Analysis of Navier–Stokes–Fokker–Planck systems for incompressible dilute polymeric fluids

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Abstract

We show the existence of large-data global-in-time weak solutions to various classes of coupled bead-spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials for incompressible dilute polymeric fluids in a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3. The governing system consists of the transport equation and the Navier–Stokes equation coupled to the Fokker–Planck equation through the elastic extra-stress tensor which is defined by the Kramers expression. The proofs are based on truncating the probability density function and approximating with Galerkin semi-discretizations on the spatial domains. We derive uniform bounds independent of the Galerkin and truncation parameters; then with weak compactness and compensated compactness techniques, we pass to the limits in the sequences of Galerkin approximations and in the truncation level. The technical tools involve using Nikolskiĭ norm estimates to derive uniform estimates for fractional time derivatives and using various generalizations of the Aubin–Lions Lemma to deduce strong convergence of approximating sequences. We also apply the Div-Curl Lemma and Vitali's Convergence Theorem to deduce the strong convergence of the sequence of approximations of the probability density function in L^1.

We first focus on homogeneous dilute polymeric fluids. The key feature is the polymer-number-density-dependent viscosity coefficient appearing in the Navier–Stokes equation. Then we move on to nonhomogeneous dilute polymeric fluids, featuring the presence of a density-dependent and polymer-number-density-dependent viscosity coefficient in the Navier–Stokes equation and a density-dependent drag coefficient in the Fokker–Planck equation. Finally, we consider nonisothermal homogeneous dilute polymeric fluids. To complete the nonisothermal Navier–Stokes–Fokker–Planck system, the temperature evolution equation is also introduced to form a thermodynamically consistent model. To simplify the nonisothermal model, we present the existence proof in the case of a corotational Fokker–Planck equation.

Contents

Chapter 1

Introduction

The Navier–Stokes equations, which were introduced by Claude–Louis Navier and George Gabriel Stokes back in the 19th century, have been of great importance in modelling various phenomena in physics and engineering. Their wide-ranging applications also motivate their study from a purely mathematical point of view. The rapid development of the mathematical theory of non-Newtonian fluids gives an insight into understanding the behaviour of more complex fluids, such as dilute polymer solutions, which are our main focus here. A kinetic model of dilute polymer solutions was proposed by Kramers [43] in the first half of the 20th century and has been widely studied ever since; see, for example, Bird et al. [13], Öttinger [54], Huilgol and Phan–Thien [42] and Lozinski et al. [48]. In this work, we mainly consider incompressible, viscous fluids and we aim to show the existence of global-in-time large-data weak solutions to coupled Navier–Stokes–Fokker–Planck systems under various conditions.

In this chapter, we first introduce the system of equations that model the incompressible dilute polymer solutions in Section 1.1. In Section 1.2, we briefly discuss the proof of the existence of weak solutions to the coupled Navier–Stokes–Fokker–Planck system, including the methodology and the difficulties. In Section 1.3, we review some relevant literature in this field. Finally, we summarize the contributions of this thesis in Section 1.4.

1.1 Governing equations

The motion of a viscous Newtonian fluid is governed by a system of nonlinear partial differential equations, known as the Navier–Stokes equations. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded open Lipschitz domain, with boundary $\partial\Omega$. Let $T \in \mathbb{R}_{>0}$ denote the length of the time interval of interest, and let $Q := \Omega \times (0,T)$ be the associated space-time domain. We consider the following system of equations:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho \boldsymbol{v}) = 0 \qquad \text{in } Q, \tag{1.1}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \mathrm{div}_x(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \mathrm{div}_x \mathcal{T} = \rho \boldsymbol{f} \qquad \text{in } Q, \tag{1.2}$$

$$\mathrm{div}_x\, \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{1.3}$$

where $\rho : Q \to \mathbb{R}$ denotes the solvent density, $\boldsymbol{v} : Q \to \mathbb{R}^d$ denotes the solvent velocity, $\boldsymbol{f} : Q \to \mathbb{R}^d$ represents the density of the external body forces, and $\mathcal{T} : Q \to \mathbb{R}^{d\times d}$ denotes the Cauchy stress. The above system of equations is derived from the laws of conservation of mass and the balance of linear momentum. For a detailed derivation of the balance equations, we refer the reader to [12] or [46]. To complete the system, we assume that the Cauchy stress $\mathcal{T}$ is decomposed as

$$\mathcal{T} := -pI + S_v + S_e,$$

where $p : Q \to \mathbb{R}$ denotes the pressure, $S_v : Q \to \mathbb{R}^{d\times d}_{sym}$ represents the viscous part of the Cauchy stress and $S_e : Q \to \mathbb{R}^{d\times d}_{sym}$ represents the elastic extra-stress tensor (i.e. the polymeric part of the Cauchy stress). Depending on the type of fluid, the viscous part of the Cauchy stress tensor S_v satisfies a certain relation, called a *constitutive relation*, involving the symmetric part of the velocity gradient $D(\boldsymbol{v})$, where

$$D(\boldsymbol{v}) := \frac{1}{2}(\nabla_x \boldsymbol{v} + (\nabla_x \boldsymbol{v})^{\mathrm{T}}).$$

We assume for simplicity that S_v is related linearly to $D(\boldsymbol{v})$, i.e.

$$S_v := 2\mu D(\boldsymbol{v}), \tag{1.4}$$

where μ is the shear viscosity coefficient that may depend on other physical parameters. Such fluids are called *Newtonian fluids.* Those fluids that do not satisfy the linear relation (1.4) are called *non-Newtonian fluids.* A more detailed discussion of non-Newtonian fluids can be found in [1].

From the microscopic point of view, in the simplest case dilute polymeric fluids can be seen as a collection of polymer molecules dissolved in a Newtonian fluid and we model the motion of polymer molecules by the classical bead-spring chain model. In a bead-spring chain model for dilute polymers, consisting of $K + 1$ beads coupled with K elastic springs to represent a polymer chain, the elastic extra-stress tensor S_e is defined by a version of the Kramers expression [54], depending on the probability density function ψ of the (random) conformation $\boldsymbol{q} := ((\boldsymbol{q}^1)^{\mathrm{T}}, \dots, (\boldsymbol{q}^K)^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{d\times K}$ of the chain, with the column vector $\boldsymbol{q}^j := (q_1^j, \dots, q_d^j)^{\mathrm{T}}$ representing the d-component conformation vector of the jth spring in the bead-spring chain. Let $D := D^1 \times \cdots \times D^K \subset \mathbb{R}^{d \times K}$ be the domain of admissible conformation vectors. Typically D^j is the whole space $\mathbb{R}^d$ or a bounded open ball centred at the origin $\mathbf{0}$ in $\mathbb{R}^d$, for each $j = 1, \dots, K$. When $K = 1$, the model is referred to as the dumbbell model.

Let $I^j \subset [0, \infty)$ denote the image of D^j under the mapping $\boldsymbol{q}^j \subset D^j \mapsto \frac{1}{2}|\boldsymbol{q}^j|^2$. Let U^j denote the *spring potential* and we assume that $U^j : I^j \to [0, \infty)$, $j = 1, \dots, K$ is a sufficiently smooth function. Then the elastic spring force $F^j : D^j \subset \mathbb{R}^d \to \mathbb{R}^d$ of the jth spring in the chain is given by

$$\boldsymbol{F}^j(\boldsymbol{q}^j) = (U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\boldsymbol{q}^j, \qquad j = 1, \dots, K.$$

Example 1.1.1. *Consider the Hookean dumbbell model with $K = 1$. The spring force is given by $F(\boldsymbol{q}) = H\boldsymbol{q}$, where H is the spring constant, with $\boldsymbol{q} \in D = \mathbb{R}^d$, corresponding to $U(s) = Hs$, $s \in I = [0, \infty)$. However, the Hookean model is physically unrealistic since it allows the spring connecting the beads to have arbitrarily large extension.*

Therefore, we focus on the finitely extensible nonlinear elastic (FENE) model where $D^j = B(0, b_j^{\frac{1}{2}})$, a ball centred at the origin $\mathbf{0}$ in $\mathbb{R}^d$ and of radius $b_j^{\frac{1}{2}}$, with $b_j > 0$ for each $j \in \{1, \dots, K\}$. The spring potential $U^j : [0, \frac{b_j}{2}) \to [0, \infty), j = 1, \dots, K$, satisfies $U^j(0) = 0$, $\lim_{s \to \frac{b_j}{2}-} U^j(s) = +\infty$.

We define the (normalized) partial Maxwellian M^j with respect to the variable $\boldsymbol{q}^j$ by

$$M^j(\boldsymbol{q}^j) = \frac{1}{Z^j} \mathrm{e}^{-U^j(\frac{1}{2}|\boldsymbol{q}^j|^2)}, \quad \text{where } Z^j := \int_{D^j} \mathrm{e}^{-U^j(\frac{1}{2}|\boldsymbol{q}^j|^2)} \, \mathrm{d}\boldsymbol{q}^j,$$

where $\mathrm{d}\boldsymbol{q}^j := \mathrm{d}q_1^j \cdots \mathrm{d}q_d^j$, $j = 1, \dots, K$. The Maxwellian in the model is then defined by

$$M(\boldsymbol{q}) := \prod_{j=1}^{K} M^j(\boldsymbol{q}^j) \qquad \forall\, \boldsymbol{q} := (\boldsymbol{q}^1, \cdots, \boldsymbol{q}^K) \in D.$$

Observe that, for $\boldsymbol{q} \in D$ and $j = 1, \dots, K$,

$$M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j}(M(\boldsymbol{q}))^{-1} = -(M(\boldsymbol{q}))^{-1}\nabla_{\boldsymbol{q}^j}M(\boldsymbol{q}) = \nabla_{\boldsymbol{q}^j}U^j\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right) = (U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\boldsymbol{q}^j,$$

and, by definition,

$$\int_D M(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} = 1.$$

In the above equalities $\nabla_{\boldsymbol{q}^j} := (\partial/\partial q^j_1, \dots, \partial/\partial q^j_d)^{\mathrm{T}}$, for $j = 1, \dots, K$. Then we define $\mathrm{div}_{\boldsymbol{q}^j} := \nabla_{\boldsymbol{q}^j}\cdot$. For a general mapping $\boldsymbol{q} \in D \mapsto B(\boldsymbol{q}) \in \mathbb{R}^{d\times K}$, we define $\mathrm{div}_{\boldsymbol{q}}\, B := \mathrm{div}_{q^1}\, B^1 + \dots + \mathrm{div}_{q^K}\, B^K$, where B^j, $j = 1, \dots, K$, denote the d-component column vectors of the matrix $B = B(\boldsymbol{q})$. We define the $d \times K$-component differential operator $\nabla_{\boldsymbol{q}} := (\nabla_{\boldsymbol{q}^1}, \dots, \nabla_{\boldsymbol{q}^K})$. Typically Maxwellians and the associated spring potentials satisfy the following properties: for $j = 1, \dots, K$ there exist constants $c_{ji} > 0$, $i = 1, 2, 3, 4$, and $\gamma_j > 1$ such that the spring potential U^j satisfies

$$c_{j1}[\mathrm{dist}(\boldsymbol{q}^j, \partial D^j)]^{\gamma_j} \le M^j(\boldsymbol{q}^j) \le c_{j2}[\mathrm{dist}(\boldsymbol{q}^j, \partial D^j)]^{\gamma_j} \qquad \forall\, \boldsymbol{q}^j \in D^j, \tag{1.5}$$

$$c_{j3} \le [\mathrm{dist}(\boldsymbol{q}^j, \partial D^j)](U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right) \le c_{j4} \qquad \forall\, \boldsymbol{q}^j \in D^j. \tag{1.6}$$

Since $[U^j(\frac{1}{2}|\boldsymbol{q}^j|^2)]^2 = (-\log M^j(\boldsymbol{q}^j) + C)^2$, it follows from (1.5) and (1.6) that

$$\int_{D^j} \left[1 + \left[U^j\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\right]^2 + \left[(U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\right]^2\right] M^j(\boldsymbol{q}^j)\,\mathrm{d}\boldsymbol{q}^j < \infty, \qquad j = 1, \dots, K.$$

Example 1.1.2. *Consider the FENE (finitely extensible nonlinear elastic) dumbbell model ($K = 1$), introduced by Warner [67]. The spring force is given by*

$$F(\boldsymbol{q}) = \left(1 - \frac{|\boldsymbol{q}|^2}{b}\right)^{-1} \boldsymbol{q}, \qquad \boldsymbol{q} \in D = B(0, b^{\frac{1}{2}}),$$

corresponding to

$$U(s) = -\frac{b}{2}\log\left(1 - \frac{2s}{b}\right), \qquad s \in I = \left[0, \frac{b}{2}\right), \ b > 2.$$

Direct calculation shows that in the case of the FENE dumbbell model, (1.5) *and* (1.6) *are satisfied with $\gamma = \frac{b}{2}$, provided that $b > 2$.*

Instead of directly assuming the properties (1.5) and (1.6) for the partial Maxwellian and the associated spring potential we shall use a simpler and less restrictive assumption on the Maxwellian in our analysis, which suffices for our purposes. In particular, we shall be assuming throughout the thesis that

$$M \in C_0(\overline{D}) \cap C^{0,1}_{loc}(D) \cap W^{1,1}_0(D),$$

which incorporates the property that FENE-type Maxwellians vanish on the boundary of the configuration space domain D.

