

On Parameterized Complexity of Group Activity Selection Problems on Social Networks

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ABSTRACT

In Group Activity Selection Problem (GASP), players form coalitions to participate in activities and have preferences over pairs of the form (activity, group size). Recently, Igarashi et al. [11] have initiated the study of group activity selection problems on social networks (gGASP): a group of players can engage in the same activity if the members of the group form a connected subset of the underlying communication structure. Igarashi et al. have primarily focused on Nash stable outcomes, and showed that many associated algorithmic questions are computationally hard even for very simple networks. In this paper we study the parameterized complexity of gGASP with respect to the number of activities as well as with respect to the number of players, for several solution concepts such as Nash stability, individual stability and core stability. The first parameter we consider is the number of activities. For this parameter, we propose an FPT algorithm for Nash stability for the case where the social network is acyclic and obtain a W[1]-hardness result for cliques (i.e., for classic GASP); similar results hold for individual stability. In contrast, finding a core stable outcome is hard even if the number of activities is bounded by a small constant, both for classic GASP and when the social network is a star. Another parameter we study is the number of players. While all solution concepts we consider become polynomial-time computable when this parameter is bounded by a constant, we prove W[1]-hardness results for cliques (i.e., for classic GASP).

Keywords

Group activity selection problems, social networks, parameterized complexity

1. INTRODUCTION

In multiagent systems, agents form coalitions to perform tasks. A useful model for analyzing how tasks can be allocated to groups of agents is the *group activity selection problem* (GASP), proposed by Darmann et al. [6]. In GASPs, participants express preferences over pairs of the form (activity, group size). The activities are then assigned to participants so as to achieve the best performance for the whole system as well as to satisfy individual agents. The key idea behind this formulation is that ideal group size depends on the task at hand: in a company, an ideal size of the sales team may differ from that of a web developers' team.

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However, there is one important feature missing from the standard GASP model, namely the *feasibility* of resulting groups. In many real-life scenarios, smooth communication among members of a group is crucial in order for different individuals to work together, and hence one needs to take into account communication structures among agents. For instance, a group of employers are unable to realize their full potential if no agent knows each other. Nevertheless, the basic GASP framework imposes no restrictions on how agents can split into different groups.

A succinct way to capture such restrictions is to represent communication structures by undirected graphs. This idea dates back to cooperative games with graph structure, proposed by Myerson [12]. Under Myerson's proposal, nodes of the graph correspond to players and edges represent communication links between them. Recently, Igarashi et al. [11] considered group activity selection on social networks (gGASP), where groups need to be connected in their underlying social network in order to achieve certain tasks. The focus of that work was on core stability and Nash stability. In contrast with similar settings in cooperative games [8, 3, 10], many of the computational problems were shown to be NP-complete for very simple network structures; in particular, deciding the existence of core and Nash stable outcomes was shown to be NP-complete even when the social network is either a path, a star, or has connected components of size at most four.

Motivated by the work of Igarashi et al. [11], we investigate the parameterized complexity of finding stable outcomes in group activity selection problems. A problem is fixed parameter tractable (FPT) with respect to a parameter k if each instance I of this problem can be solved in time $O(f(k))\text{poly}(|I|)$ where f is a computable function. We show that the problem of deciding the existence of Nash stable outcomes for gGASPs on acyclic graphs is fixed parameter tractable with respect to the number of activities, thereby solving a problem left open by Igarashi et al. [11]. For general graphs, we obtain a W[1]-hardness result, implying that this problem is unlikely to admit an FPT algorithm; in fact, our hardness result holds even for gGASPs on cliques (which correspond to the classic GASP).

We then consider a weaker stability concept, namely, *individual stability*. In the context of hedonic games, it is known that an individually stable solution always exists and can be computed in polynomial time when the communication structure is acyclic [10]. In contrast, we show that an instance of gGASP may have no individually stable outcomes, even if the underlying network is a path. We then analyze the complexity of computing individually stable solutions, and discover that from an algorithmic point of view individual stability is very similar to Nash stability.

Unfortunately, our FPT results do not extend to core stability: we prove that checking the existence of core stable assignments is NP-complete even for gGASPs on stars with two activities; for classic

Table 1: Overview of our complexity results. ‘NS’ stands for Nash stability, ‘IS’ stands for individual stability, ‘CR’ stands for core stability. All W[1]-hardness results are accompanied by XP-membership proofs except for finding an individually stable assignment in GASPs on cliques. For all ‘XP’-entries, the question whether the problem is fixed-parameter tractable remains open. $\diamond$ indicates that the result is not directly stated, but follows indirectly. New results are listed in boldface font.

Complexity (general case) few activities (FPT wrt p) few players (FPT wrt n) copyable activities				
NS cliques	NP-c. [5]	W[1]-h. (Th. 2)	W[1]-h. (Th. 15)	NP-c. [2]
acyclic	NP-c. [11]	FPT (Th. 1)	XP (Obs. 14)	poly time [11]
paths	NP-c. [11]	FPT [11]	XP (Obs. 14)	poly time [11]
stars	NP-c. [11]	FPT [11]	XP (Obs. 14)	poly time [11]
small components	NP-c. [11]	FPT [11]	XP (Obs. 14)	poly time [11] $\diamond$
IS cliques	NP-c. [5]	W[1]-h. (Th. 6)	W[1]-h. (Th. 15)	NP-c. [2]
acyclic	NP-c. (Th. 5)	FPT (Th. 7)	XP (Obs. 14)	poly time (Th. 4)
paths	NP-c. (Th. 5)	FPT (Th. 7)	XP (Obs. 14)	poly time (Th. 4)
stars	NP-c. (Th. 5)	FPT (Th. 7)	XP (Obs. 14)	poly time (Th. 4)
small components	NP-c. (Th. 5)	FPT [11] $\diamond$	XP (Obs. 14)	poly time [11] $\diamond$
CR cliques	NP-c. [5]	NP-c. for $p = 4$ (Th. 10)	W[1]-h. (Th. 15)	NP-c. [2]
acyclic	NP-c. [11]	NP-c. for $p = 2$ (Th. 9)	XP (Obs. 14)	poly time [7]
paths	NP-c. [11]	XP (Cor. 13)	XP (Obs. 14)	poly time [7]
stars	NP-c. [11]	NP-c. for $p = 2$ (Th. 9)	XP (Obs. 14)	poly time [7]
small components	NP-c. [11]	FPT [11]	XP (Obs. 14)	poly time [11] $\diamond$

GASP, we can prove that this problem is hard if there are at least four activities. On the other hand, if there is only one activity, a core stable assignment always exists and can be constructed efficiently, for any network structure.

Another parameter we consider is the number of players. Somewhat surprisingly, we show that this parameterization does not give rise to an FPT algorithm for gGASPs on general networks. Specifically, for all stability notions we consider, it is W[1]-hard to decide the existence of a stable outcome even when the underlying graph is a clique. We summarize our complexity results in Table 1.

2. MODEL

For $s, t \in \mathbb{N} \cup \{0\}$ where $s \leq t$, let $[s] = \{1, 2, \dots, s\}$ and $[s, t] = \{s, s+1, \dots, t\}$. We start by defining a group activity selection problem with a graph structure [11].

