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On inverse problems for two-dimensional steady supersonic Euler flows past curved wedges

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Abstract

We are concerned with the well-posedness of an inverse problem for determining the wedge boundary and associated two-dimensional steady supersonic Euler flow past the wedge, provided that the pressure distribution on the boundary surface of the wedge and the incoming state of the flow in the x–direction are given. We first establish the existence of wedge boundaries and associated entropy solutions of the inverse problem, when the pressure on the wedge boundary is larger than that of the incoming flow but less than a critical value, and the total variation of the incoming flow and the pressure distribution is sufficiently small. This is achieved by a careful construction of suitable approximate solutions and corresponding approximate boundaries via developing a wave-front tracking algorithm and the rigorous proof of their strong convergence subsequentially to a global entropy solution and a wedge boundary, respectively. Then we establish the L^∞–stability of the wedge boundaries, by introducing a modified Lyapunov functional for two different solutions with two distinct boundaries, each of which may contain a strong shock-front. The modified Lyapunov functional is carefully designed to control the distance

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between the two boundaries and is proved to be Lipschitz continuous with respect to the differences of the incoming flow and the pressure on the wedge, which leads to the existence of the Lipschitz semigroup as a converging limit of the approximate solutions and boundaries. Finally, when the pressure distribution on the wedge boundary is sufficiently close to that of the incoming flow, using this semigroup, we compare two solutions of the inverse problem in the respective supersonic full Euler flow and potential flow and prove that, at $x>0$, the distance between the two boundaries and the difference of the two solutions are of the same order of x multiplied by the cube of the perturbations of the initial boundary data in $L^\infty \cap BV$.

Keywords: steady Euler equations, inverse problems,
Lipschitz boundaries of wedges, pressure distribution, stability,
wave-front tracking algorithms, Glimm-type functional

1. Introduction

We are concerned with the well-posedness of an inverse problem for two-dimensional (2D) steady supersonic Euler flows past wedges; see figure 1. The inviscid compressible flows are governed by the following 2D steady Euler system:

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ \left(\rho u^2 + p\right)_x + (\rho uv)_y = 0, \\ (\rho uv)_x + \left(\rho v^2 + p\right)_y = 0, \\ \left(\rho u\left(E + \dfrac{p}{\rho}\right)\right)_x + \left(\rho v\left(E + \dfrac{p}{\rho}\right)\right)_y = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u,v)^\top$ is the velocity, p the pressure, ρ the density, and E the total energy:

$$E = \frac{1}{2}|\mathbf{u}|^2 + e(\rho,p),$$

with the internal energy e as a given function of (ρ,p).

For ideal gases, the relation between pressure p and internal energy e can be expressed as

$$p = \rho RT, \qquad e = c_\nu T \tag{1.2}$$

with T standing for the temperature, S the entropy, and $\gamma = 1 + \frac{R}{c_\nu} > 1$ for some constant $R > 0$. In particular, using the thermodynamic variables (ρ,S), we have

$$p = p(\rho,S) = \kappa \rho^\gamma e^{S/c_\nu}, \qquad e = \frac{\kappa}{\gamma-1}\rho^{\gamma-1}e^{S/c_\nu} = \frac{RT}{\gamma-1}, \tag{1.3}$$

where κ and $c_\nu > 0$ are constants. For the isentropic polytropic gas, $\gamma > 1$, while, for the isothermal flow, $\gamma = 1$. The sonic speed of the flow is $c := \sqrt{p_\rho(\rho,S)}$. For polytropic gases, $c = \sqrt{\gamma p/\rho}$.

For isentropic and irrotational flow, the governing equations form the potential flow system:

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ v_x - u_y = 0, \end{cases} \tag{1.4}$$

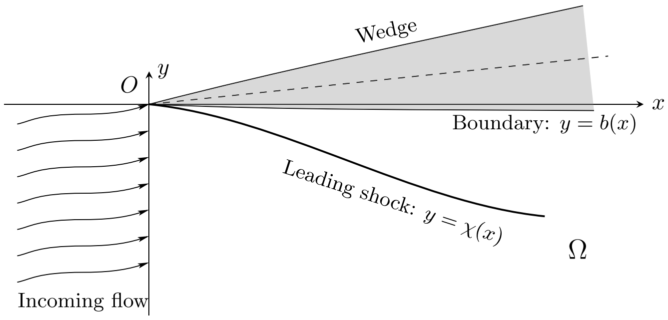


Figure 1. The inverse problem for two-dimensional steady Euler equations.

obeying the Bernoulli law:

$$\frac{1}{2}\left(u^2+v^2\right)+\frac{\gamma\rho^{\gamma-1}}{\gamma-1}=B_\infty, \tag{1.5}$$

where we have used the pressure-density relation: $p=\rho^\gamma$ without loss of generality by scaling.

The mathematical analysis for 2D steady supersonic flows past wedges initiated in the 1940s (*cf* Courant–Friedrichs [22]). As indicated in [22], when a supersonic flow passes a straight-sided wedge whose vertex angle is less than the critical angle (i.e. the sonic angle), a supersonic shock issuing from the wedge vertex can be determined, and both of the constant states connected by the shock are supersonic. Moreover, the opening angle of the wedge completely determines whether there exists such a supersonic shock. If the wedge is a perturbation of a straight-sided one, local solutions near the wedge vertex were first studied in Gu [27], Li [35], Schaeffer [45], and the references cited therein. The global existence of solutions for potential flows was obtained in [8, 16–18, 50, 51] in different types of setups. For the full Euler equations, Chen *et al* [14] first established the existence of global solutions for supersonic Euler flows past wedges and the stability of the strong shock-front attached to the vertex via a modified Glimm scheme. Later on, Chen and Li in [12] used a wave-front tracking method to establish the L^1–stability of entropy solutions with strong shock-fronts and obtained the uniform estimates for the Lipschitz semigroup S_x defined by the limit of the wave-front tracking approximate solutions, based on which the uniqueness of solutions within a broader class of viscosity solutions was proved. Recently, Chen *et al* studied the L^1–stability problem for hypersonic similarity laws for steady compressible full Euler flows over 2D Lipschitz wedges via a wavefront tracking approach and obtained the optimal convergence rate in [9, 10] with *large data*. For supersonic Euler flows over almost straight walls, the existence of the full Euler solutions and the stability of large vortex sheets and entropy waves in *BV* were first established by Chen *et al* [15], while Chen and Kukreja [11] established the well-posedness of that problem. Later on, Zhang [52] made full use of the semigroup corresponding to the Cauchy problems to prove that, at $x>0$, the solutions of the supersonic potential flow system approach that of the full Euler system at the order of x multiplied by the cube of the perturbations of the initial boundary data (also *cf* [3, 44]). See also [8, 17] and the references cited therein.

Corresponding to the stability problems for shocks, vortex sheets, and entropy waves, two inverse problems have been investigated. One of them is to determine the shape of the wedge in a 2D steady supersonic flow, provided that the location of the leading shock front is *a priori* given. This inverse problem was considered by Li-Wang in [36–38, 48, 49], in which a smooth leading shock is assumed and then the characteristic methods are applied for seeking a piecewise smooth solutions containing only one discontinuity, the leading shock; see also [34]. The other inverse problem is to determine the shape of the wedge or the cone with given pressure distribution on the wedge boundary in a 2D steady supersonic flow or an axisymmetric conical steady supersonic flow; see [42] for the inverse problem for the 2D case and see [13] for the 3D axisymmetric case. This inverse problem plays crucial roles in the aircraft design, especially in the inverse design; see [1, 2, 7, 24–26, 40, 41, 43, 47]. Though some numerical methods and linearized algorithms to deal with this problem have been developed, it seems that there is no available rigorous mathematical analysis on the well-posedness of solutions to the inverse problem for supersonic steady Euler flows past wedges.

Establishing a well-posedness theory, including the existence and stability of solutions to the inverse problem for supersonic steady Euler flows past wedges, holds transformative potential for aerospace design. A rigorous mathematical foundation not only ensures that numerical algorithms converge to physically admissible solutions but also validates the reliability of iterative methods in computational frameworks. Crucially, the mathematical proof of the stability of such an inverse problem guarantees that small perturbations of the targeted parameters (pressure distribution, Mach number, etc) do not lead to rapid oscillation or large deviations in the wing geometry, which enhances the robustness of optimization frameworks. Meanwhile, from a computational perspective, the well-posedness allows the applications of some simplified models (such as the potential flow equations as the leading shock is weak) in preliminary design phases, reducing computational costs while preserving predictive accuracy. Finally, some theoretical methods such as the Glimm scheme or wave-front tracking algorithms offer a reliable algorithmic foundation, based on which a more efficient way may be developed to achieve the inverse design task. These advantages provide a solid foundation for creating innovative and high-confidence designs for next-generation high-speed aerospace vehicles.

In this paper, for completeness, we first establish the existence of entropy solutions and wedge boundaries of the second inverse problem by employing the wave-front tracking method, given the pressure distribution (whose total variation is suitably small) on the wedge and the incoming flow (a BV perturbation of a uniform flow); see figure 1. Then we investigate the L^∞–stability of the wedge boundaries and the L^1–stability of entropy solutions via a modified Lyapunov functional. Based on these, we are able to deduce a uniformly Lipschitz semigroup $\mathfrak{S}_x$, which is defined by the converging limit of both approximate boundaries and approximate solutions generated by the wave-front tracking algorithm. Finally, we use this semigroup to compare two solutions of the inverse problem in the respective supersonic full Euler flow and potential flow, assuming the pressure distribution on the wedge boundary is sufficiently close to the pressure of the incoming flow. As a result, we prove that, at $x > 0$, the distance between two boundaries and the difference of two solutions are of the same order of x multiplied by the cube of the perturbations of initial boundary data, i.e. $O(1)x\|(\widetilde{\mathbf{u}}_\infty, \tilde{p}_b)\|^3_{L^\infty\cap BV}$.

To be precise, denote $U := (\mathbf{u}^\top, p, \rho)^\top = (u, v, p, \rho)^\top$, and consider vector functions:

$$W(U) = (\rho u, \rho u^2 + p, \rho uv, \rho u(h + \frac{u^2+v^2}{2}))^\top,$$
$$H(U) = (\rho v, \rho uv, \rho v^2 + p, \rho v(h + \frac{u^2+v^2}{2}))^\top,$$

with $h = \frac{\gamma p}{(\gamma-1)\rho}$. Then system (1.1) is reformulated into the conservative form:

$$W(U)_x + H(U)_y = 0. \tag{1.6}$$

Our problem is to seek the wedge boundary $y = b(x) < 0$ and solve (1.6) in the corresponding domain:

$$\Omega = \{(x, y) : x \geqslant 0, y < b(x)\}$$

with the upper boundary

$$\Gamma = \{(x, y) : x \geqslant 0, y = b(x)\},$$

such that

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0, \tag{1.7}$$

with

$$\mathbf{n} = \mathbf{n}(x, b(x)) = \frac{(-b'(x), 1)^{\top}}{\sqrt{1 + (b'(x))^2}}$$

as the corresponding outer normal vector to Γ when $b(x)$ is differentiable.

To solve the inverse problem, we assume that the initial-boundary data satisfy the following condition:

(C1) $U_\infty(y) := \overline{U}_\infty + \widetilde{U}_\infty(y)$ is the incoming flow at $x = 0$ such that
(a) $\overline{U}_\infty = (\overline{\mathbf{u}}_\infty^{\top}, \overline{p}_\infty, \overline{\rho}_\infty)^{\top}$ is a constant vector with

$$\begin{aligned} &\overline{u}_\infty > 0, \quad \overline{v}_\infty > 0, \quad \overline{M}_\infty := \frac{|\overline{\mathbf{u}}_\infty|}{\overline{c}_\infty} = |\overline{\mathbf{u}}_\infty| \sqrt{\frac{\overline{\rho}_\infty}{\gamma \overline{p}_\infty}} > 1, \\ &\frac{\overline{v}_\infty}{\overline{u}_\infty} = \tan(\overline{\theta}_\infty) := \frac{\frac{\overline{p}_b}{\overline{p}_\infty} - 1}{\gamma \overline{M}_\infty^2 - \frac{\overline{p}_b}{\overline{p}_\infty} + 1} \sqrt{\frac{(1 + \frac{\gamma-1}{\gamma+1})(\overline{M}_\infty^2 - 1) - (\frac{\overline{p}_b}{\overline{p}_\infty} - 1)}{\frac{\overline{p}_b}{\overline{p}_\infty} + \frac{\gamma-1}{\gamma+1}}}. \end{aligned} \tag{1.8}$$

(b) $\widetilde{U}_\infty(y) = (\widetilde{\mathbf{u}}_\infty^{\top}, \widetilde{p}_\infty, \widetilde{\rho}_\infty)^{\top}(y) \in (L^1 \cap BV)(\mathbb{R}; \mathbb{R}^4)$ is a BV perturbation at $x = 0$ with sufficiently small $\|\widetilde{U}_\infty\|_{L^\infty \cap BV} := \|\widetilde{U}_\infty\|_{L^\infty} + \text{T.V.}(\widetilde{U}_\infty)$.

(C2) $p_b(x) := \overline{p}_b + \tilde{p}_b(x)$ is the pressure distributions on the wedge boundary such that
(a) $\overline{p}_b$ a constant so that

$$\overline{p}_\infty < \overline{p}_b < p_{\text{sonic}},$$

where p_{sonic} is a critical pressure to be specified later in (2.15);
(b) $\tilde{p}_b(x) \in (L^1 \cap BV)(\mathbb{R}; \mathbb{R})$ is a BV perturbation with sufficiently small $\|\tilde{p}_b\|_{L^\infty \cap BV}$.

With these setups, it suffices to consider the following inverse problem for (U, b):

Incoming Flow Condition:

$$U|_{x=0} = U_\infty(y), \tag{1.9}$$

Free Boundary Conditions:

$$p(x, b(x)) = p_b(x), \tag{1.10}$$

$$(u, v) \cdot (-b'(x), 1) = 0. \tag{1.11}$$

We seek entropy solutions of the inverse problem (1.6)–(1.11) in the following sense:

Definition 1.1 (Entropy Solutions). A wedge boundary $\Gamma=\{(x,b(x)) : b(x)\in \mathrm{Lip}([0,\infty))\}$ and a vector function $U=(\mathbf{u}^\top,p,\rho)^\top\in BV(\Omega)$ form an entropy solution of the inverse problem (1.6)–(1.11) if they satisfy the following:

(i) U is a global weak solution of (1.6) satisfying (1.9)–(1.11) in the trace sense;
(ii) For any $a(S)\in C^1$ with $a'(S)\geqslant 0$, the entropy inequality:

$$(\rho u a(S))_x+(\rho v a(S))_y\geqslant 0 \tag{1.12}$$

holds in the distributional sense on $\Omega\cup\Gamma$.

The first main theorem of this paper is
Main Theorem I (Well-posedness). There exists $\epsilon_\infty>0$ such that, when $\|(\widetilde{U}_\infty,\tilde{p}_b)\|_{L^\infty\cap BV}<\epsilon_\infty$, the following results hold:

(i) *Global existence*: The wedge boundary $y=b(x)=\int_0^x b'_+(\xi)\,\mathrm{d}\xi$, with $b'_+(x)\in BV(\mathbb{R}_+)$ as the right-derivative, can be determined by the initial-boundary conditions (1.7)–(1.11), which is a small perturbation of the straight wedge boundary $y=b_0x$, and a corresponding global entropy solution $U(x,y)$ that satisfies the requirements of definition 1.1 can be obtained, which has bounded total variation:

$$\sup_{x>0}\mathrm{T.V.}\{U(x,y) : -\infty<y<b(x)\}<\infty, \tag{1.13}$$

and contains a strong shock $y=\chi(x)=\int_0^x s(\xi)\mathrm{d}\xi$, where $s(x)\in BV(\mathbb{R}_+)$ and $\chi(x)$ is a small perturbation of $y=s_0x$, the straight strong shock corresponding to the straight wedge.

(ii) *Existence of semigroup*: There exists $\varepsilon>0$ such that, for any $(b(0),U_\infty(\cdot+b(0)),p_b)\in\mathbb{D}^\varepsilon$ (see definition 5.1 below), the solution of the inverse problem determines a uniform Lipschitz semigroup $\mathfrak{S}_x$:

$$(b(0),U_\infty(\cdot+b(0)),p_b)\mapsto(b(x),U(x,\cdot+b(x)),\iota_x p_b)$$

satisfying that

$$\mathfrak{S}_0(b(0),U_\infty(\cdot+b(0)),p_b)=(b(0),U_\infty(\cdot+b(0)),p_b),$$
$$\mathfrak{S}_{x_1}\mathfrak{S}_{x_2}(b(0),U_\infty(\cdot+b(0)),p_b)=\mathfrak{S}_{x_1+x_2}(b(0),U_\infty(\cdot+b(0)),p_b),$$

and there exist $L^\sharp$ and $L^\flat>0$ so that, for $(b_i(0),U_{\infty,i}(\cdot+b_i(0)),p_{b,i})\in\mathbb{D}^\varepsilon$ and any $x_i\geqslant 0$, $i=1,2$,

$$\begin{aligned}&\|\mathfrak{S}_{x_1}(b_1(0),U_{\infty,1}(\cdot+b_1(0)),p_{b,1})-\mathfrak{S}_{x_2}(b_2(0),U_{\infty,2}(\cdot+b_2(0)),p_{b,2})\|_Y\\&\quad\leqslant L^\sharp\|(b_1(0),U_{\infty,1}(\cdot+b_1(0)),p_{b,1})-(b_2(0),U_{\infty,2}(\cdot+b_2(0)),p_{b,2})\|_Y+L^\flat|x_1-x_2|,\end{aligned}$$

where ι_x and $\|\cdot\|_Y$ are given in (5.1) and (5.2) below, respectively.

To compare the difference of the full Euler flows and the potential flows in solving the above inverse problem, we also need to study the inverse problem for (1.4) and (1.5). Similarly, we have parallel results for the potential flows: There exist both a wedge boundary $y=b_{\mathrm{P}}(x)<0$ and an entropy solution $\mathbf{u}_{\mathrm{P}}=(u_{\mathrm{P}},v_{\mathrm{P}})^\top$ in

$$\Omega_{\mathrm{P}}=\{(x,y) : x\geqslant 0, y<b_{\mathrm{P}}(x)\}$$

with the upper boundary:

$$\Gamma_{\mathrm{P}} = \{(x,y) : x \geqslant 0, y = b_{\mathrm{P}}(x)\},$$

satisfying

Incoming Flow Condition:

$$\mathbf{u}_{\mathrm{P}}|_{x=0} = \mathbf{u}_{\infty}, \tag{1.14}$$

Free Wedge Boundary Conditions:

$$p_{\mathrm{P}}(x, b(x)) = p_b(x), \tag{1.15}$$

$$\mathbf{u}_{\mathrm{P}} \cdot \mathbf{n}_{\mathrm{P}}|_{\Gamma_{\mathrm{P}}} = 0. \tag{1.16}$$

The initial-boundary data are assumed to satisfy the following conditions:

($\mathbf{C}_{\mathrm{P}}\mathbf{1}$) $\mathbf{u}_\infty(y) := \bar{\mathbf{u}}_\infty + \tilde{\mathbf{u}}_\infty(y)$ is the incoming flow at $x = 0$ such that
  (a) $\bar{\mathbf{u}}_\infty = (\bar{u}_\infty, 0)^\top$ is a constant vector with

$$\bar{u}_\infty > 0, \quad \frac{2(\gamma-1)B_\infty}{\gamma+1} < |\bar{u}_\infty|^2 < 2B_\infty. \tag{1.17}$$

  (b) $\tilde{\mathbf{u}}_\infty(y) = (\tilde{u}_\infty, \tilde{v}_\infty)^\top(y) \in (L^1 \cap BV)(\mathbb{R};\mathbb{R}^2)$ is a BV perturbation at $x = 0$ with sufficiently small $\|\tilde{\mathbf{u}}_\infty\|_{L^\infty \cap BV}$.

($\mathbf{C}_{\mathrm{P}}\mathbf{2}$) $p_b(x) := \bar{p}_b + \tilde{p}_b(x)$ is the pressure distribution on the wedge boundary such that
  (a) $\bar{p}_b = (\mathscr{R}(|\bar{\mathbf{u}}_\infty|))^\gamma$ with

$$\mathscr{R}(r) = \frac{\gamma-1}{\gamma}\left(B_\infty - \frac{r^2}{2}\right)^{\frac{1}{\gamma-1}}. \tag{1.18}$$

  (b) $\tilde{p}_b(x) \in (L^1 \cap BV)(\mathbb{R};\mathbb{R})$ is a BV perturbation with sufficiently small $\|\tilde{p}_b\|_{L^\infty \cap BV}$.

Meanwhile, in solving the inverse problem for the full Euler equations, we assume that
($\mathbf{C}_{\mathrm{E}}\mathbf{1}$) At $x = 0$, the incoming flow $U_\infty(y) = (\mathbf{u}_\infty^\top, (\mathscr{R}(|\mathbf{u}_\infty|))^\gamma, \mathscr{R}(|\mathbf{u}_\infty|))^\top$.
($\mathbf{C}_{\mathrm{E}}\mathbf{2}$) The pressure distribution $p_b(x) := \bar{p}_b + \tilde{p}_b(x)$ on the wedge boundary satisfies

(a) $\bar{p}_b = (\mathscr{R}(|\bar{\mathbf{u}}_\infty|))^\gamma$, where $\mathscr{R}(r)$ is defined in (1.18).
(b) $\tilde{p}_b(x) \in (L^1 \cap BV)(\mathbb{R};\mathbb{R})$ is a BV perturbation with sufficiently small $\|\tilde{p}_b\|_{L^\infty \cap BV}$.

Remark 1.1. In the above conditions, if $\bar{p}_b > (\mathscr{R}(|\bar{\mathbf{u}}_\infty|))^\gamma$ but smaller than a critical value, following the same argument as for the proof of Main Theorem I, we can also obtain the existence and stability of the solutions of the inverse problem for the potential flow equations containing a strong shock.

In particular, when only weak waves are involved, we have

Main Theorem II (Comparison of the Two Models). Assume that ($\mathbf{C}_{\mathrm{P}}\mathbf{1}$)–($\mathbf{C}_{\mathrm{P}}\mathbf{2}$) and ($\mathbf{C}_{\mathrm{E}}\mathbf{1}$)–($\mathbf{C}_{\mathrm{E}}\mathbf{2}$) hold. Let $U_{\mathrm{E}} = (\mathbf{u}_{\mathrm{E}}^\top, p_{\mathrm{E}}, \rho_{\mathrm{E}})^\top$ be the entropy solution of (1.6)–(1.11) with the corresponding boundary function $y = b_{\mathrm{E}}(x)$, and let $U_{\mathrm{P}} = (\mathbf{u}_{\mathrm{P}}^\top, (\mathscr{R}(|\mathbf{u}_{\mathrm{P}}|))^\gamma, \mathscr{R}(|\mathbf{u}_{\mathrm{P}}|))^\top$ be the entropy solution of (1.4)–(1.5) and (1.15)–(1.16) with the corresponding boundary function $y = b_{\mathrm{P}}(x)$. Then there exist $\epsilon_{\mathrm{c}} > 0$ and $C > 0$ such that, when $\|(\tilde{\mathbf{u}}_\infty, \tilde{p}_b)\|_{L^\infty \cap BV} < \epsilon_{\mathrm{c}}$, for any $x > 0$,

$$\|(b_{\mathrm{E}}(x), U_{\mathrm{E}}(x, \cdot + b_{\mathrm{E}}(x)), p_b) - (b_{\mathrm{P}}(x), U_{\mathrm{P}}(x, \cdot + b_{\mathrm{P}}(x)), p_b)\|_Y \leqslant Cx\|(\tilde{\mathbf{u}}_\infty, \tilde{p}_b)\|_{L^\infty \cap BV}^3.$$

In comparison with the previous results on the initial-boundary value problem with fixed boundary and the Cauchy problem, one of the new main difficulties in solving the inverse problem is how the unknown wedge boundary is determined, especially in the solution space of low regularity, $L^\infty \cap BV$. In most of the previous works dealing with smooth free boundary problems such as [20, 35], a transformation that flattens the boundary and changes the free boundary to a fixed boundary is introduced. Owing to the presence of the weak shock waves and the discontinuous pressure distribution, we are not able to apply such transformations directly. Instead, we adopt the idea in [42], constructing approximate boundaries and approximate leading shocks directly in the physical space and then proving that there is a converging subsequence whose limit recovers the wedge boundary and the leading shock. Moreover, we have to develop more sophisticated weighted estimates by combining the wave strength amplification on the boundary and the dissipation on the shock, making the modified Glimm-type functional decrease as x increases.

For the L^1–stability of solutions of the initial-boundary value problem with fixed boundary, the only thing that needs to be compared is the difference between two different solutions in the same fixed domain. Thus, the Lyapunov functional that measures the L^1–difference of two solutions can be designed and then, after a careful analysis of the changes of this functional near the boundary, a group of weights can be chosen to make the functional decrease, by combining the interaction estimates only involving weak waves and the fact that the total strengths of weak waves in the other families are dominated by the strength of that strong leading shock. However, for the inverse problem, new phenomena occur, mainly owing to the unknown boundaries. To establish the stability of solutions of the inverse problem, we have to determine a region on which two solutions are compared and the additional distance between the two boundaries is required to be controlled. At a glance, it is ambiguous whether a corresponding Lyapunov functional could be constructed, not to mention how it could be non-increasing.

To overcome this difficulty, our method is to consider the difference of the two solutions and the two boundaries simultaneously. We first extend two solutions to a larger domain (exactly the union of the two domains on which the two inverse problems solve). Then we construct a Lyapunov functional that controls the L^∞–norm of the difference of the two boundaries. Compared to the Lyapunov functional for the fixed boundary case (see [12]), our analysis introduces new terms, bringing additional interactions near the boundary. To make the functional decrease, we exploit the boundary condition that the pressure near the boundary aligns with the given pressure distribution on the wedge boundary. This alignment implies the additional quantitative relations, enabling the precise near-boundary estimates (see (4.14)). These estimates suggest a proper choice of weighting coefficients for the Lyapunov functional. Then, employing the interaction estimates only involving weak waves and using the strength of the leading shock to control the total strengths of weak waves in the other families, we obtain that the functional is non-increasing in the flow direction in those non-interacting region. However, the extension brings extra discontinuities (jumps) along the x-direction. So we have to manage the potential increase in the Lyapunov functional when traversing such discontinuities. Luckily, the magnitude of jumps is equivalent to the strength of the weak waves generated from the boundary, whose total variation is controlled by the decay property of the Glimm-type functional. Therefore, at each time when such a jump of strength $O(1)|\alpha|$ comes out, the Lyapunov functional experiences bounded amplification, growing by a factor of $1+O(1)|\alpha|$ (see remark 4.3), but the cumulative amplification remains controlled due to the bounded total wave strength. Ultimately, the Lyapunov functional is shown to satisfy a uniform bound proportional to its initial value, thereby establishing the stability of the solution.

Furthermore, it seems to be difficult to directly compare the two solutions of the full Euler equations and the potential flow equations. On the other hand, following [52], we can make full use of the semigroup $\mathfrak{S}_x$ after taking the boundary influence into consideration. We first compare the difference of the solutions of the Riemann-type problems including the Riemann-type inverse problems in the two different models, and then establish an estimate of the difference between the two corresponding approximate solutions locally. Combining the local estimates and properties of the semigroup (see proposition 6.7), we finally obtain our result as desired.

We organize the rest of this paper as follows: In section 2, we study some basic properties of the full Euler equations and related Riemann-type problems including the Riemann-type inverse problems. We also obtain the corresponding nonlinear wave interaction estimates. In section 3, we first introduce a wave-front tracking algorithm to construct approximate boundaries and solutions. Then an interaction potential Q is given by considering both the influence of pressure changing on the wedge boundary and the interaction estimates of wave-fronts together, based on which we design a Glimm-type functional and prove that it is non-increasing. Therefore, the global existence of entropy solutions of the inverse problem is obtained. In section 4, a modified Lyapunov functional $\mathfrak{F}$ for the two solutions is given, which is equivalent to the L^1–distance between the two solutions and the L^∞–distance between the two boundaries. Then we prove that $\mathfrak{F}$ is non-increasing as x increases, which leads to the L^1–stability of the solutions that contain a strong shock and the L^∞–stability of two boundaries. Next, in section 5, from the elementary estimates established in sections 3 and 4, we show that there exists a Lipschitz semigroup $\mathfrak{S}_x$ generating the entropy solutions and boundaries of the inverse problem. Finally, in section 6, we use this semigroup to compare the two solutions of the inverse problem in a supersonic Euler flow and a supersonic potential flow, and prove that, at $x > 0$, the distance between the two boundaries and the difference of the two solutions in L^1 are of the same order of x multiplied by the cube of the perturbation of initial boundary data, i.e. $O(1)x\|(\widetilde{\mathbf{u}}_\infty, \tilde{p}_b)\|^3_{L^\infty\cap BV}$.

