Toric Chiral Algebras

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A thesis submitted for the degree of

Doctor of Philosophy

Hilary 2017

At the same time the evil is disarmed since it is nothing, save on the plane of determinism, and since in confessing it, I posit my freedom in respect to it; my future is virgin; everything is allowed to me.
–Sartre, *Being and Nothingness.*

Dedicated to Kobi and to Dr Richard Gipps

Acknowledgements

The author is extremely grateful for the support of all those who have helped her to survive the last few years.

In particular, much thanks to Kobi Kremnitzer, whose insight, empathy, and enormous kindness have proved far greater support than this work will ever be testament to, and who, in many ways, gave me many of the things I had ever wanted or needed, including acceptance, the benefits of a very genuine warmth of character, stability, a very good role model, and, of course, the chance to study the mathematics I fell in love with.

With thanks also to my family, as well as my friends: especially Georgia, who taught me the real value in connecting with others (as well as giving me the life-changing ability to do so) and that I could be loved, and who shared and nurtured my passions; and Teresa, whose heart is truly beautiful, whose friendship has proved invaluable, and whose faith in me I have depended on far and above my own. Gareth, Charlie, and Matt have been invaluable sources of support and are really indescribably special people. Nicholas taught me the power of passion strength and resilience. Further thanks are owed to many others, including Peter, Tony, Apala, Elena, Sophie, Litsa, Mary, Jessie, Emily, Anne, Renee, and Thomas.

Much thanks to Pavel Safronov and Dario Beraldo for useful conversations.

The author is grateful for the support of an EPSRC doctoral training grant, as well as to the Mathematical Institute, and especially my office-mates and neighbours and Helen, and to Keble College, and particularly the wonderful Rich. With special thanks to Dr Richard Gipps, whose value in my life cannot be overestimated, as well as to my excellent GP.

Abstract

In this thesis, we investigate lattice chiral algebras as defined by Beilinson and Drinfeld. Given a factorisation monoid satisfying specific conditions and a super extension of this, Beilinson and Drinfeld show that one can push forward this line bundle (super extension) to give a factorisation algebra. Specifically, they describe this in the case of the factorisation monoid formed by taking Γ-valued divisors set-theoretically supported over each divisor, for Γ a lattice, as a method of constructing these lattice chiral algebras. In this work, we show that their definitions of such divisors, and of line bundles with factorisation on these, generalise to a wider class of objects given by taking coefficients in any cone, C, in a lattice. We show that, in this more general case, the functors of C-valued divisors with set-theoretic pullback contained in S are ind-schemes, and, from this, that they form a factorisation monoid. Further, we show that super line bundles with factorisation exist on this factorisation monoid, and that if we have a super line bundle with factorisation on the factorisation monoid of C-valued divisors, we can push forward such a line bundle to get a chiral (factorisation) algebra as for lattices. Hence, we obtain a new class of chiral algebras via this procedure, which we call *toric chiral algebras*.

Contents

Chapter 1

Introduction

> Winning is not some game, some result. [...] It's actually just about never actually achieving anything. It's all about the trying, the striving, the grinding on towards getting a little bit better.
>
> *Chris Packham*

Chiral algebras have their origins in the algebraic theory of vertex algebras (see [Fre02], or [FBZ01] for a more comprehensive version expanded from this). Briefly, a vertex algebra is a graded vector space V with a distinguished *vacuum vector* and a shift operator, together with a *vertex operation*, a linear map assigning formal power series over $\operatorname{End} V$ to elements of V: all of which satisfies a collection of axioms. Vertex algebras were originally defined by Borcherds in 1985 [Bor86]: a structure he extracts from a representation of a general Kac Moody algebra on a Fock space, which he extends from earlier results in physics assigning vertex operators to elements of the root lattice. Borcherds then goes on to describe further examples of his notion of vertex algebra, such as the Frenkel–Lepowsky–Meurman Moonshine representation of the Monster group.

The precise relationship between vertex and chiral algebras is explicated in [BD04], with the original vertex algebras of Borcherds corresponding to chiral algebras on $\mathbb{A}^1$ equivariant with respect to the group of translations and those of [FBZ01] arising when this group is replaced by that of affine transformations. The addition of a stress-energy tensor gives Frenkel and Ben-Zvi's *conformal vertex algebra*. Their *quasi-conformal vertex algebra* (described more below) is the data of a chiral algebra on the coordinate disc $\operatorname{Spec} \mathcal{C}[[t]]$ equivariant with respect to the action of the group ind-scheme $\mathrm{Aut}\,\mathbb{C}[[t]]$, which can, hence, be patched along a curve to assign a chiral algebra to that curve. In general, chiral algebras give a coordinate-free, geometric approach to the theory of vertex algebras.

To give more insight into the mathematics of these relationships, the operator product expansion (ope) of physicists can be encoded in the form of a map $\circ : A^l \boxtimes A^l \to \tilde{\Delta}_* A^l$, where A^l is a left $\mathcal{D}_X$-module, $\hat{\Delta}_* A^l$ is a sheaf of left $\mathcal{D}_{X\times X}$-modules supported on the diagonal $X \hookrightarrow X \times X$ which is I-adically complete and satisfies $\hat{\Delta}_* A^l / I \hat{\Delta}_* A^l = A^l$, for $I \subset \mathcal{O}_{X\times X}$ the ideal of the diagonal, and $\tilde{\Delta}_* A^l$ is the localisation of $\hat{\Delta}_* A^l$ with respect to the equation of the diagonal. The object $\hat{\Delta}_* A^l$ is defined naturally ([BD04], Section 3.5) by applying a right adjoint $\Delta_*^{\wedge}$ to the pull-back map Δ^* along the usual diagonal Δ, but of *$\mathcal{D}_P$-sheaves*: defined for any smooth variety P to be not-necessarily-quasi-coherent sheaves of left D-modules on $P_{\text{ét}}$. The composition $\mu_\circ : A^l \boxtimes A^l \to \tilde{\Delta}_* A^l \to \Delta_* A^l$, with the second map the projection $\tilde{\Delta}_* A^l \to \tilde{\Delta}_* A^l / \hat{\Delta}_* A^l = \Delta_* A^l$, then gives the binary chiral operation corresponding to the gluing map of a factorisation algebra on the complement to the diagonal, and the assignment $\circ \mapsto \mu_\circ$ gives a bijective correspondence between commutative and associative ope and chiral algebra structures on A.

From the perspective of vertex algebras (see [FBZ01] Chapter 18), let V be a quasi-conformal vertex algebra: that is, a vertex algebra with an action of $\mathrm{Der}\,\mathcal{O} = \mathbb{C}[[z]]\delta_z$, the derivations on the disc, satisfying specific commutation relations between the operators assigned to vector fields in $\mathrm{Der}\,\mathcal{O}$ and those for V (omitted here, but given on page 91 of [FBZ01]). Given a smooth curve X, one can assign a vector bundle on X associated to V, $\mathcal{V} = \mathcal{A}ut_X \times_{\mathrm{Aut}\,\mathcal{O}} V$. Here $\mathrm{Aut}\,\mathcal{O}$ is the group of continuous automorphisms of the complete topological algebra $\mathbb{C}[[z]]$, which can be identified with $\{\sum_{i\geq 1} a_i z^i \text{ with } a_1 \neq 0\}$ by taking the image of z. $\mathcal{A}ut_X$ is the principal $\mathrm{Aut}\,\mathcal{O}$-bundle whose points are pairs (x, t_x) with $x \in X$ and t_x a formal coordinate at x, forming a scheme of infinite type, and $\mathcal{V}$ has the structure of an inductive limit of vector bundles of finite rank, arising because V has a filtration by $\mathrm{Aut}\,\mathcal{O}$-submodules. The vector bundle has a canonical flat connection, giving the sheaf $\mathcal{V}^r = \mathcal{V} \otimes \Omega$ the structure of a right D-module. Denoting by $V^r \boxtimes V^r(\infty\Delta)$ the external product with poles along the diagonal, one has a map $\mu = (\mathcal{Y}^2)^r : \mathcal{V}^r \boxtimes \mathcal{V}^r(\infty\Delta) \to \Delta_! \mathcal{V}^r$, which is a chiral bracket on the sheaf $\mathcal{V}^r$. This map is the right D-module version of $\mathcal{Y}^2 : \mathcal{V} \boxtimes \mathcal{V}(\infty\Delta) \to \Delta_+ \mathcal{V} = \frac{\mathcal{O}\boxtimes\mathcal{V}(\infty\Delta)}{\mathcal{O}\boxtimes\mathcal{V}}$ defined locally on the discs D_x^2 by $\mathcal{Y}(f(z,w)A \boxtimes B) = f(z,w)Y(A, z-w)\cdot B \mod V[[z,w]]$, for $Y(-,z) : V \to \operatorname{End} V[[z^{\pm}]]$ the vertex algebra operation assigning fields to vectors, and extended to a sheaf morphism on X^2 using the fact that $\Delta_+\mathcal{V}$ is locally supported on the diagonal so any morphism to it is determined by its restriction to the formal completion of the diagonal.

One reason to be interested in such objects is that vertex algebras play a central role in two-dimensional conformal field theory (CFT), which underpins String Theory, and the study of which was initiated in by Belavin, Polyakov, and Zamolodchikov in 1984 ([BPZ84]). Specifically, the space of states of a CFT may be described as a representation of the Virasoro algebra, the unique central extension of the Witt algebra.

Our own personal motivation for studying chiral algebras originates from their fundamental role in the proof of the geometric Langlands correspondence, much work towards which was done by Dennis Gaitsgory, Dima Arinkin, and Vladimir Drinfeld. The (categorical) geometric Langlands conjecture states that there exists a unique equivalence:

$$\mathbb{L}_G : \text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}}) \to \text{D-mod}(\text{Bun}_G),$$

between the DG category $\text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}})$, on the *Galois* or *spectral* side, and the DG category of D-modules on Bun_G, $\text{D-mod}(\text{Bun}_G)$, on the *geometric* or *automorphic* side. (See [Gai15].) Here we assume the set-up that k is a field of characteristic 0, X is a smooth and complete curve over k, and G is a reductive group. Bun_G denotes the stack of G-bundles on X, and $\check{G}$ is the Langlands dual group, formed combinatorially by taking the dual of the root datum: namely, interchanging the roots and coroots, as well as the characters and cocharacters, of a maximal torus. In its original formulation, due to Beilinson and Drinfeld (with ideas from Laumon) ([Gai16]), the category on the spectral side was $\text{QCoh}(\text{LocSys}_{\check{G}})$, but this was not compatible with the functor of Eisenstein series when G was not a torus. A suitable replacement for this was found to be the category $\text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}})$ of ind-coherent sheaves on the stack of $\check{G}$-local systems on X with singular support $\text{Nilp}^{\text{glob}}_{\check{G}}$.

To clarify this, in [AG15] the notion of singular support is defined for a stack that is a derived locally complete intersection, referred to as a *quasi-smooth DG scheme*, in which paper the authors also show that $\text{LocSys}_{\check{G}}$ satisfies the conditions to be quasi-smooth. Given a quasi-smooth DG scheme Z, they use cohomological operations to define a classical stack $\text{Sing}(Z)$. The classical stack $\text{Sing}(\text{LocSys}_{\check{G}})$ classifies

Arthur parameters ([AG15], Corollary 10.4.7), which are triples $(\mathcal{P}, \nabla, A)$, with $\mathcal{P}$ a $\check{G}$-bundle on X, ∇ a connection on $\mathcal{P}$, and A a horizontal section of $\check{\mathfrak{g}}^*_{\mathcal{P}}$. In the paper, they also define a notion of singular support for a quasi-smooth DG scheme Z, which is a conical Zariski-closed subset of $\mathrm{Sing}(Z)$. Further, given a quasi-smooth Z and a conical Zariski-closed subset $Y \subset \mathrm{Sing}(Z)$, one can consider the full subcategory $\mathrm{IndCoh}_Y(Z) \subset \mathrm{IndCoh}(Z)$ of ind-coherent sheaves whose singular support is contained in Y, and this assignment establishes a bijection between such subsets Y of Z and full subcategories of $\mathrm{IndCoh}(Z)$ satisfying certain conditions. In the case of Geometric Langlands, Y is taken to be $\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}$, the subset of Arthur parameters for which A is nilpotent.

The main ideas of the proof *for the case of* GL_2 (in-progress, with some ideas left as conjectures) described in [Gai15] are as follows. The map $\mathbb{L}_G$ is induced from given maps in the following diagram (mapping the global objects through local ones), after showing that the essential images in $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ coincide:

$$\begin{array}{ccc}
\mathrm{Whit}(G,G)^{\mathrm{spec,ext}} & \xrightarrow{\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}} & \mathrm{Whit}(G,G)^{\mathrm{ext}} \\
{\scriptstyle \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})} \Big\uparrow & & \Big\uparrow {\scriptstyle \mathrm{coeff}^{\mathrm{ext}}_{G,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & & \text{D-mod}(\mathrm{Bun}_G)
\end{array}$$

To compare the essential images of $\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})$ and $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$, Gaitsgory (and collaborators) take generating sets for $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ and $\text{D-mod}(\mathrm{Bun}_G)$ that are parametrised by the same set and show that the corresponding objects (i.e. for the same $a \in A$) map to isomorphic objects in $\mathrm{Whit}(G, G)^{\mathrm{ext}}$. These generating sets are comprised of the images of the Eisenstein series functors for all proper parabolic subgroups, together with the images of global opers on X with specified singularities encoded by λ^I, $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I})$, under direct image with respect to the forgetful map $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \to \mathrm{LocSys}_{\check{G}}$ on the spectral side, and, on the geometric side, under the map $\text{q-Hitch}_{\lambda^I} : \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \to \text{D-mod}(\mathrm{Bun}_G)$, which is obtained by generalising the construction of [BD91] assigning objects in $\text{D-mod}(\mathrm{Bun}_G)$ to quasi-coherent sheaves on the scheme of opers.

The *extended Whittaker category*, $\mathrm{Whit}^{\mathrm{ext}}_{G,G}$, is a category of D-modules on a prestack stratified by prestacks parametrised by conjugacy classes of parabolics, D-modules on which form a DG diagram of functors with $\mathrm{Whit}^{\mathrm{ext}}_{G,G}$ equal to the lax limit of this (in this case for only two objects, $\mathrm{Whit}(G,G)$ and $\mathrm{Whit}(G,B)$), or *glued category* ([Ari14]). The glued categories for the parabolics can be described in terms of *factorisation categories*: a categorical analogue of chiral (factorisation) algebras, consisting of a system of categories living over the Ran space. For example, the global version of $\mathrm{Whit}(G,G)$ is equivalent to the limit (the object over the Ran space) of a unital factorisation category ([Ber14]) which Beraldo shows to be equivalent to the chiral category $\mathbf{Rep}\,\check{G}$. On the spectral side, $\mathrm{Whit}(G,G)^{\mathrm{spec,ext}}$ is the glued category $\mathrm{Glue}(p^!_{\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}_{\nabla/\mathrm{LocSys}_{\check{G}}}(\mathrm{LocSys}_{\check{B}}))$. The commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{p^!_{\mathrm{spec}}} & \mathrm{QCoh}_{\nabla/\mathrm{LocSys}_{\check{G}}}(\mathrm{LocSys}_{\check{B}})) \\ \downarrow & & \downarrow \\ \mathrm{Whit}(G,G) & \xrightarrow[i^!_B i_{G,!}]{} & \mathrm{Whit}(G,B) \end{array}$$

induces a map between lax limits: $\mathrm{Whit}(G,G)^{\mathrm{spec,ext}} \hookrightarrow \mathrm{Whit}(G,G)^{\mathrm{ext}}$, connecting the geometric and spectral sides ([Ari14]).

The geometric Langlands functor $\mathbb{L}_G$ is required to be compatible with the Hecke action, i.e. the monoidal actions of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on the categories $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ and $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$, where the action on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is given by the monoidal functor $\mathrm{Sat}(G)^{\mathrm{naive}}_{\mathrm{Ran}(X)} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})$. $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$ is a stack defined over the Ran space. Specifically, S points of $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$ are quadruples $(\underline{x}, \mathcal{P}^1_G, \mathcal{P}^2_G, \beta)$, where $\underline{x}$ is an S-point of $\mathrm{Ran}(X)$, $\mathcal{P}^1_G$ and$\mathcal{P}^2_G$ are two S-points of Bun_G, and β is the G-bundle isomorphism $\mathcal{P}^1_G \mid_{S\times X\setminus\underline{x}} \simeq \mathcal{P}^2_G \mid_{S\times X\setminus\underline{x}}$. (See [Gai15], pp. 32-33 for the construction of the Ran-space version of a prestack $\mathcal{Y}$, including symmetric monoidal structure on $\mathrm{QCoh}(\mathcal{Y}_{\mathrm{Ran}(X)})$, and p. 34 for the correspondences giving rise to the monoidal structure of $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$ and its action on $\mathrm{Bun}(G)$, which are naturally induced from the projection maps $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \to \mathrm{Bun}_G$.)

In general, chiral algebras are a general formalism to aid with the study of the representation theory of Lie algebras that have a loop component ([FG12]). Further, the process of taking chiral homology offers a powerful local-to-global principle, acting as a homotopy-theoretic generalisation of the space of conformal blocks on conformal field theory.

After some preliminary technical background, in Chapter 3 of this thesis we review the theory of chiral algebras as given by Beilinson and Drinfeld, and describe their equivalent formulation as factorisation algebras. Beilinson and Drinfeld define chiral algebras by equipping the category of D-modules on a curve X with a pseudo-tensor structure, the *chiral structure*, with chiral I-operations

$$P_I^{ch}(\{L_i\}, M) := \mathrm{Hom}_{\mathcal{M}(X^I)}(j_*^{(I)} j^{(I)*}(\boxtimes_I L_i), \Delta_*^{(I)} M).$$

Chiral algebras are then defined as algebras over the Lie operad in this pseudo-tensor category. They then develop an arguably more-intuitive formulation of these objects as factorisation algebras, existing over the Ran space. Informally, the Ran space is the set of all finite subsets of the scheme X topologised so that points with multiplicities are allowed to collide, but this prestack is neither a scheme nor an ind-scheme. More precisely, it is the direct limit in prestacks of the diagram of schemes $I \mapsto X^I$ over $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}}$. A factorisation algebra is then an $\mathcal{O}$-module over the Ran space, namely an assignment of $\mathcal{O}$-modules to the schemes X^I for $I \in \mathrm{fSet}^{\mathrm{surj}}$ compatible with the pullback maps over the surjections, in which modules are identified along the diagonals and also with the tensor product of the corresponding modules when on the complement to the diagonals. (So really all of the structure is captured by the module over X and the way in which it is glued.)

The formulation of the concept of factorisation algebra that we go on to use in our work, which is shown by Beilinson and Drinfeld to be equivalent to their original definition, is that of pairs (B, c) in which B is a morphism from the fibred category of effective divisors parametrised by Z over schemes to that of quasi-coherent $\mathcal{O}$-modules and c gives identifications for disjoint divisors $c_{S_1,S_2} : B_{S_1} \otimes B_{S_2} \xrightarrow{\sim} B_{S_1,S_2}$, together with some support conditions that capture the remainder of the data of being an object over the Ran space.

We then give an exposition of these notions as given by Francis and Gaitsgory in [FG12], which generalises the definitions to higher dimensional varieties and allows for a more conceptual appreciation of the equivalence between chiral and factorisation algebras as Koszul duality. To define these objects, Gaitsgory and Francis define the stable infinity category of D-modules on the Ran space in the way one would expect: as the natural extension of the functor of D-modules on schemes to this inductive limit. By giving lax monoidal functors, they endow this infinity category with a chiral tensor structure. Chiral algebras are then Lie algebra objects in this infinity category with respect to this which are supported on X, and factorisation algebras are given similarly as commutative coalgebra objects satisfying certain conditions.

Following this, in Chapter 4 we describe the construction of lattice chiral algebras given by Beilinson and Drinfeld, which arises from pushing forward a super line bundle with factorisation on a factorisation monoid to the space of divisors. Very similarly to the previously described, most-relevant version of the notion of factorisation algebra, a factorisation monoid assigns to each effective Cartier divisor $S \subset X \times Z/Z$ proper over Z a quasi-compact ind-algebraic space $\mathcal{G}_{S,Z}$, and to containments $S' \leq S$ a closed embedding $\mathcal{G}_{S',Z} \hookrightarrow \mathcal{G}_{S,Z}$, together with identifications c giving isomorphisms identifying disjoint divisors and their sum, such that certain conditions are satisfied. Beilinson and Drinfeld show that when one has such a structure and a super line bundle with factorisation structure on it, provided the $\mathcal{G}_{S,Z}$ may be written as an inductive limit of closed subschemes finite and flat over Z, one can define $\mathcal{O}_Z$-modules from this data, pushing the structure forward to give a factorisation algebra. They also define Γ-valued divisors for Γ a lattice and show that these form a factorisation monoid with the required properties to give a factorisation algebra via this construction.

In this thesis, we show that one can generalise the definition of Γ-divisors to give M-valued divisors for an arbitrary commutative monoid M (Section 4.2.4). We then extend this more general definition to show that one can define (super) line bundles with factorisation on these M-valued divisors. With a view to constructing new examples of chiral algebras via the construction of [BD04] described above, we consider the case in which $M = C$ is a cone in a lattice, for which we may define functors $\mathcal{D}iv(X, C)_S$. A key aspect of the construction requires that these functors be expressible as the inductive limit of their closed subschemes that are finite and flat over Z. We show that one can express the functors $\mathcal{D}iv(X, C)$ and $\mathcal{D}iv(X, C)_S$ in this way in this case (Proposition 4.2.1). Consequently, we show that the functors $\mathcal{D}iv(X, C)_S$ form a factorisation monoid (Proposition 4.4.1). In addition to this, we show the functor $\mathcal{D}iv(X, M)$ is an ind-scheme when M is a finite abelian group. We remark that one can give a more general, functorial definition of factorisation monoid, without requiring the geometric conditions, which we call a *prefactorisation monoid* (Definition 4.4.2). Further, returning to our case of cones in lattices, we prove that if one has a super line bundle with factorisation on the chiral monoid of C-valued divisors, one can push this forward to form a factorisation algebra (Theorem 4.2.2). Moreover, we show that such line bundles always exist (Proposition 4.3.2). Hence, we have constructed a new class of chiral algebras (Theorem 4.2.3), which we call *toric chiral algebras*.

Chapter 2

Preliminaries

A bird sitting on a tree is never
afraid of the branch breaking,
because her trust is not on the
branch but on her own wings.

Unknown

2.1 D-modules

The following is taken from [Gai05] and [Ber].
Let X be an algebraic variety over k, and $U \subset X$ an open affine subset. Let $C = \mathcal{O}(U)$.

A differential operator of order $\leq n$ on U is a k-linear morphism $d : C \to C$ such that $[\tilde{f}_n, \dots [\tilde{f}_1, [\tilde{f}_0, d]]] = 0$ for any $f_0, \dots, f_n \in C$, where $\tilde{f} : C \to C$ is the operator of multiplication by f. Denote by $\mathfrak{D}(U)$ the differential operators on U, and $\mathfrak{D}(U)_i$ those of order i. It is clear that $D \in \mathfrak{D}(U)_i$ if and only if for any function $f \in C$, $[D, f]$ is in $\mathfrak{D}(U)_{i-1}$, with $\mathfrak{D}(U)_{-1} := \{0\}$. One can check that this gives a filtered algebra. It is clear that $\mathcal{O}(U) \subset \mathfrak{D}(U)_0$.

We have the following proposition for understanding $\mathfrak{D}(U)$ (see [Gai05] for details):

Proposition 2.1.1. *As a ring, $\mathfrak{D}(U)$ is generated by the elements $f \in \mathcal{O}(U)$, $\xi \in T_U$, subject to the relations:*

$$f_1 \star f_2 = f_1 \cdot f_2,\ f \star \xi = f \cdot \xi,\ \xi_1 \star \xi_2 - \xi_2 \star \xi_1 = [\xi_1, \xi_2],\ \xi \star f - f \star \xi = \xi(f),$$

where, here, $\star$ denotes the multiplication in $\mathfrak{D}(U)$.

A key part of seeing this is the isomorphisms $\mathcal{O}(U) \simeq \mathfrak{D}(U)_0$ and $\mathcal{O}(U) \oplus T_U \simeq \mathfrak{D}(U)_1$. The first of these comes from the inclusion $\mathcal{O}(U) \hookrightarrow \mathfrak{D}(U)$ with inverse given by $D \mapsto D(1)$, since, by definition, $D(r \cdot 1) - r \cdot D(1) = [D, r](1) = 0$. The latter isomorphism arises from the embedding of T_U into $\mathfrak{D}(U)_1$ via seeing T_U as the derivations $D : \mathcal{O}(U) \to \mathcal{O}(U)$: Thus, $[D, \tilde{f}] = D \cdot \tilde{f} - \tilde{f} \cdot D = D(f) \in \mathcal{O}(U) = \mathfrak{D}(U)_0$. The map $\mathfrak{D}(U)_1 \to \mathcal{O}(U) \oplus T_U$ is that comprised of $D \mapsto D(1)$, from $\mathfrak{D}(U)_1 \to \mathcal{O}(U)$, and $D \mapsto [f \mapsto [D, f] \in \mathfrak{D}(U)_0 \simeq \mathcal{O}(U)]$, from $\mathfrak{D}(U)_1 \to T_U$.

Proposition 2.1.2. *Let f be a non-nilpotent function on U, and U_f the corresponding basic open subset. Then:*

$$\mathcal{O}_{U_f} \otimes_{\mathcal{O}_U} \mathfrak{D}(U)_i \simeq \mathfrak{D}(U_f)_i \simeq \mathfrak{D}(U)_i \otimes_{\mathcal{O}_U} \mathcal{O}_{U_f}.$$

If X is an arbitrary smooth algebraic variety, define the sheaf of algebras $\mathfrak{D}(X)$ on X by setting $\Gamma(U, \mathfrak{D}(X)) := \mathfrak{D}(U)$ for affine $U \subset X$. The above shows that this gives a (left and right) quasi-coherent sheaf of O_X-modules.

Definition 2.1.3. *A left (respectively, right) D-module $\mathcal{M}$ on X is a left (resp., right) module over $\mathfrak{D}(X)$.*

From the above, we see that a (left) D-module is the same as a an $\mathcal{O}_X$-module, together with an action of the Lie algebra of derivations, such that for $m \in \mathcal{M}$, $f \in \mathcal{O}_X$, and $\xi \in T_X$,

$$f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m, \text{ and } \xi \cdot (f \cdot m) - f \cdot (\xi \cdot m) = \xi(f) \cdot m.$$

The most important examples of D-modules will be the obvious modules: the two-sided module $\mathfrak{D}(X)$, and the left module $\mathcal{O}_X$ under the action $D \cdot f := D(f)$, as well as the right D-module Ω_X^n, for $n = \dim X$. Here, Ω_X is the cotangent bundle of 1-forms on X, and Ω_X^n the line bundle of n-forms on X with action $\omega \cdot f = f \cdot \omega$, and $w \cdot \xi = -\operatorname{Lie}_\xi \omega$. This module gives an equivalence of categories between left and right D-modules via the following: For a left D-module M, define the right D-module M^r by $M^r := M \otimes_{\mathcal{O}_X} \Omega_X^n$, with action:

$$(m \otimes w) \cdot f = m \otimes (f \cdot w) = f \cdot m \otimes w, \; (m \otimes w) \cdot \xi = -(\xi \cdot m) \otimes w - m \otimes \operatorname{Lie}_\xi(w).$$

(The inverse is given by $M \mapsto M \otimes (\Omega_X^n)^{-1}$.)