To close the system (1.1)–(1.4), we define the extra-stress tensor S_e by the formula:

$$S_e(x,t) := k\left(\sum_{j=1}^{K}\int_D \psi(x, \boldsymbol{q}, t)\boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} (U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\mathrm{d}\boldsymbol{q} - K\varrho(x,t) I\right), \tag{1.7}$$

with I denoting the $d \times d$ identity matrix, $\mathrm{d}\boldsymbol{q} := \mathrm{d}\boldsymbol{q}^1 \cdots \mathrm{d}\boldsymbol{q}^K$, and the density of polymer chains (referred to as *polymer number density*) located at x at time t given by

$$\varrho(x,t) := \int_D \psi(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

For the derivation of the Kramers expression (1.7), we refer the reader to [13] and [45].

The probability density function ψ satisfies the following Fokker–Planck equation:

$$\begin{aligned}\frac{\partial \psi}{\partial t} &+ \mathrm{div}_x(\boldsymbol{v}\psi) + \sum_{j=1}^{K} \mathrm{div}_{\boldsymbol{q}^j}((\nabla_x \boldsymbol{v})\boldsymbol{q}^j \psi) \\ &= \varepsilon \Delta_x \left(\frac{\psi}{\zeta(\rho)}\right) + \frac{1}{4\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\,\mathrm{div}_{\boldsymbol{q}^j}\left(M \nabla_{\boldsymbol{q}^i}\left(\frac{\psi}{\zeta(\rho) M}\right)\right),\end{aligned} \tag{1.8}$$

in $\mathcal{O} \times (0,T)$, with $\mathcal{O} := \Omega \times D$. In the above equation $\zeta(\cdot) \in \mathbb{R}_{>0}$ is a density-dependent scaled drag coefficient. Let $\partial \bar{D}^j := D^1 \times \cdots \times D^{j-1} \times \partial D^j \times D^{j+1} \times \cdots \times D^K$. We impose the following boundary conditions, for all $j = 1, \dots, K$:

$$\left[\frac{1}{4\lambda}\sum_{i=1}^{K} A_{ij} M \nabla_{\boldsymbol{q}^i}\left(\frac{\psi}{\zeta(\rho)M}\right) - (\nabla_x \boldsymbol{v})\boldsymbol{q}^j \psi\right] \cdot \boldsymbol{n}^j = 0 \quad \text{on } \Omega \times \partial \bar{D}^j \times (0,T), \tag{1.9}$$

$$\varepsilon \nabla_x \left(\frac{\psi}{\zeta(\rho)}\right) \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times D \times (0,T), \tag{1.10}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$ and $\boldsymbol{n}^j = (n_1^j, \dots, n_d^j)^{\mathrm{T}}$ is the unit outward normal vector to $\partial\bar{D}^j$, and the following initial condition:

$$\psi(x,\boldsymbol{q},0) = \psi_0(x,\boldsymbol{q}) \qquad \text{in } \mathcal{O}. \tag{1.11}$$

Remark 1.1.3. *Note that we have chosen Neumann boundary conditions for both the boundary of* Ω *and the boundary of* D*, so that the integral of* ψ *over* $\mathcal{O}$ *is kept constant. To illustrate the idea, we integrate* (1.8) *over* $\mathcal{O}$*, and apply the Divergence Theorem to obtain that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \psi\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = &-\int_D \left(\int_{\partial\Omega} \psi \boldsymbol{v}\cdot \boldsymbol{n}\,\mathrm{d}S\right)\mathrm{d}\boldsymbol{q} + \varepsilon \int_D \left(\int_{\partial\Omega} \nabla_x\left(\frac{\psi}{\zeta(\rho)}\right)\cdot \boldsymbol{n}\,\mathrm{d}S\right)\mathrm{d}\boldsymbol{q} \\ &+ \sum_{j=1}^{K}\int_\Omega \left[\int_{\partial \bar{D}^j}\left(\frac{1}{4\lambda}\sum_{i=1}^{K} A_{ij} M \nabla_{\boldsymbol{q}^i}\left(\frac{\psi}{\zeta(\rho)M}\right) - (\nabla_x \boldsymbol{v})\boldsymbol{q}^j\psi\right)\cdot \boldsymbol{n}^j\,\mathrm{d}S\right]\mathrm{d}x.\end{aligned}$$

Since we shall take the no-slip boundary condition $\boldsymbol{v} = \boldsymbol{0}$ *on* $\partial\Omega$*, then by applying the initial conditions* (1.9) *and* (1.10)*, we see that all three terms on the right-hand side* *of the above equation vanish. Therefore,*

$$\int_{\mathcal{O}} \psi(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = \int_{\mathcal{O}} \psi_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = 1, \quad \forall t \in \mathbb{R}_{\geq 0},$$

since ψ is the probability density function.

In (1.7), the dimensionless constant $k > 0$ is a constant multiple of the product of the Boltzmann constant k_B and the absolute temperature T. The centre-of-mass diffusion coefficient $\varepsilon > 0$ is defined as $\varepsilon := (l_0/L_0)^2/(4(K+1)\lambda)$ with $l_0 := \sqrt{k_B \mathsf{T}/H}$ signifying the characteristic microscopic length-scale and $\lambda := (\zeta/4H)(U_0/L_0)$, where $H > 0$ is a spring-constant. The dimensionless positive parameter λ is called the Deborah number, which characterizes the elastic relaxation property of the fluid. In the subsequent discussion, we shall simply take $\varepsilon = 1$ and $\lambda = 1/4$ since none of the results depend on the specific values of these positive parameters. Furthermore, the constant matrix $A = (A_{ij})_{1\leq i,j\leq K}$, called the Rouse matrix, is symmetric and positive definite. The detailed derivation of the Fokker–Planck equation can be found in [8].

The system (1.1)–(1.4) coupled to (1.8) via the Kramers expression (1.7) forms the general FENE-type bead-spring model for dilute polymers. Our goal is to show that global-in-time weak solutions exist for this coupled Navier–Stokes–Fokker–Planck system. In the next section, we briefly describe the main method used in this thesis and discuss the technical details.

1.2 Methodology for the existence proof

In this section, we introduce the methodology used to show the existence of weak solutions of the coupled Navier–Stokes–Fokker–Planck system. First, to complete the system (1.1)–(1.4), we comment on the boundary conditions for the momentum equation (1.2) in Subsection 1.2.1. The derivation of a formal energy identity is presented in Subsection 1.2.2, followed by an introduction to the Galerkin method in Subsection 1.2.3.

1.2.1 Boundary conditions for the momentum equation

The boundary condition for $\boldsymbol{v}$ on $\partial\Omega$ usually depends on the type of flow under consideration. First, we note that the boundary $\partial\Omega$ is assumed to be impermeable, which means that

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. For a general vector $\boldsymbol{w} \in \mathbb{R}^d$, its tangential component $\boldsymbol{w}_\tau$ is defined by $\boldsymbol{w}_\tau := \boldsymbol{w} - (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n}$. The most general boundary condition for the tangential velocity $\boldsymbol{v}_\tau$ is given by

$$\lambda(\mathcal{T}\boldsymbol{n})_\tau + (1-\lambda)\gamma_* \boldsymbol{v}_\tau = \boldsymbol{0}, \tag{1.12}$$

where γ_* is a positive constant denoting the threshold value of the friction coefficient. When $\lambda \in (0, 1]$, the condition (1.12) is called the *Navier slip* boundary condition. When $\lambda = 0$, the condition (1.12) becomes the standard *no-slip* (homogeneous Dirichlet) boundary condition.

As the Navier slip boundary condition is more general, it covers more physically interesting cases, see, for example, [36]. From a purely mathematical point of view, the advantage of taking the Navier slip boundary condition is that the pressure p can then be proved to be a globally integrable function in Q provided that the boundary $\partial\Omega \in C^{1,1}$. The details regarding the reconstruction of the pressure p can be found in [15], [18] and [20].

In contrast, for the no-slip boundary condition, the best that can be shown for weak solutions is that p is a distribution in time [65]. If one wants to increase the regularity of the pressure p, one has to work on smoother domains, for example, $\partial\Omega \in C^2$.

For simplicity, we take the no-slip boundary condition for $\boldsymbol{v}$. Therefore, the full system introduced by (1.1)–(1.4) is now given by

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho \boldsymbol{v}) = 0 \qquad \text{in } Q, \tag{1.13}$$

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \mathrm{div}_x(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \mathrm{div}_x\, \mathcal{T} = \rho \boldsymbol{f} \qquad \text{in } Q, \tag{1.14}$$

$$\mathrm{div}_x\, \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{1.15}$$

subject to the initial conditions:

$$\begin{aligned} \rho(\cdot, 0) &= \rho_0(\cdot) && \text{in } \Omega, \\ (\rho\boldsymbol{v})(\cdot, 0) &= (\rho_0 \boldsymbol{v}_0)(\cdot) && \text{in } \Omega, \end{aligned} \tag{1.16}$$

and the no-slip boundary condition:

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega \times (0, T). \tag{1.17}$$

The Cauchy stress tensor $\mathcal{T}$ is decomposed as

$$\mathcal{T} = -pI + S_v + S_e, \tag{1.18}$$

where $S_v = 2\mu D(\boldsymbol{v})$, and S_e is given by the Kramers expression (1.7). In the next section, we shall introduce the associated formal energy identity, which plays a key role in our proof of existence of solutions.

1.2.2 The formal energy identity

To derive a formal energy identity, we assume that $\boldsymbol{v}$, ρ and ψ are sufficiently smooth, and, at least for our purposes in this introductory section, ρ is nonnegative, and ψ is positive. By taking the $L^2(\Omega)$ inner product of (1.13) with $\frac{1}{2}|\boldsymbol{v}|^2$, we deduce that

$$\frac{1}{2}\int_\Omega \left(\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho \boldsymbol{v})\right)|\boldsymbol{v}|^2 \,\mathrm{d}x = 0. \tag{1.19}$$

Taking the $L^2(\Omega;\mathbb{R}^d)$ inner product of (1.14) with $\boldsymbol{v}$, we obtain that

$$\int_\Omega \frac{\partial(\rho\boldsymbol{v})}{\partial t}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathrm{div}_x(\rho\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{v}\,\mathrm{d}x - \int_\Omega \mathrm{div}_x\,\mathcal{T}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x.$$

First, we note that

$$\int_\Omega \frac{\partial(\rho\boldsymbol{v})}{\partial t}\cdot\boldsymbol{v}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\boldsymbol{v}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{\partial\rho}{\partial t}|\boldsymbol{v}|^2\,\mathrm{d}x. \tag{1.20}$$

Upon partial integration and noting the homogeneous Dirichlet boundary condition on $\boldsymbol{v}$ and the divergence-free property of $\boldsymbol{v}$, we deduce that

$$\begin{aligned}
\int_\Omega \mathrm{div}_x(\rho\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{v}\,\mathrm{d}x &= \int_\Omega [(\nabla_x\rho)\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) + \rho\,\mathrm{div}_x(\boldsymbol{v}\otimes\boldsymbol{v})]\cdot\boldsymbol{v}\,\mathrm{d}x\\
&= \int_\Omega (\nabla_x\rho)\cdot(\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}})\cdot\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \rho[(\mathrm{div}_x\,\boldsymbol{v})\boldsymbol{v}^{\mathrm{T}} + (\boldsymbol{v}\cdot\nabla_x)\boldsymbol{v}^T]\cdot\boldsymbol{v}\,\mathrm{d}x\\
&= \int_\Omega \mathrm{div}_x(\rho\boldsymbol{v})|\boldsymbol{v}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho\boldsymbol{v}\cdot\nabla_x(|\boldsymbol{v}|^2)\,\mathrm{d}x\\
&= \int_\Omega \mathrm{div}_x(\rho\boldsymbol{v})|\boldsymbol{v}|^2\,\mathrm{d}x + \frac{1}{2}\left(\int_{\partial\Omega}\rho|\boldsymbol{v}|^2(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \mathrm{div}_x(\rho\boldsymbol{v})|\boldsymbol{v}|^2\,\mathrm{d}x\right)\\
&= \frac{1}{2}\int_\Omega \mathrm{div}_x(\rho\boldsymbol{v})|\boldsymbol{v}|^2\,\mathrm{d}x.
\end{aligned} \tag{1.21}$$

Again by using (1.7), the divergence-free property of $\boldsymbol{v}$, the homogeneous Dirichlet boundary condition for $\boldsymbol{v}$ and integration by parts, we deduce that

$$\begin{aligned}
\int_\Omega \mathrm{div}_x\,\mathcal{T}\cdot\boldsymbol{v}\,\mathrm{d}x &= \int_{\partial\Omega}\mathcal{T}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_\Omega \mathcal{T}:\nabla_x\boldsymbol{v}\,\mathrm{d}x\\
&= \int_\Omega pI:\nabla_x\boldsymbol{v}\,\mathrm{d}x - \int_\Omega S_v:\nabla_x\boldsymbol{v}\,\mathrm{d}x - \int_\Omega S_e:\nabla_x\boldsymbol{v}\,\mathrm{d}x\\
&= \int_\Omega p(\mathrm{div}_x\,\boldsymbol{v})\,\mathrm{d}x - \int_\Omega 2\mu|D(\boldsymbol{v})|^2\,\mathrm{d}x\\
&\quad - k\sum_{j=1}^K \int_{\mathcal{O}} \psi\,(U^j)'\!\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\boldsymbol{q}^j\boldsymbol{q}^{j^{\mathrm{T}}}:\nabla_x\boldsymbol{v}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x + k\int_\Omega K\varrho(\mathrm{div}_x\,\boldsymbol{v})\,\mathrm{d}x\\
&= -\int_\Omega 2\mu|D(\boldsymbol{v})|^2 - k\sum_{j=1}^K \int_{\mathcal{O}} \psi\,(U^j)'\!\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\boldsymbol{q}^j\boldsymbol{q}^{j^{\mathrm{T}}}:\nabla_x\boldsymbol{v}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x,
\end{aligned} \tag{1.22}$$

since $D(\boldsymbol{v})$ is symmetric. Combining (1.20), (1.21) and (1.22) and using (1.19), we deduce the following identity:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho|\boldsymbol{v}|^2\,\mathrm{d}x &+ \int_\Omega 2\mu|D(\boldsymbol{v})|^2\,\mathrm{d}x \\ &= \int_\Omega \rho\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x - k\sum_{j=1}^{K}\int_{\mathcal{O}}\psi\,(U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\boldsymbol{q}^j{\boldsymbol{q}^j}^{\mathrm{T}} : \nabla_x\boldsymbol{v}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x.\end{aligned}\tag{1.23}$$

