Quantum groups at $q = 0$, a Tannakian reconstruction theorem for IndBanach spaces, and analytic analogues of quantum groups

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A thesis submitted for the degree of

Doctor of Philosophy

Abstract

This thesis is divided into the following three parts.

A categorical reconstruction of crystals and quantum groups at $q = 0$. The quantum co-ordinate algebra $A_q(\mathfrak{g})$ associated to a Kac-Moody Lie algebra $\mathfrak{g}$ forms a Hopf algebra whose comodules are direct sums of finite dimensional irreducible $U_q(\mathfrak{g})$ modules. In Part I we investigate whether an analogous result is true when $q = 0$. We classify crystal bases as coalgebras over a comonadic functor on the category of pointed sets and encode the monoidal structure of crystals into a bicomonadic structure. In doing this we prove that there is no coalgebra in the category of pointed sets whose comodules are equivalent to crystal bases. We then construct a bialgebra over $\mathbb{Z}$ whose based comodules are equivalent to crystals, which we conjecture is linked to Lusztig's quantum group at $v = \infty$.

A Tannakian Reconstruction Theorem for IndBanach Spaces. Classically, Tannaka-Krein duality allows us to reconstruct a (co)algebra from its category of representation. In Part II we present an approach that allows us to generalise this theory to the setting of Banach spaces. This leads to several interesting applications in the directions of analytic quantum groups, bounded cohomology and Galois descent. A large portion of Part II is dedicated to such examples.

On analytic analogues of quantum groups. In Part III we present a new construction of analytic analogues of quantum groups over non-

Archimedean fields and construct braided monoidal categories of their representations. We do this by constructing analytic Nichols algebras and use Majid's double-bosonisation construction to glue them together. We then go on to study the rigidity of these analytic quantum groups as algebra deformations of completed enveloping algebras through bounded cohomology. This provides the first steps towards a p-adic Drinfel'd-Kohno Theorem, which should relate this work to Furusho's p-adic Drinfel'd associators. Finally, we adapt these constructions to working over Archimedean fields.

Contents

0.0 Introduction

This thesis is divided into three distinct parts, all of which have the common theme of quantum groups. *Quantum groups* first appeared in the form of the *quantised enveloping algebra* $U_q(\mathfrak{sl}_2)$ in 1983 in the work of Kulish and Reshetikhin, which was later given a Hopf algebra structure by Sklyanin. This is just one of a wide range of examples of quantised enveloping algebras discovered independently by both Drinfel'd and Jimbo in 1985. From the rigidity results of Chevalley, Eilenberg and Cartan of the 1940s it was known that there were no non-trivial algebra deformations of the enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$. Nonetheless, the Drinfel'd-Jimbo quantised enveloping algebras provided deformations of $U(\mathfrak{g})$ with interesting non-cocommutative Hopf algebra structures.

Quantised enveloping algebras $U_q(\mathfrak{g})$ are deformations defined over a parameter q. The classical limit as q approaches 1 recovers the enveloping algebra $U(\mathfrak{g})$. The behaviour of $U_q(\mathfrak{g})$ as q approaches 0 was studied by Kashiwara and resulted in the theory of *crystal bases*. Treating q as a temperature parameter, Kashiwara was motivated by the idea that the representations of $U_q(\mathfrak{g})$ should be simpler at absolute zero. At $q = 0$ he discovered that representations have bases resembling the standard bases of $U_q(\mathfrak{sl}_2)$ modules that behave well under the raising and lowering operators e and f. These crystal bases then obtain a graph structure that reduces representation theoretic problems to combinatorial ones. The first part of this thesis, which appears in the preprint [40], discusses algebraic structures on crystal bases of $U_q(\mathfrak{g})$ modules, resulting in reconstructions of crystals as both coalgebras of a comonad $\mathbb{U}$ on pointed sets and as based modules of a ring $\mathbb{B}$. One of the most interesting properties of crystal bases is their monoidal structure, which we encode into both a bi-comonad structure on $\mathbb{U}$ and into a $\mathbb{Z}$-bialgebra structure on $\mathbb{B}$.

One of the most striking facts about quantised enveloping algebras is that, despite not being quasi-triangular, there exists categories of representations with an interesting braided monoidal structure. This gives a wealth of solutions to the Yang-Baxter equation, and hence representations of braid groups and the resulting invariants for knots. The third part of this thesis presents a construction of analytic analogues of quantum enveloping algebras and describes a braided monoidal category of representations with a view towards obtaining new braid group representations on Banach spaces.

Tannaka-Krein duality, or *Tannakian reconstruction*, considers when an algebra (or other algebraic structure) can be recovered from its category of representations, and which categories arise as categories of representations. The belief is that the study of algebras and of their categories of representations should be interchangeable. These ideas were introduced in the early 1930s by Pontryagin, although the formalism has been named for the work of Tannaka and Krein from 1939 and 1949 respectively. The second part of this thesis, which appears in the preprint [26], generalises this Tannakian reconstruction theory to the context of IndBanach spaces and explores some examples of applications.

Inspired by the ideas of Tannakian-Krein duality one can construct quantum enveloping algebras by deforming the categories of representations of enveloping algebras. The Drinfel'd-Kohno Theorem shows that, over $\mathbb{C}$, the category of representations of a quantised enveloping algebra can be obtained from the category of representations of the enveloping algebra by replacing the associator of the monoidal structure with the Drinfel'd associator. In Part III of this thesis we prove some rigidity results in the context of analytic quantum groups that provide the first steps towards proving a p-adic Drinfel'd-Kohno Theorem.

Part I. A categorical reconstruction of crystals and quantum groups at $q = 0$. The quantum co-ordinate algebra $A_q(\mathfrak{g})$ associated to a Kac-Moody Lie algebra $\mathfrak{g}$ forms a Hopf algebra that is dual to the quantum enveloping algebra $U_q(\mathfrak{g})$. Its comodules can be classified as direct sums of finite dimensional irreducible $U_q(\mathfrak{g})$-modules. In Part I we investigate whether a similar result is true when $q = 0$.

Crystal bases were introduced by Kashiwara as local bases of representations of $U_q(\mathfrak{g})$ as q approaches 0. They were constructed to resemble the standard bases of representations of $U_q(\mathfrak{sl}_2)$ that behave well with respect to the raising and lowering operators e and f. As a result, crystal bases are endowed with a graph structure and have interesting combinatorial properties. Perhaps the most interesting property of crystal bases is their monoidal structure. The main results of this paper show that crystal bases appear in the representation theory of a coalgebra $\mathbb{B}$, and that their monoidal structure of crystals can be encoded as a bialgebra structure on $\mathbb{B}$. This opens up new approaches to studying crystal bases through the representation theory of this bialgebra.

In Section 1.1 we recall the definitions of quantum enveloping algebras, crystal bases and Kashiwara's category of crystals, and present some basic results. In Section 1.2 we present our first result, Proposition 1.2.2, which asserts that there is an algebra structure on the crystal base $\mathcal{B}$ of $A_q(\mathfrak{g})$. This is followed by a discussion of why a lack of rigidity in pointed sets prevents us from naïvely adapting the comultiplication on $A_q(\mathfrak{g})$ to the setting of crystals.

In Section 1.3 we take a more categorical approach to the study of crystals. Using the Barr-Beck Theorem our second result classifies crystal bases as coalgebras over a comonad $\mathbb{U}$ on the category of pointed sets:

Corollary 1.3.12. There is an equivalence of categories $J_{\mathbb{U}} : Crys_{\mathfrak{g}} \to Set_{\bullet\mathbb{U}}$ between the category of crystals and the category of algebras over the comonad $\mathbb{U}$ in the category of pointed sets.

This is done in as broad generality as possible before being applied to the category of crystals. This new reconstruction of crystals provides a new context in which to study them. From this we also obtain the following result:

Corollary 1.3.13. There is no coalgebra in the category of pointed sets whose category of comodules is equivalent to $Crys_{\mathfrak{g}}$ as categories fibred over pointed sets.

We end this section by encoding the monoidal structure of the category of crystals into a bicomonadic structure on $\mathbb{U}$.

In Section 1.4 we endow the free abelian group $\mathbb{B}$ on the crystal base $\mathcal{B}$ of $A_q(\mathfrak{g})$ with the structure of a $\mathbb{Z}$-bialgebra analogous to that of $A_q(\mathfrak{g})$. In the case of $\mathfrak{sl}_2$ we give an explicit presentation of this bialgebra. The main results of this section are the following:

Theorem 1.4.16. There is an equivalence between the category of crystals and the category of based $\mathbb{B}$ comodules.

Proposition 1.4.17. The above equivalence is an equivalence of monoidal categories.

This is followed by a study of the dual algebra to $\mathbb{B}$, denoted $\dot{U}_0$. A comparison between the presentations of $A_q(\mathfrak{sl}_2)$ and $\mathbb{B}$ in the case of $\mathfrak{sl}_2$ allow us to rephrase the multiplication on $\mathbb{B}$ in terms of the global basis of $A_q(\mathfrak{sl}_2)$. This leads us to conjecture a relationship between $\dot{U}_0$ and Lusztig's quantum group at $v = \infty$ in [28].

Part II. A Tannakian reconstruction theorem for IndBanach spaces. Classically, Tannaka-Krein duality answers the questions of whether a compact topological group (or affine group scheme as in [13], [12]) can be recovered from its category of linear representations, and of when a category (with an appropriate fibre functor) is equivalent to representations of such a group. The answer to these questions can be seen as an application of the Barr-Beck Theorem, along with the fact that a cocontinuous linear functor on the category of vector spaces must be of the form $V \otimes -$ for some space V. This second point follows from the fact that any vector space is a colimit of copies of the base field.

Unfortunately, the above is not true for the category of Banach spaces. However, the contracting category of Banach spaces does have an analogous property, and so a brief investigation of contracting colimits in Section 2.1 allows us to proceed as before. We also note that the category of Banach spaces is neither complete nor cocomplete, and so we instead work in its Ind completion. Using this, we deduce an analogue of Tannaka duality for IndBanach spaces in Section 2.2. The main result of this section is:

Theorem 2.2.6. Let $\mathcal{C}$ be a locally presentable, quasi-abelian category, enriched over IndBan_k, equipped with a fibre functor $F : \mathcal{C} \to \text{IndBan}_k$. Assume further that $T = FG$ commutes with l^1 for some left adjoint G to F. Then $\mathcal{C}$ is equivalent to the category of left $\mathscr{A}$ modules in IndBan_k for some IndBanach algebra $\mathscr{A}$.

If we assume that the category $\mathcal{C}$ has sufficiently nice contracting colimits, we may improve on this result to obtain the following:

Corollary 2.2.8. Suppose $\mathcal{C}$ has constant contracting coproducts and is fibred over IndBan_k such that that the fiber functor F commutes with constant contracting coproducts. Then $\mathcal{C}$ is equivalent to the category of left $\mathscr{A}$ modules in IndBan_k for some IndBanach algebra $\mathscr{A}$.

We also obtain a dual version for recovering coalgebras from their categories of comodules:

Theorem 2.2.13. Let $\mathcal{C}$ be a locally presentable, k-linear, quasi-abelian category, equipped with a co-fibre functor $F : \mathcal{C} \to \text{IndBan}_k$. Assume further that $U = FG$ is cocontinuous and commutes with l^1, where G is some right adjoint to F. Then $\mathcal{C}$ is equivalent to the category of left $\mathscr{B}$ comodules in IndBan_k for some IndBanach coalgebra $\mathscr{B}$.

In Section 2.3 we demonstrate some examples of applications of this theory. This begins with the proof of an IndBanach version of *the fundamental theorem of coalgebras*:

Proposition 2.3.1. An IndBanach comodule of a Banach coalgebra is isomorphic to a colimit of Banach comodules.

This is followed by a short exploration of different analytic gradings, which provides the first steps towards defining analytic Nichols algebras and analytic quantum groups in Part III. The results of this subsection reconstruct certain categories of graded IndBanach spaces as comodules of bialgebras of (strictly convergent and overconvergent) analytic functions on polydisks.

Perhaps the most fruitful example in Section 2.3 involves representations of topological groups. In [9], Bühler shows that continuous bounded cohomology of a group G comes from the derived invariants functor on a quasi-ableian category which we denote $G\text{-Mod}^{\mathrm{bd}}$. In Section 2.3.6 we show that this is a category of coalgebras over a comonadic functor:

Proposition 2.3.21. $G\text{-Mod}^{\mathrm{bd}}$ is equivalent to the category of coalgebras over the monoidal comonad $C_{\mathrm{b}}^{\mathrm{lu}}(G, -)$.

or as comodules of an IndBanach bialgebra when the group is compact:

Corollary 2.3.22. In the case where G is compact, $G\text{-Mod}^{\mathrm{bd}}$ is equivalent to the category of comodules over the bialgebra $C_{\mathrm{b}}^{\mathrm{lu}}(G, k)$.

We may therefore rephrase bounded cohomology in terms of cohomology of a monoidal comonadic functor (or an IndBanach bialgebra).

We conclude with the example of Galois descent for categories of IndBanach spaces. We start by showing that, given an extension of complete valued fields $K \subset L$, we can recover an IndBanach space V over K from $\mathrm{Ind}_K^L V := L\hat{\otimes}_K V$ provided we retain the coaction of $L\hat{\otimes}_K L$ as descent data:

Proposition 2.3.36. IndBan_K is equivalent to the category of left $(L\hat{\otimes}_K L)$-comodules in IndBan_L via the induction functor $V \mapsto L\hat{\otimes}_K V$.

We then compare this descent data to a strongly continuous action of the Galois group and to the action of an Iwasawa algebra.

Part III. On analytic analogues of quantum groups. In 2007, Soibelman gave a rough introduction to p-adic analogues of quantum groups in [41] as examples of non-commutative spaces over non-Archimedean fields. Inspired by this, Lyubinin explicitly constructs a p-adic quantum hyperenveloping algebra in [29] in the case of $\mathfrak{sl}_2$. His construction involves using Skew-Tate algebras to construct completions of the positive and negative parts of the quantum enveloping algebra. The disadvantage of this construction is that it requires some work to generalise this to arbitrary Kac-Moody Lie algebras. In Part III we present an alternative construction of analytic analogues of quantum groups over non-Archimedean fields that works for any Kac-Moody Lie algebra and construct braided monoidal categories of their representations. With this we hope to exhibit interesting new analytic representations of braid groups. We then go on to use bounded cohomology to study the rigidity of these analytic quantum groups as algebra deformations of completed enveloping algebras. We hope that this will provide the first steps towards a p-adic Drinfel'd-Kohno Theorem, relating this work to Furusho's p-adic Drinfel'd associators in [14].

In [27], Lusztig constructs the positive and negative parts of quantum enveloping algebras as quotients of tensor algebras by the radical of a duality pairing. This is an example of more a general construction, a *Nichols algebra*, discussed in detail in [2]. Section 3.1 of this paper is devoted to presenting the definitions and results required to define analytic analogues of Nichols algebras. All of this is done in the categories of IndBanach spaces over both Archimedean and non-Archimedean fields.

Majid's construction in [33] brings together dually paired braided Hopf algebras

B and C with compatible respective right and left actions of a Hopf algebra H to form a new Hopf algebra $U(B, H, C)$, the *double-bosonisation*. The motivation behind this construction is that one can recover the quantum enveloping algebra $U_q(\mathfrak{g})$ from $U(B, H, C)$ if $B = U_q^+(\mathfrak{g})$ and $C = U_q^-(\mathfrak{g})$ are the respective positive and negative parts of a quantum enveloping algebra and $H = U_q^0(\mathfrak{g})$ is the Cartan part. Section 3.2 of this paper recalls and rephrases Majid's double-bosonisation construction in the context of IndBanach spaces, which will allow us to construct analytic analogues quantum enveloping algebras from analytic Nichols algebras in the subsequent sections.

In Section 3.3 we restrict ourselves to working over non-Archimedean fields. We begin Subsection 3.3.1 by proving the existence of the analytic Nichols algebras defined in Section 3.1 through two different constructions. The first, given in the proof of Proposition 3.3.4, exhibits the quotient of a completed tensor algebra by a certain universal Hopf ideal as an analytic Nichols algebra. The second, given in Proposition 3.3.10, constructs an analytic Nichols algebra as the quotient of a completed tensor algebra by the radical of a duality pairing. In particular this second construction gives the duality pairing between Nichols algebras that allows us to use Majid's double-bosonisation construction. We show in Proposition 3.3.11 that these two constructions are equivalent. In Subsection 3.3.2 we apply these constructions to obtain completions of the positive and negative parts of quantum enveloping algebras and use Majid's double-bosonisation to glue them together into completions of quantum enveloping algebras. We call the resulting IndBanach Hopf algebras *analytic quantum groups*.

Unfortunately, we see in Subsection 3.3.3 that the R-matrix of $U_q(\mathfrak{g})$ still does not converge in any of our analytic quantum groups. Nonetheless, in Subsection 3.3.4

we use an alternate description of our analytic quantum groups as quotients of a Drinfel'd doubles to obtain a braided monoidal category of representations analogous to the BGG category $\mathcal{O}$. We then present an example in the case of $\mathfrak{g} = \mathfrak{sl}_2$ of such a braided representation with no highest weight vectors. The further study of these braided representations should produce interesting new examples of braid group representation on Banach spaces and may give a new context in which some special analytic functions, such as p-adic multiple polylogarithms, naturally arise.

The classical rigidity results of Chevalley, Eilenberg and Cartan from the 1940s assert that there are no non-trivial formal deformations (as an algebra) of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$. The proof relies on the vanishing of certain Lie algebra cohomology groups. In Theorems 3.3.52 and 3.3.53 of Subsection 3.3.5 we prove an analogous result that, provided an unproven bounded cohomology vanishing result holds, any algebra deformation of a completed enveloping algebra is isomorphic to the trival one. In particular this implies Corollary 3.3.61 that asserts that, modulo a bounded cohomology vanishing result, our analytic quantum groups are isomorphic to the trivial algebra deformation of a completed enveloping algebra. Furthermore, both of these isomorphisms are unique up to conjugation. In Subsection 3.3.6 we highlight the benefits of working over formal powerseries $k[\![\hbar]\!]$ as opposed to convergent powerseries. In particular this allows us to prove Theorems 3.3.67 and 3.3.68, rigidity results analogous to Theorems 3.3.52 and 3.3.53 that require weaker assumptions on bounded cohomology.

In [23], Kassel uses algebraic analogues of the rigidity theorems of Subsections 3.3.5 and 3.3.6 to present a proof of the Drinfel'd-Khono Theorem over $\mathbb{C}$. This theorem states that the category of representations of the quantum enveloping algebra is equivalent, as a braided monoidal category, to the category of $U(\mathfrak{g})$-modules with associativity constraint given by the Drinfel'd associator and braiding given by the associated R-matrix. As a result of this, the associated braid group representations are equivalent. This can be interpreted as a statement about the monodromy of the Knizhnik-Zamolodchikov (KZ) equations that govern the Drinfel'd associator. In [14], Furusho uses p-adic multiple polylogarithms to construct solutions to the p-adic KZ equations and a p-adic Drinfel'd associator. In the future the author hopes to expand upon the work in Subsections 3.3.5 and 3.3.6 to prove a p-adic analogue of the Drinfel'd-Khono theorem and to investigate links to Furusho's work.

Finally, in Section 3.4 we adapt these constructions to working over Archimedean fields. We begin by proving the existence of some analytic Nichols algebras and then use Majid's double-bosonisation to form Archimedean analytic quantum groups. We finish by constructing a braided monoidal category of representations as in Subsection 3.3.4. Again, we hope that the further study of these representations will produce interesting new braid group representations in which we might see some special analytic functions arising, such as the quantum dilogarithms that appear in [15].

Acknowledgements

I would like to thank Kobi Kremnitzer for his expert supervision and continued support throughout this research, without which writing this thesis would not have been possible. I would also like to thank André Henriques, Dan Ciubotaru and Kevin McGerty for their invaluable insights and advice. I owe a great deal of gratitude to my fiancée, Kathryn, my mother, and my brother, Alister, for their support and encouragement, as well as to all those who have resided in Office N3.13.

This thesis is dedicated to the memory of my father.

0.1 Preliminaries and notation

We begin by setting some notation and recalling some preliminary results that will feature in various parts of this thesis.

0.1.1 Quantum groups

The following constructions of quantum groups can be seen in Kashiwara's paper [19] and in Jantzen's book [10, p. 51], or for a more detailed account, see Lusztig's book [27].

Definition 0.1.1. Let $\mathfrak{g}$ be the Lie algebra defined by the data of

i) a free $\mathbb{Z}$-module Φ, the *weight lattice*, a free submodule $\Psi \subset \Phi$, the *root lattice*, and a free basis $\{\alpha_i \mid i \in I\}$ of Ψ, the *simple roots*, indexed over some set I;

ii) a symmetric bilinear form $(\cdot,\cdot) : \Phi \times \Phi \to \mathbb{Q}$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{N}$ and $(\alpha_i, \alpha_j) \leq 0$ for $i, j \in I, i \neq j$; and

iii) *simple coroots* $\lambda_i \in \Phi^* = \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ such that $\lambda_i(\alpha) = \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}$ for $i \in I, \alpha \in \Phi$.

Then $\mathfrak{g}$ is generated over $\mathbb{Q}$ by elements e_i, f_i, h_i for $i \in I$ subject to the relations

$$[h_i, h_j] = 0, \quad [e_i, f_i] = \delta_{ij} h_i, \quad [h_i, e_j] = \lambda_i(\alpha_j) e_j, \quad [h_i, f_j] = -\lambda_i(\alpha_j) f_j \ ,$$

and for $i \neq j$,

$$(\mathrm{ad} e_i)^{1-\lambda_i(\alpha_j)} e_j = 0, \quad (\mathrm{ad} f_i)^{1-\lambda_i(\alpha_j)} f_j = 0,$$

where ad is the *adjoint map* $(\mathrm{ad} x)(y) = [x, y]$.

Definition 0.1.2. We will denote by

$$\Psi_+ = \left\{ \sum_{i \in I} n_i \alpha_i \middle| n_i \geq 0 \right\} \subset \Psi$$

the *positive roots*, and $\Psi_- = -\Psi_+$ the *negative roots*. Let

$$\Phi_+ = \{\alpha \in \Phi \mid \lambda_i(\alpha) \geq 0 \text{ for all } i \in I\}$$

be the *dominant weights*. Then Φ has a partial ordering given by $\alpha \geq \beta$ if and only if $\alpha - \beta \in \Phi_+$.

Definition 0.1.3. We define the *quantised enveloping algebra* $U_q(\mathfrak{g})$ to be the algebra generated over $\mathbb{Q}(q)$ by E_i, F_i, K_λ for $i \in I, \lambda \in \Phi^*$, with the defining relations

$$\begin{aligned}
\text{for } \lambda = 0 \quad & K_\lambda = 1, \\
\text{for } \lambda_1, \lambda_2 \in \Phi^* \quad & K_{\lambda_1} K_{\lambda_2} = K_{\lambda_1 + \lambda_2}, \\
\text{for } i \in I, \lambda \in \Phi^* \quad & K_\lambda E_i K_{-\lambda} = q^{\lambda(\alpha_i)} E_i, \\
& K_\lambda F_i K_{-\lambda} = q^{-\lambda(\alpha_i)} F_i, \\
& E_i F_i - F_i E_i = \delta_{ij} \tfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \\
\text{for } i \neq j \quad & \textstyle\sum_{k=0}^{1-\lambda_i(\alpha_j)} (-1)^k E_i^{(k)} E_j E_i^{(1-\lambda_i(\alpha_j)-k)} \\
& = \textstyle\sum_{k=0}^{1-\lambda_i(\alpha_j)} (-1)^k F_i^{(k)} F_j F_i^{(1-\lambda_i(\alpha_j)-k)} = 0
\end{aligned}$$

where $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$, $t_i = K_{\frac{(\alpha_i,\alpha_i)}{2}\lambda_i}$, $[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$, $[k]_i! = [1]_i[2]_i...[k]_i$, $F_i^{(k)} = F_i^k/[k]_i!$, and $E_i^{(k)} = E_i^k/[k]_i!$. Let us denote by $U_q^+(\mathfrak{g})$ (respectively $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$ (respectively $\{F_i \mid i \in I\}$). Similarly let $U_q^{\geq 0}(\mathfrak{g})$ (respectively $U_q^{\leq 0}(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\} \cup \{K_\lambda \mid \lambda \in \Phi^*\}$ (respectively $\{F_i \mid i \in I\} \cup \{K_\lambda \mid \lambda \in \Phi^*\}$).

Example 0.1.4. *If $\mathfrak{g} = \mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$ is the algebra generated by E, F, t, t^{-1} with defining relations*

$$tEt^{-1} = q^2 E, \;\; tFt^{-1} = q^{-2} F, \;\; EF - FE = \frac{t - t^{-1}}{q - q^{-1}}.$$

For general $\mathfrak{g}$, the subalgebras of $U_q(\mathfrak{g})$ generated by E_i, F_i, t_i, t_i^{-1}, denoted $U_q(\mathfrak{g})_i$, are *isomorphic to* $U_q(\mathfrak{sl}_2)$.

Remark The subalgebras $U_q^{\pm}(\mathfrak{g})$ naturally arise as Nichols algebras, as defined in [2]. This is the construction Lusztig gives in [27]. Given $U_q^+(\mathfrak{g})$, $U_q^-(\mathfrak{g})$ and $U_q^0(\mathfrak{g}) := \langle K_\lambda \mid \lambda \in \Phi^* \rangle$ one can construct $U_q(\mathfrak{g})$ using Majid's *double-bosonisation* construction from [33]. We shall use variations on these construction in Part III to construct analytic analogues of quantum groups.

Definition 0.1.5. We say that a left $U_q(\mathfrak{g})$ module M is *integrable* if M decomposes into *weight spaces* $M = \bigoplus_{\alpha \in \Phi} M_\alpha$,

$$M_\alpha = \{m \in M \mid K_\lambda m = q^{\lambda(\alpha)} m \text{ for all } \lambda \in \Phi^*\},$$

and for each $i \in I$, M is a locally finite dimensional $U_q(\mathfrak{g})_i$ module. We then define $\mathcal{O}_{\text{int}}(\mathfrak{g})$ to be the category of integrable left $U_q(\mathfrak{g})$ modules that are locally finite dimensional as $U_q^+(\mathfrak{g})$ modules. Likewise we define integrable right $U_q(\mathfrak{g})$ modules, and an analogous category $\mathcal{O}_{\text{int}}(\mathfrak{g}^{\text{op}})$.

Remark We borrow the notation $\mathcal{O}_{\text{int}}(\mathfrak{g})$ from Kashiwara in [19]. In [27], this category is denoted $\mathcal{C}^{\text{hi}} \cap \mathcal{C}'$. Note that this category is considerable smaller than the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ as it omits Verma modules. The following proposition characterises representations in $\mathcal{O}_{\text{int}}(\mathfrak{g})$ completely.

Proposition 0.1.6 ([27])**.** *Representations in* $\mathcal{O}_{int}(\mathfrak{g})$ *are completely reducible, and all irreducible objects are, up to isomorphism, indexed by* $\alpha \in \Phi^+$. *These irreducibles, denoted* $V(\alpha)$, *can be expressed explicitly as the representation generated by a single vector* u_α, *called the* highest weight vector, *with the defining relations*

$$E_i u_\alpha = 0 = F_i^{1+\lambda_i(\alpha)} u_\alpha, \quad K_\lambda u_\alpha = q^{\lambda(\alpha)} u_\alpha, \quad \textit{for } i \in I, \lambda \in \Phi.$$

Proof. This is Corollary 6.2.3 of [27]. □

Example 0.1.7. *In the case of* $\mathfrak{sl}_2$*, these irreducible representations are* $V(n)$ *indexed by* $n \in \mathbb{Z}_{\geq 0}$*. They have a basis* $\{u_i^{(n)} \mid 0 \leq i \leq n\}$ *of* t*-eigenvectors with*

$$tu_i^{(n)} = q^{n-2i}u_i^{(n)}, \quad Eu_i^{(n)} = [n-i+1]u_{i-1}^{(n)}, \quad Fu_i^{(n)} = [i+1]u_{i+1}^{(n)}.$$

Here we use the notation $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$*,* $[0] = 0$*. We will use the notation* $B(n) := \{u_i^{(n)} \mid 0 \leq i \leq n\} \sqcup \{0\}$*. Note that, up to scalar multiplication,* E *decreases the index* i *of basis elements* $u_i^{(n)}$*, whilst* F *increases this index. From this we define maps of sets* $B(n) \to B(n)$ *that preserve* 0*,*

$$\tilde{e} : u_k^{(n)} \mapsto \begin{cases} u_{k-1}^{(n)}, & k > 0, \\ 0 & k = 0, \end{cases} \qquad \tilde{f} : u_k^{(n)} \mapsto \begin{cases} u_{k+1}^{(n)}, & k < n, \\ 0 & k = n. \end{cases}$$

These are often referred to as the Kashiwara operators*.*

Definition 0.1.8. Let $A_q(\mathfrak{g})$ denote the *quantum co-ordinate algebra* defined as the direct sum $A_q(\mathfrak{g}) = \bigoplus_{\alpha \in \Phi_+} V(\alpha) \otimes V(\alpha)^*$, where $V(\alpha)^*$ denotes the dual vector space of $V(\alpha)$. The unit is $1 = v_0 \otimes v_0^* \in V(0) \otimes V(0)^*$ and multiplication is defined by the composition

$$\begin{aligned} V(\alpha) \otimes V(\alpha)^* \otimes V(\beta) \otimes V(\beta)^* &\xrightarrow{\sim} V(\alpha) \otimes V(\beta) \otimes V(\beta)^* \otimes V(\alpha)^* \\ &\xrightarrow{\sim} V(\alpha) \otimes V(\beta) \otimes (V(\alpha) \otimes V(\beta))^* \\ &\to \left(\bigoplus_\gamma V(\gamma) \right) \otimes \left(\bigoplus_\delta V(\delta) \right)^* \\ &\twoheadrightarrow \bigoplus_\gamma V(\gamma) \otimes V(\gamma)^* \end{aligned}$$

where the third arrow is given by the decomposition into irreducible components and the fourth projects onto corresponding pairs of components. This algebra has a comultiplication given by

$$V(\alpha) \otimes V(\alpha)^* \cong V(\alpha) \otimes k \otimes V(\alpha)^*$$
$$\to V(\alpha) \otimes (V(\alpha)^* \otimes V(\alpha)) \otimes V(\alpha)^* \hookrightarrow A_q(\mathfrak{g}) \otimes A_q(\mathfrak{g})$$

induced by the coevaluation maps $k \to V(\alpha)^* \otimes V(\alpha)$, and counit given by the evaluation maps $V(\alpha) \otimes V(\alpha)^* \to k$.

Remark By the quantum Peter-Weyl Theorem (Proposition 7.2.2 of [21]), this can be identified with a subalgebra of functions on the quantum enveloping algebra $U_q(\mathfrak{g})$, where $u \otimes v \in V(\alpha) \otimes V(\alpha)^*$ is seen as the function $x \mapsto \langle x \cdot u, v \rangle$. The image of $A_q(\mathfrak{g})$ is the subalgebra of all functions $\phi \in U_q(\mathfrak{g})^*$ such that the left and right $U_q(\mathfrak{g})$ submodules of $U_q(\mathfrak{g})^*$ generated by ϕ,

$$U_q(\mathfrak{g}) \cdot \phi = \{\phi(- \cdot x) \mid x \in U_q(\mathfrak{g})\} \text{ and } \phi \cdot U_q(\mathfrak{g}) = \{\phi(x \cdot -) \mid x \in U_q(\mathfrak{g})\},$$

are in $\mathcal{O}_{\text{int}}(\mathfrak{g})$ and $\mathcal{O}_{\text{int}}(\mathfrak{g}^{\text{op}})$ respectively.

0.1.2 (Co)Monads and the Barr-Beck Theorem

We now present some preliminaries on category theory. We recall the definitions of (co)monads, which generalised notions of (co)algebras in the setting of functors on categories. For more details see Borceaux's *Handbook of Categorical Algebra 2* [8, p. 189-197].

Definition 0.1.9. A *monad* on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : \text{id}_{\mathcal{C}} \Rightarrow T$, $\mu : T \circ T \Rightarrow T$ are natural transformations satisfying

$$\mu \circ (\text{Id} * \mu) = \mu \circ (\mu * \text{Id}) : TTT \to T$$

and

$$\mu \circ (\mathrm{Id} * \eta) = \mathrm{Id}_T = \mu \circ (\eta * \mathrm{Id}) : T \to T$$

where $*$ denotes the horisontal composition of natural transformations. An *algebra* on this monad is a pair (C, ξ) where C is an object in $\mathcal{C}$ and $\xi : T(C) \to C$ is a morphism in $\mathcal{C}$ satisfying

$$\xi \circ T(\xi) = \xi \circ \mu_C : TT(C) \to C \text{ and } \xi \circ \eta_C = \mathrm{Id}_C.$$

A *morphism of algebras* $f : (C, \xi) \to (C', \xi')$ is a morphism $f : C \to C'$ in $\mathcal{C}$ such that $f \circ \xi = \xi' \circ T(f)$. These algebras in $\mathcal{C}$ over a monad $\mathbb{T}$ form a category, denoted $\mathcal{C}^{\mathbb{T}}$, known as the *Eilenberg-Moore category* of the monad.

Definition 0.1.10. Dually, a *comonad* on a category $\mathcal{C}$ is a triple $\mathbb{U} = (U, \varepsilon, \Delta)$, where $U : \mathcal{C} \to \mathcal{C}$ is a functor, and $\varepsilon : U \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\Delta : U \Rightarrow U \circ U$ are natural transformations satisfying

$$(\mathrm{Id} * \Delta) \circ \Delta = (\Delta * \mathrm{Id}) \circ \Delta : U \Rightarrow UUU$$

and

$$(\mathrm{Id} * \varepsilon) \circ \Delta = \mathrm{Id} = (\varepsilon * \mathrm{Id}) \circ \Delta : U \Rightarrow U.$$

A *coalgebra* on this monad is a pair (D, ζ) where D is an object in $\mathcal{C}$ and $\zeta : D \to U(D)$ is a morphism in $\mathcal{C}$ satisfying

$$U(\zeta) \circ \zeta = \Delta_D \circ \zeta : D \to UU(D) \text{ and } \varepsilon_D \circ \zeta = \mathrm{Id}_D.$$

A *morphism of coalgebras* $g : (D, \zeta) \to (D', \zeta')$ is a morphism $g : D \to D'$ in $\mathcal{C}$ such that $U(g) \circ \zeta = \zeta' \circ g$. Coalgebras in $\mathcal{C}$ over a comonad $\mathbb{U}$ form a category which we shall denote $\mathcal{C}_{\mathbb{U}}$.

Example 0.1.11. *Suppose we have a pair of adjoint functors*

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G.$$

with unit $\eta : id_{\mathcal{C}} \Rightarrow G \circ F$ *and counit* $\varepsilon : F \circ G \Rightarrow id_{\mathcal{D}}$*. Then* $\mathbb{T} = (T := G \circ F, \eta, \mu)$ *defines a monad where* μ *is the horisontal composition*

$$\mu = id_G * \varepsilon * id_F : GFGF \Rightarrow G \circ id_{\mathcal{D}} \circ F = GF.$$

Similarly, $\mathbb{U} = (U := F \circ G, \varepsilon, \Delta)$ *forms a comonad where* $\Delta := id_F * \eta * id_G$*. Furthermore, we have* comparison functors $K^{\mathbb{T}} : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$*,* $J_{\mathbb{U}} : \mathcal{C} \to \mathcal{D}_{\mathbb{U}}$ *defined respectively by*

$$K^{\mathbb{T}}(A) = (G(A), G(\varepsilon_A)), \quad K^{\mathbb{T}}(f) = G(f),$$
$$J_{\mathbb{U}}(B) = (F(B), F(\eta_B)), \quad J_{\mathbb{U}}(g) = F(g),$$

for all objects A *in* $\mathcal{D}$ *and* B *in* $\mathcal{C}$*, and for all morphisms* f *in* $\mathcal{D}$ *and* g *in* $\mathcal{C}$*.*

Definition 0.1.12. A functor $G : \mathcal{D} \to \mathcal{C}$ is called *monadic* if there exists a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$ and an equivalence of categories $J : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ such that $F \circ J \cong G$, where $F : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is the forgetful functor. Equivalently, G is monadic if it has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and so the pair form a monad $\mathbb{T} = (T := G \circ F, \eta, \mu)$ on $\mathcal{C}$, and if the comparison functor $K^{\mathbb{T}} : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an equivalence of categories. Dually, a functor $F : \mathcal{C} \to \mathcal{D}$ is *comonadic* if it has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and so form a comonad $\mathbb{U} = (U := F \circ G, \varepsilon, \Delta)$ on $\mathcal{D}$, and if the comparison functor $J_{\mathbb{U}} : \mathcal{C} \to \mathcal{D}_{\mathbb{U}}$ is an equivalence of categories.

The following result, sometimes known as *Beck's Monadicity Theorem*, gives criterion for when a functor is monadic.

Theorem 0.1.13 (The Barr-Beck Theorem [8])**.** *A functor* $G : \mathcal{D} \to \mathcal{C}$ *is monadic if and only if* *i) G has a left adjoint F;*

ii) G reflects isomorphisms. That is, if $G(f)$ is an isomorphism then f is an isomorphism for all morphisms f; and

iii) given a pair $f, g : A \to B$ of morphisms in $\mathcal{D}$ such that $G(f), G(g)$ have a split coequaliser $d : G(B) \to D$ in $\mathcal{C}$ then f, g have a coequaliser $c : B \to C$ in $\mathcal{D}$ such that $G(c) = d, G(C) = D$.

Proof. This is Theorem 4.4.4 of [8]. □

A dual version of the Barr-Beck theorem then characterises comonadic functors as follows.

Theorem 0.1.14. *A functor $F : \mathcal{C} \to \mathcal{D}$ is comonadic if and only if*

i) F has a right adjoint G;

ii) F reflects isomorphisms; and

iii) given a pair $f, g : A \to B$ are morphisms in $\mathcal{C}$ such that $F(f), F(g)$ have a split equaliser $h : H \to F(A)$ in $\mathcal{D}$ then f, g have an equaliser $e : E \to A$ in $\mathcal{C}$ such that $F(e) = h, F(E) = H$.

0.1.3 Banach, IndBanach and Bornological spaces

Fix a complete valued field k with non-trivial valuation, either Archimedean or non-Archimedean.

Definition 0.1.15. Let Ban_k denote the category of k-Banach spaces, each equipped with a specific norm, and bounded linear transformations between them. If our field is non-Archimedean then Banach spaces may be defined in two ways, depending on whether we require norms to satisfy the usual triangle inequality or the strong triangle inequality. For most of this thesis we will be able to treat both of these definitions uniformly, and will refer to them as the Archimedean and non-Archimedean cases respectively when they differ.

Definition 0.1.16. Let IndBan_k be the Ind completion of Ban_k. That is, IndBan_k is the category whose objects are filtered diagrams $X : I \to \mathrm{Ban}_k$ of Banach spaces, with morphisms

$$\mathrm{Hom}(X, Y) = \mathrm{lim}_{i \in I} \mathrm{colim}_{j \in J} \mathrm{Hom}(X(i), Y(j)).$$

We think of these diagrams as formal colimits, and hence use the notation $"\mathrm{colim}"_{i \in I} X(i)$ for the diagram X. For a Banach space V we will denote by $"V"$ the object in IndBan_k represented by the constant singleton diagram at V, and often just as V when there is no ambiguity.

Definition 0.1.17. We will say that a category $\mathcal{C}$ is *locally presentable* if it is cocomplete and has a small full subcategory $\mathcal{C}_0$ of compact objects such that every object in $\mathcal{C}$ is canonically a colimit of objects in $\mathcal{C}_0$.

Proposition 0.1.18. *The category $IndBan_k$ is a complete and cocomplete, locally presentable, quasi-abelian category, and can be given a closed monoidal structure extending that of Ban_k by defining*

$$("colim"_{i \in I} X_i) \hat{\otimes} ("colim"_{j \in J} Y_j) := "colim"_{\substack{i \in I \\ j \in J}} X_i \hat{\otimes} Y_j,$$

$$\underline{Hom}("colim"_{i \in I} X_i, "colim"_{j \in J} Y_j) := lim_{i \in I}\, colim_{j \in J} \underline{Hom}(X_i, Y_j).$$

Proof. Since Ban_k has cokernels and finite direct sums, IndBan_k is cocomplete. An explicit construction of limits in IndBan_k can be found in Section 1.4.1 of [34]. Proposition 2.1.17 of [39] asserts that IndBan_k is quasi-ableian. By construction, IndBan_k is locally presentable with compact objects Ban_k. □

Remark For an account of Ind completions see [22], and more on IndBan_k can be found in [38], [4], [5] and [34] and numerous other excellent sources. A thorough exposition of quasi-abelian categories can be found in [39]. Results about locally presentable categories, including the Adjoint Functor Theorem (from which Theorem 2.2.2 in Part II is adapted), can be found in [1].

Despite their very formal definition, IndBanach spaces can still be thought of as vector spaces with some extra structure. We make this precise with the following results from [3].

Definition 0.1.19. An IndBanach space X in IndBan_k is *essentially monomorphic* if it can be written as $X \cong \text{"colim"}_{i\in I} X_i$ where $(X_i)_{i\in I}$ is a diagram of Banach spaces whose transition morphisms are all monomorphic. Let $\mathrm{IndBan}_k^{\mathrm{em}}$ denote the full subcategory of IndBan_k of essentially monomorphic IndBanach spaces.

Definition 0.1.20. A *bornology* on a set X is a collection $\mathcal{B}_X$ of subsets of X closed under taking subsets and finite unions such that $X = \bigcup_{B\in\mathcal{B}_X} B$. A set equipped with such a bornology is a *bornological set* and we call sets in $\mathcal{B}_X$ the *bounded* subsets of X. A morphism $f : X \to Y$ between bornological sets is *bounded* if $f(B) \in \mathcal{B}_Y$ for all $B \in \mathcal{B}_X$. The product of two bornological sets X and Y is $X \times Y$ with the *product bornology*

$$\mathcal{B}_{X\times Y} := \{B \times B' \mid B \in \mathcal{B}_X, B' \in \mathcal{B}_Y\}.$$

Definition 0.1.21. A *bornological vector space* is a vector space V equipped with a bornology $\mathcal{B}_V$ such that scalar multiplication $k \times V \to V$ and addition $V \times V \to V$ are bounded. We will denote by Born_k the category of bornological vector spaces with bounded linear transformations. A bornological vector space V is *complete* if there exists a filtered diagram of Banach spaces $(V_i)_{i\in I}$ in Born_k with $\mathrm{colim}_{i\in I} V_i \cong V$ in

Born_k. We will denote by CBorn_k the full subcategory of Born_k consisting of complete bornological vector spaces.

Proposition 0.1.22. *The category $CBorn_k$ is a closed, monoidal, complete and cocomplete, locally presentable, quasi-abelian category.*

Proof. This is Lemma 3.46, Proposition 3.47 and Remark 3.49 of [3]. □

Proposition 0.1.23. *The natural functor $IndBan_k \to CBorn_k$ extending the natural inclusion $Ban_k \to CBorn_k$ restricts to an equivalence of categories $IndBan_k^{em} \xrightarrow{\sim} CBorn_k$.*

Proof. This is Proposition 3.51 of [3], a restatement of Theorem 1.139 of [34]. □

Part I

A categorical reconstruction of crystals and quantum groups at $q = 0$

1.1 Crystal bases and crystals

1.1.1 The category of crystals

We now describe the category of crystals, a generalisation of crystal bases, as Kashiwara defines in [19]. See *loc. cit.* for the motivation and intuition behind the following definitions.

Definition 1.1.1. A *pointed set* is a set with a distinguished element, which we shall always denote by 0. A morphism between pointed sets is a map of sets which preserves 0. We denote this category $\mathit{Set}_\bullet$. We give this category the monoidal structure

$$A \otimes B := \{(a, b) \in A \times B \mid a \neq 0 \neq b\} \sqcup \{0\}.$$

We usually denote by $a \otimes b$ the nonzero elements (a, b) in $A \otimes B$.

Definition 1.1.2. The *category of crystals*, denoted *Crys*, has as objects pointed sets B equipped with maps

$$\tilde{e}_i : B \to B, \quad \tilde{f}_i : B \to B, \quad \mathrm{wt} : B \to \Phi,$$

for all $i \in I$ such that, for a crystal B and $b, b_1, b_2 \in B$,

i) if $\tilde{e}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$;

ii) if $\tilde{f}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$; and

iii) $b_2 = \tilde{f}_i b_1$ if and only if $b_1 = \tilde{e}_i b_2$.

We will call $\tilde{e}_i$ and $\tilde{f}_i$ the *Kashiwara operators* on B. For crystals B_1, B_2, we say that a map $\psi : B_1 \to B_2$ is a morphism of crystals if, for all $b \in B_1$ and $i \in I$, $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ and $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$. We will denote by ε and ϕ the functions $B \setminus \{0\} \to \mathbb{Z} \sqcup \{\infty\}$

given by $\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq 0\}$ and $\phi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq 0\}$ for $b \in B \setminus \{0\}$.

Remark Some definitions of crystals, for example in [19], have a broader definition of morphism and use the term *strict* to distinguish the class of morphisms we have defined above.

Definition 1.1.3. We will call a crystal *finite* if its underlying pointed set is of finite cardinality. A pointed subset of a crystal B is a *subcrystal* if it is closed under the action of $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$. We say that a crystal is *irreducible* if it has no nontrivial proper subcrystals.

Definition 1.1.4. Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}(\mathfrak{g})$. Let A_0 denote the subalgebra of $\mathbb{Q}(q)$ of rational functions without a pole at $q = 0$. A *local base* of M at $q = 0$ is a pointed subset $B \subset M$ containing 0 such that $B \setminus \{0\}$ is a free basis of the A submodule $L_B := A_0 \cdot B$ that it generates and $\mathbb{Q}(q) \otimes_{A_0} L_B \cong M$. Two local bases B and B' are equivalent if they generate the same A_0 submodule $L_B = L_{B'}$ and the base change matrix between $B \setminus \{0\}$ and $B' \setminus \{0\}$ has entries in A_0 and reduces to the identity over $A_0/qA_0 \cong \mathbb{Q}$. A local base of M at $q = 0$ is a crystal base if

i) $B = \coprod_{\alpha \in \Phi} B_\alpha$ where $B_\alpha = B \cap M_\alpha$; and

ii) for each $i \in I$ there is an isomorphism of $U_q(\mathfrak{g})_i$-modules $M \cong \bigoplus_{j \in J} V(n_j)$ for $n_j \in \mathbb{Z}_{\geq 0}$ under which B is sent to $\coprod_{j \in J} B(n_j)$.

Definition 1.1.5. Given a crystal base B of a $U_q(\mathfrak{g})$-module M, we endow B with the following crystal structure that is invariant under equivalence. Its decomposition into homogeneous subsets $B = \coprod_{\alpha \in \Phi} B_\alpha$ allows us to define a weight function $\mathrm{wt} : B \to \Phi$. For each $i \in I$ the isomorphism of $U_q(\mathfrak{g})_i$-modules $M \to \bigoplus_{j \in J} V(n_j)$ along with the Kashiwara operators from Example 0.1.7 on each $V(n_j)$ allow us to define Kashiwara operators $\tilde{e}_i, \tilde{f}_i : B \to B$ as the respective compositions

$$B \xrightarrow{\sim} \coprod_{j \in J} B(n_j) \xrightarrow{\tilde{e}} \coprod_{j \in J} B(n_j) \xrightarrow{\sim} B, \quad B \xrightarrow{\sim} \coprod_{j \in J} B(n_j) \xrightarrow{\tilde{f}} \coprod_{j \in J} B(n_j) \xrightarrow{\sim} B.$$

Theorem 1.1.6. *([20]) For each $\alpha \in \Phi$, $V(\alpha)$ has a crystal base, $B(\alpha)$, unique up to equivalence such that $B(\alpha)_\alpha = \{u_\alpha\}$. Furthermore,*

$$B(\alpha) = \{\tilde{f}_{i_1}^{n_1} \tilde{f}_{i_2}^{n_2} .. \tilde{f}_{i_k}^{n_k} u_\alpha \mid i_1, i_2, .., i_k \in I, n_1, n_2, .., n_k \geq 0\}.$$

Proof. This is Theorem 4.2 of [20]. □

Remark By Proposition 0.1.6 and Theorem 1.1.6 above, any integrable $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}(\mathfrak{g})$ gives a unique crystal in *Crys* arising as a coproduct of crystals of the form $B(\alpha)$.

Definition 1.1.7. We shall call crystals which are coproducts of crystals of the form $B(\alpha)$, as described in the previous remark, the *crystals arising from integrable $U_q(\mathfrak{g})$-modules.* We shall denote their full subcategory $\mathit{Crys}_{\mathfrak{g}}$.

Definition 1.1.8. For a crystal B, we define the crystal $B^\vee := \{b^\vee \mid b \in B\}$ such that $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$, $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$ and $\text{wt}(b^\vee) = -\text{wt}(b)$. For $\alpha \in \Phi_+$ we will denote $B(-\alpha) := B(\alpha)^\vee$.

Definition 1.1.9. For a crystal B we define its *crystal graph* to be the graph whose vertices are the nonzero points in B with arrows labeled by $i \in I$, $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$.

Remark Crystal graphs consist of disjoint unions of connected components. Since each subgraph of a crystal graph gives a subcrystal of the corresponding crystal, a crystal is irreducible if and only if its crystal graph is connected.

Example 1.1.10. *If $\mathfrak{g} = \mathfrak{sl}_2$, each irreducible $U_q(\mathfrak{sl}_2)$-module $V(n)$, $n \in \mathbb{Z}_{\geq 0}$, has a corresponding crystal $B(n) := \{u_k^{(n)}\}_{0 \leq k \leq n}$. This has crystal structure defined by*

$$\tilde{f}(u_k^{(n)}) = \begin{cases} u_{k+1}^{(n)}, & k < n, \\ 0 & k = n, \end{cases} \qquad \tilde{e}(u_k^{(n)}) = \begin{cases} u_{k-1}^{(n)}, & k > 0, \\ 0 & k = 0, \end{cases}$$

so that

$$\varepsilon(u_k^{(n)}) = k, \quad \phi(u_k^{(n)}) = n - k, \quad wt(u_k^{(n)}) = n - 2k.$$

So the crystal base of an irreducible $U_q(\mathfrak{sl}_2)$-module has crystal graph

$$\circ \overbrace{\to \circ \to \circ \ldots\ldots \circ \to \circ \to}^{\varepsilon(b)} b \overbrace{\to \circ \to \circ \ldots\ldots \circ \to \circ \to}^{\phi(b)} \circ.$$

Thus we have $B(-n) \cong B(n)$ for $n \in \mathbb{Z}_{\geq 0}$.

The following important result characterises morphisms between irreducible crystals.

Lemma 1.1.11 (Schur's Lemma for crystals,[16])**.** *A nonzero morphism between irreducible crystals in $Crys_{\mathfrak{g}}$ is an isomorphism, and there is at most one isomorphism between any two irreducible crystals.*

Proof. This is Lemma 1 of [16]. □

Kashiwara defines the following monoidal structure on *Crys* in [18].

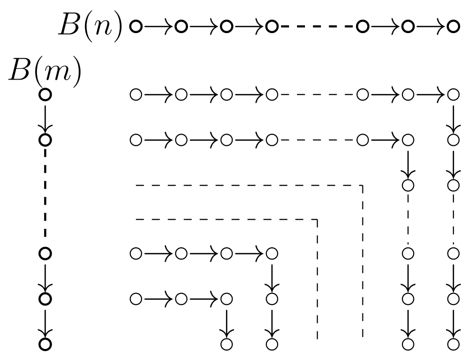
Definition 1.1.12. Let B_1, B_2 be crystals. Their tensor product is the pointed set

$B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ with

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \phi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \\ \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2). \end{aligned}$$

The unit for this monoidal structure is the crystal $\mathbb{B}(0) = \{b_0, 0\}$ where $\tilde{e}_i b_0 = 0 = \tilde{f}_i b_0$ and $\text{wt}(b_0) = 0$.

Example 1.1.13. *In the $\mathfrak{sl}_2$ case, for the crystals $B(n), B(m)$ this can be visualised as follows:*



Remark In Henriques and Kamnitzer's paper on *Crystals and Coboundary Categories* [16] they describe a *commuter for crystals* $\sigma_{B_1 \otimes B_2} : B_1 \otimes B_2 \to B_2 \otimes B_1$ for crystals B_1, B_2 in $Crys_{\mathfrak{g}}$. Given a crystal B in $Crys_{\mathfrak{g}}$ one defines a new crystal structure on $\overline{B} := \{\overline{b} \mid b \in B\}$ by

$$\tilde{e}_i(\overline{b}) = \overline{\tilde{f}_{\theta(i)}(b)}, \quad \tilde{f}_i(\overline{b}) = \overline{\tilde{e}_{\theta(i)}(b)}, \quad \text{wt}(\overline{b}) = w_0 \cdot \text{wt}(b),$$

where w_0 is the longest element of the Weyl group associated to the given root datum and $\theta : I \to I$ is the automorphism defined such that $\alpha_{\theta(i)} = -w_0 \cdot \alpha_i$. By Lemma 2 of [16] there is a canonical isomorphism of crystals $\overline{B} \cong B$, and the composition

$$\zeta : B \xrightarrow{b \mapsto \bar{b}} \overline{B} \cong B$$

gives a bijection of pointed sets known as the *Schützenberger involution*. Then the commuter of crystals is defined by $b \otimes b' \mapsto \zeta(\zeta(b') \otimes \zeta(b))$.

Proposition 1.1.14 ([19]). *For $\alpha, \beta \in \Phi_+$, there is an isomorphism of crystals*

$$B(\alpha) \otimes B(\beta) \cong \coprod B(\alpha + wt(b))$$

where the coproduct ranges over all $b \in B(\beta)$ such that $\varepsilon_i(b) \leq \lambda_i(\alpha)$ for all $i \in I$.