The pullback of (left) D-modules is constructed as follows. For $f : Y \to X$ a map of algebraic varieties, $f^* : D(X) - \text{mod} \to D(Y) - \text{mod}$ is the usual inverse image functor $\mathcal{O}_X-\text{mod} \to \mathcal{O}_Y-\text{mod}$. (That is, for a sheaf $\mathcal{G}$ on X, $f^{-1}\mathcal{G}$ is the sheafification of the presheaf given by $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, and $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$, which is also a quasi-coherent sheaf.) The action of vector fields is given by:

$$\xi \cdot (g \otimes m) = \xi(g) \otimes m + g \cdot df(\xi)(m),$$

where df is the induced map on tangent spaces, considered as a map $T_Y \to \mathcal{O}_Y \otimes_{\mathcal{O}_X} T_X$.

For a good description of many of the operations on sheaves and relevant properties that will be used greatly in the theories of chiral algebras we explain in this thesis, see the beginning of Chapter 18 of [FBZ01].

2.2 Pseudo-tensor Categories

The following is taken from [BD04].

Let $\mathcal{S}$ be the category of finite, non-empty sets and surjective maps between them. Denote by $1 \in \mathcal{S}$ the one element set, and, for any $\pi : J \twoheadrightarrow I$, and any $i \in I$, let $J_i := \pi^{-1}(i) \subset J$.

Definition 2.2.1. *A* pseudo-tensor category *is a class of objects $\mathcal{M}$ equipped with the following:*

1. *For any $I \in \mathcal{S}$, and any object $M \in \mathcal{M}$, as well as any I-family of objects $L_i \in \mathcal{M}$, there is given a set of I-*operations *$P_I^{\mathcal{M}}(\{L_i\}, M) = P_I(\{L_i\}, M)$.*
2. *For any map $J \twoheadrightarrow I$ in $\mathcal{S}$, collections of objects $\{L_i\}_{i \in I}$, $\{K_j\}_{j \in J}$, and object M, we have the composition map*

$$P_I(\{L_i\}, M) \times \prod_I P_{J_i}(\{K_j\}, L_i) \to P_J(\{K_j\}, M),\ (\phi, (\psi_i)) \mapsto \phi(\psi_i).$$

These are subjects to the following properties:

1. *Composition is associative: namely, if $H \twoheadrightarrow J$ is another map in $\mathcal{S}$, with $\{F_h\}$ an H-family of objects, $\chi_j \in P_{H_j}(\{F_h\}, K_j)$, then $\phi(\psi_i(\chi_j)) = (\phi(\psi_i))(\chi_j) \in P_H(\{F_h\}, M)$.*
2. *Identity elements: For any object M, there is an element $\mathrm{id}_M \in P_1(\{M\}, M)$ such that for any $\phi \in P_I(\{L_i\}, M)$, we have $\mathrm{id}_M(\phi) = \phi(\mathrm{id}_{L_i}) = \phi$.*

Notation: Write $P_{J/I}(\{K_j\},\{L_i\}) = P_\pi(\{K_j\},\{L_i\})$ for $\prod_I P_{J_i}(\{K_j\}, L_i)$ to write the composition map as $P_I \times P_{J/I} \to P_J$. We may also write $P_n := P_{\{1,\dots,n\}}$.

Any pseudo-tensor category $\mathcal{M}$ has an underlying category, also denoted $\mathcal{M}$, with the same objects as $\mathcal{M}$ and with morphisms $\mathrm{Hom}(L, M) := P_1(\{L\}, M)$, for $L, M \in \mathcal{M}$. The composition, and conditions of associativity and identity, for this category come from those of the original pseudo-tensor category, and, for each $I \in \mathcal{S}$, P_I becomes a functor $(\mathcal{M}^o)^I \times \mathcal{M} \to \mathrm{Set}$. A pseudo-tensor category may, thus, be considered as a category with additional *pseudo-tensor structure* of functors P_I, $|I| > 1$, together with composition morphisms. Conversely, every category has a trivial pseudo-tensor structure, given by $P_I = \emptyset$ for $|I| > 1$.

A pseudo-tensor category with one object is the same as a symmetric operad. Recall that a non-symmetric operad is given by the following datum:

- A sequence $(P(n))_{n\in\mathbb{N}}$ of sets of *n-ary operations*.
- An *identity* element $1 \in P(1)$.
- A *composition* function for any positive integers $n, k_1, \dots, k_n$:

$$\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

$$(\theta, \theta_1, \dots \theta_n) \mapsto \theta \circ (\theta_1, \dots, \theta_n),$$

  which satisfies the following identity and associativity axioms:

- Identity: $\theta \circ (1, \dots, 1) = \theta = 1 \circ \theta$.
- Associativity: $\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) =$ $(\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n})$.

It is symmetric if, in addition, each $P(n)$ has a right action $*$ of the symmetric group Σ_n with the following equivariance conditions: For any $s_i \in \Sigma_{k_i}, \in \Sigma_n$,

$$(\theta * t) \circ (\theta_{t1}, \dots, \theta_{tn}) = (\theta \circ (\theta_1, \dots, \theta_n)) * t);$$

$$\theta \circ (\theta_1 * s_1, \dots, \theta_n * s_n) = (\theta \circ (\theta_1, \dots, \theta_n)) * (s_1, \dots, s_n).$$

It is easy to see that a one-object pseudo-tensor category is a symmetric operad. The actions and equivariance come from the specific way the sets of I-operations are indexed in the composition law and its interaction with the identity axiom. We want an action of Σ_n on $P(n) = P_n(\{M_i\}, M)$, for our unique object M. So let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Then, we can embed $P_n(\{M_i\}, M)$ into $P_n(\{M\}, M) \times \prod_{\{n\}} P_{\{\sigma(i)\}}(\{M_j\}, M)$ by taking the identities in each of the copies of $P(1) = P_1(\{M\}, M)$. By the identity condition, composition with the identities in this way gives the same set $P(n)$, but, by taking the product over the permutation σ^{-1}, this composition gives a map $P_n(\{M_i\}, M) \to P_n(\{M_i\}, M) \times \prod_{\{n\}} P_{\{\sigma(i)\}}(\{M_j\}, M) \to P_n(\{M_{\sigma(i)}\}, M)$, and, from this, an action of Σ_n on $P(n)$. The first equivariance condition becomes clear:

$$\left(P_n(\{M_i\}, M) \times \prod_n P_{\{\sigma(i)\}}(\{M\}, M) \right) \times \prod_J P_{J_{\sigma(i)}}(\{M_j\}, M) =$$

$$P_n(\{M_i\}, M) \times \left(\prod_n P_{\{\sigma(i)\}}(\{M\}, M) \times \prod_J P_{J_{\sigma(i)}}(\{M_j\}, M \right)).$$

Here, the first two terms form $P(n) * \sigma$, and the third is the product of the sets $P(k_1) \times P(k_2) \times \ldots P(k_n)$ after permutation of the factors by $\sigma \in \Sigma_n$. The set J is the ordered set $\{1, \ldots, k_1, k_1 + 1, \ldots, k_1 + k_2, \ldots, k_1 + \cdots + k_n\}$, where the first k_1 map to $1 \in n$, the next k_2 to 2, and so on. The two sides are equal, by the associativity condition of composition in a pseudo-tensor category. But the left hand side is defined to be $(\theta * \sigma) \circ (\theta_{\sigma 1}, \ldots, \theta_{\sigma n})$, and the right hand side is the product $(\theta \circ (\theta_1, \ldots, \theta_n)) * \sigma)$ after applying the identity axiom.
The other equivariance condition can be formed similarly.

A *multicategory* is similar to a pseudo-tensor category, but with indices restricted to natural numbers, and, importantly, the order of the objects matters, so, in general, there isn't an action of the symmetric group. Another name for this is a coloured operad. Thus, a one-object multicategory is a plain (non-symmetric) operad.

For a field k, a *pseudo-tensor k-category* is a pseudo-tensor category $\mathcal{M}$ in which the sets of I-operations have the structure of k-vector spaces, and composition map is k-polylinear. (Thus, it can be written as a map from the tensor product.) It is *additive* if it is additive as a usual k-category, and *abelian* if it is abelian as a k-category and the polylinear functors P_I are left exact.

For pseudo-tensor categories $\mathcal{M}$ and $\mathcal{N}$, a *pseudo-tensor functor* $\tau : \mathcal{N} \to \mathcal{M}$ is a rule that assigns to any object $G \in \mathcal{N}$ an object $\tau(G) \in \mathcal{M}$, and to any family of objects $\{F_i\}_{i \in I}$ and object G in $\mathcal{N}$ a map of sets $\tau_I : P_I^{\mathcal{N}}(\{F_i\}, G) \to P_I^{\mathcal{M}}(\{\tau(F_i)\}, \tau(G))$, such that the τ_i preserve composition, and $\tau_{\cdot}(\mathrm{id}_G) = \mathrm{id}_{\tau(G)}$. Composition of pseudo-tensor functors gives a pseudo-tensor functor, and there is an obvious notion of morphism between pseudo-tensor functors. Thus, pseudo-tensor functors $\mathcal{N} \to \mathcal{M}$ form a category when $\mathcal{N}$ is equivalent to a small category. Note that any pseudo-tensor functor defines a functor of the corresponding categories of the pseudo-tensor categories, and τ is called the *pseudo-tensor extension* of this functor.

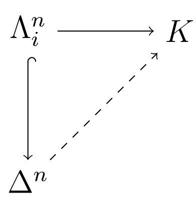
For an operad $\mathcal{B}$, and a pseudo-tensor category $\mathcal{M}$, a $\mathcal{B}$ *algebra* in $\mathcal{M}$ is a pseudo-tensor functor $\mathcal{B} \to \mathcal{M}$. This amounts to an object $L \in \mathcal{M}$ and a morphism of operads $\mathcal{B} \to P^{\mathcal{M}}(L)$. A pseudo-tensor functor $F : \mathcal{N} \to \mathcal{M}$ is k*-linear* if the maps F_I are k-linear. Analogously to the above, if $\mathcal{N} = \mathcal{B}$ is a k-operad, we get a $\mathcal{B}$ k-algebra in $\mathcal{M}$.

Recall the standard definition of an algebra over an operad $(P(n))$ (defined in the usual way): That is, a morphism of operads $P \to \mathcal{E}nd_V$, where $\mathcal{E}nd_V$ is the *endormorphism operad of a vector space* V, with $\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, the space of n-linear maps. Such a morphism consists of a collection of (linear: when the operad is an operad of vector spaces) maps $P(n) \to \mathcal{E}nd_V(n)$, compatible with composition, units, and the actions of the symmetric groups. (For a reference, see [Vor01], or [May97].)

2.3 (Stable) ∞-categories

Infinity-categories, here, will mean quasicategories, as discussed in [Lur09] and [Lur14]. We recall here some useful definitions. (See also [GJ99]).

Definition 2.3.1. *Let* Δ *be the category of finite ordinals, with order-preserving maps between them. A* simplicial set *is a functor* $\Delta^{op} \to \mathrm{Set}$*. Write* $[n]$ *or* $\mathbf{n}$ *for the ordinal in* Δ*.* Δ^n *will denote the simplicial set* $\mathrm{Hom}(-, \mathbf{n})$ *and* Λ^n_k *will denote the* kth horn *in* Δ^n*: the subfunctor of maps that map into the union of the faces* $0 \le i \le n$, $i \ne k$*. A* Kan complex *(fibrant object in simplicial sets) is a simplicial set* K *such that for any* $0 \le i \le n$ *and any diagram of solid arrows*



there exists a map $\Delta^n \to K$ making the above commute.

An ∞-category *(quasicategory) is a simplicial set K such that any map $\varphi : \Lambda^n_i \to K$ from an* inner horn *$(0 < i < n)$ admits an extension $\Delta^n \to K$. Equivalently, for each n-tuple of $(n-1)$-simplices $(\upsilon_0, \ldots, \hat{\upsilon}_k, \ldots, \upsilon_n)$ of K satisfying $d_i \upsilon_j = d_{j-1}\upsilon_i$ whenever $1 < j$, and $i, j \neq k$, there is an n-simplex υ such that $d_i \upsilon = \upsilon_i$ for all i.*

Here, d_i denotes the standard face map. We will write s_j for the degeneracy map.

One can see that the nerve of a category defines a quasicategory (and, moreover, that the extension maps are unique in this case): The natural transformation at the top of the diagram, φ say, maps the faces of Δ^n_i to a compatible collection of chains of $n-1$ morphisms in the category. Here, compatibility is ensured by the equality of maps sending points and edges into the union of the faces of $[n]$ (i.e. compositions $[0] \to [1] \to [n]$ etc. and naturality of φ, which ensures these must be preserved in K. We can extend since we can always define the extension map since we will have a linear sequence of n composable morphisms in the category. (Hence the need to only take inner horns: Outer horns would require taking inverses.)

Definition 2.3.2. *A* simplicial category *is a category enriched over simplicial sets. These can be viewed as simplicial objects in* Cat *for which the underlying map on objects is constant.*

It is well known that there is a Quillen equivalence

$$\mathrm{Set}_\Delta \underset{\mathrm{Sing}}{\overset{||}{\rightleftarrows}} \mathcal{CG}$$

between simplicial sets and compactly generated Hausdorff spaces, where $||$ denotes the geometric realisation functor and Sing the singular complex functor. Weak homotopy equivalences in simplicial sets can be thought of by considering the weak equivalences of spaces under the geometric realisation functor. It is the case that the maps $|\operatorname{Sing} X| \to X$ and $S \to \operatorname{Sing}|S|$ are weak equivalences, and this gives equivalent categories, both denoted $\mathcal{H}$, when inverting weak equivalences in $\mathcal{CG}$ and in Set_Δ (which we will return to later).

Weak equivalences in Set_Δ can also be described directly (the weak equivalences of the *classical model structure*, also referred to as the *Kan–Quillen model structure* on Set_Δ), and it is clear that this definition is the analogue of that given by taking geometric realisations ([GJ99]):

Given Kan complexes X and Y, $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all simplicial homotopy groups (so of sets for π_0). Similarly to topological spaces, the nth homotopy group at a vertex x, $\pi_n(X, x)$, is defined as the equivalence classes of maps $\alpha : \Delta^n \to X$ sending the boundary to the point x: That is, the following commutes:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow x \\ \Delta^n & \xrightarrow{\alpha} & X \end{array}$$

The equivalence is given by the following: α is equivalent to α' if they are related by a simplicial homotopy

$$\begin{array}{ccc} \Delta^n & & \\ \downarrow i_0 & \searrow^{\alpha} & \\ \Delta^n \times \Delta^1 & \xrightarrow{\eta} & X \\ \uparrow i_1 & \nearrow_{\alpha'} & \\ \Delta^n & & \end{array}$$

that fixes the boundary as in:

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^1 & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow x \\ \Delta^n \times \Delta^1 & \xrightarrow{\eta} & X \end{array}$$

The group structure is given as follows. Let $\alpha, \beta : \Delta^n \to X$. These allow one to define a map $\Lambda^{n+1}_n \to X$ by the following formula:

$$\begin{cases} v_i = s_0 \circ \cdots \circ s_0(x) & 0 \le i \le n-2, \\ v_{n-1} = \alpha & \textit{and} \\ v_{n+1} = \beta & \end{cases}$$

(since $d_i \upsilon_j = d_{j-1}\upsilon_i$ for $i < j$, $i, j \neq n$, since all faces of all simplices υ_i map through x. Thus, the Kan condition gives a corresponding $(n+1)$-simplex $\omega : \Delta^{n+1} \to X$. $[d_n\omega]$ is be the product $[\alpha] \cdot [\beta]$ in $\pi_n(X, x)$. Goerss and Jardine show that this is indeed in $\pi_n(X, x)$ (the boundary maps to x); that it is independent of the choice of representatives α and β (the corresponding maps are homotopic); and that this gives a group structure.

π_0 is the set of vertices modulo the equivalence relation $X_1 \overset{(d_1,d_0)}{\longrightarrow} X_0 \times X_0$: i.e. vertices x and y are equivalent if there is a 1-simplex υ with $d_1\upsilon = x$ and $d_0\upsilon = y$.

As referred to earlier, the *homotopy category of spaces* $\mathcal{H}$ is formed from $\mathcal{CG}$, by inverting weak equivalences (those maps which are isomorphisms on all homotopy groups π_n). This is equivalent to the analogous notion for Set_Δ, with the weak equivalences in Set_Δ taken to be those in $\mathcal{CG}$ under the geometric realisation functor or as given by the aforementioned direct definition. The functor $\theta : \mathcal{CG} \to \mathcal{H}$ preserves products. Thus, we can see that a topological category naturally gives a category enriched over $\mathcal{H}$ using the following ([Lur09] A.1.4.3):

Given a right-lax monoidal functor between monoidal categories $G : \mathcal{C} \to \mathcal{C}'$, and a category $\mathcal{D}$ enriched over $\mathcal{C}$, we can form a new category $G(\mathcal{D})$ enriched over $\mathcal{C}'$ via the obvious construction: Let the objects of $G(\mathcal{D})$ be those of $\mathcal{D}$; For $X, Y \in \mathcal{D}$, let $\mathrm{Map}_{G(\mathcal{D})}(X, Y) = G(\mathrm{Map}_{\mathcal{D}}(X, Y))$; Define the composition by $G(\mathrm{Map}_{\mathcal{D}}(Y, Z)) \otimes G(\mathrm{Map}_{\mathcal{D}}(X, Y)) \to G(\mathrm{Map}_{\mathcal{D}}(Y, Z) \otimes \mathrm{Map}_{\mathcal{D}}(X, Y)) \to G(\mathrm{Map}_{\mathcal{D}}(X, Z))$; And for any $X \in \mathcal{D}$, let the unit be $\mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{C}}) \to G(\mathrm{Map}_{\mathcal{D}}(X, X))$.

Given a simplicial category, we may form a topological category by applying the geometric realisation functor to the simplicial sets $\mathrm{Map}_{\mathcal{C}}(X, Y)$; Likewise, applying the singular complex functor to the morphism spaces of a topological category gives a simplicial category, and these two maps induce an adjunction between simplicially enriched categories Cat_Δ and topological categories $\mathrm{Cat}_{\mathrm{top}}$.

One can associate a simplicial set to a simplicial category $\mathcal{C}$ via the simplicial nerve functor, which we will describe in the following. In the case that $\mathcal{C}$ has the property that, for any $X, Y \in \mathcal{C}$, the simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex, the simplicial nerve $N(\mathcal{C})$ is an ∞-category ([Lur09], Proposition 1.1.5.10).

Let J be a finite, non-empty linearly ordered set. Define the simplicial category $\mathfrak{C}[\Delta^J]$, by letting the objects of $\mathfrak{C}[\Delta^J]$ be the elements of J, and for each $i, j \in J$,

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) := \begin{cases} \emptyset & j < i \\ N(P_{i,j}) & i \leq j \end{cases}$$

Here, $P_{i,j}$ denotes the partially ordered set $\{I \subseteq J \mid (i, j \in I) \wedge (\forall k \in I)[i \leq k \leq j]\}$. The composition of maps is given by: For $i_0 \leq i_1 \leq \cdots \leq i_n$,

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_1) \times \cdots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1}, i_n) \to \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0, i_n)$$

is the map induced by

$$P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} \to P_{i_0,i_n}$$

$$(I_1, \ldots, I_n) \to I_1 \cup \cdots \cup I_n.$$

This definition extends functorially (to a simplicially enriched functor) as follows: Let $f : J \to J'$ be a monotone map. $\mathfrak{C}[f] : \mathfrak{C}[\Delta^J] \to \mathfrak{C}[\Delta^{J'}]$ is defined on objects i by $\mathfrak{C}[f](i) := f(i)$. The map of simplicial sets on hom-sets $\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) \to \mathrm{Map}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j))$ induced (via the usual nerve functor) by $P_{i,j} \to P_{f(i),f(j)}$, $I \mapsto f(I)$. Thus, applying $\mathfrak{C}[-]$ to the ordinal category gives a cosimplicial object in simplicially enriched categories. It is this construction that is used to define the simplicial nerve of a simplicial category.

To understand the simplicial categories $\mathfrak{C}[\Delta^J]$ better, we consider here the essential example of $\mathfrak{C}[\Delta^n]$, for the ordinal $[n]$ (see [Lur09]). As a topological category under the Quillen equivalence between topological categories and simplicial categories, for $0 \leq i \leq j \leq n$, $\mathrm{Map}_{|\mathfrak{C}[\Delta^n]|}(i,j)$ is in bijection with the functions $p : \{k \in [n] \mid i \leq k \leq j\} \to [0,1]$ such that $p(i) = p(j) = 1$; that is, it is a cube, with vertices identifies with the functions from $\{i < k < j\}$ to $\{0,1\}$ and the edges (1-simplices), corresponding to the inclusions of sets (functions that change only by increasing the number of 1s) and the paths of inclusions corresponding to particular choices of paths, which are the same only up to homotopy. We can now give the definition of the simplicial nerve of a simplicial category $\mathcal{C}$.

Definition 2.3.3. *Let $\mathcal{C}$ be a simplicial category. The* simplicial nerve $N(\mathcal{C})$ *is the simplicial set given by*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Note the similarity with the definition of the nerve of an ordinary category, which is the simplicial set given by:

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N(C)) = \mathrm{Hom}_{\mathrm{Cat}}([n], C).$$

However, for a simplicial category $\mathcal{C}$, the simplicial nerve does not coincide with the ordinary nerve of the underlying ordinary category, despite the use of the same notation. As stated previously, when $\mathcal{C}$ satisfies the condition that each set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex, this definition gives a quasicategory.

The functor $\mathfrak{C}$ extends uniquely to a functor $\mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, since Δ is small and Cat_Δ is cocomplete. By [ML98] p.237: Given categories M, C, A and a functor $K : M \to C$, if $T : M \to A$ is a functor such that for each $c \in C$ the composite $(c \downarrow K) \to M \to A$ has a limit in A, then one can define the right Kan extension $R : C \to A$ via the pointwise limits in A with naturally induced maps. This has the corollary that if M is a small category and A complete, then any functor $T : M \to A$ has a right Kan extension along any $K : M \to C$. Dualising if necessary, we have that $\mathfrak{C}$ extends uniquely up to unique isomorphism to a colimit-preserving functor $\mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, also denoted $\mathfrak{C}$, which is left adjoint to the simplicial nerve by construction.

We are now able to define the homotopy category of a topological category, a simplicially enriched category, and an ∞-category.

Definition 2.3.4. *Given a topological category $\mathcal{C}$, the* homotopy category *of $\mathcal{C}$ $\mathrm{h}\mathcal{C}$ is the $\mathcal{H}$-enriched category formed by the above result applied to the map $\mathcal{CG} \to \mathcal{H}$. The homotopy category of a simplicial category $\mathcal{C}$, also denoted $\mathrm{h}\mathcal{C}$, will be the $\mathcal{H}$-enriched category formed by the composition $\mathrm{Set}_\Delta \to \mathcal{H}$ on morphism spaces, via the process of assigning a topological category to $\mathcal{C}$. Given a simplicial set S (in particular, S may be a quasicategory), the homotopy category of S $\mathrm{h}S$ is defined to be the homotopy category $\mathrm{h}\mathfrak{C}[S]$ of the simplicial category canonically assigned to S in the above.*

A map $F : \mathcal{C} \to \mathcal{D}$ of topological categories, simplicial categories, or simplicial sets is said to be an equivalence *if the induced map $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of $\mathcal{H}$-enriched categories.*

We will now describe the conditions under which an ∞-category is said to be *stable*.

Definition 2.3.5. *Let $\mathcal{C}$ be a topological/simplicial category, or a simplicial set. An* initial object *in $\mathcal{C}$ is an object X such that X is initial in the homotopy category* $\mathrm{h}\mathcal{C}$ *regarded as a category enriched over $\mathcal{H}$: That is, the mapping spaces* $\mathrm{Map}_{\mathrm{h}\mathcal{C}}(X, Y)$ *are weakly contractible for every $Y \in \mathcal{C}$. The analogous definition is made for final objects. Likewise analogously to ordinary categories, a* zero object *is an object that is both initial and final. If $\mathcal{C}$ has a zero object, it is called* pointed.

It is easily shown that an ∞-category is pointed if and only if it has an initial object $\emptyset$, a final object 1, and a map $f : 1 \to \emptyset$.

Definition 2.3.6. *An ∞-category $\mathcal{C}$ is* stable *if the following hold (*[Lur14]*):*

1. *$\mathcal{C}$ has a zero object.*
2. *Every morphism admits a fibre and cofibre.*
3. *A triangle in $\mathcal{C}$ is a fibre sequence if and only if it is a cofibre sequence.*

Definition 2.3.7. *A stable ∞-category is* presentable *if it admits small colimits and is generated (under colimits) by a set of small objects.*

In the above, in a pointed ∞-category $\mathcal{C}$, a *triangle* in $\mathcal{C}$ $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a pair of maps $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, together with a 2-simplex in $\mathcal{C}$ corresponding to a diagram

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow{h} & Z \end{array}$$

identifying h with the composition $g \cdot f$, and a 2-simplex

$$\begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ X & \xrightarrow{h} & Z \end{array}$$

a *nullhomotopy* of h.

This can alternatively be described as a map $\Delta^1 \times \Delta^1 \to \mathcal{C}$ giving a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

Such a triangle is a *fibre sequence* if it is a pullback square, and a *cofibre sequence* if it is a pushout square.

Again, let $\mathcal{C}$ be pointed. A *fibre* of $g : X \to Y$ is a fibre sequence

$$\begin{array}{ccc} W & \to & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

Dually, a *cofibre* of g is a cofibre sequence

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Given a pointed ∞-category $\mathcal{C}$ with cofibres, one can define a *suspension functor* $\Sigma : \mathcal{C} \to \mathcal{C}$. Dually, if $\mathcal{C}$ admits fibres instead of cofibres, we can define a *loop functor* $\Omega : \mathcal{C} \to \mathcal{C}$. When $\mathcal{C}$ is stable, these are mutually inverse equivalences ([Lur09]). In this case, the suspension functor allows one to define distinguished triangles on the homotopy category $\mathrm{h}\mathcal{C}$, and these endow $\mathrm{h}\mathcal{C}$ with the structure of a triangulated category according to Verdier.

The prototypical example of a stable ∞-category is the category of spectra Sp. The homotopy category $\mathrm{h}\,\mathrm{Sp}$ is the stable homotopy category.

2.4 Grothendieck Topologies, Torsors, and Fibred Categories

Here we recall the definition of a torsor.

Firstly, we recall the topological definition given in [Mit01] of a torsor, or principal G-bundle.

Definition 2.4.1. *For a topological group, G, a* left (respectively, right) G-space *is a space P equipped with a continuous left G-action $G \times P \to P$ (respectively, a right action $P \times G \to P$). For G-spaces P and P', a* G-equivariant map *(alternatively,*

equivariant*, or* G-map*) is a map* $\phi : P \to P'$ *such that* $\phi(gp) = g\phi(p)$*, for all* $g \in G$*,* $p \in P$*. Given a topological space* X*, and a right* G*-space* P*, equipped with a* G*-map* $\pi : P \to X$*, where* G *acts trivially on* X *(so* ϕ *factors uniquely through the orbit space* P/X*),* (P, π) *is a* principal G-bundle *over* X *if* π *satisfies the following local triviality condition:*
There is a covering of X *by open sets* U_i*, such that there exist* G*-equivariant homeomorphisms* $\phi_{U_i} : \pi^{-1}U_i \to U_i \times G$ *commuting in the diagram*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\phi_{U_i}} & U_i \times G \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

where $U_i \times G$ *has the obvious right action* $(u.g)h = (u, gh)$*.*

Essentially, this is made into an algebraic concept by taking all maps to be regular maps, instead of continuous, and requiring the map $P \to X$ to be flat. However, we give the formal definition according to [Mil80].