To better illustrate the idea for deriving the formal energy identity, we assume that $\zeta \equiv 1$ in (1.8). Also, to simplify the presentation of the calculation, we introduce $\hat{\psi} := \frac{\psi}{M}$. Then, we multiply the Fokker–Planck equation (1.8) by $(\log\hat{\psi} + 1)$, and integrate over $\mathcal{O}$. Let $\mathcal{F}(s) = s\log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s\to 0+}\mathcal{F}(s) = 1$. We note that $\mathcal{F}'(s) = \log s + 1$ and $\mathcal{F}''(s) = \frac{1}{s}$ for $s > 0$. Then, we obtain that

$$\begin{aligned}&\int_{\mathcal{O}}\frac{\partial(M\hat{\psi})}{\partial t}\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x + \int_{\mathcal{O}}\mathrm{div}_x(M\boldsymbol{v}\hat{\psi})\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &\quad + \sum_{j=1}^{K}\int_{\mathcal{O}}\mathrm{div}_{\boldsymbol{q}^j}((\nabla_x\boldsymbol{v})\boldsymbol{q}^j M\hat{\psi})\,\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = \int_{\mathcal{O}}\Delta_x(M\hat{\psi})\,\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}A_{ij}\int_{\mathcal{O}}\mathrm{div}_{\boldsymbol{q}^j}(M\nabla_{\boldsymbol{q}^i}\hat{\psi})\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x.\end{aligned}$$

First, we note, by the chain rule, that

$$\int_{\mathcal{O}}\frac{\partial(M\hat{\psi})}{\partial t}\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}M\mathcal{F}(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x.\tag{1.24}$$

Next, we apply partial integration, the Divergence Theorem and use the boundary condition (1.17) and the fact that $\mathrm{div}_x\,\boldsymbol{v} = 0$ to deduce that

$$\begin{aligned}&\int_{\mathcal{O}}\mathrm{div}_x(M\boldsymbol{v}\hat{\psi})\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &= \int_D\left(\int_{\partial\Omega}\mathcal{F}'(\hat{\psi})M\hat{\psi}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S\right)\,\mathrm{d}\boldsymbol{q} - \int_{\mathcal{O}}M\hat{\psi}\boldsymbol{v}\cdot\nabla_x\mathcal{F}'(\hat{\psi})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &= -\int_{\mathcal{O}}M\boldsymbol{v}\cdot\nabla_x\hat{\psi}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &= -\int_{\mathcal{O}}\mathrm{div}_x(M\hat{\psi}\boldsymbol{v})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ &= -\int_D\left(\int_{\partial\Omega}M\hat{\psi}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S\right)\,\mathrm{d}\boldsymbol{q} \\ &= 0.\end{aligned}\tag{1.25}$$

By noting the identity

$$\nabla_x \hat{\psi} = 2\sqrt{\hat{\psi}}\nabla_x \sqrt{\hat{\psi}},$$

we have, by partial integration and by using the boundary condition (1.10), that

$$\begin{aligned}
&\int_{\mathcal{O}} \Delta_x (M\hat{\psi})\, \mathcal{F}'(\hat{\psi})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = -\int_{\mathcal{O}} \nabla_x (M\hat{\psi}) \cdot \nabla_x \mathcal{F}'(\hat{\psi})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x + \int_D \left(\int_{\partial\Omega} \mathcal{F}'(\hat{\psi}) \nabla_x (M\hat{\psi}) \cdot \boldsymbol{n}\, \mathrm{d}S \right) \mathrm{d}\boldsymbol{q} \\
&\quad = -\int_{\mathcal{O}} M \nabla_x \hat{\psi} \cdot \nabla_x \hat{\psi}\, \frac{1}{\hat{\psi}}\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = -4 \int_{\mathcal{O}} M \left| \nabla_x \sqrt{\hat{\psi}} \right|^2 \mathrm{d}\boldsymbol{q}\, \mathrm{d}x.
\end{aligned} \tag{1.26}$$

Similarly as in (1.26), by using partial integration, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\mathcal{O}} \mathrm{div}_{\boldsymbol{q}^j} (M \nabla_{\boldsymbol{q}^i} \hat{\psi}) \mathcal{F}'(\hat{\psi})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\Omega} \left(\int_{\partial \bar{D}^j} M \mathcal{F}'(\hat{\psi}) \nabla_{\boldsymbol{q}^i} \hat{\psi} \cdot \boldsymbol{n}^j\, \mathrm{d}S \right) \mathrm{d}x \\
&\qquad - \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\mathcal{O}} M \nabla_{\boldsymbol{q}^i} \hat{\psi} \cdot \nabla_{\boldsymbol{q}^j} \mathcal{F}'(\hat{\psi})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\Omega} \left(\int_{\partial \bar{D}^j} M \mathcal{F}'(\hat{\psi}) \nabla_{\boldsymbol{q}^i} \hat{\psi} \cdot \boldsymbol{n}^j\, \mathrm{d}S \right) \mathrm{d}x \\
&\qquad - 4 \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\mathcal{O}} M \nabla_{\boldsymbol{q}^i} \sqrt{\hat{\psi}} \cdot \nabla_{\boldsymbol{q}^j} \sqrt{\hat{\psi}}\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x.
\end{aligned} \tag{1.27}$$

Finally, by applying partial integration, using the fact that $\mathrm{div}_x\, \boldsymbol{v} = 0$, we deduce that

$$\begin{aligned}
&\sum_{j=1}^{K} \int_{\mathcal{O}} \mathrm{div}_{\boldsymbol{q}^j} ((\nabla_x \boldsymbol{v}) \boldsymbol{q}^j M \hat{\psi}) \mathcal{F}'(\hat{\psi})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = \sum_{j=1}^{K} \int_{\Omega} \left(\int_{\partial \bar{D}^j} \mathcal{F}'(\hat{\psi}) (\nabla_x \boldsymbol{v}) \boldsymbol{q}^j M \hat{\psi} \cdot \boldsymbol{n}^j\, \mathrm{d}S \right) \mathrm{d}x \\
&\qquad - \sum_{j=1}^{K} \int_{\mathcal{O}} (\nabla_x \boldsymbol{v}) \boldsymbol{q}^j M \hat{\psi} \mathcal{F}''(\hat{\psi}) \cdot \nabla_{\boldsymbol{q}^j} \hat{\psi}\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x \\
&\quad = \sum_{j=1}^{K} \int_{\Omega} \left(\int_{\partial \bar{D}^j} \mathcal{F}'(\hat{\psi}) (\nabla_x \boldsymbol{v}) \boldsymbol{q}^j M \hat{\psi} \cdot \boldsymbol{n}^j\, \mathrm{d}S \right) \mathrm{d}x - \sum_{j=1}^{K} \int_{\mathcal{O}} M (\nabla_x \boldsymbol{v}) \boldsymbol{q}^j \cdot \nabla_{\boldsymbol{q}^j} \hat{\psi}\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}x.
\end{aligned} \tag{1.28}$$

For the last term on the right-hand side of (1.28), we again apply partial integration and use the divergence-free property of $\boldsymbol{v}$ and the fact that $M = 0$ on the boundary of D that

$$\begin{aligned}
&\sum_{j=1}^{K} \int_{\mathcal{O}} M(\nabla_x \boldsymbol{v})\boldsymbol{q}^j \cdot \nabla_{\boldsymbol{q}^j} \hat{\psi} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = \sum_{j=1}^{K} \int_{\Omega} \nabla_x \boldsymbol{v} : \left(\int_D M \boldsymbol{q}^j \nabla_{\boldsymbol{q}^j} \hat{\psi} \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x \\
&\quad = \sum_{j=1}^{K} \int_{\Omega} \nabla_x \boldsymbol{v} : \left(\int_{\partial \bar{D}^j} M \hat{\psi} \boldsymbol{q}^j \cdot \boldsymbol{n}^j \,\mathrm{d}S \right) \mathrm{d}x - \sum_{j=1}^{K} \int_{\Omega} \nabla_x \boldsymbol{v} : \left(\int_D M \hat{\psi} I \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x \\
&\qquad - \sum_{j=1}^{K} \int_{\Omega} \nabla_x \boldsymbol{v} : \left(\int_D (\boldsymbol{q}^j \otimes \nabla_{\boldsymbol{q}^j} M) \hat{\psi} \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x \\
&\quad = - \sum_{j=1}^{K} \int_{\Omega} (\mathrm{div}_x\, \boldsymbol{v}) \left(\int_D M \hat{\psi} \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x \\
&\qquad + \sum_{j=1}^{K} \int_{\mathcal{O}} M \hat{\psi} (U^j)' \left(\frac{1}{2} |\boldsymbol{q}^j|^2 \right) \boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} : \nabla_x \boldsymbol{v} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\
&\quad = \sum_{j=1}^{K} \int_{\mathcal{O}} \psi (U^j)' \left(\frac{1}{2} |\boldsymbol{q}^j|^2 \right) \boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} : \nabla_x \boldsymbol{v} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x.
\end{aligned} \tag{1.29}$$

Therefore, combining (1.24)–(1.29), using the boundary conditions (1.9) and (1.10) in (1.27) and (1.28), we get, by recalling $\hat{\psi} = \frac{\psi}{M}$, that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} M \mathcal{F}\left(\frac{\psi}{M} \right) \mathrm{d}\boldsymbol{q}\,\mathrm{d}x - \sum_{j=1}^{K} \int_{\mathcal{O}} \psi (U^j)' \left(\frac{1}{2} |\boldsymbol{q}^j|^2 \right) \boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} : \nabla_x \boldsymbol{v} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\
&\quad + 4 \int_{\mathcal{O}} M \left| \nabla_x \sqrt{\frac{\psi}{M}} \right|^2 \mathrm{d}\boldsymbol{q}\,\mathrm{d}x + 4 \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\mathcal{O}} M \nabla_{\boldsymbol{q}^i} \sqrt{\frac{\psi}{M}} \cdot \nabla_{\boldsymbol{q}^j} \sqrt{\frac{\psi}{M}} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x = 0,
\end{aligned} \tag{1.30}$$

where $\mathcal{F}(s) = s \log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s \to 0+} \mathcal{F}(s) = 1$. We now multiply (1.30) by k and add the resulting identity to (1.23) to deduce that (on assuming that $\zeta \equiv 1$),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left[\frac{1}{2} \rho |\boldsymbol{v}|^2 + k \int_D M \mathcal{F}\left(\frac{\psi}{M} \right) \mathrm{d}\boldsymbol{q} \right] \mathrm{d}x + \int_{\Omega} 2\mu |D(\boldsymbol{v})|^2 \,\mathrm{d}x \\
&+ 4k \int_{\mathcal{O}} M \left| \nabla_x \sqrt{\frac{\psi}{M}} \right|^2 \mathrm{d}\boldsymbol{q}\,\mathrm{d}x + 4k \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \int_{\mathcal{O}} M \nabla_{\boldsymbol{q}^i} \sqrt{\frac{\psi}{M}} \cdot \nabla_{\boldsymbol{q}^j} \sqrt{\frac{\psi}{M}} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\
&\quad = \int_{\Omega} \rho \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x,
\end{aligned} \tag{1.31}$$

which is the formal energy identity that is essential to our proof of existence of weak solutions to the coupled Navier–Stokes–Fokker–Planck system under consideration.

1.2.3 Galerkin approximations

To prove the existence of weak solutions to the problem described above, we use Galerkin approximations in the spatial domains, i.e. Ω for the velocity field $\boldsymbol{v}$ and $\mathcal{O}$ for the probability density function ψ. The basic idea is to project $\boldsymbol{v}$ and ψ onto suitable finite-dimensional subspaces. In this way, (1.14) and (1.8) become systems of ordinary differential equations (ODEs), for which short-time existence of solutions can be proved using Carathéodory's Existence Theorem. Then, to obtain the global-in-time existence of solutions, we shall derive uniform estimates and find weakly (and strongly) convergent subsequences by compactness arguments, through which we can pass to the relevant limits and identify the limiting equations. The detailed application of the Galerkin method to general linear second-order parabolic equations can be found in [30].

However, unlike the linear case in [30], our situation is more complicated thanks to the nonlinear convective term in (1.14). To be more specific, the weak form of the convective term is given by

$$\int_0^T \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla_x \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}t, \tag{1.32}$$

where $\boldsymbol{w}$ belongs to a suitable test function space. To simply illustrate the idea, here we assume that $\rho = 1$. The usual a priori estimates give that $\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$. Then we have $\boldsymbol{v} \in L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)$ by standard interpolation. For (1.32) to be finite, we require that the test function $\boldsymbol{w} \in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^d))$, with $p > \frac{d+2}{2}$, which then automatically excludes the velocity field $\boldsymbol{v}$ itself as a valid test function. Therefore, to maintain the energy identity, we truncate the convective term as the first level of approximation; then we also truncate the probability density function in the Fokker–Planck equation accordingly. The details on the truncations will be given in later chapters when dealing with specific models.

