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ON THE LABELLING OF POLYHEDRAL DICE

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Abstract

We extend previous work which modelled the rolling of a typical playing die using a Markov matrix. The conditional probability of a roll's result (i.e. the final uppermost face) varies according to a standard n-faced die's initial position. However, for some standard n-faced dice, we can label the faces in such a way that the conditional probability of a roll's outcome (i.e. the final score) does not vary according to the die's initial position. In such cases, we say that the die is fair under that labelling. Here, we derive general conditions that such labellings must satisfy. Using these conditions, we identify specific examples of fair polyhedral dice.

Keywords: markov process; transition matrix; platonic solids; gambling

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1. Introduction

The platonic solids are a family of regular polyhedra, each member possessing its own set of identical regular polygonal faces [8]. It is inferred from the symmetry of platonic solids that a convex regular polyhedral die should, when rolled, land on any one of its faces with equal probability [3]. Under this assumption, such dice are considered to be 'fair'. Hence, these dice are believed fair by their structure alone. We concluded that this should not be the case in a previous paper.

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1.1. Background

In [5], we modelled the rolling of a platonic solid as a Markov process. Our model considered a standard cubic playing die thrown within a structure such as a flat-bottomed bowl which, in a manner similar to the backboard of Casino Craps, would allow the direction of a die's momentum to change during a roll. We used a transition matrix $\mathbf{T} = \left[t_{i,j}\right]$ where each $t_{i,j}$ gave the probability of a transition from face i during the initial stage of a roll to face j during the next stage [10]. Consecutive stages were separated by a single tilt (where we defined a tilt to be the smallest possible movement that changes the uppermost face of a die).

By definition, a tilt results in a transition between two adjacent faces. We assumed that each of the f possible tilts of a rolling die (where f equals the number of faces adjacent to any particular face, e.g. $f = 4$ for a standard cubic die) occurs with probability $1/f$. Here, we propose that the rolling of any standard n-faced die (i.e. any which takes the form of a platonic solid) can be modelled in this way.

1.2. Conventionally labelled dice

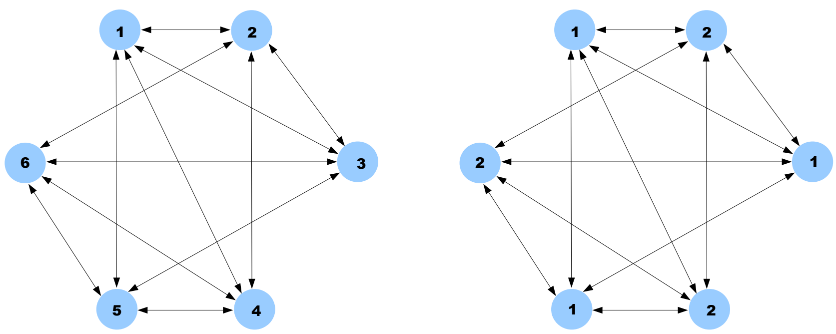

FIGURE 1: Vertices denote the 6 faces of a cubic die. If a facial transition between a and b is possible over the course of a single tilt, there exists an arc (a, b). We assume that each of these transitions is equally probable. Left: Conventional labelling of faces. Right: Novel labelling of faces proposed in [5].

In our 1-tilt transition matrices, all $t_{i,i} = 0$. Hence a single tilt of a standard n-faced die cannot result in a transition from face i to itself. When conventionally labelled, each face of a standard n-faced die bears one of n sequential labels. Hence a single tilt of a conventionally labelled standard n-faced die cannot result in a transition from score i to itself. Thus, some 1-tilt numerical transitions of a conventionally labelled standard n-faced die are more probable than others, i.e. the die's starting position influences the evolution of a roll. Therefore our model shows that the conditional probability of any roll's outcome varies according to the initial position of a conventionally labelled standard n-faced die. Given this property, which results from a combination of the die's structure and labelling, we challenge the convention of calling such a die 'fair'.