DEFINITION 1 (gGASP). *An instance of the Group Activity Selection Problem with graph structure (gGASP) is given by a finite set of players, or agents, $N = [n]$, a finite set of activities $A = A^* \cup \{a_\emptyset\}$ where $A^* = \{a_1, a_2, \dots, a_p\}$ and $a_\emptyset$ is the void activity, a profile $(\succeq_i)_{i \in N}$ of complete and transitive preference relations over the set of alternatives $X = A^* \times [n] \cup \{(a_\emptyset, 1)\}$, as well as a set of communication links between players $L \subseteq \{\{i, j\} \mid i, j \in N \wedge i \neq j\}$.*

In what follows, we will write $x \succ_i y$ or $i : x \succ y$ to indicate that player i strictly prefers alternative x to alternative y ; similarly, we will write $x \sim_i y$ or $i : x \sim y$ if i is indifferent between x and y . Also, given two sets of alternatives X, Y and a player i , we write $X \succ_i Y$ to indicate that i is indifferent among all alternatives in X as well as among all alternatives in Y , and prefers each alternative in X to each alternative in Y .

Two non-void activities a and b are said to be *equivalent* if for each player $i \in N$ and every $\ell \in [n]$ it holds that $(a, \ell) \sim_i (b, \ell)$. A non-void activity $a \in A^*$ is called *copyable* if A^* contains at least n activities that are equivalent to a (including a itself). Player $i \in N$ is said to *approve* an alternative (a, k) if $(a, k) \succ_i (a_\emptyset, 1)$. When describing a player’s preferences, by convention we only list the alternatives that she approves as well as $(a_\emptyset, 1)$.

An outcome of a gGASP is a *feasible assignment* of activities A to players N , i.e., a mapping $\pi : N \rightarrow A$ where for each

$a \in A^*$ the set $\pi^a = \{i \in N \mid \pi(i) = a\}$ of players assigned to a is connected in (N, L) . Note that we place no constraints on the set of players who can simultaneously engage in the void activity (i.e., do nothing). For $i \in N$ with $\pi(i) \neq a_\emptyset$, we let $\pi_i = \{i\} \cup \{j \in N \mid \pi(j) = \pi(i)\}$ denote the set of players assigned to the same activity as player $i \in N$; for $i \in N$ with $\pi(i) = a_\emptyset$, we set $\pi_i = \{i\}$.

A feasible assignment $\pi : N \rightarrow A$ of a gGASP is *individually rational* (IR) if each player weakly prefers her own activity to doing nothing, i.e. $(\pi(i), |\pi_i|) \succeq_i (a_\emptyset, 1)$ for all $i \in N$.

A subset $S \subseteq N$ of players is called a *coalition*, and said to be *feasible* if it is connected in the graph (N, L) .

A feasible coalition $S \subseteq N$ and an activity $a \in A^*$ *strongly block* an assignment $\pi : N \rightarrow A$ if $\pi^a \subseteq S$ and $(a, |S|) \succ_i (\pi(i), |\pi_i|)$ for all $i \in S$. A feasible assignment $\pi : N \rightarrow A$ of a gGASP is called *core stable* (CR) if it is individually rational, and there is no feasible coalition $S \subseteq N$ and activity $a \in A^*$ such that S and a strongly block π . Given a feasible assignment $\pi : N \rightarrow A$ of a gGASP, a player $i \in N$ is said to have

- an *NS-deviation* to activity $a \in A^*$ if $\pi^a \cup \{i\}$ is connected, and i strictly prefers to join the group π^a , i.e., $(a, |\pi^a|+1) \succ_i (\pi(i), |\pi_i|)$.
- an *IS-deviation* if it is an NS-deviation, and all players in π^a accept it, i.e., $(a, |\pi^a|+1) \succeq_j (a, |\pi^a|)$ for all $j \in \pi^a$.

A feasible assignment $\pi : N \rightarrow A$ of a gGASP is called *Nash stable* (NS) (respectively, *individually stable* (IS)) if it is individually rational and no player $i \in N$ has an NS-deviation (respectively, an IS-deviation) to some $a \in A^*$.

3. NASH STABILITY

The first stability concept we will consider is Nash stability. The following example, due to Igarashi et al. [11], shows that Nash stable outcomes may not exist even in very simple instances of gGASP.

EXAMPLE 1 (STALKER GAME). We consider an instance of gGASP with two players: one is a loner and another is a stalker. The loner only approves alternatives of the form $(a, 1)$, where $a \in A^*$, and the stalker only approves alternatives of the form $(a, 2)$, where $a \in A^*$. Clearly, no assignment for this instance is Nash stable: if a loner engages in an activity alone, the stalker would wish to join, and if the loner and the stalker are together, the loner prefers to

deviate to a_0 .

Igarashi et al. [11] show that it is NP-complete to decide if a given instance of gGASP admits a Nash stable outcome, even if the underlying graph (N, L) is a path or a star; on the other hand, they demonstrate that for paths and stars the problem of finding a Nash stable outcome is fixed-parameter tractable with respect to the number of activities. We will now show that this FPT result extends to arbitrary acyclic networks.

THEOREM 1. *The problem of deciding whether an instance of gGASP with $|A^*| = p$ whose underlying social network (N, L) is acyclic has a Nash stable feasible assignment and finding one if it exists is in FPT with respect to p .*

PROOF. We will first present a proof for the case where (N, L) is a tree; in the end, we will show how to extend the result to arbitrary forests. Fix an instance $(N, A, (\succeq_i)_{i \in N}, L)$ of gGASP such that (N, L) is a tree. We choose an arbitrary node in N as the root of this tree, thereby making (N, L) a rooted tree; we denote by $C(i)$ the set of children of i and by $D(i)$ the set of descendants of i (including i herself).

We process the nodes from the leaves to the root. For each $i \in N$, each $B \subseteq A^*$, each $B' \subseteq B$, each $(a, k) \in B' \times [n] \cup \{(a_0, 1)\}$ and each $t \in [k]$, we let $f_i(B, B', (a, k), t)$ be *true* if there is an assignment $\pi : N \rightarrow A$ where

- the set of activities assigned to players in $D(i)$ is exactly B' ;
- player i is assigned to a and is in a coalition with k other players;
- exactly t players in $D(i)$ belong to the same group as i ;
- the t players in $D(i) \cap \pi_i$ weakly prefer (a, k) to $(b, 1)$ for each $b \in A \setminus B$, and have no incentive to deviate to the other groups, i.e., every player in $D(i) \cap \pi_i$ whose children do not belong to π_i likes (a, k) at least as much as each of the alternatives she can deviate to;
- the players in $D(i) \setminus \pi_i$ weakly prefer their alternative under π to engaging alone in any of the activities in $A \setminus B$, have no NS deviation to activities in $B' \setminus \{a\}$, and have no incentive to deviate to i 's coalition, i.e., if $a \neq a_0$, then every player $j \in D(i) \setminus \pi_i$ whose parent belongs to π_i likes $(\pi(j), |\pi_j|)$ at least as much as $(a, k + 1)$.

Otherwise, we let $f_i(B, B', (a, k), t)$ be *false*.

By construction, our instance admits a Nash stable assignment if and only if $f_r(B, B', (a, k), k)$ is *true* for some combination of the arguments B, B' , and (a, k) , where r is the root.

