Large Markets: Asymptotic Arbitrage and Portfolio Optimisation



Alice Dub

St Anne's College

University of Oxford

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Statement of Originality

I herewith declare that I have produced this thesis without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This thesis has not previously been presented in identical or similar form to any other examination board.

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"*The day will come when, by study pursued through several ages, the things now concealed will appear with evidence; and posterity will be astonished that truths so clear had escaped us* " — Seneca on the motion on comets.

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Abstract

In this thesis, we aim to shed some light on the intricate behaviour of large, correlated financial markets, the existence or absence of asymptotic arbitrage in such a model and its connection to optimal investing. Therefore we will approximate these finite large markets by infinite-sized markets, and derive strategies describing how to invest optimally, based on the modelling coefficients. Our correlated finite real world market, spanning $n \in \mathbb{N}$ single securities as well as at most the fixed number of $J \in \mathbb{N}$, global assets is similar to the approach known from the well-established Capital Asset Pricing Model (CAPM) and can be viewed as an increasing sequence of smaller sub-markets, which can be interpreted as a collection of sectors or a set of different economies, where each is endowed with its own market risk factor.

The core problem we want to tackle in this thesis is to find out how an agent can optimally distribute his positive initial endowment over a large or infinite number of securities in order to build the optimal portfolio which can maximise his expected utility of terminal wealth. We will give both general as well as explicit solutions for a power utility function using optimal control methods.

If we are going to approximate large markets by infinite ones, we need to carefully consider the limiting arbitrage conditions, namely the asymptotic arbitrage. Our main result provides a new link between the two concepts, no asymptotic arbitrage and the existence of a finite limiting value function in the infinite market. We succeed to prove that, for measurable and adapted coefficients, there is no asymptotic arbitrage if the value function solving the optimisation problem exists and is finite in the limit for an infinite number of stocks.

In addition, we present a weak convergence result in terms of market models and look at asymptotic expansions of our optimisation results. Fundamentally, we manage to show that asymptotic arbitrage in our CAPM-style model always occurs, unless Sharpe ratios with respect to the global market of the single securities are close to the same, homogeneous value.

The thesis is concluded by a real life investment example which attempts to generate wealth by investing according to our optimal portfolio proportions.

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Chapter 1

Preliminaries

1.1 Introduction

Over the past twenty years, economic growth, faster computer networks as well as the development of the internet have dramatically increased the overall pool of investment possibilities world-wide. In addition, we can identify globalisation as the driving force behind the growing correlation between assets in financial markets. Therefore, in this thesis, we aim to shed some light on the intricate behaviour of large, correlated financial markets, the existence or absence of asymptotic arbitrage in such a model, and its connection to optimal investing. To approach this difficult problem accurately, we will approximate these finite large markets by infinite-sized markets, and derive strategies describing how to invest optimally, based on the modelling coefficients. Moreover, we will tackle the question of the possibility of the appearance of asymptotic arbitrage in connection with the solution to our optimisation when the overall market size approaches infinity.

We propose the following general setup. First, in order to capture the right level of interdependence between assets, we use a correlated factor model for our market. This is similar to the approach known from the well-established Capital Asset Pricing Model (CAPM). Here, each single stock is not only subject to an idiosyncratic risk, but is also correlated with a variety of global market risk factors, affecting the whole asset universe.

Secondly, we assume that the whole, finite real world market, spanning $n \in \mathbb{N}$ single securities as well as at most the fixed number of $J \in \mathbb{N}$ global assets, can be approximated by an increasing sequence of smaller sub-markets. This is a relatively new approach, first introduced by Kabanov and Kramkov in [KK93], when investigating the existence of asymptotic arbitrage. These sub-markets can be interpreted as, for example, a collection of sectors or a set

of different economies, where each is endowed with its own market risk factor. This is then described through a variety of global assets, driven only by a global risk factor. Of course, other drivers such as interest rate levels, GDP, economic growth, currency strength or even political stability could be used as risk factors.

The core problem we want to tackle in this thesis is to find out how an agent can optimally distribute his positive initial endowment over a large or infinite number of securities, in order to build the optimal portfolio which can maximise his expected utility of terminal wealth. The solution to this problem is the optimal value function. We aim to describe the problem setting in as much generality as possible at first and later find the explicit optimal portfolio proportions for simplified models with deterministic parameters and a limited number of risk factors. The best approach is to use optimal control methods on a finite-time horizon. In particular, as we are mainly concerned with concrete solutions, we will use the powerful methodology offered by dynamic programming and the Hamilton-Jacobi-Bellman equation. Note that it is assumed the markets are frictionless, the assets infinitely divisible and short-selling is permitted.

The second major topic we aim to cover is asymptotic arbitrage. It is well established how to identify, exclude and utilise arbitrage opportunities for comparatively small Black-Scholes-type market models. However, if we are going to approximate large markets by infinite ones, we need to carefully consider the limiting arbitrage conditions, namely the asymptotic arbitrage. After introducing this relatively recent concept, we succeed in making a previously unknown connection between the limiting optimal value function for our optimisation problem in complete markets and the existence of asymptotic arbitrage opportunities. In particular, we show that if the optimal value function exists and is finite, there is no asymptotic arbitrage in the complete infinite sized market.

Finally, the third building block of our thesis concerns the concrete behaviour of the explicit optimal investment proportions and the particular optimal value function once our market model becomes infinite. By this stage, we have managed to show that for deterministic parameters, there is no asymptotic arbitrage if and only if the value function is finite in the limit. Using asymptotic expansions and describing our model now in terms of perturbed parameters, we can describe how different levels of control on such perturbations can influence the finiteness of our value function and thus, the existence of asymptotic arbitrage.

Let us remark briefly on the general development and fundamental ideas of asymptotic arbitrage. Kabanov and Kramkov in [KK93] first formally defined asymptotic arbitrage in

complete markets and described it heuristically as the possibility of obtaining a gain, however small, with a positive probability by investing an infinitesimally small endowment. When partitioning the whole market into an increasing sequence of smaller sub-markets, it was shown that, even if there is no arbitrage in any of these 'small' markets, the overall system can still experience asymptotic arbitrage when we let the number of securities go to infinity. Schachermayer and Klein [KS96] then extended this theory to cover incomplete markets, where the martingale measures are not necessarily assumed to be unique. Later, Kabanov and Kramkov in [KK98] added further criteria for the absence of asymptotic arbitrage using the contiguity of sequences of lower and upper envelopes of martingale measures, both in the complete and incomplete setting. Extending the idea of asymptotic arbitrage, Klein, Lepinette and Ostafe in [KLO12], [LO12] also looked at general results for asymptotic arbitrage in large markets with friction or proportional transaction costs. Contrary to our setup, where the number of assets approaches infinity, Foellmer and Schachermayer in [FS08] considered asymptotic arbitrage and large deviations, focusing on the limit as the time horizon becomes infinite. They derive optimal trading strategies and growth rates for a HARA utility, using the example of Ornstein-Uhlenbeck processes as $T \to \infty$. Therefore they find explicit estimates for the boundaries for the definition of asymptotic arbitrage, showing that these are related to the large deviation estimates for the market price of risk. A very interesting general article was also written by Klein [Kle00], on the fundamental theorem of asset pricing in large markets. She generalises the notion of asymptotic arbitrage to 'no asymptotic free lunch' and shows that it cannot be replaced by the weaker condition of 'no asymptotic free lunch with bounded risk'.

To conclude this literature review, we should mention the work of Di Nunno and Baadshaug Eide, see [DNBE09], who looked at large markets from a different point of view. Contrary to Kabanov and Kramkov's initial definition, as a sequence of finite-dimensional market models converging to a large market, Di Nunno and Baadshaug Eide define a large market as a countable number of N assets, which are a sequence of price processes on one fixed probability space. Here, the discounted asset prices become martingales with respect to the physical measure. They then continue to consider what happens once N approaches infinity. In addition, Haba and Jacquier in [HJ13], have recently looked at the Heston model in large financial markets. They attempt to understand how the parameters of the model influence the asymptotic arbitrage, which can arise on an infinite horizon. Gandy and Veraart in [GV13a] on the other hand do not consider large market limits, but study the estimation of modelling parameters for investment strategies, maximising the expected log-utility of terminal wealth. In particular, they look at estimation errors, which may become interesting to us once we describe why our optimal strategies fail when applied to real trading in Chapter 5.

This thesis is organised as follows. In Chapter 1 we state our correlated, complete J- factor market model adapted from [KK93], and give our own interpretation of a large market as an increasing sequence of probability spaces. We also show that this particular structure allows the market to be successfully extended to an infinite size, by carefully taking limits as the number of assets $n \in \mathbb{N}$ approaches infinity. Thereafter we state the well-known optimisation problem, see for example [Kar97], which will be the foundation for the remainder of this dissertation. Additionally we show a convenient approach to matrix inversion in finite dimensions. In the remainder of the chapter, we introduce the concept of asymptotic arbitrage following the works of Kabanov and Kramkov, see [KK93] and [KK98], and further clarify how this is linked to contiguity of probability measures as well as the limit of an associated Hellinger process.

Chapter 2 is dedicated to our main result. This provides a new link between the two concepts, no asymptotic arbitrage and the existence of a finite limiting value function in the infinite market. To be able to verify and thoroughly explain it we first recall, briefly, the established duality and martingale methods. These are used to generate a general solution to the optimal control problem with no prior assumptions on the utility function. Thereafter we procceed to prove our theorem, which concludes that, for measurable and adapted coefficients, there is no asymptotic arbitrage if the value function solving the optimisation problem exists and is finite in the limit for an infinite number of stocks. Furthermore, we also state a corollary for deterministic parameters which says that if our modelling parameters are just functions of time, we can make the stronger assertion that the absence of asymptotic arbitrage is actually equivalent to the existence of a finite limiting value function.

In Chapter 3, we give explicit results to the optimisation problem for a complete and incomplete market model with deterministic coefficients. It is well known that the martingale and duality approach, described in Chapter 2, only gives us the general existence of the portfolio proportion and not its concrete structure. However, as we are interested in their specific shape and behaviour, it is more convenient to use the dynamic programming and Hamilton-Jacobi-Bellman equation methodology. We find explicit expressions for the investment proportion, as well as the optimal value function, assuming that the parameters are deterministic functions of time over a fixed horizon. These outcomes are stated for a power utility function, representing the agent's preference structure as a risk-averse investor. In addition to our initial, complete market model, we investigate how the behaviour changes when the agent can only invest in the basket of single securities under the assumption that the global market asset is no longer traded. Consequently we introduce a model, which is derived by omitting the market asset and, therefore, is characterised as incomplete because the number of risk sources

is higher than the number of price processes. Finally, we briefly show how those results extend to a new model with two factors for both complete and incomplete markets. Additionally, for homogeneous parameters, we manage to derive the shape of the growth rate as well as portfolio proportions for a finite number J of risk factors, where $J > 2$.

Using these explicit results, Chapter 4 explains a new way to establish particular finiteness conditions in order to avoid asymptotic arbitrage in both the complete market case, as well as the incomplete setting, once the number of assets approaches infinity. In a first step, we establish the weak convergence of our market model to a well-defined limit, with market size dependent coefficients. These coefficients are then written as a perturbed parameter model, where the small perturbations are random variables, taken from a mean-zero distribution. Secondly by restating the portfolio proportions and the optimal value function in terms of a pure correlation parameter, the Sharpe ratio, as well as a Sharpe ratio with respect to the global market, we can find a variety of limiting behaviours of the explicit portfolio proportions and value function growth rates. Different scaling of these perturbations then produce a variety of controls over the growth of the limiting value function, as well as the size of the portfolio proportions. Hence, we can establish the appropriate size of perturbations to keep the value function finite in the limit and therefore avoid asymptotic arbitrage. Fundamentally, we manage to show that asymptotic arbitrage in our CAPM-style model always occurs, unless Sharpe ratios with respect to the global market of the single securities are close to the same, homogeneous value. We also see that the correlation coefficient does not play any role in the finiteness of the value function or the existence of asymptotic arbitrage opportunities. We can also show that the limit of the value functions and portfolio parameters, as $N \to \infty$ does not equal the infinite result in general. This equality can only be established if the perturbations in the parameters are scaled in a strong manner.

Finally, Chapter 5 comprises proprietary, numerical modelling, carried out to give the reader a better intuition and understanding of the results obtained in Chapters 3 and 4. We consider two setups: first, a model with fixed homogeneous coefficients. Drawing the small perturbations from a zero-mean uniform distribution, centred around fixed parameters, we calculate realistic coefficients for our market model. Hence, we can illustrate the behaviour of the portfolio proportions and the value function growth rate. Secondly, we want to see whether the theoretical optimal portfolio strategies, derived in previous chapters, perform adequately in reality. Therefore we construct a portfolio, for a complete as well as an incomplete market setup, using US stocks according to our optimal proportion and rebalance it daily. We then compare this strategy to two simpler ones: the homogeneous parameter strategy and equally weighting the portfolio proportions of each of the N assets with $\frac{1}{N}$. Contrary to the results

in the first part of this chapter, the real trading model experiences little, to no growth. In the long run the tamer, equally weighted strategies seem to perform better than the optimal. The attempt to find an intuitive, economically-motivated explanation concludes this thesis in Chapter 6.

1.2 Market Model

Consider a market that consists of a large and in the limit possibly infinite-sized, basket of stocks with random coefficients. We shall adapt and extend Kabanov and Kramkov's findings in [KK93, KK98] by looking at a large market model represented by an increasing sequence of smaller markets. Each of these small markets has an at most finite number of assets, whose price processes admit martingale measures and are thus arbitrage free. However, as we will see in Section 1.6, asymptotic arbitrage opportunities can still arise when looking at the limit once we let $n \to \infty$. Heuristically, we can describe this financial market as follows. We have n traded stocks, which are exposed to idiosyncratic risk and are correlated with a collection of global risk factors similar to the well-known CAPM setting. We can also trade in a fixed number of additional assets, where each is only exposed to exactly one of these global risk factors. If the number of risk factors used corresponds exactly to the number of traded global assets, then we can say that our model is complete. However, if some or all of these market assets are not traded, the model becomes incomplete. This is due to the fact that we have more independent Brownian drivers than underlying securities.

Now let us explain this model with more mathematical precision. Suppose we have a sequence of probability spaces $\mathcal{B}_n = \big(\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n\big)$, $n \in \mathbb{N}$, where the filtration $\mathbb{F}_n$ satisfies the usual condition for each $n \in \mathbb{N}$. Let $\{W(t) = (W_1(t), \ldots, W_n(t)); 0 \leq t \leq T\}$ and $\{M(t) = (M^1(t), \ldots, M^J(t)); 0 \leq t \leq T\}$ be two vectors of independent standard one-dimensional Brownian motions generating these filtrations. Here the number of single securities can vary with $n \in \mathbb{N}$, and $J \in \mathbb{N}$ represents the maximal, fixed number of global risk factors. We write $\{B_n(t); 0 \leq t \leq T\}$ for simplification when we refer to the set of the $n + J$ Brownian motions $B_n(t) = (W_1(t), \ldots, W_n(t), M^1(t), \ldots, M^J(t))$. In particular, $\mathbb{F}_n = \big(\mathfrak{F}_n(t)\big)_{t\in[0,T]}$ is the filtration generated by $B_n(s)$ for $s \leq t$. It is common practise to fix a sequence of positive numbers $T_n < \infty$ for each $n \in \mathbb{N}$ as time horizons, but as we only look at the model for finite time, we will assume for simplification that all these time horizons are the same for any n and write T instead of T_n throughout.

Note that the complete n^{th} market model involves the first $n+J$ assets where the number of global securities $J \geq 0$ is fixed from the start. This emphasises that we consider the cases, where only the risky stock basket size n can become infinitely large. On the other hand, the case where $J \geq 1$ but the global market assets I_l are not available for trading, corresponds to the incomplete market model. In the case that $J = 0$ we retrieve the Black-Scholes model with n independent asset prices. The n risky assets follow the price processes given by $\{S(t) = (S_1(t), \ldots, S_n(t)), 0 \leq t \leq T\}$, $n \in \mathbb{N}$, which are described through our CAPM-style factor model and are assumed to be continuously traded. The J global market assets are denoted by $\{I(t) = (I_1(t), \ldots, I_J(t)), 0 \leq t \leq T\}$ and are also available for trading in the complete market case. Let us should point out that these securities are only governed by the market risk factor Brownian motions $M^l(t)$, $l = 1, \ldots, J$. We could think of the tradable global factors as indices.

The system of SDEs for $j = 1, \ldots, n$, $l = 1, \ldots, J$ evolves under the physical measure $\mathbb{P}_n$ as

$$\begin{aligned} dI_l(t) =& \mu_l(t) I_l(t) dt + \gamma_l(t) I_l(t) dM^l(t), \\ dS_j(t) =& \alpha_j(t) S_j(t) dt + \sigma_j(t) \sum_{l=1}^{J} \rho_{j,l} S_j(t) dM^l(t) + \sigma_j(t) \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2} S_j(t) dW_j(t). \end{aligned} \tag{1.1}$$

The risk free bank account, $\{S_0(t), 0 \leq t \leq T\}$, is

$$dS_0(t) = r(t) S_0(t) dt, \quad S_0(0) = 1, \tag{1.2}$$

and can be used as a nummëraire when changing to the equivalent martingale measures. We will return to this later in the chapter.

Before we lay out all the assumptions concerning the parameters and Brownian drivers, let us briefly comment on the financial interpretation of this model. It is obvious, that each asset can have its own, idiosyncratic risk. However, as the single stocks are all part of a larger market system, they can also experience further risk stemming from the underlying global markets. These global market risk factors can be viewed in two ways. One possibility is to assume that each of the global assets represents a market sector or industry, such as utilities, telecommunication, financials or healthcare. On the other hand, they can be viewed as attributes concerning one large market, such as GDP, financial stability or interest rate levels.

The modelling parameters are assumed to be $\mathfrak{F}_n(t)$- measurable, adapted processes, independent of the price processes and each other. Therefore, they do not anticipate the future but depend only on the past of the vector of Brownian motions B_n. We define $\{r(t), 0 \leq t \leq T\}$, $r(t) > 0$, as the risk free interest rate. In addition, we have the rate of returns of the stocks $\{\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)), 0 \leq t \leq T\}$, as well as the returns of the global market assets $\{\mu(t) = (\mu_1(t), \ldots, \mu_J(t)), 0 \leq t \leq T\}$. As the Brownian motions W_j, $j = 1, \ldots, n$, are all assumed to be independent, the volatility of the single securities $\{\sigma(t) \geq 0, 0 \leq t \leq T\}$, consists of the diagonal entries of the volatility matrix. In particular, for $i, j = 1, \ldots, N$ we have the matrix entries $\sigma_{ji} = 0$ for $j \neq i$, $i = 1, \ldots, n$ and $\sigma_{jj} = \sigma_j(t)$ represents the non-negative diagonal entries. The same argument holds for the market asset volatilities, which are given as $\{\gamma(t) \geq 0, 0 \leq t \leq T\}$ such that for $l, k = 1, \ldots J$, we have the matrix entries $\gamma_{lk} = 0$ for $l \neq k$ and $\gamma_{ll} = \gamma_l(t)$ for the diagonal elements. Finally, we have the correlation coefficients between the j^{th} stock and the risk factor from the l^{th} global asset as $\rho_{j,l}(t) \in (-1, 1)$, $j = 1, \ldots, n$, $l = 1, \ldots, J$. The coefficients satisfy the mild integrability condition

$$\int_0^T \Big(|r(t)| + |\alpha(t)| + |\mu(t)| + ||\sigma(t)||^2 + ||\gamma(t)||^2\Big) dt < \infty \quad a.s.$$

Here $|\cdot|$ is just the absolute value and $||\cdot||^2$ represents the ℓ^2-norm so that $||\sigma(t)||^2 = \sum_{j=1}^n |\sigma_j(t)|^2$ and $||\gamma(t)||^2 = \sum_{l=1}^J |\gamma_l(t)|^2$.

Hence, the symmetric covariance matrix for this system is given by

$$\Sigma = \begin{pmatrix} D & E \\ E^T & \Lambda \end{pmatrix}, \tag{1.3}$$

where

$$D = \begin{pmatrix} \gamma_1^2 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \gamma_J^2 \end{pmatrix},$$

$$E = \begin{pmatrix} \gamma_1\sigma_1 \sum_{l=1}^J \rho_{1,l} & \cdots & \cdots & \gamma_1\sigma_N \sum_{l=1}^J \rho_{N,l} \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_J\sigma_1 \sum_{l=1}^J \rho_{1,l} & \cdots & \cdots & \gamma_J\sigma_N \sum_{l=1}^J \rho_{N,l} \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_N \left(\sum_{l=1}^{J} \rho_{1,l}\rho_{N,l}\right) \\ \sigma_2\sigma_1 \left(\sum_{l=1}^{J} \rho_{2,l}\rho_{1,l}\right) & \cdots & \sigma_2\sigma_N \left(\sum_{l=1}^{J} \rho_{2,l}\rho_{N,l}\right) \\ \vdots & \ddots & \cdots \\ \sigma_N\sigma_1 \left(\sum_{l=1}^{J} \rho_{N,l}\rho_{1,l}\right) & \cdots & \sigma_N^2 \end{pmatrix}.$$

The matrix Σ is assumed to be of full rank, therefore invertible, and thus the underlying correlation structure of our asset system can be properly described.

Let us look at the dependence structure between the Brownian motions in detail. Writing $[\cdot,\cdot]$ for the quadratic covariation, we have for all $i, j = 1, \ldots, n$, $i \neq j$ and $k, l = 1, \ldots, J$, $l \neq k$, that

$$d[W_j, W_i] = 0,$$

$$d[W_j, M^l] = 0,$$

$$d[M^l, M^l] = dt,$$

$$d[W_j, W_j] = dt,$$

$$d[M^l, M^k] = 0.$$

We have now fully described our correlated J-factor model, where the idiosyncratic risk is captured through the $dW_j(t)$ term $j = 1, \ldots, n$, being unique to each asset and the overall market risk factors affecting all assets through $dM^l(t)$, $l = 1, \ldots, J$. Our general, finite market model for any fixed $n \in \mathbb{N}$ is therefore defined as the triple $\mathcal{M}_n = \Big(\mathcal{B}_n, T, (S(t), I(t))\Big)$.

Remark 1.1. *To draw a parallel to the well known Capital Asset Pricing Model, we should establish what the beta would be in the case of our modelling setup. In general, the beta of a portfolio is a widely-used measure of risk relating the movements of idiosyncratic risk factors to a global risk factor. In particular, it is defined as*

$$\beta_{j,l} = \frac{\rho_{j,l}\sigma_{j,l}}{\gamma_l},$$

$j = 1, \ldots, n$, $l = 1, \ldots, J$. *We can see that a beta of one means that the individual asset risk is exactly in line with the risk of the global asset. However, a beta smaller than one means that*

either the idiosyncratic risk of the single stock is very small in comparison with the global risk, or that its correlation with the market asset is low. On the other hand, a beta bigger than one indicates that the risk of the individual asset is substantially higher than that of a particular global risk factor, while still positively correlated with it. Applying the economic consensus that a very volatile investment should also produce a higher return, this can indicate a stock which outperforms this particular market asset.

Before moving on to the specific problem setting we want to address, let us briefly recall the concept of change of measure. Suppose $\mathbb{Q}_n$ is a probability measure on the space $(\Omega_n, \mathfrak{F}_n)$ for each $n \in \mathbb{N}$. If we have that for any event $A_n \in \mathfrak{F}_n$, that if $\mathbb{Q}_n(A_n) = 0$ when $\mathbb{P}_n(A_n) = 0$, then, $\mathbb{Q}_n$ is absolutely continuous with respect to $\mathbb{P}_n$. We write $\mathbb{Q}_n \ll \mathbb{P}_n$. If the reverse is also true, then the measures are absolutely continuous with respect to each other and are said to be equivalent, with notation $\mathbb{Q}_n \sim \mathbb{P}_n$.

For two equivalent measures $\mathbb{Q}_n \sim \mathbb{P}_n$ on the probability space $(\Omega_n, \mathfrak{F}_n(T))$ there exists an associated Radon-Nikodym density process, $Z_n(t)$, defined as

$$Z_n(t) = \left.\frac{d\mathbb{Q}_n}{d\mathbb{P}_n}\right|_{\mathfrak{F}_n(t)},$$

for each $n \in \mathbb{N}$, $t \in [0, T]$.

The process $Z_n(t)$, for all $t \in [0, T]$, is a positive martingale under $\mathbb{P}_n$ with respect to the filtration.

On the other hand, given a positive martingale $Z_n(t)$, we can find such an equivalent martingale measure $\mathbb{Q}_n$ by the relationship

$$\mathbb{Q}_n(A_n) \triangleq \mathbb{E}_{\mathbb{P}_n}[Z_n(t)\mathbb{1}_{A_n}] = \int_{A_n} Z_n(t) d\mathbb{P}_n,$$

where $A_n \in \mathfrak{F}_n(t)$ for each time $t \in [0, T]$.

We know that the Radon-Nikodym density $Z_n(t)$ is strictly positive a.s. In addition, there exists a unique continuous local martingale $L_n(t)$, with $L_n(0) = 0$, such that we have the representation

$$Z_n(t) = \exp\left\{L_n(t) - \frac{1}{2}\langle L_n(t), L_n(t)\rangle\right\} \triangleq \mathcal{E}(t, L_n(t)), \tag{1.4}$$

where $L_n(t)$, $0 \leq t \leq T$, for each $n \in \mathbb{N}$, and is formally given by

$$L_n(t) = \int_0^t \frac{1}{Z_n(s)} dZ_n(s) \quad a.s.,$$

and $\mathcal{E}(t)$, $0 \leq t \leq T$ is the Doléans-Dade exponential, see [Pha09].

For our market model, described through the set of SDEs in equation (1.1) we will define the relative risk process $\boldsymbol{\theta}_n$ by using the inverse of the covariance matrix Σ from equation (1.3).

In particular we have

$$\boldsymbol{\theta}_n \boldsymbol{\theta}_n^T = \Big((\mu_1 - r)\cdots(\mu_J - r)(\alpha_1 - r)\cdots(\alpha_n - r)\Big)\,\Sigma^{-1} \begin{pmatrix} (\mu_1 - r) \\ \vdots \\ (\mu_J - r) \\ (\alpha_1 - r) \\ \vdots \\ (\alpha_n - r) \end{pmatrix},$$

where $\boldsymbol{\theta}_n^T$ denotes the transpose. If can write $L_n(t)$ using our Brownian model as

$$L_n(t) = -\int_0^t \boldsymbol{\theta}_n(s) dB_n(s),$$

by Girsanov's theorem, then the associated exponential change of measure process $Z_n(t)$, as defined in (1.4), can be expressed using the vector $\boldsymbol{\theta}_n$ as

$$Z_n(t) \triangleq \exp\left\{-\int_0^t \boldsymbol{\theta}_n(s) dB_n(s) - \frac{1}{2}\int_0^t ||\boldsymbol{\theta}_n(s)||^2 ds\right\}, \quad 0 \leq t \leq T. \tag{1.5}$$

Let us briefly define relative risk price process in general.

Definition 1. *The relative risk process (also referred to as the market price of risk) is the continuous, $\mathfrak{F}_n$-measurable, adapted vector process $\boldsymbol{\theta}_n : [0,T] \times \Omega_n \to \mathbb{R}^{n+J}$ with $n+J$ elements, $n \in \mathbb{N}$ varying and $J \geq 0$ fixed, such that it is square integrable*

$$\int_0^T ||\boldsymbol{\theta}_n(t)||^2 dt < \infty \quad a.s.,$$

and

$$\mathbb{E}\left[\exp\left\{-\int_0^T \boldsymbol{\theta}_n(t) dB_n(t) - \frac{1}{2}\int_0^T ||\boldsymbol{\theta}_n(t)||^2 dt\right\}\right] = 1.$$

Then, under the assumption that the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||\boldsymbol{\theta}_n(t)||^2 dt\right\}\right] < \infty \tag{1.6}$$

holds, $Z_n(t)$ is a true martingale. Hence, there exists an equivalent martingale measure $\mathbb{Q}_n \sim \mathbb{P}_n$ and the market in $\mathcal{M}_n$ for each $n \in \mathbb{N}$ is arbitrage free. We can summarise this as follows:

Definition 2. *Suppose we have two sets of measures, such that for each fixed $n \in \mathbb{N}$, $\mathbb{P}_n$ and $\mathbb{Q}_n$ are absolutely continuous with respect to each other. Then, we denote by $\mathcal{P}_n$ the set of all equivalent martingale measures*

$$\mathcal{P}_n = \{\mathbb{Q}_n \sim \mathbb{P}_n | \mathbb{Q}_n \text{ is a martingale measure }\}. \tag{1.7}$$

Moreover we can make the following remark for each of the markets $\mathcal{M}_n$.

Remark 1.2. *Each market $\mathcal{M}_n$ is arbitrage free iff for each $n \in \mathbb{N}$*

$$\mathcal{P}_n \neq \emptyset.$$

It is important to emphasise that, although we have no arbitrage in each finite market $\mathcal{M}_n$, we can still experience asymptotic arbitrage in the limiting market $\mathcal{M}$ as $n \to \infty$, see Section 1.4 for the definition. The notion of asymptotic arbitrage will be formally introduced in Section 1.6 and requires a new type of no-arbitrage condition for the limiting case as we let $n \to \infty$.

Finally, by Girsanov's theorem, the process $\hat{B}_n(t)$, for $0 \leq t \leq T$, given by

$$\hat{B}_n(t) \triangleq B_n(t) + \int_0^t \boldsymbol{\theta}_n(s) ds, \tag{1.8}$$

is a Brownian motion under $\mathbb{Q}_n$, rendering our asset price dynamics (1.1) risk free if we use $S_0(t)$ from (1.2) as the numéraire.

Note that in a complete market, there exists exactly one martingale measure, i.e. $\mathcal{P}_n$ is a singleton, and there is only one risk-neutral density $Z_n(t)$, $t \in [0,T]$, thus there is a unique martingale price for each of the risky securities $S_j(t)$, $j = 1, \ldots, n$ as well as the global assets $I_l(t)$, $l = 1, \ldots, J$. However, if the model is incomplete, the set $\mathcal{P}_n$ can have infinite cardinality and we would require further constraints in order to choose the appropriate pricing measure $\mathbb{Q}_n$.

1.3 Optimisation and Stochastic Control Problems

One of the main objectives of this thesis is to study optimisation problems, where a basket of assets which is driven by Brownian motion can be characterised by its state process at any time $0 \leq t \leq T$. The aim is to control this state process, using appropriate control processes in such a way as to achieve the maximal expected utility of terminal wealth at a finite, fixed maturity $T < \infty$.

There are well-established methodologies to approach such a problem. Either, we can derive and investigate the Hamilton-Jacobi-Bellman (HJB) equation, a non-linear second order PDE, for an a priori given utility function describing the agent's preferences, see [Bjo04]. Or, in a more general setting, a solution to the optimisation problem can be found using martingale and duality arguments, which will be covered in Chapter 2. The first is a very powerful approach to find explicit solutions, by examining the local behaviour of the solution to our optimal control problem, called the value function. It is derived through the dynamic programming principle and by the application of Itô's lemma. These concrete solutions for our particular market model will be calculated in Chapter 3. The second methodology allows us to find the general optimal value function without prior knowledge of the shape of the utility function, see for example Chapter 7 in [Pha09].

In this section we cover the background for a finite market $\mathcal{M}_n$, $n \in \mathbb{N}$. We then extend these concepts to an infinite market $\mathcal{M}$ in Section 1.4. Suppose that we have the market model $\mathcal{M}_n$ as introduced in Section 1.2. Let $\pi(t) = \{(\pi_1(t), \ldots, \pi_n(t), \tilde{\pi}_1(t), \ldots, \tilde{\pi}_J(t)), 0 \leq t \leq T\}$ on $\mathbb{R}^{n+J}$, be a vector control process, that is measurable and adapted to the filtration $\mathbb{F}_n$. At every instant $t \in [0,T]$ and for a fixed number of risky stocks $n \in \mathbb{N}$ as well as a fixed number $J \geq 0$ of global assets, $\pi(t)$ represents the agent's decision on what proportion of wealth to invest in each of these assets described by the dynamics in (1.1). Hence, it is also referred to as the portfolio proportion. We then allocate the remaining funds $\pi_0 = 1 - \left(\sum_{j=1}^{n} \pi_j(t)\right) - \left(\sum_{l=1}^{J} \tilde{\pi}_l(t)\right)$ to the risk-less bond $S_0(t)$. Note that we make no further assumptions on the vector of proportions $\pi(t)$ at this moment, which means that

both short-selling as well as borrowing from the bank account are permitted.

The investor's wealth at each time $0 \leq t \leq T$ is represented by the state process $\{X_n(t); 0 \leq t \leq T\}$, also called the wealth process. In particular, for each risky stock $S_j(t)$, $j = 1, \ldots, n$ and global asset $I_l(t)$, $l = 1, \ldots, J$, we have the corresponding $\pi_j(t) = \frac{\upsilon_j S_j(t)}{X_n(t)}$ and $\tilde{\pi}_l(t) = \frac{\tilde{\upsilon}_l I_l(t)}{X_n(t)}$, where υ_j, and $\tilde{\upsilon}_l$ are the amounts of each security held by the investor. In addition, υ_0 is the amount deposited in the bank account. Therefore, the wealth process is given by

$$X_n(t) = \sum_{j=1}^{n} \upsilon_j S_j(t) + \sum_{l=1}^{J} \tilde{\upsilon}_l I_l(t) + \upsilon_0 S_0(t), \tag{1.9}$$

and assuming the investor's trading strategies are self-financing, the wealth process follows the dynamics

$$\begin{aligned} dX_n(t) &= \sum_{j=1}^{n} \upsilon_j dS_j(t) + \sum_{l=1}^{J} \tilde{\upsilon}_l dI_l(t) + \upsilon_0 dS_0(t) \\ &= \left[\sum_{j=1}^{n} (\alpha_j - r)\pi_j(t) + \sum_{l=1}^{J} (\mu_l - r)\tilde{\pi}_l(t) + r \right] X_n(t)dt \\ &\quad + \sum_{j=1}^{n} \sigma_j \pi_j(t) \sum_{l=1}^{J} \rho_{j,l} X_n(t) dM^l(t) + \sum_{l=1}^{J} \gamma_l \tilde{\pi}_l(t) X_n(t) dM^l(t) \\ &\quad + \sum_{j=1}^{n} \sigma_j \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2} \pi_j(t) X_n(t) dW_j(t), \end{aligned} \tag{1.10}$$

with the initial endowment $X_n(0) = x > 0$. To emphasise that the wealth process is controlled by the control vector $\pi(t)$ and starts at the initial capital $x > 0$ we will sometimes write $X_n(t) = X_n^{\pi,x}(t)$.

As our modelling parameters are assumed to be adapted stochastic processes, we require the following integrability conditions:

$$\begin{aligned} \int_0^T \left| \sum_{j=1}^{n} (\alpha_j(t) - r(t))\pi_j(t) + \sum_{l=1}^{J} (\mu_l(t) - r(t))\tilde{\pi}_l(t) \right| dt < \infty \quad a.s., \\ \int_0^T \left| \sum_{j=1}^{n} \sigma_j(t) \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2(t)} \pi_j(t) \right|^2 dt < \infty \quad a.s., \\ \int_0^T \left| \sum_{j=1}^{n} \sigma_j(t)\pi_j(t)(t) \sum_{l=1}^{J} \rho_{j,l}(t) + \sum_{l=1}^{J} \gamma_l(t)\tilde{\pi}_l(t) \right|^2 dt < \infty \quad a.s., \end{aligned} \tag{1.11}$$

as well as appropriate Lipschitz and linear growth assumptions, see [Kar97] or [Pha09] for more detail.

Under the above conditions, there exists a unique solution (see [Kar97, Pha09]) to the wealth SDE in (1.10), given by

$$\begin{aligned} X_n(t) =& x \exp\Bigg\{ \int_0^t \Bigg[\sum_{j=1}^{n} (\alpha_j(s) - r(s))\pi_j(s) + \sum_{l=1}^{J} (\mu_l(s) - r(s))\tilde{\pi}_l(s) - \frac{1}{2}\sum_{j=1}^{n} \sigma_j^2(s)\pi_j^2(s) \\ & - \frac{1}{2}\sum_{l=1}^{J} \gamma_l^2(s)\tilde{\pi}_l^2(s) - \sum_{j=1}^{n}\sum_{i\neq j} \sigma_j(s)\sigma_i(s)\rho_j(s)\rho_i(s)\pi_j(s)\pi_i(s) \\ & - \sum_{j=1}^{n} \sigma_j(s)\rho_j(s) \sum_{l=1}^{J} \gamma_l(s)\tilde{\pi}_l(s) \Bigg] ds \\ & + \int_0^t \left(\sum_{j=1}^{n} \sigma_j(s)\pi_j(s) \sum_{l=1}^{J} \rho_{j,l}(s) dM^l(s) + \sum_{l=1}^{J} \gamma_l(s)\tilde{\pi}_l(s) dM^l(s) \right) \\ & + \int_0^t \sum_{j=1}^{n} \sigma_j(s) \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2(s)}\, \pi_j(s) dW_j(s) \Bigg\}. \end{aligned} \tag{1.12}$$

In order to formulate the core optimisation problem, we need to define the concept of a utility function, as it describes the preference structure of our agent, i.e. how much risk he is willing to accept, see [KS99].

Definition 3. *A function $U : (0,\infty) \to (0,\infty)$ is called a utility function if, for any $x \in \mathbb{R}_+$, it satisfies that $U(x)$ is strictly increasing, strictly concave and continuously differentiable.*

Remark 1.3. *Most authors usually assume that in addition to Definition 3 the following inada conditions for the marginal utility $\frac{\partial U}{\partial x} = U'(x)$ hold (besides in the case of exponential utility, where these do not hold):*

(i) $U'(\infty) = \lim_{x\uparrow\infty} U'(x) = 0$,

(ii) $U'(0) = \lim_{x\downarrow 0} U'(x) = \infty$, *see for example [KS99].*

There are a variety of utility functions considered in the literature. To calculate a closed form value function using the HJB equation framework, it is helpful to be able to represent the value function in a separable form. This means that it is possible to separate the time and space variables. We know from the literature, see [Kar97, Pha09], that the value function inherits its shape from the utility function. Therefore, and as it allows a good interpretation from an economic point of view, we want to choose a utility function depending on the Arrow-Pratt measure of risk aversion $R(t)$, which itself is a function only dependent on time, see [Bjo04].

In that sense, a very convenient set of utility function belongs to the class of Constant Relative Risk Aversion (CRRA), where the utility functions are in general defined as:

$$U(x) = \begin{cases} \frac{x^p}{p}, \text{ if } p < 1, p \neq 0, \\ \\ \ln x, \text{ if } p = 0. \end{cases} \tag{1.13}$$

Here the parameter p measures the degree of relative risk aversion, see [Bjo04].

In particular, for utilities of the CRRA class, we have that the measure of risk aversion is just a constant and the following relation holds

$$-\frac{xU''(x)}{U'} = R. \tag{1.14}$$

Therefore, the measure of risk aversion remains the same even if the investor's demeanour changes from risk averse to risk loving as his wealth varies. Members of this class include in particular the power utility, considered by Merton [Mer69] in the original setting, as well as logarithmic utility. For a real valued parameter $p < 1$, $p \neq 0$, also called the penalty constant, the power utility function is defined as

$$U(x) = \frac{x^p}{p}, \tag{1.15}$$

which, as $p \to 0$, approaches the logarithmic utility

$$U(x) = \ln x. \tag{1.16}$$

We need one further condition on the utility function, called the asymptotic elasticity, which we assume to hold throughout this thesis. In particular, this is required to be able to find a unique and finite value function, in the case of incomplete markets.

Definition 4. *The utility function $U : (0, \infty) \to (0, \infty)$ is said to have asymptotic elasticity if*

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

Furthermore, we only consider the power utility function in our explicit calculations, which already has the property that $AE(U) = p$ where $p < 1$ by construction.

Let us comment on one final aspect of the modelling setup before we can state our optimisation problem. Optimising the expected utility of terminal wealth over the vector of portfolio proportions $\{\pi(t); 0 \leq t \leq T\}$ yields a set of solutions for these portfolio weights. Therefore, to find the concrete set of portfolio proportions best fitted for our market environment, we

have to introduce some constraints. These constraints are described by the admissibility set, as it defines which values our vector of control processes is allowed to take. In particular, we need to make sure that the policies that would allow our agent to trade when they were bankrupt, i.e. the ones which would allow our wealth process to be strictly less than zero for any time $t \in [0, T]$, are excluded. Note that bankruptcy is actually an absorbing state, which means, once it is reached, the agent has no further funds available to him for investment and hence the process stops. For more on this topic, see [Kar97].
A portfolio process $\{\pi(t); 0 \leq t \leq T\}$ is called an admissible control for the initial capital $x > 0$ if we have

$$X_n(t) \equiv X_n^{\pi,x}(t) \geq 0, \quad \forall 0 \leq t \leq T.$$

In general the class of admissible controls, for a market containing $n \in \mathbb{N}$ single stocks, is called the admissibility set and is denoted by $\mathcal{A}_n(x)$.

In our problems we will use the term admissibility set for more restrictive sets of controls incorporating utility functions.

Our main objective is to investigate the following:

Problem 1. *For a utility function as in Definition 3, we aim to maximise the expected utility of terminal wealth, given by the twice continuously differentiable, increasing and concave value function*

$$u_n(x) = \sup_{\pi \in \mathcal{A}_n(x)} \mathbb{E}\left[U(X_n^{\pi,x}(T)) | X_n^{\pi,x}(0) = x\right], \tag{1.17}$$

over the admissibility set

$$\mathcal{A}_n(x) = \{\pi : \mathbb{E}[|U(X_n^{\pi,x}(T))|] < \infty, \quad X_n^{\pi,x}(t) \geq 0, \quad t \in [0, T]\}.$$

We are primarily concerned with calculating both the explicit optimal value function $\hat{u}_n$ and the optimal portfolio proportion process $\hat{\pi}(t) \in \mathcal{A}_n(x)$, $0 \leq t \leq T$, that attains the supremum in (1.17).

We conclude this section with a brief historical comment. Over the past 50 years, a variety of methods have been developed to find a solution to this problem. One of the first studies, particularly in this economically motivated setting, was conducted by Merton [Mer69, Mer71], who solved the above problem for more simple geometric Brownian dynamics and for both a finite and an infinite time horizon in the late 1960s and early 1970s, respectively. He used the dynamic programming principle (DPP) and the Hamilton-Jacobi-Bellman equation to expand the value function around a fixed point in space and time, and thus changed the problem into

solving a partial differential equation (PDE). This was only made possible by assuming that the utility function is of a certain fixed shape, specifically CRRA. Then, the PDE simplifies to an ordinary differential equation (ODE), allowing for a straightforward solution.

A general solution, where the utility function is not specified upfront, was found considerably later by authors such as Karatzas, Shreve et al. in [KLS87, KLSS86], [Kar97] and [KS00]. They used martingale methods and duality relations between the wealth process and the risk free probability density.

In this thesis, we want to use this duality and martingale approach to obtain a link to the absence of asymptotic arbitrage. These methods, however, only show the existence of a portfolio proportion process, without providing an explicit shape for it.

On the other hand, if we aim to look at the explicit portfolio proportions in the limit as n approaches infinity, it is more convenient to apply the HJB methodology. It has the drawback of forcing us to use a particular, predetermined utility function, but gives us the ability to compute the concrete vector of strategies $\hat{\pi}(t)$ that attain the supremum in (1.17).

1.4 Extension to an Infinite Market

In the preceding sections, we have frequently referred to a large or infinite market, which we want to explain at this point. Asymptotic arbitrage only appears once we look at the limiting market. Hence we provide the required framework in this section. We will first construct the extension of our finite probability space to a countable space, then look at the limiting martingale density and provide the limiting assumptions for it to be a valid change of measure. Under these conditions we can show that there exists a well defined limiting martingale measure for our market model. Having described what we mean by an infinite market, we will then move on to define the limiting optimisation problem in the remainder of the section.