In [5], we put forward a newly labelled standard cubic die. This novel labelling uses two distinct labels (e.g. '1' and '2'), where opposite faces of the die are given different labels (see Figure 1). The rolling of this newly labelled cubic die can be modelled using the following transition matrix,

$$\begin{array}{c} \text{Score} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} \end{array} = \hat{\mathbf{T}}$$

where $\hat{\mathbf{T}} = \left[\hat{t}_{i,j}\right]$. Each $\hat{t}_{i,j}$ denotes the probability that a standard cubic die undergoes a transition from face i to face j over the course of a single tilt. Note the borders of $\hat{\mathbf{T}}$ here - these labels correspond to our proposed 'new scores' for a standard cubic die's faces. Hence the following (reduced) transition matrix can be derived from $\hat{\mathbf{T}}$,

$$\mathbf{P} = \begin{pmatrix} 1/4 + 1/4 & 1/4 + 1/4 \\ 1/4 + 1/4 & 1/4 + 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

where $\mathbf{P} = \left[p_{i,j}\right]$. Each $p_{i,j}$ gives the probability that our newly labelled standard cubic die undergoes a transition from score i to score j over the course of a single tilt. Furthermore, each entry of $\mathbf{P}^k = \left[p_{i,j}^{(k)}\right]$ (where $k \in \mathbb{Z}^+$) gives the probability of a transition from score i to score j over the course of k tilts [6] and it is clear that $\mathbf{P}^k = \mathbf{P}$ for all k. Thus, any two k-tilt numerical transitions of our newly labelled die are equally probable. Therefore the conditional probability of no roll's outcome varies according to this die's initial position. As such, we consider our newly labelled die to be fair.

To summarise, a conventionally labelled standard n-faced die's position at the start of a roll influences the final score. As such, we do not consider conventionally labelled standard n-faced dice to be fair. However, we may consider an unconventionally labelled standard n-faced die to be fair if it satisfies certain conditions. We begin by proposing these conditions and end by testing all standard n-faced dice against them.

2. Five necessary criteria for labellings of fair dice

We propose that a standard n-faced playing die is unfair if any of the following criteria are not satisfied.

(I) The die must have $m > 1$ distinct labels (because a die is used to generate random scores and therefore $m \leq 1$ is not allowed).

A roll must result in a definitive outcome. Hence, we propose that no face should be left blank or feature multiple labels. Under such a labelling, the following criterion ensures that all outcomes are, as they must be for fair dice, equally probable for any roll.

(II) The m distinct labels must be used equally, say c times each, where $c > 1$ (because $c = 1$ for conventional labellings). Hence $mc = n$ and therefore m divides n, where $m \neq n$ (because $m = n$ for conventional labellings).

It is not the case that all k-tilt numerical transitions of a conventionally labelled standard n-faced die are equally probable e.g. a 1-tilt transition between score x and itself is impossible whereas 1-tilt transitions between scores of adjacent faces are not. Hence we propose the following criterion.

(III) All k-tilt numerical transitions must be equally probable (for the same k).

2.1. Transition matrix

We propose that the rolling of any standard n-faced die that satisfies the criteria described in section 2 can be modelled as follows. The model uses a transition matrix $\mathbf{B} = \left[b_{i,j}\right]$ where each $b_{i,j}$ denotes the probability of a transition from score i to score j over the course of a single tilt. From criterion (I), it is clear that $\mathbf{B}$ is an $m \times m$ matrix. Every row of $\mathbf{B}$ must sum to one, i.e.

$$\sum_{j=1}^{m} b_{i,j} = 1 \Longrightarrow b_{i,1} + \ldots + b_{i,m} = 1$$

for all i. We can simplify the left hand side here - criterion (III) is not satisfied for $k = 1$ unless all $b_{i,j}$ are equal. Let all $b_{i,j} = \Gamma$,

$$b_{i,1} + \ldots + b_{i,m} = 1 \Longrightarrow \underbrace{\Gamma + \ldots + \Gamma}_{m \times \Gamma} = 1 \Longrightarrow m\Gamma = 1 \Longrightarrow \Gamma = \frac{1}{m}$$

i.e. all $b_{i,j} = 1/m$. Therefore criterion (III) is satisfied since $\mathbf{B} = \mathbf{B}^k$ (where entries of $\mathbf{B}^k$ are k-tilt transition probabilities) for all $k \in \mathbb{Z}^+$ (see Theorem 2.1).

Theorem 2.1. *For any positive integral power $g \geq 2$,*

$$\begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^g = \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}$$

where $\eta \geq 2$ is the side-length of these square matrices. We call this statement $P(g)$.