1.2.4 Convergence in L^1

When passing to the limits in the Galerkin approximation parameters and in the truncation level, there is another technical difficulty worth noting, that is convergence in L^1. Since for an open, bounded subset $O \subset \mathbb{R}^d$, $L^1(O)$ is not the dual space of $L^\infty(O)$, then boundedness in $L^1(O)$ does not imply weak convergence of a subsequence in $L^1(O)$, which in turn causes difficulties in deducing strong convergence of sequences in $L^1(O)$.

The key idea to solve this problem here is to apply the Div-Curl Lemma. Consider the approximate probability density function $\hat{\psi}^m$. From the energy identity (1.31), we deduce the equi-integrability of $(\hat{\psi}^m)_{m=1}^{\infty}$ by the De la Vallée Poussin's Theorem. Then the Dunford–Pettis Theorem gives the weak convergence in $L^1_{loc}(\mathcal{O} \times (0,T))$. We use the Div-Curl Lemma to show that $(\hat{\psi}^m)_{m=1}^{\infty}$ converges almost everywhere in $\mathcal{O} \times (0,T)$, followed by Vitali's Convergence Theorem, which gives the strong convergence of the sequence $(\hat{\psi}^m)_{m=1}^{\infty}$ in $L^1(0,T;L^1(\mathcal{O}))$ as $m \to \infty$.

Now we have discussed the main method used in this work and the difficulties that may arise in the proofs. In the next section, we briefly review previous existence results for Navier–Stokes–Fokker–Planck systems that arise in models of polymeric fluids.

1.3 Literature survey on existence results on polymeric fluids

In this section, we provide a brief literature survey of the study of polymeric flow models. Unless otherwise stated, the centre-of-mass diffusion term is absent from the model in the cited reference, i.e., $\varepsilon = 0$; also, unless otherwise stated, the density ρ is assumed to be constant, and in all cited references only a simple dumbbell model is considered rather than a bead-spring chain model, i.e., $K = 1$.

In [58], Renardy proved a local existence and uniqueness result for a family of Hookean-type dumbbell models. Subsequently, E, Li and Zhang [28] and Li, Zhang and Zhang [44] revisited the question of local existence of solutions for dumbbell models, while Zhang and Zhang [70] showed a local existence and uniqueness result for regular solutions to FENE-type dumbbell models. All of these papers required high regularity of the initial data. In [24], Constantin considered the Navier–Stokes equations coupled with a nonlinear Fokker–Planck equation modelling the probability distribution of particles interacting with the fluid and in [25], Constantin and Seregin proved the global regularity of solutions of the incompressible Navier–Stokes–Fokker–Planck system in $\mathbb{R}^2$ in the absence of boundaries.

In [47], Lions and Masmoudi proved the global existence of weak solutions for the corotational FENE dumbbell model and the Doi model (also called the rod model) using a propagation-of-compactness argument, i.e., the property that if one takes a sequence of weak solutions, which converges weakly and such that the initial data converge strongly, then the weak limit is also a solution. In [50], Masmoudi explored the FENE dumbbell model for a general class of potentials; he proved local existence in Sobolev spaces, global existence if the initial data are close to equilibrium, and global existence in two dimensions for the corotational FENE model.

In [3], Barrett, Schwab and Süli showed the existence of global-in-time weak solutions to the coupled microscopic-macroscopic bead-spring chain model with constant density and in the absence of the centre-of-mass diffusion term, i.e. $\varepsilon = 0$. This paper assumed a large class of potentials U, including the Hookean dumbbell model and general FENE-type dumbbell models in the general noncorotational case, however the velocity field $\boldsymbol{v}$ in the drift-term of the Fokker–Planck equation and the extra stress tensor had to be mollified.

Subsequently, in [5] and [7] Barrett and Süli managed to prove the existence of global-in-time weak solutions to general noncorotational Hookean-type bead-spring chain models and FENE-type bead-spring chain models respectively, with centre-of-mass diffusion $\varepsilon > 0$, but without mollification and in the general case of $K \geq 1$ coupled beads in the bead-spring chain. This was achieved by introducing a cut-off parameter L, discretizing the resulting model in time, and then passing to the limit as $L \to \infty$ by requiring that the time step $\Delta t = o(L^{-1})$. The papers also provided rigorous proofs, for both FENE-type and Hookean-type models, of the convergence of weak solutions to their respective equilibria: $\boldsymbol{v}_\infty = \boldsymbol{0}$ and $\psi_\infty = M$, as $t \to \infty$. A key contribution to the field has been Masmoudi's paper [51], which proved global existence of weak solutions to the FENE dumbbell model, in the absence of a centre-of-mass diffusion term.

In the above literature, the authors have assumed a linear relation between the viscous part S_v of the Cauchy stress and the symmetric part of the velocity gradient $D(\boldsymbol{v})$. In [21], Bulíček et al. introduced a general implicit constitutive equation relating S_v and $D(\boldsymbol{v})$. They considered implicit relations that generate maximal monotone graphs, and the corresponding rate of dissipation is characterized by the sum of a Young function and its conjugate depending on $D(\boldsymbol{v})$ and S_v. Bulíček et al. proved long-time and large-data existence of weak solutions by using properties of maximal monotone operators and Lipschitz approximations of Sobolev-space-valued Bochner functions via a weak compactness arguments based on the Div-Curl Lemma and Chacon's Biting Lemma.

So far, we have been reviewing the literature concerning nonisothermal fluid models. However, in some situations, the temperature change substantially influences the fluid material properties, which means that we cannot simply assume that the temperature is a constant. In such cases, fluids are referred to as the *nonisothermal* fluids.

The study of nonisothermal polymeric fluids from a microscopic-macroscopic point of view arose during the early 1970s, when Marrucci [49] first derived the kinetic model of nonisothermal polymer fluids based on the Hookean dumbbell model. The key contribution of [49] is the detailed derivation of a constitutive equation for the extra-stress tensor. Subsequently, Bird [22] generalized the constitutive relation in [49] to include nonlinear temperature dependence. The constitutive equation for the extra-stress tensor in [22] was further applied to various models, see, for example, [69] and [68]. However, the above works were only limited to fluids with low elasticity. To consider more general and physically more realistic cases, the GENERIC formulation was proposed by Öttinger and Grmela [55], Grmela and Öttinger [35].

In [27], Dostalík et al. rigorously derived a thermodynamically consistent model for the nonisothermal dilute polymeric fluids, which was similar to the model in Öttinger and Grmela [55]. However, compared to the method in [55], the arguments in [27] was simpler and more straightforward. This was achieved by identifying the correct energy storage mechanisms and the entropy production mechanisms, after which the governing equations were direct consequences of *the second law of thermodynamics.*

In the next section, we shall briefly summarise the contributions of this thesis.

1.4 Contributions of the thesis

In this thesis, we prove rigorously the existence of global-in-time weak solutions to coupled Navier–Stokes–Fokker–Planck systems modelling different types of incompressible dilute polymeric fluids, which include the homogeneous system, the nonhomogeneous system and the nonisothermal homogeneous system.

In Chapter 3, we prove the existence of global-in-time weak solutions to a model of a *homogeneous* dilute polymeric fluid. Our model is motivated by [31]. The key feature is that the viscosity coefficient μ is a C^1 function of the polymer-number-density ϱ. In [31], Feireisl et al. proved the weak sequential stability of the family of dissipative (finite-energy) weak solutions to the compressible Navier–Stokes–Fokker–Planck system provided that such weak solutions exist. However, whether such weak solutions exist it not yet known. Compared to the model in [31], we consider the simpler version by assuming the fluid is incompressible. By assuming the fluid is incompressible, the complicated assumptions which control the growth of $\mu(\varrho)$ in [31] are simplified to assuming that $\mu(\varrho)$ is bounded above and below by constants. Our proof of existence of weak solutions, motivated by the approach in [21], is based on combining truncations on the convective term and the probability density function with a Galerkin semi-discretization on the spatial domains. To identify various limits, we derive uniform a priori estimates independent of the Galerkin and truncation parameters. With the help of compact embedding theorems, we deduce the convergence of sequence of approximate solutions to a weak solution of the Navier–Stokes–Fokker–Planck system.

In Chapter 4, we show the existence of global-in-time weak solutions to a model of a *nonhomogeneous* dilute polymeric fluid. The key feature is the presence of the density-dependent drag coefficient in the Fokker–Planck equation. We extend the results in [8] to a more general class of models, where we allow the viscosity coefficient μ to depend on both the density ρ and the polymer number density ϱ. Also, compared to the method, which is based on time-discretization, in [8], the advantage here is that using the Galerkin discretization method considerably simplifies the proof. However, the introduction of the variable density and the density-dependent drag coefficient causes technical difficulties. Firstly, at the Galerkin approximation level, we do not discretize the density ρ. We apply Schauder's Fixed-Point Theorem to show the existence of solutions to the partially Galerkin-discretized system. In addition to the usual weak compactness techniques used in Chapter 3, we also apply, for example, a Nikolskiĭ norm estimate and the method of characteristics, to deduce uniform estimates and hence deduce the relevant convergence results. Finally, without further assumptions on the regularity of the data, the transport equation governing the density ρ, and subsequently the equation for the drag coefficient $\zeta(\rho)$, are only satisfied in the distributional sense. To overcome this difficulty, we approximate the initial density ρ_0 by a sequence of functions in $C^1(\overline{\Omega})$ which allows us to deduce higher regularity of $\partial_t \rho$ and $\nabla_x \rho$.

In Chapter 5, we show the existence of global-in-time weak solutions to a model of a *nonisothermal* homogeneous dilute polymeric fluid. We consider the model derived in [27]. To simplify the model, we consider the corotational Fokker–Planck equation, which means that the symmetric part of the velocity gradient $\nabla_x \boldsymbol{v}$ featuring in the drag term in the Fokker–Planck equation is assumed to vanish, i.e. $\nabla_x \boldsymbol{v}$ is replaced by $\frac{1}{2}(\nabla_x \boldsymbol{v} - (\nabla_x \boldsymbol{v})^{\mathrm{T}})$. We also include an additional cut-off parameter L in the system by introducing a cut-off function $\beta^L(\cdot)$ in the second order terms in the Fokker–Planck equation. The proof is based on combining a truncation of the convective term with a Galerkin semi-discretization of the velocity field, the probability density function and the temperature. The main difficulty lies in the fact that on the right-hand side of the temperature equation, we have $D(\boldsymbol{v}) : D(\boldsymbol{v})$ which at the level of weak solution belongs to $L^1(Q)$ only, and passage to the limit in its Galerkin approximation then requires strong convergence of the Galerkin sequence of velocity gradients. Also, since the velocity gradient $D(\boldsymbol{v})$ lacks integrability, we have to take the test function to be a fraction of the temperature θ instead of θ itself. In the final step as the truncation parameter $\ell \to \infty$, strong convergence of the velocity gradient is not possible to obtain. To circumvent this, we have to consider the total energy $E = \frac{1}{2}|\boldsymbol{v}|^2 + \theta$ and deduce the limiting equation for E instead of θ.

In Chapter 6, we briefly summarise the results obtained in this thesis and provide possible extensions.

Chapter 2

Preliminaries

In the present chapter we shall introduce the notions and results used in this thesis. In Section 2.1, we first define various function spaces. In Section 2.2, we introduce the Aubin–Lions Lemma and its extensions. Next, we state some interpolation inequalities in Section 2.3. Finally, in Section 2.4, we present some convergence results which will be useful for our analysis later.

2.1 Notation and function spaces

Here we shall summarise the definitions of Lebesgue spaces, Sobolev spaces and Bochner spaces. Let O be a measurable set in $\mathbb{R}^d$ and $p \in [1, \infty)$. The standard Lebesgue space of p-integrable functions is denoted by $(L^p(O), \|\cdot\|_{L^p(O)})$. When $p = \infty$, $(L^\infty(O), \|\cdot\|_{L^\infty(O)})$ denotes the space of essentially bounded functions. For $s \in \mathbb{N}$, let $(W^{s,p}(O), \|\cdot\|_{W^{s,p}(O)})$ be the standard Sobolev spaces and denote by $|\cdot|_{W^{s,p}(O)}$ the corresponding semi-norm. We denote the space of signed Radon measures on O with finite mass by $\mathcal{M}(O)$. Since we also need to work with Maxwellian-weighted spaces, we define, for a nonnegative weight-function $N \in L^\infty_{loc}(O)$,

$$
\begin{aligned}
L^p_N(O) &:= \{f \in L^p_{loc}(O) : \int_O N(z)\, |f(z)|^p \,\mathrm{d}z < \infty\}, \\
W^{1,p}_N(O) &:= \{f \in W^{1,p}_{loc}(O) : \int_O N(z)\, (|f(z)|^p + |\nabla_z f(z)|^p) \,\mathrm{d}z < \infty\}.
\end{aligned}
$$