The first step is to understand the underlying probability space. For a finite setting with $n + J$ assets, where n is the number of single assets and J is the number of global market assets, we have the sequence of filtered probability spaces $\mathcal{B}_n = \big(\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n\big)$ increasing in n. It is convenient to interpret Ω_n to be the product space of real-valued continuous functions, i.e. $\Omega_n = (C[0,T], \mathbb{R})^{n+J}$. Our σ-algebra is then defined as the product $\mathfrak{F}_n = \prod_{i=1}^{n+J} \mathcal{G}_i$, $i = 1, \ldots, n$, where each $\mathcal{G}_i$ is the Brownian-motion-generated σ-algebra for each one-dimensional probability space. The probability measure on $(\Omega_n, \mathfrak{F}_n)$ for each $n \in \mathbb{N}$

is $\mathbb{P}_n$. In particular, an element $\omega \in \Omega_n$ is the vector $\omega = (\omega_1, \ldots, \omega_{n+J})$, such that for an event $A \in \mathfrak{F}_n$ we have that $A = (A_1, \ldots, A_{n+J})$ and

$$A = \{\omega \in \Omega_n : \quad \omega_i \in A_i \quad \text{for} \quad i = 1, \ldots, n+J \quad A_i \in \mathcal{G}_i\}.$$

From here on we denote $\mathbb{E}_{\mathbb{P}_n}$ as $\mathbb{E}$ for simplicity.

Considering our underlying stock universe, it is more convenient to restate the $n+J$ assets consisting of the risky stocks $S_j(t)$, $j = 1, \ldots, n$ and the global market assets $I_l(t)$, $l = 1, \ldots, J$, driven by the $n+J$-length Brownian motion vector $B_n(t) = \big(W_1(t), \ldots, W_n(t), M^1(t), \ldots, M^J(t)\big)$ from equation (1.1) in terms of n correlated Brownian drivers. Therefore, our model can be rewritten as the system of SDEs:

$$\begin{aligned} dS_j(t) =& \alpha_j(t)S_j(t)dt + \sigma_j(t)S_j \left(\sum_{l=1}^{J} \rho_{j,l}(t)dM^l(t) + \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2(t)}dW_j(t) \right) \\ =& \alpha_j(t)S_j(t)dt + \sigma_j(t)S_j(t)d\tilde{B}_j(t). \end{aligned}$$

Here, $\tilde{B}_j(t) = \sum_{l=1}^{J} \rho_{j,l}(t)dM^l(t) + \sqrt{1 - \sum_{l=1}^{J} \rho_{j,l}^2(t)}dW_j(t)$, $j = 1, \ldots, n$ is the j^{th} entry to the vector of correlated Brownian motions $\tilde{B} = (\tilde{B}_1, \ldots, \tilde{B}_n)$. From now on we assume that the modelling parameters $\alpha_j, \sigma_j, \rho_{j,l}, \mu_l$ and γ_l are deterministic functions of t for all j and l.

Given such a structure and these elements of our probability space for a finite $n \in \mathbb{N}$, we need to extend this setup to a countable collection of spaces and then ultimately the limiting probability space. Suppose that the product space is extended to the limiting product space $\Omega_{\mathbb{N}} = (C[0,T], \mathbb{R})^{\mathbb{N}}$. In addition, consider the product σ-algebra $\mathfrak{F}_\infty = \prod_{n \in \mathbb{N}} \mathcal{G}_n$. Then, an element $\omega \in \Omega_{\mathbb{N}}$ is a countably infinite vector $\omega = (\omega_1, \omega_2, \ldots)$ and an event $A = (A_1, A_2, \ldots)$, where $A \in \mathfrak{F}_\infty$, has the structure

$$A = \{\omega \in \Omega_{\mathbb{N}} : \quad \omega_n \in A_n \quad \text{for} \quad n \in \mathbb{N} \quad A_i \in \mathcal{G}_i\}.$$

This assumes that we have a countable length vector of assets, which is built by adding risky stocks one by one to the finite collection described above. These are then driven by countably many independent Brownian motions.

Let us turn to the extension of the probability measure $\mathbb{P}_n$ to the limiting measure $\mathbb{P}_\infty$. According to the Kolmogorov Extension Theorem (KET), for example see Theorem 4.9, Chapter 1 in [Dur91], we can construct the extension to a probability measure $\mathbb{P}_n$ as follows. Consider the space of continuous, real valued functions $(C[0,T], \mathbb{R})^{\mathbb{N}}$. Because the asset prices $\big(S_n(t,\omega)\big)_{t \in [0,T]}$, $n \in \mathbb{N}$ are path valued random variables, such that for an

$A = (A_1, A_2, \ldots) \in \mathfrak{F}_\infty$, we have $\omega_i \in A_i$, for each $i \in \mathbb{N}$, we have that the $n+1^{st}$ asset does not affect the probabilities of the preceding assets $1, \ldots, n$. Thus, the measures are consistent, and we have that

$$\mathbb{P}_{n+1}(A_1, \ldots, A_n, C[0,T]) = \mathbb{P}_n(A_1, \ldots, A_n).$$

Applying the KET, we therefore see that there exists the limiting measure $\mathbb{P}_\infty$ on $\Omega_\mathbb{N}$ such that

$$\mathbb{P}_\infty(A_1, \ldots, A_n) = \mathbb{P}_n(A_1, \ldots, A_n).$$

Hence, restricting this measure to the first n coordinates we have

$$\mathbb{P}_\infty\Big|_{\mathfrak{F}_n} = \mathbb{P}_n.$$

Consequently, we know that our sequence of probability spaces can be extended to the limiting space $\mathcal{B}_\infty = \big(\Omega_\mathbb{N}, \mathfrak{F}_\infty, \mathbb{F}_\infty = (\mathfrak{F}_\infty(t)), \mathbb{P}_\infty\big)$, such that these probability spaces form an increasing sequence and $\mathcal{B}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Therefore, we are fully able to describe our limiting market as the union of the sequence of sub-markets such that $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$.

Let us look at the equivalent martingale measures. So far we had for each finite $n \in \mathbb{N}$, that if we can find a process

$$Z_n(t) = \exp\left\{-\int_0^t \boldsymbol{\theta}_n(s) dB_n(s) - \frac{1}{2}\int_0^t ||\boldsymbol{\theta}_n(s)||^2 ds\right\}, \quad 0 \le t \le T, \tag{1.18}$$

such that $Z_n(t)$ is a true martingale for all $t \in [0,T]$ and each $n \in \mathbb{N}$, then it is a valid change of measure. Therefore, it can be identified with the Radon-Nikodym density, such that $Z_n(t) = \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}$, in particular $\mathbb{E}_{\mathbb{P}_n}[Z_n(t)] = 1$ for all $t \in [0,T]$. Using this identity, we can construct a martingale measure $\mathbb{Q}_n$ which is equivalent to the real world measure $\mathbb{P}_n$, i.e. $\mathbb{Q}_n \in \mathcal{P}_n$ for each $n \in \mathbb{N}$. Our aim now is to find a limit of the equivalent martingale measures $\mathbb{Q}_n$, such that $\lim_{n\to\infty} \mathbb{Q}_n = \mathbb{Q}_\infty \sim \mathbb{P}_\infty$. To achieve this, we first need to show that a non-degenerate limiting process $Z_\infty(T)$, $0 \le t \le T$ exists and, secondly, that it is a true martingale for all $t \in [0,T]$.

Firstly we show that there exists a limit process $\lim_{n\to\infty} Z_n = Z_\infty$ such that $Z_\infty < \infty$. To establish this, we take a fixed time $T < \infty$ and show that the parameterised family $\{Z_n(T)\}_{n\in\mathbb{N}}$ is $\mathcal{L}^2$ bounded (and therefore a Uniformly Integrable (UI) sequence) and that for each n the random variable $Z_n(T)$ can be written as a product of independent random variables. Then, Kakutani's Theorem, see Chapter 14 Theorem 14.12 in [Wil04], establishes that $0 < Z_\infty(T) < \infty$. To start, consider the following definition.

Definition 5. *Suppose that we have the relative risk process* $\hat{\boldsymbol{\theta}} = \{\hat{\boldsymbol{\theta}}(t); t \in [0,T]\}$ *such that for each component* $\hat{\theta}^{(i)} = \{\hat{\theta}^{(i)}(t); t \in [0,T]\}$, $\hat{\boldsymbol{\theta}} = (\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(k)}, \hat{\theta}^{(k+1)}, \ldots)$. *Then, we have the space of countably infinite sequences of processes* $\ell^2(\Omega_\infty, \mathfrak{F}_\infty, \mathbb{P}_\infty)$ *with well defined limits*

$$\ell^2 \triangleq \Big\{\hat{\boldsymbol{\theta}} = (\hat{\theta}^{(i)})_{i=1}^{\infty} : \quad ||\hat{\boldsymbol{\theta}}||^2_{\ell^2} := \int_0^T \sum_{i=1}^{\infty} |\hat{\theta}^{(i)}(s)|^2 ds < \infty, \quad \mathbb{P} \quad a.s.\Big\}.$$

In particular, we can extend a finite vector $\hat{\boldsymbol{\theta}}_k = \left(\hat{\boldsymbol{\theta}}_k^{(i)}\right)_{i=1}^{k}$, *defined for a finite* $k \in \mathbb{N}$, *to* ℓ^2 *by setting the components* $\hat{\theta}_k^{(i)} = 0$ *for* $i = k+1, \ldots$, $t \in [0,T]$, *i.e.*

$$\hat{\boldsymbol{\theta}}_k = (\hat{\theta}_k^{(1)}, \ldots, \hat{\theta}_k^{(k)}, 0, 0, \ldots),$$

with entries in the first k *places and zeroes everywhere else.*

For any $\hat{\boldsymbol{\theta}} \in \ell^2$, and $n \in \mathbb{N}$, $\hat{\boldsymbol{\theta}}_n$, where $\hat{\boldsymbol{\theta}}_n = (\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(n)}, 0, 0, \ldots)$ which is $\hat{\boldsymbol{\theta}}$ restricted to the first n entries, converges to $\hat{\boldsymbol{\theta}}$ in ℓ^2. This follows as we have that the first n entries of the two vectors agree , i.e. if $\hat{\theta}_n^{(i)} = \hat{\theta}^{(i)}$ for $1 \leq i \leq n$, so that

$$\int_0^T \sum_{i=1}^{\infty} \left(\hat{\theta}_n^{(i)}(t) - \hat{\theta}^{(i)}(t)\right)^2 dt = \int_0^T \sum_{i=n+1}^{\infty} \left(\hat{\theta}^{(i)}(t)\right)^2 dt.$$

Therefore, if $||\hat{\boldsymbol{\theta}}||^2_{\ell^2} < \infty, \quad a.s.$ then

$$\int_0^T \sum_{i=n+1}^{\infty} \left(\hat{\theta}^{(i)}(t)\right)^2 dt \to 0,$$

for $n \to \infty$ and hence $\hat{\boldsymbol{\theta}}_n \to \hat{\boldsymbol{\theta}}$ in ℓ^2, almost surely.

Using the above definition of the relative risk process, it is easy to see that the process Z_n, which is driven by a vector of $n+J$ independent Brownian motions, can be written as the product of independent random variables $e^{-\int_0^T \hat{\theta}_n^{(i)}(t) dB_i(t) - \frac{1}{2}\int_0^T |\hat{\theta}_n^{(i)}|^2(t) dt}$ in $\mathcal{L}^2$, where the $\hat{\boldsymbol{\theta}}_n$ are the relative risk processes under the modelling assumptions in this section.

Now all that is left to show for the direct application of Kakutani's Theorem, which establishes the convergence of the sequence of martingales $\{Z_n(T)\}_{n\in\mathbb{N}}$ to a well-defined limit, is that this sequence is indeed UI.

Lemma 1.4. *Suppose that we have the relative risk process vectors $\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\theta}} \in \ell^2$,where $||\hat{\boldsymbol{\theta}}_n||^2_{\ell^2}$ is monotone increasing for $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \hat{\boldsymbol{\theta}}_n = \hat{\boldsymbol{\theta}}$, almost surely in ℓ^2. Assuming that*

$$\mathbb{E}\Big[e^{\int_0^T ||\hat{\boldsymbol{\theta}}(s)||^2 ds}\Big] < \infty, \tag{1.19}$$

we have that $\Big\{Z_n(T)\Big\}_{n\in\mathbb{N}}$ is a UI family of random variables and

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}_n(s)||^2 ds}\Big] = \mathbb{E}\Big[e^{\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}(s)||^2 ds}\Big] \quad a.s.$$

Proof. Consider the parameterised family $\{Z_n(T)\}_{n\in\mathbb{N}}$ and show that it is indeed UI. First let us rewrite $Z_n(T)$ as

$$\begin{aligned} Z_n(T) &= \prod_{i=1}^{n+J} e^{-\int_0^T \hat{\theta}^{(i)}(t)dB_i(t) - \frac{1}{2}\int_0^T |\hat{\theta}^{(i)}|^2(t)dt} \\ &= Z_{n-1} e^{-\int_0^T \hat{\theta}^{(n+J)}dB_{n+J} - \frac{1}{2}\int_0^T |\hat{\theta}^{(n+J)}|^2} \\ &= Z_{n-1}\chi_n \end{aligned}$$

which is a product of independent mean one random variables by our modelling assumptions. Now we need to see that $Z_n(T)$ is square integrable in n, for each $n \in \mathbb{N}$, and the sequence is bounded in $\mathcal{L}^2$, thus Uniformly Integrable. From equation (1.18) we can see by calculation that

$$\sup_n \left(\mathbb{E}[Z_n^2(T)]\right) = \sup_n \mathbb{E}\Big[e^{\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt}\Big]. \tag{1.20}$$

In particular, due to the monotonicity of the $\hat{\boldsymbol{\theta}}_n$ and the assumption (1.19), we have that

$$\sup_n \mathbb{E}\Big[e^{\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt}\Big] = \mathbb{E}\Big[e^{\int_0^T ||\hat{\boldsymbol{\theta}}(t)||^2 dt}\Big] < \infty. \tag{1.21}$$

Hence, $Z_n(T)$ is a UI product martingale for each $n \in \mathbb{N}$, and therefore, by Kakutani's Theorem $\lim_{n\to\infty} Z_n(T) = Z_\infty(T) < \infty$ exists almost surely and in $\mathcal{L}^2$. Moreover, as $||\hat{\boldsymbol{\theta}}_n||_{\ell^2}$ is monotone increasing in n and $\hat{\boldsymbol{\theta}}_n$ converges to $\hat{\boldsymbol{\theta}}$ in ℓ^2, we can interchange limits and expectations and have

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{\left\{\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}_n(s)||^2 ds\right\}}\Big] = \mathbb{E}\Big[e^{\left\{\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}(s)||^2 ds\right\}}\Big].$$

□

We can see from the statements above that the inverse $Z_n^{-1}(T)$ also exists and is finite. Then, also $Z_\infty^{-1}(T) \neq 0$ and we have that Z_∞ is non-degenerate.

Our final step in showing that Z_∞ is a valid change of measure, is to show that it is a true martingale for each $t \leq T$. So far we only know that for a fixed time T, we have the limit $\lim_{n\to\infty} Z_n(T) = Z_\infty(T)$. By the martingale property, we can retrieve $Z_\infty(t)$ for all times $t \leq T$. However, we also need to see that, in addition to being a martingale, it can actually be identified with the limiting process $\lim_{n\to\infty} Z_n(t) = Z_\infty(t)$ for all $t \in [0, T]$.

Lemma 1.5. *Suppose that for a finite $n \in \mathbb{N}$, the probability space is given by the increasing sequence $\mathcal{B}_n = (\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n)$ and can be extended to $\mathcal{B}_\infty = \big(\Omega_\mathbb{N}, \mathfrak{F}_\infty, \mathbb{F}_\infty = (\mathfrak{F}_\infty(t))_t, \mathbb{P}_\infty\big)$ in the limit. In addition, assume that the process $Z_n(t)$, given by*

$$Z_n(t) = \exp\left\{-\int_0^t \hat{\boldsymbol{\theta}}_n(s) dB_n(s) - \frac{1}{2}\int_0^t ||\hat{\boldsymbol{\theta}}_n(s)||^2 ds\right\}, \quad 0 \leq t \leq T,$$

is a true martingale for all $t \in [0, T]$, where B_n is a vector of independent Brownian motions, $n \in \mathbb{N}$. Then, for all $t \leq T$ we have

$$\lim_{n\to\infty} Z_n(t) = \mathbb{E}\Big[\lim_{n\to\infty} Z_n(T)|\mathfrak{F}_\infty(t)\Big] = Z_\infty(t) < \infty.$$

Proof. We have by the martingale property that

$$Z_n(t) = \mathbb{E}[Z_n(T)|\mathfrak{F}_n(t)],$$

for each $t \leq T$. Because we have constructed the limiting σ-algebra $\mathfrak{F}_\infty$ as the product containing all previous ones, we also have

$$Z_n(t) = \mathbb{E}[Z_n(T)|\mathfrak{F}_\infty(t)].$$

All that is left to show, is that we can interchange limits and conditional expectations in this case. This is possible by the dominated convergence theorem for conditional expectations. If we can find a random variable Y on $\mathcal{B}_\infty$, such that $\mathbb{E}[Y] < \infty$ and $|Z_n(T)| \leq Y$ on $\mathfrak{F}_\infty$, then we can dominate the exponential process and thus interchange limits and conditional expectations.

Let $Y = 1 + \sup_n Z_n(T)^2$. Then by construction $Z_n(T) \leq Y$ for each n. Then, by Doob's Submartingale Theorem and equation (1.19), we have that

$$\mathbb{E}\Big[\sup_n Z_n(T)^2\Big] \leq \sup_n 4\mathbb{E}\Big[Z_n(T)^2\Big] < \infty,$$

as Z_n is square integrable by equation (1.21).

Hence $\mathbb{E}[Y] < \infty$ and we found a dominating random variable. Thus we obtain

$$\begin{aligned}\lim_{n\to\infty} Z_n(t) &= \mathbb{E}\Big[\lim_{n\to\infty} Z_n(T)|\mathfrak{F}_\infty(t)\Big] \\ &= \mathbb{E}\Big[Z_\infty(T)|\mathfrak{F}_\infty(t)\Big].\end{aligned}$$

As we have already shown that $Z_\infty(T)$ exists, we have that

$$\mathbb{E}\Big[Z_\infty(T)|\mathfrak{F}_\infty(t)\Big] = Z_\infty(t),$$

for all $t \leq T$, the result follows. □

Remark 1.6. *Note that from now on we shall refer to the condition*

$$\mathbb{E}\left[e^{\left\{\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}(s)||^2 ds\right\}}\right] < \infty, \tag{1.22}$$

as the limiting Novikov-type condition, as it constitutes a sufficient condition for obtaining a true martingale.

We have managed to establish that there is a well-defined, non-degenerate limiting martingale Z_∞ which can be used to find a valid change of measure. Using the Radon-Nikodym theorem, we can see that this allows us to construct what we could call the 'limiting equivalent martingale measure' $\mathbb{Q}_\infty \sim \mathbb{P}_\infty$ from the identity $Z_\infty = \frac{d\mathbb{Q}_\infty}{d\mathbb{P}_\infty}$. However we cannot definitively identify $\mathbb{Q}_\infty$ the an equivalent martingale measure for $\mathbb{P}_\infty$.

To complete this section, we have to introduce the limiting equivalent to the optimal control problem posed in Section 1.3. This is essential to make a connection with asymptotic arbitrage, as it is a limiting concept and not applicable to a finite market. The main issue is that we see two ways of how to think about the concept of limiting in our case. We shall begin with the simpler interpretation.

Suppose, as in Section 1.3 that you have the portfolio process
$\pi(t) = \{(\pi_1(t), \ldots, \pi_n(t), \tilde{\pi}_1(t), \ldots, \tilde{\pi}_J(t)), 0 \leq t \leq T\}$, as well as the wealth process
$\{X_n(t); 0 \leq t \leq T\}$, as given by the equation (1.9). Then, as described in Problem 1, our value function (1.17) is given by

$$u_n(x) = \sup_{\pi\in\mathcal{A}_n(x)} \mathbb{E}\left[U(X_n^{\pi,x}(T))|X_n^{\pi,x}(0) = x\right].$$

Solving this we can, under particular conditions, find the optimal value function $u_n(x)$ which achieves the supremum. We will show in Chapter 2 and Chapter 3 how such a solution can be found in general terms and explicitly. In fact, for each $n \in \mathbb{N}$, we can find such an optimal

value function, which therefore gives us a sequence of solutions $\Big(u_i(x)\Big)_{i=1}^{n}$. Consider $u_{n+1}(x)$, if this is the optimal value function for an optimisation problem with $n+1$ risky assets, then it cannot be smaller that the value $u_n(x)$ for only n risky assets and the sequence is monotone increasing in n. If it exists, we could define $u_\infty(x)$ to be the limit of the sequence $\Big(u_i(x)\Big)_{i=1}^{n}$, i.e.

$$u_\infty(x) = \lim_{n\to\infty} u_n(x) = \sup_{\pi\in\mathcal{A}_n(x) \text{ for some n}} \mathbb{E}\big[U(X_n^{\pi,x}(T))\big|X_n^{\pi,x}(0) = x\big]. \tag{1.23}$$

Let us discuss what would be required to make sense of the second interpretation that is considering the limiting setup from the start. Similar to our setup in the finite case suppose that $\upsilon_{\infty,j}$, and $\tilde{\upsilon}_{\infty,l}$ are the amounts of the j^{th} security $S_j(t)$ or respectively the l^{th} market asset $I_l(t)$ held by the investor in the limiting setup.

Suppose that we also have the assumption

$$\sup_{t\leq T}\sum_{j=1}^{\infty}\big|\upsilon_{\infty,j}S_j(t)\big| < \infty, \quad a.s. \tag{1.24}$$

We could then formulate the limiting model version of the wealth process as

$$X_\infty(t) = \sum_{j=1}^{\infty}\upsilon_{\infty,j}S_j(t) + \sum_{l=1}^{J}\tilde{\upsilon}_{\infty,l}I_l(t) + \upsilon_0 S_0(t). \tag{1.25}$$

Let us write the countable-length portfolio proportion vector as

$$\pi_\infty(t) = \big\{(\tilde{\pi}_{\infty,1}(t),\ldots,\tilde{\pi}_{\infty,J}(t),\pi_{\infty,1}(t),\pi_{\infty,2}(t),\ldots), 0\leq t\leq T\big\}, \tag{1.26}$$

where each portfolio proportion concerning the investment in the global market asset is given by the first J entries $\tilde{\pi}_{\infty,l}(t)$, $l = 1,\ldots,J$ and the proportion of wealth to be invested in the countable collection of risky stocks $S_j(t)$ is denoted by $\pi_{\infty,j}(t)$, $j = 1,2,\ldots$.

In addition, assume that

$$\sup_{t\leq T}\sum_{j=1}^{\infty}\big|\pi_{\infty,j}(t)\big| < \infty, \quad a.s. \tag{1.27}$$

By adapting the finite definition of the portfolio proportions to our infinite setting and assuming that X_∞ not degenerate, we have $\pi_{\infty,j}(t) = \frac{\upsilon_{\infty,j}S_j(t)}{X_\infty(t)}$ and $\tilde{\pi}_{\infty,l}(t) = \frac{\tilde{\upsilon}_{\infty,l}I_l(t)}{X_\infty(t)}$. Because we only have a finite collection of global market assets, we do not need to make any further assumptions concerning $\tilde{\pi}_{\infty,l}(t)$ or $\tilde{\upsilon}_{\infty,l}I_l(t)$.

Under assumptions (1.27) and (1.24) and by construction we would expect $X_\infty(t)$ to be finite over $t \in [0,T]$ and evolve according to the following dynamics

$$\begin{aligned} dX_\infty(t) = &\left[\sum_{j=1}^{\infty}(\alpha_j - r)\pi_{\infty,j}(t) + \sum_{l=1}^{J}(\mu_l - r)\tilde{\pi}_{\infty,l}(t) + r\right]X_\infty(t)dt \\ &+ \sum_{j=1}^{\infty}\sigma_j\pi_{\infty,j}(t)\sum_{l=1}^{J}\rho_{j,l}X_\infty(t)dM^l(t) + \sum_{l=1}^{J}\gamma_l\tilde{\pi}_{\infty,l}(t)X_\infty(t)dM^l(t) \\ &+ \sum_{j=1}^{\infty}\sigma_j\sqrt{1-\sum_{l=1}^{J}\rho_{j,l}^2}\pi_{\infty,j}(t)X_\infty(t)dW_j(t), \end{aligned} \tag{1.28}$$

with the initial endowment $X_\infty(0) = x > 0$. Note that this requires us to understand stochastic integrals against an infinite sum of local martingales.

To be able to find a solution to (1.28), we would expect to require the following integrability conditions:

$$\begin{aligned} \int_0^T \left|\sum_{j=1}^{\infty}(\alpha_j(t) - r(t))\pi_{\infty,j}(t) + \sum_{l=1}^{J}(\mu_l(t) - r(t))\tilde{\pi}_{\infty,l}(t)\right|dt < \infty \quad a.s., \\ \int_0^T \left|\sum_{j=1}^{\infty}\sigma_j(t)\sqrt{1-\sum_{l=1}^{J}\rho_{j,l}^2(t)}\pi_{\infty,j}(t)\right|^2 dt < \infty \quad a.s., \\ \int_0^T \left|\sum_{j=1}^{\infty}\sigma_j(t)\pi_{\infty,j}(t)(t)\sum_{l=1}^{J}\rho_{j,l}(t) + \sum_{l=1}^{J}\gamma_l(t)\tilde{\pi}_{\infty,l}(t)\right|^2 dt < \infty \quad a.s., \end{aligned} \tag{1.29}$$

see [Kar97] or [Pha09] for the finite model assumptions and adapt for the limiting modelling setup.

Assuming that our admissibility set is now given by

$$\mathcal{A}_\infty(x) = \left\{\pi_\infty : \mathbb{E}[|U(X_\infty(T))|] < \infty, \quad X_\infty(t) \geq 0, \quad t \in [0,T]\right\},$$

the limit value function for our optimisation problem should be given by

$$\tilde{u}_\infty(x) = \sup_{\pi_\infty \in \mathcal{A}_\infty(x)} \mathbb{E}\left[U(X_\infty(T))|X_\infty(0) = x\right]. \tag{1.30}$$

If the supremum in equation (1.30) is attained for an optimal portfolio process vector $\pi_\infty^* \in \mathcal{A}_\infty(x)$, we could obtain the optimal value function $\tilde{u}_\infty(x)$.

Having described how we might understand a limiting optimisation problem and a limiting optimal value function, we want to comment on the relationship between $\tilde{u}_\infty(x)$ and $u_\infty(x)$. In particular we can see that we must have

$$\begin{aligned}\tilde{u}_\infty(x) &= \sup_{\pi_\infty \in \mathcal{A}_\infty(x)} \mathbb{E}\left[U(X_\infty(T))|X_\infty(0) = x\right] \\ &\geq u_n(x) = \sup_{\pi \in \mathcal{A}_n(x)} \mathbb{E}\left[U(X_n^{\pi,x}(T))|X_n^{\pi,x}(0) = x\right] \\ &= u_n(x),\end{aligned}$$

for all $n \in \mathbb{N}$.

As $u_n(x)$ is monotone increasing, we can also see that

$$\tilde{u}_\infty(x) \geq \lim_{n\to\infty} u_n(x) = u_\infty(x).$$

As we have seen to make this theory rigorous we need to make sense of the limiting wealth process X_∞ and therefore would need to understand an infinite sum of local martingales and be able to extend the class of admissible trading strategies to those obtainable as integrals agains infinite dimensional semimartingales. This would also enable us to give an interpretation of the set of equivalent martingale measures in the infinite case.

1.5 Some Useful Methods of Matrix Inversion

Our overall aim in this thesis is to find explicit optimal solutions in a large financial market setting, driven by the system of asset dynamics (1.1). Therefore, we require certain methods of matrix inversion for the following types of symmetric matrices. The first method, described here in detail, can also be found in [GV13b], as well as standard texts on linear algebra.

Method I

Proposition 1.7. *Suppose we have a $N \times N$ dimensional symmetric and invertible matrix Σ_N that can be expressed as*

$$\Sigma_N = D + \sum_{j=1}^{N} d_j d_j^T, \tag{1.31}$$

where D is a diagonal $N \times N$ matrix, d_j a N-length column vector and d_j^T its transpose, $j = 1, \ldots, N$.

Then, the inverse is given by

$$\Sigma_N^{-1} = D^{-1/2}\left[I_N + \sum_{j=1}^{N}\sum_{k=1}^{N}\nu_{jk}e_j e_k^T\right]D^{-1/2}, \tag{1.32}$$

where $e_j = D^{-1/2}d_j$ *for all* $j = 1, \ldots, N$ *and* ν_{jk} *is the* jk^{th} *element of the inverse matrix* Γ *such that*

$$\Gamma = \begin{pmatrix} -(1+e_1^Te_1) & -e_1^Te_2 & \cdots & -e_1^Te_N \\ -e_2^Te_1 & -(1+e_2^Te_2) & \cdots & -e_2^Te_N \\ \vdots & \vdots & \ddots & \vdots \\ -e_N^Te_1 & -e_N^Te_2 & \cdots & -(1+e_N^Te_N), \end{pmatrix}^{-1}, \tag{1.33}$$

and I_N *is the* $N \times N$ *identity matrix.*

Proof. Note that Σ_N can also be written as

$$\Sigma_N = D^{1/2}\left[I_N + \sum_{j=1}^{N} e_j e_j^T\right]D^{1/2}.$$

Let us first find the inverse to $I_N + \sum_{j=1}^{N} e_j e_j^T$. In matrix notation where $\boldsymbol{e}_1$ represents a $N \times 1$ vector, we have

$$I_N = \left[I_N + \begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_N\end{pmatrix} I_N \begin{pmatrix}\boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_N\end{pmatrix}\right]\left[I_N + \begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_N\end{pmatrix}\begin{pmatrix}\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN}\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_N\end{pmatrix}\right]$$

$$= I_N + \begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_N\end{pmatrix}\left[\begin{pmatrix}1+\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & 1+\nu_{NN}\end{pmatrix} + \begin{pmatrix}e_1^Te_N & \cdots & e_1^Te_N \\ \vdots & \ddots & \vdots \\ e_N^Te_1 & \cdots & e_N^Te_N\end{pmatrix}\begin{pmatrix}\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN}\end{pmatrix}\right]\begin{pmatrix}\boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_N\end{pmatrix}.$$

Therefore, we need to find the matrix

$$\Gamma = \begin{pmatrix}\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN}\end{pmatrix},$$

such that

$$\begin{pmatrix}1+\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & 1+\nu_{NN}\end{pmatrix} + \begin{pmatrix}e_1^Te_N & \cdots & e_1^Te_N \\ \vdots & \ddots & \vdots \\ e_N^Te_1 & \cdots & e_N^Te_N\end{pmatrix}\begin{pmatrix}\nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN}\end{pmatrix} = \begin{pmatrix}0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0\end{pmatrix}.$$

Solving the above equation we then have

$$\begin{pmatrix} \nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN} \end{pmatrix} \left[\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} e_1^T e_N & \cdots & e_1^T e_N \\ \vdots & \ddots & \vdots \\ e_N^T e_1 & \cdots & e_N^T e_N \end{pmatrix} \right] = - \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} \nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN} \end{pmatrix} = \begin{pmatrix} -(1+e_1^T e_1) & -e_1^T e_2 & \cdots & -e_1^T e_N \\ -e_2^T e_1 & -(1+e_2^T e_2) & \cdots & -e_2^T e_N \\ \vdots & \vdots & \ddots & \vdots \\ -e_N^T e_1 & -e_N^T e_2 & \cdots & -(1+e_N^T e_N). \end{pmatrix}^{-1}.$$

Multiplying

$$\left[I_N + \begin{pmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_N \end{pmatrix} \begin{pmatrix} \nu_{11} & \cdots & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NN} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_N \end{pmatrix} \right],$$

with $D^{-1/2}$ yields the required result. □

Let us briefly comment on a particular example we will require in Chapter 3, also mentioned in [GV13b]. The proof follows directly from the general case above.

Corollary 1.8. *Suppose we have the invertible $N \times N$ matrix Σ_1 such that*

$$\Sigma_1 = D + dd^T. \tag{1.34}$$

Consequently the inverse is given by

$$\Sigma_1^{-1} = D^{-1/2} \left[I_N - \frac{ee^T}{1+e^T e} \right] D^{-1/2}, \tag{1.35}$$

where $e = D^{-1/2} d$.

Method II

Now consider a more involved type of a symmetric matrix that we will also encounter in Chapter 3.

Proposition 1.9. *Given a matrix of the form*

$$\Lambda = \begin{pmatrix} D & E \\ E^T & \Sigma \end{pmatrix}, \tag{1.36}$$

where D is diagonal, E is a $N \times N$ matrix, E^T its transpose, Σ is as given in (1.31), and I_N the $N \times N$ dimensional identity matrix. Then the inverse of Λ is given by

$$\Lambda^{-1} = \begin{pmatrix} [D - E^T\Sigma^{-1}E]^{-1} & -\Sigma^{-1}E[D - E^T\Sigma^{-1}E]^{-1} \\ -[D - E^T\Sigma^{-1}E]^{-1}E^T\Sigma^{-1} & \Big[I_N + \Sigma^{-1}E[D - E^T\Sigma^{-1}E]^{-1}]E^T\Big]\Sigma^{-1} \end{pmatrix}.$$

Proof. In general, suppose that we are looking for a matrix Λ^{-1} such that

$$\begin{pmatrix} D' & H \\ H^T & \Sigma' \end{pmatrix} \begin{pmatrix} D & E \\ E^T & \Sigma \end{pmatrix} = I_{2N}. \tag{1.37}$$

From (1.37), it is evident that the particular set of equations we must solve are given by

(i) $D'D + H^TE = I_N$,

(ii) $D'E^T + E^T\Sigma = \mathbf{0}$,

(iii) $DH + \Sigma'E = \mathbf{0}$,

(iv) $HE^T + \Sigma'\Sigma = I_N$,

where $\mathbf{0}$ is the $N \times N$ dimensional zero matrix

Hence, solving for D', H, H^T and Σ' using the result for Σ^{-1} from (1.32), we obtain the required representation. □

1.6 Introduction to Asymptotic Arbitrage

Focusing on the optimisation in a financial market with a large or infinite number of securities, we are faced with a new problem: the occurrence of asymptotic arbitrage. Kabanov and Kramkov in [KK93] describe it heuristically as the possibility to achieve any gain, however small, with a positive probability starting with an infinitesimally small initial endowment. In

previous sections, in particular when describing the main model in Section 1.2, we have seen that our large financial market $\mathcal{M}$ consists of an increasing sequence of sub-markets $\mathcal{M}_n$, $n \in \mathbb{N}$.

Although we assume that each of the $\mathcal{M}_n$ is arbitrage-free, such that there exists at least one equivalent martingale measure, asymptotic arbitrage in the limiting market $\mathcal{M}$ can still arise. Therefore, we need a new no-arbitrage condition to cover the limiting case.

Kabanov and Kramkov [KK93] define asymptotic arbitrage in two ways: the asymptotic arbitrage of the first kind (AA1), and asymptotic arbitrage of the second kind (AA2). The first gives the investor the opportunity to obtain arbitrarily large profits with positive probability by risking arbitrarily small losses, i.e. with vanishing risk. Asymptotic arbitrage of the second kind is achieved by making a gain with probability arbitrarily close to one, but risking to lose only a uniformly bounded amount of money. The intuition here is that we have a chance of a very likely profit with bounded but not vanishing risk. The main difference between the two is that with AA1 one can become very rich on a set of positive probability risking nearly nothing, while with AA2 one can win a small amount with a high probability under the risk of losing a bounded amount on a set of probability of nearly zero.

This section is structured as follows. Considering an increasing sequence of market models $\mathcal{M}_n$, for $n \in \mathbb{N}$, we first define asymptotic arbitrage formally. Then we explain the link between the condition of no asymptotic arbitrage of either the first or the second kind and the concept of contiguity and the Hellinger process. Later in this section, we also briefly outline the variation between complete and incomplete models and describe the implications of these differences. See [KK98] and [KS96] for further detail.

Suppose that we have a market $\mathcal{M}_n$ as defined in Section 1.2. Assume there exists a sequence of martingale measures $(\mathbb{Q}_n)_{n\geq 1}$, such that each of them is equivalent to the corresponding real world measure in the sequence $(\mathbb{P}_n)_{n\geq 1}$, i.e. $\mathbb{Q}_n \sim \mathbb{P}_n$ for each $n \in \mathbb{N}$. Let each of these sub-markets be complete and arbitrage free, in particular $\mathbb{Q}_n \in \mathcal{P}_n$ and $\mathcal{P}_n \neq \emptyset$ is a singleton. In addition, the market is frictionless and the investor is small, therefore he cannot influence the prices, and short selling is allowed.

Define a measurable, $\mathbb{F}_n$-adapted process φ_n with values in $\mathbb{R}^{n+J}$, interpreted as a trading strategy.

Then, the stochastic integral with respect to our asset price processes $\{S_j(t); j = 1, \ldots, n \quad 0 \leq t \leq T\}$, and $\{I_l(t); l = 1, \ldots, J \quad 0 \leq t \leq T\}$ is well defined on $[0, T]$, such that

$$\varphi_n \cdot (S, I) = \int_0^t \sum_{j=1}^{n} \varphi_j(u) dS_j(u) + \int_0^t \sum_{k=1}^{J} \varphi_k(u) dI_k(u), \tag{1.38}$$

and it is a local martingale with respect to the risk neutral measure $\mathbb{Q}_n$, for each $n \in \mathbb{N}$.

For this trading strategy φ_n and initial amount $G_n(0) > 0$, the value process is then written as

$$G_n(t, \varphi_n) = G_n(0) + \int_0^t \sum_{j=1}^{n} \varphi_j(u) dS_j(u) + \int_0^t \sum_{k=1}^{J} \varphi_k(u) dI_k(u). \tag{1.39}$$

Remark 1.10. *Note that this trading strategy φ_n corresponds to introducing a control process that describes the amount of each asset in the portfolio, instead of the proportion of wealth invested in the asset. For this view on the control problem, see [Kar97]. Then, the value process that Kabanov and Kramkov in [KK93, KK98] refer to, can be identified with what we called the wealth process.*

Let us give a formal definition of asymptotic arbitrage combining the works of Kabanov and Kramkov [KK93, KK98] with the work of Klein and Schachermayer [KS96].

Definition 6. *A sequence of trading strategies φ_n realises asymptotic arbitrage of the first kind (AA1) if, for a value process $G_n(t)$ as defined in (1.39) and the real-world measure $\mathbb{P}_n$, we have*

(i) $G_n(t, \varphi_n) \geq -c_n$, $\forall\ t \in [0, T]$ *and each* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} G_n(0, \varphi_n) = 0$,

(iii) $\lim_{n\to\infty} \mathbb{P}_n(G_n(T, \varphi_n) \geq C_n) > 0$,

where $(c_n)_{n\in\mathbb{N}} > 0$, $(C_n)_{n\in\mathbb{N}} > 0$ are sequences of positive constants with $c_n \to 0$ and $C_n \to \infty$ as $n \to \infty$.

To understand the intuition behind this definition, let us elaborate briefly on its heuristic interpretation. The first condition ensures that we only experience arbitrarily small losses for any time $0 \leq t \leq T$. The second condition means that the larger the market $\mathcal{M}_n$ is, the smaller the required initial endowment is to build our portfolio. In the limit, for the market

$\mathcal{M}$, we could start with no funds and still achieve an arbitrarily large gain with strictly positive probability, using such a trading strategy.

For the asymptotic arbitrage of the second kind we have:

Definition 7. *A sequence of trading strategies φ_n realises asymptotic arbitrage of the second kind (AA2) if, for a value process $G_n(t)$ as defined in (1.39), a constant $c > 0$ and the measure $\mathbb{P}_n$, the following holds*

(i) $G_n(t, \varphi_n) \geq -c$, $\forall\ t \in [0, T]$ *and* $c > 0$,

(ii) $\lim_{n\to\infty} G_n(0, \varphi_n) > 0$,

(iii) *There exists an* $\varepsilon > 0$, *such that* $\lim_{n\to\infty} \mathbb{P}_n(G_n(T, \varphi_n) \geq \varepsilon) = 1$.

We can see that the first condition ensures that we risk at most the bounded amount $c > 0$ while the third condition ensures that we make a profit with probability arbitrarily close to 1 and the second condition keeps our initial endowment always strictly positive.

Hence, we have the following definition of no asymptotic arbitrage.

Definition 8. *An infinite security market $\mathcal{M}$ has no asymptotic arbitrage of the first, or second kind, if for any subsequence n' there are no trading strategies $\varphi_{n'}$ realising asymptotic arbitrage of the first kind, or respectively second kind, in the market $\mathcal{M}_{n'} = \left\{(\mathcal{B}_{n'}, T_{n'}, (S, I)_{n'})\right\}$.*

Kabanov and Kramkov show step-by-step how asymptotic arbitrage is governed by the contiguity of the real-world measure and the corresponding equivalent martingale measures. The link between the two concepts is provided by the Hellinger processes, as described in [JS87], and the contiguity of the martingale measure, see [KK93, JS87]. Heuristically, contiguity describes the closeness of two sequences of probability measures and can be viewed as an asymptotic version of absolute continuity. In particular, we can state the following definition.

Definition 9. *A sequence of probability measures $(\mathbb{Q}_n)_{n\geq 1}$, at a fixed time T, is contiguous with respect to the sequence $(\mathbb{P}_n)_{n\geq 1}$, with notation $\mathbb{Q}_n \triangleleft \mathbb{P}_n$, if for any sequence of sets $A_n \in \mathfrak{F}_n$ with $\mathbb{P}_n(A_n) \to 0$ we have that $\mathbb{Q}_n(A_n) \to 0$.*

The main result from Kabanov and Kramkov, see Proposition 2 [KK93], is the equivalence between asymptotic arbitrage and contiguity, provided our market is complete.

Proposition 1.11. *Suppose that, for any $n \in \mathbb{N}$, each set of equivalent martingale measures in $\mathcal{P}_n$ consists only of one point, $\mathbb{Q}_n$. We then have the following relationship between contiguity and asymptotic arbitrage:*

(i) $\mathbb{P}_n \triangleleft \mathbb{Q}_n$ *iff there is no asymptotic arbitrage of the first kind, and*

(ii) $\mathbb{Q}_n \triangleleft \mathbb{P}_n$ *iff there is no asymptotic arbitrage of the second kind.*

This is shown by contradiction using the definition of contiguity, the definition of asymptotic arbitrage and well established martingale properties.

It is clear that contiguity gives us a condition for the non-existence of asymptotic arbitrage, but no easy-to-handle criterion for it in the case of our correlated geometric Brownian factor model in (1.1). However, we can find such a criterion in terms of the Hellinger process. In particular, the Hellinger process is a condition equivalent to contiguity and therefore, can be linked to the absence of asymptotic arbitrage of both kinds.

Let us first introduce what we mean by the Hellinger process and how it is defined through Kakutani-Hellinger distances, see [JS87]. For Brownian dynamics, the Hellinger process can be written explicitly and provides an easy-to-handle condition for no asymptotic arbitrage in our large financial market. Extensive work on the use of Hellinger processes in mathematical finance and optimisation, although mostly relative to jump processes, has been done by Choulli et al., [CH01, CS05, CS06]

Hellinger processes are commonly derived through the Hellinger distance (also called the Kakutani-Hellinger distance), which allows us to understand the distance in variation between two measures on a filtered space. On an intuitive level, we can say that the Hellinger distance is a tool for quantifying the 'similarity' between two probability distributions. The idea is that we can judge how one distribution behaves in view of the other. Usually the distance ranges between one and zero, where a distance of one means that a measure $\mathbb{P}_n$ would assign the probability zero to a set $A_n \in \mathfrak{F}_n(t)$, where the martingale measure $\mathbb{Q}_n$ would assign it a positive probability.