Proof. The basis step (base case) of a proof by induction requires a proof that $P(g)$ is true for minimal g (i.e. for $g = 2$). For $P(2)$, we have the statement,

$$\begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^2 = \begin{pmatrix} \eta\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta}\right) & \cdots & \cdots & \eta\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta}\right) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \eta\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta}\right) & \cdots & \cdots & \eta\left(\frac{1}{\eta}\right)\left(\frac{1}{\eta}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}.$$

Hence we have proven the basis step - $P(2)$ is true.

The inductive step of a proof by induction requires a proof that $P(g)$ is true for $g = h+1$ if it is assumed true for $g = h$. For $P(h+1)$, we have the statement,

$$\begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^{h+1} = \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^{h} \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^{1}$$

where, under our assumption that $P(h)$ is true, it is clear that the right hand terms commute (i.e. the order of multiplication does not matter). We use our assumption to simplify this right hand side as follows,

$$\begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^{h+1} = \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}^{2} = \begin{pmatrix} \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{\eta} & \cdots & \cdots & \frac{1}{\eta} \end{pmatrix}.$$

Hence it has been shown that $P(h+1)$ holds if $P(h)$ is true. Thus, both the basis and inductive steps are proven - $P(g)$ is true for all $g \geq 2$ by induction.

We therefore propose a fourth criterion (to add to the three described in section 2) which must be satisfied by a fair standard n-faced die.

(IV) A die's roll can be modelled using a transition matrix of type $\mathbf{B} = \left[b_{i,j}\right]$.

To satisfy criterion (IV), it must be possible to describe the probabilities of all numerical transitions over each stage of a die's roll. We can do so for an unconventionally labelled standard n-faced die if and only if a condition, which we shall now derive, is met.

2.2. Model by numerical transitions

In this section, we use the following notations. Firstly, let a_r denote the distinct faces of a standard n-faced die (i.e. $r = 1 \ldots n$) and let s_r denote the labels assigned to a_r. Secondly, for any face, say a_l, we can define the set of a_r which are adjacent to a_l - we call this set A_l. Finally, let S_l define the set of s_r assigned to the elements of A_l (i.e. $|A_l| = |S_l| = f$). For example, consider $a_l = 3$ of the unconventionally labelled 6-faced die described in section 1.2. From $\hat{\mathbf{T}}$, it is clear that $A_l = \{1, 2, 5, 6\}$ and $S_l = \{1, 2, 1, 2\}$.

In this paper, we propose that the rolling of any standard n-faced die can be modelled using a transition matrix $\mathbf{Y} = \left[y_{i,j}\right]$ where each $y_{i,j}$ denotes the probability of a transition from face i to face j over the course of a single tilt. Furthermore, for some labellings, we propose that the rolling of a standard n-faced die can be modelled using a transition matrix $\mathbf{Z} = \left[z_{i,j}\right]$ where each $z_{i,j}$ denotes the probability of a transition from score i to score j over the course of a single tilt. Hence $\mathbf{Z}$ is not applicable unless the probability of a transition from score i to score j over the course of a single tilt is fixed (i.e. equal for all i-bearing faces).

This is not the case unless all a_r bearing identical s_r undergo the same set of possible numerical transitions over the course of a single tilt. The set of possible 1-tilt numerical transitions undergone by any face, say a_l, is determined by the elements of S_l. Therefore $\mathbf{Z}$ is not applicable unless S_r is equal for all a_r bearing identical s_r.

Example 2.1. ($\mathbf{Z}$ *is applicable.*) Let us consider a standard cubic die which uses just three distinct labels (e.g. '1', '2' and '3'). Each label is used twice - opposing faces are assigned identical labels, non-opposing faces are assigned distinct labels. Is $\mathbf{Z}$ applicable in this case?

For both a_r bearing $s_r = 1$, $S_r = \{2, 2, 3, 3\}$. For both a_r bearing $s_r = 2$, $S_r = \{1, 1, 3, 3\}$. For both a_r bearing $s_r = 3$, $S_r = \{1, 1, 2, 2\}$. Hence the rolling of our proposed die can be modelled using some $\mathbf{Z}$. It is clear that, for this example,

$$\mathbf{Z} = \begin{pmatrix} 0 & 1/4 + 1/4 & 1/4 + 1/4 \\ 1/4 + 1/4 & 0 & 1/4 + 1/4 \\ 1/4 + 1/4 & 1/4 + 1/4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

since our model assumes that the f possible tilts of a rolling die occur with probability $1/f$ (and $1/f = 1/4$ for this example).