For any pair of functions f, g, with $f \in L^p(O)$ and $g \in L^{p'}(O)$, where $1/p + 1/p' = 1$ and $p, p' \in [1, \infty]$, we set

$$
(f, g)_O := \int_O f(z)\, g(z) \,\mathrm{d}z.
$$

Note that we set $1' = \infty$ and $\infty' = 1$. We adopt analogous notation for vector-valued and tensor-valued functions. We shall specify the dimension of vector-valued and tensor-valued functions by $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ respectively. If $O = \Omega$, we omit the subscript Ω from the inner product $(f, g)_\Omega$ for simplicity. For a general Banach space $(X, \|\cdot\|_X)$, the dual space consisting of all continuous linear functionals on X is denoted by X' and the dual pairing is denoted by $\langle f, g\rangle_X$ if $f \in X'$ and $g \in X$. When it is clear from the context which space X is intended, the subscript $_X$ will be omitted from the notation of the duality pairing, and we shall, for the sake of notational simplicity, write $\langle f, g\rangle$ instead of $\langle f, g\rangle_X$. For the Sobolev space $W^{1,p}(O)$ where $1 < p < \infty$, we denote its dual space by $(W^{1,p}(O))'$. We define the negative order Sobolev space $W^{-1,1}(O)$ by $W^{-1,1}(O) := \{\operatorname{div} \boldsymbol{f} : \boldsymbol{f} \in L^1(O; \mathbb{R}^d)\}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain, with $d = 2$ or 3, and let $C(\overline{\Omega})$ denote the set of all continuous real-valued functions on $\overline{\Omega}$. Let $C^\infty(\Omega)$ be the set of all smooth functions on Ω and let $C_0^\infty(\Omega)$ be the set of all functions in $C^\infty(\Omega)$ that are compactly supported in Ω. Then we define for $p \in (1, \infty)$ the following function spaces:

$$\begin{aligned}
W_0^{1,p}(\Omega; \mathbb{R}^d) &:= \overline{C_0^\infty(\Omega; \mathbb{R}^d)}^{\|\cdot\|_{W^{1,p}(\Omega;\mathbb{R}^d)}},\\
W_{0,\mathrm{div}}^{1,p}(\Omega; \mathbb{R}^d) &:= \overline{\{\boldsymbol{v} \in C_0^\infty(\Omega; \mathbb{R}^d) : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}}^{\|\cdot\|_{W^{1,p}(\Omega;\mathbb{R}^d)}},\\
L_{0,\mathrm{div}}^{2}(\Omega; \mathbb{R}^d) &:= \overline{W_{0,\mathrm{div}}^{1,2}(\Omega; \mathbb{R}^d)}^{\|\cdot\|_{L^{2}(\Omega;\mathbb{R}^d)}},\\
W^{-1,p'}(\Omega; \mathbb{R}^d) &:= (W_0^{1,p}(\Omega; \mathbb{R}^d))',\\
W_{\mathrm{div}}^{-1,p'}(\Omega; \mathbb{R}^d) &:= (W_{0,\mathrm{div}}^{1,p}(\Omega; \mathbb{R}^d))'.
\end{aligned}$$

We for $p \in [1, \infty]$ denote by $L^p(0, T; X)$ the standard Bochner space of p-integrable for $p \in [1, \infty)$, and essentially bounded, when $p = \infty$, X-valued functions defined on the interval $(0, T)$. If X is separable and reflexive and $p \in (1, \infty)$, then $L^p(0, T; X)$ is separable and reflexive and $(L^p(0, T; X))' = L^{p'}(0, T; X')$ (see [23]).

2.2 Compactness theorems in Banach spaces

In this section, we introduce the Aubin–Lions Lemma and its extensions, which will be helpful in proving strong convergence results.

Lemma 2.2.1. *(Theorem 2.1 in [65]) Let X_0, X, X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \le p, q \le \infty$, let*

$$\mathcal{Y} := \left\{ f \in L^p(0, T; X_0), \frac{\mathrm{d}f}{\mathrm{d}t} \in L^q(0, T; X_1) \right\}.$$

1. *if $p < \infty$, then the embedding of $\mathcal{Y}$ into $L^p(0,T;X)$ is compact,*
2. *if $p = \infty$ and if $q > 1$, then the embedding of $\mathcal{Y}$ into $C([0,T];X)$ is compact.*

The above lemma can be generalized to the case where $\frac{\mathrm{d}f}{\mathrm{d}t}$ is a measure.

Lemma 2.2.2. *(Corollary 7.9 in [60]) Let X_0, X, X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Suppose also that X_0 is reflexive, and the Banach space X_1 has a predual space E, in the sense that $X_1 = E'$. For $1 < p < \infty$, let*

$$\mathcal{Y} := \left\{ f \in L^p(0,T;X_0), \frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{M}(0,T;X_1) \right\}.$$

Then the embedding of $\mathcal{Y}$ into $L^p(0,T;X)$ is compact.

In certain cases, we cannot derive a bound on the full time derivative of f. We then have to consider a fractional time derivative. Let X be a Banach space. For $1 \le p < \infty$ and $0 < \gamma < 1$, we define the Nikolskiĭ spaces $N_p^\gamma(0,T;X)$ by

$$N_p^\gamma(0,T;X) := \left\{ f \in L^p(0,T;X), \sup_{0<h<T} \frac{\|f(\cdot + h) - f(\cdot)\|_{L^p(0,T-h;X)}}{h^\gamma} < \infty \right\},$$

and for $f \in N_p^\gamma(0,T;X)$, we introduce the following norm:

$$\|f\|_{N_p^\gamma(0,T;X)} := \left(\|f\|_{L^p(0,T;X)}^p + \sup_{0<h<T} \left(\frac{\|f(\cdot + h) - f(\cdot)\|_{L^p(0,T-h;X)}}{h^\gamma} \right)^p \right)^{\frac{1}{p}}.$$

We introduce the following results concerning the compact embedding of the Nikolskiĭ spaces.

Lemma 2.2.3. *(Simon, [62]) Let X_0, X, X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Then, for all $1 \le p < \infty$ and $0 < \gamma < 1$, the embedding*

$$L^p(0,T;X_0) \cap N_p^\gamma(0,T;X_1) \hookrightarrow L^p(0,T;X)$$

is compact.

2.3 Interpolation inequalities

In this section, we present some interpolation results which will be of frequent use in what follows. The first is called the Gagliardo–Nirenberg multiplicative embedding inequality.

Lemma 2.3.1. *(Theorem I.2.1 in [26]) For given $p, q \in [1, \infty)$, there exists a positive constant C depending only on d, p, q such that*

$$\|f\|_{L^s(\Omega)} \leq C \|\nabla f\|^{\gamma}_{L^q(\Omega;\mathbb{R}^d)} \|f\|^{1-\gamma}_{L^p(\Omega)} \tag{2.1}$$

for all $f \in W_0^{1,q}(\Omega) \cap L^p(\Omega)$, whenever $s \geq 1$ and $\gamma \in [0, 1]$ satisfy

$$\gamma = \frac{\frac{1}{p} - \frac{1}{s}}{\frac{1}{d} - \frac{1}{q} + \frac{1}{p}}. \tag{2.2}$$

For $d \geq 2$, the admissible range for $p, q, s \in [1, \infty)$ and $\gamma \in [0, 1]$ is given by

$$\text{if } 1 \leq q < d, \gamma \in [0,1], \text{ and } s \in \begin{cases} \left[p, \frac{dq}{d-q}\right], & \text{if } p \in \left[1, \frac{dq}{d-q}\right], \\ \left[\frac{dq}{d-q}, p\right], & \text{if } p \in \left[\frac{dq}{d-q}, \infty\right); \end{cases}$$

$$\text{if } d \leq q < \infty, s \in [p, \infty), \text{ and } \gamma \in \left[0, \frac{dq}{dq + p(q-d)}\right).$$

We reproduce the following results from [66].

Lemma 2.3.2. *(Parabolic Interpolation, Lemma 2.4 in [66]) Let $p, q, s \in [1, \infty)$ and $\gamma \in [0, 1]$ satisfying (2.2) and assume that $s \in [p, \infty)$. Then there exists a positive constant C such that*

$$\|f\|_{L^r(0,T;L^s(\Omega))} \leq C \|\nabla f\|^{\gamma}_{L^q(Q;\mathbb{R}^d)} \|f\|^{1-\gamma}_{L^\infty(0,T;L^p(\Omega))}, \tag{2.3}$$

for all $f \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\infty(0, T; L^p(\Omega))$, with

$$r := \frac{s(q(p + d) - dp)}{(s - p)d} \in (1, \infty]. \tag{2.4}$$

Corollary 2.3.3. *(Corollary 2.5 in [66]) If $q \geq \frac{2d}{d+2}$, then there exists a positive constant C depending only on q such that*

$$\|f\|_{L^{\frac{q(d+2)}{d}}(Q)} \leq C \|\nabla f\|^{\gamma}_{L^q(Q;\mathbb{R}^d)} \|f\|^{1-\gamma}_{L^\infty(0,T;L^2(\Omega))}, \tag{2.5}$$

for all $f \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ with $\gamma = \frac{d}{d+2}$.

We also need the following function space interpolation which is a consequence of the Hölder's inequality.

Lemma 2.3.4. *(Corollary 2.10 in [52]) Assume that $\infty \geq p_1 \geq p \geq p_2 \geq 1$ and $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$. Then*

$$\|f\|_{L^p(\Omega)} \leq \|f\|^{\alpha}_{L^{p_1}(\Omega)} \|f\|^{1-\alpha}_{L^{p_2}(\Omega)}, \tag{2.6}$$

where $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$, $\alpha \in [0, 1]$.

2.4 Some convergence results

2.4.1 Vitali's convergence theorem

In order to characterize compactness in L^1, we start with the definition of equi-integrability (c.f., Definition 2.23 in [34]).

Definition 2.4.1. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space. A family* $\mathcal{F}$ *of measurable functions* $f : X \to [-\infty, \infty]$ *is said to be equi-integrable if for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $\int_E |f| d\mu \leq \varepsilon$ *for all* $f \in \mathcal{F}$ *and for every measurable set* $E \subset X$ *with* $\mu(E) \leq \delta$*.*

The equi-integrability can be deduced via the following De la Vallée Poussin's Theorem.

Theorem 2.4.2. *(De la Vallée Poussin, Theorem 2.29 in [34]) Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $\mathcal{F} \subset L^1(X)$ *be a bounded set. Then the following conditions are equivalent:*

1. $\mathcal{F}$ *is equi-integrable;*
2. *there exists an increasing function* $\gamma : [0, \infty) \to [0, \infty]$*, with*

$$\lim_{t \to \infty} \frac{\gamma(t)}{t} = \infty$$

such that

$$\sup_{f \in \mathcal{F}} \int_X \gamma(|f|) \, \mathrm{d}\mu < \infty.$$

With the notion of equi-integrability, the following Dunford–Pettis Theorem can be used to prove weak convergence in L^1.

Theorem 2.4.3. *(Dunford–Pettis, Theorem 2.54 in [34]) Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $\mathcal{F} \subset L^1(X)$*. Then* $\mathcal{F}$ *is weakly sequentially precompact if and only if*

1. $\mathcal{F}$ *is bounded in* $L^1(X)$*;*
2. $\mathcal{F}$ *is equi-integrable and for every* $\varepsilon > 0$ *there exists* $E \subset X$ *with* $E \in \mathfrak{M}$ *such that* $\mu(E) < \infty$ *and*

$$\sup_{f \in \mathcal{F}} \int_{X \setminus E} |f| \, \mathrm{d}\mu \leq \varepsilon.$$

Next, we state the Vitali Convergence Theorem as in [34], which gives strong convergence in L^1.

Theorem 2.4.4. *(Theorem 2.24 in [34]). Let* $(X, \mathfrak{M}, \mu)$ *be a measure space, and let* $f_n, f : X \to \mathbb{R}$ *be measurable functions. Then,* $(f_n)_{n=1}^{\infty}$ *converges to* f *strongly in* $L^1(X)$ *if and only if*

1. $f_n \to f$ *a.e. in* X*, and*
2. $(f_n)_{n=1}^{\infty}$ *is equi-integrable.*

2.4.2 Weak lower semi-continuity

The following lemma recalls the fact that convex (concave) functions are lower (upper) semi-continuous with respect to weak convergence.

Lemma 2.4.5. *[29] If* $F : \mathbb{R} \to \mathbb{R}$ *is convex and* $f_n \rightharpoonup f$ *in* $L^1(\Omega)$*, then*

$$\int_\Omega F(f)dx \leq \liminf_{n\to\infty} \int_\Omega F(f_n)dx.$$

If $F : \mathbb{R} \to \mathbb{R}$ *is concave and* $f_n \rightharpoonup f$ *in* $L^1(\Omega)$*, then*

$$\int_\Omega F(f)dx \geq \limsup_{n\to\infty} \int_\Omega F(f_n)dx.$$

The next result characterizes pointwise convergence from convexity and lower semi-continuity.

Theorem 2.4.6. *(Theorem 10.20 in [32]) Let* $O \subset \mathbb{R}^d$ *be a measurable set and* $(f_n)_{n=1}^{\infty}$ *be a sequence in* $L^1(O)$ *such that* f_n *converges to* f *weakly in* $L^1(O)$*. Let* $F : \mathbb{R} \to \mathbb{R}$ *be a lower semi-continuous convex function such that* $F(f_n) \in L^1(O)$ *for any* n*, and* $F(f_n)$ *converges weakly to* $\overline{F(f)}$ *weakly in* $L^1(O)$*. Then*

$$F(f) \leq \overline{F(f)} \qquad \text{a.e. in } O.$$

If, moreover, F *is strictly convex on an open convex subset* $U \subset \mathbb{R}^d$*, and* $F(f) = \overline{F(f)}$ *a.e. in* O*, then there exists a subsequence, which is not relabelled, such that*

$$f_n(x) \to f(x) \qquad \text{for a.e. } x \in \{x \in O | f(x) \in U\}.$$

2.4.3 Div-curl lemma

We introduce the Div-Curl Lemma which will be useful for deducing the limit of the sequence of Galerkin approximations to the probability density function ψ.