Proof. This is Lemma 4.1 of [19]. □

Corollary 1.1.15. *For $\alpha, \beta \in \Phi_+$, $B(\alpha+\beta)$ appears in the decomposition of $B(\alpha) \otimes B(\beta)$ into irreducible components.*

Proof. This follows from Proposition 1.1.14, since $\varepsilon_i(u_\beta) = 0 \leq \lambda_i(\alpha)$ for each i. □

1.2 The crystal algebra $\mathcal{B}$

1.2.1 The crystal associated to $A_q(\mathfrak{g})$

Recall the definition of the quantum co-ordinate ring $A_q(\mathfrak{g})$ from Section 0.1.1. It is known that its comodules are precisely the representations of $U_q(\mathfrak{g})$ in $\mathcal{O}_{\text{int}}(\mathfrak{g})$. This is because, as a coalgebra, $A_q(\mathfrak{g})$ is a direct sum of coalgebras $V(\alpha) \otimes V(\alpha)^*$ whose comodules are precisely direct sums of copies of $V(\alpha)$. The focus of this section is to investigate whether an analogous result is true in the setting of crystal bases. We will consider the corresponding crystal

$$\mathcal{B} := \coprod_{\alpha \in \Phi^+} B(\alpha) \otimes B(-\alpha).$$

Definition 1.2.1. Let $\mathcal{B}$ be the crystal $\mathcal{B} := \coprod_{\alpha\in\Phi_+} B(\alpha)\otimes B(-\alpha)$. For $\alpha, \beta \in \Phi$, we will denote by $b \cdot b'$ the image of $b \otimes b'$ in $B(\alpha)\otimes B(\beta)$ under the decomposition into irreducible components

$$B(\alpha)\otimes B(\beta) \cong \coprod_{\gamma\in\Gamma_{\alpha,\beta}} B(\gamma).$$

We then define maps

$$\mu_{\alpha,\beta} : B(\alpha)\otimes B(-\alpha)\otimes B(\beta)\otimes B(-\beta) \to \coprod_{\gamma\in\Gamma_{\alpha,\beta}} B(\gamma)\otimes B(-\gamma)$$

by mapping $b\otimes b'^{\vee}\otimes d\otimes d'^{\vee}$ to $(b\cdot d)\otimes(b'\cdot d')^{\vee}$ whenever $b\otimes d$ and $b'\otimes d'$ lie in the same irreducible component of $B(\alpha)\otimes B(\beta)$ and to 0 if not. Collectively, these induce a map $\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$, which we will denote μ. Let η denote the embedding

$$\eta : B(0) \cong B(0)\otimes B(0) \hookrightarrow \mathcal{B}.$$

Note that the maps $\mu_{\alpha,\beta}$ above are not morphisms of crystals, just of pointed sets.

Proposition 1.2.2. *The maps μ and η define an algebra structure on $\mathcal{B}$ in $Set_\bullet$.*

Proof. Let $\alpha, \beta, \gamma \in \Phi_+$ and let $b, b' \in B(\alpha)$, $c, c' \in B(\beta)$, $d, d' \in B(\gamma)$. By the associativity of the monoidal structure in Definition 1.1.12, $(b\cdot c)\cdot d$ corresponds to $b\cdot(c\cdot d)$ under the decomposition of $(B(\alpha)\otimes B(\beta))\otimes B(\gamma) \cong B(\alpha)\otimes(B(\beta)\otimes B(\gamma))$ into irreducible components. Thus both $(\mu\otimes \mathrm{Id})\circ\mu$ and $(\mathrm{Id}\otimes\mu)\circ\mu$ map $b\otimes b'^{\vee}\otimes c\otimes c'^{\vee}\otimes d\otimes d'^{\vee}$ to $(b\cdot c\cdot d)\otimes(b'\cdot c'\cdot d')^{\vee}$ if $b\otimes c\otimes d$ and $b'\otimes c'\otimes d'$ lie in the same irreducible component of $B(\alpha)\otimes B(\beta)\otimes B(\gamma)$ or 0 otherwise. So μ is an associative multiplication. Since $B(0)\otimes B(\alpha) \xrightarrow{\sim} B(\alpha)$, $b_0\otimes x\mapsto x$, is an isomorphism, and likewise $B(\alpha)\otimes B(0)\xrightarrow{\sim} B(\alpha)$, η is a unit for this multiplication. □

Definition 1.2.3. For a symmetric monoidal category $\mathcal{C}$ with monoidal unit $\mathbb{I}$, we say that an object $A^\vee$ is *dual* to an object A in $\mathcal{C}$ if there exist maps

$$\iota_A : \mathbb{I} \to A^\vee \otimes A, \quad \epsilon_A : A \otimes A^\vee \to \mathbb{I},$$

called the coevaluation and evaluation respectively, such that the composition

$$A \cong \mathbb{I} \otimes A \xrightarrow{\mathrm{Id}\otimes\iota_A} A \otimes A^\vee \otimes A \xrightarrow{\epsilon_A\otimes\mathrm{Id}} A \otimes \mathbb{I} \cong A$$

is the identity on A. We will say that A is *dualisable* if such a dual $A^\vee$ exists.

Recall that, in Definition 0.1.8, the comultiplication of $A_q(\mathfrak{g})$ is induced by coevaluation maps $k \to V(\alpha)^* \otimes V(\alpha)$. These exists since each $V(\alpha)$ is dualisable in the category of vector spaces, with dual $V(\alpha)^*$. We do not, however, have dualisability for $B(\alpha)$ in $Set_\bullet$.

Lemma 1.2.4. *The pointed set $B(\alpha)$ is not dualisable in the symmetric monoidal category $Set_\bullet$ for nonzero $\alpha \in \Phi_+$.*

Proof. Suppose we have a pointed set A along with evaluation and coevalutation maps ϵ and ι that exhibit A as a dual to $B(\alpha)$. The monoidal unit in $Set_\bullet$ is the pointed set $\mathbb{I} = \{1, 0\}$, and so the map ι is given by an element $\iota(1) = a \otimes b \in A \otimes B(\alpha)$. Then for any $b' \in B(\alpha)$

$$1 \otimes b' = \epsilon(a \otimes b) \otimes b \in \mathbb{I} \otimes B(\alpha)$$

so $b = b'$. But this gives a contradiction as $B(\alpha)$ has more than one non-zero element for $\alpha \neq 0$. □

The above lemma means we cannot proceed in direct analogy to $A_q(\mathfrak{g})$ to give $\mathcal{B}$ a bialgebra structure. In Section 1.4 we will work in $\mathbb{Z}$-modules instead of pointed sets where we regain dualisability and can construct a bialgebra structure on $\mathbb{B} := \mathbb{Z}\mathcal{B}$.

Before that, we will use a categorical approach to determine that $Crys_{\mathfrak{g}}$ cannot be reconstructed as comodules over a coalgebra in $Set_{\bullet}$ but can be reconstructed as coalgebras over a comonad on this category.

1.3 A functorial approach to crystals

1.3.1 Crystals as coalgebras of a comonad

Definition 1.3.1. Suppose we have a set $\mathbb{X}$ and, for each $x \in \mathbb{X}$, a pointed set $B(x)$. Let $\mathcal{C}_{\mathbb{X}}$ denote the category whose objects are sets A equipped with a map $\pi_A : A \to \mathbb{X}$. Morphisms in this category are defined to be morphisms of pointed sets $A \sqcup \{0\} \xrightarrow{\psi} A' \sqcup \{0\}$ such that $\pi_A(a) = \pi_{A'}\phi(a)$ whenever $\phi(a) \neq 0$. We will denote by F the functor

$$\mathcal{C}_{\mathbb{X}} \to Set_{\bullet}, \quad (A \xrightarrow{\pi_A} \mathbb{X}) \mapsto \coprod_{a \in A} B(\pi_A(a)).$$

For a morphism $A \sqcup \{0\} \xrightarrow{\psi} A' \sqcup \{0\}$ in $\mathcal{C}_{\mathbb{X}}$, $F(\phi)$ maps $B(\pi_A(a))$ isomorphically to $B(\pi_{A'}\phi(a))$ whenever $\phi(a) \neq 0$, and maps $B(\pi_A(a))$ to 0 when $\phi(a) = 0$.

Lemma 1.3.2. *If $\mathbb{X} = \Phi_+$ and $B(\alpha)$ are as previously defined for $\alpha \in \Phi_+$ then $\mathcal{C}_{\mathbb{X}} \cong Crys_{\mathfrak{g}}$. Furthermore, under this equivalence, F is the forgetful functor to pointed sets.*

Proof. This equivalence is given by

$$\mathcal{C}_{\mathbb{X}} \to Crys_{\mathfrak{g}}, \quad (A \xrightarrow{\pi_A} \mathbb{X}) \mapsto \coprod_{a \in A} B(\pi_A(a)),$$

where a morphism $\psi : A \sqcup \{0\} \to A' \sqcup \{0\}$ in $\mathcal{C}_{\mathbb{X}}$ is mapped to the morphism of crystals $\coprod_{a \in A} B(\pi_A(a)) \to \coprod_{a' \in A'} B(\pi_{A'}(a'))$ where $B(\pi_A(a))$ is mapped isomorphically to $B(\pi_{A'}\phi(a))$ whenever $\phi(a) \neq 0$, and to 0 when $\phi(a) = 0$. Its quasi-inverse is given by the functor

$$Crys_{\mathfrak{g}} \to \mathcal{C}_{\mathbb{X}}, \quad \coprod_{j \in J} B(\alpha_j) \mapsto (J, j \mapsto \alpha_j).$$

By Lemma 1.1.11, a morphism of crystals $\coprod_{j\in J} B(\alpha_j) \to \coprod_{j'\in J'} B(\beta_{j'})$ maps each $B(\alpha_j)$ either isomorphically to some $B(\beta_{j'})$, where $\alpha_j = \beta_{j'} \in \mathbb{X}$, or to 0. The resulting map $J \sqcup \{0\} \to J' \sqcup \{0\}$ maps $j \mapsto j'$ in the former case and $j \mapsto 0$ in the latter. $\square$

Definition 1.3.3. Let G denote the functor

$$G : Set_{\bullet} \to \mathcal{C}_{\mathbb{X}}, \quad X \mapsto G(X)$$

where $G(X)$ is the set $\coprod_{x\in\mathbb{X}} (\mathrm{Hom}(B(x), X) \setminus \{0\})$ equipped with the map $\pi_{G(X)}$ which takes $f \in \mathrm{Hom}(B(x), X) \setminus \{0\}$ to x. A map of pointed sets $\psi : X \to Y$ gives the map $G(\psi) : G(X) \to G(Y)$ taking $f \in \mathrm{Hom}(B(x), X)$ to $\phi \circ f \in \mathrm{Hom}(B(x), Y)$.

Proposition 1.3.4. *The functor G is right adjoint to F.*

Proof. Suppose we have a morphism $(A \xrightarrow{\pi_A} \mathbb{X}) \xrightarrow{f} (GX \xrightarrow{\pi_{GX}} \mathbb{X})$ in $\mathcal{C}_{\mathbb{X}}$. Each $a \in A$ is either mapped to 0 or to an element of

$$\pi_{GX}^{-1}(\pi_A(a)) = \mathrm{Hom}(B(\pi_A(a)), X) \setminus \{0\}.$$

That is, a is mapped to a function $f_a \in \mathrm{Hom}(B(\pi_A(a)), X)$, which allows us to define a map $FA = \coprod_{a\in A} B(\pi_A(a)) \to X$. Conversely, a map of pointed sets $FA = \coprod_{a\in A} B(\pi_A(a)) \xrightarrow{g} X$ is given by a collection of maps

$$g_a \in \mathrm{Hom}(B(\pi_A(a)), X) = (\mathrm{Hom}(B(\pi_A(a)), X) \setminus \{0\}) \sqcup \{0\}.$$

Thus we get a map

$$A \sqcup \{0\} \to \coprod_{a \in A} (\mathrm{Hom}(B(\pi_A(a)), X) \setminus \{0\}) \sqcup \{0\} = GX \sqcup \{0\}$$

by taking $a \in A$ to the function g_a. These mutual inverses give the adjunction

$$\mathrm{Hom}_{Set_\bullet}(FA, X) \cong \mathrm{Hom}_{\mathcal{C}_\mathbb{X}}(A, GX).$$

Note that the unit of this adjunction is given by maps

$$FG(X) = \coprod_{x \in \mathbb{X}} \coprod_{\mathrm{Hom}(B(x),X)} B(x) \to X,$$

for pointed sets X, where the copy of $B(x)$ indexed by $f \in \mathrm{Hom}(B(x), X)$ is mapped to X via f. The counit of the adjunction is given by maps

$$A \to GF(A) = \coprod_{x \in \mathbb{X}} \mathrm{Hom}(B(x), \coprod_{a \in A} B(\pi_A(a))),$$

for (A, π_A) in $\mathcal{C}_\mathbb{X}$, where $a \in A$ is mapped to the inclusion of $B(\pi_A(a))$ into $\coprod_{a' \in A} B(\pi_A(a'))$. □

Lemma 1.3.5. *The functor F reflects isomorphisms.*

Proof. Let $\phi : A \sqcup \{0\} \to A' \sqcup \{0\}$ be a morphism in $\mathcal{C}_\mathbb{X}$, and suppose that $F(\phi) : FA \to FA'$ is an isomorphism. This precisely means that $\phi(a) \neq 0$ for all $a \in A$ and each $B(\pi_{A'}(a'))$ for $a' \in A'$ is the image of some $B(\pi_A(a))$ for some $a \in A$. Thus ϕ is both monic and epic, hence an isomorphism. Its inverse it just its inverse as a map of pointed sets. □

Lemma 1.3.6. *$\mathcal{C}_\mathbb{X}$ has, and F preserves, all equalisers.*

Proof. Suppose we have parallel maps $f, g : A \sqcup \{0\} \to A' \sqcup \{0\}$ in $\mathcal{C}_\mathbb{X}$. Their equaliser is just $\{a \in A \mid f(a) = g(a)\}$. Likewise, the equaliser of $F(f)$ and $F(g)$

in $Set_\bullet$ is $\{b \in FA \mid F(f)(b) = F(g)(b)\}$. On each component $B(\pi_A(a))$ of FA, if $f(a) = g(a) \neq 0$ then they are both the same isomorphism

$$B(\pi_A(a)) \xrightarrow{\sim} B(\pi_{A'}(f(a))) = B(\pi_{A'}(g(a))).$$

If $f(a) = 0 = g(a)$ then $F(f)$ and $F(g)$ are both the zero map. Otherwise $F(f)$ and $F(g)$ must disagree on all non-zero elements of the component. Hence the equaliser of $F(f)$ and $F(g)$ is the union of $B(\pi_A(a))$ where $f(a) = g(a)$. This is $F(\{a \in A \mid f(a) = g(a)\})$, and the lemma is proved. □

Theorem 1.3.7. *For the comonad* $\mathbb{U} = (U = F \circ G, \eta, \mu)$ *the comparison functor* $J_\mathbb{U} : \mathcal{C}_\mathbb{X} \to Set_{\bullet\mathbb{U}}$ *is an equivalence of categories between* $\mathcal{C}_\mathbb{X}$ *and the category of algebras over the comonad* $\mathbb{U}$ *in* $Set_\bullet$*.*

Proof. This follows from Theorem 0.1.13, using Proposition 1.3.4, Lemma 1.3.5 and Lemma 1.3.6. □

Corollary 1.3.8. *Setting* $\mathbb{X} = \Phi_+$ *and* $B(\alpha)$ *are as previously defined for* $\alpha \in \Phi_+$*, the comparison functor* $J_\mathbb{U} : Crys_\mathfrak{g} \to Set_{\bullet\mathbb{U}}$ *is an equivalence of categories between* $Crys_\mathfrak{g}$ *and the category of algebras over the comonad* $\mathbb{U} = (U = F \circ G, \eta, \mu)$ *in* $Set_\bullet$*.*

Proof. This follows from Theorem 1.3.7 and Lemma 1.3.2. □

Remark Explicitly, from the proof of Proposition 1.3.4, we see that

$$U = FG : A \mapsto \coprod_{\alpha \in \Phi_+} \coprod_{\substack{f \in \mathrm{Hom}(FB(\alpha), A) \\ f \neq 0}} FB(\alpha)_f$$

with

$$\eta_{B(\alpha)} : B(\alpha) \to \coprod_{\beta \in \Phi_+} \coprod_{\substack{f \in \mathrm{Hom}(FB(\beta), FB(\alpha)) \\ f \neq 0}} B(\beta)_f,$$

$$b \mapsto (b)_{\mathrm{id}_{FB(\alpha)}} \in B(\alpha)_{\mathrm{id}_{FB(\alpha)}},$$

and

$$\varepsilon_A : \coprod_{\alpha\in\Phi_+} \coprod_{\substack{f\in\mathrm{Hom}(FB(\alpha),A)\\ f\neq 0}} F(B(\alpha)_f) \to A,$$

$$(b)_f \mapsto f(b).$$

So

$$\Delta_A : \coprod_{\alpha\in\Phi_+} \coprod_{\substack{f\in\mathrm{Hom}(FB(\alpha),A)\\ f\neq 0}} FB(\alpha)_f \to \coprod_{\beta\in\Phi_+} \coprod_{\substack{g\in\mathrm{Hom}(FB(\beta),FG(A))\\ g\neq 0}} FB(\beta)_g$$

maps the copy of $FB(\alpha)$ indexed by $f : FB(\alpha) \to A$ isomorphically to the copy of $FB(\alpha)$ indexed by $FB(\alpha) \cong FB(\alpha)_f \hookrightarrow FG(A)$. From here we can explicitly see the coalgebra structure of each $B(\alpha)$ over FG is given by a map

$$\zeta : F(B(\alpha)) \to FG(F(B(\alpha)),\ \ b \mapsto (b)_{\mathrm{id}_{F(B(\alpha))}}$$

which extends to the coalgebra structure of a general crystal $X = \coprod_{j\in J} B(\beta_j)$ as follows:

$$\zeta : F(X) \to FG(F(X)),\ \ b \mapsto (b)_{(F(B(\beta_j))\hookrightarrow FX)} \text{ for } b \in F(B(\beta_j)).$$

Corollary 1.3.9. *There is no coalgebra in $Set_\bullet$ whose category of comodules is equivalent to $Crys_{\mathfrak{g}}$ as categories over $Set_\bullet$.*

Proof. Suppose there is a coalgebra C in $Set_\bullet$ whose category of comodules is equivalent to $Crys_{\mathfrak{g}}$, and suppose this equivalence preserves the forgetful functor to $Set_\bullet$. Then the right adjoint to this forgetful functor, G, would be isomorphic to $C \otimes - : Set_\bullet \to C\text{-comod} \cong Crys_{\mathfrak{g}}$. Then $U \cong C \otimes -$, as a functor on $Set_\bullet$, preserves coproducts. However, by the explicit description of U, this is not the case and we reach a contradiction. □

1.3.2 Recovering the crystal structure

Given a pointed set A with a coalgebra structure (A, ζ_A) over our comonad $U = FG$, we know from the above that A carries a crystal structure that has been forgotten by the functor F. In fact, there is a way of recovering this crystal structure from the coalgebra structure.

Proposition 1.3.10. *We regain the Kashiwara operator $\tilde{f}_i$ (and similarly $\tilde{e}_i$) on a U-coalgebra A via the following composition:*

$$A \xrightarrow{\zeta_A} FG(A) \xrightarrow{F\tilde{f}_i} FG(A) \xrightarrow{\varepsilon_A} A.$$

We also regain the weight function via

$$A \to FG(A) \to \Phi$$

where the last arrow is the map $(b)_f \mapsto wt(b)$.

Proof. This follows from the explicit description of the U-coaction on a crystal as described in the remark following Corollary 1.3.8. □

1.3.3 The monoidal structure of U

Recall that a bialgebra is simultaneously an algebra and a coalgebra where the structure maps are compatible. In the setting of functors, there is no analogous notion of a bimonad. The subtlety comes from the lack of symmetry when composing functors. There is no natural twist $A \circ B \Rightarrow B \circ A$ for functors A, B on a category $\mathcal{C}$, and so, whilst the tensor product of two algebras in a symmetric monoidal category again gives an algebra, the composition of two monads does not naturally give a monad. So, if a functor T on a category is both a monad and a comonad, we cannot simply ask that the comultiplication map $T \Rightarrow TT$ be a morphism of monads. Recall that, for a bialgebra H, the categories of comodules of H inherit a monoidal structure. We wish to generalise this property of bialgebras that allows us to encode a monoidal structure on comodules. To generalise this, we recall the definition of a monidal functor. For more on these notions see [36], [7], [6] and [30].

Definition 1.3.11. Let $T : \mathcal{C} \to \mathcal{D}$ be a functor between monoidal categories. T is said to be *monoidal* if we equip T with the data of a natural tranformation

$$\chi_{A,B} : T(A) \otimes T(B) \Rightarrow T(A \otimes B)$$

and a morphism $\mathbb{I} \to T(\mathbb{I})$, where $\mathbb{I}$ is taken to be the identity of the tensor product, satisfying the diagram

$$\begin{array}{ccc} T(A) \otimes (T(B) \otimes T(C)) & \xrightarrow{\sim} & (T(A) \otimes T(B)) \otimes T(C) \\ \downarrow \mathrm{Id} \otimes \chi_{B,C} & & \downarrow \chi_{A,B} \otimes \mathrm{Id} \\ T(A) \otimes T(B \otimes C) & & T(A \otimes B) \otimes T(C) \\ \downarrow \chi_{A,B\otimes C} & & \downarrow \chi_{A\otimes B,C} \\ T(A \otimes (B \otimes C)) & \xrightarrow{\sim} & T((A \otimes B) \otimes C) \end{array},$$

and such that the compositions

$$T(A) \cong T(A) \otimes \mathbb{I} \to T(A) \otimes T(\mathbb{I}) \to T(A \otimes \mathbb{I}) \cong T(A),$$

$$T(A) \cong \mathbb{I} \otimes T(A) \to T(\mathbb{I}) \otimes T(A) \to T(\mathbb{I} \otimes A) \cong T(A),$$

are the identity on $T(A)$ for all A, B and C in $\mathcal{C}$. We say that T is *strong monoidal* if $\chi_{A,B}$ and $\mathbb{I} \to T(\mathbb{I})$ are isomorphisms. If T is a monoidal comonad on a monoidal category, we will call it is a *bicomonad* if the diagram

$$\begin{array}{ccccc} T(A) \otimes T(B) & & \xrightarrow{\chi_{A,B}} & & T(A \otimes B) \\ \Delta_A \otimes \Delta_B \downarrow & & & & \Delta_{A\otimes B} \downarrow \\ TT(A) \otimes TT(B) & \xrightarrow[\chi_{T(A),T(B)}]{} & T(T(A) \otimes T(B)) & \xrightarrow[T(\chi_{A,B})]{} & TT(A \otimes B) \end{array},$$

commutes and $\chi_{A,B} \circ \epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B$ as maps $T(A) \otimes T(B) \to A \otimes B$ for all A and B in $\mathcal{C}$.

Remark For a comonad $\mathbb{U}$, Proposition 1.4 of [36] shows that the property of being a bicomonad gives a monoidal structure on the category of coalgebras. The coaction on a tensor product of two coalgebras is given by the composition

$$A \otimes B \to T(A) \otimes T(B) \to T(A \otimes B)$$

where the first arrow is given by the respective coactions of A and B, and the second given by χ. In fact, Moerdijk proves the following in [36].

Theorem 1.3.12 ([36])**.** *Let $\mathbb{U} = (U, \Delta, \epsilon)$ be a comonad on a monoidal category $\mathcal{C}$. Then monoidal structures on $\mathcal{C}_{\mathbb{U}}$ such that the forgetful functor F to $\mathcal{C}$ is strong monoidal correspond to bicomonad structures on $\mathbb{U}$.*

Proof. This is Theorem 7.1 of [36]. Suppose we have endowed $\mathcal{C}_{\mathbb{U}}$ with a monoidal structure $(\otimes, \mathbb{I})$ such that F is strong monoidal. Let $G : \mathcal{C} \to \mathcal{C}_{\mathbb{U}}$, $C \mapsto (UC, \Delta_C)$ denote the right adjoint to F mapping an object of $\mathcal{C}$ to its free coalgebra. Then $U = FG$ and we obtain $\chi_{A,B} : U(A) \otimes U(B) \to U(A \otimes B)$ as the image of $\epsilon_A \otimes \epsilon_B$ under the composition

$$\begin{aligned}
\mathrm{Hom}(FGA \otimes FGB, A \otimes B) &\cong \mathrm{Hom}(F(GA \otimes GB), A \otimes B)\\
&\cong \mathrm{Hom}(GA \otimes GB, G(A \otimes B))\\
&\to \mathrm{Hom}(F(GA \otimes GB), FG(A \otimes B))\\
&\cong \mathrm{Hom}(FGA \otimes FGB, FG(A \otimes B)).
\end{aligned}$$

The morphism $U(\mathbb{I}) \to \mathbb{I}$ is given by the counit $\epsilon_{\mathbb{I}}$. □

Proposition 1.3.13. *The monoidal structure on $\mathbb{U}$ corresponding to the monoidal structure on $Crys_{\mathfrak{g}}$ under the equivalence in Corollary 1.3.8 is given as follows. For* *each* $b \otimes b' \in FB(\alpha)_f \otimes FB(\beta)_g \subset U(A) \otimes U(B)$ *indexed by* $f : FB(\alpha) \to A$ *and* $g : FB(\beta) \to B$ *there is some* $\gamma_{b,b'} \in \Gamma_{\alpha,\beta}$ *such that the image* $b \cdot b'$ *of* $b \otimes b'$ *in the decomposition* $B(\alpha) \otimes B(\beta) \cong \coprod_{\gamma \in \Gamma_{\alpha,\beta}} B(\gamma)$ *lies in the component* $B(\gamma_{b,b'})$*. Then we define*

$$\chi_{A,B} : U(A) \otimes U(B) \to U(A \otimes B)$$

by mapping $b \otimes b'$ *to* $b \cdot b'$ *in the copy of* $B(\gamma_{b,b'})$ *indexed by the map*

$$B(\gamma_{b,b'}) \hookrightarrow \coprod_{\gamma \in \Gamma_{\alpha,\beta}} B(\gamma) \cong B(\alpha) \otimes B(\beta) \xrightarrow{f \otimes g} A \otimes B.$$

We define a map $\mathbb{I} \to U(\mathbb{I})$*, where* $\mathbb{I} = \{0, 1\}$ *is the monoidal unit in* $Set_\bullet$*, by mapping* 1 *to* $b_0 \in B(0)$ *indexed by the map* $FB(0) \xrightarrow{\sim} \mathbb{I}$*,* $b_0 \mapsto 1$*.*

Proof. This result follows from the proof of Theorem 1.3.12. The image of the map $\varepsilon_A \otimes \varepsilon_B$ under the isomorphism

$$\begin{aligned} \mathrm{Hom}(FGA \otimes FGB, A \otimes B) &\cong \mathrm{Hom}(F(GA \otimes GB), A \otimes B) \\ &\cong \mathrm{Hom}(GA \otimes GB, G(A \otimes B)) \end{aligned}$$

is the map taking $b \otimes b' \in B(\alpha)_f \otimes B(\beta)_g \subset G(A) \otimes G(B)$ indexed by $f : FB(\alpha) \to A$ and $g : FB(\beta) \to B$ to $b \cdot b'$ in the copy of the irreducible crystal $B(\gamma_{b,b'})$ indexed by the composition

$$B(\gamma_{b,b'}) \hookrightarrow \coprod_{\gamma \in \Gamma_{\alpha,\beta}} B(\gamma) \cong B(\alpha) \otimes B(\beta) \xrightarrow{f \otimes g} A \otimes B.$$

□

1.4 A crystal bialgebra

1.4.1 The bialgebra $\mathbb{B}$

Definition 1.4.1. For $\alpha \in \Phi_+$ let $\mathbb{B}(\alpha)$ be the free abelian group $\mathbb{Z}B(\alpha)$. Let ι_α and ϵ_α denote the homomorphisms

$$\begin{aligned}
\iota_\alpha &: \mathbb{Z} \to \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha), \quad 1 \mapsto \textstyle\sum_{b \in B(\alpha)} b^\vee \otimes b, \\
\varepsilon_\alpha &: \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \to \mathbb{Z}, \quad b \otimes b' \mapsto \delta_{b,b'^\vee},
\end{aligned}$$

called the *coevaluation* and *evaluation* respectively.

Proposition 1.4.2. *The composition*

$$\mathbb{B}(\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \xrightarrow{id \otimes \iota_\alpha} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \xrightarrow{\varepsilon_\alpha \otimes id} \mathbb{B}(\alpha)$$

agrees with the identity. Hence $\mathbb{B}(-\alpha)$ is dual to $\mathbb{B}(\alpha)$ in the category of free abelian groups.

Proof. This follows since the image of $b \otimes b' \in B(\alpha) \otimes B(-\alpha)$ under this composition is $\sum_{d \in B(\alpha)} \delta_{b,d} d \otimes b' = b \otimes b'$. □

Definition 1.4.3. Let $\mathbb{B}$ denote the free abelian group on the crystal $\mathcal{B}$,

$$\mathbb{B} := \mathbb{Z}\mathcal{B} = \oplus_{\alpha \in \Phi_+} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha).$$

Let $\Delta : \mathbb{B} \to \mathbb{B} \otimes \mathbb{B}$ denote the homomorphism defined on each summand $\mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \otimes \mathbb{B}(-\alpha)$ by $\mathrm{id} \otimes \iota_\alpha \otimes \mathrm{id}$, let $\varepsilon : \mathbb{B} \to \mathbb{Z}$ be the sum of the maps ε_α. Let $\mu : \mathbb{B} \otimes \mathbb{B} \to \mathbb{B}$ and $\eta : \mathbb{Z} \cong \mathbb{B}(0) \otimes \mathbb{B}(-0) \hookrightarrow \mathbb{B}$ be the homomorphisms induced by the multiplication and unit in Definition 1.2.1.

Proposition 1.4.4. *The maps η, μ, ε and Δ make $\mathbb{B}$ a $\mathbb{Z}$-bialgebra.*

Proof. The fact that $(\mathbb{B}, \mu, \eta)$ forms a $\mathbb{Z}$-algebra follows from Proposition 1.2.2. Both $(\Delta \otimes \mathrm{Id}) \circ \Delta$ and $(\mathrm{Id} \otimes \Delta) \circ \Delta$ can be identified with the map $\mathrm{Id} \otimes \iota_\alpha \otimes \iota_\alpha \otimes \mathrm{Id}$ on

$$\mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{B}(-\alpha),$$

so the comultiplication is coassociative. Furthermore,

$$\begin{aligned} \textstyle\sum_{d \in B(\alpha)} \varepsilon(b \otimes d^\vee)\ d \otimes b' &= \textstyle\sum_{d \in B(\alpha)} \delta_{b,d}\ d \otimes b' \\ &= b \otimes b' \\ &= \textstyle\sum_{d \in B(\alpha)} \delta_{d,b'^\vee}\ b \otimes d^\vee \\ &= \textstyle\sum_{d \in B(\alpha)} \varepsilon(d \otimes b')\ b \otimes d^\vee, \end{aligned}$$

so ε acts as a counit. It remains to verify that Δ and ε are $\mathbb{Z}$-algebra homomorphisms. Let $b, b' \in B(\alpha)$ and $d, d' \in B(\beta)$. If $(b \cdot d)$ and $(d' \cdot b')^\vee$ lie in different irreducible crystals then $\Delta \circ \mu((b \otimes b') \otimes (d \otimes d')) = 0$. Also,

$$(b \otimes b') \otimes (d \otimes d') \xmapsto{\mu_{\mathbb{B}\otimes\mathbb{B}} \circ \Delta_{\mathbb{B}\otimes\mathbb{B}}} \sum_{\substack{b'' \in B(\alpha) \\ d'' \in B(\beta)}} \mu(b \otimes b''^\vee \otimes d \otimes d''^\vee) \otimes \mu(b'' \otimes b' \otimes d'' \otimes d'),$$

the nonzero terms of which only occur when both $b \otimes d$ and $(d''^\vee \otimes b''^\vee)^\vee = b'' \otimes d''$ lie in the same component, and $b'' \otimes d''$ and $(d' \otimes b')^\vee$ lie in the same component. Since this never occurs, this sum must also be zero. Now suppose that $(b \otimes d)$ and $(d' \otimes b')^\vee$ do lie in the same irreducible component, $B(\gamma)$ say. In this case we have

$$(b \otimes b') \otimes (d \otimes d') \overset{\mu}{\mapsto} (b \cdot d) \otimes (d' \cdot b') \overset{\Delta}{\mapsto} \sum_{c \in B(\gamma)} ((b \cdot d) \otimes c^\vee) \otimes (c \otimes (d' \cdot b'))$$

whilst

$$
\begin{aligned}
(b \otimes b') \otimes (d \otimes d') &\xmapsto{\Delta\otimes\Delta} \sum_{\substack{b'' \in B(\alpha)\\ d'' \in B(\beta)}} b \otimes b''^{\vee} \otimes b'' \otimes b' \otimes d \otimes d''^{\vee} \otimes d'' \otimes d' \\
&\xmapsto{\mu_{\mathbb{B}\otimes\mathbb{B}}} \sum_{\substack{b'' \in B(\alpha)\\ d'' \in B(\beta)}} \mu(b \otimes b''^{\vee} \otimes d \otimes d''^{\vee}) \otimes \mu(b'' \otimes b' \otimes d'' \otimes d') \\
&= \sum_{\substack{b'' \in B(\alpha)\\ d'' \in B(\beta)\\ b''\cdot d'' \in B(\gamma)}} (b \cdot d) \otimes (b'' \cdot d'')^{\vee} \otimes (b'' \cdot d'') \otimes (d' \cdot b') \\
&= \sum_{c \in B(\gamma)} (b \cdot d) \otimes c^{\vee} \otimes c \otimes (d' \cdot b').
\end{aligned}
$$

So Δ is an algebra homomorphism. Similarly, if we say $b \otimes d$ and $b'^{\vee} \otimes d'^{\vee}$ lie in the same component,

$$
\begin{aligned}
\epsilon((b \otimes b') \cdot (d \otimes d')) &= \epsilon(b \cdot d \otimes d' \cdot b') = \delta_{(b\cdot d)^{\vee}, d' \cdot b'} \\
&= \delta_{d^{\vee} \cdot b^{\vee}, d' \cdot b'} = \delta_{d^{\vee}, d'} \delta_{b^{\vee}, b'} \\
&= \epsilon(b \otimes b') \epsilon(d \otimes d').
\end{aligned}
$$

since $d^{\vee} \cdot b^{\vee} = d' \cdot b'$ if and only if $d^{\vee} = d'$ and $b^{\vee} = b'$. The case when they do not lie in the same component is trivial, hence ϵ is an algebra homomorphism too. Thus we have our result. □

Definition 1.4.5. Let $\mathbb{B}_{\lambda} = \mathrm{Span}_{\mathbb{Z}}\{b \otimes b' \in \mathcal{B} \mid \mathrm{wt}(b) + \mathrm{wt}(b') = \lambda\}$ for $\lambda \in \Phi$.

Proposition 1.4.6. *We have* $\mathbb{B} = \bigoplus_{\lambda \in \Phi} \mathbb{B}_{\lambda}$ *with* $\mathbb{B}_{\lambda} \cdot \mathbb{B}_{\lambda'} \subset \mathbb{B}_{\lambda+\lambda'}$ *and* $\Delta(\mathbb{B}_{\lambda}) \subset \bigoplus_{\lambda=\lambda'+\lambda''} \mathbb{B}_{\lambda'} \otimes \mathbb{B}_{\lambda''}$, *so* $\mathbb{B}$ *is a graded bialgebra.*

Proof. Let $b \otimes b' \in B(\alpha) \otimes B(-\alpha)$ and $d \otimes d' \in B(\beta) \otimes B(-\beta)$. Then their product is either 0 or $(b \cdot d) \otimes (d' \cdot b')$, and

$$
\mathrm{wt}(b \cdot d) + \mathrm{wt}(b' \cdot d') = \mathrm{wt}(b) + \mathrm{wt}(d) + \mathrm{wt}(b') + \mathrm{wt}(d').
$$

So $\mathbb{B}_\lambda \cdot \mathbb{B}_{\lambda'} \subset \mathbb{B}_{\lambda+\lambda'}$. Also, $\Delta(b \otimes b') = \sum_{d \in B(\alpha)} b \otimes d \otimes d^\vee \otimes b'$ and

$$\begin{aligned} \mathrm{wt}(b) + \mathrm{wt}(d) + \mathrm{wt}(d^\vee) + \mathrm{wt}(b') &= \mathrm{wt}(b) + \mathrm{wt}(d) - \mathrm{wt}(d) + \mathrm{wt}(b') \\ &= \mathrm{wt}(b) + \mathrm{wt}(b'). \end{aligned}$$

So $\Delta(\mathbb{B}_\lambda) \subset \bigoplus_{\lambda=\lambda'+\lambda''} \mathbb{B}_{\lambda'} \otimes \mathbb{B}_{\lambda''}$. □

Proposition 1.4.7. *If we take a basis of Φ of fundamental weights $\{\Lambda_i \mid i \in I\}$ then $\mathbb{B}$ is generated as an algebra by the $B(\Lambda_i) \otimes B(-\Lambda_i)$ for $i \in I$.*

Proof. For each $\sum_i n_i\Lambda_i \in \Phi_+$, $B(\sum_i n_i\Lambda_i)$ appears in the decomposition of $B(\Lambda_1)^{\otimes n_1} \otimes \ldots \otimes B(\Lambda_k)^{\otimes n_k}$ into irreducible crystals. Hence the basis $B(\sum_i n_i\Lambda_i) \otimes B(-\sum_i n_i\Lambda_i)$ of $\mathbb{B}(\sum_i n_i\Lambda_i) \otimes \mathbb{B}(-\sum_i n_i\Lambda_i)$ is in the image of $\bigotimes_{i=1}^k (B(\Lambda_i) \otimes B(-\Lambda_i))^{\otimes n_i}$ under multiplication. □

Proposition 1.4.8. *Suppose $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathbb{B}$ is the quotient of the free algebra $\mathbb{Z}\langle a, b, c, d\rangle$ by the relations*

$$cb = bc = db = dc = ba = ca = 0, \quad da = 1$$

with comultiplication

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d. \end{aligned}$$

Proof. In the case of $\mathfrak{sl}_2$, the fundamental weight is $1 \in \mathbb{N}$, and $B(1)$ has crystal graph $u_0^{(1)} \to u_1^{(1)}$. So we have four generators in $B(1) \otimes B(-1)$, namely

$$\begin{aligned} a &= u_1^{(1)} \otimes (u_1^{(1)})^\vee, & b &= u_0^{(1)} \otimes (u_1^{(1)})^\vee, \\ c &= u_1^{(1)} \otimes (u_0^{(1)})^\vee, & d &= u_0^{(1)} \otimes (u_0^{(1)})^\vee. \end{aligned}$$

These generators have the described comultiplication. It follows from Definiton 1.1.12

and the diagram in Example 1.1.13 that

$$u_p^{(n)} \cdot u_q^{(m)} = \begin{cases} u_p^{(m+n-2q)} \in B(m+n-2q) & \text{if } p+q \leq n, \\ u_{2p+q-n}^{(m-n-2p)} \in B(m-n-2p) & \text{if } p+q < n. \end{cases}$$

Then $B(1)^{\otimes n} \twoheadrightarrow B(n)$ maps $(u_1^{(1)})^{\otimes k} \otimes (u_0^{(1)})^{\otimes n-k}$ to $u_k^{(n)}$. From this it follows that

$$u_k^{(n)} \otimes (u_l^{(n)})^\vee \quad = \quad \begin{cases} a^l c^{k-l} d^{n-k} & \text{if } k \geq l, \\ a^k b^{l-k} d^{n-l} & \text{if } k \leq l. \end{cases}$$

Furthermore, the multiplication in $\mathcal{B}$ can be computed as

$$(u_p^{(n)} \otimes (u_q^{(n)})^\vee) \cdot (u_r^{(m)} \otimes (u_s^{(m)})^\vee) = \begin{cases} u_p^{(m+n-2r)} \otimes (u_q^{(m+n-2r)})^\vee & \substack{\text{if } p+r \leq n, \\ q+s \neq n, \\ r=s,} \\ u_p^{(m+n-2r)} \otimes (u_{2q+s-n}^{(m+n-2r)})^\vee & \substack{\text{if } p+r \leq n, \\ q+s > n, \\ r=n+q,} \\ u_{2p+r-n}^{(m-n-2p)} \otimes (u_{2q+s-n}^{(m-n-2p)})^\vee & \substack{\text{if } p+r > n, \\ q+s > n, \\ p=q,} \\ u_{2p+r-n}^{(m-n-2p)} \otimes (u_q^{(m-n-2p)})^\vee & \substack{\text{if } p+r > n, \\ q+s \neq n, \\ n+p=s,} \\ 0 & \text{otherwise.} \end{cases}$$

Rewriting this in terms of the generators a, b, c and d this becomes

$$(a^i c^j d^k) \cdot (a^r c^s d^t) = \begin{cases} a^i c^j d^{t+k-r} & \text{if } s = 0, r \leq k \\ a^{i+r-k} c^s d^t & \text{if } j = 0, r \geq k \\ a^i c^{j+s} d^t & \text{if } r = k \\ 0 & \text{otherwise,} \end{cases}$$

$$(a^i c^j d^k) \cdot (a^r b^s d^t) = \begin{cases} a^i c^j d^{t+k-r} & \text{if } s = 0, r \leq k \\ a^{i+r-k} b^s d^t & \text{if } j = 0, r \geq k \\ 0 & \text{otherwise,} \end{cases}$$

$$(a^i b^j d^k) \cdot (a^r c^s d^t) = \begin{cases} a^i b^j d^{t+k-r} & \text{if } s = 0, r \leq k \\ a^{i+r-k} c^s d^t & \text{if } j = 0, r \geq k \\ 0 & \text{otherwise,} \end{cases}$$

$$(a^i b^j d^k) \cdot (a^r b^s d^t) = \begin{cases} a^i b^j d^{t+k-r} & \text{if } s = 0, r \leq k \\ a^{i+r-k} b^s d^t & \text{if } j = 0, r \geq k \\ a^i b^{j+s} d^t & \text{if } r = k \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the multiplication is completely determined by the given relations.

□

Remark The presentation above for $\mathbb{B}$ is closely related to a presentation of the quantum coordinate ring, which we discuss at the end of Part I.

1.4.2 The comodules of $\mathbb{B}$

Definition 1.4.9. For each $\alpha \in \Phi_+$ we can give $\mathbb{B}(\alpha)$ a $\mathbb{B}$-comodule structure via the following map:

$$\mathbb{B}(\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \to \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \hookrightarrow \mathbb{B} \otimes \mathbb{B}(\alpha)$$

$$b \mapsto \sum_{b' \in B(\alpha)} b \otimes b'^{\vee} \otimes b'.$$

This induces a functor $Crys_{\mathfrak{g}} \to \mathbb{B}$-comod.

Definition 1.4.10. Let M be a $\mathbb{B}$-comodule. For each $\alpha \in \Phi_+$ and each $b, b' \in B(\alpha)$ let us denote by $A^{\alpha}_{b,b'}$ the $\mathbb{Z}$-linear endomorphism of M defined uniquely by the property that

$$\Delta_M(m) = \sum\nolimits_{\substack{\alpha\in\Phi_+ \\ b,b'\in B(\alpha)}} b \otimes b'^{\vee} \otimes A^{\alpha}_{b,b'}(m) \quad \text{for all } m \in M.$$

Let us denote by M^{α}_b the image of M under $A^{\alpha}_{b,b}$ for $b \in B(\alpha)$, $\alpha \in \Phi_+$, and let $M^{\alpha} = \sum_{b\in B(\alpha)} M^{\alpha}_b$.

Lemma 1.4.11. *With notation as in Definition 1.4.10,*

$$A^{\beta}_{d,d'} A^{\alpha}_{b,b'} = \delta_{\alpha,\beta}\delta_{b',d} A^{\alpha}_{b,d'}, \quad \sum_{\alpha\in\Phi_+} \sum_{b\in B(\alpha)} A^{\alpha}_{b,b} = Id_M,$$

as automorphisms of M for all $\alpha, \beta \in \Phi_+$, $b, b' \in B(\alpha)$, $d, d' \in B(\beta)$. This latter relation makes sense since, for each $m \in M$, $A^{\alpha}_{b,b'}(m) = 0$ for all but finitely many b, b', α. Hence

$$M = \bigoplus\nolimits_{\alpha\in\Phi_+} M^{\alpha} = \bigoplus\nolimits_{\substack{\alpha\in\Phi_+ \\ b\in B(\alpha)}} M^{\alpha}_b$$

and $A^{\alpha}_{b,b'}$ restrict to isomorphisms $M^{\alpha}_b \to M^{\alpha}_{b'}$.

Proof. Since M is a comodule, we have

$$\sum_{\alpha\in\Phi_+}\sum_{b,b'\in B(\alpha)}\sum_{\beta\in\Phi}\sum_{d,d'\in B(\beta)} b\otimes b'^{\vee}\otimes d\otimes d'^{\vee}\otimes A^{\beta}_{d,d'}A^{\alpha}_{b,b'}(m)$$

$$=\sum_{\alpha\in\Phi_+}\sum_{b,b'\in B(\alpha)}\sum_{d\in B(\alpha)} b\otimes d\otimes d^{\vee}\otimes b'^{\vee}\otimes A^{\alpha}_{b,b'}(m)$$

and

$$m=\sum_{\alpha\in\Phi_+}\sum_{b\in B(\alpha)} A^{\alpha}_{b,b}(m)$$

for all $m\in M$, from which the relations follow. These imply that $A^{\alpha}_{b,b}$ form a set of perpendicular idempotents, which give the direct sum decomposition. Also, $A^{\alpha}_{b',b}A^{\alpha}_{b,b'} = A^{\alpha}_{b,b}$, which is the identity on M^{α}_b, and if $m = A^{\alpha}_{b,b}(m) \in M^{\alpha}_b$ then $A_{b,b'}(m) = A_{b',b'}A_{b,b'}(m) \in M^{\alpha}_{b'}$. □

Definition 1.4.12. Let us denote by $\mathbb{B}$-comod$^{\text{free}}$ the category whose objects are $\mathbb{B}$ comodules M such that each M^{α}_b is a free $\mathbb{Z}$-modules.

Lemma 1.4.13. *For M in $\mathbb{B}$-comod^{free} and $\alpha\in\Phi_+$ we may endow the pointed set*

$$\mathcal{C}(M^{\alpha}) := \left(\coprod_{b\in B(\alpha)} M^{\alpha}_b\setminus\{0\}\right)\sqcup\{0\}$$

with the structure of a crystal such that, for $m\in M^{\alpha}_b$,

$$\tilde{f}_i(m)=\begin{cases}A^{\alpha}_{\tilde{f}_ib,b}(m) & \text{if } \tilde{f}_ib\neq 0,\\ 0 & \text{otherwise,}\end{cases}\qquad \tilde{e}_i(m)=\begin{cases}A^{\alpha}_{\tilde{e}_ib,b}(m) & \text{if } \tilde{e}_ib\neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

and $wt(m) = wt(b)$. Hence we may endow the pointed set $\mathcal{C}(M) := \coprod_{\alpha\in\Phi_+}\mathcal{C}(M^{\alpha})$ with a crystal structure. Furthermore, $\mathcal{C}$ gives a functor $\mathbb{B}$-comod$^{free}\to Crys_{\mathfrak{g}}$ by *restricting a morphism $M \to M'$ in $\mathbb{B}$-comod to the components*

$$\coprod_{b \in B(\alpha)} M_b^\alpha \setminus \{0\} \to \coprod_{b \in B(\alpha)} M_b^\alpha \setminus \{0\}.$$

Proof. The fact that we indeed have a crystal structure on $\mathcal{C}(M^\alpha)$ follows from the observation that we may identify $\coprod_{M_{b_\alpha}^\alpha \setminus \{0\}} B(\alpha) \cong \mathcal{C}(M^\alpha)$ under the mapping that takes b in the copy of $B(\alpha)$ indexed by a nonzero $m \in M_{b_\alpha}^\alpha$ to the nonzero $A_{b_\alpha,b}^\alpha(m) \in M_b^\alpha$. Since comodule homomorphisms commute with the operators $A_{b,b'}^\alpha$ it follows that the restriction of such a homomorphism gives a morphism of crystals. □

Theorem 1.4.14. *The functor $Crys_{\mathfrak{g}} \to \mathbb{B}$-comod^{free} described in Definition 1.4.9 is essentially surjective. Furthermore it has a right adjoint*

$$\mathcal{C} : \mathbb{B}\text{-}comod^{free} \to Crys_{\mathfrak{g}}.$$

Proof. We first prove essential surjectivity. Let M be in $\mathbb{B}$-comod$^{\text{free}}$, and let $X_{b_\alpha}^\alpha$ be a free basis of $M_{b_\alpha}^\alpha$ for each $\alpha \in \Phi_+$. Given $\alpha \in \Phi_+$ and $b \in B(\alpha)$ let $X_b^\alpha = A_{b_\alpha,b}^\alpha X_{b_\alpha}^\alpha$, which is a free basis of M_b^α. We may endow

$$X := \left(\coprod_{\alpha \in \Phi_+} \coprod_{b \in B(\alpha)} X_b^\alpha \right) \sqcup \{0\}$$

with a crystal structure by viewing it as a subset of $\mathcal{C}(M)$ closed under the action of the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$. Under the identification of $\mathcal{C}(M)$ with $\coprod_{\alpha \in \phi_+} \coprod_{M_{b_\alpha}^\alpha \setminus \{0\}} B(\alpha)$, X corresponds to the disjoint union of copies of $B(\alpha)$ indexed over elements of $X_{b_\alpha}^\alpha$. Its image in $\mathbb{B}$-comod$^{\text{free}}$ under the functor in Definition 1.4.9 is the free abelian group $\mathbb{Z}X \cong M$ with the $\mathbb{B}$-coaction

$$x \mapsto \sum_{b' \in B(\alpha)} b \otimes b'^{\vee} \otimes A_{b,b'}(x)$$

for $x \in X_b^\alpha$. This is just the usual coaction on M. So M is the image of X under the functor in Definition 1.4.9.

We now prove the adjuction. It is enough to exhibit a natural isomorphism

$$\mathrm{Hom}(\mathbb{B}(\alpha), M) \cong \mathrm{Hom}(B(\alpha), \mathcal{C}(M))$$

for each $\alpha \in \Phi_+$. First, note that a morphism f of $\mathbb{B}$ comodules $f : \mathbb{B}(\alpha) \to M$ commutes with the operators $A^\alpha_{b,b'}$. Thus f maps $\mathbb{B}(\alpha)^\alpha_b$ to M^α_b. In fact, f is entirely determined by the restriction

$$f_{b_\alpha} : \mathbb{Z} \cong \mathbb{B}(\alpha)^\alpha_{b_\alpha} \to M^\alpha_{b_\alpha}$$

since, on $\mathbb{B}(\alpha)^\alpha_b$, f is given by $A^\alpha_{b_\alpha,b} f_{b_\alpha} A^\alpha_{b,b_\alpha}$. Thus f amounts to a choice of element in $M^\alpha_{b_\alpha}$. Likewise, since $\mathcal{C}(M) \cong \coprod_{\alpha\in\Phi_+} \coprod_{M^\alpha_{b_\alpha}\setminus\{0\}} B(\alpha)$, a morphism of crystals $B(\alpha) \to \mathcal{C}(M)$ is either 0 or corresponds to an element of $M^\alpha_{b_\alpha} \setminus \{0\}$. This correspondence gives our natural isomorphism as required. □

Remark As a comodule, $\mathbb{B} \cong \bigoplus_\alpha \bigoplus_{b'\in B(-\alpha)} \mathbb{B}(\alpha)$ via $b \otimes b' \mapsto (b)_{b'}$ in the copy of $\mathbb{B}(\alpha)$ indexed by $b' \in B(-\alpha)$. Under this isomorphism, multiplication becomes $(b)_{b'} \cdot (d)_{d'} = (b \cdot d)_{d' \cdot b'}$ whenever this is well defined, and 0 otherwise, and comultiplication becomes $(b)_{b'} \mapsto \sum_{b''\in B(\alpha)} (b)_{b''^\vee} \otimes (b'')_{b'}$.

Definition 1.4.15. A *based* $\mathbb{B}$*-comodule* is a pair (M, X) such that

i) M is in $\mathbb{B}$-comod$^{\text{free}}$, and X is a pointed set such that $X \setminus \{0\}$ is a free basis of M;

ii) $X = \coprod_{\alpha\in\Phi_+} \coprod_{b\in B(\alpha)} X^\alpha_b$ where $X^\alpha_b = X \cap M^\alpha_b$; and

iii) each $A^\alpha_{b,b'}$ restricts to a bijection between the sets $X^\alpha_b \to X^\alpha_{b'}$.

A morphism of based comodules $(M, X) \to (N, Y)$ is a morphism of comodules $f : M \to N$ such that $f(X) \subset Y$. This forms a category which we denote $\mathbb{B}\text{-comod}^{\text{based}}$. The direct sum of two based comodules is $(M, X) \oplus (N, Y) = (M \oplus N, X \sqcup Y)$ and their tensor product is $(M, X) \otimes (N, Y) = (M \otimes N, X \otimes Y)$.

Remark The data of a basis X in the above definition is equivalent to having chosen a basis $X^{\alpha}_{b_\alpha}$ for each $M^{\alpha}_{b_\alpha}$ for $\alpha \in \Phi_+$.

Theorem 1.4.16. *The functor*

$$Crys_{\mathfrak{g}} \to \mathbb{B}\text{-}comod^{based}, \quad B(\alpha) \mapsto (\mathbb{B}(\alpha), B(\alpha)),$$

is an equivalence of categories.

