex null direction $[v] \in Q$ one has

$$[v] = \pi_* H_{[v]} = \pi_* \alpha_* H_{[v]} = * \bar{H}_{\alpha([v])} = \overline{\alpha([v])}.$$

As for the second conformal fundamental form, we regress to our special local coordinate calculation, taking $\{e_i\}$ to again be real and $\tilde{I}\tilde{I}$ to be trace-free; our former calculation, stood on its head, shows that

$$\tilde{I}\tilde{I}(\bar{w}, \bar{w}) = \overline{\tilde{I}\tilde{I}(w, w)}$$

for all complex null vectors w , which shows that $\tilde{I}\tilde{I}$ (and hence $\hat{I}\hat{I}$) is real, since it is trace-free. \square

Now for our non-embeddability result. Nirenberg [14] has shown that there exist smooth CR-structures on \mathbb{R}^3 such that no neighbourhood of the origin can be CR embedded in \mathcal{C}^N for any N ; any CR function on such an

example would have to vanish to infinite order at the origin, which implies the non-embeddability. Now suppose that w is a non-vanishing nowhere real complex vector field on \mathbb{R}^3 which is a section of the said CR distribution. Let $e_1 = \operatorname{Re} w$, $e_2 = \operatorname{Im} w$, and take e_3 to be some other non-vanishing real vector field which is everywhere linearly independent from e_1 and e_2 ; define a smooth positive definite metric on \mathbb{R}^3 by taking $\{e_i\}$ to be an orthonormal frame. In our local coordinates for the quadric of this metric, $\zeta = 0$ corresponds to the null direction field spanned by w , and for some real conformal second fundamental form (e.g. that corresponding to the conformal torsion of the torsion metric connexion such that $\{e_i\}$ is a covariantly constant frame), $\tilde{\mathbb{I}}(w, w)$ vanishes, and $\zeta = 0$ becomes a CR submanifold *which is isomorphic* (via the canonical projection) to the Nirenberg example. It follows that *our CR 5-manifold cannot be embedded*. We've demonstrated the following

Fact (3.5.III). There exist smooth 3-manifolds with (positive definite) conformal metric and (real) conformal second fundamental form, which cannot be realised as totally real submanifolds of a complex 4-fold with half-flat conformal structure.

If such an embedding exists, however, it is locally unique. This follows from the fact that any two realisations of a CR manifold with mixed non-degenerate Levi-form as a real hypersurface in complex manifolds are equivalent in a neighbourhood of the hypersurface, as can be derived from a generalised version [18] of an extension theorem of Lewy.

§3.6 REAL SLICES OF "HEAVEN-ON-EARTH"

Let M_0 be a real analytic 3-fold with real-analytic positive-definite metric g_0 and (real) real-analytic symmetric form II_0 . By analytic continuation, we construct a (locally unique) complex 3-fold M which comes equipped with an anti-holomorphic involution $Q: M \rightarrow M$ (complex conjugation) and holomorphic metric g and symmetric form II such that $\bar{g} = Q^*g$, $\bar{II} = Q^*II$, as well as an identification of M_0 with the fixed-point set of Q such that $g_0 = \frac{1}{2}(g + \bar{g})|_{M_0}$ and $II_0 = \frac{1}{2}(II + \bar{II})|_{M_0}$. If we define a connexion ∇ on M by adding

$$\Delta_{jk}^i = e_{j\ell}^i II_k^\ell$$

to the torsion-free metric connexion associated with g , where e is obtained by analytic continuation of the metric volume element (for some orientation) of M_0 , then $Q_*\nabla = \bar{\nabla}$, where the anti-holomorphic connexion $\bar{\nabla}$ is defined by

$$\bar{\nabla}V := \overline{(\nabla\bar{V})}$$

for any anti-holomorphic vector field V on M ; thus, if $V \lrcorner \nabla V = 0$ one has

$$\overline{Q_*V \lrcorner \nabla(Q_*V)} = \overline{Q_*(V \lrcorner \nabla V)} = 0$$

and so Q sends geodesics of ∇ (with holomorphic affine parameters) to geodesics of $\bar{\nabla}$ (with anti-holomorphic affine parameters), in particular, Q induces an anti-holomorphic involution α of the space $N(M)$ of null geodesics of M (which restricts to the CR manifold corresponding to M_0 to give the anti-CR involution constructed in §3.5).