Example 2.2. ($\mathbf{Z}$ *is not applicable.*) Let us consider another standard cubic die which uses just three distinct labels (e.g. '1', '2' and '3'). Each label is used twice - each face and one of its adjacent faces are assigned an identical label. Is $\mathbf{Z}$ applicable in this case?

For the a_r bearing $s_r = 1$, $S_r = \{1, 2, 3, 3\}$ or $S_r = \{1, 3, 2, 2\}$. For the a_r bearing $s_r = 2$, $S_r = \{2, 1, 3, 3\}$ or $S_r = \{2, 3, 1, 1\}$. For the a_r bearing $s_r = 3$, $S_r = \{3, 1, 2, 2\}$ or $S_r = \{3, 2, 1, 1\}$. Hence the rolling of our proposed die cannot be modelled using some $\mathbf{Z}$. This is confirmed by $\mathbf{Y}$ where, for this example,

$$\begin{array}{c c}
\text{Score} & \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{3} & \mathbf{2} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{3} \\ \mathbf{2} \end{array} & \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} = \mathbf{Y}.
\end{array}$$

From this $\mathbf{Y}$ it is clear that, for example, the probability of a transition from $s_r = 1$ to $s_r = 3$ over the course of a single tilt is not fixed. Therefore the rolling of such a die cannot be modelled using some $\mathbf{Z}$.

Like each $z_{i,j}$, each $b_{i,j}$ (see criterion (IV) of section 2.1) denotes the probability of a transition from score i to score j over the course of a single tilt. Hence $\mathbf{B}$ is a special case of $\mathbf{Z}$. Furthermore, our condition for the applicability of $\mathbf{B}$ is a special case of our condition for the applicability of $\mathbf{Z}$.

Theorem 2.2. $\mathbf{B}$ *is applicable* $\Longleftrightarrow$ S_r *is equal for all* a_r.

Proof. Let $P_k(A|B)$ = the probability of a transition from B to A over the course of k tilts.

Assume $\mathbf{B} = \left[b_{i,j}\right]$ is applicable.

Then, for any i and j,

$$P_{k=1}\left(\text{score } j \mid \text{score } i\right) = 1/m.$$

Hence, for any score, say $\hat{s}$,

$$P_{k=1}\left(\hat{s} \mid a_{f_1}\right) = P_{k=1}\left(\hat{s} \mid a_{f_2}\right) \tag{1}$$

where a_{f_1} and a_{f_2} are any two faces of the standard n-faced die.

Our model assumes that, for any face, say a_l,

$$P_{k=1}(s \mid a_l) = \begin{cases} \mu\left(\frac{1}{f}\right) & \text{if } s \in S_l \\ 0 & \text{if } s \notin S_l \end{cases} \tag{2}$$

where μ element(s) of set S_l are equal to s. Hence equation (1) is not satisfied unless set S_{f_1} is equal to set S_{f_2}.

Therefore, if **B** is applicable then S_r is equal for all a_r.

Assume S_r is equal for all a_r.

Then, for any score of the standard n-faced die, say $\hat{s}$,

$$P_{k=1}\left(\hat{s} \mid a_{g_1}\right) = P_{k=1}\left(\hat{s} \mid a_{g_2}\right)$$

where a_{g_1} and a_{g_2} are any two a_r (see equation (2)). Hence,

$$P_{k=1}(\text{score } j \mid \text{score } i)$$

is fixed, i.e. equal for any i and j. Hence $\mathbf{Z} = \left[z_{i,j}\right]$ is applicable where all $z_{i,j}$ are equal. Thus **B** is applicable.

Therefore, if S_r is equal for all a_r then **B** is applicable.

Therefore **B** is applicable if and only if S_r is equal for all a_r.

Therefore, we propose a fifth criterion for a fair standard n-faced die.

(V) S_r is equal for all a_r.

Let us now test all standard n-faced dice against our criteria.

3. Standard polyhedral dice

Standard n-faced dice are in common usage, e.g. in the board game, 'Dungeons and Dragons'. In testing such dice against our criteria, let us begin with a revealing example - an octahedral die.

3.1. Octahedral die

A standard 8-faced die has the form of a regular octahedron [11]. To satisfy both criteria (I) and (II), m must equal either 2 or 4. Let us first consider the case for $m = 4$.