Proof. It is clear that $X \mapsto (\mathbb{Z}X, X)$ is functorial. We shall construct a quasi-inverse as follows. Given a based comodule (M, X), the pointed set X is a subset of $\mathcal{C}(M)$ invariant under the Kashiwara operators, hence naturally forms a subcrystal. Given a morphism of based comodules $(M, X) \to (M', X')$ the restriction to X maps to X' and commutes with the operators $A^{\alpha}_{b,b'}$, hence with the Kashiwara operators, so gives a morphism of crystals. By the proof of Theorem 1.4.14, the composition

$$\mathbb{B}\text{-comod}^{\text{based}} \to Crys_{\mathfrak{g}} \to \mathbb{B}\text{-comod}^{\text{based}}$$

is naturally isomorphic to the identity. By construction, the composition

$$Crys_{\mathfrak{g}} \to \mathbb{B}\text{-comod}^{\text{based}} \to Crys_{\mathfrak{g}}$$

is naturally isomorphic to the identity. Hence we have an equivalence. □

Proposition 1.4.17. *The functor in Theorem 1.4.16 gives an equivalence of monoidal categories.*

Proof. For $\alpha, \beta \in \Phi_+$, the comodule structure of $\mathbb{B}(\alpha) \otimes \mathbb{B}(\beta)$ under the monoidal structure of $\mathbb{B}$-Comod$^{\text{based}}$ is

$$b \otimes d \mapsto \sum (b \cdot d) \otimes (d'^{\vee} \cdot b'^{\vee}) \otimes (b' \otimes d') = \sum (b \cdot d) \otimes (b' \cdot d')^{\vee} \otimes (b' \otimes d')$$

where both summations are taken over all $b' \in B(\alpha)$ and $d' \in B(\beta)$ such that $b \otimes d$ and $(d'^{\vee} \otimes b'^{\vee})^{\vee} = b' \otimes d'$ lie in the same connected component, $B(\gamma)$ say. Since all terms of this connected component appear uniquely as some product $b' \cdot d'$, we can then rewrite this as $\sum_{c \in B(\gamma)} (b \cdot d) \otimes c^{\vee} \otimes c$. This is the same comultiplication of $b \otimes d$ as when viewed as an element of $\mathbb{Z}(B(\alpha) \otimes B(\beta))$ under its decomposition into irreducible components. Our result then follows. □

1.4.3 Relation to the crystal functor

Recall from Corollary 1.3.8 that $Crys_{\mathfrak{g}}$ is equivalent to the category of coalgebras of the comonad

$$U : Set_{\bullet} \to Set_{\bullet},\ X \mapsto \coprod_{\alpha \in \Phi_+} \coprod_{\substack{f : FB(\alpha) \to X \\ f \neq 0}} FB(\alpha).$$

Definition 1.4.18. For pointed sets A, B, we define $\underline{\text{Hom}}_{Set_{\bullet}}(A, B)$ to be pointed set

$$\underline{\text{Hom}}_{Set_{\bullet}}(A, B) = \{f : A \to B \mid f \neq 0\} \sqcup \{0 : A \to B\}.$$

Proposition 1.4.19. *The comonad U is isomorphic to*

$$U' : X \mapsto \coprod_{\alpha \in \Phi_+} FB(\alpha) \otimes \underline{Hom}_{Set_{\bullet}}(FB(\alpha), A).$$

Under this identification, the comultiplication on U', $\Delta : U' \Rightarrow U'U'$, becomes $b \otimes f \mapsto b \otimes f^{\sim}$ where $f^{\sim}(b') = b' \otimes f \in U'(A)$.

Proof. The isomorphism is given my the maps

$$FB(\alpha) \otimes \underline{\mathrm{Hom}}_{Set_\bullet}(FB(\alpha), A) \to \coprod_{\substack{f:FB(\alpha)\to X \\ f\neq 0}} FB(\alpha)$$

taking $b \otimes f$ to b in the copy of $FB(\alpha)$ indexed by f. □

Proposition 1.4.20. *The comonad $\mathbb{B} \otimes -$ is isomorphic to*

$$A \mapsto \bigoplus_{\alpha\in\Phi_+} \mathbb{B}(\alpha) \otimes \underline{Hom}_{\mathbb{Z}}(\mathbb{B}(\alpha), A).$$

Under this identification, the comultiplication on $\mathbb{B}$ becomes $b \otimes f \mapsto b \otimes f^{\sim}$ where again $f^{\sim}(b') = b' \otimes f$.

Proof. For a free abelian group A, we have

$$\mathbb{B} \otimes A = \bigoplus_{\alpha\in\Phi_+} \mathbb{B}(\alpha) \otimes \mathbb{B}(\alpha)^{\vee} \otimes A \cong \bigoplus_{\alpha\in\Phi_+} \mathbb{B}(\alpha) \otimes \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), A)$$

given by $b \otimes b' \otimes a \mapsto b \otimes [x \mapsto \epsilon(x \otimes b')a]$. □

Remark Note that the functor U' clearly does not preserve coproducts, whilst the functor in Proposition 1.4.20 does. As a result, the latter is isomorphic to tensoring with a coalgebra whilst the former is not.

1.4.4 The dual bialgebra

Definition 1.4.21. Let

$$\mathbb{B}^* := \underline{\mathrm{Hom}}(\mathbb{B}, \mathbb{Z}) \cong \prod\nolimits_{\alpha\in\Phi_+} \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$$

be the dual of $\mathbb{B}$. Let $\{\phi_{b,b'} \mid b \in B(\alpha), b' \in B(-\alpha)\} \subset \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$ denote the dual $\mathbb{Z}$-basis to $B(\alpha) \otimes B(-\alpha)$, $\phi_{b,b'}(d \otimes d') = \delta_{b,d}\delta_{b',d'}$ for $d \in B(\alpha)$, $d' \in B(-\alpha)$.

We shall denote elements of this dual by formal sums $\sum_{b,b'} a_{b,b'}\phi_{b,b'}$ ranging over all $b \in B(\alpha)$, $b' \in B(-\alpha)$ and $\alpha \in \Phi_+$.

Lemma 1.4.22. *The coalgebra structure on $\mathbb{B}$ induces an algebra structure on the dual $\mathbb{B}^*$ given by*

$$\left(\sum_{b,b'} a_{b,b'}\phi_{b,b'}\right)\cdot\left(\sum_{b,b'} a'_{b,b'}\phi_{b,b'}\right) = \sum_{\substack{\alpha\in\Phi_+\\ b\in B(\alpha)\\ b'\in B(-\alpha)}} \left(\sum_{d\in B(\alpha)} a_{b,d^\vee}a'_{d,b'}\right)\phi_{b,b'},$$

$$1 := \sum_{\substack{\alpha\in\Phi_+\\ b\in B(\alpha)}} \widehat{b\otimes b^\vee}.$$

Each $\mathbb{B}(\alpha)$ is a $\mathbb{B}^*$-module where

$$\phi_{d,d'^\vee}\cdot b = \sum_{c\in B(\alpha)} \phi_{d,d'^\vee}(b\otimes c^\vee)\ c = \delta_{d,b}d'$$

for $b \in B(\alpha)$. Furthermore, the algebra structure on $\mathbb{B}$ induces algebra homomorphisms

$$\Delta : \mathbb{B}^* \to (\mathbb{B}\otimes\mathbb{B})^* \cong \prod\nolimits_{\alpha,\beta\in\Phi_+} \mathbb{B}(\alpha)^*\otimes\mathbb{B}(-\alpha)^*\otimes\mathbb{B}(\beta)^*\otimes\mathbb{B}(-\beta)^*$$

and $\epsilon : \mathbb{B}^* \to \mathbb{Z}$ given by

$$\Delta\left(\sum_{b,b'} a_{b,b'}\phi_{b,b'}\right) = \sum_{\substack{\alpha\in\Phi_+\\ b\in B(\alpha)\\ b'\in B(-\alpha)}}\ \sum_{\substack{\beta\in\Phi_+\\ d\in B(\beta)\\ d'\in B(-\beta)}} a_{b\cdot d,d'\cdot b'}\ \phi_{b,b'}\otimes\phi_{d,d'},$$

$$\epsilon\left(\sum_{b,b'} a_{b,b'}\phi_{b,b'}\right) = a_{b_0,b_0^\vee},$$

where $B(0) = \{b_0\}$. Here, the algebra structure on $(\mathbb{B}\otimes\mathbb{B})^*$ is induced by the coalgebra structure on $\mathbb{B}\otimes\mathbb{B}$.

Proof. This is a straightforward verification given Definition 1.2.1 and Definition 1.4.3. $\square$

Definition 1.4.23. Let $\mathscr{A}$ be a $\mathbb{Z}$-algebra and let $\mathbf{1}_{\mathscr{A}}$ be a subset of $\mathscr{A}$. We will say that $\mathbf{1}_{\mathscr{A}}$ forms a *generalised unit* for $\mathscr{A}$ if, for each $a \in \mathscr{A}$, there is a finite subset $X \subset \mathbf{1}_{\mathscr{A}}$ such that

i) $a = (\sum_{x\in X} x) \cdot a$; and

ii) $x \cdot a = 0$ if $x \in \mathbf{1}_{\mathscr{A}} \setminus X$.

We will say that an $\mathscr{A}$-module M is *unital* if, for any $m \in M$, there is a finite subset $X \subset \mathbf{1}_{\mathscr{A}}$ such that

i) $m = (\sum_{x\in X} x) \cdot m$;

ii) $x \cdot m = 0$ if $x \in \mathbf{1}_{\mathscr{A}} \setminus X$; and additionally

iii) $x \cdot M$ is a free abelian group for all $x \in \mathbf{1}_{\mathscr{A}}$.

Definition 1.4.24. Let $\dot{U}_0$ be the additive subgroup $\bigoplus_{\alpha\in\Phi_+} \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$ of $\mathbb{B}^*$. Let $\mathbf{1}$ denote the collection $\{1_\alpha \mid \alpha \in \Phi_+\} \subset \dot{U}_0$ where $1_\alpha = \sum_{b\in B(\alpha)} \phi_{b,b^\vee}$.

Lemma 1.4.25. *$\dot{U}_0$ is an ideal in $\mathbb{B}^*$, and hence both a non-unital subalgebra and a $\mathbb{B}^*$-bimodule. Furthermore, the collection $\mathbf{1}$ forms a generalised unit in $\dot{U}_0$.*

Proof. Given $\sum_{b,b'} a_{b,b'^\vee}\phi_{b,b'^\vee} \in \mathbb{B}^*$ and $\phi_{d,d'^\vee} \in \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$ we have

$$\left(\sum_{b,b'} a_{b,b'^\vee}\phi_{b,b'^\vee}\right) \cdot \phi_{d,d'^\vee} = \sum_{b,b'} a_{b,b'^\vee}\delta_{b',d}\phi_{b,d'^\vee} = \sum_{b\in B(\alpha)} a_{b,d^\vee}\phi_{b,d'^\vee}$$

which is in $\dot{U}_0$. Likewise, $\phi_{d,d'^\vee} \cdot (\sum_{b,b'} a_{b,b'^\vee}\phi_{b,b'^\vee}) \in \dot{U}_0$. Furthermore, multiplication by 1_α is the identity on $\mathbb{B}(\beta)^* \otimes \mathbb{B}(-\beta)^*$ when $\beta = \alpha$ and is 0 otherwise. From this it follows that $\mathbf{1}$ is a generalised unit. $\square$

Remark We use the notation $\dot{U}_0$ to highlight the similarity with Lusztig's construction of $\dot{U}$ in [27, p. 183]. In the subsequent subsection we shall conjecture a more precise relationship between these two constructions.

Remark Note that Δ from Lemma 1.4.22 does not restrict to a comultiplication on $\dot{U}_0$. For example, $\Delta(\phi_{b_0,b_0^\vee}) \in (\mathbb{B} \otimes \mathbb{B})^*$ takes the value 1 on each $b \otimes b^\vee \otimes d \otimes d^\vee$ for b a highest weight element and d a lowest weight element in $B(\alpha)$, $\alpha \in \Phi_+$, of which there are infinitely many. This is because $b \otimes d$ corresponds to b_0 in a copy of $B(0)$ under the decomposition of $B(\alpha) \otimes B(\alpha)$ into irreducible components. However, we do have the following collection of maps.

Definition 1.4.26. For $\alpha, \beta, \gamma \in \Phi_+$ let

$$\Delta^{\alpha}_{\beta,\gamma} : \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^* \to \mathbb{B}(\beta)^* \otimes \mathbb{B}(-\beta)^* \otimes \mathbb{B}(\gamma)^* \otimes \mathbb{B}(-\gamma)^*,$$

$$\phi_{b,b'} \mapsto \sum_{\substack{\alpha\in\Phi_+ \\ d\in B(\alpha) \\ d'\in B(-\alpha)}} \sum_{\substack{\beta\in\Phi_+ \\ d''\in B(\beta) \\ d'''\in B(-\beta)}} \delta_{b,d\cdot d''}\delta_{b',d'''\cdot d'} \; \phi_{d,d'} \otimes \phi_{d'',d'''}.$$

Let ε be the restriction of the counit from Lemma 1.4.22 to $\dot{U}_0$.

Remark The maps in Definition 1.4.26 can be considered as a single map $\dot{U}_0 \to (\mathbb{B} \otimes \mathbb{B})^*$ that agrees with the restriction of the comultiplication in Lemma 1.4.22. They are therefore associative in the sense that

$$\sum_{\beta'\in\Phi_+} (\Delta^{\beta'}_{\beta,\delta} \otimes \mathrm{Id}) \circ \Delta^{\alpha}_{\beta',\gamma} = \sum_{\gamma'\in\Phi_+} (\mathrm{Id} \otimes \Delta^{\gamma'}_{\delta,\gamma}) \circ \Delta^{\alpha}_{\beta,\gamma'}$$

for all $\alpha, \beta, \gamma, \delta \in \Phi_+$.

Proposition 1.4.27. *The maps $\Delta^{\alpha}_{\beta,\gamma}$ and ε induce a monoidal structure on the category of unital $\dot{U}_0$-modules where $x \in \mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$ acts on $(\mathbf{1}_\beta \otimes \mathbf{1}_\gamma)(M \otimes N)$ as $\Delta^{\alpha}_{\beta,\gamma}(x)$ for unital modules M and N.*

Proof. We first note that $\mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^*$ are unital subalgebras of $\dot{U}_0$ with unit 1_α, and that $\Delta^\alpha_{\beta,\gamma}$ are algebra homomorphisms as a result of Proposition 1.4.4. Also, since $M = \bigoplus_{\alpha\in\Phi_+} \mathbf{1}_\alpha M$ and $N = \bigoplus_{\beta\in\Phi_+} \mathbf{1}_\beta N$, we obtain a well defined action of $\dot{U}_0$ on $M \otimes N$. The associativity constraint observed in the previous remark ensures that the monoidal structure in associative. Furthermore, the fact that the compositions

$$\mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^* \xrightarrow{\Delta^\alpha_{\beta,\gamma}} \mathbb{B}(\beta)^* \otimes \mathbb{B}(-\beta)^* \otimes \mathbb{B}(\gamma)^* \otimes \mathbb{B}(-\gamma)^* \xrightarrow{\mathrm{Id}\otimes\varepsilon} \mathbb{B}(\beta)^* \otimes \mathbb{B}(-\beta)^*$$

and

$$\mathbb{B}(\alpha)^* \otimes \mathbb{B}(-\alpha)^* \xrightarrow{\Delta^\alpha_{\beta,\gamma}} \mathbb{B}(\beta)^* \otimes \mathbb{B}(-\beta)^* \otimes \mathbb{B}(\gamma)^* \otimes \mathbb{B}(-\gamma)^* \xrightarrow{\varepsilon\otimes\mathrm{Id}} \mathbb{B}(\gamma)^* \otimes \mathbb{B}(-\gamma)^*$$

are the identity when $\alpha = \beta$ or $\alpha = \gamma$ respectively or 0 otherwise ensures that $\mathbb{Z}$ with the action of $\dot{U}_0$ given by ε is a monoidal unit. $\square$

Definition 1.4.28. A *based $\dot{U}_0$-module* is a pair (M, X) such that

i) M is a $\dot{U}_0$-module and X is a pointed set such that $X \setminus \{0\}$ is a free $\mathbb{Z}$-basis of M;

ii) $X = \coprod_{\alpha\in\Phi_+} \coprod_{b\in B(\alpha)} X^\alpha_b$ where $X^\alpha_b := X \cap \phi_{b,b^\vee} \cdot M$ for $b \in B(\alpha)$; and

iii) the action of each $\phi_{b,b'^\vee}$ restricts to a bijection between the sets $X^\alpha_b \to X^\alpha_{b'}$.

A morphism of based modules is a module homomorphism that preserves the free basis. The tensor product of two based modules is $(M, X) \otimes (N, Y) = (M \otimes N, X \otimes Y)$.

Proposition 1.4.29. *The monoidal category of based unital $\dot{U}_0$-modules is equivalent to $Crys_\mathfrak{g}$.*

Proof. By Theorem 1.4.16, it is enough to show that based unital $\dot{U}_0$-modules are equivalent to based $\mathbb{B}$-comodules.

Given a based unital $\dot{U}_0$-module (M, X), the maps

$$M \to \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes M, \quad m \mapsto \sum_{b,b' \in B(\alpha)} b \otimes b'^{\vee} \otimes (\phi_{b,b'^{\vee}} m),$$

are non-zero for only finitely many $\alpha \in \Phi_+$. Summing these gives a coaction of $\mathbb{B}$ on M. This makes (M, X) a based $\mathbb{B}$-comodule.

Conversely, given a based $\mathbb{B}$-comodule (M, X) we obtain a $\dot{U}_0$-module structure via the composition

$$\dot{U}_0 \otimes M \xrightarrow{\Delta_M} \dot{U}_0 \otimes \mathbb{B} \otimes M \xrightarrow{\langle -,- \rangle \otimes \mathrm{Id}} \mathbb{Z} \otimes M \cong M.$$

For each $m \in M$, the coaction $\Delta_M(m)$ is a finite sum in the free basis $\mathcal{B} \otimes X$. Hence there is a finite subset $A \subset \Phi_+$ such that the non-zero coefficients are for basis elements in $(\coprod_{\alpha \in A} B(\alpha) \otimes B(-\alpha)) \otimes X \subset \mathcal{B} \otimes X$. Thus $(\sum_{\alpha \in A} 1_\alpha) m = m$ under this action of $\dot{U}_0$ and $1_\beta \cdot m = 0$ for $\beta \notin A$. So (M, X) becomes a based unital $\dot{U}_0$-modules.

These mutually inverse functors give the stated equivalence. The monoidal structure on $\dot{U}_0$-modules is induced by the comultiplication maps $\Delta^{\alpha}_{\beta,\gamma}$, whilst the monoidal structure on $\mathbb{B}$-comodules is induced by the multiplication μ on $\mathbb{B}$. The duality between μ and the maps $\Delta^{\alpha}_{\beta,\gamma}$ ensures that this equivalence is monoidal. □

Definition 1.4.30. For $i \in I$ and $\alpha \in \Phi_+$, let

$$\tilde{e}_{\alpha,i} = \sum_{\substack{b \in B(\alpha) \\ \tilde{e}_i b \neq 0}} \phi_{\tilde{e}_i b, b^{\vee}}, \quad \tilde{f}_{\alpha,i} = \sum_{\substack{b \in B(\alpha) \\ \tilde{f}_i b \neq 0}} \phi_{\tilde{f}_i b, b^{\vee}},$$

in $\dot{U}_0$.

Lemma 1.4.31. *For all $i \in I$ and $\alpha \in \Phi_+$, $\tilde{e}_{\alpha,i}$ and $\tilde{f}_{\alpha,i}$ act as $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbb{B}(\alpha)$, and by zero on $\mathbb{B}(\beta)$ for $\beta \neq \alpha$.*

Proof. Recall that for $b_0 \in B(\beta)$, $\tilde{e}_{\alpha,i}$ acts as

$$\sum_{\substack{b\in B(\alpha)\\ \tilde{e}_i b\neq 0}} \phi_{\tilde{e}_i b_0, b^\vee} \cdot b = \sum_{\substack{b\in B(\alpha)\\ \tilde{e}_i b\neq 0}} \delta_{\tilde{e}_i b_0, b} b$$

which is $\tilde{e}_i b_0$ if $\alpha = \beta$ and 0 otherwise. Likewise for the action of $\tilde{f}_{\alpha,i}$. $\square$

Proposition 1.4.32. *$\dot{U}_0$ is generated as an algebra by*

$$\{\tilde{e}_{\alpha,i}, \tilde{f}_{\alpha,i} \mid i \in I, \alpha \in \Phi_+\}$$

along with the generalised unit elements $\mathbf{1} = \{1_\alpha \mid \alpha \in \Phi_+\}$.

Proof. Fix $\alpha \in \Phi_+$. For $i \in I$, $1_\alpha - \tilde{f}_{\alpha,i}\tilde{e}_{\alpha,i}$ is the sum of $\phi_{b,b^\vee}$ such that $\tilde{e}_i b = 0$. So, for any ordering of I, $\prod_{i\in I}(1_\alpha - \tilde{f}_{\alpha,i}\tilde{e}_{\alpha,i})$ is the sum of $\phi_{b,b^\vee}$ where $\tilde{e}_i b = 0$ for all $i \in I$. That is,

$$\phi_{b_\alpha, b_\alpha^\vee} = \prod_{i\in I}(1_\alpha - \tilde{f}_{\alpha,i}\tilde{e}_{\alpha,i}).$$

The result then follows from the fact that if $b = \tilde{f}_{i_1}\tilde{f}_{i_2}..\tilde{f}_{i_n} b_\alpha$ and $b' = \tilde{f}_{j_1}\tilde{f}_{j_2}..\tilde{f}_{j_m} b_\alpha$ then

$$\begin{aligned}\phi_{b,b'^\vee} &= \tilde{f}_{\alpha,i_1}\tilde{f}_{\alpha,i_2}..\tilde{f}_{\alpha,i_n}(\phi_{b_\alpha,b_\alpha^\vee})\tilde{e}_{\alpha,j_1}\tilde{e}_{\alpha,j_2}..\tilde{e}_{\alpha,j_m} \\ &= \tilde{f}_{\alpha,i_1}..\tilde{f}_{\alpha,i_n}\left(\prod_{i\in I}(1_\alpha - \tilde{f}_{\alpha,i}\tilde{e}_{\alpha,i})\right)\tilde{e}_{\alpha,j_1}..\tilde{e}_{\alpha,j_m}.\end{aligned}$$

$\square$

1.4.5 Relation to global bases

In [20] Kashiwara shows that the equivalence class of a crystal base $B(\alpha)$ contains a canonical $\mathbb{Q}(q)$-basis of the representation $V(\alpha)$, known as the associated *global basis.* Using these bases, we obtain a global base of $A_q(\mathfrak{g})$ from $\mathcal{B}$.

Example 1.4.33. *Recall from Proposition 1.4.8 that, in the case of $\mathfrak{sl}_2$, the bialgebra*

$\mathbb{B}$ is generated by a, b, c, d which satisfy the relations

$$cb = bc = db = dc = ba = ca = 0, \;\; da = 1,$$

with comultiplication

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.$$

Similarly, the quantum coordiante ring $A_q(\mathfrak{sl}_2)$ can be realised as a quotient of the fee algebra $k\langle a, b, c, d\rangle$ by the relations

$$cb = bc = qad - q1, \quad db = qbd, \qquad dc = qcd,$$
$$ba = qab, \qquad ca = qac, \quad da = qcb + 1,$$

again viewed as a bialgebra with an analogous comultiplication to the above. Kashiwara shows in [21] that

$$\{a^i c^j d^k \mid i, j, k \geq 0\} \cup \{a^i b^j d^k \mid i, j, k \geq 0, j \neq 0\} \subset A_q(\mathfrak{sl}_2)$$

is the global basis of $A_q(\mathfrak{sl}_2)$ given by the cystal base $\mathcal{B}$, where $a^i c^j d^k = u_{i+j}^{(i+j+k)} \otimes (u_i^{(j+k)})^\vee$ and $a^i b^j d^k = u_i^{(j+k)} \otimes (u_{i+j}^{(k)})^\vee$ in $\mathcal{B}$. Thus, for global basis elements x and y, their product in $A_q(\mathfrak{sl}_2)$ can be written as

$$x \cdot y = \sum_z p_z^{x,y}(q) z, \quad p_z^{x,y}(q) \in \mathbb{Q}(q),$$

where z ranges through the global basis. It follows from Proposition 1.4.8 that in fact $p_z^{x,y}(q) \in \mathbb{Z}[q]$ and the product of x and y in $\mathbb{B}$ is

$$x \cdot y = \sum_z p_z^{x,y}(0) z.$$

It is a goal of future work by the author to investigate whether this phenomenon is exclusive to $\mathfrak{sl}_2$. In [28], Lusztig uses a similar procedure of multiplying global basis elements (or *canonical basis* elements in his terminology) of a modified version of $U_q(\mathfrak{g})$, denoted $\dot{U}$, to construct a bialgebra. He refers to his construction as a quantum group at $v = \infty$, or $q = v^{-1} = 0$ in our notation. Since $\dot{U}$ is dual to $A_q(\mathfrak{g})$, this bialgebra at $q = 0$ should be dual to $\mathbb{B}$. We conjecture that it is $\dot{U}_0$. This should give some way of describing the (co)multiplication of $\mathbb{B}$ in terms of the (co)multiplication of global basis elements of $A_q(\mathfrak{g})$ in general.

Part II

A Tannakian reconstruction theorem for IndBanach spaces

2.1 Contracting (co)products

Fix a complete valued field k with non-trivial valuation, either Archimedean or non-Archimedean. Recall the definitions of Section 0.1.3.

Definition 2.1.1. Let $\mathrm{Ban}_k^{\leq 1}$ denote the wide subcategory of Ban_k whose morphisms are bounded linear transformations of norm at most 1, the *contracting category of Banach spaces.* By *wide* we mean that $\mathrm{Ban}_k^{\leq 1}$ contains all objects of Ban_k.

Definition 2.1.2. Let $(V_i)_{i\in I}$ be a family of Banach spaces. Let us define the *contracting product* of this family as the Banach space

$$\prod_{i\in I}^{\leq 1} V_i = \{(v_i)_{i\in I} \in \times_{i\in I} V_i \mid \mathrm{Sup}_{i\in I}\|v_i\| \leq \infty\}$$

with norm $\|(v_i)\| = \mathrm{Sup}_{i\in I}\|v_i\|$ in both the Archimedean and non-Archimedean cases, and the *contracting coproduct* as the Banach space

$$\coprod_{i\in I}^{\leq 1} V_i = \{(v_i)_{i\in I} \in \times_{i\in I} V_i \mid \sum_{i\in I} \|v_i\| \leq \infty\}$$

with norm $\|(v_i)\| = \sum_{i\in I} \|v_i\|$ in the Archimedean case and

$$\coprod_{i\in I}^{\leq 1} V_i = \{(v_i)_{i\in I} \in \times_{i\in I} V_i \mid \lim_{i\in I}\|v_i\| = 0\}$$

with norm $\|(v_i)\| = \mathrm{Sup}_{i\in I}\|v_i\|$ in the non-Archimedean case.

Proposition 2.1.3. *The category* $\mathrm{Ban}_k^{\leq 1}$ *has small limits and colimits.*

Proof. Indeed, it has kernels and cokernels inhereted from Ban_k, and it is straightforward to check that Definition 2.1.2 describes products and coproducts in this category. □

Definition 2.1.4. Limits and colimits in $\mathrm{Ban}_k^{\leq 1}$ give objects in Ban_k. We shall refer to them as *contracting limits and colimits* respectively, and denotate them by $\lim_I^{\leq 1}$ and $\mathrm{colim}_I^{\leq 1}$.

Remark Note that filtered contracting colimits are not left exact. For example, the maps $k \to k_{\frac{1}{n}}$ are all isomorphisms in Ban_k (and bimorphisms in $\mathrm{Ban}_k^{\leq 1}$) but taking contracting colimits over $n \geq 1$ we obtain the morphism $k \to \{0\}$.

Contracting (co)products have the following universal property in Ban_k.

Lemma 2.1.5. *For all collections of morphisms $\{f_i : U \to V_i\}_{i\in I}$ (respectively $\{g_i : V_i \to W\}_{i\in I}$) such that $\{\|f_i\|\}_{i\in I}$ is bounded (respectively $\{\|g_i\|\}_{i\in I}$ is bounded) by some $M > 0$, there exists a unique map $U \to \prod_{i\in I}^{\leq 1} V_i$ (respectively $\coprod_{i\in I}^{\leq 1} V_i \to W$) of norm as most M such that f_i is the composite $U \to \prod_{j\in I}^{\leq 1} V_j \to V_i$ (respectively g_i is the composite $V_i \to \coprod_{j\in I}^{\leq 1} V_j \to U$). That is,*

$$\underline{Hom}(U, \prod\nolimits_{i\in I}^{\leq 1} V_i) \cong \prod\nolimits_{i\in I}^{\leq 1} \underline{Hom}(U, V_i)$$

and

$$\underline{Hom}(\coprod\nolimits_{i\in I}^{\leq 1} V_i, W) \cong \prod\nolimits_{i\in I}^{\leq 1} \underline{Hom}(V_i, W).$$

Proof. As the valuation on our field is assumed to be non-trivial, we may take $M \in |k^\times|$ without loss of generality, so there is $\lambda \in k^\times$ with $|\lambda| = M$. Then we may rescale our family of morphisms to $\{\frac{f_i}{\lambda}\}_{i\in I}$ in $\mathrm{Ban}_k^{\leq 1}$. By the universal property we get a map $\phi : U \to \prod_{i\in I}^{\leq 1} V_i$ of modulus at most 1. Scaling by λ gives our desired map, $\lambda \cdot \phi$. The proof for contracting coproducts is similar. □

Definition 2.1.6. For a set I, let $\mathrm{Ban}_k^{I,\mathrm{bd}}$ be the category whose objects are collections $(V_i)_{i\in I}$ of Banach spaces V_i indexed by $i \in I$ and whose morphisms are uniformly bounded,

$$\mathrm{Hom}((V_i)_{i\in I}, (V_i')_{i\in I}) := \prod\nolimits_{i\in I}^{\leq 1} \underline{\mathrm{Hom}}(V_i, V_i').$$

It follows from Lemma 2.1.5 that $\prod_{i\in I}^{\leq 1}$ and $\coprod_{i\in I}^{\leq 1}$ define functors from $\mathrm{Ban}_k^{I,\mathrm{bd}}$ to Ban_k. Furthermore, contracting products are right adjoints to the diagonal functors

$$\Delta^I : \mathrm{Ban}_k \to \mathrm{Ban}_k^{I,\mathrm{bd}}, \quad V \mapsto (V)_{i\in I},$$

and likewise contracting coproducts are left adjoints to Δ^I.

Remark Note that contracting products and contracting coproducts do not necessarily commute. For example the natural map

$$\coprod_{i\in\mathbb{Z}}^{\leq 1} \prod_{j\in\mathbb{Z}}^{\leq 1} k \to \prod_{j\in\mathbb{Z}}^{\leq 1} \coprod_{i\in\mathbb{Z}}^{\leq 1} k$$

is not surjective, as $(\delta_{i,j})_{i,j\in\mathbb{Z}}$ is not in the image.

Definition 2.1.7. We extend the definition of contracting (co)products to IndBan_k as follows. The contracting product and coproduct functors

$$\prod_I^{\leq 1}, \coprod_I^{\leq 1} : \mathrm{Ban}_k^{I,\mathrm{bd}} \to \mathrm{Ban}_k$$

induce functors from the Ind completion of $\mathrm{Ban}_k^{I,\mathrm{bd}}$,

$$\mathrm{IndBan}_k^{I,\mathrm{bd}} := \mathrm{Ind}(\mathrm{Ban}_k^{I,\mathrm{bd}}),$$

to IndBan_k, which we will continue to denote as $\prod_I^{\leq 1}$ and $\coprod_I^{\leq 1}$ respectively. There is a faithful diagonal embedding functor $\Delta^I : \mathrm{IndBan}_k \to \mathrm{IndBan}_k^{I,\mathrm{bd}}$ induced by $\Delta^I : \mathrm{Ban}_k \to \mathrm{Ban}_k^{I,\mathrm{bd}}$.

Remark The embedding $\mathrm{Ban}_k^{I,\mathrm{bd}} \hookrightarrow \mathrm{Ban}_k^I$ induces a faithful embedding $\mathrm{IndBan}_k^{I,\mathrm{bd}} \to \mathrm{Ind}(\mathrm{Ban}_k^I) \cong \mathrm{IndBan}_k^I$. This allows us to think of objects of $\mathrm{IndBan}_k^{I,\mathrm{bd}}$ as certain collections of IndBanach spaces indexed over I.

Proposition 2.1.8. *There are adjunctions*

$$Hom(\coprod\nolimits_I^{\leq 1} X_I, Y) \cong Hom(X_I, \Delta^I Y)$$

and

$$Hom(Y, \prod\nolimits_I^{\leq 1} X_I) \cong Hom(\Delta^I Y, X_I)$$

for X_I in $IndBan_k^{I,bd}$ and Y in $IndBan_k$.

Proof. The result follows from the adjunction given in Lemma 2.1.5 by taking filtered colimits. □

Definition 2.1.9. We will say that a functor $\mathcal{F} : \mathrm{IndBan}_k \to \mathrm{IndBan}_k$ *commutes with contracting coproducts* if the functors $\mathcal{F}^I : \mathrm{IndBan}_k^I \to \mathrm{IndBan}_k^I$ restrict to functors $\mathcal{F}^I : \mathrm{IndBan}_k^{I,\mathrm{bd}} \to \mathrm{IndBan}_k^{I,\mathrm{bd}}$ under the embedding $\mathrm{IndBan}_k^{I,\mathrm{bd}} \hookrightarrow \mathrm{IndBan}_k^I$ such that the diagram of functors

$$\begin{array}{ccc} \mathrm{IndBan}_k^{I,\mathrm{bd}} & \xrightarrow{\mathcal{F}^I} & \mathrm{IndBan}_k^{I,\mathrm{bd}} \\ {\scriptstyle \coprod_I^{\leq 1}} \downarrow & \swarrow\!\!\swarrow & \downarrow {\scriptstyle \coprod_I^{\leq 1}} \\ \mathrm{IndBan}_k & \xrightarrow{\mathcal{F}} & \mathrm{IndBan}_k \end{array}$$

commutes up to a natural isomorphism.

Remark It is important to note that, since contracting coproducts are not functorial on IndBan_k^I, or even on the full subcategory on the essential image of $\mathrm{IndBan}_k^{I,\mathrm{bd}}$, the statement of whether or not a functor commutes with contracting coproducts is not invariant under isomorphism. However, the following weaker notion is invariant under isomorphism of functors.

Definition 2.1.10. For a set S we will denote by $l^1(S)$ the contracting coproduct $l^1(S) := \coprod_S^{\leq 1} k$. We will say that a functor F *commutes with* l^1 if the natural map

$$\coprod\nolimits_S^{\leq 1} F(k) \xrightarrow{\sim} F(l^1(S))$$

is an isomorphism. This map is the image of the identity under the composition

$$\begin{aligned}\mathrm{Hom}(\textstyle\coprod_S^{\leq 1} k, \coprod_S^{\leq 1} k) &\cong \textstyle\prod_S^{\leq 1} \mathrm{Hom}(k, \coprod_S^{\leq 1} k) \\ &\to \textstyle\prod_S^{\leq 1} \mathrm{Hom}(F(k), F(\coprod_S^{\leq 1} k)) \\ &\cong \mathrm{Hom}(\textstyle\coprod_S^{\leq 1} F(k), F(\coprod_S^{\leq 1} k)).\end{aligned}$$

2.2 Categories of IndBanach (co)modules

2.2.1 IndBanach modules of IndBanach algebras

Definition 2.2.1. Let $\mathcal{C}$ be a locally presentable, quasi-abelian category enriched over IndBan_k and let $F : \mathcal{C} \to \mathrm{IndBan}_k$ be an enriched functor. We say that F is a *fibre functor* over IndBan_k if F is bicontinuous, strongly exact, faithful and reflects strict morphisms.

The following adaptation of the Adjoint Functor Theorem for locally presentable categories (see [1]) tells us when an enriched adjoint functor exists.

Theorem 2.2.2 (Enriched Adjoint Functor Theorem [24]). *Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a functor between locally presentable categories, enriched over IndBan_k. Then $\mathscr{F}$ has an enriched right adjoint if and only if it preserves all small colimits. If $\mathscr{C}$ is complete and $\mathscr{F}$ also preserves all small limits then $\mathscr{F}$ has an enriched left adjoint.*

Proof. This follows directly from Theorem 5.32 and Theorem 5.33 in [24]. □

This gives us the following Lemma.

Lemma 2.2.3. *Let $\mathcal{C}$ be a locally presentable, quasi-abelian category, and let $F : \mathcal{C} \to \mathrm{IndBan}_k$ be a fibre functor over IndBan_k. Then F satisfies the conditions of Barr-Beck (Theorem 0.1.13), so $\mathcal{C}$ is equivalent to the category of algebras of a monadic functor T on IndBan_k.*

Proof. By Theorem 2.2.2, since IndBan_k is locally presentable and a fibre functor F is both continuous and cocontinuous it has a left adjoint, G. Hence property (i) of Theorem 0.1.13 is satisfied. For property (ii), if $f : A \to B$ is a morphism in $\mathcal{C}$ such that Ff is an isomorphism then it fits into a strictly coexact sequence $A \xrightarrow{f} B \to \text{Coker}(f)$, the image of which under F is then also strictly coexact, so $F(\text{Coker}(f)) = 0$. A similar argument shows $F(\text{Ker}(f)) = 0$. Since F is faithful, this means that f has trivial kernel and cokernel. It then follows from the fact that F reflects strictness that f is also an isomorphism. $\mathcal{C}$ is quasi-abelian and hence has equalisers, and so (iii) follows from the strong exactness of F. Thus, by Theorem 0.1.13, F is monadic and $\mathcal{C}$ is equivalent to the category of algebras of $T = FG$. □

Lemma 2.2.4. *With conditions as in the previous lemma, the monad T is cocontinuous.*

Proof. This follows from the fact that F is assumed to be cocontinuous and G is a left adjoint, hence also cocontinuous. □

Lemma 2.2.5. *A functor $\mathscr{V} : \text{IndBan}_k \to \text{IndBan}_k$ is naturally isomorphic to one of the form $V\hat{\otimes}-$ for an IndBanach space V if and only if $\mathscr{V}$ is enriched over IndBan_k, cocontinuous and commutes with l^1.*

Proof. For a Banach space V, $V\hat{\otimes}-$ is a left adjoint on both Ban_k and $\text{Ban}_k^{\leq 1}$ hence is cocontinuous and commutes with contracting coproducts. Since contracting coproducts commute with colimits, this is also true for any IndBanach space V. Hence $V\hat{\otimes}-$ commutes with l^1.

Conversely, suppose $\mathscr{V} : \text{IndBan}_k \to \text{IndBan}_k$ is enriched, cocontinuous and commutes with l^1. Let W be a Banach space which, by Lemma A.39 of [5], can be written as the cokernel of a morphism

$$f : P(W') \to P(W)$$

where

$$P(X) := \coprod_{\substack{x\in X\\ \|x\|=1}}^{\leq 1} k = l^1(\{x \in X \mid \|x\| = 1\})$$

for any Banach space X, and W' is the kernel of the natural map $I(W) \twoheadrightarrow W$. But, since $\mathscr{V}$ commutes with l^1,

$$\mathscr{V}(P(X)) \cong \coprod_{\substack{x\in X\\ \|x\|=1}}^{\leq 1} \mathscr{V}(k) \cong \mathscr{V}(k)\hat{\otimes}P(X)$$

for all sets X. The map f is induced by uniformly bounded maps $f_x : k \to P(W)$ indexed over $x \in W'$ with $\|x\| = 1$. Each f_x is a convergent sum $\sum_y a_{x,y}\iota_y$, $a_{x,y} \in k$, indexed over $y \in W$ with $\|y\| = 1$, where ι_y injects the copy of k indexed by y into $P(W)$. Since $\mathscr{V}$ is enriched, if $\mathscr{V}(k) = \text{"colim"}_{i\in I}X_i$ and $\mathscr{V}(P(W)) = \text{"colim"}_{j\in J}Y_j$, then the map

$$\mathrm{Hom}(k, P(W)) \xrightarrow{\mathscr{V}} \mathrm{Hom}(\mathscr{V}(k), \mathscr{V}(P(W)))$$

is given by a compatible collection of continuous maps of Banach spaces

$$\mathrm{Hom}(k, P(W)) \xrightarrow{\mathscr{V}_i} \mathrm{Hom}(X_i, Y_{j_i})$$

for each $i \in I$ and for some corresponding $j_i \in J$. Then

$$\mathscr{V}_i(f_x) = \mathscr{V}_i(\sum_y a_{x,y}\iota_y) = \sum_y a_{x,y}\mathscr{V}_i(\iota_y)$$

as maps $X_i \to Y_{j_i}$ for each $i \in I$. By construction of the morphism in Definition 2.1.10, the map $\mathscr{V}(\iota_y)$ is equal to the composition

$$\mathscr{V}(k) \cong \mathscr{V}(k)\hat{\otimes}k \xrightarrow{\mathrm{Id}\otimes\iota_y} \mathscr{V}(k)\hat{\otimes}P(W) \cong \mathscr{V}(P(W)).$$

By potentially replacing each j_i with another element in the filtered set J, we may assume that the isomorphism $\mathscr{V}(k)\hat{\otimes}P(W) \xrightarrow{\sim} \mathscr{V}(P(W))$ is given by a collection of maps $X_i \hat{\otimes} P(W) \to Y_{j_i}$, where the composition

$$X_i \cong X_i \hat{\otimes} k \xrightarrow{\mathrm{Id} \otimes \iota_y} X_i \hat{\otimes} P(W) \to Y_{j_i}$$

is equal to $\mathscr{V}_i(\iota_y)$. Thus the diagram

$$\begin{array}{ccc} X_i \hat{\otimes} k & \xrightarrow{\sim} & X_i \\ {\scriptstyle \mathrm{Id} \otimes f_x} \downarrow & & \downarrow {\scriptstyle \mathscr{V}_i(f_x)} \\ X_i \hat{\otimes} P(W) & \longrightarrow & Y_{j_i} \end{array}$$

commutes, and hence so does the diagram

$$\begin{array}{ccccc} \mathcal{V}(k) \hat{\otimes} k & \xrightarrow{\sim} & \mathcal{V}(k) & \xrightarrow{\mathscr{V}(\iota_x)} & \mathcal{V}(P(W')) \\ {\scriptstyle \mathrm{Id} \otimes f_x} \downarrow & & \downarrow {\scriptstyle \mathscr{V}(f_x)} & & \downarrow {\scriptstyle \mathscr{V}(f)} \\ \mathcal{V}(k) \hat{\otimes} P(W) & \xrightarrow{\sim} & \mathcal{V}(P(W)) & \longrightarrow & \mathcal{V}(P(W)). \end{array}$$

Thus the diagram

$$\begin{array}{ccc} \mathcal{V}(k) \hat{\otimes} P(W') & \xrightarrow{\sim} & \mathcal{V}(P(W')) \\ {\scriptstyle \mathrm{Id} \otimes f} \downarrow & & \downarrow {\scriptstyle \mathscr{V}(f)} \\ \mathcal{V}(k) \hat{\otimes} P(W) & \xrightarrow{\sim} & \mathcal{V}(P(W)) \end{array}$$

must also commute. From this we have that

$$\begin{aligned} \mathscr{V}(W) &\cong \mathscr{V}(\mathrm{Coker}(P(W') \xrightarrow{f} P(W))) \\ &\cong \mathrm{Coker}(\mathscr{V}(P(W')) \xrightarrow{\mathscr{V}(f)} \mathscr{V}(P(W))) \\ &\cong \mathrm{Coker}(\mathscr{V}(k) \hat{\otimes} P(W') \xrightarrow{\mathrm{Id} \otimes f} \mathscr{V}(k) \hat{\otimes} P(W)) \\ &\cong \mathscr{V}(k) \hat{\otimes} \mathrm{Coker}(P(W') \xrightarrow{f} P(W)) \\ &\cong \mathscr{V}(k) \hat{\otimes} W \end{aligned}$$

for any Banach space W. Since any IndBanach space can be written as a colimit of Banach spaces, and since both $\mathscr{V}$ and $\mathscr{V}(k) \hat{\otimes} -$ are cocontinuous, $\mathscr{V}$ is isomorphic to the functor $V \hat{\otimes} -$ for $V = \mathscr{V}(k)$. □

Theorem 2.2.6. *Let $\mathcal{C}$ be a locally presentable, quasi-abelian category, enriched over $IndBan_k$, equipped with a fibre functor $F : \mathcal{C} \to IndBan_k$ as in Definition 2.2.1. Assume further that $T = FG$ commutes with l^1, as in Definition 2.1.10, for some left adjoint G to F. Then there exists an algebra $\mathscr{A}$ in $IndBan_k$ such that $\mathcal{C}$ is equivalent to the category of left $\mathscr{A}$ modules in $IndBan_k$.*

Proof. By Lemma 2.2.3, $\mathcal{C}$ is equivalent to the category of alegbras of T in IndBan_k. By Lemma 2.2.4, Lemma 2.2.5 and our assumption that T commutes with l^1, T is isomorphic to $\mathscr{A}\hat{\otimes}-$ for $\mathscr{A} = T(k)$. Then the fact that T is a monad is equivalent to $\mathscr{A}$ being an algebra, and the category of T algebras in IndBan_k is then just the category of $\mathscr{A}$ modules. □

Definition 2.2.7. Let $\mathcal{C}$ be a category enriched over IndBan_k. We will say that $\mathcal{C}$ has *constant contracting coproducts* if, for each set S, there is a functor

$$\coprod_S^{\leq 1} : \mathcal{C} \to \mathcal{C}$$

and, for each map of sets $S' \to S$, there is a natural transformation

$$\coprod_S^{\leq 1} \Rightarrow \coprod_{S'}^{\leq 1}$$

such that

i) $\underline{\mathrm{Hom}}_{\mathcal{C}}(\coprod_S^{\leq 1} X, Y) \cong \prod_S^{\leq 1} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ for all X and Y in $\mathcal{C}$;

ii) the assignment $S \mapsto \coprod_S^{\leq 1}$ is contravariantly functorial.

By property (i), if such functors exist then they exist uniquely. We will say that a functor $F : \mathcal{C} \to \mathcal{C}'$ between categories with constant contracting coproducts commutes with constant contracting coproducts if we have a collection of natural isomorphisms $F \circ \coprod_S^{\leq 1} \cong \coprod_S^{\leq 1} \circ F$ compatible with the functoriality in S.

Remark In the case where $\mathcal{C} = \mathrm{IndBan}_k$, the functor $\coprod_S^{\leq 1}$ is the composition

$$\mathrm{IndBan}_k \xrightarrow{\Delta^S} \mathrm{IndBan}_k^{S,\mathrm{bd}} \xrightarrow{\coprod_S^{\leq 1}} \mathrm{IndBan}_k.$$

Corollary 2.2.8. *Suppose we have a category $\mathcal{C}$ with constant contracting coproducts as defined above that is fibred over IndBan$_k$. Suppose further that the fiber functor F commutes with constant contracting coproducts. Then there exists an algebra $\mathscr{A}$ in IndBan$_k$ such that $\mathcal{C}$ is equivalent to the category of left $\mathscr{A}$ modules in IndBan$_k$.*

Proof. Let G denote the left adjoint to F, which exists by Lemma 2.2.3. We have that

$$\begin{aligned}
\underline{\mathrm{Hom}}(\textstyle\coprod_S^{\leq 1} G(X), Y) &\cong \textstyle\prod_S^{\leq 1} \underline{\mathrm{Hom}}(G(X), Y) \\
&\cong \textstyle\prod_S^{\leq 1} \underline{\mathrm{Hom}}(X, F(Y)) \\
&\cong \underline{\mathrm{Hom}}(\textstyle\coprod_S^{\leq 1} X, F(Y)) \\
&\cong \underline{\mathrm{Hom}}(G(\textstyle\coprod_S^{\leq 1} X), Y)
\end{aligned}$$

for all X in IndBan$_k$ and Y in $\mathcal{C}$, hence $G(\coprod_S^{\leq 1} X) \cong \coprod_S^{\leq 1} G(X)$ naturally for all X in IndBan$_k$. The result then follows from Theorem 2.2.6. □

We may, in fact, give an alternate and perhaps more explicit description of the algebra $\mathscr{A}$ from Theorem 2.2.6 and Corollary 2.2.8.

Definition 2.2.9. Let $\mathscr{F} : \mathcal{C} \to \mathrm{IndBan}_k$ be a functor. As IndBan$_k$ is closed, we may define the *internal natural transformations* $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{F})$ from $\mathscr{F}$ to itself as the end

$$\int_{V \in \mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V) = \mathrm{eq}\left(\prod_{V \in \mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V) \rightrightarrows \prod_{V \to V'} \underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V') \right).$$

The maps $k \to \underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V)$ picking out the identity in $\underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V)$ induce a unit map

$$k \to \int_{V \in \mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V, \mathscr{F}V) = \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{F})$$

and the compositions $\underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V)\hat{\otimes}\underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V) \to \underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V)$ induce a multiplication

$$\left(\int_{V\in\mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V)\right)\hat{\otimes}\left(\int_{V\in\mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V)\right) \to \int_{V\in\mathcal{C}} \underline{\mathrm{Hom}}(\mathscr{F}V,\mathscr{F}V)$$

from which we give $\underline{\mathrm{Hom}}(\mathscr{F},\mathscr{F})$ the expected IndBanach algebra structure.

Proposition 2.2.10. *Let $\mathscr{A}$ be an IndBanach algebra, let $\mathcal{C}$ be the category of its IndBanach modules and let F be the forgetful functor to IndBanach spaces. Then $\mathscr{A} \cong \underline{Hom}(F,F)$ as IndBanach algebras.*

Proof. $\mathscr{A}$ naturally gives an object of $\mathcal{C}$, and $F \cong \underline{\mathrm{Hom}}(\mathscr{A},-)$. So, by the enriched Yoneda Lemma (see Section 2.4 of [24]), $\mathscr{A} \cong \underline{\mathrm{Hom}}(F,F)$ cannonically. It is clear from construction that this is an isomorphism of IndBanach algebras. □

Remark Suppose $\mathcal{C}$ is the category of IndBanach modules over an IndBanach algebra $\mathscr{A}$. Let F denote the forgetful functor to IndBan_k, G its left adjoint, and $T = FG \cong \mathscr{A}\hat{\otimes}-$. Moerdijk proves in [36] that monoidal structures on $\mathcal{C}$ for which F is strong monoidal correspond to comonoidal structures on T, which in turn correspond to coalgebra structures on $\mathscr{A}$. For any given monoidal structure on $\mathcal{C}$ with F strong monoidal, the counit of the adjunction gives us a morphism $T(k) \to k$. The image of $\eta_V\hat{\otimes}\eta_W$ under

$$\begin{aligned}\mathrm{Hom}(A\hat{\otimes}B, FGA\hat{\otimes}FGB) &\cong \mathrm{Hom}(A\hat{\otimes}B, F(GA\hat{\otimes}GB))\\ &\cong \mathrm{Hom}(G(A\hat{\otimes}B), GA\hat{\otimes}GB).\end{aligned}$$

gives a natural transfromation $G(-\hat{\otimes}-) \Rightarrow G(-)\hat{\otimes}G(-)$. Then the composite $T(-\hat{\otimes}-) \Rightarrow F(G(-)\hat{\otimes}G(-)) \cong T(-)\hat{\otimes}T(-)$ makes T comonoidal. This gives $\mathscr{A}$ a comultiplication compatible with its multiplication, from which the monoidal structure of $\mathcal{C}$ comes.

2.2.2 IndBanach comodules of IndBanach coalgebras

Classical Tannaka-Krein duality asks when a category $\mathcal{C}$ is a category of comodules over a coalgebra, which we aim to provide an analytic analogue of here.

Definition 2.2.11. Let $\mathcal{C}$ be a locally presentable, quasi-abelian category, enriched over IndBan_k, and let $F : \mathcal{C} \to \text{IndBan}_k$ be an enriched functor. We say that F is a *co-fibre functor* if it is cocontinuous, strongly exact, faithful and reflects strict morphisms.

Lemma 2.2.12. *Let $\mathcal{C}$ be a locally presentable, quasi-abelian category, enriched over $IndBan_k$, and let $F : \mathcal{C} \to IndBan_k$ be a co-fibre functor over $IndBan_k$. Then F satisfies the dual conditions of Barr-Beck (Theorem 0.1.14), so $\mathcal{C}$ is equivalent to the category of coalgebras of a comonadic functor U in $IndBan_k$.*

Proof. The proof is analogous to the proof of Lemma 2.2.3. □

Remark Since G is a right adjoint, it is not necessarily true that G or U is cocontinuous.

Theorem 2.2.13. *Let $\mathcal{C}$ be a locally presentable, k-linear, quasi-abelian category, equipped with a co-fibre functor $F : \mathcal{C} \to IndBan_k$. Assume further that $U = FG$ is cocontinuous and commutes with l^1, where G is some right adjoint to F. Then there exists a coalgebra $\mathscr{B}$ in $IndBan_k$ such that $\mathcal{C}$ is equivalent to the category of left $\mathscr{B}$ comodules in $IndBan_k$.*

Proof. The proof is analogous to that of Theorem 2.2.6. □

Remark At first sight this result is less satisfying than Theorem 2.2.6 or Corollary 2.2.8, as U is not automatically cocontinuous and the assumption that it commutes with l^1 is not automatic from a good notion of contracting coproducts. However, in applications, the adjoint G, and hence the comonad U, can often be described explicitly and checked for (contracting) cocontinuity.

Remark Let $\mathcal{C}$ be the category of IndBanach comodules over an IndBanach coalgebra $\mathscr{B}$, and let us denote by F the forgetful functor to IndBan_k, G its right adjoint, and $U = FG \cong \mathscr{B}\hat{\otimes}-$. As before, monoidal structures on $\mathcal{C}$ for which F is strong monoidal were shown by Moerdijk in [36] to correspond directly to monoidal structures on U, which in turn correspond to algebra structures on $\mathscr{B}$. This correspondence is dual to the one outlined in the final Remark of Subsection 2.2.1.

2.2.3 Simultaneous modules and comodules

In the case where we have both left and right adjoints G and G' as described in Theorems 2.2.6 and 2.2.13, we relate $\mathscr{A} = FG(k)$ and $\mathscr{B} = FG'(k)$ as follows.

Proposition 2.2.14. *$\mathscr{A}$ is dualisable with dual $\mathscr{B}$ in IndBan_k.*

Proof. The adjunction gives an adjunction between $T = FG$ and $U = FG'$

$$\mathrm{Hom}(TV, W) \cong \mathrm{Hom}(GV, G'W) \cong \mathrm{Hom}(V, UW).$$

Then the unit and counit of this adjunction at k give the described duality. $\square$

Remark Conversely, suppose that $\mathscr{A}$ is a dualisable IndBanach algebra with dual $\mathscr{B}$. Then $\mathscr{B}$ forms an IndBanach coalgebra, and there is an adjunction as above between the functors $T = \mathscr{A}\hat{\otimes}-$ and $U = \mathscr{B}\hat{\otimes}-$. It then follows that the category of IndBanach $\mathscr{A}$ modules and IndBanach $\mathscr{B}$ comodules are equivalent in a way compatible with the forgetful functor.

2.3 Examples

We now present some examples to highlight the possible applications of this theory.

2.3.1 Comodules of a Banach coalgebra

Proposition 2.3.1. *Let B be a Banach coalgebra, viewed as an IndBanach space, and let M be an IndBanach B-comodule. Then M is isomorphic to a colimit of Banach comodules of B.*

Proof. Let $\mathcal{C}$ be the Ind completion of the category of Banach B-comodules. The forgetful functor from the category of Banach B-comodules to Ban_k induces a co-continuous, strongly exact, faithful functor $F : \mathcal{C} \to \mathrm{IndBan}_k$ that reflects strict morphisms. Hence F is a co-fibre functor and $\mathcal{C}$ is equivalent to the category of coalgebras over a comonad U. The right adjoint of the forgetful functor $B\hat{\otimes}-$ from Ban_k to Banach B-comodules induces a right adjoint G to F, and FG is isomorphic to the functor $B\hat{\otimes}-$. So it follows that $\mathcal{C}$ is equivalent to the category of B-comodules in IndBan_k, from which the proposition follows. □

2.3.2 Analytic gradings

This example is motivated by the prospect of defining analytic analogues of quantum groups. In constructing the positive part of the quantum group through Nicholls algebras, one works with graded vector spaces. The following gives an analytic analogue of such a grading which we will use in Part III to define analytic Nichols algebras.