If i is a leaf, we set $f_i(B, B', (a, k), t)$ to *true* if $B' = \{a\}$, $t = 1$, and i weakly prefers (a, k) to every alternative $(b, 1)$ such that $b \in A \setminus B$; otherwise, we set $f_i(B, B', (a, k), t)$ to *false*. Now, consider the case where i is an internal vertex. We first check whether i strictly prefers some alternative $(b, 1)$ such that $b \in A \setminus B$ to (a, k) ; if so, we set $f_i(B, B', (a, k), t)$ to *false*. Otherwise, we proceed and check for each partition $\mathcal{P}$ of $B' \setminus \{a\}$ whether there is an allocation of each activity set $P \in \mathcal{P}$ to some subtree rooted at i 's child that gives rise to an assignment with the conditions described above.

We do this by using the color-coding technique; we ‘color’ each child of i using colors from $\mathcal{P}$ independently and uniformly at random. Suppose that π is an assignment satisfying the properties described above where each activity set $P \in \mathcal{P}$ is assigned to $D(j)$ for some $j \in C(i)$; we denote by S the set of i 's children whose subtrees are assigned to some $P \in \mathcal{P}$. Then, the probability that the subtrees associated with S are assigned to these activities by a coloring χ chosen at random is $|\mathcal{P}|^{-|S|}$, since there are $|\mathcal{P}|^{|C(i)|}$ possible colorings, and $|\mathcal{P}|^{|C(i)| - |S|}$ of them coincide with π on S .

We can then derandomize our algorithm using a family of k -perfect hash functions [1].

Now, let us fix a coloring $\chi : C(i) \rightarrow \mathcal{P}$. For each $P \in \mathcal{P}$, we denote by $C_P = \{j \in C(i) \mid \chi(j) = P\}$ the set of i 's children of color P . We seek to assign each subtree rooted at the j -th element of $C(i)$ to the activities $\chi(j) \cup \{a, a_0\}$ in such a way that exactly one subtree of each color $P \in \mathcal{P}$ uses all the activities in P . We will show that there exists an efficient algorithm that finds an assignment compatible with χ , or determines that no such assignment exists.

We will first determine for each color $P \in \mathcal{P}$, each $j \in C_P$, and each $\ell \in [0, k - 1]$ whether the subtree rooted at j can be assigned to the activity set P , and exactly ℓ players in the subtrees of C_P can be assigned to a ; we refer to this subproblem by $f_P(j, \ell)$. We initialize $f_P(j, \ell)$ to *true* if

- $\ell = 0$, and i and j can be separated from each other, that is, there exists an alternative $(b, \ell) \in P \times [n] \cup \{(a_0, 1)\}$ such that (i.) $f_j(B, P, (b, \ell), \ell)$ is *true*, (ii.) $b = a_0$ or i weakly prefers (a, k) to $(b, \ell + 1)$, and (iii.) $a = a_0$ or j weakly prefers (b, ℓ) to $(a, k + 1)$; or
- $\ell \geq 1$, and $f_j(B, P \cup \{a\}, (a, k), \ell)$ is *true*.

We set $f_P(j, \ell)$ to *false* otherwise. Then, we iterate through all the subtrees associated with players in $C_P \setminus \{j\}$ and update $f_P(j, \ell)$: for each $j' \in C_P \setminus \{j\}$ and for $\ell = k - 1, k - 2, \dots, 0$, we set $f_P(j, \ell)$ to *true* if

- $\ell \geq 2$, and, moreover, there exists an $x \in [\ell]$ such that both $f_{j'}(B, \{a\}, (a, k), x)$ and $f_P(j, \ell - x)$ are *true*; or
- both $f_{j'}(B, \emptyset, (a_0, 1), 1)$ and $f_P(j, \ell)$ are *true*, and $a = a_0$ or player j' weakly prefers $(a_0, 1)$ to $(a, k + 1)$;

we set $f_P(j, \ell)$ to *false* otherwise.

In a similar manner, we determine whether exactly ℓ successors of i can be assigned to activity a ; we denote this problem by $f(\ell)$. We initialize $f(1)$ to *true* and $f(\ell)$ to *false* for each $\ell \in [2, t]$. We iterate through all the colors in $\mathcal{P}$ and update $f(\ell)$ one by one; for each color $P \in \mathcal{P}$ and each number $\ell = t, t - 1, \dots, 1$, we set $f(\ell)$ to *true* if there exists $x \in [0, t - 1]$ such that $f_P(j, x)$ is *true* for some $j \in C_P$ and $f(\ell - x)$ is *true*, and we set $f(\ell)$ to *false* otherwise. Finally, we reject the coloring χ if $f(t)$ is *false*. Clearly, if there exists a Nash stable feasible assignment that is compatible with χ , then the algorithm does not reject the coloring. We omit the proof for the bound on the running time.

Now, if (N, L) is a forest, we can combine the algorithm described above with the algorithm for graphs with small connected components proposed by Igarashi et al. [11]. The running time of the latter algorithm is a product of the time required to find a Nash stable assignment for a single connected component and the time required to combine solutions for different components; in the analysis of Igarashi et al., the former is $O(p^c)$, where c is the maximum component size and the latter is $O(8^p n^3)$. In our case, each connected component is a tree, so instead of the $O(p^c)$ algorithm for general graphs we can use our FPT algorithm for trees. This shows that our problem is in FPT for arbitrary forests. $\square$

It is unlikely that Theorem 1 can be extended to general graphs or even cliques: our next result shows that the problem of finding a Nash stable outcome is W[1]-hard with respect to the number of activities even for ‘vanilla’ GASP, i.e., when the social network imposes no constraints on possible coalitions.

THEOREM 2. *The problem of determining whether an instance of gGASP admits a Nash stable assignment is W[1]-hard with respect to the number of activities, even if the underlying graph $G = (N, L)$ is a clique.*

PROOF. We reduce from CLIQUE on regular graphs, which is

known to be W[1]-hard (see e.g. Theorem 13.4 in the book of Cygan [4]). Given a graph $G = (V, E)$ and an integer k , where $|V| = n$, $|E| = m$, and each vertex of G has degree $\delta \geq k - 1$, we create an instance of gGASP whose underlying graph is a clique, as follows.

We define the set of activities as

$$A = A' \cup B' \cup \{x, a_\emptyset\},$$

where $|A'| = k$, $|B'| = k(k-1)/2$. Notice that $|A^*| = |A| - 1 = 1 + k(k+1)/2$.

For each $v \in V$, we create a vertex player v , and for each edge $e = \{u, v\} \in E$, we create two edge players e_{uv} and e_{vu} . Let $X = \{j(k+3)+2+\delta \mid j \in [n]\}$, and let $\alpha : V \rightarrow X$ be a bijection that assigns a distinct number in X to each vertex $v \in V$. Note that $\alpha(u) < \alpha(v)$ implies that the intervals $[\alpha(u), \alpha(u) + k + 1]$ and $[\alpha(v) - 1, \alpha(v) + k]$ are disjoint. Similarly, let $Y = \{1 + 2j \mid j \in [m]\}$ and let $\beta : E \rightarrow Y$ be a bijection that assigns a distinct number in Y to each edge $e \in E$. For each $v \in V$ we construct a set of $\alpha(v) - \delta + k - 2$ dummy players D_v ; similarly, for each $e \in E$ we construct a set of $\beta(e) - 2$ dummy players D_e . We let

$$N_G = V \cup \bigcup_{v \in V} D_v \cup \bigcup_{e = \{u, v\} \in E} \{e_{uv}, e_{vu}\} \cup D_e.$$

Finally, we create five additional players $\{b_1, b_2, c_1, c_2, g\}$ and let

$$N = N_G \cup \{b_1, b_2, c_1, c_2, g\};$$

intuitively, b_1, b_2 and c_1, c_2 will form two instances of the stalker game, with g used to stabilize the latter. Let L be the set of pairs of distinct players of N . Note that $|A^*|$ depends on k , but not on n , and the size of our instance of gGASP is bounded by $O(n^3 + m^2)$.