If \tilde{M} is the half-conformally-flat 4-fold in which M is embedded so as to realise the conformal parts of (g, II) as the conformal fundamental forms of M , then α constructs for us a real positive definite slice of \tilde{M} (cf. [28]) containing M_0 . Namely, since the complex analytic family \tilde{M} of \mathbb{P}_1 's in $N(M)$ completing the family M is locally unique, α sends \mathbb{P}_1 's of \tilde{M} to

\mathbb{P}_1 's of \tilde{M} since it preserves the family M ; thus α induces an antiholomorphic map \tilde{Q} on M , which will have a real-analytic 4-fold \tilde{M}_0 as its fixed-point set. Moreover, \tilde{Q} takes twistor surfaces to twistor surfaces, since if two \mathbb{P}_1 's in $N(M)$ belonging to the family \tilde{M} meet, then their images certainly do also; thus \tilde{Q} carries the conformal metric on \tilde{M} to its complex conjugate. Therefore the induced conformal metric on \tilde{M}_0 is *real*. But since α has no fixed points, the conformal metric on \tilde{M}_0 is positive-definite.

Theorem (3.6.I). Every real-analytic pair of positive-definite conformal metric and real second conformal fundamental form specified on a real-analytic 3-fold can be (locally) uniquely realized as those of a real-analytic hypersurface in a real-analytic 4-fold with half-flat positive-definite conformal metric. □

Similarly, it seems that one can realize a real-analytic Lorentzian ($++-$) conformal metric, specified along with a real real-analytic second conformal fundamental form as the conformal fundamental forms of an analytic hypersurface in a real-analytic 4-fold with half-flat conformal metric of signature ($++--$):

We may also consider the embedding problem dealt with in (3.6.I) with the supposition of data that is only *smooth*. In fact, if one can realize the data in a *complex* 4-fold (cf. §3.5), then a generalization [18] of the Hans Lewy extension theorem for CR functions allows one to extend the anti-CR real-structure map of §3.5 to an anti-holomorphic real-structure map without fixed points on some neighbourhood of the CR hypersurface Σ in the ambient twistor space, which then produces a real positive-definite slice of the complex 4-fold in the manner previously explained. Our previous results then show that general smooth data *cannot* be realized for smooth hypersurfaces in *real-analytic* 4-folds with half-flat positive-definite conformal metrics. However, any smooth half-flat positive-definite metric *is* real-analytic with respect to some analytic structure [28], because one

can directly construct the twistor space as a \mathbb{P}_1 -bundle over such a 4-fold, complete with an *integrable* almost complex structure, which then introduces a notion of real-analyticity via the Newlander-Nirenberg theorem. (The construction of this almost complex resembles and is closely related to our construction of CR manifolds from fundamental forms in §3.5). Thus we have the

Theorem (3.6.II). Theorem 3.6.I becomes false if "real-analytic" is replaced by "smooth".

A particular consequence of these results is that the "heaven-on-earth" construction, applied to an analytic hypersurface of a Riemannian 4-manifold, yields a half-flat 4-fold with positive definite slice. Notice that one gets a positive-definite slice when applying the construction to a positive-definite hypersurface in a *Lorentzian* 4-manifold precisely when the hypersurface is umbilic; by our conventions, Lorentzian second fundamental forms are imaginary.

§3.7 POSITIVE-DEFINITE HEAVEN-WITH-A-COSMOLOGICAL-CONSTANT

Let M_0 be a real-analytic 3-fold with real-analytic positive-definite conformal structure, and let the complex 3-fold M , equipped with an anti-holomorphic complex conjugation Q and a holomorphic ^{conformal structure preserved} by Q , be the analytic continuation of M_0 . The annihilator distribution of the canonical contact form θ on the space $N(M)$ of torsion-free null geodesics of M is taken onto its complex conjugate by the anti-holomorphic involution $\alpha: N(M) \rightarrow N(M)$ induced by Q because this involution takes \mathcal{P}_γ 's of the family M to other \mathcal{P}_γ 's of the family, and the annihilator distribution of θ is precisely spanned by the tangent spaces of these \mathcal{P}_γ 's. Thus

$$\alpha^*\theta = a \theta$$

for some holomorphic function a on $N(M)$; but such a function is necessarily a constant. (There is actually a sign ambiguity in the complex conjugate of θ , since θ takes values in $K^{-\frac{1}{2}}$, but this will prove to be no difficulty). Hence