We first want to give a formal definition of the Kakutani-Hellinger distance, followed by a brief definition of a Hellinger process (for a more detailed description, see [JS87]). In fact, we will use this to determine that, if the sequence of Hellinger processes is finite, then the real-world measure is contiguous with respect to the martingale measure and vice versa. Thus, if the distance is small, there is no possibility of experiencing asymptotic arbitrage, as we only have

the possibility to generate a risk-free return almost surely. For now, let us assume that we only act on one fixed finite market and omit the subscript n as well as the dependence on time t. In addition, only mentioning the background critical to our particular modelling setup, we assume that the measures are equidistant, and here the distance factor is set to $1/2$.

Suppose $(\Omega, \mathfrak{F}, \mathbb{F})$ is a filtered space with the probability measures $\mathbb{P}, \mathbb{Q}$ and $\mathbb{O} = \frac{\mathbb{P}+\mathbb{Q}}{2}$. We have that $\mathbb{P}$ and $\mathbb{Q}$ are absolutely continuous with respect to $\mathbb{O}$, such that

$$\mathbb{P} \ll \mathbb{O}, \quad \mathbb{Q} \ll \mathbb{O}. \tag{1.40}$$

Consider the Radon-Nikodym derivatives

$$Z = \frac{d\mathbb{P}}{d\mathbb{O}}, \quad Z' = \frac{d\mathbb{Q}}{d\mathbb{O}}.$$

Then, the Kakutani-Hellinger distance $\varrho(\mathbb{P}, \mathbb{Q})$ between the measures $\mathbb{P}$ and $\mathbb{Q}$ is a number greater than or equal to zero, and its square is defined by

$$\varrho^2(\mathbb{P}, \mathbb{Q}) = \frac{1}{2} \int_\Omega \left(\sqrt{\frac{d\mathbb{P}}{d\mathbb{O}}} - \sqrt{\frac{d\mathbb{Q}}{d\mathbb{O}}} \right)^2 d\mathbb{O} = \frac{1}{2} \mathbb{E}_\mathbb{O} \left[\left(\sqrt{Z} - \sqrt{Z'} \right)^2 \right].$$

We can rewrite this in terms of the Hellinger integral of order $\frac{1}{2}$, as

$$\varrho^2(\mathbb{P}, \mathbb{Q}) = 1 - H(1/2, \mathbb{P}, \mathbb{Q}),$$

where $H(1/2, \mathbb{P}, \mathbb{Q})$ is defined as the expectation $H(1/2, \mathbb{P}, \mathbb{Q}) = \mathbb{E}_\mathbb{O} \left[\sqrt{ZZ'} \right]$ and $Z + Z' = 2$ by construction and the choice of $\mathbb{O}$. Here $H(1/2, \mathbb{P}, \mathbb{Q})$ is symmetric in $\mathbb{P}$,$\mathbb{Q}$ and quantifies the degree of separation between these measures.

Therefore, we can state the general definition from [KK93] and [JS87].

Definition 10. *For a positive, bounded process $Y(t, 1/2) = Z^{\frac{1}{2}} Z'^{\frac{1}{2}}$, which is a $\mathbb{O}$-supermartingale, the Hellinger process $h(t, 1/2)$ is a predictable increasing process defined through the decomposition*

$$Y(t, 1/2) = \mathcal{R}(t, 1/2) \mathcal{E}(-h(t, 1/2)), \tag{1.41}$$

and $\mathcal{R}(t, 1/2)$ is a local martingale with $\mathcal{R}(0, 1/2) = 0$, such that $\mathcal{E}(-h(t, 1/2))$ is the Doleans exponential, being the density, and solves the equation

$$\mathcal{E}(t, -h(t, 1/2)) = 1 - \int_0^t \mathcal{E}(s, -h(t, 1/2)) dh(s, 1/2), \quad \text{if} \quad h(t, 1/2) < \infty. \tag{1.42}$$

Therefore, the Hellinger process is defined up to the time where $ZZ' = 0$.

Heuristically, we can say that the Hellinger process measures the rate of separation of the martingale measure $\mathbb{Q}_n$ from the physical measure $\mathbb{P}_n$ for each $n \in \mathbb{N}$.

To provide a link between the Hellinger process, the contiguity of two measues and, thus, to asymptotic arbitrage, we need the following lemma. We omit the proof, which can be found in [KP91]. Let us return to our notation using a market $\mathcal{M}$ on the space $\mathcal{B}_\infty$, which can be represented by an increasing sequence of markets $\mathcal{M}_n$ on the underlying probability spaces $\mathcal{B}_n = (\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n)$ for each $n \in \mathbb{N}$. Considering the equidistant Hellinger process on each of these spaces, which we shall denote using the shorter notation $h_n(t)$ from now on, $t \in [0, T]$, we have the following result.

Lemma 1.12. *For a fixed time horizon T, the following conditions are equivalent:*

(i) $\lim_{n\to\infty} ||\mathbb{P}_n(T) - \mathbb{Q}_n(T)|| = 0$,

(ii) $\lim_{n\to\infty} \mathbb{P}_n[h_n(T) \geq \varepsilon] = 0$, $\forall\ \varepsilon > 0$.

To show the explicit structure of $h_n(t)$ for each $n \in \mathbb{N}$ and $t \in [0, T]$, we will use [KLS86, KP91]. Suppose, in general, that we have a system of SDEs, such that

$$dQ(t) = o(t)dt + z(t)dB(t), \quad Q(0) = 0$$

$$dQ'(t) = o'(t)dt + z(t)dB(t), \quad Q'(0) = 0, \tag{1.43}$$

where $B(t)$ is a Brownian motion, o, o' are $\mathfrak{F}_n(t)$-measurable, adapted, n-dimensional random vectors representing the drift, and $z(t)$ is a $\mathfrak{F}_n(t)$-measurable, adapted, $n \times n$ volatility matrix.

The solution to the system of SDEs exists and is unique, if the parameters satisfy the Lipschitz and linear growth conditions, the matrix $z(t)z(t)^T$ is non-degenerate and

$$\int_0^T \Big(||o(t)|| + ||o'(t)|| + ||z(t)||^2\Big)dt < \infty.$$

Then, we have the following

Corollary 1.13. *Suppose that for the above system of dynamics (1.43) we have $\mathbb{P}_n$- almost surely*

$$\int_0^T o^T(t)\Big(z(t)z^T(t)\Big)^{-1}o(t)dt < \infty,$$
$$\int_0^T o'^T(t)\Big(z(t)z^T(t)\Big)^{-1}o'(t)dt < \infty,$$

then the explicit Hellinger process is given by

$$h_n(t) := \frac{1}{8}\int_0^T \Big(o(t) - o^T(t)')^T(z(t)z^T(t)\Big)^{-1}\Big(o(t) - o'(t)\Big)dt. \tag{1.44}$$

This can be shown by deriving the Radon-Nikodym densities Z, Z' for our equivalent measures with respect to a reference measure, as before, and inserting them into the process $Y = Z^{1/2}Z'^{1/2}$, see [KP91].

Kabanov and Kramkov state two criteria for contiguity, see [JS87, KK93, KP91]. However, as the Hellinger process for our market model is continuous, based on the behaviour of the parameters and the sequence of measures are equidistant by assumption, we only state the most applicable criterion.

Lemma 1.14. *Suppose that the Hellinger process $h_n(T)$ is continuous, then two measures $\mathbb{Q}_n$ and $\mathbb{P}_n$ are contiguous, we have that*

$$\mathbb{Q}_n(T) \triangleleft \mathbb{P}_n(T) \quad \textit{iff} \quad \lim_{c\to 0} \limsup_{n\to\infty} \mathbb{Q}_n\left(h_n(T) \geq c\right) = 0.$$

In particular, using both [KK93] and [KP91], we can deduce that the concrete Hellinger process for our system asset dynamics in (1.1) is the Radon-Nikodym density derived in Section 1.3. Explicitly, the criterion for no asymptotic arbitrage for our system is the finiteness of the Hellinger process $h_n(t)$ in the limit as $n \to \infty$.

Corollary 1.15. *Suppose we have the martingale density process $Z_n(t) = \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}$, with*

$$Z_n(t) = \exp\left\{-\int_0^t \boldsymbol{\theta}_n(s)dB_n(s) - \frac{1}{2}\int_0^t ||\boldsymbol{\theta}_n(s)||^2 ds.\right\}$$

Recall that the relative risk price process $\boldsymbol{\theta}_n(t)$ is deterministic, meaning being given as in Definition 1 but with deterministic parameters. Then, for $0 \leq t \leq T$, the explicit Hellinger process can be written as

$$h_n(t) = \frac{1}{8}\int_0^t ||\boldsymbol{\theta}_n(s)||^2 ds, \tag{1.45}$$

and the no asymptotic arbitrage condition becomes

$$\lim_{n\to\infty} h_n(t) < \infty, \quad \textit{for} \quad 0 \leq t \leq T. \tag{1.46}$$

See [KK93] and [KK98] for the derivation. This result gives us the limiting condition to ensure that there is no asymptotic arbitrage in the infinite market $\mathcal{M}$.

Let us now move to the incomplete market case. So far, we have assumed that the equivalent martingale measure is unique in each of the submarkets $\mathcal{M}_n$, and our market is complete. What if this is not the case? It is clear from the derivation of the Hellinger process that this can be accommodated. As we mentioned previously, the general definition of the Hellinger

process does not depend on the choice of reference measure.

Schachermayer and Klein [KS96] show that the above criteria for the asymptotic arbitrage of the first kind also work if the market is incomplete, by selecting an appropriate sequence $(\mathbb{Q}_n)_{n\geq 1}$. This can also be used as the criterion for asymptotic arbitrage of the second kind under certain ε-δ convergence conditions.

For an incomplete model, i.e. each $\mathcal{P}_n \neq \emptyset$, but not a singleton for each $n \in \mathbb{N}$, we have the following theorem, see [KS96].

Theorem 1. *We can see that:*

(i) there is no asymptotic arbitrage of the first kind iff there exists a sequence $(\mathbb{Q}_n)_{n\geq 1}$ such that $\mathbb{Q}_n \in \mathcal{P}_n$ for all $n \in \mathbb{N}$, such that $(\mathbb{P}_n)_{n\geq 1} \lhd (\mathbb{Q}_n)_{n\geq 1}$, and

(ii) there is no asymptotic arbitrage of the second kind iff for each $\varepsilon > 0$ there exist measures $\mathbb{Q}_n \in \mathcal{P}_n$ and $\delta > 0$, such that for any set $A_n \in \mathfrak{F}_n$ with $\mathbb{P}_n(A_n) < \delta$ we have that $\mathbb{Q}_n(A_n) < \varepsilon$.

In particular, we see that the condition for no asymptotic arbitrage of the first kind remains valid as before. However, to have no asymptotic arbitrage of the second kind, we need to apply stricter tightness assumptions on the sequences of equivalent measures. This theorem states that for each $\varepsilon > 0$, there is a particular sequence, such that for each $\mathbb{Q}_n \in \mathcal{P}_n$, which depends on the epsilon, we can find a $\delta > 0$ that dominates the sequence $\mathbb{P}_n(A_n)$ for large n and even in the limit. This is a technical assumption and the condition on no asymptotic arbitrage of the second kind by Kabanov and Kramkov fails in an unnatural counter-example constructed in [KS96]. We would have to construct a particular random variable with an economically unnatural structure as well as a filtration generated by such a random variable to show the contradiction. Since in our case the incompleteness stems from deleting market assets from our portfolio, we could say that our model is 'very close' to being complete. A more detailed discussion of this topic is outside the scope of this thesis, and we shall incoprorate Klein's and Schachermayer's condition on no asymptotic arbitrage of the second kind when dealing with incomplete markets in Chapter 2.

Chapter 2

Linking General Utility Maximisation to Asymptotic Arbitrage

This chapter serves two purposes. First, we want to discuss the martingale and duality approach used to resolve a class of stochastic optimal control problems, which aim to maximise the expected value of the utility of terminal wealth. The major advantage of this methodology is that we do not have to specify the shape of the utility function, however we can still obtain a general existence and characterisation result for the value function as well as the optimal portfolio proportions. The only drawback here is that we cannot derive the concrete shape of such portfolio proportions in general. We will focus on the complete market case and only briefly touch on the solution in incomplete markets. Section 2.1 is largely based on well established theory which can be found in [Kar97] or [Pha09] for example.

In Section 2.2, we want to present the key theorem of this thesis. Here we show that there is a link between the limiting optimisation result for an infinite-sized, complete market and asymptotic arbitrage. In particular, for adapted coefficients we see that the existence of a finite, limiting value function implies no asymptotic arbitrage on the complete, infinite market. On the other hand, if we reduce the modelling parameters to be deterministic functions of time, we can also show the opposite direction where no asymptotic arbitrage implies that a finite limiting value function exists.

2.1 Duality and Martingale Approach to Optimisation

Duality and martingale methods represent a powerful tool to address optimal control problems. We can obtain the existence and general shape of the value function, which is the solu-

tion to the maximisation problem, without specifying a utility function, see [Kar97, Pha09].

Previously, we have stated our finite control problem as an optimisation over a vector of admissible portfolio processes $\pi(t) \in \mathcal{A}_n(x)$, $n \in \mathbb{N}$. The entries of $\pi(t)$ are measurable and adapted processes. Here we want to restate this problem as a maximisation over the state variable $\{X_n(t); 0 \leq t \leq T\}$, identified as the wealth process, using superreplication and the martingale representation theorem. This is then called the primal problem. Subsequently, we can formulate the dual minimisation problem by Legendre transform and use Lagrange methods to find its solution. We then retrieve the primal value function using convex conjugate relations.

As in Chapter 1 we denote $\mathbb{E}_{\mathbb{P}_n}$ as $\mathbb{E}$ for simplicity.

Let us briefly recap our market setup and the optimisation problem outlined in Chapter 1. Recall from Section 1.2 that we have the increasing sequence of probability spaces $\mathcal{B}_n = \big(\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)\big), \mathbb{P}_n)$, $n \in \mathbb{N}$. Then in a market $\mathcal{M}_n$, we have a collection of risky assets $\{S(t) = (S_1(t), \ldots, S_n(t)), 0 \leq t \leq T\}$ and market assets $\{I(t) = (I_1(t), \ldots, I_J(t)), 0 \leq t \leq T\}$ with dynamics given in equation (1.1), as well as the risk-free bank account $S_0(t)$. The control vector $\{\pi(t) = (\pi_1(t), \ldots, \pi_n(t), \tilde{\pi}_1(t), \ldots, \tilde{\pi}_J(t)); 0 \leq t \leq T\}$ on $\mathbb{R}^{n+J}$, represents the proportion of wealth invested in each asset. Then the wealth process (state process) $\{X_n(t), 0 \leq t \leq T\}$ evolves as:

$$\begin{aligned} dX_n(t) = &\left[\sum_{j=1}^{n}(\alpha_j - r)\pi_j(t) + \sum_{l=1}^{J}(\mu_l - r)\tilde{\pi}_l(t) + r\right]X_n(t)dt \\ &+ \sum_{j=1}^{n}\sigma_j\pi_j(t)\sum_{l=1}^{J}\rho_{j,l}dM^l(t)X_n(t) + \sum_{l=1}^{J}\gamma_l\tilde{\pi}_l(t)dM^l(t)X_n(t) \\ &+ \sum_{j=1}^{n}\sigma_j\sqrt{1 - \sum_{l=1}^{J}\rho_{j,l}^2}\pi_j(t)X_n(t)dW_j(t). \end{aligned}$$

See equation (1.10) for the explicit derivation.

Given an increasing and concave utility function $U : (0, \infty) \to (0, \infty)$, the initial optimisation problem (1.17) over the admissible controls $\pi(t) \in \mathcal{A}_n(x)$ is given by:

$$u_n(x) = \sup_{\pi \in \mathcal{A}_n(x)} \mathbb{E}\left[U(X_n^{\pi,x}(T))|X_n^{\pi,x}(0) = x\right], \tag{2.1}$$

where the admissibility set is defined as

$$\mathcal{A}_n(x) = \{\pi : \mathbb{E}[|U(X_n^{\pi,x}(T))|] < \infty, X_n^{\pi,x}(t) \geq 0, \quad t \in [0, T]\}.$$

We call $X_n^{\pi,x}$ the wealth process controlled by the vector $\pi(t)$ starting at the positive initial endowment $X_n(0) = x > 0$.

Recall from Section 1.2 that for each n we have the martingale density

$$Z_n(t) \triangleq \exp\left\{-\int_0^t \boldsymbol{\theta}_n(s)dB_n(s) - \frac{1}{2}\int_0^t ||\boldsymbol{\theta}_n(s)||^2 ds\right\}, \quad 0 \leq t \leq T.$$

where the vector $\boldsymbol{\theta}_n$, for a finite n, is the relative risk process for our market as in Definition 1 and $B_n(t) = \{W_1(t), \ldots, W_n(t), M^1(t), \ldots, M^J(t)\}$ is the vector of $n + J$ independent one-dimensional Brownian motions. We have also seen that if $\mathbb{E}[Z_n(T)] = 1$, for each n and $t \in [0, T]$ the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||\boldsymbol{\theta}_n(t)||^2 dt\right\}\right] < \infty.$$

is sufficient to see that it is a true martingale. Therefore it is a valid change of measure and can be identified with the Radon-Nikodym density

$$Z_n(t) = \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}\Big|_{\mathbb{F}_t}.$$

This gives us the tools to construct the martingale measure $\mathbb{Q}_n$ on $\mathcal{B}_n$ which is equivalent to $\mathbb{P}_n$. The set of all such equivalent martingale measures $\mathcal{P}_n$ is defined as

$$\mathcal{P}_n = \{\mathbb{Q}_n \sim \mathbb{P}_n | \mathbb{Q}_n \text{ is a martingale measure } \}.$$

In a complete market the set $\mathcal{P}_n$ only contains one martingale measure $\mathbb{Q}_n$ for each n. We have already discussed that $\mathcal{P}_n \neq \emptyset$ for each $n \in \mathbb{N}$ is equivalent to no arbitrage for each sub-market $\mathcal{M}_n$, see [Pha09]. In particular, we cannot find an admissible portfolio policy such that we can make a gain with positive probability starting with zero capital almost surely.

For the extension to the infinite market, recall that under the conditions set out in Section 1.4, $\Big(Z_\infty(t)\Big)_{0\leq t\leq T} < \infty$ exists and therefore it is possible to construct what we could call a 'limiting equivalent martingale measure' $\mathbb{Q}_\infty \sim \mathbb{P}_\infty$. In order to prove that this is an equivalent martingale measure, we would require both an appropriate infinite dimensional version of Girsanov's Theorem and of the Martingale Representation Theorem, whose derivation is beyond the scope of this thesis.

We want to refer to Chapter 7, [Pha09] and Chapter 2, [Kar97] for a detailed demonstration of the following theory. Suppose that we have the controlled wealth process $X_n^{\pi,x}(t)$, $t \in [0, T]$ starting with the initial endowment $X_n^{\pi,x}(0) = x$ for each $n \in \mathbb{N}$. In addition, assume that for each $n \in \mathbb{N}$ we have the $\mathfrak{F}_n(T)$-measurable random variable $X_n(T)$, which is a contingent

claim at maturity $T < \infty$. Then, the superreplication problem of $X_n(T)$ is to find the minimal initial capital x_n such that we can achieve at least $X_n(T)$ at maturity.

Then x_n is called the superreplication cost of $X_n(T)$ for each n. By $L^0_+(\Omega_n, \mathfrak{F}_n(T), \mathbb{P}_n)$ denote the space of $\mathfrak{F}_n(T)$-measurable non-negative random variables. For any $x \in \mathbb{R}_+$ we have the set of contingent claims

$$\mathcal{C}_n(x) = \{X_n(T) \in L^0_+(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n) : X_n^{\pi,x}(T) \geq X_n(T) \quad \mathbb{P}_n - a.s. \quad \text{for some} \quad \pi \in \mathcal{A}_n\}, \tag{2.2}$$

which can be dominated by the initial capital $x > 0$ and using an admissible portfolio strategy $\pi \in \mathcal{A}_n$.

For an equivalent martingale measure $\mathbb{Q}_n \in \mathcal{P}_n$ and an admissible portfolio proportion vector $\pi \in \mathcal{A}_n$, we can see that that $X_n^{\pi,x}(T) - x$ is a lower bounded local martingale under $\mathbb{Q}_n$ and by Fatou's lemma a $\mathbb{Q}_n$-supermartingale. Therefore we have that

$$\mathbb{E}_{\mathbb{Q}_n}[X_n^{\pi,x}(T) - x] \leq 0.$$

As the set of martingale measures $\mathcal{P}_n$ is non-empty, we cannot find an admissible portfolio proportion vector such that we can reach a non-negative wealth almost surely at T, starting with zero capital, for each $n \in \mathbb{N}$. Using the martingale representation theorem on $X_n^{\pi,x}(T)$ as well as considering the dual representation for the contingent claim $X_n(T) \in L^0_+(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$ given by

$$X_n(t) = ess \sup_{\mathbb{Q}_n \in \mathcal{P}_n} \mathbb{E}_{\mathbb{Q}_n}[X_n(T)|\mathfrak{F}_n(t)], \quad 0 \leq t \leq T,$$

we can rewrite the superreplication cost as

$$x_n = \sup_{\mathbb{Q}_n \in \mathcal{P}_n} \mathbb{E}_{\mathbb{Q}_n}[X_n(T)]. \tag{2.3}$$

If we identify the terminal wealth with the contingent claim $X_n(T)$, we have the budget constraint

$$\mathbb{E}_{\mathbb{Q}_n}[X_n(T)] \leq x_n = x, \tag{2.4}$$

where equality holds if the supremum is attained in equation (2.3). In particular, as $X_n(t)$ is a martingale under $\mathbb{Q}_n$, this condition ensures that we cannot expect to have a wealth higher than the initial endowment $x > 0$ at maturity T.

Therefore for each $x \in \mathbb{R}^+$, the terminal wealth must belong to the space

$$\mathcal{C}_n(x) = \{X_n(T) \in L^0_+(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n) : \sup_{\mathbb{Q}_n \in \mathcal{P}_n} \mathbb{E}_{\mathbb{Q}_n}[X_n(T)] \leq x\}. \tag{2.5}$$

Having set up the foundation, we now want to derive the primal problem. We know the fact that under the real life measure $\mathbb{P}_n$, the constraint (2.4) can be restated using the change of measure $Z_n = \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}$ as

$$\mathbb{E}_{\mathbb{P}_n}[Z_n(T)X_n(T)] \leq x. \tag{2.6}$$

Hence we can restate (2.1) as a concave maximisation problem over $\mathcal{C}_n(x)$

$$u_n(x) = \sup_{X_n(T)\in\mathcal{C}_n(x)} \mathbb{E}_{\mathbb{P}_n}[U(X_n(T))], \tag{2.7}$$

with the initial endowment $x > 0$ and optimal solution $\hat{X}_n^x(T) \in \mathcal{C}_n(x)$.

If this value function is non-degenerate and finite, i.e. $u_n(x) < \infty$, we can use the fact that it inherits its shape from the utility function. Thus, by its concavity, we can show that there exists a unique solution $\hat{X}_n^x(T)$ to the problem (2.7), see [Pha09] Section 7.3.3.

Let us now derive the dual minimisation problem. For $X_n(T) \in L^0_+(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$ and $Z_n = \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}$, we have by the dual characterisation of $\mathcal{C}_n$ in equation (2.5), that

$$X_n(T) \in \mathcal{C}_n(x) \Leftrightarrow \mathbb{E}[Z_n(T)X_n(T)] \leq x, \tag{2.8}$$

for all $Z_n(T)$, as we assumed that the budget constraint (2.6) is satisfied. In particular there is an equivalence between our terminal wealth lying in $L^0_+(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$, thus becoming a martingale under $\mathbb{Q}_n$, to the budget constraint weighted by the dual variable $Z_n(T)$. We will use the notation $Z_n(T) \in \mathcal{P}_n$ from now on to represent that actually $\mathbb{Q}_n \in \mathcal{P}_n$ and $\mathbb{Q}_n = \int Z_n(T)d\mathbb{P}_n$.

From here on we assume that the Inada conditions on marginal utility, described in Section 1.2 hold. As a result we can apply the Legendre transform to the utility function U and, for a $y > 0$, its convex conjugate is

$$V(y) = \sup_{x>0}[U(x) - xy], \tag{2.9}$$

with $V(y) < \infty$. In addition we also need to define the inverse of the marginal utility $\mathcal{I}$ on $(0, \infty)$ as

$$\mathcal{I} = (U')^{-1}, \tag{2.10}$$

with $\mathcal{I}(0) = \infty$ and $\mathcal{I}(\infty) = 0$, therefore

$$V' = -(U')^{-1} = -\mathcal{I}.$$

Then the supremum in (2.9) is attained by $x = \mathcal{I}(y) > 0$, in particular we have that:

$$V(y) = U(\mathcal{I}(y)) - y\mathcal{I}(y), \quad y > 0. \tag{2.11}$$

By the conjugate relation we can see that:

$$U(x) = \inf_{y>0}[V(x) + xy], \quad x > 0 \tag{2.12}$$

and (2.12) reaches the infimum at $y = U'(x)$.

Having introduced the Legendre transform, we now come to the crucial part of the duality methods. For all $x > 0$, $X_n(T) \in \mathcal{C}_n(x)$ and $Z_n(T) \in \mathcal{P}_n$, using (2.8), the Legendre transform, and Lagrange optimisation with multiplier $y > 0$, we have that:

$$\begin{aligned}
\mathbb{E}[U(X_n(T))] &\leq \mathbb{E}[U(X_n(T))] + y\Big(x - \mathbb{E}[Z_n(T)X_n(T)]\Big) \\
&= \mathbb{E}[U(X_n(T)) - yZ_n(T)X_n(T)] + xy \\
&\leq \mathbb{E}[V(yZ_n(T))] + xy.
\end{aligned} \tag{2.13}$$

Let this be the dual minimisation problem:

$$v_n(y) = \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))], \quad y > 0, \tag{2.14}$$

then the second line of (2.13) as well as the convex conjugate relation (2.12), we have

$$\begin{aligned}
u_n(x) &= \sup_{X_n(T)\in\mathcal{C}_n(x)} \mathbb{E}[U(X_n(T))] \\
&\leq \inf_{y>0}[v_n(y) + xy] \\
&= \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T)) + xy],
\end{aligned}$$

for all $y > 0$.

We will briefly outline the steps required to solve this minimisation problem and retrieve the solution to the primal problem from it. For the detailed discussion, see [Pha09] Section 7.3.3. The structure is as follows. First, we have to show that there is an optimal pair $(\hat{y}, \hat{Z}_n(T))$ depending on $x > 0$ solving the dual minimisation problem, for $y > 0$

$$v_n(y) = \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))], \tag{2.15}$$

such that $\hat{y} > 0$ attains the minimum in

$$\inf_{y>0}[v_n(y) + xy].$$

Then we set

$$\hat{X}_n^x(T) = \mathcal{I}(\hat{y}\hat{Z}_n(T)), \tag{2.16}$$

where $U'(\hat{X}_n^x(T)) = \hat{y}\hat{Z}_n(T)$ and see that this implies

$$\hat{X}_n^x(T) \in \mathcal{C}_n(x)$$

therefore

$$\mathbb{E}[\hat{Z}_n(T)\hat{X}_n^x(T)] = x.$$

Using (2.11), we have that

$$\mathbb{E}[U(\hat{X}_n^x(T))] = \mathbb{E}[V(\hat{y}\hat{Z}_n(T))] + x\hat{y}, \tag{2.17}$$

which gives us

$$\hat{u}_n(x) = \mathbb{E}[U(\hat{X}_n^x(T))],$$

and by conjugate duality relations between the dual and the primal value function, we have

$$\begin{aligned} u_n(x) &= \inf_{y>0}[v_n(y) + xy] \\ &= v_n(\hat{y}) + x\hat{y}. \end{aligned} \tag{2.18}$$

Let us make a couple of comments. The crucial point here is the existence of a solution to the dual problem $v_n(y)$ for each $n \in \mathbb{N}$ and $y > 0$. The set of martingale measures $\mathcal{P}_n$ over which we are optimising is included, but not compact in $\mathcal{L}^1(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$. However, as $\mathcal{P}_n$ is not compact in $\mathcal{L}^1(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$ we need to make sure that this limit is $\hat{Z}_n(T) \in \mathcal{P}_n$ and therefore is the optimal solution to our maximisation problem.

Consider the following set:

$$\mathcal{D}_n = \big\{Y_n(T) \in \mathcal{L}^0(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n) : \exists (Z_{n_k})_{k\geq 1} \in \mathcal{P}_n, \quad Y_n(T) \leq \lim_{k\to\infty} Z_{n_k}(T)\big\}$$

which is the smallest, closed subset in $\mathcal{L}^0(\Omega_n, \mathfrak{F}_n, \mathbb{P}_n)$ and $Y_n(T)$ is the state price proces dominated by the limit $\lim_{k\to\infty} Z_{n_k}(T)$. Then the set $\mathcal{C}_n$, using equation (2.8) can be written in duality relation to $\mathcal{D}_n$ for each n as

$$X_n(T) \in \mathcal{C}_n(x) \Leftrightarrow \mathbb{E}[Y_n(T)X_n(T)] \leq x, \quad \forall Y_n(T) \in \mathcal{D}_n. \tag{2.19}$$

We can now consider the dual problem

$$v_n(y) = \inf_{Y_n(T)\in\mathcal{D}_n} \mathbb{E}\big[V(yY_n(T))\big] \quad y > 0, \tag{2.20}$$

which is consistent with the definition in equation (2.15) if the infimum would be taken over $\mathcal{P}_n$ or $\mathcal{D}_n$.

To be able to cover both the complete and incomplete problem, we need one further condition outlined by Pham in [Pha09] as well as by Kramkov and Schachermayer in [KS99]. We assume that the condition of reasonable asymptotic elasticity, given by

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \tag{2.21}$$

is satisfied throughout.

Remark 2.1. *In addition, to ensure that for the dual value function we have*

$$v_n(y) < \infty, \quad \text{for all } y > 0$$

we need to have that for all $y > 0$, there exists a sequence $\left\{Z_n(T)\right\}_n$, such that $Z_n(T) \in \mathcal{P}_n$ and $\mathbb{E}[V(yZ_n(T))] < \infty$. However, this is a condition which can be fulfilled if we make the following assumption on the utility function U.

Suppose there exists a $p \in (0,1)$ and two positive constants k_1, k_2 and each $Z_n(T) \in \mathcal{P}_n$ such that for each n

$$\begin{aligned} U^+(x) &\leq k_1 x^p + k_2, \quad \forall x > 0, \\ \mathbb{E}[Z_n^{-q}(T)] &< \infty, \text{ where } q = \frac{p}{1-p} > 0 \end{aligned} \tag{2.22}$$

then we have

$$V(y) \leq \sup_{x>0} \left[k_1 x^p - xy\right] + k_2 = (k_1 p)^{\frac{1}{1-p}} \frac{y^{-q}}{q} + k_2,$$

for all $y > 0$, then $\mathbb{E}[V(yZ_n(T))] < \infty$ is satisfied and the dual value function is finite, see [Pha09].

Let us summarise the results of duality theory briefly by stating Pham's Theorem 7.3.5, which can be found (including the explicit proof which we will omit) on p.187 in [Pha09].

Theorem 2. *Consider $U : (0,\infty) \to (0,\infty)$ to be a utility function satisfying the Inada conditions as in Definition 3 and the condition outlined above in (2.21). Then*

(i) *The function $u_n(x)$ is finite, differentiable, strictly concave on $(0,\infty)$ and there exists a unique solution $\hat{X}_n^x(T) \in \mathcal{C}_n(x)$ to $u_n(x)$ for all $x > 0$, achieving the supremum in (2.7).*

(ii) *The function $v_n(x)$ is finite, differentiable, strictly convex on $(0,\infty)$ and there exists a unique solution $\hat{Y}_n^y(T) \in \mathcal{D}_n(x)$ to $v_n(x)$ for all $y > 0$, achieving the infinum in (2.20).*

(iii) *For all $x > 0$, we have*

$$\hat{X}_n^x(T) = \mathcal{I}(\hat{y}\hat{Y}_n^y(T)), \quad \textit{i.e. } U'(\hat{X}_n^x(T)) = \hat{y}\hat{Y}_n^y(T),$$

where $\hat{Y}_n^y(T) \in \mathcal{D}_n$ is the solution to $v_n(\hat{y})$ with $\hat{y} = u_n^{'}(x)$ the unique solution to $argmin_{y>0}[v(y) + xy]$, satisfying

$$\mathbb{E}[Y_n(T)X_n(T)] = x.$$

In addition we have the conjugate duality relations

$$\begin{aligned} u_n(x) &= \min_{y>0}[v_n(y) + xy] \quad \forall x > 0, \\ v_n(y) &= \max_{x>0}[u_n(x) - xy] \quad \forall y > 0. \end{aligned} \tag{2.23}$$

(iv) *If there exists a $y > 0$, such that $\inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))] < \infty$, then*

$$v_n(y) = \inf_{Y_n(T)\in\mathcal{D}_n} \mathbb{E}[V(yY_n(T))] = \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))].$$

Proof. To ensure that there exists an optimal solution to the primal problem we had to assume that $u_n(x) < \infty$ for $x > 0$. Now, after going through the dual methodology, we have seen that we also require that the dual value function $v_n(y) < \infty$ for all $y > 0$, which is satisfied once we can ensure the existence of a martingale density $Z_n(T) \in \mathcal{P}_n$ such that $\mathbb{E}[V(yZ_n(T))] < \infty$.

Finally, the optimal wealth process $\hat{X}_n^x(t)$ associated with the value function $u_n(x)$ in the case where the dual problem has the solution $\hat{Z}_n(T) \in \mathcal{P}_n$, with corresponding measure $\hat{\mathbb{Q}}_n$ and such that $\mathbb{E}_{\hat{\mathbb{Q}}_n}[\hat{X}_n^x(T)] = x$, is a local martingale under $\hat{\mathbb{Q}}_n$, such that

$$\hat{X}_n^x(t) = \mathbb{E}_{\hat{\mathbb{Q}}_n}[I(\hat{y}\hat{Z}_n(T))|\mathfrak{F}_n(t)] \quad 0 \le t \le T, \tag{2.24}$$

and the optimal portfolio proportion $\hat{\pi}_n$ can be found by the optimal decomposition of $\hat{X}_n^{\pi,x}(t)$ once we use a particular utility function. □

So far this section was only concerned with the recollection of the finite optimisation problem. On the other hand, as we have already mentioned on several occasions, asymptotic arbitrage is a limiting concept. Therefore we need to tie the above concepts in with this framework.

Extending the assumption of a finite value function to $u_\infty(x) < \infty$, we need to be able to apply Theorem 2 and duality methodology to the infinite optimisation problem. The first step is to show that the limit of the dual value functions:

$$\lim_{n\to\infty} v_n(y) = v_\infty(y)$$

exists and is finite. Recall that from equation (2.15), we have for $y > 0$ that

$$v_n(y) = \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))],$$

for each n.
We have seen that for each n, the dual value function $v_n(y) < \infty$, for all $y > 0$ if we have that $\mathbb{E}[V(yZ_n(T))] < \infty$ for all $y > 0$. Therefore, to see that

$$\lim_{n\to\infty} v_n(y) = \lim_{n\to\infty} \inf_{Z_n(T)\in\mathcal{P}_n} \mathbb{E}[V(yZ_n(T))] < \infty$$

we will show in the following section that for each fixed $y > 0$

$$\lim_{n\to\infty} \mathbb{E}[V(yZ_n(T))] < \infty,$$

for some sequence $\Big\{Z_n(T)\Big\}_n$, with $Z_n(T) \in \mathcal{P}_n$.

2.2 Asymptotic Arbitrage and the Optimal Value Function

We want to show that asymptotic arbitrage in large financial markets is directly related to the existence of an optimal limiting value function. Here, we want to formulate and prove this fully for complete markets and then briefly remark on incomplete markets.

Recall that we act on a large market $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ with the sequence of underlying probability spaces $\mathcal{B}_n = \big(\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n\big)$, such that $\mathcal{B}_\infty = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Then for each finite $n \in \mathbb{N}$, the set of equivalent martingale measures $\mathcal{P}_n$ is given by

$$\mathcal{P}_n = \{\mathbb{Q}_n \sim \mathbb{P}_n | \mathbb{Q}_n \text{ is a martingale measure } \} \neq \emptyset.$$

We make the following observations for both complete as well as incomplete large markets under the notation and assumptions introduced in Section 1.4 concerning the limiting behaviour.

Theorem 3. *Suppose that we have a sequence of martingale measures $(\mathbb{Q}_n)_{n\geq 1}$ equivalent to the sequence of measures $(\mathbb{P}_n)_{n\geq 1}$ for each $n \in \mathbb{N}$ in a large market $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ as in Section 1.4.*

If there exists an optimal, finite limiting value function $u_\infty(x) < \infty$, such that

$$u_\infty(x) = \lim_{n\to\infty} u_n(x).$$

and which is strictly concave, strictly increasing and continuously differentiable solving the primal optimisation problem (2.7), then there is no asymptotic arbitrage of the first or second kind.

Proof. First, we need to show that our infinite problem satisfies the conjugate duality relations, outlined in the previous section. Then, in a second step, we will show how this ties in with asymptotic arbitrage.

Recall from (1.23) that $u_\infty(x)$ is defined as the limit of the sequence of optimal value functions

$$u_\infty(x) = \lim_{n\to\infty} u_n(x) < \infty.$$

We first need to see that the dual value function v_n is monotone increasing in n. Consider

$$\begin{aligned} v_n(y) &= \max_{x>0} \big(u_n(x) - xy\big) \\ &\leq \max_{x>0} \big(u_{n+1}(x) - xy\big) \\ &= v_{n+1}(y) \end{aligned}$$

therefore

$$v_\infty(y) = \lim_{n\to\infty} v_n(y).$$

We should briefly remark that as $u_\infty(x)$ is the limit of finite, differentiable strictly concave functions, in general the limit will not be strictly concave however we have assumed that it is. Similarly $v_\infty(y)$ will be a convex function.

Let u^* be the conjugate function to v_∞, that is

$$u^*(x) = \inf_{y>0} \Big(v_\infty(y) + xy\Big).$$

Similarly, v^* is the conjugate of u_∞, i.e.

$$v^*(y) = \sup_{x>0} \Big(u_\infty(x) - xy \Big).$$

Then we can see from

$$\begin{aligned} v_n(y) &= \sup_{x>0} \Big(u_n(x) - xy \Big) \\ &\leq \sup_{x>0} \Big(u_\infty(x) - xy \Big) \\ &= v^*(y), \end{aligned}$$

that

$$\lim_{n\to\infty} v_n(y) = v_\infty(y) \leq v^*(y).$$

In addition we see that

$$\begin{aligned} u_n(x) &= \inf_{y>0} \Big(v_n(y) + xy \Big) \\ &\leq \inf_{x>0} \Big(v_\infty(y) + xy \Big) \\ &= u^*(x), \end{aligned}$$

therefore

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) \leq u^*(x).$$

On the other hand we have

$$\begin{aligned} v^*(y) &= \sup_{x>0} \Big(u_\infty(x) - xy \Big) \\ &\leq \sup_{x>0} \Big(u^*(x) - xy \Big) \\ &= \sup_{x>0} \Big(\inf_{\zeta>0} \Big(v_\infty(\zeta) + \zeta x \Big) - xy \Big). \end{aligned}$$

Let $\zeta = y$, then

$$\begin{aligned} v^*(y) &\leq \sup_{x>0} v_\infty(y) \\ &= v_\infty(y), \end{aligned}$$

therefore $v^*(y) = v_\infty(y)$. Thus v^* is the conjugate dual to u^*.

Now,

$$\begin{aligned} u^*(x) &= \inf_{y>0} \Big(v^*(y) + xy \Big) \\ &= \inf_{y>0} \Big(\sup_{\zeta>0} \Big(u_\infty(\zeta) - y\zeta \Big) + xy \Big) \end{aligned}$$

As u^* is strictly concave and continuously differentiable, the supremum

$$\sup_{\zeta>0}\Big(u_\infty(\zeta)-y\zeta\Big)$$

is achieved at a point $0<\zeta_y<\infty$, such that

$$u'_\infty(\zeta_y)=y.$$

Note ζ_y is a continuous function of y, thus

$$\begin{aligned}u^*(x)&=\inf_{y>0}\Big(u_\infty(\zeta_y)-y\zeta_y+xy\Big)\\&\leq u_\infty(x),\end{aligned}$$

by choosing y, such that $\zeta_y=x$.

Therefore

$$u_\infty(x)=u^*(x),$$

and we have established the conjugate duality relations.

As as consequence

$$v_\infty(y)<\infty,\ \text{for all } y.$$

Now let us cover the second part of the proof, establishing the link between duality and asymptotic arbitrage.

As we have a complete market, there exists a sequence Z_n such that

$$\lim_{n\to\infty}\mathbb{E}[V(yZ_n)]=v_\infty(y)<\infty,\ \text{for all } y>0.$$

Observe that by the product structure of the Z_n, as described in Section 1.4 equation (1.20), and the convexity of V, we have that

$$\begin{aligned}\mathbb{E}[V(yZ_{n+1})|\mathfrak{F}_n(T)]&\geq\mathbb{E}[V(yZ_n\chi_{n+1})|\mathfrak{F}_n(T)]\\&\geq V\Big(yZ_n\mathbb{E}[\chi_{n+1}]\Big)\\&=V(yZ_n),\end{aligned}$$

hence $V(yZ_n)$ is a submartingale. As

$$\sup_n\mathbb{E}[V(yZ_n)]=v_\infty(y)<\infty,$$

we have by the Submartingale Convergence Theorem, setting $y = 1$, that

$$V(Z_n) \to M_\infty,$$

for some random variable M_∞.

Therefore

$$Z_n \to V^{-1}(M_\infty),$$

by the continuity and monotonicity of V, $Z_\infty = V^{-1}(M_\infty) > 0$, $\mathbb{P}_\infty$- a.s.

By the definition of the Z_n, we have that

$$\lim_{n\to\infty} \exp\left\{ -\sum_{i=1}^{n+J} \int_0^t \hat{\boldsymbol{\theta}}^{(i)} dB_i - \frac{1}{2}\int_0^t ||\hat{\boldsymbol{\theta}}_n||^2 ds \right\} = Z_\infty.$$

In particular, the limit

$$\lim_{n\to\infty} -\sum_{i=1}^{n+J} \int_0^T \hat{\boldsymbol{\theta}}^{(i)} dB_i - \frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds$$

exists.

We now show that $\lim_{n\to\infty} \frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds$ is finite, almost surely, by contradiction.

Assume that $\lim_{n\to\infty} \int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds = \infty$, then we must have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n+J} \int_0^T \hat{\boldsymbol{\theta}}^{(i)} dB_i}{\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds} > 0, a.s. \tag{2.25}$$

Now

$$\begin{aligned}
&\mathbb{P}_\infty \left(\frac{\sum_{i=1}^{n+J} \int_0^T \hat{\boldsymbol{\theta}}^{(i)} dB_i}{\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds} > \lambda \right) \\
&= \mathbb{E}_{\mathbb{P}_\infty}\Big[\mathbb{P}_\infty\Big(\sum_{i=1}^{n+J} \int_0^T \hat{\boldsymbol{\theta}}^{(i)} dB_i > \lambda \int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds \Big| \hat{\boldsymbol{\theta}} \Big) \Big] \\
&\leq \mathbb{E}_{\mathbb{P}_\infty} \left[\frac{\mathbb{E}_{\mathbb{P}_\infty}\big[\sum_{i=1}^{n+J} \int_0^T \hat{\boldsymbol{\theta}}^{(i)} dB_i \Big| \hat{\boldsymbol{\theta}} \big]^2}{\lambda^2 \big(\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds \big)^2} \right] \\
&= \frac{1}{\lambda^2} \mathbb{E}_{\mathbb{P}_\infty} \left[\frac{1}{\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2} \right],
\end{aligned}$$

by Itô isometry and the Chebychev inequality.