2. Steady Euler equations and Riemann problems

In this section, we first give some basic properties of system (1.6) and then study nonlinear waves and related interaction estimates that are used in the subsequent development.

When $u > c$, system (1.6) has four eigenvalues:

$$\begin{aligned} &\lambda_j = \frac{uv + (-1)^j c\sqrt{u^2+v^2-c^2}}{u^2-c^2} \qquad \text{for } j = 1, 4,\\ &\lambda_k = \frac{v}{u} \qquad \text{for } k = 2, 3, \end{aligned} \tag{2.1}$$

and corresponding eigenvectors:

$$\begin{aligned} &\mathbf{r}_j = \mathfrak{k}_j\Big(-\lambda_j, 1, \rho(\lambda_j u - v), \frac{\rho(\lambda_j u - v)}{c^2}\Big)^\top \qquad \text{for } j = 1, 4,\\ &\mathbf{r}_2 = (u, v, 0, 0)^\top, \qquad \mathbf{r}_3 = (0, 0, 0, \rho)^\top, \end{aligned} \tag{2.2}$$

where

$$\mathfrak{k}_j = \mathfrak{k}_j(U) = \frac{2}{\gamma+1}\frac{(u^2-c^2)\lambda_j - uv}{(1+\lambda_j^2)(\lambda_j u - v)} \qquad \text{for } j = 1, 4. \tag{2.3}$$

It is direct to see that $\mathbf{r}_j \cdot \nabla\lambda_j = 1$ for $j = 1, 4$, and $\mathbf{r}_k \cdot \nabla\lambda_k = 0$ for $k = 2, 3$.

The discontinuous wave curves of (1.6) must satisfy the Rankine–Hugoniot conditions:

$$s\,[W(U)] = [H(U)]\,, \tag{2.4}$$

where s is the discontinuity speed.

The contact Hugoniot curves $C_k(U_0), k = 2,3$, through U_0 are

Vortex sheets:

$$C_2\,(U_0): \qquad s = \frac{v}{u} = \frac{v_0}{u_0}, \quad p = p_0, \quad \rho = \rho_0; \tag{2.5}$$

Entropy waves:

$$C_3\,(U_0): \qquad s = \frac{v}{u} = \frac{v_0}{u_0}, \quad \mathbf{u} = \mathbf{u}_0, \quad p = p_0. \tag{2.6}$$

Corresponding to the repeated eigenvalues $\lambda_2 = \lambda_3 = \frac{v}{u}$, there are two linearly independent eigenvectors. Hence, in the physical (x, y)–plane, the vortex sheet and the entropy wave appear as one characteristic discontinuity, while in the phase space, they need to be determined by two parameters independently.

The nonlinear j-waves, $j = 1, 4$, are shock waves or rarefaction waves. In the state space, the rarefaction wave curves $R_j^+(U_0)$ through U_0 are given by

$$R_j^+\,(U_0): \quad \mathrm{d}u = -\lambda_j \mathrm{d}v, \quad \rho\,(\lambda_j u - v)\,\mathrm{d}v = \mathrm{d}p, \quad \mathrm{d}p = c^2 \mathrm{d}\rho \quad \text{ for } \begin{cases} \rho < \rho_0, & j = 1, \\ \rho > \rho_0, & j = 4. \end{cases} \tag{2.7}$$

The speeds of shock waves are

$$s_j := \frac{u_0 v_0 + (-1)^j\,\bar{c}_0 \sqrt{u_0^2 + v_0^2 - \bar{c}_0^2}}{u_0^2 - \bar{c}_0^2} \qquad \text{for } j = 1, 4, \tag{2.8}$$

where $\bar{c}_0^2 = \frac{c_0^2}{\mathfrak{b}_0}\frac{\rho}{\rho_0}$ and $\mathfrak{b}_0 = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}\frac{\rho}{\rho_0}$. Substituting s_j into (2.4) leads to the j-Hugoniot curve $S_j(U_0)$ across U_0:

$$S_j\,(U_0): \qquad [u] = -s_j\,[v]\,, \quad [p] = \frac{c_0^2}{\mathfrak{b}_0}\,[\rho]\,, \quad \rho_0\,(s_j u_0 - v_0)\,[v] = [p] \qquad \text{for } j = 1,4. \tag{2.9}$$

For a piecewise smooth solution U that contains a shock, each of the following conditions is equivalent to (1.12) (see also [12, 14]):

(i) The density increases across the shock along the flow direction:

$$\rho_{\text{back}} > \rho_{\text{front}}. \tag{2.10}$$

(ii) The speed s_j of the jth-shock satisfies

$$\lambda_j\,(\text{above}) < s_j < \lambda_j\,(\text{below}) \;\; \text{for } j = 1, 4, \quad s_1 < \lambda_{2,3}\,(\text{above})\,, \quad \lambda_{2,3}\,(\text{above}) < s_4. \tag{2.11}$$

Here, the state above the 1-shock refers to its back state, while the state above the 4-shock refers to its front state.

In the phase space, we define $S_1^-(U_0)$ as the subset of $S_1(U_0)$ with $\rho > \rho_0$, and $S_4^-(U_0)$ as the subset of $S_4(U_0)$ with $\rho < \rho_0$. In the (x, y)–plane, any state on $S_j^-(U_0)$ leads to a shock connecting to the below state U_0 satisfying the entropy condition (1.12) so that $S_j^-(U_0)$ are called shock curves. Moreover, curves $S_j^-(U_0)$ coincide with $R_j^+(U_0)$ at state U_0 up to the second order for $j = 1, 4$.

As in [5, 23, 46], we parameterize $R_j(U_0)$ and $S_j(U_0)$ by $\alpha_j \mapsto R_j(\alpha_j)(U_0)$ and $\alpha_j \mapsto S_j(\alpha_j)(U_0)$, respectively, such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} R_j(\alpha_j)(U_0)\bigg|_{\alpha_j=0} = \frac{\mathrm{d}}{\mathrm{d}\alpha} S_j(\alpha_j)(U_0)\bigg|_{\alpha_j=0} = \mathbf{r}_j(U_0).$$

Then we define the nonlinear wave curves $T_j(U_0) = R_j^+(U_0) \cup S_j^-(U_0)$ and parameterize $T_j(U_0)$ by $\alpha_j \mapsto \Phi_j(\alpha_j;U_0)$ so that

$$\Phi_j(\alpha_j;U_0) = \begin{cases} R_j(\alpha_j)(U_0), & \alpha_j \geqslant 0, \\ S_j(\alpha_j)(U_0), & \alpha_j < 0, \end{cases} \qquad \text{for } j = 1, 4.$$

For the linearly degenerate case, $T_k(U_0) = C_k(U_0)$, and we choose parameter $\alpha_k \mapsto \Phi_k(\alpha_k;U_0)$ such that

$$\frac{\mathrm{d}\Phi_k(\alpha_k;U_0)}{\mathrm{d}\alpha} = \mathbf{r}_k(U_0) \qquad \text{for } k = 2, 3. \tag{2.12}$$

With these, we define

$$\Phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_0) = \Phi_4(\alpha_4;\Phi_3(\alpha_3;\Phi_2(\alpha_2;\Phi_1(\alpha_1;U_0)))), \tag{2.13}$$

and denote the ith component of $\Phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_0)$ by $\Phi^{(i)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_0)$, $i = 1, 2, 3, 4$.

2.1. Riemann-type problems and Riemann solutions

We now investigate several Riemann-type problems and their solutions, which are essential in the construction of approximate solutions for the inverse problem (1.6)–(1.11) when carrying out the front tracking algorithm.

Inverse Riemann problem. Consider an inverse Riemann problem with a boundary Γ to be determined:

$$\begin{cases} (1.6), \\ U|_{\{x=\bar{x},\, y<\bar{y}\}} = U_-, \\ p|_\Gamma = p_+, \quad \mathbf{u}\cdot\mathbf{n}|_\Gamma = 0, \end{cases} \tag{2.14}$$

where $U_- = (\mathbf{u}_-^\top, p_-, \rho_-)^\top$ satisfies $|\mathbf{u}_-| > \sqrt{\frac{\gamma p_-}{\rho_-}}$, and Γ starts at $(\bar{x},\bar{y})$. Set $U_0 = (|\mathbf{u}_-|, 0, p_-, \rho_-)^\top$ and denote the shock polar through U_0 by $S(U_0) = S_1(U_0) \cup S_4(U_0)$. For any state $U = (u,v,p,\rho)^\top$ on the shock polar $S(U_0)$, we use $\theta = \arctan(\frac{v}{u})$ to denote the angle of the flow direction. Then, from [22] (also see [19]), we know that there is a critical angle $\theta_{\text{sonic}} < 0$ such that $\theta_{\text{sonic}} < \theta < 0$. On the (u,v)–plane, ray $v = u\tan\theta$ with $u \geqslant 0$ intersects with curve $S_1^+(U_0)$ at a supersonic state $U = (u,v,p,\rho)^\top$, $u^2+v^2 > c^2 := \frac{\gamma p}{\rho}$; see figure 2. Furthermore, from the relation (see [19, 22]):

$$\tan\theta = -\frac{\frac{p}{p_-}-1}{\gamma M_0^2 - \frac{p}{p_-}+1}\sqrt{\frac{(1+\frac{\gamma-1}{\gamma+1})(M_0^2-1)-(\frac{p}{p_-}-1)}{\frac{p}{p_-}+\frac{\gamma-1}{\gamma+1}}}, \qquad M_0^2 = \frac{|\mathbf{u}_-|^2\rho_-}{\gamma p_-},$$

for the critical angle θ_{sonic}, we can find a corresponding critical pressure:

$$p_{\text{sonic}} = p_{\text{sonic}}(|\mathbf{u}_-|, p_-, \rho_-) \tag{2.15}$$

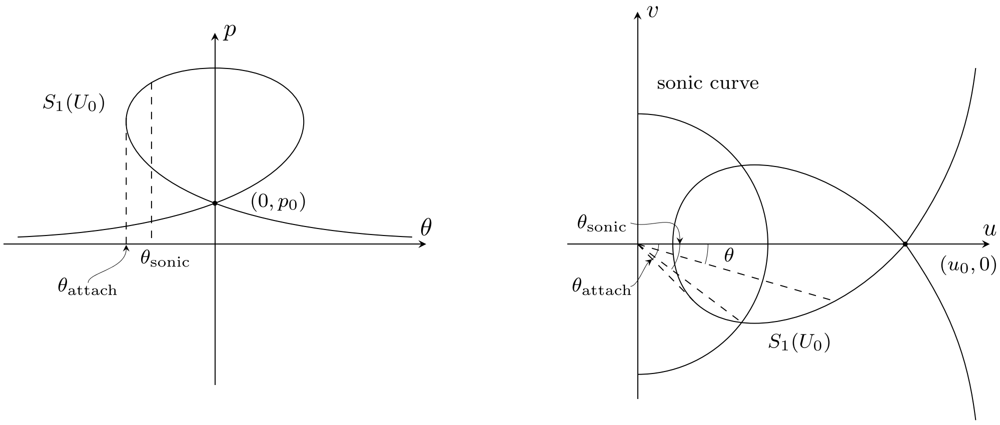


Figure 2. Shock polar and critical angle.

such that, given $p_- < p < p_{\text{sonic}}$, there is a unique supersonic state $U = (u,v,p,\rho)^\top \in S_1^+(U_0)$.

Moreover, as in [22] (see also [19]), recalling (1.8), it can be shown that, when

$$(\bar{x},\bar{y}) = (0,0)\,, \quad p_+ = \bar{p}_b, \quad U_- = \overline{U}_\infty, \tag{2.16}$$

there is a unique entropy solution of the above inverse problem, consisting of two constant states U_- and $U_+ = (u_+,0,p_+,\rho_+)^\top$ with $u_\pm > c_\pm > 0$, connected by a 1-shock wave of speed s_0, and the boundary is thus determined as $\Gamma = \{(x,0) : x > 0\}$; see figure 3.

Riemann problem (*only weak waves*): Consider the Riemann problem:

$$\begin{cases} (1.6), \\ U|_{x=\bar{x}} = \begin{cases} U_l & \text{for } y < \bar{y}, \\ U_r & \text{for } y > \bar{y}, \end{cases} \end{cases} \tag{2.17}$$

where the constant states U_l and U_r are the below state and the above state with respect to line $y = \bar{y}$, respectively. Then there exists $\epsilon_{\rm w} > 0$ such that, for U_l, $U_r \in O_{\epsilon_{\rm w}}(U_-)$, or U_l, $U_r \in O_{\epsilon_{\rm w}}(U_+)$, problem (2.17) has a unique admissible solution containing at most four waves $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ that connect U_l and U_r by $U_r = \Phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_l)$.

Riemann problem (*containing strong 1-shock*): As in [12, 14], we have

Lemma 2.1. *If U_+ is connected with U_- through a 1-shock wave of speed s_0 with $\rho_+ > \rho_-$, i.e.*

$$s_0\left(W(U_-) - W(U_+)\right) = H(U_-) - H(U_+)\,, \tag{2.18}$$

then

$$\begin{aligned} &s_0 < 0, \quad u_+ < u_- < \left(1 + \frac{1}{\gamma}\right)u_+, \\ &\det(\nabla_U H(U_+) - s_0 \nabla_U W(U_+)) > 0. \end{aligned}$$

In addition, there exists $\epsilon_{\rm s} > 0$ such that, for any $U_0 \in O_{\epsilon_{\rm s}}(U_-)$, $S_1^+(U_0) \cap O_{\epsilon_{\rm s}}(U_+)$ can be parameterized by the shock speed s as: $s \mapsto G(s;U_0)$ near (s_0,U_-) with $G = (G^{(1)},G^{(2)},G^{(3)},G^{(4)})^\top \in C^2$ and $G(s_0;U_-) = U_+$.

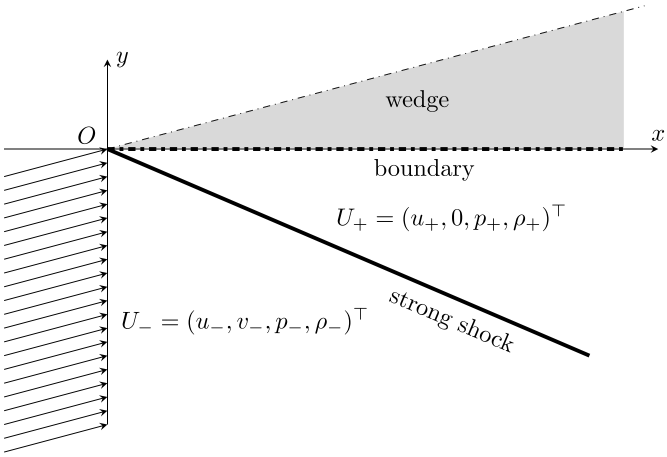


Figure 3. Background solutions.

Now we give an explicit formula of the solution of the Riemann problem (2.17) in a forward neighbourhood $B_+((\bar{x},\bar{y}),r)$ of $(\bar{x},\bar{y})$:

$$B_+\left((\bar{x},\bar{y}),r\right) = \left\{(x,y)\,:\,(x-\bar{x})^2+(y-\bar{y})^2<r^2 \text{ and } x>\bar{x} \text{ for some } r>0\right\}.$$

When only weak waves are involved, assume that $U_r = \Phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_l)$. Then any solution of this Riemann problem generally has the following form:

$$U_{Rw}(x,y) = \begin{cases} U_{m_0} = U_l, & \xi < \sigma_1^-, \\ \Phi_1\left(\xi-\lambda_1\left(U_{m_0}\right);U_{m_0}\right), & \sigma_1^- < \xi < \sigma_1^+, \\ U_{m_1}, & \sigma_1^+ < \xi < \sigma_2^-, \\ U_{m_3}, & \sigma_3^+ < \xi < \sigma_4^-, \\ \Phi_4\left(\xi-\lambda_4\left(U_{m_3}\right);U_{m_3}\right), & \sigma_4^- < \xi < \sigma_4^+, \\ U_{m_4} = U_r, & \sigma_4^+ < \xi, \end{cases} \tag{2.19}$$

where

$$\begin{aligned} &\xi = \frac{y-\bar{y}}{x-\bar{x}}, \qquad U_{m_1} = \Phi\left(\alpha_1,0,0,0;U_l\right), \qquad U_{m_3} = \Phi\left(0,\alpha_2,\alpha_3,0;U_{m_1}\right), \\ &\sigma_j^+ = \sigma_j^- = \sigma_j, \qquad \sigma_j \text{ is the speed of the weak shock } \alpha_j \text{ when } \alpha_j<0 \text{ for } j=1,4, \\ &\sigma_j^- = \lambda_j\left(U_{m_{j-1}}\right), \quad \sigma_j^+ = \lambda_j\left(U_{m_j}\right) \qquad \text{when } \alpha_j>0 \text{ for } j=1,4, \\ &\sigma_2^- = \sigma_3^+ = \lambda_2\left(U_{m_1}\right) = \lambda_3\left(U_{m_3}\right). \end{aligned} \tag{2.20}$$

When a strong 1-shock wave is involved, saying $U_r = \Phi(0,\alpha_2,\alpha_3,\alpha_4;G(s;U_l))$, it suffices to let $U_{m_1} = G(s;U_l)$ and $\sigma_1^+ = \sigma_1^- = s$ in (2.19).

2.2. Wave interaction and reflection estimates

In the following estimates, $O(1)$ is denoted to be bounded so that the bound of $|O(1)|$ depends only on U_-, U_+, and system (1.1). Firstly, we have estimates of the interactions among weak waves, which are classical Glimm's wave interaction estimates inside the domain used in case 1; see [12, 14].

Lemma 2.2. *There is a positive constant* ε_w *such that, for three constant states* U_l, U_m, $U_r \in O_{\varepsilon_w}(U_-)$, *or* U_l, U_m, $U_r \in O_{\varepsilon_w}(U_+)$, *with* $U_m = \Phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_l)$ *and* $U_r = \Phi(\beta_1,\beta_2,\beta_3,\beta_4;U_m)$, *we can find* $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ *such that* $U_r = \Phi(\gamma_1,\gamma_2,\gamma_3,\gamma_4;U_l)$ *and*

$$\gamma_i = \alpha_i + \beta_i + O(1)\triangle(\alpha,\beta) \qquad \text{for } i = 1,2,3,4,$$

where $\triangle(\alpha,\beta) = \sum_{1\leqslant j<i\leqslant 4}|\alpha_i||\beta_j| + \sum_{k=1}^{4}\triangle_k(\alpha,\beta)$ *with*

$$\triangle_k(\alpha,\beta) = \begin{cases} 0 & \text{when } \alpha_k \geqslant 0 \text{ and } \beta_k \geqslant 0, \\ |\alpha_k||\beta_k| & \text{otherwise.} \end{cases}$$

To balance the alteration of the pressure distribution on the boundary, we allow 1-waves emanating from the boundary when solving the following initial-boundary value problem:

$$\begin{cases} (1.6), \\ U|_{\{x=\bar{x},\,y<\bar{y}\}} = U_1, \\ p|_\Gamma = p_2, \ \mathbf{u}\cdot\mathbf{n}|_\Gamma = 0. \end{cases}$$

When only weak waves involve, we have

Lemma 2.3. *There is a positive constant* ε_p *such that, for any* $U_1 = (u_1,v_1,p_1,\rho_1)^\top \in O_{\varepsilon_p}(U_+)$ *and* p_1, $p_2 \in O_{\varepsilon_p}(p_+)$, *the equation*:

$$\Phi^{(3)}(\delta_1,0,0,0;U_1) = p_2 \tag{2.21}$$

determines a unique twice differentiable function $\delta_1 = \delta_1(p_2,U_1)$. *Furthermore, there exists a bounded quantity* K_b, *whose bound is independent of* δ_1 *and* $p_1 - p_2$, *such that*

$$\delta_1 = K_b(p_2 - p_1).$$

Proof. Since $\Phi^{(3)}(0,0,0,0;U_1) = p_1$, differentiating (2.21) with respect to δ_1, we have

$$\frac{\partial\Phi^{(3)}(\delta_1,0,0,0;U_1)}{\partial\delta_1}\Big|_{\{\delta_1=0,\,U_1=U_+\}} = \mathfrak{k}_1(U_+)\rho_+(\lambda_1(U_+)u_+ - v_+) \neq 0.$$

Then the implicit function theorem gives the result. Furthermore, the bound of K_b depends only on U_+ and system (1.1). □

Remark 2.1. Lemma 2.3 implies that a discontinuity in the pressure distribution on the wedge generates a weak 1-wave, whose strength is governed by the magnitude of the jump in the pressure distribution. This estimate will be applied in case 4 in the proof of proposition 3.1.

Also, with the presence of a strong 1-shock wave issuing from the boundary (see [12, 14]), we have

Lemma 2.4. *There exists* $\varepsilon_s > 0$ *such that, when* $p_2 \in O_{\varepsilon_s}(p_+)$ *and* $U_1 \in O_{\varepsilon_s}(U_-)$, *the equation*:

$$G^{(3)}(s;U_1) = p_2 \tag{2.22}$$

determines a unique twice differentiable function $s = s(p_2,U_1)$ *with*

$$s = s_0 + K_{bs}(|p_2 - p_+| + |p_1 - p_-|),$$

where $|K_{bs}|$ *has a bound depending only on* U_-, U_+, *and system (1.1).*

Next, we give the estimates of reflections of weak waves on the boundary.

Lemma 2.5. *There exists a positive constant* ε_r *such that, for any* U_1, $U_2 \in O_{\varepsilon_r}(U_+)$ *with* $U_2 = \Phi(0,\beta_2,\beta_3,\beta_4;U_1)$, *the equation*:

$$\Phi^{(3)}(\delta_1,0,0,0;U_1) = \Phi^{(3)}(0,\beta_2,\beta_3,\beta_4;U_1) \tag{2.23}$$

determines a unique twice differentiable function $\delta_1 = \delta_1(\beta_2,\beta_3,\beta_4,U_1)$ *satisfying*

$$\delta_1 = K_{b2}\beta_2 + K_{b3}\beta_3 + K_{b4}\beta_4,$$

where K_{bi}, $i = 2, 3, 4$, *are* C^2*–functions of* $(\beta_2,\beta_3,\beta_4,U_1)$ *satisfying*

$$\begin{cases} K_{b2}|_{\{\beta_2=\beta_3=\beta_4=0,\,U_1=U_+\}} = K_{b3}|_{\{\beta_2=\beta_3=\beta_4=0,\,U_1=U_+\}} = 0, \\ K_{b4}|_{\{\beta_2=\beta_3=\beta_4=0,\,U_1=U_+\}} = -1. \end{cases}$$

Proof. The existence of δ_1 can be proved analogously as in lemma 2.3. Differentiating (2.23) with respect to β_i gives

$$\frac{\partial\Phi^{(3)}(\delta_1,0,0,0;U_1)}{\partial\delta_1}K_{bi} = \frac{\Phi^{(3)}(0,\beta_2,\beta_3,\beta_4;U_1)}{\partial\beta_i}.$$

Using $U_+ = (u_+,0,p_+,\rho_+)^\top$, (2.2)–(2.3), and (2.12)–(2.13), we obtain our results. □

Remark 2.2. Lemma 2.5 shows that, when a weak 4-wave impinges on the boundary, a weak 1-wave is reflected, with its strength approximately matching that of the incident weak 4-wave. This estimate will be used in case 5 in the proof of proposition 3.1.

Furthermore, the following estimates of interactions with the presence of a strong shock are needed; see [12, 14] for their proofs.

Lemma 2.6. *There exists a positive constant* ε_1 *such that, for any* $U_l \in O_{\varepsilon_1}(U_-)$ *and* U_m, $U_r \in O_{\varepsilon_1}(U_+)$ *with* $U_m = \Phi(0,\alpha_2,\alpha_3,\alpha_4;G(s;U_l))$ *and* $U_r = \Phi(\beta_1,0,0,0;U_m)$, *the equation*:

$$\Phi(0,\delta_2,\delta_3,\delta_4;G(s';U_l)) = \Phi(\beta_1,0,0,0;\Phi(0,\alpha_2,\alpha_3,\alpha_4;G(s;U_l))) \tag{2.24}$$

determines twice differentiable functions $(s',\delta_2,\delta_3,\delta_4)$ *with*

$$s' = s + K_{s1}\beta_1, \qquad \delta_2 = \alpha_2 + K_{s2}\beta_1, \qquad \delta_3 = \alpha_3 + K_{s3}\beta_1, \qquad \delta_4 = \alpha_4 + K_{s4}\beta_1.$$

Moreover,

$$|K_{s4}|\big|_{\{\alpha_4=\beta_1=0,\,s=s_0,\,U_l=U_-\}} = \left|\frac{\lambda_1(U_+)-s_0}{\lambda_4(U_+)-s_0}\right|\left|\frac{s_0u_-N-u_+\lambda_4(U_+)M}{s_0u_-N+u_+\lambda_4(U_+)M}\right| < 1,$$

and $|K_{si}|$ *are bounded for* $i = 1, 2, 3$, *where*

$$M = \frac{c_+^2}{\gamma-1}(2u_+-u_-) + u_+^2(u_+-u_-), \qquad N = -\frac{c_+^2}{\gamma-1} < 0.$$

In particular, the following holds:

$$|K_{s4}|\big|_{\{\alpha_4=\beta_1=0,\,s=s_0,\,U_l=U_-\}}\left|\frac{\lambda_4(U_+)-s_0}{\lambda_1(U_+)-s_0}\right| = \left|\frac{s_0u_-N-u_+\lambda_4(U_+)M}{s_0u_-N+u_+\lambda_4(U_+)M}\right| < 1.$$

Remark 2.3. Lemma 2.6 implies that, when a weak 1-wave interacts with the approximate leading shock from above, a weak 4-wave is reflected, with its strength attenuated during the interaction. This estimate will be applied in case 2 in the proof of proposition 3.1.

Lemma 2.7. *There exists a positive constant ε_2 such that, for any U_l, $U_m \in O_{\varepsilon_2}(U_-)$ and $U_r \in O_{\varepsilon_2}(U_+)$ with $U_m = \Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U_l)$ and $U_r = \Phi(0, \beta_2, \beta_3, \beta_4; G(s; U_m))$, the equation:*

$$\Phi(0, \delta_2, \delta_3, \delta_4; G(s'; U_l)) = \Phi(0, \beta_2, \beta_3, \beta_4; G(s; \Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U_l))) \tag{2.25}$$

determines twice differentiable functions $(s', \delta_2, \delta_3, \delta_4)$ with

$$s' = s + \sum_{i=1}^{4} \hat{K}_{1i}\alpha_i, \quad \delta_2 = \beta_2 + \sum_{i=1}^{4} \hat{K}_{2i}\alpha_i, \quad \delta_3 = \beta_3 + \sum_{i=1}^{4} \hat{K}_{3i}\alpha_i, \quad \delta_4 = \beta_4 + \sum_{i=1}^{4} \hat{K}_{4i}\alpha_i,$$

where $|\hat{K}_{ji}|$, $i, j = 1, 2, 3, 4$, are bounded, depending only on U_-, U_+ and system (1.1).

3. Construction of approximate solutions

In this section, a modified wavefront tracking algorithm is developed to construct approximate boundaries and solutions. Moreover, some necessary estimates are given for the boundary value problem (1.1) and (1.6)–(1.11).

We choose

$$\hat{\lambda} > \sup\{\lambda_4(U) : U \in O_{\varepsilon_0}(U_-) \cup O_{\varepsilon_0}(U_+)\}, \tag{3.1}$$

where ε_0 is a small constant satisfying

$$0 < \varepsilon_0 < \min\{\varepsilon_p, \varepsilon_s, \varepsilon_r, \varepsilon_1, \varepsilon_2\}. \tag{3.2}$$

For any $\mu > 0$, there exist $\Delta x > 0$ and $\Delta y = 2\hat{\lambda}\Delta x$ such that p_b and U_∞ are approached by piecewise constant functions $p_b^{\mu,\Delta x}$ and $U_\infty^{\mu,\Delta x}$, respectively:

$$\begin{aligned} &p_b^{\mu,\Delta x}(x) = p_b^{\mu,\Delta x,h} && \text{for } (h-1)\Delta x \leqslant x < h\Delta x,\ h = 1, 2, \cdots, N_0, \\ &U_\infty^{\mu,\Delta x}(y) = U_\infty^{\mu,\Delta x,l} && \text{for } \Delta y \leqslant x < (l+1)\Delta y,\ l = -1, -2, \cdots, -N_1, \end{aligned}$$

with

$$\begin{aligned} &\text{T.V.}(p_b^{\mu,\Delta x}) \leqslant \text{T.V.}(p_b) = \text{T.V.}(\tilde{p}_b), && \|p_b^{\mu,\Delta x}(\cdot) - p_b(\cdot)\|_{L^1(\mathbb{R}_+)} < \mu, \\ &\text{T.V.}(U_\infty^{\mu,\Delta x}) \leqslant \text{T.V.}(U_\infty) = \text{T.V.}(\widetilde{U}_\infty), && \|U_\infty^{\mu,\Delta x}(\cdot) - U_\infty(\cdot)\|_{L^1(\mathbb{R}_-)} < \mu. \end{aligned}$$

We construct approximate solutions $U^{\mu,\Delta x}(x,y)$ of the Riemann problem in a forward neighborhood of the origin. In our construction, we also obtain an approximate strong shock $\mathrm{S}^{\mu,\Delta x} = \{(x, \chi^{\mu,\Delta x}) : x \geqslant 0\}$ and an approximate boundary $\Gamma^{\mu,\Delta x} = \{(x, b^{\mu,\Delta x}) : x \geqslant 0\}$ in the forward neighborhood of the origin.

