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Bounds for finite primitive complex linear groups

Michael J. Collins [a,b]

[a] University College, Oxford OX1 4BH, England, UK
[b] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA [1]

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In memoriam Walter Feit 1930–2004

Abstract

In 1878, Jordan showed that a finite complex linear group must possess a normal abelian subgroup whose index is bounded by a function of the degree n alone. In this paper, we study primitive groups; when $n > 12$, the optimal bound is $(n + 1)!$, achieved by the symmetric group of degree $n + 1$. We obtain the optimal bounds in smaller degree also. Our proof uses known lower bounds for the degrees of the faithful representations of each quasisimple group, for which the classification of finite simple groups is required. In a subsequent paper [M.J. Collins, On Jordan's theorem for complex linear groups, J. Group Theory 10 (2007) 411–423] we will show that $(n + 1)!$ is the optimal bound in general for Jordan's theorem when $n \geqslant 71$.

Keywords: Finite primitive complex linear groups

1. Introduction

It is well known that the complex representation of the symmetric group S_n of degree $n - 1$ that occurs as a constituent of the standard permutation representation has the smallest degree amongst all its faithful representations in characteristic 0; equivalently, S_{n+1} is the largest symmetric group that can be embedded in the matrix group $GL(n, \mathbb{C})$. In a paper directed towards

E-mail address: mjc@herald.ox.ac.uk.
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an application to linear differential equations [13], Jordan showed that, modulo an abelian normal subgroup, the order of a finite group embedded in $GL(n,\mathbb{C})$ can be bounded by a function of n. Jordan did not give any explicit bound, establishing only existence by induction on n; later Frobenius [11] and Schur [16] gave explicit bounds. These, though, prove to be of orders of magnitude that we will show in the final section are far too large.

Here we will show, using the classification of finite simple groups, that the group S_{n+1} does in fact determine the correct bound generically for primitive groups; in a subsequent paper [6] we will extend this result to drop the assumption of primitivity while in a third paper [7] we will look at analogues for representations over fields of nonzero characteristic.

Previously, Boris Weisfeiler [21] had announced a result that gave bounds of approximately the right asymptotic order of magnitude where he too assumed the classification. Sadly he disappeared in 1985, leaving a near-complete manuscript in which he obtained a result close to that which we will obtain — since he was interested also in the corresponding problem for linear groups in nonzero characteristic p, he needed to allow for arbitrarily large groups of Lie type in that characteristic, and also for the fact that S_n has an irreducible representation of degree $n-2$ when p divides n and so gave a generic bound of $(n+2)!$. Some years ago, Walter Feit asked me to prepare Weisfeiler's work for publication. However, while Weisfeiler relied heavily on studying some quite delicate functions in order to obtain overarching bounds (and some of that work is in a missing appendix), a deeper analysis of the group theoretic structure of, in particular, primitive groups allows for a proper understanding of what is happening, and for a very precise description of the small obstructions to the "generic bound"; that part dealing with primitive groups will be presented here.

Much of the technical argument lies in determining lower bounds for the degrees of faithful projective representations for every finite simple group, and this is where the classification is used; this work has been carried out by others although we should remark that, except for a detailed analysis of small cases, quite crude bounds would actually suffice for our purpose, and only for the convenience of their expression will we use more recently obtained results in place of those already available to Weisfeiler.

Prior to the classification of finite simple groups, Feit himself had studied primitive complex linear groups of small degree. Jordan in his original paper had given a list of the finite linear groups of degrees 2 and 3, and Blichfeldt gave the primitive groups of degree 4 [4].[2] Using methods based on modular representation theory and studying specifically groups of prime degree, Brauer had determined the finite linear groups of degree 5, followed by Wales' thesis on groups of degree 7 and Lindsay's on groups of degree 6; Feit then worked on groups of degrees 8, 9 and 10. (See [10] and references cited therein.) Interestingly, the highest degree studied in that "programme" was for $n = 11$ by Robinson in his thesis [15] when, as we will see, the final obstruction is the pair of dual 12-dimensional representations of the 6-fold cover of Suzuki's sporadic simple group.

Our goal in this paper is to establish the generic bound for primitive complex groups, and to describe the small exceptions. In particular, it will be seen that the generic bound for primitive complex groups will be satisfied as soon as $n > 12$, and Feit used this in unpublished work on representations over cyclotomic fields.

In the process, we will obtain some general structure theorems that are independent of the characteristic of the representation so that these may be applied when we later consider imprim-

[2] Though both Jordan and Blichfeldt missed groups.

itive groups and groups of nonzero characteristic [6,7]. Indeed, we will see then that the generic bound that we obtain here holds for *all* complex groups provided that $n \geqslant 71$.

Recall that (in arbitrary characteristic) an irreducible representation of a group G is said to be *primitive* if the underlying vector space cannot be decomposed as a direct sum of proper subspaces permuted under the action of G. It follows from Nakayama's generalisation of the Frobenius reciprocity theorem (see, for example, [5, Chapter 1, Theorem 35]) that an irreducible representation is primitive if and only if it is not induced from that of any proper subgroup. We will say that the group G is primitive when G is a subgroup of $GL(n,k)$ and the representation so afforded is primitive.

Theorem A. *Let G be a finite primitive subgroup of $GL(n,\mathbb{C})$ and suppose that $n > 1$. Then $[G:Z(G)]$ is bounded. If the bound is achieved, then $G' \cong A_{n+1}$ and $G/Z(G) \cong S_{n+1}$, with the following exceptions*:

n	$[G:Z(G)]$	$[G:Z(G)]/(n+1)!$	H
2	60	10	$2.A_5$
3	360	15	$3.A_6$
4	25920	216	$Sp_4(3)$
5	25920	36	$PSp_4(3)$
6	6531840	1296	$6_1.U_4(3).2_2$
7	1451520	36	$Sp_6(2)$
8	348368800	960	$2.O_8^+(2).2$
9	4199040	1.157	$3^{1+4}.Sp_4(3)$
12	448345497600	72	$6.Suz$

where H is a group uniquely determined up to isoclinism,[3] *and $G = Z(G).H$.*

In particular, in general $[G:Z(G)]!(n+1)!$ if $n > 12$.