As we have assumed that $\lim_{n\to\infty}\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds = \infty$, we must have that

$$\lim_{n\to\infty}\frac{1}{\lambda^2}\mathbb{E}_{\mathbb{P}_\infty}\left(\frac{1}{\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2}\right) = 0,$$

for all $\lambda > 0$, contradicting (2.25) and therefore

$$\lim_{n\to\infty}\frac{1}{2}\int_0^T ||\hat{\boldsymbol{\theta}}_n||^2 ds < \infty, \quad \mathbb{P}_\infty \quad a.s.$$

By Corollary 1.15 this is the condition for no asymptotic arbitrage, and thus we obtain the required result. □

Remark 2.2. *As we assume that our parameters and therefore the relative risk price process vector $\hat{\boldsymbol{\theta}}_n \in \ell^2$, for all n, are $\mathfrak{F}_n(t)$-measurable and adapted vector processes, we can only show this conjecture in the above direction. In particular, the Novikov condition required is stronger than the assumption that $\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt < \infty$, and therefore the equivalence fails. However, if we assume that our parameters are deterministic functions of time, we can make an if and only if statement.*

Corollary 2.3. *Suppose now that our modelling coefficients are deterministic functions only dependent on time and that all the other assumptions of Theorem 3 hold. In addition we assume that $V(y) \leq C_1 y^{-\gamma} + C_2$ for constants C_1, C_2, γ, $C_2 > 0$. Then, if there is no asymptotic arbitrage, the optimal value function exists and is finite.*

Proof. To show the equivalence in this case, we now need to show that the statement works in both directions. By Kabanov and Kramkov in [KK93], we have that there is no asymptotic arbitrage if for a collection of equivalent martingale measures $\mathbb{Q}_n \sim \mathbb{P}_n$, for each $n \in \mathbb{N}$, the following holds

$$\lim_{n\to\infty} 8h_n(T) < \infty. \tag{2.26}$$

Therefore by Corollary 1.15 in Chapter 1, we have that the particular structure of the Hellinger process is given by

$$\lim_{n\to\infty}\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt < \infty,$$

for the relative risk price process $\hat{\boldsymbol{\theta}}_n(t) \in \ell^2$, $t \in [0, T]$, defined in Definition 1.

As we have assumed that $\hat{\boldsymbol{\theta}}_n(t)$ is just a deterministic function only dependent on time, we have by the no asymptotic arbitrage condition, that

$$\lim_{n\to\infty}\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt < \infty,$$

so

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_\infty} \exp\Big\{ \int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt \Big\} < \infty.$$

We look at the complete market case. then the exist a unique Z_n for each n . By the product structure of Z_n, as described in Section 1.4 we have that $Z_{n-1}\chi_n = Z_n$ and $\lim_{n\to\infty} Z_n = Z_\infty$, almost surely.

As already seen, we have that $v_n(y) \uparrow v_\infty(y)$. By assumption, we have that

$$V(y) \leq C_1 y^{-\gamma} + C_2,$$

then

$$\mathbb{E}_{\mathbb{P}_\infty} V(yZ_n) \leq C_1 y^{-\gamma} \mathbb{E}_{\mathbb{P}_\infty} Z_n^{-\gamma} + C_2.$$

Assuming that the modelling parameters are deterministic, we can compute $\mathbb{E}_{\mathbb{P}_\infty} Z_n^{-\gamma}$directly using the decomposition of Z_n from Section 1.4 equation (1.20) as

$$\mathbb{E}_{\mathbb{P}_\infty} Z_n^{-\gamma} = e^{\frac{\gamma}{2}(\gamma+1)\int_0^T ||\hat{\boldsymbol{\theta}}_n(t)||^2 dt} \leq K < \infty,$$

for a constant K which is independent of n.

Then by monotonicity of the dual value function we have that $v_n(y) \uparrow v_\infty(y) < \infty$ for all $y > 0$ and it follows that $u_\infty(x) < \infty$ for all $x > 0$. □

Throughout this section we only considered the modelling setup for complete markets and managed to show a link between the existence of a finite value function and asymptotic arbitrage in this case. For incomplete markets, this setup is more tricky, as we don't have one sequence Z_n but in fact an infinite number of sequences. If the product structure of the Z_n holds for all sequences, we could apply the approach we have taken here. However we will not pursue this further.

Chapter 3

Explicit Optimisation Results

Let us begin with some general remarks regarding the modelling setup. The main objective in the following chapter is to calculate concrete solutions to the finite, large market optimisation problem, which was formally introduced in Section 1.3. We look at both the complete market where all the assets are traded, and the incomplete market, where the global asset is not traded and thus cannot be invested in directly. In this section we fix $J = 1$ and use N instead of n to signify that we are looking at a finite set of assets. Our finite size market is denoted by $\mathcal{M}_{N+1} = \bigcup_{j=1}^{N+1} \mathcal{M}_j$, where the subscript N signifies that we only trade in a basket of up to N stocks $S_j(t)$, $j = 1, \ldots, N$ and at most one market asset $I(t)$.

Our ultimate goal is to derive the vectors of the optimal portfolio proportions given by $\pi^*(t) = (\pi_1^*(t), \ldots, \pi_N^*(t), \tilde{\pi}^*(t))$, $0 \leq t \leq T$ for the complete model and $\hat{\pi}(t) = (\hat{\pi}_1(t), \ldots, \hat{\pi}_N(t))$, $0 \leq t \leq T$ for the incomplete model, as well as the explicit value functions $u_N^*(x)$ and $\hat{u}_N(x)$. Throughout this chapter we will use a power utility function $U : (0, \infty) \to (0, \infty)$, with shape $U(x) = \frac{x^p}{p}$, $p < 1, p \neq 0$, satisfying the assumptions of Definition 3.

Focusing on the calculation of particular results, we want to simplify the previously random modelling parameters from Section 1.2 to be deterministic functions, only dependent on the time $t \in [0, T]$. As $J = 1$ we will omit the second subscript in our notation so that from now on, let $\alpha_j(t) \in \mathbb{R}^N$ be the rate of return and $\sigma_j(t) \in \mathbb{R}_+^N$ be the volatility of each single stock $S_j(t)$. Furthermore, $\mu(t) \in \mathbb{R}$ is the return and $\gamma(t) \in \mathbb{R}_+$ the volatility of the market asset $I(t)$. The correlation coefficient with the overall market, for each security $S_j(t)$, $j = 1, \ldots, N$, is given by $\rho_j(t) \in (-1, 1)$, and the risk free interest rate is $r(t) \in \mathbb{R}_+$. Although the modelling parameters in this Chapter are time-dependent, we will not display it to simplify notation, and only show the time dependence when particularly required in the argument.

On a filtered probability space $\mathcal{B}_N = \big(\Omega_N, \mathfrak{F}_N, \mathbb{F}_N = (\mathfrak{F}_N(t)), \mathbb{P}_N\big)$, we have the diffusion processes $\{S_j(t); 0 \leq t \leq T\}$ for the risky assets, and $\{I(t); 0 \leq t \leq T\}$ for the global market. Under the real world measure $\mathbb{P}_N$, they satisfy the following general system of equations

$$\begin{aligned} dS_j(t) &= \alpha_j S_j(t)dt + \sigma_j \rho_j S_j(t)dM(t) + \sigma_j \sqrt{1-\rho_j^2} S_j(t)dW_j(t), \\ dI(t) &= \mu I(t)dt + \gamma I(t)dM(t), \end{aligned} \tag{3.1}$$

for $t \in [0, T]$, $j = 1, \ldots, N$.

The risk-free bank account, $\{S_0(t), 0 \leq t \leq T\}$ evolves as

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1.$$

Here, the vector $\{W(t) = (W_1(t), \ldots, W_N(t)); 0 \leq t \leq T\}$ of one-dimensional Brownian motions is associated with the idiosyncratic risk of each stock, and $\{M(t); 0 \leq t \leq T\}$ is driving the global risk factor. They are assumed to be independent Brownian motions, defined on the probability space $\mathcal{B}_N$. See Section 1.2 for further details.

As previously mentioned, we are interested in two different setups: first, the complete market model as given above, secondly the incomplete version of it, obtained from the system (3.1) by omitting the market asset $I(t)$. We have seen in Chapter 2 that the martingale and duality approach only yields general results, hence in this chapter we will rely on the more convenient dynamic programming methodology for the calculation of our concrete solutions. It should be emphasised that we are only dealing with a fixed finite number N of risky stocks throughout this chapter.

The core problem is then posed as follows. Find the the admissible, optimal portfolio proportions that maximise the expected utility of terminal wealth, described by the value function

$$u_N(x) = \sup_{\pi \in \mathcal{A}_N(x)} \mathbb{E}\left[U(X_N^{\pi,x}(T)) | X_N^{\pi,x}(0) = x\right].$$

The appropriate admissibility set in this case is

$$\mathcal{A}_N(x) = \{\pi(t) : \mathbb{E}[|U(X_N^{\pi,x}(T))|] < \infty, X_N^{\pi,x}(t) \geq 0, \quad \forall t \in [0, T]\}.$$

Applying the well-known dynamic programming principle, together with Itô's lemma, leads to the Hamilton-Jacobi-Bellman equation, a nonlinear second-order PDE, describing the local behaviour of the value function $u_N(x)$ for every fixed point $(t, x) \in [0, T] \times \Omega_N$ in time and space. The solution to this HJB equation, the optimal value function, which is at least twice

continuously differentiable, then attains the supremum for the optimal portfolio proportion vector. We will only state the HJB equations, without showing the rather long derivation or detailed verification, as similar types of equations have been previously studied in well established literature such as [Bjo04], [Pha09], or [Kar97].

In the final section of this chapter, we briefly outline how these particular results can be extended to a setup with two global risk factors for a complete and incomplete market model with deterministic parameters, as well as an arbitrary number of $J > 2$ risk factors in the simpler case of homogeneous parameters.

3.1 Complete One-Factor Model

Let us now demonstrate the necessary steps to finding the explicit optimal portfolio proportion, as well as the optimal value function for the complete one-factor market model using a power utility function.

Consider the complete market that is given by the system (3.1). Recall that it was derived from the general set of equations (1.1) by assuming that $J = 1$ and that the parameters are deterministic functions of time.

As before $\{S_j(t); 0 \leq t \leq T\}$, $j = 1, \ldots, N$, is a model for the correlated stock price while $\{I(t); 0 \leq t \leq T\}$ is the traded global market asset.

Assume, that the wealth process $\{X_N(t); 0 \leq t \leq T\}$ represents the agent's total holdings at a time $t \in [0, T]$ and is comprised of the sum over the holdings in the N stocks, the holding in the global market asset and an amount of the risk-free asset.

It can be derived from the general form given in (1.10), as

$$\begin{cases} dX_N(t) = & \left[\sum_{j=1}^{N}(\alpha_j - r)\pi_j(t) + (\mu - r)\tilde{\pi}(t) + r\right]X_N(t)dt \\ & + \left[\sum_{j=1}^{N}\sigma_j\rho_j\pi_j(t) + \gamma\tilde{\pi}(t)\right]X_N(t)dM(t) \\ & + \sum_{j=1}^{N}\sigma_j\sqrt{1-\rho_j^2}\pi_j(t)X_N(t)dW_j(t), \\ X_N(0) = & x, \end{cases} \tag{3.2}$$

where $\tilde{\pi}(t) = \frac{\tilde{\upsilon} I(t)}{X_N(t)}$ is the proportion of wealth invested in the global asset, and $\pi_j(t) = \frac{\upsilon_j S_j(t)}{X_N(t)}$ the proportions of wealth invested in each single security. We can identify $X_N(t)$ with the controlled wealth process starting at $X_N(0) = x > 0$. Whenever it is useful to emphasise this fact, we will write $\{X_N^{\pi,x}(t); 0 \leq t \leq T\}$.

To be able to state the results in a financially meaningful and concise way, let us start by introducing a list of new notation. For each security $S_j(t)$, $j = 1, \ldots, N$, $t \in [0, T]$ we want to look at:

(i) the correlation ratio

$$\xi_j = \frac{\rho_j}{\sqrt{1 - \rho_j^2}},$$

(ii) the Sharpe ratio

$$\lambda_j = \frac{\alpha_j - r}{\sigma_j},$$

(iii) the Sharpe ratio with respect to the market

$$\eta_j = \frac{\lambda_j}{\rho_j},$$

(iv) the convenient, non-negative constant

$$K_N = 1 + \sum_{j=1}^{N} \xi_j^2(t),$$

(v) and the Sharpe ratio for the global market asset $I(t)$

$$\lambda_0 = \frac{\mu - r}{\gamma}.$$

The following theorem gives a summary of the results for complete markets.

Theorem 4. *Suppose that our investment universe is given by (**??**) and we have a power utility function of the form*

$$U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0.$$

The objective of maximising the expected utility of terminal wealth is represented by the value function $u_N(x)$, with general form

$$u_N(x) = \sup_{\pi \in \mathcal{A}_N(x)} \mathbb{E}\left[U(X_N^{\pi,x}(T)) | X_N^{\pi,x}(0) = x\right],$$

over the vector of admissible controls $\pi(t) = (\tilde{\pi}(t), \pi_1(t), \ldots, \pi_N(t))$, $0 \leq t \leq T$, with admissibility set

$$\mathcal{A}_N(x) = \{\pi(t) : \mathbb{E}[|U(X_N^{\pi,x}(T))|] < \infty, X_N^{\pi,x}(t) \geq 0, \quad \forall t \in [0, T]\}.$$

Then, we have the following results:

(i) *the optimal investment policies* $\pi^*(t) = (\pi_1^*(t), \ldots, \pi_N^*(t), \tilde{\pi}^*(t))$*, attaining the supremum in* $u_N(x)$*, are given by*

$$\begin{aligned} \tilde{\pi}^*(t) &= \frac{1}{(1-p)\gamma}\left(\lambda_0 - \sum_{k=1}^{N} \xi_k^2(\eta_k - \lambda_0)\right), \\ \pi_j^*(t) &= \frac{\xi_j^2}{(1-p)\sigma_j\rho_j}\left(\eta_j - \lambda_0\right), \end{aligned} \tag{3.3}$$

for $j, k = 1, \ldots, N$ *and each* $t \in [0, T]$*.*

(ii) *Therefore, the optimal value function is calculated to be*

$$u_N^*(x) = \exp\left\{\int_0^t \left[\frac{\Theta_N^*(s)}{1-p} + r(s)\right] pds\right\} \frac{x^p}{p}, \tag{3.4}$$

with the value function growth rate

$$\Theta_N^*(t) = \frac{1}{2}\left(\lambda_0^2 + \sum_{j=1}^{N} \xi_j^2(\eta_j - \lambda_0)^2\right). \tag{3.5}$$

Proof. We want to briefly highlight the crucial steps of the proof. A complete proof, in particular the derivation of the HJB equation and the verification theorem, can be adapted directly from the classical Merton case. Refer for example to Bjork [Bjo04] for a more detailed account.

Suppose that the value function $u_N(x)$ is an at least twice continuously differentiable function increasing in x and the vector of portfolio proportions is given by $\pi(t) = (\pi_1(t), \ldots, \pi_N(t), \tilde{\pi}(t))$.

Fix the coordinates $(t, x) \in [0, T] \times \mathbb{R}^{N+1}$ and use the dynamic programming principle together with Itô's lemma to evaluate the value function around this point. This results in the following HJB equation:

$$\begin{aligned} 0 = \sup_{\pi \in \mathcal{A}(x)} \Bigg\{ & \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x}\left(\sum_{j=1}^{N}(\alpha_j - r)\pi_j(t) + (\mu - r)\tilde{\pi}(t) + r\right) \\ & + \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\Bigg(\sum_{j=1}^{N}\sigma_j^2\pi_j^2(t) + \gamma^2\tilde{\pi}^2(t) + \sum_{j=1}^{N}\sum_{i\neq j}\sigma_j\sigma_i\rho_j\rho_i\pi_j(t)\pi_i(t) \\ & + 2\sum_{j=1}^{N}\gamma\sigma_j\rho_j\tilde{\pi}(t)\pi_j(t)\Bigg)\Bigg\}. \end{aligned} \tag{3.6}$$

To construct the candidate optimal portfolio proportions $\pi^*(t) = (\pi_1^*(t), \ldots, \pi_N^*(t), \tilde{\pi}^*(t))$, we differentiate (3.6) with respect to $\pi_j(t)$ for each $j = 1, \ldots, N$, and then with respect to $\tilde{\pi}(t)$, equating the results to zero. Thus, we have to solve the following system of equations:

$$\begin{aligned} \sigma_j^2 \pi_j^*(t) + \sum_{i \neq j} \sigma_j \sigma_i \rho_j \rho_i \pi_i^*(t) + \gamma \sigma_j \rho_j \tilde{\pi}^*(t) &= -\frac{\frac{\partial u}{\partial x}(\alpha_j - r)}{x \frac{\partial^2 u}{\partial x^2}}, \\ \gamma^2 \tilde{\pi}^*(t) + \sum_{j=1}^{N} \sigma_j \gamma \rho_j \pi_j^*(t) &= -\frac{\frac{\partial u}{\partial x}(\mu - r)}{x \frac{\partial^2 u}{\partial x^2}}. \end{aligned} \tag{3.7}$$

As the value function is concave by assumption, we already know that the solution to the above system of equations describes a maximum.

It is convenient to rewrite the above system in matrix notation for further clarity to be

$$\begin{pmatrix} \gamma^2 & \gamma\sigma_1\rho_1 & \cdots & \cdots & \gamma\sigma_N\rho_N \\ \gamma\sigma_1\rho_1 & \sigma_1^2 & \sigma_1\sigma_2\rho_1\rho_2 & \cdots & \sigma_1\sigma_N\rho_1\rho_N \\ \vdots & \vdots & \ddots & \vdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma\sigma_N\rho_N & \cdots & \cdots & \cdots & \sigma_N^2 \end{pmatrix} \begin{pmatrix} \tilde{\pi}^* \\ \pi_1^* \\ \vdots \\ \vdots \\ \pi_N^* \end{pmatrix} = \begin{pmatrix} -\frac{\frac{\partial u}{\partial x}(\mu - r)}{x \frac{\partial^2 u}{\partial x^2}} \\ -\frac{\frac{\partial u}{\partial x}(\alpha_1 - r)}{x \frac{\partial^2 u}{\partial x^2}} \\ \vdots \\ \vdots \\ -\frac{\frac{\partial u}{\partial x}(\alpha_N - r)}{x \frac{\partial^2 u}{\partial x^2}} \end{pmatrix}. \tag{3.8}$$

Under the assumption that the covariance matrix

$$\tilde{\Sigma} = \begin{pmatrix} \gamma^2 & \gamma\sigma_1\rho_1 & \cdots & \cdots & \gamma\sigma_N\rho_N \\ \gamma\sigma_1\rho_1 & \sigma_1^2 & \sigma_1\sigma_2\rho_1\rho_2 & \cdots & \sigma_1\sigma_N\rho_1\rho_N \\ \vdots & \vdots & \ddots & \vdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma\sigma_N\rho_N & \cdots & \cdots & \cdots & \sigma_N^2 \end{pmatrix}, \tag{3.9}$$

is invertible, symmetric and of full rank the above system (3.7) has a unique solution.

In the next step, we need to find the inverse of (3.9) by applying the matrix inversion methodology II, as described in Section 1.5, to $\tilde{\Sigma}$. After performing a number of purely algebraic manipulations, we obtain:

$$\tilde{\Sigma}^{-1} = \begin{pmatrix} \frac{1+\sum_{i=1}^{N} \xi_i^2}{\gamma^2} & -\frac{\rho_1}{(1-\rho_1^2)\gamma\sigma_1} & -\frac{\rho_2}{(1-\rho_2^2)\gamma\sigma_2} & \cdots & \cdots & -\frac{\rho_N}{(1-\rho_N^2)\gamma\sigma_N} \\ -\frac{\rho_1}{(1-\rho_1^2)\gamma\sigma_1} & \frac{1}{(1-\rho_1^2)\sigma_1^2} & 0 & \cdots & \cdots & 0 \\ -\frac{\rho_2}{(1-\rho_2^2)\gamma\sigma_2} & 0 & \frac{1}{(1-\rho_2^2)\sigma_2^2} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ -\frac{\rho_N}{(1-\rho_N^2)\gamma\sigma_N} & 0 & 0 & \cdots & 0 & \frac{1}{(1-\rho_N^2)\sigma_N^2} \end{pmatrix}. \tag{3.10}$$

Multiplying equation (3.8) by $\tilde{\Sigma}^{-1}$ from the left yields the required vector of optimal investment proportions:

$$\begin{aligned}\tilde{\pi}^*(t) &= -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}\left[\frac{1}{\gamma}\left(\lambda_0 + \sum_{k=1}^{N}\xi_k^2(\eta_k - \lambda_0)\right)\right], \\ \pi_j^*(t) &= -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}\left[\frac{\xi_j^2}{\sigma_j\rho_j}(\eta_j - \lambda_0)\right],\end{aligned} \tag{3.11}$$

for $j = 1, \ldots, N$.

Substituting the results from (3.11) into the HJB equation (3.6), we can see that the supremum is attained and it reduces to the simpler non-linear PDE

$$\begin{aligned}0 &= \frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{\frac{\partial^2 u}{\partial x^2}}\left[\frac{1}{2}\left(\lambda_0^2 + \sum_{j=1}^{N}\xi_j^2(\eta_j - \lambda_0)^2\right)\right] \\ &= \frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{\frac{\partial^2 u}{\partial x^2}}\Theta_N^*(t),\end{aligned} \tag{3.12}$$

yielding the required the growth rate

$$\Theta_N^*(t) = \frac{1}{2}\left(\lambda_0^2 + \sum_{j=1}^{N}\xi_j^2(\eta_j - \lambda_0)^2\right).$$

The key to finding the exact value function $u_N^*(t,x)$, and therefore the concrete control processes, is that the value function inherits its structure from our chosen utility function, see [Pha09]. Using this fact, we can find an ansatz for the value function, which has to be written in separable form. This means that it is possible to divide it into a time- and a space-dependent part. In particular assume that we have

$$u_N^*(t,x) = g(t)\frac{x^p}{p},$$

for some continuously differentiable function $g(t)$, $g : \mathbb{R} \to \mathbb{R}$ and a penalty factor $p < 1$, $p \neq 0$. Calculating the differentials $\frac{\partial u_N^*}{\partial x}$, $\frac{\partial^2 u_N^*}{\partial x^2}$ and $\frac{\partial u_N^*}{\partial t}$, and inserting them back into (3.12), the PDE (3.12) reduces to the ordinary differential equation (ODE)

$$0 = \frac{1}{p}\frac{dg(t)}{dt} + \left(\frac{\Theta_N^*}{1-p} + r\right).$$

Its solution is easily found and concludes the proof. □

Remark 3.1. *To gain some intuitive insight into the behaviour of the portfolio proportion and value function growth rate, we should also mention the case of homogeneous parameters.*

Suppose that for each asset $S_j(t)$, $j = 1, \ldots, N$, we now have $\alpha_j(t) = \alpha(t)$, $\sigma_j(t) = \sigma(t)$ and $\rho_j(t) = \rho(t)$. Then, in the case of a power utility function, the optimal investment proportions in (3.3) simplify to

$$\begin{aligned} \tilde{\pi}^{*,h}(t) &= \frac{1}{(1-p)\gamma}\Big(\lambda_0 + N\xi^2(\eta - \lambda_0)\Big), \\ \pi_j^{*,h}(t) &= \frac{\xi^2}{(1-p)\sigma\rho}(\eta - \lambda_0). \end{aligned} \tag{3.13}$$

Under these assumptions the corresponding growth rate of the value function can be directly derived from (3.34) to be

$$\Theta^{*,h} = \frac{1}{2}\Big(\lambda_0^2 + N\xi^2(\eta - \lambda_0)^2\Big). \tag{3.14}$$

Note that the above quantities (3.13) and (3.14) are strongly dependent on the behaviour of the difference between the Sharpe ratio with respect to the market of each security, denoted by $\eta(t)$ and the Sharpe ratio of the global asset $\lambda_0(t)$. Later in Chapter 4, when we discuss the asymptotic behaviour of these quantities, we will need to assume that this difference goes to zero as $\frac{c}{N^s}$, $c \in \mathbb{R}$, for an appropriate choice of the positive real constant s. The main reason for this lies in a particular no arbitrage argument, which in financial terms means that all assets, including the global market asset, should have the same market price of risk in the limit. More on this topic will be mentioned in Chapter 4.

3.2 Incomplete One-Factor Model

In this section we calculate the explicit results for a market where the global asset is no longer available for trading. Such a model is considered to be incomplete, as the number of Brownian-motion-driven sources of risk is higher than the number of price processes describing the available stocks. In particular, we are curious to discover how the optimal investment proportions behave in comparison to the outcomes in the complete model and what value we could generate, if we only invest in the N stocks and the risk free bank account.

Consider the following system of dynamics describing our incomplete market:

$$dS_j(t) = \alpha_j S_j(t)dt + \sigma_j\rho_j S_j(t)dM(t) + \sigma_j\sqrt{1-\rho_j^2}S_j(t)dW_j(t), \tag{3.15}$$

for $t \in [0, T]$, $j = 1, \ldots, N$, and

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

for the bank account.

Structurally, we will proceed as in the previous section, stating the concrete solutions under the assumption that a power utility function describes the investor's preferences. Applying the notation introduced in Section 3.1, we can summarise the results as follows.

Theorem 5. *Suppose that our investment universe is governed by the collection of SDEs given by equation (3.15). We assume that we have a power utility function of the form*

$$U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0.$$

Our objective is to maximise the expected utility of terminal wealth, which is represented by the value function $u_N(x)$ of the form

$$u_N(x) = \sup_{\pi \in \mathcal{A}_N(x)} \mathbb{E}\left[U(X_N^{\pi,x}(T))|X_N^{\pi,x}(0) = x\right],$$

over the vector of admissible controls $\pi(t) = (\pi_1(t), \ldots, \pi_N(t))$, with admissibility set

$$\mathcal{A}_N(x) = \{\pi(t) : \mathbb{E}[|U(X_N^{\pi,x}(T))|] < \infty, X_N^{\pi,x}(t) \geq 0, \quad \forall t \in [0,T]\}.$$

Then,

(i) *the optimal investment policies $\hat{\pi}(t) = (\hat{\pi}_1(t), \ldots, \hat{\pi}_N)$ are*

$$\hat{\pi}_j(t) = \frac{1}{1-p}\left[\frac{1}{K_N \sigma_j \rho_j}\left(\eta_j \xi_j^2 + \sum_{k=1}^{N} \xi_k^2 \xi_j^2 (\eta_j - \eta_k)\right)\right], \tag{3.16}$$

for $j, k = 1, \ldots, N$,

(ii) *and the optimal solution to the value function is given by*

$$\hat{u}_N(x) = \exp\left\{\int_0^t \left[\frac{\hat{\Theta}_N(s)}{1-p} + r(s)\right] p ds\right\} \frac{x^p}{p}, \tag{3.17}$$

with growth rate

$$\hat{\Theta}_N(t) = \frac{1}{2}\sum_{j=1}^{N} \xi_j^2 \eta_j^2 - \frac{1}{2K_N}\left(\sum_{j=1}^{N} \xi_j^2 \eta_j\right)^2. \tag{3.18}$$

Proof. Our underlying state price process $\{X_N(t); 0 \leq t \leq T\}$, starting from the initial value $X_N(0) = x > 0$, evolves as

$$\begin{cases} dX_N(t) = & \left[\sum_{j=1}^N (\alpha_j - r)\pi_j(t) + r\right] X_N(t)dt + \sum_{j=1}^N \sigma_j \sqrt{1-\rho_j^2}\pi_j(t)X_N(t)dW_j(t) \\ & \quad + \sum_{j=1}^N \sigma_j \rho_j \pi_j(t) X_N(t) dM(t), \\ X_N(0) = & x, \end{cases} \tag{3.19}$$

for $j = 1, \ldots, N$.

We identify $X_N(t)$ with the controlled wealth process starting at $x > 0$ and therefore write $\{X_N^{\pi,x}(t), 0 \leq t \leq T\}$ whenever this fact requires emphasising. Assume, that the value function (1.17) is smooth enough, i.e. at least twice continuously differentiable, then the HJB equation can be derived as

$$\begin{aligned} 0 = \sup_{\pi \in \mathcal{A}(x)} \Bigg\{ & \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} x \left(\sum_{j=1}^N (\alpha_j - r)\pi_j(t) + r \right) \\ & + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} x^2 \left(\sum_{j=1}^N \sigma_j^2 \pi_j^2(t) + \sum_{j=1}^N \sum_{i \neq j} \sigma_i \sigma_j \rho_i \rho_j \pi_i(t) \pi_j(t) \right) \Bigg\}. \end{aligned} \tag{3.20}$$

Differentiating (3.20) with respect to each of the control variables $\pi_j(t)$, $j = 1, \ldots, N$, and equating the result to zero, yields the following system of equations for the candidate optimal portfolio proportions:

$$\sigma_j^2 \hat{\pi}_j(t) + \sum_{i \neq j} \sigma_i \sigma_j \rho_i \rho_j \hat{\pi}_i(t) = -\frac{\frac{\partial u}{\partial x}(\alpha_j - r)}{x \frac{\partial^2 u}{\partial x^2}}. \tag{3.21}$$

By concavity of the value function, we already know that the solution to the above system of equations describes a maximum.

To find the concrete solution, we need to invert the covariance matrix underlying our asset universe and then solve the HJB equation to find the explicit value function. First, rewrite (3.21) in matrix notation

$$\begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_1\rho_2 & \cdots & \sigma_1\sigma_N\rho_1\rho_N \\ \sigma_1\sigma_2\rho_1\rho_2 & \sigma_2^2 & \cdots & \sigma_2\sigma_N\rho_2\rho_N \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_N\rho_1\rho_N & \cdots & \cdots & \sigma_N^2 \end{pmatrix} \begin{pmatrix} \hat{\pi}_1 \\ \vdots \\ \vdots \\ \hat{\pi}_N \end{pmatrix} = \begin{pmatrix} -\frac{\frac{\partial u}{\partial x}(\alpha_1 - r)}{x\frac{\partial^2 u}{\partial x^2}} \\ \vdots \\ \vdots \\ -\frac{\frac{\partial u}{\partial x}(\alpha_N - r)}{x\frac{\partial^2 u}{\partial x^2}} \end{pmatrix}. \tag{3.22}$$

Secondly, we have to find the inverse Σ^{-1} to the $N \times N$ dimensional, invertible and symmetric covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_1\rho_2 & \cdots & \sigma_1\sigma_N\rho_1\rho_N \\ \sigma_1\sigma_2\rho_1\rho_2 & \sigma_2^2 & \cdots & \sigma_2\sigma_N\rho_2\rho_N \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_N\rho_1\rho_N & \cdots & \cdots & \sigma_N^2 \end{pmatrix}. \tag{3.23}$$

Using the matrix inversion method I, given in Section 1.5, we can calculate this inverse as:

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{(1-\rho_1^2)\sigma_1^2}\left(1-\frac{\xi_1^2}{K_N}\right) & \cdots & -\frac{\xi_1\xi_N}{\sqrt{(1-\rho_1^2)(1-\rho_N^2)}\sigma_1\sigma_N K_N} \\ \vdots & \ddots & \vdots \\ -\frac{\xi_1\xi_N}{\sqrt{(1-\rho_1^2)(1-\rho_N^2)}\sigma_1\sigma_N K_N} & \cdots & \frac{1}{(1-\rho_N^2)\sigma_N^2}\left(1-\frac{\xi_N^2}{K_N}\right) \end{pmatrix}. \tag{3.24}$$

Resolving equation (3.22), using (3.24), yields a general set of optimal strategies

$$\hat{\pi}_j(t) = -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}\left[\frac{1}{K_N\sigma_j\rho_j}\left(\eta_j\xi_j^2 + \sum_{k=1}^{N}\xi_k^2\xi_j^2(\eta_j-\eta_k)\right)\right], \tag{3.25}$$

for $j,k = 1,\ldots,N$.

Inserting (3.25) back into (3.20), the supremum is attained and the Hamilton-Jacobi-Bellman equation reduces to

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}rx - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{\frac{\partial^2 u}{\partial x^2}}\left[\frac{1}{2}\sum_{j=1}^{N}\xi_j^2\eta_j^2 - \frac{1}{2K_N}\left(\sum_{j=1}^{N}\xi_j^2\eta_j\right)^2\right] \\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}rx - \frac{\left(\frac{\partial u}{\partial x}\right)^2}{\frac{\partial^2 u}{\partial x^2}}\hat{\Theta}_N(t). \end{aligned} \tag{3.26}$$

The growth rate is then

$$\hat{\Theta}_N(t) = \frac{1}{2}\sum_{j=1}^{N}\xi_j^2\eta_j^2 - \frac{1}{2K_N}\left(\sum_{j=1}^{N}\xi_j^2\eta_j\right)^2,$$

as required.

Applying the standard methodology, as used in the previous section, under the premise that the value function inherits the shape of the power utility function, see [DN90], [Bjo04] or [Kar97], and can be written in separable form, we obtain all required results and conclude the proof of the theorem. □

Remark 3.2. *Similar to the complete case, we want to consider homogeneous parameters. In particular, assume that every single stock in the basket described in (3.15) has the constant coefficients* $\alpha_j(t) = \alpha(t)$, $\sigma_j(t) = \sigma(t)$ *and* $\rho_j(t) = \rho(t) \in (-1, 1)$ *for each* $j = 1, \dots, N$. *Then the optimal proportion of total wealth to be invested in each asset* $S_j(t)$ *is*

$$\hat{\pi}^h(t) = \frac{1}{N}\left(\frac{\lambda}{(1-p)(1-\rho^2)(\frac{1}{N}+\xi^2)\sigma}\right). \tag{3.27}$$

We can see that for this choice of coefficients the $\hat{\pi}^h(t)$ *are of size* $\mathcal{O}\left(\frac{1}{N}\right)$. *The corresponding growth rate for the value function (3.18) then becomes*

$$\hat{\Theta}^h_N(t) = \frac{\eta^2\xi^2}{2(\frac{1}{N}+\xi^2)}. \tag{3.28}$$

These results will be of interest when considering the limiting properties and asymptotic expansions of (3.16) and (3.43) in Chapter 4, where they will be identified as the leading terms of the expansions.

Let us briefly comment on the differences between the homogeneous, one-factor, complete market value function growth rate from (3.14) and incomplete market value function growth rate shown in equation (3.28). Subtracting the one from the other, we see for finite N that

$$\begin{aligned}\Theta^{*,h}(t) - \hat{\Theta}^h_N(t) &= \frac{1}{2}\Big(\lambda_0^2 + N\xi^2(\eta-\lambda_0)^2\Big) - \frac{\eta^2\xi^2}{2(\frac{1}{N}+\xi^2)} \\ &= \frac{1}{2}\left(\frac{\big(\lambda_0 - \xi^2 N(\eta-\lambda_0)\big)^2}{1+N\xi^2}\right).\end{aligned}$$

Note that if $\eta - \lambda_0 \not\to 0$ as $N \to \infty$, than the difference is of size $\mathcal{O}(N)$.

As expected, this expression is positive for all choices of parameters. Therefore, we can say that it is more profitable to invest in a combination of single securities and the global asset for a finite basket size N. This is exactly in line with our intuition. The parameters are homogeneous, so consequently all stocks follow the same dynamics. Therefore the complete model just offers us one additional different asset to invest in and such a portfolio should outgrow the one where the global asset is unavailable, as is the case for the incomplete setting. If this combined portfolio would not generate superior returns, the strategy would not be optimal and we would drop the market asset, making at least as much as is possible with just the single stocks.

In general, we can see that in the complete case, the optimal growth rate Θ^*_N is growing very fast as $\mathcal{O}(N)$, however if the global market asset is not available for trading, the value function growth $\hat{\Theta}_N(t)$ is of size $\mathcal{O}(1)$.

3.3 Extension to Two Factors

The purpose of this section is to illustrate how the results from Section 3.1 and Section 3.2 for one global economic factor can be extended to the case of two factors i.e. if $J = 2$.

The idea behind this is to show the implications of the optimal investment behaviour when we assume that the stocks are correlated with two market risk factors. A reasonable economic interpretation is that the global market is divided into two sectors, which exhibit different risk-return profiles and each stock is correlated differently with them. A different possibility is that we consider the GDP and political stability as two distinct risk factors influencing our overall system.

First, consider the complete setting. The general system of price processes for a finite basket of N risky stocks $\{S_j(t); 0 \leq t \leq T\}$, $j = 1, \ldots, N$ and two market assets $\{I_l(t); 0 \leq t \leq T\}$, $l = 1, 2$ can be derived from (1.1) by assuming that $J = 2$. In particular we have the dynamics

$$\begin{aligned} dS_j(t) =& \alpha_j S_j(t)dt + \sigma_j S_j(t)[\rho_{j,1}dM^1(t) + \rho_{j,2}dM^2(t)] + \sigma_j S_j(t)\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}dW_j(t), \\ dI_1(t) =& \mu_1 I_1(t)dt + \gamma_1 I_1(t)dM^1(t), \\ dI_2(t) =& \mu_2 I_2(t)dt + \gamma_2 I_2(t)dM^2(t). \end{aligned} \tag{3.29}$$

The risk-free bank account evolves as

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

for $0 \leq t \leq T$.

Here, $\rho_{j,l}(t) \in (-1, 1)$ is the correlation of the stock $S_j(t)$ for each $j = 1, \ldots, N$ with the market risk $\{M^l(t); 0 \leq t \leq T\}$, $l = 1, 2$, where the latter are two independent Brownian motions.

Assume that the proportion of total wealth in each singe stock is $\pi_j(t) = \frac{\upsilon_j S_j(t)}{X_N(t)}$, $j = 1, \ldots, N$, and that we have $\tilde{\pi}_l(t) = \frac{\tilde{\upsilon}_l I_l(t)}{X_N(t)}$, $l = 1, 2$ for the two global assets. The proportion in the bank account is then $\left(1 - \sum_{j=1}^N \pi_j - \sum_{l=1}^2 \tilde{\pi}_l\right)$. Note, that we make no assumption that these portfolio proportions are positive. This means, that we can either deposit or withdraw from the bank account at each time $t \in [0, T]$ and that short-selling is allowed.

Let us state some further notation to give a clearer structure to the results. Suppose that in the case of two factors, the correlation ratio has to be adjusted to be

$$\xi_{j,l}(t) = \frac{\rho_{j,l}}{\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}},$$

for $l = 1, 2$ and $0 \leq t \leq T$. We want to define the following additional quantities:

$$\begin{aligned}
C_{jk} &= \xi_{j,1}\xi_{k,1}\left(1+\sum_{j=1}^{N}\xi_{j,2}^2\right) + \xi_{j,2}\xi_{k,2}\left(1+\sum_{j=1}^{N}\xi_{j,1}^2\right) - (\xi_{j,1}\xi_{k,2}+\xi_{k,1}\xi_{j,2})\left(\sum_{j=1}^{N}\xi_{j,2}\xi_{j,1}\right), \\
\zeta_j &= \sum_{k=1}^{N}(\xi_{k,1}+\xi_{k,2})C_{jk}, \\
O_N &= \sum_{j=1}^{N}(\xi_{j,1}+\xi_{j,2})\Big[(\xi_{j,1}+\xi_{j,2}) - \zeta_j\Big],
\end{aligned}$$

for $j, k = 1, \ldots, N$.

We should emphasise that besides C_{jk}, which can be seen as an appropriate extension to K_N from the two previous sections, the notation above is merely applied to shorten and clarify the display of the results and has no obvious financial interpretation.

Theorem 6. *Suppose the investment universe is given by (3.29). Assume further, that we have a power utility function of the form*

$$U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0,$$

and that our objective is to maximise the expected utility of terminal wealth, represented by the value function $u_N(x)$ with general form

$$u_N(x) = \sup_{\pi \in \mathcal{A}_N(x)} \mathbb{E}\left[U(X_N^{\pi,x}(T))|X_N^{\pi,x}(0) = x\right],$$

over the vector of admissible controls $\pi(t) = (\tilde{\pi}_1(t), \tilde{\pi}_2(t), \pi_1(t), \ldots, \pi_N(t)) \in \mathcal{A}_N(x)$, where the admissibility set is

$$\mathcal{A}_N(x) = \{\pi(t) : \mathbb{E}[|U(X_N^{\pi,x}(T))|] < \infty, X_N^{\pi,x}(t) \geq 0, \quad \forall t \in [0,T]\}.$$

Then, we have the following results:

(i) *the optimal investment proportions are*

$$\begin{aligned}\tilde{\pi}_1^{**}(t) =& \frac{1}{\gamma_1(1-2O_N)(1-p)}\left(\frac{\mu_1-r}{\gamma_1}-\frac{O_N((\mu_1-r)-(\mu_2-r))}{\gamma_1-\gamma_2}\right.\\ &\left.-\sum_{j=1}^{N}\frac{(\xi_{j,1}+\xi_{j,2}-\zeta_j)(\alpha_j-r)}{\sigma_j\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}}\right)\\ =& \frac{1}{1-p}\tilde{\Pi}_1^*,\end{aligned} \tag{3.30}$$

$$\begin{aligned}\tilde{\pi}_2^{**}(t) =& \frac{1}{\gamma_2(1-2O_N)(1-p)}\left(\frac{\mu_2-r}{\gamma_2}-\frac{O_N((\mu_2-r)-(\mu_1-r))}{\gamma_2-\gamma_1}\right.\\ &\left.-\sum_{j=1}^{N}\frac{(\xi_{j,1}+\xi_{j,2}-\zeta_j)(\alpha_j-r)}{\sigma_j\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}}\right)\\ =& \frac{1}{1-p}\tilde{\Pi}_2^*,\end{aligned} \tag{3.31}$$

and for each $j=1,\ldots,N$:

$$\begin{aligned}\pi_j^{**}(t) =& \frac{1}{1-p}\left(\frac{\alpha_j-r}{\sigma_j^2(1-\rho_{j,1}^2-\rho_{j,2}^2)}-\frac{(\xi_{j,1}+\xi_{j,2}-\zeta_j)(\alpha_j-r)}{\sigma_j\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}(1-2O_N)\left(\frac{((\mu_2-r)+(\mu_1-r))}{\gamma_2+\gamma_1}\right)}\right.\\ &\left.-\sum_{k=1}^{N}\frac{(\alpha_k-r)C_{jk}(1-2O_N)+2(\xi_{j,1}+\xi_{j,2}-\zeta_j)(\xi_{k,1}+\xi_{k,2}-\zeta_k)}{\sigma_j\sigma_k(1-2O_N)\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}\sqrt{1-\rho_{k,1}^2-\rho_{k,2}^2}}\right)\\ =& \frac{1}{1-p}\Pi_j^*.\end{aligned} \tag{3.32}$$

(ii) *The optimal solution to the value function is given by*

$$u_N^{**}(x)=\exp\left\{\int_0^t\left[\frac{\Theta_N^*(s)}{1-p}+r(s)\right]pds\right\}\frac{x^p}{p}, \tag{3.33}$$

with growth rate

$$\begin{aligned}\Theta_N^{**} =& \sum_{j=1}^{N}(\alpha_i-r)\Pi_j^*+\sum_{l=1}^{2}(\mu_l-r)\tilde{\Pi}_l^*\\ &+\frac{1}{2}\sum_{j=1}^{N}\sigma_j^2\Pi_j^*+\frac{1}{2}\sum_{j=1}^{N}\sum_{i\neq j}\sigma_i\sigma_j\Pi_j^*\Pi_i^*(\rho_{i,1}\rho_{j,1}+\rho_{i,2}\rho_{j,2})\\ &+\frac{1}{2}\sum_{l=1}^{2}\gamma_l^2\tilde{\Pi}_l^*+2\sum_{j=1}^{N}\sigma_j\hat{\Pi}_j^*\Big[\sum_{l=1}^{2}\gamma_l\rho_{j,l}\tilde{\Pi}_l^*\Big].\end{aligned} \tag{3.34}$$

Proof. The methodology for proving the above results does not differ from that applied in Theorem 4, hence we only identify the key steps.