We will now define the players' preferences. Each vertex player $v \in V$ and the players in D_v approve each alternative in $A' \times [\alpha(v), \alpha(v) + k + 1]$. Each edge player e_{vu} approves each alternative in $A' \times [\alpha(v), \alpha(v) + k + 1]$ as well as each alternative in $B' \times \{\beta(e)\}$, whereas its dummies only approve the alternatives in $B' \times \{\beta(e)\}$. All of these players are indifferent among all alternatives they approve. The stabilizer g approves each alternative of the form (a, s) with $a \in A'$, $s \in \bigcup_{v \in V} [\alpha(v) + 2, \alpha(v) + k + 2]$ and is indifferent among them; she also approves $(x, 3)$, but likes it less than all other approved alternatives. Also, we have

$$\begin{aligned} c_1 : (x, 1) \sim (x, 3) \succ (a_\emptyset, 1), \quad c_2 : (x, 2) \sim (x, 3) \succ (a_\emptyset, 1), \\ b_1 : A' \times \{1\} \succ (a_\emptyset, 1), \quad b_2 : A' \times \{2\} \succ (a_\emptyset, 1). \end{aligned}$$

We will now argue that the graph G contains a clique of size k if and only if there exists a Nash stable assignment for our instance of gGASP. Suppose that G contains a clique S of size k . We construct an assignment π as follows. We establish a bijection η between S and A' , and for each $v \in V$ we form a coalition of size $\alpha(v)$ that engages in $\eta(v)$: this coalition consists of v , all players in D_v , and all edge players e_{vu} such that $u \notin S$. Also, we establish a bijection ξ between the edge set $\{\{u, v\} \in E \mid u, v \in S\}$ and B' , and assign the activity $\xi(e)$ to the edge players e_{uv}, e_{vu} as well as to all players in D_e . Finally, we set $\pi(c_1) = \pi(c_2) = \pi(g) = x$, and assign the void activity to the remaining players. We will now argue that the resulting assignment π is Nash stable.

Clearly, no player assigned to an activity in A' or B' wishes to deviate. Now, consider a player $v \in N$ with $\pi(v) = a_\emptyset$; by construction, v only wants to join a coalition if it engages in an activity in A' and its size is in the interval $[\alpha(v) - 1, \alpha(v) + k]$, and no such coalition exists. The same argument applies to players in D_v . Similarly, consider an edge player e_{vu} with $\pi(e_{vu}) = a_\emptyset$. We have $u, v \notin S$, and therefore e_{vu} does not want to join any of

the existing coalitions; the same argument applies to all dummies of e_{vu} . Further, b_1 and b_2 do not want to deviate since each activity $a \in A'$ is assigned to at least three players, and players c_1 and c_2 do not want to deviate since they are allocated one of their top choices. Finally, the stabilizer g does not want to deviate, since there is no coalition of size $s \in \bigcup_{v \in V} [\alpha(v) + 1, \alpha(v) + k + 1]$ that engages in an activity $a \in A'$. Hence, π is Nash stable.

Conversely, suppose that there exists a Nash stable feasible assignment π . Notice that π cannot allocate an activity $a \in A'$ to b_1 or b_2 , or leave it unallocated, since no such assignment can be Nash stable. Thus, each activity in A' is allocated to a coalition whose size lies in the interval $[\alpha(v), \alpha(v) + k + 1]$ for some $v \in V$. Further, Nash stability implies that π allocates x to c_1, c_2 and g . Now, if some activity $a \in A'$ is assigned to s players, where $s \in [\alpha(v) + 1, \alpha(v) + k + 1]$ for some $v \in V$, the stabilizer g would then deviate to that coalition; hence, for each $a \in A'$ we have $|\pi^a| = \alpha(v)$ for some $v \in V$. Now, let $S = \{v \in V \mid \alpha(v) = |\pi^a| \text{ for some } a \in A'\}$. By construction, $|S| = k$. We will show that S is a clique.

Consider a player $v \in S$, and let $a(v)$ be the activity assigned to $\alpha(v)$ players under π . We have $\pi(v) = a(v)$, since otherwise v could deviate to a ; similarly, all players in D_v are assigned to a . The only other players who approve $(a(v), \alpha(v))$ are edge players e_{vu} . Thus, $\delta - k + 1$ such players must be assigned to $a(v)$, and the remaining $k - 1$ of these players must be assigned to some activity $b \in B'$, since otherwise they would deviate to $a(v)$. Now, consider an edge player e_{vu} with $\pi(e_{vu}) = b$ for some $b \in B'$. By individual rationality we have $\pi^b = \beta(\{u, v\})$ and hence π^b consists of e_{vu}, e_{uv} and all dummies of the edge $\{u, v\}$.

For each $v \in S$, consider the set of players $E_{\pi, v} = \{e_{vu}, e_{uv} \mid v \in S, \pi(e_{vu}) \neq \pi(v)\}$. The size of this set is $2(k - 1)$, and we have argued that all players in such sets need to be assigned to activities in B' . However, $|B'| = k(k - 1)/2$, and each activity in B' can be assigned to at most two edge players simultaneously. Thus, $|\bigcup_{v \in S} E_{\pi, v}| = k(k - 1)$. As each edge player $e_{uv} \in \bigcup_{v \in S} E_{\pi, v}$ can belong to at most two sets $E_{\pi, v}$, it follows that it belongs to exactly two such sets, namely, $E_{\pi, u}$ and $E_{\pi, w}$. Thus, both u and w are in S , i.e., S is a clique, which is what we wanted to prove. $\square$

On the positive side, for gGASPs on cliques, we can place the problem of finding a Nash stable assignment in the complexity class XP with respect to the number of activities. However, it is not clear if this result can be extended to general gGASPs.

THEOREM 3. *There exist an algorithm that, given an instance $(N, A, (\succeq_i)_{i \in N}, L)$ of gGASP with $|N| = n$, $|A| = p + 1$ such that (N, L) is a clique, determines whether it admits a Nash stable assignment in time $(n + 1)^p \text{poly}(n)$.*

PROOF. For every mapping $f : A^* \rightarrow [0, n]$, we will check if there is a Nash stable assignment such that $|\pi^a| = f(a)$ for each $a \in A^*$. There are at most $(n + 1)^p$ such mappings; hence, it remains to show that each check will take at most $\text{poly}(n)$ steps.