Here, in the final column, we have followed the ATLAS [9] to describe the normal structure of a group H with $|Z(H)|$ minimal and to describe *which* extensions (central, or by an automorphism) occur. We have used the notation of the tables in the ATLAS for the simple groups, except for *Sp* and *PSp* as the symplectic groups and their simple quotients since they arise with their natural actions in Sections 2 and 3. We will also use "full" linear notation when discussing classical groups in Section 4. However, we note that here $O_8^+(2)$ is the simple group more often written as $\Omega_8^+(2)$ $(= O^+(8,2)')$. (See also Chapter 2 of [9].)

One may reasonably ask *why* one gets these small exceptions, other than by small numerical accident. While this undoubtedly has a role, it would appear that the answer may be partly geometric too. The generic examples of the symmetric groups are, of course, Coxeter groups and hence groups generated by reflections, but we observe too that the Weyl group $W(E_8)$ has a normal structure $2.O_8^+(2).2$, while $W(E_7) \cong Z_2 \times Sp_6(2)$ and $W(E_6) \cong Aut(PSp_4(3))$. Furthermore, the stated groups in degrees 4, 5 and 6 may be realised as complex reflection groups, while $Z_2 \times 3.A_6$ also occurs as a (3-dimensional) complex reflection group.

The isomorphism $W(E_6) \cong Aut(PSp_4(3))$ has a further interesting consequence. The group $U_4(3)$ contains two conjugacy classes of maximal subgroups isomorphic to $PSp_4(3)$, and these fuse under an outer automorphism of order 4, with an embedding $Aut(PSp_4(3)) \hookrightarrow U_4(3).2_2$.

[3] This is relevant only in degrees 6 and 8, where there are two possible extensions of the derived group, and either may be taken. Recall that two groups that are isoclinic possess central extensions that are isomorphic.

As we will see in Theorem 8, $PSp_4(3)$ has an irreducible representation of degree 6. This is stable under an outer automorphism and leads to an embedding $Aut(PSp_4(3)) \hookrightarrow 6_1.U_4(3).2_2$; since $Sp_4(3)$ does not have a 6-dimensional representation, $PSp_4(3)$ remains split in the central extension $6_1.U_4(3)$.

The group 6.*Suz* is a subgroup of the full Conway group *Co* of automorphisms of the 24-dimensional Leech lattice, where its 24-dimensional orthogonal representation decomposes as the sum of two conjugate 12-dimensional representations. We will show in Proposition 7 that $3^{1+4}.Sp_4(3)$ defines a unique group; it also occurs as the centraliser of a particular element of order 3 in Conway's first simple group Co_1. Indeed, the extension is split and the restriction of this 9-dimensional representation to $Sp_4(3)$ splits as the direct sum of four- and five-dimensional primitive representations (the latter being a representation of $PSp_4(3)$). We refer to [9] for some further related descriptions, though we highlight one historic realisation: $PSp_4(3)$ is the group of the 27 lines on a cubic surface.

We should remark too that the groups given in Theorem A are "largest possible" only in that they exhibit the bounds; they are not maximal in any "inclusive" sense. Theorem 8 will include a list of all quasisimple groups G for which there is a faithful irreducible representation of degree n for which $[G : Z(G)] > (n+1)!$. Amongst those groups is the simple group $L_3(2)$ of order 168 which has a representation of degree 3, yet clearly there can be no "inclusion" in the largest group $3.A_6$ of dimension 3.

2. The structure of primitive groups

In this section, we will study primitive groups over algebraically closed fields of arbitrary characteristic. We start with an immediate consequence of Clifford's theorem.

Lemma 1. *Let G be a finite group having a faithful primitive representation. Then every abelian normal subgroup of G is cyclic and central.*

Proof. Let V be a module affording the representation and let N be an abelian normal subgroup of G. Since G acts faithfully, if N were noncyclic or cyclic and noncentral, then $V|_N$ would be inhomogeneous and the homogeneous components would form a system of imprimitivity for G. □

Remark. It is a consequence of this lemma that in Theorem A we were able to replace the abelian normal subgroup of Jordan's theorem by the centre. Also, the assumption of algebraic closure is used crucially here to ensure absolute irreducibility; over an arbitrary field, centrality need not hold. This lemma, in the complex case, was actually known to Blichfeldt, and generalises to show that the restriction to any normal subgroup is homogeneous — i.e., that G is *quasiprimitive*.

Let $O_p(G)$ denote the largest normal p-subgroup of a group G. Lemma 1 together with a theorem of Philip Hall determines the precise structure of $O_p(G)$ for each prime divisor of $|G|$.

Lemma 2. *Let G be a finite group having a faithful primitive irreducible representation. Let $P = O_p(G)$. Then either P is cyclic, or else P contains an extraspecial subgroup E such that $P = Z(P).E$. If p is odd, then E may be chosen to have exponent p, with $E = \Omega_1(P)$.*

Proof. Certainly $Z(P)$ is cyclic, by the previous lemma; so assume that P is nonabelian. Hall classified all finite p-groups in which every characteristic abelian subgroup is cyclic (see [12, Theorem 5.4.9]); since we have further that such subgroups are even central, only the stated possibilities can occur (and Hall's argument may be simplified). □

It is now convenient to make the following definition.

Definition. Let G be a finite group having a faithful primitive irreducible representation and let p be a prime divisor of $|G|$. Suppose that $P = O_p(G)$ is noncyclic. Then P will be called a *quasicomponent*.

We show next that quasicomponents will play the same role in the structure of G as components; indeed, they have the stronger property of being normal (and, indeed, characteristic).

Recall that a *component* in a finite group is a subnormal subgroup which is quasisimple — i.e., the perfect central extension of a nonabelian finite simple group — and that the *Bender subgroup* $E(G)$ is the subgroup of G generated by the components, and that this is actually a central product of them. The *generalised Fitting subgroup* of G is the (characteristic) subgroup $F^*(G) = F(G).E(G)$ where $F(G)$ is the Fitting subgroup. The crucial property of $F^*(G)$ is that

$$C_G\big(F^*(G)\big) \subseteq F^*(G),$$

mirroring the role of the Fitting subgroup for soluble groups (see Chapter 11 of [2] for a fuller discussion), and we now give a different factorisation of $F^*(G)$ for our particular situation.

Definition. For a primitive group G, let $E_1(G)$ be the (central) product of the components and quasicomponents of G.