Then the approximate solutions are piecewise constant vector-valued functions that are separated by wavefronts. We let every wavefront travel freely until it collides with other wavefronts or the boundary. A new Riemann problem arises when different wavefronts collide and interact at some point. Also, on the approximate boundary, due to the change of $p_b^{\mu,\Delta x}$, new Riemann problems come out at $x = h\Delta x$ for $h \in \mathbb{N}_+$.

To construct approximate solutions to those new Riemann problems, following [5, 19, 29], we introduce two kinds of Riemann solvers, each of which contains shocks, contact discontinuities, rarefaction fronts, and non-physical fronts.

Let $\delta = \delta(\mu) > 0$ be a parameter that is larger than the maximum strength of rarefaction fronts, whose value is determined later. Moreover, we define non-physical fronts to be of family 5 of speed $\hat{\lambda}$ and use $U_r = T_5(\epsilon)(U_l)$ to indicate that the below state U_l is connected with the above state U_r by a non-physical front of strength $\epsilon = |U_r - U_l|$. Non-physical waves also belong to weak waves.

- *Accurate Riemann solver*. The accurate Riemann solver provides us with an approximate solution of the Riemann problem (1.6), where the rarefaction region in the real solution of the Riemann problem is replaced by piecewise constants that are separated by rarefaction wavefronts. To be specific, suppose that a j-wave α_j, originated at $(\bar{x}, \bar{y})$, is a rarefaction wave connecting two constant states $U_{m_{j-1}}$ and U_{m_j}, $j = 1, 4$. Taking the minimal $n \in \mathbb{N}_+$ such that $\alpha_j < n\delta$, we then define

$$
\begin{aligned}
&U^{\delta}_{j,\alpha_j}(x,y)\\
&:= U_{m_{j-1}} + \sum_{i=1}^{n} (U_{m_{j-1},i} - U_{m_{j-1},i-1}) H(y - \bar{y} - \lambda_j(U_{m_{j-1},i-1})(x - \bar{x})), \qquad j = 1,4,
\end{aligned}
$$

where

$$
U_{m_{j-1},i} = R_j(\frac{i\alpha_j}{n})(U_{m_{j-1}}), \qquad i = 0,1,\cdots,n,\ j = 1,4,
$$

and H is the Heaviside function. To obtain an accurate Riemann solver $U^{\delta}_{A}(U_l, U_r; x, y)$, we use $U^{\delta}_{j,\alpha_j}(x,y)$ to substitute $R_j(\frac{x}{t} - \lambda_j(U_{m_{j-1}}))(U_{m_{j-1}})$ in the real solution (2.19) when (x,y) belongs to the rarefaction region $\sigma_j^-(x - \bar{x}) < y - \bar{y} < \sigma_j^+(x - \bar{x})$ for $j = 1, 4$.

- *Simplified Riemann solver*. We have several different cases.

Case 1: Interactions of weak waves. Assume that two weak waves collide at $(\bar{x}, \bar{y})$ with below state U_l, middle state $U_m = \Phi_i(\beta; U_l)$, and above state $U_r = \Phi_j(\alpha; U_m)$, $i, j = 1,2,3,4$. We introduce the auxiliary above state

$$
U_r' = \begin{cases} \Phi_j(\beta; \Phi_i(\alpha; U_l)), & j > i, \\ \Phi_j(\alpha + \beta; U_l), & j = i. \end{cases}
$$

Then the simplified Riemann solver in a forward neighbourhood of $(\bar{x}, \bar{y})$ is defined as

$$
U^{\delta}_{S}(x,y) = \begin{cases} U^{\delta}_{A}(U_l, U_r'; x, y), & y - \bar{y} < \hat{\lambda}(x - \bar{x}), \\ U_r, & y - \bar{y} > \hat{\lambda}(x - \bar{x}). \end{cases}
$$

Case 2: Interactions of a non-physical wave with another weak wave. Assume that a non-physical wave collides another weak wave at $(\bar{x}, \bar{y})$ with below state U_l, middle state $U_m = \Phi_5(\epsilon; U_l)$, and above state $U_r = \Phi_j(\alpha; U_m)$ for $j = 1,2,3,4$. We choose the auxiliary above state as

$$
U_r' = \Phi_j(\alpha; U_l),
$$

and the simplified Riemann solver as

$$U_S^\delta(x,y) = \begin{cases} U_A^\delta(U_l, U_r'; x, y), & y - \bar{y} < \hat{\lambda}(x - \bar{x}), \\ U_r, & y - \bar{y} > \hat{\lambda}(x - \bar{x}). \end{cases}$$

Case 3: Interactions of a weak wave with the strong shock from above. Assume that a weak wave collides the strong shock at $(\bar{x}, \bar{y})$ with below state U_l, middle state $U_m = G(s; U_l)$, and above state $U_r = \Phi_j(\alpha; U_m)$ for $j = 1, 2, 3, 4$. We define the simplified Riemann solver as

$$U_S^\delta(x,y) = \begin{cases} U_l, & y - \bar{y} < s(x - \bar{x}), \\ U_m, & \hat{\lambda}(x - \bar{x}) > y - \bar{y} > s(x - \bar{x}), \\ U_r, & y - \bar{y} > \hat{\lambda}(x - \bar{x}). \end{cases}$$

Case 4: Interactions of a weak wave with the strong shock from below. Assume that a weak wave collides the strong shock at $(\bar{x}, \bar{y})$ with below state U_l, middle state $U_m = \Phi_j(\alpha; U_l)$, and above state $U_r = G(s; U_m)$ for $j = 1, 2, 3, 4$. We define the simplified Riemann solver as

$$U_S^\delta(x,y) = \begin{cases} U_l, & y - \bar{y} < s(x - \bar{x}), \\ G(s; U_l), & \hat{\lambda}(x - \bar{x}) > y - \bar{y} > s(x - \bar{x}), \\ U_r, & y - \bar{y} > \hat{\lambda}(x - \bar{x}). \end{cases}$$

Now we can define the approximate solutions inductively. For simplicity of notation, in the section, we omit superscript δ and always take $\|(\tilde{p}_b, \widetilde{U}_\infty)\|_{L^\infty \cap BV}$ small enough such that the conditions of lemmas 2.1–2.7 are satisfied (as we will prove later). Given $p_b^{\mu,\Delta x,1}$, lemma 2.4 provides us with $s_1^{\mu,\Delta x}$ such that

$$G^{(3)}(s_1^{\mu,\Delta x}; U_\infty^{\mu,\Delta x,-1}) = p_b^{\mu,\Delta x,1}.$$

For $x \in [0, \Delta x)$, define

$$U_b^{\mu,\Delta x}(x) = G(s_1^{\mu,\Delta x}; U_\infty^{\mu,\Delta x,-1}), \qquad b^{\mu,\Delta x}(x) = \int_0^x \frac{v_b^{\mu,\Delta x}(t)}{u_b^{\mu,\Delta x}(t)}\,\mathrm{d}t,$$
$$s^{\mu,\Delta x}(x) = s_1^{\mu,\Delta x}, \qquad \chi^{\mu,\Delta x}(x) = \int_0^x s^{\mu,\Delta x}(t)\,\mathrm{d}t.$$

Then we solve the standard Riemann problems at points $(0, l\Delta y)$, $l \in \mathbb{N}_-$, with $U_\infty^{\mu,\Delta x,l}$ and $U_\infty^{\mu,\Delta x,l-1}$ being the corresponding above and below states. Thus, we can construct the approximate solutions and the approximate strong shocks on the region:

$$\Omega^{\mu,\Delta x,1} = \left\{(x,y) : y < b^{\mu,\Delta x}(x),\ x \in [0, \Delta x)\right\}.$$

Suppose now that our approximate solutions $U^{\mu,\Delta x}(x,y)$ are constructed on

$$\bigcup_{k=1}^{h} \Omega^{\mu,\Delta x,k} := \bigcup_{k=1}^{h} \left\{(x,y) : y < b^{\mu,\Delta x}(x),\ x \in [(k-1)\Delta x, k\Delta x)\right\},$$

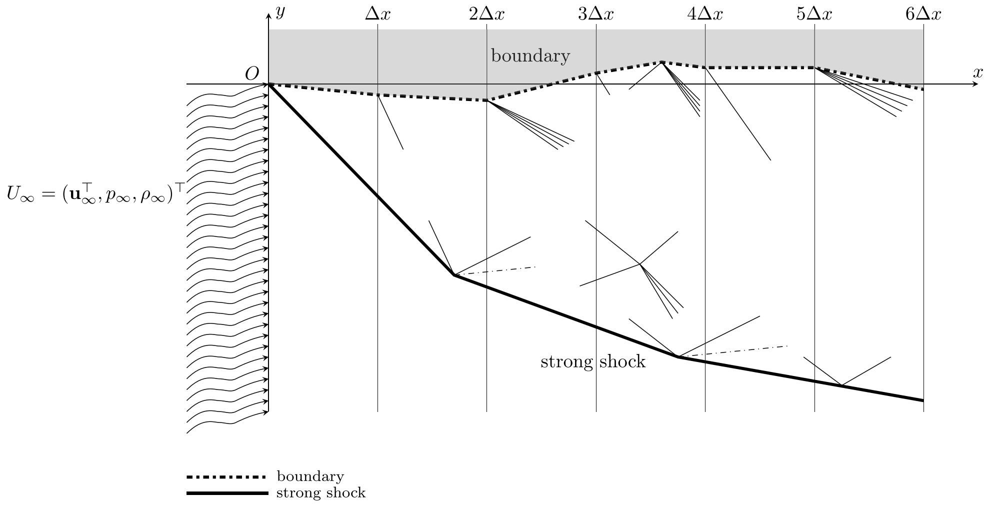


Figure 4. Approximate solutions.

and contain the jumps of rarefaction fronts $\mathcal{R}$, weak shock fronts $\mathcal{S}$, contact discontinuities $\mathcal{C}$, non-physical fronts $\mathcal{NP}$, and strong shock fronts $\mathcal{S}_s$; see figure 4. We also suppose that the approximate solutions are defined on the approximate boundaries $b^{\mu,\Delta x}(x)$ as $U_b^{\mu,\Delta x} = U^{\mu,\Delta x}(x, b^{\mu,\Delta x}(x))$ for $x \in [0, h\Delta x)$. Moreover, in $\Omega^{\mu,\Delta x,k}$, the number of these wavefronts is finite for $k = 1, \cdots, h$.

To extend our solutions, we generally let wavefronts travel ahead until they collide with another wavefront. In order to avoid the cases that more than two wavefronts collide at one point and that more than one wavefront interact with the boundary or the strong shock, we can modify the speeds of these wavefronts slightly such that the difference between the modified speeds and the original speeds is no more than μ.

When two wavefronts collide at some point $(\bar{x}, \bar{y})$ for $\bar{x} \in [h\Delta x, (h+1)\Delta x)$, three states separated by the wavefronts, from below to above, are labeled as U_l, U_m, and U_r. To make the number of wavefronts remain finite on region $\{x < h\Delta x,\ h \in \mathbb{N}_+\}$, we need to use the simplified Riemann solver to continue our construction; see [5].

Let $\nu > 0$ be a threshold parameter to determine when an accurate or a simplified Riemann solver is applied. With little abuse of notation, a front itself is denoted by α, while its strength is denoted by $|\alpha|$.

Case 1: α and β are weak waves colliding at $(\bar{x}, \bar{y})$. We solve the Riemann problem (2.17) as follows:

- When α and β are physical with $|\alpha\beta| > \nu$, we apply the accurate Riemann solver.
- When α and β are physical with $|\alpha\beta| \leqslant \nu$, or one of them is non-physical, we apply the simplified Riemann solver.

Case 2: A weak wave α interacts with a strong shock s at $(\bar{x}, \bar{y})$. We solve the corresponding Riemann problem as follows:

- When α is physical with $|\alpha| > \nu$, we apply the accurate Riemann solver.
- When α is non-physical or $|\alpha| \leqslant \nu$, we apply the simplified Riemann solver.

Case 3: When the approximate pressure changes at $(h\Delta x, b^{\mu,\Delta x}(h\Delta x))$, *we solve the Riemann problem (2.17) as follows*: Let

$$U_1 = U^{\mu,\Delta x}\left(h\Delta x-, b^{\mu,\Delta x}(h\Delta x-)-\right), \qquad p_2 = p_b^{\mu,\Delta x,h}.$$

Then lemma 2.3 provides us with δ_1 such that

$$\Phi^{(3)}(\delta_1, 0, 0, 0; U_l) = p_b^{\mu,\Delta x,h}.$$

We define $U_b^{\mu,\Delta x} = \Phi(\delta_1, 0, 0, 0; U_l)$ and always apply the accurate Riemann solver.
Case 4: When a weak physical wave α *separating states* U_r *(the above one) and* U_l *hits the boundary*:

- If $|\alpha| > \nu$, let

$$U_1 = U_l, \quad p_2 = p_b^{\mu,\Delta x,h}.$$

  Then lemma 2.5 provides us with δ_1 such that

$$\Phi^{(3)}(0,0,0,\alpha; U_l) = \Phi^{(3)}(\delta_1, 0, 0, 0; U_l).$$

  Define $U_b^{\mu,\Delta x} = \Phi(\delta_1, 0, 0, 0; U_l)$, where an accurate Riemann solver is used.
- If $|\alpha| \leqslant \nu$, we let α cross the boundary and change $U_b^{\mu,\Delta x}$ from U_r to U_l.

Case 5: When U_r *(the above) and* U_l *are separated by a non-physical weak wave that hits the boundary at* $(x, b^{\mu,\Delta x}(x))$ *for* $x \in ((h-1)\Delta x, h\Delta x)$, *we allow it cross the boundary and define* $U_b^{\mu,\Delta x} = U_l$.

Finally, let

$$\chi^{\mu,\Delta x}(x) = \int_0^x s^{\mu,\Delta x}(t)\mathrm{d}t, \; b^{\mu,\Delta x}(x) = \int_0^x \frac{v_b^{\mu,\Delta x}}{u_b^{\mu,\Delta x}}(t)\,\mathrm{d}t \qquad \text{for } x \in [h\Delta x, (h+1)\Delta x),$$

and let our approximate solutions $U^{\mu,\Delta x}$ have been constructed on the region

$$\Omega^{\mu,\Delta x,h} := \left\{(x,y) : y < b^{\mu,\Delta x}(x), \; x \in [h\Delta x, (h+1)\Delta x)\right\}$$

with approximate strong shocks $\mathtt{S}^{\mu,\Delta x}$, where $y = b^{\mu,\Delta x}(x)$ is our approximate boundary. In addition, on approximate boundaries, we define

$$U^{\mu,\Delta x}(x, b^{\mu,\Delta x}(x)) = U_b^{\mu,\Delta x}(x) \qquad \text{for } x \in [0, (h+1)\Delta x).$$

To show these approximate solutions can be well defined in

$$\Omega^{\mu,\Delta x} \cup \Gamma^{\mu,\Delta x} := \bigcup_{h=1}^{\infty} \Omega^{\mu,\Delta x,h} \cup \bigcup_{h=1}^{\infty} \Gamma^{\mu,\Delta x,h},$$

via the steps exhibited above, we need to prove that there exists a uniform bound, where $\Gamma^{\mu,\Delta x,h} = \{(x, b^{\mu,\Delta x}(x)) : (h-1)\Delta x \leqslant x < h\Delta x\}$ for $h \in \mathbb{N}_+$. Assume that $U^{\mu,\Delta x}$ has been defined on $\left(\bigcup_{k=1}^{h} \Omega^{\mu,\Delta x,k}\right) \bigcup \left(\bigcup_{k=1}^{h} \Gamma^{\mu,\Delta x,h}\right)$ and, furthermore, the following conditions hold:

$\mathrm{H}_1(\mathrm{h})$: There is a strong 1-shock

$$\mathrm{S}^{\mu,\Delta x,k} = \left\{(x, \chi^{\mu,\Delta x}(x)) : (k-1)\Delta x \leqslant x < k\Delta x\right\}$$

in $\Omega^{\mu,\Delta x,k}$ for $1 \leqslant k \leqslant h$, dividing $\Omega^{\mu,\Delta x,k}$ into $\Omega_-^{\mu,\Delta x,k}$ and $\Omega_+^{\mu,\Delta x,k}$, where $\chi^{\mu,\Delta x}(x) = \int_0^x s^{\mu,\Delta x}(t)\,\mathrm{d}t$, and $\Omega_+^{\mu,\Delta x,k}$ is the part bounded by $\mathrm{S}^{\mu,\Delta x}$ and $\Gamma^{\mu,\Delta x}$;

$\mathrm{H}_2(\mathrm{h})$: $U^{\mu,\Delta x}|_{\Omega_-^{\mu,\Delta x,k}} \in O_{\varepsilon_0}(U_-)$ and $U^{\mu,\Delta x}|_{\Omega_+^{\mu,\Delta x,k}} \in O_{\varepsilon_0}(U_+)$ in each $\Omega^{\mu,\Delta x,k}$ for $1 \leqslant k \leqslant h$, and $U^{\mu,\Delta x}|_{\Gamma^{\mu,\Delta x}} = U_b^{\mu,\Delta x}(x) \in O_{\varepsilon_0}(U_+)$ for $x \in ((h-1)\Delta x, h\Delta x)$, where $\varepsilon_0 > 0$ is introduced in (3.2);

$\mathrm{H}_3(\mathrm{h})$: $U^{\mu,\Delta x}|_{\bigcup_{k=1}^h \Omega^{\mu,\Delta x,k}}$ is piecewise constant and contains the jumps of rarefaction fronts $\mathcal{R}$, weak shock fronts $\mathcal{S}$, contact discontinuities $\mathcal{C}$, non-physical fronts $\mathcal{NP}$, and the number of these wavefronts is finite, saying N_h.

It suffices to prove that $U^{\mu,\Delta x}$ can be extended to $\Omega^{\mu,\Delta x,h+1}$ and satisfy $\mathrm{H}_1(\mathrm{h}+1)$, $\mathrm{H}_2(\mathrm{h}+1)$, and $\mathrm{H}_3(\mathrm{h}+1)$. To this end, we introduce a Glimm-type functional and prove that it is non-increasing, which ensures the smallness of total variations of approximate solutions. The following lemmas are needed in our proofs.

Lemma 3.1. *The following statements hold:*

(i) *If* $U_2 = \Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U_1)$ *with* U_1, $U_2 \in O_{\varepsilon_0}(U_-)$ *or* U_1, $U_2 \in O_{\varepsilon_0}(U_+)$, *then*

$$|U_1 - U_2| \leqslant C_1\left(|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|\right);$$

(ii) *If* $U_2 = \Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U_1)$, $U_3 = G(s; U_1)$, *and* $U_4 = G(s; U_2)$ *with* U_1, $U_2 \in O_{\varepsilon_0}(U_-)$ *and* U_3, $U_4 \in O_{\varepsilon_0}(U_+)$, *then*

$$|U_3 - U_4| \leqslant C_2\left(|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|\right),$$

where $C_1 > 0$ *and* $C_2 > 0$ *are constants depending only on* U_-, U_+, *and system (1.1).*

Lemma 3.2. *For any* U_1, U_2, $U_3 \in O_{\varepsilon_0}(U_-)$ *or* U_1, U_2, $U_3 \in O_{\varepsilon_0}(U_+)$, *suppose that*

(i) $U_2 = \Phi_i(\beta_j; \Phi_i(\alpha_i; U_1))$ *and* $U_3 = \Phi_i(\alpha_i + \beta_j; U_1)$ *for* $i \in \{1, 2, 3, 4\}$,
(ii) $U_2 = \Phi_j(\beta_j; \Phi_i(\alpha_i; U_1))$ *and* $U_3 = \Phi_i(\alpha_i; \Phi_j(\beta_j; U_1))$ *for* $i \neq j$ *and* $i, j \in \{1, \cdots, 5\}$.

Then $|U_3 - U_2| = O(1)|\alpha_i||\beta_j|$.

Definition 3.1 (Approaching). (i) Suppose that two fronts α and β are located at points $y_\alpha < y_\beta$ and belong to the characteristic families i_α, $i_\beta \in \{1, \cdots, 5\}$, respectively. Then we say that they are approaching if $i_\alpha > i_\beta$ or if $i_\alpha = i_\beta$ and one of them is a shock. In this case, we denote the approaching relation by $(\alpha, \beta) \in \mathcal{A}$.
(ii) For any weak front $\alpha \in \bigcup_{h=1}^\infty \Omega_{h,+}^{\mu,\Delta x}$ of family 1, or $\alpha \in \bigcup_{h=1}^\infty \Omega_{h,-}^{\mu,\Delta x}$ of family i, $i \in \{1, \cdots, 5\}$, we say that it approaches the strong shock and write $\alpha \in \mathcal{A}_s$ in this case.
(iii) For any weak front α of family 4 or 5, we say that it approaches the boundary and write $\alpha \in \mathcal{A}_b$.

Similar to [12], we define the following Glimm-type functionals. Let the weighted strength for an i-weak wave α be

$$b_\alpha = \begin{cases} K_-\alpha & \text{for } \alpha \in \cup_{h=1}^\infty \Omega_-^{\mu,\Delta x}, \\ \alpha & \text{for } \alpha \in \cup_{h=1}^\infty \Omega_+^{\mu,\Delta x}, \end{cases} \tag{3.3}$$

where $K_{-} = 2\max_{1\leqslant i\leqslant 4, 2\leqslant j\leqslant 4}\{|\hat{K}_{ij}|\} + 4C_2$ with coefficients $\hat{K}_{ij}$ in lemma 2.7. Then, for each $x \in [h\Delta x, (h+1)\Delta x)$ with $h \in \mathbb{N}_+$, the weighted total strength of weak waves in $U^{\mu,\Delta x}(x,\cdot)$ is defined to be

$$L(x) = \sum_{\alpha} |b_\alpha|.$$

The interaction potential is defined as

$$Q(x) = K_0 \sum_{i>h} \omega_i + K_s \sum_{\alpha\in\mathcal{A}_s} |b_\alpha| + \sum_{\beta\in\mathcal{A}_b} |b_\beta| + K \sum_{(\alpha,\beta)\in\mathcal{A}} |b_\alpha||b_\beta|, \tag{3.4}$$

where $\omega_i = |p_b^{\mu,\Delta x,i+1} - p_b^{\mu,\Delta x,i}|$, K_0, K_s, and K are constants that need to be specified later.

Definition 3.2. For each $x \in [h\Delta x, (h+1)\Delta x)$ with $h \in \mathbb{N}_+$, define

$$F(x) = L(x) + \mathcal{K}Q(x) + |U_\diamond(x) - U_\infty^-| + |U^\diamond(x) - U_\infty^+|$$

with $\mathcal{K} > 0$ to be determined later, where vector $U_\diamond$ ($U^\diamond$) is the below (above) state of the large shock at *time* x, and U_∞^- (U_∞^+) is the below (above) state of the large shock at $x = 0$.

When two wavefronts collide (a wavefront hits the boundary or the boundary pressure changes) at (τ,ξ), we have the following proposition:

Proposition 3.1. *There are constants K_0, K_s, K, and $\mathcal{K}$ so that there exists $\varepsilon > 0$ such that, when $F(\tau-) \leqslant \varepsilon$,*

$$F(\tau+) \leqslant F(\tau-).$$

Proof. Assume that, on $x = \tau$, only one interaction happens. Our proof is divided into the following five cases, according to the location where an interaction takes place. Let $C > 0$ be a universal constant that may vary at different occurrences, depending only on U_-, U_+, and system (1.1).

Case 1. *Interior interactions between weak waves*. A weak wavefront α_i of family i interacts the other weak wavefront β_j of j-family at (τ,ξ) for $i,j \in \{1,\cdots,5\}$; see figure 5. Then lemma 2.2 leads to

$$L(\tau+) - L(\tau-) \leqslant C|\alpha_i||\beta_j|,$$
$$\mathcal{K}Q(\tau+) - \mathcal{K}Q(\tau-) \leqslant \mathcal{K}\big(-K(1 - CL(\tau-)) + (K_s+1)C\big)|\alpha_i||\beta_j|,$$

which implies

$$F(\tau+) - F(\tau-) \leqslant \mathcal{K}\big(-K(1 - CL(\tau-)) + C(K_s+1)\big)|\alpha_i||\beta_j| + C|\alpha_i||\beta_j|.$$

Case 2. *A weak physical wavefront collides the strong shock*. A weak physical wavefront β_1 of 1-family interacts the shock wavefront s at (τ,ξ) from above; see figure 6.

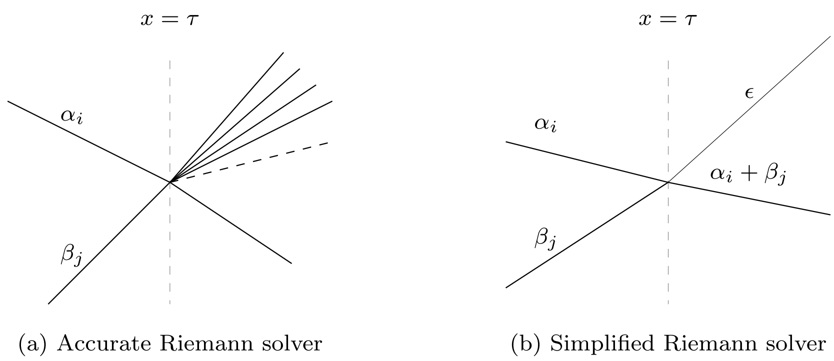


Figure 5. Case 1. *Interior interactions between weak waves.*

- When an accurate Riemann solver is used, then lemmas 2.6 and 3.1 imply that

$$L(\tau+) - L(\tau-) \leqslant \sum_{i=2}^{4} |K_{si}||\beta_1| - |\beta_1|,$$

$$\mathcal{K}Q(\tau+) - \mathcal{K}Q(\tau-) \leqslant \mathcal{K}\Big(KL(\tau-)\sum_{i=2}^{4}|K_{si}| - K_s + |K_{s4}|\Big)|\beta_1|,$$

$$|U_\diamond(\tau-) - U_\infty^-| + |U^\diamond(\tau-) - U_\infty^+| - |U_\diamond(\tau+) - U_\infty^-| - |U^\diamond(\tau+) - U_\infty^+|$$

$$\leqslant |U_\diamond(\tau-) - U_\diamond(\tau+)| + |U^\diamond(\tau-) - U^\diamond(\tau+)| \leqslant C_1\Big(\sum_{i=2}^{4}|K_{si}| + 1\Big)|\beta_1|.$$

Thus, we have

$$F(\tau+) - F(\tau-) \leqslant C|\beta_1| - \mathcal{K}\Big(K_s - |K_{s4}| - KL(\tau-)\sum_{i=2}^{4}|K_{si}|\Big)|\beta_1|.$$

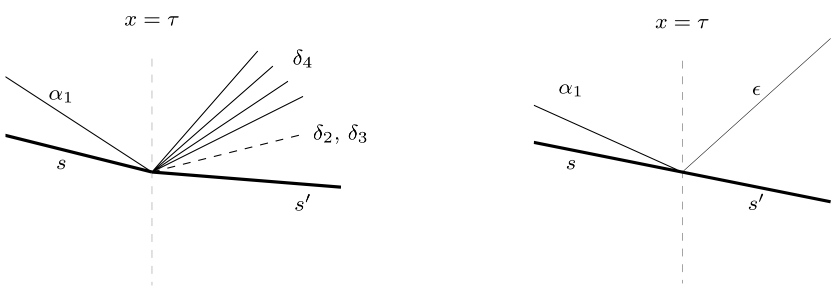
- When a simplified Riemann solver is used, then lemma 3.1 indicates

$$L(\tau+) - L(\tau-) = -|\beta_1| + C_1|\beta_1|,$$
$$\mathcal{K}Q(\tau+) - \mathcal{K}Q(\tau-) \leqslant -\mathcal{K}K_s|\beta_1|,$$

which gives

$$F(\tau+) - F(\tau-) \leqslant -(\mathcal{K}K_s + 1 - C_1)|\beta_1|.$$

Case 3. *A weak wavefront collides the strong shock*. A weak wavefront α_i of i-family interacts the strong shock wavefront s at (τ, ξ) from below; see figure 7.