Fix a mapping $f : A^* \rightarrow [0, n]$. We construct an instance of the network flow problem as follows. We introduce a source s , a sink t , a node i for each player $i \in N$, and a node a for each activity $a \in A^*$. We create an arc with unit capacity from the source s to each player, and an arc with capacity $f(a)$ from node $a \in A^*$ to the sink t . Then, for each $i \in N$ we create an arc of unit capacity from player i to an activity $a \in A^*$ if and only if i weakly prefers $(a, f(a))$ to $(a_\emptyset, 1)$ and to all pairs of the form $(b, f(b) + 1)$, where $b \in A^* \setminus \{a\}$. It can be easily verified that an integral flow of size $\sum_{a \in A^*} f(a)$ in this network corresponds to a Nash stable assignment where exactly $f(a)$ players are engaged in each activity

$a \in A^*$. It remains to note that one can check in polynomial time whether a given network admits a flow of a given size. $\square$

4. INDIVIDUAL STABILITY

We will next consider a less stringent stability requirement, namely, individual stability. Igarashi and Elkind [10] showed that in the context of hedonic games, acyclicity is sufficient for individually stable outcomes to exist: an individually stable partition of players always exists and can be computed in polynomial time. In contrast, it turns out that for gGASPs this is not the case: an individually stable outcome may fail to exist even if the underlying social network is a path; moreover, this may happen even if there are only three players and their preferences are strict.

EXAMPLE 2. Consider a gGASP with $N = \{1, 2, 3\}$, $A^* = \{x, y, z\}$, $L = \{\{1, 2\}, \{2, 3\}\}$, where players' preferences are as follows:

- 1 : $(b, 2) \succ (a, 1) \succ (c, 3) \succ (c, 2) \succ (c, 1) \succ (a_\emptyset, 1)$
- 2 : $(c, 3) \succ (c, 2) \succ (a, 2) \succ (b, 2) \succ (b, 1) \succ (a_\emptyset, 1)$
- 3 : $(c, 3) \succ (a, 2) \succ (a, 1) \succ (a_\emptyset, 1)$

We will argue that each individually rational feasible assignment π admits an IS-deviation. Indeed, if $\pi(1) = a_\emptyset$ then no player is engaged in c and hence player 1 can deviate to c . Similarly, if $\pi(2) = a_\emptyset$ then no player is engaged in b and hence player 2 can deviate to b . There are 9 individually rational feasible assignments where $\pi(1) \neq a_\emptyset$, $\pi(2) \neq a_\emptyset$; for each of them we can find an IS deviation as follows (we write $i \rightarrow x$ to indicate that player i has an IS-deviation to activity x):

- (1) $\pi(1) = a, \pi(2) = b, \pi(3) = a_\emptyset$: $1 \rightarrow b$;
- (2) $\pi(1) = b, \pi(2) = b, \pi(3) = a_\emptyset$: $3 \rightarrow a$;
- (3) $\pi(1) = b, \pi(2) = b, \pi(3) = a$: $2 \rightarrow a$;
- (4) $\pi(1) = c, \pi(2) = a, \pi(3) = a$: $2 \rightarrow c$;
- (5) $\pi(1) = c, \pi(2) = b, \pi(3) = a_\emptyset$: $3 \rightarrow a$;
- (6) $\pi(1) = c, \pi(2) = b, \pi(3) = a$: $2 \rightarrow a$;
- (7) $\pi(1) = c, \pi(2) = c, \pi(3) = a_\emptyset$: $3 \rightarrow a$;
- (8) $\pi(1) = c, \pi(2) = c, \pi(3) = a$: $3 \rightarrow c$;
- (9) $\pi(1) = c, \pi(2) = c, \pi(3) = c$: $1 \rightarrow a$.

In Example 2 all activities are distinct. On the other hand, if all activities are copyable, an individually stable outcome is guaranteed to exist. Moreover, we can adapt the result of Igarashi and Elkind [10] for hedonic games and obtain an efficient algorithm for computing an individually stable outcome.

THEOREM 4. *Each instance of gGASP where each activity $a \in A^*$ is copyable and (N, L) is acyclic admits an individually stable feasible assignment; moreover, such an assignment can be found in polynomial time.*

PROOF SKETCH. The algorithm is similar to the one for hedonic games [10]. The basic idea is to create a rooted tree for each connected component and construct an assignment for every subtree in a bottom-up manner. When considering a subtree rooted at player i , we start with the assignment obtained by combining the previously constructed assignments for the subtrees rooted at the children of i and assigning i to the void activity. We then let player

i join the most preferred activity among those to which she has an IS deviation. After that we keep adding players to i 's coalition as long as the resulting coalition remains feasible, the player being added is willing to move, and such a deviation is acceptable for all members of i 's coalition. Similarly to the proof in Theorem 1 of Igarashi and Elkind [10], one can show that the resulting assignment is individually stable. $\square$

In contrast, when activities are not copyable, finding an individually stable feasible assignment is hard, even for very simple social networks. Indeed, for every class of simple social networks for which Igarashi et al. [11] show that finding a Nash stable outcome is NP-complete, finding an individually stable outcome is NP-complete as well. The proof of the following theorem is similar to the proofs of the respective results of Igarashi et al. [11]: essentially, we have to replace each instance of the stalker game in these proofs with an instance of the game from Example 2.

THEOREM 5. *Given an instance of gGASP whose underlying graph is a path, a star, or has connected components whose size is bounded by a constant, it is NP-complete to determine whether it has an individually stable feasible assignment.*

Moreover, the problem of finding individually stable feasible assignments remains hard even if the number of activities is small, and even if the social network is a clique. The proof is similar to that of Theorem 2 and is omitted due to space constraints.

THEOREM 6. *The problem of determining whether an instance of gGASP admits an individually stable assignment is W[1]-hard with respect to the number of activities, even if the underlying graph $G = (N, L)$ is a clique.*

However, just as in the case of Nash stability, if we both restrict the structure of the social network and assume that the number of activities is small, we can obtain positive algorithmic results. The proof of the next theorem is similar to that of Theorem 1; again, we omit it due to space constraints.

THEOREM 7. *The problem of deciding whether an instance of gGASP with $|A^*| = p$ whose underlying social network (N, L) is acyclic has an individually stable feasible assignment and finding one if it exists is in FPT with respect to p .*

The results for individual stability presented so far indicate that from the complexity perspective it is very similar to Nash stability. However, it is not clear if the XP algorithm presented in Theorem 3 extends to individual stability. The difficulty is that, to determine whether an agent i has an IS deviation to an activity a , it is not sufficient to know how many players engage in a : knowing their preferences is important to decide whether i 's deviation will be vetoed by one of the players currently assigned to a . Another important difference concerns copyable activities and games on acyclic graphs: in this setting, individually stable outcomes always exist (Theorem 4), whereas for Nash stable outcomes this is not the case, as illustrated by the stalker game.

5. CORE STABILITY

Igarashi et al. [11] have demonstrated that the core can be empty even for gGASPs on paths with 3 players and 2 activities. We reproduce their example below.

EXAMPLE 3 ([11]). Consider a gGASP with $N = \{1, 2, 3\}$, $A^* = \{a, b\}$, $L = \{\{1, 2\}, \{2, 3\}\}$, where agents' preferences $(\succeq_i)_{i \in N}$ are as follows:

$$\begin{aligned} 1 : (b, 2) &\succ_1 (a, 3) \succ_1 (a_\emptyset, 1) \\ 2 : (a, 2) &\succ_2 (b, 2) \succ_2 (a, 3) \succ_2 (a_\emptyset, 1) \\ 3 : (a, 3) &\succ_3 (b, 1) \succ_3 (a, 2) \succ_3 (a_\emptyset, 1) \end{aligned}$$

It can be shown that this instance admits no core stable assignment [11].

On the positive side, we can show that checking whether a given feasible assignment is core stable is easy, irrespective of the structure of the social network. The proposition below generalizes Theorem 11 of Darmann [5] and Theorem 12 of Igarashi et al. [11].