Clearly $F^*(G) = Z(G).E_1(G)$ since $F(G)$ is just the direct product of those subgroups $O_p(G)$ that are cyclic and the quasicomponents; for the remainder of this section, we examine how $E_1(G)$ controls $[G : Z(G)]$.

Proposition 4. *Let G be a primitive group. If P is a quasicomponent and $P = Z(P).E$ with E extraspecial of order p^{2m+1}, then*

(i) *the stabiliser of the chain $P \supset Z(P) \supset 1$ is $P.C_G(P)$,*

and

(ii) *$G/P.C_G(P)$ is isomorphic to a subgroup of $Sp_{2m}(p)$.*

Proof. Let H be the stabiliser of the chain $P \supseteq Z(P) \supseteq 1$. Then $P.C_G(P) \subseteq H \triangleleft G$. Put $\hat{H} = H/C_G(P)$.

Assume first that $|Z(P)| = p$. Then $P = E$ and $Z(P) = P'$, and P is generated by a set of $2m$ elements lying outside $Z(P)$. In this case, any automorphism of P that stabilises the chain $P \supset Z(P) \supset 1$ sends each generator to an element lying in the same coset of $Z(P)$, so that the number of such automorphisms is at most p^{2m}. On the other hand, this *is* the number of inner automorphisms, so that all appear this way and $H = P.C_G(P)$.

Suppose then that $|Z(P)| > p$. We imitate the above argument. Choose a set of $2m + 1$ generators for P by taking a generator for $Z(P)$ together with $2m$ generators for E. We may replace each of the latter by its product with an element of $Z(P)$ of order at most p^2 without affecting the alternating form on $P/Z(P)$ induced by commutation. Consequently, we may assume that E has been chosen so that it can be generated by a set of $2m$ elements of order p. Conjugation by an element of H must now map each such generator of E to another element of order p which then lies in the same coset of P' $(= Z(E))$, rather than merely of $Z(P)$. Thus, as previously, we have $|\hat{H}| \leqslant p^{2m}$ so that there is equality, with $\hat{H}$ realised by all the inner automorphisms of P, and again $H = P.C_G(P)$.

In either case, G preserves the alternating bilinear form on $P/Z(P)$ induced by conjugation, and this can be naturally identified with the corresponding form on $E/Z(E)$. The group G/H acts faithfully on $P/Z(P)$ since $Z(P) \subseteq Z(G)$ so that it embeds into $Sp_{2m}(p)$. □

Proposition 4 gives us the control of G that we need in order to bound $[G : Z(G)]$ in terms of the components and quasicomponents.

Theorem 5. *Let G be a nonabelian primitive group with quasicomponents $P_1, \ldots, P_r$ and components $E_1, \ldots, E_s$. For each i, put $|P_i/Z(P_i)| = p_i^{2n_i}$ and let $N = \bigcap_{j=1}^{s} N_G(E_j)$. Then*

(i) *there is a monomorphism from $N/F^*(G)$ into the direct product*

$$Sp_{2n_1}(p_1) \times \cdots \times Sp_{2n_r}(p_r) \times Out_c(E_1) \times \cdots \times Out_c(E_s),$$

and

(ii) *G/N is isomorphic to a subgroup of a direct product $S_{l_1} \times \cdots \times S_{l_t}$ of symmetric groups where $l_1, \ldots, l_t$ are the sizes of the distinct isomorphism classes of components of G.*

Remark. Here, Out_c denotes the subgroup of the outer automorphism group that is the image of the group Aut_c of automorphisms that act trivially on the centre. For quasicomponents E that are extraspecial, $Out_c(E) \cong Sp_{2m}(p)$ only if E has exponent p (for p odd) or if E is a quaternion group; the argument in the proof of Proposition 4 for quasicomponents P that are not extraspecial easily extends to show that the full symplectic group will occur in all these cases.

Proof of Theorem 5. Since the quasicomponents are normal in G, certainly there is a homomorphism from N into the direct product

$$Aut(P_1) \times \cdots \times Aut(P_r) \times Aut(E_1) \times \cdots \times Aut(E_s)$$

whose kernel is $C_G(E_1(G))$, i.e., $Z(G)$. Now $F^*(G)$ is the preimage of

$$Inn(P_1) \times \cdots \times Inn(P_r) \times Inn(E_1) \times \cdots \times Inn(E_s)$$

and (i) follows from Proposition 4 and the fact that $Z(E_i) \subseteq Z(E_1(G)) \subseteq Z(G)$ for each component E_i.

G permutes the components under conjugation, and N is precisely the kernel of this action. So (ii) holds. □

Now we turn to the representation of G. As remarked after Lemma 1, G is quasiprimitive and the restriction to $E_1(G)$ is homogeneous. Finding an upper bound on the group order modulo the centre for a fixed degree representation is equivalent to determining a minimal degree projective representation for a fixed group. Thus, to achieve the generic bound, we lose nothing by assuming that the restriction to $E_1(G)$ is irreducible[4] — we effectively compare the bound given by Theorem 5 with $(m+1)!$ where m is the degree of an irreducible representation for $E_1(G)$, and any small exception can then be considered by hand. Since $E_1(G)$ is a central product of components and quasicomponents, the irreducible representations are obtained as tensor products. In particular, their degrees are the products of the individual degrees. In the case of components, we must consider all possible central extensions of the simple quotient to seek the smallest degree; for quasicomponents, we can take any faithful representation of degree $p_i^{n_i}$.

There are a number of reductions that we can make since our key target is the generic bound. First, we can replace any quasicomponent or component by another group of the same degree, but which contributes more to the bound on the order of $N/Z(G)$, provided that we replace an entire conjugacy class. Generically, this replacement will be a symmetric group. But then also we may have been able to increase the contribution to the index $[G:N]$ by enlarging the permutation orbit sizes. (*Warning.* In examining bounds we will not necessarily assume either that corresponding groups exist or that, if they do, the representations exist.)

In order to carry out this process, we must first determine the maximum individual contributions, and these correspond to the case of a single component or quasicomponent.