$$a^2 \theta \wedge d\theta = \overline{\alpha^*(\theta \wedge d\theta)}.$$

But $\theta \wedge d\theta$ is just an ordinary complex number, and so

$$\alpha^*(\theta \wedge d\theta) = \theta \wedge d\theta$$

and $a^2 = \overline{\theta \wedge d\theta} / \theta \wedge d\theta$; therefore, by normalizing θ so that $\theta \wedge d\theta$ is real, we may assume that

$$\theta = \overline{\alpha^*\theta}.$$

Now suppose that γ is a \mathcal{P}_γ of the family \tilde{M} completing M such that $\alpha[\gamma] = \gamma$. The tangent vectors to the real slice \tilde{M}_0 of M at the point corresponding to γ will be represented by sections of $D|\gamma$ which are taken onto their own complex conjugates by α , and such a section can be written as

$$\xi \otimes \pi + \overline{\alpha_* \xi} \otimes \overline{\alpha_* \pi}$$

where $\xi \in H^0(\gamma, \mathcal{O}(D \otimes K^{\frac{1}{4}}))$, $\pi \in H^0(\gamma, \mathcal{O}(K^{-\frac{1}{4}}))$, corresponding to the fact that any tangent vector to the positive-definite slice is the sum of a complex null vector and its complex conjugate. The length of such a vector, as defined by the canonical Einstein metric, is

$$\begin{aligned} & d \theta (\xi, \overline{\alpha_* \xi}) \cdot \theta^{-1} (\pi d \overline{\alpha_* \pi} - \overline{\alpha_* \pi} d\pi) = \\ & d \alpha^* \theta (\alpha_* \xi, \overline{\xi}) \cdot (\alpha^* \theta)^{-1} (\alpha_* \pi d\overline{\pi} - \overline{\pi} d \alpha_* \pi) = \\ & \overline{d \theta (\overline{\alpha_* \xi}, \xi)} \cdot \overline{\theta^{-1} (\overline{\alpha_* \pi} d\pi - \pi d \overline{\alpha_* \pi})} \end{aligned}$$

and so the Einstein metric is *real* on the positive definite slice $\tilde{M}_0 - M_0$. It remains only to notice that picking the metric to be positive- (as opposed to negative-) definite forces the cosmological constant (the scalar curvature) to be *negative*, since this is the case for the "flat" example of Lobachevskian geometry with S^3 as the conformal infinity, and since the entire construction will vary continuously with the initial data. Thus we have the

Theorem (3.7.I). Every real-analytic positive-definite three dimensional conformal structure bounds a unique positive-definite solution of Einstein's equations with cosmological constant -1 as its (umbilic) conformal infinity.

§4. PROBLEMS AND PROSPECTS

There remain any number of open questions as to the full ramifications of the results presented in this thesis. We end this work on an unresolved chord by stating a few of these.

a) Suppose M is a strongly geodesically convex complex n -manifold with projective connexion. If $S \subset L(M)$ is a complex submanifold biholomorphically equivalent to \mathbb{P}_{n-1} and with normal bundle isomorphic to $T^*\mathbb{P}_{n-1} \otimes H^{-1}$, is S the submanifold canonically associated to a point of M ? (This is the case, for example, if M is a region in \mathbb{C}^2 with its usual flat projective connexion). If not, what if one posits that S is a member of a connected analytic family containing a submanifold of the family M ?

Analogous questions may be asked regarding complex n -manifolds, $n \geq 4$, with conformal connexion.

b) Isenberg, Yasskin and Green [29] have shown that general solutions of the Yang-Mills equations on \mathbb{C}^4 (with its usual flat conformal structure) correspond precisely to holomorphic vector bundles over the space of (torsion-free) null geodesics that extend to the third-order formal neighbourhood of the canonical embedding of this space in the product of the twistor and "dual" twistor spaces of \mathbb{C}^4 . (Each null line lies in precisely one self-dual plane and one anti-self-dual plane; one can thus identify $N(\mathbb{C}^4)$ with an open dense subset of the variety $\{([z^\alpha], [w_\beta]), \alpha, \beta = 1, \dots, 4 \mid z^\alpha w_\beta = 0\} \subset \mathbb{P}_3 \times \mathbb{P}_3$). Is there a generalization of this to some reasonably general class of conformal torsion structures? One expects that the conformal torsion, at least, must vanish, as there should be an analogue of the contact form $Z^\alpha dW_\alpha = -W_\alpha dZ^\alpha$ when the analogue of such a formal neighbourhood can be defined; conversely, it would seem that the contact form of a space of torsion-free geodesics should define an analogue of at least the first-order neighbourhood.