(a) Accurate Riemann solver (b) Simplified Riemann solver

Figure 6. Case 2. *A weak physical wave collides the strong shock from above.*

- When an accurate Riemann solver is used, then lemmas 2.7 and 3.1 lead to

$$L(\tau+) - L(\tau-) \leqslant \sum_{i=2}^{4} |\hat{K}_{ij}||\alpha_i| - |b_{\alpha_i}|,$$

$$\mathcal{K}Q(\tau+) - \mathcal{K}Q(\tau-) \leqslant \mathcal{K}\Big(KL(\tau-)\sum_{j=2}^{4} |\hat{K}_{ji}||\alpha_i| - |b_{\alpha_i}| + |\hat{K}_{4i}||\alpha_i|\Big),$$

$$|U_\diamond(\tau-) - U_\infty^-| + |U^\diamond(\tau-) - U_\infty^+| - |U_\diamond(\tau+) - U_\infty^-| - |U^\diamond(\tau+) - U_\infty^+|$$

$$\leqslant |U_\diamond(\tau-) - U_\diamond(\tau+)| + |U^\diamond(\tau-) - U^\diamond(\tau+)| \leqslant C_1\Big(\sum_{j=2}^{4} |\hat{K}_{ji}| + 1\Big)|\alpha_i|.$$

Then we obtain

$$F(\tau+) - F(\tau-) \leqslant -\Big(K_- - \sum_{i=2}^{4} |\hat{K}_{ij}|\Big)|\alpha_i| - \mathcal{K}\Big(K_- - KL(\tau-)\sum_{j=2}^{4} |\hat{K}_{ji}| - |\hat{K}_{4i}|\Big)|\alpha_i| + C_1\Big(\sum_{j=2}^{4} |\hat{K}_{ji}| + 1\Big)|\alpha_i|.$$

- When a simplified Riemann solver is used, then lemma 3.1 indicates

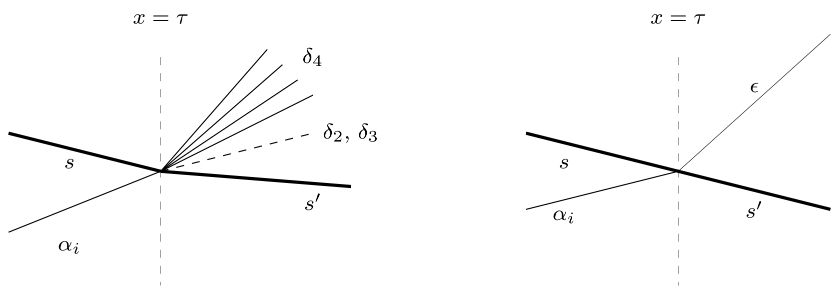
$$L(\tau+) - L(\tau-) = -K_-|\alpha_i|,$$

$$\mathcal{K}Q(\tau+) - \mathcal{K}Q(\tau-) \leqslant \mathcal{K}\big(KL(\tau-)C_2 - K_sK_- + C_2\big)|\alpha_i|,$$

$$|U_\diamond(\tau-) - U_\infty^-| + |U^\diamond(\tau-) - U_\infty^+| - |U_\diamond(\tau+) - U_\infty^-| - |U^\diamond(\tau+) - U_\infty^+|$$
$$\leqslant |U_\diamond(\tau-) - U_\diamond(\tau+)| + |U^\diamond(\tau-) - U^\diamond(\tau+)| \leqslant (C_2 + C_1)|\alpha_i|,$$

which gives

$$F(\tau+) - F(\tau-) \leqslant -K_-|\alpha_i| - \mathcal{K}\big(K_-K_s - KL(\tau-)C_2 - C_2\big)|\alpha_i| + (C_2 + C_1)|\alpha_i|.$$



(a) Accurate Riemann solver (b) Simplified Riemann solver

Figure 7. Case 3. *A weak wavefront collides the strong shock from below.*

Case 4. *A weak physical wavefront is generated from the boundary.* A weak physical wavefront α_1 of 1-family is generated from the boundary at $(\tau,\xi)=(h\Delta x, b^{\mu,\Delta x}(h\Delta x))$; see figure 8.

- When the accurate Riemann solver is applied, then lemma 2.3 leads to

$$\begin{aligned}
L(\tau+)-L(\tau-) &= |\alpha_1| \leqslant |K_b|\omega_h,\\
\mathcal{K}Q(\tau+)-\mathcal{K}Q(\tau-) &\leqslant \mathcal{K}\big(-K_0\omega_h+KL(\tau-)\,|\alpha_1|+K_s|\alpha_1|\big)\\
&\leqslant \mathcal{K}\big(-K_0+KL(\tau-)\,|K_b|+K_s|K_b|\big)\omega_h.
\end{aligned}$$

Therefore, we have

$$F(\tau+)-F(\tau-) \leqslant |K_b|\omega_h-\mathcal{K}\big(K_0-KL(\tau-)\,|K_b|-K_s|K_b|\big)\omega_h.$$

Case 5. *A weak physical wavefront hits the boundary.* A weak physical wavefront β_4 of 4-family hits the boundary at (τ,ξ).

- When an accurate Riemann solver is used, then lemma 2.5 implies

$$\begin{aligned}
L(\tau+)-L(\tau-) &= |\delta_1|-|\beta_4| \leqslant \big(|K_{b4}|-1\big)|\beta_4|,\\
\mathcal{K}Q(\tau+)-\mathcal{K}Q(\tau-) &\leqslant \mathcal{K}\big(KL(\tau-)\,|\delta_1|+K_s|\delta_1|-|\beta_4|\big)\\
&\leqslant \mathcal{K}\big(KL(\tau-)\,|K_{b4}|+K_s|K_{b4}|-1\big)|\beta_4|,
\end{aligned}$$

which leads to

$$F(\tau+)-F(\tau-) \leqslant \big(|K_{b4}|-1\big)|\beta_4|-\mathcal{K}\big(1-K_s|K_{b4}|-KL(\tau-)\,|K_{b4}|\big)|\beta_4|.$$

To conclude, from lemma 2.5, when ε is small enough, we may take

$$\max\{\frac{1}{2},|K_{s4}\} < K_s < \min\{1,\frac{1}{|K_{b4}|}\}.$$

Moreover, we take K_0, K, and $\mathcal{K}$ large enough, and then take ε smaller if necessary to obtain from the above estimates:

$$F(\tau+) \leqslant F(\tau-).$$

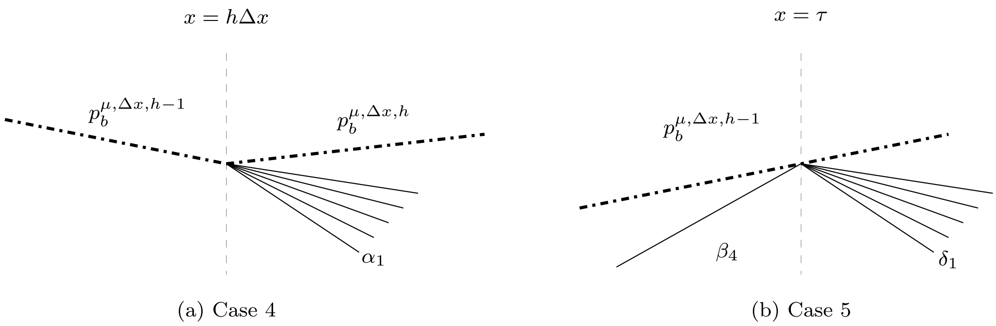


Figure 8. Cases 4 & 5. *Interaction involving a weak physical wave and the boundary.*

This completes the proof.

□

Remark 3.1. Indeed, in the proof of proposition 3.1, we apply the estimates in lemma 2.3 to control the increase of the Glimm-type functional near the boundary (see remark 2.1) and apply the estimates in lemma 2.6 (and then choose proper weights) to balance the total strength of weak 1-waves and 4-waves between the wedge boundary and the leading shock (see also remarks 2.2 and 2.3).

Using a similar argument as in [12], we can obtain

Corollary 3.1. *If $F(0) < \varepsilon$, then $F(x) < \varepsilon$ for any $x > 0$.*

Next, we give an estimate of the non-physical waves. Let $\epsilon(t)$ be a non-physical wave of $U^{\mu,\Delta}$ crossing $x = t$. Then we can obtain the following estimates of the total strength of non-physical waves, whose proof will be given in appendix A.1.

Proposition 3.2. *For every $x > 0$, there exists $\nu_0 > 0$ such that, when the threshold parameter $\nu \in (0, \nu_0)$,*

$$\sum_{\epsilon \in \mathcal{NP}} \epsilon(x) < \mu.$$

By corollary 3.1, applying a similar argument as in [12], we obtain

Proposition 3.3. *There exists $\tilde{\varepsilon} > 0$ such that, given $\|(\tilde{p}_b, \widetilde{U}_\infty)\|_{L^\infty \cap BV} < \tilde{\varepsilon}$, then, for every μ, Δx, and $\delta > 0$, the modified wave-front tracking algorithm provides global approximate solutions $U^{\mu,\Delta x}$ and the corresponding approximate boundaries $\Gamma^{\mu,\Delta x}$ and 1-strong shocks $\mathrm{S}^{\mu,\Delta x}$ in $\Omega^{\mu,\Delta x}$ satisfying all $H_1(h)$–$H_3(h)$ for any $h \geqslant 1$,*

$$\text{T.V.}\{U^{\mu,\Delta x}(x,\cdot) : (-\infty, \chi^{\mu,\Delta x}(x))\} < C\text{T.V.}\{(\tilde{p}_b, \widetilde{U}_\infty)\},$$
$$\text{T.V.}\{U^{\mu,\Delta x}(x,\cdot) : (\chi^{\mu,\Delta x}(x), b^{\mu,\Delta x}(x))\} < C\text{T.V.}\{(\tilde{p}_b, \widetilde{U}_\infty)\},$$

and

$$|b^{\mu,\Delta x}(x+t) - b^{\mu,\Delta x}(x)| \leqslant \left(\left|\frac{v_+}{u_+}\right| + \mathcal{E}\right)|t|,$$

$$|\chi^{\mu,\Delta x}(x+t)-\chi^{\mu,\Delta x}(x)|\leqslant(|s_0|+\mathcal{E})\,|t|,$$

for any $x\geqslant 0$ and $t>0$, where $\mathcal{E}>0$ and $C>0$ are constants depending only on U_-, U_+, and system (1.1).

Furthermore, we now give estimates about the strong 1-shock and the boundary.

Proposition 3.4. *There is a constant $\bar{M}>0$ such that*

$$T.V.\big\{\sigma^{\mu,\Delta x}(\cdot)\,:\,[0,\infty)\big\}=\sum_{\tau\in\Lambda}|\sigma^{\mu,\Delta x}(\tau+)-\sigma^{\mu,\Delta x}(\tau-)|\leqslant\bar{M},$$
$$T.V.\big\{U_b^{\mu,\Delta x}(\cdot)\,:\,[0,\infty)\big\}=\sum_{\tau\in\Lambda}|U_b^{\mu,\Delta x}(\tau+)-U_b^{\mu,\Delta x}(\tau-)|\leqslant\bar{M}(1+\mu),$$

where Λ is a set consisting of all the x-coordinates at which a colliding happens, and $\bar{M}$ is independent of μ and Δx.

Proof. For any $\tau\in\Lambda$, we define

$$E_{U^{\mu,\Delta x}}(\tau):=\begin{cases}\eta_1|\alpha_i||\beta_j| & \text{for Case 1,}\\ \eta_1|\beta_1| & \text{for Case 2,}\\ \eta_1|\alpha_i| & \text{for Case 3,}\\ \eta_1|\omega_h| & \text{for Case 4,}\\ \eta_1|\beta_4| & \text{for Case 5,}\end{cases}\tag{3.5}$$

for some $\eta_1>0$. From the proof of proposition 3.1, when η_1 is sufficiently small, it can be verified that

$$\sum_{\tau\in\Lambda}E_{U^{\mu,\Delta x}}(\tau)\leqslant\sum_{\tau\in\Lambda}\eta_1^{-1}\big(F(\tau-)-F(\tau+)\big)\leqslant\eta_1^{-1}F(0)\,.$$

Meanwhile, lemmas 2.4 and 2.6 imply that

$$\begin{aligned}\sum_{\tau\in\Lambda}|s^{\mu,\Delta x}(\tau+)-s^{\mu,\Delta x}(\tau-)|&\leqslant\sum_{\tau\in\Lambda}\big(|K_{bs}|+|K_{s1}|\big)E_{U^{\mu,\Delta x}}(\tau)\\&\leqslant\big(\sup|K_{bs}|+\sup|K_{s1}|\big)\eta_1^{-1}F(0)\,.\end{aligned}$$

As for the estimates of the boundary, we need to note that there are errors produced due to non-physical waves. However, those non-physical waves do not change, once they hit the boundary. Hence, before $x=t$, the total strength of non-physical waves that hit the boundary is less than the total strength of all the non-physical waves at $x=t$, which is less than μ, by proposition 3.2. Then similar argument shows

$$\sum_{\tau\in\Lambda}|U_b^{\mu,\Delta x}(\tau+)-U_b^{\mu,\Delta x}(\tau-)|\leqslant C\mu+\sum_{\tau\in\Lambda}CE_{U^{\mu,\Delta x}}(\tau)=C\mu+C\eta_1^{-1}F(0)\,.$$

Combining these estimates yields the result. □

Combining propositions 3.2–3.4, we now conclude one of our main theorems (see [14, 15, 51]). For completeness, a proof is provided in appendix A.2.

Theorem 3.1. *There exists $\tilde{\varepsilon} > 0$ such that, when $\|(\widetilde{U}_\infty, \tilde{p}_b)\|_{L^\infty \cap BV} < \tilde{\varepsilon}$, there is a subsequence $\{\mu_l\}_{l=1}^\infty$ and corresponding $\{\Delta x_l\}_{l=1}^\infty$ so that*

(i) *In any bounded x-interval, $(b^{\mu_l,\Delta x_l}, \chi^{\mu_l,\Delta x_l})$ converges to (b,χ) uniformly;*
(ii) *$U_b^{\mu_l,\Delta x_l}$ converges to $U_b \in \mathrm{BV}([0,\infty))$ a.e. with $\dot{b}(x) = \frac{v_b(x)}{u_b(x)}$ a.e. while $s^{\mu_l,\Delta x_l}$ converges to $s \in \mathrm{BV}([0,\infty))$ a.e. with $\dot{\chi}(x) = s(x)$ a.e.;*
(iii) *For every $x > 0$, $U^{\mu_l,\Delta x_l}$ converges to U in $L^1_{\mathrm{loc}}\big((-\infty, b(x)); \mathbb{R}^4\big)$, which is an entropy solution to problem (1.1) in $\Omega = \{(x,y) : x \geqslant 0, y < b(x)\}$ satisfying (1.6)–(1.11).*

4. A modified Lyapunov functional

Relying on the analysis in section 3, we now analyze the well-posedness of this system. Compared to the problem with the given wedge boundary, our solutions may be constructed on the different regions. As a result, it seems hard to measure the distance of two weak solutions U_1 and U_2 in L^1 directly. However, in this paper, we extend two different solutions to the same domain and compare the corresponding L^1–norm of two extended solutions in this domain. With this setup, we can introduce a Lyapunov-type functional to measure the new L^1–distance, as well as the L^∞–distance between the two boundaries. We first extend $U^{\mu,\Delta x}(x,y)$ to

$$U_E^{\mu,\Delta x}(x,y) = \begin{cases} U^{\mu,\Delta x}(x,y) & \text{for all } y \leqslant b^{\mu,\Delta x}(x), \\ U^{\mu,\Delta x}(x, b^{\mu,\Delta x}(x)) & \text{for all } y > b^{\mu,\Delta x}(x), \end{cases}$$

for all $x > 0$, where $b^{\mu,\Delta x}(x)$ is the corresponding boundary. As a result, our weak solutions are extended as

$$U_E(x,y) = \begin{cases} U(x,y) & \text{for all } y \leqslant b(x), \\ U(x, b(x)) & \text{for all } y > b(x). \end{cases}$$

Notably, the extended functions $U_E(x,y)$ are not weak solutions to the problem in general; nevertheless, this does not compromise the subsequent estimates.

To simplify the notation, we omit subscript $_E$ and superscript $^{\mu,\Delta x}$, and use U to denote an extended approximate solution. Given two suitable initial data functions $U_{\infty,1}$ and $U_{\infty,2}$ with corresponding pressure distributions $p_{b,1}$ and $p_{b,2}$ on the boundaries, according to our previous construction, for every $\mu > 0$, there are two μ-approximate solutions U_1 and U_2 with approximate boundaries $y = b_1(x)$ and $y = b_2(x)$, respectively. Set

$$b_{\max}(x) := \max\{b_1(x), b_2(x)\}, \qquad b_{\min}(x) := \min\{b_1(x), b_2(x)\},$$

and call

$$\Gamma_{\max} := \{(x, b_{\max}(x)) : x \geqslant 0\}, \qquad \Gamma_{\min}(x) = \{(x, b_{\min}(x)) : x \geqslant 0\}$$

the outer and inner boundary of U_1 and U_2, respectively. Similar to [5, 6, 12, 28, 31–33, 39], fixing x and given U_1 *and* U_2, the scalar functions $h_i(y)$ are implicitly defined by

- $U_2(x,y) = H\big(h_1(y), h_2(y), h_3(y), h_4(y); U_1(x,y)\big)$, $U_2 \in O_{\varepsilon_0}(U_+)$, and $U_1 \in O_{\varepsilon_0}(U_-) \cup O_{\varepsilon_0}(U_+)$;
- $U_1(x,y) = H\big(h_1(y), h_2(y), h_3(y), h_4(y); U_2(x,y)\big)$, $U_1 \in O_{\varepsilon_0}(U_+)$, and $U_2 \in O_{\varepsilon_0}(U_-)$,

where

$$H(h_1,h_2,h_3,h_4;U) := S_4(h_4)(\Phi_3(h_3;\Phi_2(h_2;S_1(h_1)(U)))),$$

and S_i are i-Hugoniot curves, $i=1,2,3,4$. Moreover, $h_{b,i}(x)$, $i=1,2,3,4$, are implicitly defined by

$$U_2(x,b_{\max}(x)) = H(h_{b,1}(x),h_{b,2}(x),h_{b,3}(x),h_{b,4}(x);U_1(x,b_{\min}(x))). \tag{4.1}$$

Furthermore, for an i-wave that connects two states on the i-Hugoniot curve, let $\lambda_i(x)$ be its speed. Then we define the weighted L^1–strengths:

$$q_i(y) = \begin{cases} c_i^u h_i(y) & \text{if } U_1,\ U_2 \in O_{\varepsilon_0}(U_+), \\ c_i^m h_i(y) & \text{if } U_1,\ U_2 \text{are in different domains}, \\ c_i^l h_i(y) & \text{if } U_1,\ U_2 \in O_{\varepsilon_0}(U_-), \end{cases}$$
$$q_{b,i}(x) = c_i^u h_{b,i}(x), \tag{4.2}$$

where c_i^u, c_i^m, $c_i^l < 1$ are constants that remain to be determined based on the interaction and reflection estimates obtained in lemmas 2.2–2.7. In addition, when h_1 is a large shock that connects a state in $O_{\varepsilon_0}(U_-)$ and the other state in $O_{\varepsilon_0}(U_+)$, we set $q_1 = B$, where $B<1$ is a constant larger than the total strength of small waves of the other families, which is regarded as the 'strength' of the strong shock.

For each $x>0$, the set of all weak waves in U_1 and U_2 is denoted by $\mathcal{J} = \mathcal{J}(U_1)\cup\mathcal{J}(U_2)$, and the strength of a k_α-wave $\alpha\in\mathcal{J}$, located at point y_α, is denoted by $|\alpha|$. Then we define the following quantities:

$$B_i(y) = \Bigg(\sum_{\substack{\alpha\in\mathcal{J}(U_1)\\ y_\alpha<y,\, i<k_\alpha\leqslant 4}} + \sum_{\substack{\alpha\in\mathcal{J}(U_2)\\ y_\alpha<y,\, i<k_\alpha\leqslant 4}} + \sum_{\substack{\alpha\in\mathcal{J}(U_1)\\ y_\alpha>y,\, 1\leqslant k_\alpha<i}} + \sum_{\substack{\alpha\in\mathcal{J}(U_2)\\ y_\alpha>y,\, 1\leqslant k_\alpha<i}}\Bigg)|\alpha|,$$

$$C_i(y) = \begin{cases} \left(\sum_{\alpha\in\mathcal{J}(U_1),y_\alpha<y,k_\alpha=i} + \sum_{\alpha\in\mathcal{J}(U_2),y_\alpha>y,k_\alpha=i}\right)|\alpha| & \text{if } q_i(y)<0, \\ \left(\sum_{\alpha\in\mathcal{J}(U_2),y_\alpha<y,k_\alpha=i} + \sum_{\alpha\in\mathcal{J}(U_1),y_\alpha>y,k_\alpha=i}\right)|\alpha| & \text{if } q_i(y)>0, \end{cases}$$

$$D_i(y) = B - D_i^c(y),$$

$$D_i^c(y) = \begin{cases} B & \text{if } i=1 \text{ and } U_1,U_2 \text{ are in different domains} \\ & \quad \text{or } i=2,3 \text{ and } U_1,U_2\in O_{\varepsilon_0}(U_+), \\ 0 & \text{other cases}, \end{cases}$$

$$F_i(y) = \Bigg(\sum_{\substack{\alpha\in\mathcal{J}\\ y_\alpha>y,\, k_\alpha=1\\ \text{both states joined by } \alpha\\ \text{are located in } O_{\varepsilon_0}(U_+)}} + \sum_{\substack{\alpha\in\mathcal{J}\\ y_\alpha<y,\, k_\alpha=1\\ \text{both states joined by } \alpha\\ \text{are located in } O_{\varepsilon_0}(U_-)}}\Bigg)|\alpha|.$$

Let

$$A_i(y) = B_i(y) + D_i(y) + \begin{cases} C_i(y) & \text{if } q_i(y) \text{ is small}, \\ F_i(y) & \text{if } i=1 \text{ and } q_i(y)=B \text{ is large}, \end{cases} \tag{4.3}$$

where the 'small' and the 'large' mean a wave connecting both states in either $O_{\varepsilon_0}(U_+)$ or $O_{\varepsilon_0}(U_-)$ and a strong shock wave connecting a state in $O_{\varepsilon_0}(U_-)$ and the other in $O_{\varepsilon_0}(U_+)$, respectively. Thus, $A_i(y)$ equals to the total strength of the waves in U_1 and U_2 approaching the i-wave $q_i(y)$. We now introduce the following modified Lyapunov functional:

$$\mathfrak{F}(U_1(x), U_2(x)) = c_b \int_0^x \left(|h_{b,1}(\tau)| + |h_{b,4}(\tau)|\right) \mathrm{d}\tau + \sum_{i=1}^{4} \int_{-\infty}^{b_{\max}} W_i(y) |q_i(y)| \,\mathrm{d}y, \tag{4.4}$$

where $c_0 > 0$ to be chosen and

$$W_i(y) = 1 + \kappa_1 A_i(y) + \kappa_2 (Q(U_1) + Q(U_2)),$$

with the two constants κ_1 and κ_2 to be determined later and the interaction potential Q introduced in (3.4). Taking the initial value of the Glimm functional $F(0)$ small enough, we can prove

$$1 \leqslant W_i(y) \leqslant C_0 \qquad \text{for } i = 1,2,3,4,$$

where C_0 is independent of x and μ.

Remark 4.1. The construction of the Lyapunov functional (4.4) involves two stability components:

(i) L^1-stability for entropy solutions with a strong shock—a standard component as in [12, 33];
(ii) L^∞-stability for distinct boundary configurations. We extend the framework by introducing the term:

$$c_b \int_0^x \left(|h_{b,1}(\tau)| + |h_{b,4}(\tau)|\right) \mathrm{d}\tau$$

in (4.4), which quantifies the L^∞-distance between the boundaries.

Proposition 4.1. *Let $F_j(t)$ denote the Glimm-type functional of U_{j}, $j = 1,2$. For $F_j(0)$ suitably small and κ_2 suitably large, at each $x > 0$ where two fronts of U_1 or U_2 interact, or one of the approximate pressure corresponding to the outer boundary changes, or a physical wavefront of U_1 or U_2 hits the outer boundary, then*

$$\mathfrak{F}(U_1(x+), U_2(x+)) \leqslant \mathfrak{F}(U_1(x-), U_2(x-)).$$

Proof. A direct computation leads to

$$\mathfrak{F}(U_1(x+), U_2(x+)) - \mathfrak{F}(U_1(x-), U_2(x-)) = \sum_{i=1}^{4} \int_{-\infty}^{b_{\max}} \left(W_i(x+,y) - W_i(x-,y)\right) |q_i(y)| \,\mathrm{d}y.$$

Then propositions 3.1 and 3.4 yield

$$Q(U_j(x+)) - Q(U_j(x-)) \leqslant -\frac{1}{4} E_{U_j}(x),$$
$$A_i(x+,y) - A_i(x-,y) = O(1) \left(E_{U_1}(x) + E_{U_2}(x)\right),$$

where E_{U_j}, $j = 1, 2$, are given in (3.5). Therefore, if κ_2 is large enough, all the weight functions $W_i(y)$, $i = 1,2,3,4$, decrease. □

Proposition 4.2. *There are suitable constants κ_1, κ_2, B, c_i^u, c_i^m, c_i^l, and $c_b > 0$ such that the following estimate holds: If there are no interactions at $x > 0$, and $F_j(0)$ are sufficiently small for $j = 1, 2$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{F}\left(U_1\left(x\right), U_2\left(x\right)\right) \leqslant O\left(1\right)\mu + O\left(1\right)\left|p_{b,2}^{\mu,\Delta x}\left(x\right) - p_{b,1}^{\mu,\Delta x}\left(x\right)\right|. \tag{4.5}$$

Proof. We divide the proof into three steps.

1. Denote the speed of the i-wave $q_i(x)$ and $q_{b,i}(x)$ by λ_i and $\lambda_{b,i}$, respectively. Taking the derivative of $\mathfrak{F}(U_1(x), U_2(x))$ with respect to x leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{F}\left(U_1\left(x\right), U_2\left(x\right)\right)\\
&= c_b\big(|h_{b,1}\left(x\right)| + |h_{b,4}\left(x\right)|\big) + \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{4}\big(|q_i^{\alpha-}|W_i^{\alpha-} - |q_i^{\alpha+}|W_i^{\alpha+}\big)\dot{y}_\alpha\\
&\quad + \sum_{i=1}^{4}|q_{b,i}\left(x\right)|W_i\left(b_{\max}\left(x\right)\right)\dot{b}_{\max}\left(x\right)\\
&= c_b\big(|h_{b,1}\left(x\right)| + |h_{b,4}\left(x\right)|\big) + \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{4}\big(|q_i^{\alpha-}|W_i^{\alpha-}(\dot{y}_\alpha - \lambda_i^{\alpha-}) - |q_i^{\alpha+}|W_i^{\alpha+}(\dot{y}_\alpha - \lambda_i^{\alpha+})\big)\\
&\quad + \sum_{i=1}^{4}|q_{b,i}\left(x\right)|W_i\left(b_{\max}\left(x\right)\right)\big(\dot{b}_{\max}\left(x\right) - \lambda_{b,i}\left(x\right)\big),
\end{aligned}$$

where $\dot{y}_\alpha$ stands for the speed of $\alpha \in \mathcal{J}$, and $q_i^{\alpha\pm} = q_i\big(y_\alpha\pm\big)$, $W_i^{\alpha\pm} = W_i\big(y_\alpha\pm\big)$, $\lambda_i^{\alpha\pm} = \lambda_i\big(y_\alpha\pm\big)$, and $\lambda_{b,i}(x) = \lambda_i(b_{\max}(x)-)$ for $\alpha \in \mathcal{J}(U_j)$. Define

$$\begin{aligned}
E_{\alpha,i} &= |q_i^+|W_i^+\left(\lambda_i^+ - \dot{y}_\alpha\right) - |q_i^-|W_i^-\left(\lambda_i^- - \dot{y}_\alpha\right),\\
E_{b,i} &= |q_{b,i}\left(x\right)|W_i\left(b_{\max}\left(x\right)\right)\big(\dot{b}_{\max}\left(x\right) - \lambda_{b,i}\left(x\right)\big),
\end{aligned}$$

with $q_i^\pm = q_i^{\alpha\pm}$, $W_i^\pm = W_i^{\alpha\pm}$, and $\lambda_i^\pm = \lambda_i^{\alpha\pm}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{F}\left(U_1\left(x\right), U_2\left(x\right)\right) = c_b\big(|h_{b,1}\left(x\right)| + |h_{b,4}\left(x\right)|\big) + \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{4}E_{\alpha,i} + \sum_{i=1}^{4}E_{b,i}. \tag{4.6}$$

2. We need the following lemma to complete the proof.

Lemma 4.1. *There are suitable constants κ_1, κ_2, B, c_i^u, c_i^m, c_i^l, and $c_b > 0$ such that the following estimates hold:*

$$\alpha \text{ is a strong shock}: \sum_{i=1}^{4}E_{\alpha,i} \leqslant 0, \tag{4.7}$$

$$\alpha \text{ is a non-physical wave}: \sum_{i=1}^{4}E_{\alpha,i} \leqslant O\left(1\right)|\alpha|, \tag{4.8}$$

$$\alpha \text{ is a weak wave}: \sum_{i=1}^{4}E_{\alpha,i} \leqslant O\left(1\right)\mu|\alpha|, \tag{4.9}$$

$$\text{on the boundary}: \sum_{i=1}^{4} E_{b,i} + c_0\left(|h_{b,1}(x)| + |h_{b,4}(x)|\right) \leqslant O(1)\,|p_{b,2}^{\mu,\Delta x}(x) - p_{b,1}^{\mu,\Delta x}(x)| + O(1)\,\mu, \tag{4.10}$$

where $c_0 > 0$ *is a sufficiently small constant, independent of the waves and the boundary.*

Proof. There are five cases.