PROPOSITION 8. *Given an instance $(N, (\succeq_i)_{i \in N}, A, L)$ of gGASP and a feasible assignment π for that instance, we can decide in polynomial time whether π is core stable.*

PROOF. Let $A = A^* \cup \{a_\emptyset\}$ and let $n = |N|$. By scanning the assignment π and the players' preferences, we can check whether π is individually rational. Now, suppose that this is the case. Then, for each $a \in A^*$ and each $s \in [n]$ we can check if there is a deviation by a connected coalition of size s that engages in a . To this end, we consider the set $S_{a,s}$ of all players who strictly prefer (a, s) to their assignment under π and verify whether $S_{a,s}$ has a connected component of size at least s that contains π^a ; if this is the case, π^a (which is itself connected or empty) could be extended to a connected coalition of size exactly s that strongly blocks π . If no such deviation exists, π is core stable. $\square$

However, core stability turns out to be more computationally challenging than Nash stability and individual stability when the number of activities is small: we will now show that core stable assignments are hard to find even if there are only two activities and the underlying graph is a star (and thus one cannot expect an FPT result with respect to the number of activities for this setting). Later, we will see that this hardness result can be extended to the case where (N, L) is a clique, i.e., to classic GASP, thereby solving a problem left open by the work of Darmann [5].

THEOREM 9. *It is NP-complete to determine whether an instance of gGASP admits a core stable assignment even when the underlying graph is a star and the number of non-void activities is 2.*

PROOF. Our problem is in NP by Proposition 8. To establish NP-hardness, we reduce from the NP-complete problem HITTING SET [9]. An instance of HITTING SET is a family $\mathcal{H} = \{V_1, V_2, \dots, V_m\}$ of subsets of a finite set V and an integer k with $k < |V|$. It is a "yes"-instance if $\mathcal{H}$ admits a hitting set $V' \subseteq V$ of size at most k , i.e., $|V'| \leq k$ and $V' \cap V_i \neq \emptyset$ for each $V_i \in \mathcal{H}$; otherwise, it is a "no"-instance.

Given an instance of HITTING SET, we can create three disjoint copies of $(V, \mathcal{H})$: for each $v \in V$ we create elements x_v, y_v and z_v , set $W = \{x_v, y_v, z_v \mid v \in V\}$ and for each $i \in [m]$ we let $X_i = \{x_v \mid v \in V_i\}$, $Y_i = \{y_v \mid v \in V_i\}$, $Z_i = \{z_v \mid v \in V_i\}$ and set $\mathcal{G} = \{X_i, Y_i, Z_i \mid i \in [m]\}$. In what follows, we will use the fact that $(W, \mathcal{G})$ admits a hitting set of size $3k$ if and only if $(V, \mathcal{H})$ admits a hitting set of size k .

Consider an instance $(V, \mathcal{H}, k)$ of HITTING SET and construct the pair $(W, \mathcal{G})$ as described above. For readability, we renumber the elements of $\mathcal{G}$ as $W_1, \dots, W_{3m}$.

We construct an instance of gGASP as follows. We define the set of activities to be $A^* = \{a, b\}$. We introduce a center player

c , two players s_1 and s_2 , and a player w for each $w \in W$. Also, for each $i \in [3m]$, we let $t_i = i + |W| + 1$ and create a set $D_i = \{d_i^{(1)}, d_i^{(2)}, \dots, d_i^{(t_i - |W_i| - 1)}\}$ of dummy players. We then attach s_1, s_2 , each player $w \in W$ and each of the dummies to the center. Formally, the graph (N, L) is given by

$$N = \{c, s_1, s_2\} \cup W \cup \bigcup_{i=1}^{3m} D_i \text{ and } L = \{\{c, x\} \mid x \in N \setminus \{c\}\}.$$

Intuitively, for each $i \in [3m]$ the number t_i is the target coalition size when all players in W_i are engaged in activity b , together with c and the players in D_i .

The agents' preferences over alternatives are defined as follows. We let $A' = \{a\} \times [4, 3k + 3]$. For each $w \in W$, we let $B_w = \{b\} \times \{t_i \mid w \in W_i\}$; also, set $B = \bigcup_{w \in W} B_w$. The preferences of each player $w \in W$ are given by

$$w : A' \succ B_w \succ (a_\emptyset, 1).$$

For each $i \in [3m]$ the dummy players in D_i only approve the alternative (b, t_i) . The preferences of the center player c are given by

$$c : (a, 2) \succ (b, 2) \succ (a, 3) \succ B \succ A' \succ (a_\emptyset, 1).$$

Finally, the preferences of players s_1 and s_2 are given by

$$\begin{aligned} s_1 : A' &\succ (b, 2) \succ (a, 3) \succ (a_\emptyset, 1) \\ s_2 : A' &\succ (a, 3) \succ B \succ (b, 1) \succ (a, 2) \succ (a_\emptyset, 1). \end{aligned}$$

Note that the preferences of c, s_1 and s_2 , when restricted to $A \times [1, 2, 3]$, form an instance of gGASP with an empty core (Example 3).

We will show that $(V, \mathcal{H})$ admits a hitting set of size at most k if and only if there exists a core stable feasible assignment.

Let V' be a hitting set of size at most k in $(V, \mathcal{H})$. Set $W' = \{x_v, y_v, z_v \mid v \in V'\}$; we have $|W'| \leq 3k$, and, by construction, W' is a hitting set for $(W, \mathcal{G})$. Then, we construct a feasible assignment π by assigning activity a to c, s_1, s_2 , and the players $w \in W'$, and assigning the void activity to the remaining players. Clearly, π is individually rational, since $(a, |W'| + 3) \in A'$. Further, notice that no connected subset S together with activity a strongly blocks π : every such subset has to contain players s_1 and s_2 , who are currently enjoying one of their top alternatives. It remains to show that no connected subset S together with activity b strongly blocks π . Suppose towards a contradiction that such a subset S exists; as s_1 and s_2 are not interested in deviating, it must be the case that $|S| = t_i$ for some $i \in [3m]$ and hence S consists of agents who approve (b, t_i) , i.e., $S = \{c\} \cup W_i \cup D_i$ for some $i \in [3m]$. However, since W' is a hitting set, there is an agent $j \in W' \cap W_i$ with $\pi(j) = a$, and this agent prefers $(a, |W'| + 3)$ to (b, t_i) , a contradiction. Hence, π is core stable.

Conversely, suppose that there exists a core stable feasible assignment π and let $W' = \{w \in W \mid \pi(w) = a\}$.

We will first argue that $\pi(c) = a$. Indeed, if $\pi(c) = a_\emptyset$, then $\pi^a = \emptyset$ and agents c, s_1 and s_2 can deviate to a . If $\pi(c) = b$ and the only other agent to engage in b is s_1 , then $\pi^a = \emptyset$ and c and s_2 can deviate to a . If $\pi(c) = b$ and $|\pi^b| = t_i$ for some $i \in [3m]$ then $\pi^a = \emptyset$, and c, s_1 and s_2 can deviate to a . It follows that $\pi(c) = a$.