Definition 2.3.2. Let Γ be a discrete group with identity e. Let $\mathrm{Gr}_\Gamma\mathrm{IndBan}_k$ be the category of IndBanach spaces of the form $\coprod_\Gamma^{\leq 1} M_\Gamma$ for M_Γ in $\mathrm{IndBan}_k^{\Gamma,\mathrm{bd}}$ with morphisms that preserve this grading, that is,

$$\underline{\mathrm{Hom}}_{\mathrm{Gr}_\Gamma}(\coprod_\Gamma^{\leq 1} M_\Gamma, \coprod_\Gamma^{\leq 1} M'_\Gamma) = \lim_{i\in I}\mathrm{colim}_{j\in J}\prod_{g\in\Gamma}^{\leq 1}\underline{\mathrm{Hom}}(M_g(i), M'_g(j))$$

where M_Γ = "colim"$_{i\in I}(M_g(i))_{g\in\Gamma}$ and M'_Γ = "colim"$_{j\in J}(M'_g(j))_{g\in\Gamma}$ in $\mathrm{IndBan}_k^{\Gamma,\mathrm{bd}}$. Let F be the forgetful functor to IndBan_k which maps morphisms via the natural maps

$$\begin{aligned}\prod_{g\in\Gamma}^{\leq 1}\underline{\mathrm{Hom}}(M_g(i), M'_g(j)) &\to \prod_{g\in\Gamma}^{\leq 1}\coprod_{g'\in\Gamma}^{\leq 1}\underline{\mathrm{Hom}}(M_g(i), M'_{g'}(j)) \\ &= \underline{\mathrm{Hom}}(\coprod_{g\in\Gamma}^{\leq 1} M_g(i), \coprod_{g'\in\Gamma}^{\leq 1} M'_{g'}(j)).\end{aligned}$$

$\mathrm{Gr}_\Gamma\mathrm{IndBan}_k$ is monoidal, with tensor product

$$\left(\coprod_\Gamma^{\leq 1} M_\Gamma\right)\hat{\otimes}\left(\coprod_\Gamma^{\leq 1} M'_\Gamma\right) = \mathrm{colim}_{\substack{i\in I\\ j\in J}}\coprod_{g\in\Gamma}^{\leq 1}\left(\coprod_{g'g''=g}^{\leq 1} M_{g'}(i)\hat{\otimes}M'_{g''}(j)\right).$$

Proposition 2.3.3. *$\mathrm{Gr}_\Gamma\mathrm{IndBan}_k$ is equivalent to the monoidal category of $k\{\Gamma\}$ comodules in IndBan_k, where $k\{\Gamma\}$ is the bialgebra $\coprod_{g\in\Gamma}^{\leq 1} k\cdot t^g$. Here, $k\{\Gamma\}$ has the comultiplication $t^g\mapsto t^g\otimes t^g$, with counit $t^g\mapsto 1$, and multiplication $t^g\cdot t^{g'} = t^{gg'}$, with unit t^e.*

Proof. Since

$$\underline{\mathrm{Hom}}(F(\coprod_\Gamma^{\leq 1} M_\Gamma), X) = \lim_{i\in I}\mathrm{colim}_{j\in J}\prod_{g\in\Gamma}^{\leq 1}\underline{\mathrm{Hom}}(M_g(i), X(j))$$

for M_Γ = "colim"$_{i\in I}(M_g(i))_{g\in\Gamma}$ and X = "colim"$_{j\in J}X(j)$, we see that F is left adjoint to the functor $G:\mathrm{IndBan}_k\to\mathrm{Gr}_\mathbb{Z}\mathrm{IndBan}_k$ that takes X to the contracting coproduct $\coprod_{g\in\Gamma}^{\leq 1} X = \mathrm{colim}_j\coprod_{g\in\Gamma}^{\leq 1} X(j)$. Then $U = FG$ is cocontinuous and commutes with contracting colimits, so is isomorphic to the functor $k\{\Gamma\}\hat{\otimes}-$. It is clear that the monoidal comonadic structure on U induces the above bialgebra structure on $k\{\Gamma\}$. □

Remark If we take $\Gamma = \mathbb{Z}^n$ we obtain a completion of $k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, the coalgebra of analytic functions on the unit sphere in k^n.

2.3.3 Gradings arising from strictly convergent and overconvergent powerseries on the unit polydisk

In the previous example, we showed that analytically $\mathbb{Z}^n$-graded IndBanach spaces are comodules over the bialgebra of analytic functions on the unit sphere in k^n. There are, of course, other bialgebras of analytic functions, and these give rise to other analytic gradings.

Definition 2.3.4. Let $\mathrm{Gr}_{\mathbb{N}^N}\mathrm{IndBan}_k$ be the category whose objects are IndBanach spaces of the form $\coprod_{\mathbb{N}^N}^{\leq 1} M$ for $M \in \mathrm{IndBan}_k^{\mathbb{N}^N,\mathrm{bd}}$ with morphisms that respect the grading

$$\mathrm{Hom}_{\mathrm{Gr}_{\mathbb{N}^n}}(\coprod_{\mathbb{N}^N}^{\leq 1} M, \coprod_{\mathbb{N}^N}^{\leq 1} M') = \lim_{i\in I}\mathrm{colim}_{j\in J} \prod_{\underline{n}\in\mathbb{N}^N}^{\leq 1} \mathrm{Hom}(M_{\underline{n}}(i), M'_{\underline{n}}(j))$$

where $M = \text{"colim"}_{i\in I}(M_{\underline{n}}(i))_{\underline{n}\in\mathbb{N}^N}$ and $M' = \text{"colim"}_{j\in J}(M'_{\underline{n}}(j))_{\underline{n}\in\mathbb{N}^N}$ in $\mathrm{IndBan}_k^{\mathbb{N}^N,\mathrm{bd}}$. Let us denote by F the forgetful functor to IndBan_k. Then $\mathrm{Gr}_{\underline{r}}\mathrm{IndBan}_k$ is monoidal, with

$$\left(\coprod_{\mathbb{N}^N}^{\leq 1} M\right) \hat{\otimes} \left(\coprod_{\mathbb{N}^N}^{\leq 1} M'\right) = \mathrm{colim}_{\substack{i\in I\\ j\in J}} \coprod_{\underline{n}}^{\leq 1} \left(\coprod_{\underline{m}+\underline{m}'=\underline{n}}^{\leq 1} M_{\underline{m}}(i)\hat{\otimes} M'_{\underline{m}'}(j)\right).$$

Proposition 2.3.5. *The category $\mathrm{Gr}_{\mathbb{N}^N}\mathrm{IndBan}_k$ is equivalent to the category of $k\{\underline{t}\} = k\{t_1, \ldots, t_N\} := \coprod_{\underline{n}\in\mathbb{N}^N}^{\leq 1} k\cdot \underline{t}^{\underline{n}}$ comodules, where the comultiplication maps $\underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{n}} \otimes \underline{t}^{\underline{n}}$ and the counit is $\underline{t}^{\underline{n}} \mapsto 1$, and the multiplication maps $\underline{t}^{\underline{m}} \otimes \underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{m}+\underline{n}}$ with unit $\underline{t}^{\underline{0}}$.*

Proof. This is just a variation of Proposition 2.3.3. □

Remark $k\{\underline{t}\}$ is the bialgebra of strictly convergent powerseries on the polydisk of radius 1, $\{\underline{a} = (a_1, .., a_N) \in k^N \mid |a_i| \leq 1\}$. Note that strictly convergent powerseries on a polydisk of polyradius $\underline{r}$ does not have a bounded comultiplication unless all $r_i \leq 1$, and the counit is only bounded if all $r_i \geq 1$, hence we are restricted to the unit polydisk.

Definition 2.3.6. Let $\mathrm{Gr}^{\dagger,1}_{\mathbb{N}^N}\mathrm{IndBan}_k$ be the category whose objects are IndBanach spaces of the form

$$M = \text{"colim"}_{\substack{\underline{r}>1 \\ i\in I}} \coprod\nolimits^{\leq 1}_{\underline{n}\in\mathbb{N}^N} M(\underline{n}, i)_{\underline{r}^{\underline{n}}}$$

for $\text{"colim"}_{i\in I}(M(\underline{n}, i))_{\underline{n}\in\mathbb{N}^N}$ in $\mathrm{IndBan}_k^{\mathbb{N}^N,\mathrm{bd}}$. Here we use the notation $M(\underline{n}, i)_{\underline{r}^{\underline{n}}}$ for the Banach space $M(\underline{n}, i)$ with the norm scaled by $\underline{r}^{\underline{n}} > 0$. Morphisms are defined by

$$\mathrm{Hom}_{\mathrm{Gr}^{\dagger,1}_{\mathbb{N}^N}}(M, M') = \lim_{\underline{r}<1}\lim_{i\in I}\mathrm{colim}_{j\in J} \prod\nolimits^{\leq 1}_{\underline{n}\in\mathbb{N}^n} \mathrm{Hom}(M(\underline{n}, i), M'(\underline{n}, j))_{\underline{r}^{\underline{n}}},$$

for

$$M = \text{"colim"}_{\underline{r}>1, i\in I} \coprod\nolimits^{\leq 1}_{\underline{n}\in\mathbb{N}^N} M(\underline{n}, i)_{\underline{r}^n}$$

and

$$M' = \text{"colim"}_{\underline{r}'>1, j\in J} \coprod\nolimits^{\leq 1}_{\underline{n}'\in\mathbb{N}^N} M'(\underline{n}', j)_{\underline{r}'^{n'}}.$$

Here, colimits and limits are taken over polyradii $\underline{r} = (r_1, .., r_N)$ with $1 < r_i$ and $1 > r_i$ respectively for $i = 1, .., N$. The category $\mathrm{Gr}^{\dagger,1}_{\mathbb{N}^N}\mathrm{IndBan}_k$ is monoidal, with

$$M\hat{\otimes}M' = \text{"colim"}_{\substack{\underline{r}>1 \\ i\in I \\ j\in J}} \coprod\nolimits^{\leq 1}_{\underline{n}} \left(\coprod\nolimits^{\leq 1}_{\underline{m}+\underline{m}'=\underline{n}} M(\underline{m}, i)\hat{\otimes}M'(\underline{m}', j)\right)_{\underline{r}^{\underline{n}}}.$$

Proposition 2.3.7. *The category $Gr^{\dagger,1}_{\mathbb{N}^N}IndBan_k$ is equivalent to the monoidal category of $k\{\underline{t}\}^\dagger :=$ "colim"$_{\underline{r}>1} \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} k_{\underline{r}^{\underline{n}}}$ comodules. The algebra structure comes from that of each $k\{\frac{\underline{t}}{\underline{r}}\} = \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} k_{\underline{r}^{\underline{n}}}$, whilst the counit and comultiplication are induced by* *the maps*

$$k\{\tfrac{\underline{t}}{\underline{r}}\} \to k, \quad \underline{t}^{\underline{n}} \mapsto 1, \quad k\{\tfrac{\underline{t}}{\underline{r}^2}\} \to k\{\tfrac{\underline{t}}{\underline{r}}\}\hat{\otimes} k\{\tfrac{\underline{t}}{\underline{r}}\}, \quad \underline{t}^{\underline{n}} \mapsto \underline{t}^{\underline{n}} \otimes \underline{t}^{\underline{n}}.$$

Proof. For $M = \text{"colim"}_{\underline{r}>1, i\in I} \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^N} M(\underline{n}, i)_{\underline{r}^{\underline{n}}}$ and for $X = \text{colim}_{j\in J} X(j)$ an Ind-Banach space we have

$$\begin{aligned}
&\text{Hom}_{\text{Gr}^{\dagger}_{\mathbb{N}^N}}(M, \text{"colim"}_{\substack{\underline{r}>1\\ j\in J}} \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} X(j)_{\underline{r}^{\underline{n}}})\\
&\quad = \lim_{\substack{i\in I\\ \underline{r}<1}} \text{colim}_{j\in J} \prod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} \text{Hom}(M(\underline{n}, i), X(j))_{\underline{r}^{\underline{n}}}\\
&\quad = \lim_{\substack{i\in I\\ \underline{r}>1}} \text{colim}_{j\in J} \prod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} \text{Hom}(M(\underline{n}, i), X(j))_{(1/\underline{r})^{\underline{n}}}\\
&\quad = \lim_{\substack{i\in I\\ \underline{r}>1}} \text{colim}_{j\in J} \prod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} \text{Hom}(M(\underline{n}, i)_{\underline{r}^{\underline{n}}}, X(j))\\
&\quad = \text{Hom}(FM, X),
\end{aligned}$$

and so $X \mapsto \text{"colim"}_{\underline{r}>1, j\in J} \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} X(j)_{\underline{r}^{\underline{n}}}$ is right adjoint to the forgetful functor. The associated comonad is then isomorphic to $k\{\underline{t}\}^{\dagger}\hat{\otimes}-$. The monoidal structure on $\text{Gr}^{\dagger}_{\mathbb{N}^N}\text{IndBan}_k$ gives $k\{\underline{t}\}^{\dagger}$ the described bialgebra structure. □

Remark $k\{\underline{t}\}^{\dagger}$ is referred to as the bialgebra of *overconvergent powerseries* on the polydisk of radius 1. For similar reasons to the case of strictly convergent powerseries, we are restricted on our choice of polyradius. Alongside the previous example of radius 1, we also have the following at radius 0, where we consider germs of analytic functions at 0.

Definition 2.3.8. Let $\text{Gr}^{\dagger,0}_{\mathbb{N}^N}\text{IndBan}_k$ be the category whose objects are IndBanach spaces of the form

$$M = \text{"colim"}_{\substack{\underline{r}>0\\ i\in I}} \coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^N} M(\underline{n}, i)_{\underline{r}^{\underline{n}}}$$

for "colim"$_{i\in I}(M(\underline{n},i))_{\underline{n}\in\mathbb{N}^N}$ in $\mathrm{IndBan}_k^{\mathbb{N}^N,\mathrm{bd}}$. Morphisms are defined by

$$\mathrm{Hom}_{\mathrm{Gr}^{\dagger,0}_{\mathbb{N}^N}}(M,M') = \lim_{\underline{r}>0}\lim_{i\in I}\mathrm{colim}_{j\in J}\prod\nolimits^{\leq 1}_{\underline{n}\in\mathbb{N}^n}\mathrm{Hom}(M(\underline{n},i),M'(\underline{n},j))_{\underline{r}^{\underline{n}}},$$

for

$$M = \text{"colim"}_{\underline{r}>0,i\in I}\coprod\nolimits^{\leq 1}_{\underline{n}\in\mathbb{N}^N} M(\underline{n},i)_{\underline{r}^n}$$

and

$$M' = \text{"colim"}_{\underline{r'}>0,j\in J}\coprod\nolimits^{\leq 1}_{\underline{n'}\in\mathbb{N}^N} M'(\underline{n'},j)_{\underline{r'}^{n'}}.$$

As before, the category $\mathrm{Gr}^{\dagger,0}_{\mathbb{N}^N}\mathrm{IndBan}_k$ is monoidal.

Proposition 2.3.9. *The category $Gr^{\dagger,0}_{\mathbb{N}^N}IndBan_k$ is equivalent to the monoidal category of $k\{\frac{t}{0}\}^{\dagger} :=$ "colim"$_{\underline{r}>0}\coprod^{\leq 1}_{\underline{n}\in\mathbb{N}^n} k_{\underline{r}^{\underline{n}}}$ comodules.*

Proof. This follows as in the proof of Proposition 2.3.7. □

2.3.4 Non-example: Contracting products

Let $\mathcal{C}$ be the category of IndBanach spaces of the form "colim"$_{i\in I}\prod^{\leq 1}_{n\in\mathbb{Z}} M_n(i)$, for "colim"$_{i\in I}(M_n(i))_{n\in\mathbb{N}}$ in $\mathrm{IndBan}_k^{\mathbb{N},\mathrm{bd}}$, with morphisms similar to $\mathrm{Gr}_{\mathbb{Z}}\mathrm{IndBan}_K$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\text{"colim"}_{i\in I}\prod\nolimits^{\leq 1}_{n\in\mathbb{Z}} M_n(i), \text{"colim"}_{j\in J}\prod\nolimits^{\leq 1}_{n\in\mathbb{Z}} M'_n(j))\\ = \lim_{i\in I}\mathrm{colim}_{j\in J}\prod\nolimits^{\leq 1}_{n\in\mathbb{Z}}\mathrm{Hom}(M_n(i),M'_n(j)),\end{aligned}$$

and again let F be the forgetful functor to IndBan_k. Then as

$$\mathrm{Hom}(X,FM)\cong \lim_{j\in J}\mathrm{colim}_{i\in I}\prod\nolimits^{\leq 1}_{n\in\mathbb{Z}}\mathrm{Hom}(X(j),M_n(i))$$

for $X =$ "colim"$_{j\in J}X(j)$ in IndBan_k and $M =$ "colim"$_{i\in I}\prod^{\leq 1}_{n\in\mathbb{Z}} M_n(i)$, we see that

F has as left adjoint the functor $G' : X \mapsto \prod_{n\in\mathbb{Z}}^{\leq 1} X$. However $T = FG'$ does not commute with contracting coproducts, and so is not isomorphic to taking the tensor product with an IndBanach algebra.

2.3.5 Representations of discrete groups

Definition 2.3.10. Let Γ be a discrete group and let Γ-IndBan$_k$ be the category of representations of Γ on IndBanach spaces. This has the obvious forgetful functor F to IndBan$_k$ forgetting the action of Γ. With the diagonal action of Γ, $\mathcal{C}$ is monoidal and F is strong monoidal.

Lemma 2.3.11. *F has a left adjoint $G : X \mapsto \coprod_{g\in\Gamma} X$ where $h \in \Gamma$ acts on GX by mapping the copy of X indexed by g isomorphically to the copy indexed by hg.*

Proof. The isomorphism $\mathrm{Hom}(X, FY) \cong \mathrm{Hom}_\Gamma(\coprod_{g\in\Gamma} X, Y)$, for X in IndBan$_k$ and Y in Γ-IndBan$_k$, that gives this adjunction takes $f : X \to Y$ to the morphism defined on the copy of X indexed by g as $X \xrightarrow{f} Y \xrightarrow{g\cdot} Y$. The inverse to this map just restricts a morphism $\coprod_{g\in\Gamma} X \to Y$ to the copy of X indexed by the identity $1 \in G$. □

Proposition 2.3.12. *Γ-IndBan$_k$ is equivalent to the monoidal category of $\mathscr{A} = \coprod_{g\in\Gamma} k$ modules in IndBan$_k$. Here, the multiplication on $\mathscr{A}$ is determined by mapping isomorphically the tensor product $k\hat{\otimes}k$ of the copies of k indexed by g and g' to the gg' copy of k in $\mathscr{A}$, with the unit being the map from k to the copy of k indexed by e. The comultiplication on $\mathscr{A}$ maps the copy of k indexed by g isomorphically to the tensor product $k\hat{\otimes}k$ of the copies of k indexed by g and g in $\mathscr{A}\hat{\otimes}\mathscr{A}$.*

Proof. This follows from Theorem 2.2.13, noting that $FG \cong \mathscr{A}\hat{\otimes}-$. □

Definition 2.3.13. Let Γ-IndBan$_k^{\leq 1}$ be the full subcategory of Γ-IndBan$_k$ consisting of IndBanach spaces V with a bounded action of Γ. That is, there is an inductive system of Banach spaces $(V_i)_{i\in I}$ and a map $I \to I$, $i \mapsto i'$, such that $V \cong$ "colim"$_{i\in I} V_i$

and the action of Γ is determined by maps $\Gamma \to \mathrm{Hom}(V_i, V_{i'})$ for each $i \in I$ with $\{\|g\cdot : V_i \to V_{i'}\| \mid g \in \Gamma\}$ bounded for each $i \in I$. We will continue to denote the restriction of F to Γ-$\mathrm{IndBan}_k^{\leq 1}$ as F. Then Γ-$\mathrm{IndBan}_k^{\leq 1}$ is again monoidal, and F is strong monoidal.

Lemma 2.3.14. *The forgetful functor F again has a left adjoint, $G' : X \mapsto \coprod_{g\in\Gamma}^{\leq 1} X$, where $h \in \Gamma$ acts on $G'X$ by mapping the copy of X indexed by g isomorphically to the copy indexed by hg.*

Proof. The proof is similar to the proof of Lemma 2.3.11, with contracting coproducts in place of coproducts. □

Proposition 2.3.15. *Γ-$\mathrm{IndBan}_k^{\leq 1}$ is equivalent to $l^1(\Gamma)$ modules for the Banach bialgebra $l^1(\Gamma) := \coprod_{g\in\Gamma}^{\leq 1} k \cdot t^g$, with bialgebra structure defined by $t^g \cdot t^{g'} = t^{gg'}$, $\Delta(t^g) = t^g \otimes t^g$.*

Proof. As with Proposition 2.3.12, this follows from Theorem 2.2.13, noting that $FG' \cong l^1(\Gamma)\hat{\otimes}-$. □

Remark $l^1(\Gamma)$ is often referred to as the *Banach group algebra* of Γ.

Remark Note that the forgetful functors from Γ-IndBan_k and Γ-$\mathrm{IndBan}_k^{\leq 1}$ also have right adjoints, $X \mapsto \prod_\Gamma X$ and $X \mapsto \prod_\Gamma^{\leq 1} X$, with similar Γ-actions to $G(X)$ and $G'(X)$. However these functors are not cocontinuous, so our monad is not isomorphic to tensoring with a coalgebra, unless Γ is finite. There are still natural morphisms $\coprod_\Gamma \prod_\Gamma X \to X$ and $X \to \prod_\Gamma \coprod_\Gamma X$, and $\coprod_\Gamma^{\leq 1} \prod_\Gamma^{\leq 1} X \to X$ and $X \to \prod_\Gamma^{\leq 1} \coprod_\Gamma^{\leq 1} X$, exhibiting adjunctions. If Γ is finite then $\mathscr{A} = \mathscr{A}' = l^1(\Gamma)$ is dualisable, with dual $l^\infty(\Gamma) = \prod_\Gamma^{\leq 1} k$.

2.3.6 Representations of topological groups

Definition 2.3.16. For a locally compact topological group H and a Banach space V let us denote by $C(H, V)$ the topological vector space of continuous functions

$H \to V$, with the topology of uniform convergence on compact subsets. Let us denote by $C_b(H,V)$ the closed subspace of functions which are bounded on H, which forms a Banach space with the supremum norm. Note that if H is compact then all continuous functions are bounded, so $C_b(H,V) = C(H,V)$. We will use $C_b^{\mathrm{lu}}(H,V)$ to denote the closed subspace of *left uniformly continuous* functions. That is, the subspace of functions $f : H \to V$ such that, for each net $(h_\lambda)_{\lambda\in\Lambda}$ in H converging to the identity, $\mathrm{Sup}_{x\in H}\|f(h_\lambda x) - f(x)\|$ converges to 0.

Remark It was remarked to the author by Anton Lyubinin that if H is compact then all continuous functions must be left uniformly continuous, by a variation of the Heine-Cantor Theorem, in which case $C_b^{\mathrm{lu}}(H,V) = C_b(H,V) = C(H,V)$.

We fix a locally compact topological group G.

Topological groups with a bounded continuous action

Definition 2.3.17. Let $G\text{-Mod}^{\mathrm{bd}}$ be the category of strongly continuous bounded IndBanach G modules V. That is, there is an inductive system of Banach spaces $(V_i)_{i\in I}$ and a map $I \to I$, $i \mapsto i'$, such that $V \cong \text{"colim"}_{i\in I} V_i$ and the action of G is determined by continuous maps $G \to \mathrm{Hom}(V_i, V_{i'})$ for each $i \in I$, where $\mathrm{Hom}(V_i, V_{i'})$ is given the strong operator topology, and $\{\|g\cdot : V_i \to V_{i'}\| \mid g \in G\}$ bounded for each $i \in I$. The diagonal action of G makes $G\text{-Mod}^{\mathrm{bd}}$ monoidal. We denote by F the forgetful functor to IndBan_k.

Definition 2.3.18. For a Banach space V, let $C_{\mathrm{b}}(G,V)$ be the Banach space of bounded continuous functions from G to V, and let $C_{\mathrm{b}}^{\mathrm{lu}}(G,V)$ be the closed subspace of left uniformly continuous functions. For a general IndBanach space $V = \text{"colim"}_{i\in I} V_i$ we set $C_{\mathrm{b}}^{\mathrm{lu}}(G,V) = \text{"colim"}_{i\in I} C_{\mathrm{b}}^{\mathrm{lu}}(G,V_i)$.

Remark In Definition 2.3.17, our representations are in some sense locally Banach. Likewise, our definition of $C_{\mathrm{b}}^{\mathrm{lu}}(G,-)$ in Definition 2.3.18 as a functor is in some sense local.

Lemma 2.3.19 ([9]). *The functor $C_b^{lu}(G,-)$ is right adjoint to the forgetful functor F.*

Proof. This is proved by Bühler in [9] for Banach spaces but follows for IndBanach spaces too. □

Proposition 2.3.20. *G-Mod^{bd} is equivalent to the category of coalgebras over the monoidal comonad $C_b^{lu}(G,-)$.*

Proof. This follows from Lemma 2.2.12. □

Corollary 2.3.21. *In the case where G is compact, G-Mod^{bd} is equivalent to the category of comodules over the bialgebra $C_b^{lu}(G,k)$. Here, the multiplication is pointwise, and the comultiplication is given by the composition*

$$C_b^{lu}(G,k) \xrightarrow{\Delta} C_b^{lu}(G, C_b^{lu}(G,k)) \cong C_b^{lu}(G,k)\hat{\otimes} C_b^{lu}(G,k),$$

with $\Delta(f)(g)(g') = f(gg')$.

Proof. If G is compact, $C_{\mathrm{b}}^{\mathrm{lu}}(G,-)$ is cocontinuous and commutes with contracting colimits, so is isomorphic to $C_{\mathrm{b}}^{\mathrm{lu}}(G,k)\hat{\otimes}-$ by Lemma 2.2.5, and G-Mod$^{\mathrm{bd}}$ is equivalent to IndBanach $C_{\mathrm{b}}^{\mathrm{lu}}(G,k)$-comodules. Then the monoidal structure gives $C_{\mathrm{b}}^{\mathrm{lu}}(G,k)$ the usual algebra structure arising from pointwise multiplication. □

Topological Groups with a continuous action, not necessarily bounded

We now consider a wider class of representations of a topological group. Suppose, for simplicity, that we can write G as a directed union of compact open subgroups $G = \bigcup_{i\in I} G_i$.

Definition 2.3.22. Let G-Mod be the category of k-IndBanach spaces V with a *strongly continuous* action of G. By this we mean an IndBanach space V such that, for each $i \in I$ there is an inductive system of Banach spaces $(V_j)_{j \in J}$ and map $J \to J$, $j \mapsto j'$, such that $V \cong \text{"colim"}_{j \in J} V_j$ and the action of G on V is induced by continuous maps $G_i \to \mathrm{Hom}(V_j, V_{j'})$ where $\mathrm{Hom}(V_j, V_{j'})$ is given the strong operator topology. We will denote by F the forgetful functor from G-Mod to the category of IndBanach spaces. The diagonal action of G makes G-Mod monoidal, with trivial action on the monoidal unit k, and F is strong monoidal.

Remark If $V \in G$-Mod is a Banach space then this just means that the action by G is strongly continuous in the usual sense.

Remark Note that $G\text{-Mod}^{\mathrm{bd}}$ sits as a full subcategory of G-Mod.

Definition 2.3.23. For any $i \in I$ and for any Banach space V, $C^{\mathrm{lu}}(G_i, V)$ is a Banach space. For a general IndBanach space $V = \text{"colim"}_{j \in J} V_j$ we can view $C^{\mathrm{lu}}(G_i, V)$ as the colimit $\text{"colim"}_{j \in J} C^{\mathrm{lu}}(G_i, V_j)$ in IndBan_k, and we view $C^{\mathrm{lu}}(G, V)$ as the limit $\lim_{i \in I} C^{\mathrm{lu}}(G_i, V)$. $C^{\mathrm{lu}}(G, V)$ has a left action of $g \in G$ induced by the right regular actions of G_i on $C^{\mathrm{lu}}(G_i, V_j)$.

Lemma 2.3.24. *$C^{lu}(G, V)$ can be expressed as the colimit of spaces*

$$\left\{ (f_i)_{i \in I} \in \prod_{i \in I}^{\leq 1} C^{lu}(G_i, V_{j_i})_{r_i} \,\middle|\, \begin{matrix} \text{For all } i \geq i' \text{ there exists } j \geq j_i, j_{i'} \\ \text{with } \phi_{j_i, j} \circ f_i|_{G_{i'}} = \phi_{j_{i'}, j} \circ f_{i'} \end{matrix} \right\}$$

indexed over pairs $((j_i)_{i \in I}, (r_i)_{i \in I})$ where $(j_i)_{i \in I}$ is a collection of indecies in J and $(r_i)_{i \in I}$ is a collection of positive real numbers, both indexed over I. Here, $\phi_{j,j'} : V_j \to V_{j'}$ are the transition maps in the inductive system $(V_j)_{j \in J}$.

Proof. Firstly, note that $C^{\mathrm{lu}}(G, V)$ is the kernel of the map

$$\phi : \prod_{i \in I} C^{\mathrm{lu}}(G_i, V) \to \prod_{(i \leq i') \in I} C^{\mathrm{lu}}(G_{i'}, V)$$

defined by $\pi_{i,i'} \circ \phi = \pi_{i'} - \rho_{i,i'} \circ \pi_i$ where $\pi_{i,i'}$ and π_i are the respective projections and $\rho_{i,i'} : C^{\mathrm{lu}}(G_{i'}, V) \to C^{\mathrm{lu}}(G_i, V)$ is the restriction map. By the explicit description of limits in [34],

$$\prod_{i \in I} C^{\mathrm{lu}}(G_i, V) = \text{"colim"}_{(j_i)_{i\in I},(r_i)_{i\in I}} \prod\nolimits_{i\in I}^{\leq 1} C^{\mathrm{lu}}(G_i, V_{j_i})_{r_i}$$

and likewise

$$\prod_{(i \leq i') \in I} C^{\mathrm{lu}}(G_{i'}, V) = \text{"colim"}_{\substack{(j_{i,i'})_{(i\leq i')\in I} \\ (r_{i,i'})_{(i\leq i')\in I}}} \prod\nolimits_{(i\leq i')\in I}^{\leq 1} C^{\mathrm{lu}}(G_{i'}, V_{j_{i,i'}})_{r_{i,i'}}.$$

The result then follows by direct computation of this kernel, again using *loc. cit.* □

Proposition 2.3.25. *The action of G on $C^{lu}(G, V)$ is strongly continuous for any IndBanach space V.*

Proof. Note that, for any fixed $i_0 \in I$, we may replace I with $I_{\geq i_0}$. In which case, G_{i_0} has a strongly continuous action on the spaces describes in Lemma 2.3.24. □

Definition 2.3.26. For V in G-Mod with action maps $\pi_V^{i,j,j'} : G_i \to \mathrm{Hom}(V_j, V_{j'})$ we get a collection of bounded linear map $V_j \to C^{\mathrm{lu}}(G_i, V_{j'})$, $v \mapsto \pi_V^{i,j,j'}(-)(v)$, where $V =$ "colim"$_{i\in I} V_i$. These then induce morphisms $V \to C^{\mathrm{lu}}(G_i, V)$ in IndBan_k, inducing in turn a map $\pi_V^* : V \to C^{\mathrm{lu}}(G, V)$, the *adjoint* of the action.

Lemma 2.3.27. *The forgetful functor F : G-Mod $\to$ $IndBan_k$ has a right adjoint $C^{lu}(G, -)$.*

Proof. For an object V of G-Mod, with underlying IndBanach space FV, and an IndBanach space W, there is a natural map

$$\mathrm{Hom}_{\mathrm{IndBan}_k}(FV, W) \to \mathrm{Hom}_G(V, C^{\mathrm{lu}}(G, W)),$$

taking f to the composite $V \xrightarrow{\pi_V^*} C^{\mathrm{lu}}(G,V) \xrightarrow{f\circ -} C^{\mathrm{lu}}(G,W)$. Given $i \in I$, the restriction of the map

$$\mathrm{Hom}_{\mathrm{IndBan}}(V, C^{\mathrm{lu}}(G,W)) \to \mathrm{Hom}_{\mathrm{IndBan}}(V, C^{\mathrm{lu}}(G_i,W)) \to \mathrm{Hom}_{\mathrm{IndBan}}(V,W)$$

to $\mathrm{Hom}_G(V, C^{\mathrm{lu}}(G,W))$ provides an inverse, where the first arrow is induced by the restriction map $C^{\mathrm{lu}}(G,W) \to C^{\mathrm{lu}}(G_i,W)$ and the second arrow is induced by the map $C^{\mathrm{lu}}(G_i,W) \to W$ that evaluates a function at $1 \in G_i \subset G$ (coming from the maps $C^{\mathrm{lu}}(G_i,W_j) \to W_j$ for $W = \text{"colim"}_{j\in J} W_j$). Hence $\mathrm{Hom}_{\mathrm{IndBan}_k}(V,W) \cong \mathrm{Hom}_G(V, C^{\mathrm{lu}}(G,W))$. □

Proposition 2.3.28. *G-Mod is equivalent to the category of IndBanach spaces with a coaction of the comonad $C^{lu}(G,-)$.*

Proof. This follows from Lemma 2.2.12. □

Remark Here, the comultiplication $\Delta_V : C^{\mathrm{lu}}(G,V) \to C^{\mathrm{lu}}(G, C^{\mathrm{lu}}(G,V))$ can be thought of as $\Delta(f)(g)(g') = f(gg')$ with counit $f \mapsto f(1)$.

Corollary 2.3.29. *If G is compact then G-Mod is equivalent to the monoidal category of IndBanach $C^{lu}(G,k)$-comodules. Here, the multiplication on $C^{lu}(G,k)$ is pointwise.*

Proof. If G is compact, this monad is isomorphic to $C^{\mathrm{lu}}(G,k)\hat{\otimes}_k -$. □

Remark The above Corollary is not true if G is not assumed to be compact, and $C^{\mathrm{lu}}(G,k)$ is not *a priori* a coalgebra.

2.3.7 Analytic Galois descent

Let $K \subset L$ be two complete valued fields, let IndBan_K and IndBan_L be their respective categories of IndBanach spaces, let $\mathrm{Hom}_K(-,-)$ and $\mathrm{Hom}_L(-,-)$ be their morphisms, and let $\hat{\otimes}_K$ and $\hat{\otimes}_L$ be their monoidal structures. We assume throughout the following that L and K are non-Archimedean, which ensures that all IndBanach spaces over either of them are flat by Lemma 3.49 of [4].

Definition 2.3.30. Let $\mathrm{Res}_K^L : \mathrm{IndBan}_L \to \mathrm{IndBan}_K$ be the restriction functor that restricts L-IndBanach spaces to K-IndBanach spaces, and let $\mathrm{Ind}_K^L : \mathrm{IndBan}_K \to \mathrm{IndBan}_L$ be the induction functor $X \mapsto L\hat{\otimes}_K X$.

Lemma 2.3.31. *Ind_K^L and Res_K^L form an adjunction,*

$$Hom_L(Ind_K^L X, Y) \cong Hom_K(X, Res_K^L Y),$$

for each K-IndBanach space X and L-IndBanach space Y.

Proof. This adjunction is clear when we restrict X and Y to being Banach spaces. Taking colimits then gives the result. ☐

Remark From the above Lemma we obtain a monad $\mathrm{Rest}_K^L \mathrm{Ind}_K^L \cong L\hat{\otimes}_K-$ on IndBan_K, where the resulting K-algebra structure on L is the obvious one. It is clear that the restriction functor satisfies the conditions of Barr-Beck. So, unsurprisingly, IndBan_L is equivalent to the category of K-IndBanach spaces with an action of L. This adjunction also gives the following result.

Proposition 2.3.32. *$IndBan_K$ is equivalent to objects in $IndBan_L$ with a coaction by $U \cong L\hat{\otimes}_K-$ via the functor $X \mapsto L\hat{\otimes}_K X$ for K-IndBanach spaces X.*

Proof. We obtain a comonad $U = \mathrm{Ind}_K^L \mathrm{Rest}_K^L = L\hat{\otimes}_K-$ on IndBan_L from the adjunction in Lemma 2.3.31. The comonad structure on U has comultiplication given by the composition $L\hat{\otimes}_K Y \cong L\hat{\otimes}_K K\hat{\otimes}_K Y \to L\hat{\otimes}_K L\hat{\otimes}_K Y$, with counit given by scalar multiplication by L on each L-IndBanach space Y. By Lemma 3.49 of [4], L is flat over K. Furthermore, if $f : V \to W$ is a morphism of K-IndBanach spaces then the diagram

$$\begin{array}{ccccc} V & \xrightarrow{\sim} & K\hat{\otimes}_K V & \longrightarrow & \mathrm{Ind}_K^L V \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow {\scriptstyle \mathrm{Ind}_K^L f} \\ W & \xrightarrow{\sim} & K\hat{\otimes}_K W & \longrightarrow & \mathrm{Ind}_K^L W. \end{array}$$

commutes, and the horisontal morphisms are strict monomorphisms. So if $\mathrm{Ind}_K^L f$ is strict then so is f. The proof of Lemma 2.2.12 then gives our result. □

Remark Note that this differs from the theory outlined in Section 2.2 since U is not L-linear, only K-linear. Thus we introduce the following framework to deal with this.

Definition 2.3.33. For algebras $\mathcal{R}$ and $\mathcal{S}$ in IndBan_K, let us denote by $\mathcal{R}$-$\mathcal{S}$-IndBan_K the category of K-IndBanach spaces with a left action by $\mathcal{R}$ and right action by $\mathcal{S}$ that are compatible. Then, for K-IndBanach algebras $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and objects $M \in \mathcal{R}$-$\mathcal{S}$-IndBan_K and $N \in \mathcal{S}$-$\mathcal{T}$-IndBan_K we obtain an object $M\hat{\otimes}_{\mathcal{S}}N$ in $\mathcal{R}$-$\mathcal{T}$-IndBan_K as the coequaliser of the two maps $M\hat{\otimes}_K\mathcal{S}\hat{\otimes}_K N \rightrightarrows M\hat{\otimes}_K N$. In particular, this gives $\mathcal{R}$-$\mathcal{R}$-IndBan_K a monoidal structure, $\hat{\otimes}_{\mathcal{R}}$. Suppose now that $\mathcal{R}$ and $\mathcal{S}$ are commutative. For left $\mathcal{R}$ modules (respectively right $\mathcal{S}$ modules) M and N we may view $M \otimes_K N$ as a left $\mathcal{R}$ module (resp. right $\mathcal{S}$ module) in two ways depending on whether we act on M or N. Thus, for $M, N \in \mathcal{R}$-$\mathcal{S}$-IndBan_K, there are four morphisms $\mathcal{R}\hat{\otimes}_K(M\hat{\otimes}_K N)\hat{\otimes}_K\mathcal{S} \to M\hat{\otimes}_K N$. The coequaliser of these four maps, which we denote by $M\hat{\otimes}_{\mathcal{R}\text{-}\mathcal{S}}N$, has a natural left action by $\mathcal{R}$ and right action by $\mathcal{S}$, hence gives an object in $\mathcal{R}$-$\mathcal{S}$-IndBan_K. In particular, this gives $\mathcal{R}$-$\mathcal{R}$-IndBan_K a second monoidal structure, which we shall denote by $\hat{\otimes}_{\mathcal{R}\text{-}\mathcal{R}}$.

Lemma 2.3.34. *A functor $\mathscr{V} : \mathrm{IndBan}_L \to \mathrm{IndBan}_L$ is isomorphic to one of the form $V\hat{\otimes}_L-$ for some $V \in L$-L-IndBan_K if and only if is it a cocontinuous functor, enriched over IndBan_K, that commutes with l^1.*

Proof. This is entirely similar to the proof of Lemma 2.2.5. The main difference is that $\mathscr{V}_i(a_{x,y}\iota_y)$ is not equal to $a_{x,y}\mathscr{V}_i(\iota_y)$ with the usual left L action on Y_{j_i}. As a result

$V = \mathscr{V}(L)$ now has two actions of L. On the left, $\lambda \in L$ acts by $\lambda \cdot \mathrm{id}_{\mathscr{V}(L)}$, whilst on the right λ acts by $\mathscr{V}(\lambda \cdot \mathrm{id}_L)$. □

Proposition 2.3.35. *IndBan$_K$ is equivalent to the category of left $(L\hat{\otimes}_K L)$-comodules in IndBan$_L$ via the induction functor. Here, $(L\hat{\otimes}_K L)$ is not a bialgebra in IndBan$_L$ but instead in L-L-IndBan$_K$ with respect to the monoidal structure $\hat{\otimes}_L$. The comultiplication on $(L\hat{\otimes}_K L)$ is given by*

$$(a \otimes b) \mapsto (a \otimes 1) \otimes (1 \otimes b)$$

and the counit is just multiplication in L.

Proof. This follows from Proposition 2.3.32 and Lemma 2.3.34. □

Remark The previous proposition says that we can recover an IndBanach space V over K from $\mathrm{Ind}_K^L V$ provided we retain the coaction of $L\hat{\otimes}_K L$ as descent data.

Remark In [12], Deligne refers to objects such as $(L\hat{\otimes}_K L)$ as *groupoides*, or, in this particular case, *cogebroides.*

Proposition 2.3.36. *With respect to the equivalence in the above Proposition, the monoidal structure of IndBan$_K$ corresponds to the algebra structure on $(L\hat{\otimes}_K L)$ given by $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$, with unit $1 \otimes 1$. Note that this algebra structure is with respect to the tensor product $\hat{\otimes}_{L\text{-}L}$ on L-L-IndBan$_K$.*

Proof. This is a trivial computation. □

Definition 2.3.37. Consider $\mathrm{Hom}_K(L, L)$ as an object of L-L-IndBan$_K$ with left action $(\lambda \cdot f)(a) = \lambda f(a)$ and right action $(f \cdot \lambda)(a) = f(\lambda \cdot a)$ for $\lambda, a \in L$, $f \in \mathrm{Hom}_K(L, L)$. Then composition gives $\mathrm{Hom}_K(L, L)$ an algebra structure with respect to $\hat{\otimes}_L$.

Proposition 2.3.38. *We have a non-degenerate pairing*

$$Hom_K(L, L)\hat{\otimes}_L(L\hat{\otimes}_K L) \to L, \ \ \langle f, a \otimes b\rangle = f(a)b,$$

of an algebra with a coalgebra. That is, with the induced pairing between $Hom_K(L, L)\hat{\otimes}_L Hom_K(L, L)$ *and* $(L\hat{\otimes}_K L)\hat{\otimes}_L(L\hat{\otimes}_K L)$ *given by*

$$\langle f \otimes f', (a \otimes b) \otimes (a' \otimes b')\rangle = \langle f\langle f', a \otimes b\rangle, a' \otimes b'\rangle = \langle f, \langle f', a \otimes b\rangle a' \otimes b'\rangle,$$

we have that $\langle f \circ g, a \otimes b\rangle = \langle f \otimes g, \Delta(a \otimes b)\rangle$.

Proof. $\langle f \circ g, a \otimes b\rangle = f(g(a))b = \langle f, (g(a) \cdot 1)1 \otimes b\rangle = \langle f \otimes g, \Delta(a \otimes b)\rangle$. □

Definition 2.3.39. Let $\Delta : \mathrm{Hom}_K(L, L) \to \mathrm{Hom}_K(L\hat{\otimes}_K L, L)$ be the L-linear bounded map $\Delta(f)(a \otimes b) = f(ab)$. If L/K is finite then

$$\mathrm{Hom}_K(L\hat{\otimes}_K L, L) \cong \mathrm{Hom}_K(L, L)\hat{\otimes}_{L\text{-}L}\mathrm{Hom}_K(L, L)$$

and so Δ can be viewed as a comultiplication.

Proposition 2.3.40. *We can pair* $Hom_K(L\hat{\otimes}_K L, L)$ *with* $(L\hat{\otimes}_K L)\hat{\otimes}_{L\text{-}L}(L\hat{\otimes}_K L)$, $\langle f, (a\otimes b) \otimes (a' \otimes b')\rangle = f(a \otimes a')bb'$, $f \in Hom_K(L\hat{\otimes}_K L, L)$, $a, a', b, b' \in L$. *In which case* $\langle \Delta(f), (a \otimes b) \otimes (a' \otimes b')\rangle = \langle f, (a \otimes b) \cdot (a' \otimes b')\rangle$.

Proof. $\langle \Delta(f), (a \otimes b) \otimes (a' \otimes b')\rangle = f(aa')bb' = \langle f, (a \otimes b) \cdot (a' \otimes b')\rangle$. □

Remark As a bialgebra, $L\hat{\otimes}_K L$ can be thought of as dual to $\mathrm{Hom}_K(L, L)$. Since the Galois group, $\Gamma = \Gamma_{L/K}$, sits as the group-like elements within $\mathrm{Hom}_K(L, L)$, we may think of $L\hat{\otimes}_K L$ as functions on the Galois group. We shall make this more precise. Since Γ is a profinite, hence compact, topological group, its strongly continuous L-IndBanach representations should fit in the framework of Section 2.3.6. Since Γ does not act L-linearly, only K-linearly, we must modify the example slightly.

Definition 2.3.41. Let Γ-Mod_L be the category of L-IndBanach spaces V with a strongly continuous action on $\mathrm{Res}_K^L(V)$ as in Definition 2.3.22, given by $\pi_{V,i,i'} : \Gamma \to \mathrm{Hom}_K(V_i, V_{i'})$ for $V \cong "\mathrm{colim}"_{i\in I} V_i$, such that

$$\pi_{V,i,i'}(\sigma)(\lambda v) = \sigma(\lambda)\pi_{V,i,i'}(v) \text{ for } \lambda \in L, v \in V_i, \sigma \in \Gamma.$$

Let F be the forgetful functor to IndBan_L. The diagonal action of Γ makes Γ-Mod_L monoidal, with F strong monoidal. Let, for a Banach space W, $\tilde{C}^{\mathrm{lu}}(\Gamma, W)$ be the K-Banach space of left uniformly continuous functions from Γ to W extended to an L-Banach space with the twisted action $(\lambda \cdot f)(\sigma) = \sigma(\lambda)f(\sigma)$ for $\lambda \in L$ and $f \in \tilde{C}^{\mathrm{lu}}(\Gamma, W)$. For $W = "\mathrm{colim}"_{i\in I} W_i$ an IndBanach space we define $\tilde{C}^{\mathrm{lu}}(\Gamma, W) = "\mathrm{colim}"_{i\in I} \tilde{C}^{\mathrm{lu}}(\Gamma, W_i)$.

Lemma 2.3.42. *The forgetful functor F has a left adjoint $\tilde{C}^{lu}(\Gamma, -)$.*

Proof. The K-linear adjoint map $\pi_V^* : V \to C^{\mathrm{lu}}(\Gamma, V)$ extends to an L-linear map $\pi_V^* : V \to \tilde{C}^{\mathrm{lu}}(\Gamma, V)$. The rest follows as in the proof of Lemma 2.3.27. □

Proposition 2.3.43. *The category Γ-Mod_L is equivalent to monoidal category of left $C^{lu}(\Gamma, L)$-comodules in $IndBan_L$. Here, $C^{lu}(\Gamma, L)$ is an object of L-L-$IndBan_K$ with left action by L as described for $\tilde{C}^{lu}(\Gamma, L)$ and right action by L the usual pointwise action on $C^{lu}(\Gamma, L)$. The multiplication is pointwise, and with respect to $\hat{\otimes}_{L\text{-}L}$, and comultiplication given by the composition*

$$C^{lu}(\Gamma, L) \xrightarrow{\Delta} C^{lu}(\Gamma, C^{lu}(\Gamma, L)) \cong C^{lu}(\Gamma, L)\hat{\otimes}_L C^{lu}(\Gamma, L)$$

where $\Delta(f)(\sigma)(\tau) = f(\tau\sigma)$ for $f \in C^{lu}(\Gamma, L)$, $\sigma, \tau \in \Gamma$.

Proof. This follows from Lemma 2.3.42, Lemma 2.2.12 and Lemma 2.3.34. □

Lemma 2.3.44. *There is a morphism $\phi : L\hat{\otimes}_K L \to C^{lu}(\Gamma, L)$, given by*

$$\phi(a \otimes b)(\sigma) = \sigma(a)b,$$

that is compatible with the multiplication and comultiplication, and has norm $\|\phi\| = 1$.

Proof. Firstly, the fact that $\phi(a \otimes b)$ is left uniformly continuous is straightforward to prove. In fact, if $(x_\lambda)_{\lambda\in\Lambda}$ is a net converging to $1 \in \Gamma$ then $\mathrm{Sup}_{\sigma\in\Gamma}|\phi(a\otimes b)(x_\lambda\sigma) - \phi(a \otimes b)(\sigma)|$ eventually becomes constant at 0. Secondly,

$$\phi(\lambda \cdot (a \otimes b) \cdot \mu)(\sigma) = \sigma(\lambda)\sigma(a)b\mu = (\lambda \cdot \phi(a \otimes b) \cdot \mu)(\sigma),$$

$$\phi((a \otimes b)(a' \otimes b'))(\sigma) = \sigma(a)\sigma(a')bb' = (\phi(a \otimes b) \cdot \phi(a' \otimes b'))(\sigma),$$

and

$$\Delta(\phi(a \otimes b))(\sigma)(\tau) = \tau\sigma(a)b = (\sigma(a) \cdot \phi(1 \otimes b))(\tau) = (\phi(a \otimes 1) \otimes \phi(1 \otimes b))(\sigma)(\tau)$$

for $a, b, a', b'\lambda, \mu \in L$ and $\sigma, \tau \in \Gamma$. Also,

$$|\phi(\sum\nolimits_i a_i \otimes b_i)(\sigma)| = |\sum\nolimits_i \sigma(a_i)b_i| \leq \mathrm{Sup}_i\{|\sigma(a_i)||b_i|\} = \mathrm{Sup}_i\{|a_i||b_i|\}$$

for all $a_i, b_i \in L$ and $\sigma \in \Gamma$, hence

$$\mathrm{Sup}_{\sigma\in\Gamma}\{|\phi(\alpha)(\sigma)|\} \leq \mathrm{Inf}\left\{\mathrm{Sup}_i\{|a_i||b_i|\}\middle|\alpha = \sum\nolimits_i a_i \otimes b_i\right\}$$

for all $\alpha \in L\hat{\otimes}_K L$. That is, $\|\phi\| \leq 1$. The fact that $\|\phi\| = 1$ follows since ϕ preserves the unit, which is of norm 1 in both spaces. $\square$

Lemma 2.3.45. *Let L/K be an extension of complete valued fields such that the algebraic elements are dense in L. Then $L \cong colim^{\leq 1}_{K\subset L'\subset L} L'$, where this is the contracting* *colimit taken in Ban_K over all finite extensions $K \subset L'$ contained in L.*

Proof. We have strict contracting monomorphisms $L' \hookrightarrow L$ for all finite extensions $K \subset L'$ contained in L. Suppose we are given a compatible collection of bounded linear maps $\{f_{L'} : L' \to V\}_{K \subset L' \subset L}$ such that $\{\|f_{L'}\|\}_{K \subset L' \subset L}$ is bounded by some $M > 0$. Then we obtain a well defined bounded linear map $f : \bigcup_{K \subset L' \subset L} L' \to V$ defined on each L' by $f_{L'}$. The compatibility of the collection $\{f_{L'}\}_{K \subset L' \subset L}$ ensures that this is well defined. By assumption, $\bigcup_{K \subset L' \subset L} L'$ is dense in L, hence we may extend f to a unique map $L \to V$ such that $f_{L'}$ is the composition $L' \hookrightarrow L \to V$. Clearly $\|f\| \leq M$. □

Lemma 2.3.46. *For an extension of complete valued fields, L/K, such that the algebraic elements are dense in L, there is an isomorphism $L \hat{\otimes}_K L \cong colim^{\leq 1}_{K \subset L' \subset L} L' \hat{\otimes}_K L'$.*

Proof. This follows from Lemma 2.3.45. □

Lemma 2.3.47. *For G a profinite group and V a Banach space, the subspace of locally constant functions is dense in $C^{lu}(G, V)$.*

Proof. Let $f : G \to V$ be a left uniformly continuous function. For a fixed $g_0 \in \Gamma$, suppose, for a contradiction, that the net

$$\left(\mathrm{Sup}_{g \in g_0 N} \|f(g) - f(g_0)\|\right)_{\substack{N \trianglelefteq G \\ [G;N] < \infty}}$$

does not converge to 0. Hence there is a net $(g_N)_{N \trianglelefteq G}$ converging to g_0 such that $\|f(g_N) - f(g_0)\|$ does not converge to 0, which contradicts left uniform continuity of f. Thus for all $\varepsilon > 0$ there exists $N_{g_0} \trianglelefteq G$ such that $\mathrm{Sup}_{g \in g_0 N_{g_0}} \|f(g) - f(g_0)\| < \varepsilon$. This means that, by looking at $\{g_0 N_{g_0} \mid g_0 \in G\}$ and $f(g_0) \in V$, for each $\varepsilon > 0$ there exists a cover $\mathcal{U}_\varepsilon$ of compact open subsets which has the property that each $U \in \mathcal{U}$ has some $\lambda_U \in V$ for which $\mathrm{Sup}_{g \in U} \|f(g) - \lambda_U\| < \varepsilon$. By compactness of G we may assume that $\mathcal{U}_\varepsilon$ is finite, and furthermore we can take the sets in $\mathcal{U}_\varepsilon$ to be pairwise disjoint. We then have that the locally constant function $\sum_{U\in\mathcal{U}} \lambda_U \chi_U$ approximates f, $\|f - \sum_{U\in\mathcal{U}} \lambda_U \chi_U\| \leq \varepsilon$, in $C^{\mathrm{lu}}(G, V)$. □

Lemma 2.3.48. *Let L/K be an extension of complete valued fields such that the algebraic elements are dense in L and form a Galois extension over K. Then there is an isomorphism $\mathrm{colim}^{\leq 1}_{H \trianglelefteq \Gamma} C^{lu}(\Gamma/H, L) \xrightarrow{\sim} C^{lu}(\Gamma, L)$, where this is the contracting colimit taken in Ban_K over all finite index normal subgroups $H \trianglelefteq \Gamma$.*

Proof. A proof similar to that of Lemma 2.3.45 shows that the Banach space $\mathrm{colim}^{\leq 1}_{H \trianglelefteq \Gamma} C^{\mathrm{lu}}(\Gamma/H, L)$ is isomorphic to the closure of $\bigcup_{H \trianglelefteq \Gamma} C^{\mathrm{lu}}(\Gamma, L)^H$, since the image of $C^{\mathrm{lu}}(\Gamma/H, L)$ in $C^{\mathrm{lu}}(\Gamma, L)$ is just the H invariant subspace. It follows from the definition of the profinite topology on Γ that a function is locally constant if and only if it lies in one of these invariant subspaces. By Lemma 2.3.47 this subspace is dense. □

Lemma 2.3.49. *For an extension of complete valued fields, L/K, such that the algebraic elements are dense in L and form a Galois extension over K, there is an isomorphism $\mathrm{colim}^{\leq 1}_{H \trianglelefteq \Gamma} C^{lu}(\Gamma/H, L^H) \xrightarrow{\sim} C^{lu}(\Gamma, L)$, where the contracting colimit is taken in Ban_K over all finite index normal subgroups $H \trianglelefteq \Gamma$.*

Proof. This follows from Lemma 2.3.45, Lemma 2.3.48, the fact that $C^{\mathrm{lu}}(G, -)$ commutes with contracting colimits for finite discrete groups G, and the fact that all finite Galois extensions over K in L are of the form L^H for $H \trianglelefteq \Gamma$ of finite index. □

Lemma 2.3.50. *If L/K is a finite Galois extension then the morphism ϕ in Lemma 2.3.44 is an isomorphism.*

Proof. By the open mapping theorem and Lemma 2.3.44, it is enough to show that ϕ is a bijection. First, by the Normal Basis Theorem, we may take be a normal basis B of L over K. That is, B is a basis of L over K comprised of a single orbit of the Galois group Γ. Taking a basis $\{b \otimes 1 \mid b \in B\}$ of $L \hat{\otimes}_K L$ over L (with its right action) and the basis $\{\sigma \mapsto \delta_{\sigma,\tau} \mid \tau \in \Gamma\}$ of $C^{\mathrm{lu}}(\Gamma, L)$ over L (with its right action) we see that ϕ is given by the matrix with entries $(\tau(b))_{(b,\tau)\in B\times\Gamma}$ indexed over $B \times \Gamma$. The columns of this matrix are all linearly independent since Γ permutes B simply transitively, hence it is invertible and so is ϕ. □

Remark It is not clear whether ϕ is an isometry in the above finite dimensional case. This means that the norm of ϕ^{-1} might become arbitrarily large as we range over an infinite collection of such extensions. Hence ϕ may not remain an isomorphism after taking contracting colimits over infinitely many of these finite extensions (using Lemmas 2.3.46 and 2.3.49). We do, however, have the following result.