Case 1: The states connected by α *is a weak wave* (*either physical or non-physical*). In this case, we choose B suitably small and take κ_1 large enough so that the estimates can be obtained by following Bressan-Liu-Yang [6].

Case 2: α *is the below strong shock in* U_1 *or* U_2; see also [12, 33]. Then we have

$$\begin{aligned} E_{\alpha,1} &= BW_1^+\left(\lambda_1^+ - \dot{y}_\alpha\right) - |q_1^-|W_1^-\left(\lambda_1^- - \dot{y}_\alpha\right) \\ &\leqslant O(1)\,B\sum_{i=1}^{4}|q_1^-| - \kappa_1 B|\lambda_1^- - \dot{y}_\alpha|, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=2}^{4} E_{\alpha,i} &= \sum_{i=2}^{4}\left(|q_i^-|(\lambda_i^- - \dot{y}_\alpha)(W_i^+ - W_i^-) + W_i^-\left(|q_i^+|(\lambda_i^+ - \dot{y}_\alpha) - |q_i^-|(\lambda_i^- - \dot{y}_\alpha)\right)\right) \\ &\leqslant \sum_{i=2}^{4} O(1)\,\kappa_1 B|q_i^+||\lambda_i^+ - \dot{y}_\alpha| - \frac{3}{4}\sum_{i=2}^{4}\kappa_1 B|q_i^-||\lambda_i^- - \dot{y}_\alpha|. \end{aligned}$$

When κ_1 is large enough, we have

$$\sum_{i=1}^{4} E_{\alpha,i} = \sum_{i=2}^{4} O(1)\,\kappa_1 B|q_i^+||\lambda_i^+ - \dot{y}_\alpha| - \frac{1}{2}\sum_{i=1}^{4}\kappa_1 B|q_i^-||\lambda_i^- - \dot{y}_\alpha|.$$

Recall that the wave curve and the Hugoniot curve passing through U_0 have the same curvature at U_0. Carrying out similar arguments as in lemma 2.7 (by letting $\alpha_i = h_i^-$, $\beta_i = 0$, and $\delta_i = h_i^+$), we obtain

$$|h_k^+| = O(1)\sum_{i=1}^{4}|h_i^-| \qquad \text{for } k = 2, 3, 4.$$

In (4.2), we take c_i^l, $i = 1, 2, 3, 4$, larger enough than c_i^m, $i = 2, 3, 4$, to complete the proof of (4.7).

Case 3: α *is a weak wave lying between strong shocks*; see also [12, 33]. When $i = 1$, we have

$$\begin{aligned} E_{\alpha,1} &= B\left((W_1^+ - W_1^-)(\lambda_1^\pm - \dot{y}_\alpha) + W_1^\mp(\lambda_1^+ - \lambda_1^-)\right) \\ &\leqslant B\left(-\kappa_1|\alpha||\lambda_1^\pm - \dot{y}_\alpha| + O(1)\,|\alpha|\right). \end{aligned}$$

As for $i = 2, 3, 4$, we can obtain

$$\begin{aligned} E_{\alpha,i} &= |q_i^\pm|\left(W_i^+ - W_i^-\right)\left(\lambda_i^\pm - \dot{y}_\alpha\right) + W_i^\mp\left(|q_i^+|\left(\lambda_i^+ - \dot{y}_\alpha\right) - |q_i^-|\left(\lambda_i^- - \dot{y}_\alpha\right)\right) \\ &\leqslant O(1)\left(O(1) + \kappa_1 B\right)\left(|q_i^+ - q_i^-| + |q_i^-||\alpha|\right) + O(1)\,|\alpha|. \end{aligned}$$

As a result, summing all the estimates obtained above, we conclude

$$\begin{aligned}\sum_{i=1}^{4} E_{\alpha,i} &\leqslant B\left(-\kappa_1|\alpha||\lambda_1^{\pm}-\dot{y}_\alpha|+O(1)|\alpha|\right)+(O(1)+\kappa_1 B)O(1)\left(|q_i^+-q_i^-|+|q_i^-||\alpha|\right)+O(1)|\alpha| \\ &\leqslant \Big(-\kappa_1 cB+O(1)\Big)|\alpha|+B\kappa_1 O(1)\left(\sum_{i=2}^{4}|q_i^+-q_i^-|+\sum_{i=2}^{4}|q_i^-||\alpha|\right) \\ &= \Big(-\frac{\kappa_1 cB}{2}+O(1)\Big)|\alpha|+B\Big(-\frac{\kappa_1 c}{2}|\alpha|+\kappa_1 O(1)\left(\sum_{i=2}^{4}|q_i^+-q_i^-|+\sum_{i=2}^{4}|q_i^-||\alpha|\right)\Big),\end{aligned}$$

where $c>0$ is a lower bound of the difference between the speeds of the strong 1-shock and a weak shock. Choose κ_1 large enough and all the weights c_i^m sufficiently small to obtain

$$\sum_{i=1}^{4} E_{\alpha,i} \leqslant 0.$$

Case 4: α is the above strong shock in U_1 or U_2; see also [12, 33]. Similar arguments as in lemma 2.6 yield

$$h_4^- = K_{s4}h_1^+ + h_4^+. \tag{4.11}$$

Due to lemma 2.6, when $F_j(0)$, $j=1,2$, are sufficiently small, we can choose c_1^u and c_4^u such that

$$\begin{aligned}&\frac{c_1^u}{c_4^u}<1, \\ &|K_{s4}|\frac{c_4^u}{c_1^u}\frac{\lambda_4(U_+)-s_0}{\lambda_1(U_+)-s_0}<\gamma_0<1.\end{aligned} \tag{4.12}$$

Then, when $i=1$, we obtain

$$\begin{aligned}E_{\alpha,1} &= -BW_1^-\left(\lambda_1^- - \dot{y}_\alpha\right)+|q_1^+|W_1^+\left(\lambda_1^+-\dot{y}_\alpha\right) \\ &\leqslant O(1)B|q_1^+|-\kappa_1 B|q_1^+||\lambda_1^+-\dot{y}_\alpha| \\ &\leqslant O(1)B|q_1^+|-\kappa_1 Bc_1^u|h_1^+||\lambda_1^+-\dot{y}_\alpha|.\end{aligned}$$

When $i=2,3$, we have

$$\begin{aligned}E_{\alpha,i} &= |q_i^-|\left(W_i^+-W_i^-\right)\left(\lambda_i^--\dot{y}_\alpha\right)+W_i^+\left(|q_i^+|(\lambda_i^+-\dot{y}_\alpha)-|q_i^-|(\lambda_i^--\dot{y}_\alpha)\right) \\ &\leqslant -\kappa_1 B|q_i^-|\left(\lambda_i^--\dot{y}_\alpha\right)+O(1)|q_i^+| \\ &\leqslant -\kappa_1 B|q_i^-|\left(\lambda_i^--\dot{y}_\alpha\right)+O(1)\left(|q_i^-|+|q_1^+|\right).\end{aligned}$$

By (4.11) and (4.12),

$$\begin{aligned}E_{\alpha,4} &= W_4^+|q_4^+|\left(\lambda_4^+-\dot{y}_\alpha\right)-|q_4^-|W_4^-\left(\lambda_4^--\dot{y}_\alpha\right) \\ &\leqslant (\kappa_1 B+O(1))|q_4^+|\left(\lambda_4^+-\dot{y}_\alpha\right)-\kappa_1 B|q_4^-|\left(\lambda_4^--\dot{y}_\alpha\right) \\ &\leqslant (\kappa_1 B+O(1))c_4^u\left(|h_4^-|+K_{s4}|h_1^+|\right)\left(\lambda_4^+-\dot{y}_\alpha\right)-\kappa_1 Bc_4^m|h_4^-|\left(\lambda_4^--\dot{y}_\alpha\right) \\ &\leqslant (\kappa_1 B+O(1))\Big(c_4^u|h_4^-|\left(\lambda_4^+-\dot{y}_\alpha\right)+\gamma_0 c_1^u|h_1^+||\lambda_1^+-\dot{y}_\alpha|-c_4^m|h_4^-|\left(\lambda_4^--\dot{y}_\alpha\right)\Big).\end{aligned}$$

Choosing c_4^u relatively smaller than c_4^m and κ_1 suitably large, we obtain

$$\begin{aligned}\sum_{i=1}^{4} E_{\alpha,i} \leqslant & -(1-\gamma_0)\,\kappa_1 B|q_1^+||\lambda_1^+ - \dot{y}_\alpha| + O(1)\,|q_1^+||\lambda_1^+ - \dot{y}_\alpha| + O(1)\,|q_1^+| \\ & + (\kappa_1 B + O(1))\left(c_4^u|h_4^-|\left(\lambda_4^+ - \dot{y}_\alpha\right) - c_4^m|h_4^-|\left(\lambda_4^- - \dot{y}_\alpha\right)\right) \\ & + \sum_{i=2}^{3}\left(-\kappa_1 B|q_i^-|\left(\lambda_i^- - \dot{y}_\alpha\right) + O(1)\,|q_i^-|\right) \\ \leqslant & \ 0.\end{aligned}$$

Case 5: Near the boundary. First of all, let $U=(u,v,p,\rho)^\top$ be in a small neighborhood of U_+. Consider the following equation:

$$H^{(3)}\left(h_1,h_2,h_3,h_4;U\right) = p + r, \tag{4.13}$$

where $H^{(3)}(h_1,h_2,h_3,h_4;U)$ is the third component of $H(h_1,h_2,h_3,h_4;U)$. Similar to lemma 2.5, we deduce from (4.13) that

$$\begin{aligned} h_4 &= h_4\left(h_1,h_2,h_3,r\right) - h_4\left(0,h_2,h_3,r\right) + h_4\left(0,h_2,h_3,r\right) - h_4\left(0,h_2,h_3,0\right) \\ &= h_1\int_0^1 \frac{\partial h_4}{\partial h_1}\left(\lambda h_1,h_2,h_3,r\right)\mathrm{d}\lambda + O(1)\,r \end{aligned} \tag{4.14}$$

for $|r|$ small enough. In addition, we have

$$\left|\int_0^1 \frac{\partial h_4}{\partial h_1}\left(\lambda h_1,h_2,h_3,r\right)\mathrm{d}\lambda\right| \to 1 \qquad \text{as } (h_1,h_2,h_3,r)\to 0 \text{ and } U\to U_+.$$

From (4.1), considering the effect of non-physical waves on the boundary, we have

$$\begin{aligned} & H^{(3)}\left(h_{b,1}(x),h_{b,2}(x),h_{b,3}(x),h_{b,4}(x);U_1\left(x,b_1(x)\right)\right) \\ & = p_{b,1}^{\mu,\Delta x_1}(x) + p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x) + O(1)\,\mu. \end{aligned}$$

The former arguments tell us that

$$|h_{b,1}(x)| \leqslant |h_{b,4}(x)|\eta + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x) + \mu\right)$$

for some η close to 1. Note that, crossing a vortex sheet and an entropy wave, the flow direction does not change. Then we have

$$|\dot{b}_1(x) - \dot{b}_2(x)| = \left|\frac{v_2\left(x,b_2(x)\right)}{u_2\left(x,b_2(x)\right)} - \frac{v_1\left(x,b_1(x)\right)}{u_1\left(x,b_1(x)\right)}\right| = O(1)\left(|h_{b,1}(x)| + |h_{b,4}(x)|\right). \tag{4.15}$$

Therefore, when the initial values of the Glimm functionals $F_j(0), j = 1, 2$, are small enough, we obtain

$$\begin{aligned} E_{b,1} &= |q_{b,1}(x)|W_1\left(b_{\max}(x)\right)\left(\dot{b}_{\max}(x) - \lambda_{b,1}(x)\right) = c_1^u|h_{b,1}|\kappa_1 B|\lambda_{b,1}| + O(1)\,|h_{b,1}| \\ &\leqslant \kappa_1 B c_1^u\,\eta|h_{b,4}||\lambda_{b,1}| + O(1)\,|h_{b,4}| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x) + \mu\right), \\ E_{b,k} &= |q_{b,k}(x)|W_k(b_{\max}(x))\left(\dot{b}_{\max}(x) - \lambda_{b,k}(x)\right) \end{aligned}$$

$$
\begin{aligned}
&= c_k^u|h_{b,k}|O(1)\,|\dot{b}_1-\lambda_{b,k}| = O(1)\,(|h_{b,1}|+|h_{b,4}|)\\
&\leqslant O(1)\,|h_{b,4}| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right) \qquad \text{for } k=2,3,\\
E_{b,4} &= |q_{b,4}(x)|W_4(b_{\max}(x))\left(\dot{b}_{\max}(x)-\lambda_{b,4}(x)\right)\\
&= -c_4^u|h_{b,4}|\kappa_1 B|\lambda_{b,1}| + c_4^u|h_{b,4}|\kappa_1 B(|\lambda_{b,1}|-\lambda_{b,4}) + O(1)|h_{b,4}|\\
&\leqslant -c_4^u|h_{b,4}|\kappa_1 B|\lambda_{b,1}| + O(1)|h_{b,4}|.
\end{aligned}
$$

In (4.12), we have chosen $c_1^u < c_4^u$ with κ_1 suitably large, we conclude

$$
\begin{aligned}
\sum_{i=1}^{4} E_{b,i} &\leqslant \kappa_1 B c_1^u \eta |h_{b,4}||\lambda_{b,1}| + O(1)\,|h_{b,4}|\\
&\quad + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right) - c_4^u|h_{b,4}|\kappa_1 B|\lambda_{b,1}|\\
&\leqslant (c_1^u\eta - c_4^u)\,\kappa_1 B|h_{b,4}||\lambda_{b,1}| + O(1)\,|h_{b,4}| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right)\\
&\leqslant -\zeta|h_{b,4}| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right)
\end{aligned}
$$

for some $\zeta>0$. When c_0 is sufficiently small, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{4} E_{b,i} + c_0\left(|h_{b,1}(x)| + |h_{b,4}(x)|\right)\\
&\leqslant -\zeta|h_{b,4}| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right)\\
&\quad + c_0\left((\eta+1)|h_{b,4}(x)| + O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right)\right)\\
&\leqslant O(1)\left(p_{b,2}^{\mu,\Delta x_2}(x) - p_{b,1}^{\mu,\Delta x_1}(x)+\mu\right).
\end{aligned}
$$

This concludes lemma 4.1.

□

3. By lemma 4.1, (4.15), and (4.6), as long as $F_1(0), F_2(0)$, and c_b are chosen small enough, we obtain (4.5). This completes the proof of proposition 4.2. □

Remark 4.2. The proof of lemma 4.1 makes a strategic selection of the weights in the Lyapunov functional, based on the following criteria:

(i) The nature of the discontinuity (leading shock or weak wave),
(ii) The spatial position of the discontinuity,
(iii) The sign of the wave strength h_i,
(iv) Interaction estimates between waves.

Our approach synthesizes the techniques from [6, 12, 33], together with the boundary condition for the pressure in (4.13). This enables the derivation of (4.10), mirroring the role of the slope boundary conditions in [12, proposition 5.2].

Proposition 4.3. *If the approximate pressures $p_{b,i}$, $i=1,2$, corresponding to the inner boundary, change at some $x>0$, or there is a reflection on the inner boundary at x, then*

$$
\mathfrak{F}(U_1(x+), U_2(x+)) \leqslant (1+O(1)\,|\alpha|)\,\mathfrak{F}(U_1(x-), U_2(x-)),
$$

where $|\alpha| = |p_{b,i}(x+) - p_{b,i}(x-)|$ or $|\alpha|$ denotes the strength of the incoming wavefront that hits the inner boundary.

Proof. With out loss of generality, denote $b_{\max}(x) = b_1(x)$ and $b_{\min}(x) = b_2(x)$. Then, in both cases,

$$|U_2(x+) - U_2(x-)| \leqslant O(1)|\alpha|.$$

Then

$$\begin{aligned}
&\mathfrak{F}(U_1(x+), U_2(x+)) - \mathfrak{F}(U_1(x-), U_2(x-)) \\
&\quad = \sum_{i=1}^{4} \int_{-\infty}^{b_{\max}(x)} W_i(x+, y)|q_i(x+, y)|\,\mathrm{d}y - \sum_{i=1}^{4} \int_{-\infty}^{b_{\max}(x)} W_i(x-, y)|q_i(x-, y)|\,\mathrm{d}y \\
&\quad = \sum_{i=1}^{4} \int_{b_{\min}(x)}^{b_{\max}(x)} \big(W_i(x+, y)|q_i(x+, y)| - W_i(x-, y)|q_i(x-, y)|\big)\,\mathrm{d}y \\
&\qquad + \sum_{i=1}^{4} \int_{-\infty}^{b_{\min}(x)} \big(W_i(x+, y)|q_i(x+, y)| - W_i(x-, y)|q_i(x-, y)|\big)\,\mathrm{d}y \\
&\quad \leqslant \sum_{i=1}^{4} \int_{b_{\min}(x)}^{b_{\max}(x)} \big(W_i(x+, y)|q_i(x+, y)| - W_i(x-, y)|q_i(x-, y)|\big)\,\mathrm{d}y \\
&\quad \leqslant O(1)|U_2(x+) - U_2(x-)|\,|b_{\max}(x) - b_{\min}(x)| \\
&\quad \leqslant O(1)|\alpha|\,|b_{\max}(x) - b_{\min}(x)|.
\end{aligned}$$

Using the boundary condition (1.7), we have

$$\begin{aligned}
|b_{\max}(x) - b_{\min}(x)| &\leqslant \int_0^x \Big|\frac{v_1}{u_1}(s) - \frac{v_2}{u_2}(s)\Big|\,\mathrm{d}s \leqslant O(1)\int_0^x \big(|h_{b,1}(s)| + |h_{b,4}(s)|\big)\,\mathrm{d}s \\
&\leqslant O(1)\,\mathfrak{F}(U_1(x-), U_2(x-)).
\end{aligned}$$

Therefore, we conclude

$$\mathfrak{F}(U_1(x+), U_2(x+)) \leqslant (1 + O(1)|\alpha|)\,\mathfrak{F}(U_1(x-), U_2(x-)).$$

□

Remark 4.3. Proposition 4.3 implies that the extension of the approximate solution introduces the discontinuities in the x-direction, resulting in the non-differentiable singularities. Therefore, at each time where such a discontinuity forms, the Lyapunov functional experiences bounded amplification, growing by a factor of $1 + O(1)|\alpha|$, where $|\alpha|$ denotes the magnitude of the jump of the pressure or the strength of the incoming wavefront that hits the inner boundary.

To conclude, we have

Proposition 4.4. *Under the assumptions of propositions 4.1 and 4.2, for any $x > 0$,*

$$\begin{aligned}
&\int_0^x |\dot{b}_1(t) - \dot{b}_2(t)|\,\mathrm{d}t + \int_{-\infty}^{b_{\max}(x)} |U_1(x, y) - U_2(x, y)|\,\mathrm{d}y \\
&\quad \leqslant O(1)x\mu + O(1)\|p_{b,2}^{\mu,\Delta x_2} - p_{b,1}^{\mu,\Delta x_1}\|_{L^1(0,x)} + O(1)\|U_{\infty,2}^{\mu,\Delta x_2} - U_{\infty,1}^{\mu,\Delta x_1}\|_{L^1((-\infty,0))}.
\end{aligned}$$

Proof. For any fixed $x > 0$, let

$$P_d = \{\tau \in [0,x) : \text{an interaction occurs at } x = \tau\}, \qquad P_c = \{\tau \in [0,x) : \tau \notin P_d\}.$$

Then P_d is finite, which is written as

$$P_d = \{x_i, i \in \mathbb{N} : 0 = x_0 < x_1 < \cdots < x_n \leqslant x < x_{n+1}\}.$$

We may assume that, for each $x = x_i$, there is an interaction or a nonphysical wave $\alpha(i)$ cross the boundary, or there is a physical wave $\alpha(i)$ hitting the boundary, or the variance of the approximate pressure on the inner boundary is equal to $|\alpha(i)|$. If there is no nonphysical wave or only a nonphysical wave crossing the boundary, we set $\alpha(i) = 0$. Then propositions 4.1–4.3 imply

$$\begin{aligned}
&\mathfrak{F}(U_1(x), U_2(x)) - \mathfrak{F}(U_1(0), U_2(0)) \\
&\quad = \int_{x_n}^{x} \frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{F}(U_1(s), U_2(s))\,\mathrm{d}s + \mathfrak{F}(U_1(x_n), U_2(x_n)) - \mathfrak{F}(U_1(0), U_2(0)) \\
&\quad \leqslant O(1)(x - x_n)\mu + O(1)\|p_{b,2}^{\mu,\Delta x_2} - p_{b,1}^{\mu,\Delta x_1}\|_{L^1((x_n,x))} \\
&\qquad + \exp(O(1)|\alpha_n|)\mathfrak{F}(U_1(x_n-), U_2(x_n-)) - \mathfrak{F}(U_1(0), U_2(0)) \\
&\quad \leqslant O(1)\exp\Big(O(1)\sum_{i=1}^{n}|\alpha(i)|\Big)x\mu + O(1)\exp\Big(O(1)\sum_{i=1}^{n}|\alpha(i)|\Big)\|p_{b,2}^{\mu,\Delta x_2} - p_{b,1}^{\mu,\Delta x_1}\|_{L^1(0,x)} \\
&\qquad + \Big(\exp\Big(O(1)\sum_{i=1}^{n}|\alpha(i)|\Big) - 1\Big)\mathfrak{F}(U_1(0), U_2(0)).
\end{aligned}$$

Since propositions 3.2–3.4 give an upper bound of $\sum_{i=1}^{n}|\alpha(i)|$, which is independent of our approximate solutions, we obtain

$$\mathfrak{F}(U_1(x), U_2(x)) \leqslant O(1)x\mu + O(1)\|p_{b,2}^{\mu,\Delta x_2} - p_{b,1}^{\mu,\Delta x_2}\|_{L^1(0,x)} + O(1)\,\mathfrak{F}(U_1(0), U_2(0)).$$

Finally, the construction of our Lyapunov functional leads to

$$\int_0^x |\dot{b}_1(t) - \dot{b}_2(t)|\,\mathrm{d}t + \int_{-\infty}^{b_{\max}(x)} |U_1(x,y) - U_2(x,y)|\,\mathrm{d}y \leqslant O(1)\,\mathfrak{F}(U_1(x,\cdot), U_2(x,\cdot)).$$

This completes the proof. □

Remark 4.4. As indicated in remark 4.3, the Lyapunov functional has bounded amplifications at some *time*. However, proposition 4.4 states that the magnitude of each discontinuity is controlled by $|\alpha(i)|$, the magnitude of the jump in the pressure distribution, the strength of a physical wave or a non-physical wave that hits the boundary, whose total variation is bounded by the initial value of the Glimm-type functional. This incremental amplification remains globally manageable due to the *a priori* bounds on $\sum|\alpha(i)|$.

Corollary 4.1. *Under the assumptions of propositions 4.1 and 4.2, for any $x > 0$,*

$$\begin{aligned}
&|b_1(x) - b_2(x)| + \int_{-\infty}^{b_{\max}(x)} |U_1(x,y) - U_2(x,y)|\,\mathrm{d}y \\
&\quad \leqslant O(1)x\mu + O(1)\|p_{b,2}^{\mu,\Delta x_2} - p_{b,1}^{\mu,\Delta x_1}\|_{L^1(0,x)} + O(1)\,\mathfrak{F}(U_1(x,\cdot), U_2(x,\cdot)).
\end{aligned}$$

5. Existence of the semigroup

Combining all the analysis in sections 3 and 4, we now establish the existence of the semigroup that generates the solution of the inverse problem.

First we introduce the following definitions:

(i) For $f \in L^1_{\text{loc}}(x,\infty)$, define

$$\iota_x: \quad L^1_{\text{loc}}(x,\infty) \to L^1_{\text{loc}}(\mathbb{R}_+), \quad (\iota_x f)(\theta) := f(\theta + x); \tag{5.1}$$

(ii) for $(b_j, \check{U}_j^\top, p_j)^\top \in \mathbb{R} \times L^1_{\text{loc}}(\mathbb{R}_-) \times L^1_{\text{loc}}(\mathbb{R}_+)$, $j = 1, 2$, define

$$\begin{aligned}&\left\|(b_1, \check{U}_1^\top, p_1)^\top - (b_2, \check{U}_2^\top, p_2)^\top\right\|_Y \\ &:= |b_1 - b_2| + \|\check{U}_1 - \check{U}_2\|_{L^1(\mathbb{R}_-)} + \|p_1 - p_2\|_{L^1(\mathbb{R}_+)}.\end{aligned} \tag{5.2}$$

Definition 5.1. Given $\varepsilon > 0$, define

$$\mathbb{D}^\varepsilon = \text{cl}\left\{(b, \check{U}^\top, p)^\top \,:\, b \in \mathbb{R}, \begin{array}{l}\check{U} \in \mathbf{PWC},\ \check{U} - \bar{U}_\flat \in L^1(\mathbb{R}_-;\mathbb{R}^4),\\ p \in \mathbf{PWC},\ p - \bar{p}_b \in L^1(\mathbb{R}_+;\mathbb{R}),\end{array} F(0; b, \check{U}^\top, p) \leqslant \varepsilon\right\},$$

where **PWC** stands for the piecewise constant functions (vectors), $\bar{U}_\flat(y)$ satisfies

$$\bar{U}_\flat(y) = \begin{cases} U_- & \text{for } y < \chi_\flat, \\ U_+ & \text{for } \chi_\flat < y < 0,\end{cases}$$

for some $\chi_\flat \leqslant 0$, $F(0; b, \check{U}^\top, p)$ is the Glimm-type functional corresponding to the initial data $\check{U}$ and the pressure distribution p_b (see definition 3.2), and cl represents the closure in $\|\cdot\|_Y$.

We also need the following lemma (see lemma 2.3 in [5]).

Lemma 5.1. *If $U : \mathbb{R} \to \mathbb{R}^n$ has bounded total variation, then*

$$\int_{-\infty}^{\infty} |U(x+t) - U(x)|\,\mathrm{d}x \leqslant t\,\text{T.V.}(U) \qquad \textit{for any } t > 0.$$

We now establish the existence theorem of the semigroup that generates the solution of this inverse problem.