Every face of an octahedral die is an equilateral triangle (i.e. $f = 3$). For a single tilt of an octahedral die, transitions between adjacent faces are possible, whilst transitions between non-adjacent faces are impossible. Hence criterion (III) is not satisfied for $k = 1$ if, as is the case for this example, m exceeds f. Therefore, we do not consider a standard 8-faced die to be fair if $m = 4$ and, furthermore, we propose that a standard n-faced die is unfair unless the die has $m \leq f$ distinct labels. In fact, we can refine this rule as follows.

Theorem 3.1. *m must divide f for a fairly labelled standard n-faced die.*

Proof. Let $P_k(A|B)$ = probability of a transition from B to A over the course of k tilts.

When applicable, our model for the rolling of a standard n-faced die uses some $\mathbf{Z} = \left[z_{i,j}\right]$ to describe the probabilities of numerical transitions over each stage of the roll such that, if G and H are scores assigned to adjacent faces,

$$P_{k=1}(G|H) = \mu(1/f)$$

for some $\mu \in \mathbb{Z}^+$ (see Theorem 2.2). Moreover, we have proposed that,

$$P_{k=1}(J|K) = 1/m$$

if J and K are scores assigned to adjacent faces of a fair standard n-faced die (reflected by entries of $\mathbf{B}$ - see criterion (IV)). Hence we do not consider a standard n-faced die to be fair unless,

$$\mu(1/f) = 1/m$$

for some $\mu \in \mathbb{Z}^+$. Therefore,

$$\mu/f = 1/m \Longrightarrow \mu m = f$$

i.e. m divides f (since $m, f \in \mathbb{Z}^+$).

Therefore we propose a sixth criterion for a fair standard n-faced die.

(VI) m divides f.

Let us now return to the octahedral die and consider the case of $m = 2$. For an octahedral die, $f = 3$. Hence criterion (VI) is not satisfied for $m = 2$. Thus, we do not consider a standard 8-faced die to be fair if $m = 2$. Therefore we do not consider a standard 8-faced die to be fair under any labelling.

3.2. Remaining Platonic solids

Platonic solid	n	f	**Factors of** n	**Factors of** f	**Potential** m
Equilateral Tetrahedron	4	3	1, 2, 4	1, 3	-
Regular Dodecahedron	12	5	1, 2, 3, 4, 6, 12	1, 5	-
Regular Icosahedron	20	3	1, 2, 4, 5, 10, 20	1, 3	-
Regular Hexahedron	6	4	1, 2, 3, 6	1, 2, 4	2

Table 1: Properties of 4 Platonic solids

We can now test the other four Platonic solids against our six criteria. The key properties needed for this testing are contained in Table 1: the fourth column relates to criterion (II), the fifth column relates to criterion (VI) and these criteria are used (in conjunction with criterion (I)) to compute the sixth column.

A standard 4-faced die has the form of an equilateral tetrahedron (or equilateral triangular pyramid) [1]. There are two typical labelling schemes - labels are assigned either to every face (in which case a roll's outcome is determined by the lowermost face at the end of the roll) or to every apex (in which case a roll's outcome is determined by the uppermost apex at the end of the roll). In both cases, we can model the roll using an identical $\mathbf{Y} = \left[y_{i,j}\right]$ since every face / apex has three adjacent faces / apices (i.e. $f = 3$ in both cases). From Table 1, we see that no value of m satisfies our six criteria. Therefore we do not consider a standard 4-faced die to be fair under any labelling.

A standard 12-faced die and a standard 20-faced die have the forms of a regular dodecahedron [2] and a regular icosahedron [7] respectively. From Table 1, we see that no value of m satisfies our six criteria for either of these dice. Therefore we do not consider either a standard 12-faced die or a standard 20-faced die to be fair under any labelling.

A standard 6-faced die has the form of a cube (or regular hexahedron) [4]. As Table 1 shows, we do not consider a standard 6-faced die to be fair if $m \neq 2$ (noting that we have demonstrated in section 1.2 that a fair die exists where $m = 2$). We conclude that this is the only platonic solid which can be labelled fairly.

3.3. Other possibilities

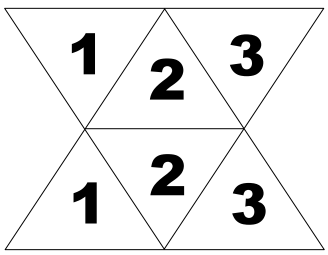

FIGURE 2: Net of a fairly labelled die which takes the form of a unit equilateral triangular dipyramid (i.e. two equilateral tetrahedra joined at their bases).