Proposition 2.3.51. *Let L/K be an extension of complete valued fields such that the algebraic elements, L^a, are dense in L and form a Galois extension over K. Then ϕ restricts to a continuous bijection between the dense subspaces $L^a \otimes_K L \subset L\hat{\otimes}_K L$ (the algebraic tensor product of L^a with L) and the subspace of locally constant functions in $C^{lu}(\Gamma, L)$.*

Proof. By Lemma 2.3.45, there is an isomorphism

$$\mathrm{colim}^{\leq 1}_{K\subset L'\subset L} L'\hat{\otimes}_K L \cong L\hat{\otimes}_K L$$

in IndBan_L under which the algebraic tensor product $L^a \otimes_K L$ is the union of the images of $L'\hat{\otimes}_K L = L' \otimes_K L$. By Lemma 2.3.48 there is an isomorphism

$$\mathrm{colim}^{\leq 1}_{K\subset L'\subset L} C^{\mathrm{lu}}(\Gamma_{L'/K}, L) = \mathrm{colim}^{\leq 1}_{H\trianglelefteq\Gamma} C^{\mathrm{lu}}(\Gamma/H, L) \cong C^{\mathrm{lu}}(\Gamma, L)$$

under which the union of the images of $C^{\mathrm{lu}}(\Gamma_{L'/K}, L)$ is the subspace of locally constant functions. The result then follows from Lemma 2.3.50 applied to the maps from each $L' \otimes_K L$ to the corresponding $C^{\mathrm{lu}}(\Gamma_{L'/K}, L)$. □

Remark The above proposition says precisely that $L\hat{\otimes}_K L$ is a completion of the space of locally constant functions with respect to a stronger topology than that inherited from $C^{\text{lu}}(\Gamma, L)$. It is in this way that we may think of $L\hat{\otimes}_K L$ as functions on the Galois group Γ.

Definition 2.3.52. Let L/K be an extension of complete valued fields such that the algebraic elements, L^a, are dense in L and form a Galois extension over K with Galois group Γ. We think of L^a as a formal colimit over finite extensions of K in L in IndBan_K, hence as a K-IndBanach algebra. We define the IndBanach space of locally constant L-valued functions on G, $C^{\text{lc}}(\Gamma, L)$, to be the colimit

$$C^{\text{lc}}(\Gamma, L) := \text{"colim"}_{N \trianglelefteq \Gamma} C^{\text{lu}}(\Gamma/N, L)$$

taken over finite index normal subgroups of Γ. Similarly we define the IndBanach algebraic tensor product, $L^a \otimes L$, to be the colimit

$$L^a \otimes_K L := \text{"colim"}_{K \subset L' \subset L} L' \hat{\otimes}_K L$$

taken over finite extensions L' of K in L. We may also define

$$C^{\text{lc}}(\Gamma, L^a) := \text{"colim"}_{N \trianglelefteq \Gamma} C^{\text{lu}}(\Gamma/N, L^N) = \text{"colim"}_{\substack{N \trianglelefteq \Gamma \\ K \subset L' \subset L}} C^{\text{lu}}(\Gamma/N, L')$$

and

$$L^a \otimes_K L^a := \text{"colim"}_{K \subset L' \subset L} L' \hat{\otimes}_K L' = \text{"colim"}_{\substack{K \subset L' \subset L \\ K \subset L'' \subset L}} L' \hat{\otimes}_K L''$$

in a similar way.

Proposition 2.3.53. *There is a commutative diagram*

$$\begin{array}{ccc} C^{la}(\Gamma, L) & \xrightarrow{\phi} & L\hat{\otimes}_K L \\ \uparrow & & \uparrow \\ C^{lc}(\Gamma, L) & \xrightarrow{\sim} & L^a \otimes_K L \\ \uparrow & & \uparrow \\ C^{lc}(\Gamma, L^a) & \xrightarrow{\sim} & L^a \otimes_K L^a \end{array}$$

whose vertical arrows are bimorphisms.

Proof. This is just rephrasing Proposition 2.3.51. □

Definition 2.3.54. Let Ind_ϕ be the induction functor

$$\mathrm{Ind}_\phi : (L\hat{\otimes}_K L)\text{-Comod} \to C^{\mathrm{lu}}(G, k)\text{-Comod} \cong \Gamma\text{-Mod}_L, \;\; M \mapsto \mathrm{Ind}_\phi M,$$

from the category of $L\hat{\otimes}_K L$ comodules in IndBan_L to Γ-Mod_L, where $\mathrm{Ind}_\phi M$ has the same underlying IndBanach space as M but with the coaction

$$M \to (L\hat{\otimes}_K L)\hat{\otimes}_L M \xrightarrow{\phi \otimes \mathrm{id}_M} C^{\mathrm{lu}}(G, k)\hat{\otimes}_L M.$$

Lemma 2.3.55. *The induction functor Ind_ϕ is exact and fully faithful.*

Proof. Exactness and faithfulness follows from the fact that the forgetful functors from these categories are faithful and reflect exactness, and that composition of Ind_ϕ with the forgetful functor from Γ-Mod_L gives the forgetful functor from $(L\hat{\otimes}_K L)$-Comod. If $f : \mathrm{Ind}_\phi M \to \mathrm{Ind}_\phi N$ is a morphism of $C^{\mathrm{lu}}(G, k)$ comodules, where M and N are $(L\hat{\otimes}_K L)$ comodules with respective coactions Δ_M and Δ_N then

$$(\phi \otimes \mathrm{Id}) \circ \Delta_N \circ f = (\mathrm{Id} \otimes f) \circ (\phi \otimes \mathrm{Id}) \circ \Delta_M = (\phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes f) \circ \Delta_M.$$

By Lemma 3.49 of [4], $C^{\mathrm{lu}}(G, k)$ is a flat IndBanach space and so $\phi \otimes \mathrm{Id}$ is monic. Hence $f : M \to N$ is a morphism of $(L\hat{\otimes}_K L)$ comodules. □

Remark The previous Lemma says that we can retain a strongly continuous action of the Galois group Γ as descent data to recover a K-IndBanach space V from $\mathrm{Ind}_K^L V$.

Definition 2.3.56. Let G be a profinite group, k be a complete valued field and A be an IndBanach algebra over k. We define the IndBanach (or Bornological) Iwasawa algebra, $\Lambda_A^{\mathrm{Born}}(G)$, to be the limit

$$\Lambda_A^{\mathrm{Born}}(G) = \lim_{N \trianglelefteq G} A\{G/N\}$$

in IndBan_k taken over all open normal subgroups of G, where $A\{G/N\}$ is the Banach group algebra $\coprod_{G/N}^{\leq 1} A$ over A defined similarly to the algebra in Proposition 2.3.15. If A is a Banach algebra then we define the Banach Iwasawa algebra, $\Lambda_k^{\mathrm{Ban}}(G)$, as the contracting limit

$$\Lambda_A^{\mathrm{Ban}}(G) = \lim_{N \trianglelefteq G}^{\leq 1} A\{G/N\}$$

in Ban_k.

Proposition 2.3.57. *Let L/K be an extension of complete valued fields such that the algebraic elements, L^a, are dense in L and form a Galois extension over K with Galois group Γ. Then, as IndBanach spaces over L, $\Lambda_L^{Ban}(\Gamma)$ is dual to $C^{lu}(\Gamma, L)$, and, as L^a modules in $IndBan_K$, $\Lambda_{L^a}^{Born}(\Gamma)$ is dual to $C^{lc}(\Gamma, L^a) \cong L^a \otimes_K L^a$.*

Proof. The first statement follows from the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_L(C^{\mathrm{lu}}(\Gamma, L), L) &= \mathrm{Hom}_L(\mathrm{colim}_{N \trianglelefteq \Gamma}^{\leq 1} C^{\mathrm{lu}}(\Gamma/N, L), L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \mathrm{Hom}_L(C^{\mathrm{lu}}(\Gamma/N, L), L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \mathrm{Hom}_L(\textstyle\coprod_{\Gamma/N}^{\leq 1} L, L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \textstyle\coprod_{\Gamma/N}^{\leq 1} \mathrm{Hom}_L(L, L) \\
&\cong \lim_{N \trianglelefteq \Gamma}^{\leq 1} \textstyle\coprod_{\Gamma/N}^{\leq 1} L = \Lambda_L^{\mathrm{Ban}}(\Gamma).
\end{aligned}$$

The second follows from

$$\begin{aligned}\mathrm{Hom}_{L^a}(C^{\mathrm{lu}}(\Gamma, L^a), L^a) &= \mathrm{Hom}_{L^a}(L^a \hat{\otimes} C^{\mathrm{lu}}(\Gamma/N, K), L^a) \\ &\cong \mathrm{Hom}_K(C^{\mathrm{lu}}(\Gamma, K), L^a)\end{aligned}$$

and a similar argument to the above. □

Remark The above isomorphisms are not isomorphisms of algebras. The multiplications on $\Lambda_L^{\mathrm{Ban}}(\Gamma)$ and $\Lambda_{L^a}^{\mathrm{Born}}(\Gamma)$ induced by the respective comultiplications on $C^{\mathrm{lu}}(\Gamma, L)$ and $C^{\mathrm{lc}}(\Gamma, L^a)$ are twisted by the actions of Γ on L and L^a. Since there is a faithful embedding of Γ-Mod_L, viewed as $C^{\mathrm{lu}}(\Gamma, L)$-comodules, into modules over the twisted Iwasawa algebra $\Lambda_L^{\mathrm{Ban}}(\Gamma)$ we may alternatively take this action as our descent data.

Part III

On analytic analogues of quantum groups

3.1 Braided IndBanach Hopf algebras, analytic gradings and Nichols algebras

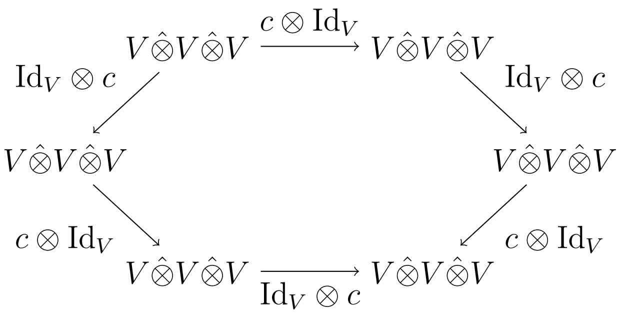
Let k be a complete valued field throughout Part III. Recall Definitions 0.1.15 and 2.1.1 of Ban_k and $\mathrm{Ban}_k^{\leq 1}$ and the results of Section 2.1 in Part II on contracting (co)products. The results in Part III will differ depending on whether we are working in a non-Archimedean or an Archimedean setting. We shall therefore distinguish between the following two cases:

(NA) k is non-Archimedean and we require all norms in Ban_k to satisfy the strong triangle inequality; and

(A) k is not necessarily non-Archimedean and we only require norms in Ban_k to satisfy the weak triangle inequality.

3.1.1 Braided IndBanach Hopf algebras

Definition 3.1.1. Let V be an IndBanach space. We say that a morphism $c : V\hat{\otimes}V \to V\hat{\otimes}V$ is a *pre-braiding* on V if it satisfied the *hexagon axiom*, *i.e.* the diagram



commutes. We say that the pair (V, c) is a *pre-braided IndBanach space*. If c is an isomorphism then c is a *braiding* and (V, c) is a *braided IndBanach space*.

Definition 3.1.2. Let V be a Banach space. We define the *Banach tensor algebra*, $T(V)$, to be the contracting coproduct

$$T(V) := \coprod_{n\in\mathbb{Z}_{\geq 0}}^{\leq 1} V^{\hat{\otimes} n}$$

where $V^{\hat{\otimes} 0} := k$. For $r > 0$ we will use the notation $T_r(V)$ for the Banach space $T(V_r)$, where V_r is the Banach space V with its norm rescaled by r. For $r \leq r'$ there is a natural map $T_{r'}(V) \to T_r(V)$, and for all $\rho \geq 0$ we will denote by $T_\rho(V)^\dagger$ the colimit "colim"$_{r>\rho} T_r(V)$ of this system. We will call this the *dagger tensor algebra* or *overconvergent tensor algebra* of radius ρ.

Lemma 3.1.3. *Given a pre-braiding c of a Banach space V, there is an induced map $c_{n,m} : V^{\hat{\otimes} n}\hat{\otimes} V^{\hat{\otimes} m} \to V^{\hat{\otimes} m}\hat{\otimes} V^{\hat{\otimes} n}$ with $\|c_{n,m}\| \leq \|c\|^{mn}$ for each $n, m \geq 0$ satisfying the commutative diagram below:*

$$\begin{array}{ccc} & V^{\hat\otimes l}\hat\otimes V^{\hat\otimes m}\hat\otimes V^{\hat\otimes n} \xrightarrow{c_{l,m}\otimes Id_V} V^{\hat\otimes m}\hat\otimes V^{\hat\otimes l}\hat\otimes V^{\hat\otimes n} & \\ Id_V\otimes c_{m,n} \swarrow & & \searrow Id_V\otimes c_{l,n} \\ V^{\hat\otimes l}\hat\otimes V^{\hat\otimes n}\hat\otimes V^{\hat\otimes m} & & V^{\hat\otimes m}\hat\otimes V^{\hat\otimes n}\hat\otimes V^{\hat\otimes l} \\ c_{l,n}\otimes Id_V \searrow & & \swarrow c_{m,n}\otimes Id_V \\ & V^{\hat\otimes n}\hat\otimes V^{\hat\otimes l}\hat\otimes V^{\hat\otimes m} \xrightarrow[Id_V\otimes c_{l,m}]{} V^{\hat\otimes n}\hat\otimes V^{\hat\otimes m}\hat\otimes V^{\hat\otimes l} & \end{array}$$

Hence if $\|c\| \leq 1$ then there is an induced pre-braiding $\tilde{c}$ on $T(V)$ with $\|\tilde{c}\| = \|c\|$. Furthermore, $\tilde{c}$ is a braiding if and only if c is an isometry.

Proof. Applying successively $\mathrm{Id}_V^{\otimes n-i} \otimes c \otimes \mathrm{Id}_V^{\otimes i-1}$ for $i = 1, \ldots, n$ we obtain a map $c_n : V^{\hat{\otimes} n}\hat{\otimes} V \to V\hat{\otimes} V^{\hat{\otimes} n}$ with $\|c_n\| \leq \|c\|^n$. Then successive applications of $\mathrm{Id}_V^{\otimes i-1} \otimes c_n \otimes \mathrm{Id}_V^{\otimes m-i}$ for $i = 1, \ldots, m$ gives a map $c_{n,m} : V^{\hat{\otimes} n}\hat{\otimes} V^{\hat{\otimes} m} \to V^{\hat{\otimes} m}\hat{\otimes} V^{\hat{\otimes} n}$ with $\|c_{n,m}\| \leq \|c_n\|^m \leq \|c\|^{nm}$. The commutativity of the given diagram follows from repeated applications of the hexagon axiom from Definition 3.1.1. $\square$

Definition 3.1.4. A *pre-braided IndBanach bialgebra* is an IndBanach space A with both the structure of an algebra, (A, μ, η), and a coalgebra, (A, Δ, ε), and equipped with a pre-braiding c on A such that

$$c \circ (\eta \otimes \mathrm{Id}) = \mathrm{Id} \otimes \eta, \quad c \circ (\mathrm{Id} \otimes \eta) = \eta \otimes \mathrm{Id},$$

$$(\mathrm{Id} \otimes \mu)(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c) = c \circ (\mu \otimes \mathrm{Id}), \quad (\mu \otimes \mathrm{Id})(\mathrm{Id} \otimes c)(c \otimes \mathrm{Id}) = c \circ (\mathrm{Id} \otimes \mu),$$

$$(\varepsilon \otimes \mathrm{Id}) \circ c = \mathrm{Id} \otimes \varepsilon, \quad (\mathrm{Id} \otimes \varepsilon) \circ c = \varepsilon \otimes \mathrm{Id},$$

$$(c \otimes \mathrm{Id})(\mathrm{Id} \otimes c)(\Delta \otimes \mathrm{Id}) = (\mathrm{Id} \otimes \Delta) \circ c, \quad (\mathrm{Id} \otimes c)(c \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta) = (\Delta \otimes \mathrm{Id}) \circ c,$$

the diagram

$$\begin{array}{ccc}
A\hat{\otimes}A & \xrightarrow{\Delta \otimes \Delta} & A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \\
 & & \big\downarrow \mathrm{Id} \otimes c \otimes \mathrm{Id} \\
\mu \big\downarrow & & A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \\
 & & \big\downarrow \mu \otimes \mu \\
A & \xrightarrow{\quad\Delta\quad} & A\hat{\otimes}A
\end{array}$$

commutes and

$$\Delta \circ \eta = \eta \otimes \eta, \quad \varepsilon \circ \eta = \mathrm{Id}_k, \quad \varepsilon \circ \mu = \varepsilon \circ \varepsilon.$$

If c is an isomorphism then A is a *braided IndBanach bialgebra.*

Remark For any IndBanach algebra A with a (pre-) braiding c, we may define morphisms

$$A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \xrightarrow{\mathrm{Id}\otimes c\otimes \mathrm{Id}} A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \xrightarrow{\mu\otimes\mu} A\hat{\otimes}A,$$

$$A\hat{\otimes}A \xrightarrow{\Delta\otimes\Delta} A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \xrightarrow{\mathrm{Id}\otimes c\otimes \mathrm{Id}} A\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A.$$

The first four relations in Definition 3.1.4 are equivalent to the first of these maps making $A\hat{\otimes}A$ an associative algebra with unit $\eta \otimes \eta$. The next four relations are equivalent to the second of these maps making $A\hat{\otimes}A$ an associative coalgebra with counit $\varepsilon\otimes\varepsilon$. Then the diagram and final three relations are equivalent to Δ and ε being algebra homomorphisms, or equivalently μ and η being coalgebra homomorphisms.

Proposition 3.1.5. *Assume* **(NA)**. *For V a Banach space, $T_r(V)$ is a Banach algebra with multiplication μ induced by the maps $V_r^{\hat{\otimes}n}\hat{\otimes}V_r^{\hat{\otimes}m} \xrightarrow{\sim} V_r^{\hat{\otimes}n+m}$. Given a pre-braiding c on V with $\|c\| \leq 1$, $T_r(V)$ is a pre-braided Banach bialgebra with the comultiplication Δ uniquely determined by $\Delta(v) = 1\otimes v+v\otimes 1$ for all $v \in V_r = V_r^{\hat{\otimes}1} \subset T_r(V)$ and the pre-braiding $\tilde{c}$ induced by c. Furthermore, Δ has norm at most 1.*

Proof. The fact that $T_r(V)$ forms a Banach algebra is clear from construction, with unit $k \xrightarrow{\sim} V^{\hat{\otimes}0}$. We have a map $\Delta_1 : V_r^{\hat{\otimes}1} \to T_r(V)\hat{\otimes}T_r(V)$, given by $v \mapsto 1\otimes v+v\otimes 1$, with $\|\Delta_1\| \leq 1$. Suppose we have a map $\Delta_n : V_r^{\hat{\otimes}n} \to T_r(V)\hat{\otimes}T_r(V)$ with $\|\Delta_n\| \leq \|c\|^{n-1}$. Then we define Δ_{n+1} as the composition

$$\begin{aligned} V_r^{\hat{\otimes}n+1} = V_r^{\hat{\otimes}n}\hat{\otimes}V_r &\xrightarrow{\Delta_n\otimes\Delta_1} T_r(V)\hat{\otimes}T_r(V)\hat{\otimes}T_r(V)\hat{\otimes}T_r(V) \\ &\xrightarrow{\mathrm{Id}\otimes\tilde{c}\otimes\mathrm{Id}} T_r(V)\hat{\otimes}T_r(V)\hat{\otimes}T_r(V)\hat{\otimes}T_r(V) \\ &\xrightarrow{\mu\otimes\mu} T_r(V)\hat{\otimes}T_r(V). \end{aligned}$$

Then since the multiplication on $T_r(V)$ is of norm 1, we have $\|\Delta_{n+1}\| \leq \|\Delta_n\| \cdot \|\Delta_1\| \cdot \|\tilde{c}\| \leq \|c\|^n$ □

Proposition 3.1.6. *Assume* **(A)**. *For V a Banach space, $T_r(V)$ is a Banach algebra with multiplication μ induced by the maps $V_r^{\hat{\otimes}n}\hat{\otimes}V_r^{\hat{\otimes}m} \xrightarrow{\sim} V_r^{\hat{\otimes}n+m}$. Given a pre-braiding c on V with $\|c\| \leq 1$, $T_0(V)^\dagger$ is a pre-braided IndBanach bialgebra with comultiplication Δ whose restriction to V is*

$$\text{"colim"}_{r>0}V_r \cong V \to T(V)\hat{\otimes}T(V) \to T_0(V)^\dagger\hat{\otimes}T_0(V)^\dagger, \quad v \mapsto 1 \otimes v + v \otimes 1,$$

for the pre-braiding $\tilde{c}$ induced by c.

Proof. The given map $V \to T(V)\hat{\otimes}T(V)$ induces maps $V_r \to T_{\frac{r}{2}}(V)\hat{\otimes}T_{\frac{r}{2}}(V)$ of norm at most 1. By the same construction as in the proof of the previous proposition, we obtain maps $T_r(V) \to T_{\frac{r}{2}}(V)\hat{\otimes}T_{\frac{r}{2}}(V)$ which induce the desired comultiplication on $T_0(V)^\dagger$. $\square$

Remark Note that $V \mapsto T(V)$ is only functorial on the contracting category $\mathrm{Ban}_k^{\leq 1}$. So, in the **(NA)** case, the diagonal embedding $V \to V \oplus V$ induces the map $T(V) \to T(V \oplus V) \cong T(V)\hat{\otimes}T(V)$. However, in the **(A)** case, the diagonal embedding is not contracting, thus $T(V)$ does not form a coalgebra.

Definition 3.1.7. Let A be a (pre-) braided IndBanach bialgebra. We say that A is a (pre-) braided IndBanach Hopf algebra if the identity on A is convolution invertible. That is, Id_A is invertible with respect to the convolution product $*$ on $\mathrm{Hom}(A, A)$,

$$f * g : A \xrightarrow{\Delta} A\hat{\otimes}A \xrightarrow{f\otimes g} A\hat{\otimes}A \xrightarrow{\mu} A$$

for $f, g \in \mathrm{Hom}(A, A)$ whose unit is $\eta \circ \varepsilon$. We will call the convolution inverse of Id_A the antipode, and often denote it by S or S_A.

Lemma 3.1.8. *Assume* **(NA)**. *For a Banach space V with a pre-braiding c on V with $\|c\| \leq 1$, we have that, in $T(V)$, $\Delta(V^{\hat{\otimes}n}) \subset \sum_{i=0}^{n} V^{\hat{\otimes}i}\hat{\otimes}V^{\hat{\otimes}n-i}$, and so in particular*

$$\Delta\left(\coprod_{i\leq n}^{\leq 1} V^{\hat{\otimes}i}\right) \subset \left(\coprod_{i\leq n-1}^{\leq 1} V^{\hat{\otimes}i}\right)\hat{\otimes}T(V) + T(V)\hat{\otimes}\left(\coprod_{i\leq n-1}^{\leq 1} V^{\hat{\otimes}i}\right)$$

for all $n \geq 0$.

Proof. This follows by induction using the proof of Proposition 3.1.5. $\square$

Proposition 3.1.9. *Assume* **(NA)**. *Given a Banach space V with a pre-braiding c on V with $\|c\| \leq 1$, $T(V)$ is a pre-braided Banach Hopf algebra.*

Proof. We proceed as in Takeuchi's proof of Lemma 5.2.10 presented in [37] to find a convolution inverse to $\mathrm{Id}_{T(V)}$. Let $\gamma = \eta \circ \varepsilon - \mathrm{Id}_{T(V)}$. Then $\gamma|_{V^{\hat{\otimes} 0}} = 0$. Now suppose that $\gamma^{*n}|_{\coprod^{\leq 1}_{i \leq n-1} V^{\hat{\otimes} i}} = 0$ for some $n \geq 1$ and let $x \in \coprod^{\leq 1}_{i \leq n} V^{\hat{\otimes} i}$. Here we use the notation γ^{*n} for the n-fold product of γ under the convolution product. Since

$$\Delta(x) \in (\coprod^{\leq 1}_{i \leq n-1} V^{\hat{\otimes} i}) \hat{\otimes} T(V) + T(V) \hat{\otimes} (\coprod^{\leq 1}_{i \leq n-1} V^{\hat{\otimes} i}),$$

by Lemma 3.1.8, and since $\gamma^{*(n+1)} = \gamma^{*n} * \gamma = \gamma * \gamma^{*n}$ we see that $\gamma^{*n+1}(x) = 0$. Hence, inductively, $\gamma^{*n+1}|_{\coprod^{\leq 1}_{i \leq n} V^{\hat{\otimes} i}} = 0$ for all $n \geq 0$. It follows that $\sum_{n=0}^{\infty} \gamma^{*n}$ is well defined on the direct sum of vector spaces $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^{\hat{\otimes} n}$. Furthermore, since $\|c\| \leq 1$ and so $\|\Delta\| \leq 1$ we see that $\|\gamma\| \leq 1$, $\|\gamma^{*n}\| \leq 1$ and so $\| \sum_{n=0}^{N} \gamma^{*n}\| \leq 1$ for all $N \geq 0$. It then follows that $\sum_{n=0}^{\infty} \gamma^{*n}$ converges to a well defined function on $T(V)$, which is convolution inverse to $\mathrm{Id}_{T(V)}$ since

$$\begin{aligned} \mathrm{Id}_{T(V)} * \textstyle\sum_{n=0}^{\infty} \gamma^{*n} &= (\eta \circ \varepsilon) * \textstyle\sum_{n=0}^{\infty} \gamma^{*n} - \gamma * \textstyle\sum_{n=0}^{\infty} \gamma^{*n} \\ &= \textstyle\sum_{n=0}^{\infty} \gamma^{*n} - \textstyle\sum_{n=1}^{\infty} \gamma^{*n} \\ &= \eta \circ \varepsilon. \end{aligned}$$

□

Definition 3.1.10. Assume **(NA)**. Given a (pre-) braided Banach space (V, c) with $\|c\| \leq 1$ and $r > 0$ we denote by $T^c_r(V)$ the (pre-) braided Banach Hopf algebra described in Proposition 3.1.5 and Proposition 3.1.9, or just $T_r(V)$ if the (pre-) braiding is implicit.

Proposition 3.1.11. *Assume* **(A)**. *Given a Banach space V with a braiding c on V with $\|c\| \leq 1$, $T_0(V)^\dagger$ is a pre-braided Banach Hopf algebra.*

Proof. We take a slightly different approach to the proof of Proposition 3.1.9. The linear maps $S_r : V_r \to T_r(V)$ defined by $v \mapsto -v$ determine a unique algebra homomorphisms $S : T_r(V) \to T_r(V)^{\mathrm{op}}$, where $T_r(V)^{\mathrm{op}}$ is the opposite algebra whose multiplication is $\mu \circ \tilde{c}$. The compositions

$$T_r(V) \xrightarrow{\Delta} T_{\frac{r}{2}}(V)\hat{\otimes}T_{\frac{r}{2}}(V) \xrightarrow{S_{\frac{r}{2}}\otimes \mathrm{Id}} T_{\frac{r}{2}}(V)\hat{\otimes}T_{\frac{r}{2}}(V) \to T_{\frac{r}{2}}(V)$$

agree with $\eta \circ \varepsilon$ when restricted to V. It is then easy to check that they agree on the subalgebra generated by V, which is dense in $T_r(V)$. So they agree. Likewise this is true for $\mathrm{Id} \otimes S_{\frac{r}{2}}$ in place of $S_{\frac{r}{2}} \otimes \mathrm{Id}$. Taking colimits we obtain the antipode $S : T_0(V)^\dagger \to T_0(V)^\dagger$. □

Definition 3.1.12. Assume **(A)**. Given a (pre-) braided Banach space (V, c) with $\|c\| \leq 1$ we denote by $T_0^c(V)^\dagger$ the (pre-) braided Banach Hopf algebra described in Proposition 3.1.6 and Proposition 3.1.11, or just $T_0(V)^\dagger$ if the (pre-) braiding is implicit.

Remark The main distinction between the cases **(NA)** and **(A)** is that the tensor Hopf algebra can be defined on any radius in the non-Archimedean setting but in the Archimedean setting can only be defined at radius 0. This is not entirely unexpected. Analogously we see that, for a non-Archimedean field k, the balls of each radius in k form additive subgroups, however the same cannot be said for Archimedean fields.

3.1.2 Analytic gradings

Definition 3.1.13. Let C be an IndBanach bialgebra. We will say that an IndBanach space V is *graded by* C if it is a C-comodule. An IndBanach space V with a (pre-) braiding c is a *graded (pre-) braided IndBanach space* (graded by C) if c is a morphism of $C\hat{\otimes}C$-comodules. A *graded (pre-) braided IndBanach bialgebra* is a (pre-) braided IndBanach bialgebra A such that A is a C-comodule, η, μ, ε and Δ are C-comodule homomorphisms, and c is a $C\hat{\otimes}C$-comodule homomorphism. If, in addition, the identity on A has a convolution inverse S that is a C-comodule homomorphism then A is a *graded (pre-) braided IndBanach Hopf algebra*.

The results in Sections 2.3.2 and 2.3.3 of Part II justify our definition of grading above.

Definition 3.1.14. We say that an IndBanach space is *analytically* $\mathbb{N}^N$*-graded*, or just *analytically graded* if N is implicit, if it is graded over $k\{\underline{t}\}$. Likewise, we say that an IndBanach space is *dagger-1* $\mathbb{N}^N$*-graded*, or just *dagger-1 graded*, (respectively *dagger-0* $\mathbb{N}^N$*-graded*, or *dagger-0 graded*) if it is graded over $k\{\underline{t}\}^\dagger$ (respectively over $k\{\underline{t}/0\}^\dagger$). We will occasionally just use the term *dagger graded* when there is no ambiguity.

Lemma 3.1.15. *Assuming* **(NA)***, for each* $0 < r$ *and each pre-braided Banach space* (V, c) *with* $\|c\| \leq 1$ *the Banach tensor algebra* $T_r^c(V)$ *is naturally an analytically graded pre-braided Hopf algebra. For any* $0 < \rho$, $T_\rho^c(V)^\dagger := "colim"_{r>\rho} T_r^c(V)$ *is naturally a dagger-1 graded pre-braided Hopf algebra, and* $T_0^c(V)^\dagger := "colim"_{r>0} T_r^c(V)$ *is a dagger-0 graded pre-braided Hopf algebra. Likewise, assuming* **(A)***, for each pre-braided Banach space* (V, c) *with* $\|c\| \leq 1$ *the dagger tensor algebra* $T_0^c(V)^\dagger$ *is naturally a dagger-0 graded pre-braided Hopf algebra.*

Proof. We define $T_r(V) \to k\{\frac{t}{s}\}\hat{\otimes} T_{r'}(V)$ by $x \mapsto t^n \otimes x$ for $x \in V^{\hat{\otimes} n}$ whenever $r \geq r's$. For $r = r'$, $s = 1$, this gives $T_r(V)$ an analytic grading for which it is an analytically graded braided Hopf algebra. For fixed $\rho > 0$ and each $r > \rho$ there exists $s > 1$ such that $r > s\rho$, so we may define a map $T_\rho(V)^\dagger \to k\{t\}^\dagger \hat{\otimes} T_\rho(V)^\dagger$. Likewise we can define a map $T_0(V)^\dagger \to k\{t/0\}^\dagger \hat{\otimes} T_0(V)^\dagger$. These give $T_\rho(V)^\dagger$ and $T_0(V)^\dagger$ dagger-1 and dagger-0 gradings respectively. By construction all of the structure maps for the pre-braided Hopf algebras $T_r^c(V)$, $T_\rho^c(V)^\dagger$ and $T_0^c(V)^\dagger$ are graded. □

Definition 3.1.16. Let C be an IndBanach coalgebra. A *generalised element* of C is a morphism $\lambda : k \to C$. We say that a generalised element is *grouplike* if it is a coalgebra homomorphism. Let $G(C)$ denote the set of grouplike generalised elements of C.

Lemma 3.1.17. *Let C be an IndBanach coalgebra. Then $G(C)$ forms a monoid under the composition law where $\lambda * \lambda'$ is the composition $k \cong k\hat{\otimes}k \xrightarrow{\lambda\otimes\lambda'} C\hat{\otimes}C \to C$. If C is an IndBanach Hopf algebra then $G(C)$ is a group.*

Proof. Note that this is just the restriction of the convolution product on $\mathrm{Hom}(k, C)$. As with Hopf algebras over vector spaces, it is clear that the inverse to λ is the composition $k \xrightarrow{\lambda} C \xrightarrow{S} C$. □

Remark Note that $G(k\{\underline{t}\}) \cong G(k\{\underline{t}\}^\dagger) \cong G(k\{\underline{t}/0\}^\dagger) \cong \mathbb{N}^N$ as monoids.

Definition 3.1.18. Let C be an IndBanach coalgebra and V be an IndBanach space graded over C. Given $\lambda \in G(C)$ we define $V(\lambda) = \mathrm{eq}(V \rightrightarrows C\hat{\otimes}V)$ as the equaliser of the given coaction of C on V and the map

$$V \cong k\hat{\otimes}V \xrightarrow{\lambda\otimes\mathrm{Id}_V} C\hat{\otimes}V.$$

We think of this as the degree λ graded piece. If η_C is the unit of C then we use the notation $V(0) := V(\eta_C)$.

Lemma 3.1.19. *Let C be an IndBanach coalgebra and (V, c) a C-graded, pre-braided IndBanach space and let $\lambda, \lambda' \in G(C)$. Assume further that V and $V(\lambda')$ are flat as IndBanach spaces. Then $(V\hat{\otimes}V)(\lambda\otimes\lambda') = V(\lambda)\hat{\otimes}V(\lambda')$ where $\lambda\otimes\lambda'$ is the morphism $k \cong k\hat{\otimes}k \xrightarrow{\lambda\otimes\lambda'} C \otimes C$.*

Proof. The map $V(\lambda)\hat{\otimes}V(\lambda') \to V\hat{\otimes}V$ induces a map

$$V(\lambda)\hat{\otimes}V(\lambda') \to (V\hat{\otimes}V)(\lambda\otimes\lambda').$$

Suppose we are given a morphism $f : W \to V\hat{\otimes}V$ such that the compositions

$$W \xrightarrow{f} V\hat{\otimes}V \cong k\hat{\otimes}k\hat{\otimes}V\hat{\otimes}V \xrightarrow{\lambda\otimes\lambda'\otimes\mathrm{Id}\otimes\mathrm{Id}} C\hat{\otimes}C\hat{\otimes}V\hat{\otimes}V$$

and

$$W \xrightarrow{f} V\hat{\otimes}V \xrightarrow{\Delta_V\otimes\Delta_V} C\hat{\otimes}V\hat{\otimes}C\hat{\otimes}V \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}} C\hat{\otimes}C\hat{\otimes}V\hat{\otimes}V$$

agree. Postcomposing both with $\varepsilon \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id}$ we see that the compositions

$$W \xrightarrow{f} V\hat{\otimes}V \cong V\hat{\otimes}k\hat{\otimes}V \xrightarrow{\mathrm{Id}\otimes\lambda'\otimes\mathrm{Id}} V\hat{\otimes}C\hat{\otimes}V$$

and

$$W \xrightarrow{f} V\hat{\otimes}V \xrightarrow{\mathrm{Id}\otimes\Delta_V} V\hat{\otimes}C\hat{\otimes}V$$

agree, so we obtain a map from W to the equaliser of $\mathrm{Id}\otimes\Delta_V$ and $\mathrm{Id}\otimes\lambda'\otimes\mathrm{Id}$. Since V is flat, this equaliser is $V\hat{\otimes}V(\lambda')$. Thus we have a unique map $f' : W \to V\hat{\otimes}V(\lambda')$ such that the compositions

$$W \xrightarrow{f'} V\hat{\otimes}V(\lambda') \cong k\hat{\otimes}V\hat{\otimes}V(\lambda') \xrightarrow{\lambda\otimes\mathrm{Id}\otimes\mathrm{Id}} C\hat{\otimes}V\hat{\otimes}V(\lambda')$$

and

$$W \xrightarrow{f'} V\hat{\otimes}V(\lambda') \xrightarrow{\Delta_V\otimes\mathrm{Id}} C\hat{\otimes}V\hat{\otimes}V(\lambda')$$

agree. Again, since $V(\lambda')$ is assumed to be flat, the equaliser of $\lambda \otimes \mathrm{Id} \otimes \mathrm{Id}$ and $\Delta_V \otimes \mathrm{Id}$ is $V(\lambda)\hat{\otimes}V(\lambda')$ and we obtain a unique map $f'' : W \to V(\lambda)\hat{\otimes}V(\lambda')$. This exhibits $V(\lambda)\hat{\otimes}V(\lambda')$ as the equaliser $(V\hat{\otimes}V)(\lambda\otimes\lambda')$ as required. □

Lemma 3.1.20. *Let C be an IndBanach coalgebra and (V, c) a C-graded, (pre-) braided IndBanach space and let $\lambda \in G(C)$. Assume further that V and $V(\lambda)$ are flat as an IndBanach space. Then c restricts to a (pre-) braiding of $C(\lambda)$.*

Proof. The braiding c restricts to a morphism $(V\hat{\otimes}V)(\lambda\otimes\lambda) \to (V\hat{\otimes}V)(\lambda\otimes\lambda)$. By Lemma 3.1.19, this gives a braiding $V(\lambda)\hat{\otimes}V(\lambda) \to V(\lambda)\hat{\otimes}V(\lambda)$. □

3.1.3 Analytic and dagger Nichols algebras

Theorem 4.3 of [2] shows that the positive and negative parts of quantum enveloping algebras arise as Nichols algebras in the category of vector spaces. In this section we introduce analytic and dagger analogues of Nichols algebras that will allow us to construct the positive and negative parts of analytic quantum groups in Sections 3.3 and 3.4.

Definition 3.1.21. Given an IndBanach Hopf algebra H, we denote by $P(H)$ the equaliser of the two maps $H \to H\hat{\otimes}H$, given by the comultiplication Δ on H and the sum of the maps

$$H \cong k\hat{\otimes}H \xrightarrow{\eta\otimes\mathrm{Id}_H} H\hat{\otimes}H \text{ and } H \cong H\hat{\otimes}k \xrightarrow{\mathrm{Id}_H\otimes\eta} H\hat{\otimes}H,$$

the *primitive subspace* of H.

Definition 3.1.22. Given an IndBanach algebra A and an IndBanach space V equipped with a strict monomorphism $V \hookrightarrow A$, we say that A *is generated by* V if, for any diagram

$$\begin{array}{ccc} V & \longrightarrow & A \\ & \searrow & \uparrow f \\ & & A' \end{array}$$

where A' is an IndBanach algebra and f is a morphism of algebras, f is an epimorphism.

Lemma 3.1.23. *Given an IndBanach algebra A and an IndBanach space V equipped with a strict monomorphism $V \hookrightarrow A$, A is generated by V if and only if the induced map $\coprod_{n\geq 0} V^{\hat{\otimes}n} \to A$ is an epimorphism.*

Proof. If V generates A then the induced algebra map $\coprod_{n\geq 0} V^{\hat{\otimes} n} \to A$ is epic by assumption. Conversely, suppose that we have an epimorphism $\coprod_{n\geq 0} V^{\hat{\otimes} n} \to A$ and a diagram

$$\begin{array}{ccc} V & \longrightarrow & A \\ & \searrow & \uparrow f \\ & & A' \end{array}$$

as in Definition 3.1.22. Then for each $n \geq 0$ we obtain a commutative diagram

$$\begin{array}{ccccc} V^{\hat{\otimes} n} & \longrightarrow & A^{\hat{\otimes} n} & \longrightarrow & A \\ & \searrow & \uparrow f^{\hat{\otimes} n} & & \uparrow f \\ & & (A')^{\hat{\otimes} n} & \longrightarrow & A' \end{array}$$

which induces a diagram

$$\begin{array}{ccc} \coprod_{n\geq 0} V^{\hat{\otimes} n} & \longrightarrow & A \\ & \searrow & \uparrow f \\ & & A' \end{array}$$

which ensures that f is an epimorphism since the map $\coprod_{n\geq 0} V^{\hat{\otimes} n} \to A$ is epic. $\square$

Definition 3.1.24. Let V be a flat IndBanach space with pre-braiding c. Fix an IndBanach bialgebra C and a grouplike generalised element $\lambda : k \to C$. Then a flat braided graded IndBanach Hopf algebra R, graded over C, is called an *IndBanach Nichols algebra* of V if $R(0) \cong k$, $P(R) = R(\lambda) \cong V$ as a braided IndBanach space, and R is generated by $R(\lambda)$. If $C = k\{t\}$, $\lambda : 1 \mapsto t$, we say that R is an *analytic Nichols algebra*, and likewise if $C = k\{t\}^{\dagger}$ (respectively $C = k\{t/0\}^{\dagger}$), $\lambda : 1 \mapsto t$, we say that R is a *dagger-1 Nichols algebra* (respectively a *dagger-0 Nichols algebra*).

Remark We require flatness in the above definition so that the braiding on R automatically restricts to $R(\lambda)$ by Lemma 3.1.20. In the **(NA)** case flatness is automatic by Lemma 3.49 of [4].

We will prove existence of these Nichols algebras in Sections 3.3.1 and 3.4.1, following a discussion of how to form quantum groups using Majid's double-bosonisation construction.

3.2 Double-bosonisation

In [33], Majid introduces a construction, *double-bosonisation*, which he uses to reconstruct Lusztig's form of the quantum enveloping algebra $U_q(\mathfrak{g})$. We present here an adaptation of this construction to the setting of IndBanach spaces. For more on these ideas, see [31], a brief review of which is given in [33].

Definition 3.2.1. Let H and A be IndBanach Hopf algebras. A duality pairing is a bilinear form $\langle -,- \rangle : H\hat{\otimes}A \to k$ such that the following diagrams commute:

$$\begin{CD}
H\hat{\otimes}H\hat{\otimes}A @>{\mathrm{Id}\otimes\mathrm{Id}\otimes\Delta_A}>> H\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A \\
@V{\tau\otimes\mathrm{Id}}VV @VV{\mathrm{Id}\otimes\langle -,-\rangle\otimes\mathrm{Id}}V \\
H\hat{\otimes}H\hat{\otimes}A @. H\hat{\otimes}A \\
@V{\mu_H\otimes\mathrm{Id}}VV @VV{\langle -,-\rangle}V \\
H\hat{\otimes}A @>{\langle -,-\rangle}>> k
\end{CD}
\qquad
\begin{array}{ccc}
H & \xrightarrow{\eta_A} & H\hat{\otimes}A \\
 & \overset{\varepsilon_H}{\searrow} & \downarrow{\scriptstyle\langle -,-\rangle} \\
 & & k
\end{array}$$

$$\begin{CD}
H\hat{\otimes}A\hat{\otimes}A @>{\Delta_H\otimes\mathrm{Id}\otimes\mathrm{Id}}>> H\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A \\
@V{\mathrm{Id}\otimes\tau}VV @VV{\mathrm{Id}\otimes\langle -,-\rangle\otimes\mathrm{Id}}V \\
H\hat{\otimes}A\hat{\otimes}A @. H\hat{\otimes}A \\
@V{\mathrm{Id}\otimes\mu_A}VV @VV{\langle -,-\rangle}V \\
H\hat{\otimes}A @>{\langle -,-\rangle}>> k
\end{CD}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta_H} & H\hat{\otimes}A \\
 & \overset{\varepsilon_A}{\searrow} & \downarrow{\scriptstyle\langle -,-\rangle} \\
 & & k
\end{array}$$

$$\begin{CD}
H\hat{\otimes}A @>{\mathrm{Id}\otimes S_A}>> H\hat{\otimes}A \\
@V{S_H\otimes\mathrm{Id}}VV @VV{\langle -,-\rangle}V \\
H\hat{\otimes}A @>>{\langle -,-\rangle}> k
\end{CD}$$

If H and A have respective right and left actions ζ_H and ζ_A of an algebra $\mathcal{A}$ then we say that $\langle -,- \rangle$ is $\mathcal{A}$-equivariant if the diagram

$$
\begin{array}{ccc}
H\hat{\otimes}\mathcal{A}\hat{\otimes}A & \xrightarrow{\zeta_H \otimes \mathrm{Id}} & H\hat{\otimes}A \\
{\scriptstyle \mathrm{Id}\otimes\zeta_A}\downarrow & & \downarrow{\scriptstyle \langle -,-\rangle} \\
H\hat{\otimes}A & \xrightarrow[\langle -,-\rangle]{} & k
\end{array}
$$

commutes. Likewise, if H and A have respective left and right coactions ζ_H and ζ_A of a coalgebra C then we say that $\langle -,-\rangle$ is C-equivariant if the diagram

$$
\begin{array}{ccc}
H\hat{\otimes}A & \xrightarrow{\zeta_H \otimes \mathrm{Id}} & C\hat{\otimes}H\hat{\otimes}A \\
{\scriptstyle \mathrm{Id}\otimes\zeta_A}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\langle -,-\rangle} \\
H\hat{\otimes}A\hat{\otimes}C & \xrightarrow[\langle -,-\rangle\otimes\mathrm{Id}]{} & C
\end{array}
$$

commutes.

Definition 3.2.2. Let H and A be IndBanach Hopf algebras with a duality pairing $\langle -,-\rangle : H\hat{\otimes}A \to k$. Suppose we have a pair of convolution invertible maps $\mathscr{R}$ and $\overline{\mathscr{R}}$ in $\mathrm{Hom}(A,H)$ that are both algebra homomorphisms and anti-coalgebra homomorphisms, such that the following diagrams commute:

$$
\begin{array}{ccc}
A\hat{\otimes}A & \xrightarrow{\mathrm{Id}\otimes\mathscr{R}^{-1}} & A\hat{\otimes}H \\
 & & \tau\downarrow \\
{\scriptstyle \overline{\mathscr{R}}\otimes\mathrm{Id}}\downarrow & & H\hat{\otimes}A \\
 & & \downarrow \\
H\hat{\otimes}A & \xrightarrow{\langle -,-\rangle} & k
\end{array}
$$

$$
\begin{array}{ccc}
H\hat{\otimes}A & \xrightarrow{\Delta_H\otimes((\mathrm{Id}\otimes\Delta_A)\circ\Delta_A)} & H\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \\
 & & {\scriptstyle \mathrm{Id}\otimes\tau\otimes\mathrm{Id}\otimes\mathrm{Id}}\downarrow \\
 & & H\hat{\otimes}A\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A \\
 & & {\scriptstyle \tau\otimes\tau\otimes\mathrm{Id}}\downarrow \\
{\scriptstyle \Delta_H\otimes\mathrm{Id}}\downarrow & & A\hat{\otimes}H\hat{\otimes}A\hat{\otimes}H\hat{\otimes}A \\
 & & {\scriptstyle \mathscr{R}\otimes\langle -,-\rangle\otimes\mathrm{Id}\otimes\mathscr{R}^{-1}}\downarrow \\
 & & H\hat{\otimes}H\hat{\otimes}H \\
 & & {\scriptstyle \mu_H\circ(\mu_H\otimes\mathrm{Id})}\downarrow \\
H\hat{\otimes}H\hat{\otimes}A & \xrightarrow[\mathrm{Id}\otimes\langle -,-\rangle]{} & H
\end{array}
$$

$$
\begin{array}{ccc}
H\hat{\otimes}A & \xrightarrow{\Delta_H \otimes ((\mathrm{Id}\otimes\Delta_A)\circ\Delta_A)} & H\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A\hat{\otimes}A \\
 & & \big\downarrow {\scriptstyle ((\tau\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau))\otimes\mathrm{Id}\otimes\mathrm{Id}} \\
{\scriptstyle \Delta_H\otimes\mathrm{Id}}\big\downarrow & & A\hat{\otimes}H\hat{\otimes}H\hat{\otimes}A\hat{\otimes}A \\
H\hat{\otimes}H\hat{\otimes}A & & \big\downarrow {\scriptstyle \overline{\mathscr{R}}\otimes\mathrm{Id}\otimes\langle -,-\rangle\otimes\overline{\mathscr{R}}^{-1}} \\
 & & H\hat{\otimes}H\hat{\otimes}H \\
{\scriptstyle \tau\otimes\mathrm{Id}}\big\downarrow & & \big\downarrow {\scriptstyle \mu_H\circ(\mu_H\otimes\mathrm{Id})} \\
H\hat{\otimes}H\hat{\otimes}A & \xrightarrow[\mathrm{Id}\otimes\langle -,-\rangle]{} & H
\end{array}
$$

In this case we call H and A a *weakly quasi-triangular dual pair.*

Definition 3.2.3. For an IndBanach Hopf algebra H, let us denote by H-Mod and Mod-H the categories of left and right H-modules respectively. Given a dual pair of IndBanach Hopf algebras, H and A, let us denote by A-Comod and Comod-A the categories of left and right A-comodules respectively. There are faithful functors A-Comod $\to$ Mod-H and Comod-$A \to H$-Mod induced by the duality pairing.

Proposition 3.2.4. *For a weakly quasi-triangular dual pair of IndBanach Hopf algebras, H and A, A-Comod and Comod-A are braided monoidal, with braidings given by the compositions*

$$M\hat{\otimes}M' \longrightarrow A\hat{\otimes}M\hat{\otimes}A\hat{\otimes}M' \xrightarrow{\mathscr{R}\otimes Id\otimes Id\otimes Id} H\hat{\otimes}M\hat{\otimes}A\hat{\otimes}M' \longrightarrow M\hat{\otimes}M' \xrightarrow{\tau} M'\hat{\otimes}M,$$

where the third map is $(\langle -,-\rangle \otimes Id)\circ(Id\otimes\tau\otimes Id)$*, and*

$$N\hat{\otimes}N' \xrightarrow{\tau} N'\otimes N \longrightarrow N'\hat{\otimes}A\hat{\otimes}N\hat{\otimes}A \xrightarrow{Id\otimes\mathscr{R}\otimes Id\otimes Id} N'\hat{\otimes}H\hat{\otimes}N\hat{\otimes}A \longrightarrow N'\hat{\otimes}N,$$

where the last map is $(Id\otimes\langle -,-\rangle)\circ(Id\otimes\tau\otimes Id)$*, for left A-comodules M and M' and right A-comodules N and N'.*

Proof. This follows from Theorem 1.16 of [31], using the remark from the preliminary section of [33] that A is dual quasi-triangular under the composition

$$A\hat{\otimes}A \xrightarrow{\mathscr{R}\otimes\mathrm{Id}} H\hat{\otimes}A \xrightarrow{\langle -,-\rangle} k.$$

□

Proposition 3.2.5. *For a weakly quasi-triangular dual pair of IndBanach Hopf algebra, H and A, and an algebra B in Comod-A there is an algebra structure on $B\hat{\otimes}H$ with multiplication defined by*

$$\begin{aligned} B\hat{\otimes}H\hat{\otimes}B\hat{\otimes}H &\xrightarrow{Id\otimes\Delta_H\otimes Id\otimes Id} B\hat{\otimes}H\hat{\otimes}H\hat{\otimes}B\hat{\otimes}H \\ &\xrightarrow{Id\otimes Id\otimes\tau\otimes Id} B\hat{\otimes}H\hat{\otimes}B\hat{\otimes}H\hat{\otimes}H \\ &\xrightarrow{Id\otimes\zeta_B\otimes Id\otimes Id} B\hat{\otimes}B\hat{\otimes}H\hat{\otimes}H \\ &\xrightarrow{\mu_M\otimes\mu_H} B\hat{\otimes}H. \end{aligned}$$

Here, ζ_B is the left action of H on B. Furthermore, if B is a braided IndBanach Hopf algebra then we can give $B\hat{\otimes}H$ a braided IndBanach Hopf algebra structure with comultiplication defined by

$$B\hat{\otimes}H \xrightarrow{\Delta_B\otimes\Delta_H} B\hat{\otimes}B\hat{\otimes}H\hat{\otimes}H \xrightarrow{Id\otimes\Psi_{B,H}\otimes Id} B\hat{\otimes}H\hat{\otimes}B\hat{\otimes}H$$

where Ψ is the braiding on Comod-A. Likewise, for C in Mod-H there is a (Hopf) algebra structure on $H\hat{\otimes}C$ defined analogously.