Theorem 5.1. *Suppose that $\varepsilon > 0$ is sufficiently small. Then, for any $(b(0), \check{U}_\infty^\top, p_b)^\top \in \mathbb{D}^\varepsilon$, corresponding to the initial data $U_\infty^\top(y) = \check{U}_\infty^\top(y - b(0))$ for $y < b(0)$ and the pressure distribution p_b, there is a subsequence of μ-approximate solutions $(b^{\mu,\Delta x}, U^{\mu,\Delta x})$ converging to a unique solution (b, U) as $\mu \to 0$. The map:*

$$(b(0), \check{U}_\infty^\top, p_b, x)^\top \mapsto (b(x), U^\top(x, \cdot + b(x)), \iota_x p_b)^\top := \mathfrak{S}_x(b(0), \check{U}_\infty^\top, p_b)^\top$$

is a semigroup that generates the solution of the inverse problem so that, for any

$$(b(0), \check{U}_\infty^\top, p_b)^\top,\ (b_i(0), \check{U}_{\infty,i}^\top, p_{b,i})^\top \in \mathbb{D}^\varepsilon,$$

and $x_i \geqslant 0$, $i = 1, 2$,

$$\begin{aligned}&\mathfrak{S}_0(b(0), \check{U}_\infty^\top, p_b)^\top = (b(0), \check{U}_\infty^\top, p_b)^\top,\\ &\mathfrak{S}_{x_1}\mathfrak{S}_{x_2}(b(0), \check{U}_\infty^\top, p_b)^\top = \mathfrak{S}_{x_1+x_2}(b(0), \check{U}_\infty^\top, p_b)^\top.\end{aligned}$$

Moreover, there are constants $L^\sharp > 0$ and $L^\flat > 0$ such that

$$\begin{aligned}&\big\|\mathfrak{S}_{x_1}(b_1(0),\check{U}^\top_{\infty,1}(\cdot),p_{b,1})^\top - \mathfrak{S}_{x_2}(b_2(0),\check{U}^\top_{\infty,2}(\cdot),p_{b,2})^\top\big\|_Y \\ &\quad\leqslant L^\sharp\big\|(b_1(0),\check{U}^\top_{\infty,1}(\cdot),p_{b,1})^\top - (b_2(0),\check{U}^\top_{\infty,2}(\cdot),p_{b,2})^\top\big\|_Y + L^\flat|x_1-x_2|.\end{aligned}$$

Proof. For any $\mu_1,\mu_2>0$, let U^{μ_1} and U^{μ_2} be the μ_j-approximate solutions of (1.1) and (1.6)–(1.11), whose initial data are $U^{\mu_1}_\infty$ and $U^{\mu_2}_\infty$, and pressure distributions are $p^{\mu_1}_b$ and $p^{\mu_2}_b$, respectively. Fixing $x>0$, by propositions 4.1–4.3 and lemma 5.1, we have

$$\begin{aligned}&\big\|(b^{\mu_1}(x),(U^{\mu_1}(x,\cdot+b^{\mu_1}(x)))^\top,\iota_x p^{\mu_1}_b)^\top - (b^{\mu_2}(x),(U^{\mu_2}(x,\cdot+b^{\mu_2}(x)))^\top,\iota_x p^{\mu_2}_b)^\top\big\|_Y \\ &\quad\leqslant O(1)\,\mathfrak{F}(U^{\mu_1}(x),U^{\mu_2}(x)) + \|p^{\mu_1}_b - p^{\mu_2}_b\|_{L^1(x,\infty)} \\ &\quad\leqslant O(1)\,\mathfrak{F}(U^{\mu_1}(0),U^{\mu_2}(0)) + O(1)\|p^{\mu_1}_b - p^{\mu_2}_b\|_{L^1(0,x)} \\ &\qquad + O(1)\max\{\mu_1,\mu_2\}x + \|p^{\mu_1}_b - p^{\mu_2}_b\|_{L^1(x,\infty)} \\ &\quad\leqslant C\big\|(b^{\mu_1}(0),(\check{U}^{\mu_1}_\infty)^\top(\cdot),p^{\mu_1}_b)^\top - (b^{\mu_2}(0),(\check{U}^{\mu_2}_\infty)^\top(\cdot),p^{\mu_2}_b)^\top\big\|_Y + C\max\{\mu_1,\mu_2\}x.\end{aligned}$$

Thus, as μ_1, $\mu_2\to 0$, $\|(b^{\mu_1}(x),(U^{\mu_1}(x,\cdot+b^{\mu_1}(x)))^\top,p^{\mu_1}_b)^\top - (b^{\mu_2}(x),(U^{\mu_2}(x,\cdot+b^{\mu_2}(x)))^\top,p^{\mu_2}_b)^\top\|_Y$ tends to zero, which implies that the sequence is a Cauchy sequence converging to a unique limit, saying $(b(x),U^\top(x,\cdot+b(x)),\iota_x p_b)^\top$. Then the semigroup properties follow from the uniqueness.

Finally, for $\mu>0$, let U^μ_1 and U^μ_2 be μ-approximate solutions to (1.1) and (1.6)–(1.11) satisfying

$$\begin{aligned}&\|U^\mu_1(0,\cdot+b_1(0)) - \check{U}_{\infty,1}(\cdot)\|_{L^1(\mathbb{R}_-)}<\mu, &&\|U^\mu_2(0,\cdot+b_2(0)) - \check{U}_{\infty,2}(\cdot)\|_{L^1(\mathbb{R}_-)}<\mu,\\ &\|p^\mu_{b,1}(\cdot)-p_{b,1}(\cdot)\|_{L^1(\mathbb{R}_+)}<\mu, &&\|p^\mu_{b,2}(\cdot)-p_{b,2}(\cdot)\|_{L^1(\mathbb{R}_+)}<\mu.\end{aligned}$$

Then

$$\begin{aligned}&\big\|(b^\mu_1(x),(U^\mu_1(x,\cdot+b^\mu_1(x)))^\top,\iota_x p^\mu_{b,1}) - (b^\mu_2(x),(U^\mu_2(x,\cdot+b^\mu_2(x)))^\top,\iota_x p^\mu_{b,2})\big\|_Y \\ &\quad\leqslant O(1)\,\mathfrak{F}(U^\mu_1(x),U^\mu_2(x)) + \|p^\mu_{b,1}-p^\mu_{b,2}\|_{L^1(x,\infty)} \\ &\quad\leqslant O(1)\,\mathfrak{F}(U^\mu_1(0),U^\mu_2(0)) + O(1)\|p^\mu_{b,1}-p^\mu_{b,2}\|_{L^1(0,x)} + O(1)\mu x + \|p^\mu_{b,1}-p^\mu_{b,2}\|_{L^1(x,\infty)} \\ &\quad\leqslant C\|(b^\mu_1(0),(\check{U}^\mu_{\infty,1})^\top(\cdot),p^\mu_{b,1})^\top - (b^\mu_2(0),(\check{U}^\mu_{\infty,2})^\top(\cdot),p^\mu_{b,2})^\top\|_Y + C\mu x.\end{aligned}$$

Taking $\mu\to 0$, we obtain

$$\begin{aligned}&\big\|(b_1(x),U_1^\top(x,\cdot+b_1(x)),\iota_x p_{b,1})^\top - (b_2(x),U_2^\top(x,\cdot+b_2(x)),\iota_x p_{b,2})^\top\big\|_Y \\ &\quad\leqslant L^\sharp\big\|(b_1(0),U^\top_{\infty,1}(\cdot),p_{b,1})^\top - (b_2(0),U^\top_{\infty,2}(\cdot),p_{b,2})^\top\big\|_Y\end{aligned}$$

for some $L^\sharp>0$, which gives the Lipschitz continuity. Moreover, from proposition 3.3, (A.13), and the fact that

$$\|\iota_x p_{b,1} - \iota_x p_{b,2}\|_{L^1(\mathbb{R}_+)} \leqslant \|p_{b,1}-p_{b,2}\|_{L^1(\mathbb{R}_+)},$$

we conclude the Lipschitz continuity on x.

□

Main Theorem I is a direct corollary of theorems 3.1 and 5.1.

6. Approximate the full Euler equations by the potential flow equations

This section focuses on the comparison between the two solutions of the inverse problem, which are obtained by solving the full Euler equations and the potential flow equations, respectively. If there is no strong shock, at *time* x, we show that the difference of two solutions in the norm $\|\cdot\|_Y$ is up to the third order of the total variation of the initial boundary data, multiplying x.

6.1. Existence and stability of the potential flow equations

Regarding x as *time*, when $u^2+v^2<2B_\infty$ and $u>c_*=\sqrt{2B_\infty(\gamma-1)/(\gamma+1)}$, system (1.4) and (1.5) is strictly hyperbolic, whose eigenvalues are

$$\lambda_j(\mathbf{u})=\frac{uv+(-1)^j c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \qquad j=1,2, \tag{6.1}$$

with corresponding eigenvectors (see [50])

$$r_j(\mathbf{u})=\frac{(-\lambda_j,1)^\top}{(-\lambda_j(\mathbf{u}),1)\cdot\nabla\lambda_j(\mathbf{u})}, \qquad j=1,2. \tag{6.2}$$

Using the same method as we have developed in the previous sections, we have

Theorem 6.1. *Let* $(\mathbf{C}_\mathrm{P}\mathbf{1})$*–*$(\mathbf{C}_\mathrm{P}\mathbf{2})$ *hold. Then there is* $\breve{\varepsilon}>0$ *such that, when* $\|(\widetilde{\mathbf{u}}_\infty,\tilde{p}_b)\|_{L^\infty\cap BV}<\breve{\varepsilon}$*, there exist subsequences* $\{\mu_l\}_{l=1}^\infty$ *and* $\{\Delta x_l\}_{l=1}^\infty$ *so that*

(i) $b^{\mu_l,\Delta x_l}$ *converges uniformly to b on any compact subset contained in the x-axis*:
(ii) $\mathbf{u}_b^{\mu_l,\Delta x_l}$ *converges to* u_b *in* $\mathrm{BV}([0,\infty))$*, and* $\dot{b}(x)=\frac{v_b(x)}{u_b(x)}$ *a.e.*:
(iii) *for any* $x>0$*,* $\mathbf{u}^{\mu_l,\Delta x_l}$*converges to u in* $L^1_{\mathrm{loc}}\big((-\infty,b(x));\mathbb{R}^2\big)$*, and u is an entropy solution of equations (1.4) and (1.5) satisfying (1.15)–(1.16).*

Definition 6.1. Let $(b_j,\check{\mathbf{u}}_j^\top,p_j)^\top\in\mathbb{R}\times L^1_{\mathrm{loc}}(\mathbb{R}_-)\times L^1_{\mathrm{loc}}(\mathbb{R}_+), j=1,2$. Define

$$\big\|(b_1,\check{\mathbf{u}}_1^\top,p_1)^\top-(b_2,\check{\mathbf{u}}_2^\top,p_2)^\top\big\|_{Y_\mathrm{P}}=|b_1-b_2|+\|\check{\mathbf{u}}_1-\check{\mathbf{u}}_2\|_{L^1(\mathbb{R}_-)}+\|p_1-p_2\|_{L^1(\mathbb{R}_+)}.$$

Definition 6.2. Given $\hat{\varepsilon}>0$, define

$$\mathbb{D}_\mathrm{P}^{\hat{\varepsilon}}=\mathrm{cl}\left\{(b,\check{\mathbf{u}}^\top,p)^\top\,:\,b\in\mathbb{R},\ \begin{array}{l}\check{\mathbf{u}}\in\mathbf{PWC},\ \ \check{\mathbf{u}}-\overline{\mathbf{u}}_0\in L^1\left(\mathbb{R}_-;\mathbb{R}^2\right),\\ p\in\mathbf{PWC},\ \ p-\bar{p}_b\in L^1\left(\mathbb{R}_+;\mathbb{R}\right),\end{array}\ F_\mathrm{P}(0;b,\check{\mathbf{u}}^\top,p)\leqslant\hat{\varepsilon}\right\},$$

where $F_\mathrm{P}(0;b,\check{\mathbf{u}}^\top,p)$ is the Glimm-type functional corresponding to the initial data $\check{\mathbf{u}}$ and the pressure distribution p_b for the potential flow equations, and cl stands for the closure with respect to $\|\cdot\|_{Y_\mathrm{P}}$.

Furthermore, we have

Theorem 6.2. *For* $\hat{\varepsilon}>0$ *sufficiently small, take* $(b(0),\check{\mathbf{u}}_\infty^\top,p_b)^\top\in\mathbb{D}_\mathrm{P}^{\hat{\varepsilon}}$*. Then, with the initial value* $\mathbf{u}_\infty^\top(y)=\check{\mathbf{u}}_\infty^\top(y-b(0))$ *for* $y<b(0)$ *and the pressure distribution* p_b*, a subsequence of* μ*-approximate solutions* $(b^{\mu,\Delta x},\mathbf{u}^{\mu,\Delta x})$ *converges to a unique limit* $(b,\mathbf{u})$*, as* $\mu\to0$*. As a result,*

$$(b(0),\check{\mathbf{u}}_\infty^\top,p_b,x)^\top\mapsto(b(x),\mathbf{u}^\top(x,\cdot+b(x)),\iota_x p_b)^\top:=\mathfrak{S}_x^\mathrm{P}(b(0),\check{\mathbf{u}}_\infty^\top,p_b)^\top$$

is a semigroup that generates the solution to the inverse problem for the potential flow system: *If*

$$(b(0),\check{\mathbf{u}}_\infty^\top,p_b)^\top,\ (b_i(0),\check{\mathbf{u}}_{\infty,i}^\top,p_{b,i})^\top\in\mathbb{D}_{\mathrm{P}}^{\hat{\varepsilon}},$$

and $x_i\geqslant 0$, $i=1,2$, *then*

$$\begin{aligned}&\mathfrak{S}_0^{\mathrm{P}}(b(0),\check{\mathbf{u}}_\infty^\top(\cdot),p_b)^\top=(b(0),\check{\mathbf{u}}_\infty^\top(\cdot),p_b)^\top,\\&\mathfrak{S}_{x_1}^{\mathrm{P}}\mathfrak{S}_{x_2}^{\mathrm{P}}(b(0),\check{\mathbf{u}}_\infty^\top(\cdot),p_b)^\top=\mathfrak{S}_{x_1+x_2}^{\mathrm{P}}(b(0),\mathbf{u}_\infty^\top(\cdot),p_b)^\top.\end{aligned}$$

Moreover, there exist $K^\sharp>0$ *and* $K^\flat>0$ *such that*

$$\begin{aligned}&\|\mathfrak{S}_{x_1}^{\mathrm{P}}(b_1(0),\mathbf{u}_{\infty,1}^\top(\cdot),p_{b,1})^\top-\mathfrak{S}_{x_2}^{\mathrm{P}}(b_2(0),\mathbf{u}_{\infty,2}^\top(\cdot),p_{b,2})^\top\|_{Y_{\mathrm{P}}}\\&\leqslant K^\sharp\,\|(b_1(0),\mathbf{u}_{\infty,1}^\top(\cdot),p_{b,1})^\top-(b_2(0),\mathbf{u}_{\infty,2}^\top(\cdot),p_{b,2})^\top\|_{Y_{\mathrm{P}}}+K^\flat\,|x_1-x_2|.\end{aligned}$$

6.2. Comparison of the wave curves and Riemann-type problems

It follows from $(\mathbf{C}_{\mathrm{P}}\mathbf{1})$–$(\mathbf{C}_{\mathrm{P}}\mathbf{2})$ and $(\mathbf{C}_{\mathrm{E}}\mathbf{1})$–$(\mathbf{C}_{\mathrm{E}}\mathbf{2})$ that $\overline{\mathbf{u}}_\infty=(\overline{u}_\infty,0)^\top$ and $\overline{U}_\infty=(\overline{u}_\infty,0,(\mathscr{R}(|\overline{\mathbf{u}}_\infty|))^\gamma,\mathscr{R}(|\overline{\mathbf{u}}_\infty|))^\top$. According to [30] (see also [14, 46]), there exists $\delta_1>0$ such that, in the neighborhood $O_{\delta_1}(\overline{U}_\infty)$ of $\overline{U}_\infty=(\overline{\mathbf{u}}_\infty^\top,(\mathscr{R}(|\overline{\mathbf{u}}_\infty|))^\gamma,\mathscr{R}(|\overline{\mathbf{u}}_\infty|))^\top$, the jth wave curve of the full Euler equations through $U_l\in O_{\delta_1}(\overline{U}_\infty)$ is parameterized as

$$\alpha_j\mapsto\Phi_{\mathrm{E},j}(\alpha_j;U_l),\qquad j=1,2,3,4,$$

in the neighborhood $O_{\delta_1}(\overline{\mathbf{u}}_\infty)$ of $\overline{\mathbf{u}}_\infty$, the jth wave curve of the potential flow equations through $\mathbf{u}_l\in O_{\delta_1}(\overline{\mathbf{u}}_\infty)$ is parameterized as

$$\alpha_j\mapsto\Phi_{\mathrm{P},j}(\alpha_j;\mathbf{u}_l),\qquad j=1,2,$$

such that

$$\frac{\mathrm{d}\Phi_{\mathrm{P},j}(\alpha_j;\mathbf{u}_l)}{\mathrm{d}\alpha_j}=r_j(\mathbf{u}_l).$$

We denote

$$\begin{aligned}&\Phi_{\mathrm{E}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4;U_l)=\Phi_{\mathrm{E},4}(\alpha_4;\Phi_{\mathrm{E},3}(\alpha_3;\Phi_{\mathrm{E},2}(\alpha_2;\Phi_{\mathrm{E},1}(\alpha_1;U_l)))),\\&\Phi_{\mathrm{P}}(\alpha_1,\alpha_2;\mathbf{u}_l)=\Phi_{\mathrm{P},2}(\alpha_2;\Phi_{\mathrm{P},1}(\alpha_1;\mathbf{u}_l)),\end{aligned}$$

and define

$$\begin{aligned}&D_{\mathrm{E}}:=\left\{(\mathbf{u}^\top,p,\rho)^\top\in O_{\delta_1}(\overline{U}_\infty):|\mathbf{u}|>c,p=\rho^\gamma,\frac{|\mathbf{u}|^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}=B_\infty\right\},\\&D_{\mathrm{P}}:=\left\{\mathbf{u}\in O_{\delta_1}(\overline{\mathbf{u}}_\infty):\frac{2(\gamma-1)B_\infty}{\gamma+1}<|\mathbf{u}|^2<2B_\infty\right\}.\end{aligned}$$

For $\mathbf{u}_l\in D_{\mathrm{P}}$, define

$$\begin{aligned}&\Psi_j(\alpha_j;\mathbf{u}_l):=(\Phi_{\mathrm{P},j}(\alpha_j;\mathbf{u}_l)^\top,(\mathscr{R}(|\Phi_{\mathrm{P},j}(\alpha_j;\mathbf{u}_l)|))^\gamma,\mathscr{R}(|\Phi_{\mathrm{P},j}(\alpha_j;\mathbf{u}_l)|))^\top,\\&\Psi(\alpha_1,\alpha_2;\mathbf{u}_l):=\Psi_2(\alpha_2;\Psi_1(\alpha_1;\mathbf{u}_l)),\end{aligned}$$

where $\mathscr{R}(r)$ is given in (1.18). We now compare between the wave curves of the full Euler equations and the potential flow equations. For any $U_l \in D_{\mathrm{E}}$, denote

$$\Phi_{\mathrm{E},j}(\alpha_j;U_l) := \left(u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l), p_{\mathrm{E},j}(\alpha_j;U_l), \rho_{\mathrm{E},j}(\alpha_j;U_l)\right)^{\top}.$$

First, for $\Phi_{\mathrm{E},j}$, $j = 1,4$, we have the following property:

Lemma 6.1. *For $\alpha_j \geqslant 0$ for $j = 1,4$,*

$$p_{\mathrm{E},j} = (\rho_{\mathrm{E},j})^{\gamma}, \qquad \frac{(u_{\mathrm{E},j})^2 + (v_{\mathrm{E},j})^2}{2} + \frac{\gamma(\rho_{\mathrm{E},j})^{\gamma-1}}{\gamma-1} = B_{\infty}. \tag{6.3}$$

Proof. Let $\lambda_{\mathrm{E},j}$ be the jth eigenvalue of the full Euler equations. A direct computation leads to

$$\begin{aligned}\frac{\mathrm{d}\left(p_{\mathrm{E},j} - (\rho_{\mathrm{E},j})^{\gamma}\right)}{\mathrm{d}\alpha_j} &= \mathfrak{k}_j\rho_{\mathrm{E},j}(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j}) - \gamma(\rho_{\mathrm{E},j})^{\gamma-1}\mathfrak{k}_j\frac{(\rho_{\mathrm{E},j})^2(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}{\gamma p_{\mathrm{E},j}}\\ &= \frac{\mathfrak{k}_j\rho_{\mathrm{E},j}(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}{\gamma p_{\mathrm{E},j}}\left(p_{\mathrm{E},j} - (\rho_{\mathrm{E},j})^{\gamma}\right).\end{aligned}$$

Notice that the first part of (6.3) holds when $\alpha_j = 0$ and $p_l - \rho_l^{\gamma} = 0$. Then, since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha_j}\left(\frac{(u_{\mathrm{E},j})^2 + (v_{\mathrm{E},j})^2}{2} + \frac{\gamma p_{\mathrm{E},j}}{(\gamma-1)\rho_{\mathrm{E},j}}\right) &= -u_{\mathrm{E},j}\mathfrak{k}_j\lambda_{\mathrm{E},j} + \mathfrak{k}_j v_{\mathrm{E},j} + \frac{\gamma\mathfrak{k}_j\rho_{\mathrm{E},j}(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}{(\gamma-1)\rho_{\mathrm{E},j}}\\ &\quad - \frac{\gamma p_{\mathrm{E},j}}{(\gamma-1)(\rho_{\mathrm{E},j})^2}\frac{\rho_{\mathrm{E},j}\mathfrak{k}_j(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}{(c_{\mathrm{E},j})^2} = 0,\end{aligned}$$

and $p_{\mathrm{E},j} = (\rho_{\mathrm{E},j})^{\gamma}$, we obtain

$$\begin{aligned}\frac{(u_{\mathrm{E},j})^2 + (v_{\mathrm{E},j})^2}{2} + \frac{\gamma(\rho_{\mathrm{E},j})^{\gamma-1}}{\gamma-1} &= \frac{(u_{\mathrm{E},j})^2 + (v_{\mathrm{E},j})^2}{2} + \frac{\gamma p_{\mathrm{E},j}}{(\gamma-1)\rho_{\mathrm{E},j}}\\ &= \frac{u_l^2 + v_l^2}{2} + \frac{\gamma p_l}{(\gamma-1)\rho_l} = B_{\infty}.\end{aligned}$$

This completes the proof. □

Next, we have (see also [52])

Lemma 6.2. *For $U_l = (\mathbf{u}_l^{\top}, p_l, \rho_l)^{\top} \in D_{\mathrm{E}}$ and $\mathbf{u}_l \in D_{\mathrm{P}}$, when $\alpha_j \geqslant 0$, then*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_j}\left(u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l)\right)^{\top} = \frac{\mathrm{d}\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)}{\mathrm{d}\alpha_j}, \qquad j = 1,4. \tag{6.4}$$

Proof. By (2.3), it is direct to see

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\alpha_j}\left(u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l)\right)^{\top}\\ &\quad = \mathfrak{k}_j(-\lambda_{\mathrm{E},j},1)^{\top} = \frac{2}{\gamma+1}\frac{(u_{\mathrm{E},j}^2 - c_{\mathrm{E},j}^2)\lambda_{\mathrm{E},j} - u_{\mathrm{E},j}v_{\mathrm{E},j}}{(1+\lambda_{\mathrm{E},j}^2)(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}(-\lambda_{\mathrm{E},j},1)^{\top}.\end{aligned}$$

Form lemma 6.1, (2.1)–(2.3), and (6.1)–(6.2), we obtain

$$\lambda_{\mathrm{E},j}(\Phi_{\mathrm{E},j}(\alpha_j;U_l)) = \lambda_{\mathrm{P},\sqrt{j}}((u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l))^\top),$$

where $\lambda_{\mathrm{P},\sqrt{j}}(\mathbf{u})$ is the $\sqrt{j}$th eigenvalue of the potential flow equations. By lemma 6.1, we see that $c^2_{\mathrm{E},j} = \gamma(\rho_{\mathrm{E},j})^{\gamma-1}$ so that

$$\begin{aligned}
&\frac{2}{\gamma+1}\frac{(u^2_{\mathrm{E},j} - c^2_{\mathrm{E},j})\lambda_{\mathrm{E},j} - u_{\mathrm{E},j}v_{\mathrm{E},j}}{(1+\lambda^2_{\mathrm{E},j})(\lambda_{\mathrm{E},j}u_{\mathrm{E},j} - v_{\mathrm{E},j})}(-\lambda_{\mathrm{E},j},1)^\top \\
&\quad = \frac{(-\lambda_{\mathrm{P},\sqrt{j}}((u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l))^\top),1)^\top}{(-\lambda_{\mathrm{P},\sqrt{j}}((u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l))^\top),1)\cdot\nabla\lambda_{\mathrm{P},\sqrt{j}}((u_{\mathrm{E},j}(\alpha_j;U_l), v_{\mathrm{E},j}(\alpha_j;U_l))^\top)}.
\end{aligned}$$

Note that

$$\frac{\mathrm{d}\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)}{\mathrm{d}\alpha_j} = \frac{(-\lambda_{\mathrm{P},\sqrt{j}}(\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)),1)^\top}{(-\lambda_{\mathrm{P},\sqrt{j}}(\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)),1)\cdot\nabla\lambda_{\mathrm{P},\sqrt{j}}(\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l))},$$

and $(u_{\mathrm{E},j}(0;U_l), v_{\mathrm{E},j}(0;U_l))^\top = \mathbf{u}_l = \Phi_{\mathrm{P},\sqrt{j}}(0;\mathbf{u}_l)$. Therefore, we obtain (6.4). □

By lemmas 6.1 and 6.2, noting that the wave curves $\Phi_{\mathrm{E},j}$ and Ψ_j are C^2 functions, we have (see also [52])

Lemma 6.3. *For $U_l = (\mathbf{u}_l^\top, p_l, \rho_l)^\top \in D_{\mathrm{E}}$ and $\mathbf{u}_l \in D_{\mathrm{P}}$, if $\alpha_j \geqslant 0$, then, for $j = 1,4$,*

$$\begin{aligned}
&\Phi_{\mathrm{E},j}(\alpha_j;U_l) = \Psi_j(\alpha_j;\mathbf{u}_l),\\
&\frac{\partial\Psi_j(0;\mathbf{u}_l)}{\partial\alpha_j} = \mathbf{r}_j(U_l), \qquad \frac{\partial^2\Phi_{\mathrm{E},j}(0;U_l)}{\partial\alpha_j^2} = \frac{\partial^2\Psi_j(0;\mathbf{u}_l)}{\partial\alpha_j^2}.
\end{aligned}$$

Proposition 6.1. *Assume that $U_l = (\mathbf{u}_l^\top, p_l, \rho_l)^\top \in D_{\mathrm{E}}$ and $\mathbf{u}_l \in D_{\mathrm{P}}$. For α_j sufficiently small, $j = 1,4$, the equation:*

$$\Phi_{\mathrm{E}}(\beta_1,\beta_2,\beta_3,\beta_4;U_l) = (\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l), (\mathscr{R}(|\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)|)^\gamma), \mathscr{R}(|\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)|))^\top \tag{6.5}$$

has a unique solution $(\beta_1,\beta_2,\beta_3,\beta_4)$ satisfying

$$\begin{aligned}
&\beta_j = \alpha_j + O(1)|\alpha_j^-|^3,\\
&\beta_k = O(1)|\alpha_j^-|^3, \qquad k \neq j,
\end{aligned}$$

where $a^- = \min\{a,0\}$ and the bound of O(1) is independent of α_j and u_l.

Proof. Note that

$$\det\left(\frac{\partial\Phi_{\mathrm{E}}}{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}\right)\Bigg|_{\beta_1=\beta_2=\beta_3=\beta_4=0} = \det(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \neq 0.$$

Then the implicit function theorem implies that there exists a unique solution $(\beta_1,\beta_2,\beta_3,\beta_4)$, $\beta_k = \beta_k(\alpha_j,\mathbf{u}_l) \in C^2$, of equation (6.5) with $\beta_k|_{\alpha_j=0} = 0$.

In order to obtain the expansion of β_k, we first differentiate equation (6.5) with respect to α_j and then let $\alpha_j = 0$. By lemma 6.3, we obtain

$$\sum_{k=1}^{4} \frac{\partial \beta_k}{\partial \alpha_j}\bigg|_{\alpha_j=0} \mathbf{r}_k(U_l) = \mathbf{r}_j(U_l), \qquad j = 1, 4,$$

which imply

$$\frac{\partial \beta_k}{\partial \alpha_j}\bigg|_{\alpha_j=0} = \delta_{jk}.$$

Next, we take the second-order derivatives in equation (6.5) with respect to α_j and then let $\alpha_j = 0$. Therefore, from lemma 6.3 and (6.2), we have

$$\sum_{k=1}^{4} \frac{\partial^2 \beta_k}{\partial \alpha_j^2}\bigg|_{\alpha_j=0} \mathbf{r}_k(U_l) + \frac{\partial^2 \Phi_{\mathrm{E},j}(0;U_l)}{\partial \beta_j^2} = \frac{\partial^2 \Psi_j(0;\mathbf{u}_l)}{\partial \alpha_j^2}, \qquad j = 1, 4,$$

which, together with lemma 6.3, leads to

$$\frac{\partial^2 \beta_k}{\partial \alpha_j^2}\bigg|_{\alpha_j=0} = 0. \tag{6.6}$$

Thus, when $\alpha_j < 0$, from (6.2) and (6.6), we obtain the result. When $\alpha_j \geqslant 0$, from lemma 6.3 and the uniqueness of $(\beta_1, \beta_2, \beta_3, \beta_4)$, we see that $\beta_j = \delta_{kj}\alpha_j$. This completes the proof. □

Proposition 6.2. *For* $\mathbf{u}_l, \mathbf{u}_r \in D_{\mathrm{P}}$ *satisfying*

$$\mathbf{u}_r = \Phi_{\mathrm{P}}(\alpha_1, \alpha_4; \mathbf{u}_l),$$
$$U_r = \Phi_{\mathrm{E}}(\beta_1, \beta_2, \beta_3, \beta_4; U_l),$$

with $U_r = (\mathbf{u}_r^\top, (\mathscr{R}(|\mathbf{u}_r|))^\gamma, \mathscr{R}(|\mathbf{u}_r|))^\top$ *and* $U_l = (\mathbf{u}_l^\top, (\mathscr{R}(|\mathbf{u}_l|))^\gamma, \mathscr{R}(|\mathbf{u}_l|))^\top$. *Then*

$$\beta_j = \alpha_j + O(1)\left(|\alpha_1^-| + |\alpha_4^-|\right)^3, \qquad j = 1, 4,$$
$$\beta_k = O(1)\left(|\alpha_1^-| + |\alpha_4^-|\right)^3, \qquad k = 2, 3,$$

where $a^- = \min\{a, 0\}$, *and the bound of O(1) is independent of* u_l *and* α_j, $j = 1, 2, 3, 4$.