Lemma 2.4.7. *(Proposition 3.3 in [32]). Let* $\mathcal{D} \subset \mathbb{R}^d$ *be a bounded open Lipschitz domain and* $d \in \mathbb{N}_{\geq 2}$*. Let* $W^{-1,s}(\mathcal{D})$ *and* $W^{-1,s}(\mathcal{D};\mathbb{R}^{d\times d})$ *denote the dual of the Sobolev spaces* $W^{1,\frac{s}{s-1}}(\mathcal{D})$ *and* $W^{1,\frac{s}{s-1}}(\mathcal{D};\mathbb{R}^{d\times d})$ *respectively, for any* $s > 1$*. Assume that*

$$\begin{aligned} \boldsymbol{H}_n &\rightharpoonup \boldsymbol{H} \quad \textit{weakly in } L^p(\mathcal{D};\mathbb{R}^d), \\ \boldsymbol{Q}_n &\rightharpoonup \boldsymbol{Q} \quad \textit{weakly in } L^q(\mathcal{D};\mathbb{R}^d), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$*. Suppose also that there exists a real number* $s > 1$ *such that*

$$\begin{aligned} &\operatorname{div} \boldsymbol{H}_n \equiv \nabla \cdot \boldsymbol{H}_n && \textit{is precompact in } W^{-1,s}(\mathcal{D}), \\ &\operatorname{curl} \boldsymbol{Q}_n \equiv (\nabla \boldsymbol{Q}_n - (\nabla \boldsymbol{Q}_n)^T) && \textit{is precompact in } W^{-1,s}(\mathcal{D};\mathbb{R}^{d\times d}), \end{aligned}$$

Then,

$$\boldsymbol{H}_n \cdot \boldsymbol{Q}_n \rightharpoonup \boldsymbol{H} \cdot \boldsymbol{Q} \quad \textit{weakly in } L^r(\mathcal{D}).$$

Chapter 3

Existence of weak solutions to a model of a homogeneous dilute polymeric fluid

This chapter is devoted to studying homogeneous incompressible dilute polymers. The model is a simpler version of the model introduced in Chapter 1. We assume that the density ρ and the drag coefficient ζ are constants identically equal to 1. Our goal is to prove the existence of global-in-time weak solutions to the Navier–Stokes–Fokker–Planck system that models homogeneous incompressible dilute polymers.

In Section 3.1, we first state the system of equations under consideration with certain assumptions on the initial data. In Section 3.2, we formulate the main result to be proved in this chapter. In Section 3.3, we begin our proof by first truncating the convective term and the extra-stress tensor S_e with a truncation parameter ℓ in Subsection 3.3.1. To preserve the energy identity, we also truncate the probability density function in the drag term in the Fokker–Planck equation; then we truncate the initial condition accordingly. In Subsection 3.3.2, we introduce a two-stage Galerkin approximation of the velocity field and the probability density function with parameters n and m. In Subsections 3.3.3 – 3.3.5, we derive uniform estimates independent of n, m and ℓ, respectively, which enable us to pass to the limit as $n, m, \ell \to \infty$.

3.1 Statement of the problem

We consider the following system of nonlinear partial differential equations, which governs the evolution of the velocity field $\boldsymbol{v}(x,t)$, the probability density function

$\psi(x,\boldsymbol{q},t)$ and the polymer number density $\varrho(x,t)$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{div}_x(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}_x \mathcal{T} = \boldsymbol{f} \qquad \text{in } Q, \tag{3.1}$$

$$\operatorname{div}_x \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{3.2}$$

and

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \operatorname{div}_x(\boldsymbol{v}\psi) + \sum_{j=1}^{K} \operatorname{div}_{\boldsymbol{q}^j}((\nabla_x \boldsymbol{v})\boldsymbol{q}^j \psi) \\ &\quad = \Delta_x \psi + \sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij} \operatorname{div}_{\boldsymbol{q}^j}\left(M \nabla_{\boldsymbol{q}^i}\left(\frac{\psi}{M}\right)\right) \qquad \text{in } \mathcal{O}\times(0,T). \end{aligned} \tag{3.3}$$

The equation (3.1) is supplemented by the following initial and boundary conditions

$$\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0(\cdot) \qquad \text{in } \Omega, \tag{3.4}$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega\times(0,T). \tag{3.5}$$

The Cauchy stress is decomposed as

$$\mathcal{T} = -pI + S_v + S_e. \tag{3.6}$$

The viscous part S_v of the Cauchy stress $\mathcal{T}$ is defined by

$$S_v \coloneqq \mu(\varrho)\left(\frac{\nabla_x \boldsymbol{v} + (\nabla_x \boldsymbol{v})^{\mathrm{T}}}{2}\right) = \mu(\varrho) D(\boldsymbol{v}), \tag{3.7}$$

where $D(\boldsymbol{v})$ denotes the symmetric part of the velocity gradient and the shear viscosity coefficient μ is a function of the polymer number density ϱ which is given by

$$\varrho(x,t) \coloneqq \int_D \psi(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{3.8}$$

The extra-stress tensor S_e is given by the Kramers expression:

$$S_e(x,t) \coloneqq k\left(\sum_{j=1}^{K}\int_D \psi(x,\boldsymbol{q},t)\boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} (U^j)'\left(\frac{1}{2}|\boldsymbol{q}^j|^2\right)\mathrm{d}\boldsymbol{q} - K\varrho(x,t) I\right). \tag{3.9}$$

To simplify the presentation of the Fokker–Planck equation (3.3), we define

$$\hat{\psi} \coloneqq \frac{\psi}{M}, \quad \hat{\psi}_0 \coloneqq \frac{\psi_0}{M}.$$

Moreover, we associate the Rouse matrix A in (3.3) with the linear mapping $\mathbb{A} : \mathbb{R}^{d\times K} \to \mathbb{R}^{d\times K}$ defined, for any $B = (B_i^j)_{i=1,\dots,d}^{j=1,\dots,K} \in \mathbb{R}^{d\times K}$, by $(\mathbb{A}(B))_i^j \coloneqq \sum_{k=1}^{K} B_i^k A_{kj}$, and let $\mathbb{A}^j : \mathbb{R}^{d\times K} \to \mathbb{R}^d$ be the linear mapping defined by $(\mathbb{A}^j(B))_i := (\mathbb{A}(B))_i^j$, for $i = 1, \ldots, d$ and $j = 1, \ldots, K$. By the positive definiteness of the Rouse matrix $A \in \mathbb{R}^{K\times K}_{sym}$, there exist positive constants C_1 and C_2 such that

$$C_1|B|^2 \leq \mathbb{A}(B) : B \leq C_2|B|^2 \qquad \forall B \in \mathbb{R}^{d\times K}. \tag{3.10}$$

The extra-stress tensor S_e can be rewritten as

$$S_e(x,t) = k\sum_{j=1}^{K}\int_D M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j}\hat{\psi}(x,\boldsymbol{q},t)\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}. \tag{3.11}$$

The Fokker–Planck equation (3.3) becomes

$$\begin{aligned}\frac{\partial(M\hat{\psi})}{\partial t} &+ \mathrm{div}_x(M\hat{\psi}\boldsymbol{v}) + \mathrm{div}_{\boldsymbol{q}}\left(M\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q}\right)\\ &- \Delta_x(M\hat{\psi}) - \mathrm{div}_{\boldsymbol{q}}\,\mathbb{A}(M\nabla_{\boldsymbol{q}}\hat{\psi}) = 0 \qquad \text{in } \mathcal{O}\times(0,T).\end{aligned} \tag{3.12}$$

The above equation is supplemented by the following initial and boundary conditions:

$$\hat{\psi}(x,\boldsymbol{q},0) = \hat{\psi}_0(x,\boldsymbol{q}) \qquad \text{in } \mathcal{O}, \tag{3.13}$$

$$M\nabla_x\hat{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\times D\times(0,T), \tag{3.14}$$

$$\left(M\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q}^j - \mathbb{A}^j(M\nabla_{\boldsymbol{q}}\hat{\psi})\right)\cdot\boldsymbol{n}^j = 0 \qquad \text{on } \Omega\times\partial\bar{D}^j\times(0,T), \tag{3.15}$$

for all $j = 1, \ldots, K$.

By formally integrating the partial differential equation (3.12) over D and using the boundary condition (3.15), and by integrating the boundary condition (3.14) over D, we deduce the following partial differential equation for the polymer number density ϱ:

$$\frac{\partial\varrho}{\partial t} + \mathrm{div}_x(\boldsymbol{v}\varrho) = \Delta_x\varrho \qquad \text{in } Q, \tag{3.16}$$

with the following initial and boundary conditions:

$$\varrho(x,0) = \varrho_0(x) \qquad \text{in } \Omega, \tag{3.17}$$

$$\nabla_x\varrho\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{3.18}$$

where

$$\varrho_0(x) := \int_D M(\boldsymbol{q})\hat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}. \tag{3.19}$$

Next, we impose some assumptions on the data. We assume that $\partial\Omega \in C^{0,1}$. For the Maxwellian M, we assume that

$$M \in C_0(\overline{D}) \cap C^{0,1}_{loc}(D) \cap W^{1,1}_0(D). \tag{3.20}$$

For the initial velocity $\boldsymbol{v}_0$, we assume that

$$\boldsymbol{v}_0 \in L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d). \tag{3.21}$$

For the initial value of the probability density function $\hat{\psi}_0$, we assume that

$$\hat{\psi}_0 \geq 0 \qquad \text{a.e. in } \mathcal{O}, \qquad \hat{\psi}_0 \log \hat{\psi}_0 \in L^1_M(\mathcal{O}). \tag{3.22}$$

For the polymer number density, we require the initial data ϱ_0 to satisfy

$$\varrho_0 \in L^\infty(\Omega). \tag{3.23}$$

We shall suppose that μ is a strictly positive continuous function and we adopt the following assumption: there exist positive constants c_1 and c_2 such that, for any $\varrho \geq 0$,

$$0 < c_1 \leq \mu(\varrho) \leq c_2. \tag{3.24}$$

Finally, we assume that

$$\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{3.25}$$

In the next section, we formulate the main result to be proved in this chapter.

3.2 The main result

We first state the definition of weak solutions to the incompressible Navier–Stokes–Fokker–Planck system.

Definition 3.2.1. *We say that the triple* $(\boldsymbol{v},\hat{\psi},\varrho)$ *is a weak solution to the system of nonlinear partial differential equations* (3.1)–(3.19)*, if the following conditions hold:*

(i) The functions $(\boldsymbol{v},\hat{\psi},\varrho)$ *belong to the following function spaces:*

$$\begin{aligned}
\boldsymbol{v} &\in L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\\
\hat{\psi} &\in L^\infty(\Omega\times(0,T);L^1_M(D)) \cap L^1(0,T;W^{1,1}_M(\mathcal{O})), \quad \hat{\psi} \geq 0 \textit{ a.e. in } \mathcal{O}\times(0,T),\\
\varrho &\in L^\infty(\Omega\times(0,T)) \cap L^2(0,T;W^{1,2}(\Omega)),\\
\partial_t\boldsymbol{v} &\in L^p(0,T;W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \textit{ for all } p \in \left(1,\frac{d+2}{d}\right],\\
\partial_t(M\hat{\psi}) &\in L^p(0,T;W^{-1,1}(\mathcal{O})) \textit{ for all } p \in [1,2),\\
\partial_t\varrho &\in L^2(0,T;(W^{1,2}(\Omega))'),
\end{aligned}$$

where $\mathcal{O} := \Omega \times D$*, and*

$$\varrho(x,t) = \int_D M(\boldsymbol{q})\hat{\psi}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q} \quad \textit{for a.e. } (x,t) \in \Omega\times(0,T).$$

(ii) The equation (3.1) *is satisfied in the following sense:*

$$\begin{aligned}&\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle \,\mathrm{d}t + \int_0^T [-(\boldsymbol{v}\otimes\boldsymbol{v}, \nabla_x \boldsymbol{w}) + (S_v, \nabla_x \boldsymbol{w})]\,\mathrm{d}t \\ &= \int_0^T [-(S_e, \nabla_x \boldsymbol{w}) + (\boldsymbol{f}, \boldsymbol{w})]\,\mathrm{d}t, \quad \textit{for all } \boldsymbol{w} \in L^\infty(0,T; W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\end{aligned} \tag{3.26}$$

where the viscous part S_v of the Cauchy stress is given by

$$S_v(x,t) = \mu(\varrho)D(\boldsymbol{v}) \quad \textit{for a.e. } (x,t) \in \Omega\times(0,T), \tag{3.27}$$

and the extra-stress tensor S_e is given by

$$S_e(x,t) = k\sum_{j=1}^K \int_D M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j}\hat{\psi}(x,\boldsymbol{q},t)\otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q} \quad \textit{for a.e. } (x,t)\in\Omega\times(0,T). \tag{3.28}$$

The Fokker–Planck equation (3.12) *is satisfied in the following sense:*

$$\begin{aligned}&\int_0^T \left[\left\langle \partial_t(M\hat{\psi}), \varphi\right\rangle_{\mathcal{O}} - \left(M\boldsymbol{v}\hat{\psi}, \nabla_x\varphi\right)_{\mathcal{O}} - \left(M\hat{\psi}(\nabla_x \boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\right]\mathrm{d}t \\ &\quad + \int_0^T \left[\left(M\nabla\hat{\psi}, \nabla_x\varphi\right)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}), \nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\right]\mathrm{d}t = 0, \\ &\quad \textit{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})).\end{aligned} \tag{3.29}$$

The evolution equation (3.16) for ϱ is satisfied in the following sense:

$$\begin{aligned}&\int_0^T [\langle \partial_t \varrho, \bar{\varphi}\rangle - (\boldsymbol{v}\varrho, \nabla_x\bar{\varphi}) + (\nabla_x\varrho, \nabla_x\bar{\varphi})]\,\mathrm{d}t = 0, \\ &\qquad\qquad\qquad\qquad \textit{for all } \bar{\varphi} \in L^2(0,T;W^{1,2}(\Omega)).\end{aligned} \tag{3.30}$$