Proof. This follows from Theorem 2.1 of [32]. □

Definition 3.2.6. We denote by $B \rtimes H$ the IndBanach (Hopf) algebra on $B\hat{\otimes}H$ as described above, and likewise we denote by $H \ltimes C$ the (Hopf) algebra on $H\hat{\otimes}C$. These are the *bosonisations* of B and H or H and C respectively.

Lemma 3.2.7. *Let H and A be a weakly quasi-triangular dual pair of IndBanach Hopf algebras, with maps $\mathscr{R}_{H,A}$, $\overline{\mathscr{R}}_{H,A}$ giving the weakly quasi-triangular structure.*

Let C be a braided IndBanach Hopf algebra in Comod-A with invertible antipode S_C. Then there is a weakly quasi-triangular dual pair $\overline{H}$ and $\overline{A}$ with the same Hopf algebra structures as H and A but with weakly quasi-triangular structure given by $\mathscr{R}_{\overline{H},\overline{A}} = \overline{\mathscr{R}}^{-1}_{H,A}$ and $\overline{\mathscr{R}}_{\overline{H},\overline{A}} = \mathscr{R}^{-1}_{H,A}$. Furthermore, there is a braided IndBanach Hopf algebra $\overline{C}$ in Comod-$\overline{A}$ with the same algebra structure as C but with the opposite comultiplication $\Delta_{\overline{C}} = \Psi^{-1}_{C,C} \circ \Delta_C$, where Ψ is the braiding on Comod-A, and antipode $S_{\overline{C}} = S_C^{-1}$.

Proof. This follows from Lemma 3.1 in [33], which was proven in [31], using Remark 3.9 of [33]. □

Proposition 3.2.8. *Let H and A be a weakly quasi-triangular dual pair of IndBanach Hopf algebras, let B be a braided IndBanach Hopf algebra in A-Comod and let C be a braided IndBanach Hopf algebra in Comod-A with respective induced right and left actions ζ_B and ζ_C of H. Suppose further that we have a H-equivariant duality pairing between B and C, $\langle -,-\rangle : B\hat{\otimes}C \to k$. Then there is an IndBanach Hopf algebra structure on $C\hat{\otimes}H\hat{\otimes}B$ such that the maps $\overline{C} \rtimes \overline{H} \to C\hat{\otimes}H\hat{\otimes}B$ and $H \ltimes B \to C\hat{\otimes}H\hat{\otimes}B$ are morphisms of Hopf algebras, and the multiplication restricts to the composition*

$$\begin{aligned}
B\hat{\otimes}\overline{C} &\longrightarrow B\hat{\otimes}B\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}\overline{C}\hat{\otimes}\overline{C}\\
&\xrightarrow{\tau_{(1\ 2\ 4)(3\ 6\ 5)}} \overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\\
&\xrightarrow{Id\otimes Id\otimes Id\otimes Id\otimes S_{\overline{C}}\otimes Id} \overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\\
&\xrightarrow{Id\otimes R_{B,\overline{C}}\otimes R^{-1}_{B,\overline{C}}\otimes Id} \overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}B\\
&\xrightarrow{\langle -,-\rangle\otimes Id\otimes Id\otimes\langle -,-\rangle} \overline{C}\hat{\otimes}B \longrightarrow C\hat{\otimes}H\hat{\otimes}B
\end{aligned}$$

between $B \hookrightarrow C\hat{\otimes}H\hat{\otimes}B$ and $C \hookrightarrow C\hat{\otimes}H\hat{\otimes}B$. Here, the first map is induced by the coproducts Δ_B and $\Delta_{\overline{C}}$, $\tau_{(1\ 2\ 4)(3\ 6\ 5)}$ is a reordering given by the permutation

$(1\ 2\ 4)(3\ 6\ 5) \in S_6$, *and* $R_{B,\overline{C}}$ *and* $R^{-1}_{B,\overline{C}}$ *are the respective compositions*

$$B\hat{\otimes}\overline{C} \to A\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}A \xrightarrow{\mathscr{R}\otimes Id\otimes Id\otimes\overline{\mathscr{R}}} H\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}H \to B\hat{\otimes}\overline{C}$$

and

$$B\hat{\otimes}\overline{C} \to A\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}A \xrightarrow{\mathscr{R}^{-1}\otimes Id\otimes Id\otimes\overline{\mathscr{R}}^{-1}} H\hat{\otimes}B\hat{\otimes}\overline{C}\hat{\otimes}H \to B\hat{\otimes}\overline{C}$$

where the first maps are the coactions of A *and the last maps are the induced actions of* H*.*

Proof. This follows from Theorem 3.2, along with Remark 3.9, of [33]. □

Definition 3.2.9. We will denote by $U(C, H, B)$ the Banach Hopf algebra $C\hat{\otimes}H\hat{\otimes}B$ as described in Proposition 3.2.8, the *double bosonisation* of B, H and C over A.

Definition 3.2.10. Let H be an IndBanach Hopf algebra. An *R-matrix* for H is a convolution invertible generalised element of $H\hat{\otimes}H$, $\mathscr{R} : k \to H\hat{\otimes}H$, such that $(\Delta \otimes \mathrm{Id}) \circ \mathscr{R} = \mathscr{R}_{13} * \mathscr{R}_{23}$, $(\mathrm{Id} \otimes \Delta) \circ \mathscr{R} = \mathscr{R}_{13} * \mathscr{R}_{12}$, and $\tau \circ \Delta = \mathscr{R}(\Delta)\mathscr{R}^{-1}$. Here we use the notation $\mathscr{R}_{12}$ for the composition

$$k \xrightarrow{\mathscr{R}} H\hat{\otimes}H \cong H\hat{\otimes}H\hat{\otimes}k \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\eta_H} H\hat{\otimes}H\hat{\otimes}H$$

and likewise for $\mathscr{R}_{13}$ and $\mathscr{R}_{23}$, and $\mathscr{R}(\Delta)\mathscr{R}^{-1}$ for the composition

$$H \cong k\hat{\otimes}H\hat{\otimes}k \xrightarrow{\mathscr{R}\otimes\Delta_H\otimes\mathscr{R}^{-1}} (H\hat{\otimes}H)\hat{\otimes}(H\hat{\otimes}H)\hat{\otimes}(H\hat{\otimes}H) \to H\hat{\otimes}H$$

where the last map is the multiplication on $H\hat{\otimes}H$. A *quasi-triangular IndBanach Hopf algebra* is an IndBanach Hopf algebra H equipped with an R-matrix $\mathscr{R}$.

Lemma 3.2.11. *Let* H *and* A *be Hopf algebras with a dual pairing* $\langle -, - \rangle : H\hat{\otimes}A \to k$*. Suppose further that* H *is quasi-triangular, with R-matrix* $\mathscr{R}' : k \to H\hat{\otimes}H$*. Then* *the maps*

$$\mathscr{R} : A \cong k\hat{\otimes}A \xrightarrow{\mathscr{R}'\otimes Id} H\hat{\otimes}H\hat{\otimes}A \xrightarrow{Id\otimes\tau} H\hat{\otimes}A\hat{\otimes}H \xrightarrow{\langle -,-\rangle\otimes Id} k\hat{\otimes}H \cong H$$

and

$$\overline{\mathscr{R}} : A \cong k\hat{\otimes}A \xrightarrow{(\mathscr{R}')^{-1}\otimes Id} H\hat{\otimes}H\hat{\otimes}A \xrightarrow{Id\otimes\langle -,-\rangle} H\hat{\otimes}k \cong H$$

induce a weak quasi-triangular structure on the dual pair H and A.

Proof. This follows from the remarks in the preliminary section of [33]. □

Proposition 3.2.12. *For a quasi-triangular IndBanach Hopf algebra H, H-Mod and Mod-H are braided monoidal. The braiding on M and M' in H-Mod is given by the composition*

$$M\hat{\otimes}M' \cong k\hat{\otimes}M\hat{\otimes}M' \xrightarrow{\mathscr{R}\otimes Id\otimes Id} H\hat{\otimes}H\hat{\otimes}M\hat{\otimes}M' \xrightarrow{\tau\circ(\zeta_M\otimes\zeta_{M'})\circ(Id\otimes\tau\otimes Id)} M'\hat{\otimes}M,$$

whilst the braiding on N and N' in Mod-H is given by the composition

$$N\hat{\otimes}N' \xrightarrow{\tau} N'\hat{\otimes}N \xrightarrow{Id\otimes Id\otimes\mathscr{R}} N'\hat{\otimes}N\hat{\otimes}H\hat{\otimes}H \xrightarrow{(\zeta_{N'}\otimes\zeta_N)\circ(Id\otimes\tau\otimes Id)} N'\hat{\otimes}N.$$

Proof. This is Theorem 1.10 of [31]. □

3.3 Non-Archimedean analytic quantum groups

We will assume **(NA)** throughout this section. The **(A)** case will be covered in the subsequent section.

3.3.1 Constructing non-Archimedean Nichols algebras

Definition 3.3.1. Let V be an analytically $\mathbb{N}$-graded Banach space, $V(n) := V(t^n)$, $V = \coprod_{n\in\mathbb{N}}^{\leq 1} V(n)$. For $W \subset V$ a subspace, not necessarily closed, let $W(n) = V(n) \cap W$. We say that W is homogeneous if

$$\bigoplus_{n\in\mathbb{N}} W(n) \subset W \subset \coprod_{n\in\mathbb{N}}^{\leq 1} W(n).$$

Note that $W = \coprod_{n\in\mathbb{N}}^{\leq 1} W(n)$ if and only if it is closed and homogeneous.

Lemma 3.3.2. *Let V be an analytically graded Banach space as above. Suppose that $V(n)$ is finite dimensional for each $n \geq 0$. Then any closed homogeneous subspace is complemented by a closed homogeneous subspace.*

Proof. Let $W = \coprod_{n\in\mathbb{N}}^{\leq 1} W(n) \subset V$. Then each $W(n)$ is complemented in $V(n)$, by $W'(n)$ say. Then W is complemented by $W' = \coprod_{n\in\mathbb{N}}^{\leq 1} W'(n) \subset V$. $\square$

Lemma 3.3.3. *Let V be a Banach space with pre-braiding c of norm at most 1. Suppose we have closed homogeneous coideals $I \subset J \subset T(V)$. If the induced map $T(V)/I \to T(V)/J$ is injective on $P(T(V)/I)$ then $I = J$.*

Proof. We proceed similarly to Lemma 5.3.3 of [37]. Let us denote by $R = T(V)/I$, $R' = T(V)/J$, by $R(n)$, $R'(n)$ their respective nth graded pieces, and by $R(\leq n) = \bigoplus_{i\leq n} R(i)$ and $R'(\leq n) = \bigoplus_{i\leq n} R'(i)$. Since $I \subset J$ we have a strict graded epimorphism $f : R \to R'$ which restricts to an isometry $R(1) \to R'(1)$. Suppose that we know that f restricts also to an isometry $R(\leq n) \to R'(\leq n)$, and let $x \in R(\leq n+1)$. We know that, by Lemma 3.1.8,

$$\begin{aligned}\Delta(R(n+1)) &\subset \textstyle\sum_{i=0}^{n+1} R(i)\hat{\otimes}R(n-i) \\ &\subset R(n+1)\hat{\otimes}k + k\hat{\otimes}R(n+1) + R(\leq n)\hat{\otimes}R(\leq n),\end{aligned}$$

so $\Delta(x) = y \otimes 1 + 1 \otimes y' + z$ for some $y, y' \in R(n+1)$, $z \in R(\leq n)\hat{\otimes}R(\leq n)$.

But then $x - y = (\mathrm{Id} \otimes \varepsilon)\Delta(x) - y = \varepsilon(y) \cdot 1 + (\mathrm{Id} \otimes \varepsilon)(z) \in R(\leq n)$ and likewise $x - y' \in R(\leq n)$. So $\Delta(x) = x \otimes 1 + 1 \otimes x + z'$ for some $z' \in R(\leq n)\hat{\otimes}R(\leq n)$. If $f(x) = 0$ then $(f \otimes f)(z') = 0$, but, by assumption and by Lemma 3.49 of [4], $f \otimes f$ is injective on $R(\leq n)\hat{\otimes}R(\leq n)$. So $z' = 0$ and x is primitive, hence $x = 0$. Thus f is an isometry $R(\leq n) \to R'(\leq n)$ since the norms on $R(\leq n)$ and $R'(\leq n)$ are the quotient norms from $\coprod_{i\leq n}^{\leq 1} V^{\hat{\otimes} i}$. Taking contracting colimits over n we see that f is isometric. Hence $I = J$. □

Proposition 3.3.4. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1. Then an analytic Nichols algebra of V exists.*

Proof. Let $\mathcal{I}(V)$ be the set of all homogeneous ideals of $T(V)$ contained in $\coprod_{n\geq 2}^{\leq 1} V^{\hat{\otimes} n}$ that are also coideals. Let $\mathcal{I}'(V)$ be the subset of ideals I in $\mathcal{I}(V)$ for which $\overline{I}\hat{\otimes}T(V) + T(V)\hat{\otimes}\overline{I}$ is preserved by c. Let $I(V)$ and $I'(V)$ be the sums of all ideals in $\mathcal{I}(V)$ and $\mathcal{I}'(V)$ respectively, and let $\overline{I}(V)$ and $\overline{I}'(V)$ be their respective closures. Clearly then $\overline{I}(V)$ is a homogeneous ideal contained in $\coprod_{n\geq 2}^{\leq 1} V^{\hat{\otimes} n}$. Also, $\Delta(\overline{I}(V)) \subset \overline{\Delta(I(V))}$ is in the closure of $T(V)\hat{\otimes}\overline{I}(V) + \overline{I}(V)\hat{\otimes}T(V)$. By Lemma 3.3.3 we have that $\overline{I}(V)$ is complemented in $T(V)$ by some W, so

$$T(V)\hat{\otimes}\overline{I}(V) + \overline{I}(V)\hat{\otimes}T(V) = W\hat{\otimes}\overline{I}(V) \oplus \overline{I}(V)\hat{\otimes}\overline{I}(V) \oplus \overline{I}(V)\hat{\otimes}W$$

is closed. So $\overline{I}(V)$ is also a coideal in $T(V)$, and hence is also in $\mathcal{I}(V)$. So $\overline{I}(V) = I(V)$ is closed. Likewise $I'(V)$ is closed. We must check that

$$P(T(V)/I(V)) = (T(V)/I(V))(1).$$

This follows as in Lemma 5.3.3 of [2] since the closed ideal in $T(V)$ generated by $I(V)$ and

$$\left\{x \in \coprod_{n\geq 2}^{\leq 1} V^{\hat{\otimes} n} \middle| \Delta(x) \in x \otimes 1 + 1 \otimes x + I(V) \otimes T(V) + T(V) \otimes I(V)\right\}$$

must be in $\mathcal{I}(V)$. Likewise $P(T(V)/I'(V)) = (T(V)/I'(V))(1)$. But $I'(V) \subset I(V)$, so by Lemma 3.3.3 we have $I'(V) = I(V)$. Hence c descends to a braiding on $T(V)/I(V)$. We then have that $T(V)/I(V)$ is an analytically graded braided IndBanach Hopf algebra with $(T(V)/I(V))(0) \cong k$ and generated by $(T(V)/I(V))(1) \cong V$. $\square$

Definition 3.3.5. For a finite dimensional Banach space with pre-braiding c of norm at most 1 we will denote by $\mathfrak{B}^c(V)$, or $\mathfrak{B}(V)$ when the braiding is implicit, the Banach Nichols algebra defined in the proof of Proposition 3.3.4. For $0 < r$, let us denote by $\mathfrak{B}_r(V)$, or $\mathfrak{B}_r^c(V)$, the analytically graded braided Banach Hopf algebra $\mathfrak{B}^c(V_r)$. For $0 \leq \rho$, let us denote by $\mathfrak{B}_\rho(V)^\dagger$, or $\mathfrak{B}_\rho^c(V)^\dagger$, the dagger graded braided Banach Hopf algebra "colim"$_{r>\rho}\mathfrak{B}_r^c(V)$.

Proposition 3.3.6. *Let $0 < r$, $0 \leq \rho$ and let V be a finite dimensional Banach space with braiding c of norm at most 1. Then $\mathfrak{B}_r^c(V)$ is an analytic Nichols algebra of V and $\mathfrak{B}_\rho^c(V)^\dagger$ is a dagger Nichols algebra of V.*

Proof. The fact that $\mathfrak{B}_r^c(V)$ is an analytic Nichols algebra follows from Proposition 3.3.4. If $R = \mathfrak{B}_\rho(V)^\dagger$ then it follows from Proposition 3.3.4 that $R(0) =$ "colim"$_{r>\rho}k_r \cong k$ and $P(R) = R(1) =$ "colim"$_{r>\rho}V_r \cong V$. It remains to check that $R(0)$ generates R. This follows since, for each $r > \rho$, the composition $\coprod_{n\geq 0} V^{\hat{\otimes} n} \to T_r(V) \to \mathfrak{B}_r^c(V)$ is an epimorphism. $\square$

Proposition 3.3.7. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1. Let R be an analytically graded pre-braided Banach Hopf algebra with contracting multiplication (*i.e *of norm at most 1) such that $R(0) \cong k$, $R(1) \cong V$ as pre-braided Banach spaces, and R is generated as an algebra by $R(1)$. Then there is an epimorphism of analytically graded braided Hopf algebras $\mathfrak{B}_r^c(V) \to R$ extending $V \xrightarrow{\sim} R(1)$ where r is the norm of this isomorphism.*

Proof. Since $V \xrightarrow{\sim} R(0)$ is of norm r, the map $V_r \xrightarrow{\sim} R(0)$ is of norm 1. Since the multiplication is contracting, we obtain contracting maps $V_r^{\hat{\otimes} n} \to R^{\hat{\otimes} n} \to R$ which induce an algebra homomorphism $T_r(V) \to R$ through which $V \cong R(0) \to R$ factors. Since $R(0)$ generates R, this morphism is epic. Furthermore, by construction of the analytically graded pre-braided bialgebra structure on $T_r(V)$ and since

$$\Delta(R(1)) \subset R(0)\hat{\otimes}R(1) + R(1)\hat{\otimes}R(0)$$

implies that $R(1) \subset P(R)$, the map $T_r(V) \to R$ is a morphism of bialgebras. Hence the kernel of this map is a closed homogeneous ideal and coideal, so is contained in $I(V_r)$. Hence we obtain an epimorphism $\mathfrak{B}^c_r(V) = T(V_r)/I(V_r) \to R$. $\square$

Remark Note that the assignment $(V, c) \mapsto \mathfrak{B}^c_r(V)$ is functorial if we restrict to morphisms $(V, c) \to (V', c')$ given by maps $V \to V'$ of norm at most 1 that respect the braiding. Hence different norms on finite dimensional Banach spaces and different values of r may give non-isomorphic analytic Nichols algebras. This differs from the algebraic case where Nichols algebras always exist uniquely.

Definition 3.3.8. A *bilinear form* on a braided IndBanach space (V, c) is a morphism $\langle -, - \rangle : V\hat{\otimes}V \to k$ such that the diagram

$$\begin{array}{ccccc} V\hat{\otimes}V & \longrightarrow & V^*\hat{\otimes}V^* & \longrightarrow & (V\hat{\otimes}V)^* \\ c\big\downarrow & & & & \big\downarrow c^* \\ V\hat{\otimes}V & \longrightarrow & V^*\hat{\otimes}V^* & \longrightarrow & (V\hat{\otimes}V)^* \end{array}$$

commutes for both of the induced maps $V \to V^*$. A bilinear form is *symmetric* if $\langle -, - \rangle \circ \tau = \langle -, - \rangle$, and is *non-degenerate* if the induced maps $V \to V^*$ are both injective.

Lemma 3.3.9. *Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, - \rangle : V\hat{\otimes}V \to k$ of norm $C > 0$. Then for each $r, s > 0$ with $C \leq rs$ there is a unique extension of this bilinear form to a dual pairing of Hopf algebras $T^c_r(V)\hat{\otimes}T^c_s(V) \to k$. Furthermore, the restriction to $V^{\hat{\otimes}n}\hat{\otimes}V^{\hat{\otimes}m}$ is symmetric if $n = m$ and is zero if $n \neq m$.*

Proof. This is proved as in [27]. We construct this extension as follows. Our bilinear form on V induces a continuous map $V_r \to V_s^*$ of norm $\frac{C}{rs}$, whilst the natural projection $T_s^c(V) \to V_s$ induces a map $V_s^* \to T_s^c(V)^*$ of norm 1, and composition gives us a map $V_r \to T_s^c(V)^*$ of norm at most $\frac{C}{rs} \leq 1$. The coalgebra structure on $T_s^c(V)$ induces an algebra structure on $T_s^c(V)^*$ whose multiplication is contracting, and we get a unique continuous algebra homomorphism $T_r^c(V) \to T_s^c(V)^*$ extending $V_r \to T_s^c(V)^*$. This gives us our desired bilinear form on $T_r^c(V)\hat{\otimes}T_s^c(V)$. For this to be a dual pairing we must also check that the diagrams

$$\begin{array}{ccc} T_r^c(V) & \longrightarrow & T_s^c(V)^* \\ \Delta\downarrow & & \downarrow\mu^* \\ T_r^c(V)\hat{\otimes}T_r^c(V) & \longrightarrow & (T_s^c(V)\hat{\otimes}T_s^c(V))^* \end{array} \qquad \begin{array}{ccc} T_r^c(V) & \longrightarrow & T_s^c(V)^* \\ \varepsilon\downarrow & & \downarrow\eta^* \\ k & \xrightarrow{\sim} & k^* \end{array}$$

commute. By assumption all of these morphisms are algebra homomorphisms, and so it is enough to check these diagrams commute on V, which is trivial. Since the algebra homomorphism $T_r^c(V) \to T_s^c(V)^*$ is graded we have $\langle V^{\hat{\otimes}n}, V^{\hat{\otimes}m}\rangle = \{0\}$ for $n \neq m$. Since this is a duality pairing, symmetry on $V^{\hat{\otimes}n}\hat{\otimes}V^{\hat{\otimes}n}$ can be reduced to the case where $n = 1$ where it is true by assumption. □

Proposition 3.3.10. *Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, -\rangle : V\hat{\otimes}V \to k$ of norm C. Then for each $0 < r$, let I_r be the radical in $T_r^c(V)$ of the extension of this bilinear form to $T_r^c(V)\hat{\otimes}T_s^c(V)$ for some $s > 0$ such that $C \leq rs$. Then I_r is a closed ideal and coideal of $T_r^c(V)$, independent of the choice of s, and $P(T_r^c(V)/I_r) = V$. Hence $T_r^c(V)/I_r$ is an analytic Nichols algebra of V.*

Proof. The fact that I_r is a closed homogeneous ideal and coideal of $T_r^c(V)$ follows from Lemma 3.3.9. It is independent of the choice of s since the vector subspace

$\bigoplus_{n\geq 0} V^{\otimes n}$ is dense in $T^c_s(V)$ for all s, hence

$$I_r = \left\{ x \in T^c_r(V) \middle| \langle x, y\rangle = 0 \text{ for all } y \in \bigoplus\nolimits_{n\geq 0} V^{\otimes n} \right\}.$$

Since the bilinear form on V is non-degenerate, $I_r \subset \coprod^{\leq 1}_{n\geq 2} V_r^{\hat{\otimes} n}$. Clearly the quotient $T_r(V)/I_r$ is generated by V, and so it remains to check that the subspace of primitive elements is just V. Given $x \in T^c_r(V)$ homogeneous of degree $n \geq 2$ (*i.e.* in $V_r^{\hat{\otimes} n}$) such that its image in $T^c_r(V)/I_r$ is primitive, we must have that $\langle x, yy'\rangle = \langle 1, y\rangle\langle x, y'\rangle + \langle x, y\rangle\langle 1, y'\rangle = 0$ for all y, y' homogeneous of degree at least 1. It then follows that x must be in the radical I_r since $\langle x, z\rangle = 0$ for any z homogeneous of degree at most 1. By the assumption in Definition 3.3.8 the diagram

$$\begin{array}{ccc} T_r(V)\hat{\otimes}T_r(V) & \longrightarrow & (T_s(V)\hat{\otimes}T_s(V))^* \\ \tilde{c}\downarrow & & \downarrow\tilde{c}^* \\ T_r(V)\hat{\otimes}T_r(V) & \longrightarrow & (T_s(V)\hat{\otimes}T_s(V))^* \end{array}$$

commutes, and so $I_r\hat{\otimes}T_r(V) + T_r(V)\hat{\otimes}I_r$ is preserved by $\tilde{c}$ and the braiding on $T^c_r(V)$ descends to a braiding of $T^c_r(V)/I_r$. Hence this is an analytic Nichols algebra of V. □

Proposition 3.3.11. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most* 1*, and suppose we have a non-degenerate symmetric bilinear form $\langle -, -\rangle : V\hat{\otimes}V \to k$. Retaining the notation of Proposition 3.3.10, the induced map $\mathfrak{B}^c_r(V) \to T^c_r(V)/I_r$ is an isomorphism for each $0 < r$. In particular, I_r is independent of the choice of bilinear form.*

Proof. This follows from Lemma 3.3.3, noting that $I_r \subset I(V_r)$. □

Proposition 3.3.11 above justifies the following extension of notation.

Definition 3.3.12. Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, -\rangle : V\hat{\otimes}V \to k$.

For each $0 < r$ we will denote by $\mathfrak{B}^c_r(V)$ the quotient $T^c_r(V)/I_r$ of Proposition 3.3.10. We will also define $\mathfrak{B}^c_\rho(V)^\dagger = "\text{colim}"_{r>\rho}\mathfrak{B}^c_r(V)$ for $0 \leq \rho$ in this situation.

Proposition 3.3.13. *Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, - \rangle : V\hat{\otimes}V \to k$. Then for each $0 \leq \rho$, $\mathfrak{B}^c_\rho(V)^\dagger$ is a dagger Nichols algebra of V.*

Proof. This follows as in the proof of Proposition 3.3.6. □

Proposition 3.3.14. *Let V be a Banach space with a pre-braiding c of norm at most 1 that restricts to an algebraic braiding $V \otimes V \to V \otimes V$ of vector spaces, and let $\langle -, - \rangle : V\hat{\otimes}V \to k$ be a non-degenerate symmetric bilinear form. Then the algebraic Nichols algebra of V, as defined in Definition 2.1 of [2], is dense in the Banach space $\mathfrak{B}^c_r(V)$ for each $r > 0$.*

Proof. The algebraic Nichols algebra can be constructed as the quotient of the tensor algebra of vector spaces, $\bigoplus_{n\geq 0} V^{\otimes n}$, by the radical of the restriction of the induced bilinear form on $T_r(V)\hat{\otimes}T_s(V)$ for some sufficiently large $s > 0$. The result then follows since $\bigoplus_{n\geq 0} V^{\otimes n}$ is dense in $T_r(V)$ for all $r > 0$. □

Proposition 3.3.15. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1, and suppose that R is a Banach analytic Nichols algebra of V. Then the algebraic Nichols algebra of V, as defined in Definition 2.1 of [2], is dense in R. Furthermore, $R(n)$ is isomorphic as a vector space to the nth graded piece of the algebraic Nichols algebra.*

Proof. In the category of vector spaces, there is a unique braided $\mathbb{N}$-graded Hopf algebra structure on $\bigoplus_{n\geq 0} V^{\otimes n}$ for which V is primitive and the braiding restricts to c. The inclusion $V \hookrightarrow R$ induces a morphism of braided graded Hopf algebras $\bigoplus_{n\geq 0} V^{\otimes n} \to R$. The image of this morphism, which we denote by R_0, is a braided graded Hopf algebra, generated by V, that is dense in R. Then $V \subset P(R_0) \subset P(R) =$

V and so $P(R_0) = V$. Likewise $R_0(0) = k$ and $R_0(1) = V$. Thus R_0 is an algebraic Nichols algebra. Furthermore, each $R_0(n)$ must be dense in $R(n)$. Since these are finite dimensional, they must be equal. $\square$

3.3.2 Constructing non-Archimedean analytic quantum groups

The main motivation behind Majid's work in [33], which we summarised in Section 3.2, is the reconstruction of Lusztig's from of the quantum enveloping algebra. We will use the same technique to construct analytic analogues of these quantum enveloping algebras.

Throughout the following, we will fix an element $q \in k \setminus \{0\}$ of norm 1 that is not a root of unity. We fix the root datum of a Lie algebra as in Definition 0.1.1.

Definition 3.3.16. Let $H = \coprod^{\leq 1}_{\lambda \in \Phi^*} k \cdot K_\lambda$ be the Banach group Hopf algebra of Φ^* with

$$K_\lambda \cdot K_{\lambda'} = K_{\lambda+\lambda'}, \quad \Delta_H(K_\lambda) = K_\lambda \otimes K_\lambda \quad \text{and} \quad S(K_\lambda) = K_{-\lambda}.$$

We use the notation

$$t_i := K_{\frac{(\alpha_i,\alpha_i)}{2}\lambda_i}$$

which we borrow from [19]. Let H' be the closed sub-Hopf algebra generated by $\{t_i \mid i \in I\}$, $H' = \coprod^{\leq 1}_{\underline{n} \in \mathbb{Z}^I} k \cdot \underline{t}^{\underline{n}}$.

Lemma 3.3.17. *There is a duality pairing $H \hat{\otimes} H' \to k$ defined by*

$$K_\lambda \otimes \underline{t}^{\underline{n}} \mapsto q^{\lambda(\sum n_i \alpha_i)}$$

and simultaneous algebra homomorphisms and coalgebra anti-homomorphisms

$$\mathscr{R} : H' \to H, t_i \mapsto t_i, \;\; \overline{\mathscr{R}} : H' \to H, t_i \mapsto t_i^{-1},$$

for $i \in I$, making H and H' a weakly quasi-triangular dual pair.

Proof. This is Lemma 4.1 of [33]. □

Definition 3.3.18. We will say that a Banach H-modules M is a *Banach weight space* of weight $\alpha \in \Phi$ if $K_\lambda \cdot m = q^{\lambda(\alpha)} m$ for all $m \in M$. We say that an Banach H-module M *decomposes into Banach weight space* if it is a contracting coproduct of Banach weight spaces M_α of weights $\alpha \in \Phi$, $M \cong \coprod_{\alpha \in \Phi}^{\leq 1} M_\alpha$. The weights of M will be those $\alpha \in \Phi$ such that $M_\alpha \neq 0$. We will say that an IndBanach H-module M *decomposes locally into Banach weight spaces* if it can be written as a colimit of Banach H-modules that decompose into Banach weight spaces. Given a subset $X \subset \Phi$ we denote by $H\text{-Mod}_X$ the full subcategory of H-Mod consisting of modules that decompose locally into Banach weight spaces with weights in X.

Remark Lemma 3.3.17 above, along with Proposition 3.2.4 and Proposition 2.3.5 of Part II, says precisely that $H\text{-Mod}_\Psi$ is braided monoidal. In the next section we shall extend this braiding, under certain conditions, to $H\text{-Mod}_\Phi$.

Definition 3.3.19. Let $V = \coprod_{i \in I}^{\leq 1} k \cdot v_i$ with basis $\{v_i \mid i \in I\}$. This is a left H'-comodule with coaction $v_i \mapsto t_i \otimes v_i$. The weakly quasi-triangular structure of H and H' then induces a braiding c on V given by

$$c(v_i \otimes v_j) = q^{(\alpha_i, \alpha_j)} v_j \otimes v_i$$

of norm $\|c\| = 1$. Let $\langle -, - \rangle$ be the non-degenerate bilinear form on V defined by

$$\langle v_i, v_j \rangle = \delta_{i,j} \frac{1}{(q_i - q_i^{-1})} \quad \text{for} \quad q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}.$$

Given $0 < r$ or $0 \leq \rho$ we denote by $\mathbf{f}_r^{\mathrm{an}}$ and $\mathbf{f}_\rho^\dagger$ the Nichols algebras $\mathfrak{B}_r^c(V)$ and $\mathfrak{B}_\rho^c(V)^\dagger$ respectively. We will also use the notation $U_q^+(\mathfrak{g})_r^{\mathrm{an}} = U_q^-(\mathfrak{g})_r^{\mathrm{an}} = \mathbf{f}_r^{\mathrm{an}}$ and

$U_q^+(\mathfrak{g})_\rho^\dagger = U_q^-(\mathfrak{g})_\rho^\dagger = \mathbf{f}_\rho^\dagger$. Note that these are braided IndBanach Hopf algebras in H'-Comod $=$ Comod-H'.

Lemma 3.3.20. *For each $0 < r$, the positive part of the quantum group, as defined in [27] and [33], is dense in the Banach space $\mathbf{f}_r^{an}$.*

Proof. This follows from Proposition 3.3.15 along with Theorem 4.2 of [2], which is a restatement of the constructions in Chapter 1 of [27]. □

Lemma 3.3.21. *Suppose $0 < r, s$ such that $1 \leq |q_i - q_i^{-1}|rs$ for all $i \in I$. Then there is a duality pairing $\mathbf{f}_s^{an}\hat{\otimes}\mathbf{f}_r^{an} \to k$ as Banach Hopf algebras in H-Mod $=$ Mod-H extending $\langle -, - \rangle$ in Definition 3.4.12. Likewise, for $0 \leq \rho, \sigma$ with $1 \leq |q_i - q_i^{-1}|\rho\sigma$ for all $i \in I$ there is a duality pairing $\mathbf{f}_\sigma^\dagger\hat{\otimes}\mathbf{f}_\rho^\dagger \to k$.*

Proof. This follows from Lemma 3.3.9. □

Definition 3.3.22. For $0 < r, s$ with $1 \leq |q_i - q_i^{-1}|rs$ for all $i \in I$ we denote by $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$ the *analytic quantum group* $U(\mathbf{f}_r^{\mathrm{an}}, H, \mathbf{f}_s^{\mathrm{an}})$. We will denote by $U_q^{\leq 0}(\mathfrak{g})_r^{\mathrm{an}}$ and $U_q^{\geq 0}(\mathfrak{g})_r^{\mathrm{an}}$ the respective sub-Hopf algebras $\mathbf{f}_r^{\mathrm{an}} \rtimes H$ and $\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}}$ of $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$. Let us denote by F_i the element $v_i \otimes 1 \in \mathbf{f}_r^{\mathrm{an}} \rtimes H$, and by E_i the element $1 \otimes v_i \in \overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}}$ for $i \in I$. For $0 \leq \rho, \sigma$ with $1 \leq |q_i - q_i^{-1}|\rho\sigma$ for all $i \in I$ we denote by $U_q(\mathfrak{g})_{\rho,\sigma}^\dagger$ the *dagger quantum group* $U(\mathbf{f}_\rho^\dagger, H, \mathbf{f}_\sigma^\dagger)$. We will denote by $U_q^{\leq 0}(\mathfrak{g})_\rho^\dagger$ and $U_q^{\geq 0}(\mathfrak{g})_\sigma^\dagger$ the respective sub-Hopf algebras $\mathbf{f}_\rho^\dagger \rtimes H$ and $\overline{H} \ltimes \overline{\mathbf{f}_\sigma^\dagger}$ of $U_q(\mathfrak{g})_{\rho,\sigma}^\dagger$.

Proposition 3.3.23. *$U_q(\mathfrak{g})_{r,s}^{an}$ and $U_q(\mathfrak{g})_{\rho,\sigma}^\dagger$ are analytically graded by $\mathbb{Z}I \cong \Psi$ (*i.e *are graded by the Banach group Hopf algebra of Ψ).*

Proof. Note that $\mathbf{f}_r^{\mathrm{an}}$ and $\mathbf{f}_s^{\mathrm{an}}$ are H'-comodules, both left and right since H' is cocommutative. If we give H the trivial H'-coaction,

$$H \cong k\hat{\otimes}H \overset{\eta_{H'}\otimes \mathrm{Id}_H}{\longrightarrow} H'\hat{\otimes}H,$$

then all of the morphisms involved in defining $U(\mathbf{f}_r^{\mathrm{an}}, H, \mathbf{f}_s^{\mathrm{an}})$ are H'-comodule homomorphisms. The result then follows since H' is isomorphic to the Banach group Hopf algebra of Φ. □

Proposition 3.3.24. *For each $0 < r, s$ with $1 \leq |q_i - q_i^{-1}| rs$ for all $i \in I$, the quantum enveloping algebra $U_q(\mathfrak{g})$ is dense in the Banach space $U_q(\mathfrak{g})^{an}_{r,s}$, and the Hopf structure on $U_q(\mathfrak{g})^{an}_{r,s}$ restricts to the usual Hopf structure on $U_q(\mathfrak{g})$.*

Proof. This follows from Proposition 4.3 of [33], Lemma 3.3.20 and the triangular decomposition of $U_q(\mathfrak{g})$ given by the Poincaré-Birkhoff-Witt Theorem. □

Proposition 3.3.25. *Suppose $\mathcal{U}$ is a Banach Hopf algebra in which $U_q(\mathfrak{g})$ is a dense sub-Hopf algebra, with $\mathcal{U}$ analytically $\mathbb{Z}I$ graded (*i.e *are graded by the Banach group Hopf algebra of Ψ) extending the grading on $U_q(\mathfrak{g})$. Then there is an epimorphism of Banach Hopf algebras*

$$U_q(\mathfrak{g})^{an}_{r,s} \to \mathcal{U}$$

for some $r, s > 0$ with $|q_i - q_i^{-1}| rs \geq 1$ for all $i \in I$.

Proof. Let $\mathcal{U}^-$, $\mathcal{U}^0$, and $\mathcal{U}^+$ be the respective closured of $U_q^-(\mathfrak{g})$, $U_q^0(\mathfrak{g})$, and $U_q^+(\mathfrak{g})$ in $\mathcal{U}$. The $\mathbb{Z}I$ grading on $U_q^+(\mathfrak{g})$ is concentrated in degrees in the submonoid $\mathbb{N}I$, and hence so is the analytic grading on $\mathcal{U}^+$. The homomorphism of monoids from this submonoid to $\mathbb{N}$, $\sum_{i \in I} n_i \cdot i \mapsto \sum_{i \in I} n_i$ gives $\mathcal{U}^+$ an analytic $\mathbb{N}$ grading compatible with that of the algebraic Nichols algebra $U_q^+(\mathfrak{g})$. Since $\mathcal{U}^+$ is a completion of the algebraic Nichols algebra $U_q^+(\mathfrak{g})$ it must be an analytic Nichols algebra of V as in Definition 3.4.12, and hence by Proposition 3.3.25 we have an epimorphism $\mathbf{f}_s \to \mathcal{U}^+$ for some $s > 0$. Likewise we obtain an epimorphism $\mathbf{f}_r \to \mathcal{U}^-$ for some $r > 0$, and without loss of generality we may take r and s such that $rs|q_i - q_i^{-1}| \geq 1$ for all $i \in I$. Finally, each $K_\lambda \in \mathcal{U}_0$ is grouplike, so since comultiplication on $\mathcal{U}_0$ is bounded by some constant $C > 0$ we have $\|K_\lambda\|^2 = \|K_\lambda \otimes K_\lambda\| \leq C \|K_\lambda\|$, so $\|K_\lambda\| \leq C$. Thus we can define a bounded epimorphism $H \to \mathcal{U}^0$ by mapping each K_λ to the associated element of $U_q^0(\mathfrak{g})$ inside $\mathcal{U}^0$. This gives us an epimorphism

$$U_q(\mathfrak{g})^{\mathrm{an}}_{r,s} = \mathbf{f}^{\mathrm{an}}_r \hat{\otimes} H \hat{\otimes} \mathbf{f}^{\mathrm{an}}_s \twoheadrightarrow \mathcal{U}^- \hat{\otimes} \mathcal{U}^0 \hat{\otimes} \mathcal{U}^+ \overset{\mu}{\twoheadrightarrow} \mathcal{U}$$

that restricts to a Hopf algebra homomorphism on the dense subspace $U_q(\mathfrak{g})$. □

3.3.3 Quasi-triangularity and the R-matrix

Suppose throughout this section that the symmetrised Cartan matrix associated to our root datum in Definition 0.1.1, $A = ((\alpha_i, \alpha_j))_{i,j \in I}$, is invertible (over $\mathbb{Q}$). Suppose further that $q = \exp(\hbar)$ for some $\hbar \in k$ of sufficiently small norm such that $\frac{1}{|n!|}(|\hbar| \cdot \max_{i,j} |A^{-1}_{i,j}|)^n$ converges to 0, where $A^{-1}_{i,j}$ are the entries of the inverse of the Cartan matrix.

Remark If k is an extension of $\mathbb{Q}_p$, the requirement that $\frac{1}{|n!|}(|\hbar| \cdot \max_{i,j} |A^{-1}_{i,j}|)^n$ converges to 0 is equivalent to $|\hbar| \cdot \max_{i,j} |A^{-1}_{i,j}| < p^{\frac{1}{1-p}}$. If k is such that $|n| = 1$ for all integers $n \in \mathbb{Z}$ then this is just the assumption that $|\hbar| < 1$.

Definition 3.3.26. Let us denote by $\mathscr{H}$ the Banach space $\mathscr{H} := \coprod^{\leq 1}_{\alpha \in \mathbb{Z}I} k \cdot H_\alpha$. This has an algebra structure induced by the group structure of $\mathbb{Z}I$, and we make $\mathscr{H}$ a Hopf algebra by defining

$$\Delta_{\mathscr{H}}(H_i) = 1 \otimes H_i + H_i \otimes 1 \quad \text{for} \quad i \in I.$$

We will denote by $e : H' \to \mathscr{H}$ the morphism of Hopf algebras determined by

$$t_i \mapsto \exp\left(\frac{(\alpha_i, \alpha_i)}{2} \hbar H_i\right) = \sum_{n \geq 0} \frac{(\alpha_i, \alpha_i)^n \hbar^n}{2^n \cdot n!} H_{ni}.$$

Lemma 3.3.27. *Under our assumptions, $\mathscr{H}$ is quasi-triangular with R-matrix*

$$\mathscr{R}_{\mathscr{H}} = exp(\hbar \sum_{i,j} A^{-1}_{i,j} H_i \otimes H_j).$$

Proof. Since $\mathscr{H}$ is both commutative and cocommutative, it is trivial to check that $\mathscr{R}_{\mathscr{H}}$ is an R-matrix. It remains only to check that this converges, which follows from our assumptions since

$$\|\hbar \sum_{i,j} A^{-1}_{i,j} H_i \otimes H_j\| \leq |\hbar| \cdot \max_{i,j} |A^{-1}_{i,j}|.$$

□

Proposition 3.3.28. *There is a braiding on H-Mod$_\Phi$ extending that of Lemma 3.3.17.*

Proof. Suppose M_α and $N_{\alpha'}$ are Banach H-modules, and hence H'-modules, of weights α and α' in Φ respectively. Then M_α is a $\mathscr{H}$-module where H_i acts by the scalar (α_i, α), and likewise so is $N_{\alpha'}$. The action of $\mathscr{R}_{\mathscr{H}}$ induces the braiding

$$M_\alpha \hat{\otimes} N_{\alpha'} \to N_{\alpha'} \hat{\otimes} M_\alpha, \quad m \otimes n \mapsto q^{\sum_{i,j} A^{-1}_{i,j}(\alpha_i,\alpha)(\alpha_j,\alpha')} n \otimes m.$$

This braiding commutes not only with the action of H' but also with that of H. If $\alpha = \sum n_i \alpha_i$ and $\alpha' = \sum n'_i \alpha_i$ in Ψ then

$$\begin{aligned} \textstyle\sum_{i,j} A^{-1}_{i,j}(\alpha_i,\alpha)(\alpha_j,\alpha') &= \textstyle\sum_{i,j,i',j'} A^{-1}_{i,j} n_{i'} n'_{j'} (\alpha_i, \alpha_{i'})(\alpha_j, \alpha_{j'}) \\ &= \textstyle\sum_{i,j,i',j'} A^{-1}_{i,j} n_{i'} n'_{j'} (\alpha_i, \alpha_{i'}) A_{j,j'} \\ &= \textstyle\sum_{i,i'} n_{i'} n'_i (\alpha_i, \alpha_{i'}) = (\alpha, \alpha'), \end{aligned}$$

so the braiding restricts to the one given by Lemma 3.3.17 on H-Mod$_\Psi$. □

Proposition 3.6 of [33] gives criterion for when a double-bosonisation is quasi-triangular. The Banach version of this would be the following.

Theorem. Let H be a quasi-triangular Banach Hopf algebra, let B be a braided Banach Hopf algebra B in H-Mod and let C be a braided Banach Hopf algebra in Mod-H, equipped with a duality pairing between B and C, $\langle -, - \rangle : B\hat{\otimes}C \to k$. Suppose there exists linearly independent subsets $\{b_n \mid n \in \mathbb{N}\} \subset B$ and $\{c_n \mid n \in \mathbb{N}\} \subset C$ that span dense subspaces and satisfy

$$\langle b_n, c_m \rangle = \delta_{n,m} \quad \text{and} \quad \|b_n\| \cdot \|c_n\| \to 0.$$

Then $U(C, H, B)$ is quasi-triangular with R-matrix

$$\mathscr{R}_{U(C,H,B)} := \exp_{C,B} \cdot \mathscr{R}_H = \sum_{x \in X} (c_x \otimes \mathscr{R}_2^{(1)} \mathscr{R}_1^{(1)} \otimes 1) \otimes (1 \otimes \mathscr{R}_1^{(2)} \otimes S_B(b_x) \cdot \mathscr{R}_2^{(2)})$$

where $\exp_{C,B} = \sum_{x \in X} c_x \otimes S_B(b_x)$, and $\mathscr{R}_1 = \sum \mathscr{R}_1^{(1)} \otimes \mathscr{R}_1^{(2)}$ and $\mathscr{R}_2 = \sum \mathscr{R}_2^{(1)} \otimes \mathscr{R}_2^{(2)}$ are copies of the R-matrix $\mathscr{R}_H$ on H.

Unfortunately, as in [33], the conditions required for this theorem do not hold if B and C are infinite dimensional, hence it does not apply to $U(\mathbf{f}_r^{\mathrm{an}}, \mathscr{H}, \mathbf{f}_s^{\mathrm{an}})$. Indeed, if such sets were to exist for infinite dimensional B and C then $\sum b_n \otimes c_n$ would converge in $B \otimes C$ but its image under $\langle -, - \rangle$ would not.

If we allow ourselves to work over formal powerseries in $\hbar$ then we may define an R-matrix for some analytic quantum groups, which we will see in Section 3.3.6.

3.3.4 Quantum groups as Drinfel'd doubles

Lemma 3.3.29. *Suppose we have IndBanach Hopf algebras B and C with a duality pairing $\langle -, - \rangle : B\hat{\otimes}C^{op} \to k$. Then there is a Hopf algebra structure on $C\hat{\otimes}B$ such* *that $B \to C\hat{\otimes}B$ and $C \to C\hat{\otimes}B$ are morphisms of Hopf algebras and the multiplication restricts to the composition*

$$\begin{aligned} B\hat{\otimes}C &\longrightarrow B\hat{\otimes}B\hat{\otimes}B\hat{\otimes}C\hat{\otimes}C\hat{\otimes}C \\ &\longrightarrow B\hat{\otimes}C\hat{\otimes}C\hat{\otimes}B\hat{\otimes}B\hat{\otimes}C \\ &\longrightarrow C\hat{\otimes}B \end{aligned}$$

where the first morphism is given by the respective comultiplications, the second is a reordering given by the permutation $(2 \quad 4)(3 \quad 5) \in S_6$ and the last is given by $\langle -, - \rangle \otimes Id \otimes Id \otimes \langle -, - \rangle$.

Proof. This construction is outlined in Section 8.2.1 of [25] and Section IX.5 of [23] for (finite dimensional) vector spaces, but is easily generalised to the category of IndBanach spaces. □

Definition 3.3.30. We will denote by $D(B, C)$ the Hopf algebra on $C\hat{\otimes}B$ described in the previous lemma, the *relative Drinfel'd double* of B and C.

Lemma 3.3.31. *There is a duality pairing $(\overline{H} \ltimes \overline{\mathbf{f}_s^{an}})\hat{\otimes}(\mathbf{f}_r^{an} \rtimes H')^{op} \to k$ such that*

$$\begin{aligned} K_\lambda \otimes t_j &\mapsto q^{-\lambda(\alpha_j)}, \quad & K_\lambda \otimes F_j &\mapsto 0, \\ E_i \otimes t_j &\mapsto 0, & E_i \otimes F_j &\mapsto -\delta_{i,j}\tfrac{1}{q_i - q_i^{-1}}. \end{aligned}$$

Proof. This follows as in Section 6.3.1 of [25]. We first define bounded algebra homomorphisms ϕ_i and ψ_i from $H \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}}$ to k such that

$$\phi_i(K_\lambda \otimes v_i) = \frac{1}{q_i^{-1} - q_i}, \quad \phi_i(K_\lambda \otimes x) = 0$$

and

$$\psi_i(K_\lambda \otimes y) = q^{-\lambda(\alpha_i)}\varepsilon(y)$$

for all $i \in I$, $x \in \mathbf{f}_s^{\mathrm{an}}(\alpha)$, $\alpha_i \neq \alpha \in \Psi$, and $y \in \mathbf{f}_s^{\mathrm{an}}$. Here ε is the counit on $\mathbf{f}_s^{\mathrm{an}}$.

We define contracting morphisms $V_r \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$ and $\coprod_{i\in I}^{\leq 1} t_i \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$ defined respectively by $v_i \mapsto \phi_i$ and $t_i \mapsto \psi_i$. These induce algebra homomorphisms $T_r(V) \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$, $H' \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$. It follows from the proof of Proposition 34 in *loc. cit.* that the map $T_r(V) \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$ factors as a composition $T_r(V) \to \mathbf{f}_r^{\mathrm{an}} \to (H \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$. From this we obtain a morphism $\mathbf{f}_r^{\mathrm{an}} \rtimes H' \to (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})^*$ and hence a bilinear form as desired. The fact that this is a duality pairing is checked on a dense subspace in [25], and extends by continuity. $\square$

Remark It is shown in Proposition 34 of [25] that this pairing can be expressed as the composition

$$H\hat{\otimes}\mathbf{f}_s^{\mathrm{an}}\hat{\otimes}\mathbf{f}_r^{\mathrm{an}}\hat{\otimes}H' \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes S\otimes\mathrm{Id}} H\hat{\otimes}\mathbf{f}_s^{\mathrm{an}}\hat{\otimes}\mathbf{f}_r^{\mathrm{an}}\hat{\otimes}H' \xrightarrow{\mathrm{Id}\otimes\langle -,-\rangle\otimes\mathrm{Id}} H\hat{\otimes}k\hat{\otimes}H' \xrightarrow{\mathrm{Id}\otimes S} H\hat{\otimes}H' \xrightarrow{\langle -,-\rangle} k.$$

Definition 3.3.32. For an IndBanach Hopf algebra C, we will denote by ${}_C\mathrm{Cross}^C$ the category of IndBanach spaces V equipped with both a left action and right coaction of C, $\mu_V : C\hat{\otimes}V \to V$ and $\Delta_V : V \to V\hat{\otimes}C$, such that the following diagram commutes:

$$\begin{array}{ccc}
C\hat{\otimes}V & \xrightarrow{\Delta_C\otimes\mathrm{Id}} & C\hat{\otimes}C\hat{\otimes}V \\
\downarrow \Delta\otimes\Delta_V & & \downarrow \mathrm{Id}\otimes\mu_V \\
C\hat{\otimes}C\hat{\otimes}V\hat{\otimes}C & & C\hat{\otimes}V \\
\downarrow \mathrm{Id}\otimes\tau\otimes\mathrm{Id} & & \downarrow \tau \\
C\hat{\otimes}V\hat{\otimes}C\hat{\otimes}C & & V\hat{\otimes}C \\
\downarrow \mu_V\otimes\mu_C & & \downarrow \Delta_V\otimes\mathrm{Id} \\
V\hat{\otimes}C & \xleftarrow[\mathrm{Id}\otimes\mu_C]{} & V\hat{\otimes}C\hat{\otimes}C.
\end{array}$$

We will refer to these as *crossed bimodules*, although they are sometimes also referred to as *Yetter-Drinfel'd modules.*

Lemma 3.3.33. *The category ${}_C$Cross^C is pre-braided monoidal with braiding given by*

$$M\hat{\otimes}N \xrightarrow{\tau} N\hat{\otimes}M \xrightarrow{\Delta_N\otimes Id} N\hat{\otimes}C\hat{\otimes}M \xrightarrow{Id\otimes\mu_M} N\hat{\otimes}M$$

for M and N in ${}_C\mathit{Cross}^C$. If C has an invertible antipode then this is a braiding.

Proof. This is Theorem 5.2 and Theorem 7.2 of [42]. □

Lemma 3.3.34. *There is a faithful functor ${}_C\mathit{Cross}^C \to D(B,C)$-Mod.*

Proof. This follows from the proof of Proposition 6 in Section 13.1 of [25]. An object V of ${}_C\mathrm{Cross}^C$ already has an action of C. It gains an action of B using the duality pairing, giving a map

$$\mu'_V : B\hat{\otimes}V \xrightarrow{\mathrm{Id}\otimes\Delta_V} B\hat{\otimes}V\hat{\otimes}C \xrightarrow{(\mathrm{Id}\hat{\otimes}\langle -,-\rangle)\circ(\tau\otimes\mathrm{Id})} V.$$

Then B and C generate $D(B,C)$, and the fact that μ_V and μ'_V together give a well defined action of $D(B,C) = C\hat{\otimes}B$ on V (given by $\mu_V \circ (\mathrm{Id}\otimes\mu'_V)$) follows from the commutativity of the diagram in Definition 3.3.32 as in the proof of Proposition 6 of *loc. cit.* □

Proposition 3.3.35. *There is a strict epimorphism of braided analytically graded Banach Hopf algebras*

$$D(\overline{H} \ltimes \overline{\mathbf{f}_s^{an}}, \mathbf{f}_r^{an} \rtimes H') \to U_q(\mathfrak{g})_{r,s}^{an}$$

whose kernel is the closed two sided ideal generated by

$$\{1\otimes t_i\otimes 1\otimes 1 - 1\otimes 1\otimes t_i\otimes 1 \mid i\in I\} \subset \mathbf{f}_r^{an}\hat{\otimes}H\hat{\otimes}H'\hat{\otimes}\mathbf{f}_s^{an}.$$

Proof. This strict epimorphism can be written as

$$\mathbf{f}_r^{\mathrm{an}}\hat{\otimes}H'\hat{\otimes}H\hat{\otimes}\mathbf{f}_s^{\mathrm{an}} \xrightarrow{\mathrm{Id}\otimes\mu_H\otimes\mathrm{Id}} \mathbf{f}_r^{\mathrm{an}}\hat{\otimes}H\hat{\otimes}\mathbf{f}_s^{\mathrm{an}}.$$

Corollary 15 of Section 8.2.4 of [25], along with Proposition 3.3.24, show that this restricts to a morphism of braided graded Hopf algebras between the dense subspaces

$$U_q^{\leq 0}(\mathfrak{g}) \otimes U_q^{\geq 0}(\mathfrak{g}) \to U_q(\mathfrak{g})$$

whose kernel is generated by $t_i \otimes 1 - 1 \otimes t_i$. The result then follows by continuity. $\square$

Definition 3.3.36. Let us denote by $\mathcal{C}$ the full subcategory of ${}_{(\mathbf{f}_r^{\mathrm{an}} \rtimes H')}\mathrm{Cross}^{(\mathbf{f}_r^{\mathrm{an}} \rtimes H')}$ consisting of IndBanach spaces V equipped with both a left action and right coaction of $\mathbf{f}_r^{\mathrm{an}} \rtimes H'$, $\mu_V : (\mathbf{f}_r^{\mathrm{an}} \rtimes H')\hat{\otimes} V \to V$ and $\Delta_V : V \to V \hat{\otimes} (\mathbf{f}_r^{\mathrm{an}} \rtimes H')$, such that the following additional diagram commutes:

$$\begin{array}{ccc} H'\hat{\otimes} V & \longrightarrow & (\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}})\hat{\otimes} V \\ \downarrow & & \downarrow \mu'_V \\ (\mathbf{f}_r^{\mathrm{an}} \rtimes H')\hat{\otimes} V & \xrightarrow{\mu_V} & V \end{array}$$

where μ'_V is the action described in the proof of Lemma 3.3.34.