Proof. It suffices to solve

$$\Phi_{\mathrm{E}}(\beta_1, \beta_2, \beta_3, \beta_4; U_l) = (\Phi_{\mathrm{P}}(\alpha_1, \alpha_4; \mathbf{u}_l)^\top, (\mathscr{R}(|\Phi_{\mathrm{P}}(\alpha_1, \alpha_4; \mathbf{u}_l)|))^\gamma, \mathscr{R}(|\Phi_{\mathrm{P}}(\alpha_1, \alpha_4; \mathbf{u}_l)|))^\top$$

for β_k, $k = 1, 2, 3, 4$. Carrying out a similar procedure as in the proof of proposition 6.1, there exist C^2 functions $\beta_k = \beta_k(\alpha_1, \alpha_4, \mathbf{u}_l), k = 1, 2, 3, 4$, such that $(\beta_1, \beta_2, \beta_3, \beta_4)$ solves the above system.

As for estimates on β_k, we let

$$\mathbf{u}_m = \Phi_{\mathrm{P}}(\alpha_1, 0; \mathbf{u}_l), \quad U_m = (\mathbf{u}_m^\top, p_m, \rho_m)^\top = (\mathbf{u}_m^\top, (\mathscr{R}(|\mathbf{u}_m|))^\gamma, \mathscr{R}(|\mathbf{u}_m|)),$$

and consider the equations:

$$U_m = \Phi_{\mathrm{E}}(\beta_1^{'}, \beta_2^{'}, \beta_3^{'}, \beta_4^{'}; U_l),$$
$$U_r = \Phi_{\mathrm{E}}(\beta_1^{''}, \beta_2^{''}, \beta_3^{''}, \beta_4^{''}; U_m).$$

Thanks to proposition 6.1, we have

$$\beta_k^{'} = \alpha_1 \delta_{k1} + O(1)\,|\alpha_1^-|^3, \qquad \beta_k^{''} = \alpha_4 \delta_{k4} + O(1)\,|\alpha_4^-|^3, \tag{6.7}$$

where δ_{ij} is the Kronecker symbol. Hence, by Glimm's interaction estimates (see lemma 2.2), (6.7) gives the corresponding estimates for β_k, $k = 1, 2, 3, 4$.

□

Proposition 6.3. *Suppose that* $\mathbf{u}_l \in D_{\mathrm{P}}$ *and that* p_{r} *satisfies*

$$p_r = \left(\mathscr{R}\left(|\Phi_{\mathrm{P}}(\alpha_1, 0; \mathbf{u}_l)|\right)\right)^{\gamma},$$
$$p_r = \Phi_{\mathrm{E}}^{(3)}(\beta_1, 0, 0, 0; U_l),$$

with $U_l = (\mathbf{u}_l^{\top}, (\mathscr{R}(|\mathbf{u}_l|))^{\gamma}, \mathscr{R}(|\mathbf{u}_l|))^{\top}$. *Then*

$$\beta_1 = \alpha_1 + O(1)\,|\alpha_1^-|^3 = \alpha_1 + O(1)\,|\omega^-|^3,$$

where $\omega = p_r - p_l$, $a^- = \min\{a, 0\}$, *and the bound of O(1) is independent of* α_1 *and* u_l.

Proof. When $\omega \geqslant 0$, α_1 and β_1 are rarefaction waves. According to lemma 6.1, it suffices to consider the equations:

$$\frac{|\Phi_{\mathrm{P}}(\alpha_1, 0; \mathbf{u}_l)|^2}{2} + \frac{\gamma(\omega + p_l)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} = B_\infty, \tag{6.8}$$

$$\frac{|\Xi(\beta_1, 0, 0, 0; U_l)|^2}{2} + \frac{\gamma(\omega + p_l)^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} = B_\infty, \tag{6.9}$$

where $\Xi(\beta_1, 0, 0, 0; U_l) = (u_{\mathrm{E}}(\beta_1, 0, 0, 0; U_l), v_{\mathrm{E}}(\beta_1, 0, 0, 0; U_l))^{\top}$. We first take the derivative with respect to ω and then let $\omega = 0$ to obtain

$$\mathbf{u}_l \cdot \frac{\partial \Phi_{\mathrm{P}}(0, 0; \mathbf{u}_l)}{\partial \alpha_1} \left.\frac{\mathrm{d}\alpha_1}{\mathrm{d}\omega}\right|_{\omega=0} + (p_l)^{-\frac{1}{\gamma}} = 0,$$
$$\mathbf{u}_l \cdot \frac{\partial \Xi_{\mathrm{E}}(0, 0, 0, 0; U_l)}{\partial \beta_1} \left.\frac{\mathrm{d}\beta_1}{\mathrm{d}\omega}\right|_{\omega=0} + (p_l)^{-\frac{1}{\gamma}} = 0.$$

Thus, by lemma 6.3, we have

$$\left.\frac{\mathrm{d}\alpha_1}{\mathrm{d}\omega}\right|_{\omega=0} = \left.\frac{\mathrm{d}\beta_1}{\mathrm{d}\omega}\right|_{\omega=0} \neq 0. \tag{6.10}$$

Next, we first take the second-order derivative with respect to ω in equations (6.8) and (6.9), respectively, and then let $\omega = 0$ to obtain

$$\left(\frac{\partial \Phi_{\mathrm{P}}(0, 0; \mathbf{u}_l)}{\partial \alpha_1} \cdot \frac{\partial \Phi_{\mathrm{P}}(0, 0; \mathbf{u}_l)}{\partial \alpha_1} + \mathbf{u}_l \cdot \frac{\partial^2 \Phi_{\mathrm{P}}(0, 0; \mathbf{u}_l)}{\partial \alpha_1^2}\right) \left.\left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}\omega}\right)^2\right|_{\omega=0}$$
$$+ \mathbf{u}_l \cdot \frac{\partial \Phi_{\mathrm{P}}(0, 0; \mathbf{u}_l)}{\partial \alpha_1} \left.\frac{\mathrm{d}^2\alpha_1}{\mathrm{d}\omega^2}\right|_{\omega=0} - \frac{1}{\gamma}(p_l)^{-\frac{1+\gamma}{\gamma}} = 0,$$

$$\left(\frac{\partial\Xi(0,0,0,0;U_l)}{\partial\beta_1}\cdot\frac{\partial\Xi(0,0,0,0;U_l)}{\partial\beta_1}+U_l\cdot\frac{\partial^2\Xi(0,0,0,0;U_l)}{\partial\beta_1^2}\right)\left(\frac{\mathrm{d}\beta_1}{\mathrm{d}\omega}\right)^2\bigg|_{\omega=0}$$
$$+\mathbf{u}_l\cdot\frac{\partial\Xi(0,0,0,0;U_l)}{\partial\beta_1}\left.\frac{\mathrm{d}^2\beta_1}{\mathrm{d}\omega^2}\right|_{\omega=0}-\frac{1}{\gamma}(p_l)^{-\frac{1+\gamma}{\gamma}}=0.$$

Combining lemma 6.3 with (6.10), we see that

$$\left.\frac{\mathrm{d}^2\alpha_1}{\mathrm{d}\omega^2}\right|_{\omega=0}=\left.\frac{\mathrm{d}^2\beta_1}{\mathrm{d}\omega^2}\right|_{\omega=0}. \tag{6.11}$$

Hence, when $\omega<0$, since α_1, β_1, and ω are the quantities of the same order, we conclude the result; when $\omega\geqslant 0$, from the uniqueness of $\beta_1(\omega)$ and lemma 6.3, we also obtain the result. □

Remark 6.1. Informally, lemmas 6.1–6.3 show that the rarefaction waves for the full Euler system and the potential flow system coincide. Moreover, since the Hugoniot curve and the rarefaction wave have a tangency of second-order, propositions 6.1–6.3 demonstrate that the Hugoniot waves of the two systems differ at third order in the jump strength parameter.

6.3. *The proof of Main Theorem II*

To compare the μ-approximate solutions, we need to establish the estimate involving different types of wave fronts. To this end, it suffices to consider the case that there is only one wave-front. Let b and p_b be constant functions, and let U be a piecewise constant vector. For any $W(x_0,\cdot)\in\mathbb{D}^\varepsilon$ with $W(x_0,y)=(b(x_0),U(x_0,y+b(x_0)),\iota_{x_0}p_b)$, when $x>x_0$ is sufficiently close to x_0, $(\mathfrak{R}_{x-x_0}W)(\cdot+(\mathfrak{S}_{x-x_0}W)^{(1)})$ is an entropy solution that connects all the solutions of the Riemann-type problems solved at which U has a discontinuity, where

$$\mathfrak{R}_xW=((\mathfrak{S}_{x-x_0}W)^{(2)},(\mathfrak{S}_{x-x_0}W)^{(3)},(\mathfrak{S}_{x-x_0}W)^{(4)},(\mathfrak{S}_{x-x_0}W)^{(5)})^\top,$$

and $(\mathfrak{S}_xW)^{(i)}$ stands for the ith components of $\mathfrak{S}_xW$ for $i=1,\cdots,6$.

Case 1: *Suppose that* $\mathbf{u}_l,\mathbf{u}_r\in D_{\mathrm{P}}$ *with* $\mathbf{u}_l$ *and* $\mathbf{u}_r$ *sufficiently close to each other*. Let $U_l=\Psi(0,0;\mathbf{u}_l)$ and $U_r=\Psi(0,0;\mathbf{u}_r)$. For any $s\in\mathbb{R}$, $x_0\geqslant 0$, and $y_0<b$, let

$$U(x,y)=\begin{cases}U_l & \text{for } y-y_0<s(x-x_0),\\ U_r & \text{for } y-y_0>s(x-x_0).\end{cases}$$

In what follows, we denote $U(\tau)=U|_{x=\tau}$ and $U(\tau)(y)=U(\tau,y)$. Then we assume that

$$U(x_0)(y)=\begin{cases}U_r & \text{for } y>y_0,\\ U_l & \text{for } y<y_0,\end{cases}$$

and $p_b|_{x=x_0}=p_r$.

Proposition 6.4. *Suppose that there exists* α_j *such that* $\mathbf{u}_r=\Phi_{\mathrm{P},\sqrt{j}}(\alpha_j;\mathbf{u}_l)$ *for each* $j=1,4$. *Let s be the speed of α_j. Then, for $h>0$,*

(i) *if* $\alpha_j<0$,

$$\|\mathfrak{S}_hW(x_0)-W(x_0+h)\|_Y\leqslant O(1)h|\alpha_j|^3;$$

(ii) *if* $\alpha_j \geqslant 0$,

$$\|\mathfrak{S}_h W(x_0) - W(x_0+h)\|_Y \leqslant O(1)\, h|\alpha_j|^2,$$

where the bound of O(1) is independent of x_0*, h, and* α_j.

Proof. We only consider the case when $j=1$, since the proof for $j=4$ is similar.
(a). $\alpha_1 < 0$. Then α_1 is a shock with speed s. Let $(\beta_1,\beta_2,\beta_3,\beta_4)$ be the solution to the equation:

$$U_r = \Phi_{\mathrm{E}}\left(\beta_1,\beta_2,\beta_3,\beta_4; U_l\right).$$

$\mathfrak{R}_h W(x_0)$ contains waves β_k, $k = 1, 2, 3, 4$. According to proposition 6.1,

$$\beta_k = \alpha_j \delta_{1k} + O(1)\,|\alpha_1|^3, \qquad k = 1, 2, 3, 4,$$

which gives $\beta_1 < 0$.

In addition, since

$$\sigma = \lambda_{\mathrm{E},1}\left(U_l\right) + \frac{1}{2}\beta_1 + O(1)\,|\beta_1|^2 = \lambda_{\mathrm{E},1}\left(U_l\right) + \frac{1}{2}\alpha_1 + O(1)\,|\alpha_1|^2,$$
$$s = \lambda_{\mathrm{P},1}\left(\mathbf{u}_l\right) + \frac{1}{2}\alpha_1 + O(1)\,|\alpha_1|^2,$$

we obtain

$$\sigma = s + O(1)\,|\alpha_1|^2.$$

Denote $q_1 = \min\{s,\sigma\}$, $q_2 = \max\{s,\sigma\}$, and $\lambda_M = \max\{|\lambda_{\mathrm{E},4}(U)| : U \in D_{\mathrm{E}}\} + 1$. Therefore, q_1 and q_2 are bounded, and

$$q_2 - q_1 = O(1)\,|\alpha_1|^2.$$

Furthermore, for $\varpi(h,\cdot) := \mathfrak{R}_h W(x_0)$, we can obtain

(i) if $y < q_1 h + y_0$, then $\varpi(h,y) = U(x_0+h,y)$,
(ii) if $q_1 h + y_0 < y < q_2 h + y_0$, then $\varpi(h,y) - U(x_0+h,y) = O(1)|\alpha_1|$,
(iii) if $q_2 h + y_0 < y < \lambda_M h + y_0$, then $U(x_0+h,y) = U_r$, and

$$\varpi(h,y) - U(x_0+h,y) = \varpi(h,y) - U_r = O(1)\left(|\beta_2| + |\beta_3| + |\beta_4|\right) = O(1)\,|\alpha_1|^3,$$

(iv) if $y > \lambda_M h + y_0$, then $\varpi(h,y) = U(x_0+h,y) = U_r$.

Now, the boundary corresponding to $\mathfrak{S}_h W(x_0)$ and the boundary corresponding to $W(x_0 + h)$ coincide, and the pressure on the boundary is p_r. Then

$$\begin{aligned}\|\mathfrak{S}_h W(x_0) - W(x_0+h)\|_Y &= \int_{q_2 h + y_0}^{\lambda_M h + y_0} |\varpi(h,y) - U(x_0+h,y)|\,\mathrm{d}y \\ &\quad + \int_{q_1 h + y_0}^{q_2 h + y_0} |\varpi(h,y) - U(x_0+h,y)|\,\mathrm{d}y \\ &\leqslant O(1)\,|\alpha_1|^3 |\lambda_M - q_2| h + O(1)\,|\alpha_1||q_2 - q_1| h \\ &\leqslant O(1)\,|\alpha_1|^3 h.\end{aligned}$$

(b). $\alpha_1 \geqslant 0$. Then α_1 is a rarefaction wavefront with characteristic speed s. Let $(\beta_1, \beta_2, \beta_3, \beta_4)$ be the solution of the equation:

$$U_r = \Phi_{\mathrm{E}}(\beta_1, \beta_2, \beta_3, \beta_4; U_l).$$

By proposition 6.1, we have

$$\beta_1 = \alpha_1, \qquad \beta_2 = \beta_3 = 0,$$

which means that $\mathfrak{R}_h W(x_0)$ contains a fan-shaped wave β_1. In addition, at $x_0 + h$, the width of the fan-shaped wave β_1 has an upper bound $O(1)h|\alpha_1|$. Therefore, we obtain

$$\|\mathfrak{S}_h W(x_0) - W(x_0 + h)\|_Y \leqslant O(1)h|\alpha_j|^2,$$

which gives the result. □

Proposition 6.5. *If $U(x,y)$ contains nonphysical waves only, then*

$$\|\mathfrak{S}_h W(x_0) - W(x_0 + h)\|_Y \leqslant O(1)h|\mathbf{u}_r - \mathbf{u}_l|,$$

where O(1) has an upper bound independent of x_0, h, and the strength of the wave.

Proof. From the construction of the μ-approximate solutions, we have

$$(\mathfrak{R}_h W(x_0))(y) = U(x_0 + h, y + (\mathfrak{S}_h W(x_0))^{(1)}) = U_r \qquad \text{for } y > y_0 + sh - (\mathfrak{S}_h W(x_0))^{(1)}.$$

Thus, by proposition 6.2, carrying out similar arguments as in the proof of proposition 6.4 leads to the result. □

Case 2: *Suppose that $\mathbf{u}_l$, $\mathbf{u}_r \in D_{\mathrm{P}}$ with $\mathbf{u}_l$ and $\mathbf{u}_r$ sufficiently close each other*. Let $U_l = \Psi(0,0;\mathbf{u}_l)$ and $U_r = \Psi(0,0;\mathbf{u}_r)$. For any $s < \frac{v_r}{u_r}$ and $x_0 \geqslant 0$, set

$$U(x,y) = \begin{cases} U_l & \text{for } y - b < s(x - x_0), \\ U_r & \text{for } y - b > s(x - x_0). \end{cases}$$

Moreover, we assume that

$$U(x_0, y) = U_l \qquad \text{for } y < b,$$

and $p_b|_{x=x_0} = p_r$.

Proposition 6.6. *Suppose that there exists α_1 such that $\mathbf{u}_r = \Phi_{\mathrm{P},1}(\alpha_1; \mathbf{u}_l)$ and that the speed of α_1 is s. Then, for $h > 0$,*

(i) *if $\alpha_1 < 0$, then*
$$\|\mathfrak{S}_h W(x_0) - W(x_0 + h)\|_Y \leqslant O(1)h|\alpha_1|^3;$$
(ii) *if $\alpha_1 \geqslant 0$, then*
$$\|\mathfrak{S}_h W(x_0) - W(x_0 + h)\|_Y \leqslant O(1)h|\alpha_1|^2,$$

where the bound of O(1) is independent of x_0, h, and α_j.

Proof. We divide the proof into two steps.
1. Case: $\alpha_1 < 0$. Then α_1 is a shock whose speed is denoted by s. Let β_1 be the solution of

$$p_r = \Phi^{(3)}_{\mathrm{E},1}(\beta_1; U_l).$$

Using proposition 6.3, we obtain

$$\beta_1 = \alpha_1 + O(1)|\alpha_1^-|^3,$$

which implies $\beta_1 < 0$.

For the estimates of σ and β_1, since

$$\sigma = \lambda_{\mathrm{E},1}(U_l) + \frac{1}{2}\beta_1 + O(1)|\beta_1|^2 = \lambda_{\mathrm{E},1}(U_l) + \frac{1}{2}\alpha_1 + O(1)|\alpha_1|^2,$$
$$s = \lambda_{\mathrm{P},1}(\mathbf{u}_l) + \frac{1}{2}\alpha_1 + O(1)|\alpha_1|^2,$$

we obtain

$$\sigma = s + O(1)|\alpha_1|^2.$$

Let $q_1 = \min\{s,\sigma\}$, $q_2 = \max\{s,\sigma\}$, and $\lambda_M = \max\{|\lambda_{\mathrm{E},4}(U)| : U \in D_{\mathrm{E}}\} + 1$. Then q_1 and q_2 are bounded and satisfy

$$q_2 - q_1 = O(1)|\alpha_1|^2.$$

Furthermore, denote $\varpi(h,\cdot) := \mathfrak{R}_h W(x_0)$. Using lemma 6.3 and proposition 6.3 yields

(i) if $y < q_1 h + y_0$, then $\varpi(h,y) = U(x_0+h,y)$;
(ii) if $q_1 h + y_0 < y < q_2 h + y_0$, then $\varpi(h,y) - U(x_0+h,y) = O(1)|\alpha_1|$;
(iii) if $q_2 h + y_0 < y < b + \frac{v_r}{u_r}h$, then $U(x_0+h,y) = U_r$,

$$\varpi(h,y) - U(x_0+h,y) = \varpi(h,y) - U_r = O(1)|\alpha_1|^3,$$
$$\left|\frac{\Phi^{(2)}_{\mathrm{E},1}(\beta_1;U_l)}{\Phi^{(1)}_{\mathrm{E},1}(\beta_1;U_l)} - \frac{v_r}{u_r}\right| = O(1)|a_1|^3.$$

Set $\dot{b}_1 := \min\{\frac{\Phi^{(2)}_{\mathrm{E},1}(\beta_1;U_l)}{\Phi^{(1)}_{\mathrm{E},1}(\beta_1;U_l)}, \frac{v_r}{u_r}\}$ and $\dot{b}_2 := \max\{\frac{\Phi^{(2)}_{\mathrm{E},1}(\beta_1;U_l)}{\Phi^{(1)}_{\mathrm{E},1}(\beta_1;U_l)}, \frac{v_r}{u_r}\}$. Then, by lemma 5.1, similar to the proof of proposition 6.4, we have

$$\begin{aligned}
&\|\mathfrak{S}_h W(x_0) - W(x_0+h)\|_Y \\
&\leqslant O(1)h(\dot{b}_2 - \dot{b}_1) + O(1)|\alpha_1||q_2-q_1|h + \int_{q_2h+b}^{\dot{b}_1 h+b} |\varpi(h,y) - U(x_0+h,y)|\,\mathrm{d}y \\
&\leqslant O(1)|\alpha_1|^3 h.
\end{aligned}$$

2. Case: $\alpha_1 \geqslant 0$. Then α_1 is a rarefaction wavefront whose speed is s. Let β_1 solve the equation:

$$p_r = \Phi^{(3)}_{\mathrm{E},1}(\beta_1; U_l).$$

Using proposition 6.3, we have

$$\beta_1 = \alpha_1, \qquad \frac{\Phi^{(2)}_{\mathrm{E},1}(\beta_1; U_l)}{\Phi^{(1)}_{\mathrm{E},1}(\beta_1; U_l)} = \frac{v_r}{u_r}.$$

That is, $\mathfrak{R}_h W(x_0)$ contains a fan-shaped wave β_1. Furthermore, at $x_0 + h$, the width of the fan-shaped wave β_1 has an upper bound $O(1)h|\alpha_1|$. Thus,

$$\|\mathfrak{S}_h W(x_0) - W(x_0 + h)\|_Y \leqslant O(1)\, h|\alpha_j|^2.$$

This completes the proof. □

With those preparations, we are about to prove the following estimates for $\mathfrak{S}$.

Proposition 6.7. *Suppose that* $\mathbf{u}^\mu$ *is a* μ*-approximate solution to the potential flow equations (1.4) and (1.5) satisfying (1.15)–(1.16) and* b^μ_{P} *is an approximate boundary. Let* $W^\mu_{\mathrm{P}}(x) = (b^\mu_{\mathrm{P}}(x), \Psi(0,0;\mathbf{u}^\mu(x,\cdot))^\top, \iota_x p_b)$*. Then, at* $x > 0$*,*

$$\liminf_{h\to 0+} \frac{1}{h}\|\mathfrak{S}_h W^\mu_{\mathrm{P}}(x) - W^\mu_{\mathrm{P}}(x+h)\|_Y \leqslant O(1)\left(\mathrm{T.V.}^-(\mathbf{u}^\mu(x,\cdot))\right)^3 + O(1)\,\mu,$$

where $\mathrm{T.V.}^-(\mathbf{u}^\mu(x,\cdot))$ *is the total strength of the shock waves, and the bound of O(1) is independent of x and* μ*.*

Proof. Note that $\mathbf{u}^\mu$ is a piecewise constant vector, and the number of all wavefronts is finite. From propositions 6.2–6.3 and 6.4–6.6, when h is small enough, we have

$$\begin{aligned}
&\frac{1}{h}\|\mathfrak{S}_h W^\mu_{\mathrm{P}}(x) - W^\mu_{\mathrm{P}}(x+h)\|_Y \\
&\quad\leqslant O(1)\sum_{\alpha \text{ is a shock at } x}|\alpha(x)|^3 + O(1)\sum_{\beta \text{ is a rarefaction wave at } x}|\beta(x)|^2 \\
&\qquad + O(1)\sum_{\epsilon(x) \text{ is a nonphysical wave at } x}|\epsilon(x)| \\
&\quad\leqslant O(1)\left(\left(\mathrm{T.V.}^-(\mathbf{u}^\mu(x,\cdot))\right)^3 + \mathrm{T.V.}^+(\mathbf{u}^\mu(x,\cdot))\,\mu + \mu\right),
\end{aligned}$$

where $\mathrm{T.V.}^+(\mathbf{u}^\mu(x,\cdot))$ is the total variation of the rarefaction wavefronts in $\mathbf{u}^\mu(x,\cdot)$, and $|\alpha(x)|$, $|\beta(x)|$, and $|\epsilon(x)|$ are the strengths of waves α, β, and ϵ at x, respectively. This completes the proof. □

Definition 6.3. For an interval I,

$$\mathrm{Lip}_\varepsilon(I;Y) = \left\{ W : W(t) \in \mathbb{D}^\varepsilon,\ t \in I,\ \sup_{t\neq\tau}\frac{\|W(t) - W(\tau)\|_Y}{|t-\tau|} < \infty \right\}.$$

To obtain a uniform estimate for the μ-approximate solution, we need the following proposition (*cf* [21, lemma 6.2] and [5, theorem 2.9]):

Proposition 6.8. *For* $T^* > 0$*, assume that* $w : [0, T^*] \to \mathbb{D}^\varepsilon$ *is a map such that* $w - (0, \overline{U}^\top_\infty, \bar{p}_b)^\top \in \mathrm{Lip}_\varepsilon([0,T^*];Y)$*. Then there exists* $L > 0$*, independent of* ε*, such that*

$$\|\mathfrak{S}_T w(0) - w(T)\|_Y \leqslant L\int_0^T \liminf_{h\to 0+}\frac{\|w(t+h) - \mathfrak{S}_h w(t)\|_Y}{h}\,\mathrm{d}t \qquad \textit{for any } T \in [0, T^*].$$

By the construction of approximate solutions, we know that

$$W_{\mathrm{P}}^{\mu}-(0,\overline{U}_{\infty}^{\top},\bar{p}_b)^{\top}\in \mathrm{Lip}_{\varepsilon}([0,\infty);Y),\qquad W_{\mathrm{P}}^{\mu}(x)\in\mathbb{D}^{\varepsilon}\quad \text{for } x\geqslant 0.$$

Therefore, we obtain

Theorem 6.3. *For any $T>0$, $W_{\mathrm{P}}^{\mu}(x)\in\mathbb{D}^{\varepsilon}$ satisfies*

$$\begin{aligned}&\|\mathfrak{S}_T W_{\mathrm{P}}^{\mu}(0)-W_{\mathrm{P}}^{\mu}(T)\|_Y\\&\leqslant O(1)\,T\Big(\big(\mathrm{T.V.}^{-}(\mathbf{u}^{\mu}(0,\cdot))+\mathrm{T.V.}(p_b^{\mu}(\cdot))+|p_b(0+)-p^{\mu}(0,0-)|\big)^3+\mu\Big),\end{aligned}\tag{6.12}$$

where the bound of O(1) is independent of T and μ.

Proof. From (6.7) and propositions 6.7 and 6.8, we have

$$\|\mathfrak{S}_T W_{\mathrm{P}}^{\mu}(0)-W_{\mathrm{P}}^{\mu}(T)\|_Y\leqslant O(1)\int_0^T\Big(\big(\mathrm{T.V.}^{-}(\mathbf{u}^{\mu}(x,\cdot))\big)^3+\mu\Big)\,\mathrm{d}x.\tag{6.13}$$

Note that

$$\mathrm{T.V.}^{-}(\mathbf{u}^{\mu}(x,\cdot))\leqslant O(1)\big(\mathrm{T.V.}(\mathbf{u}^{\mu}(0,\cdot))+\mathrm{T.V.}(p_b^{\mu}(\cdot))+|p_b(0+)-p^{\mu}(0,0-)|+\mu\big),\tag{6.14}$$

and the result follows from (6.13) and (6.14). □

Finally, we have the following theorem.

Theorem 6.4. *Assume that (C_{P}**1**)–(C_{P}**2**) and (C_{E}**1**)–(C_{E}**2**) hold and that $U_1=(\mathbf{u}_1^{\top},p_1,\rho_1)^{\top}$ is an entropy solution of (1.6) satisfying (1.9)–(1.11). Let $U_2=(\mathbf{u}_2^{\top},(\mathscr{R}(|\mathbf{u}_2|))^{\gamma},\mathscr{R}(|\mathbf{u}_2|))^{\top}$ be an entropy solution of (1.5)–(1.4) satisfying (1.16)–(1.15). Moreover, let b_1 and b_2 be corresponding boundaries. Then there exist $\varepsilon_c>0$ and $C>0$ such that, when $\|(\widetilde{\mathbf{u}}_{\infty},\tilde{p}_b)\|_{L^{\infty}\cap BV}<\varepsilon_c$,*

$$\|(b_1(x),U_1(x,\cdot),\iota_x p_b)-(b_2(x),U_2(x,\cdot),\iota_x p_b)\|_Y\leqslant Cx\|(\widetilde{\mathbf{u}}_{\infty},\tilde{p}_b)\|^3_{L^{\infty}\cap BV}\quad \textit{for any } x>0.$$

Proof. Using $U_1(x,\cdot+b_1(x))=\mathfrak{R}_x U_{\infty}$ and the properties of μ-approximate solutions, and then letting $\mu\to 0+$ in (6.12), we obtain the result by carrying out a similar argument as in [4–6, 52]. □

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

No new data were created or analysed in this study.