3. Quasicomponents

Suppose that $E_1(G)$ consists of a single quasicomponent. Then, by Theorem 5,

$$\left|G/Z(G)\right| \leqslant p^{2n}\left|Sp_{2n}(p)\right|$$

for some prime p and natural number[5] n, while a lower bound for the degree of the representation is p^n. So we must compare $p^{2n}|Sp_{2n}(p)|$ with $(p^n+1)!$. Note that

$$\left|Sp_{2n}(p)\right| = p^{n^2}\prod_{i=1}^{n}\left(p^{2i}-1\right) \leqslant p^{2n^2+n}.$$

Theorem 6. $p^{2n}|Sp_{2n}(p)| \leqslant (p^n+1)!$ *unless* $p^n = 2, 3, 4, 5, 8$ *or* 9.

Proof. We claim that $p^{2n^2+3n} \leqslant (p^n+1)!$ bar the stated exceptions. If $p \geqslant 7$, then $p^5 \leqslant (p+1)!$ and we proceed by induction on n. We need only show that

$$p^{2(n+1)^2+3(n+1)-2n^2-3n} = p^{4n+5} \leqslant \left(p^{n+1}+1\right)\cdots\left(p^n+2\right)$$

[4] Since p-groups are monomial, a quasicomponent can never be primitive. Thus we cannot assume that $E_1(G)$ is primitive.

[5] In this and the following section, n will be defined by the particular context, and will not necessarily be the degree of the underlying linear group (except in the stated theorems).

and this is true whenever $4n + 5 \leqslant (p^{n+1} - p^n)n = p^n(p-1)n$, which is true for all n since $p^n(p-1) \geqslant 9$.

The same inductive argument holds for $p \leqslant 5$ once we can establish the base cases, and these occur at 16, 27 and 25 for the primes 2, 3 and 5, respectively. □

Remark. The exceptions remain even with exact computation. Specifically, we have the values

$$\begin{aligned}
p^n &= 2, \quad p^{2n}\left|Sp_{2n}(p)\right| = 24,\\
p^n &= 3, \quad p^{2n}\left|Sp_{2n}(p)\right| = 216,\\
p^n &= 4, \quad p^{2n}\left|Sp_{2n}(p)\right| = 2^8 \cdot 3^2 \cdot 5 = 11520,\\
p^n &= 5, \quad p^{2n}\left|Sp_{2n}(p)\right| = 2^3 \cdot 3 \cdot 5^3 = 3000,\\
p^n &= 8, \quad p^{2n}\left|Sp_{2n}(p)\right| = 2^{15} \cdot 3^4 \cdot 5 \cdot 7 = 92897280,\\
p^n &= 9, \quad p^{2n}\left|Sp_{2n}(p)\right| = 2^7 \cdot 3^8 \cdot 5 = 4199040.
\end{aligned}$$

Only the last group will have a significant role to play in characteristic zero,[6] and we therefore examine it more carefully.

The following result is well known, so we will just sketch the proof.

Proposition 7. *Let E be an extraspecial group of order p^{2n+1} and exponent p for p an odd prime. Then there is a split extension X of E by $Sp_{2n}(p)$ with $Z(X) = Z(E)$ having a faithful primitive complex representation of degree p^n.*

Proof. A linear map on $E/Z(E)$ need only preserve the alternating bilinear form induced by commutation to induce an automorphism in $Aut_c(E)$ so that the full symplectic group $Sp_{2n}(p)$ occurs as $Out_c(E)$. Put $\bar{E} = E/Z(E)$ and identify $\bar{E}$ with the group of inner automorphisms $Inn(E)$. Let $\langle t\rangle = Z(Out_c(E))$. Then $\langle t\rangle$ splits over $\bar{E}$, and $Aut_c(E)$ splits over $\bar{E}$ as $Aut_c(E) = \bar{E}.C_{Aut_c(E)}(t) \cong \bar{E}.Sp_{2n}(p)$. So we may take $Out_c(E)$ as a subgroup of $Aut_c(E)$, determined up to conjugacy by an inner automorphism, and form the semidirect product $X = E.Sp_{2n}(p)$ in which $Z(X) = Z(E)$.

We now construct an extension of E by $Sp_{2n}(p)$ inside $GL(p^n, \mathbb{C})$. Take any faithful irreducible representation ρ of E. Let $\mathcal{C}$ be any conjugacy class of noncentral p'-elements in $Sp_{2n}(p)$; then, excluding the case when $n = 1$ and $p = 3$, $\mathcal{C}$ generates $Sp_{2n}(p)$. For $\sigma \in \mathcal{C}$, form the natural semidirect product $X_\sigma = E\langle\sigma\rangle$; this group depends up to isomorphism only on the choice of σ in X/E. Now σ stabilises ρ. Suppose that σ has order m. Since $(p, m) = 1$, ρ has m extensions to X_σ and we may choose the (unique) extension for which $\det(\rho(\sigma)) = 1$. Let

$$\hat{X} = \left\langle \rho(X_\sigma) \;\middle|\; \sigma \in \mathcal{C}\right\rangle \subseteq GL\left(p^n, \mathbb{C}\right)$$

and put $N = C_{\hat{X}}(\rho(E))$. Then N consists of scalar matrices by Schur's lemma since ρ is irreducible. Now $\hat{X}/N.\rho(E) \cong Sp_{2n}(p)$ and hence, since $Sp_{2n}(p)$ has trivial Schur multiplier, $\hat{X}' \cong X$.

6 The cases $p^n = 2, 3, 5$ and 7 will arise in nonzero characteristic. The first group has the structure $2.S_4$; Proposition 7 covers the other cases also.

The case $n = 1$, $p = 3$ needs special treatment since $Sp_2(3)(\cong SL_2(3))$ is soluble, and $SL_2(3) \cong Q_8.Z_3$. The argument above allows us to construct four extensions of a faithful representation of E to $H = E.Q_8$. An element of order 3 in $X \setminus H$ permutes these, in fact stabilising just the unimodular extension, and this extends to three different representations of X.