Recall that a *Grothendieck topology*, or *site*, is given by the following data ([Mil13]): A category C together with, for each object U in C, a distinguished set of families of maps $(U_i \to U)_{i \in I}$, the *coverings* of U, subject to the following conditions:

1. For any covering $(U_i \to U)_{i \in I}$ and morphism $V \to U$ in C, the fibre products $U_i \times_U V$ exist, and $(U_i \times_U V \to V)_{i \in I}$ is a covering of V;
2. If $(U_i \to U)_{i \in I}$ is a covering of U, and for each $i \in I$, we have a covering $(V_{ij} \to U_i)_{j \in J_i}$, then the family $(V_{ij} \to U)_{i,j}$ is a covering of U;
3. For any U in C, the family consisting of the single map $(U \xrightarrow{\mathrm{id}} U)$ is a covering of U.

(Technically speaking, the system of coverings is the Grothendieck topology, and the category C together with the topology is the site.)

The underlying category of a site $\mathbf{T}$ is denoted by $\mathrm{Cat}(\mathbf{T})$.

The prototypical example of a site is that for which X is a topological space, and C is the category whose objects are the open subsets of X, and morphisms the inclusion maps. The coverings are given by the families $(U_i \to U)_{i \in I}$ for which $(U_i)_{i \in I}$

is an open covering of U. Then the fibre products $U \times_V U'$ are equal to the intersections $U \cap U'$. A main consideration of sites is that of sheaves on the site (defined in the following paragraphs), and the site in this primary example leads to the usual geometric definition of a sheaf.

A morphism of algebras $f : A \to B$ is *flat* if the functor $M \mapsto B \otimes_A M$ from $A - \mathbf{mod}$ to $B - \mathbf{mod}$ is exact. One may say also that B is a *flat A-algebra.* One can easily see that this equates to B being a flat A-module. One may also show that, given such a ring morphism $A \to B$, and a B-module M, M is a flat A-module if and only if the localisation $M_{\mathfrak{q}}$ is a flat $A_{f^{-1}(\mathfrak{q})}$-module for all prime ideals $\mathfrak{q}$ of B. (See [TS16][Tag 00HT].) Similarly, we can restrict this to maximal ideals, in that such an M is a flat A-module if and only if $M_{\mathfrak{m}}$ is a flat $A_{f^{-1}(\mathfrak{m})}$-module for all maximal ideals $\mathfrak{m}$ of B. Taking $M = B$, we see that to show flatness of $f : A \to B$, it is sufficient to check that the morphisms $A_{f^{-1}(\mathfrak{m})} \to B_{\mathfrak{m}}$ are flat for all maximal ideals $\mathfrak{m}$ in B.

A morphism $\phi : Y \to X$ of schemes (or varieties) is *flat* if the local homomorphisms $\mathcal{O}_{X,\phi(y)} \to \mathcal{O}_{Y,y}$ are flat for all $y \in Y$. From the above, we see it is only necessary to check this for the closed points of Y.

The following can be found at [TS16][Tag 01T0]. A morphism of rings $f : A \to B$ is of *finite type* if B is isomorphic to a quotient of $A[x_1, \ldots, x_n]$ as an A-algebra. Let $\phi : Y \to X$ be a morphism of schemes. ϕ is *of finite type at $y \in Y$* if there is an affine open neighbourhood $U = \mathrm{Spec}(B)$ of y, $U \subset Y$, and another of $f(y)$, $\mathrm{Spec}(A) \subset X$, such that $f(U) \subset V$, with the induced ring map $A \to B$ of finite type. We say ϕ is *locally of finite type* if it is of finite type at every $y \in Y$, and we say ϕ is *of finite type* if it is locally of finite type and quasi-compact. Here, a morphism $\phi : Y \to X$ of schemes is quasi-compact if X can be covered by open affine subschemes V_i, such that the preimages $f^{-1}(V_i)$ are (quasi-)compact topological spaces.

Lemma 2.4.2. ([TS16][Tag 01T0]) *Let $\phi : Y \to X$ be a morphism of schemes. Then the following are equivalent:*

1. *ϕ is locally of finite type.*
2. *For every open affine $U \subset Y$, $V \subset X$ with $f(U) \subset V$, the ring map $\mathcal{O}_{X,V} \to \mathcal{O}_{Y,U}$ is of finite type.*

3. *There exists an open covering* $X = \cup_{j \in J} V_j$ *and open coverings* $\phi^{-1}(V_j) = \cup_{i \in I_j} U_i$ *such that each morphism* $U_i \to V_j$*,* $j \in J$*,* $i \in I_j$*, is locally of finite type.*

4. *There exists an affine open covering* $X = \cup_{j \in J} V_j$ *and affine open coverings* $\phi^{-1}(V_j) = \cup_{i \in I_j} U_i$ *such that the ring maps* $\mathcal{O}_X(V_j) \to \mathcal{O}_Y(V_i)$ *are of finite type, for all* $j \in J$*,* $i \in I_j$*.*

A ring is local if it has a unique maximal ideal. A homomorphism $A \to B$ of local rings is a map of rings that maps $\mathfrak{m}_A$ into $\mathfrak{m}_B$. This corresponds to a continuous map of the induced topological spaces. A local morphism $f : A \to B$ of local rings is *unramified* if $B/f(\mathfrak{m}_A)B$ is a finite separable field extension of $A/\mathfrak{m}_A$, or equivalently:

- $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, and
- the field $B/\mathfrak{m}_B$ is finite and separable over $A/\mathfrak{m}_A$.

A morphism $\phi : Y \to X$ of schemes is *unramified* if it is of finite type and if the maps $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are unramified for all $y \in Y$. It suffices to check this condition for all of the closed points of Y. Further, if $\phi : Y \to X$ is a morphism of finite type, the condition that ϕ is unramified is equivalent to that of the sheaf $\Omega^1_{Y/X}$ being zero (See [Mil13]).

A morphism $\phi : Y \to X$ of schemes is *étale* if it is flat and unramified. In particular, ϕ is of finite type.

A morphism of rings $f : A \to B$ is étale if $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale. Equivalently,

- B is a finitely-generated A-algebra;
- B is a flat A-algebra;
- For all maximal ideals $\mathfrak{n}$ of B, $B_\mathfrak{n}/f(\mathfrak{p})B_\mathfrak{n}$ is a finite separable field extension of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{n})$.

Common examples of sites:

- The *Zariski site on* X, denoted X_{zar} is the site described above, with objects the open subsets of X, morphisms inclusions, and coverings given by the open coverings, where X is endowed with the Zariski topology, in the usual geometric sense.

- The *étale site on* X, X_{et}, is the site with underlying category Et/X, which has objects the étale morphisms $U \to X$ and arrows the X-morphisms $\phi : U \to V$, and with coverings the surjective families of étale morphisms $U_i \to U$) in Et/X.

- The *flat site on* X, X_{Fl}, is the site with underlying category Sch/X, and coverings the surjective families of X-morphisms $(U_i \xrightarrow{\phi_i} U)$ with each ϕ_i flat and of finite type.
  An fppf covering of a scheme S will be a family of morphisms $\{f_i : S_i \to S\}_{i \in I}$ of schemes with each f_i flat and locally of finite presentation, and $S = \bigcup_i f_i(S_i)$. ([TS16] [Tag 021L]). The site with these coverings will be used later. (There are several notions of "flat" site and Milne has chosen one. Note that a locally Noetherian scheme is locally of finite presentation if and only if it is locally of finite type.)

A *presheaf of sets* on a site $\mathbf{T}$ is a contravariant functor $\mathcal{F} : \text{Cat}(\mathbf{T}) \to \text{Set}$. Note that the notation $a \mapsto a \mid U$ may be used to denote the map $\mathcal{F}(\phi) : \mathcal{F}(V) \to \mathcal{F}(U)$ even though there may be several maps $\phi : U \to V$. It is a *sheaf* on $\mathbf{T}$ if the following is exact for every covering $(U_i \to U)$:

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

This equates to saying that the map $f \mapsto (f \mid U_i) : \mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ identifies $\mathcal{F}(U)$ with the subset of the product consisting of families (f_i) such that $f_i \mid U_i \times_U U_j = f_j \mid U_i \times_U U_j$ for all $i, j \in I \times I$.

A morphism of presheaves/sheaves is simply a natural transformation.

The following is taken from [Mil80], p. 120.

Let $\mathbf{T}$ be the flat site on Sch/X. Groups schemes over X will be flat and of finite type, but not necessarily commutative. The existence of the identity section (the identity map from the final object $\text{id} : X \to X$ to G) means that any such group scheme is faithfully flat over X. Given this data, an action of G on an X-scheme S is a morphism $S \times_X G \to S$ that induces a group action of each group $G(T)$ on the corresponding set $S(T)$ for any X-scheme T. The most basic example of such an action is the multiplication $G \times G \to G$.

Lemma 2.4.3. *Let G be a group scheme over a scheme X, acting on a scheme S as above. The following are equivalent:*

1. *S is faithfully flat and locally of finite type over X, and the map $(s, g) \mapsto (s, sg)$: $S \times_X G \to S \times_X S$ is an isomorphism.*

2. *There is a covering $(U_i \to X)$ for the flat topology on X, such that, for each i, $S_{(U_i)}$ is isomorphic with its $G_{(U_i)}$-action to $G_{(U_i)}$.*

Here, $S_{(U_i)}$ denotes the fibre product of the map $S \to X$ with the given map $U_i \to X$: that is, the fibre of S over U_i.

Proof. The second condition follows immediately from the first by taking the cover to be $(S \to X)$ with the given isomorphism.

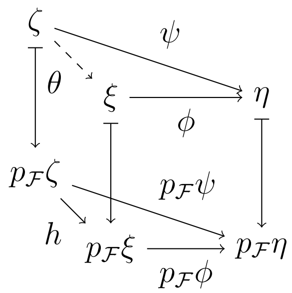
For the harder direction, let $U = \amalg U_i$. Then, U is faithfully flat and of finite type over X, and $S_{(U)} \cong G_{(U)}$. So $S_{(U)}$ is faithfully flat and of finite type over U, since $G_{(U)}$ is by the properties of pullbacks. Descent theory allows us to bring this forward through the pullback diagram to see that S has these properties over X. Further, $(S \times_X G)_{(U)} = (S \times_X G) \times_X U = S \times_X (G \times_X U) \cong S \times_X (S \times_X U) = (S \times_X S) \times_X U = (S \times_X S)_{(U)}$, so $(S \times_X G)_{(U)} \cong (S \times_X S)_{(U)}$, which, again by descent theory, gives an isomorphism $S \times_X G \to S \times_X S$, as required. □

Definition 2.4.4. *Let G, S, and X be as in the above lemma. Then such a scheme S is called a* principal homogeneous space *or* torsor *for G over X. A torsor S that is isomorphic to G acting on itself is called a* trivial *torsor.*

2.4.5 Fibred Categories

We recall, here, the notion of a fibred category, and the equivalent notion of a pseudo-functor, according to Vistoli's article [Vis05] and, originally, Chapter VI of [GR04].

Definition 2.4.6. *Let $\mathcal{C}$ be a category. A* category over $\mathcal{C}$ *is a category $\mathcal{F}$ together with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$.*
If $\mathcal{F}$ is a category over $\mathcal{C}$, then a cartesian arrow *is an arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ such that for any $\psi : \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h : p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ in $\mathcal{C}$ such that $p_{\mathcal{F}}\phi \cdot h = p_{\mathcal{F}}\psi$, there is a unique map $\theta : \zeta \to \xi$ with $p_{\mathcal{F}}\theta = h$ and $\phi \cdot \theta = \psi$:*



In this situation, ξ *is a* pullback of η to $U = p_{\mathcal{F}}\xi$ *(along* $p_{\mathcal{F}}\phi$*).*

Definition 2.4.7. *A* fibred category over $\mathcal{C}$ *is a category* $\mathcal{F}$ *over* $\mathcal{C}$ *such that for any* $f : U \to V$ *in* $\mathcal{C}$ *and any object* η *in* $\mathcal{F}$ *with* $p_{\mathcal{F}}\eta = V$*, there is a cartesian arrow* $\phi : \xi \to \eta$ *with* $p_{\mathcal{F}}\phi = f$*.*

A morphism of categories fibred over $\mathcal{C}$ *is a functor* $F : \mathcal{F} \to \mathcal{G}$ *such that* $p_{\mathcal{G}} \cdot F = p_{\mathcal{F}}$ *and* F *maps cartesian arrows to cartesian arrows.*

Given such a fibred category, the fibre $\mathcal{F}(U)$ *of* $\mathcal{F}$ *over* U *is the subcategory of* $\mathcal{F}$ *with objects the objects* ξ *of* $\mathcal{F}$ *with* $p_{\mathcal{F}}\xi = U$ *and arrows the maps* ϕ *with* $p_{\mathcal{F}}\phi = \mathrm{id}_U$*. A* base-preserving natural transformation *of morphisms of fibred categories* $F, G : \mathcal{F} \to \mathcal{G}$ *is a natural transformation* $\alpha : F \to G$ *such that each* $\alpha_\xi : F(\xi) \to G(\xi)$ *is in* $\mathcal{G}(U)$*, where* $\xi \in \mathcal{F}(U)$*. There are obvious notions of isomorphisms of morphisms of fibred categories and equivalences of fibred categories.*

These notions of morphism and (base-preserving) natural transformation equip fibred categories with the structure of a 2-category.

We will also require the notion of a *cleavage* of a fibred category:

Definition 2.4.8. *Let* $\mathcal{F} \to C$ *be a fibred category. A* cleavage *of* $\mathcal{F} \to C$ *is a class* K *of Cartesian arrows in* $\mathcal{F}$ *such that for any map* $f : U \to V$ *in* C *and any object* η *in* $\mathcal{F}(V)$*, there is a unique arrow in* K *above* f *with target* η*.*

Any fibred category has a choice of cleavage by the Axiom of Choice.

It is often useful to work, instead, with the notion of pseudo-functor, which is equivalent to the notion of a fibred category with choice of cleavage (Proposition 2.4.10):

Definition 2.4.9. *A* pseudo-functor*, or* lax 2-functor*, Φ on $\mathcal{C}$ is the assignment of a category $\Phi(U)$ to each object U of $\mathcal{C}$ and a functor $\Phi(f) : \Phi(V) \to \Phi(U)$ (or f^*) for each $f : U \to V$ in $\mathcal{C}$ such that Φ is a functor up to natural isomorphism; that is, there exist natural isomorphisms $\epsilon_U : id_U^* \simeq id_{\Phi U}$ of functors $\Phi U \to \Phi U$ for each U in $\mathcal{C}$ and $\alpha_{f,g} : f^*g^* \simeq (gf)^* : \Phi W \to \Phi U$ for any $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathcal{C}$. These must satisfy some associativity and compatibility conditions:*

1. *For any $f : U \to V$ in $\mathcal{C}$ and η in ΦV, $\alpha_{\mathrm{id}_U,f}(\eta) = \epsilon_U(f^*\eta) : \mathrm{id}_U^* f^*\eta \to f^*\eta$ and $\alpha_{f,\mathrm{id}_V} = f^*\epsilon_V(\eta) : f^*id_V^*\eta \to f^*\eta$.*

2. *For any $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ in $\mathcal{C}$ and θ in ΦT, the following commutes:*

$$\begin{array}{ccc} f^*g^*h^*\theta & \xrightarrow{\alpha_{f,g}(h^*\theta)} & (gf)^*h^*\theta \\ {\scriptstyle f^*\alpha_{g,h}(\theta)}\downarrow & & \downarrow{\scriptstyle \alpha_{gf,h}(\theta)} \\ f^*(hg)^*\theta & \xrightarrow[\alpha_{f,hg}(\theta)]{} & (hgf)^*\theta \end{array}$$

Proposition 2.4.10. *Fix a category C. A fibred category over C with a choice of cleavage defines a pseudo-functor on C. Conversely, a pseudo-functor on C gives rise to a fibred category with a cleavage. Given a category $\mathcal{F}$ fibred over C, if we associate the corresponding pseudofunctor then assign to this a fibred category with cleavage, the resultant fibred category is isomorphic to $\mathcal{F}$. Likewise, if we begin with a pseudo-functor. Thus, the notions of fibred category with cleavage and pseudo-functor can be considered to be equivalent.*

Proof. (Idea)
Let $\mathcal{F} \to \mathcal{C}$ be a fibred category with a choice of cleavage K. We obtain a pseudo-functor F as follows: Set $F(U) := \mathcal{F}(U)$ for each object U of $\mathcal{C}$, the fibre over U, and define the maps $f^* = F(f) : F(V) \to F(U)$ for $f : U \to V$ in C by the unique choice of pullback of each object by the Cartesian arrow over f. More precisely, by definition of a cleavage of a fibred category, for each η in V, there is a unique Cartesian arrow $f^*\eta \to \eta$ in K mapping to f in C and with target η. To extend f^* to morphisms β in $F(V)$, use the unique map induced by applying the property of a Cartesian arrow to the diagram:

$$\begin{array}{ccc} f^*\eta & \longrightarrow & \eta \\ {\scriptstyle f^*\beta}\,\downarrow & & \downarrow\,{\scriptstyle \beta} \\ f^*\eta' & \longrightarrow & \eta' \end{array}$$

That this makes f^* into a functor, together with conditions 1 and 2 in Definition 2.4.9, can be seen by applying the defining property of a Cartesian arrow to appropriate choices of pairs of pullbacks.

Instead, suppose we have a pseudo-functor Φ on $\mathcal{C}$. Define $\mathcal{F}$ by letting the objects be pairs (ξ, U), with ξ in ΦU, and maps $(a, f) : (\xi, U) \to (\eta, V)$, for maps $a : \xi \to \Phi(f)(\eta)$ in ΦU, with $f : U \to V$ a map in C. The reason for this is that, since all maps in a fibred category factor through the choice of Cartesian arrows K by an arrow over an identity in C, the maps can be encoded as these Cartesian arrows (given by the pseudo-functor) and the maps in the categories ΦU. For $a : \xi \to \Phi(f)(\eta)$ and $b : \eta \to \Phi(g)(\zeta)$, define $(b, g) \cdot (a, f) := (b \star a, g \cdot f)$, where $b \star a := \alpha_{f,g}(\xi) \cdot \Phi(f)(b) \cdot a : \xi \to (gf)^*\zeta$. Identities are given by $\mathrm{id}_{(\xi,U)} := (\epsilon_U(\xi)^{-1}, \mathrm{id}_U)$ and Cartesian arrows by $(\mathrm{id}_{\Phi(f)(\eta)}, f) : (\Phi(f)(\eta), U) \to (\eta, V)$. $F : \mathcal{F} \to \mathcal{C}$ is given by $(\xi, U) \mapsto U$, $(a, f) \mapsto f$. (This simplifies considerably if Φ is a functor.)

One can check these are mutually inverse constructions. For further details, we refer the reader to [Vis05].

□

Using this proposition in some cases, we see that examples of fibred categories include (omitting details): any presheaf of sets, a surjective group homomorphism $G \to H$ (considered as one object categories), vector bundles over topological spaces with a choice of pullback, quasi-coherent sheaves on schemes over a base scheme.

2.4.11 Faithfully flat descent

The notion of faithfully flat descent will be useful later in understanding the affine Grassmannian. We record it here as given in [TS16][Tag 0306].

Definition 2.4.12. *Let T be a scheme and $\mathcal{U} := \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. A* descent datum *for quasi-coherent sheaves with respect to $\mathcal{U}$ is a family $(\mathcal{F}_i, \varphi_{i,j})_{i,j \in I}$, such that, for each i, $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i, and for all $i, j \in I$, the map φ_{ij} gives an isomorphism $\mathrm{pr}_0^* \mathcal{F}_i \cong \mathrm{pr}_1^* \mathcal{F}_j$ on $T_i \times_T T_j$, such that for all triples i, j, k the diagrams*

$$\begin{array}{ccc} \mathrm{pr}_0^* \mathcal{F}_i & \xrightarrow{\mathrm{pr}_{01}^* \varphi_{ij}} & \mathrm{pr}_1^* \mathcal{F}_j \\ & {\scriptstyle \mathrm{pr}_{02}^* \varphi_{ik}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} & \\ & \mathrm{pr}_2^* \mathcal{F}_k & \end{array}$$

commute on $T_i \times_T T_j \times_T T_k$.
A descent datum is effective *if there exists a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^* \mathcal{F} \cong \mathcal{F}_i$ satisfying the cocycle condition:*

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

The following result, which we will quote without proof, will be used later.

Theorem 2.4.13. *Let $\mathcal{V} = \{T_i \to T\}_{i \in I}$ be an fpqc covering. Then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.*

Another way to say this is that the fibred category of quasi-coherent sheaves is a stack on the fpqc site. (See [Vis05], Chapter 4, for a good explanation of descent data for fibred categories.)

2.4.13.1 Cartier Divisors

We will at various points use the following interpretation of *relative Cartier divisor* from [MFK94], p. 24.

Suppose $f : X \to Y$ is a flat morphism and $D \subset X$ an effective Cartier divisor. Then the following are equivalent:

1. D is flat over Y;

2. For all $y \in Y$, the induced subscheme $D_y \subset X_y$ is a Cartier divisor.

Thus, we have a family of Cartier divisors in the fibres of f over Y. The above conditions give the notion of *relative Cartier divisor*. Moreover, for any fibre product

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow \\ Y' & \rightarrow & Y \end{array}$$

then $g'^{-1}(D)$ is a relative Cartier divisor on X' over Y'.

Chapter 3

Chiral and Factorisation Algebras

Failure gave me an inner security that I had never attained by passing examinations. Failure taught me things about myself that I could have learned no other way.

J. K. Rowling

3.1 Chiral Algebras and Factorisation Algebras

Here we give a description of chiral and factorisation algebras according to [Gai99].

An axiomatic definition of a chiral algebra can be given as follows:

Definition 3.1.1. *Let X be a smooth curve, and A^l a left D-module on X, with corresponding right D-module $A = A^l \otimes \Omega$. A chiral algebra structure on A consists of a D-module map (*chiral bracket*):*

$$j_* j^*(A \boxtimes A) \xrightarrow{\{,\}} \Delta_!(A),$$

satisfying the following conditions:

- *Antisymmetry:*
  If $f(x,y)$ is a local section of $\mathcal{O}_{X\times X}$ (but pulled back and pushed forward along the open embedding: so it is a rational function on $X \times X$ allowing poles along the diagonal), and a and b local sections of A, so $f(x,y) \cdot a \boxtimes b$ is a section of $j_* j^*(A \boxtimes A)$, then

  $$\{f(x,y) \cdot a \boxtimes b\} = -\sigma_{1,2}(\{f(y,x) \cdot b \boxtimes a\}),$$

  where $\sigma_{1,2}$ is the lift of the transposition on $X \times X$, acting on $\Delta_!(A)$.

- *Jacobi Identity:*
  Let $a \boxtimes b \boxtimes c \cdot f(x,y,z)$ *be a section of* $A \boxtimes A \boxtimes A$ *restricted to the complement of the divisor of diagonals in* $X \times X \times X$*. Then*

  $$\{\{f(x,y,z)\cdot a\boxtimes b\}\boxtimes c\}+\sigma_{1,2,3}\{\{f(z,x,y)\cdot c\boxtimes a\}\boxtimes b\}+\sigma_{1,2,3}^2\{\{f(y,z,x)\cdot b\boxtimes c\}\boxtimes a\}=0$$

  as a section of $\Delta_{x=y=z!}A$*.*

For a D-module A, have a short exact sequence:

$$0 \to \Omega \boxtimes A \to j_* j^*(\Omega \boxtimes A) \to \Delta_!(A) \to 0. \tag{3.1.1}$$

(See equation 3.1.2 for an explanation of this.)

In the case when $A = \Omega$, this gives a bracket $\mathrm{can}_\Omega : j_* j^*(\Omega \boxtimes \Omega) \to \Delta_!(\Omega)$, giving the most basic example of a chiral algebra.

A *unit* is a map **unit** $: \Omega \to A$ making the following commute:

$$\begin{array}{ccc} j_* j^*(\Omega \boxtimes A) & \xrightarrow{\mathbf{unit} \boxtimes \mathrm{id}} & j_* j^*(A \boxtimes A) \\ \mathrm{can}_{\mathcal{A}} \downarrow & & \downarrow \{,\} \\ \Delta_!(A) & \xrightarrow[\mathrm{id}]{} & \Delta_!(A) \end{array}$$

where the left vertical arrow is the canonical map given in the short exact sequence 3.1.1.

3.1.2 Chiral algebras via pseudo-tensor structures

We relay here some geometric intuition for the pseudo-tensor structures on the category of D-modules on a smooth curve, due to [BD04]. There exists a *compound tensor structure* on the category of right (or left) D-modules on X: that is, the data of a tensor and pseudo-tensor structure with suitable compatibilities. This translates to basic differential geometry, with the geometric objects the $\mathcal{D}_X$-schemes and multiplication of "functions" the D-module tensor product, while the action of "operators" arises from the $*$ pseudo-tensor structure. Moreover, the functor h from right D-modules to sheaves on X, given by $h(M) := M \otimes_{\mathcal{D}_X} \mathcal{O}_X$, sends Lie* algebras to the sheaves of Lie algebras, since h commutes with the pushforward along immersions and the external tensor product.

In the following X will be a fixed smooth algebraic curve over $\mathbb{C}$. For a finite non-empty set I, let $U^{(I)} := \{(x_i) \in X^I : x_{i_1} \neq x_{i_2} \text{for every } i_1 \neq i_2\}$, the complement to the diagonal divisor for $|I| > 1$, with $j^{(I)} : U^{(I)} \hookrightarrow X^{(I)}$ the open embedding. Write $U^{(n)}$ for $U^{(\{1,\dots,n\})}$. Denote by $\mathcal{M}_{\mathcal{O}}(X)$ the category of $\mathcal{O}$-modules on X (that is, quasi-coherent sheaves of $\mathcal{O}_X$-modules), and $\mathcal{M}(X) = \mathcal{M}^r(X)$ (respectively, $\mathcal{M}^l(X)$) the category of right (resp. left) D-modules on X.

We will also use the following notation. $\Delta^{(\pi)} = \prod_I \Delta^{(J_i)} : X^I \hookrightarrow X^J$, where $\Delta^{(J)} : X \hookrightarrow X^J$ is the diagonal map: That is to say, $\Delta^{(\pi)} : X^I \hookrightarrow X^J$, $(x_i) \mapsto (x_{\pi(j)})$. Similarly, $j^{(\pi)}$ will denote the open embedding $j^{(\pi)} : U^{(\pi)} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \hookrightarrow X^J$.

There exists an abelian pseudo-tensor structure on $\mathcal{M}(X)$, called the *chiral structure* and denoted $\mathcal{M}(X)^{ch}$, which we will describe here. For $I \in \mathcal{S}$, and objects $L_i, i \in I$, and an object M in $\mathcal{M}(X)$, the vector space of *chiral I-operations* will be

$$P_I^{ch}(\{L_i\}, M) := \mathrm{Hom}_{\mathcal{M}(X^I)}(j_*^{(I)} j^{(I)*}(\boxtimes_I L_i), \Delta_*^{(I)} M).$$

Write P_n^{ch} for $P_{\{1,\dots,n\}}^{ch}$. The elements of P_2^{ch} are *chiral pairings*.