c) If M is an oriented Lorentzian 4-manifold, and $S \subset M$ is a smooth Cauchy surface, then the CR manifold that we constructed from the space of complex null directions tangent to S can be identified with the space $N(M)$ of null geodesics of any conformal connexion on M . (For reasons detailed below, the only sensible choice of the conformal connexion will be the one without torsion). Namely, a null geodesic meets S in a single point and, choosing a time orientation, has a unique future-pointing tangent n with unit component normal to S ; letting v denote the future pointing normal to S , there is, up to multiplication by a non-zero complex number, a unique complex null vector tangent to S which is orthogonal to n and which satisfies

$$i\epsilon(n, v, w, \bar{w}) > 0$$

where ϵ is a volume element representing the chosen orientation; and every complex null direction tangent to S arises in this fashion. The underlying real distribution $(H + \bar{H}) \cap TN(M)$ of the CR tangent space H consists of those Jacobi fields whose values at the Cauchy surface S are orthogonal to the null geodesic along which they are defined; we saw in §2.5 that this distribution is independent of S precisely if the conformal torsion vanishes, as we shall henceforth proceed to assume.

We can describe the CR structure defined on $N(M)$ by S as follows: a Jacobi field orthogonal to the null geodesic along which it is defined is taken by the action of the CR structure tensor ("multiplication by i ") to the Jacobi field orthogonal to the geodesic which has initial value and first derivative at S obtained from the initial value and first derivative of the first field by a "right-handed" rotation of 90° , relative to the future-pointing tangents to the geodesic. This is clear if the initial value vanishes, since our prescription for the CR structure gives the "celestial sphere" of null geodesics through a point of S the complex structure such that $PL(2, \mathbb{C})$ acts by Lorentz $(O(1,3))$ transformations; on the other hand, the CR structure also acts by right orthogonal rotation on

the horizontal lifts of vectors tangent to S and in the orthogonal space to the given null geodesic, and it is rather easy to see that such horizontal vectors correspond precisely to those Jacobi fields having vanishing initial derivative. Notice that the CR tangent space at any null geodesic depends on no other data than the point at which the geodesic meets S ; the surprising fact from this point of view is that the CR tangent bundle is involutory for any choice of smooth Cauchy surface.

We have seen that the CR structure, along with its foliation by \mathcal{P}_1 's representing the points of S , determines the conformally invariant part of the metric and second fundamental form of a hypersurface; but the metric and second fundamental form, provided they satisfy certain constraint equations, are just the data needed for the initial value formulation of Einstein's equations. Indeed, York [30] has shown that for a *compact* Cauchy surface, the conformally invariant part of the data specifies the rest via the constraint equations; such a formulation loses, however, the local character that we normally associate with physical initial-value problems, and for this reason one may think it unsatisfactory. At any rate, it seems that a formulation of the (Lorentzian) general relativistic initial-value problem is possible within the framework of CR manifolds, and could be very interesting. Presumably, one would need some abstract criterion for determining whether a foliated CR manifold corresponds to data which are real in the *Lorentzian* sense, which is to say that the conformal metric should be positive-definite and the second fundamental form, if defined according to the conventions of Chapter 3, should be purely imaginary; this would be some sort of analogue of the anti-CR involution encountered in the positive-definite case, but would be of necessity rather different in detail.

d) Can one show that an embedding as a hypersurface of a CR 5-manifold with non-degenerate Levi-form and a smooth foliation by \mathcal{P}_1 's necessarily

gives the \mathcal{P}_1 's normal bundle $2H$? (It does follow from the non-degeneracy of the Levi-form that the normal bundle will have Chern class 2). If so, our arguments against the possibility of performing certain smooth embeddings might be turned around to construct certain classes of non-embeddable CR manifolds, because one should expect that the general umbilic in a positive-definite space with self-dual conformal structure will admit a real-analytic structure such that the induced conformal metric is real-analytic.

e) The most important question that remains to be answered is by far the most daunting. Namely, what extra structure on a complex 5-manifold of torsion-free null geodesics corresponds to the existence of an Einstein metric representing the conformal structure of the corresponding 4-fold? It remains to be seen whether or not this problem admits an elegant solution, but in a way it would seem rather odd if it did not.