Finally, we must show that the extension $\tilde{\rho}$ of ρ given by any embedding of X into $GL(p^n, \mathbb{C})$ is primitive. If not, the underlying space V decomposes as a direct sum

$$V = V_1 \oplus \cdots \oplus V_r$$

of proper subspaces permuted transitively under the action of X and also under the action of E since E acts irreducibly. If $H = Stab_X(V_1)$, then

$$Z(E) \subseteq H \cap E \lhd H$$

and $X = HE$. Thus $H \cap E \lhd X$, which is impossible since X/E acts irreducibly on $E/Z(E)$. So primitivity is established. □

Remark. In the case $p^n = 9$, the restriction of ρ to $Sp_4(3)$ decomposes as the sum of the 4- and 5-dimensional primitive representations of $Sp_4(3)$ and $PSp_4(3)$ that appear in Theorem A.

4. The case $E_1(G)$ quasisimple

Suppose that $E_1(G) = E$ is quasisimple. It is here that we invoke the classification of finite simple groups in order to examine all possibilities systematically. We want to bound $|Aut_c(E)|$ in terms of the minimal degree for a faithful representation of E. Since $Z(E)$ is cyclic and central in G, it is enough to consider $|Aut(\bar{E})|$ where $\bar{E} = E/Z(E)$. As we will see, the calculation in the previous section is a prototype for handling the general case when $\bar{E}$ is of Lie type.

Lower bounds for the degrees of representations of E for $\bar{E}$ of Lie type were first given by Landazuri and Seitz [14]; most of their results are close to best possible (see, for example, [20]). Bounds for the minimal degrees for the alternating groups and their covering groups were established by Schur [17]. For the sporadic groups and explicit values for some smaller groups of Lie type, we refer to the ATLAS of finite simple groups [9]; in particular, we employ the ATLAS notation in describing the relevant groups.

Theorem 8. *Let G be a subgroup of $GL(n, \mathbb{C})$ with $F^*(G)$ quasisimple and irreducible. Suppose that $Z(E(G)) = Z(G)$ and that $[G : Z(G)] \geqslant (n+1)!$. Then either $E(G) \cong A_{n+1}$ or one of the following holds, where the second column lists groups that actually attain the maximum value for $|G/Z(G)|$.*

n	G	$\max \lvert G/Z(G)\rvert$	*Other possible* $E(G)$
2	$2.A_5$	60	
3	$3.A_6$	360	A_5, $L_3(2)$
4	$2.PSp_4(3)$	25920	$2.L_3(2)$, $2.A_6$, $2.A_7$
5	$PSp_4(3)$	25920	
6	$6_1.U_4(3).2_2$	6531840	$U_3(3)$, $6.L_3(4)$, $PSp_4(3)$, $2.J_2$
7	$Sp_6(2)$	1451520	
8	$2.O_8^+(2).2$	348368800	$2.Sp_6(2)$
12	$6.Suz$	448345497600	

It is convenient to prove two variants simultaneously. The first will provide the appropriate working bounds for the proof of Theorem A via Hypothesis I in the next section; we note, in particular, that no further groups occur.

Theorem 9. *Let E be a quasisimple irreducible subgroup of $GL(n,\mathbb{C})$. If $[E:Z(E)].|Out_c(E)| > (n+1)!$, then $n \leqslant 8$ or $n = 12$ and, when maximality is achieved, $E = E(G)$ where G is the group listed in Theorem 8, and either*

(i) *the bound is as in Theorem 8,*

or

(ii) *$n = 4$ or 5, and $[E:Z(E)] \cdot |Out_c(E)| = 51840$.*

The second variant is not required in this paper, but we will need it to prove modular analogues of these results; we include it here since the core arguments when we examine simple groups of Lie type hold in every nonzero characteristic different from that of the underlying group also.

Proposition 10. *Let G be a finite group with $F^*(G)$ a quasisimple group of Lie type and characteristic p and $Z(E(G)) = Z(G)$. Suppose that $F^*(G)$ has a faithful irreducible representation of degree n over some algebraically closed field of characteristic different from p and that $[G:Z(G)] > (n+1)!$. Then $n \leqslant 8$ and either $E(G)$ is one of the groups listed in the conclusion of Theorem 8 (other than $2.A_7$, $2.J_2$ or $6.Suz$), or $n = 5$ and $G \cong Aut(A_6)$.*

Remark. The only "new" group that occurs here is the alternating group A_6 in its alternative guises as $L_2(9)$ or $Sp_4(2)'$, in view of the outer automorphism of S_6. The group $Aut(A_6)$ does not feature in Theorem 8 since here we no longer require G to have a representation of degree n. (It should be noted, too, that the alternating group A_5 appears in Proposition 10 both as $SL_2(4)$ and as $L_2(5)$.)

We shall now prove these three results.

That the groups appearing have the claimed properties can be seen by inspection of the ATLAS. The information there includes the representations for every central extension of a simple group, and also enables one to tell when an automorphism of the simple group extends to an automorphism of a central extension, *and* its action on the centre. Thus we can check that the representations of $E(G)$ for those groups given in column 2 of Theorem 8 cannot be extended to permit any further automorphisms, and that no other central extensions of sporadic simple groups or small alternating groups can occur. At the same time, we can see the additional cases that arise in Theorem 9.

We need now to show that no other alternating group or group of Lie type can occur under the hypothesis of Theorem 8. We can establish this under the weaker hypothesis that $G = E(G)$ and, putting $\bar{G} = G/Z(G)$, that $|Aut(\bar{G})| \geqslant (n+1)!$, and this will give Theorem 9. Proposition 10 will follow too since our arguments for groups of Lie type permit any coprime characteristic.

Schur showed that the minimal degree for a faithful representation of the alternating group A_m in characteristic 0 is $m-1$ if $m \geqslant 6$; the covering groups $2.A_8$ and $2.A_9$ both have representations of degree 8 which cannot be extended to representations of $2.S_8$ or $2.S_9$, while for $m \geqslant 10$ the minimal degree for $2.A_m$ is at least $2^{(m-2)/2}$ [17, §44].

This leaves the groups of Lie type. Here we can treat all coprime characteristic representations simultaneously. We will first consider the classical groups and then the exceptional groups and remaining twisted groups.

Classical groups

For each of the families of groups, we work by induction on dimension for each field order q. In the tables below, $q = p^m$, the factor m in the bound for $|Aut(\bar{G})|$ (or $2m$ for unitary groups or orthogonal groups $\Omega_{2n}^-(q)$) comes from the field automorphisms, with the remaining constant factor arising from possible graph automorphisms (though the full factor 6 for the orthogonal groups $\Omega_{2n}^+(q)$ can occur only when $n = 4$); at this stage we are making no assertions about the existence of an extension of the representation of $E(G)$ nor, in taking these bounds, we are restricting our attention to $Out_c(E(G))$.