The P_I^{ch} are left exact k-polylinear functors on $\mathcal{M}(X)$. The composition of I-operations is defined as follows. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$ and $\{K_j\} \in \mathcal{M}(X)$, $j \in J$, the map

$$P_I^{ch}(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\}, L_i)) \to P_J^{ch}(\{K_j\}, M),$$

sending $\varphi \otimes (\otimes \psi_I)$ to $\varphi(\psi_I)$ is given by:

$$\begin{aligned} j_*^{(J)} j^{(J)*}(\boxtimes_J K_j) &= j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I (j_*^{(J_i)} j^{(J_i)*} \boxtimes_{J_i} K_j)) \xrightarrow{\boxtimes \psi_i} j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I \Delta_*^{(J_i)} L_i) \\ &= \Delta_*^{(\pi)} j_*^{(I)} j^{(I)*}(\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M. \end{aligned}$$

To see what is happening here, we describe the slightly simpler $*$ pseudo-tensor structure. For $I \in \mathcal{S}$ and an I-family L_i of D-modules, as well as a D-module M,

$$P_I^*(\{L_i\}, M) := \mathrm{Hom}(\boxtimes L_i, \Delta_*^{(I)} M).$$

The elements of P_I^* are called $*$ *I-operations* and have the étale local nature: We have a sheaf $\underline{P}_I^*(\{L_i\}, M)$, $U \mapsto P_I^*(\{L_i \mid_U\}, M \mid_U)$ on the étale topology of X. For $\pi : J \twoheadrightarrow I$, the composition

$$P_I^*(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^*(\{K_j\}, L_i)) \to P_J^*(\{K_j\}, M)$$

sending $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ is given by:

$$\boxtimes_J K_j \xrightarrow{\boxtimes \psi_i} \boxtimes_I \Delta_*^{(J_i)} L_i = \Delta_*^{(\pi)}(\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M.$$

The composition is associative, so the P_I^* define an abelian pseudo-tensor structure on $\mathcal{M}(X)$, $\mathcal{M}(X)^*$. The pseudo-tensor categories $\mathcal{M}(U)^*$, $U \in X_{ét}$, form a sheaf of pseudo-tensor categories on the étale topology of X.

Returning to the chiral pseudo-tensor structure, we find the same holds: The composition is also seen to be associative, so the P_I^{ch} define an abelian pseudo-tensor category structure on $\mathcal{M}(X)$, denoted $\mathcal{M}(X)^{ch}$. The pseudo-tensor categories $\mathcal{M}(U)^{ch}$, $U \in X_{ét}$ form a sheaf of pseudo-tensor categories $\mathcal{M}(X_{ét})^{ch}$ on the étale topology of X.

For $M \in \mathcal{M}(X)$ the *unit operation* $\epsilon_M \in P_2^{ch}(\{\omega_X, M\}, M)$ is the composition

$$j_* j^* \omega_X \boxtimes M \to (j_* j^* \omega_X \boxtimes M)/\omega_X \boxtimes M = \Delta_*(\omega_X \otimes^! M) = \Delta_* M, \tag{3.1.2}$$

where the last equality comes from the canonical isomorphism $\omega_X \otimes^! M = M$.

Definition 3.1.3. *Let $\mathcal{L}ie$ be the Lie operad. (So algebras over $\mathcal{L}ie$ are Lie algebras in that category.) A chiral algebra is a $\mathcal{L}ie$ algebra in the pseudo-tensor category $\mathcal{M}(X)^{ch}$. The category of such objects is denoted Lie^{ch}.*

For a Lie^{ch} algebra A, denote by $\mu = \mu_A \in P_2^{ch}(\{A, A\}, A)$ the commutator (the *chiral product*).

Let A be a Lie^{ch} algebra. A *unit* in A is a morphism of D-modules $1 = 1_A : \omega_X \to A$ such that $\mu_A(1, \mathrm{id}_A) = \epsilon_A$ in $P_2^{ch}(\{\omega_X, A\}, A)$. It is unique if it exists, and such a chiral algebra is said to be *unital*.

We will unravel the definitions of (unital) chiral algebras in a later section. We note here that one defines the notion of Lie* algebra, which we refer to at multiple points in order to set the material here in a wider context, analogously to the notion of chiral algebra, but with respect to the $*$ pseudo-tensor structure: namely, a Lie* algebra is a D-module L equipped with a $*$-pairing $[\,] \in P_2^*(\{L, L\}, L)$, the *Lie* bracket*, satisfying skew-symmetry and the Jacobi identity.

3.1.4 Viewing chiral algebras as factorization structures on the Ran space

Possibly the most fundamental result underlying the theory of chiral algebras is the statement that the category of chiral algebras on X is equivalent to the category of factorisation algebras on X ([BD04], Theorem in Section 3.4.9). This is realised as a form of Koszul duality in [FG12]. Later we will construct new examples of chiral algebras in the form of factorisation algebras arising from line bundles on divisors.

We return here to the notion of the Ran space on a topological space X via [BD04]. Given a topological space X, its *exponential* $\mathcal{E}xp(X)$ is the set of all finite subsets of X topologised with the strongest topology such that the obvious maps $r_I : X^I \to \mathcal{E}xp(X)$ are continuous, for all finite sets I. (Hence, the maps r_n, and the equivalent to the Ran space, are open maps.) For $S \subset X$, denote by $[S]$ the corresponding element of $\mathcal{E}xp(X)$. The *Ran space*, $\mathcal{R}(X)$, is the subspace of $\mathcal{E}xp(X)$ not including $[\emptyset]$.

Note that both $\mathcal{E}xp(X)$ and $\mathcal{R}$ have increasing filtrations with $\mathcal{E}xp(X)_n$ (respectively, $\mathcal{R}(X)_n$) those S with $|S| \leq n$. So $\mathcal{R}(X)_0 = \emptyset$, $\mathcal{R}(X)_1 = X$, and $\mathcal{R}(X)_2 = \mathrm{Sym}^2(X)$. More generally, $\mathcal{R}(X)^0_n := \mathcal{R}(X)_n \setminus \mathcal{R}(X)_{n-1}$ is the configuration space of n points in X (which is the complement to the diagonals in $\mathrm{Sym}^n(X)$.

$\mathcal{R}(X)$ can be given as a functor of points via $\mathcal{R}(X)(Z) = \mathcal{R}(X(Z)) = \mathcal{R}(\mathrm{Hom}(Z, X))$.

It is helpful to view $\mathcal{R}(X)$ as an ind-object as follows. (It will be an ind-scheme when X is a curve.) For any surjection $J \twoheadrightarrow I$, we have the diagonal embedding $\Delta^{(J/I)} : X^I \to X^J$. Clearly, $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{E}xp(X)$ (respectively, $\mathcal{R}(X)$) is the inductive limit of the topological spaces X^I over all diagonal embeddings (respectively, over all with $I \neq \emptyset$).

It is easy to see that $\mathcal{R}(X)$ and $\mathcal{E}xp(X)$ are semigroups under $[S_1] \circ [S_2] := [S_1 \cup S_2]$, with $\emptyset$ making $\mathcal{E}xp(X)$ into a monoid.

A factorization algebra on X is a sheaf B of vector spaces on $\mathcal{E}xp(X)$ with natural identifications of the stalks $B_{[S_1]} \otimes B_{[S_2]} \xrightarrow{\sim} B_{[S_1 \circ S_2]}$, for *disjoint* S_1 and S_2, satisfying obvious associativity and commutativity relations. Let B_X denote the restriction of

B to $X \subset \mathcal{E}xp(X)$. Morphisms between factorization algebras are completely determined (via the *factorization structure*) to their restrictions to the B_Xs, and such a sheaf B can be thought of as the sheaf B_X equipped with the data of a factorization structure.

The restriction of B to $\mathcal{E}xp(X)_i^0$ is equal to the restriction of $\mathrm{Sym}^i(B_X)$ to $\mathrm{Sym}^i(X)\setminus$ (diagonals).

3.1.4.1 Factorisation algebras

Now let X be a curve. We will give the definition of factorisation algebras on X, via $\mathcal{O}$-modules and D-modules on the Ran space of X, according to [BD04].

Definition 3.1.5. *An $\mathcal{O}$-module on $\mathcal{R}(X)$, or $\mathcal{O}_{\mathcal{R}(X)}$-module, (respectively, a left D-module on $\mathcal{R}(X)$) is an assignment, for every finite non-empty set I, of a quasi-coherent $\mathcal{O}_{X^I}$-module F_{X^I} (respectively, a left D-module on X^I), and, for every $\pi : J \twoheadrightarrow I$, an isomorphism*

$$\nu^{(\pi)} = \nu_F^{(\pi)} = \nu^{(J/I)} : \Delta^{(\pi)*} F_{X^J} \xrightarrow{\sim} F_{X^I}$$

compatible with the composition of surjections. We also require the condition that the F_{X^I} have no non-zero local sections supported on the diagonal divisor.

There is a k-category of $\mathcal{O}_{\mathcal{R}(X)}$-modules, $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$, which is exact, with short exact sequences defined to be those for which all the corresponding sequences of $\mathcal{O}_{X^n}$-modules are exact. $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ naturally forms a tensor k-category, with unit object $\mathcal{O}_{\mathcal{R}(X)}$, the $\mathcal{O}$-module with components $\mathcal{O}_{X^I}$. Similarly, we have an exact tensor k-category of left D-modules on the Ran space, $\mathcal{M}^l(\mathcal{R}(X))$, with unit $\mathcal{O}_{\mathcal{R}(X)}$.

Now we give the definition of a factorisation structure.

Definition 3.1.6. *Let $\pi : J \twoheadrightarrow I$, and let $U^{[J/I]} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2}$ if $\pi(j_1) \neq \pi(j_2)\}$, with $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ the corresponding open embedding. Let B be an $\mathcal{O}$-module on $\mathcal{R}(X)$.*
A factorisation structure *on B is a rule that assigns to every $J \twoheadrightarrow I$ an isomorphism of $\mathcal{O}_{U^{[J/I]}}$-modules*

$$c_{[J/I]} : j^{[J/I]*}(\boxtimes_I B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^I}.$$

These should be compatible with compositions and the maps ν:

- *Given another map $K \twoheadrightarrow J$, the isomorphism $c_{[K/J]}$ should coincide with the composition of isomorphisms $c_{[K/I]}(\boxtimes c_{[K_i/J_i]})$.*

  Here, the map $K \twoheadrightarrow J$ partitions K according to J as $K = K_{j_1} \cup K_{j_2} \cup \dots K_{j_{|J|}}$, and the map $J \twoheadrightarrow I$ groups together some of the K_i according to those for which the j_l maps to the same element of i, giving a partition of K into fewer groups, each of larger, or equal, size. The maps $c_{[K_i/J_i]}$ group together the X^{K_j} to form each larger group X^{K_i}, then the map $c_{[K/I]}$ groups these to form X^K.

- *For every composition of surjections $J \twoheadrightarrow J' \twoheadrightarrow I$, we have*

$$\nu^{(J/J')}\Delta^{(J/J')*}(c_{[J/I]}) = c_{[J'/I]}(\boxtimes \nu^{(J_i/J_i')}).$$

$\mathcal{O}$-modules on $\mathcal{R}(X)$ equipped with a factorisation structure form a tensor category, although it is not additive.

Now consider the case of $J \twoheadrightarrow J$, i.e. splitting J into a partition of sets of size one, with $j^{(J)}$ the embedding of the complement to the diagonal divisor. By its definition, the sheaf $j_*^{(J)} j^{(J)*} B_{X^J}$ contains, as well as the sections of B_{X^J}, those with poles along the diagonals, so we have an embedding $B_{X^J} \hookrightarrow j_*^{(J)} j^{(J)*} B_{X^J}$. Using the identification from the factorisation structure, $c^{[J/J]} : j^{(J)*} B_X^{\boxtimes J} \xrightarrow{\sim} j^{(J)*} B_{X^J}$, we have a canonical embedding

$$B_{X^J} \hookrightarrow j_*^{(J)} j^{(J)*} B_X^{\boxtimes J}.$$

We see, using these canonical embeddings, that B is determined by the sheaf B_X and the factorisation morphisms, so the functor $B \mapsto B_X$ is faithful.

Definition 3.1.7. *We say that an $\mathcal{O}$-module B with factorisation structure is* unital *if there is a global section $1 = 1_B$ of B_X, such that for every $f \in B_X$, $1 \boxtimes f \in B_{X^2} \subset j_*^{(J)} j^{(J)*} B_X^{\boxtimes 2}$, and $\Delta^*(1 \boxtimes f) = f$. Such an $\mathcal{O}$-module is referred to as a* factorisation algebra*.*

It is easy to see, using the standard uniqueness proofs in algebra, that a *unit*, as described in the above definition, must be unique if it exists.

Factorisation algebras, with unit-preserving morphisms, form a tensor category, denoted $\mathcal{FA}(X)$. They have the X-local nature, so we have a sheaf of categories $\mathcal{FA}(X_{ét})$ on the étale topology of X. This category has unit given by the trivial factorisation algebra $\mathcal{O}$.

3.1.7.1 Factorisation algebras via Cartier divisors

We will require an equivalent notion of factorisation algebra in terms of divisors, which we give here, also from [BD04]. This will require the following definition of the fibred category $\mathcal{C}(X)$, which will be of importance in the remainder of this thesis.

Recall that, for an affine scheme Z, an effective Cartier divisor in $X \times Z/Z$ proper over Z is the same as a subscheme $S \subset X \times Z$ which is finite and flat over Z.

Definition 3.1.8. *The category $\mathcal{C}(X)$, fibred over affine schemes, is defined as follows. Let Z be an affine scheme. Define the fibre over Z, $\mathcal{C}(X)_Z$, to be the set of effective Cartier divisors in $X \times Z/Z$ proper over Z under the equivalence relation that two divisors S and S' are equivalent if $S_{red} = S'_{red}$. This is a category by considering it as the preorder given by $S' \leq S$ if $S'_{red} \subset S_{red}$.*

Given a map of affine schemes $f : Z \to Z'$, we can pull back the subscheme $S' \subset X \times Z'$ to a subscheme $S' \times_{X \times Z'} X \times Z \subset X \times Z$, finite and flat over Z:

$$\begin{array}{ccc} S & \longrightarrow & X \times Z \\ \downarrow & & \downarrow 1 \times f \\ S' & \hookrightarrow & X \times Z' \end{array}$$

So we have a functor $\mathcal{C}(X)_{Z'} \to \mathcal{C}(X)_Z$, giving $\mathcal{C}(X)$ the structure of a fibred category over affine schemes.

A word on how to view this: A divisor of $X \times Z/Z$ proper over Z looks like divisors in X varying in a way that is parametrised by Z. Since X is a curve, a divisor S in X is a $\mathbb{Z}$-valued finite sum of points, with all values non-negative in the case of an effective divisor. The reduced part S_{red} will be the sum of points with single multiplicity. Hence, $S_{red} = S'_{red}$ if and only if $S \subset nS'$ for some $n \in \mathbb{N}$. Thus, any $\mathcal{C}(X)_Z$ is the localisation of the category of Cartier divisors in $X \times Z/Z$ proper over Z and inclusions with respect to the family of all morphisms $S \subset nS$, $n \geq 1$.

We will consider pairs (B, c) (which turn out to be the same as factorisation algebras), such that B is a morphism of fibred categories from $\mathcal{C}(X)$ to that of quasi-coherent $\mathcal{O}$-modules. That is, B assigns to every $S \in \mathcal{C}(X)_Z$ an $\mathcal{O}_Z$-module $B_S = B_{S,Z}$, and for every inclusion $S' \leq S$, we have a morphism $B_{S'} \to B_S$. This should be compatible with the base change, i.e. the maps between the categories over schemes: That is a map $Z \to Z'$ corresponding to a map of rings $f : S' \to S$, $Z = \operatorname{Spec} S$, $Z' = \operatorname{Spec} S'$, should give a functor $\mathcal{C}(Z') \to \mathcal{C}(Z)$, taking an S'-module M' to $M' \otimes_{f(S')} S$, in the structure of the fibred category of $\mathcal{O}$-modules. The assignment B should be compatible with these maps.

Of note is that for each $n \geq 0$ the universal effective divisor of degree n is assigned an $\mathcal{O}_{\operatorname{Sym}^n X}$-module $B_{\operatorname{Sym}^n(X)}$. The universal effective divisor of degree n is the divisor in $X \times \operatorname{Sym}^n X / \operatorname{Sym}^n X$ such that for any divisor $D \in \operatorname{Sym}^n X$, $B_{\operatorname{Sym}^n(X)}$ cuts X over D on exactly D.

c is a rule that assigns to any pair of mutually disjoint divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c_{S_1,S_2} : B_{S_1} \otimes B_{S_2} \xrightarrow{\sim} B_{S_1+S_2}$, in a way that is commutative and associative, and compatible with the morphisms for B and base change.

Assume also that the $\mathcal{O}$-modules $B_{\operatorname{Sym}^n(X)}$ on $\operatorname{Sym}^n(X)$ have no non-zero local sections supported at the discriminant divisor, and that $B_{\operatorname{Sym}^0 X} \neq 0$.

Such pairs (B, c) form a category $\mathcal{FA}(X)'$, with a tensor structure given by $(\otimes B_i)_S = \otimes(B_{iS})$.

Proposition 3.1.9. *There is a canonical equivalence of tensor categories $\mathcal{FA}(X)' \xrightarrow{\sim} \mathcal{FA}(X)$.*

Proof. We give the functors defining the equivalence to sketch the idea of the proof. First we give the functor $F : \mathcal{FA}(X)' \to \mathcal{FA}(X)$.

Let $(B, c) \in \mathcal{FA}(X)'$. Then assign to X^I the $\mathcal{O}_{X^I}$-module labelled by the Cartier divisor S, where S is the union of the subschemes $x = x_i$ of $X \times X^I$, $i \in I$. This is a Cartier divisor of degree $|I|$ in $X \times X^I / X^I$. This module B_S, which we may also refer to as B_{X^I} is the pullback of the universal effective divisor (confusingly denoted $B_{\operatorname{Sym}^{|I|} X}$, but sitting over $\operatorname{Sym}^{|I|} X$) by the projection $p_{|I|} : X^I \to \operatorname{Sym}^{|I|} X$. The maps $v^{(\pi)}$, gluing along the diagonal maps, arise immediately from the localisations along $S_{red} = S'_{red}$, since the diagonal maps keeps the same underlying reduced scheme.

To get the maps $c_{[J/I]} : j^{[J/I]*}(\boxtimes_I B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^I}$—which we will consider here for the case $J = I_1 \sqcup I_2$ for simplicity—we will apply the isomorphisms c_{S_1,S_2} :

$B_{S_1} \otimes B_{S_2} \xrightarrow{\sim} B_{S_1+S_2}$. To do this, we need to pull back the sheaves on divisors S_1 over X^{I_1} and S_2 over X^{I_2} to $X^{J=I_1 \sqcup I_2}$, along the projections $X^J \to X^{I_j}$. This gives divisors over the same scheme, and the isomorphisms $c_{S_1,S_2} : B_{S_1} \otimes B_{S_2} \xrightarrow{\sim} B_{S_1+S_2}$ are precisely those we need: If $\tilde{B_{S_j}}$ denotes the pullback of the sheaf over X^{I_j} to X^J, or, better, $U^{[J/I]}$, then $\tilde{B_{S_1}} \otimes \tilde{B_{S_2}}$ is the pullback of $B_1 \boxtimes B_2$ to $U^{[J/I]}$. Note that, since the divisors S_1 and S_2 are taken to be disjoint, we have a module over $U^{[J/I]}$.

To construct the inverse functor, it suffices to construct the $\mathcal{O}$-modules over $B_{\mathrm{Sym}^n X}$ corresponding to the universal divisors (since the compatibility conditions require that everything be determined by these). Let p_n be the projection $p_n : X^n \to \mathrm{Sym}^n(X)$. The symmetric group Σ_n acts in the obvious ways on X^n and on B_{X^n}. $B_{\mathrm{Sym}^n X}$ is defined to be the invariants of the action on the pushforward of B_{X^n}, namely $(p_{n*}B_{X^n})^{\Sigma_n}$.

It can be shown that the canonical map $p_n^* B_{\mathrm{Sym}^n X} \to B_{X^n}$ is an isomorphism: Thus, this does give an inverse to the functor $\mathcal{FA}(X)' \to \mathcal{FA}(X)$, after extending to general divisors.

So let S be a Cartier divisor in $X \times Z/Z$ proper over Z. The problem is Z-local, so we may assume the degree n of S over Z is constant. Then such as S amounts to a morphism $Z \to \mathrm{Sym}^n X$, and, necessarily, $B_{S,Z}$ is the pullback of $B_{\mathrm{Sym}^n X}$.

Likewise, to define the maps $B_{S'} \to B_S$ for $S' \subset S$, it suffices to define these for the universal divisors. Note that, if $S' \subset S$, then $S = S' + T$ for some effective Cartier divisor T. Let $Z = \mathrm{Sym}^n X \times \mathrm{Sym}^m X$, and S, T be the universal divisors of orders n and m respectively. Let $q^{(n)} : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^n X$ be the projection, and let $p : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{n+m} X$ be the addition map. The structure morphism $\nu^{(\pi)}$ for $\pi : \{1, \dots, n\} \hookrightarrow \{1, \dots, n+m\}$ gives a $\Sigma_n \times \Sigma_m$-equivariant morphism $B_{X^n} \boxtimes \mathcal{O}_{X^m} \to B_{X^{n+m}}$, so using the isomorphisms $p_n^* B_{\mathrm{Sym}^n X} \to B_{X^n}$, we get a map $q^{(n)*} B_{\mathrm{Sym}^n X} \to p^* B_{\mathrm{Sym}^{n+m}} X$ as required.

□

3.2 θ-data

Lattice chiral algebras provide an example of non-commutative chiral algebras that do not arise as enveloping algebras of Lie*-algebras. We will give a detailed explication of these in Chapter 4, from which we will obtain our new examples of chiral algebras. Here, we define combinatorial information referred to as θ-data, which we will see later gives a combinatorial characterisation of lattice chiral algebras via the equivalence of the Picard groupoid of θ-data with that of super line bundles with factorisation on Γ-valued divisors (Proposition 4.1.11).

Let X be a smooth curve. In the following, Γ will denote a lattice (a free abelian group of finite rank), $\Gamma^\vee$ the dual lattice, $\mathrm{Hom}(\Gamma, \mathbb{Z})$, and $\langle , \rangle : \Gamma \times \Gamma^\vee \to \mathbb{Z}$ the dual pairing. Denote the corresponding tori by $T := \mathbb{G}_m \otimes \Gamma = \mathrm{Spec}(k[\Gamma^\vee])$ ($\simeq (\mathbb{G}_m)^n$), for the lattice $\mathbb{Z}^n$, via a choice of basis) and $T^\vee := \mathbb{G}_m \otimes \Gamma^\vee = \mathrm{Spec}(k[\Gamma])$. Recall here that the regular function ring $k[\Gamma]$ is the group algebra on the lattice Γ with basis the group elements, i.e. it looks like k-linear sums of products $x_1^{k_1} \cdots x_n^{k_n}$, $k_i \in \mathbb{Z}$ for all i, where each x_i denotes the choice of ith basis element of the lattice $\mathbb{Z}^n$. Let $\mathrm{Tors}(X, T)$ and $\mathrm{Tors}(X, T^\vee)$ denote the Picard groupoids of T-and $T^\vee$-torsors on X, respectively.

We refer the reader to Definition 4.1.9 for the definition of *super line bundle*. In the following, X is considered as the functor $\mathrm{hom}(-, X)$ on AffSch, or, equivalently for $X = \mathrm{Spec}\, S$ affine, $\mathrm{hom}(S, -)$ on $\mathrm{Alg_k}$, and, by definition, a super line bundle on X is an assignment to each pair $(X(Z), s \in X(Z))$, for $Z = \mathrm{Spec}\, S'$, of an odd or even S'-module (a quasi-coherent sheaf) that is a line bundle, with appropriate compatibilities.

Definition 3.2.1. *A* θ-datum *for* Γ *on* X *is a triple* $\theta = (\kappa, \lambda, c)$*, where* $\kappa : \Gamma \times \Gamma \to \mathbb{Z}$ *is a symmetric bilinear form,* λ *assigns to each* $\gamma \in \Gamma$ *a super line bundle on* X*,* λ^γ*, and* c *assigns to any* $\gamma_1, \gamma_2 \in \Gamma$ *an isomorphism* $c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2} \otimes \omega^{\otimes \kappa(\gamma_1,\gamma_2)}$*.* *c should satisfy the following:*

1. *(Associativity) The isomorphisms* $c^{\gamma_1,\gamma_2+\gamma_3}(\mathrm{id}_{\lambda^{\gamma_1}} \otimes c^{\gamma_2,\gamma_3})$ *and* $c^{\gamma_1+\gamma_2,\gamma_3}(c^{\gamma_1,\gamma_2} \otimes \mathrm{id}_{\lambda^{\gamma_3}}) : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \lambda^{\gamma_3} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2+\gamma_3} \otimes \omega^{\kappa(\gamma_1,\gamma_2)+\kappa(\gamma_2,\gamma_3)+\kappa(\gamma_1,\gamma_3)}$ *coincide.*

2. *((Twisted) commutativity)* $c^{\gamma_1,\gamma_2} = (-1)^{\kappa(\gamma_1,\gamma_2)} c^{\gamma_2,\gamma_1}\sigma$*, where* σ *is the commutativity constraint: That is, given two line bundles* l_1 *and* l_2 *with parities in* $\{0, 1\}$*, and* $v \in l_1, w \in l_2$*,* $\sigma_{v,w} : v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$*, where* $\deg(v)$ *denotes the parity of* v*.*

Taking, duplicated, the single point v in a line bundle λ^γ in the second condition gives that $(-1)^{\kappa(\gamma,\gamma)}(-1)^{\deg(v)\deg(v)} = 1$, i.e. $\kappa(\gamma,\gamma) \equiv \deg(v) \bmod 2$.

The datum $\theta = (\kappa, \lambda, c)$ will sometimes be referred to as a *κ-twisted* θ-datum, $\theta^\kappa = (\lambda, c)$. The θ-data form a Picard groupoid $\mathcal{P}^\theta(X,\Gamma)$ with the product $(\kappa,\lambda,c) \otimes (\kappa',\lambda',c') = (\kappa+\kappa', \lambda\otimes\lambda', c\otimes c')$, where $(\lambda\otimes\lambda')^\gamma := \lambda^\gamma \otimes \lambda'^\gamma$ and $(c\otimes c')^{\gamma_1,\gamma_2} := c^{\gamma_1,\gamma_2} \otimes c'^{\gamma_1,\gamma_2}$. Write $\mathcal{P}^\theta(X,\Gamma)^\kappa$ for the κ-twisted θ-data.

3.3 Another Approach: Extension to Higher Dimensional Varieties

In [FG12], the authors extend the theory of chiral algebras to higher dimensional varieties, by developing a homotopy theory for chiral and factorization structures analogous to Quillen's homotopy theory of differential graded Lie algebras. They also realise the equivalence between higher dimensional chiral and factorization algebras as a chiral form of Koszul duality.

In the following, X will be a scheme of finite type.