Lemma 3.3.37. *There is a fully faithful functor $\mathcal{C} \to U_q(\mathfrak{g})_{r,s}^{an}$-Mod.*

Proof. By Lemma 3.3.34 and Proposition 3.3.35 there is a faithful functor $\mathcal{C} \to U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$-Mod. Let $f : M \to N$ be a morphism in $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$-Mod between two objects in the image of $\mathcal{C}$. Then the action of $\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}}$ on both M and N is induced by a coaction of $\mathbf{f}_r^{\mathrm{an}} \rtimes H'$ via their dual pairing. Furthermore, f commutes with the action of $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$, and hence with the actions of $\mathbf{f}_r^{\mathrm{an}} \rtimes H'$ and $\overline{H} \ltimes \overline{\mathbf{f}_s^{\mathrm{an}}}$. The morphism f must therefore preserve the locally Banach weight space decompositions of M and N, so it preserves their respective coactions of H'. Since the pairing of $\mathbf{f}_r^{\mathrm{an}}$ and $\mathbf{f}_s^{\mathrm{an}}$ is non-degenerate, f must also preserve the coaction of $\mathbf{f}_r^{\mathrm{an}}$ that induces the action of $\mathbf{f}_s^{\mathrm{an}}$. Hence f preserves the coaction of $\mathbf{f}_r^{\mathrm{an}} \rtimes H'$, so comes from a morphism in $\mathcal{C}$. So the functor is fully faithful. $\square$

Definition 3.3.38. We will denote by $\mathcal{O}_\Psi$ the essentail image of $\mathcal{C}$ in $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$-Mod. This is the full subcategory of $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$ modules whose action of H gives a representation in Comod-$H' = H$-Mod$_\Psi$ and whose action of $U_q^-(\mathfrak{g})_s^{\mathrm{an}}$ comes from a coaction of $\mathbf{f}_r^{\mathrm{an}}$ via their dual pairing. Let us denote by $\mathcal{O}$ the full subcategory of $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$-Mod consisting of modules whose action of $U_q^-(\mathfrak{g})_s^{\mathrm{an}}$ comes from a coaction of $\mathbf{f}_r^{\mathrm{an}}$ and whose action of H gives a representation in H-Mod$_\Phi$. Note that $\mathcal{O}_\Psi \subset \mathcal{O}$.

Remark Note that the conditions for a $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$ module to be in $\mathcal{O}$ resemble the conditions for a $U_q(\mathfrak{g})$ module to be integrable, with the requirement that the action of $U_q^+(\mathfrak{g})_s^{\mathrm{an}}$ comes from a coaction of $\mathbf{f}_r^{\mathrm{an}}$ taking the place of a locally finite dimensional action of $U_q^+(\mathfrak{g})$.

Corollary 3.3.39. *The category $\mathcal{O}_\Psi$ is braided monoidal.*

Proof. This is a consequence of Lemma 3.3.33 and Lemma 3.3.37. □

Remark A short computation shows that the braiding on $\mathcal{O}_\Psi$ can be expressed as performing the braiding on H'-comodules given by Lemma 3.3.17 followed by the action of $\exp_{\mathbf{f}_r^{\mathrm{an}},\mathbf{f}_r^{\mathrm{an}}}$ as described at the end of Section 3.3.3, which is well defined despite $\exp_{\mathbf{f}_r^{\mathrm{an}},\mathbf{f}_r^{\mathrm{an}}}$ not converging. This is expected given the description of the R-matrix at the end of Section 3.3.3.

Proposition 3.3.40. *Suppose that the symmetrised Cartan matrix associated to our root datum, $A = ((\alpha_i, \alpha_j))_{i,j\in I}$, has an inverse over $\mathbb{Q}$ with entries $A_{i,j}^{-1}$. Suppose further that $q = exp(\hbar)$ for some $\hbar \in k$ of sufficiently small norm such that $\frac{1}{|n!|}(|\hbar| \cdot max_{i,j}|A_{i,j}^{-1}|)^n$ converges to 0. Then there is a braiding on the category $\mathcal{O}$ extending that of $\mathcal{O}_\Psi$.*

Proof. Given M and N in $\mathcal{O}$, the braiding is given by the composition

$$M\hat{\otimes}N \xrightarrow{\tau} N\hat{\otimes}M \longrightarrow N\hat{\otimes}\mathbf{f}_r^{\mathrm{an}}\hat{\otimes}M \longrightarrow N\hat{\otimes}M \xrightarrow{\mathscr{R}\mathscr{H}} N\hat{\otimes}M$$

where the second morphism is the coaction of $\mathbf{f}_r^{\mathrm{an}}$ on N, the third is its action on M, and the last in the action of the R-matrix as in the proof of Proposition 3.3.28.

The computations in the proof of Proposition 3.6 of [33], alongside the previous remark, ensure that this is a braiding. This is an extension of the braiding in Lemma 3.3.33. □

Example 3.3.41. *Let $\mathfrak{g} = \mathfrak{sl}_2$, and fix $r, s > 0$ with $|q_i - q_i^{-1}|rs \geq 1$ for all $i \in I$. Let $\lambda \in \Phi \cong \mathbb{Z}$, and let $W_\lambda = k\{\frac{x}{r}\} = \coprod_{n\geq 0}^{\leq 1} k_{r^n} \cdot x^n$ with the action of $U_q(\mathfrak{sl}_2)_{r,s}^{an}$ given by*

$$K \cdot x^n = q^{\lambda - 2n} x^n, \quad F \cdot x^n = x^{n+1}, \quad E \cdot x^n = [n][\lambda - (n-1)]x^{n-1}$$

where $x^{-1} := 0$. Alternatively, taking $y^n = \frac{x^n}{[n]!}$, $W = \{\sum a_n y^n \mid \frac{|a_n| r^n}{[n]!} \to 0\}$ with action

$$K \cdot y^n = q^{\lambda - 2n} y^n, \quad F \cdot y^n = [n+1] y^{n+1}, \quad E \cdot y^n = [\lambda - (n-1)]y^{n-1}$$

where $y^{-1} := 0$. Note that W_λ is isomorphic to the quotient of $U_q(\mathfrak{sl}_2)_{r,s}^{an}$ by the closed left ideal generated by E and $K - q^\lambda$, and so we call it an analytic Verma module *of weight λ. This gives a representation in $\mathcal{O}$ where the coaction of $\mathbf{f}_r^{an}$ is given by*

$$\begin{aligned} x^n &\longmapsto \textstyle\sum (-1)^k \frac{(q-q^{-1})^k}{[k]!} F^k \otimes E^k x^n \\ &\quad = \textstyle\sum_{k=0}^n (-1)^k (q-q^{-1})^k \frac{[n]![\lambda - n + k]!}{[k]![n-k]![\lambda - n]!} F^k \otimes x^{n-k}, \\ y^n &\longmapsto \textstyle\sum_{k=0}^n (-1)^k (q-q^{-1})^k \frac{[\lambda - n + k]!}{[k]![\lambda - n]!} F^k \otimes y^{n-k}. \end{aligned}$$

Given $\lambda, \lambda' \in \Phi$ the braiding $W_\lambda \hat{\otimes} W_{\lambda'} \to W_{\lambda'} \hat{\otimes} W_\lambda$ is given by

$$\begin{aligned} x^n \otimes x^m &\longmapsto \textstyle\sum_{k=0}^n (-1)^k (q-q^{-1})^k \frac{[n]![\lambda - n + k]!}{[k]![n-k]![\lambda - n]!} q^{2(m+k)(n-k)} \, x^{m+k} \otimes x^{n-k}, \\ y^n \otimes y^m &\longmapsto \textstyle\sum_{k=0}^n (-1)^k (q-q^{-1})^k \frac{[m+k]![\lambda - n + k]!}{[k]![m]![\lambda - n]!} q^{2(m+k)(n-k)} \, y^{m+k} \otimes y^{n-k}. \end{aligned}$$

Example 3.3.42. *Note that objects in $\mathcal{O}^{an}$ are not necessarily generated by highest weight vectors. We give an example of a representation with no such highest weights. Let $\mathfrak{g} = \mathfrak{sl}_2$, fix $r, s > 0$ with $|q_i - q_i^{-1}|rs \geq 1$ for all $i \in I$, and suppose that $q = exp(\hbar)$*

for $|\hbar| \ll 1$. *Then*

$$q - q^{-1} = \left(\sum_{n=0}^{\infty} \frac{(\hbar)^n}{n!} \right) - \left(\sum_{n=0}^{\infty} \frac{(-\hbar)^n}{n!} \right) = 2\hbar \left(\sum_{k=0}^{\infty} \frac{(\hbar)^{2k}}{k!} \right)$$

so $|q - q^{-1}| = |2\hbar|$, *and*

$$\begin{aligned} [n] - n &= q^{n-1} + q^{n-3} + ... + q^{-n+1} - n \\ &= \hbar \textstyle\sum_{k=1}^{\infty} \frac{1}{k!}[(n-1)^k + (n-3)^k + ... + (-n+1)^k]\hbar^{k-1} \end{aligned}$$

has norm strictly smaller than 1 for $|\hbar|$ *sufficiently small, so* $|[n]| = |n|$. *Fix some* $\lambda \in \mathbb{Z}_{\geq 0}$. *Let us define*

$$M_\lambda := \coprod\nolimits_{i,j \geq 0}^{\leq 1} k_{r^{j-i}} \cdot x_{i,j} = \left\{ \sum \alpha_{i,j} x_{i,j} \middle| |\alpha_{i,j}| r^{j-i} \to 0 \right\}.$$

Then M_λ *becomes a* $U_q(\mathfrak{sl}_2)^{an}_{r,s}$ *module with*

$$\begin{aligned} K \cdot x_{i,j} &= q^{\lambda+2i-2j} x_{i,j}, \\ E \cdot x_{i,j} &= x_{i+1,j}, \\ F \cdot x_{0,j} &= x_{i,j+1}, \\ F \cdot x_{i,j} &= x_{i,j+1} - [i][\lambda + i - 1 - 2j] x_{i-1,j} \quad \textit{for } i > 0, \end{aligned}$$

so that $x_{i,j} = E^i F^j x_{0,0}$ *and* $x_{0,0}$ *is of weight* λ. *Note that this action is bounded since*

$$\|K \cdot x_{i,j}\| = \|x_{i,j}\|,$$

$$\|E \cdot x_{i,j}\| = r^{j-i-1} = \frac{1}{rs} \|E\| \cdot \|x_{i,j}\| \leq \|E\| \cdot \|x_{i,j}\|,$$

$$\|F \cdot x_{i,j}\| \leq max\{1, |[i][\lambda + i - 1 - 2j]|\} r^{j+1-i} \leq r^{j+1-i} = \|F\| \cdot \|x_{i,j}\|.$$

The map $M_\lambda \to \mathbf{f}_r^{an}\hat{\otimes} M_\lambda$ *given by*

$$x_{i,j} \mapsto \sum_{k\geq 0} (-1)^k \frac{(q-q^{-1})^k}{[k]!} F^k \otimes x_{i+k,j}$$

is well defined and bounded since $|\frac{(q-q^{-1})^k}{[k]!}| r^k r^{j-i-k} = \frac{|2\hbar|^k}{|k!|} \to 0$*. This shows that* M_λ *is indeed in* $\mathcal{O}^{an}$*, since*

$$\sum_{k\geq 0} (-1)^k \frac{(q-q^{-1})^k}{[k]!} \langle E^n, F^k \rangle \otimes x_{i+k,j} = x_{i+n,j} = E^n \cdot x_{i,j}.$$

3.3.5 Rigidity results

Classical rigidity results of Chevalley, Eilenberg and Cartan from the 1940s assert that there are no non-trivial formal deformations (as an algebra) of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$. The proof relies on the vanishing of the second Lie algebra cohomology group. In this section we prove an analogous result that relies on a bounded cohomology vanishing result that has yet to be proven. We proceed as in Chapter XVIII of [23].

We fix a set of root datum as in Definition 0.1.1.

Definition 3.3.43. Let $\mathscr{H}_0 := \coprod_{\alpha\in\mathbb{Z}I}^{\leq 1} kH_\alpha$ be the Banach Hopf algebra generated by H_i for $i \in I$, where

$$\Delta_{\mathscr{H}_0}(H_i) = 1 \otimes H_i + H_i \otimes 1.$$

Let $V_0 := \coprod_{i\in I}^{\leq 1} k \cdot v_i$ and let $T_r(V_0) = \coprod_{n\geq 0}^{\leq 1} (V_0^{\hat{\otimes} n})_{r^n}$. For $r > 0$ we will denote by both $U^-(\mathfrak{g})_r^{\mathrm{an}}$ and $U^+(\mathfrak{g})_r^{\mathrm{an}}$ the quotient of $T_r(V_0)$ by the closed ideal generated by the

Serre relations

$$\sum_{k=0}^{1-(\alpha_i,\alpha_j)} (-1)^k \binom{1-(\alpha_i,\alpha_j)}{k} v_i^{1-(\alpha_i,\alpha_j)-k} v_j v_i^k = 0$$

for $i \neq j$.

Theorem 3.3.44. *Let $r, s > 0$ such that $1 \leq rs$. Then there is a Banach algebra structure on*

$$U(\mathfrak{g})_{r,s}^{an} := U^-(\mathfrak{g})_r^{an} \hat{\otimes} \mathscr{H}_0 \hat{\otimes} U^+(\mathfrak{g})_s^{an}$$

such that $U^-(\mathfrak{g})_r^{an}$, $\mathscr{H}_0$ and $U^+(\mathfrak{g})_s^{an}$ are all subalgebras, and

$$[H_i, E_j] = (\alpha_i, \alpha_j)E_j, \quad [H_i, F_j] = -(\alpha_i, \alpha_j)F_j, \quad [E_i, F_j] = \delta_{i,j}H_i,$$

where $F_i = v_i \otimes 1 \otimes 1$ and $E_i = 1 \otimes 1 \otimes v_i$. This becomes a Banah Hopf algebra with $\mathscr{H}_0$ as a sub-Hopf algebra and

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + 1 \otimes E_i, \quad S(E_i) = -E_i, \\ \Delta(F_i) &= F_i \otimes 1 + 1 \otimes F_i, \quad S(F_i) = -F_i. \end{aligned}$$

Proof. By construction and the triangular decomposition given by the Poincaré-Birkhoff-Witt Theorem the enveloping algebra $U(\mathfrak{g})$ sits as a dense subspace of $U(\mathfrak{g})_{r,s}^{\text{an}}$ on which this Hopf algebra structure is well defined. It is therefore enough to check that it extends continuously. Since

$$\begin{aligned} \|H_i \cdot F_j\| &= \|F_j H_i - (\alpha_i, \alpha_j)F_j\| \leq r = \|H_i\|\|F_j\| \\ \|E_i \cdot H_j\| &= \|H_j E_i + (\alpha_i, \alpha_j)E_j\| \leq s = \|E_i\|\|H_j\| \\ \|E_i \cdot F_j\| &= \|F_j E_i - \delta_{i,j}H_i\| \leq rs = \|E_i\|\|F_j\| \end{aligned}$$

and since

$$\|\Delta(x)\| = \|1 \otimes x + x \otimes 1\| \leq \|x\|$$

for each generator $x \in \{F_i, H_i, E_i \mid i \in I\}$ we may define bounded linear transformations

$$\left(T_r(V_0)\hat{\otimes}\mathscr{H}_0\hat{\otimes}T_s(V_0)\right)\hat{\otimes}\left(T_r(V_0)\hat{\otimes}\mathscr{H}_0\hat{\otimes}T_s(V_0)\right) \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}$$

$$T_r(V_0)\hat{\otimes}\mathscr{H}_0\hat{\otimes}T_s(V_0) \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\mathrm{an}}_{r,s}$$

which descend to the described multiplication and comultiplication maps. □

Definition 3.3.45. We will denote by $\mathfrak{g}_{r,s}$ the closed Lie subalgebra of $U(\mathfrak{g})^{\mathrm{an}}_{r,s}$ generated by $\{F_i, H_i, E_i \mid i \in I\}$. This is a Banach Lie algebra with $\|[x,y]\| \leq \|x\| \cdot \|y\|$ for $x, y \in \mathfrak{g}_{r,s}$.

Proposition 3.3.46. *Suppose we have a Banach algebra A and a morphism of Banach Lie algebras $\mathfrak{g}_{r,s} \to A$ of norm at most 1. Then this extends to a unique contracting morphism of Banach algebras $U(\mathfrak{g})^{an}_{r,s} \to A$. In particular, given a Banach $\mathfrak{g}_{r,s}$ module M whose action satisfies $\|x \cdot m\| \leq \|x\| \cdot \|m\|$ for $x \in \mathfrak{g}$, $m \in M$, this extends to a unique action of $U(\mathfrak{g})^{an}_{r,s}$ on M such that the morphism $U(\mathfrak{g})^{an}_{r,s}\hat{\otimes}M \to M$ is contracting.*

Proof. Taking the images of F_i, H_i and E_i in A gives us a contracting map

$$(T_r(V_0)\hat{\otimes}\mathscr{H}_0\hat{\otimes}T_s(V_0)) \to A.$$

The images of F_i and E_i must satisfy the Serre relations in A, and hence this descends to a map $U(\mathfrak{g})_{r,s} \to A$. This restricts to a well defined algebra homomorphism on the dense subspace $U(\mathfrak{g})$, hence the result follows by continuity. Applying this to a morphism $\mathfrak{g}_{r,s} \to \mathrm{Hom}(M, M)$ gives the rest of this result. □

Definition 3.3.47. Let M be a left Banach $\mathfrak{g}_{r,s}$ module whose action satisfies $\|x \cdot m\| \leq \|x\| \cdot \|m\|$ for $x \in \mathfrak{g}_{r,s}$, $m \in M$. Then we define the complex

$$C^n_b(\mathfrak{g}_{r,s}, M) := \mathrm{Hom}(\Lambda^n \mathfrak{g}_{r,s}, M),$$

the space of bounded antisymmetric n-linear maps from $\mathfrak{g}_{r,s}$ to M, and let δ_n : $C_b^n(\mathfrak{g}_{r,s}, M) \to C_b^{n+1}(\mathfrak{g}_{r,s}, M)$ be the map

$$\begin{aligned}\delta_n(f)(x_1,..,x_{n+1}) &= \sum_{1\leq i\leq j\leq n+1}(-1)^{i+j} f([x_i,x_j],x_1,..,\hat{x_i},..,\hat{x_j},..,x_{n+1}) \\ &\quad + \sum_{1\leq i\leq n+1}(-1)^{i+1} x_i f(x_1,..,\hat{x_i},..,x_{n+1}).\end{aligned}$$

Note that $\|\delta_n\| \leq 1$ and $\delta_{n+1}\circ\delta_n = 0$ for all $n \geq 0$. We define $H_b^n(\mathfrak{g}_{r,s}, M)$ to be the seminormed space $\mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n-1})$, the *nth bounded Lie algebra cohomology of* $\mathfrak{g}_{r,s}$ *with coefficients in* M. We will say that $C_b^\bullet(\mathfrak{g}_{r,s}, M)$ is *strictly exact* at $C_b^n(\mathfrak{g}_{r,s}, M)$ if $H_b^n(\mathfrak{g}_{r,s}, M) = 0$ and the induced map

$$C_b^{n-1}(\mathfrak{g}_{r,s}, M)/\mathrm{Ker}(\delta_n) \to \mathrm{Im}(\delta_{n-1}) = \mathrm{Ker}(\delta_n)$$

is an isomorphism.

Definition 3.3.48. Let M be as above. Then we say that a strict epimorphism $p : \tilde{\mathfrak{g}} \to \mathfrak{g}_{r,s}$ of Banach Lie algebras is an extension with kernel M if $\mathrm{Ker}(p) \cong M$ as $\mathfrak{g}_{r,s}$ modules, where $\mathrm{Ker}(p) \subset \tilde{\mathfrak{g}}$ has the adjoint action. This extension is split if there exists a morphism of Banach Lie algebras $s : \mathfrak{g}_{r,s} \to \tilde{\mathfrak{g}}$ with $p \circ s = \mathrm{Id}$.

Lemma 3.3.49. *If* $H_b^2(\mathfrak{g}_{r,s}, M) = 0$ *then any extension* $p : \tilde{\mathfrak{g}} \to \mathfrak{g}_{r,s}$ *with kernel* M *and* $\|p\| \leq 1$ *which is already split in* Ban_k *is split as an extension of Lie algebras. Moreover, if* $C_b^\bullet(\mathfrak{g}_{r,s}, M)$ *is strictly exact at* $C_b^2(\mathfrak{g}_{r,s}, M)$*, the splitting* s *has norm* $\|s\| \leq C$ *where* C *is the norm of the isomorphism* $Ker(\delta_2) = Im(\delta_1) \xrightarrow{\sim} C_b^1(\mathfrak{g}_{r,s}, M)/Ker(\delta_1)$.

Proof. This proceeds as in Proposition XVIII.1.2. in [23]. Fix a splitting $\tilde{\mathfrak{g}} \cong \mathfrak{g}_{r,s} \oplus M$ as Banach spaces, and let

$$f(x, y) = p([(x, 0), (y, 0)]) \text{ for } x, y \in \mathfrak{g}_{r,s},$$

so that $f \in C_b^2(\mathfrak{g}_{r,s}, M)$ and $\|f\| \leq 1$. Then it is easily checked that $f \in \mathrm{Ker}(\delta_2)$

and hence $f = \delta_1(\alpha)$ for some $\alpha \in C^1_b(\mathfrak{g}_{r,s}, M)$. If $C^\bullet_b(\mathfrak{g}_{r,s}, M)$ is strictly exact at $C^2_b(\mathfrak{g}_{r,s}, M)$ then $\|\alpha\| \le C\|f\| \le C$. Then, as in *loc. cit.*, $s(x) = (x, -\alpha(x))$ gives a splitting, and $\|s\| \le \max\{C, 1\}$. Note that, as the inverse $C^1_b(\mathfrak{g}_{r,s}, M)/\mathrm{Ker}(\delta_1) \xrightarrow{\sim} \mathrm{Ker}(\delta_2)$ is contracting, we automatically have $C \ge 1$. □

Definition 3.3.50. Let M be a Banach $\mathfrak{g}_{r,s}$ bimodule. Then we denote by $\overline{M}$ the $\mathfrak{g}_{r,s}$ module whose underlying Banach space is M with action $x \cdot m := xm - mx$.

Lemma 3.3.51. *Let M be a Banach $\mathfrak{g}_{r,s}$ bimodule with $\|xm\| \le \|x\|\|m\|$ and $\|mx\| \le \|x\|\|m\|$ for all $x \in \mathfrak{g}_{r,s}$, $m \in M$. Let $f : U(\mathfrak{g})^{an}_{r,s}\hat{\otimes}U(\mathfrak{g})^{an}_{r,s} \to M$ be a bounded linear map such that*

$$f(1, x) = f(x, 1) = 0, \quad xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z = 0,$$

for all $x, y, z \in U(\mathfrak{g})^{an}_{r,s}$, and $\|f\| \le 1$. Suppose that $H^2_b(\mathfrak{g}_{r,s}, \overline{M}) = 0$. Then there is a bounded bilinear map $\alpha : U(\mathfrak{g})^{an}_{r,s} \to M$ such that

$$\alpha(1) = 0 \text{ and } f(x, y) = x\alpha(y) - \alpha(xy) + \alpha(x)y \text{ for all } x, y \in U(\mathfrak{g})^{an}_{r,s}.$$

Furthermore, if we assume that $C^\bullet_b(\mathfrak{g}_{r,s}, \overline{M})$ is strictly exact at $C^2_b(\mathfrak{g}_{r,s}, \overline{M})$, and the isomorphism Ker(δ_2) = Im$(\delta_2) \xrightarrow{\sim} C^1_b(\mathfrak{g}_{r,s}, \overline{M})/Ker(\delta_1)$ is contracting, then we can take α such that $\|\alpha\| \le 1$.

Proof. We proceed as in Proposition XVIII.1.3. in [23]. We may define a contracting multiplication on $U(\mathfrak{g})^{\mathrm{an}}_{r,s} \oplus M$ by

$$(x, m) \cdot (y, n) = (xy, xn + my + f(x, y)) \text{ for } x, y \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}, m, n \in M.$$

Under the commutator bracket, $\tilde{\mathfrak{g}} = \mathfrak{g}_{r,s} \oplus M \subset U(\mathfrak{g})^{\mathrm{an}}_{r,s} \oplus M$ is a Banach Lie algebra, and an extension of $\mathfrak{g}_{r,s}$ with kernel $\overline{M}$. Thus by Lemma 3.3.49, there exists $s : \mathfrak{g}_{r,s} \to$

$\tilde{\mathfrak{g}}$ splitting the projection $p : \tilde{\mathfrak{g}} \to \mathfrak{g}_{r,s}$, with $\|s\| \leq 1$ under the stronger assumption. Then the map $\mathfrak{g}_{r,s} \xrightarrow{s} \tilde{\mathfrak{g}} \to U(\mathfrak{g})^{\text{an}}_{r,s} \oplus M$ induces a unique algebra homomorphism $s' : U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s} \oplus M$ which splits the first projection. This map must be of the form $s'(x) = (x, -\alpha(x))$ for some $\alpha : U(\mathfrak{g})^{\text{an}}_{r,s} \to M$. But then, for all $x, y \in U(\mathfrak{g})^{\text{an}}_{r,s}$,

$$(xy, -\alpha(xy)) = s'(xy) = s'(x)s'(y) = (xy, -x\alpha(y) - \alpha(x)y + f(x,y))$$

which completes the proof. □

Theorem 3.3.52. *Let $\mathfrak{g}_{r,s}$ and $\mathfrak{g}'_{r',s'}$ be Banach Lie algebras, each coming from some root datum. Let $1 > \varepsilon > 0$. Suppose we have two morphisms of $k\{\frac{\hbar}{\epsilon}\}$ algebras $\alpha, \alpha' : U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\} \to U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon}\}$ such that $\alpha \equiv \alpha'$ modulo $\hbar$ and $\|\alpha\| \leq 1$, $\|\alpha'\| \leq 1$. Suppose that $C^\bullet_b(\mathfrak{g}_{r,s}, U(\mathfrak{g}')^{an}_{r',s'})$ is strictly exact at $C^1_b(\mathfrak{g}_{r,s}, U(\mathfrak{g}')^{an}_{r',s'})$ and the isomorphism*

$$Ker(\delta_1) = Im(\delta_0) \xrightarrow{\sim} U(\mathfrak{g}')^{an}_{r',s'}/Ker(\delta_0)$$

is contracting. Then, for any $\varepsilon > \varepsilon' > 0$, there exists an invertible $F \in U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon'}\}$ such that $\alpha'(x) = F\alpha(x)F^{-1}$ for all $x \in U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$ where we now view α and α' as maps from $U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon'}\} = U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}k\{\frac{\hbar}{\varepsilon'}\}$ to $U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon'}\} = U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon}\}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}k\{\frac{\hbar}{\varepsilon'}\}$.

Proof. We proceed as in Theorem XVIII.2.1 of [23]. Fix $1 > \varepsilon > \varepsilon' > 0$ and a sequence $(\varepsilon_n)_{n\geq 0}$ with

$$\varepsilon > \varepsilon_1 > \varepsilon_2 > ... > \varepsilon_n > \varepsilon_{n+1} > ... > \varepsilon'.$$

Since α is $k\{\frac{\hbar}{\varepsilon}\}$-linear, it is uniquely determined by its restriction to $U(\mathfrak{g})^{\text{an}}_{r,s}$. We may write α in the form

$$\alpha(x) = \sum \alpha_i(x)\hbar^i \text{ for } x \in U(\mathfrak{g})^{\text{an}}_{r,s}$$

for $\alpha_i : U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g}')^{\text{an}}_{r',s'}$, $\|\alpha_i\|\varepsilon^i \leq 1$. Now, suppose we have $u_0, u_1, ..., u_n \in$

$U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ such that $U_i\alpha \equiv \alpha_0 U_i$ modulo $\hbar^{i+1}$, where

$$U_i = (1 + u_i\hbar^i)(1 + u_{i-1}\hbar^{i-1})...(1 + u_0),$$

and $\|u_i\|\varepsilon_i^i < 1$. Note that $(1 + u_i\hbar^i)$ is then invertible in $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon_n}\}$ for each $i = 0, .., n$, with inverse $\sum_{j\geq 0}(-u_i\hbar^i)^j$, and hence so is U_i, in which case

$$\alpha^{(i)}(x) := U_i\alpha(x)U_i^{-1} \equiv \alpha_0(x) \text{ modulo } \hbar^{i+1}$$

for all $x \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}$. Note also that

$$\|1 + u_i\hbar^i\| = \|(1 + u_i\hbar^i)^{-1}\| = \|U_i\| = \|U_i^{-1}\| = 1,$$

and so $\|\alpha^{(n)}\| \leq \|\alpha\|$ as maps $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon_n}\} \to U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon_n}\}$. Again, $\alpha^{(n)}$ is $k\{\frac{\hbar}{\varepsilon_n}\}$-linear, and we may write the restricted map $\alpha^{(n)} : U(\mathfrak{g})^{\mathrm{an}}_{r,s} \to U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon_n}\}$ as

$$\alpha^{(n)}(x) = \sum \alpha_i^{(n)}(x)\hbar^i \text{ for } x \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}$$

for $\alpha_i^{(n)} : U(\mathfrak{g})^{\mathrm{an}}_{r,s} \to U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$, $\|\alpha_i^{(n)}\|\varepsilon_n^i \leq \|\alpha^{(n)}\| \leq \|\alpha\| \leq 1$. By assumption, $\alpha_0^{(n)} = \alpha_0$ and $\alpha_i^{(n)} = 0$ for $i = 1, ..., n$. Looking at the $\hbar^{n+1}$ coefficient of $\alpha^{(n)}(xy) = \alpha^{(n)}(x)\alpha^{(n)}(y)$ we see that

$$\begin{aligned}\alpha_{n+1}^{(n)}(xy) &= \alpha_0^{(n)}(x)\alpha_{n+1}^{(n)}(y) + \alpha_{n+1}^{(n)}(x)\alpha_0^{(n)}(y) \\ &= \alpha_0(x)\alpha_{n+1}^{(n)}(y) + \alpha_{n+1}^{(n)}(x)\alpha_0(y)\end{aligned}$$

for all $x, y \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}$. Thus

$$\alpha_{n+1}^{(n)}([x, y]) = [\alpha_0(x), \alpha_{n+1}^{(n)}(y)] - [\alpha_0(y), \alpha_{n+1}^{(n)}(x)]$$

for all $x, y \in \mathfrak{g}_{r,s}$. Given that $x \in \mathfrak{g}_{r,s}$ acts on $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ via $[\alpha_0(x), -]$, this is precisely the fact that $\alpha_{n+1}^{(n)}$ restricted to $\mathfrak{g}_{r,s}$ is in $\mathrm{Ker}(\delta_1)$. Hence there is a $u_{n+1} \in U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ such that $\alpha_{n+1}^{(n)}(x) = [\alpha_0(x), u_{n+1}]$ for all $x \in \mathfrak{g}_{r,s}$ and $\|u_{n+1}\| \leq \|\alpha_{n+1}^{(n)}\|$, so that $\|u_{n+1}\|\varepsilon_n^{n+1} \leq 1$. Then, in $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon_{n+1}}\}$, $\|u_{n+1}\|\varepsilon_{n+1}^{n+1} < 1$ and $(1 + u_{n+1}\hbar^{n+1})$ is invertible with

$$\begin{aligned} \alpha^{(n+1)}(x) &:= (1 + u_{n+1}\hbar^{n+1})\alpha(x)(1 + u_{n+1}\hbar^{n+1})^{-1} \\ &\equiv \alpha_0(x) + (u_{n+1}\alpha_0(x) - \alpha_0(x)u_{n+1} + \alpha_{n+1}^{(n)}(x))\hbar^{n+1} \quad \mod \hbar^{n+2} \\ &\equiv \alpha_0(x) \quad \mod \hbar^{n+2}. \end{aligned}$$

Taking $u_0 := 0$ as our base case, we obtain inductively sequences $(u_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ in $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon'}\}$. The sequence $(U_n)_{n\geq 0}$ converges to $U := 1 + \sum_{n=1}^{\infty} v_n \hbar^n$ where $v_n = \sum u_{i_1}u_{i_2}...u_{i_k}$ whose sum is taken over all finite sequences $i_1 > i_2 > ... > i_k$ with $i_1 + i_2 + ... + i_k = n$. Since $\|v_n\|(\varepsilon')^n < 1$ for all $n \geq 0$, U is invertible in $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon'}\}$ with inverse $U^{-1} = \sum_{i=0}^{\infty}(-\sum_{n=1}^{\infty} v_n\hbar^n)^i$. It follows from the fact that

$$U_n\alpha(x)U_n^{-1} \equiv \alpha_0(x) \text{ modulo } \hbar^{n+1}$$

for each $n \geq 0$ that $U\alpha(x)U^{-1} = \alpha_0(x)$ for each $x \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$. Similarly, there exist mutual inverses U', U'^{-1} in $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}\{\frac{\hbar}{\varepsilon'}\}$ such that $U'\alpha'(x)U'^{-1} = \alpha_0'(x) = \alpha_0(x)$ for each $x \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$. Thus taking $F = U'^{-1}U$ gives our result. ☐

Theorem 3.3.53. *Let $\varepsilon > 0$. Suppose A is a Banach $k\{\frac{\hbar}{\varepsilon}\}$ algebra with contracting multiplication such that there is a bounded $k\{\frac{\hbar}{\varepsilon}\}$-linear isomorphism of Banach spaces $A \cong U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\}$ that preserves the unit, and that $A/\hbar A \cong U(\mathfrak{g})^{an}_{r,s}$ is an isomorphism of Banach algebras. Suppose that $C_b^\bullet(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ is strictly exact at $C_b^2(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ and the isomorphism*

$$Ker(\delta_2) = Im(\delta_1) \xrightarrow{\sim} C_b^1(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})/Ker(\delta_1)$$

is contracting. Then for any $\varepsilon > \varepsilon' > 0$, there is an isomorphism of $k\{\frac{\hbar}{\varepsilon'}\}$ algebras

$$\alpha : A_{\varepsilon'} := A\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}} k\{\tfrac{\hbar}{\varepsilon'}\} \xrightarrow{\sim} U(\mathfrak{g})^{an}_{r,s}\{\tfrac{\hbar}{\varepsilon'}\}$$

inducing the given isomorphism $A_{\varepsilon'}/\hbar A_{\varepsilon'} \cong A/\hbar A \cong U(\mathfrak{g})^{an}_{r,s}$.

Proof. We proceed as in Theorem XVIII.2.2 of [23]. As before, fix $1 > \varepsilon > \varepsilon' > 0$ and a sequence $(\varepsilon_n)_{n\geq 0}$ with

$$\varepsilon > \varepsilon_1 > \varepsilon_2 > ... > \varepsilon_n > \varepsilon_{n+1} > ... > \varepsilon'.$$

The given isomorphism $A \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon}\}$ induces a multiplication

$$\mu : (U(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\mathrm{an}}_{r,s})\{\tfrac{\hbar}{\varepsilon}\} \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon}\}\hat{\otimes}U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon}\} \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon}\}$$

which is $k\{\frac{\hbar}{\varepsilon}\}$-linear and hence is determined by its restriction to $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\mathrm{an}}_{r,s}$. We may write this restriction as

$$\mu = \sum \mu_i\hbar^i, \quad \mu_i : U(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\mathrm{an}}_{r,s} \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}, \quad \|\mu_i\|\varepsilon^i \leq 1.$$

Suppose we have maps $\alpha_0, \alpha_1, ..., \alpha_n : U(\mathfrak{g})^{\mathrm{an}}_{r,s} \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}$ such that $\|\alpha_i\|\varepsilon_i^i < 1$ and $V_i(\mu(x,y)) \equiv \mu_0(V_i(x), V_i(y))$ modulo $\hbar^{i+1}$ for all $x, y \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}$, where V_i is the endomorphism of $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon}\}$ with

$$V_i|_{U(\mathfrak{g})^{\mathrm{an}}_{r,s}} = (\mathrm{Id} + \hbar^i\alpha_i)(\mathrm{Id} + \hbar^{i-1}\alpha_{i-1})...(\mathrm{Id} + \alpha_0).$$

Note that, since $\|\alpha_i\|\varepsilon_n^i < \|\alpha_i\|\varepsilon_i^i \leq 1$, the endomorphism of $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon_n}\}$ whose restriction to $U(\mathfrak{g})^{\mathrm{an}}_{r,s}$ is $(\mathrm{Id} + \hbar^i\alpha_i)$ is then invertible for each $i = 0, .., n$, with inverse

$\sum_{j\geq 0}(-\hbar^i\alpha_i)^j$, and hence so is V_i as an endomorphism of $U(\mathfrak{g})^{\text{an}}_{r,s}\{\frac{\hbar}{\varepsilon_n}\}$, in which case

$$\mu^{(i)}(x,y) := V_i(\mu(V_i^{-1}(x), V_i^{-1}(y))) \equiv \mu_0(x) \text{ modulo } \hbar^{i+1}$$

for all $x, y \in U(\mathfrak{g})^{\text{an}}_{r,s}$. Note also that

$$\|1+\hbar^i\alpha_i\| = \|(1+\hbar^i\alpha_i)^{-1}\| = \|V_i\| = \|V_i^{-1}\| = 1,$$

and so $\|\mu^{(n)}\| \leq \|\mu\|$ as multiplication maps on $U(\mathfrak{g})^{\text{an}}_{r,s}\{\frac{\hbar}{\varepsilon_n}\}$. Again, $\mu^{(n)}$ is $k\{\frac{\hbar}{\varepsilon_n}\}$-linear, and we may write

$$\mu^{(n)}(x,y) = \sum \mu_i^{(n)}(x,y)\hbar^i \text{ for } x, y \in U(\mathfrak{g})^{\text{an}}_{r,s}$$

where $\mu_i^{(n)} : U(\mathfrak{g})^{\text{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s}$, $\|\mu_i^{(n)}\|\varepsilon_n^i \leq \|\mu^{(n)}\| \leq \|\mu\| \leq 1$. By assumption, $\mu_0^{(n)} = \mu_0$ and $\mu_i^{(n)} = 0$ for $i = 1, ..., n$. Looking at the $\hbar^{n+1}$ coefficient of $\mu^{(n)}(\mu^{(n)}(x,y),z) = \mu^{(n)}(x,\mu^{(n)}(y,z))$ we see that

$$\mu_{n+1}^{(n)}(xy,z) + \mu_{n+1}^{(n)}(x,y)z = \mu_{n+1}^{(n)}(x,yz) + x\mu_{n+1}^{(n)}(y,z)$$

for all $x, y, z \in U(\mathfrak{g})^{\text{an}}_{r,s}$. Here, we are using the simplified notation $xy := \mu_0(x,y)$. So, by Lemma 3.3.51, there exists $\alpha_{n+1} : U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s}$ such that $\|\alpha_{n+1}\| \leq \|\mu_{n+1}^{(n)}\|$, $\alpha_{n+1}(1) = 0$ and

$$\mu_{n+1}^{(n)}(x,y) = x\alpha_{n+1}(y) - \alpha_{n+1}(xy) + \alpha_{n+1}(x)y.$$

Setting V_{n+1} as the endomorphism of $U(\mathfrak{g})^{\text{an}}_{r,s}\{\frac{\hbar}{\varepsilon_{n+1}}\}$ whose restriction to $U(\mathfrak{g})^{\text{an}}_{r,s}$ is $(\text{Id} + \hbar^{n+1}\alpha_{n+1})V_n$, we have that V_{n+1} is invertible since $\|\alpha_{n+1}\|\varepsilon_{n+1}^{n+1} < 1$. Let

$$\mu^{(n+1)}(x,y) := V_{n+1}(\mu(V_{n+1}^{-1}(x), V_{n+1}^{-1}(y))).$$

Then, modulo $\hbar^{n+2}$,

$$\begin{aligned}\mu^{(n+1)}(x,y) &\equiv (\mathrm{Id}+\alpha_{n+1}\hbar^{n+1})\circ(\mu_0+\mu^{(n)}_{n+1}\hbar^{n+1})\\ &\qquad (x-\alpha_{n+1}(x)\hbar^{n+1}, y-\alpha_{n+1}(y)\hbar^{n+1})\\ &\equiv xy+(\alpha_{n+1}(xy)+\mu^{(n)}_{n+1}(x,y)-\alpha_{n+1}(x)y-x\alpha_{n+1}(y))\hbar^{n+1}\\ &\equiv xy.\end{aligned}$$

Taking $\alpha_0 = 0$ as a base case, we inductively obtain sequences $(\alpha_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$. The sequence $(V_n)_{n\geq 0}$ converges on $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$ to $V := \mathrm{Id} + \sum_{n=1}^{\infty}\hbar^n\beta_n$ where $\beta_n = \sum \alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}$ whose sum is taken over all finite sequences $i_1 > i_2 > ... > i_k$ with $i_1 + i_2 + ... + i_k = n$. Since $\|\alpha_n\|(\varepsilon')^n < 1$, V is invertible on $U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$ with inverse

$$V^{-1} = \sum_{i=0}^{\infty}(-\sum_{n=1}^{\infty}\hbar^n\beta_n)^i.$$

It follows from the fact that $V_n\mu(V_n^{-1}(x), V_n^{-1}(y)) \equiv \mu_0(x,y)$ modulo $\hbar^{n+1}$ for each $n \geq 0$ that $V\mu(V^{-1}(x), V^{-1}(y)) = \mu_0(x,y)$ for each $x, y \in U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$. It then follows that

$$A_{\varepsilon'} \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon'}\} \xrightarrow{V} U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon'}\}$$

is our desired isomorphism of algebras. $\square$

The following are slight variations of the above theorems.

Corollary 3.3.54. *Let $\mathfrak{g}_{r,s}$ and $\mathfrak{g}'_{r',s'}$ be Banach Lie algebras, each coming from some root datum. Let $\varepsilon \geq 0$. Suppose we have two morphisms of $k\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ algebras $\alpha, \alpha' : U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\}^{\dagger} \rightrightarrows U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ such that $\alpha \equiv \alpha'$ modulo $\hbar$. Suppose that $C^{\bullet}_b(\mathfrak{g}_{r,s}, U(\mathfrak{g}')^{an}_{r',s'})$ is strictly exact at $C^1_b(\mathfrak{g}_{r,s}, U(\mathfrak{g}')^{an}_{r',s'})$ and the isomorphism*

$$Ker(\delta_1) = Im(\delta_0) \xrightarrow{\sim} U(\mathfrak{g}')^{an}_{r',s'}/Ker(\delta_0)$$

is contracting. Then there exists a convolution invertible generalised element $F : k \to$

$U(\mathfrak{g}')^{an}_{r',s'}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ *such that* $\alpha' = F * \alpha * F^{-1}$.

Proof. The morphisms α and α' are $k\{\frac{\hbar}{\varepsilon}\}^{\dagger}$-linear, so are determined by their restrictions $U(\mathfrak{g})^{\text{an}}_{r,s} \rightrightarrows U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$. Since $U(\mathfrak{g})^{\text{an}}_{r,s}$ is Banach, there is $\varepsilon' > \varepsilon$ such that the restrictions of α and α' are determined by morphisms of Banach algebras $U(\mathfrak{g})^{\text{an}}_{r,s} \rightrightarrows U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon'}\}$. By the proof of Theorem 3.3.52 there is $\varepsilon' > \varepsilon'' > \varepsilon$ and an invertible element $F \in U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon''}\}$ such that $\alpha'(x) = F\alpha(x)F^{-1} \in U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon''}\}$ for all $x \in U(\mathfrak{g})^{\text{an}}_{r,s}$. It then follows that $\alpha = F * \alpha' * F^{-1}$ as maps $U(\mathfrak{g})^{\text{an}}_{r,s}\{\frac{\hbar}{\varepsilon}\}^{\dagger} \rightrightarrows U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$, where we denote by F the generalised element $k \xrightarrow{1 \mapsto F} U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon''}\} \to U(\mathfrak{g}')^{\text{an}}_{r',s'}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$. □

Corollary 3.3.55. *Let* $\varepsilon \geq 0$. *Suppose we have a Banach* $k\{\frac{\hbar}{\varepsilon'}\}$*-algebra* $A_{\varepsilon'}$, *for some* $\varepsilon' > \varepsilon$, *equipped with a* $k\{\frac{\hbar}{\varepsilon'}\}$*-linear isomorphism* $A_{\varepsilon'} \cong U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon'}\}$ *that preserves the unit such that* $A_{\varepsilon'}/\hbar A_{\varepsilon'} \cong U(\mathfrak{g})^{an}_{r,s}$ *is an isomorphism of Banach algebras. Suppose further that* $C^{\bullet}_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ *is strictly exact at* $C^2_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ *and the isomorphism*

$$Ker(\delta_2) = Im(\delta_1) \xrightarrow{\sim} C^1_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})/Ker(\delta_1)$$

is contracting. Then there is an isomorphism of $k\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ *algebras*

$$\alpha : A \xrightarrow{\sim} U(\mathfrak{g})^{an}_{r,s}\{\tfrac{\hbar}{\varepsilon}\}^{\dagger},$$

where $A = k\{\frac{\hbar}{\varepsilon}\}^{\dagger}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon'}\}}A_{\varepsilon'}$, *inducing the given isomorphism* $A/\hbar A \cong A_{\varepsilon'}/\hbar A_{\varepsilon'} \cong U(\mathfrak{g})^{an}_{r,s}$.

Proof. By Theorem 3.3.53 there is $\varepsilon' > \varepsilon'' > \varepsilon$ and an isomorphism of $k\{\frac{\hbar}{\varepsilon''}\}$-algebras

$$\alpha_{\varepsilon''} : A_{\varepsilon''} \xrightarrow{\sim} U(\mathfrak{g})^{\text{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon''}\},$$

where $A_{\varepsilon''} = k\{\frac{\hbar}{\varepsilon''}\}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon'}\}}A_{\varepsilon'}$, inducing the given isomorphism $A_{\varepsilon''}/\hbar A_{\varepsilon''} \cong A_{\varepsilon'}/\hbar A_{\varepsilon'} \cong$

$U(\mathfrak{g})^{\mathrm{an}}_{r,s}$. Then α is the composition

$$A \cong k\{\tfrac{\hbar}{\varepsilon}\}^{\dagger}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon''}\}}A_{\varepsilon''} \xrightarrow{\mathrm{Id}\otimes\alpha_{\varepsilon''}} k\{\tfrac{\hbar}{\varepsilon}\}^{\dagger}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon''}\}}U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon''}\} \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}\{\tfrac{\hbar}{\varepsilon}\}^{\dagger}.$$

□

Remark If we let $k = \mathbb{C}$ with the trivial valuation then $k\{\frac{\hbar}{\varepsilon}\} = k\{\frac{\hbar}{\varepsilon}\}^{\dagger} = \mathbb{C}[\![\hbar]\!]$ and we recover the classical rigidity results as stated in Section XVIII.2 of [23].

We may construct analytic quantum groups over $k\{\frac{\hbar}{\varepsilon}\}$ and $k\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ for a formal parameter $\hbar$, where $q = e^{\hbar}$, to which the above deformation theory applies. For the remainder of this section, assume that $\varepsilon > 0$ is sufficiently small such that $\exp(\hbar)$ converges in $k\{\frac{\hbar}{\varepsilon}\}$.

Definition 3.3.56. Let $\mathscr{H}_{\frac{\hbar}{\varepsilon}} := \coprod^{\leq 1}_{\alpha\in\mathbb{Z}I} k\{\frac{\hbar}{\varepsilon}\}\cdot H_{\alpha}$ be the Banach Hopf algebra over $k\{\frac{\hbar}{\varepsilon}\}$ generated by H_i for $i \in I$, where

$$\Delta_{\mathscr{H}}(H_i) = 1 \otimes H_i + H_i \otimes 1.$$

Let $V_{\frac{\hbar}{\varepsilon}} := \coprod^{\leq 1}_{i\in I} k\{\frac{\hbar}{\varepsilon}\}\cdot v_i$ and define

$$c : V_{\frac{\hbar}{\varepsilon}}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}V_{\frac{\hbar}{\varepsilon}} \to V_{\frac{\hbar}{\varepsilon}}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}V_{\frac{\hbar}{\varepsilon}}, \quad v_i \otimes v_j \mapsto q^{\lambda_i(\alpha_j)}v_j \otimes v_i,$$

where $q = e^{\hbar}$. Let $T_r(V_{\frac{\hbar}{\varepsilon}})$ be the resulting braided analytically graded Hopf algebra on $\coprod^{\leq 1}_{n\geq 0}(V_{\frac{\hbar}{\varepsilon}}^{\hat{\otimes}n})_{r^n}$ defined using the tensor product $\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}$ over $k\{\frac{\hbar}{\varepsilon}\}$. We define a bilinear form $\langle -,-\rangle : V_{\frac{\hbar}{\varepsilon}}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}V_{\frac{\hbar}{\varepsilon}} \to k\{\frac{\hbar}{\varepsilon}\}$ by

$$\langle v_i, v_j\rangle = \delta_{i,j}\frac{\hbar}{(q_i - q_i^{-1})}$$

where $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$. By Lemma 3.3.9 this extends to a bilinear form on $T_r(V_{\frac{\hbar}{\varepsilon}})$. Let $\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}})$ be the quotient of $T_r(V_{\frac{\hbar}{\varepsilon}})$ by the radical of this bilinear form, which again is a braided analytically graded Banach Hopf algebra.

Remark Note that

$$q_i - q_i^{-1} = (\alpha_i, \alpha_i)\hbar \left(\sum_{k=0}^{\infty} \frac{(\frac{(\alpha_i,\alpha_i)}{2}\hbar)^{2k}}{k!} \right)$$

is not invertible in $k\{\frac{\hbar}{\varepsilon}\}$, but $\frac{1}{\hbar}(q_i - q_i^{-1})$ is. Thus we have had to rescale the inner product from Definition 3.4.12 in order to define it over $k\{\frac{\hbar}{\varepsilon}\}$.

Theorem 3.3.57. *Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$. Then there is a Banach algebra structure on*

$$U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s} := \mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}}) \hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}} \mathscr{H}_{\frac{\hbar}{\varepsilon}} \hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}} \mathfrak{B}_s(V_{\frac{\hbar}{\varepsilon}})$$

such that $\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}})$, $\mathscr{H}_{\frac{\hbar}{\varepsilon}}$ and $\mathfrak{B}_s(V_{\frac{\hbar}{\varepsilon}})$ are all subalgebras, and

$$[H_i, E_j] = (\alpha_i, \alpha_j)E_j, \quad [H_i, F_j] = -(\alpha_i, \alpha_j)F_j, \quad [E_i, F_j] = \delta_{i,j}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},$$

where $F_i = v_i \otimes 1 \otimes 1$, $E_i = 1 \otimes 1 \otimes v_i$ and $t_i = exp(\frac{(\alpha_i,\alpha_i)}{2}\hbar H_i)$. This becomes a Banah Hopf algebra with $\mathscr{H}_{\frac{\hbar}{\varepsilon}}$ as a sub-Hopf algebra and

$$\begin{aligned} \Delta(E_i) &= E_i \otimes t_i + 1 \otimes E_i, & S(E_i) &= -E_i t_i^{-1}, \\ \Delta(F_i) &= F_i \otimes 1 + t_i^{-1} \otimes F_i, & S(F_i) &= -t_i F_i. \end{aligned}$$

Proof. The fact that this is a Hopf algebra is checked on a dense subspace in Proposition 7 of Section 6.1.3 of [25]. We must check that it extends continuously to $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s}$.