Suppose that we have the system with dynamics (3.29) and the controlled wealth process $\{X_N^{\pi,x}(t); 0 \leq t \leq T\}$ starting at the initial endowment $x \in \mathbb{R}_+$. Then the HJB equation can be written as

$$\begin{aligned} 0 = \sup_{\pi \in \mathcal{A}(x)} \Bigg\{ & \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x}\left(\sum_{j=1}^{N}(\alpha_j - r)\pi_i(t) + \sum_{l=1}^{2}(\mu_l - r)\tilde{\pi}_l(t) + r\right) \\ & + \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\left(\sum_{j=1}^{N}\sigma_j^2\pi_j^2(t) + \sum_{j=1}^{N}\sum_{i\neq j}\sigma_i\sigma_j\pi_i(t)\pi_j(t)(\rho_{i,1}\rho_{j,1} + \rho_{i,2}\rho_{j,2})\right. \\ & \left.\left. + \sum_{l=1}^{2}\gamma_l^2\tilde{\pi}_l^2(t) + 2\sum_{j=1}^{N}\sigma_j\pi_j(t)\left[\sum_{l=1}^{2}\gamma_l\rho_{j,l}\tilde{\pi}_l\right]\right)\right\}. \end{aligned} \tag{3.35}$$

Differentiating (3.35) first with respect to each of the π_j, $j = 1, \ldots, N$, $\tilde{\pi}_1$, then $\tilde{\pi}_2$ and equating the results to zero, we obtain the optimal values for the $N+2$ portfolio proportions. By concavity of the value function the solution to these set of equations describes a maximum.

In particular, to find the concrete solutions for the investment vector, we are faced with the problem of inverting the symmetric covariance matrix

$$\tilde{A} = \begin{pmatrix} D & e \\ (e')^T & L \end{pmatrix}, \tag{3.36}$$

where

$$D = \begin{pmatrix} \gamma_1^2 & 0 \\ 0 & \gamma_2^2 \end{pmatrix},$$

$$e = \begin{pmatrix} \gamma_1\sigma_1(\rho_{1,1} + \rho_{1,2}) & \cdots & \gamma_1\sigma_N(\rho_{N,1} + \rho_{N,2}) \\ \gamma_2\sigma_1(\rho_{1,1} + \rho_{1,2}) & \cdots & \gamma_2\sigma_N(\rho_{N,1} + \rho_{N,2}) \end{pmatrix},$$

$(e')^T$ its transpose, and

$$L = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_N(\rho_{1,1}\rho_{N,1} + \rho_{1,2}\rho_{N,2}) \\ \sigma_1\sigma_2(\rho_{1,1}\rho_{2,1} + \rho_{1,2}\rho_{2,2}) & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_N(\rho_{1,1}\rho_{N,1} + \rho_{1,2}\rho_{N,2}) & \cdots & \sigma_N^2 \end{pmatrix}.$$

This can be done using matrix inversion method II, as described in Chapter 1, which yields the result

$$\tilde{A}^{-1} = \begin{pmatrix} D' & e \\ e^T & L' \end{pmatrix}, \tag{3.37}$$

such that

$$D' = \begin{pmatrix} \frac{1-O_N}{\gamma_1^2(1-2O_N)} & \frac{O_N}{\gamma_1\gamma_2(1-2O_N)} \\ \frac{O_N}{\gamma_1\gamma_2(1-2O_N)} & \frac{1-O_N}{\gamma_2^2(1-2O_N)} \end{pmatrix},$$

$$e = \begin{pmatrix} -\frac{\xi_{1,1}+\xi_{1,2}-\zeta_1}{\gamma_1\sigma_1\sqrt{1-\rho_{1,1}^2-\rho_{1,2}^2}} & \cdots & -\frac{\xi_{N,1}+\xi_{N,2}-\zeta_N}{\gamma_1\sigma_N\sqrt{1-\rho_{N,1}^2-\rho_{N,2}^2}} \\ -\frac{\xi_{1,1}+\xi_{1,2}-\zeta_1}{\gamma_2\sigma_1\sqrt{1-\rho_{1,1}^2-\rho_{1,2}^2}} & \cdots & -\frac{\xi_{N,1}+\xi_{N,2}-\zeta_N}{\gamma_2\sigma_N\sqrt{1-\rho_{N,1}^2-\rho_{N,2}^2}} \end{pmatrix},$$

e^T the transpose, and

$$L' = \begin{pmatrix} \frac{(1-C_{11})(1-2O_N)+2(\xi_{1,1}+\xi_{1,2}-\zeta_1)^2}{\sigma_1^2(1-\rho_{1,1}^2-\rho_{1,2}^2)(1-2O_N)} & \cdots & -\frac{(1-C_{1N})(1-2O_N)+2(\xi_{1,1}+\xi_{1,2}-\zeta_1)(\xi_{N,1}+\xi_{N,2}-\zeta_N)}{\sigma_1\sigma_N\sqrt{1-\rho_{1,1}^2-\rho_{1,2}^2}\sqrt{1-\rho_{N,1}^2-\rho_{N,2}^2(1-2O_N)}} \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \frac{(1-C_{NN})(1-2O_N)+2(\xi_{N,1}+\xi_{N,2}-\zeta_N)^2}{\sigma_N^2(1-\rho_{N,1}^2-\rho_{N,2}^2)(1-2O_N)} \end{pmatrix}.$$

Therefore we can calculate the candidate optimal portfolio proportions. Using these, we can solve the HJB equation to find the optimal value function and its growth rate. Assuming an a priori fixed power utility function, we obtain the explicit investment weights consisting of our modelling parameters. □

Unfortunately, we were not able to find a more concise way to display the growth rate Θ_N^{**} and have to leave it as shown in (3.34) to maintain clarity of the presentation.

Remark 3.3. *It is possible to extend the solution to the homogeneous parameter model to the more general case of $J > 2$ risk factors. Using the HJB equation (3.35) for two factors, we can see that for homogeneous coefficients its extension to an arbitrary number of J factors can be written as*

$$0 = \sup_{\pi\in\mathcal{A}(x)} \left\{ \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x}\left(N(\alpha - r)\pi_j + \sum_{l=1}^{J}(\mu_l - r)\pi_l + r \right) + \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\left(N\sigma^2\pi_j^2 + \sum_{l=1}^{J}\gamma_l^2\pi_l^2 + N(N-1)\sigma^2\pi_j^2\sum_{l=1}^{J}\rho_l^2 + 2N\pi_j\sigma\sum_{l=1}^{J}\gamma_l\rho_l\pi_l \right) \right\}. \tag{3.38}$$

Once again, π_j is the proportion of wealth invested in each of the N securities $S_j(t)$, $j = 1, \ldots, N$, then the π_l are the proportions for each of the global market assets $I_l(t)$, $l = 1, \ldots, J$ and ρ_l is the correlation of each asset $S_j(t)$ with the market risk from $I_l(t)$. The notation ρ_l emphasises that this homogeneous correlation coefficient is the same for each $j = 1, \ldots, N$ and only changes with a varying global asset $l = 1, \ldots, J$.

Differentiating the above equation first with respect to π_j and then each of the π_l, equating the resulting expressions to zero and rearranging the $J+1$ equations slightly, we can calculate the following optimal proportions for a power utility

$$\begin{aligned} \hat{\pi}_j^{J,h} &= \frac{1}{(1-p)\sigma\left(1-\sum_{l=1}^{J}\rho_l^2\right)}\left[\lambda - \sum_{l=1}^{J}\lambda_l\rho_l\right], \\ \hat{\pi}_l^{J,h} &= \frac{1}{(1-p)\gamma_l}\left[\lambda_l - N\frac{\rho_l}{1-\sum_{l=1}^{J}\rho_l^2}\left(\lambda - \sum_{l=1}^{J}\lambda_l\rho_l\right)\right], \end{aligned} \tag{3.39}$$

for $l = 1, \ldots, J$, with the homogeneous Sharpe ratio $\lambda = \frac{\alpha - r}{\sigma}$ and varying Sharpe ratio $\lambda_l = \frac{\mu_l - r}{\gamma_l}$.

Inserting $\hat{\pi}_j^{J,h}$ and $\hat{\pi}_l^{J,h}$ back into the HJB equation (3.38), we can find the optimal value function growth rate as:

$$\hat{\Theta}^{J,h} = \frac{1}{2}\left(\sum_{l=1}^{J}\lambda_l^2 + N\frac{\left(\lambda - \sum_{l=1}^{J}\lambda_l\rho_l\right)^2}{1-\sum_{l=1}^{J}\rho_l^2}\right). \tag{3.40}$$

When comparing this to the results for the one factor homogeneous model, we can see that they nicely display the idea of how our two factor model can be extended to $J > 2$ risk factors. However exact calculations with varying parameters are too involved and beyond the scope of this thesis.

Let us briefly turn to the incomplete model. First, we want to introduce some new notation in addition to what we used for the complete model setup above. Let us define the analogue to K_N in the incomplete two factor model as

$$P_N = \left(1 + \sum_{j=1}^{N}\xi_{j,1}^2\right)\left(1 + \sum_{j=1}^{N}\xi_{j,2}^2\right) - \left(\sum_{j=1}^{N}\xi_{j,2}\xi_{j,1}\right)^2,$$

which unfortunately does not seem to have an interesting financial interpretation.

In addition, for each $j = 1, \ldots, N$, we will use

$$\kappa_j = \frac{\lambda_j}{\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}} P_N - \sum_{k=1}^{N} \frac{\lambda_k}{\sqrt{1-\rho_{k,1}^2-\rho_{k,2}^2}} \Bigg(\xi_{j,2}\xi_{k,2} + \xi_{j,1}\xi_{k,1} + \sum_{i=1}^{N} (\xi_{k,1}\xi_{i,2} - \xi_{i,1}\xi_{k,2})(\xi_{j,1}\xi_{i,2} - \xi_{j,2}\xi_{i,1}) \Bigg).$$

Theorem 7. *Suppose our investment universe is given by the set of SDEs:*

$$dS_j(t) = \alpha_j S_j(t)dt + \sigma_j S_j(t)\Big(\rho_{j,1}dM^1(t) + \rho_{j,2}dM^2(t)\Big) + \sigma_j\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}S_j(t)dW_j(t), \tag{3.41}$$

for $0 \le t \le T$, $j = 1, \ldots, N$.

Assume that we have a power utility function of the shape

$$U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0,$$

and the objective of maximising the expected utility of terminal wealth is represented by the value function $u_N(x)$, with general form

$$u_N(x) = \sup_{\pi \in \mathcal{A}_N(x)} \mathbb{E}\left[U(X_N^{\pi,x}(T)) | X_N^{\pi,x}(0) = x\right],$$

over the vector of admissible controls $\pi(t) = (\pi_1(t), \ldots, \pi_N(t)) \in \mathcal{A}_N(x)$.

Then we have the following results:

(i) *The optimal investment policies are*

$$\pi_j^{***}(t) = \frac{1}{1-p}\left[\frac{\alpha_j - r}{\sigma_j^2(1-\rho_{j,1}^2-\rho_{j,2}^2)} - \sum_{k=1}^{N} \frac{(\alpha_k - r)C_{jk}}{\sigma_j\sigma_k\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}\sqrt{1-\rho_{k,1}^2-\rho_{k,2}^2}P_N}\right], \tag{3.42}$$

for $j, k = 1, \ldots, N$.

(ii) *The optimal solution to the value function is given by*

$$u_N^{***}(x) = \exp\left\{\int_0^t \left[\frac{\Theta_N^{**}(s)}{1-p} + r(s)\right] pds\right\} \frac{x^p}{p}, \tag{3.43}$$

with growth rate

$$\Theta_N^{***}(t) = \frac{1}{P_N}\sum_{j=1}^{N} \frac{\lambda_j\kappa_j}{\sqrt{1-\rho_{j,1}^2-\rho_{j,2}^2}} + \frac{1}{P_N^2}\left[\sum_{j=1}^{N}\kappa_j^2 + \left(\sum_{j=1}^{N}\xi_{j,1}\kappa_j\right)^2 + \left(\sum_{j=1}^{N}\xi_{j,2}\kappa_j\right)^2\right]. \tag{3.44}$$

Proof. As previously, we can derive the HJB equation to be

$$0 = \sup_{\pi \in \mathcal{A}(x)} \left\{ \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} \left(\sum_{j=1}^{N} (\alpha_j - r)\pi_j + r \right) + \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2} \left(\sum_{j=1}^{N} \sigma_j^2 \pi_j^2 + \sum_{j=1}^{N} \sum_{i \neq j} \sigma_i \sigma_j \pi_i \pi_j (\rho_{i,1}\rho_{j,1} + \rho_{i,2}\rho_{j,2}) \right) \right\}. \tag{3.45}$$

Then, by differentiating with respect to each of the N the portfolio policies and equating the results to zero, we obtain a set of equations governing the optimal proportions.

To find the exact values for these investment policies, we first need to invert the covariance matrix

$$A^{**} = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_N(\rho_{1,1}\rho_{N,1} + \rho_{1,2}\rho_{N,2}) \\ \sigma_1\sigma_2(\rho_{1,1}\rho_{2,1} + \rho_{1,2}\rho_{2,2}) & \cdots & \sigma_2\sigma_N\rho_2\rho_N \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_N(\rho_{1,1}\rho_{N,1} + \rho_{1,2}\rho_{N,2}) & \cdots & \sigma_N^2 \end{pmatrix}. \tag{3.46}$$

Using the fact that (3.46) can be decomposed into a diagonal matrix and the sum of the matrices $d_1 d_1^T$ and $d_2 d_2^T$, we can apply the matrix inversion method I from Chapter 1.

Hence, the inverse is

$$(A^{**})^{-1} = \begin{pmatrix} A_{11}^{**} & \cdots & \cdots & A_{1N}^{**} \\ A_{21}^{**} & A_{22}^{**} & \cdots & A_{2N}^{**} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}^{**} & \cdots & \cdots & A_{NN}^{**}, \end{pmatrix} \tag{3.47}$$

such that for $j, k = 1, \ldots, N$ we have the diagonal entries

$$A_{jj}^{**} = \frac{1}{\sigma_j^2(1 - \rho_{j,1}^2 - \rho_{j,2}^2)} \left[1 - \frac{C_{jj}}{P_N} \right],$$

and all other entries have the structure

$$A_{jk}^{**} = -\frac{C_{jk}}{\sigma_j\sigma_k \sqrt{1 - \rho_{j,1}^2 - \rho_{j,2}^2}\sqrt{1 - \rho_{k,1}^2 - \rho_{k,2}^2} P_N}.$$

Applying the same methodology as for the incomplete one-factor model, gives the required result.

□

To conclude this section, let us make some final interesting comments.

Remark 3.4. *If we reduce our view to the homogeneous parameter case, we can easily see how the investment proportions and the value function growth rate behave in general for an arbitrary number of J global risk factors. First, we need to clarify some notation. For the use in this remark ρ_l is the homogeneous correlation coefficient of all the securities $S_j(t)$, $j = 1, \ldots, N$, with the l^{th} risk factor, for $l = 1, \ldots, J$. The squared correlation ratio then becomes $\xi_l^2(t) = \frac{\rho_l^2}{1-\sum_{k=1}^{J}\rho_k^2}$.*

Similar to the structure of (3.45), we can derive the following HJB equation

$$0 = \sup_{\pi \in \mathcal{A}(x)} \left\{ \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x}\Big(N(\alpha - r)\pi_j^{J,h} + r\Big) + \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\left(N\sigma^2\big(\pi_j^{J,h}\big)^2 + N(N-1)\sigma^2\big(\pi_j^{J,h}\big)^2\sum_{k=1}^{J}\rho_k^2\right)\right\}. \tag{3.48}$$

Differentiating (3.48) with respect to $\pi_j^{J,h}(t)$ and equating it to zero we can calculate the optimal portfolio proportion invested in each of the risky securities, to be

$$\pi_j^{*,J,h}(t) = \frac{1}{N}\left(\frac{\alpha - r}{(1-p)\sigma^2\big(\frac{1}{N} + \sum_{l=1}^{J}\xi_l^2\big)\big(1 - \sum_{k=1}^{J}\rho_k^2\big)}\right). \tag{3.49}$$

Applying the usual methodology, we can then find the optimal growth rate of the value function as

$$\Theta^{*,J,h}(t) = \frac{(\alpha - r)^2}{2\sigma^2\Big(\frac{1}{N} + \sum_{l=1}^{J}\xi_l^2\Big)\Big(1 - \sum_{k=1}^{J}\rho_k^2\Big)}. \tag{3.50}$$

If we compare the results in (3.49) and in (3.50) to the one factor case, described in Section 3.2, we immediately see the similarities in the structure. Therefore, we can conclude that to add further global factors to the dynamics, it suffices to just take their correlation coefficients into account.

This is a very intuitive outcome, because here we have used homogeneous returns and volatilities throughout the whole asset basket, and therefore it is sensible to assume that the Sharpe ratio will remain constant. However, as more and more global risk factors are added, only the correlation ratios should change significantly.

Chapter 4

Finiteness Conditions, Asymptotic Expansions and Arbitrage Arguments

In this section, we explore the limiting behaviour of the value function and of the explicit portfolio proportions for both the complete and incomplete one-factor model, as introduced in Chapter 3. In particular, we want to find conditions such that the growth rate of the value function stays finite in the limit for an infinite asset universe. We will also explore the size of the required total investment outlay, which can achieve that. Ultimately, this will help us to determine what restrictions we need to impose on the modelling parameters to avoid asymptotic arbitrage in the limiting market $\mathcal{M}$. It is important to point out that in this chapter, contrary to the previous chapters, all modelling parameters are assumed to be constant in time.

Our approach is structured as follows. First, as in Chapter 3, we want to look at the sequence of markets $\mathcal{M}_N$, $N \in \mathbb{N}$ which is determined by a system of CAPM style, correlated single stocks $S_j(t)$, $j = 1, \ldots, N$ with varying coefficients and in the complete case the traded market asset $I(t)$, $t \in [0, T]$. We rewrite the model introduced in Chapter 3 in terms of perturbed parameters and show that for $N \to \infty$ this sequence of markets weakly converges to a limiting market with constant, homogeneous coefficients. However, this only tells us that a well-defined limit to our expanding asset basket exists. The weak convergence does not imply that the limit of the solutions of the optimisation problems on each of the finite markets, i.e. the value functions $u_N(x)$ do not necessarily converge to the value function $u_\infty(x)$ for the countable optimisation problem. The same holds true for the optimal portfolio proportions. We will see in the more detailed calculation carried out in Sections 4.1 and 4.2 that in fact the vector of optimal portfolio proportions for each $N \in \mathbb{N}$ only converge to the optimal

limiting homogeneous parameter portfolio proportion if we scale the varying parameters appropriately.In the case that such scaling is not sufficient, we see that $\lim_{N\to\infty} u_N(x) \neq u_\infty(x)$.

Secondly, we want to rewrite our results in terms of convenient groups of parameters, the Sharpe ratios with respect to the market η_j, and the correlation ratios ξ_j, $j = 1, \ldots, N$, perturb them, and consider the limiting behaviour of the optimal portfolio proportions and the value function growth rates.

The most interesting conclusion of this section is, that unless the Sharpe ratios with respect to the market for each of our CAPM style assets are close to the homogeneous constant coefficient ratio, asymptotic arbitrage will always occur. It is also worth mentioning, that it is sufficient to scale only the perturbations of these Sharpe ratios with respect to the market and it does not influence the conclusions if the correlation ratios remain heterogeneous.

We want to begin by showing the weak convergence of our varying markets to a limiting constant coefficient market. In Chapter 1 and Chapter 2 we assumed that the market is growing by adding further assets one by one to the already existing basket where the parameters of the previous stocks remain the same. Now, we want to introduce a slight modification to the idea of how the asset universe grows. Here the parameters vary with the market size $N \in \mathbb{N}$. Assume that we have the N-dependent dynamics

$$dS_j^{(N)}(t) = \alpha_j^{(N)} S_j^{(N)}(t)dt + \sigma_j^{(N)}\rho_j^{(N)} S_j^{(N)}(t)dM(t) + \sigma_j^{(N)}\sqrt{1-\left(\rho_j^{(N)}\right)^2} S_j^{(N)}(t)dW_j^{(N)}(t), \tag{4.1}$$

for $j = 1, \ldots, N$, $N \in \mathbb{N}$ and $t \in [0, T]$. We intentionally omit the global market assets. This does not mean that they are not available for trading, but because we have at most a finite number of them, they are irrelevant to the proof of weak convergence.

Suppose that we have an additional probability space $(\Omega, \mathfrak{B}, \mathfrak{P})$. For example we can assume that $\mathfrak{B}$ is the Borel σ-algebra generated by the sets in $(-\infty, \infty)$ and $\mathfrak{P} : \mathfrak{B} \to [0, 1]$ the probability measure.

Consider $\hat{\alpha}_j^{(N)}$, $\hat{\sigma}_j^{(N)}$ and $\hat{\rho}_j^{(N)}$, $j = 1, \ldots, N$, $N \in \mathbb{N}$, which are real valued, independent, bounded random variables with mean zero on this probability space. In particular, fixing N, for each $1 \leq j \leq N$ the triples $\psi_j^{(N)} = \left(\hat{\alpha}_j^{(N)}, \hat{\sigma}_j^{(N)}, \hat{\rho}_j^{(N)}\right)$ are independent and identically distributed and bounded. In addition we have the constants $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and $\rho \in (-1, 1)$. Denote by $\left(\psi_j^{(N)}\right)_{j=1}^N = \left(\hat{\alpha}_j^{(N)}, \hat{\sigma}_j^{(N)}, \hat{\rho}_j^{(N)}\right)_{j=1}^N$ the sequence of N- dependent perturbation triples.

We want to rewrite the parameters in equation (4.1) for $j = 1, \ldots, N$ in terms of their fixed, homogeneous and random perturbation parts

$$\begin{aligned} \alpha_j^{(N)} &= \alpha + \hat{\alpha}_j^{(N)}, \\ \sigma_j^{(N)} &= \sigma + \hat{\sigma}_j^{(N)}, \\ \rho_j^{(N)} &= \rho + \hat{\rho}_j^{(N)}. \end{aligned} \tag{4.2}$$

In particular because $\rho_j^{(N)} \in (-1, 1)$, we have the natural condition $-1 - \rho < \hat{\rho}_j^{(N)} < 1 - \rho$. Note that the parameters are fixed in time and we have dropped the dependence in the equations to emphasise this.

Let us provide further insight into how this N-dependence can be interpreted. First, we fix some real valued, constant coefficients $\alpha, \sigma > 0$ and $\rho \in (-1, 1)$. Then for each $N \in \mathbb{N}$, we generate a set of random triples $\left(\psi_j^{(N)}\right)_{j=1}^{N}$ on the probability space $(\Omega, \mathfrak{B}, \mathfrak{P})$ and then calculate the parameters according to equation (4.2) for each level N. If we wish to include an additional stock, expanding our basket to $N+1$ assets of type $S_j^{(N+1)}(t)$, $j = 1, \ldots, N+1$, we need to generate an entirely new set of random variables

$$\left(\psi_j^{(N+1)}\right)_{j=1}^{N+1} = \left(\hat{\alpha}_j^{(N+1)}, \hat{\sigma}_j^{(N+1)}, \hat{\rho}_j^{(N+1)}\right)_{j=1}^{N+1}.$$

We can see that our market does not just grow by mere addition of new stocks, but completely changes for each $N \in \mathbb{N}$. Therefore it is important to show that this sequence of market models converges weakly to the limiting market

$$dS_j(t) = \alpha S_j(t)dt + \sigma\rho S_j(t)dM(t) + \sigma\sqrt{1-\rho^2}S_j(t)dW_j(t), \tag{4.3}$$

with constant, homogeneous coefficients for every $j \in \mathbb{N}$. We want to emphasise that although the parameters vary with basket size, the Brownian driver $M(t)$ stays the same for all models. Later we will see that the sequence of markets converges to a finite limit, however, the value functions do not, unless the scaling of the perturbed parameters is strong enough.

To be able to show weak convergence of our market models, we use Ethier and Kurtz [EK86]. Let us briefly summarise the statements they make in Lemma 4.3 together with Proposition 4.6, Chapter 3, [EK86] concerning countably infinite product spaces.

Suppose we have a sequence of complete and separable metric spaces $(\mathfrak{S}_i, d_i)$, $i = 1, 2, \ldots$, with a metric d_i, for example $(\mathfrak{S}_i, d_i) = (C[0,T], ||\cdot||_\infty)$, where $||\cdot||_\infty$ denotes the supremum norm and $C[0,T]$ is the space of continuous functions. Then we can define the product space

by $\mathfrak{S} = \prod_{i=1}^{\infty} \mathfrak{S}_i$, in particular $(\mathfrak{S}, d) = (C[0,T], ||\cdot||_\infty))^{\mathbb{N}}$. We know that $(\mathfrak{S}, d)$ is separable and complete if $(\mathfrak{S}_i, d_i)$ is separable and complete.

In addition, for $\bar{C}[0,T]$, the space of bounded continuous functions, we consider a set of bounded continuous functions of continuous functions $\mathfrak{M}_i \subset \bar{C}[0,T] \subset C[0,T]$of bounded continuous functions of continuous functions taking values in $\mathfrak{S}_i$. Then an $f_i \in \mathfrak{M}_i$ and $f_i : \bar{C}[0,T] \to \mathbb{R}$, $i = 1, 2, \ldots$, is a bounded continuous function of continuous functions. For an infinite vector of continuous functions $x = (x_1, x_2, \ldots)$ let

$$\mathfrak{M} = \left\{ f(x) = \prod_{i=1}^{n} f_i(x_i) : \quad n \geq 1, \quad f_i \in \mathfrak{M}_i \cup \{1\}, \quad i = 1, \ldots, n \right\},$$

then if $\mathfrak{M}_i$ is convergence determining, so is $\mathfrak{M}$. To show weak convergence in $\mathfrak{M}$ it is therefore enough to look at the $\mathfrak{M}_i$ for $i = 1, \ldots, n$ for arbitrary finite n. Then the sequence x_n lives on the first n components of x. Then, the sequence of probability measures $\mathbb{P}_n$ on $\mathfrak{M}_i$ converges weakly to a limit measure $\mathbb{P}$ on the space $\mathfrak{M}$ as $n \to \infty$ meaning that

$$\mathbb{E}_{\mathbb{P}_n}[f_n(x)] \to \mathbb{E}_{\mathbb{P}}[f(x)],$$

for $n \to \infty$. We can use these results to show weak convergence of our particular markets.

Let us make some brief remarks prior to summarising our convergence results in a theorem. We have already mentioned, that the triples of perturbations $\psi_j^{(N)}$ are independent and identically distributed for each $j \in \mathbb{N}$. For each new asset we generate a new set, therefore they behave exactly like a triangular array. Therefore we will have to make some strong assumptions about their behaviour, as we later require that these triples converge to zero as $N \to \infty$.

We begin by establishing the right probability space and a suitable probability measure to be able to take expectations. We have already seen, that our continuous, bounded functions lie in the product space $(\mathfrak{S}, d) = (C[0,T], ||\cdot||_\infty)^{\mathbb{N}}$. However, we also have the triples of real valued random perturbations. Therefore, we want to work with the product space $(\mathfrak{S}, d) = (C[0,T], ||\cdot||_\infty))^{\mathbb{N}} \times \mathbb{R}^3$ from now on. The perturbations, which have to be drawn from a mean zero distribution, thus determine the distribution applied and this, in turn, gives us a new probability measure $\hat{\mathbb{P}}$. To simplify notation, we will write $\hat{\mathbb{E}}$ instead of $\mathbb{E}_{\hat{\mathbb{P}}}$.

Theorem 8. *Consider the probability space* $(C[0,T],||\cdot||_\infty)^{\mathbb{N}}\times\mathbb{R}^3$. *Suppose that the sequence of markets with varying coefficients as in equation (4.1) can be written as the vector*

$$S^{(N)}=\Big(S_1^{(N)},\dots,S_N^{(N)},0,0,\dots\Big).$$

In addition we have the corresponding countable vector of limiting assets with constant coefficients (4.3)

$$S=(S_1,S_2,\dots).$$

If we have a set of continuous, bounded functions $f_i\in\mathfrak{M}_i\subset\bar{C}[0,T]$, $i=1,2,\dots$ *and triples of bounded random variables* $\Big(\psi_j^N\Big)_{j=1}^N=\Big(\hat{\alpha}_j^{(N)},\hat{\sigma}_j^{(N)},\hat{\rho}_j^{(N)}\Big)_{j=1}^N$ *for each* $N\in\mathbb{N}$ *that weakly converge to zero, i.e.*

$$\psi_j^{(N)}=\Big(\hat{\alpha}_j^{(N+1)},\hat{\sigma}_j^{(N+1)},\hat{\rho}_j^{(N+1)}\Big)\Rightarrow(0,0,0)=\psi_j,\quad as\quad N\to\infty,$$

for each j, *then we have the weak convergence of the markets in that*

$$S^{(N)}\Rightarrow S,\ as\ N\to\infty.$$

Proof. A key element of this proof is to use the continuity and boundedness of the functions f_j together with the results of Ethier and Kurtz to show the required market convergence.

Suppose that we have the underlying space $(C[0,T],||\cdot||_\infty)^{\mathbb{N}}\times\mathbb{R}^3$ equipped with a new probability measure $\hat{\mathbb{P}}$, we need to show overall market convergence. Here we can use the continuous mapping theorem for continuous bounded functions. In particular, if we have for each j that

$$S_j^{(N)}\Rightarrow S_j,$$

then the functions

$$f_j\Big(S_j^{(N)}\Big)\Rightarrow f_j(S_j),$$

also converge, because the f_j are bounded and continuous by assumption, and we have component wise weak convergence for each j:

$$\hat{\mathbb{E}}\bigg[f_j\Big(S_j^{(N)}\Big)\bigg]\to\hat{\mathbb{E}}\Big[f_j(S_j)\Big],$$

as $n\to\infty$.

To be able to show that the whole system converges to the constant coefficient market we now follow the work outlined in [EK86] of Ethier and Kurtz. We start by rewriting $\hat{\mathbb{E}}\big[f\big(S^{(N)}\big)\big]$, for $f = \prod_{i=1}^{n} f_i \in \mathfrak{M}$ as

$$\hat{\mathbb{E}}\big[f\big(S^{(N)}\big)\big] = \begin{cases} \hat{\mathbb{E}} \prod_{i=1}^{n} f_i\Big(S_i^{(N)}\Big), & \text{if } n < N \\ \hat{\mathbb{E}} \prod_{i=1}^{N} f_i\Big(S_i^{(N)}\Big) \prod_{m=N+1}^{n} f_i(0), & \text{if } n > N, \end{cases} \tag{4.4}$$

where $\prod_{i=N+1}^{n} f_i(0)$ is an extension to the 0 paths. Then once N is large, enough we only need to consider $\hat{\mathbb{E}} \prod_{i=1}^{n} f_i\Big(S_i^{(N)}\Big)$ for a finite number of functions $n < N$.

Consider the solutions to the asset price processes (4.1) and (4.3) and denote them in the following way:

$$\begin{aligned} S_i^{(N)}(t) &= S_i^{(N)}(0) e^{\left(\alpha_i^{(N)} - \frac{1}{2}\left(\sigma_i^{(N)}\right)^2\right)t + \sigma_i^{(N)} \rho_i^{(N)} M(t) + \sigma_i^{(N)} \sqrt{1-(\rho_i^{(N)})^2} W_i^{(N)}(t)} \\ &:= e^{(N)}\Big(M, W_i^{(N)}\Big) \\ S_i(t) &= S_i(0) e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma\rho M(t) + \sigma\sqrt{1-\rho^2} W_i(t)} \\ &:= e(M, W_i), \end{aligned}$$

assuming that both processes start at the same point $S_i^{(N)}(0) = S_i(0)$.

To see that in fact $\lim_{N\to\infty} \hat{\mathbb{E}}\big[f\big(S^{(N)}\big)\big] = \hat{\mathbb{E}}[f(S)]$ we need to look at the difference $\Big|\hat{\mathbb{E}}\big[f\big(S^{(N)}\big) - f(S)\big]\Big|$ for each $N \in \mathbb{N}$. For a given n take $N > n$.

Then

$$\begin{aligned} &\Big|\hat{\mathbb{E}}\big[f\big(S^{(N)}\big) - f(S)\big]\Big| \\ &\leq \left|\hat{\mathbb{E}}\Big|\prod_{i=1}^{n} f_i\Big(S_i^{(N)}\Big) - \prod_{i=1}^{n} f_i(S_i)\Big|\right| \\ &\leq \sum_{j=1}^{n} \hat{\mathbb{E}}\left|\prod_{i=1}^{j-1}\Big|f_i\Big(S_i^{(N)}\Big)\Big|\Big|f_j\Big(S_j^{(N)}\Big) - f(S_j)\Big| \prod_{i=j+1}^{n} |f_i(S_i)|\right| \\ &\leq c^{n-1} \sum_{j=1}^{n} \hat{\mathbb{E}}\left[\prod_{i=1}^{j-1} f_i\Big(S_i^{(N)}\Big) \prod_{i=j+1}^{n} f_i(S_i)\Big|f_j\Big(S_j^{(N)}\Big) - f_j(S_j)\Big|\middle|\psi_i^{(N)}, M, W_i^{(N)}, i \neq j\right] \\ &= \sum_{j=1}^{n} \hat{\mathbb{E}}\left[g\Big(S_i^N, S_i\Big)\hat{\mathbb{E}}\left[f_j\Big(e^{(N)}(M, W_j^{(N)})\Big) - f(e(M, W_j))\right]\middle|M\right], \end{aligned} \tag{4.5}$$

assuming that $\prod_{i=1}^{0} f_i\Big(S_i^{(N)}\Big) = 1$ and $\prod_{i=n+1}^{n} f_i(S_i) = 1$.

To establish that we have

$$\hat{\mathbb{E}}\left[\hat{\mathbb{E}}[f(e(M, W_j^N))|M]\right] \Rightarrow \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[f_j(e_j(M, W_j))|M]\right],$$

as $n \to \infty$ for all $j \in \mathbb{N}$, we need to make sure that it is possible to interchange limits and conditional expectations in the bottom line of equation (4.5). We use the dominated convergence theorem. As we know that both functions g and f are bounded by construction, there exists a constant C, such that $g, f \leq C$, therefore we have

$$\begin{aligned}
&\lim_{N\to\infty} c^{n-1} \sum_{j=1}^{n} \hat{\mathbb{E}}\left[\left|\hat{\mathbb{E}} f_j\Big(e^{(N)}\Big(M, W_j^{(N)}\Big)\Big) - f(e(M, W_j))\right|\right] \\
&= \sum_{j=1}^{n} \hat{\mathbb{E}} \lim_{N\to\infty} \left[\left|\hat{\mathbb{E}}\Big[f_j\Big(e^{(N)}\Big(M, W_j^{(N)}\Big)\Big) - f(e(M, W_j))\Big]\right|\right].
\end{aligned}$$

Now as we know that $\psi_j^N \Rightarrow 0$ for any fixed j and because ψ is bounded, we can establish weak convergence conditioning on the M. Recall that we also have weak convergence path wise, i.e.

$$\hat{\mathbb{E}}\Big[f_j\Big(e^{(N)}\Big(M, W_j^{(N)}\Big)\Big) - \hat{\mathbb{E}}\big[f_j\big(e\big(M, W_j\big)\big)\big]\Big] \to 0.$$

Then we can see that

$$\sum_{j=1}^{n} \hat{\mathbb{E}} \lim_{N\to\infty} \left[g\Big(S_i^N, S_i\Big)\hat{\mathbb{E}}\Big[f_j\Big(e^{(N)}\Big(M, W_j^{(N)}\Big)\Big) - f(e(M, W_j))\Big]\Big|M\right] \to 0.$$

and hence

$$\mathbb{E}\Big[f\big(S^{(N)}\big) - f(S)\Big] \to 0,$$

as $N \to \infty$ and therefore we have the required weak convergence. □

A brief final remark. Throughout this section we have assumed that the perturbation random variables $\Big(\hat{\alpha}_j^{(N)}, \hat{\sigma}_j^{(N)}, \hat{\rho}_j^{(N)}\Big)$ are bounded for any $N \in \mathbb{N}$ and $j = 1, \ldots, N$.

We have now established that the varying coefficients market models converge weakly to a finite constant coefficient market in the limit as $N \to \infty$. However, this does not shed any light on the behaviour of the portfolio proportions, the value function growth rates and asymptotic arbitrage, yet. This is the topic in the remainder of Chapter 4. In Chapter 3 we have seen that it is a lot more convenient and economically intuitive to express our results in terms of

the combined parameters, in particular the Sharpe ratio with respect to the market and the correlation ratio. Therefore, we need to bridge the gap between the market models described above in equations (4.1) , (4.3) and the notation in Chapter 3.

Recall the we had correlation ratio $\xi_j = \frac{\rho_j}{\sqrt{1-\rho_j^2}}$. For our N-dependent parameter model, we can express it as follows:

$$\xi_j^{(N)} = \xi + w_j^{(N)}, \tag{4.6}$$

where

$$w_j^{(N)} = \frac{\rho + \hat{\rho}_j^{(N)}}{\sqrt{1-\rho^2-2\rho\hat{\rho}_j^{(N)} - \hat{\rho}_j^{(N)})^2}} - \frac{\rho}{\sqrt{1-\rho^2}},$$

and $\rho \in (-1,1)$ is a constant.

Similarly for the Sharpe ratio with respect to the market $\eta_j = \frac{\alpha_j - r}{\sigma_j \rho_j}$ we now have

$$\eta_j^{(N)} = \eta + z_j^{(N)}, \tag{4.7}$$

where

$$z_j^{(N)} = \frac{\alpha + \hat{\alpha}_j^{(N)} - r}{(\rho + \hat{\rho}_j^{(N)})(\sigma + \hat{\sigma}_j^{(N)})} - \frac{\alpha - r}{\rho\sigma},$$

and α, σ, ρ are real valued constants.

Therefore, for each choice of distribution for the random variable triples $\left(\psi_j^{(N)}\right)_{j=1}^N$ we immediately get a distribution for $z^{(N)}$ and $w^{(N)}$. However we do not want to discuss their intricate relationship in this thesis but just determine conditions on the unbounded random variables $z^{(N)}$ and $w^{(N)}$. Again, we should emphasise that the bounded triple $\left(\psi_j^{(N)}\right)_{j=1}^N$ can produce unbounded random variables when aggregated as in equations (4.6) and (4.7). To be able to make direct distribution assumptions on these new random variables and not worry about the interaction with the initial perturbations to the single parameters. From now on consider the model for the perturbed correlation ratio to be

$$\eta_j^{(N)} = \eta + b_j^{(N)}, \tag{4.8}$$

as well as the perturbed Sharpe ratio with respect to the market

$$\xi_j^{(N)} = \xi(t) + a_j^{(N)}, \tag{4.9}$$

for $j = 1, \ldots, N$, $N \in \mathbb{N}$. Although not fitting precisely with assumptions on $\alpha^{(N)}$, $\sigma^{(N)}$ and $\rho^{(N)}$ we assume that the $\left(a_j^{(N)}, b_j^{(N)}\right)$ for $j = 1, \ldots, N$ are a sequence of not necessary

independent pairs of random variables with mean zero on the additional space $(\Omega, \mathfrak{B}, \mathfrak{P})$.

Note that the dependence of the random variables $a_j^{(N)}$ and $b_j^{(N)}$ stems from the fact that the same correlation coefficient that is present in both the correlation ratio and the Sharpe ratio with respect to the market. However, for two different securities such that $i \neq j$ for $i, j = 1, \ldots, N$ we have that $a_j^{(N)}$ is independent of $b_i^{(N)}$. We assume our sample space to be unbounded such that we can capture the behaviour of $\xi_j^{(N)}$ for both, $\rho_j^{(N)}$ being close to -1, i.e. the stocks almost completely negatively correlated with the market, or close to 1, such that we experience almost full correlation with the market risk in global asset $I(t)$. In addition, this allows us to include the case that the Sharpe ratio with respect to the market, $\eta_j^{(N)}$, has a correlation coefficient $\rho_j^{(N)}$ close to zero, i.e. the assets are almost uncorrelated with the global risk factor.

We will be applying the Strong Law of Large Numbers (SLLN) and therefore need further assumptions on the existence of moments of our random variables $a_j^{(N)}$ and $b_j^{(N)}$. We are generating the set of the $a_j^{(N)}$ and $b_j^{(N)}$ anew for each $N \in \mathbb{N}$ which then forms a triangular array. Therefore, in addition with the following minimal integrability condition

$$\sup_{j,N} \mathbb{E}\Big[\Big(\big|a_j^{(N)}\big| + \big|b_j^{(N)}\big|\Big)^{24}\Big] < \infty, \quad j = 1, \ldots, N. \tag{4.10}$$

Then all the moments that are required later in this chapter exist. Recall that $a_j^{(N)}$ and $b_j^{(N)}$ are drawn from a mean zero distribution, so $\mathbb{E}\Big[a_j^{(N)}\Big] = 0$ and $\mathbb{E}\Big[b_j^{(N)}\Big] = 0$. Then from (4.10) we directly see that in particular the second moments $\mathbb{E}\Big[\big(a_j^{(N)}\big)^2\Big] < \infty$ and $\mathbb{E}\Big[\big(b_j^{(N)}\big)^2\Big] < \infty$ exist. To further simplify notation, we will drop the subscript j and write $\mathbb{E}[a]$ as well as $\mathbb{E}[b]$ when taking the expected value and if the assignment to a particular stock $S_j(t)$ is not relevant. In addition we also will not write the superscript (N) when describing parameters for the rest of the thesis to obtain more clarity in notation, however they are still dependent on the market size $N \in \mathbb{N}$.

We aim to expand the explicit portfolio proportions, calculated in Chapter 3, around the homogeneous parameter ratios, to derive the leading term and the size of the error terms. Having done that, we show that the sum of the portfolio proportions over all assets is finite in the limit, as we let $N \to \infty$. On the other hand we notice that the shape of the optimal proportion suggests that the investor has to take both large long and substantial short positions in the securities, as well as the global asset to achieve the optimal growth rate. Therefore, we will show that the total position of the agent, described by the sum of the absolute values of the portfolio proportions, can become infinite as we let $N \to \infty$. In particular, we find that

it is sufficient to only scale the random variables b_j, stemming from the decomposition of the Sharpe ratio with respect to the market, with N^{-k} for an appropriate choice of $k \in \mathbb{R}_+$ to gain the required control over this sum, referred to as total outlay. We are able to identify three distinct regimes of behaviour depending on the constant k. A similar type of scaling will be applied to the growth rate of the value function. However, we will see that we need to choose a different size of k to ensure finiteness of the growth rate in the limit. We then discuss these explicit results in light of asymptotic arbitrage.

4.1 Complete Model

In this section, we want to investigate the complete market model using the results from Section 3.1. Recall, that the explicit vector of portfolio proportions, as well as the value function and its growth rate for a power utility have already been stated in Theorem 4. In this section, we aim to investigate them further.

The structure of our argument is as follows. First, we want to examine each of the portfolio proportions by expanding them and using the decomposition of parameter ratios from equation (4.8) as well as equation (4.9), so we can derive the particular leading and error terms. These terms are dependent on the realisations of the unbounded random variables a_j and b_j and can therefore become very large. Therefore, we propose to scale these random variables in a certain way to be able to control their size in the limit as $N \to \infty$. To keep our control minimal, but strong enough to avoid infinite sized portfolio proportions, we have discovered that it is enough to scale the b_j for each $j = 1, \dots, N$ with N^{-k} for a constant $k \in \mathbb{R}_+$. Scaling the random variables a_j, $j = 1, \dots, N$, would on the other hand not produce a finite portfolio proportion. Depending on the size of this constant k, we illustrate various limiting behaviour, ranging from the constant coefficient case for $k > 1$ up to an infinite investment proportion for $k < 1$. As previously mentioned, we might expect such a result because the weak convergence of the sequence of market models does not imply that the optimisation results converge.