We take the lower bounds for minimal degrees from [20] as it is convenient for arithmetic reasons to use the best bounds available. These bounds apply over all fields of characteristic different from the defining characteristic of the group. There are some exceptions for small groups, but these occur only for groups already listed as exceptions in Theorem 8 and so can be excluded from our consideration. (The isomorphisms $PSp(4,2)' \cong A_6$ and $U_4(2) \cong PSp_4(3)$ should be noted in particular when considering base cases. Also $\Omega_{2n+1}(q) \cong Sp_{2n}(q)$ when q is even.)

$\bar{G}$	Upper bound on $\lvert Aut(\bar{G})\rvert$	Lower bound for min. degree
$PSL_2(q)$	$m.q(q^2-1)$	$\begin{cases} q-1, & q \text{ even},\\ (q-1)/2, & q \text{ odd}\end{cases}$
$PSL_n(q), n \geqslant 3$	$2m.q^{n(n-1)/2}\prod_{i=1}^{n-1}(q^{i+1}-1)$	$\begin{cases} 26 & q=3,\ n=4,\\ (q^n-1)/(q-1)-1, & \text{otherwise}\end{cases}$
$PSp_{2n}(q), n \geqslant 2$	$2m.q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	$\begin{cases} (q^n-1)/2 & q \text{ odd},\\ (q^n-1)(q^n-q)/2(q+1), & q \text{ even}\end{cases}$
$P\Omega_{2n+1}(q), n \geqslant 3$	$m.q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	$\begin{cases} (q^{2n}-1)/(q^2-1)-2, & q>3,\\ (q^n-1)(q^n-q)/(q^2-1) & q=3,\ n>3,\\ 27 & q=3,\ n=3\end{cases}$
$P\Omega_{2n}^+(q), n \geqslant 4$	$6m.q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\begin{cases} (q^n-1)(q^{n-1}+q)/(q^2-1)-2, & q>3,\\ (q^n-1)(q^{n-1}-1)/(q^2-1), & q \leqslant 3\end{cases}$
$P\Omega_{2n}^-(q), n \geqslant 4$	$2m.q^{n(n+1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$(q^n+1)(q^{n-1}-q)/(q^2-1)-1$
$U_n(q), n \geqslant 3$	$2m.q^{n(n+1)/2}\prod_{i=1}^{n-1}(q^{i+1}-(-1)^{i+1})$	$\begin{cases} (q^n-1)/(q+1), & n \text{ even},\\ (q^n-q)/(q+1), & n \text{ odd}\end{cases}$

For each family of groups, let $N(q,n)$ denote the upper bound in the second column and $D(q,n)$ the lower bound in the third. Following the technique of the proof of Theorem 6, we fix each q and need only establish the inequality

$$N(q,n) < \big(D(q,n)+1\big)!$$

for a base case in each family and then apply induction on n using inequalities

$$N(q,n+1)/N(q,n) \leqslant \big(D(q,n+1)+1\big)!/\big(D(q,n)+1\big)! \qquad (*)$$

for the successive ratios. The right-hand side of this inequality, of course, consists of a product of $D(q,n+1) - D(q,n)$ terms, each greater than $D(q,n)$, and hence grows faster than $q^{q^{cn}}$ while an estimate for the left-hand side is given by a power of q that is linear in n.

For example, taking the groups $P\Omega_{2n}^{-}(q)$ we have

$$N(q,n+1)/N(q,n) = q^{2n+2}\big(q^{n+1}+1\big)\big(q^n-1\big) \leqslant q^{4n+3}$$

while

$$D(q,n+1)!/D(q,n)! > D(q,n)^{q^{n-1}(q^n-q+1)} \geqslant q^{(2n-3)q^{n-1}(q^n-q+1)};$$

the inequality $(*)$ will hold if $4n+3 \leqslant (2n-3)q^{n-1}(q^n-q+1)$, which is certainly true since $n \geqslant 4$ and $q \geqslant 2$. As for the base case, $q^{(2n-3)}$is a lower bound for $D(q,n)$ so that the base cases, when $n=4$, follow from the inequalities $N(q,4) \leqslant 2mq^{26} \leqslant 2q^{27}$ and $D(q,4)! \geqslant q^5! \geqslant q^{4(q^5-q^4)} \geqslant q^{64}$. So all groups $P\Omega_{2n}^{-}(q)$ satisfy the inequality $N(q,n) < (D(q,n)+1)!$.

Inequalities for the other families may be computed similarly.

Exceptional and remaining twisted groups

We will not give exact group orders since we are in each case working with a fixed rank, and the estimates we give will suffice; bounds are in every case given by the highest power of q in the polynomial for the group order. Our notation here for groups of Lie type follows [12]; in particular, we take q as the order of the *fixed* field, not the defining field. The bounds for degrees are those given by [18] since they are easier to work with than the better, more recent, bounds given in [20]; as for the classical groups, they hold for all characteristics different from that of the group. (For example, the exceptional bound for the smallest Suzuki group $^2B_2(8)$ is relevant only in characteristic 5.)

$\bar{G}$	Upper bound on $\lvert Aut(\bar{G})\rvert$	Lower bound for min. degree
$G_2(3)$	8491392	14
$G_2(4)$	503193600	12
$G_2(q), q \geqslant 5$	$2mq^{14}$	$q(q^2-1)$
$F_4(2)$	2^{53}	44
$F_4(q), q \geqslant 3$	$2mq^{52}$	$\begin{cases} q^4(q^6-1), & q \text{ odd}, \\ q^7(q^3-1)(q-1)/2, & q \text{ even} \end{cases}$
$E_6(q)$	$2mq^{78}$	$q^9(q^2-1)$
$E_7(q)$	mq^{133}	$q^{15}(q^2-1)$
$E_8(q)$	mq^{248}	$q^{27}(q^2-1)$
$^2B_2(8)$	87360	8
$^2B_2(q), q=2^{2l+1}, l>1$	mq^5	$(q-1)\sqrt{\frac{q}{2}}$
$^3D_4(q)$	$3mq^{28}$	$q^3(q^2-1)$
$^2G_2(q), q=3^{2l+1}$	mq^7	$q(q-1)$
$^2F_4(2)'$	2^{25}	26
$^2F_4(q), q=2^{2l+1}$	mq^{26}	$q^4(q-1)\sqrt{\frac{q}{2}}$
$^2E_6(q)$	mq^{78}	$q^9(q^2-1)$

Since ranks are fixed, every case is a "base case," and verification of the inequality $N(q) \leqslant (D(q)+1)!$ where $N(q)$ and $D(q)$ are the values in the second and third columns for every case is a triviality which can be estimated in a similar fashion to that for $P\Omega_8^-(q)$. We can omit $G_2(2)$ since $G_2(2)' \cong U_3(3)$.