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Conflict of interest

The authors declare that they have no conflict of interest. The authors also declare that this manuscript has not been previously published, and will not be submitted elsewhere before your decision.

Appendix. Proofs of proposition 3.2 and theorem 3.1

In this appendix, we give the proofs of proposition 3.2 and theorem 3.1, respectively.

A.1. Proof of proposition 3.2

We first show that the magnitude of each non-physical wave satisfies

$$\epsilon = O(1)\nu, \qquad \epsilon \in \mathcal{NP}.$$

Indeed, a new non-physical wave ϵ comes out at $x = t$, when a weak physical wave α interacts with another physical wave β with $|\alpha||\beta| < \nu$, or a weak 1-wave α collides the strong shock with $|\alpha| < \nu$. In both cases, the interaction estimates show that $\epsilon = O(1)\nu$. Consider the quantity

$$L_\epsilon(x) := \sum_{\alpha\in\mathcal{A}(\epsilon)} |\alpha|,$$

where $\mathcal{A}(\epsilon)$ is the set of wavefronts which approaches ϵ. Suppose that the interaction occurs at $x = t$.

Case 1: The interaction does not involve ϵ. From the interaction estimates, we conclude

$$\Delta\epsilon(t) = 0, \quad \Delta L_\epsilon(t) + \mathcal{K}\Delta Q(t) \leqslant 0. \tag{A.1}$$

Case 2: The non-physical wave ϵ collides another weak wave α. Again the interaction estimates imply

$$\Delta L_\epsilon(t) = -|\alpha|, \quad \Delta Q(t) < 0, \quad \Delta\epsilon(t) \leqslant O(1)|\epsilon(t-)||\alpha|. \tag{A.2}$$

By (A.1) and (A.2), the map:

$$x \mapsto \epsilon(x)\exp\{C(L_\epsilon(x) + \mathcal{K}Q(x))\}$$

is non-increasing in x, where $M_1 > |O(1)|$ is a constant. Therefore, for $x > x_0$, we have

$$\begin{aligned}\epsilon(x) &\leqslant \epsilon(x_0)\exp\{M_1(L_\epsilon(x_0) + \mathcal{K}Q(x_0))\}\\ &\leqslant O(1)\nu\exp\{M_1(L_\epsilon(0) + \mathcal{K}Q(0))\}\\ &\leqslant O(1)e^{M_1\varepsilon}\nu\\ &\leqslant M_2\nu,\end{aligned} \tag{A.3}$$

where $\varepsilon > 0$ is the constant given in proposition 3.1.

Next, we assign each wavefront in $U^{\mu,\Delta x}$ an integer number, counting the number of interactions that occur to give birth to such a front. To be specific, we define the generation order $Od(\alpha)$ of a front α inductively as follows:

- Fronts generated from $x = 0$ and on the boundary (on points $(h\Delta x, b^{\mu,\Delta x}(h\Delta x))$ for $h \in \mathbb{N}_+$) have generation order $m = 1$.

- The boundary and the strong shock are always attached with generation order $m=1$.
- When two incoming fronts (including the boundary and the strong shock) of the families $i_1, i_2 \in \{1,\cdots,5\}$ and of generation orders m_1 and m_2 interact, then the outgoing fronts have a generation order

$$m=\begin{cases} m_1, & i=i_1, i_1\neq i_2,\\ m_2, & i=i_2, i_1\neq i_2,\\ \max\{m_1,m_2\}+1, & i\neq i_1, i\neq i_2,\\ \min\{m_1,m_2\}, & i=i_1=i_2,\end{cases} \tag{A.4}$$

where $i\in\{1,\cdots,5\}$ indicates the ith family of the outgoing wave-fronts.

For $m\geqslant 1$, let

$$L_m(t)=\sum_{\substack{\alpha\,\text{crosses}\,x=t,\\ Od(\alpha)\geqslant m}} b_\alpha,$$

and let

$$L_{i,m}(t)=\sum_{\substack{\alpha\,\text{is of family}\,i\\ \text{and crosses}\,x=t,\\ Od(\alpha)\geqslant m}} b_\alpha \qquad \text{for } i=1,2,3,4.$$

Moreover, we write $L_0(t)=L(t)$. For $m\geqslant 1$, define

$$Q_m(t)=K_s\sum_{\substack{\alpha\in\mathcal{A}_s,\\ Od(\alpha)\geqslant m}}|b_\alpha|+\sum_{\substack{\beta\in\mathcal{A}_b,\\ Od(\beta)\geqslant m}}|b_\beta|+K\sum_{\substack{(\alpha,\beta)\in\mathcal{A},\\ \max\{Od(\alpha),Od(\beta)\}\geqslant m}}|b_\alpha||b_\beta|,$$

and $Q_0(t)=Q(t)$. In addition, when $m\geqslant 1$, let I_m be the set of x-coordinates at which two waves of order m_α and m_β interact, with $\max\{m_\alpha,m_\beta\}=m$, and let $I_0=\{h\Delta x : h\in\mathbb{N}\}$. Similar to the proof in proposition 3.1, tracking the generation order of the wavefronts in a subsequent of interactions, we obtain

$$\begin{aligned}
&\Delta L_m(x)=0, && x\in I_0\cup\cdots\cup I_{m-2},\ m\geqslant 2,\\
&\Delta L_m(x)+\mathcal{K}\Delta Q_{m-1}(x)\leqslant 0, && x\in I_{m-1}\cup I_m\cup I_{m+1}\cdots,\ m\geqslant 1,\\
&\Delta Q_m(x)+\mathcal{K}C\Delta Q(x)L_m(x-)\leqslant 0, && x\in I_0\cup\cdots\cup I_{m-2},\ m\geqslant 2,\\
&\Delta Q_m(x)+\mathcal{K}C\Delta Q_{m-1}(x)L(x-)\leqslant 0, && x\in I_{m-1},\ m\geqslant 1,\\
&\Delta Q_m(x)\leqslant 0, && x\in I_m\cup I_{m+1}\cdots,\ m\geqslant 0.
\end{aligned} \tag{A.5}$$

Estimates (A.5) imply

$$\begin{aligned}
L_m(x)&\leqslant \mathcal{K}\sum_{0<t\leqslant x}(\Delta Q_{m-1}(t))^-,\\
Q_m(x)&\leqslant \sum_{0<t\leqslant x}(\Delta Q_m(t))^+\\
&\leqslant \mathcal{K}C\sum_{0<t\leqslant x}(\Delta Q(t))^-\sup_x L_m(x)+\mathcal{K}C\sum_{0<t\leqslant x}(\Delta Q_{m-1}(t))^-\sup_x L(x),
\end{aligned} \tag{A.6}$$

both of which are valid for every $x > 0$ and $m \geqslant 1$. Furthermore, we have

$$0 \leqslant Q_m(x) = \sum_{0<t\leqslant x} \left\{ (\Delta Q_m(t))^+ - (\Delta Q_m(t))^- \right\}.$$

Since $F(x) := L(x) + \mathcal{K}Q(x) + |U_\diamond(x) - U_\infty^-| + |U^\diamond(x) - U_\infty^+|$ is non-increasing, we have

$$L_m(x) \leqslant L(x) \leqslant F(x) \leqslant F(0), \qquad \sum_{0<t<\infty} (\Delta Q(t))^- \leqslant Q(0) \leqslant F(0). \quad \text{(A.7)}$$

Recalling

$$\widetilde{Q}_m := \sum_{x>0} (\Delta Q_m(x))^+, \qquad \widetilde{L}_m := \sup_{x>0} L_m(x),$$

from (A.4)–(A.7), we deduce the sequence of the inequalities (valid for $m \geqslant 1$):

$$\begin{cases} \widetilde{L}_m \leqslant \mathcal{K}\widetilde{Q}_{m-1}, \\ \widetilde{Q}_m \leqslant C\mathcal{K}F(0)\widetilde{L}_m + C\mathcal{K}\widetilde{Q}_{m-1}F(0) \leqslant C\left(\mathcal{K}^2 + \mathcal{K}\right)F(0)\widetilde{Q}_{m-1}. \end{cases} \quad \text{(A.8)}$$

For $F(0)$ sufficiently small, we obtain

$$\eta := C\left(\mathcal{K}^2 + \mathcal{K}\right)F(0) < 1. \quad \text{(A.9)}$$

In this case, for every $x > 0$ and $m \geqslant 1$, by induction, (A.8)–(A.9) yield

$$Q_m(x) \leqslant \widetilde{Q}_m \leqslant \varepsilon\eta^m, \qquad L_m(x) \leqslant \widetilde{L}_m \leqslant \mathcal{K}\varepsilon\eta^{m-1}. \quad \text{(A.10)}$$

Meanwhile, the number of wavefronts of the mth generation can be counted as follows: Since the wavefronts of generation 1 are generated at $x = 0$, as well as from the change of the pressure distribution, the number of the first-order fronts is less than N_μ. From each interaction between the fronts of first-order, recalling that each of the rarefaction fronts has size $< \delta$, the second-order front is generated and its number is less than $O(1)\delta^{-1}$. Therefore, the number of second-order wavefronts is $O(1)(N_\mu)^2\delta^{-1}$. Inductively, it is clear that the number of fronts of order $\leqslant m$ is bounded by some polynomial function of (N_μ, δ^{-1}), say

$$P_m(N_\mu, \delta^{-1}). \quad \text{(A.11)}$$

The particular form of P_m is not interested here.

Next, we establish the total strength estimates of non-physical waves. We track of the fronts of generation order $> m$ and $\leqslant m$, separately. Using (A.3) and (A.10)–(A.11), we obtain

$$\begin{aligned} &\sum_{\epsilon\in\mathcal{NP}, Od(\epsilon)>m} \epsilon + \sum_{\epsilon\in\mathcal{NP}, Od(\epsilon)\leqslant m} \epsilon \\ &\quad \leqslant [\text{total strength of all fronts of order} > m] \\ &\qquad + [\text{number of all fronts of order} \leqslant m] \times [\text{maximum strength of non physical fronts}] \\ &\quad \leqslant \mathcal{K}\varepsilon\eta^m + M_2\nu P_m(N_\mu, \delta^{-1}). \end{aligned} \quad \text{(A.12)}$$

For any $\mu > 0$, recalling $\eta < 1$, take m large enough such that $\mathcal{K}\varepsilon\eta^m < \frac{\mu}{2}$. Then we choose ν small enough so that $M_2\nu P_m(N_\mu, \delta^{-1}) < \mu$. For all $x > 0$, we conclude

$$\sum_{\epsilon\in\mathcal{NP}, Od(\epsilon)>m} \epsilon + \sum_{\epsilon\in\mathcal{NP}, Od(\epsilon)\leqslant m} \epsilon < \mu,$$

This completes the proof. □

A.2. Proof of theorem 3.1

The proof is completed as follows: Results(i)–(iv) are deduced easily from propositions 3.3 and 3.4, together with the Arzelà–Ascoli Theorem.

We now prove (v). Firstly, from the previous analysis, we have

$$\begin{aligned}&\int_{-\infty}^{0}\left|U^{\mu,\Delta x}(x_1,y+b^{\mu,\Delta x}(x))-U^{\mu,\Delta x}(x_2,y+b^{\mu,\Delta x}(x))\right|\mathrm{d}y\\&\leqslant O(1)\,|x_1-x_2|\,[\text{maximum speed}]\times[\text{total strength of wave fronts}]\\&\leqslant L|x_1-x_2|,\end{aligned}\tag{A.13}$$

with L being a constant independent of μ. Then there exists a subsequence (still denoted) $\{U^{\mu,\Delta x}\}$ that converges to a limit function $U\in L^1_{\mathrm{loc}}\big((-\infty,b(x));\mathbb{R}^4\big)$, guaranteed by the Helly Theorem.

To show that U is a weak solution, it suffices to prove that, for every $\phi\in C^1_{\mathrm{c}}(\mathbb{R}^2)$ and $\psi\in C^1_{\mathrm{c}}(\Omega)$,

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^{b(x)}(\phi_x\rho u+\phi_y\rho v)\,\mathrm{d}y\mathrm{d}x+\int_{-\infty}^{0}\phi(0,y)\,u_\infty\rho_\infty\,\mathrm{d}y=0,\\&\int_0^\infty\int_{-\infty}^{b(x)}\big(\psi_x(\rho u^2+p)+\psi_y\rho uv\big)\,\mathrm{d}y\mathrm{d}x+\int_{-\infty}^{0}\psi(0,y)\,\big(\rho_\infty u_\infty^2+p_\infty\big)\,\mathrm{d}y=0,\\&\int_0^\infty\int_{-\infty}^{b(x)}\big(\psi_x\rho uv+\psi_y(\rho v^2+p)\big)\,\mathrm{d}y\mathrm{d}x+\int_{-\infty}^{0}\psi(0,y)\,\rho_\infty u_\infty v_\infty\,\mathrm{d}y=0,\\&\int_0^\infty\int_{-\infty}^{b(x)}\Big(\psi_x\rho u\big(\frac{u^2+v^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}\big)+\psi_y\rho v\big(\frac{u^2+v^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}\big)\Big)\mathrm{d}y\mathrm{d}x\\&\quad+\int_{-\infty}^{0}\psi(0,y)\,\big(\frac{u_\infty^2}{2}+\frac{\gamma p_\infty}{(\gamma-1)\,\rho_\infty}\big)\,\mathrm{d}y=0.\end{aligned}\tag{A.14}$$

We only give a proof of the first equality in (A.14), since the remaining part can be obtained analogously.

Since ϕ is compactly supported, it is required to verify

$$\int_0^X\int_{-\infty}^{b(x)}(\phi_x\rho u+\phi_y\rho v)\,\mathrm{d}y\mathrm{d}x+\int_{-\infty}^{0}\phi(0,y)\,u_\infty\rho_\infty\,\mathrm{d}y=0\tag{A.15}$$

for some $X>0$. To calculate the term

$$\int_0^X\int_{-\infty}^{b^{\mu,\Delta x}(x)}\big(\phi_x\rho^{\mu,\Delta x}u^{\mu,\Delta x}+\phi_y\rho^{\mu,\Delta x}v^{\mu,\Delta x}\big)\,\mathrm{d}y\mathrm{d}x,\tag{A.16}$$

we fix μ and assume that, on any level set $x=t$, $y_\alpha(t)$ is a jump for $\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}$. Let

$$\begin{aligned}\Delta(\rho^{\mu,\Delta x}u^{\mu,\Delta x})&:=(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(t,y_\alpha+)-(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(t,y_\alpha-),\\\Delta(\rho^{\mu,\Delta x}v^{\mu,\Delta x})&:=(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(t,y_\alpha+)-(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(t,y_\alpha-).\end{aligned}$$

Observe that the polygonal lines $y = y_\alpha(x)$ divide stripe $[0,X]\times\mathbb{R}$ into regions D_j on which $U^{\mu,\Delta x}$ is constant. Define

$$\Phi_s := \left(\phi\rho^{\mu,\Delta x}u^{\mu,\Delta x},\, \phi\rho^{\mu,\Delta x}v^{\mu,\Delta x}\right).$$

By the divergent theorem, (A.16) can be written as

$$\sum_j \iint_{D_j} \mathrm{div}\Phi_s(x,y)\,\mathrm{d}y\mathrm{d}x = \sum_j \int_{\partial D_j} \Phi_s\cdot\mathbf{n}\,\mathrm{d}s, \tag{A.17}$$

where ∂D_j is the boundary and $\mathbf{n}$ is the unit outer normal. Since, on the polygonal line $y = y_\alpha(x)$, $\mathbf{n}\mathrm{d}s = \pm(\dot{y}_\alpha,-1)\mathrm{d}x$, while $\phi(x,y)=0$ on line $x=X$. Therefore, (A.17) is computed by

$$\begin{aligned}
&\int_0^X \sum_\alpha \left(\dot{y}_\alpha(x)\,\Delta(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,y_\alpha) - \Delta(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,y_\alpha)\right)\phi(x,y_\alpha(x))\,\mathrm{d}y\mathrm{d}x\\
&\quad+\int_0^X \Big(\dot{b}^{\mu,\Delta x}(x)(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x)) - (\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x))\Big)\,\phi(x,b^{\mu,\Delta x}(x))\,\mathrm{d}x\\
&\quad-\int_{-\infty}^0 (\rho_\infty^{\mu,\Delta x}u_\infty^{\mu,\Delta x})(y)\,\phi(0,y)\,\mathrm{d}y.
\end{aligned}\tag{A.18}$$

If the discontinuity α is physical, then

$$\dot{y}_\alpha(x)\,\Delta(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,y_\alpha) - \Delta(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,y_\alpha) = O(1)\,\mu\,|\alpha|.$$

On the other hand, if the wave at y_α is non-physical, then

$$\dot{y}_\alpha(x)\,\Delta(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,y_\alpha) - \Delta(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,y_\alpha) = O(1)\,|\alpha|.$$

Moreover, on the approximate boundary, from the construction, we obtain

$$\dot{b}^{\mu,\Delta x}(x)(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x)) - (\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x)) = 0.$$

Therefore, by proposition 3.2, we have

$$\begin{aligned}
&\limsup_{\mu\to0}\Big|\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}} \left(\dot{y}_\alpha(x)\,\Delta(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,y_\alpha) - \Delta(\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,y_\alpha)\right)\phi(x,y_\alpha(x))\Big|\\
&\qquad\leqslant \max_{x,y}|\phi(x,y)|\,\limsup_{\mu\to0}\left\{\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}} O(1)\,\mu|\alpha| + \sum_{\mathcal{NP}} O(1)\,|\alpha|\right\} = 0.
\end{aligned}\tag{A.19}$$

Moreover, from (i), (iii), and the L^1_{loc} convergence of $\{U^{\mu,\Delta x}\}$, the Dominated Convergence Theorem implies

$$\begin{aligned}
&\limsup_{\substack{\Delta x\to0,\\ \mu\to0}} \int_0^X \Big(\dot{b}^{\mu,\Delta x}(x)\,(\rho^{\mu,\Delta x}u^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x)) - (\rho^{\mu,\Delta x}v^{\mu,\Delta x})(x,b^{\mu,\Delta x}(x))\Big)\,\phi(x,b^{\mu,\Delta x}(x))\mathrm{d}x\\
&\quad=\int_0^X \Big(\dot{b}(x)\,\rho u(x,b(x)) - \rho v(x,b(x))\Big)\,\phi(x,b(x))\,\mathrm{d}x.
\end{aligned}\tag{A.20}$$

Again, the Dominated Convergence Theorem yields

$$\limsup_{\substack{\Delta x \to 0,\\ \mu \to 0}} \int_0^X \int_{-\infty}^{b(x)} \left(\phi_x \rho^{\mu,\Delta x} u^{\mu,\Delta x} + \phi_y \rho^{\mu,\Delta x} v^{\mu,\Delta x}\right) \mathrm{d}y\mathrm{d}x$$
$$= \int_0^\infty \int_{-\infty}^{b(x)} (\phi_x \rho u + \phi_y \rho v)\,\mathrm{d}y\mathrm{d}x. \tag{A.21}$$

Finally, noting (A.18)–(A.21), we conclude

$$\int_0^\infty \int_{-\infty}^{b(x)} (\phi_x \rho u + \phi_y \rho v)\,\mathrm{d}y\mathrm{d}x + \int_{-\infty}^0 \phi(0,y)\, u_\infty \rho_\infty \mathrm{d}y$$
$$= \limsup_{\substack{\Delta x \to 0,\\ \mu \to 0}} \int_0^X \int_{-\infty}^{b^{\mu,\Delta x}(x)} \left(\phi_x \rho^{\mu,\Delta x} u^{\mu,\Delta x} + \phi_y \rho^{\mu,\Delta x} v^{\mu,\Delta x}\right) \mathrm{d}y\mathrm{d}x$$
$$+ \limsup_{\Delta x \to 0} \int_{-\infty}^0 \phi(0,y)\, u_\infty^{\mu,\Delta x} \rho_\infty^{\mu,\Delta x}\,\mathrm{d}y. \tag{A.22}$$

By (A.18) and (A.19), we obtain (A.15).

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References

[1] Abbott I H and von Doenhoff A E 1959 *Theory of Wing Sections: Including a Summary of Airfoil Data* (Dover Publications, Inc.)
[2] Abzalilov D F 2005 Minimization of the wing airfoil drag coefficient using the optimal control method *Izv. Ross. Akad. Nauk Mekh. Zhidk. Gaza* **40** 173–9
[3] Bianchini S and Colombo R M 2002 On the stability of the standard Riemann semigroup *Proc. Am. Math. Soc.* **130** 1961–73
[4] Bressan A 1995 The unique limit of the Glimm scheme *Arch. Ration. Mech. Anal.* **130** 205–30
[5] Bressan A 2000 *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem* (Oxford University Press) (https://doi.org/10.1093/oso/9780198507000.001.0001)
[6] Bressan A, Liu T-P and Yang T 1999 L^1 stability estimates for $n \times n$ conservation laws *Arch. Ration. Mech. Anal.* **149** 1–22
[7] Caramia G and Dadone A 2019 A general use adjoint formulation for compressible and incompressible inviscid fluid dynamic optimization *Comput. Fluids* **179** 289–300
[8] Chen G-Q and Feldman M 2018 *Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures* (*Research Monograph, Annals of Mathematics Studies*) vol 197 (Princeton University Press) (https://doi.org/10.2307/j.ctt1jktq4b)
[9] Chen G-Q, Kuang J, Xiang W and Zhang Y 2024 Hypersonic similarity for steady compressible full Euler flows over two-dimensional Lipschitz wedges *Adv. Math.* **451** 109782
[10] Chen G-Q, Kuang J, Xiang W and Zhang Y 2025 Convergence rate of the hypersonic similarity for two-dimensional steady potential flows with large data *Nonlinearity* **38** 045013
[11] Chen G-Q and Kukreja V 2023 L^1-stability of vortex sheets and entropy waves in steady supersonic Euler flows over Lipschitz walls *Discrete Contin. Dyn. Syst.* **43** 1239–68
[12] Chen G-Q and Li T-H 2008 Well-posedness for two-dimensional steady supersonic Euler flows past a Lipschitz wedge *J. Differ. Equ.* **244** 1521–50

[13] Chen G-Q, Pu Y and Zhang Y 2025 Stability of inverse problems for steady supersonic flows past Lipschitz perturbed cones *Archive for Rational Mechanics and Analysis* (available at: https://arxiv.org/abs/2310.17815)
[14] Chen G-Q, Zhang Y and Zhu D 2006 Existence and stability of supersonic Euler flows past Lipschitz wedges *Arch. Ration. Mech. Anal.* **181** 261–310
[15] Chen G-Q, Zhang Y and Zhu D 2007 Stability of compressible vortex sheets in steady supersonic Euler flows over Lipschitz walls *SIAM J. Math. Anal.* **38** 1660–93
[16] Chen S 1998 Supersonic flow past a concave double wedge *Sci. China* A **41** 39–47
[17] Chen S 1998 Asymptotic behavior of supersonic flow past a convex combined wedge *Chin. Ann. Math.* B **19** 255–64
[18] Chen S 1998 Global existence of supersonic flow past a curved convex wedge *J. Partial Differ. Equ.* **11** 73–82
[19] Chen S 2020 *Mathematical Analysis of Shock Wave Reflection* (*Series in Contemporary Mathematics*) vol 4 (Shanghai Scientific and Technical Publishers, Springer) (https://doi.org/10.1007/978-981-15-7752-9)
[20] Chen S and Li D 2000 Supersonic flow past a symmetrically curved cone *Indiana Univ. Math. J.* **49** 1411–35
[21] Colombo R M and Corli A 2004 A semilinear structure on semigroups in a metric space *Semigroup Forum* **68** 419–44
[22] Courant R and Friedrichs K O 1948 *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc.)
[23] Dafermos C M 2016 *Hyperbolic Conservation Laws in Continuum Physics* 4th edn (Springer) (https://doi.org/10.1007/978-3-662-49451-6)
[24] Goldsworthy F A 1952 Supersonic flow over thin symmetrical wings with given surface pressure distribution *Aeronaut. Q.* **3** 263–79
[25] Golubkin V N and Negoda V V 1988 Calculation of the hypersonic flow over the upwind side of a wing of small span at high angles of attack *Zh. Vychisl. Mat. i Mat. Fiz.* **28** 1586–94
[26] Golubkin V N and Negoda V V 1994 Optimization of hypersonic wings *Zh. Vychisl. Mat. i Mat. Fiz.* **34** 446–60
[27] Gu C 1962 A method for solving the supersonic flow past a curved wedge *Fudan J.* **7** 11–14
[28] Holden H and Risebro N H 2002 *Front Tracking for Hyperbolic Conservation Laws* (Springer) (https://doi.org/10.1007/978-3-642-56139-9)
[29] Kuang J and Zhao Q 2020 Global existence and stability of shock front solution to 1-D piston problem for exothermically reacting Euler equations *J. Math. Fluid Mech.* **22** 22
[30] Lax P D 1957 Hyperbolic systems of conservation laws. II *Commun. Pure Appl. Math.* **10** 537–66
[31] Lewicka M 2000 L^1 stability of patterns of non-interacting large shock waves *Indiana Univ. Math. J.* **49** 1515–37
[32] Lewicka M 2001 Stability conditions for patterns of noninteracting large shock waves *SIAM J. Math. Anal.* **32** 1094–116
[33] Lewicka M and Trivisa K 2002 On the L^1 well posedness of systems of conservation laws near solutions containing two large shocks *J. Differ. Equ.* **179** 133–77
[34] Li Q and Zhang Y 2022 An inverse problem for supersonic flow past a curved wedge *Nonlinear Anal. Real World Appl.* **66** 103541
[35] Li T 1980 On a free boundary problem *Chin. Ann. Math.* **1** 351–8
[36] Li T and Wang L 2006 Global exact shock reconstruction for quasilinear hyperbolic systems of conservation laws *Discrete Contin. Dyn. Syst.* **15** 597–609
[37] Li T and Wang L 2007 Existence and uniqueness of global solution to an inverse piston problem *Inverse Problems* **23** 683–94
[38] Li T and Wang L 2009 *Global Propagation of Regular Nonlinear Hyperbolic Waves* (*Progress in Nonlinear Differential Equations and Their Applications*) vol 76 (Birkhäuser Boston, Ltd)
[39] Lien W and Liu T-P 1999 Nonlinear stability of a self-similar 3-dimensional gas flow *Commun. Math. Phys.* **204** 525–49
[40] Maddalena F, Mainini E and Percivale D 2020 Euler's optimal profile problem *Calc. Var. PDE* **59** 56
[41] Mohammadi B and Pironneau O 2010 Applied shape optimization for fluids *Numerical Mathematics and Scientific Computation* (The Clarendon, Oxford University Press, Oxford Science Publications) (https://doi.org/10.1093/acprof:oso/9780199546909.001.0001)

[42] Pu Y and Zhang Y 2023 An inverse problem for determining the shape of the wedge in steady supersonic potential flow *J. Math. Fluid Mech.* **25** 25
[43] Robinson A and Laurmann J A 1956 *Wing Theory* (Cambridge University Press)
[44] Saint-Raymond L 2000 Isentropic approximation of the compressible Euler system in one space dimension *Arch. Ration. Mech. Anal.* **155** 171–99
[45] Schaeffer D G 1976 Supersonic flow past a nearly straight wedge *Duke Math. J.* **43** 637–70
[46] Smoller J 1994 *Shock Waves and Reaction-Diffusion Equations* 2nd edn (Springer) (https://doi.org/10.1007/978-1-4612-0873-0)
[47] Vorobev N F 1998 On an inverse problem in the aerodynamics of a wing in a supersonic flow *J. Appl. Mech. Tech. Phys.* **39** 86–91
[48] Wang L 2014 An inverse piston problem for the system of one-dimensional adiabatic flow *Inverse Problems* **30** 085009
[49] Wang L and Wang Y 2019 An inverse Piston problem with small BV initial data *Acta Appl. Math.* **160** 35–52
[50] Zhang Y 1999 Global existence of steady supersonic potential flow past a curved wedge with a piecewise smooth boundary *SIAM J. Math. Anal.* **31** 166–83
[51] Zhang Y 2003 Steady supersonic flow past an almost straight wedge with large vertex angle *J. Differ. Equ.* **192** 1–46
[52] Zhang Y 2007 On the irrotational approximation to steady supersonic flow *Z. Angew. Math. Phys.* **58** 209–23