By construction

$$U_{\frac{\hbar}{\varepsilon}}^{\leq 1}(\mathfrak{g})_{r,s}^{\mathrm{an}} := \mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}}) \rtimes \mathscr{H}_{\frac{\hbar}{\varepsilon}} \quad \text{and} \quad U_{\frac{\hbar}{\varepsilon}}^{\geq 1}(\mathfrak{g})_{r,s}^{\mathrm{an}} := \mathscr{H}_{\frac{\hbar}{\varepsilon}} \ltimes \mathfrak{B}_s(V_{\frac{\hbar}{\varepsilon}})$$

sit as sub-Hopf algebras in $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})_{r,s}^{\mathrm{an}}$, it is enough to check that the restriction of the multiplication map

$$\mathfrak{B}_s(V_{\frac{\hbar}{\varepsilon}})\hat{\otimes}\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}}) \to U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})_{r,s}^{\mathrm{an}}, \quad E_i \otimes F_j \mapsto F_i E_j + \delta_{i,j}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},$$

is continuous. This follows from the assumption that

$$\left\| \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \right\| = |q_i - q_i^{-1}|^{-1} \leq rs.$$

□

Theorem 3.3.58. *Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$. Then there is an isomorphism of $k\{\frac{\hbar}{\varepsilon}\}$-modules $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})_{r,s}^{an} \cong U(\mathfrak{g})_{r,s}^{an}\{\frac{\hbar}{\varepsilon}\}$ that descends to an isomorphism of Banach Hopf algebras $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})_{r,s}^{an}/\hbar U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})_{r,s}^{an} \cong U(\mathfrak{g})_{r,s}^{an}$.*

Proof. By Theorem 33.1.3 of [27] and Proposition 3.3.14, $\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}})$ is the quotient of $T_r(V_{\frac{\hbar}{\varepsilon}})$ by the closed homogeneous ideal generated by the quantum Serre relations

$$\sum_{k=0}^{1-(\alpha_i,\alpha_j)} (-1)^k \frac{[1-(\alpha_i,\alpha_j)]_q}{[k]_q[1-(\alpha_i,\alpha_j)-k]_q} v_i^{1-(\alpha_i,\alpha_j)-k} v_j v_i^k = 0$$

for $i \neq j$. Since $V_{\frac{\hbar}{\varepsilon}} \to V_0\{\frac{\hbar}{\varepsilon}\}$, $v_i \mapsto v_i$, is an isomorphism there is an isomorphism of $k\{\frac{\hbar}{\varepsilon}\}$-modules $T_r(V_{\frac{\hbar}{\varepsilon}}) \cong T_r(V_0)\{\frac{\hbar}{\varepsilon}\}$. By the Poincaré-Birkhoff-Witt Theorem this descends to isomorphisms between the graded pieces $\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}})(n) \cong U^-(\mathfrak{g})_r^{\mathrm{an}}(n)\{\frac{\hbar}{\varepsilon}\}$, hence $\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}}) \cong U^-(\mathfrak{g})_r^{\mathrm{an}}\{\frac{\hbar}{\varepsilon}\}$. Likewise $\mathscr{H}_{\frac{\hbar}{\varepsilon}} \cong \mathscr{H}_0\{\frac{\hbar}{\varepsilon}\}$. So, as Banach spaces,

$$\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}})/\hbar\mathfrak{B}_r(V_{\frac{\hbar}{\varepsilon}}) \cong U^-(\mathfrak{g})_r^{\mathrm{an}} \quad \text{and} \quad \mathscr{H}_{\frac{\hbar}{\varepsilon}}/\hbar\mathscr{H}_{\frac{\hbar}{\varepsilon}} \cong \mathscr{H}_0,$$

and so $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s}/\hbar U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s} \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}$. By Remark 4 of Section 6.1.3 this restricts to a Hopf algebra isomorphism on a dense subspace, hence is a Hopf algebra isomorphism by continuity. □

Definition 3.3.59. Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$ and let $\varepsilon > 0$. Then for any $\varepsilon' > \varepsilon$ we define

$$U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s} := U_{\frac{\hbar}{\varepsilon'}}(\mathfrak{g})^{\mathrm{an}}_{r,s} \hat{\otimes}_{k\{\frac{\hbar}{\varepsilon'}\}} k\{\hbar/\varepsilon\}^{\dagger}.$$

Corollary 3.3.60. *Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$. Then there is an isomorphism of $k\{\frac{\hbar}{\varepsilon}\}^{\dagger}$-modules $U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$ that descends to an isomorphism of Banach Hopf algebras $U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s}/\hbar U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}$.*

Proof. This follows from Theorem 3.3.58. □

Corollary 3.3.61. *Suppose that the complex $C^{\bullet}_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ is strictly exact at both $C^1_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$ and $C^2_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})$, and that the isomorphisms*

$$Ker(\delta_1) = Im(\delta_0) \xrightarrow{\sim} U(\mathfrak{g})^{an}_{r,s}/Ker(\delta_0)$$

and

$$Ker(\delta_2) = Im(\delta_1) \xrightarrow{\sim} C^1_b(\mathfrak{g}_{r,s}, U(\mathfrak{g})^{an}_{r,s})/Ker(\delta_1)$$

are contracting. Then there is an isomorphism of algebras

$$U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s} \xrightarrow{\sim} U(\mathfrak{g})^{an}_{r,s}\{\tfrac{\hbar}{\varepsilon}\}^{\dagger}$$

that induces the isomorphism $U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s}/\hbar U^{\dagger}_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}$ of Corollary 3.3.60. Furthermore, this isomorphism is unique up to conjugation by a convolution invertible generalised element of $U(\mathfrak{g})^{an}_{r,s}\{\frac{\hbar}{\varepsilon}\}^{\dagger}$.

Proof. This follows from Corollary 3.3.54, Corollary 3.3.55 and Corollary 3.3.60. □

If $\mathfrak{g}$ is finite dimensional then, as vector spaces, bounded Lie algebra cohomology $H_b^n(\mathfrak{g}_{r,s}, M)$ agrees with the unbounded Lie algebra cohomology $H^n(\mathfrak{g}, M)$ as defined in Chapter XVIII of [23]. Furthermore, if we also assume that k is algebraically closed and $\mathfrak{g}$ is semisimple then $H^n(\mathfrak{g}, U(\mathfrak{g})) = 0$ by Corollary XVIII.3.3 of [23] for $n = 1, 2$. Unfortunately, it is unclear whether the same argument can be extended to show that $H_b^n(\mathfrak{g}_{r,s}, U(\mathfrak{g})_{r,s}^{\mathrm{an}})$ vanishes when $n = 1, 2$, or whether we have the strict exactness required for Corollary 3.3.61 to hold. This is a goal of future work by the author. At the end of the following section we show that if we allow ourselves to work over formal powerseries in $\hbar$ then we may weaken these assumptions on bounded cohomology to remove the requirement of strict exactness.

In [23], Kassel uses an algebraic analogue of the rigidity theorems of this Section, alongside another based on results of Drinfel'd from 1989 regarding quasi-Hopf algebra structures on universal enveloping algebra, to present a proof of the Drinfel'd-Khono theorem. This theorem states that the category of representations of the quantum enveloping algebra is equivalent, as a braided monoidal category, to the category of $U(\mathfrak{g})$-modules with associativity constraint given by the Drinfel'd associator and braiding given by the associated R-matrix. As a result of this, the associated braid group representations are equivalent. This can be interpreted as a statement about the monodromy of the Knizhnik-Zamolodchikov (KZ) equations that govern the Drinfel'd associator. In [14], Furusho uses p-adic multiple polylogarithms to construct solutions to the p-adic KZ equations and a p-adic Drinfel'd associator. In the future the author hopes to prove a p-adic analogue of the Drinfel'd-Khono theorem and to investigate this link to Furusho's work.

3.3.6 Analytic quantum groups over $k[\![\hbar]\!]$

Definition 3.3.62. Let $k[\![\hbar]\!]$ denote the IndBanach algebra of powerseries in $\hbar$,

$$k[\![\hbar]\!] := \lim_{n\geq 0} k[\hbar]/(\hbar^n) = \prod_{n\geq 0} k\cdot \hbar^n.$$

For an IndBanach space V let $V[\![\hbar]\!] := \prod_{n\geq 0} V\cdot\hbar^n$, which forms an IndBanach algebra if V is an algebra.

Lemma 3.3.63. *Let $(V(n))_{n\in\mathbb{N}}$ be a countable collection of Banach spaces, and W be a Banach space. Then the natural map*

$$\left(\prod_{n\geq 0} V(n)\right)\hat{\otimes} W \to \prod_{n\geq 0}\left(V(n)\hat{\otimes} W\right)$$

is an isomorphism.

Proof. Fix a summable sequence of positive real numbers $a_n \in \mathbb{R}_{>0}$. By the explicit description of products in Section 1.4.1 in [34], we have that

$$\prod\nolimits_{n\geq 0} V(n) \cong \operatorname{colim}_{r_n>0} \prod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n}.$$

The maps

$$\coprod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n} \to \prod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n}, \quad (v_n)_{n\geq 0}\mapsto (v_n)_{n\geq 0},$$

and

$$\prod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n} \to \coprod\nolimits_{n\geq 0}^{\leq 1} V(n)_{a_n r_n}, \quad (v_n)_{n\geq 0}\mapsto (v_n)_{n\geq 0},$$

induce an isomorphism

$$\operatorname{colim}_{r_n>0} \prod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n} \xrightarrow{\sim} \operatorname{colim}_{r_n>0} \coprod\nolimits_{n\geq 0}^{\leq 1} V(n)_{r_n}.$$

Hence

$$\begin{aligned}\left(\prod_{n\geq 0} V(n)\right)\hat{\otimes} W &\cong \left(\operatorname{colim}_{r_n>0}\prod^{\leq 1}_{n\geq 0} V(n)_{r_n}\right)\hat{\otimes} W\\ &\cong \left(\operatorname{colim}_{r_n>0}\coprod^{\leq 1}_{n\geq 0} V(n)_{r_n}\right)\hat{\otimes} W\\ &\cong \operatorname{colim}_{r_n>0}\coprod^{\leq 1}_{n\geq 0}(V(n)\hat{\otimes} W)_{r_n}\\ &\cong \operatorname{colim}_{r_n>0}\prod^{\leq 1}_{n\geq 0}(V(n)\hat{\otimes} W)_{r_n}\\ &\cong \prod_{n\geq 0}(V(n)\hat{\otimes} W).\end{aligned}$$

□

Corollary 3.3.64. *For a Banach space V, $V[\![\hbar]\!] \cong V\hat{\otimes}_k k[\![\hbar]\!]$. Hence*

$$V[\![\hbar]\!]\hat{\otimes}_{k[\![\hbar]\!]} W[\![\hbar]\!] \cong (V\hat{\otimes} W)[\![\hbar]\!]$$

for all Banach spaces V and W.

Proof. This follows from Lemma 3.3.63. □

Definition 3.3.65. Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$. Then for any sufficiently small $\varepsilon > 0$ we define

$$U_{[\![\hbar]\!]}(\mathfrak{g})^{\mathrm{an}}_{r,s} := U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}} k[\![\hbar]\!]$$

as an IndBanach Hopf algebra over $k[\![\hbar]\!]$.

Quasi-triangularity

The following is essentially a restatement of a well known result of Drinfel'd.

Proposition 3.3.66. *Suppose that $\mathfrak{g}$ is a simple Lie algebra. Then the IndBanach Hopf algebra $U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s}$ is quasi-triangular.*

Proof. Fix $\varepsilon > 0$. By the proof of Lemma 3.3.63 we may write $U_{[\![\hbar]\!]}(\mathfrak{g})^{\mathrm{an}}_{r,s}$ as the colimit

$$U_{[\![\hbar]\!]}(\mathfrak{g})^{\mathrm{an}}_{r,s} = \text{"colim"}_{(\varepsilon_n)}\, U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}\left(\coprod^{\leq 1}_{n\geq 0} k_{\varepsilon_n}\cdot \hbar^n\right)$$

where the colimit is taken over all sequences of positive real numbers $(\varepsilon_n)_{n\geq 0}$ such that $\varepsilon^n\varepsilon_{n'} \geq \varepsilon_{n+n'}$ for all $n, n' \geq 0$. Note that the requirement on $(\varepsilon_n)_{n\geq 0}$ ensures that $\coprod_{n\geq 0}^{\leq 1} k_{\varepsilon_n} \cdot \hbar^n$ is naturally a $k\{\frac{\hbar}{\varepsilon}\}$-module. Theorem 17 of Section 8.3.2 of [25] exhibits a formula for an R-matrix in the $\hbar$-adic qunatum enveloping algebra over $\mathbb{C}$. As in *loc. cit.* this formula naturally converges in $U_{\frac{\hbar}{\varepsilon}}(\mathfrak{g})^{\mathrm{an}}_{r,s}\hat{\otimes}_{k\{\frac{\hbar}{\varepsilon}\}}\left(\coprod_{n\geq 0}^{\leq 1} k_{\varepsilon_n} \cdot \hbar^n\right)$ for some sufficiently small sequence $(\varepsilon_n)_{n\geq 0}$, which gives a generalised element of $U_{[\![\hbar]\!]}(\mathfrak{g})^{\mathrm{an}}_{r,s}$ that makes $U_{[\![\hbar]\!]}(\mathfrak{g})^{\mathrm{an}}_{r,s}$ quasi-triangular. □

Rigidity

Theorem 3.3.67. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Banach Lie algebras. Suppose we have two morphisms of $k[\![\hbar]\!]$ algebras $\alpha, \alpha' : U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!] \to U(\mathfrak{g}')^{an}_{r',s'}[\![\hbar]\!]$ such that $\alpha \equiv \alpha'$ modulo $\hbar$. Suppose that $H^1_b(\mathfrak{g}, U(\mathfrak{g}')^{an}_{r',s'}) = 0$. Then there exists a convolution invertible generalised element $F : k \to U(\mathfrak{g}')^{an}_{r',s'}[\![\hbar]\!]$ such that $\alpha' = F * \alpha * F^{-1}$.*

Proof. This follows as in the proof of Theorem 3.3.52. By Corollary 3.3.64, α is uniquely determined by its restriction to $U(\mathfrak{g})^{\mathrm{an}}_{r,s}$, which may be written as a formal sum $\sum \hbar^i\alpha_i$ for $\alpha_i : U(\mathfrak{g})^{\mathrm{an}}_{r,s} \to U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ bounded. We suppose we have $u_0, u_1, ..., u_n \in U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ such that $U_i\alpha \equiv \alpha_0 U_i$ modulo $\hbar^{i+1}$ as before, where $U_i = (1 + u_i\hbar^i)(1 + u_{i-1}\hbar^{i-1})...(1 + u_0)$ is now a generalised element of $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}[\![\hbar]\!]$. Since we are working with formal powerseries, both $(1 + u_i\hbar^i)$ and U_i are automatically convolution invertible. Again, if $\alpha^{(n)} := U_n * \alpha * U_n^{-1}$ whose restriction to $U(\mathfrak{g})^{\mathrm{an}}_{r,s}$ is given by the formal sum $\sum \hbar^i\alpha^{(n)}_i$, then $\alpha^{(n)}_{n+1}$ restricts to a bounded 1-cocycle on $\mathfrak{g}$, hence is a 1-coboundary. We therefore obtain $u_{n+1} \in U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}$ such that $\alpha^{(n)}_n(x) = [\alpha_0(x), u_{n+1}]$ for all $x \in \mathfrak{g}_{r,s}$. Then $(1+u_{n+1}\hbar^{n+1})$ is an invertible generalised element of $U(\mathfrak{g}')^{\mathrm{an}}_{r',s'}[\![\hbar]\!]$ and as in the proof of Theorem 3.3.52 we have

$$(1 + u_{n+1}\hbar^{n+1})\alpha(x)(1 + u_{n+1}\hbar^{n+1})^{-1} \equiv \alpha_0(x) \bmod \hbar^{n+2}.$$

Taking $u_0 := 0$ as our base case, as before, inductively gives a sequence of convolution invertible generalised elements $(U_n)_{n\geq 0}$ that converge in each $U(\mathfrak{g})^{\text{an}}_{r,s} \cdot \hbar^n$ to give $U : k \to U(\mathfrak{g}')^{\text{an}}_{r',s'}[\![\hbar]\!]$ such that $U*\alpha*U^{-1} = \alpha_0$. Similarly there is a convolution invertible generalised element U' of $U(\mathfrak{g}')^{\text{an}}_{r',s'}[\![\hbar]\!]$ such that $U' * \alpha' * U'^{-1} = \alpha'_0 = \alpha_0$. Taking $F = U'^{-1} * U$ gives our result. $\square$

Theorem 3.3.68. *Suppose A is an IndBanach $k[\![\hbar]\!]$ algebra equipped with a $k[\![\hbar]\!]$-linear isomorphism $A \cong U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!]$ that preserves the unit, and that the induced isomorphism $A/\hbar A \cong U(\mathfrak{g})^{an}_{r,s}$ is an isomorphism of algebras. Suppose that $H^2_b(\mathfrak{g}, U(\mathfrak{g})^{an}_{r,s}) = 0$. Then there is an isomorphism of $k[\![\hbar]\!]$-algebras $A \cong U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!]$ inducing the given algebra isomorphism modulo $\hbar$.*

Proof. This proof follows as for Theorem 3.3.53. The alternate multiplication μ on $U(\mathfrak{g})^{\text{an}}_{r,s}$ induced by A is again determined by its restriction to $U(\mathfrak{g})^{\text{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\text{an}}_{r,s}$, which may written as a formal sum $\mu = \sum \hbar^i \mu_i$ for

$$\mu_i : U(\mathfrak{g})^{\text{an}}_{r,s}\hat{\otimes}U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s}.$$

We again assume we have maps $\alpha_0, \alpha_1, ..., \alpha_n : U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s}$ such that $V_i(\mu(x,y)) \equiv \mu_0(V_i(x), V_i(y))$ modulo $\hbar^{i+1}$ for all $x, y \in U(\mathfrak{g})^{\text{an}}_{r,s}$, where

$$V_i = (\text{Id} + \hbar^i\alpha_i)(\text{Id} + \hbar^{i-1}\alpha_{i-1})...(\text{Id} + \alpha_0) : U(\mathfrak{g})^{\text{an}}_{r,s}[\![\hbar]\!] \to U(\mathfrak{g})^{\text{an}}_{r,s}[\![\hbar]\!].$$

Again, since we are working with formal powerseries, each V_i is automatically invertible and give new multiplication maps $\mu^{(i)} := V_i \circ \mu \circ (V_i^{-1} \otimes V_i^{-1})$ on $U(\mathfrak{g})^{\text{an}}_{r,s}[\![\hbar]\!]$. We write the restriction of $\mu^{(i)}$ to $U(\mathfrak{g})^{\text{an}}_{r,s}$ as a formal sum $\sum \hbar^j \mu^{(i)}_j$. As in the proof of Theorem 3.3.53, $\mu^{(n)}_{n+1}$ satisfies the conditions of Lemma 3.3.51, so gives $\alpha_{n+1} : U(\mathfrak{g})^{\text{an}}_{r,s} \to U(\mathfrak{g})^{\text{an}}_{r,s}$ such that

$$(\text{Id} + \alpha_{n+1}\hbar^{n+1}) \circ \mu^{(n)} \circ \big((\text{Id} + \alpha_{n+1}\hbar^{n+1})^{-1} \otimes (\text{Id} + \alpha_{n+1}\hbar^{n+1})^{-1}\big)$$

is equivalent to μ_0 modulo $\hbar^{n+1}$. Taking $\alpha_0 = 0$ as a base case, we inductively obtain a sequence of morphisms $(V_n)_{n\geq 0}$. Their restrictions to $U(\mathfrak{g})^{\mathrm{an}}_{r,s}$ converges in each $U(\mathfrak{g})^{\mathrm{an}}_{r,s} \cdot \hbar^n$ to give a morphism $V : U(\mathfrak{g})^{\mathrm{an}}_{r,s}[\![\hbar]\!] \to U(\mathfrak{g})^{\mathrm{an}}_{r,s}[\![\hbar]\!]$. It then follows that

$$A \cong U(\mathfrak{g})^{\mathrm{an}}_{r,s}[\![\hbar]\!] \xrightarrow{V} U(\mathfrak{g})^{\mathrm{an}}_{r,s}[\![\hbar]\!]$$

is our desired isomorphism of algebras. □

Lemma 3.3.69. *Let $r, s > 0$ such that $1 \leq |q_i - q_i^{-1}|rs$. Then there is an isomorphism of $k[\![\hbar]\!]$-modules $U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!]$ that descends to an isomorphism of Banach Hopf algebras $U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s}/\hbar U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}$.*

Proof. This follows from Theorem 3.3.58. □

Corollary 3.3.70. *Suppose that*

$$H^1_b(\mathfrak{g}, U(\mathfrak{g})^{an}_{r,s}) = 0 = H^2_b(\mathfrak{g}, U(\mathfrak{g})^{an}_{r,s}).$$

Then there is an isomorphism of algebras

$$U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s} \xrightarrow{\sim} U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!]$$

that induces the isomorphism $U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s}/\hbar U_{[\![\hbar]\!]}(\mathfrak{g})^{an}_{r,s} \cong U(\mathfrak{g})^{an}_{r,s}$. Furthermore, this isomorphism is unique up to conjugation by a convolution invertible generalised element of $U(\mathfrak{g})^{an}_{r,s}[\![\hbar]\!]$.

Proof. This follows from Theorem 3.3.67, Theorem 3.3.68 and Corollary 3.3.69. □

3.4 Archimedean analytic quantum groups

Throughout this section we shall assume **(A)**.

3.4.1 Constructing Archimedean analytic Nichols algebras

Lemma 3.4.1. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1. Suppose we have closed homogeneous subspaces $I_r \subset J_r \subset T_r(V)$ for each $r > 0$ such that, whenever $r \geq r'$, $T_r(V) \to T_{r'}(V)$ maps I_r and J_r to $I_{r'}$ and $J_{r'}$ respectively. Suppose further that $\Delta(I_r) \subset I_{\frac{r}{2}} \hat{\otimes} T_{\frac{r}{2}}(V) + T_{\frac{r}{2}}(V) \hat{\otimes} I_{\frac{r}{2}}$ and $\Delta(J_r) \subset J_{\frac{r}{2}} \hat{\otimes} T_{\frac{r}{2}}(V) + T_{\frac{r}{2}}(V) \hat{\otimes} J_{\frac{r}{2}}$. If the induced map*

$$P(colim_{r>0} T_r(V)/I_r) \to colim_{r>0} T_r(V)/I_r \to colim_{r>0} T_r(V)/J_r$$

is injective then $I_r = J_r$ for all $r > 0$.

Proof. This is similar to Lemma 3.3.3. We will denote by $R_r = T_r(V)/I_r$, $R'_r = T_r(V)/J_r$, by $R_r(n)$, $R'_r(n)$ their respective nth graded pieces, and by $R_r(\leq n) = \bigoplus_{i \leq n} R_r(i)$ and $R'_r(\leq n) = \bigoplus_{i \leq n} R'_r(i)$. Since $I_r \subset J_r$ we have strict analytically graded epimorphisms $f_r : R_r \to R'_r$ which restricts to isometries $R_r(1) \to R'_r(1)$. Suppose that f_r restricts also to an isometry $R_r(\leq n) \to R'_r(\leq n)$ for all $r > 0$. Let $r > 0$ and $x \in R_r(\leq n+1)$. By the proof of Lemma 3.1.8, we see that

$$\begin{aligned} \Delta(R_r(n+1)) &\subset \textstyle\sum_{i=0}^{n+1} R_{\frac{r}{2}}(i) \hat{\otimes} R_{\frac{r}{2}}(n-i) \\ &\subset R_{\frac{r}{2}}(n+1) \hat{\otimes} k + k \hat{\otimes} R_{\frac{r}{2}}(n+1) + R_{\frac{r}{2}}(\leq n) \hat{\otimes} R_{\frac{r}{2}}(\leq n), \end{aligned}$$

so $\Delta(x) = y \otimes 1 + 1 \otimes y' + z$ for some $y, y' \in R_{\frac{r}{2}}(n+1)$, $z \in R_{\frac{r}{2}}(\leq n) \hat{\otimes} R(\leq n)$. But then $x - y = (\mathrm{Id} \otimes \varepsilon)\Delta(x) - y = \varepsilon(y) \cdot 1 + (\mathrm{Id} \otimes \varepsilon)(z) \in R_{\frac{r}{2}}(\leq n)$ and likewise $x - y' \in R_{\frac{r}{2}}(\leq n)$. So $\Delta(x) = x \otimes 1 + 1 \otimes x + z'$ for some $z' \in R_{\frac{r}{2}}(\leq n) \hat{\otimes} R_{\frac{r}{2}}(\leq n)$. If $f_r(x) = 0$ then $(f_{\frac{r}{2}} \otimes f_{\frac{r}{2}})(z') = 0$, but, since each $R_{\frac{r}{2}}(\leq n)$ is finite dimensional and hence flat, $f_{\frac{r}{2}} \otimes f_{\frac{r}{2}}$ is injective on $R_{\frac{r}{2}}(\leq n) \hat{\otimes} R_{\frac{r}{2}}(\leq n)$. So $z' = 0$ and x is primitive, hence $x = 0$. Thus f_r is an isometry $R_r(\leq n) \to R'_r(\leq n)$ since the norms on $R_r(\leq n)$ and $R'_r(\leq n)$ are the quotient norms from $\coprod_{i \leq n}^{\leq 1} V_r^{\hat{\otimes} i}$. Taking contracting colimits over n we see that f_r is isometric. Hence $I_r = J_r$. $\square$

Lemma 3.4.2. *Let $W = \coprod_{n\geq 0}^{\leq 1} W(n)$ be a Banach space where each $W(n)$ is finite dimensional. Then W is flat.*

Proof. Let $f : A \to B$ be a morphism of Banach spaces. Then

$$\begin{aligned}
\operatorname{Ker}\left(W\hat{\otimes}A \xrightarrow{\mathrm{Id}\otimes f} W\hat{\otimes}B\right) &\cong \operatorname{Ker}\left(\coprod^{\leq 1}(W(n)\hat{\otimes}A) \to \coprod^{\leq 1}(W(n)\hat{\otimes}B)\right)\\
&\cong \coprod^{\leq 1}\operatorname{Ker}\left(W(n)\hat{\otimes}A \xrightarrow{\mathrm{Id}\otimes f} W(n)\hat{\otimes}B\right)\\
&\cong \coprod^{\leq 1}\left(W(n)\hat{\otimes}\operatorname{Ker}\left(A \xrightarrow{f} B\right)\right)\\
&\cong W\hat{\otimes}\operatorname{Ker}\left(A \xrightarrow{f} B\right),
\end{aligned}$$

where the second isomorphism is an easy computation and the third is because each $W(n)$ is finite dimensional. Hence W is flat. □

Proposition 3.4.3. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most* 1*. Then a dagger-0 Nichols algebra of V exists.*

Proof. As a variation on the proof of Proposition 3.3.4, let $\mathcal{I}(V)$ be the set consisting of collections $(I_r)_{r\in\mathbb{R}_{>0}}$ where each I_r is a homogeneous ideals of $T_r(V)$ contained in $\coprod_{n\geq 2}^{\leq 1} V_r^{\hat{\otimes}n}$ such that the maps $T_r(V) \to T_{r'}(V)$ for $r > r'$ send I_r to a subspace of $I_{r'}$ and the maps $\Delta : T_r(V) \to T_{\frac{r}{2}}(V)\hat{\otimes}T_{\frac{r}{2}}(V)$ giving the comultiplication send I_r to a subspace of $I_{\frac{r}{2}}\hat{\otimes}T_{\frac{r}{2}}(V) + T_{\frac{r}{2}}(V)\hat{\otimes}I_{\frac{r}{2}}$. Let $\mathcal{I}'(V)$ be the subset of $\mathcal{I}(V)$ consisting of $(I_r)_{r\in\mathbb{R}_{>0}}$ for which $\overline{I_r}\hat{\otimes}T(V) + T(V)\hat{\otimes}\overline{I_r}$ is preserved by $\tilde{c}$. Let $I_r(V)$ be the sum of all ideals I_r for $(I_r)_{r\in\mathbb{R}_{>0}} \in \mathcal{I}(V)$, which is closed by a similar argument to the proof of Proposition 3.3.4. Likewise let $I'_r(V)$ be the sum of all ideals I'_r for $(I'_r)_{r\in\mathbb{R}_{>0}} \in \mathcal{I}'(V)$, which is again closed. We will denote by $I(V)$ and $I'(V)$ the collection $(I_r(V))_{r\in\mathbb{R}_{>0}}$ and $(I'_r(V))_{r\in\mathbb{R}_{>0}}$, and let $T_0(V)^\dagger/I(V) := \text{"colim"}_{r>0}T_r(V)/I_r(V)$ and $T_0(V)^\dagger/I'(V) := \text{"colim"}_{r>0}T_r(V)/I'_r(V)$. We must check that

$$P(T_0(V)^\dagger/I(V)) = (T_0(V)^\dagger/I(V))(1).$$

As before, the closed ideals generated by $I_r(V)$ and

$$\left\{x \in \coprod_{n\geq 2}^{\leq 1} \middle| \substack{\Delta(x)\in x\otimes 1+1\otimes x+I_{r'}(V)\otimes T_{r'}(V)+T_{r'}(V)\otimes I_{r'}(V) \\ \text{for some sufficiently small } \frac{r}{2}\geq r'>0}\right\}$$

form an element of $\mathcal{I}(V)$. So $P(T_0(V)^\dagger/I(V)) = (T_0(V)^\dagger/I(V))(1)$, and likewise $P(T_0(V)^\dagger/I'(V)) = (T_0(V)^\dagger/I'(V))(1)$. So, by Lemma 3.4.1, we have $I_r(V) = I'_r(V)$ for all $r > 0$, and so the braiding $\tilde{c}$ descends to braidings of $T_r(V)/I_r(V)$. Hence $T_0(V)^\dagger/I(V) :=$ "colim"$_{r>0}T_r(V)/I_r(V)$ forms a $k\{t\}^\dagger$-graded braided IndBanach Hopf algebra with $(T_0(V)^\dagger/I(V))(0) \cong k$ and generated by $(T_0(V)^\dagger/I(V))(1) \cong V$. Finally, by Lemma 3.4.2, $T_r(V)/I_r(V)$ is flat for each $r > 0$. Hence the filtered colimit "colim"$_{r>0}T_r(V)/I_r(V)$ is flat. □

Definition 3.4.4. Let $\mathfrak{B}_r(V) := T_r(V)/I_r(V)$. We shall use the notation $\mathfrak{B}_0(V)^\dagger$ for the dagger Nichols algebra $\text{colim}_{r>0}\mathfrak{B}_r(V)$ defined in the proof of Proposition 3.4.3.

Remark Note that, in the proof of Proposition 3.4.3, each $I_r(V)$ is a closed ideal of $T_r(V)$, and hence each $\mathfrak{B}_r(V)$ is a Banach algebra. So $\mathfrak{B}_0(V)^\dagger$ is locally Banach as an algebra.

Lemma 3.4.5. *Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a symmetric non-degenerate bilinear form $\langle -,-\rangle : V\hat{\otimes}V \to k$. Then this extends to a dual pairing $(\coprod_{n\geq 0} V^{\hat{\otimes}n})\hat{\otimes}T_0^c(V)^\dagger \to k$ such that the composition*

$$V^{\hat{\otimes}n}\hat{\otimes}V^{\hat{\otimes}m} \longrightarrow \left(\coprod_{n\geq 0} V^{\hat{\otimes}n}\right)\hat{\otimes}T_0^c(V)^\dagger \longrightarrow k$$

is symmetric when $n = m$ and is 0 if $n \neq m$.

Proof. Again, we proceed as in the proof of Proposition 1.2.3 of [27], as we did for Lemma 3.3.9. The given bilinear form induces a morphism $V \to V^*$ whilst the projections $T_r(V) \twoheadrightarrow V$ induce a morphism $V^* \to (T_0(V)^\dagger)^*$. The coalgebra structure on $T_0(V)^\dagger$ gives $(T_0(V)^\dagger)^*$ an algebra structure, and so the composition $V \to$

$V^* \to (T_0(V)^\dagger)^*$ induces a unique algebra homomorphism $\coprod_{n\geq 0} V^{\hat{\otimes} n} \to (T_0(V)^\dagger)^*$ that gives our bilinear form. The fact that this is is a dual pairing that is symmetric on $V^{\hat{\otimes} n}\hat{\otimes}V^{\hat{\otimes} n}$ with $V^{\hat{\otimes} n}$ perpendicular to $V^{\hat{\otimes} m}$ for $m \neq n$ follows as in the proof of Lemma 3.3.9. □

Remark Suppose we have a Banach space V with pre-braiding c of norm at most 1 and a non-degenerate symmetric bilinear form $\langle -,-\rangle : V\hat{\otimes}V \to k$. Let $r, s > 0$ with $\|\langle -,-\rangle\| \leq rs$. The n-fold comultiplication $T_r(V) \to T_{r/2^n}(V)^{\hat{\otimes} n}$ given by Proposition 3.1.6 induces an n-fold multiplication

$$(T_{r/2^n}(V)^*)^{\hat{\otimes} n} \to T_r(V)^*.$$

Also, for each $n > 0$ the bilinear form $\langle -,-\rangle$ and the projection $T_{r/2^n}(V) \twoheadrightarrow V_{r/2^n}$ induce contracting homomorphisms $V_{2^n s} \to V^*_{r/2^n}$ and $V^*_{r/2^n} \twoheadrightarrow T_{r/2^n}(V)^*$. Then the compositions

$$V_{2^n s}^{\hat{\otimes} n} \to (V^*_{r/2^n})^{\hat{\otimes} n} \to (T_{r/2^n}(V)^*)^{\hat{\otimes} n} \to T_r(V)^*$$

induce a bilinear pairing $\mathscr{T}_s(V)\hat{\otimes}T_r(V) \to k$, where

$$\mathscr{T}_s(V) := \coprod_{n\geq 0}^{\leq 1} (V_{2^n s})^{\hat{\otimes} n} = \left\{ \sum x_n \,\middle|\, x_n \in V^{\hat{\otimes} n} \text{ and } \sum_n \|x\| 2^{n^2} s^n < \infty \right\},$$

and hence a bilinear pairing $\mathscr{T}_\infty(V)\hat{\otimes}T_0(V)^\dagger \to k$, where $\mathscr{T}_\infty(V) := \lim_{s>0}\mathscr{T}_s(V)$. Furthermore, the restriction of these bilinear forms to $\bigoplus_{n\geq 0} V^{\otimes n}$ is a dual pairing by Proposition 1.2.3 of [27]. Unfortunately, the algebra structure of $\bigoplus_{n\geq 0} V^{\otimes n}$ neither extends to $\mathscr{T}_s(V)$ nor $\mathscr{T}_\infty(V)$. It does, however, extend to

$$\lim_{s_n>0} \coprod_{n\geq 0}^{\leq 1} (V_{s_n})^{\hat{\otimes} n} = \lim_{s_n>0} \left\{ \sum x_n \,\middle|\, x_n \in V^{\hat{\otimes} n} \text{ and } \sum_n \|x\| s_n^n < \infty \right\}$$

which we may pair with $T_0(V)^\dagger$ via the composition

$$\left(\lim_{s_n>0}\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}\right)\hat{\otimes}T_0(V)^\dagger \to \mathscr{T}_\infty(V)\hat{\otimes}T_0(V)^\dagger \to k.$$

The following lemma shows that $\lim_{s_n>0}\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}$ is just $\coprod_{n\geq 0}V^{\hat{\otimes}n}$ as in the previous lemma.

Lemma 3.4.6. *The natural morphism*

$$\coprod_{n\geq 0}V^{\hat{\otimes}n} \to lim_{s_n>0}\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}$$

is an isomorphism.

Proof. As in the proof of Lemma 3.3.63, for a sequence of positive real numbers $a_n > 0$ such that $(a_n^n)_{n\geq 0}$ is summable, the maps

$$\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n} \to \prod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}, \quad (x_n)_{n\geq 0}\mapsto (x_n)_{n\geq 0},$$

and

$$\prod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n} \to \coprod_{n\geq 0}^{\leq 1}(V_{a_n s_n})^{\hat{\otimes}n}, \quad (x_n)_{n\geq 0}\mapsto (x_n)_{n\geq 0},$$

induce an isomorphism

$$\lim_{s_n>0}\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n} \cong \lim_{s_n>0}\prod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}.$$

For a Banach space W,

$$\begin{aligned}
\mathrm{Hom}(W,\lim_{s_n>0}\textstyle\coprod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}) &\cong \mathrm{Hom}(W,\lim_{s_n>0}\textstyle\prod_{n\geq 0}^{\leq 1}(V_{s_n})^{\hat{\otimes}n}) \\
&\cong \lim_{s_n>0}\textstyle\prod_{n\geq 0}^{\leq 1}\mathrm{Hom}(W,(V_{s_n})^{\hat{\otimes}n}) \\
&= \left\{(f_n)_{n\geq 0}\,\middle|\, \begin{smallmatrix} f_n:W\to V^{\hat{\otimes}n} \text{ and } (\|f_n\|s_n)_{n\geq 0} \\ \text{is bounded for all } s_n>0\end{smallmatrix}\right\} \\
&= \left\{(f_n)_{n\geq 0}\,\middle|\, \begin{smallmatrix} f_n:W\to V^{\hat{\otimes}n} \text{ and} \\ f_n=0 \text{ for } n\gg 0\end{smallmatrix}\right\}
\end{aligned}$$

where the last equality is because if $s_n := \frac{n}{\|f_n\|}$ when $f_n \neq 0$ and $s_n := 1$ if $f_n = 0$ then $(\|f_n\|s_n)_{n\geq 0}$ is only bounded if $f_n = 0$ for $n \gg 0$. Hence the map of sets

$$\mathrm{Hom}(W, \coprod_{n\geq 0} V^{\hat{\otimes} n}) \cong \bigoplus_{n\geq 0} \mathrm{Hom}(W, V^{\hat{\otimes} n}) \to \mathrm{Hom}(W, \lim_{s_n>0} \coprod\nolimits_{n\geq 0}^{\leq 1} (V_{s_n})^{\hat{\otimes} n})$$

is bijective. Since this holds for all Banach spaces W we must have

$$\coprod_{n\geq 0} V^{\hat{\otimes} n} \cong \lim_{s_n>0} \coprod\nolimits_{n\geq 0}^{\leq 1} (V_{s_n})^{\hat{\otimes} n}.$$

□

Proposition 3.4.7. *Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate bilinear form $\langle -, - \rangle : V \hat{\otimes} V \to k$. Then for each $0 < r$, let I_r be the radical in $T_r^c(V)$ of*

$$\left(\coprod_{n\geq 0} V^{\hat{\otimes} n}\right) \hat{\otimes} T_r^c(V) \longrightarrow \left(\coprod_{n\geq 0} V^{\hat{\otimes} n}\right) \hat{\otimes} T_0^c(V)^{\dagger} \longrightarrow k.$$

Then the I_r are closed homogeneous ideals in $T_r^c(V)$ such that $T_r(V) \to T_{r'}(V)$ maps I_r to $I_{r'}$ and $\Delta : T_r(V) \to T_{\frac{r}{2}}(V) \hat{\otimes} T_{\frac{r}{2}}(V)$ maps I_r to $I_{\frac{r}{2}} \hat{\otimes} T_{\frac{r}{2}}(V) + T_{\frac{r}{2}}(V) \hat{\otimes} I_{\frac{r}{2}}$. Furthermore, $P(\mathrm{colim}_{r>0} T_r^c(V)/I_r) = V$, hence $\mathrm{colim}_{r>0} T_r^c(V)/I_r$ is a dagger Nichols algebra of V.

Proof. This is analogous to the proof of Proposition 3.3.10. The fact that I_r are closed homogeneous ideals compatible under restrictions to smaller radii with $\Delta(I_r) \subset I_{\frac{r}{2}} \hat{\otimes} T_{\frac{r}{2}}(V) + T_{\frac{r}{2}}(V) \hat{\otimes} I_{\frac{r}{2}}$ follows from Lemma 3.4.5. As the bilinear form on V is non-degenerate, $I_r \subset \coprod_{n\geq 2}^{\leq 1} V_r^{\hat{\otimes} n}$, and the quotient $T_r(V)/I_r$ is generated by V. It remains to check that the subspace of primitive elements is just V, which follows from the same argument as for Proposition 3.3.10. Again, $\tilde{c}$ preserves $I_r \hat{\otimes} T_r(V) + T_r(V) \hat{\otimes} I_r$ for all $r > 0$, so the braidings on $T_r^c(V)$ descend to braidings of $T_r^c(V)/I_r$ and $\mathrm{colim}_{r>0} T_r^c(V)/I_r$ is a dagger Nichols algebra of V. □

Proposition 3.4.8. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1. Let $R = colim_{r>0} \coprod^{\leq 1} R(n)_{r^n}$ be a dagger graded pre-braided IndBanach Hopf algebra. Suppose further that the algebra structure on R is determined by algebra structures on $\coprod^{\leq 1} R(n)_{r^n}$ for each $r > 0$ which are generated by $R(1)_r$. Then, if $R(0) \cong k$, $P(R) = R(1) \cong V$ as pre-braided Banach spaces, there is an epimorphism of dagger graded braided Hopf algebras $\mathfrak{B}_0^c(V) \to R$ extending $V \xrightarrow{\sim} R(1)$.*

Proof. Without loss of generality, we may assume that the isomorphism $V \xrightarrow{\sim} R(1)$ is of norm at most 1. Hence we obtain morphisms of Banach algebras $T_r(V) \to \coprod_{n\geq 0}^{\leq 1} R(n)_{r^n}$ that are epic since $R(1)_r$ generate $\coprod_{n\geq 0}^{\leq 1} R(n)_{r^n}$. Since $P(R) = R(1)$, this induces a morphism of Hopf algebras $T_0(V)^\dagger \to R$. Hence the kernels of the maps $T_r(V) \to \coprod_{n\geq 0}^{\leq 1} R(n)_{r^n}$ form an element of $\mathcal{I}(V)$, in the notation of the proof of Proposition 3.4.3. Thus we obtain a well-defined epimorphism $\mathfrak{B}_0^c(V) \to R$. □

Proposition 3.4.9. *Let V be a finite dimensional Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, - \rangle : V \hat{\otimes} V \to k$. Retaining the notation of Proposition 3.4.7, there is an isomorphism $\mathfrak{B}_0^c(V)^\dagger \to colim_{r>0} T_r^c(V)/I_r$.*

Proof. By Proposition 3.4.8 there is an epimorphism $\mathfrak{B}_0^c(V) \to \mathrm{colim}_{r>0} T_r^c(V)/I_r$. By Lemma 3.4.1 the maps $\mathfrak{B}_r^c(V) \to T_r^c(V)/I_r$ are all isomorphisms. Hence $\mathfrak{B}_0^c(V) \to \mathrm{colim}_{r>0} T_r^c(V)/I_r$ is an isomorphism. □

Definition 3.4.10. Let V be a Banach space with pre-braiding c of norm at most 1, and suppose we have a non-degenerate symmetric bilinear form $\langle -, - \rangle : V \hat{\otimes} V \to k$. As justified by the previous proposition, we will extend the notation $\mathfrak{B}_r^c(V)$ to the algebra $T_r^c(V)/I_r$ from Proposition 3.4.7 when V is not necessarily finite dimensional. We will then use the notation $\mathfrak{B}_0^c(V)^\dagger$ for the dagger Nichols algebra $\mathrm{colim}_{r>0} \mathfrak{B}_r^c(V)$.

For each $n \geq 0$ let $I(n)$ be the radical in $V^{\hat{\otimes} n}$ of the composition

$$V^{\hat{\otimes} n} \hat{\otimes} T_0^c(V)^\dagger \longrightarrow \left(\coprod_{n \geq 0} V^{\hat{\otimes} n} \right) \hat{\otimes} T_0^c(V)^\dagger \longrightarrow k$$

and let $\mathfrak{B}^c_{\text{alg}}(V)$ be the braided graded Hopf algebra $\coprod_{n \geq 0} V^{\hat{\otimes} n}/I(n)$. When V is finite dimensional this is the algebraic Nichols algebra of V, as defined in [2], by Proposition 2.10 of *loc. cit.* By Lemma 3.4.5 there is a dual pairing

$$\mathfrak{B}^c_{\text{alg}}(V) \hat{\otimes} \mathfrak{B}^c_0(V)^\dagger \to k,$$

which we also denote by $\langle -, - \rangle$.

3.4.2 Constructing Archimedean analytic quantum groups

Again, we will use q to denote an element of $k \setminus \{0\}$ of norm 1 and we fix root datum as in Definition 0.1.1 for a Lie algebra $\mathfrak{g}$.

Definition 3.4.11. Let $H = \coprod^{\leq 1}_{\lambda \in \Phi^*} k \cdot K_\lambda$ denote the Banach group Hopf algebra of Φ^* with

$$K_\lambda \cdot K_{\lambda'} = K_{\lambda + \lambda'}, \quad \Delta_H(K_\lambda) = K_\lambda \otimes K_\lambda \quad \text{and} \quad S(K_\lambda) = K_{-\lambda}.$$

We continue to use the notation

$$t_i := K_{\frac{(\alpha_i, \alpha_i)}{2} \lambda_i}$$

and let H' be the closed sub-Hopf algebra generated by $\{t_i \mid i \in I\}$. As before, there is a duality pairing $H\hat{\otimes}H' \to k$, which we continue to denote by $\langle -,-\rangle$, defined by

$$K_\lambda \otimes \underline{t}^{\underline{n}} \mapsto q^{\lambda(\sum n_i\alpha_i)}$$

and simultaneous algebra homomorphisms and coalgebra anti-homomorphisms

$$\mathscr{R}: H' \to H,\ t_i \mapsto t_i, \quad \overline{\mathscr{R}}: H' \to H,\ t_i \mapsto t_i^{-1},$$

for $i \in I$, making H and H' a weakly quasi-triangular dual pair.

Remark Proposition 4.4 of [26] says that the category of IndBanach H-modules that decompose locally into Banach weight spaces with weights in the root lattice $\Psi \subset \Phi$, which we will continue to denote by H-Mod_Ψ as before, is braided.

Definition 3.4.12. Let $V = \coprod_{i\in I}^{\leq 1} k \cdot v_i$ with basis $\{v_i \mid i \in I\}$ have the H'-coaction $v_i \mapsto t_i \otimes v_i$. Then V is a H-module with braiding $c(x_i \otimes x_j) = q^{(\alpha_i,\alpha_j)} x_j \otimes x_i$. Let $\langle -,-\rangle$ be the non-degenerate bilinear form on V where

$$\langle v_i, v_j\rangle = \delta_{i,j}\frac{1}{(q_i - q_i^{-1})} \quad \text{for} \quad q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}.$$

Given $0 < r$ we denote by $\mathbf{f}_r^{\mathrm{an}}$ the algebras $\mathfrak{B}_r^c(V)$. We then use the notation $\mathbf{f}_0^\dagger$ for the dagger Nichols algebra $\mathfrak{B}_0^c(V)^\dagger$ and $\mathbf{f}$ for the algebraic Nichols algebra $\mathfrak{B}_{\mathrm{alg}}^c(V)$.

Lemma 3.4.13. *For each $0 < r$, the positive part of the quantum group is dense in the Banach space $\mathbf{f}_r^{an}$.*

Proof. The proof of this is the same as for Lemma 3.3.20. □

Lemma 3.4.14. *Suppose $0 < r$. Then there is a H'-equivariant dual pairing $\langle -,-\rangle$: $\mathbf{f}_0^\dagger\hat{\otimes}\mathbf{f} \to k$ extending $\langle -,-\rangle$ in Definition 3.4.12 such that $\langle \mathbf{f}_0^\dagger(n), \mathbf{f}(m)\rangle = \{0\}$ for $n \neq m$.*

Proof. This follows from Lemma 3.4.5. □

Definition 3.4.15. We denote by $U_q(\mathfrak{g})_0^\dagger$ the double bosonisation $U(\mathbf{f}_0^\dagger, H, \mathbf{f})$. Let us denote by F_i the generalised element in $\mathbf{f}_0^\dagger \rtimes H$ represented by $v_i \otimes 1 \in \mathbf{f}_r^{\mathrm{an}} \hat{\otimes} H$, and by E_i the generalised element in $\overline{H} \ltimes \overline{\mathbf{f}}$ represented by $1 \otimes v_i \in H \hat{\otimes} V$ for $i \in I$. We retain this notation when viewing $\mathbf{f}_0^\dagger \rtimes H$ and $\overline{H} \ltimes \overline{\mathbf{f}}$ as sub-Hopf algebras of $U_q(\mathfrak{g})_0^\dagger$.

Proposition 3.4.16. *$U_q(\mathfrak{g})_{0,\infty}^{an}$ is analytically graded by $\mathbb{Z}I \cong \Psi$.*

Proof. As in the proof of Proposition 3.3.23 we have that $\mathbf{f}_r^{\mathrm{an}}$, $\mathbf{f}_0^\dagger$ and $\mathbf{f}$ are H'-comodules, and if we give H the trivial H'-coaction $H \cong k \hat{\otimes} H \xrightarrow{\eta_{H'} \otimes \mathrm{Id}_H} H' \hat{\otimes} H$ then all of the morphisms involved in defining $U(\mathbf{f}_0^\dagger, H, \mathbf{f})$ are H'-comodule homomorphisms.

□

3.4.3 Quantum groups as Drinfel'd doubles and braided monoidal representations

Lemma 3.4.17. *There is a duality pairing*

$$\langle -, - \rangle : (\overline{H} \ltimes \overline{\mathbf{f}}) \hat{\otimes} (\mathbf{f}_0^\dagger \rtimes H')^{op} \to k$$

given by the composition

$$\begin{aligned} H \hat{\otimes} \mathbf{f}_{\frac{s}{4}}^{an} \hat{\otimes} \mathbf{f}_{\frac{r}{4}}^{an} \hat{\otimes} H' &\xrightarrow{Id \otimes Id \otimes S \otimes Id} H \hat{\otimes} \mathbf{f}_{\frac{s}{4}}^{an} \hat{\otimes} \mathbf{f}_{\frac{r}{4}}^{an} \hat{\otimes} H' \xrightarrow{Id \otimes \langle -,- \rangle \otimes Id} H \hat{\otimes} k \hat{\otimes} H' \\ &\xrightarrow{Id \otimes S} H \hat{\otimes} H' \xrightarrow{\langle -,- \rangle} k. \end{aligned}$$

Proof. As with Lemma 3.3.31, this follows from Proposition 34 of Section 6.3.1 of [25]. □

Definition 3.4.18. We will denote by $D(\overline{H} \ltimes \overline{\mathbf{f}}, \mathbf{f}_0^\dagger \rtimes H')$ the relative Drinfel'd double of $\overline{H} \ltimes \overline{\mathbf{f}}$ and $\mathbf{f}_0^\dagger \rtimes H'$ as constructed in Lemma 3.3.29.

Recall the definition of crossed bimodules from Definition 3.3.32.

Lemma 3.4.19. *There is a fully faithful functor*

$$ {}_{\mathbf{f}_0^\dagger \rtimes H'}\mathit{Cross}^{\mathbf{f}_0^\dagger \rtimes H'} \to D(\overline{H} \ltimes \overline{\mathbf{f}}, \mathbf{f}_0^\dagger \rtimes H')\text{-}\mathit{Mod}. $$

Proof. This functor is constructed in Lemma 3.3.34. Given a morphism $M \xrightarrow{f} N$ in $D(\overline{H} \ltimes \overline{\mathbf{f}}, \mathbf{f}_0^\dagger \rtimes H')$-Mod between objects in the image of ${}_{\mathbf{f}_0^\dagger \rtimes H'}\mathrm{Cross}^{\mathbf{f}_0^\dagger \rtimes H'}$ then f commutes with both the action of H' and $\mathbf{f}$. Hence f must preserve the locally Banach weight space decomposition, hence commutes with the coaction of H'. Also, since the bilinear pairing between the corresponding graded pieces of $\mathbf{f}$ and $\mathbf{f}_0^\dagger$ is nondegenrate, f must also commute with the coaction of $\mathbf{f}_0^\dagger$. Hence f is a morphism in ${}_{\mathbf{f}_0^\dagger \rtimes H'}\mathrm{Cross}^{\mathbf{f}_0^\dagger \rtimes H'}$. □

Proposition 3.4.20. *There is a strict epimorphism*

$$ D(\overline{H} \ltimes \overline{\mathbf{f}}, \mathbf{f}_0^\dagger \rtimes H') \to U_q(\mathfrak{g})_0^\dagger $$

whose kernel is

$$ \mathbf{f}_0^\dagger \hat{\otimes} \overline{\langle t_i \otimes 1 - 1 \otimes t_i \mid i \in I \rangle} \hat{\otimes} \mathbf{f} \hookrightarrow \mathbf{f}_0^\dagger \hat{\otimes} H' \hat{\otimes} H \hat{\otimes} \mathbf{f}. $$

Proof. This follows as in the proof of Proposition 3.3.35. Again, this morphism can be written as

$$ \mathbf{f}_0^\dagger \hat{\otimes} H' \hat{\otimes} H \hat{\otimes} \mathbf{f} \xrightarrow{\mathrm{Id} \otimes \mu_H \otimes \mathrm{Id}} \mathbf{f}_0^\dagger \hat{\otimes} H \hat{\otimes} \mathbf{f}. $$

By Proposition 3.4.2, $\mathbf{f}_0^\dagger$ is flat. Also, since $\mathbf{f}$ is a colimit of finite dimensional spaces it is also flat. The result then follows from the fact that $\overline{\langle t_i \otimes 1 - 1 \otimes t_i \mid i \in I \rangle}$ is the kernel of $\mu_H : H' \hat{\otimes} H \to H$. □

Definition 3.4.21. Let us denote by $\mathcal{C}$ the full subcategory of ${}_{(\mathbf{f}_0^\dagger \rtimes H')}\mathrm{Cross}^{(\mathbf{f}_0^\dagger \rtimes H')}$

consisting of IndBanach spaces V equipped with both a left action and right coaction of $\mathbf{f}_0^\dagger \rtimes H'$, $\mu_V : (\mathbf{f}_0^\dagger \rtimes H')\hat{\otimes} V \to V$ and $\Delta_V : V \to V \hat{\otimes} (\mathbf{f}_0^\dagger \rtimes H')$, such that the diagram

$$\begin{array}{ccc} H'\hat{\otimes}V & \longrightarrow & (\overline{H} \ltimes \overline{\mathbf{f}})\hat{\otimes}V \\ \downarrow & & \downarrow \mu'_V \\ (\mathbf{f}_0^\dagger \rtimes H')\hat{\otimes}V & \xrightarrow{\mu_V} & V \end{array}$$

commutes, where μ'_V is the action of $\overline{H} \ltimes \overline{\mathbf{f}}$ on V induced by Δ_V.

Lemma 3.4.22. *There is a fully faithful functor $\mathcal{C} \to U_q(\mathfrak{g})_0^\dagger$-Mod.*

Proof. This follows from Lemma 3.3.34 and Proposition 3.4.20. □

Definition 3.4.23. Let us denote by $\mathcal{O}_\Psi$ the essential image of $\mathcal{C}$ in $U_q(\mathfrak{g})_0^\dagger$-Mod. By the previous Lemma this is precisely the full subcategory of $U_q(\mathfrak{g})_{r,s}^{\mathrm{an}}$-Mod consisting of modules, M, such that the action of H gives a module in H-Mod$_\Psi$ and the action of $\mathbf{f}$ is induced by a coaction of $\mathbf{f}_0^\dagger$ via their pairing.

Corollary 3.4.24. *The category $\mathcal{O}_\Psi$ is braided.*

Proof. This follows from Lemma 3.3.33. □

In time, the author hopes to study the representations in $\mathcal{O}_\Psi$ further and compute examples of the braiding. The hope is that this will produce interesting new braid group representations in which we might see some special analytic functions arising. For example, in [15], Goncharov exhibits an automorphism of a Schwarz space using the quantum dilogarithm that satisfies a pentagon relation. This Schwarz space is equipped with an action of the algebra of regular functions on a quantised cluster variety, and this automorphism of the Schwarz space intertwines an automorphism of this algebra of regular functions. The action of these regular functions gives a natural action of the positive part of $U_q(\mathfrak{sl}_2)$. It would be interesting to see whether we can use this to obtain a representation in $\mathcal{O}_\Psi$ for $\mathfrak{g} = \mathfrak{sl}_2$ whose braiding is related to the quantum dilogarithm and this work of Goncharov.

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