Secondly, we are interested in the sum over all the portfolio proportions. Here we notice that it is always finite in the limit, and the resulting expression is of order $\mathcal{O}(1)$. This is due to the fact that we take large positive and negative positions in the various assets and therefore cancellations occur. However, because we know the investor actually has to take out large opposite positions in the securities and the global asset to act optimally, we have the suspicion that the sum of the absolute values of the portfolio proportions (also referred to as total outlay) grows with N. We then aim to understand how fast the total outlay can grow for

different choices of the scaling exponent $k \geq 0$ and if there is a regime where it can be kept finite. We find that there are three regimes, depending on the choice of scaling parameter, ranging from a finite result identical to the homogeneous parameter result to an infinite total outlay for insufficiently strong scaling.

Having looked at the portfolio proportions and the initial outlay of the agent in detail we then want to understand what growth rate of the value function we can expect. Therefore we look at Θ_N^* and expand it using the same parameter decomposition as introduced in (4.8) and (4.9). We observe that the result is structured as a leading term, behaving like the homogeneous constant coefficient growth in the limit, as well as error terms depending on the unbounded random variables a_j and b_j. Once again, scaling all b_j, $j = 1, \ldots, N$ with N^{-k} for some $k \in \mathbb{R}_+$ we have three different results. For strong scaling, i.e. $k > \frac{1}{2}$, Θ_N^* converges to the finite homogeneous rate $\Theta_N^{*,h}$ as $N \to \infty$. In the intermediate regime for $k = \frac{1}{2}$, we have that Θ_N^* stays bounded and converges to an explicit limit, depending on the expected values of combinations of the random variables a_j, b_j. This rate outperforms $\Theta_N^{*,h}$. Finally, for the choice $0 \leq k < \frac{1}{2}$ the growth becomes infinite once we let $N \to \infty$. The fact, that we can obtain finite limits by replacing the infinite random variables b_j with their scaled counterparts is due to the earlier standing assumption that the second moments are finite. In particular, we have the stronger assumption that $\mathbb{E}[(|a_j| + |b_j|)^{24}] < \infty$ as well as the stronger triangular array assumptions.

Our results present an intriguing picture. First, we have found that the sum over the absolute value of the portfolio proportions is finite if we choose $k \geq 1$, but becomes infinite in the limit if $k < 1$. However, the results for the growth rate are different. Here, we only need to have $k \geq \frac{1}{2}$ to keep $\lim_{N\to\infty} \Theta_N^*$ bounded. We can only achieve infinite growth for the value function in the limit if we choose $k < \frac{1}{2}$. This means that we have three distinct cases. For $k \geq 1$ we achieve a finite growth taking out a finite total outlay in the limit. On the other hand for $k < \frac{1}{2}$, the sum of the absolute values of the portfolio proportions becomes infinite in the limit, as does the value function. The most interesting case, however, is the intermediate region. For $\frac{1}{2} \leq k < 1$, the total outlay of our agent is actually infinite in the limit, but the optimal growth of the value function that can be achieved in the limit, is finite. This means that by taking an infinite risk using the optimal portfolio proportions, the maximal reward is bounded. This gives rise to the idea that although the portfolio proportions derived are optimal for our model, they would not be a robust investment option in reality. In particular, we suspect that it would be reasonable to impose further constraints on the investment proportions, such as to forbid short-selling. However, this is beyond the scope of this thesis.

The second, and probably even more interesting conclusion, concerns the existence of asymptotic arbitrage. In the last paragraph of this section, we will see that the condition on asymptotic arbitrage, namely the explicit Hellinger process, behaves exactly like the value function growth rate Θ_N^*. Therefore, we can only guarantee that there is no asymptotic arbitrage in our CAPM style model if in the limit the individual, varying Sharpe ratios with respect to the market for each asset in our portfolio converge to the constant coefficient case with a small finite error term or even completely without one. Therefore, we can say that if the differences between the parameters of each asset become smaller and smaller with growing N, then there is no asymptotic arbitrage. Note that the size of the correlation ratio, depending on the random variable a_j, does not change this behaviour and we do not need to impose scaling on the a_j for any $j = 1, \ldots, N$.

Let us begin by looking at the portfolio proportions (3.3) presented in Theorem 4,

$$\tilde{\pi}^* = \frac{1}{(1-p)\gamma} \left(\lambda_0 - \sum_{j=1}^{N} \xi_j^2 (\eta_j - \lambda_0) \right),$$
$$\pi_j^* = \frac{\xi_j^2}{(1-p)\sigma_j \rho_j} (\eta_j - \lambda_0),$$

and investigate their behaviour for a finite collection of N stocks, as well as in the limit as $N \to \infty$.

Recall from Section 3.1, that the homogeneous constant coefficient parameter portfolio proportions have the structure

$$\tilde{\pi}^{*,h} = \frac{1}{(1-p)\gamma} \left(\lambda_0 - N\xi^2 (\eta - \lambda_0) \right),$$
$$\pi_j^{*,h} = \frac{\xi^2}{(1-p)\sigma\rho} (\eta - \lambda_0), \quad \forall j = 1, \ldots, N,$$

where λ_0 is the Sharpe ratio for the global market asset and η is the Sharpe ratio with respect to the market, if the coefficients for each of the assets $j = 1, \ldots, N$ are assumed to be homogeneous.

Note that the proportion of wealth invested in the overall market asset $\tilde{\pi}^*$, as well as its homogeneous parameter counterpart $\tilde{\pi}^{*,h}$, grow with N. If we want to keep both proportions finite for $N \to \infty$, we need to make an additional assumption on the behaviour of λ compared to η. We want the market price of risk to be the same for all assets in our homogeneous constant coefficients market once we reach the limit. If this is not be the case, it would be

possible to build an arbitrage portfolio using such an asset, which allows a higher return for the same amount of risk taken. Therefore, assume for constants $c, s \in \mathbb{R}^+$ that

$$\lambda_0 = \eta + \frac{c}{N^s}. \tag{4.11}$$

In addition, we will assume that $s > 0$ so that we have the desired limiting result

$$\lim_{N\to\infty} \lambda_0 = \eta.$$

This means that we can rewrite $\tilde{\pi}^{*,h}$ and $\pi^{*,h}$ for all $j = 1, \ldots, N$ as

$$\begin{aligned} \tilde{\pi}^{*,h} &= \frac{1}{(1-p)\gamma}\left(\lambda_0 + \xi^2 \frac{c}{N^{s-1}}\right), \\ \pi_j^{*,h} &= \frac{\xi^2 c}{(1-p)\sigma\rho N^s} \quad \forall j = 1, \ldots, N, \end{aligned} \tag{4.12}$$

and their limit for an arbitrary choice of $s \geq 1$ is

$$\begin{aligned} \lim_{N\to\infty} \tilde{\pi}^{*,h} &= \frac{\eta}{(1-p)\sigma}, \\ \lim_{N\to\infty} \pi^{*,h} &= 0 \quad \forall j = 1, \ldots, N. \end{aligned}$$

We see that, in the limit as $N \to \infty$, all the funds are allocated to the global market asset. On an intuitive level, we can say that for homogeneous, constant coefficients, we distribute the agent's money over a very large collection of increasingly similar assets, which in the limit spans the whole global market. Therefore, the above results confirm that in the limit all funds should be invested in the market asset and the proportion in each single stock has to approach zero.

Let us return to the investigation of the varying coefficient proportion. We now want to rewrite $\tilde{\pi}^*$ and π_j^* using the perturbation model introduced in equations (4.8) and (4.9). We have seen that we can decompose the correlation proportion as $\xi_j = \xi + a_j$ and the Sharpe ratio of each security with respect to the market as $\eta_j = \eta + b_j$, for each $j = 1, \ldots, N$. Recall that for each market size N, we draw the random variables a_j, b_j from the same mean zero distribution and thus they are identically distributed. In addition, for $j \neq i$, a_j is independent of b_i. For clarity, we will adopt the notation $\mathcal{S}_{a^n b^m} = \sum_{j=1}^N a_j^n b_j^m$ for the sum of products of

the perturbations. Then, assuming that $s \geq 1$, we have

$$\begin{aligned}
\tilde{\pi}^*(a,b) &= \frac{1}{\gamma(1-p)}\left(\lambda_0 - \sum_{j=1}^{N}\left(\xi^2 + 2a_j\xi + a_j^2\right)\left(\eta + b_j - \eta - \frac{c}{N^s}\right)\right) \\
&= \frac{1}{\gamma(1-p)}\left(\eta + \frac{c}{N^s} - \xi^2\mathcal{S}_b - 2\xi\mathcal{S}_{ab} - \mathcal{S}_{a^2b} + \xi^2\frac{c}{N^{s-1}} + 2\xi\frac{c}{N^s}\mathcal{S}_a + \frac{c}{N^s}\mathcal{S}_{a^2}\right), \\
\pi_j^*(a,b) &= \frac{1}{\sigma_j\rho_j(1-p)}\left(\left(\xi^2 + 2a_j\xi + a_j^2\right)\left(b_j - \frac{c}{N^s}\right)\right) \\
&= \frac{1}{\sigma_j\rho_j(1-p)}\left(\xi^2 b_j + 2\xi a_j b_j + b_j a_j^2 - \xi^2\frac{c}{N^s} - 2\xi a_j\frac{c}{N^s} - a_j^2\frac{c}{N^s}\right),
\end{aligned} \tag{4.13}$$

for all $j = 1, \ldots, N$.

To simplify future calculations, we will actually use the weighted portfolio proportions $\gamma\tilde{\pi}^*(a,b)$ and $\sigma_j\rho_j\pi_j^*(a,b)$. This merely clarifies the display of the results but does not alter their statement and the conclusions drawn from them. We will comment further on this at the appropriate point.

It is clear from (4.13) that the proportion invested in the global assets $\tilde{\pi}^*(a,b)$ is of order N and could therefore become infinitely large in the limit. Our objective is to keep the portfolio proportions finite, thus we need to impose a certain scaling on the random perturbations to control their growth. We propose to have minimal influence on the model, to only scale the terms involving the random variables b_j, $j = 1, \ldots, N$. We will replace each b_j with $\frac{1}{N^k}$ for a positive, real constant k. For a different choice of the constant k we can now observe a range of limiting results.

Let us now summarise the limiting behaviour of the expanded portfolio proportions (4.13). From now on, we will use the notation $\tilde{\pi}^*(a, b/N^k)$ and $\pi_j^*(a, b/N^k)$ if we are scaling the b_j with a factor $\frac{1}{N^k}$.

Theorem 9. *Suppose that the expanded optimal portfolio proportions $\tilde{\pi}^*(a,b)$ and $\pi_j^*(a,b)$, $j=1,\ldots,N$, for the complete market model are given by (4.13). In addition, assume that the real constant $s>1$. Then, independent of the choice of $k\in\mathbb{R}_+$, we have the following limit for the weighted proportion invested in each asset S_j:*

$$\lim_{N\to\infty}\sigma_j\rho_j\pi_j^*(a,b/N^k)=0\quad a.s, \tag{4.14}$$

for all $j=1,\ldots,N$. The limit of the weighted proportion to be invested in the market asset admits the following cases, almost surely:

$$\lim_{N\to\infty}\gamma\tilde{\pi}^*(a,b/N^k)=\begin{cases}\frac{\eta}{1-p} & \text{for } k>1\\ \frac{\eta}{1-p}+\frac{1}{1-p}\Big(\mathbb{E}[ab]+\mathbb{E}[a^2b]\Big) & \text{for } k=1\\ \infty & \text{for } k<1.\end{cases} \tag{4.15}$$

Proof. First, consider the scaled weighted proportion to be invested in each single stock $\sigma_j\rho_j\pi_j^*(a,b/N^k)$. Looking at its expansion

$$\sigma_j\rho_j\pi_j^*(a,b/N^k)=\frac{1}{(1-p)}\left(\xi^2\frac{1}{N^k}b_j+2\xi a_j\frac{1}{N^k}b_j+\frac{1}{N^k}b_ja_j^2-\xi^2\frac{c}{N^s}-2\xi a_j\frac{c}{N^s}-a_j^2\frac{c}{N^s}\right), \tag{4.16}$$

it is clear that $\sigma_j\rho_j\pi_j^*(a,b/N^k)\to 0$ almost surely as $N\to\infty$ for any choice of $k>0$ under the assumption that $s>0$.

(i) Suppose that $k>1$ and $s>1$, then we have the following scaled expansion for the proportion of wealth to be invested in the global asset

$$\begin{aligned}\gamma\tilde{\pi}^*(a,b/N^k)=&\frac{1}{(1-p)}\left(\eta-\xi^2\frac{1}{N^k}\mathcal{S}_b-2\xi\frac{1}{N^k}\mathcal{S}_{ab}-\frac{1}{N^k}\mathcal{S}_{a^2b}\right.\\ &\left.+\frac{c}{N^s}+\xi^2\frac{c}{N^{s-1}}+2\xi\frac{c}{N^s}\mathcal{S}_a+\frac{c}{N^s}\mathcal{S}_{a^2}\right).\end{aligned} \tag{4.17}$$

Then we have

$$\gamma\tilde{\pi}^*(a,b/N^k)\to\frac{\eta}{1-p}\quad a.s.,$$

for $N\to\infty$, because $\frac{1}{N}\mathcal{S}_{ab}\to\mathbb{E}[ab]$, $\frac{1}{N}\mathcal{S}_{a^2b}\to\mathbb{E}[a^2b]$ and $\frac{1}{N}\mathcal{S}_{a^2}\to\mathbb{E}[a^2]$ as $N\to\infty$ by the Strong Law of Large Numbers. Therefore, all terms inside the bracket except for the first go to zero, almost surely.

(ii) Let us just look at the proportion in the global asset from now on. Assuming that $k=1$ and using the scaled expansion (4.17), we have by the strong Law of Large Numbers the proposed limit, almost surely.

(iii) On the other hand for $k < 1$, we see that the expansion becomes

$$\begin{aligned}\gamma\tilde{\pi}^*(a, b/N^k) =& \frac{1}{(1-p)}\Bigg(\eta - \xi^2 N^{1-k}\frac{1}{N}\mathcal{S}_b - 2\xi N^{1-k}\frac{1}{N}\mathcal{S}_{ab} - N^{1-k}\frac{1}{N}\mathcal{S}_{a^2b} \\ &+ \frac{c}{N^s} + \xi^2\frac{c}{N^{s-1}} + 2\xi\frac{c}{N^s}\mathcal{S}_a + \frac{c}{N^s}\mathcal{S}_{a^2}\Bigg).\end{aligned} \tag{4.18}$$

Then $\gamma\tilde{\pi}^*(a, b/N^k)$ a.s. goes to infinity as $N^{1-k} > 0$, because $\frac{1}{N}\mathcal{S}_{ab} \to \mathbb{E}[ab]$ and $\frac{1}{N}\mathcal{S}_{a^2b} \to \mathbb{E}[a^2b]$ in the limit for $N \to \infty$, by the Strong Law of Large Numbers.

□

Theorem 9 shows that it is enough to control the size of the perturbations b_j, $j = 1, \ldots, N$, of the Sharpe ratio with respect to the market in order to prevent the optimal proportion from growing as N and therefore becoming infinite in the limit as $N \to \infty$. This means that we do not have to scale the perturbations a_j, $j = 1, \ldots, N$, arising from the correlation proportion. Economically speaking, this means that it is enough to control the risk- return profiles of the asset universe and by scaling them strongly enough, ensure that they are approaching the Sharpe ratio of the global asset in the limit. However, we can see that although the sequence of markets converges weakly to a constant coefficient limit, that we do not always have a finite limit of the portfolio proportions.

This clearly shows that the vector of portfolio proportions for the limiting optimisation only corresponds to the limit of the vectors for each single small market if we assume that the perturbations to the Sharpe ratio with respect to the market scales with $\frac{1}{N^k}$ for $k > 1$.

Having established the shape and limiting behaviour of the weighted single portfolio proportions, we now turn to their sum. In particular, we want to work towards establishing the total outlay of our investor, represented by the sum of the absolute values of the portfolio proportions in the complete market.

Summing over all the portfolio polices for the securities $j = 1, \ldots, N$ and adding the proportion to be invested in the global asset we have the expression

$$\begin{aligned}\sum_{j=1}^{N}\sigma_j\rho_j\pi_j^* + \gamma\tilde{\pi}^* &= \frac{1}{(1-p)}\left[\sum_{j=1}^{N}(\eta_j - \lambda_0) + \left(\lambda_0 - \sum_{j=1}^{N}\xi_j^2(\eta_j - \lambda_0)\right)\right] \\ &= \frac{1}{(1-p)}\left[\eta + \frac{c}{N^s}\right].\end{aligned} \tag{4.19}$$

In particular, as we assumed that if $s > 1$, we have the constant limit

$$\lim_{N\to\infty} \sum_{j=1}^{N} \sigma_j \rho_j \pi_j^* + \gamma\tilde{\pi}^* = \frac{\eta}{(1-p)} \quad a.s., \tag{4.20}$$

which only depends on the homogeneous, constant Sharpe ratio with respect to the market.

Clearly, the sum over all policies is finite, as we expected. In particular, the above equation suggests that we have to be either long in the assets and short in the market or the other way around to achieve optimal growth. However, we can also see that the individual positions can become very large. In particular, as we have shown in Theorem 9, the portfolio proportions can be unbounded if we do not apply sufficient control through scaling. The natural question to consider is how the sum of the absolute values of the portfolios proportions behaves in the limit. This quantity represents the total outlay of an investor and will help us to determine what growth rate corresponds to what magnitude of outlay from the agent.

Theorem 10. *Consider the agent's total outlay, represented by the sum of absolute values of the weighted portfolio proportions*

$$\sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a,b)| + |\gamma\tilde{\pi}^*(a,b)| = \frac{1}{|1-p|}\left[\sum_{j=1}^{N} \xi_j^2 \left|b_j - \frac{c}{N^s}\right| + \left|\eta + \frac{c}{N^s} - \sum_{j=1}^{N} \xi_j^2 \left(b_j - \frac{c}{N^s}\right)\right|\right], \tag{4.21}$$

scale the random variables b_j with $\frac{1}{N^k}$ for some real constant $k > 0$ and suppose that $s > 1$. Note that for simplification we write $\xi_j = \xi + a_j$.

Then for different choices of k, we have the following limiting behaviour:

(i) *For $k > 1$ we have*

$$\lim_{N\to\infty} \sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a, b/N^k)| + |\gamma\tilde{\pi}^*(a, b/N^k)| \leq \frac{|\eta|}{|1-p|} \quad a.s. \tag{4.22}$$

(ii) *Then for $k = 1$*

$$\begin{aligned} &\lim_{N\to\infty} \sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a, b/N^k)| + |\gamma\tilde{\pi}^*(a, b/N^k)| \\ \leq &\frac{1}{|1-p|}\left[\xi^2 \mathbb{E}[|b|] + 2|\xi|\mathbb{E}[|ab|] + \mathbb{E}[a^2] + |\eta| + \left|2\xi\mathbb{E}[ab] + \mathbb{E}[a^2 b]\right|\right] \quad a.s. \end{aligned} \tag{4.23}$$

(iii) *And finally for $k < 1$*

$$\lim_{N\to\infty} \sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a, b/N^k)| + |\gamma\tilde{\pi}^*(a, b/N^k)| = \infty \quad a.s. \tag{4.24}$$

Let us show this by finding the required upper and lower bounds for the limits above.

Proof. **(i)** Suppose that we choose $k > 1$. Then we want to show that the sum of the absolute values stays bounded from above in the limit. Expanding the expression and looking for an upper bound we have

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a, b/N^k)| + |\gamma \tilde{\pi}^*(a, b/N^k)| \\
&= \lim_{N\to\infty} \frac{1}{|1-p|} \left[\sum_{j=1}^{N} \left| \frac{b_j \xi_j^2}{N^k} - \frac{c\xi_j^2}{N^s} \right| + \left| \eta + \frac{c}{N^s} - \frac{1}{N^k} \sum_{j=1}^{N} \xi_j^2 b_j + \frac{c}{N^s} \sum_{j=1}^{N} \xi_j^2 \right| \right] \\
&\leq \lim_{N\to\infty} \frac{1}{|1-p|} \left[\sum_{j=1}^{N} \left| \frac{b_j \xi_j^2}{N^k} \right| + \sum_{j=1}^{N} \left| \frac{c\xi_j^2}{N^s} \right| + |\eta| + \frac{|c|}{N^s} + \frac{1}{N^k} \left| \sum_{j=1}^{N} \xi_j^2 b_j \right| + \frac{|c|}{N^s} \sum_{j=1}^{N} \xi_j^2 \right] \\
&= \frac{|\eta|}{|1-p|} \quad a.s.,
\end{aligned}$$

which follows by the Strong Law of Large Numbers and the assumption that a_j and b_j are mean zero random variables, as well as the fact that $s > 1$.

(ii) For $k = 1$, using the above result for the expanded upper bound as well as the Law of Large Numbers, we have the result. Note that the ξ_j still contain the random variable a_j. However, the size of the a_j does not play a role in obtaining the limit and therefore we prefer to use the quantity ξ_j for a cleaner notation. To be sure that terms such as $\frac{c\xi_j^2}{N^s}$ actually converge to zero for $s > 1$ and any unbounded random variable a_j, recall the assumption in(4.10).

(iii) To establish the divergence of the sum as $N \to \infty$ when $k < 1$ we ned to find a lower bound. By the definition of the π^* and $\tilde{\pi}^*$ we have

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{j=1}^{N} |\sigma_j \rho_j \pi_j^*(a, b/N^k)| + |\gamma \tilde{\pi}^*(a, b/N^k)| \\
&\quad = \lim_{N\to\infty} \frac{1}{1-p} \sum_{j=1}^{N} \left| \frac{b_j \xi_j^2}{N^k} - \frac{c\xi_j^2}{N^s} \right| + \left| \eta + \frac{c}{N^s} - \frac{1}{N^k} \sum_{j=1}^{N} \xi_j^2 b_j + \frac{c}{N^s} \sum_{j=1}^{N} \xi_j^2 \right| \\
&\quad \geq \lim_{N\to\infty} \sum_{j=1}^{N} \frac{\xi_j^2}{N^k} \left| b_j - \frac{c}{N^{s-k}} \right| \\
&\quad \geq \lim_{N\to\infty} \sum_{j=1}^{N} \frac{\xi_j^2}{N^k} \left| b_j - \frac{c}{N^{s-k}} \right| \mathbb{1}_{b_j > \varepsilon},
\end{aligned}$$

for any $\varepsilon > 0$.

For this $\varepsilon > 0$ there exists an N_0 such that for $N > N_0$ we have $\frac{c}{N^{s-k}} < \frac{\varepsilon}{2}$, and hence

$$\geq \lim_{N\to\infty} \sum_{j=1}^{N} \frac{\xi_j^2}{N^k} \left| b_j - \frac{\varepsilon}{2} \right| \mathbb{1}_{b_j > \varepsilon},$$

As $\left| b_j - \frac{\varepsilon}{2} \right| \mathbb{1}_{b_j > \varepsilon}$ are positive random variables we have by the Strong Law of Large Numbers that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \xi_j^2 \left| b_j - \frac{\varepsilon}{2} \right| \mathbb{1}_{b_j > \varepsilon} > 0.$$

Therefore, as $k < 1$ we have

$$\lim_{N\to\infty} \frac{1}{N^k} \sum_{j=1}^{N} \xi_j^2 \left| b_j - \frac{\varepsilon}{2} \right| \mathbb{1}_{b_j > \varepsilon} = \infty.$$

□

We have shown that if the random variables perturbing the Sharpe ratio with respect to the market are not scaled sufficiently, the total outlay becomes infinite as the number of assets approaches infinity. This means that the total position, which the investor takes, can be infinitely large and therefore the agent is possibly exposed to very high risk using this investment strategy. It is important to determine what growth rate of the value function $\Theta^*_{N,k}(t)$ we can achieve using these optimal policies and how this growth rate behaves with a growing number of securities to be able to judge if this possibly very risky investment would make economic sense. In addition, this examination of the value function growth rate will give us clues about the absence or existence of asymptotic arbitrage in this complete market in the limit.

Now we want to consider two things. First, we establish the finiteness conditions of the value function given by (3.33) in Theorem 4 by looking at its growth rate. Here, we want to discuss what magnitude of growth can be achieved for different scaling of the perturbation parameters. Secondly, we want to relate this to our no asymptotic arbitrage conditions, as discussed in Chapter 2.

Recall that growth rate Θ^*_N of the exponential value function is given by

$$\Theta^*_N(a,b) = \frac{1}{2} \left(\lambda_0^2 + \sum_{j=1}^{N} \xi_j^2 (\eta_j - \lambda_0)^2 \right),$$

where $\Theta^*_N(a,b)$ represents the dependence on the perturbations a_j and b_j, $j = 1, \ldots, N$.

We have used the assumption $\lambda_0 = \eta + \frac{c}{N^s}$ for the real constants c, s where $s > 1$, and therefore $\lim_{N\to\infty} \lambda_0 = \eta$. Then, for two sets of random variables a_j and b_j, $j = 1, \ldots, N$, drawn from the same distribution with mean zero, the growth rate Θ_N^* can be expanded as

$$\begin{aligned}\Theta_N^* =& \frac{1}{2}\Bigg(\eta^2 + \frac{2\eta c}{N^s} + \frac{c^2}{N^{2s}} + \xi^2\mathcal{S}_{b^2} + 2\xi\mathcal{S}_{ab^2} + \mathcal{S}_{a^2b^2} - \frac{2\xi^2 c}{N^s}\mathcal{S}_b - \frac{4\xi c}{N^s}\mathcal{S}_{ab} - \frac{2c}{N^s}\mathcal{S}_{a^2b} \\ &+ \frac{\xi^2 c^2}{N^{2s-1}} + \frac{2\xi c^2}{N^{2s}}\mathcal{S}_a - \frac{c^2}{N^{2s}}\mathcal{S}_{a^2}\Bigg).\end{aligned} \tag{4.25}$$

Because we assume that $s > 1$, we can use an approximation to (4.25) as all terms in $\frac{c}{N^s}$ converge to zero with a rate of at least $\frac{1}{N}$ almost surely. To be able to compare the growth rate that can be achieved by investing optimally with the overall outlay of the agent as computed above, we only want to scale the set of random variables b_j, $j = 1, \ldots, N$ with $\frac{1}{N^k}$ for some real $k \geq 0$. Hence, consider the following k-dependent expression for the value function growth

$$\Theta_N^*(a, b/N^k) \sim \frac{1}{2}\left(\eta^2 + \xi^2\frac{1}{N^{2k}}\mathcal{S}_{b^2} + 2\xi\frac{1}{N^{2k}}\mathcal{S}_{ab^2} + \frac{1}{N^{2k}}\mathcal{S}_{a^2b^2}\right), \tag{4.26}$$

where '$\sim$' denotes the approximation made by omitting all terms of order $\frac{1}{N}$ and smaller.

We show in the following that there are three distinct regimes of behaviour, depending on our choice of $k \geq 0$.

Theorem 11. *Let $\Theta_N^*(a, b/N^k)$ be the value function growth rate as in (4.26), then we can observe three distinct cases of limiting behaviour depending on $k \geq 0$, such that almost surely its limit is:*

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \begin{cases} \frac{1}{2}\eta^2 & \text{for } k > \frac{1}{2} \\ \frac{1}{2}\left(\eta^2 + \xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\right) & \text{for } k = \frac{1}{2} \\ \infty & \text{for } k < \frac{1}{2}. \end{cases} \tag{4.27}$$

Proof. Let us briefly show the crucial steps for each region depending on our choice of the real constant $k \geq 0$.

(i) For $k > \frac{1}{2}$, we can see from equation(4.26) that $\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \frac{1}{2}\eta^2$ because all terms in the brackets, besides the first one, tend to zero almost surely applying the Strong Law of Large Numbers and under the assumption that the required moments exist, in particular $\mathbb{E}[(|a| + |b|)^4] < \infty$.

(ii) Inserting $k = \frac{1}{2}$ into the approximation of $\Theta_N^*(a, b/N^k)$ in (4.26), we have

$$\Theta_N^*(a, b/N^k) = \frac{1}{2}\left(\eta^2 + \xi^2 \frac{1}{N}\mathcal{S}_{b^2} + 2\xi \frac{1}{N}\mathcal{S}_{ab^2} + \frac{1}{N}\mathcal{S}_{a^2b^2}\right).$$

If $\mathbb{E}[(|a| + |b|)^4] < \infty$, we can use the Strong Law of Large Numbers to obtain the finite limit

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \frac{1}{2}\left(\eta^2 + \xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\right) \quad a.s.$$

Here $\Theta_N^*(a, b/N^k)$ is of $\mathcal{O}(1)$.

(iii) Finally for $k < \frac{1}{2}$, we have that $\Theta_N^*(a, b/N^k)$ becomes infinite in the limit, as the scaling is not sufficient to control the perturbed parameters. In particular, we can look at the following approximation using the Strong Law of Large Numbers for very large N

$$\Theta_N^*(a, b/N^k) = \frac{1}{2}\left(\eta^2 + N^{1-2k}\xi^2\mathbb{E}[b^2] + 2\xi N^{1-2k}\mathbb{E}[ab^2] + N^{1-2k}\mathbb{E}[a^2b^2]\right),$$

and because $1 - 2k > 0$ we get the required result and

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \infty \quad a.s.$$

□

We need to make a couple of brief comments at this point. First, from the particular shape of $\Theta_N^*(a, b/N^k)$ in equation (4.26), together with the above theorem, we can see that additional scaling of the set of random variables a_j, $j = 1, \ldots, N$ would only alter the results very mildly. We would still have convergence to the homogeneous, constant coefficient growth for $k > \frac{1}{2}$, and $\Theta_N^*(a, b/N^k)$ would still become infinite in the limit for $k < \frac{1}{2}$, almost surely, for $N \to \infty$. The only difference would be in the error term of the intermediate regime for $k = \frac{1}{2}$. This error would not completely vanish, however depending on the size of the scaling of the a_j, we could rid ourselves of the last or the final two terms. We will not explicitly calculate those cases as they are not relevant to our investigation. On the other hand, it is interesting to see that the growth of the value function is not highly dependent on the perturbation term of the correlation ratio. This is an interesting result, as it tells us that correlation is not the crucial parameter in our CAPM style large market, and the agent's investment does not depend strongly on its size.

Secondly, we have seen that the total outlay of the agent, represented by the sum of the absolute values of the portfolio proportions, behaves quite differently for various choices of the scaling exponent. If we scale strongly, i.e. $k > 1$, we just obtain the homogeneous constant

coefficient result, which is finite in the limit. If we relax our scaling, assuming that $k = 1$, the investor still risks only a finite, although higher amount in total. For $k < 1$, we have shown that it becomes infinite once we let $N \to \infty$, almost surely. However, for the growth rate of the value function, that can be achieved using optimal investment policies, we have a different result. As the perturbed parameters b_j present in squares, we only need $k \geq \frac{1}{2}$ to keep the limiting growth finite. Once we assume that $k < \frac{1}{2}$, we can achieve unlimited growth. This is an interesting result. We see that there is a region for $\frac{1}{2} \leq k < 1$, where we would have to invest an infinite amount of money, taking out large positive and negative positions in the assets, but can only achieve a possibly large, but finite growth rate of the value function in the limit. This means that although we have seen that the portfolio proportions of the single stocks go to zero for $N \to \infty$, the revenue obtained from them is still big enough to generate a significant total value once the growth rate is inserted into the value function. The natural question arising here is whether we can impose constraints on the portfolio proportions such as forbidding short-selling to limit our exposure to the market, but still have a higher growth than for homogeneous parameters. Unfortunately we were not able to answer this to date. On the other hand, for $k > 1$, the total sum invested is a finite and possibly small amount based on the Sharpe ratio with respect to the market, but by construction the value function grows faster and we can achieve a higher value with a comparably low risk.

Let us now look closer at asymptotic arbitrage in this particular complete market setting. We have already seen at the beginning of Chapter 4, that the sequence of market models converges weakly to a finite, constant coefficient limit. However, asymptotic arbitrage could still arise. We have discussed in Chapter 1 and Chapter 2 that a convenient condition on no asymptotic arbitrage, in particular for deterministic modelling parameters is the finiteness of the Hellinger process in the limit. Using [KK93] we can restate the Hellinger process for such parameters in our notation as

$$\lim_{N\to\infty} h_N = \frac{1}{8} \lim_{N\to\infty} T\left(\lambda_0^2 + \sum_{j=1}^{N} \xi_j^2 (\eta_j - \lambda_0)^2\right). \tag{4.28}$$

Comparing this to the value function growth rate Θ_N^*, it is evident that we have

$$h_N = \frac{1}{4}\Theta_N^*. \tag{4.29}$$

Because we have assumed that $T < \infty$, the condition on no asymptotic arbitrage is equivalent to the finiteness of the value function growth rate Θ_N^*. Hence we can see that there is no asymptotic arbitrage if, as described in Theorem 11, we scale the perturbations b_j stemming from the Sharpe ratio with respect to the market with at least $\frac{1}{N^k}$, for a constant $k \geq \frac{1}{2}$. Heuristically speaking, we have found that the Sharpe ratios with respect to the market of

all the assets in our basket have to converge sufficiently quickly towards one and the same constant value to avoid asymptotic arbitrage. Intuitively, this means that our whole universe has to contract and all the parameters of the stocks have become sufficiently close to each other. If on the other hand we would still experience high variation within the parameters, then it would be possible to build a portfolio which achieves infinite value and thus realises arbitrage asymptotically.

To conclude this section we want to state one final intriguing remark.

Remark 4.1. *Having established the finiteness conditions as well as the asymptotic behaviour of the expanded growth rate for different scaling parameters $k \geq 0$, we want to show by how much, approximately, the optimal value function can outperform the simpler homogeneous one, stated in Section 3.1. Therefore, we look at the difference of the approximate growth rate $\Theta_N^*(a, b/N^k)$ from (4.26) and the homogeneous growth $\Theta_N^{*,h}$*

$$\Theta_N^*(a, b/N^k) - \Theta_N^{*,h} = \frac{1}{2}\left(\xi^2 \frac{1}{N^{2k}} \mathcal{S}_{b^2} + 2\xi \frac{1}{N^{2k}} \mathcal{S}_{ab^2} + \frac{1}{N^{2k}} \mathcal{S}_{a^2b^2}\right). \tag{4.30}$$

Then we have the following three cases in the limit as $N \to \infty$, almost surely by the Strong Law of Large Numbers, and as the required moments exist by assumption

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) - \Theta_N^{*,h} = \begin{cases} 0 & \text{for } k > \frac{1}{2} \\ \frac{1}{2}\left(\xi^2 \mathbb{E}[b^2] + 2\xi \mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\right) & \text{for } k = \frac{1}{2} \\ \infty & \text{for } k < \frac{1}{2}. \end{cases} \tag{4.31}$$

These results will be shown graphically in Chapter 5.

4.2 Incomplete Model

Let us take a look at the incomplete market model as introduced in Section 3.2. We have already stated the explicit optimisation results for a power utility function in Theorem 5. Now we want to look at their asymptotic behaviour. First, we will expand the portfolio proportions around the homogeneous parameters similar to the methodology used in Section 4.1 and then consider their limit as $N \to \infty$. We also want to see how large the sum of the absolute values of the portfolio proportions (the total outlay of the investor) is. We suspect, as seen for the complete market model, that the optimal investment proportions make the agent take out large long positions in some assets, while shorting others. This means however that the total exposure to the market can be very large increasing the risk profile of the investment. To

quantify this we look at the sum of the absolute values of the portfolio proportions as the number of securities approaches infinity.

Heuristically, we can summarise our results as follows. We know that the optimal portfolio proportions $\hat{\pi}_j$ are finite for any $j = 1, \dots, N$. Moreover, its homogeneous parameter counterparts $\hat{\pi}_j^h$ are $\mathcal{O}\big(N^{-1}\big)$ for $N \to \infty$. Therefore, the sum over the $\hat{\pi}_j^h(t)$ is $\mathcal{O}(1)$. However, when we look at the sum of the absolute values of the expanded $\hat{\pi}_j(a, b)$, $j = 1, \dots, N$, we see that unless there is particular scaling of the perturbations to the Sharpe ratios with respect to the market we can not keep the sum bounded.

On the other hand, we want to derive a finiteness condition for the growth rate $\hat{\Theta}_N$, which plays a role in in the exponential of the optimal value function $\hat{u}_N(x)$. We expand this quantity as in Section 4.1 and derive explicit scaling parameters to understand its behaviour as $N \to \infty$. In particular, we are interested in the comparison between the total position of the investor and the maximal growth he can achieve using these portfolio proportions.

The result is similar to Section 4.1, where we have seen that there is a region where the exposure to the market becomes infinite in the limit, however the value which can be achieved stays finite.

Since we are only looking at the deterministic parameter case, we know by Corollary 2.3 that the finiteness of the value function in the limit is equivalent to no asymptotic arbitrage in the infinite market. At the end of this section, we derive the structure of the Hellinger process for this particular incomplete market setup, as this has not been stated previously.

We will apply the same notation as well as similar arguments and methodology as in the previous section on complete markets without further explanation.

Recall from Theorem 5 in Chapter 3, that the general proportion of wealth (3.16) to be invested in each asset $S_j(t)$, for $j = 1, \dots, N$ is

$$\hat{\pi}_j = \frac{1}{1-p}\left[\frac{1}{K_N \sigma_j \rho_j}\left(\eta_j \xi_j^2 + \sum_{k=1}^{N} \xi_k^2 \xi_j^2 (\eta_j - \eta_k)\right)\right].$$

If we assume that all coefficients are homogeneous throughout our stock basket, the above proportion becomes

$$\hat{\pi}_j^h = \frac{1}{N}\left(\frac{\lambda}{(1-p)(1-\rho^2)(\frac{1}{N}+\xi^2)\sigma}\right) \quad \forall j = 1,\dots,N, \tag{4.32}$$

as in (3.27).

Clearly the expression in (4.32) is of order $\mathcal{O}(N^{-1})$ and converges to zero almost surely when we let $N \to \infty$. Essentially the homogenous portfolio proportion (4.32) represents a way to build an equally weighted portfolio of the approximate form $\frac{C}{N}$ for the constant $C = \frac{\lambda}{(1-p)(1-\rho^2)(\xi^2)\sigma}$. Observe that this result is different from what we have seen in the complete model, where we had the possibility to invest in the global market assets as well as the single securities. Here the global market asset is not available for trading, which explains the different result.

To understand the quantity fully, we want to expand $\hat{\pi}_j(t)$ around the homogeneous parameters ξ and η. Recall from the beginning of this chapter, that for two sets of random variables a_j and b_j, $j = 1,\dots,N$, on a probability space $(\Omega, \mathfrak{B}, \mathfrak{P})$ drawn from a mean zero distribution, we have $\eta_j = \eta + b_j$ and $\xi_j = \xi + a_j$.

As in Section 4.1 consider the weighted portfolio proportions $\sigma_j\rho_j\hat{\pi}_j$, $j = 1,\dots,N$ for convenience. We have the following expansion for each of the N stocks:

$$\begin{aligned}\sigma_j\rho_j\hat{\pi}_j(a,b) =& \frac{1}{(1-p)(\frac{1}{N}+\xi^2+\frac{2\xi}{N}\mathcal{S}_a+\frac{1}{N}\mathcal{S}_{a^2})}\Bigg[\frac{\eta\xi^2}{N}+\xi^2 b_j+2\xi^3 a_j b_j+(\xi^2 b_j+2\xi a_j b_j)\frac{1}{N}\mathcal{S}_{a^2}\\ &-(\xi+a_j^2)^2\frac{1}{N}\mathcal{S}_{a^2b}-(2\xi a_j^2+2\xi+4\xi^2 a_j)\frac{1}{N}\mathcal{S}_{ab}-(\xi^3+2\xi^2 a_j)\frac{1}{N}\mathcal{S}_b\\ &+(2\xi^3 b_j+4\xi^2 a_j b_j)\frac{1}{N}\mathcal{S}_a+\frac{1}{N}\Big(2\eta\xi a_j+\eta a_j^2+b_j\xi^2+2\xi b_j a_j+b_j a_j^2\Big)\Bigg],\end{aligned} \tag{4.33}$$

where $\hat{\pi}_j(a,b)$ represents the dependence on the perturbations a_j and b_j.

Taking the limit, we have

$$\begin{aligned}\lim_{N\to\infty}\sigma_j\rho_j\hat{\pi}_j(a,b) =& \frac{1}{(1-p)(\xi^2+\mathbb{E}[a^2])}\Big[\xi^2 b_j+2\xi^3 a_j b_j+(\xi^2 b_j+2\xi a_j b_j)\mathbb{E}[a^2]\\ &-(\xi+a_j^2)^2\mathbb{E}[a^2 b]-(2\xi a_j^2+2\xi+4\xi^2 a_j)\mathbb{E}[ab]\Big] \quad a.s.,\end{aligned} \tag{4.34}$$

using the Strong Law of Large Numbers, and the assumption that $\mathbb{E}[a] = 0$, $\mathbb{E}[b] = 0$. We see that $\sigma_j \rho_j \hat{\pi}_j(a, b)$ is a finite number in the limit for any choice of a_j and b_j. In particular, the leading term is of order $\mathcal{O}(N^{-1})$ and therefore goes to zero in the limit as expected from the shape of the homogeneous parameter proportion. Therefore, the expression in (4.34) represents the limit of the error terms. In particular, this shows that the optimal portfolio proportion has a more complicated structure than its homogeneous counterpart. At the end of this chapter, we will see that the growth rate achieved by investing as $\hat{\pi}_j(a, b)$ outperforms using $\hat{\pi}^h$.

Let us now turn to the discussion of the total outlay of the investor, which is represented by the absolute value of the sum over all weighted portfolio proportions. First, consider the sum of the weighted portfolio proportions

$$\begin{aligned}
\lim_{N\to\infty} \sum_{j=1}^{N} \sigma_j \rho_j \hat{\pi}_j(a, b) &= \lim_{N\to\infty} \frac{1}{(1-p)K_N} \left[\sum_{j=1}^{N} \eta_j \xi_j^2 + \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_k^2 \xi_j^2 (\eta_j - \eta_k) \right] \\
&= \lim_{N\to\infty} \frac{1}{(1-p)K_N} \sum_{j=1}^{N} \eta_j \xi_j^2 \\
&= \frac{1}{(1-p)(\xi^2 + \mathbb{E}[a^2])} \Big(\eta\xi^2 + \eta\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab] + \mathbb{E}[ab^2] \Big) \qquad (4.35) \\
&< \infty \quad a.s.,
\end{aligned}$$

by the Strong Law of Large Numbers. Even without expanding, we can see that the term is of order $\mathcal{O}(1)$ and therefore finite.

We have no restrictions on short selling or borrowing from the bank account, therefore the $\hat{\pi}_j$ can be both negative as well as positive. Summing over them averages the total position and therefore (4.35) stays bounded in the limit as $N \to \infty$.

If we want to see how exposed we actually are to the risky securities in our incomplete market, we need to consider the sum of the absolute values of the positions. Using the same expansion techniques as above, we also need to impose scaling on the random perturbations to gain control over these sums in the limit. As in Section 4.1, we will see that it suffices to scale the perturbations to the Sharpe ratio with respect to the market, b_j, with a factor $\frac{1}{N^k}$ for a real constant $k \geq \frac{1}{2}$. Again, once we apply the scaling we will write $\hat{\pi}_j(a, b/N^k)$ to symbolise the dependence on $\frac{1}{N^k}$. We can summarise our results depending on the choice of k as follows.