5. The proof of Theorem A

Fix $n \geqslant 2$. Since an imprimitive subgroup of $GL(n, \mathbb{C})$ necessarily has a homomorphic image isomorphic to a nonidentity subgroup of $S_{n'}$ for some $n' \leqslant n$, for $n \geqslant 5$ the alternating group A_{n+1} acts primitively as then does the symmetric group S_{n+1}. For $n \leqslant 4$, the groups of Theorem 8 are primitive since they have no proper normal subgroups of index less than $n!$. So there is a primitive subgroup G of $GL(n, \mathbb{C})$ with $[G : Z(G)] \geqslant (n+1)!$. We will apply Theorem 5 to bound and then maximise the index $[G : Z(G)]$.

$E_1(G)$ must act homogeneously on the underlying vector space V; assume initially that its action is irreducible. If $E_1(G)$ consists of either a single quasicomponent or a single component, then the claimed bound follows immediately from Theorems 6 and 8. So suppose otherwise. Then V is a tensor product of spaces on each of which one quasicomponent or component in turn acts irreducibly and the rest trivially. Denote the quasicomponents and components (now without distinguishing between them) by $E_1, \ldots, E_s$ and suppose that the corresponding spaces $V_1, \ldots, V_s$ have dimensions $n_1, \ldots, n_s$, respectively, which we call the *subdegrees*; then $\dim V = n_1 \cdots n_s$. For each i, we define the *contribution* of E_i to a bound for $[G : Z(G)]$ to be

$$c_i = \left|E_i/Z(E_i)\right|.\left|Out_c(E_i)\right|.$$

Then, by Theorem 5, we have

$$\left[G : Z(G)\right] \leqslant c_1 \cdots c_s \prod_{j=1}^{t} l_j!$$

where $l_1, \ldots, l_t$ are the sizes of the distinct isomorphism classes of components of G. By Theorems 6 and 9, the contribution corresponding to each possible subdegree is bounded, so $[G : Z(G)]$ is bounded.[7]

Suppose now that $E_1(G)$ acts reducibly and that U is an irreducible $\mathbb{C}E_1(G)$-submodule of V. Theorem 5 describes only the group theoretic structure of G; thus we will get precisely the same bound as above if we replace V by U and let $n_1, \ldots, n_s$ be the subdegrees of the action of $E_1(G)$ on U; then, putting

$$m = \dim U = n_1 \cdots n_s,$$

m divides n, and possibly $s = 1$.

Thus we are interested in the inequality

$$c_1 \cdots c_s \prod_{j=1}^{t} l_j! \geqslant (n+1)!$$

[7] Recall that we do not require that these contributions can be achieved — our sole interest is in their contribution to a bound.

where $m = n_1 \cdots n_s$ and m divides n, with a view to maximising the left-hand side. Without loss, we can assume that for each i the contribution c_i is maximal given the subdegree n_i, and so study this inequality under the following hypothesis.

Hypothesis I. Suppose that $m = n_1 \cdots n_s$ and m divides n. The values of c_i in the inequality

$$c_1 \cdots c_s \prod_{j=1}^{t} l_j! \geqslant (n+1)!$$

make the left-hand side maximal over all choices of $n_1, \ldots, n_s$ and satisfy either

(i) $c_i = (n_i + 1)!$ if $n_i > 12$ or if $n_i = 10$ or 11,

or

(ii) c_i is given by the following table, where E is a component or quasicomponent that yields that contribution.

n_i	c_i	$k_i = c_i/(n_i+1)!$	E	$\lvert Out_c(E)\rvert$
2	120	20	$2.A_5$	2
3	720	30	$3.A_6$	2
4	51840	432	$2.PSp_4(3)$	2
5	51840	72	$PSp_4(3)$	2
6	6531840	1296	$6_1.U_4(3)$	2
7	1451520	36	$Sp_6(2)$	1
8	348368800	960	$2.O_8^+(2)$	2
9	4199040	1.157	3^{1+4}	$\lvert Sp_4(3)\rvert$
12	448345497600	144	$6.Suz$	1

Remark. (i) The values for c_i in the table are given by the maximal values of $[E : Z(E)] \cdot |Out_c(E)|$ from Theorems 6 and 9.

(ii) For all $n_i \geqslant 6$, the contribution c_i can be achieved by a group having a primitive representation of degree n_i.

(iii) For each degree r, let H_r be a group for which $E_1(H_r)$ is the component or quasicomponent listed, and $H_r/E_1(H_r) \cong Out_c(E_1(H_r))$; such a group does exist (although it may not have a representation of degree r). Up to isoclinism, H_r is uniquely determined, and we may therefore assume that, if $c_i = c_{i'}$, then (in the earlier notation) $E_i \cong E_{i'}$.

(iv) We will show that Hypothesis I can be satisfied only with $s = 1$, and then appeal to Theorems 6 and 8 to complete the proof of Theorem A. Although we are no longer concerned whether there are groups that realise these contributions, either individually or jointly, we will still use actual groups to describe "components" that give rise to particular contributions.

In view of (iii), for each value r taken by some n_i, we can define the *total contribution* of the subdegree to be $(N_r)^{l_r}.(l_r)!$, where $N_r = c_i$ for $r = n_i$, given by a central product of l_r

isomorphic groups each having smallest degree representations of degree r, extended by the full symmetric group S_r of degree r permuting them.[8]

Lemma 11. $(N_r)^{l_r} \cdot (l_r)! < (r^{l_r} + 1)!$ *if* $l_r > 1$ *for all* $r \geqslant 3$ *and for* $l_r > 3$ *if* $r = 2$.