3.3.0.1 Constructing chiral algebras

Chiral algebras according Francis and Gaitsgory ([FG12]) are (a subcategory of) Lie algebra objects in the category of D-modules on the Ran space of X equipped with the chiral monoidal structure. Intuitively, the Ran space of X is the set of all non-empty finite subsets, i.e. the union of all configuration spaces of finitely many unordered points in X, topologised so that points collide on moving between different spaces: that is, $X^I \to \operatorname{Ran} X$ is continuous for all I. Formally, it is defined as the direct limit in prestacks of the diagram of schemes $I \mapsto X^I$ over $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}}$. The functor defining $\operatorname{Ran} X$ is not representable by a scheme or an ind-scheme. (In fact, it is a pseudo-indscheme. The problem here is that it is not a *filtered* colimit.) However, we can define D-modules on this functor $\mathfrak{D}(\operatorname{Ran} X)$ as an appropriate limit in presentable stable ∞-categories. That is, an object of $\mathfrak{D}(\operatorname{Ran} X)$ will be a collection of objects $M^I \in \mathfrak{D}(X^I)$ for each finite set I and a homotopy equivalence $\Delta(\pi)^!(M^I) \simeq M^J$ for every surjection $\pi : I \twoheadrightarrow J$, with $\Delta^\pi : X^J \to X^I$ the corresponding map. This is the expected definition extending the functor assigning D-modules to schemes to a direct limit of a diagram of schemes.

Denote by $\mathrm{fSet}^{\mathrm{surj}}$ the category of non-empty finite subsets with surjective functions. ∞-Cat will denote the $(\infty, 1)$-category of ∞-categories, and $\infty\text{-Cat}^{\mathrm{st}}$ the non-full subcategory of ∞-Cat consisting of stable ∞-categories and exact functors. $\infty\text{-Cat}_{\mathrm{pres}}$ will be the full subcategory of ∞-Cat consisting of presentable ∞-categories, with $\infty\text{-Cat}_{\mathrm{pres,L}} \subset \infty\text{-Cat}_{\mathrm{pres}}$ the non-full subcategory with functors restricted to those commuting with colimits. Note that a functor in $\infty\text{-Cat}_{\mathrm{pres}}$ preserves colimits if and only if it admits a right adjoint. $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres}}$ will be the full subcategory of $\infty\text{-Cat}^{\mathrm{st}}$ with objects presentable stable ∞-categories, i.e. the preimage of $\infty\text{-Cat}_{\mathrm{pres}} \subset \infty\text{-Cat}$ under the forgetful functor $\infty\text{-Cat}^{\mathrm{st}} \to \infty\text{-Cat}$. Further, $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,L}}$ will be the non-full subcategory of $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres}}$ equal to the preimage of $\infty\text{-Cat}_{\mathrm{pres,L}} \subset \infty\text{-Cat}_{\mathrm{pres}}$ under this forgetful functor, and $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ will also denote $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,L}}$.

We will define $\mathfrak{D}(\mathrm{Ran}\, X)$ by constructing several functors $\mathrm{Sch} \to \infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ or $\mathrm{Sch}^{\mathrm{op}} \to \infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ and then defining monoidal structures on $\mathfrak{D}(\mathrm{Ran}\, X)$.

Let X be a separated scheme of finite type. If X is smooth, assign to X the stable ∞-category $\mathfrak{D}(X)$ as follows: Let $\mathfrak{D}(X)^{\heartsuit}$ be the abelian category of right D-modules on X. Take the DG quotient of the DG category of complexes over $\mathfrak{D}(X)^{\heartsuit}$ by the subcategory of acyclic complexes, and canonically form the corresponding simplicial category. (Methods of doing this are given in [Tab10] and [Coh13].) $\mathfrak{D}(X)$ is the ∞-category associated to this, as described in the previous section. The resulting ∞-category is cocomplete and compactly generated.

In the case that X is not smooth, one can define the category of right D-modules over the ring of differential operators on X by locally embedding X into a smooth scheme and using Kashiwara's theorem. (See [BD04], Section 2.1.3).

There are two ways to directly extend this assignment to a functor from Sch or $\mathrm{Sch}^{\mathrm{op}}$ to $\infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. Let $f : X \to Y$ in Sch. To obtain a functor $\mathfrak{D}^* : \mathrm{Sch} \to \infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$, one can define the functor corresponding to f $\mathfrak{D}(X) \to \mathfrak{D}(Y)$ to be the D-module pushforward at the level of homotopy categories. Alternatively, to obtain a functor $\mathfrak{D}^! : \mathrm{Sch}^{\mathrm{op}} \to \infty\text{-Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$, one takes the functor $\mathfrak{D}(Y) \to \mathfrak{D}(X)$ to be the D-module pullback $f^!$ at the level of homotopy categories. Francis and Gaitsgory define these by considering the category of schemes with correspondences and defining these functors on non-full subcategories, as explicated in the following.

The $(2,1)$-category $\mathrm{Sch}^{\mathrm{corr}}$ has objects schemes of finite type and morphisms in $\mathrm{Hom}_{\mathrm{Sch}^{\mathrm{corr}}}(Y_1, Y_2)$ correspondences

$$Y_1 \xleftarrow{f^l} Z \xrightarrow{f^r} Y_2,$$

denoted (f_l, Z, f_r). Maps in the groupoid $\mathrm{Hom}_{\mathrm{Sch}^{\mathrm{corr}}}(Y_1, Y_2)$ are defined naturally. Compositions of correspondences are given by taking pullbacks, and identities $Y \to Y$ occur when f_l and f_r are both isomorphisms. $\mathrm{Sch}^{\mathrm{corr}}$ has a natural symmetric monoidal structure given by products. It also has non-full subcategories Sch^* and $\mathrm{Sch}^!$ equivalent to Sch and $\mathrm{Sch}^{\mathrm{op}}$ respectively, and formed by taking f^l (respectively, f^r) to be an isomorphism.

This notation Sch^* and $\mathrm{Sch}^!$ comes from the following symmetric monoidal functor $\mathfrak{D}^\blacklozenge : \mathrm{Sch}^{\mathrm{corr}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ assigning to X the category $\mathfrak{D}(X)$ and to a map $Y_1 \xleftarrow{f^l} Z \xrightarrow{f^r} Y_2$ the functor $\mathfrak{D}(Y_1) \to \mathfrak{D}(Y_2)$ given by $f_\blacklozenge := (f^r)_* \circ (f^l)^!$. Up to homotopy theory, functoriality is give by the base change theorem:
For a Cartesian square in Sch

$$\begin{array}{ccc} Y' & \xrightarrow{g_Y} & Y \\ \pi' \downarrow & & \downarrow \pi \\ X' & \xrightarrow[g_X]{} & X \end{array}$$

there is a canonical homotopy equivalence $g^! \circ \pi_* \simeq \pi'_* \circ g^!_Y$.

Restricting $\mathfrak{D}^\blacklozenge$ to Sch^* and $\mathrm{Sch}^!$ gives the aforementioned symmetric monoidal functors $\mathfrak{D}^*$ and $\mathfrak{D}^!$.

Let $X^{\mathrm{fSet}^{\mathrm{surj}}}$ denote the functor $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Sch}$ given by $I \rightsquigarrow X^I$. The functor $\mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}}) : \mathrm{fSet}^{\mathrm{surj}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ is defined by pre-composing $\mathfrak{D}^! : \mathrm{Sch}^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ with the functor $X^{\mathrm{fSet}^{\mathrm{surj}}}$. The ∞-category $\mathfrak{D}(\mathrm{Ran}\, X)$ is the limit of $\mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}})$ in $\infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$.

Definition 3.3.1. *The ∞-category $\mathfrak{D}(\mathrm{Ran}\, X)$ is the limit of $\mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}})$ in $\infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$.*

It will be useful to consider $\mathfrak{D}(\operatorname{Ran} X)$ as a colimit, which we can do as follows. As with the definition of $\mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}})$, we may define a functor $\mathfrak{D}^*(X^{\mathrm{fSet}^{\mathrm{surj}}}) : (\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$ given by the composition $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \xrightarrow{X^{\mathrm{fSet}^{\mathrm{surj}}}} \mathrm{Sch} \xrightarrow{\mathfrak{D}^*} \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. Apply the following result:
Let K be a small category, and let $\Phi : K \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres}}$ be a functor. Assume that for every arrow $\alpha : k_1 \to k_2$ in K, the corresponding functor $\Phi_\alpha : \Phi_{k_1} \to \Phi_{k_2}$ admits a left adjoint (which is automatically a 1-morphism in $\infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$). Then the assignment

$$i \rightsquigarrow \Phi_i, (\alpha : k_1 \to k_2) \rightsquigarrow (\Phi_\alpha)^L$$

extends to a functor $\Phi^L : K^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. Further, there is a canonical equivalence $\lim_K \Phi \simeq \operatorname{colim}_{K^{\mathrm{op}}} \Phi^L$.

Francis and Gaitsgory show that the D-module theory given via correspondences satisfies the $(i_*, i^!)$ adjunction, for i a closed embedding, and the $(j^!, j_*)$ adjunction, for j an open embedding. Thus, in the case of the functor $\Phi = \mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}})$ and $K = \mathrm{fSet}^{\mathrm{surj}}$, we get that

$$\mathfrak{D}(\operatorname{Ran} X) := \lim_{\mathrm{fSet}^{\mathrm{surj}}} \mathfrak{D}^!(X^{\mathrm{fSet}^{\mathrm{surj}}}) = \operatorname{colim}_{(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}}} \mathfrak{D}^*(X^{\mathrm{fSet}^{\mathrm{surj}}}).$$

For a finite set I, $(\Delta^I)^!$ will denote the tautological functor $\mathfrak{D}(\operatorname{Ran} X) \to \mathfrak{D}(X^I)$ corresponding to evaluation on I, and $(\Delta^I)_*$ will be the tautological functor $\mathfrak{D}(X^I) \to \mathfrak{D}(\operatorname{Ran} X)$, which is left adjoint to $(\Delta^I)^!$. When $I = \mathrm{pt}$, write $(\Delta^{\mathrm{main}})^! := (\Delta^I)^!$ and $(\Delta^{\mathrm{main}})_* := (\Delta^I)_*$.

Monoidal structures on $\mathfrak{D}(\operatorname{Ran} X)$

To construct monoidal products on $\mathfrak{D}(\operatorname{Ran} X)$, we use a result that if K is a small symmetric monoidal category, and $\mathcal{A}$ is a symmetric monoidal category closed under colimits, with $\Psi : K \to \mathcal{A}$ a right lax symmetric monoidal functor, then $\operatorname{colim}_K \Psi \in \mathcal{A}$ is a commutative algebra object in $\mathcal{A}$. So it suffices to define monoidal structures on $\mathrm{fSet}^{\mathrm{surj}}$ and right lax monoidal functors to $\mathrm{Sch}^{\mathrm{corr}}$, and use the composition with $\mathfrak{D}_\blacklozenge$ to extend these to right lax monoidal functors to $\infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. Then $\mathfrak{D}(\operatorname{Ran} X)$ will be this commutative algebra object, by the above.

We explicate here this product on such a colimit. Let X be of the form $\operatorname{colim}_K \Psi(k)$, where K, Ψ, and $\mathcal{A}$ are as above. Then we require a map $X \otimes X \to X$. The natural transformation in the definition of the lax monoidal structure on $\mathcal{A}$ gives a map of diagrams between $\Psi \otimes \Psi \to \Psi \circ (- \otimes -)$. This induces a map between colimits $\mathrm{colim}_{K\times K}(\Psi(k) \otimes \Psi(k')) \to \mathrm{colim}_{K\times K}(\Psi(l) \otimes \Psi(l'))$. The diagram of objects $\Psi(l) \otimes \Psi(l')$ and corresponding maps over $K \times K$ forms part of the diagram for the $\mathrm{colim}_K \Psi(k)$, giving an induced map $\mathrm{colim}_{K\times K}(\Psi(l \otimes l')) \to \mathrm{colim}_K \Psi(k)$. This gives a map (the composition) $\mathrm{colim}_{K\times K}(\Psi(k) \otimes \Psi(k')) \to \mathrm{colim}_K \Psi(l)$. By definition of a monoidal structure, we have a map $\Psi \times \Psi \to \Psi \otimes \Psi$, inducing a map $\mathrm{colim}_{K\times K} \Psi(k) \times \Psi(k') \to \mathrm{colim}_{K\times K} \Psi(k) \otimes \Psi(k')$, which we can precompose the above composition with. We want a map $\mathrm{colim}_K \Psi(k) \times \mathrm{colim}_K \Psi(k) \to \mathrm{colim}_K \Psi(k)$. To obtain this from the above, we use the fact that, in our case of $\mathcal{A} = \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$, $\mathcal{A}$ is a closed symmetric monoidal $(\infty, 1)$-category (an analogue of the notion of closed symmetric monoidal category). The existence of the right adjoint $(\infty, 1)$-functor to the (tensor) product means that this functor preserves colimits: in particular, $\mathrm{colim}_K \Psi(k) \times \mathrm{colim}_K \Psi(k) \simeq \mathrm{colim}_{K\times K}(\Psi(k) \times \Psi(k'))$.

The monoidal structure on $\mathrm{fSet}^{\mathrm{surj}}$ will be given by taking disjoint unions.

We will define two right lax symmetric monoidal functors: $(\mathrm{X}^{\mathrm{fSet}^{\mathrm{surj}}})^\star$ and $(\mathrm{X}^{\mathrm{fSet}^{\mathrm{surj}}})^{\mathrm{ch}}$: $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{corr}}$, respectively. The compositions with $\mathfrak{D}_\blacklozenge$ will define the right lax monoidal functors to $\infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. $(\mathrm{X}^{\mathrm{fSet}^{\mathrm{surj}}})^\star$ will be the assignment

$$X^{\mathrm{fSet}^{\mathrm{surj}}} : (\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Sch} \simeq \mathrm{Sch}^* \hookrightarrow \mathrm{Sch}^{\mathrm{corr}},$$

with the natural monoidal structure, which is actually monoidal, rather than merely right lax monoidal ($X^I \times X^J \simeq X^{I\sqcup J}$). To define the chiral product, we consider surjections in $\mathrm{fSet}^{\mathrm{surj}}$, equivalently, as collections I_J of finite sets parametrised by another finite set $J : j \rightsquigarrow I_j$. (So, $\sqcup_{j\in J} I_j =: I \overset{\pi}{\twoheadrightarrow} J$.) Let $U(\pi) \subset X^I$ be the open subset of points such that $x_{i_1} \neq x_{i_2}$ if $\pi(i_1) \neq \pi(i_2)$, and let $j(\pi) : U(\pi) \hookrightarrow X^I$ be the open embedding (of the complement to the diagonal). $(\mathrm{X}^{\mathrm{fSet}^{\mathrm{surj}}})^{\mathrm{ch}}$ will be the same underlying functor as $(\mathrm{X}^{\mathrm{fSet}^{\mathrm{surj}}})^\star$, but with lax symmetric monoidal structure maps $\prod_{j\in J} X^{I_j} \to X^I \in \mathrm{Sch}^{\mathrm{corr}}$ given by the correspondence

$$\prod_{j\in J} X^{I_j} \overset{j(\pi)}{\leftarrow} U(\pi) \overset{j(\pi)}{\to} X^I.$$

Under the functor $\mathfrak{D}_\blacklozenge$, this gives two right lax symmetric monoidal functors $(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}_{\mathrm{pres,cont}}$. Such functors are each given by a natural transformation (part of the data of a right lax monoidal functor)

$$\left(I_J \rightsquigarrow \otimes_{j\in J} \mathfrak{D}(X^{I_j})\right) \Rightarrow \left(I_J \rightsquigarrow \mathfrak{D}(X^{\sqcup I_j})\right),$$

since the right hand side is the image under the composition of functors of the tensor product in $\mathrm{fSet}^{\mathrm{surj}}$. In the case of the $\star$ symmetric monoidal structure $\otimes^\star$ on $\operatorname{Ran} X$, the required natural transformation will be the external tensor product:

$$\left(M^{I_j} \in \mathfrak{D}(X^{I_j})\right) \rightsquigarrow \left(\boxtimes_j M^{I_j} \in \mathfrak{D}(X^I)\right).$$

Now, let $M_j \in \mathfrak{D}(\operatorname{Ran} X)$, $j \in J$, and let $\pi : I \twoheadrightarrow J$. As explained above, the map $\otimes^\star : \mathfrak{D}(\operatorname{Ran} X) \times \mathfrak{D}(\operatorname{Ran} X) \to \mathfrak{D}(\operatorname{Ran} X)$ is induced by the map $\operatorname{colim}_{K\times K}(\Psi(k) \times \Psi(k')) \to \operatorname{colim}_{K\times K} \Psi(k) \otimes^\star \Psi(k') \to \Psi(k \otimes k')$, i.e. it factors through the diagram of maps $\Psi(k) \times \Psi(k') \to \Psi(k) \otimes^\star \Psi(k')$, using the fact that $\operatorname{colim}_{K\times K}(\Psi(k) \times \Psi(k')) \simeq \operatorname{colim}_K \Psi(k) \times \operatorname{colim}_K \Psi(k)$. Since we have a map $\mathfrak{D}(X^{I_1}) \times \cdots \times \mathfrak{D}(X^{I_{|J|}})$ into $\mathfrak{D}(\operatorname{Ran} X) \times \ldots \mathfrak{D}(\operatorname{Ran} X)$, composing this with $\otimes^\star$ on $\mathfrak{D}(\operatorname{Ran} X)$ will factor it through the map $\mathfrak{D}(X^{I_1}) \times \cdots \times \mathfrak{D}(X^{I_{|J|}}) \to \mathfrak{D}(X^{I_1}) \otimes^\star \cdots \otimes^\star \mathfrak{D}(X^{I_{|J|}})$, which will give a map $\boxtimes_{j\in J}\left((\Delta^{I_j})^!(M_j)\right) \to \otimes^\star_{j\in J} M_j$, mapping into the component over X^I. That is, we have a canonical map

$$\boxtimes_{j\in J}\left((\Delta^{I_j})^!(M_j)\right) \to (\Delta^I)^!\left(\otimes^\star_{j\in J} M_j\right).$$

Basically, all this says is that we defined the monoidal product $\otimes^\star$ on $\mathfrak{D}(\operatorname{Ran} X)$ to commute with the diagrams of colimits and tensor products.

Similarly, the chiral symmetric monoidal structure $\otimes^{\mathrm{ch}}$ is given by

$$\left(M^{I_j} \in \mathfrak{D}(X^{I_j})\right) \rightsquigarrow \left(j(\pi)_* \circ j(\pi)^* \left(\boxtimes_j M^{I_j}\right) \in \mathfrak{D}(X^I)\right).$$

As with $\otimes^\star$, for objects $M_j \in \mathfrak{D}(\operatorname{Ran} X)$, $j \in J$, and a finite set I and surjection $\pi : I \twoheadrightarrow J$, there is a canonical map

$$j(\pi)_* \circ j(\pi)^* \left(\boxtimes_{j\in J}\left((\Delta^{I_j})^!(M_j)\right)\right) \to (\Delta^I)^!\left(\otimes^{\mathrm{ch}}_{j\in J} M_j\right).$$

Lemma 3.3.2. *For $M_j \in \mathfrak{D}(\operatorname{Ran} X)$, $j \in J$ and I and J as in the set up above, the resulting map*

$$\oplus_\pi j(\pi)_* \circ j(\pi)^* \left(\boxtimes_{j\in J}\left((\Delta^{I_j})^!(M_j)\right)\right) \to (\Delta^I)^!\left(\otimes^{\mathrm{ch}}_{j\in J} M_j\right)$$

is a homotopy equivalence, where the direct sum is taken over all surjections $\pi : I \twoheadrightarrow J$.

Chiral and factorisation algebras

Definition 3.3.3. *Let* $\text{Lie-alg}^{\text{ch}}(\operatorname{Ran} X)$ *and* $\text{Lie-alg}^{\star}(\operatorname{Ran} X)$ *be the* ∞*-categories of Lie algebras in the* ∞*-category* $\mathfrak{D}(\operatorname{Ran} X)$ *equipped with the chiral and* $\star$ *symmetric monoidal structures, respectively. The* ∞*-categories of chiral Lie and* $\star$*-Lie algebras on* X *are the full* ∞*-subcategories*

$$\text{Lie-alg}^{\text{ch}}(X) \subset \text{Lie-alg}^{\text{ch}}(\operatorname{Ran} X) \textit{ and } \text{Lie-alg}^{\star}(X) \subset \text{Lie-alg}^{\star}(\operatorname{Ran} X),$$

respectively, spanned by objects for which the underlying D-module is supported on X*: that is, it lies in the essential image of the functor* $(\Delta^{\text{main}})_* : \mathfrak{D}(X) \to \mathfrak{D}(\operatorname{Ran} X)$*.*

Beilinson and Drinfeld refer to chiral Lie algebras as *chiral algebras* and to $\star$-Lie algebras as *Lie***-algebras.*

We will denote by $\text{Com-coalg}^{\text{ch}}(\operatorname{Ran} X)$ the ∞-category of (non-unital) chiral commutative coalgebras for the chiral monoidal structure on $\mathfrak{D}(\operatorname{Ran} X)$. By Lemma 3.3.2, if $B \in \text{Com-coalg}^{\text{ch}}(\operatorname{Ran} X)$ and $\pi : I \twoheadrightarrow J$ is a surjection, we have a map

$$(\Delta^J)^!(B) \to j(\pi)_* \circ j(\pi)^* \left(\boxtimes_{j\in J} \left((\Delta^{I_j})^!(B)\right)\right),$$

and by adjunction a map

$$j(\pi)^* \left((\Delta^J)^!(B)\right) \to j(\pi)^* \left(\boxtimes_{j\in J} \left((\Delta^{I_j})^!(B)\right)\right). \tag{3.3.1}$$

Definition 3.3.4. *Such a* $B \in \text{Com-coalg}^{\text{ch}}(\operatorname{Ran} X)$ *is a* factorisation coalgebra *if the above map (3.3.1) is a homotopy equivalence for all* I *and* π*.*

Write $\text{Fact}(X) := \text{Com-coalg}^{\text{ch}}_{\text{Fact}}(\operatorname{Ran} X)$ for the full subcategory of $\text{Com-coalg}^{\text{ch}}(\operatorname{Ran} X)$ spanned by factorisation coalgebras. $\text{Fact}(X)$ corresponds to Beilinson and Drinfeld's $\mathcal{FA}(X)$, with factorisation coalgebras their notion of factorisation algebras.

3.4 Loop Groups and the Affine Grassmannian

The construction of lattice chiral algebras due to Beilinson and Drinfeld that we will proceed to generalise in this work is centred around the chiral monoid of Γ-valued divisors, for Γ a lattice. One can describe this chiral monoid, instead, as that of the affine Grassmannian, under the relationship between Γ-valued divisors and T-torsors. Another way to view the affine Grassmannian is as the fpqc quotient of loop groups. We describe here the affine Grassmannian, together with some properties of loop group functors. The affine Grassmannian is widely discussed in the literature: For example, [BD91], [Góm10],

Definition 3.4.1. *Let X be a scheme. The loop functor $X((t))$ is the functor from commutative k algebras to sets given by*

$$X((t))(A) := \mathrm{Hom}(\mathrm{Spec}(A((t))), X),$$

where $A((t)) := A \otimes k((t))$, and $k((t))$ denotes Laurent series in t with only finitely many non-zero terms in negative powers of t.

In [GR09], the loop functor $X((t))$, X a scheme (affine, of finite type), is shown to be representable by an ind-scheme, through considering the truncated arc functors $X(k[t]/t^n)$. Here we show that, for an algebraic group G, $G[t, t^{-1}]$ is an ind-scheme. The functors $G((t))$, $G[t]$ and $G[[t]]$ are quotients or completions of $G[t, t^{-1}]$ and may be understood in relation to this.

Definition 3.4.2. *Let X be a scheme. The functor $X[t, t^{-1}]$ is the functor from commutative k algebras to sets given by*

$$X[t, t^{-1}](A) := \mathrm{Hom}(\mathrm{Spec}(A[t, t^{-1}]), X),$$

where $A[t, t^{-1}] := A \otimes k[t, t^{-1}]$.
Analogously, the functors $X[[t]]$, $X[t]$, and $X(k[t]/t^n)$ are defined as sending an algebra A to $\mathrm{Hom}(\mathrm{Spec}(A[[t]]), X)$, $\mathrm{Hom}(\mathrm{Spec}(A[t]), X)$ and $\mathrm{Hom}(\mathrm{Spec}(A \otimes k[t]/t^n), X)$ respectively.

In [GR09], it is shown that the truncated arcs functors $X(k[t]/t^n)$, and the arcs functor $X[[t]]$, are representable by schemes, and consequently that the loops functor $\mathrm{Hom}(\mathrm{Spec}(A((t))), X)$ is representable by an ind-scheme when X is affine. (Through identifying representable functors with schemes, we may refer to these simply as *being* schemes and ind-schemes respectively.) Here we show that the loops $X[t, t^{-1}]$ are also given by an ind-scheme.

Proposition 3.4.3. *Let X be an affine scheme of finite type. Then $X[t, t^{-1}]$ is an ind-scheme.*

Remark
Note that by Lemma 2.4.2, part 2, an affine scheme is of finite type if and only if it is of the form $\mathrm{Spec}(k[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n))$ (since the map $f : X = \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ must be of finite type, and $X = \mathrm{Spec}(A) \subseteq f^{-1}(\mathrm{Spec}(k))$ is one such affine open set mapping to an affine open of $\mathrm{Spec}(k)$).

Proof. Firstly, $\mathrm{Hom}(R, -)$ is left exact so preserves finite limits, and any affine scheme of finite type is a finite limit of copies of the affine space $\mathbb{A}^1$, so it suffices to show the result holds for $X = \mathbb{A}^1$. But, when $X = \mathbb{A}^1$, we have the following:

$$X[t, t^{-1}](A) = \mathrm{Hom}(\mathrm{Spec}(A[t, t^{-1}]), \mathbb{A}^1) = \mathrm{Hom}(k[t], A[t, t^{-1}]) \simeq A[t, t^{-1}].$$

We can understand this last term, $A[t, t^{-1}]$, in the following way. For any n, $\mathrm{Hom}(k[x_1, ..., x_n], A)$ picks out n elements of A. As sets, $A[t, t^{-1}]$ is a union over all finite m of all possible choices of m elements of A (as coefficients in the Laurent polynomials). But for a partially ordered system of sets under inclusion, the inductive limit is equal to the union. $A[t, t^{-1}] = \varinjlim \mathrm{Hom}(k[x_1, \ldots, x_n, y_1, \ldots, y_n], A) = \varinjlim \mathrm{Hom}(\mathrm{Spec}(A), \mathbb{A}^{2n})$ an inductive limit of representable presheaves.

Given the above remark, it is easy to see that one can write an affine scheme of finite type as a fibre product in the following way:

$$\begin{array}{ccc}
\mathrm{Spec}\left(\frac{k[x_1,\ldots,x_n]}{(f_1(x_1,\ldots,x_n),\ldots,f_r(x_1,\ldots,x_n))}\right) & \xrightarrow{i} & \mathbb{A}^n \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle (f_1(x_1,\ldots,x_n),\ldots,f_r(x_1,\ldots,x_n))} \\
* & \xrightarrow{i=0} & \mathbb{A}^r
\end{array}$$

For a fixed scheme Y, $\mathrm{Hom}(Y, -)$ preserves (inverse) limits, and a pullback is an inverse limit. Further, affine n-space is a trivial pullback (direct product) of copies of $\mathbb{A}^1$, so we can pull the limit out to get a limit of affine schemes, which is also an affine scheme. Thus, we can carry the result across and get that $X[t, t^{-1}]$ is an ind-scheme whenever X is an affine scheme of finite type.