Theorem 12. *Consider the sum of the absolute values of the weighted portfolio proportions*

$$\sum_{j=1}^{N} \left| \sigma_j \rho_j \hat{\pi}_j(a, b/N^k) \right|,$$

then assuming that we have two random variables a_j and b_j drawn from a mean zero distribution such that $\xi_j = \xi + a_j$ and $\eta_j = \eta + \frac{b_j}{N^k}$, for a real positive constant k and sufficient moments exist with the standing assumption that $\mathbb{E}[(|a| + |b|)^{24}] < \infty$. Then we have the following limiting behaviour:

(i) *for $k > 1$*

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left| \sigma_j \rho_j \hat{\pi}_j(a, b/N^k) \right| = \frac{|\eta|}{|1-p|} \quad a.s., \tag{4.36}$$

(ii) *for $k = 1$ the lower and upper bounds, where $\hat{\xi}$ is the unexpanded quantity and ξ the homogeneous correlation ratio, converge almost surely as*

$$\begin{aligned}
\lim_{N\to\infty} \sum_{j=1}^{N} \left| \sigma_j \rho_j \hat{\pi}_j(a, b/N^k) \right| &> \frac{1}{|1-p|(\xi^2 + \mathbb{E}[a^2])} \left[\mathbb{E}[\hat{\xi}^2] \left(\mathbb{E}[\hat{\xi}^2|b|] - \left| \mathbb{E}[\hat{\xi}^2|b|] - \eta \right| \right) \right], \\
\lim_{N\to\infty} \sum_{j=1}^{N} \left| \sigma_j \rho_j \hat{\pi}_j(a, b/N^k) \right| &\leq \frac{1}{|1-p|(\xi^2 + \mathbb{E}[a^2])} \left[\mathbb{E}[\hat{\xi}^2] \left(\mathbb{E}[\hat{\xi}^2|b|] + \left| \mathbb{E}[\hat{\xi}^2|b|] - \eta \right| \right) \right],
\end{aligned} \tag{4.37}$$

(iii) *and finally for $k < 1$ we have*

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left| \sigma_j \rho_j \hat{\pi}_j(a, b/N^k) \right| = \infty \quad a.s. \tag{4.38}$$

Proof. Let us briefly show the above arguments. First a couple of comments. We will write $\hat{\xi}$ in the expected values when we do not decompose the correlation ratio in its homogeneous part ξ and the random perturbation term, i.e. $\mathbb{E}[\hat{\xi}]$ is the limit of $\frac{1}{N}\sum_{j=1}^{N} \xi_j$ as $N \to \infty$. In addition, we want to start by showing the result for $k = 1$ as the other two regimes follow directly from it.

(i) Suppose $k = 1$, then we scale the random perturbations b_j with $\frac{1}{N}$. Therefore we have, using the Law of Large Numbers under the assumption that $\mathbb{E}[|\hat{\xi}^2 b|] < \infty$ and thus $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \xi_j^2 = \mathbb{E}[\hat{\xi}^2]$ as well as $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \xi_j^2 b_j = \mathbb{E}[\hat{\xi}^2 b]$, the following

approximation symbolised by '$\sim$'

$$\sum_{j=1}^{N}\Big|\sigma_j\rho_j\hat{\pi}_j(a,b/N^k)\Big| = \frac{1}{|1-p|K_N}\sum_{j=1}^{N}\left|\xi_j^2\eta + \frac{b_j\xi_j^2}{N} + \xi_j^2\frac{1}{N}\sum_{k=1}^{N}\xi_k^2(b_j-b_k)\right|$$
$$\sim \frac{1}{|1-p|K_N}\sum_{j=1}^{N}\left|b_j\xi_j^2\mathbb{E}[\hat{\xi}^2] - \xi_j^2\Big(\mathbb{E}[\hat{\xi}^2 b]-\eta\Big)\right|.$$

To be able to see that the above approximation is valid, we first have to show that for

$$\chi_j(N) = \left|\xi_j^2\eta + \frac{b_j\xi_j^2}{N} + \xi_j^2\frac{1}{N}\sum_{k=1}^{N}\xi_k^2(b_j-b_k)\right|,$$

and

$$\chi_j = \left|b_j\xi_j^2\mathbb{E}[\hat{\xi}^2] - \xi_j^2\Big(\mathbb{E}[\hat{\xi}^2 b]-\eta\Big)\right|,$$

we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\chi_j(N) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\chi_j \quad a.s.$$

It is enough to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\big|\chi_j(N)-\chi_j\big| = 0 \quad a.s.$$

By the first Borel-Cantelli lemma this will hold, provided for all $\varepsilon > 0$

$$\sum_{j=1}^{\infty}\mathbb{P}_N\Big(\frac{1}{N}\sum_{j=1}^{N}\big|\chi_j(N)-\chi_j\big| > \varepsilon\Big) < \infty.$$

In order to establish this, we use Markov's inequality and monotonicity of $\mathcal{L}^p$ norms to see

$$\mathbb{P}_N\Big(\frac{1}{N}\sum_{j=1}^{N}\big|\chi_j(N)-\chi_j\big| > \varepsilon\Big) \le \frac{1}{\varepsilon^4}\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}\big|\chi_j(N)-\chi_j\big|\right)^4$$
$$\le \frac{1}{\varepsilon^4}\mathbb{E}\big|\chi_j(N)-\chi_j\big|^4. \tag{4.39}$$

Now

$$\mathbb{E}\big|\chi_j(N)-\chi_j\big|^4$$
$$\le \mathbb{E}\left|\xi_j^2\eta + \frac{b_j\xi_j^2}{N} + \xi_j^2\frac{1}{N}\sum_{k=1}^{N}\xi_k^2(b_j-b_k) - b_j\xi_j^2\mathbb{E}[\hat{\xi}^2] + \xi_j^2\Big(\mathbb{E}[\hat{\xi}^2 b]-\eta\Big)\right|^4,$$

as the absolute value is Lipschitz and using Cauchy-Schwartz

$$
\begin{aligned}
&\leq \mathbb{E}\left| b_j\xi_j^2\Big(\frac{1}{N}\Big(1+\sum_{k=1}^{N}\xi_k^2\Big)-\mathbb{E}[\hat{\xi}^2]\Big)+\mathbb{E}[\hat{\xi}^2 b]-\frac{1}{N}\sum_{k=1}^{N}\xi_k^2 b_k\right|^4 \\
&\leq 64\left(\mathbb{E}\left|\frac{b_j\xi_j^2}{N}\right|^4+\mathbb{E}\big|b_j\hat{\xi}^2\big|^4\left|\frac{1}{N}\sum_{k=1}^{N}\xi_k^2-\mathbb{E}[\hat{\xi}^2]\right|^4+\mathbb{E}\left|\frac{1}{N}\sum_{k=1}^{N}\xi_k^2 b_k-\mathbb{E}[\hat{\xi}^2 b]\right|^4\right) \\
&\leq 64\Bigg(\frac{\mathbb{E}[b^4\hat{\xi}^8]}{N^4}+\big(\mathbb{E}[b^8\hat{\xi}^{16}]\big)^{1/2}\left(\frac{1}{N^3}\mathbb{E}[\hat{\xi}^{16}]+\frac{N(N-1)}{N^4}\mathbb{E}[\hat{\xi}^8]\right)^{1/2} \\
&\qquad\qquad +\frac{1}{N^3}\mathbb{E}[b^4\hat{\xi}^8]+\frac{N(N-1)}{N^4}\big(\mathbb{E}[b^2\hat{\xi}^4]\big)^2\Bigg) \\
&\leq \frac{c}{N^2}.
\end{aligned}
$$

provided $\mathbb{E}[(|a|+|b|)^{24}]<\infty$.

Hence (4.39) is summable and by the Borel-Cantelli lemma we have the equivalence. Using the above expression, we find the limiting lower bound to be

$$
\begin{aligned}
&\lim_{N\to\infty}\sum_{j=1}^{N}\Big|\sigma_j\rho_j\hat{\pi}_j(a,b/N^k)\Big| \\
\geq &\lim_{N\to\infty}\frac{1}{|1-p||\frac{1}{N}+\mathbb{E}[\hat{\xi}^2]|}\left[\Big(\mathbb{E}[\hat{\xi}^2]+\frac{1}{N}\Big)\frac{1}{N}\sum_{j=1}^{N}|b_j\xi_j^2|-\Big|\mathbb{E}[\hat{\xi}^2 b]-\eta\Big|\frac{1}{N}\sum_{j=1}^{N}|\xi_j^2|\right] \\
= &\frac{1}{|1-p|(\xi^2+\mathbb{E}[a^2])}\left[\mathbb{E}[\hat{\xi}^2]\left(\mathbb{E}[\hat{\xi}^2|b|]-\Big|\mathbb{E}[\hat{\xi}^2|b|]-\eta\Big|\right)\right] \quad a.s.
\end{aligned}
$$

On the other hand, for the upper bound we have

$$
\begin{aligned}
&\lim_{N\to\infty}\sum_{j=1}^{N}\Big|\sigma_j\rho_j\hat{\pi}_j(a,b/N^k)\Big| \\
\leq &\lim_{N\to\infty}\frac{1}{|1-p||\frac{1}{N}+\mathbb{E}[\hat{\xi}^2]|}\left[\Big(\mathbb{E}[\hat{\xi}^2]+\frac{1}{N}\Big)\frac{1}{N}\sum_{j=1}^{N}|b_j\xi_j^2|+\Big|\mathbb{E}[\hat{\xi}^2 b]-\eta\Big|\frac{1}{N}\sum_{j=1}^{N}|\xi_j^2|\right] \\
= &\frac{1}{|1-p|(\xi^2+\mathbb{E}[a^2])}\left[\mathbb{E}[\hat{\xi}^2]\left(\mathbb{E}[\hat{\xi}^2|b|]+\Big|\mathbb{E}[\hat{\xi}^2|b|]-\eta\Big|\right)\right] \quad a.s.
\end{aligned}
$$

(ii) Now for $k > 1$, we see using the above results but altering the scaling accordingly, that the lower bound is

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{j=1}^{N} \Big|\sigma_j \rho_j \hat{\pi}_{j,k}(t)\Big| \\
>&\frac{1}{|1-p||\frac{1}{N} + \mathbb{E}[\hat{\xi}^2]|} \left[\frac{1}{N}\Big(N^{1-k}\mathbb{E}[\hat{\xi}^2] + N^{1-k}\frac{1}{N}\Big)\sum_{j=1}^{N}|\xi_j^2 b_j| - \Big|N^{1-k}\mathbb{E}[\hat{\xi}^2 b] - \eta\Big|\frac{1}{N}\sum_{j=1}^{N}|\xi_j^2|\right] \\
=&\frac{|\eta|}{|1-p|},
\end{aligned}$$

almost surely, by the Strong Law of Large Numbers.

On the other hand, we have for the upper bound that

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{j=1}^{N} \Big|\sigma_j \rho_j \hat{\pi}_j(a, b/N^k)\Big| \\
\geq&\frac{1}{|1-p||\frac{1}{N} + \mathbb{E}[\hat{\xi}^2]|} \left[\frac{1}{N^k}\Big(\mathbb{E}[\hat{\xi}^2] + \frac{1}{N}\Big)\sum_{j=1}^{N}|\xi_j^2 b_j| + \Big|N^{1-k}\mathbb{E}[\hat{\xi}^2 b] - \eta\Big|\frac{1}{N}\sum_{j=1}^{N}|\xi_j^2|\right] \\
=&\frac{|-\eta|}{|1-p|} = \frac{|\eta|}{|1-p|},
\end{aligned}$$

almost surely, therefore both bounds coincide. Hence

$$\lim_{N\to\infty} \sum_{j=1}^{N} \Big|\sigma_j \rho_j \hat{\pi}_j(a, b/N^k)\Big| = \frac{|\eta|}{|1-p|} \quad a.s.,$$

in this case.

(iii) Finally, for $k < 1$ it is enough to show the lower bound. We can easily see using the lower bound in (i), that for a scaling of $k < 1$ we have

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{j=1}^{N} \Big|\sigma_j \rho_j \hat{\pi}_j(a, b/N^k)\Big| \\
\geq& \lim_{N\to\infty} \frac{1}{|1-p||\frac{1}{N} + \mathbb{E}[\hat{\xi}^2]|} \Bigg[N^{1-k}\Big(\mathbb{E}[\hat{\xi}^2] + \frac{1}{N}\Big)\frac{1}{N}\sum_{j=1}^{N}|b_j \xi_j^2| \\
&\qquad\qquad - \Big|N^{1-k}\mathbb{E}[\hat{\xi}^2 b] - \eta\Big|\frac{1}{N}\sum_{j=1}^{N}|\xi_j^2|\Bigg],
\end{aligned}$$

which is of order $\mathcal{O}(N^{1-k})$ and goes to infinity in the limit, almost surely because $1 - k > 0$ by assumption.

□

We have shown that unless we choose a scaling exponent of $k \geq 1$, the limit of the sums of the absolute values of the portfolio proportions (i.e. the total outlay) becomes infinite. Heuristically, this means that the larger the securities basket we can invest in, the more we are exposed to the risks in such a market. Now the question is how much we could gain from such an investment. This is quantified by the optimal value function $\hat{u}_N(x)$, in particular its core growth rate $\hat{\Theta}_N$. We will also see under what conditions asymptotic arbitrage can be excluded.

Theorem 13. *Suppose that the value function $\hat{u}_N(x)$ has the growth rate $\hat{\Theta}_N$. By Theorem 2.3 there is no asymptotic arbitrage in a deterministic parameter setting iff*

$$\limsup_{N\to\infty} \hat{u}_N(x) < \infty \quad a.s.$$

For the incomplete one-factor market model, the necessary and sufficient condition therefore is:

$$\lim_{N\to\infty} \frac{1}{2}\left(\sum_{j=1}^{N} \eta_j^2 \xi_j^2 - \frac{1}{K_N} \left(\sum_{j=1}^{N} \eta_j \xi_j^2 \right)^2 \right) < \infty \quad a.s. \tag{4.40}$$

Let us briefly prove this.

Proof. Recall from Theorem 5 in Chapter 3, that the optimal growth rate for this market model is given explicitly as

$$\hat{\Theta}_N(t) = \frac{1}{2} \sum_{j=1}^{N} \xi_j^2 \eta_j^2 - \frac{1}{2K_N} \left(\sum_{j=1}^{N} \xi_j^2 \eta_j \right)^2,$$

and that the value function is given by:

$$\hat{u}_N(x) = \exp\left\{ \int_0^t \left[\frac{\hat{\Theta}_N(s)}{1-p} + r(s) \right] p ds \right\} \frac{x^p}{p}.$$

From the structure of (3.18), we then have using the Cauchy-Schwarz inequality that

$$\left(\sum_{j=1}^{N} \eta_j^2 \xi_j^2 \right)^2 \leq \sum_{j=1}^{N} \eta_j^2 \xi_j^2 (K_N - 1).$$

Then

$$\frac{1}{2K_N} \left(\sum_{j=1}^{N} \eta_j \xi_j^2 \right)^2 \leq \frac{1}{2} \sum_{j=1}^{N} \eta_j^2 \xi_j^2 - \frac{1}{2K_N} \sum_{j=1}^{N} \eta_j^2 \xi_j^2,$$

and thus

$$\frac{1}{2K_N} \left(\sum_{j=1}^{N} \eta_j \xi_j^2 \right)^2 \leq \frac{1}{2} \sum_{j=1}^{N} \eta_j^2 \xi_j^2.$$

Hence for finiteness of the growth rate, and therefore the value function, we need to ensure that the difference

$$\frac{1}{2}\left(\sum_{j=1}^{N}\eta_j^2\xi_j^2-\frac{1}{K_N}\left(\sum_{j=1}^{N}\eta_j\xi_j^2\right)^2\right)<\infty,$$

stays finite as $N\to\infty$, almost surely. This holds as long as the first term is dominated by the second, squared term. □

We have seen what the general condition on no asymptotic arbitrage is. However, we need to expand $\hat{\Theta}_N$ around the homogeneous correlation ratio as well as the Sharpe ratio with respect to the market to gain further insight into the asymptotic approximations.

Let us start with a simplification of the problem. Consider the homogeneous parameter growth rate as in (3.28)

$$\hat{\Theta}_N^h=\frac{\eta^2\xi^2}{2\left(\frac{1}{N}+\xi^2\right)}.$$

Its shape gives us an indication of the size of the leading term we are to expect. In addition, we clearly see that it remains finite in the limit, in particular

$$\lim_{N\to\infty}\hat{\Theta}_N^h=\frac{\eta^2}{2}\quad a.s. \tag{4.41}$$

Now let us turn to the expression $\hat{\Theta}_N$. Decomposing the parameters ξ_j and η_j into the homogeneous and the random perturbation part, we obtain the expansion

$$\begin{aligned}\hat{\Theta}_N=&\frac{1}{2(\frac{1}{N}+\xi^2+2\xi\frac{1}{N}\mathcal{S}_a+\frac{1}{N}\mathcal{S}_{a^2})}\Bigg[\eta^2\xi^2+\xi^4\mathcal{S}_{b^2}+2\xi^3\mathcal{S}_{ab^2}+\xi^2\mathcal{S}_{a^2b^2}+2\xi^3\frac{1}{N}S_{b^2}\mathcal{S}_a\\&+4\xi^4\frac{1}{N}\mathcal{S}_a\mathcal{S}_{ab^2}+2\xi\frac{1}{N}\mathcal{S}_a\mathcal{S}_{a^2b^2}+\xi^2\frac{1}{N}\mathcal{S}_{a^2}\mathcal{S}_{b^2}+2\xi\frac{1}{N}\mathcal{S}_{a^2}\mathcal{S}_{ab^2}+\frac{1}{N}\mathcal{S}_{a^2}\mathcal{S}_{a^2b^2}\\&-4\xi^3\frac{1}{N}\mathcal{S}_bS_{ab}-2\xi^2\frac{1}{N}\mathcal{S}_b\mathcal{S}_{ba^2}-4\xi\frac{1}{N}\mathcal{S}_{ab}\mathcal{S}_{ba^2}-4\xi^2\frac{1}{N}(\mathcal{S}_{ab})^2-\frac{1}{N}(\mathcal{S}_{ba^2})^2\\&+\frac{1}{N}\Big(2\xi\eta^2\mathcal{S}_a+\eta^2\mathcal{S}_{a^2}+2\eta\xi^2\mathcal{S}_b+4\eta\xi\mathcal{S}_{ab}+2\eta\mathcal{S}_{ba^2}+(\xi^2-\xi^4)\mathcal{S}_{b^2}\\&\qquad+2\xi\mathcal{S}_{ab^2}+\mathcal{S}_{a^2b^2}\Big)\Bigg].\end{aligned} \tag{4.42}$$

Note that (4.42) consists of a finite leading term

$$\frac{\eta^2\xi^2}{2(\frac{1}{N}+\xi^2+2\xi\frac{1}{N}\mathcal{S}_a+\frac{1}{N}\mathcal{S}_{a^2})},$$

as well as expressions of order $\mathcal{O}(N)$ and further error terms of order $\mathcal{O}(1)$. This makes it clear that we have to scale particular perturbations to keep $\hat{\Theta}_N$ bounded in the limit.

Suppose, as before, that we scale the b_j with $\frac{1}{N^k}$ for a real constant $k \geq 0$. We want to summarise the asymptotic behaviour for varied choices of k. From here on, we will display the dependence of the growth rate on k by writing $\hat{\Theta}_N(a, b/N^k)$.

Theorem 14. *Let $\hat{\Theta}_N(a, b/N^k)$ be the scaled value function growth, such that the random perturbations b_j to the Sharpe ratio with respect to the market are scaled with $\frac{1}{N^k}$ for a real constant $k \geq 0$. Then almost surely we have the following limiting results:*

$$\lim_{N\to\infty} \hat{\Theta}_N(a, b/N^k) = \begin{cases} \frac{\eta^2}{2} & \text{for } k > \frac{1}{2}, \\ \frac{\eta^2}{2} + \frac{1}{2}\Big(\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Big) - \frac{\left(2\xi\mathbb{E}[ab]+\mathbb{E}[a^2b]\right)^2}{2(\xi^2+\mathbb{E}[a^2])} & \text{for } k = \frac{1}{2}, \\ \infty & \text{for } k < \frac{1}{2}. \end{cases} \tag{4.43}$$

Proof. First, recall that a_j and b_j are random variables drawn from a mean zero distribution and that $\mathbb{E}[(|a| + |b|)^{24}] < \infty$, so that all the required moments exist. Then, we look at the three cases separately.

(i) For $k > \frac{1}{2}$, we have that

$$\begin{aligned}
\hat{\Theta}_N(a, b/N^k) =& \frac{1}{2(\frac{1}{N} + \xi^2 + 2\xi\frac{1}{N}\mathcal{S}_a + \frac{1}{N}\mathcal{S}_{a^2})}\Bigg[\eta^2\xi^2 + \frac{1}{N}\Big(2\xi\eta^2\mathcal{S}_a + \eta^2\mathcal{S}_{a^2}\Big) \\
&+ \frac{1}{N^{2k}}\Big(\xi^4\mathcal{S}_{b^2} + 2\xi^3\mathcal{S}_{ab^2} + \xi^2\mathcal{S}_{a^2b^2}\Big) \\
&+ \frac{1}{N^{1+2k}}\Bigg(2\xi^3 S_{b^2}\mathcal{S}_a + 4\xi^4\mathcal{S}_a\mathcal{S}_{ab^2} + 2\xi\mathcal{S}_a\mathcal{S}_{a^2b^2} + \xi^2\mathcal{S}_{a^2}\mathcal{S}_{b^2} \\
&\qquad + 2\xi\mathcal{S}_{a^2}\mathcal{S}_{ab^2} + \mathcal{S}_{a^2}\mathcal{S}_{a^2b^2} - 4\xi^3\mathcal{S}_b S_{ab} - 2\xi^2\mathcal{S}_b\mathcal{S}_{ba^2} \\
&\qquad - 4\xi\mathcal{S}_{ab}\mathcal{S}_{ba^2} - 4\xi^2(\mathcal{S}_{ab})^2 - (\mathcal{S}_{ba^2})^2\Bigg) \\
&+ \frac{1}{N^{1+k}}\Bigg(2\eta\xi^2\mathcal{S}_b + 4\eta\xi\mathcal{S}_{ab} + 2\eta\mathcal{S}_{ba^2}\Bigg) \\
&+ \frac{1}{N^{1+2k}}\Bigg(\xi^2\mathcal{S}_{b^2} + 2\xi\mathcal{S}_{ab^2} + \mathcal{S}_{a^2b^2} - \xi^4\mathcal{S}_{b^2}\Bigg)\Bigg] \\
\sim& \frac{1}{2(\frac{1}{N} + \xi^2 + \mathbb{E}[a^2])}\Bigg[\eta^2(\xi^2 + \mathbb{E}[a^2]) \\
&+ \frac{1}{N^{2k-1}}\Bigg((\xi^2 + \mathbb{E}[a^2])[\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] \\
&\qquad + \mathbb{E}[a^2b^2]] - (2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b])^2\Bigg) \\
&+ \frac{1}{N^k}\Bigg(4\eta\xi\mathbb{E}[ab] + 2\eta\mathbb{E}[ba^2]\Bigg) \\
&+ \frac{1}{N^{2k}}\Bigg((\xi^2 - \xi^4)\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Bigg)\Bigg],
\end{aligned}$$

where '$\sim$' is a notation for the approximation made to the growth rate by omitting terms of mean zero and terms smaller than $\frac{1}{N}$. Then we have the limit

$$\lim_{N\to\infty} \hat{\Theta}_N(a, b/N^k) = \frac{\eta^2}{2} \quad a.s.,$$

as the highest order is $\mathcal{O}(\frac{1}{N^{2k-1}})$ using the Strong Law of Large Numbers.

(ii) For $k = \frac{1}{2}$, we have finiteness in the limit for $N \to \infty$ and obtain, using the expansion above, the order $\mathcal{O}(1)$ expression

$$\begin{aligned}
\lim_{N\to\infty} \hat{\Theta}_N(a, b/N^k) =& \frac{\eta^2}{2} + \frac{1}{2}\Big(\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Big) \\
&- \frac{\Big(2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b]\Big)^2}{2(\xi^2 + \mathbb{E}[a^2])} \\
&< \infty \quad a.s.,
\end{aligned}$$

using the Strong Law of Large Numbers. In particular we have that all three terms of the limit are positive, because the second term $\frac{1}{2}\Big(\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Big)$ can be rewritten as $\frac{1}{2}\mathbb{E}[(\xi + a)^2b^2]$.

(iii) Finally, for $k < \frac{1}{2}$:

$$\begin{aligned}\hat{\Theta}_N(a, b/N^k) \sim &\frac{\eta^2(\xi^2 + \mathbb{E}[a^2])}{2(\frac{1}{N} + \xi^2 + \mathbb{E}[a^2])} \\ &+ \frac{1}{2(\frac{1}{N} + \xi^2 + \mathbb{E}[a^2])}\Bigg[N^{2(1-k)}\Bigg(\xi^2 + \mathbb{E}[a^2])[\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] \\ &\qquad + \mathbb{E}[a^2b^2] - (2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b])^2\Bigg) \\ &\quad + \frac{1}{N^{1/2+k}}\Bigg(4\eta\xi\mathbb{E}[ab] + 2\eta\mathbb{E}[ba^2]\Bigg) \\ &\quad + \frac{1}{N^{2k-1}}\Bigg((\xi^2 - \xi^4)\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Bigg)\Bigg] \quad a.s.,\end{aligned}$$

which approaches infinity for $N \to \infty$ and the highest order is $\mathcal{O}(N^{2(1-k)})$.

□

A couple of brief comments. If we impose strong scaling, with $k > \frac{1}{2}$, then the growth rate converges to the homogeneous rate $\hat{\Theta}^h_N$. On the other hand, if we relax our control on the random perturbations slightly, the growth that can be generated is higher than the homogeneous parameters expression by the positive amount

$$\frac{1}{2}\mathbb{E}\big[(\xi + a)^2b^2\big] + \frac{\Big(2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b]\Big)^2}{2(\xi^2 + \mathbb{E}[a^2])}.$$

We have now seen that the scaling parameters used to keep the growth rate finite, differ significantly to the one used for the portfolio proportions or on the sum of their absolute values. In particular we see three general regions. For $k \geq 1$, both the total outlay as well as the growth that can be achieved stay finite. However, as we actually take the exponential of the growth rate in the value function, the actual gain is significantly higher.

On the other hand, if we would choose $k < \frac{1}{2}$, both the total position and the value function are infinite in the limit, technically allowing the possibility of asymptotic arbitrage but at a possibly infinitely high risk.

The most interesting regime is when we choose $\frac{1}{2} \leq k < 1$. Here, the total investment is still infinite, however the growth we can achieve is bounded. Heuristically speaking, this means

that the total position, taking both short and long positions in the stocks, is infinite. This means that the investment is prone to infinite risk. But the gain is, although exponential, at most a finite amount. Economically, this makes little sense and gives rise to further questions. In particular, as a next step, it would be interesting to see if it would be possible to construct a policy from the optimal one, which is simpler and has less exposure but still outperforms the homogeneous investment. On the other hand, one could try and constrain the admissibility set of the portfolio proportions by prohibiting for example short-selling. Both ideas, however, are outside the scope of this thesis.

Remark 4.2. *Taking all the results into account, it is evident that the optimal growth rate of the value function outperforms its equivalent for homogeneous modelling parameters, unless we scale b_j with $\frac{1}{N^k}$ for a $k > \frac{1}{2}$ when it is equal in the limit, a.s. We will illustrate this difference in the numerical model in Chapter 5.*

Finally let us derive the explicit Hellinger process for our incomplete market model.

Corollary 4.3. *The general parameter Hellinger process for the incomplete market given by the system of dynamics (3.15) is given by*

$$h_N(t) = \frac{1}{8}\int_0^t \frac{1}{K_N}\left[K_N \sum_{j=1}^N \eta_j(s)^2\xi_j^2(s) - \left(\sum_{j=1}^N \eta_j(s)\xi_j^2(s)\right)^2\right]ds. \tag{4.44}$$

Therefore, the Kabanov and Kramkov criterion for no asymptotic arbitrage,

$$\begin{aligned}\lim_{N\to\infty} h_N(T) &= \lim_{N\to\infty}\frac{1}{8}\int_0^T \frac{1}{K_N}\left[K_N \sum_{j=1}^N \eta_j^2(t)\xi_j^2(t) - \left(\sum_{j=1}^N \eta_j(t)\xi_j^2(t)\right)^2\right]dt \\ &< \infty,\end{aligned}$$

holds in particular if

$$\limsup_{N\to\infty}\left[\sum_{j=1}^N \eta_j^2\xi_j^2 - \frac{1}{K_N}\left(\sum_{j=1}^N \eta_j\xi_j^2\right)^2\right] < \infty \quad a.s., \tag{4.45}$$

for $0 \le t \le T$ and $j = 1, \ldots, N$.

We will briefly show how to derive this particular form of the Hellinger process, as the second part follows along the lines of the proof of Theorem 13.

Proof. Recall from Corollary 1.15 in Chapter 1, that the Hellinger process can be derived by calculating

$$8h_N(t) = \int_0^t ||\boldsymbol{\theta}_N(s)||^2 ds.$$

Then for the relative risk price process $\boldsymbol{\theta}_N(s)$, we can explicitly calculate the change of measure matrix

$$\boldsymbol{\theta}_N^T\boldsymbol{\theta}_N = \begin{pmatrix}(\alpha_1 - r) & \cdots & (\alpha_N - r)\end{pmatrix}\Sigma^{-1}\begin{pmatrix}(\alpha_1 - r)\\ \vdots \\ (\alpha_N - r)\end{pmatrix},$$

where $\boldsymbol{\theta}_N^T$ is the transpose of the N dimensional column vector $\boldsymbol{\theta}_N$ and

$$\Sigma = \begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_1\rho_2 & \cdots & \cdots & \sigma_1\sigma_N\rho_1\rho_N\\ \sigma_1\sigma_2\rho_1\rho_2 & \sigma_2^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_1\sigma_N\rho_1\rho_N & \cdots & \cdots & \cdots & \sigma_N^2\end{pmatrix}, \tag{4.46}$$

is the covariance matrix for our asset prices (1.1).

Then we can find the inverse Σ^{-1}, which is a direct application of inversion method I in Section 1.5. The result is displayed in equation (3.24) in Chapter 3.

Therefore

$$\boldsymbol{\theta}_N^T\boldsymbol{\theta}_N = \sum_{j=1}^{N}\left[\frac{1}{(1-\rho_j^2)}\frac{1}{1+\sum_{i=1}^{N}\xi_j^2}\right.$$
$$\left.\times\left(\frac{(\alpha_j - r)^2\big(1+\sum_{k\neq j}\xi_k^2\big)}{\sigma_j^2} - \sum_{k\neq j}\frac{\rho_j\rho_k(\alpha_k - r)(\alpha_j - r)}{(1-\rho_k^2)\sigma_j\sigma_k}\right)\right],$$

and we obtain

$$h_N(t) = \frac{1}{8}\int_0^t \frac{1}{K_N}\left[\sum_{j=1}^{N}\eta_j(s)\left(\eta_j(s)\xi_j^2(s) + \sum_{k=1}^{N}\xi_j^2(s)\xi_k^2(s)(\eta_j(s) - \eta_k(s))\right)\right]ds.$$

Some rearranging yields the required expression. □

It is evident, when comparing $h_N(t)$ to $\hat{\Theta}_N(t)$ that $h_N(t) = \frac{1}{4}\int_0^t \hat{\Theta}_N(s)ds$ and therefore has to be finite under the same scaling conditions as introduced in the previous section.

4.3 Optimal Wealth Solution for Incomplete Markets

In this section, we aim to understand the behaviour of the optimal controlled wealth process $\hat{X}_N(t)$ for our incomplete asset system. In addition to knowing what the agent's optimal wealth is at any time $0 \leq t \leq T$, this can also give us some idea on how the optimal strategy performs in comparison to equally weighting our stocks using the homogeneous parameter solution. We will see that for scaling the random variables b_j with N^{-k}, for $k > \frac{1}{2}$ perturbing the correlated Sharpe ratio, the optimal wealth behaves as the homogeneous parameter wealth in the limit. On the other hand, if we relax the scaling slightly and assuming $k = \frac{1}{2}$, the limiting wealth process obtained if the agent behaves optimally, is actually a random process driven by the Brownian motion of the global risk factor as well as a time changed Brownian motion arising from the limit of the idiosyncratic risk.

Recall the solution of the general wealth process (1.10) is (1.12). Assuming that $J = 1$, i.e., we only have one global risk factor, and under the integrability conditions

$$\int_0^T \left| \sum_{j=1}^N (\alpha_j - r)\pi_j^*(t) \right| dt < \infty,$$
$$\int_0^T \left| \sum_{j=1}^N \sigma_j \rho_j \pi_j^* \right|^2 dt < \infty,$$
$$\int_0^T \left| \sum_{j=1}^N \sigma_j \sqrt{1-\rho_j^2}\pi_j^* \right|^2 dt < \infty,$$

we obtain the solution to the optimal wealth process $\hat{X}_N(t)$ by inserting the optimal portfolio proportions $\hat{\pi}_{j,N}$, $j = 1, \ldots, N$, into (1.12) as

$$\begin{aligned} \hat{X}_N(t) = x \exp \Bigg\{ & \int_0^t \Bigg[\sum_{j=1}^N (\alpha_j - r)\hat{\pi}_j - \frac{1}{2}\sum_{j=1}^N \sigma_j^2 (\hat{\pi}_j)^2 \\ & - \frac{1}{2}\sum_{j=1}^N \sum_{i \neq j} \sigma_j \sigma_i \rho_j \rho_i \hat{\pi}_j \hat{\pi}_i \Bigg] ds \\ & + \int_0^t \sum_{j=1}^N \sigma_j \sqrt{1-\rho_j^2}\hat{\pi}_j dW_j(s) + \int_0^t \sum_{j=1}^N \sigma_j \rho_j \hat{\pi}_j dM(s) \Bigg\}. \end{aligned} \tag{4.47}$$

Here, $\{(M(t), W_1(t), \ldots, W_N(t)); 0 \leq t \leq T\}$ is the vector with $N+1$ one-dimensional, independent Brownian motions.

Using the particular $\hat{\pi}_{j,N}$, $j = 1, \ldots, N$ we can rewrite (4.47) as

$$\begin{aligned}\hat{X}_N(t) =& x \exp\left\{ \int_0^t \left[\frac{1}{(1-p) - 2(1-p)^2} \left(\sum_{j=1}^N \eta_j^2 \xi_j^2 - \frac{1}{K_N} \left(\sum_{j=1}^N \eta_j \xi_j^2 \right)^2 \right) \right] ds \right. \\ &+ \int_0^t \frac{1}{K_N(1-p)} \Big(K_N + \sum_{k=1}^N \xi_k \Big) \sum_{j=1}^N \eta_j \xi_j^2 dW_j(s) \\ &\left. + \int_0^t \frac{1}{K_N(1-p)} \sum_{j=1}^N \eta_j \xi_j^2 dM(s) \right\}. \end{aligned} \tag{4.48}$$

Performing the same type of asymptotic expansion as in Chapter 4, we obtain the following the results.

Theorem 15. *Let $\hat{X}_N(t)$ in described by the process (4.47) be the optimal wealth process for an incomplete market with N securities. If we scale the set of random variables b_j with N^{-k} for some real non-negative k, we obtain three different limits for $N \to \infty$ depending on the choice of:*

(i) *For $k > \frac{1}{2}$*

$$\lim_{N\to\infty} \hat{X}_N(t) = x \exp\left\{ \int_0^t \frac{\eta^2}{(1-p) - 2(1-p)^2} ds + \int_0^t \frac{\eta}{(1-p)} dM(s) \right\} \quad a.s. \tag{4.49}$$

(ii) *For $k = \frac{1}{2}$*

$$\begin{aligned}\lim_{N\to\infty} \hat{X}_N(t) =& x \exp\left\{ \int_0^t \frac{\eta^2}{(1-p) - 2(1-p)^2} \right. \\ &+ \frac{1}{\Big((1-p) - 2(1-p)^2\Big)\Big(\xi^2 + \mathbb{E}[a^2]\Big)} \Big(\xi^4 \mathbb{E}[b^2] \\ &\qquad + 2\xi^3 \mathbb{E}[ab^2] + \xi^2 \mathbb{E}[a^2 b^2] \\ &\qquad + \xi^2 \mathbb{E}[a^2]\mathbb{E}[b^2] + 2\xi \mathbb{E}[a^2]\mathbb{E}[ab^2] \\ &\qquad\quad \left. + \mathbb{E}[a^2]\mathbb{E}[a^2 b^2] - (2\xi \mathbb{E}[ab] + \mathbb{E}[a^2 b])^2 \Big) ds \right. \\ &+ \int_0^t \frac{\eta}{(1-p)} dM(s) \\ &+ \int_0^t \Big(\xi^2 \sqrt{\mathbb{E}[a^2 b^2]} - \xi \mathbb{E}[a^2] \sqrt{\mathbb{E}[a^2 b^2]} + \mathbb{E}[ab] \sqrt{\mathbb{E}[a^4]} \Big) dW(s) \quad a.s., \end{aligned} \tag{4.50}$$

under the assumption that the required moments exist.

(iii) *Finally for* $k < \frac{1}{2}$

$$\lim_{N\to\infty} \hat{X}_N(t) = \infty \quad a.s., \tag{4.51}$$

Proof. The proof for (i) and (iii) can be found using the same methodology and ideas as in Sections 4.1 and 4.2 and is omitted. However, we need to take a more careful look at (ii). In particular in the expansion we have three terms $\int_0^t \xi \frac{1}{N} \sum_{i=1}^N a_i^2 \frac{1}{\sqrt{N}} \sum_{j=1}^N b_j dW_j$, $\int_0^t \frac{1}{N} \sum_{i=1}^N a_i b_i \frac{1}{\sqrt{N}} \sum_{j=1}^N a_j^2 dW_j$ and $\int_0^t \xi^2 \frac{1}{\sqrt{N}} \sum_{j=1}^N a_j b_j dW_j$, which require our attention. Because they can be treated in a similar fashion, let us look at the last term as an example and adopt the method to the other two terms. We recognise that we are dealing with the martingale $\tilde{M}_{ab}^{(N)}(t) = \int_0^t \xi^2 \frac{1}{\sqrt{N}} \sum_{j=1}^N a_j b_j dW_j$. Looking at the quadratic variation, we have

$$\left\langle \tilde{M}_{ab}^{(N)} \right\rangle_t = \int_0^t \xi^4 \frac{1}{N} \sum_{j=1}^N a_j^2 b_j^2 ds,$$

and therefore by our moment assumption

$$\lim_{N\to\infty} \left\langle \tilde{M}_{ab}^{(N)} \right\rangle_t = \int_0^t \xi^4 \mathbb{E}[a^2 b^2] ds.$$

Hence we get:

$$\lim_{N\to\infty} \tilde{M}_{ab}^{(N)}(t) = \int_0^t \xi^2 \sqrt{\mathbb{E}[a^2 b^2]} dW(s).$$

Using this result, we obtain the other two terms above. We can swap the integration and limits using the dominated convergence theorem, as we have the approximation $\left|\xi^4 \frac{1}{N} \sum_{j=1}^N a_j^2 b_j^2\right| \leq \xi^4 \mathbb{E}[a^2 b^2]$ for $j = 1, \ldots, N$ as $\xi^4 \frac{1}{N} \sum_{j=1}^N a_j^2 b_j^2 \to \xi^4 \mathbb{E}[a^2 b^2]$ for $N \to \infty$ by the Strong Law of Large Numbers. □

On the other hand, for the purpose of comparison, let us look at the wealth produced by the homogeneous investment strategy. Assume in (4.47) that the parameters are homogeneous over the whole basket and use the optimal portfolio proportion

$$\hat{\pi}^h = \frac{1}{N} \left(\frac{\eta}{(1-p)(1-\rho^2)(\frac{1}{N} + \xi^2)\sigma} \right),$$

then the result of such an equally weighted investment in the limit is

$$\lim_{N\to\infty} \hat{X}_N^h(t) = x \exp\left\{ \int_0^t \frac{\eta^2}{(1-p) - 2(1-p)^2} ds + \int_0^t \frac{\eta}{(1-p)} dM(s) \right\} \quad a.s. \tag{4.52}$$

Comparing the limiting wealth (4.52) to the results obtained for the more sophisticated optimal portfolio proportion in Theorem 15 directly above it, we observe that for strong scaling, i.e. $k > \frac{1}{2}$, they coincide. Therefore, all the securities in our asset universe converge quickly

towards the same Sharpe ratio with respect to the market and can essentially be treated as a homogeneous asset basket. Hence there is no outperformance in the limit as $N \to \infty$.

On the other hand, if we relax the scaling, we can see directly that the optimal investment achieves superior wealth to the equally weighted strategy.

4.4 Comparison Between Complete Market and Incomplete Market Growth Rate

So far, we have considered both, the complete and incomplete market models separately. In this section, we aim to provide a comparison of their value function growth rates. In particular, we know that for N stocks, Θ_N^* differs significantly from $\hat{\Theta}_N$, however the question we consider is, if scaled, do they converge to the same limit? Considering $\Theta_N^*(a, b/N^k)$ and $\hat{\Theta}_N(a, b/N^k)$, we want to understand if the influence of the global asset persists into the limit. Therefore we will see if the agent can achieve a higher gain by investing in both, the single stocks and the market asset simultaneously, in comparison to just allocating his wealth into the single securities only. This heuristically answers the question whether the incomplete market model can be viewed as 'asymptotically complete'. Here we will give some evidence comparing the scaled value function growth rates calculated in Sections 4.1 and 4.2, for various scaling parameters and then give a brief comment on the equivalent martingale measures and their limits.

Suppose that we want to look at the complete market growth rate Θ_N^* and the incomplete market growth rate $\hat{\Theta}_N$. We have already found that scaling the random variables b_j for $j = 1, \ldots, N$, stemming from the Sharpe ratios with respect to the market, in both models, with N^{-k} for some real, positive k, yields a variety of limiting results depending on k. We only want to look at the cases where $k = \frac{1}{2}$ and $k > \frac{1}{2}$, as we have already shown that in both markets, choosing $k < \frac{1}{2}$ cannot control the parameters sufficiently and the growth rates are infinite in the limit as $N \to \infty$, almost surely.

Theorem 16. *Let $\Theta_N^*(a, b/N^k)$ and $\hat{\Theta}_N(a, b/N^k)$ be as described in Sections 4.1 and 4.2. Suppose we have a real positive constant k, then the difference in the growth rates has the following behaviour:*

(i) *For $k > \frac{1}{2}$, we have*

$$\lim_{N\to\infty} \Big(\Theta_N^*(a, b/N^k) - \hat{\Theta}_N(a, b/N^k)\Big) = 0 \quad a.s.$$

(ii) *Then for $k = \frac{1}{2}$, we obtain*

$$\lim_{N\to\infty} \Big(\Theta_N^*(a, b/N^k) - \hat{\Theta}_N(a, b/N^k)\Big) = \frac{\Big(2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b]\Big)^2}{2(\xi^2 + \mathbb{E}[a^2])} > 0 \quad a.s.,$$

Proof. We will briefly mention the cornerstones of the proof, as the majority of results have already been shown and the techniques, as well as the notation, have been thoroughly explained in Section 4.1 and 4.2.

For $k > \frac{1}{2}$ we have that

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \frac{\eta^2}{2} \quad a.s.,$$

as well as

$$\lim_{N\to\infty} \hat{\Theta}_N(a, b/N^k) = \frac{\eta^2}{2} \quad a.s.,$$

Therefore, the difference goes to zero in the limit, almost surely.