Now, if $\pi^a = \{c, s_2\}$ then $\pi^b = \emptyset$ and agent s_2 can deviate to b . Similarly, if $\pi^a = \{c, s_1, s_2\}$ then $\pi^b = \emptyset$, and c and s_1 can deviate to b . It follows that $|\pi^a| \in [4, 3k + 3]$ and hence $|W'| \leq 3k + 2$; also, either $\pi^b = \{s_2\}$ or $\pi^b = \emptyset$. Now, if W' is not a hitting set for $(W, \mathcal{G})$, there exists a set $W_i \in \mathcal{G}$ such that all players in W_i are assigned to the void activity. Then if $\pi^b = \emptyset$ the coalition

$\{c\} \cup W_i \cup D_i$ together with the activity b strongly blocks π , and if $\pi^b = \{s_2\}$ the coalition $\{c, s_2\} \cup W_i \cup D_i \setminus \{d_i^{(1)}\}$ together with the activity b strongly blocks π . In either case, we obtain a contradiction with the stability of π . Thus, W' is a hitting set for $(W, \mathcal{G})$. At least one of the three sets $X' = \{x_v \mid x_v \in W'\}$, $Y' = \{y_v \mid y_v \in W'\}$, and $Z' = \{z_v \mid z_v \in W'\}$ contains at most k elements; assume without loss of generality that $|X'| \leq k$. By construction, $\{v \in V \mid x_v \in X'\}$ is a hitting set for $(V, \mathcal{H})$. Thus, $(V, \mathcal{H})$ admits a hitting set of size at most k , which is what we wanted to prove. $\square$

The hardness result in Theorem 9 immediately generalizes to instances of gGASP with more than two activities: we can modify the construction in our hardness reduction by introducing additional activities that no player wants to engage in.

Also, Theorem 9 can be extended from stars to cliques; however, our proof for cliques relies on having at least four non-void activities. It remains an interesting open problem whether core stable outcomes of gGASPs on cliques can be found efficiently if the number of activities does not exceed 3. We conjecture that the answer is ‘no’, i.e., the problem of computing core stable outcomes remains NP-hard for $|A^*| = 2, 3$.

THEOREM 10. *It is NP-complete to determine whether an instance of gGASP admits a core stable assignment even when the underlying graph is a clique and the number of activities is 4.*

In contrast, checking the existence of core stable assignments in gGASP is easy if $|A^*| = 1$, irrespective of the structure of the social network.

PROPOSITION 11. *Every instance of gGASP with $A = \{a, a_\emptyset\}$ admits a core stable assignment; moreover, such an assignment can be computed in polynomial time.*

PROOF. Consider an instance $(N, (\succeq_i)_{i \in N}, A, L)$ of gGASP with $A = \{a, a_\emptyset\}$, and let $n = |N|$. For each $s \in [n]$, let S_s be the set of all players who weakly prefer (a, s) to $(a_\emptyset, 1)$. If for each $s \in [n]$ the largest connected component of S_s with respect to (N, L) contains fewer than s agents, then no outcome in which a non-empty set of agents engages in a is individually rational, whereas the assignment π given by $\pi(i) = a_\emptyset$ for all $i \in N$ is core stable. Otherwise, consider the largest value of s such that S_s has a connected component of size at least s . Find a connected subset of S_s of size exactly s , and assign the agents in this subset to a ; assign the remaining agents to $a_\emptyset$. To see that this assignment is core stable, note that for every deviating coalition S we would have $|S| > s$, and hence such a coalition is either disconnected or some players in S prefer $(a_\emptyset, 1)$ to $(a, |S|)$. $\square$

It is not clear if the problem of finding core stable assignments in gGASP is in FPT with respect to the number of activities when the underlying graph is a path. However, we can place this problem in XP with respect to this parameter. In fact, we have the following more general result for graphs with few connected coalitions (see the work of Elkind [8] for insights on the structure of such graphs).

PROPOSITION 12. *Given an instance $(N, (\succeq_i)_{i \in N}, A, L)$ of gGASP with $|N| = n$, $|A| = p + 1$, such that the number of non-empty connected subsets of (N, L) is κ , we can decide in time $O(\kappa^p) \cdot \text{poly}(n, p)$ whether this instance admits a core stable assignment, and find one such assignment if it exists.*

PROOF. Let $C_1, \dots, C_\kappa$ be the list of all non-empty connected subsets of (N, L) . Since each assignment of players to activities

has to assign a connected (possibly empty) subset of players to each activity, we can bound the number of possible assignments by $(\kappa + 1)^p$: each of the p non-void activities is assigned to a coalition in our list or to no player at all, and the remaining players are assigned the void activity. We can then generate all these assignments one by one and check if any of them is core stable; by Proposition 8, the stability check can be performed in time polynomial in n and p . $\square$

If the social network (N, L) is a path, it has $O(|N|^2)$ connected subsets. Thus, we obtain the following corollary.

COROLLARY 13. *The problem of deciding whether a given instance of gGASP whose underlying graph is a path admits a core stable assignment is in XP with respect to the number of activities.*

6. FEW PLAYERS

In the previous sections, the parameter that we focused on was the number of activities p . Although we expect this parameter to be small in many realistic settings, there are also situations where players can choose from a huge variety of possible activities. It is then natural to ask if stability-related problems for gGASPs are tractable in the number of players n is small. This is the question that we consider in this section.

We first observe that for all stability concepts considered in this paper the problem of finding a stable feasible assignment is in XP with respect to n : we can simply guess the activity of each player (there are at most $(p + 1)^n$ possible guesses) and check whether the resulting assignment is feasible and stable.

OBSERVATION 14. *The problem of deciding whether a given instance of gGASP admits a core stable, Nash stable, or individually stable assignment is in XP with respect to the number of players n .*

The following theorem shows that gGASP is unlikely to be fixed-parameter tractable with respect to n . This is somewhat surprising, because an FPT algorithm with respect to n could afford to iterate through all possible partitions of the players into coalitions. Interestingly, in the following hardness reduction there is a unique partition of players into coalitions that can lead to a stable assignment; that is, computational hardness comes solely from assigning known coalitions of players to activities.

THEOREM 15. *The problem of determining whether an instance of gGASP whose underlying graph is a clique admits a core stable (Nash stable, or individually stable) assignment is W[1]-hard with respect to the number of players n .*

PROOF. Due to space constraints, we only provide a proof for Nash stability. We describe a parameterized reduction from the W[1]-hard MULTICOLORED CLIQUE problem. An instance of this problem is given by an undirected graph $G = (V, E)$, a positive integer $h \in \mathbb{N}$, and a vertex coloring $\phi: V \rightarrow \{1, 2, \dots, h\}$. It is a ‘yes’-instance if G admits a colorful h -clique, that is, a size- h vertex subset $Q \subseteq V$ such that the vertices in Q are pairwise adjacent and have pairwise distinct colors. Without loss of generality, we assume that there are exactly q vertices of each color for some $q \in \mathbb{N}$, and that there are no edges between vertices of the same color.

Let (G, h, ϕ) be an instance of MULTICOLORED CLIQUE with $G = (V, E)$. For every $i \in [h]$, we write $V^{(i)} = \{v_1^{(i)}, \dots, v_q^{(i)}\}$ to denote the set of vertices of color i . For each vertex v , we write $E(v) = \{e_1^{(v)}, \dots, e_{|E(v)|}^{(v)}\}$ to denote the set of edges incident to v . We construct our gGASP instance as follows. We have one vertex activity $a_\ell^{(i)}$ for each vertex $v_\ell^{(i)} \in V^{(i)}$, $i \in [h]$, one edge activity $b_\ell^{(v)}$ for each edge $e_\ell^{(v)} \in E$, two color activities c_i and c'_i

for each color $i \in [h]$, and one *colorpair activity* $c_{i,j}$ for each color pair $i, j \in [h], i \neq j$.