BIBLIOGRAPHY

1. ATIYAH, M. "Complex Analytic Connexions in Fibre Bundles", Trans. Am. Math. Soc., 85, (1957), 181-207.
2. BOTT, R. "Homogeneous Vector Bundles", Ann. Maths., 66, (1957), 203-247.
3. GRIFFITHS, P. and HARRIS, J. Principles of Algebraic Geometry, Wiley-Interscience, New York (1978).
4. GROTHENDIECK, A. "Sur la Classification des Fibres Holomorphes sur la Sphere de Riemann", Am. J. Maths., 79, (1957), 121-138.
5. GUNNING, R.C. On the Uniformization of Complex Manifolds: The Role of Connections, Princeton U.P., Princeton, N.J. (1978).
6. HÖRMANDER, L. An Introduction to Complex Analysis in Several Variables, North-Holland, Amsterdam (1973).
7. HICKS, N. Notes on Differential Geometry, Van Nostrand Reinhold, London (1971).
8. KOBAYASHI, S. and NAGANO, T. "On Projective Connections", J. Maths. and Mechs., 13, (1964), 215-235.
9. KODAIRA, K. "A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds", Ann. Maths., 75, (1962), 146-162.
10. KODAIRA, K. "On Stability of Compact Submanifolds of Complex Manifolds", Am. J. Maths., 85, (1963), 79-94.
11. KODAIRA, K. and MORROW, J. Complex Manifolds, Holt, Rinehart and Winston, New York (1971).
12. KODAIRA, K. and SPENCER, D.C. "On Deformations of Complex Analytic Structures", Ann. Maths., 67, (1958), 328-465.
13. FOLLAND, G.B. and KOHN, J.J. The Neumann Problem for the Cauchy-Riemann Complex, Princeton U.P., Princeton, N.J. (1972).

14. NIRENBERG, L. Lectures on Linear Partial Differential Equations, American Mathematical Society, Providence, R.I. (1973).
15. PENROSE, R. "Nonlinear Gravitons and Curved Twistor Theory", Gen. Relativity and Gravitation, 7, (1976), 31-52.
16. SPANIER, E. Algebraic Topology, McGraw-Hill, New York (1966).
17. WELLS, R.O. Jr. Differential Analysis on Complex Manifolds, Prentice-Hall, Englewood Cliffs, N.J. (1973).
18. WELLS, R.O. Jr. "Local Holomorphic Nulls of Real Submanifolds in Several Complex Variables", Proc. Am. Math. Soc., 17, (1966), 1272-1275.
19. VEBLEN, O. "Projective and Affine Geometry of Paths", Proc. Nat. Acad. Sci. USA, 8, (1922), 347-350.
20. WHITEHEAD, J.H.C. "Convex Regions in the Geometry of Paths", Quarterly J. Maths. (Oxford), 3, (1932), 33-42.
21. WARD, R. Curved Twistor Spaces, D.Phil. Thesis, Oxford (1977).
22. PENROSE, R. "Twistor Theory: its Aims and Achievements", in Quantum Gravity (Ed. C. Isham et al.), Oxford U.P. (1975), 268-407.
23. PENROSE, R. "Structure of Space-Time", in Battelle Rencontres (Eds. C. De Witt and J. Wheeler), Benjamin, New York (1968), 121-235.
24. YAU, S.T. "Calabi's Conjecture and Some New Results in Algebraic Geometry", Proc. Nat. Acad. Sci. USA, 74, (1977), 1798-1799.
25. STERNBERG, S. Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, N.J. (1964).
26. WARD, R. "On Self-Dual Gauge Fields", Phys. Lett., 61A, (1977), 81-82.
27. KO, M., NEWMAN, E.T. and TOD, K.P. "H-Space and Null Infinity", in Asymptotic Structure of Space-Time (Eds. Esposito, F.P. and Witten, L.), Plenum Press, New York (1977).
28. ATIYAH, M., HITCHIN, N. and SINGER, I.M. "Self-Duality in Four-

- Dimensional Riemannian Geometry", Proc. Roy. Soc. Lond. A, 362, (1978), 425-461.
29. ISENBERG, J., YASSKIN, P.B. and GREEN, P.S. "Non-Self-Dual Gauge Fields", Phys. Lett., 78B, (1978), 462-464.
30. YORK, J.W. Jr. "The Role of Conformal 3-Geometry in the Dynamics of Gravitation", Phys. Rev. Lett., 28, (1972), 1082-1085.
31. GRIFFITHS, P. "The Extension Problem in Complex Analysis II: Embeddings with Positive Normal Bundle", Am. J. Maths., 88, (1966), 366-446.