(iii) The initial data are attained in the following sense:

$$\lim_{t\to 0+} \|\boldsymbol{v}(\cdot,t)-\boldsymbol{v}_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\hat{\psi}(\cdot,t)-\hat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})} + \|\varrho(\cdot,t)-\varrho_0(\cdot)\|_{L^1(\Omega)} = 0. \tag{3.31}$$

Theorem 3.2.2. *Let $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$, be a bounded open Lipschitz domain. Let $K\in\mathbb{N}$ be arbitrary and let $D^i\subset\mathbb{R}^d$, $i=1,\ldots,K$, be bounded open balls centred at the origin. Suppose that $\boldsymbol{f}\in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. Assume that the map $\mathbb{A}: B\in\mathbb{R}^{d\times K}\mapsto \mathbb{A}(B)\in\mathbb{R}^{d\times K}$ is linear and satisfies (3.10), the Maxwellian $M: D\to\mathbb{R}$ satisfies (3.20), the shear viscosity coefficient $\mu(\cdot)$ satisfies* (3.24) *and the initial data $(\boldsymbol{v}_0, \hat{\psi}_0, \varrho_0)$ satisfy (3.21)–(3.23). Then, there exists $(\boldsymbol{v}, \hat{\psi}, \varrho)$ which is a weak solution* *to the system* (3.1)–(3.19) *in the sense of Definition 3.2.1. Moreover, for a.e.* $t \in (0,T)$*, we have the following energy inequality:*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{v}(\cdot,t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \varrho(\cdot,t)^2 \,\mathrm{d}x + k\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot,t)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + \int_0^t \int_\Omega \mu(\varrho)|D(\boldsymbol{v})|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Omega |\nabla_x \varrho|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + 4kC \int_0^t \int_{\mathcal{O}} M \left|\nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&\leq \frac{1}{2}\int_\Omega |\boldsymbol{v}_0|^2 \,\mathrm{d}x + \int_\Omega \varrho_0^2 \,\mathrm{d}x + k\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_0^t \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \,\mathrm{d}x \,\mathrm{d}s,
\end{aligned}$$

where $\mathcal{F}(s) := s\log s + 1$ *for* $s > 0$ *and* $\mathcal{F}(0) := \lim_{s\to 0+} \mathcal{F}(s) = 1$.

Remark 3.2.3. *We comment here that it is not possible for us to recover the homogeneous Neumann boundary conditions for* $\hat{\psi}$ *and* ϱ *in terms of traces on the boundary since weak solutions do not possess sufficient regularity to talk about traces of normal derivatives on the boundary. For this to be possible, the relevant function should at least belong to* $W^{2,2}$*. The situation is similar for Chapter 4 and Chapter 5.*

In the next section, we will prove this theorem by first truncating the Navier–Stokes–Fokker–Planck system with a parameter ℓ, and then approximating the truncated system using the Galerkin method with parameters n and m. To deduce the existence of global-in-time weak solutions to the problem, we derive uniform estimates independent of each parameter and pass to the limits as $n, m, \ell \to \infty$.

3.3 Existence proof

3.3.1 The first level of approximation: truncation

To approximate the Navier–Stokes–Fokker–Planck system, as in [21], we begin by introducing a smooth nonnegative function $\Gamma \in C_0^\infty((-2,2))$, such that $\Gamma(s) = 1$ for all $s \in [-1,1]$, and for an arbitrary $\ell \in \mathbb{N}$, we define $\Gamma_\ell(s) := \Gamma(\frac{s}{\ell})$. As in [21], the primitive function to Γ_ℓ is given by

$$T_\ell(s) := \int_0^s \Gamma_\ell(r) \,\mathrm{d}r.$$

We define the ℓ-th approximation of (3.1) and (3.2) as follows:

$$\begin{aligned}
\frac{\partial \boldsymbol{v}^\ell}{\partial t} + \mathrm{div}_x(\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell) - \mathrm{div}_x\, S_v^\ell &= -\nabla_x p^\ell + \mathrm{div}_x\, S_e^\ell + \boldsymbol{f} \qquad \text{in } Q, \\
\mathrm{div}_x\, \boldsymbol{v}^\ell &= 0 \qquad \text{in } Q,
\end{aligned}$$

with initial and boundary conditions given by

$$\begin{aligned} \boldsymbol{v}^\ell(\cdot,0) &= \boldsymbol{v}_0(\cdot) && \text{in } \Omega,\\ \boldsymbol{v}^\ell &= \boldsymbol{0} && \text{on } \partial\Omega\times(0,T). \end{aligned}$$

The viscous part S_v^ℓ of the Cauchy stress is given by

$$S_v^\ell := \mu(\varrho^\ell)D(\boldsymbol{v}^\ell),$$

where the ℓ-th approximation ϱ^ℓ of the polymer number density ϱ is defined by

$$\varrho^\ell(x,t) := \int_D M(\boldsymbol{q})\hat{\psi}^\ell(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

The ℓ-th approximation S_e^ℓ of the extra-stress tensor S_e is given by

$$S_e^\ell(x,t) := k\sum_{j=1}^{K}\int_D M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j}T_\ell(\hat{\psi}^\ell(x,\boldsymbol{q},t))\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q},$$

where $\hat{\psi}^\ell$ is the solution of the initial-boundary-value problem (3.32) and (3.33) stated below.

By integration by parts, we have

$$S_e^\ell(x,t) = -k\int_D\left[KM(\boldsymbol{q})T_\ell(\hat{\psi}^\ell(x,\boldsymbol{q},t))I + \sum_{j=1}^{K}T_\ell(\hat{\psi}^\ell(x,\boldsymbol{q},t))\nabla_{\boldsymbol{q}^j}M(\boldsymbol{q})\otimes\boldsymbol{q}^j\right]\mathrm{d}\boldsymbol{q}$$

on observing that the boundary term on ∂D vanishes since $M = 0$ on ∂D. This process can be justified using Lemma 3.1 in [8]. We shall also modify the Fokker–Planck equation (3.12) so that the energy identity is preserved during the truncation process. We first set

$$\Lambda_\ell(s) := s\Gamma_\ell(s).$$

Then the ℓ-approximation of (3.12) is given by

$$\begin{aligned} \frac{\partial(M\hat{\psi}^\ell)}{\partial t} + \mathrm{div}_x(M\hat{\psi}^\ell\boldsymbol{v}^\ell) + \mathrm{div}_{\boldsymbol{q}}\Big(M\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x\boldsymbol{v}^\ell)\boldsymbol{q}\Big)&\\ - \Delta_x(M\hat{\psi}^\ell) - \mathrm{div}_{\boldsymbol{q}}\,\mathbb{A}(M\nabla_{\boldsymbol{q}}\hat{\psi}^\ell) = 0 &\qquad \text{in } \mathcal{O}\times(0,T), \end{aligned} \tag{3.32}$$

and is supplemented by the following Neumann boundary conditions:

$$\begin{aligned} M\nabla_x\hat{\psi}^\ell\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega\times D\times(0,T),\\ \Big(M\hat{\psi}^\ell(\nabla_x\boldsymbol{v}^\ell)\boldsymbol{q}^j - \mathbb{A}^j(M\nabla_{\boldsymbol{q}}\hat{\psi}^\ell)\Big)\cdot\boldsymbol{n}^j &= 0 && \text{on } \Omega\times\partial\bar{D}^j\times(0,T), \end{aligned}$$

for all $j = 1, \ldots, K$. We also truncate the initial condition for $\hat{\psi}^\ell$ as follows:

$$\hat{\psi}^\ell(x, \boldsymbol{q}, 0) = T_\ell(\hat{\psi}_0(x, \boldsymbol{q})) \qquad \text{for } (x, \boldsymbol{q}) \in \mathcal{O}. \tag{3.33}$$

The ℓ-approximation of (3.16) is given by

$$\frac{\partial \varrho^\ell}{\partial t} + \mathrm{div}_x(\boldsymbol{v}^\ell \varrho^\ell) = \Delta_x \varrho^\ell \qquad \text{in } Q,$$

and is supplemented by the following boundary condition:

$$\nabla_x \varrho^\ell \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times (0, T).$$

The initial condition for ϱ^ℓ is given by

$$\varrho^\ell(x, 0) = \int_D M(\boldsymbol{q}) \hat{\psi}^\ell(x, \boldsymbol{q}, 0) \, \mathrm{d}\boldsymbol{q}.$$

For simplicity, we shall omit the superscript ℓ temporarily in the following discussions; we shall reinstate it later and will then pass to the limit as $\ell \to \infty$. We need to show first, however, that this approximating problem has a solution $(\boldsymbol{v}^\ell, \hat{\psi}^\ell, \varrho^\ell)$ for each $\ell \geq 1$; we shall do so by constructing a two-level Galerkin approximation to it and passing to the limits in the sequences of Galerkin approximations.

3.3.2 A two-stage Galerkin approximation

In this section, we define a Galerkin approximation to the truncated Navier–Stokes–Fokker–Planck system. First, as in [21], we define an approximate Maxwellian M^m by fixing a sequence of functions $(\overline{M}^m)_{m\in\mathbb{N}} \subset C_0^{0,1}(\overline{D})$ such that for each compact set $\varkappa \subset D$ the following holds:

$$\lim_{m\to\infty} \|\overline{M}^m - M\|_{C(\overline{D}) \cap W_0^{1,1}(D)} + \|(\overline{M}^m)^{-1} - M^{-1}\|_{C(\varkappa)} = 0. \tag{3.34}$$

To this end the approximate Maxwellian M^m is defined by

$$M^m := \overline{M}^m + \frac{1}{m}, \quad \text{for } m = 1, 2, \ldots. \tag{3.35}$$

Next, we shall introduce the Galerkin basis functions for the velocity field and the probability density function respectively. The Hilbert space $W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega; \mathbb{R}^d)$, equipped with the inner product of $W^{d+1,2}(\Omega; \mathbb{R}^d)$ is compactly and densely embedded in the Hilbert space $L^2_{0,\mathrm{div}}(\Omega; \mathbb{R}^d)$. Hence by the Hilbert–Schmidt Theorem (c.f., Lemma 5.1 and Lemma 5.2 in [33]), there exists a countable set $(\boldsymbol{w}_i)_{i\in\mathbb{N}}$ of eigenfunctions in $W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega; \mathbb{R}^d)$ whose linear span is dense in $L^2_{0,\mathrm{div}}(\Omega; \mathbb{R}^d)$, such that the $\boldsymbol{w}_i$, $i = 1, 2, \dots$, are orthogonal in the inner product of $W^{d+1,2}(\Omega; \mathbb{R}^d)$ and orthonormal in the inner product of $L^2(\Omega; \mathbb{R}^d)$. Similarly, for each $m \in \mathbb{N}$ we find a countable set $(\varphi_i^m)_{i\in\mathbb{N}}$ of eigenfunctions in $W^{1,2}(\mathcal{O})$ that are orthogonal in $W^{1,2}_{M^m}(\mathcal{O})$ and orthonormal in $L^2_{M^m}(\mathcal{O})$. Finally, we fix $m, n \in \mathbb{N}$ and look for $(\boldsymbol{v}^{m,n}, \hat{\psi}^{m,n}, \varrho^{m,n})$ given by

$$\boldsymbol{v}^{m,n}(x,t) := \sum_{i=1}^{m} c_i^{m,n}(t)\boldsymbol{w}_i(x), \tag{3.36}$$

$$\hat{\psi}^{m,n}(x,\boldsymbol{q},t) := \sum_{i=1}^{n} d_i^{m,n}(t)\varphi_i^m(x,\boldsymbol{q}), \tag{3.37}$$

$$\varrho^{m,n}(x,t) := \int_D M^m(\boldsymbol{q})\hat{\psi}^{m,n}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}, \tag{3.38}$$

that solve

$$\begin{aligned}(\partial_t \boldsymbol{v}^{m,n}, \boldsymbol{w}_i) &- (\Gamma_\ell(|\boldsymbol{v}^{m,n}|)\boldsymbol{v}^{m,n}\otimes\boldsymbol{v}^{m,n}, \nabla_x \boldsymbol{w}_i) + (S_v^{m,n}, \nabla_x\boldsymbol{w}_i)\\ &= -(S_e^{m,n}, \nabla_x \boldsymbol{w}_i) + (\boldsymbol{f}, \boldsymbol{w}_i) \quad \text{for all } i = 1,\dots,m \text{ and a.e. } t\in(0,T),\end{aligned} \tag{3.39}$$

$$\begin{aligned}\Big(\partial_t(M^m\hat{\psi}^{m,n}), \varphi_i^m\Big)_{\mathcal{O}} &- \Big(M^m\boldsymbol{v}^{m,n}\hat{\psi}^{m,n}, \nabla_x\varphi_i^m\Big)_{\mathcal{O}}\\ &- \Big(M\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x\boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}} + (M^m\nabla_x\hat{\psi}^{m,n}, \nabla_x\varphi_i^m)_{\mathcal{O}}\\ &+ \Big(M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}), \nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}} = 0 \quad \text{for all } i = 1,\dots,n \text{ and a.e. } t\in(0,T),\end{aligned} \tag{3.40}$$

and by choosing $\varphi_i^m(x,\boldsymbol{q}) := \bar{\varphi}(x)$ in (3.40), we get the equation satisfied by $\varrho^{m,n}$:

$$(\partial_t\varrho^{m,n}, \bar{\varphi}) - (\boldsymbol{v}^{m,n}\varrho^{m,n}, \nabla_x\bar{\varphi}) + (\nabla_x\varrho^{m,n}, \nabla_x\bar{\varphi}) = 0 \tag{3.41}$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$. The initial data are given by

$$\begin{aligned}\boldsymbol{v}^{m,n}(x,0) = \boldsymbol{v}_0^m(x) &:= \sum_{i=1}^{m}(\boldsymbol{v}_0, \boldsymbol{w}_i)\boldsymbol{w}_i(x),\\ \hat{\psi}^{m,n}(x,\boldsymbol{q},0) = \hat{\psi}_0^{m,n}(x,\boldsymbol{q}) &:= \sum_{i=1}^{n}(T_\ell(\hat{\psi}_0^m), \varphi_i^m)_{\mathcal{O}}\varphi_i^m(x,\boldsymbol{q}),\\ \varrho^{m,n}(x,0) = \varrho_0^{m,n}(x) &:= \int_D M^m(\boldsymbol{q})\hat{\psi}_0^{m,n}(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q},\end{aligned}$$

where

$$\hat{\psi}_0^m := \hat{\psi}_0\frac{M}{M^m}.$$

Furthermore, the expressions $S_e^{m,n}$ and $S_v^{m,n}$ are defined as follows:

$$S_v^{m,n} := \mu([\varrho^{m,n}]_+)D(\boldsymbol{v}^{m,n}) \qquad \text{a.e. in } Q, \tag{3.42}$$

$$S_e^{m,n} := -k\int_D\left[KMT_\ell(\hat{\psi}^{m,n})I + \sum_{j=1}^{K}T_\ell(\hat{\psi}^{m,n})\nabla_{\boldsymbol{q}^j}M\otimes\boldsymbol{q}^j\right]\mathrm{d}\boldsymbol{q} \qquad \text{a.e. in } Q, \tag{3.43}$$

where, for a real number s, $[s]_+ := \max(0, s)$.