Proof. An easy inductive argument on l_r based on ratios for each r, similar to that employed in the proofs of Theorems 6 and 8, establishes this inequality, once the base cases are established.

If $r \geqslant 3$, then the product $(r^2 + 1) \cdots (r + 2)$ has more than $r + 2$ factors, each greater than $r + 1$ so that $2((r + 1)!)^2 < (r^2 + 1)!$, giving the required inequality when $N_r = (r + 1)!$, while for the remaining $r > 2$ the inequality follows from direct calculation. If $r = 2$, only the values $l_r = 2$ and $l_r = 3$ provide exceptions. □

As a consequence, we can establish

Lemma 12. *If Hypothesis* I *is satisfied, then G has at most one component or quasicomponent for each subdegree.*

Proof. If there were more than one component or quasicomponent of a given degree $r \geqslant 3$, then by Lemma 11 their total contribution would be increased by replacing them by a single symmetric group. The same is true when $r = 2$ and $l_2 > 3$.

If $r = 2$ and $2 \leqslant l_2 \leqslant 3$, we have $2 \cdot (120)^2 < 51840$ and $6 \cdot (120)^3 < 348368800$ so that we should replace the two or three components isomorphic to $2.A_5$ by a single component isomorphic to $2.PSp_4(3)$ or $2.O_8^+(2)$, respectively. □

Remark. The replacement argument of this lemma does not assert that when such a replacement is carried out, the new group, even if it exists, is primitive; we are concerned solely with bounds and the inequality of Hypothesis I. The next lemma removes this obstruction.

Lemma 13. *The inequality of Hypothesis* I *can be satisfied with* $c_1 \cdots c_s \prod_{j=1}^{t} l_j!$ *maximal only when* $s = 1$ *and* $m = n$. *If* $n \geqslant 6$, *there is a primitive group of degree n whose order equals this bound.*

Proof. We need to show that $N_p N_q < N_{pq}$ whenever $p < q$ for then we could replace any two components or quasicomponents with a single term yielding a greater total contribution. This can be seen by direct calculation if $q \leqslant 12$ and, when $p \leqslant 12$ and $q \geqslant 13$, from the inequalities

$$\frac{N_{pq}}{N_q} \geqslant \frac{N_{2q}}{N_q} \geqslant (2q + 1) \cdots (q + 2) \geqslant (15)^{13} > N_p.$$

If $13 \leqslant p < q$, then a similar argument yields the inequality

$$(p + 1)!(q + 1)! < (pq + 1)!.$$

8 At this stage, we are looking only at bounds; if we maximise each contribution, the corresponding tensor product construction will not necessarily yield a faithful representation of the central product when components have nontrivial centres.

Now, if $n = km$ with $k > 1$, then $c_k N_m > N_m$, contradicting maximality.

The only cases where a single contribution does not equal the order of a known primitive group are when $2 \leqslant n \leqslant 5$. $\square$

We may now complete the proof of Theorem A. Lemma 13 together with Theorem 8 establishes the bounds claimed in Theorem A provided that $n \geqslant 6$. However, an examination of the possibilities for $E_1(G)$ using Theorem 8 establishes the bound of $c/2$ where c is the contribution in Hypothesis I for $n \leqslant 5$. At the same time, Lemma 13 shows that, if G is any primitive subgroup of $GL(n, \mathbb{C})$ with $[G : Z(G)]$ maximal, then $E_1(G)$ can consist only of a single component or quasicomponent acting irreducibly. Further, for any n, the maximal index is achieved under the assumptions of Hypothesis I, and $E_1(G)$ is uniquely determined up to isomorphism, except for its centre when $n = 9$. It remains just to consider the further structural claim.

If $E_1(G) \cong A_{n+1}$ (when $n \geqslant 10$, $n \neq 12$), then clearly $G' = E_1(G)$ and $G/Z(G) \cong S_{n+1}$. In the remaining cases, if $n \neq 9$ and $|Out_c(E)| = 1$ in Hypothesis I, then $E(G) \cong E(H)$ and $G = F^*(G) = Z(G).E(G) \cong Z(G).H$; if $n = 9$, then Theorem 5 shows that G has the claimed normal structure (since we do not specify H up to isomorphism). In the cases $n = 6$ or 8, we have

$$G/Z(G) \cong Aut_c\big(E(H)/Z\big(E(H)\big)\big) \cong H/Z(H)$$

and

$$G' = F^*(G)' = E(G) \cong E(H) \cong Z(G).H',$$

establishing the isoclinism.

6. Asymptotic behaviour

In a subsequent paper [6], we will show that the generic bound $(n + 1)!$ that we have obtained here for primitive groups will hold for arbitrary linear groups provided that $n \geqslant 71$ when we consider the index $[G : N]$ for some abelian (not necessarily central) subgroup N. Thus, generically, the bound for linear groups is actually achieved by a primitive group. An estimate related to Stirling's formula [1, Theorem 15.18] states that

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + C + E(n)$$

where C is a constant and $0 < E(n) < \frac{1}{8n}$. This yields an asymptotic bound for the index $[G : N]$ of the form

$$\mu(n + 1)^{n+\frac{3}{2}} e^{-n}$$

for some constant μ, and thus a bound of order of magnitude $O(n^{cn})$ for c arbitrarily close to 1.

The bound[9] given by Frobenius [11] (see also [3] and [19, Satz 196]) for this index was $n! \cdot 12^{n(\pi(n+1)+1)}$ which asymptotically has order of magnitude

$$O\left(n^{c(n/\log n)^2}\right).$$

In the more general setting of periodic linear groups, Schur [16, §3] gave a specific bound of $(\sqrt{(8n)}+1)^{2n^2}$ by studying the geometry of the embedding $G \hookrightarrow U_n(\mathbb{C}) \hookrightarrow \mathbb{R}^{2n^2}$; this bound has order of magnitude $O(n^{n^2})$. No asymptotic improvements on these bounds were published until Weisfeiler first announced a bound of the form

$$(n+1)! \cdot n^{\alpha \log n + \beta}$$

in [21]. In his unpublished work, Weisfeiler went on to analyse the nature of the asymptotic bounds; in particular, he showed that this bound is approached only in the presence of large alternating groups as components. We will explore these and related questions in [8].

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