□

Proposition 3.4.4. *Let X be an affine scheme of finite type, as above. Then $X[[t]]$ is a pro-object in affine schemes of finite type.*

Proof. As before, if $X = \mathbb{A}^1$ we have:

$$X[[t]](A) = \mathrm{Hom}(\mathrm{Spec}(A[[t]]), \mathbb{A}^1) = \mathrm{Hom}(k[t], A[[t]]) \simeq A[[t]].$$

We fill in the details, which are omitted in [GR09]. $A[[t]] = \lim_n A[t]/(t^n)$, where the limit is taken over the cofiltered diagram with truncation maps between polynomials of differing lengths, and the maps from power series are likewise truncation maps. As given in [GR09], $A[t]/(t^n) \simeq A^{\oplus n} \simeq \mathbb{A}^n(A)$. Thus, we see that

$\mathbb{A}^1[[t]](A) = \lim_i \mathbb{A}^n(A)$, a pro-object in affine schemes, which is, again, an affine scheme. One moves from the case of $\mathbb{A}^1$ to a general affine scheme of finite type exactly as above.

□

Definition 3.4.5. *The affine Grassmannian is the fpqc quotient $G(K)/G(O)$, where $O = \mathbb{C}[[t]]$ and $K = \mathbb{C}((t))$.*

This is naturally isomorphic to the functor $S \mapsto$

$$\{P, \alpha : P \text{ is a principal } G \text{ bundle on } X \times S \text{ and}$$

$$\alpha \text{ is an isomorphism between } P|_{X\setminus x\times S} \text{ and the trivial bundle on } X \setminus x \times S\},$$

for some (any) $x \in X$.

To see this, [GR09] argues (in a version of the Beauville-Laszlo theorem for vector bundles, given in [BL95]) that for Noetherian commutative k-algebras A, a smooth curve X, and a point $x \in X$, the map $\mathrm{Spec}(A[[t]]) \to \mathrm{Spec}(A) \times X$ is flat, and, thus, the disjoint union of $\mathrm{Spec}(A[[t]])$ and $\mathrm{Spec}(A) \times (X \setminus x)$ form a faithfully flat cover of $\mathrm{Spec}(A) \times X$. Further,

$$(\mathrm{Spec}(A[[t]])) \times_{\mathrm{Spec}(A)\times X} (\mathrm{Spec}(A) \times (X \setminus x)) \simeq \mathrm{Spec}(A((t))).$$

Thus, we can apply faithfully flat descent (Theorem 2.4.13) to obtain that quasi-coherent sheaves $\mathcal{F}$ on $\mathrm{Spec}(A[[t]])$ are the same data as triples $(\mathcal{F}_1, \mathcal{F}_2, \alpha)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent sheaves on $\mathrm{Spec}(A[[t]])$ and $\mathrm{Spec}(A) \times (X \setminus x)$ respectively, and α is an isomorphism between their pullbacks to the fibre product, which, in this case, is $\mathrm{Spec}(A((t)))$.

To transform this into a result about G-bundles, use the following Tannakian formalism for geometric stacks of [Lur05]:

Theorem 3.4.6. *Let $(S, \mathcal{O}_S)$ be a ringed topos which is local for the étale topology and that X is a geometric stack. Then there is an equivalence of categories*

$$T : \mathrm{Hom}(S, X) \to \mathrm{Hom}_{\otimes}(\mathrm{QC}_X, \mathcal{M}_{\mathcal{O}_S}),$$

where QC_X denotes the category of quasi-coherent sheaves on the stack X and $\mathcal{M}_{\mathcal{O}_S}$ is the category of modules over the sheaf of rings on S, $\mathcal{O}_S$.

Letting $S = \operatorname{Spec}(A[[t]])$ and $X = BG$, the above gives that $\operatorname{Hom}(\operatorname{Spec}(A[[t]]), BG) \simeq \operatorname{Funct}_{\otimes}(QC(BG), QC(\operatorname{Spec}(A[[t]]))) = \operatorname{Funct}_{\otimes}(\operatorname{Rep}(G), QC(\operatorname{Spec}(A[[t]])))$. The object $\operatorname{Hom}(\operatorname{Spec}(A[[t]]), BG)$ is the space of principal G-bundles on $\operatorname{Spec}(A[[t]])$. (Note, Gaitsgory defines families of G-bundles to be these exact tensor functors, avoiding the need to invoke this formalism.) From this, we see that the data of a G-bundle on $\operatorname{Spec}(A) \times X$ is the same data as the gluing data consisting of a G-bundle on $U = \operatorname{Spec}(A) \times (X \setminus x)$, another G-bundle on $V = \operatorname{Spec}(A) \times D_x \simeq \operatorname{Spec} A \otimes \mathbb{C}[[t]] = \operatorname{Spec}(A[[t]])$, where D_x is a small analytic disk around x, both of which are necessarily trivial ([Kum97]), plus an isomorphism of these bundles on the intersection $\operatorname{Spec} A \otimes \mathbb{C}((t))$, where $\operatorname{Spec} C((t))$ is the annulus around x.

An isomorphism of trivial bundles is a map

$$\begin{array}{ccc} (U \cap V) \times G & \xrightarrow{\sim} & (U \cap V) \times G \\ & \searrow \quad \swarrow & \\ & U \cap V & \end{array}$$

Such a map is determined, for each $v \in U \cap V$, by where it sends $(v, 1)$, so this is the same as a map $U \cap V \to G$. That is to say, this data is the same as the space $\operatorname{Hom}(\operatorname{Spec}(A \times \mathbb{C}((t))), G) =: G((t))$. This corresponds to the moduli space of G-bundle on X with given trivialisations on the open subsets D_x and $X \setminus x$. To get the affine Grassmannian from this, note that this is defined to be pairs (P, α), with P a principal G-bundle trivialised on $X \setminus x$ by the given map α, leaving $\operatorname{Hom}(\operatorname{Spec}(A) \times D_x, G) = G[[t]]$ trivialisations on D_x. So the affine Grassmannian is given by $\operatorname{Hom}(U \cap V, G)/\operatorname{Hom}(V, G) = G((t))/G[[t]]$ as required. Similarly, one finds that the moduli space of G-bundles on X is given by

$$_{\operatorname{Hom}(U,G)}\backslash^{\operatorname{Hom}(U\cap V,G)}/_{\operatorname{Hom}(V,G)} = _{G[[t]]}\backslash^{G((t))}/_{G[[t]]}.$$

Chapter 4

New Examples

> A *picture* held us captive. And we could not get outside it, for it lay in our language and language seemed to repeat it to us inexorably.
>
> *Ludwig Wittgenstein*

In this chapter, we recall an explicit construction of lattice chiral algebras from super line bundles with factorisation on the factorisation monoid of Γ-valued divisors, due to Beilinson and Drinfeld. We then demonstrate that one can generalise their definition of Γ-valued divisors to allow Γ to be replaced by an arbitrary commutative monoid M (Section 4.2.4), which extends to a definition of $\mathcal{D}iv(X, C)_S$ when $M = C$ is a cone in a lattice. We show that one can define (super) line bundles with factorisation on these M-valued divisors. We go on to show that, when M is a cone in a lattice, or M is a finite abelian group, the functors $\mathcal{D}iv(X, M)$ are ind-schemes (Proposition 4.2.1), and that, in the case of a cone, the functors $\mathcal{D}iv(X, M)_S$ are also representable by ind-schemes. Hence, in the case of $M = C$ a cone, the functors $\mathcal{D}iv(X, M)_S$ form a factorisation monoid (Proposition 4.4.1). We then note that there is a natural functorial generalistion of factorisation monoid, which we define and refer to as a *prefactorisation monoid* (Definition 4.4.2). Further, returning to the cases of a cone in a lattice, given a super line bundle with factorisation on the factorisation monoid of C-valued divisors, we show that one can push this forward to form a chiral (factorisation) algebra (Theorem 4.2.2). Further, we prove that such super line bundles always exist (Proposition 4.3.2) on C-valued divisors. It follows that we have constructed a new class of chiral algebras for our more general case of cones in lattices (See our main theorem: Theorem 4.2.3), which we call *toric chiral algebras*.

In this chapter X will be a curve, Γ will always denote a lattice, C a cone in a lattice, and M a commutative monoid.

4.1 Constructing Chiral Algebras from Chiral Monoids

4.1.1 The General Idea

We will describe here the general idea of the construction of Beilinson and Drinfeld ([BD04], Section 3.10.16).

4.1.1.1 Some Terminology

First, recall the notion of *algebraic space*, a generalisation of a scheme ([Pre09], [TS16] [Tag 025R]).

Definition 4.1.2. *Let* Sch/S *denote affine schemes over some base scheme* S*. (The case without a base scheme is covered by taking* $S = k$*.) Let* $\mathcal{F}, \mathcal{G} : \mathrm{Sch}^{op} \to \mathrm{Set}$ *be presheaves, and* $\phi : \mathcal{F} \to \mathcal{G}$*. Then* ϕ *is called* schematic*, a* relative scheme*, or* representable *if for every scheme* T *and every* $\alpha \in \mathrm{Hom}(T, \mathcal{G})$*, the fibre product* $\mathcal{F} \times_{\mathcal{G}} T$ *is a scheme.*

Let S *be a scheme. An fppf sheaf* $\mathcal{F} : \mathrm{Sch}/S \to \mathrm{Set}$ *is an* algebraic space *if the following hold:*

1. *The diagonal morphism* $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ *is representable.*

2. *There exists a scheme* U *over* S *and a surjective schematic étale map* $U \to \mathcal{F}$*.*

The algebraic space $\mathcal{F}$ *is quasi-compact if* U *can be found to be quasi-compact. (*[TS16] [Tag 03E2]*).*

Conceptually, schemes are glued from affine schemes in the Zariski topology, whilst algebraic spaces are glued in the finer étale topology. They retain some of the properties of schemes, but allow for a wider variety of constructions in the category and give a geometry to a wider class of objects: For example, every Moishezon manifold is an algebraic space, but not all are schemes. (However, one still needs to move to stacks to consider problems involving automorphisms.) We will only really work with schemes, which are examples of algebraic spaces.

Definition 4.1.3. *An* ind-algebraic space*, according to* [BD04]*, is functor on affine schemes that is representable by an inductive limit of a directed system of quasi-compact algebraic spaces with closed embeddings. For a quasi-compact scheme Z, an* ind-algebraic Z-space *is an ind-object in algebraic spaces over Z: that is, an ind-algebraic space $\mathcal{G}$ together with a morphism $\mathcal{G} \to Z$ induced by the diagram of algebraic spaces with maps to Z.*

Let $\mathcal{J}_Z$ denote the category of ind-algebraic Z-spaces with closed embeddings, and let $\mathcal{J}$ be the fibred category over quasi-compact schemes formed from the $\mathcal{J}_Z$ with fibre products as pullback maps.

We now give the definition of a *chiral monoid*, which will be crucial in what follows.

Definition 4.1.4. *A* chiral monoid*, or* factorisation monoid*, is a pair $(\mathcal{G}, c)$, where $\mathcal{G}$ is a morphism of fibred categories $\mathcal{C}(X) \to \mathcal{J}$: that is, an assignment to every $S \in \mathcal{C}(X)_Z$ an ind-algebraic Z-space $\mathcal{G}_S = \mathcal{G}_{S,Z}$, and to every $S' \leq S$ a canonical closed embedding $\mathcal{G}_{S',Z} \hookrightarrow \mathcal{G}_{S,Z}$, all compatible with base change. Analogously to a factorisation algebra, c is a rule that assigns to every pair of mutually disjoint divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c_{S_1,S_2} : \mathcal{G}_{S_1,Z} \times_Z \mathcal{G}_{S,Z} \xrightarrow{\sim} \mathcal{G}_{S_1+S_2,Z}$, which must be commutative, associative, and compatible with the morphisms of $\mathcal{G}$.*

Additionally, the following conditions must hold: For every n, the closure in $\mathcal{G}_{\mathrm{Sym}^n X}$ of the complement to the preimage of the discriminant divisor in $\mathrm{Sym}^n X$ equals $\mathcal{G}_{\mathrm{Sym}^n X}$ (Equivalently, $\mathcal{G}_{\mathrm{Sym}^n X}$ can be represented as the inductive limit of a directed family $\{\mathcal{G}_\alpha\}$ of algebraic $\mathrm{Sym}^n X$-spaces and closed embeddings such that each $\mathcal{G}_\alpha$ has no non-zero local functions supported over the discriminant divisor.); $\mathcal{G}_{\mathrm{Sym}^0_X} \neq \emptyset$; and the ind-algebraic spaces $\mathcal{G}_{\mathrm{Sym}^n_X}$ are separable.

These technical conditions account for the use of divisors in the definition rather than finite sets of points and are required to ensure that we have an object defined over the Ran space. (See the proof that factorisation algebras are equivalent to the alternative notion in terms of divisors in [BD04] Section 3.46 for more details on this.)

Definition 4.1.5. *For a chiral monoid $\mathcal{G}$, a* line bundle *on $\mathcal{G}$ is a rule that assigns to any $S \in \mathcal{C}(X)_Z$ a line bundle $\lambda_{\mathcal{G}_S}$ on $\mathcal{G}_S$, and to any morphism $(S', Z') \to (S, Z)$*

in $\mathcal{C}(X)$ an identification of $\lambda_{\mathcal{G}_{S'}}$ and the pullback of $\lambda_{\mathcal{G}_S}$ by the map $\mathcal{G}_{S'} \to \mathcal{G}_S$, compatible with composition.
For such a λ, a factorisation structure *on λ is a rule that assigns to any disjoint $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c : \lambda_{\mathcal{G}_{S_1}} \otimes \lambda_{\mathcal{G}_{S_2}} \xrightarrow{\sim} c^*_{S_1,S_2}\lambda_{\mathcal{G}_{S_1+S_2}}$, with these isomorphisms compatible with the line bundle morphisms and commutative and associative. A line bundle on $\mathcal{G}$ with a factorisation structure is also called a* $\mathbb{G}_m$-extension *of $\mathcal{G}$.*

One can, instead, take a super line bundle on a chiral monoid to get a notion of *super $\mathbb{G}_m$-extension.*

4.1.5.1 Constructing Modules from a Super Extension

The general idea for the construction of a chiral algebra from a super line bundle on a chiral monoid is as follows. Let $\mathcal{G}$ be a chiral monoid with super extension λ. Now suppose that each $\mathcal{G}_{S,Z}$ is the inductive limit of its closed subschemes that are finite and flat over Z, for Z affine and $S \in \mathcal{C}(X)_Z$. That is, $\mathcal{G} = \operatorname{Spf} R$ for $R = \varprojlim R_\alpha$, where the R_α are $\mathcal{O}_Z$-algebras which are locally free $\mathcal{O}_Z$-modules of finite rank. The structure of a factorisation algebra (of the form described in Section 3.1.7.1) is given by setting

$$A^l_{S,Z} := \lambda \otimes_R R^* := \bigcup \lambda \otimes_R \mathcal{H}om_{\mathcal{O}_Z}(R_\alpha, \mathcal{O}_Z).$$

The above formula allows the taking of a direct limit of modules, which gives a well-defined module. This is the reason for the dualising.

4.1.6 Γ-valued Divisors, Line Bundles, and Factorisation

We will proceed to describe in greater detail the application of the above construction to the specific example of lattice chiral algebras via super line bundles on the chiral monoid of divisors in order that we may adapt this to new cases. First, we define the objects involved.

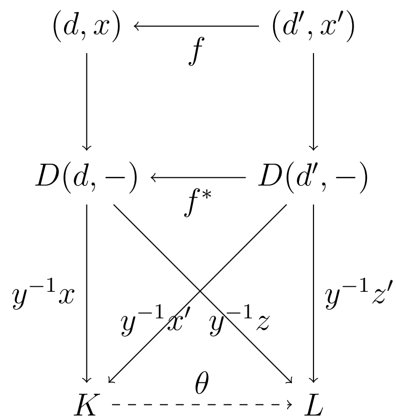
4.1.6.1 Quasi-coherent Sheaves on Prestacks

In order to work with the constructions in Beilinson and Drinfeld, we will need to recall the construction of quasi-coherent sheaves on prestacks.

We use the following standard result, as given in [ML98], p. 76:

Theorem 4.1.7. *Any functor $K : D \to \mathrm{Sets}$ from a small category D to the category of sets can be represented (in a canonical way) as a colimit of a diagram of representable functors $\hom(d, -)$ for objects d in D.*

For such a functor K, the following diagram depicts the way in which K is expressible as such a colimit:



Here y^{-1} is the Yoneda isomorphism, $y^{-1} : K(d) \to \mathrm{Nat}(D(d, -), K)$, and K is the colimit of the $D(d, -)$ over the diagrams of pairs (d, x), $d \in D$, $x \in K(d)$. L and θ show how the depicted cone for K is a universal object.

This comes with the dual result that any contravariant functor $D^{\mathrm{op}} \to \mathrm{Set}$ can be represented as a colimit of a diagram of representable functors $\mathrm{Hom}(-, d)$.

To define quasi-coherent sheaves on a functor F, we write F as a colimit as described above, and define the quasi-coherent sheaves on it to be the colimit of quasi-coherent sheaves on representable functors, i.e. affine schemes: namely, modules over the corresponding rings. So let $F : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf on affine schemes (over k here). Then F is the same as a functor $\mathrm{Alg_k} \to \mathrm{Set}$. So, as in the above diagram, F is the colimit, over pairs (S, x) with S a ring and $x \in F(S)$, of hom-sets $\mathrm{Hom}_{\mathrm{Alg_k}}(S, -)$.

Thus, quasi-coherent sheaves on F should be given by the colimit over pairs (S, f), where $f : \mathrm{Spec}(S) \to F$ is a natural transformation, i.e. an element in $F(S)$ by Yoneda's lemma, of a diagram of modules: an S-module assigned to each such pair, for the appropriate ring S. This must be subject to the compatibility conditions given by the colimit diagram. So, for any

$$\begin{array}{ccc}
\operatorname{Spec} S_1 & \xrightarrow{f_1} & F \\
{\scriptstyle g}\downarrow & \nearrow{\scriptstyle f_2} & \\
\operatorname{Spec} S_2 & &
\end{array}$$

with S_i-modules M_i assigned to (S_i, f_i), for $i \in \{1, 2\}$, we have $M_1 \cong M_2 \otimes_{S_2} S_1$. Note that $g : \operatorname{Spec} S_1 \to \operatorname{Spec} S_2$ corresponds to a map $S_2 \to S_1$. We may sometimes denote the component of the quasi-coherent sheaf M on F over (S, f) by f^*M.

This construction forms a Kan extension (See section 2.3 or [ML98] p.236.):

$$\begin{array}{ccc}
C & \xrightarrow{F} & D \\
{\scriptstyle y}\downarrow & \nearrow{\scriptstyle \tilde{F}} & \\
\mathrm{PSh}(C) & &
\end{array}$$

Here, y is the Yoneda embedding.

Note that, included in this is the statement that when $F = \operatorname{Spec} S$ for some S, the quasi-coherent sheaf defined by the colimit construction is the same as a quasi-coherent sheaf on $\operatorname{Spec} S$: namely, an S-module.

Given another functor G and a natural transformation $h : G \to F$, we can pull back a quasi-coherent sheaf on F to one on G since for any map $f : \operatorname{Spec} S \to G$, the composition $h \cdot f : \operatorname{Spec} S \to F$ is already part of the diagram of maps to F taken in the colimit, so we can assign the modules for these maps and they will already satisfy the appropriate compatibilities to form a quasi-coherent sheaf on G.

Note that we could equally well have taken the category of all schemes in these definitions, instead of that of affine schemes (since any scheme is a limit of affine schemes of some type). The results would have been the same.

4.1.7.1 Line bundles on Divisors

We review here the main definitions that will be used in the construction.

Definition 4.1.8. *Define the presheaf on* Sch*, or* AffSch*,* $\mathcal{D}iv(X)$*, as follows.* $\mathcal{D}iv(X)$ *is the functor that assigns to a quasi-compact scheme* Z *the group of Cartier divisors in* $X \times Z/Z$ *proper over* Z*. For a lattice* Γ*, let* $\mathcal{D}iv(X, \Gamma) := \mathcal{D}iv(X) \otimes \Gamma$*. Its points are called* Γ-divisors*.*

$\mathcal{D}iv(X)$ forms a sheaf with respect to the flat topology. One can define line bundles on the functors $\mathcal{D}iv(X, \Gamma)$ using the previously described method for defining the Kan extension of quasi-coherent sheaves. We will recall here an explicit description here as to what is meant by a *super line bundle* on a presheaf $F : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Set}$.

First, we give the definition of a super line bundle on an affine scheme Y. Consider the category of $\mathbb{Z}_2$-graded coherent $\mathcal{O}_Y$ modules, i.e. $\mathbb{Z}_2$-graded A modules $F_0 \oplus F_1$, where $Y = \operatorname{Spec} A$. This is monoidal, with $(F_0 \oplus F_1) \otimes (G_0 \oplus G_1) = H_0 \oplus H_1$, where $H_0 = (F_0 \otimes G_0) \oplus (F_1 \otimes G_1)$ and $H_1 = (F_0 \otimes G_1) \oplus (F_1 \otimes G_0)$. Thus, the unit is given by A in even degree. The super commutativity constraint is given by $\sigma(v \otimes w) = (-1)^{\deg(v)\deg(w)} w \otimes v$ for homogeneous elements v, w of degrees $\deg(v)$ and $\deg(w)$ respectively. (So odd elements anticommute, but all other combinations of parities of homogeneous elements give usual commutativity.) Morphisms are maps of $\mathbb{Z}_2$-graded $\mathcal{O}_Y$-modules.

Definition 4.1.9. *Let* Y *be an affine scheme as above. A* super line bundle *on* Y *will be an invertible* $\mathbb{Z}_2$*-graded coherent* $\mathcal{O}_Y$*-modules: namely, a line bundle in either degree* 0 *or degree* 1*.*

Now let $F : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Set}$ *be any presheaf. Hence, by* [ML98] *p.237, the assignment of super line bundles to affine schemes has a Kan extension to presheaves given by pointwise limits (see also Theorem 4.1.7 and the subsequent section), allowing the following definition: A* super line bundle *on* F *will be an assignment to every pair* (S, F)*, with* S *a ring and* $f : \operatorname{Spec} S \to F$*, of a super line bundle on* $\operatorname{Spec} S$ *such that the assignments are compatible with the pullbacks when* F *is realised as a colimit. Note that the maps are now required to respect the* $\mathbb{Z}_2$*-grading, i.e. the parity.*

Thus, we see that a super line bundle on $\mathcal{D}iv(X, \Gamma)$ will be an assignment to any Z and Γ-divisor $D \in \mathcal{D}iv(X, \Gamma)(Z)$ a super line bundle λ_D on Z in a way that is compatible with base change.

We remark here that we believe that super line bundles are used, rather than merely line bundles, for increased generality and to cover examples that my be applicable to physics. As an example which requires odd line bundles, Beilinson and Drinfeld give a proof of the boson-fermion correspondence ([BD04], Section 3.10.10) between a lattice vertex superalgebra and the free fermionic vertex superalgebra, a Clifford algebra. For more on this correspondence, see [FBZ01], p.78.

We now give the definition of a factorisation structure on a super line bundle.

Definition 4.1.10. *Given such a super line bundle λ, a* factorisation structure *on λ is a rule that assigns to any Z and any pair of Γ-divisors $D_1, D_2 \in \mathcal{D}iv(X,\Gamma)(Z)$ whose supports do not intersect an isomorphism $c_{D_1,D_2} : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_1+D_2}$, such that c is compatible with base change, as well as associative and commutative as in the following:*

1. *For any $D_1, D_2, D_3 \in \mathcal{D}iv(X,\Gamma)(Z)$ with pairwise disjoint supports, the morphisms $c_{D_1+D_2,D_3}(c_{D_1,D_2} \otimes \mathrm{id}_{\lambda_{D_3}})$ and $c_{D_1,D_2+D_3}(\mathrm{id}_{\lambda_{D_1}} \otimes c_{D_2,D_3})$:*
$\lambda_{D_1} \otimes \lambda_{D_2} \otimes \lambda_{D_3} \xrightarrow{\sim} \lambda_{D_1+D_2+D_3}$ coincide.

2. *$c_{D_1,D_2} = c_{D_2,D_1} \cdot \sigma$, where $\sigma : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_2} \otimes \lambda_{D_1}$ is the commutativity constraint.*

With the tensor product $(\lambda_1, c_1) \otimes (\lambda_2, c_2) := (\lambda_1 \otimes \lambda_2, c_1 \otimes c_2)$, the pairs (λ, c) form a Picard groupoid: A small monoidal groupoid, in which every object is invertible under the tensor product. Denote this by $Pic^f(\mathcal{D}iv(X,\Gamma))$. (Clearly, as λ is a line bundle, one can define an inverse to any (λ, c) by $(\lambda, c)^{-1} := (\lambda^{-1}, c^{-1})$, where c^{-1} is used as formal notation here for the isomorphism $c^{\lambda^{-1}}_{D_1,D_2} : \lambda^{-1}_{D_1} \otimes \lambda^{-1}_{D_2} \xrightarrow{\sim} \lambda^{-1}_{D_1+D_2}$.)

We give here a result from [BD04], p.262, relating this Picard groupoid of line bundles with factorisation to the Picard groupoid of θ-data defined in Definition 3.2.1. We will use this in Section 4.3 in the proof of Proposition 4.3.2.

Proposition 4.1.11. *There is a natural equivalence of Picard groupoids:*

$$Pic^f(\mathcal{D}iv(X,\Gamma)) \xrightarrow{\sim} \mathcal{P}^\theta(X,\Gamma).$$

For each $\gamma \in \Gamma$, set i^γ to be the map of presheaves on AffSch given, on pairs $(X(\operatorname{Spec} k), x \in X(*))$, by $x \mapsto (x) \otimes \gamma$. So, more generally, for pairs $X(Z)$ and $y \in X(Z)$, i.e. $y \in \hom(Z, X)$, $i^\gamma : X(Z) \to \mathcal{D}iv(X,\Gamma)(Z)$ is given by $y \mapsto \Gamma_y \otimes \gamma$, where $\Gamma_y \subset X \times Z$ is the graph of y. This definition clearly defines a natural transformation of functors, for each γ. This allows one to pull back quasi-coherent sheaves on $\mathcal{D}iv(X, \Gamma)$ to those on $X = \hom(-, X)$ by setting $\lambda^\gamma := i^{\gamma *}\lambda$.

We need to give the isomorphisms $c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2} \otimes \omega^{\otimes \kappa(\gamma_1,\gamma_2)}$. The factorisation structure on our super line bundle λ on $\mathcal{D}iv(X, \Gamma)$ gives us isomorphisms $c_{D_1,D_2} : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_1+D_2}$, for divisors with non-intersecting supports. On the complement U to the diagonal Δ, we can pull this back in two ways to $X \times X$ to get an isomorphism: $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2} \mid_U \xrightarrow{\sim}$ "$\lambda^{\gamma_1+\gamma_2}$" $\mid_U$. Let $i^{\gamma_1,\gamma_2} : X \times X \to \mathcal{D}iv(X, \Gamma)$ be the map given by $(x_1, x_2) \mapsto i^{\gamma_1}(x_1) + i^{\gamma_2}(x_2)$, generalised for $X(Z)$, $Z \neq k$, as above. Then, on U, pulling this back via i^{γ_1,γ_2} will give a line bundle on $U \subset X \times X$, denoted $\lambda^{\gamma_1,\gamma_2}$, which we will pull back to X to be $\lambda^{\gamma_1+\gamma_2}$. Via the isomorphism c for $Pic^f(\mathcal{D}iv(X, \Gamma))$, this is the same as pulling back the tensor products for the bundles on the divisors separately: So we can pull this back via the product $i^{\gamma_1 *} \times i^{\gamma_2 *}$ to get $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2}$ on U.