On the other hand, for $k = \frac{1}{2}$, the limiting results are

$$\lim_{N\to\infty} \Theta_N^*(a, b/N^k) = \frac{\eta^2}{2} + \frac{1}{2}\left(\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\right) \quad a.s.,$$

as well as

$$\begin{aligned}\lim_{N\to\infty} \hat{\Theta}_N(a, b/N^k) =& \frac{\eta^2}{2} + \frac{1}{2}\Big(\xi^2\mathbb{E}[b^2] + 2\xi\mathbb{E}[ab^2] + \mathbb{E}[a^2b^2]\Big) \\ &- \frac{\Big(2\xi\mathbb{E}[ab] + \mathbb{E}[a^2b]\Big)^2}{2(\xi^2 + \mathbb{E}[a^2])} \quad a.s.\end{aligned}$$

Subtracting the one from the other gives us the result, which is positive for any random variable a_j and b_j, as the denominator is a square and the numerator involves the variance of the a_j which is always positive.

□

We can draw the following conclusions from the result. Scaling the random variable associated with the Sharpe ratio with respect to the market, means that we let the asset universe contract with a growing number of stocks in the market. This results in a case where the risk-return profiles of the single stocks converge to the homogeneous parameters almost surely, once we let $N \to \infty$. Now, if our choice of scaling is strong enough, in particular for $k > \frac{1}{2}$, it is converging fast, with a rate of N^{-1} and both growth rates $\Theta_N^*(a, b/N^k)$ and $\hat{\Theta}_N(a, b/N^k)$ approach the same limit, namely the homogeneous parameter growth rate $\frac{\eta^2}{2}$. In such a case, we can heuristically say that the incomplete market is asymptotically complete. However, if we choose a lower level of control, i.e. $k = \frac{1}{2}$, as described in the previous Sections 4.2 and 4.3, it is still sufficient to ensure finiteness and the absence of asymptotic arbitrage. We have already seen that the markets do not contract fast enough in this case and there is a difference in growth between the complete and the incomplete markets in the limit. In particular, we can see that it is better to invest in both the market asset and the single stocks to achieve a higher growth rate than by just investing in a portfolio of single securities. This is exactly the result we have expected, since in the limit both models are close to the homogeneous parameter case and the complete market model offers the agent one more asset to invest in. This should always yield a higher overall return, because the value function is monotone increasing with the number of assets N.

The above result is only obtained through heuristic intuition and some brief calculations. In the future it would be interesting to further explore the concept of asymptotic completeness at a more rigorous level.

One difference between the complete and incomplete market worth mentioning is the following in the complete case we have exactly one equivalent martingale measure, whereas in the incomplete the set of equivalent martingale measures can be of infinite cardinality.

Consider the complete one-factor market model on our probability space $\mathcal{B}_n = \big(\Omega_n, \mathfrak{F}_n, \mathbb{F}_n = (\mathfrak{F}_n(t)), \mathbb{P}_n\big)$, such that we have the increasing sequence $\mathcal{B}_\infty = \big(\mathcal{B}_n\big)_{n=1}^{\infty}$. As already described in Chapter 1, Section 1.4, we have for each $n \in \mathbb{N}$ exactly one equivalent martingale measure $\mathbb{Q}_n^* \sim \mathbb{P}_n$. If we have the limiting exponential martingale representing the density

$$Z_\infty = \frac{d\mathbb{Q}_\infty^*}{d\mathbb{P}_\infty},$$

with $0 < Z_\infty < \infty$ as in Section 1.4, then we even find a unique equivalent martingale measure $\mathbb{Q}_\infty^* \sim \mathbb{P}_\infty$ for the complete market case and therefore the limiting market $\mathcal{M}$ is also complete.

On the other hand, if our market is incomplete, then for each $n \in \mathbb{N}$ we have an infinite number of martingale measures $\hat{\mathbb{Q}}_n$, such that each is equivalent to $\mathbb{P}_n$. We have already seen in Chapter 2 that the only way to deal with such a market, is to choose a particular sequence of equivalent martingale measures under certain constraints and to obtain the required results. This is used both for optimisation using duality and martingales and also when considering the existence of asymptotic arbitrage for incomplete markets in a general setting. The question, still unanswered at the moment, is the following. Is it possible to choose a particular tight subsequence $\hat{\mathbb{Q}}_{n_k}$, as a sequence of incomplete markets such that under certain compactness conditions, it would converge to the same limiting measure $\mathbb{Q}^*_\infty$ as in the complete case and therefore make the limiting market asymptotically complete? This is still open to discussion. However, the fact that it is possible to obtain one and the same limiting optimal utility function growth a.s. for both markets, under strong scaling of the perturbation parameters, is a sign that there might be such a possibility.

Chapter 5

Numerical modelling

In the previous chapters we have looked at the structure as well as the behaviour of the optimal investment strategies of an agent and the value he can generate by allocating his funds according to these strategies, considering a one-factor complete and one-factor incomplete market model.

In this chapter, we want to illustrate those results graphically by generating the parameters using the assumptions from Chapter 4 and looking at both the portfolio proportions as well as the growth rates with or without scaling.

We are also concerned with a more realistic trading example. In Section 5.2 we will see how the theoretically obtained optimal strategies would perform in real markets, when trading in the chosen assets occurs daily.

As a consequence, we see that investing according to our explicit optimal portfolio proportions does not bankrupt the agent, but it does fail to deliver a higher growth rate than less involved strategies, which are only utilising portfolio proportions consisting of homogeneous coefficients.

In particular, we observe different outcomes from the real data trading depending on the level of the risk-free rate $r > 0$. For a very small risk-free rate, the value either remains close to the initial endowment or even declines. On the other hand, if the risk-free rate is high, we experience exponential growth. We will attempt to give intuitive reasons for this behaviour.

We want to emphasise that this entire chapter should be viewed mostly as an illustration, giving the reader an intuitive understanding of the rigorous findings preceding it.

5.1 Generated Parameters

Using fixed parameters and randomly generated perturbations, we want to illustrate the complete and incomplete portfolio proportions as well as the value function growth calculated in Chapter 3 and compare these results to the expanded and scaled quantities for a large number of assets from Chapter 4.

We observe that if we do not scale the perturbations, there is an infinite growth rate, whereas strong scaling produces a growth rate that quickly converges to the homogeneous parameter rate.

The model is set up as follows: For simplicity, we want to look at only one timestep and 100 as well as 100000 securities, which will be represented on the x-axis, unless otherwise stated.

Suppose that the underlying homogeneous parameters used are fixed and derived from commonly used market assumptions. Therefore, assume throughout this section that for all assets $S_j(t)$, $j = 1, \ldots, N$, the rate of return is $\alpha = 0.04$, the standard deviation $\sigma = 0.15$, the risk free rate $r = 0.002$ and the correlation with the market asset $\rho = 0.1$. We can then compute the deterministic correlation ratio $\xi = \frac{\rho}{\sqrt{1-\rho^2}} = 0.1054$ and the Sharpe ratio with respect to the market $\eta = \frac{\alpha - r}{\sigma\rho} = 2.53$.

Expanding the portfolio proportions and the value function growth rate asymptotically in Chapter 4, we made the assumption that the actual parameters, varying with each asset, are of the form:

$$\xi_j = \xi + a_j,$$

$$\eta_j = \eta + b_j.$$

Previously a_j and b_j were random variables on the probability space $(\Omega, \mathfrak{B}, \mathfrak{P})$, drawn from a mean zero distribution. However, as we want to have some easy to model values, we will take a_j and b_j from a Uniform distribution on bounded intervals centred around zero, such that we have the desired property of $\mathbb{E}[a_j] = 0$ and $\mathbb{E}[b_j] = 0$.

Recall that we restricted the explicit results to the power utility function $U(x) = \frac{x^p}{p}$, $p \neq 0$, $p < 1$ and the penalty factor is taken as $p = -0.5$.

5.1.1 Portfolio Proportions

Consider the results from Chapter 3 and Chapter 4. We start by illustrating the size of the portfolio proportions for a complete one-factor market. First, we want to determine the sizes of

$$\pi_j^* = \frac{\xi_j^2}{(1-p)\sigma_j\rho_j}\,(\eta_j - \lambda_0)$$

and

$$\tilde{\pi}^* = \frac{1}{(1-p)\gamma}\left(\lambda_0 - \sum_{k=1}^{N}\xi_k^2(\eta_k - \lambda_0)\right)$$

for a basket of 100 stocks and $100,000$ stocks, each time comparing the optimal to the homogeneous proportions. Note that we do take the assumption of $\lambda_0 \to \eta$ as $N \to \infty$ into account.

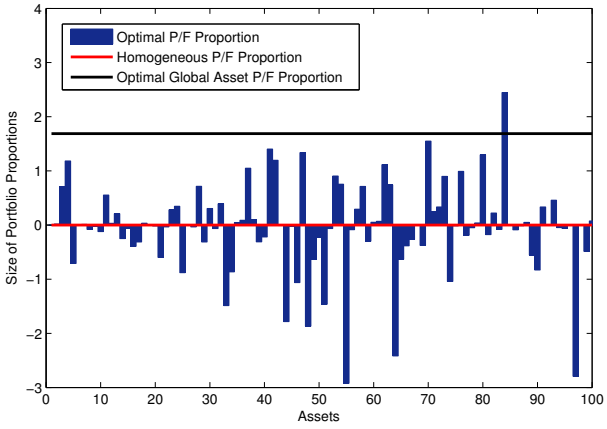


Figure 5.1: Optimal vs. homogeneous portfolio proportions in a complete market for 100 securities.

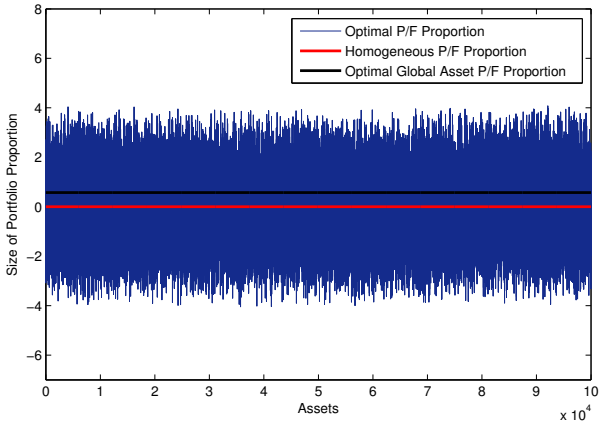


Figure 5.2: Optimal vs. homogeneous portfolio proportions in a complete market for 100,000 securities.

We can compare this to the size of the a_j and b_j used in this example. They are only shown for 100 stocks for clarity, as there is no substantial difference if we would model them for more assets.

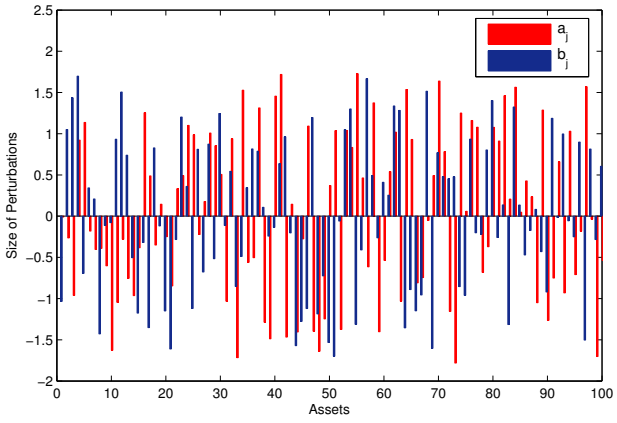


Figure 5.3: Size of the modelled a_j and b_j for 100 securities.

To summarise, it is visible that even for 100 stocks, the homogeneous rate, which is calculated to be

$$\pi_j^{*,h} = \frac{\xi^2}{(1-p)\sigma\rho}(\eta - \lambda_0)$$

is close to zero as it is of order $\mathcal{O}(\frac{1}{N})$ and the proportion to be invested in the market asset is slightly above that value. However, the proportion π_j^* to be invested in each asset S_j exhibits rapid changes and the more assets we have, the higher are these peaks. This coincides with our findings that without scaling, the weights in each stock can grow with initial basket size.

On the other hand, if we scale the b_j in the expanded proportions with $\frac{1}{N^k}$, first assuming $k = 1$ and then that $k = \frac{1}{2}$, we observe that $\pi_j^*(a, b/N^k)$, given in Chapter 4, groups closer together and each proportion is smaller in magnitude. Therefore we can see the results for different strength of control over the parameters. We omit $\tilde{\pi}^*(a, b/N^k)$ as it is constant and would just move closer to zero when we scale the parameters.

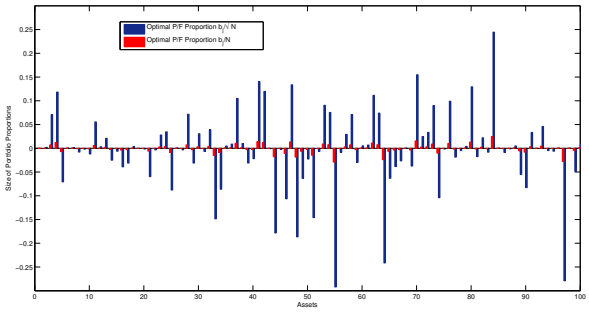


Figure 5.4: Scaled optimal vs. homogeneous portfolio proportions in a complete market for 100 securities.

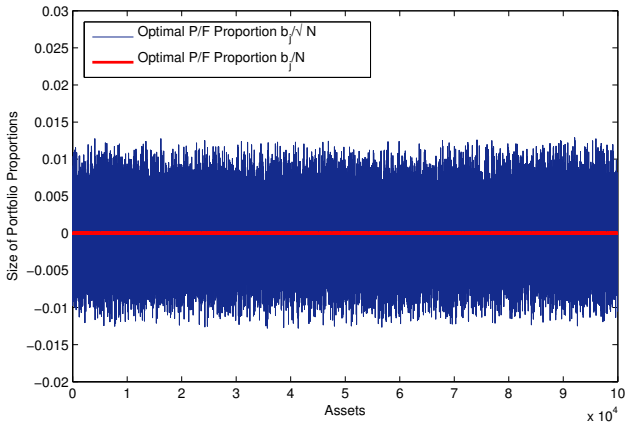


Figure 5.5: Scaled optimal vs. homogeneous portfolio proportions in a complete market for 100000 securities.

Compare this to the distribution of the b_j scaled with both $\frac{1}{N}$ and $\frac{1}{\sqrt{N}}$ generated for the purpose of this example. Again we consider only the perturbation sizes for 100 assets, however it is evident that they would just become even smaller for a higher number of stocks.

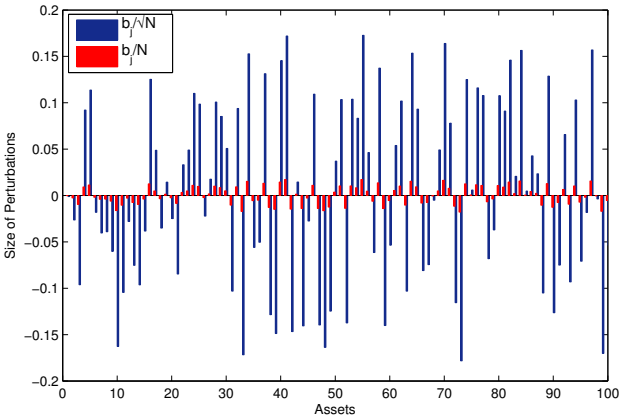


Figure 5.6: b_j scaled both with $\frac{1}{N}$ and $\frac{1}{\sqrt{N}}$ for 100 securities.

In particular, observe that the stronger the scaling and the larger our portfolio, the closer we get to the homogeneous parameter proportion. For $100,000$ assets and $k = 1$, the result is visually indistinguishable from the homogeneous rate.

Now take a look at the incomplete model for one market risk factor, as described in Section 3.2. We want to illustrate how this behaves in comparison to the homogeneous parameter proportion, as well as its expanded and scaled perturbation analogue.

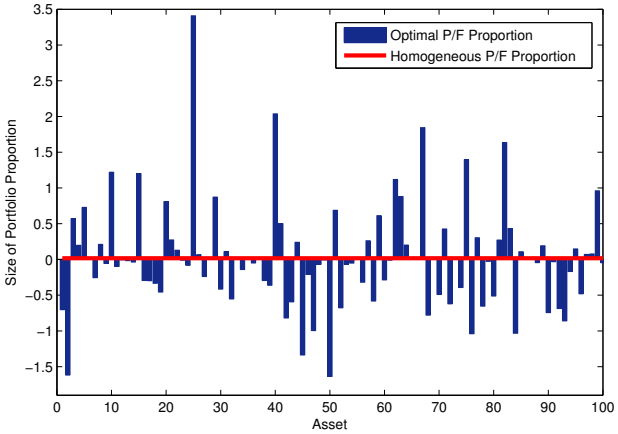


Figure 5.7: Optimal vs. homogeneous portfolio proportions in an incomplete market for 100 securities.

For further information we can refer to the distribution of the a_j and b_j displayed above.

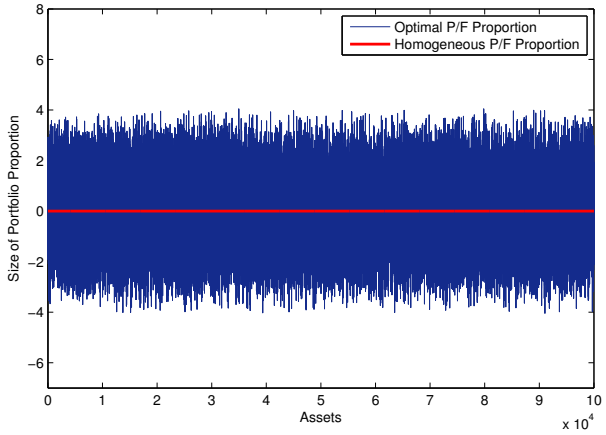


Figure 5.8: Optimal vs. homogeneous portfolio proportions in an incomplete market for 100,000 securities.

We can see that the optimal portfolio proportions vary around the homogeneous proportion. These spikes become more pronounced, the higher the number of assets is. The homogeneous proportion, on the other hand, is very small but positive being a constant comprised of homogeneous parameters over the number of securities.

Now, expanding $\sigma_j \rho_j \tilde{\pi}^*(a, b/N^k)$ as in Chapter 4 and scaling b_j with $\frac{1}{N^k}$, we will compare the optimal strategy to the results for $k = 1$ and $k = 1/2$.

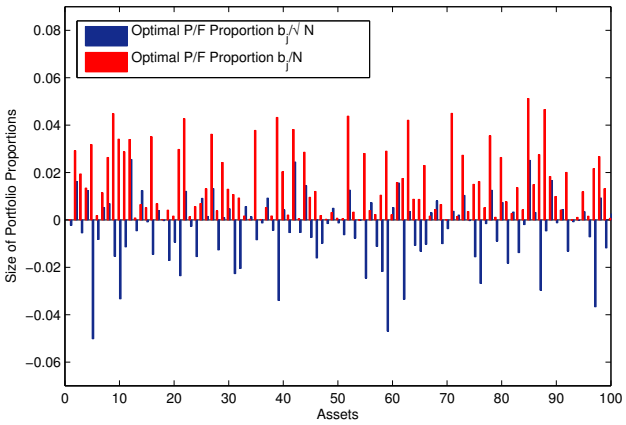


Figure 5.9: Scaled optimal vs. homogeneous portfolio proportions in an incomplete market for 100 securities.

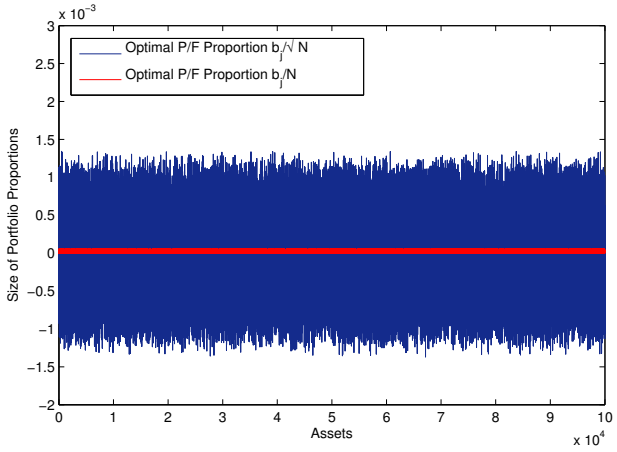


Figure 5.10: Scaled optimal vs. homogeneous portfolio proportions in an incomplete market for 100,000 securities.

Scaling has the effect of diminishing the portfolio proportion in each asset. However, when there are only 100 stocks we can see almost no change in the outcomes for the different scaling exponents k. On the other hand, for a significantly larger number of assets, stronger scaling keeps the proportion close to the level of the homogeneous coefficients. This effect results from the fact that the stronger the scaling and the more stocks we can have to choose from, the closer together their parameters are. In a sense, our universe becomes 'tighter' and the proportion of each single stock does not vary significantly from the other.

5.1.2 Growth Rate

We now want to briefly outline the modelling results for the value function growth rate, starting with the complete case. We saw the explicit results in Chapter 3. Now, without scaling, we can see how the optimal growth rate given by:

$$\Theta_N^*(t) = \frac{1}{2}\left(\lambda_0^2 + \sum_{j=1}^{N} \xi_j^2(\eta_j - \lambda_0)^2\right),$$

becomes infinitely large.

Here, the x-axis describes the number of stocks in our basket.

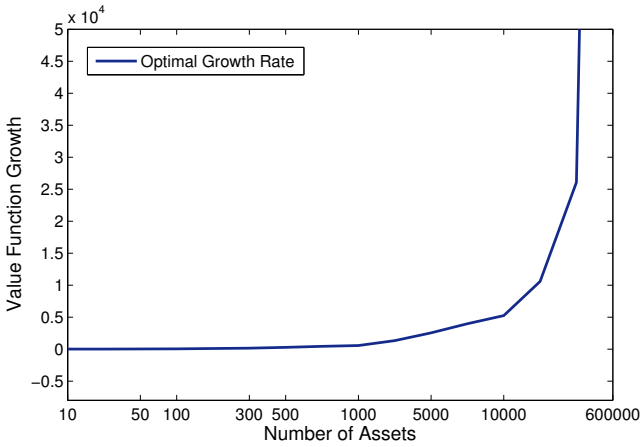


Figure 5.11: Optimal value function growth in a complete market.

Recall that we had to impose some control on the parameters, in particular we had to scale the random variables b_j, which perturb the Sharpe ratio with respect to the market with $\frac{1}{N^k}$. For weak scaling, i.e. $k = \frac{1}{2}$, the expanded value function growth $\Theta_N^*(a, b/N^k)$ behaves almost stochastically for a small number of assets, but then approaches the limit given in equation (4.27) as the number of assets grows.

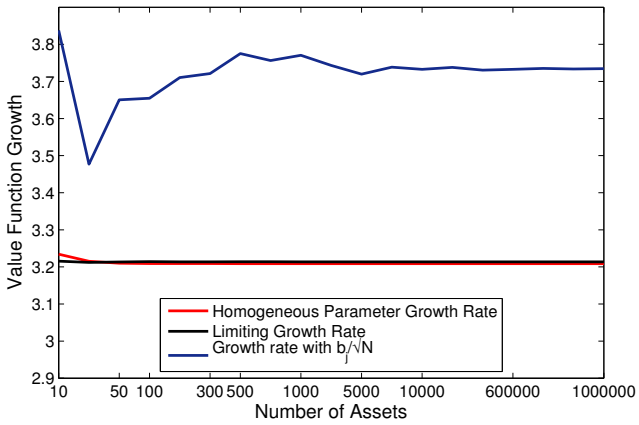


Figure 5.12: Optimal growth scaled with $\frac{1}{\sqrt{N}}$ vs. scaled limiting approximation and homogeneous parameter growth in a complete market.

On the other hand, if we scale the b_j with $\frac{1}{N}$, for $k = 1$, we have fast convergence to the limiting rate. Thus we see that the scaled optimal growth always outperforms the growth when using homogeneous coefficients. However, both approach the homogeneous limit of $\frac{\eta}{2}$ at some point.

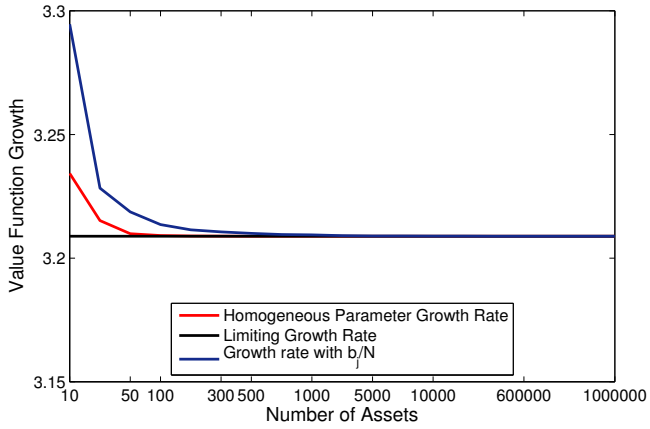


Figure 5.13: Optimal growth scaled with $\frac{1}{N}$ vs. scaled limiting approximation and homogeneous parameter growth in a complete market.

Comparing the results from the above figures with the unscaled results, we can see that the limit is close to 3.2, which is a large number, meaning heuristically, that the investment in total would evolve as $e^{3.2}$ in each time period.

Consider the incomplete market setup. We see a very similar picture for the optimal growth rate, which without any restrictions is growing almost exponentially, although slower than in the complete market case, as shown in figure 5.11:

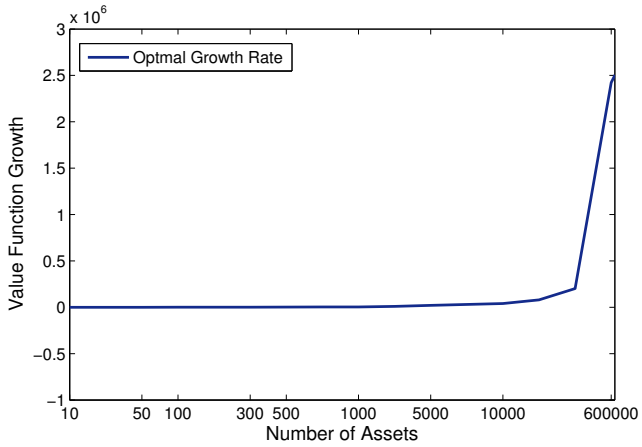


Figure 5.14: Optimal value function growth in an incomplete market.

Thus, scaling is necessary to stay bounded. Again, we look at two cases: first scaling b_j with $\frac{1}{N}$ and then with $\frac{1}{\sqrt{N}}$. We can see that they outperform the growth rate when only using homogeneous parameters, as expected.

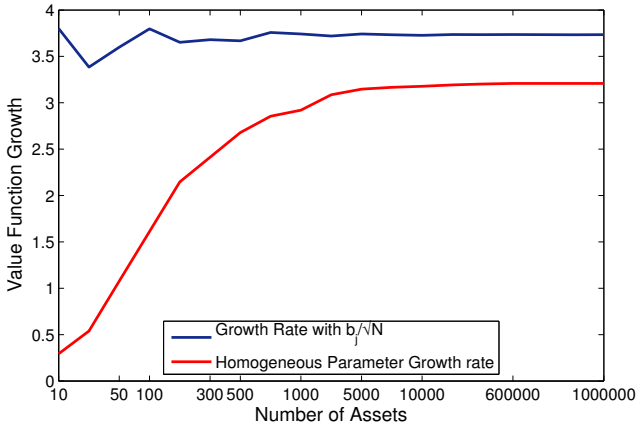


Figure 5.15: Optimal growth scaled with $\frac{1}{\sqrt{N}}$ vs. scaled limiting approximation and homogeneous parameter growth in an incomplete market.

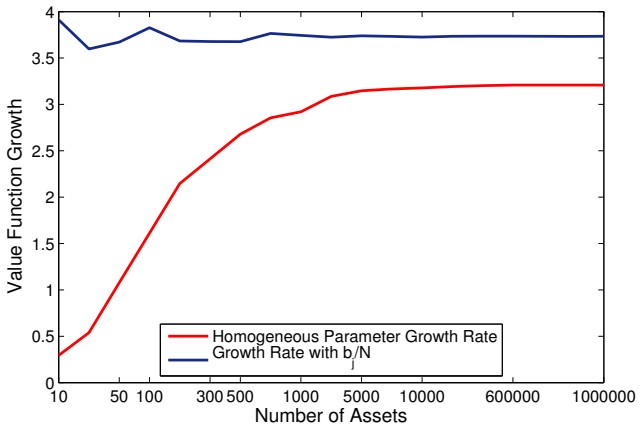


Figure 5.16: Optimal growth scaled with $\frac{1}{N}$ vs. scaled limiting approximation and homogeneous parameter growth in an incomplete market.

Again, as in the complete case, we see that the growth rate for our chosen parameters roughly approaches 3.2, which is due to the chosen, fixed parameter values.

5.2 A Real Trading Example

While describing our models for complete and incomplete markets, we only considered optimal behaviour in a Black-Scholes-type market. In particular, we assumed that our assets are driven by geometric Brownian motions and therefore the returns are log-normally distributed. This assumption leads to the belief that very high returns and large losses, have a small probability of occurring. This is not the case in reality. We see, that we can achieve maximal portfolio growth modelled through the value function, if we distribute our funds according to the optimal portfolio proportions that we have calculated explicitly in Chapter 3, in our theoretical setup.

On the other hand, recent stock market crashes give strong evidence that this is not the case in reality. Stock returns actually exhibit what is commonly referred to as 'fat tails'. This means that they are not log-normally distributed and the probability of abnormal losses, as well as gains, is much higher than such a distribution would imply.

Therefore the following question arises: if we use the optimal portfolio proportions to trade on a real market, where the stock prices are quoted daily through an exchange, do we achieve superior growth to the less involved strategy of equally weighting all assets in our portfolio?

To answer this question, we propose the following setup. As we have looked at large finite markets at the beginning of this thesis and later approximated them by infinite sized markets, we need to construct a similar setup in reality. The Wilshire 5000 index, covering almost all traded stocks in the US, is a good proxy. Therefore, we will use it as a proxy for the global market asset $I(t)$. The assets $S_j(t)$, $j = 1, \ldots, N$, are then taken from this index. We will look at the last 10 years of opening and closing data for certain stocks and build a portfolio according to the findings in Chapter 3, rebalancing daily. In our case daily rebalancing is a sensible approach, as we do not incur trading cost. Trading weekly or even monthly yields the same overall result with less granularity.

Suppose we look at 373 stocks in our large market represented by the Wilshire 5000, with prices quoted twice daily, observed over the last 10 years. Taking out holidays and weekends, this leaves us with $2,519$ trading days. This restriction mainly stems from our limited access to data and is based on the assets, which have observable opening and closing prices quoted every day over this period of time.

We derive the return by calculating the log return of the historic prices. The volatility can be calculated from the same data but is always done so over a 30-day period, using a rolling standard deviation to obtain an estimate. Similarly, we find the correlation of each stock return with the overall market returns on a rolling 30-day basis. The risk-free rate will be chosen specifically for each scenario as, surprisingly, we find that it does influence the growth significantly. We use power utility to model our agent's preference structure and assume that the penalty constant is given by $p = -0.5$.

We will only discuss the incomplete market setup, as we found the results to be similar in the complete setting and therefore it is not necessarily to displayed them here.

The general methodology is as follows: each morning, we observe the opening stock prices for each of the 373 stocks. From these, we calculate the modelling parameters: the return, volatility and correlation coefficients and use them to compute the portfolio proportions (3.3) and (3.16). Then, we consider these to invest in the assets using the evening prices, as we only have the opening and closing prices available. In addition, we also invest in the risk-free asset, modelled through a fixed risk free-rate. The next morning, we clear our accounts and collect all the gains or losses together to be invested again in the evening. We repeat the above algorithm every day i.e. at each of the $2,519$ trading days available in the 10 year period. If we run out of investment money, i.e. hit the absorbing state of bankruptcy, the trading stops.

Suppose now that we distribute our initial endowment of $\$1,000$ over the 373 stocks as well as investing a proportion in the risk-free money market. Then, we observe the following results, varying substantially with the choice of the risk free rate $r > 0$.

For a large, annualised risk-free rate of $r = 0.05$, we experience almost exponential growth using the optimal strategies obtained in the previous chapters. This is shown in the following figure:

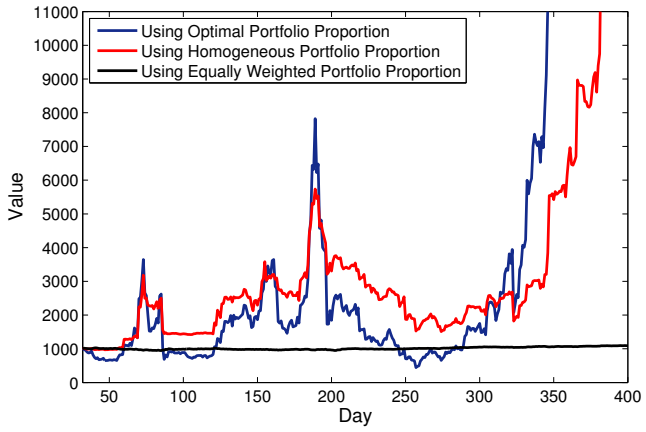


Figure 5.17: Value using optimal portfolio proportions and $r = 0.05$

One explanation for this phenomenon is that, as we invest the proportion $1 - \sum_{j=1}^{373} \pi_j^*$ in the risk-free bank account (and due to the fact that the optimal behaviour consists of the large positive and negative allocation in the risky assets), the bond investment acts as a buffer for the low stock returns. A value of $r = 0.05$ is higher than any of the risky return rates and therefore helps us to achieve a particularly high profit. In addition, we see that the growth rate, when we use homogeneous coefficients, calculated as the average parameter over all stocks, is similar to the growth in the optimal case, but smaller.

Recently, the risk-free rate has been observed to be very small, actually almost zero. Therefore we want to look at the unusually small rate of $r = 0.0002$ just for comparison.

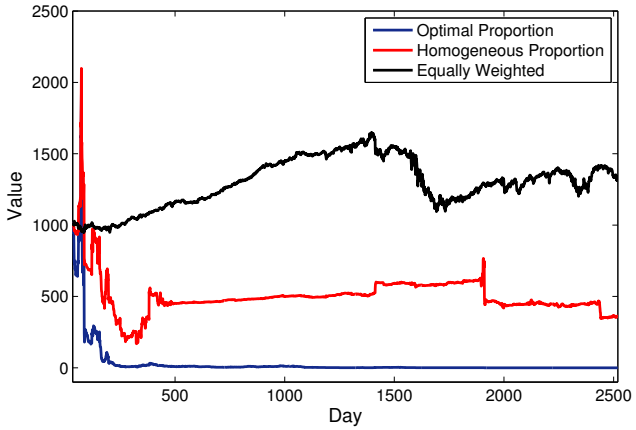


Figure 5.18: Value using optimal portfolio proportions and $r = 0.0002$

Here, we see a very contradictory picture. The equally weighed strategy performs reasonably well, as does the one using only homogeneous parameters. However, the optimal strategy fails. We have to infer that the investment in the bank account acts as a buffer to even out the major spikes in the single portfolio proportions. Thus, when there is very little growth in the risk free asset, we suffer more from market changes.

Looking back at the explicit portfolio proportions and the numerical modelling based on theoretical markets, as described in the previous sections, we know that the optimal portfolios tell us to invest heavily in some stocks while shorting others. Now, as we do not trade instantaneously but overnight, the prices in the market can change substantially and sometimes, due to high volatility, can even completely reverse direction a couple of times over the course of a day. If our strategy tells us to invest heavily in one asset, which while we are holding it becomes inferior, we may encounter high losses that are not sufficiently covered by the gains from our other investments. This, in turn, shows up in the poor performance of the growth rate.

Taking our strong modelling assumptions into account, it is clear that the results are prone to some modelling error. In particular, as is well known, it is impossible to properly estimate the rate of return from the historical data available to us. Therefore, the natural question would be to investigate whether these strategies, combined with other even heuristic portfolio investment techniques, such as limiting the volatility of the portfolio, or a higher investment in the risk-free asset could present a more stable picture. In addition, it would be interesting to see if some further constraints on the portfolio proportion, such as prohibition of short-selling could achieve an improved investment strategy. Unfortunately, this is beyond the scope of this thesis.

Chapter 6

Outlook

We have studied optimal behaviour in large markets as well as their approximation as the number of securities approaches infinity. In this, we have been able to find explicit optimal portfolio proportions as well as determining the value function of such investments assuming a priori that the agent's preferences can be described through a power utility function. In addition, we have introduced the notion of asymptotic arbitrage and linked the no asymptotic arbitrage condition to the existence of such an optimal value function and its finiteness. In this final section, we want to discuss possible ideas and question arising from our research.

We were able to set up the optimisation problem, as well as general no asymptotic arbitrage conditions, for a market with J risk factors, which can be interpreted as the existence of J sectors in the global market in practice. In addition, we obtained explicit results concerning general parameters for up to two risk factors and proposed extensions to J factors in the case of homogeneous parameters. However, it would be interesting to see what the explicit portfolio proportions and the value function, as well as their expansions, would look like for a general number of global risk factors. We laid the foundations for this by showing how to invert particular matrices for any dimension. The general result, however, would require computerised calculations.

Another possible extension to the presented research, which has not been our focus here, would be to assume that the market risk factors are not independent, but correlated. This could be seen as a more realistic approach, if we think that that each of the risk factors stems from a particular sector of the global market. This, in turn, implies that they must be correlated somehow, even if this correlation is very small.

We did not look at this in our thesis, as the main objective was to first establish the no asymptotic arbitrage conditions and then to find and investigate optimal portfolio proportions and optimal value functions in a large market. Additional correlation coefficients would be problematic to deal with, using the HJB approach.

In Chapter 4, we have seen how to approximate a finite sized market by an infinite one and derived particular scaling conditions to keep both, the portfolio proportions as well as the value function growth rate finite as N approaches infinity, for both a complete and incomplete one-factor market model. However, we have discovered that for a particular region of scaling, the total outlay of our investor, calculated as the sum of the absolute values of the portfolio proportions, can be infinite while the growth generated with such a strategy stays finite. This gives rise to the following question. If we would omit the order $\mathcal{O}(1)$ parts of the optimal portfolio proportion and only keep the parts which are of order $\frac{1}{N}$, can we generate a higher value than the homogeneous parameter proportion in the limit without having infinitely large exposure in the market, which is not possible. If we only leave the $\mathcal{O}(N^{-1})$ parts, their sum of absolute values is finite, however they generate the same limiting growth as in the homogeneous coefficient case.

Now a different idea is to constrain the portfolio proportions through their admissibility set during the optimisation. ln particular, it would be interesting to see if prohibiting short-selling, which results in having only positive portfolio proportions, can generate a finite value with finite exposure or even admit arbitrage, in the sense that for relaxed scaling condition a finite initial outlay could lead to an infinite value as $N \to \infty$.

Another open question arises in Chapter 5. Here we used our optimal strategies to trade on a real market. However, it is clear from the results that, even though these portfolio proportions achieve the required growth in theory, they do not outperform the growth generated using only homogeneous parameters in reality. In particular, the portfolio based on the optimal proportions performs rather poorly on real markets. This suggests that on top of the expected modelling and parameter estimation risk, this model is not robust in reality.

Therefore, as we have already suggested above, it would be interesting to see if the optimal strategy with appropriate constraints could produce a more stable model. In particular, as at the moment our optimal weights are very large with either positive or negative values, it would be reasonable to constrain them to a certain band around the homogeneous parameter proportion.

Finally one further strategy, also widely utilised in the banking system, is to either discard very volatile assets when rebalancing or have an overall volatility threshold for the whole portfolio. If the estimated volatility rises over a certain limit, usually represented as a percentage, then the investment is locked into the risk free bank account until the next rebalancing date, where its volatility profile is reconsidered. All these strategies would be good starting points when trying to take this research further to attempt to develop a robust trading model, outperforming the commonly utilised strategies.

Bibliography

[Bjo04] Thomas Bjork. *Arbitrage Theory in Continuous Time*. Oxford University Press, 2004.

[CH01] Tahir Choulli and T.R. Hurd. The portfolio selection problem via Hellinger processes. *Entropy*, 3:150–161, 2001.

[CS05] Tahir Choulli and Ch. Stricker. Minimal Entropy-Hellinger martingale measure in incomplete markets. *Mathematical Finance*, 15:464–490, 2005.

[CS06] Tahir Choulli and Ch. Stricker. More on minimal Entropy-Hellinger martingale measure. *Mathematical Finance*, 16:1–19, 2006.

[DN90] M.H.A. Davis and A.R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[DNBE09] Giulia Di Nunno and Inga Baadshaug Eide. Minimal-variance hedging in large financial markets: Random fields approach. *Stochastic Analysis and Applications*, 28:54–85, 2009.

[Dur91] Richard Durett. *Probability Theory and Examples*. Brooks Cole Publishing Company, 1991.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterisation and Convergence*. John Wiley and Sons, New Jersey, 1986.

[FS08] H Foellmer and W Schachermayer. Asymptotic arbitrage and large deviations. *Mathematical and Financial Economics*, 1:213–249, 2008.

[GV13a] Axel Gandy and Luitgard A. M. Veraart. The effect of estimation in high-dimensional portfolios. *Mathematical Finance*, 23:531–559, 2013.

[GV13b] Axel Gandy and Luitgard A. M. Veraart. The effect of estimation in high-dimensional portfolios. *Mathematical Finance*, 23(3):531–559, 2013.

[HJ13] Fatma Haba and Antoine Jacquier. Asymptotic arbitrage in the heston model. Technical report, Imperial College London, Submitted 2013.

[JS87] Jean Jacod and Albert N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Stuttgart, 2th edition edition, 1987.

[Kar97] Ioannis Karatzas. *CRM Monograph Series-Lectures on the Mathematics of Finance*. American Mathematical Society, New York, 1997.

[KK93] Yu. M. Kabanov and Dmitry Kramkov. Large financial markets: Asymptotic arbitrage and contiguity. *Theory and Applied Probability*, 39(1):182–187, 1993.

[KK98] Yu. M. Kabanov and Dmitry Kramkov. Asymptotic arbitrage in large financial markets. *Finance and Stochastics*, 2:143–172, 1998.

[Kle00] Irene Klein. A fundamental theorem of asset pricing for large financial markets. *Mathematical Finance*, 10(4):443–458, 2000.

[KLO12] Irene Klein, Emmanuel Lepinette, and Lavinia Ostafe. Large financial markets and asymptotic arbitrage with small transaction costs. Technical report, Cornell University Library, Submitted 2012.

[KLS86] Yu. M. Kabanov, R. Sh. Lipster, and A. N. Shiryaev. On the variation distance for probability measures defined on a filtered space. *Probability Theory and Related Fields*, 71:19–35, 1986.

[KLS87] Ioannis Karatzas, John P. Lechoczky, and Steven E. Shreve. Optimal portfolio and consumption decisions for a "small investor " on a finite horizon. *SIAM J. Control and Optimazation*, 25(6):1557–1586, November 1987.

[KLSS86] Ioannis Karatzas, John P. Lehoczky, Suresh P. Sethi, and Steven E. Shreve. Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research*, 11(2):261–294, May 1986.

[KP91] Yu. M. Kabanov and S. M. Pergamenshchikov. Optimal control of singularly perturbed stochastic linear systems. *Stochastics and Stochastics Reports*, 36:109–135, 1991.

[KS96] Irene Klein and Walter Schachermayer. Asymptotic arbitrage in non-complete large financial markets. *Theory of Probability and its Applications*, 41(4):927–934, 1996.

[KS99] Dmitry Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9:904–950, 1999.

[KS00] Ioannis Karatzas and Steven Shreve. *Methods of Mathematical Finance*. Springer Applications of Mathematics, Stuttgart, 2000.

[LO12] Emmanuel Lepinette and Lavinia Ostafe. Asymptotic arbitrage in large financial markets with friction. *Mathematical and Financial Economics*, 6:313–335, 2012.

[Mer69] Robert C. Merton. Lifetime portfolio selection under uncertainity: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, August 1969.

[Mer71] Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

[Pha09] Huyen Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer, Paris, France, 2009.

[Wil04] David Williams. *Probability with Martingales*. Cambridge Mathematical Textbook. Cambridge University Press, Cambridge, United Kingdom, 2004.