Unifying criteria (I), (II) and (VI), we propose that a standard n-faced die is unfair unless $m > 1$ is a common factor of f and n. For the platonic solids, we have seen that this condition is exclusively satisfied for the cubic die described in section 1.2 (see Figure 1). However, if we widen our definition of standard n-faced dice beyond the platonic solids, we can find more dice which satisfy our six criteria.

The platonic solids are members of a larger polyhedral class called isohedra. A polyhedron is isohedral if, by some rotation or reflection, any face can be mapped directly onto any other face. It has been argued that dice are 'fair' (i.e. there is an equally likely chance of landing on any face) if and only if they are isohedral [9]. In Table 2, we report properties of the 15 isohedra provided by [12].

Isohedron	n	f	**Potential** m
Deltoidal Hexecontahedron / Trapezoidal Hexecontahedron	60	4	2, 4
Deltoidal Icositetrahedron / Trapezoidal Icositetrahedron	24	4	2, 4
Unit Equilateral Triangular Dipyramid	6	3	3
Unit Equilateral Pentagonal Dipyramid	10	3	-
Disdyakis Dodecahedron / Hexakis Octahedron	48	3	3
Disdyakis Triacontahedron / Hexakis Icosahedron	120	3	3
Pentagonal Hexecontahedron	60	5	5
Pentagonal Icositetrahedron	24	5	-
Pentakis Dodecahedron	60	3	3
Rhombic Dodecahedron	12	4	2, 4
Rhombic Triacontahedron	30	4	2
Tetrakis Hexahedron	24	3	3
(Small) Triakis Octahedron	24	3	3
Triakis Icosahedron	60	3	3
Triakis Tetrahedron	12	3	3

Table 2: Properties of 15 Isohedra

For 12 of the 15 isohedra in Table 2, $m = f$ satisfies our condition above (and hence satisfies criteria (I), (II) and (VI)). If $m = f$, a possible labelling is described by the rule '$S_r = \{1, \ldots, m\}$ for all a_r' (where S_r and a_r are defined in section 2.2). Under such a labelling, a die satisfies criterion (V) and therefore also criterion (IV) (see Theorem 2.2). Furthermore, criterion (III) holds when criterion (IV) is satisfied (since, for any k, all $b_{i,j}^{(k)}$ are equal). Therefore, when labelled under the rule above, these 12 isohedra satisfy our six criteria (see Figure 2 for an example).

4. Conclusions

We propose a model for the rolling of a standard n-faced die within a structure such as a flat-bottomed bowl. This model shows that the conditional probability of a roll's outcome varies according to the initial position of a conventionally labelled die. However, for some unconventionally labelled dice, the starting position does not influence a roll's outcome in such a manner. In this paper, we have proposed six criteria that such labellings must satisfy - these criteria can be reduced to the following two conditions.

(i) $m > 1$ is a common factor of f and n.

(ii) S_r is equal for all a_r.

Condition (i) unifies criteria (I), (II) and (VI), as proposed in section 3.3. Condition (ii) is just criterion (V) which is sufficient to satisfy criteria (III) and (IV), as proposed in section 3.3. Thus, a standard n-faced die which possesses the physical attributes described by conditions (i) and (ii) will, when rolled, exhibit the probabilistic attributes of a fair die (described by criteria (III) and (IV)).

It is generally assumed that the rolling of a conventionally labelled standard n-faced die yields any one of the die's scores with equal probability. However, it is accepted that this assumption is less realistic for less vigorous rolls. It is therefore noteworthy that, when rolling a standard n-faced die which satisfies our two conditions, all k-tilt numerical transition probabilities are equal. Hence the conditional probability of no roll's outcome (whatever the roll's length) varies according to the initial position of such a die. Therefore the use of such a die would prevent unfair play as a player could no longer intentionally decrease the fairness of their roll through lack of vigor.

The most widely-used playing dice take the form of platonic solids. Interestingly, four of the five platonic solids do not satisfy both of our two conditions under any labelling. However, both of these conditions are satisfied by many other isohedra under some labelling. We conclude that, for general usage, such isohedra seem preferable to the four intrinsically unfair platonic solids.

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