Using the result that, for line bundles L_1 and L_2 on a space Y, if $L_1 \otimes L_2^\vee \cong \mathcal{O}$ on $Y \backslash D$, for a divisor D, then $L_1 \otimes L_2^\vee \cong \mathcal{O}(nD)$, the line bundle associated to the divisor, on the whole space, we find that, on $X \times X$, we have $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1\gamma_2}(-\kappa(\gamma_1, \gamma_2)\Delta)$ for some bilinear form κ. Pulling this back via the diagonal map gives the isomorphism on X: $c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2} \otimes \omega^{\otimes \kappa(\gamma_1,\gamma_2)}$, which is what we wanted.

The functor preserves the Picard groupoid structure and Beilinson and Drinfeld proceed to show that it is fully faithful and essentially surjective.

4.1.12 Constructing Lattice Chiral Algebras

Here we explicate in detail the method of Section 3.10.8 of [BD04] of constructing factorisation algebras by pushing forward line bundles on the factorisation monoid of Γ-valued divisors to a factorisation algebra on the space of effective divisors.

Let X be a smooth curve. Let Z be a quasi-compact scheme and S an effective Cartier divisor in $X \times Z/Z$ proper over Z.

Definition 4.1.13. *Define $\mathcal{D}iv(X)_S$ as the functor on the category of quasi-compact Z-schemes that assigns to Y/Z the subgroup $\mathcal{D}iv(X)_S(Y) \subset \mathcal{D}iv(X)(Y)$ of all Cartier*

divisors whose support is contained set-theoretically in the pull-back of S.
Let $\mathcal{D}iv(X,\Gamma)_S = \mathcal{D}iv(X,\Gamma)_{S,Z} := \mathcal{D}iv(X)_S \otimes \Gamma$.

The reason for defining the functors $\mathcal{D}iv(X,\Gamma)_S$ is that we would like an object that lives over the Ran space, i.e. is dependent only on the reduced part of a divisor S. We require an assignment of an ind-algebraic space to each of the objects in the fibred category $\mathcal{C}(X)$ (Definition 4.1.4), which is localised along maps between divisors with equal reduced part. The family of functors $\mathcal{D}iv(X,\Gamma)_S$ depends only on the reduced part of S, and so respects this localisation, whereas the groups assigned by the functors $\mathcal{D}iv(X,\Gamma)$ or $\mathcal{D}iv(X,\Gamma)_S$ depend on the divisor.

Beilinson and Drinfeld state that because the set of all effective Cartier divisors in $X \times Y/Y$ of degree n is representable by the scheme $\operatorname{Sym}^n X$, it can be argued that $\mathcal{D}iv(X)_S$ and $\mathcal{D}iv(X,\Gamma)_S$ are formally smooth ind-schemes over Z, which can be represented as the inductive limit of a directed family of subschemes which are finite and flat over Z. We will show how to do this in Section 4.4. This gives the required expression of the functor as an ind-scheme: $\mathcal{D}iv(X,\Gamma)_S = \varinjlim \operatorname{Spec} R_\alpha$, for $\{R_\alpha\}$ a directed system of $\mathcal{O}_Z$-algebras connected by surjections with the R_α locally free $\mathcal{O}_Z$-modules of finite rank. We see that the $\mathcal{D}iv(X,\Gamma)_S$ form a chiral monoid, as given in Definition 4.1.4, with each $\mathcal{G}_{S,Z} := \mathcal{D}iv(X,\Gamma)_S$ the inductive limit of closed subschemes that are finite and flat over Z, and we may define the modules for a factorisation algebra as given in Section 4.1.5.1 by:

$$A^l_{S,Z} := \mathcal{H}om_{\mathcal{O}_Z}(\lambda^*_{S,Z}, \mathcal{O}_Z) = \bigcup \lambda_{R_\alpha} \otimes_{R_\alpha} \mathcal{H}om_{\mathcal{O}_Z}(R_\alpha, \mathcal{O}_Z).$$

Here, the notation $\mathcal{H}om$ denotes the sheaf hom: Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, the sheaf hom is the assignment $U \mapsto \operatorname{Hom}(\mathcal{F}\mid_U, \mathcal{G}\mid_U)$, where this latter Hom is taken in the category of presheaves. By taking the union here, we mean taking the inductive (direct) limit over the system of inclusions $\mathcal{H}om_{\mathcal{O}_Z}(R_\alpha, \mathcal{O}_Z) \hookrightarrow \mathcal{H}om_{\mathcal{O}_Z}(R_{\alpha'}, \mathcal{O}_Z)$ arising from the surjections $R_{\alpha'} \twoheadrightarrow R_\alpha$. This functor is well-defined in terms of being independent of the choice of inductive system $\{R_\alpha\}$.

4.1.13.1 The Factorisation Structure

This construction pushes the line bundle on Γ-valued divisors forward to get a factorisation algebra of modules over divisors. We describe here how the maps c from the line bundle (λ, c) carry across to give the required isomorphisms for a factorisation structure on the A^l.

The line bundle (λ, c) comes equipped with factorisation isomorphisms: the data c, $c : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_1+D_2}$, for D_1, D_2 in $X \times Z/Z$ tensored with Γ whose supports do not intersect, defined for each Z. The factorisation maps required are those of Section 3.1.7.1 ([BD04] Section 3.4.6): For S_1, S_2 disjoint, and assigned $\mathcal{O}_Z$-modules $B_{S_i,Z}$ respectively, we require isomorphisms $c_{S_1,S_2} : B_{S_1} \otimes B_{S_2} \xrightarrow{\sim} B_{S_1+S_2}$. Here, the modules $B_{S,Z} = A^l_{S,Z} = \mathcal{H}om_{\mathcal{O}_Z}(\lambda^*_{S,Z}, \mathcal{O}_Z)$. Although this is defined in terms of the system $\{R_\alpha\}$, the well-definedness of the functor means we only need consider the isomorphisms at the level of the functors $\lambda_{S,Z}$ (since the construction from then is a canonical taking of a (double) dual via dualising the $\lambda_{S,Z}$ then taking the hom-sheaf to the structure sheaf.

The $\lambda_{S,Z}$ here are the functors defined by taking the pullback of the quasi-coherent sheaf λ on the functor $\mathcal{D}iv(X, \Gamma)$ to a quasi-coherent sheaf on the functor $\mathcal{D}iv(X, \Gamma)_S$ for fixed (S, Z). As in 4.1.6.1, such a λ on $\mathcal{D}iv(X, \Gamma)_S$ can be considered in terms of compatible assignments of (invertible) $\mathcal{O}_Z$-modules for each divisor contained in the pullback of S. But the factorisation structure c gives us isomorphisms between these for disjoint S_1, S_2 (and thus for any D_1, D_2 contained in the pullbacks of these to any Z'), which are compatible with pullbacks, so this defines natural isomorphisms between the functors of $A^l_{S_1,Z}$ and $A^l_{S_2,Z}$ for S_1 and S_2 disjoint.

4.2 Constructing New Examples

We will now extend the definitions of the functors $\mathcal{D}iv(X, \Gamma)$ and $\mathcal{D}iv(X, \Gamma)_S$, as well as super line bundles with factorisation on these, to our more general case of cones in lattices. In fact, the functor $\mathcal{D}iv(X, M)$ may defined for an arbitrary commutative monoid M.

We then prove the following proposition:

Proposition 4.2.1. *Let C be a cone in a lattice. Then the functors $\mathcal{D}iv(X, C)$ and $\mathcal{D}iv(X, C)_S$ are ind-schemes. When M is a finite abelian group, the functor $\mathcal{D}iv(X, M)$ is also an ind-scheme.*

In particular, this allows us to show that the functors $\mathcal{D}iv(X, C)_S$ form a factorisation monoid, for C a cone in a lattice (Proposition 4.4.1). It also leads to the following key theorem:

Theorem 4.2.2. *In the case that C is a cone in a lattice, given a super line bundle on the factorisation monoid of C-valued divisors, we can push the line bundle forward to form a factorisation algebra.*

We show in Section 4.3 that when C is a cone in a lattice there exist super line bundles on the factorisation monoid of C-valued divisors (Proposition 4.3.2). Consequently, we have our main result:

Theorem 4.2.3. *There exists a new class of factorisation algebras given by pushing forward super line bundles on the factorisation space of C-valued divisors, for C a cone in a lattice.*

4.2.4 M-valued Divisors

We will find it makes better sense to work with the functor assigning to a scheme the subscheme of $\mathcal{D}iv(X)$ of *effective* Cartier divisors, and to build up the other functors from this, so we define here $\mathcal{D}iv_e(X)$ to be that subscheme, and $\mathcal{D}iv_e(X)_S$ the analogous object in relation to $\mathcal{D}iv(X)_S$. (This gives us the functor we would like when we tensor $\mathcal{D}iv(X)$ with $\mathbb{N}$ rather than $\mathbb{Z}$.)

Definition 4.2.5. *Let M be an arbitrary commutative monoid. As above, $\mathcal{D}iv(X, M)$ will denote the functor $\mathcal{D}iv_e(X) \otimes M$, where $\mathcal{D}iv_e(X)$ is as above, and $\mathcal{D}iv(X)$ as before.*
In the case that $M = C$ is a cone in a lattice, we can also define the functors $\mathcal{D}iv(X, C)_S$ to be $\mathcal{D}iv_e(X)_S \otimes C$.

Here, the tensor product on the category of commutative monoids M and N is the expected generalisation of that for abelian groups: Take the free commutative monoid on elements of $M \times N$ and quotient by the relations:

$$(m, n) + (m', n) = (m + m', n), \qquad (m, n) + (m, n') = (m, n + n'),$$

$$(m, 0) = (0, n) = (0, 0).$$

This has the required property that monoid homomorphisms $M \otimes N \to K$ are in bijection with maps $M \times N \to K$ that are monoid maps in each variable.

4.2.5.1 Line Bundles with Factorisation on $\mathcal{D}iv(X, M)$

We will show now that we can define super line bundles on $\mathcal{D}iv(X, M)$ with factorisation structure, for any commutative monoid M. We will then argue that, in the case of C a cone in a lattice, given a super line bundle with factorisation on $\mathcal{D}iv(X, C)$, we can pull this back to the $\mathcal{D}iv(X, C)_S$ to give rise to a super line bundle with factorisation structure on the chiral monoid formed by the $\mathcal{D}iv(X, C)_S$, in much the same way as is done in the case of a lattice.

Defining a line bundle on $\mathcal{D}iv(X, M)$ amounts to defining an invertible quasi-coherent sheaf, which we can do for any functor. Thus, we need only check that we can define the maps c to give a factorisation structure: namely, maps c_{D_1,D_2} : $\lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_1+D_2}$, for any Z and pair of M-divisors $D_1, D_2 \in \mathcal{D}iv(X, M)$, with appropriate compatibility relations. Note that the definition of the M-divisor $D_1 + D_2$ makes good sense over any commutative semi-group. We remind the reader of the associativity and commutative conditions on c given in Definition 4.1.10:

1. For any $D_1, D_2, D_3 \in \mathcal{D}iv(X, M)(Z)$ with pairwise disjoint supports, the morphisms $c_{D_1+D_2,D_3}(c_{D_1,D_2} \otimes \mathrm{id}_{\lambda_{D_3}})$ and $c_{D_1,D_2+D_3}(\mathrm{id}_{\lambda_{D_1}} \otimes c_{D_2,D_3})$ from $\lambda_{D_1} \otimes \lambda_{D_2} \otimes \lambda_{D_3} \xrightarrow{\sim} \lambda_{D_1+D_2+D_3}$ coincide.

2. $c_{D_1,D_2} = c_{D_2,D_1} \cdot \sigma$, where $\sigma : \lambda_{D_1} \otimes \lambda_{D_2} \to \lambda_{D_2} \otimes \lambda_{D_1}$ is the commutativity constraint.

The first of these uses additivity in any $\mathcal{D}iv(X, M)(Z)$, which we have because we have just replaced the group tensor product with a lattice with the tensor product with a commutative monoid, giving a functor in commutative monoids. The second condition does not use the structure on Γ at all. Thus, we may define (super) line bundles with factorisation on our new functor $\mathcal{D}iv(X, M)$ according to Definition 4.1.10.

Now let us consider the case of C a cone in a lattice. Given a super line bundle on $\mathcal{D}iv(X, C)$, we may pull this back to the functors $\mathcal{D}iv(X, C)_S$ as for the lattice ([BD04], Section 3.10.8). To see that we obtain a factorisation structure on the induced line bundle on the chiral monoid of spaces $\mathcal{D}iv(X, C)_S$ when the original line bundle has factorisation, we use the same argument we described briefly in Section 4.1.13.1. (See Section 4.1.5, or [BD04], p.273, for the definition of a line bundle with factorisation on a chiral monoid.) That is to say, the factorisation maps are naturally induced between these functors for disjoint $S, S' \in \mathcal{C}(X)_Z$ by those pointwise maps we have for the component line bundles on disjoint pairs of individual divisors D_1 and D_2 with the pullback of D_i contained in S_i, $i = 1, 2$. This carries across to the cone as it relies only on the additive structure, which the cone is closed under.

4.3 The Existence of Line Bundles on Divisors

We show here that, for C a cone in a lattice, we can always construct super line bundles with factorisation on $\mathcal{D}iv(X, C)$, using the existence of θ-data in the case of a lattice and the equivalence between super line bundles on Γ-valued divisors and the Picard groupoid of θ-data for the case of lattices. Thus, when we have shown that the functors $\mathcal{D}iv(X, C)_S$ can be expressed as ind-schemes, as we will proceed to do, it will follow that there exist new examples of chiral algebras by pushing forward these line bundles on the factorisation monoid formed by the $\mathcal{D}iv(X, C)_S$.

We record here the following result from [BD04] p.259 that θ-data exist for a lattice Γ:

Lemma 4.3.1. *For any symmetric bilinear form κ on Γ one has $\mathcal{P}^\theta(X, \Gamma)^\kappa \neq \emptyset$.*

Proof. The idea is to define the θ-datum $\theta^\kappa = (\lambda, c) \in \mathcal{P}^\theta(X, \Gamma)^\kappa$ by $\lambda^\gamma := (\omega^{\otimes 1/2})^{\otimes -\kappa(\gamma,\gamma)} \otimes \epsilon^\gamma$ for $\gamma \in \Gamma$, where $\omega^{\otimes 1/2}$ is a line bundle of half-forms, with the map c given by the product multiplied by c_ϵ. Here, ϵ_γ and c_ϵ are part of the data of a system of super lines on Γ with factorisation isomorphisms, which they construct. This data (referred to as a *central super $\mathbb{G}_m$-extension of* Γ and denoted Γ^ϵ) is the assignment of an odd or even 1-dimensional k-vector space ϵ_γ for each $\gamma \in \Gamma$, with isomorphisms $c_\epsilon^{\gamma_1,\gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \xrightarrow{\sim} \epsilon^{\gamma_1+\gamma_2}$ satisfying associativity: $c_\epsilon^{\gamma_1,\gamma_2+\gamma_3}(\mathrm{id}_{\epsilon^{\gamma_1}} \otimes c_\epsilon^{\gamma_2,\gamma_3}) = c_\epsilon^{\gamma_1+\gamma_2,\gamma_3}(c_\epsilon^{\gamma_1,\gamma_2} \otimes \mathrm{id}_{\epsilon^{\gamma_3}})$. They choose the commutator pairing $\Gamma \times \Gamma \to \mathbb{G}_m$ to be $\gamma_1, \gamma_2 \mapsto (-1)^{\kappa(\gamma_1,\gamma_2)}$, giving $c_\epsilon^{\gamma_1,\gamma_2} = (-1)^{\kappa(\gamma_1,\gamma_2)} c_\epsilon^{\gamma_2,\gamma_1}\sigma$, where σ denotes the commutativity constraint.

Beilinson and Drinfeld explain how one can always construct such a Γ^ϵ via a particular choice of map $\Gamma \to \mathbb{Z}/2$, together with a bilinear pairing $\Gamma \times \Gamma \to \mathbb{Z}/2$, which they specify.

□

Under the isomorphism of Proposition 4.1.11, we see that there exist super line bundles with factorisation on Γ-valued divisors, for Γ a lattice. However, as explained in Section 4.2.5.1, the factorisation structure uses only the addition on the monoid, so we see that if we start with values of γ within the cone in the lattice, the maps will only involve values from within the cone. That is, the line bundle restricts to the cone.

Thus, we have shown the following:

Proposition 4.3.2. *There exist super line bundles with factorisation on the functors* $\mathcal{D}iv(X, C)$*, and hence, on the chiral monoid of functors* $\mathcal{D}iv(X, C)_S$*, when* C *is a cone in a lattice.*

4.4 Ind-algebraic Spaces

The key problem in proving we have new examples of chiral algebras is that of showing that the functors $\mathcal{D}iv(X, C)_S$ can be expressed as ind-schemes (more generally, ind-algebraic spaces, but we will only need to consider ind-schemes here). In this section, we show how the functors $\mathcal{D}iv(X, \Gamma)$ and $\mathcal{D}iv(X, \Gamma)_S$ can be expressed as such in the original case of a lattice (as considered by Beilinson and Drinfeld), and then proceed to prove this is also the case for $\mathcal{D}iv(X, C)$ and $\mathcal{D}iv(X, C)_S$, with C a cone in a lattice. We will also show that the functor $\mathcal{D}iv(X, M)$ is an ind-scheme, when M is a finite abelian group. This will prove Proposition 4.2.1.

A straightforward consequence of this is the following proposition:

Proposition 4.4.1. *Let* C *be a cone in a lattice. Then the functors* $\mathcal{D}iv(X, C)_S$ *of divisors set-theoretically contained in the pullback of* S *form a factorisation monoid.*

Proof. The main point here is that the divisors form ind-algebraic Z-spaces (in this case, ind-schemes). The rest follows immediately from the definition of a chiral (factorisation) monoid (Definition 4.1.4) as in the case of lattices, with the embeddings the restrictions of those for the lattice case. The factorisation map $\mathcal{D}iv(X, M)_{S_1,Z} \times_Z \mathcal{D}iv(X, M)_{S_2,Z} \to \mathcal{D}iv(X, M)_{S_1+S_2,Z}$, $(\sum a_i x_i, \sum b_j y_j) \mapsto \sum a_i x_i + \sum b_j x_j$ only uses the commutative addition on the divisors. □

Remark: There is clearly a more general, functorial definition of factorisation monoid, if we are prepared to forgo the geometric conditions of requiring the functors to be ind-algebraic spaces. This leads us to introduce the following:

Definition 4.4.2. *A* prefactorisation monoid *will be a pair* $(\mathcal{G}_{\text{pre}}, c)$*, where* $\mathcal{G}_{\text{pre}}$ *is an assignment to every* $S \in \mathcal{C}(X)_Z$ *of a functor on affine schemes* $F_{S,Z} : \text{Sch} \to \text{Set}$*, and for every* $S' \leq S$ *a map* $F_{S',Z} \hookrightarrow F_{S,Z}$ *which is an inclusion of a subfunctor, such that everything is compatible with base change, and* c *is a rule that assigns to mutually disjoint divisors* $S_1, S_2 \in \mathcal{C}(X)_Z$ *an identification* $c_{S_1,S_2} : F_{S_1,Z} \times_Z F_{S_2,Z} \xrightarrow{\sim} F_{S_1+S_2,Z}$*. These maps are required to be commutative and associative and compatible with the maps from* $\mathcal{G}_{\text{pre}}$*.*

We note here that Beilinson and Drinfeld use the term *pre-factorisation algebra* to mean an alternatively weakened notion of factorisation algebra to the way in which our notion of prefactorisation monoid is a generalisation of a factorisation monoid ([BD04], Section 3.4.14). We use the terminology *pre-* here because it is seems most apt for indicating our intention in making this definition.

4.4.3 The Case of a Lattice

As stated previously, Beilinson and Drinfeld make the claim that the functors $\mathcal{D}iv(X)_S$ and $\mathcal{D}iv(X, \Gamma)_S$, for Γ a lattice, are formally smooth ind-schemes which can be represented as the inductive limit of a directed family of subschemes that are finite and flat over Z because the functor assigning the set of all effective Cartier divisors of degree n is representable by $\text{Sym}^n(X)$. First, we expound this argument for this in the case they considered: that is of a lattice $\Gamma \simeq \mathbb{Z}^r$.

$\mathcal{D}iv(X)$ is the group of Cartier divisors, so for each Z gives a group freely generated by points of X and indexed by Z. Thus, $\mathcal{D}iv(X) \otimes \mathbb{Z}$ does nothing, since the addition in the $\mathbb{Z}$ component commutes across the tensor product with the group addition in $\mathcal{D}iv(X)$. In $\mathcal{D}iv(X) \otimes \mathbb{Z}^2$, we can write any $\sum x_i \otimes (a_i, b_i)$ as $\sum x_i \otimes (a_i, 0) + \sum x_i \otimes (0, b_i)$, so we see that we really get two copies of $\mathcal{D}iv(X)$. Likewise, we can see that $\mathcal{D}iv(X) \otimes \mathbb{Z}^r$ gives us r copies of $\mathcal{D}iv(X)$, and that such a functor is an ind-scheme by taking the finite product of r copies of that for $\mathcal{D}iv(X)$. (See Section 4.4.3.1 for seeing products of ind-schemes as ind-schemes.) Similarly, the same arguments can be made of $\mathcal{D}iv_e(X)$ in relation to tensoring with copies of $\mathbb{N}$.

When we consider our first example of cones in a lattice, we need to be careful about how the limit is constructed for $\mathbb{Z}$. It should be a union over $n \in \mathbb{Z}$ of $\text{Sym}^{|n|} X$, but we need to be careful about how this is taken for negative n. When we restrict to $\mathbb{N} \subset \mathbb{Z}$, our simplest example of a cone, we would then get the union of $\text{Sym}^n X$

for positive n, which would still give an ind-scheme. Then it immediately follows that when M is a finite product of copies of $\mathbb{Z}$ with copies of $\pm\mathbb{N}$, we would get an inductive limit of finite products of schemes. We will try to make this precise in the following.

We give here the details omitted by Beilinson and Drinfeld. We look at the case of $\mathcal{D}iv_e(X)$. As the authors claim, effective Cartier divisors of degree n are representable by $\operatorname{Sym}^n X$, so divisors of degree at most N are representable by the finite disjoint union $\sqcup_{k=0}^n \operatorname{Sym}^k X$, where we write $\operatorname{Sym}^0 X$ here in place of a point, the zero divisor. We want a union $\bigcup_{n=0}^\infty \sqcup_{k=0}^n \operatorname{Sym}^k X$ for effective Cartier divisors. So this colimit, under the natural inclusions (i.e. the union), would give us the case of $\mathcal{D}iv_e(X)$, or $\mathcal{D}iv_e(X) \otimes \mathbb{N}$. For $\mathcal{D}iv(X) \otimes \mathbb{Z} = \mathcal{D}iv_e(X) \otimes \mathbb{Z}$, we want another copy of $\operatorname{colim} \sqcup_{k=1}^n \operatorname{Sym}^k X$ for the negative coefficients. We can't take a union because this would not allow divisors of the form $\{x, -y\}$, $x \neq y$: It would only allow *all* coefficients positive, or *all* coefficients negative. We can instead consider the product $(\bigcup_{n=0}^\infty \sqcup_{k=0}^n \operatorname{Sym}^k X) \times (\bigcup_{n=0}^\infty \sqcup_{k=0}^n \operatorname{Sym}^k X)$. However, this would allow divisors of the form $(x, -x)$, where we need to collect all terms in one point to one coefficient. We need pairs $((x_1, \ldots, x_k), (y_1, \ldots, y_l))$, in which the x_i can occur with multiplicities in the first bracket, accounting for divisors with positive coefficients, and, likewise, the y_j may occur with multiplicities in the second bracket (each occurrence here counting as $(-1)y_j$) for points with negative coefficients, and *all* points in the first set of brackets distinct from *all* points in the second brackets.

To do this, we consider $U_{k,l} \subset X^k \times X^l$, the open subset of points $((x_1, \ldots, x_k), (y_1, \ldots, y_l))$ for which $x_i \neq y_j$ for any i, j. This comes with an obvious action of the finite group $G_{k,l} := \operatorname{Sym}_k \times \operatorname{Sym}_l$, induced from permuting the coordinates of $X^k \times X^l$, with Sym_k acting on X^k and Sym_l on X^l: specifically, $(\sigma, \tau) \cdot ((x_1, \ldots, x_k), (y_1, \ldots, y_l)) = (\sigma(\underline{x}), \tau(\underline{y})) = ((x_{\sigma(1)}, \ldots, x_{\sigma(k)}), (y_{\tau(1)}, \ldots, y_{\tau(l)}))$. Let $U_{k,l}^{G_{k,l}}$ be the quotient of $U_{k,l}$ by this group action, i.e. the orbit spaces of points. We then take unions over k and l, and require the colimit of these as they tend to infinity as follows:

$$\operatorname{colim}_{m_1,m_2} \left(\sqcup_{k=0}^{m_1} \sqcup_{l=0}^{m_2} U_{k,l}^{G_{k,l}} \right),$$

where one can equally well take $m_1 = m_2$. Here, the degenerate cases with $k = 0$ or $l = 0$ (or both) account for $U^G = \operatorname{Sym}^l X$, corresponding to divisors with no positive coefficients, $U^G = \operatorname{Sym}^k X$, with no negative coefficients, and, in the case of $k = l = 0$, the zero divisor. Note that we have a natural inclusion of $\mathcal{D}iv(X, \mathbb{N}) \hookrightarrow \mathcal{D}iv(X, \mathbb{Z})$

given by the sub-ind-scheme of terms $\mathrm{colim}_{m_1}\left(\sqcup_{k=0}^{m_1} U_{k,0}^{G_{k,0}}\right)$ with $U_{k,0}^{G_{k,0}}$ representing the degenerate case of $\mathrm{Sym}^k X$ as just remarked.

Note that the $U_{k,l}^{G_{k,l}}$ are indeed schemes. X is a smooth algebraic curve, i.e. a smooth algebraic variety of dimension 1. Any smooth curve is a quasi-projective variety ([Vak99], or [Oss09]). The product of two (so finitely many) quasi-projective varieties is quasi-projective, using the Segre embedding ([Mor13], solving Problem 3.16 of Hartshorne's *Algebraic Geometry*). Thus, each $X^k \times X^l$ is quasi-projective, and, hence, the open subschemes $U_{k,l}$ are also. For an action of a finite group G on a scheme Y in which every $y \in Y$ has an affine open neighbourhood preserved by the G-action, one can define the quotient scheme by gluing the natural quotients on the affine open subsets U, on each of which one takes $\mathrm{Spec}(A^G)$, where A is the algebra with $U = \mathrm{Spec}(A)$ ([Mus11], following SGA1). In the case that Y is a quasi-projective scheme, one can always find such open affines (ibid.).

Further, a finite disjoint union of schemes is a scheme, since disjoint union is the coproduct in schemes, corresponding to taking the product of algebras in the affine case (so locally). We take a colimit over the resulting inductive system.

Now we examine the case of $\mathcal{D}iv_e(X)_S \otimes \Gamma$ for $\Gamma = \mathbb{N}$ or $\Gamma = \mathbb{Z}$. Fix $S \subset X \times Z/Z$.