Idea. We will have one *color gadget* for each color $i \in [h]$ and one *colorpair gadget* for each color pair $\{i, j\}, i \neq j$. For most of the possible assignments, these gadgets will be unstable, unless the following holds:

1. For each color $i \in [h]$ the first two players from the color gadget select a vertex of color i (by being assigned together to the corresponding vertex activity).
2. For each color pair $\{i, j\} \in [h], i \neq j$ the first two players of the colorpair gadget select an edge connecting one vertex of color i and one vertex of color j (by being assigned together to the corresponding edge activity).
3. Every selected edge for the color pair $\{i, j\}$ must be incident to both vertices selected for color i and color j .

If the three conditions above hold, then the assignment must encode a colorful h -clique.

Construction details. The color gadget $C(i), i \in [h]$ consists of the following four players.

$$\begin{aligned}
p_1^{(i)} : (a_1^{(i)}, 2) &\succ B(1, i, 3) \succ (a_2^{(i)}, 2) \succ B(2, i, 3) \succ \\
&\dots \succ (a_q^{(i)}, 2) \succ B(q, i, 3) \succ (c_i, 1) \succ (a_0, 1) \\
p_2^{(i)} : (a_q^{(i)}, 2) &\succ B(q, i, 3) \succ (a_{q-1}^{(i)}, 2) \succ B(q-1, i, 3) \succ \\
&\dots \succ (a_1^{(i)}, 2) \succ B(1, i, 3) \succ (c'_i, 1) \succ (a_0, 1) \\
p_3^{(i)} : (c_i, 2) &\succ (a_0, 1) \\
p_4^{(i)} : (c'_i, 2) &\succ (a_0, 1)
\end{aligned}$$

with $B(\ell, i, z) = (b_1^{(v_\ell^{(i)})}, z) \succ (b_2^{(v_\ell^{(i)})}, z) \dots \succ (b_{|E(v_\ell^{(i)})|}^{(v_\ell^{(i)})}, z)$

The colorpair gadget $C(\{i, j\}), \{i, j\} \subseteq [h], i \neq j$, consists of three players. Players $p_1^{\{i,j\}}$ and $p_2^{\{i,j\}}$ approve all edge activities corresponding to edges between vertices of color i and j with size two and the colorpair activity $c_{i,j}$ with size two and one, and strictly prefer the former alternatives to the latter. Player $p_3^{\{i,j\}}$ approves only the colorpair activity $c_{i,j}$ with size three.

Correctness. We will now argue that the graph G admits a colorful clique of size h if and only if our instance of gGASP admits a Nash stable assignment.

Suppose that there exists a colorful clique H of size h . Assign the first two players of the color gadget $C(i)$ to the activity corresponding to the vertex of color i from H . Assign the first two players of the colorpair gadget $C(\{i, j\})$ to the activity corresponding to the edge between the vertices of color i and j in H . Assign all other players to the void activity. Clearly, the resulting assignment is feasible. Observe that for a successful Nash deviation a player must join an existing non-empty coalition, because no player prefers a size-one activity to the currently assigned one. By construction, the last two players of each color gadget and the last player of the colorpair gadget cannot deviate (no other players engage in an approved activity). Consider the first two players of color gadget $C(i)$. They cannot deviate to a vertex activity, because their current activity is their only approved vertex activity that has some players assigned to it. They cannot deviate to an edge activity either, because they would only prefer edge activities corresponding to edges that are not incident to the vertex of color i that is part of colorful clique H ; these activities, however, have no players assigned to them. Finally, the first two players of a colorpair gadget $C(\{i, j\})$ cannot deviate

since no player is assigned to any of their more preferred activities. Thus, we have a Nash stable feasible assignment.

Suppose that there exists a Nash stable feasible assignment π . Observe that the first two players of each colorpair gadget $C(\{i, j\})$ are assigned to the same edge activity corresponding to some edge $e_{\ell_{i,j}} \in E$. Otherwise, they would need to be assigned to the colorpair activity $c_{i,j}$ and be stalked by the third player of the gadget (similarly to the Stalker game from Example 1). We say that these players “select edge $e_{\ell_{i,j}}$ ”. By similar reasoning the first two players of each color gadget $C(i)$ are assigned to the same vertex activity corresponding to some vertex $v_{\ell_i}^{(i)}$. They cannot be assigned to an edge activity, because the third player would necessarily be a player from a colorpair gadget, and these players do not want to engage in activities with two other players. We say that these players “select vertex $v_{\ell_i}^{(i)}$ ”. (Note that at this point, the coalition structure of any stable outcome is already fixed: The first two players of each color gadget and of each colorpair gadget must form a coalition of size two, respectively, and all other players must be assigned to the void activity.)

Now, assume towards a contradiction that the selected vertices and edges do not form a colorful clique of size h . The size and colorfulness are clear from the construction. Hence, there must be some pair $\{v_{\ell_i}^{(i)}, v_{\ell_j}^{(j)}\}$ of selected vertices that is not adjacent. However, this would imply that colorpair gadget $\{i, j\}$ selected an edge that is either not incident to vertex $v_{\ell_i}^{(i)}$ or not incident to vertex $v_{\ell_j}^{(j)}$. Without loss of generality let it be non-adjacent to vertex $v_{\ell_i}^{(i)}$. Now, there are two cases. First, if $e_{\ell_{i,j}} \in E(v_{\ell_i}^{(i)})$ with $x < \ell_i$, then player $p_1^{(i)}$ would deviate to the activity corresponding to $e_{\ell_{i,j}}$. Second, if $e_{\ell_{i,j}} \in E(v_{\ell_i}^{(i)})$ with $x > \ell_i$, then player $p_2^{(i)}$ would deviate to the activity corresponding to $e_{\ell_{i,j}}$. In both cases we have a contradiction to the assumption that π is Nash stable. $\square$

Note that although we showed W[1]-hardness for each of the parameters p and n , parameterizing by the combined parameter $p+n$ immediately gives fixed-parameter tractability, since the input size is trivially upper-bounded by $n^2 \cdot p$.

7. CONCLUSION

We have investigated the parameterized complexity of computing stable outcomes of group activity selection problems on networks, with respect to two natural parameters. Many of our hardness results hold for the classic GASP problem, where there are no constraints on possible coalitions; however, some of our positive results only hold for acyclic graphs. Interestingly, one of our tractability results holds for GASPs, but it is not clear if it can be extended to gGASPs; thus, while simple networks may decrease complexity, allowing for arbitrary networks may have the opposite effect.

In some of our hardness reductions players have non-strict preferences. In some cases, it is easy to modify our gadgets to rely on strict preferences only; in other cases (notably, in individual stability proofs) our arguments make crucial use of indifferences.

Confirming our intuition, we showed that finding core stable assignments tends to be computationally more difficult than finding Nash stable or individually stable assignments; we note that this distinction only becomes visible from the parameterized complexity perspective, as all of these problems are NP-hard.

Our results for games with few players are somewhat preliminary: we have only investigated the complexity of our problems with respect to this parameter for general networks and for cliques, and have not considered simple networks such as stars and paths. These questions offer an interesting direction for future work.

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