For ease of notation, write W_α for $\sqcup_{k=0}^{m_1} \sqcup_{l=0}^{m_2} U_{k,l}^{G_{k,l}}$, with α indicating the multiple indices. Then the functors $W_\alpha \times Z$ with the projection maps $p_Z : W_\alpha \times Z \to Z$ form a directed system of schemes over Z. Let I_S^α denote the ideal sheaf that cuts out the closed subscheme $\tilde{S}_\alpha \subset W_\alpha \times Z$, where $\tilde{S}$ is S considered as the graph of a map $Z \to W_\alpha$. Then, for a scheme Y over Z, via the map $f_Y : Y \to Z$, divisors in $\mathcal{D}iv_e(X)_S \otimes \Gamma(Y)$ contained set-theoretically in the pullback of S are precisely those maps $D_Y : Y \to W_\alpha$ that factor through

$$Y \xrightarrow{(D_Y, f_Y)} \tilde{S}_{\alpha,r} \subset W_\alpha \times Z$$

where $\tilde{S}_{\alpha,r}$ denotes the closed subscheme given by the relative Spec of $\mathcal{O}_{W_\alpha \times Z} \big/ (I_S^\alpha)^r$ for some power r of the ideal. The introduction of these powers allow us to look at divisors supported on the formal completion along S, giving the required set-theoretic containment, which translates to taking an additional inductive limit over this directed system formed by the different powers r.

We may apply the same technique to acquire the functors $\mathcal{D}iv(X)_S \otimes \mathbb{N}$ by taking $W_\alpha = W_k$ to be $\sqcup_{l=0}^{k} \operatorname{Sym}^l X$.

4.4.3.1 Direct Products of Ind-Schemes:

We explain here how to take a direct product of ind-schemes, which we needed in the case of lattices above and will need in our new examples.

A product of filtered categories $C \times D$ is filtered in a natural way: For any two pairs in the product (c_1, d_1) and (c_2, d_2), there exist objects $c_3 \in C$ and $d_3 \in D$ with $c_1, c_2 \to c_3$ and $d_1, d_2 \to d_3$, so $(c_1, d_1), (c_2, d_2) \to (c_3, d_3)$. Likewise, coequalisers of parallel maps exist as the products of the coequalisers in C and D. Thus, we define the product of ind-schemes to be the inductive limit diagram produced in this way. It is clear from the nature of a directed system that one could take particular subcategories and still produce the same colimit object where one exists, so any such diagrams will be considered equivalent.

The Case of Finite Abelian Groups:

We will construct the functors $\mathcal{D}iv_e(X) \otimes \mathbb{Z}_n$. $\mathcal{D}iv_e(X) \otimes \{0\} = \{0\}$, a point. $\mathcal{D}iv_e(X) \otimes (\mathbb{Z}/2\mathbb{Z})$ gives divisors of any positive degree, but in which each point can only occur once. (It is the free $\mathbb{Z}_2$ module generated by the distinct points in X.) Thus we want to take $\operatorname{colim}_n(\sqcup_{k=0}^{n} Y_k)$, where Y_k is the scheme that gives divisors of degree k in which all pairs of points are distinct ($Y_0 = \text{pt}$). But this is the open subscheme of $\operatorname{Sym}^k X$ given by the complement to the union of the divisors cut out by the finitely many equations $x_i = x_j$: that is, we take $U_k \subset X^k$ cut out by these equations, and then take the quotient by the action of Sym_k. To get $\mathcal{D}iv_e(X) \otimes (\mathbb{Z}/r\mathbb{Z})$, up to $r-1$ points can be equal, so the schemes Y_k in the above must be replaced by $\operatorname{Sym}^k X$ for $k < r$ and, for $k \geq r$, the quotient by the $\operatorname{Sym}^k X$-action of the open subscheme of X^k given by the complement to the finite union of closed subschemes cut out by the equations $x_{i_1} = x_{i_2} = \cdots = x_{i_r}$ (each of which is a finite union of subschemes that are the intersection of finitely many closed subschemes of the form $x_i = x_j$).

By the Structure Theorem for finite abelian groups, any finite abelian group is a product of the form $\prod_{i=1}^{r}(\mathbb{Z}/c_i\mathbb{Z})$ with $c_i \mid c_{i+1}$ for all i. As with $\mathbb{Z}^r$, we can split the coordinates of the product and we get here a product of the functors $\mathcal{D}iv_e(X) \otimes (\mathbb{Z}/c_i\mathbb{Z})$. Thus, we get the inductive limit of the products of the schemes we have for each of these.

The Case of Cones in Lattices:

We use the following definition of a cone in a lattice:

Definition 4.4.4. *Let V be a real vector space of dimension n, and let $\varphi_1, \dots, \varphi_n \in V^*$ form a linearly independent set (so a basis). Then the cone defined by these dual vectors will be the set $\{v \in V \mid \varphi_1(v) \geq 0, \dots \varphi_n(v) \geq 0\} \cap \mathbb{Z}^n$.*

Such a cone defines a monoid M. Not all of these will be finitely generated, e.g. if the equations define hyperplanes with irrational slopes. We will first restrict our attention to consider the case of a rational cone. Without loss of generality, by scaling the maps φ_i, we may assume the maps φ_i have integer coefficients.

Let φ be a linear map $\mathbb{Z}^n \to \mathbb{Z}$ with integer coefficients. Then φ defines a map of divisors $\varphi_* : \mathcal{D}iv(X, \mathbb{Z}^n) \to \mathcal{D}iv(X, \mathbb{Z})$ given on k points by $\sum_{i=1}^{k} \gamma_i x_i \mapsto \sum_{i=1}^{k} \varphi(\gamma_i) x_i$. Thus, we see that the points in a cone (as defined above) would be exactly those points in the preimage of $\mathcal{D}iv(X, \mathbb{N})$ for all of the maps $\varphi_1, \dots, \varphi_n$. Consequently, if we can find the sub-ind-schemes of $\mathcal{D}iv(X, \mathbb{Z}^n)$ mapping into $\mathcal{D}iv(X, \mathbb{N})$ under each map φ_i, their intersection should give the required functor $\mathcal{D}iv(X, M)$ for the cone as an ind-scheme.

Thus, the object we want to study is the fibre product in presheaves $\mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi_{\geq 0}}$ as given by the diagram:

$$\begin{array}{ccc}
\mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi_{\geq 0}} & \longrightarrow & \mathcal{D}iv(X, \mathbb{N}) \\
{\scriptstyle f}\downarrow & & \downarrow \\
\mathcal{D}iv(X, \mathbb{Z}^n) & \xrightarrow[\varphi_*]{} & \mathcal{D}iv(X, \mathbb{Z})
\end{array}$$

Since we have shown in Section 4.4.3 above that $\mathcal{D}iv(X, \mathbb{Z}^n)$ is an ind-scheme, it will suffice to show that f is a closed immersion: that is, we want to show that

$\mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi \geq 0}$ is a union of objects whose images under f intersect the schemes in the directed system for $\mathcal{D}iv(X, \mathbb{Z}^n)$ in closed subschemes (so the maps on individual schemes are closed immersions). The map f will naturally be an inclusion of a subfunctor because the map $\mathcal{D}iv(X, \mathbb{N}) \hookrightarrow \mathcal{D}iv(X, \mathbb{Z})$ is. First, we note that the inclusion $\mathcal{D}iv(X, \mathbb{N}) \hookrightarrow \mathcal{D}iv(X, \mathbb{Z})$ is a closed immersion. As noted above, this is the inclusion of the sub-ind-scheme formed by the colimit $\mathrm{colim}_{m_1} \left(\sqcup_{k=0}^{m_1} U_{k,0}^{G_{k,0}} \right)$ of terms with $l = 0$, $U_{k,0}^{G_{k,0}} = \mathrm{Sym}^k X$, in $\mathrm{colim}_{m_1,m_2} \left(\sqcup_{k=0}^{m_1} \sqcup_{l=0}^{m_2} U_{k,l}^{G_{k,l}} \right)$. So we see that the intersection of each of the schemes $\sqcup_{k=0}^{m_1} U_{k,0}^{G_{k,0}}$ with the schemes in the image is either the whole scheme, a point, or some disjoint union comprising part of a larger disjoint union.

By [TS16] [Tag 01JU], the property of being a closed immersion (for schemes) is stable under base change: that is, for schemes X, Y, and S, with $f : X \to S$ and $g : Y \to S$ maps of schemes, if f is a closed immersion, then $X \times_S Y \to Y$ is a closed immersion. Hence, it suffices to show that $\mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi \geq 0}$ is a union of the fibre products of the individual schemes comprising the ind-schemes $\mathcal{D}iv(X, \mathbb{N})$ and $\mathcal{D}iv(X, \mathbb{Z}^n)$ over those in $\mathcal{D}iv(X, \mathbb{Z})$ (since the scheme maps $\sqcup_{k=0}^{m_1} U_{k,0}^{G_{k,0}} \hookrightarrow \sqcup_{k=0}^{m_1} \sqcup_{l=0}^{m_2} U_{k,l}^{G_{k,l}}$ are closed immersions). But finite limits commute with filtered colimits in Set ([ML98], p.215) (so in presheaves, since limits here are taken pointwise), so this is the case.

Thus, $\mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi \geq 0}$ is a closed sub-ind-scheme of $\mathcal{D}iv(X, \mathbb{Z}^n)$ and, hence, for our monoid M corresponding to the cone cut out by the inequalities $\varphi_1 \geq 0, \ldots, \varphi_n \geq 0$, $\mathcal{D}iv(X, M) = \bigcap_{i=1}^n \mathcal{D}iv(X, \mathbb{Z}^n)_{\varphi_{i_{\geq 0}}}$ is an ind-scheme.

When σ is not a *rational* cone—that is, when it is cut out by equations, some of which have irrational coefficients—we can approximate the equations by equations with all rational coordinates (equivalently, integral coordinates), such that we approximate the cone from the inside. We can then express the functors as the colimit of the above ind-schemes Y_i for each rational cone σ_i over the resultant directed system of inclusions of these.

As before, to get the ind-scheme $\mathcal{D}iv_S(X, M)$, letting W_α again denote the terms in our inductive limit, we need to replace the W_α with the closed subschemes of $W_\alpha \times Z$ exactly as defined earlier for lattices and take the colimit of the ind-schemes over the inverse system of algebras $\mathcal{O}_{W_\alpha \times Z}/(I_S^\alpha)^r$ (i.e. the direct system of closed subschemes indexed by r).

Thus, we have shown that the functors $\mathcal{D}iv(X, C)$ and $\mathcal{D}iv_S(X, C)$ are ind-schemes when C is a cone in a lattice, as well as $\mathcal{D}iv(X, M)$ for M a finite abelian group.

4.4.5 Obtaining Chiral Algebras

We can now prove Theorem 4.2.2. Let C be a cone in a lattice, and suppose we have a super line bundle with factorisation (λ, c) on the factorisation monoid of C-valued divisors. We now need to argue why we can push forward such a line bundles with factorisation to give a factorisation algebra in much the same way as Beilinson and Drinfeld do for lattice algebras, as described in Section 4.1.5.1. However, careful examination of Section 4.1.5.1 shows that the structure of Γ is not used, beyond where it is used in defining the factorisation structure c on the original line bundle. Thus, since we have such a line bundle with factorisation (λ, c) on the cone, and we have the required ind-scheme structures to construct the dual module structures for the $A^l_{S,Z}$, we can define these modules, and push forward the factorisation structure, for our new examples of cones in lattices, giving new examples of factorisation algebras, i.e. chiral algebras.

Finally, we can prove Theorem 4.2.3, on the existence of a new class of chiral algebras when C is a cone in a lattice. By Theorem 4.2.2, if we have a super line bundle on the factorisation monoid of C-valued divisors with factorisation, then we can push it forward to form a factorisation algebra. But by Proposition 4.3.2, there exists super line bundles with factorisation on this chiral monoid. This concludes the proof.

Definition 4.4.6. *The chiral algebras given in Theorem 4.2.3, for C a cone in a lattice, will be referred to as* toric chiral algebras*.*

4.4.7 Some Remarks on Toric Chiral Algebras

Firstly, we note that we have constructed chiral algebras from cones in lattices, and, hence, there should be a natural relationship between the toric chiral algebra and the associated lattice chiral algebra for the relevant lattice. We believe that this is indeed the case, with this taking the form of a map from the factorisation algebra defined for the cone to that defined for the lattice, and that this is most naturally seen when viewed with the language of ind-coherent sheaves. We are grateful to Dario Beraldo for explaining to us some of the functoriality of this theory, and we refer the reader to [Gai13] for a comprehensive exposition.

We note that we have the following commutative diagram of prestacks:

$$\begin{array}{ccc} & & \lambda_L \\ & & \downarrow \\ \mathcal{X} & \overset{i}{\hookrightarrow} & \mathcal{Y} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & Z & \end{array}$$

where $S \subset Z$ are the divisor and scheme we have fixed for the functor $\mathcal{D}iv_S(X, C)$, $C \subset L$ will be the cone and lattice, and $\mathcal{X} = \mathcal{G}_{S,Z,C}$ and $\mathcal{Y} = \mathcal{G}_{S,Z,L}$ the ind-schemes representing the functors $\mathcal{D}iv_S(X, C)$ and $\mathcal{D}iv_S(X, L)$ respectively. In order to apply the theory of ind-coherent sheaves, it will suffice that the map i be a closed immersion. But this was a fact that we demonstrated and used in our proof that the functors $\mathcal{D}iv_S(X, C)$ are ind-schemes.

Beilinson and Drinfeld give an explicit construction of the modules as $A^l_{S,Z} := \lambda \otimes_R R^* := \bigcup \lambda \otimes_R \mathcal{H}om_{\mathcal{O}_Z}(R_\alpha, \mathcal{O}_Z)$ ([BD04], page 274), in work which predates the language of ind-coherent sheaves. However, they describe it as a pushforward of the line bundle to Z given by $A^l_{S,Z} = \pi_!(\lambda_{\mathcal{G}_S} \otimes \pi^! \mathcal{O}_Z)$. Since $\mathcal{Y} = \operatorname{Spf} R$, where $R = \lim R_\alpha$, we would want to define a quasi-coherent sheaf on Z by taking the inverse limit of the λ_{R_α} pushed forward to Z. However, an inverse limit of quasi-coherent sheaves is not in general quasi-coherent. Thus, the explicit construction is the functor of converting the ind-coherent sheaf into a quasi-coherent sheaf and then pushing it forward to Z (or vice-versa, since the functors commute).

All of our spaces are smooth, so $\otimes \pi^! \mathcal{O}_Z$ gives the functor Ψ_Z from ind-coherent sheaves to quasi-coherent sheaves defined as the tautological extension of the inclusion Coh $\hookrightarrow$ QCoh. This is given in [Gai13], Lemma 1.4.2: specifically, the statement that, on objects $\varepsilon \in \mathrm{QCoh}(Z)$ and $\mathcal{F} \in \mathrm{IndCoh}(Z)$, Ψ_Z gives a canonical isomorphism $\Psi_Z(\varepsilon \otimes \mathcal{F}) \simeq \varepsilon \otimes \Psi_Z(\mathcal{F})$, where the tensor products are the actions as $\mathrm{QCoh}(Z)$-module categories. In this, we take $\mathcal{F} = \mathcal{O}_Z$ and use the fact that Ψ_Z is an equivalence in this case, by Lemma 1.1.6, as Z is a scheme. We believe that the formulae given by

Beilinson and Drinfeld (of tensoring with the dualising sheaf) are exactly the functor of this action of QCoh(Z) in giving the functor Ψ_Z : IndCoh → QCoh, followed by the pushforward of the resulting quasi-coherent sheaf on $\mathcal{Y}$ (respectively, $\mathcal{X}$) to Z. Alternatively, by [Gai13] Proposition 3.1.1, Ψ_Z commutes with the functor of taking pushforwards (as ind-coherent or quasi-coherent sheaves as appropriate), so we can, instead, take the ind-coherent sheaf pushforward and then apply the functor to obtain a quasi-coherent sheaf.

Then the module $A^l_{S,Z,L}$ is, by the above formula, the pushforward along the map $\mathcal{Y} \to Z$ of $\Psi_Z(\lambda_L)$: namely, $(\pi_{\mathcal{Y}})_!(\Psi_Z(\lambda_L))$. Using the fact that we defined $\lambda_C = i^*\lambda_L$, the module we defined for the cone is $(\pi_{\mathcal{X}})_!(\Psi_{\mathcal{Z}}(i^*\lambda_L))$. But $\Psi_Z(i^*\lambda_L) = i^!\Psi_Z(\lambda_L)$, and $(\pi_{\mathcal{X}})_!(i^!\Psi_Z(\lambda_L)) = (\pi_{\mathcal{Y}})_!(i_*i^!\Psi_Z\lambda_L)$ by commutativity. So we are comparing $A^l_{S,Z,L} = (\pi_{\mathcal{Y}})_!(\Psi_Z(\lambda_L))$ and $A^l_{S,Z,C} = (\pi_{\mathcal{Y}})_!(i_*i^!\Psi_Z(\lambda_L))$ and we see easily that we have a map $A^l_{S,Z,C} \to A^l_{S,Z,L}$ given by the counit of the adjunction.

Another point of interest is that it would be good to demonstrate that our new examples of chiral algebras are indeed distinct from those constructed from lattices by Beilinson and Drinfeld. One way to do this might be to use the Γ-grading inherited from $\mathcal{O}_X[\Gamma]$-coaction ([BD04] Section 3.10.5). By definition, a lattice chiral algebra is a $\mathcal{T}^\vee_X$-module, where $\mathcal{T}^\vee_X = \operatorname{Spec} F^l_X = \mathcal{J}T^\vee_X$, the jet $\mathcal{D}_X$-scheme of the group scheme $T^\vee_X := T^\vee \times X$. These F_X-cotorsors have a coaction of $\mathcal{O}_X[\Gamma]$, giving a lattice chiral algebra A a Γ-grading $A = \bigoplus A^\gamma$ that is compatible with the chiral product ([BD04], Section 3.10.6): μ_A sends $j_*j^*A^{\gamma_1} \boxtimes A^{\gamma_2}$ to $\Delta_*A^{\gamma_1+\gamma_2}$.

Firstly, we note that, since the chiral product is compatible with the grading, there exists a distinct chiral subalgebra of A given by taking only the graded pieces for the γ in a given cone in the lattice. This will be distinct from the whole lattice algebra as, by [BD04] Section 3.10.6 (referencing Section 3.10.5), there exists a super line bundle in each graded degree of the lattice chiral algebra. So we know that distinct subalgebras exist. We believe that our construction gives these subalgebras.

Given a lattice chiral algebra, we see from [BD04] p. 261 that the line bundles found to exist in each degree $\lambda^\gamma_A := A^{l_\gamma}_0 \subset A^{l_\gamma}$, as described in the previous paragraph, form a θ-datum, together with maps c and κ, which they define. However, on p.264, Beilinson and Drinfeld show that the lattice chiral algebras they construct from θ-data give rise to a θ-datum canonically identified with the θ-datum they started with (for example, by moving through the equivalence of Proposition 4.1.11). So the line bundles should exist within the graded pieces. However, when we examine the maps in Proposition 4.1.11 ([BD04], p.262), we see that the line bundle λ^γ is pulled back exactly from that over tensors in $\mathcal{D}iv(X, \Gamma)$ of the form $- \otimes \gamma$. We hope that if we have restricted to only those γ in the cone, we will only obtain line bundles in the degrees for which we have divisors. Hence, only in the graded pieces for γ within the cone.

Chapter 5

Future Directions

There is freedom waiting for you,
On the breezes of the sky,
And you ask "What if I fall?"
Oh but my darling,
What if you fly?

Erin Hanson

Our main aim now is to compute the chiral homology of our new examples of chiral algebras. In Chapter 4 of [BD04], the authors define chiral homology and give specific resolutions for computing the chiral homology of chiral algebras. Specifically, in Section 4.2.11, they define chiral homology for a not-necessarily-unital chiral algebra A on X as the de Rham cohomology of the associated Chevalley-Cousin complex, $C(A)$. In actual fact, they give the definitions for the more-general, DG setting. (See also [FG12], Section 6.3). Beilinson and Drinfeld also show that (Equation (4.9.1.2) of the Proposition in Section 4.9.1), for the case of a torus T, there exists the following equivalence of line bundles on torsors with those on divisors:

$$\mathcal{P}ic(\mathcal{T}ors(X,T)) \xrightarrow{\sim} \mathcal{P}ic(\mathcal{D}iv(X,\Gamma)).$$

They then show the following result linking chiral homology of a lattice chiral algebra to the right derived global sections functor of line bundles on torsors:

Theorem 5.0.1. *Let A be a lattice chiral algebra, λ a super line bundle on $\mathcal{T}ors(X,T)$, and $\lambda^* = \lambda^{\otimes -1}$ the dual bundle. Then there is a canonical quasi-isomorphism*

$$C^{ch}(X,A) \xrightarrow{\sim} R\Gamma(\mathcal{T}ors(X,T),\lambda^*)^* = \oplus_{\gamma\in\Gamma} R\Gamma(\mathcal{T}ors(X,T)_\gamma,\lambda^*)^*,$$

compatible with the Γ-gradings.

We may be able to use this to understand chiral homology for our new chiral algebras by looking at the subobjects of $\mathcal{T}ors(X,T)$ given by the restricting the map from $\mathcal{D}iv(X,\Gamma)$ to divisors within the cone in the lattice given in [BD04], p.358, by $D \otimes \gamma \mapsto \mathcal{O}(D)^{\otimes \gamma}$. By the Tannakian formalism for geometric stacks (See Theorem 3.4.6, or [Lur05]), $\mathcal{T}ors(X,T) = \mathrm{Hom}(X, BG) \simeq \mathrm{Hom}_{\otimes}(\mathrm{Rep}(G), \mathrm{QC}(X))$. Thus, we would aim to consider the image of the cone in this space of tensor functors.

It would also be interesting to understand how our new examples of chiral algebras relate to the construction of lattice vertex algebras, from which lattice chiral algebras originate. (See [FBZ01], p.80.) There is an interesting construction of a super vertex algebra for the intersection of a lattice with the union of the positive and negative quadrants in [DH17], but we are largely unfamiliar with the vertex algebra literature.

Another direction would be to extend our new examples to a case of finite abelian groups. We considered previously divisors valued in finite abelian groups. This does not allow us to define a factorisation monoid via a map of fibred categories as in [BD04]. We remark here how one could consider, instead, divisors that should correspond to torsors over these, which should lead to another application of the construction of [BD04].

Firstly, note that $\mathbb{G}_m$-torsors are equivalent to line bundles (by removing the zero section and looking at transition functions $U_i \cap U_j \to \mathbb{G}_m = \mathrm{GL}_1$), which are equivalent to the groupoid of divisors modulo principal divisors. Further, the category of $\mathbb{G}_m$-torsors trivialised away from a finite set of points is equivalent to that of divisors. We have an evident short exact sequence of groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Taking group algebras and dualising gives that $\mathrm{Spec}\left(k[t] \big/ (t^n - 1)\right) = \mu_n$, the group of nth roots of unity, is the kernel of the map $\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[t])$ induced by $t \mapsto t^n$: that is, we have:

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{a \mapsto a^n} \mathbb{G}_m \longrightarrow 0.$$

Letting $\{U_i\}_{i \in I}$ be a cover for X, we can see easily by considering transition functions $U_i \cap U_j \to \mu_n \hookrightarrow \mathbb{G}_m$ that μ_n-torsors are precisely $\mathbb{G}_m$-torsors whose nth power is trivial. These should correspond to divisors D for which nD is principal. Since divisors for which nD is principal form a closed subgroup, this should give rise to a prefactorisation monoid, and, we belive, a factorisation monoid. We would hope to then apply the construction of [BD04] to these divisors.

We could also look to define a suitable notion of principal bundles over commutative monoids. Such a definition might allow us to define chiral algebras for a wider class of examples. After initial consideration, one finds that just taking the obvious definition of what would be a "reasonable" definition of torsors over a general monoid gives torsors over the group completion of the monoid, which is not the object we seek, e.g. it gives the result for $\mathbb{Z}$ as the corresponding result for $\mathbb{N}$ also. However, if we could look to find a suitable replacement for the object $\mathcal{T}ors(X,T)$ for more general monoids, or, at least, our examples of cones in lattices, we may be able to understand chiral homology like this. Likewise, understanding chiral homology for our examples may give us more insight into the geometry of these objects.

Further, we note that the definition of θ-data naturally restricts to values of γ within a cone in a lattice, so it would be an interesting and logical question to ask if we can characterise our new examples of chiral algebras via θ-data.

Another option, although we have not thought much about this, would be to try to extend our examples to when Γ is a simplicial abelian group and compute the chiral cohomology for these.

A further option would be to look more at the relationship between divisors and torsors and to try to define chiral algebras for actions of the torus twisted by the action of an algebraic group.

Let H be an algebraic group, $\mathcal{F}$ a sheaf or torsor on X and $\alpha_{x,h} : \mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{hx}$, the data of an action of H on the sheaf or torsor. In the case of an equivariant action, we have diagrams of the following form:

$$H \times H \times X \underset{1 \times \text{act}}{\overset{\text{mult} \times 1}{\rightrightarrows}} H \times X \underset{\text{proj.}}{\overset{\text{act}}{\rightrightarrows}} X$$

$$\underset{\mathcal{F}_{hx}}{\text{act}^* \mathcal{F}} \quad \xleftarrow{\sim} \quad \underset{\mathcal{F}_x}{\text{proj.}^* \mathcal{F}} : \alpha,$$

where the two pullbacks of α to $H \times H \times X$ must be equal. On the level of fibres, this gives:

$$\mathcal{F} \xrightarrow[\sim]{\alpha_h} h^*\mathcal{F} \xrightarrow[\sim]{\alpha_{h'}} h'^*(h^*\mathcal{F})$$
$$\xrightarrow[\alpha_{h'h}]{}$$

$$h^*\mathcal{F} \overset{\sim}{\leftarrow} \mathcal{F} : \alpha_{h'}.$$

We can, instead, look at twisted equivariance, in which the T action on a T-torsor $\mathcal{F}$ interacts with the action of H. Suppose H acts on T by automorphisms. Write $\mathcal{F}^{i(h)}$ for the twisting of $\mathcal{F}$ by the action of $h \in H$: That is, each element $t \in T$ acts by $i(h)(t)$. We now have maps $\alpha : \mathcal{F} \overset{\sim}{\to} (h^*\mathcal{F})^{i(h)}$, which must satisfy the twisted equivariance condition:

$$\mathcal{F} \xrightarrow[\sim]{\alpha_h} (h^*\mathcal{F})^{i(h)} \xrightarrow[\sim]{\alpha_{h'}} \left(h'^*\left(h^*\mathcal{F}\right)^{i(h)}\right)^{i(h')}$$
$$\xrightarrow[\alpha_{h'h}]{}$$

One possibility for further research could be to look at constructing and classifying H-equivariant lattice chiral algebras, for either straightforward equivariance of the group action, or, more generally, for twisted actions.

Another direction would be to try to relate the chiral homology for the chiral algebras associated to the cones to the toric varieties associated to the affine semigroups given by the cones ([CLS11], p. 17):

Proposition 5.0.2. *Let S be an affine semigroup. Then:*

1. $\mathbb{C}[S]$ *is an integral domain and finitely generated as a* $\mathbb{C}$*-algebra.*
2. $\mathrm{Spec}(\mathbb{C}[S])$ *is an affine toric variety whose torus has character lattice* $\mathbb{Z}S$*, and if* $S = \mathbb{N}\mathcal{A}$ *for a finite set* $\mathcal{A} \subseteq M$*, then* $\mathrm{Spec}(\mathbb{C}[S]) = Y_{\mathcal{A}}$.

Here, $\mathbb{C}[S]$ denotes the monoid algebra on the commutative monoid S and $Y_{\mathcal{A}}$ is the affine toric variety canonically defined for the finite set $\mathcal{A}$.

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