Now we have constructed the sequence of approximations. Note that the short-time existence of solutions to (3.36)–(3.43) directly follows from Carathéodory's Existence Theorem. The global-in-time existence of solutions to (3.36)–(3.43) is a consequence of the uniform estimates derived in the following sections.

3.3.3 Passage to the limit with n

This section is devoted to deriving estimates that are independent of n and passing to the limit as $n \to \infty$.

Thanks to our assumptions on M and the presence of the cut-off function T_ℓ in (3.43), we have that

$$|S_e^{m,n}| \leq C(\ell, M).$$

Therefore, there exists a subsequence, which we do not relabel, such that as $n \to \infty$,

$$S_e^{m,n} \rightharpoonup^* S_e^m \qquad \text{weak* in } L^\infty(Q; \mathbb{R}^{d\times d}). \tag{3.44}$$

We multiply the i-th equation in (3.39) by $c_i^{m,n}(t)$ and sum with respect to $i = 1, \ldots, m$ to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)} + (S_v^{m,n}, D(\boldsymbol{v}^{m,n})) = -(S_e^{m,n}, D(\boldsymbol{v}^{m,n})) + (\boldsymbol{f}, \boldsymbol{v}^{m,n}). \tag{3.45}$$

Note that the convective term vanishes since we have the following identity:

$$\begin{aligned}
&\int_\Omega \Gamma_\ell(|\boldsymbol{v}^{m,n}|)(\boldsymbol{v}^{m,n} \otimes \boldsymbol{v}^{m,n}) : \nabla_x \boldsymbol{v}^{m,n} \,\mathrm{d}x \\
&= \int_\Omega \Gamma_\ell(|\boldsymbol{v}^{m,n}|)\, |\boldsymbol{v}^{m,n}|\, \nabla_x |\boldsymbol{v}^{m,n}| \cdot \boldsymbol{v}^{m,n} \,\mathrm{d}x \\
&= \int_\Omega \nabla_x \left(\int_0^{|\boldsymbol{v}^{m,n}|} \Gamma_\ell(s) s \,\mathrm{d}s \right) \cdot \boldsymbol{v}^{m,n} \,\mathrm{d}x \\
&= -\int_\Omega \left(\int_0^{|\boldsymbol{v}^{m,n}|} \Gamma_\ell(s) s \,\mathrm{d}s \right) \mathrm{div}_x\, \boldsymbol{v}^{m,n} \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Hence, using Gronwall's inequality, we deduce from (3.45) that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_0^T \int_\Omega \mu([\varrho^{m,n}]_+)|D(\boldsymbol{v}^{m,n})|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(\ell, M, \boldsymbol{v}_0, \boldsymbol{f}). \tag{3.46}$$

Since, by assumption $\mu(\cdot)$ is bounded from above and below by positive constants, it then follows from (3.24) and Korn's inequality that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_0^T \|\boldsymbol{v}^{m,n}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \le C(\ell, M, \boldsymbol{v}_0, \boldsymbol{f}). \tag{3.47}$$

In particular, using the definition of $\boldsymbol{v}^{m,n}$ and the orthogonality of the basis we get that

$$\sup_{t\in(0,T);\, i=1,\dots,m} |c_i^{m,n}(t)| \le C(m, \ell, M, \boldsymbol{v}_0, \boldsymbol{f}). \tag{3.48}$$

From the definition of $\boldsymbol{v}^{m,n}$, we see that

$$\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}(x,t) = \sum_{j=1}^m \frac{\mathrm{d}c_j^{m,n}(t)}{\mathrm{d}t}\boldsymbol{w}_j(x).$$

Substituting into (3.39), we have, thanks to the orthogonality of the basis, that for a fixed i,

$$\begin{aligned}\left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| &\le |(\Gamma_\ell(|\boldsymbol{v}^{m,n}|)\boldsymbol{v}^{m,n}\otimes\boldsymbol{v}^{m,n}, \nabla_x \boldsymbol{w}_i)| + |(S_v^{m,n}, \nabla_x \boldsymbol{w}_i)| + |(S_e^{m,n}, \nabla_x \boldsymbol{w}_i)| \\ &\quad + |(\boldsymbol{f}, \boldsymbol{w}_i)| \\ &\le C(m,\ell)\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}.\end{aligned}$$

Next, we shall prove that $\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(m)$. From the Hilbert–Schmidt Theorem, we have

$$\sum_{|\alpha|=d+1} (D^\alpha \boldsymbol{w}_i, D^\alpha \boldsymbol{e}) = \lambda_i(\boldsymbol{w}_i, \boldsymbol{e}) \qquad \forall \boldsymbol{e} \in W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d), \quad \forall i = 1,2,\dots,m.$$

By choosing $\boldsymbol{e} = \boldsymbol{w}_i$, we get

$$|\boldsymbol{w}_i|^2_{W^{d+1,2}(\Omega;\mathbb{R}^d)} = \sum_{|\alpha|=d+1} \|D^\alpha \boldsymbol{w}_i\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} = \lambda_i \|\boldsymbol{w}_i\|^2_{L^2(\Omega;\mathbb{R}^d)} = \lambda_i.$$

Note that $\|\cdot\|^2_{W^{d+1,2}(\Omega)} \sim \|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{W^{d+1,2}(\Omega)}$; thus $\|\boldsymbol{w}_i\|^2_{W^{d+1,2}(\Omega;\mathbb{R}^d)} \sim 1 + \lambda_i$. Since $W^{d+1,2}(\Omega;\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\Omega;\mathbb{R}^d)$, we have

$$\|\boldsymbol{w}_i\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(1+\lambda_i) \le C(m).$$

Thus, we have shown that $\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(m)$. Finally, we have

$$\sup_{t\in(0,T); i=1,\dots,m} \left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| \le C(m,\ell). \tag{3.49}$$

Therefore, from (3.48) and (3.49), we see that there exists a subsequence, which we do not relabel, such that

$$c_i^{m,n} \rightharpoonup^* c_i^m \qquad \text{weak* in } W^{1,\infty}((0,T)), \tag{3.50}$$

$$c_i^{m,n} \to c_i^m \qquad \text{strongly in } C([0,T]). \tag{3.51}$$

Meanwhile, by the definition of $\boldsymbol{v}^{m,n}$ and (3.48), it is easy to see that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}(t)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \leq C(m,\ell). \tag{3.52}$$

It then follows from the above bounds and the definition of $\boldsymbol{v}^{m,n}$ (3.36) that

$$\boldsymbol{v}^{m,n} \to \boldsymbol{v}^m \qquad \text{strongly in } C([0,T]; W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)). \tag{3.53}$$

Similarly, we multiply the i-th equation in (3.40) by $d_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, n$. Again using the fact that $\mathrm{div}_x\, \boldsymbol{v}^{m,n} = 0$, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + \|\nabla_x\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O};\mathbb{R}^d)} + \Big(M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}), \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\Big)_{\mathcal{O}} \\ = \Big(M\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x\boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\Big)_{\mathcal{O}}.\end{aligned} \tag{3.54}$$

From the definitions of M^m and Λ_ℓ together with Hölder's inequality, we get that

$$\begin{aligned}&\Big(M\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x\boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\Big)_{\mathcal{O}} \\ &\quad\leq C\int_{\mathcal{O}} M^m|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + C(\ell)\|\nabla_x\boldsymbol{v}^{m,n}\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\|\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} \\ &\quad\leq C\int_{\mathcal{O}} M^m|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + C(m,\ell)\|\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}.\end{aligned} \tag{3.55}$$

Inserting (3.55) into (3.54) and recalling (3.10), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + \int_{\mathcal{O}} M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C(m,\ell)\|\hat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}. \tag{3.56}$$

Noting that $M^m \geq 1/m$ together with the definition of $\hat{\psi}_0^{m,n}$, we apply Gronwall's inequality to deduce that

$$\sup_{t\in(0,T)}\|\hat{\psi}^{m,n}\|^2_{L^2(\mathcal{O})} + \int_0^T \|\hat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \leq C(m,\ell). \tag{3.57}$$

Since M^m is Lipschitz-continuous, we can further deduce that

$$\int_0^T \|M^m\hat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \leq C(m,\ell). \tag{3.58}$$

It remains to obtain a uniform bound on the time derivative of $\hat{\psi}^{m,n}$. Using (3.40) and the uniform bounds (3.52) and (3.57), we get by an application of the Hölder's inequality, that,

$$
\begin{aligned}
\int_0^T \|\partial_t(M^m\hat{\psi}^{m,n})\|^2_{(W^{1,2}(\mathcal{O}))'}\,\mathrm{d}t &= \int_0^T \left(\sup_{\varphi\in W^{1,2}(O)} \frac{|(\partial_t(M^m\hat{\psi}^{m,n}),\varphi)_{\mathcal{O}}|}{\|\nabla_{x,\boldsymbol{q}}\varphi\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}} \right)^2 \mathrm{d}t \\
&\leq C\int_0^T \left(\sup_{\varphi\in W^{1,2}(O)} \frac{|(M^m\boldsymbol{v}^{m,n}\hat{\psi}^{m,n},\nabla_x\varphi)_{\mathcal{O}}|}{\|\nabla_x\varphi\|_{L^2(\mathcal{O};\mathbb{R}^d)}} \right)^2 \mathrm{d}t \\
&\quad + C\int_0^T \left(\sup_{\varphi\in W^{1,2}(O)} \frac{|(M\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x\boldsymbol{v}^{m,n})\boldsymbol{q},\nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}|}{\|\nabla_{\boldsymbol{q}}\varphi\|_{L^2(\mathcal{O};\mathbb{R}^{K+1})}} \right)^2 \mathrm{d}t \\
&\quad + C\int_0^T \left(\sup_{\varphi\in W^{1,2}(O)} \frac{|(M^m\nabla_x\hat{\psi}^{m,n},\nabla_x\varphi)_{\mathcal{O}}|}{\|\nabla_x\varphi\|_{L^2(\mathcal{O};\mathbb{R}^d)}} \right)^2 \mathrm{d}t \\
&\quad + C\int_0^T \left(\sup_{\varphi\in W^{1,2}(O)} \frac{|(M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}),\nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}}|}{\|\nabla_{\boldsymbol{q}}\varphi\|_{L^2(\mathcal{O};\mathbb{R}^{K+1})}} \right)^2 \mathrm{d}t \\
&\leq C(\ell,M)\int_0^T \|\boldsymbol{v}^{m,n}\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^d)}\|\hat{\psi}^{m,n}\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t \\
&\quad + C(M)\int_0^T \|\nabla_x\hat{\psi}^{m,n}\|^2_{L^2(\mathcal{O};\mathbb{R}^d)}\,\mathrm{d}t + C(M)\int_0^T \|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\|^2_{L^2(\mathcal{O};\mathbb{R}^{K+1})}\,\mathrm{d}t \\
&\leq C(m,\ell).
\end{aligned}
\tag{3.59}
$$

Thus, from the n-independent estimates (3.57), (3.58) and (3.59) and by applying the Aubin–Lions Lemma, we see that there exists a subsequence, which we do not relabel, such that as $n\to\infty$,

$$
\begin{aligned}
\hat{\psi}^{m,n} &\rightharpoonup \hat{\psi}^m && \text{weakly in } L^2(0,T;W^{1,2}(\mathcal{O})), && (3.60)\\
\partial_t(M^m\hat{\psi}^{m,n}) &\rightharpoonup \partial_t(M^m\hat{\psi}^m) && \text{weakly in } L^2(0,T;(W^{1,2}(\mathcal{O}))'), && (3.61)\\
\hat{\psi}^{m,n} &\to \hat{\psi}^m && \text{strongly in } L^2(0,T;L^2(\mathcal{O})). && (3.62)
\end{aligned}
$$

We first derive uniform estimates for $\varrho^{m,n}$ from the convergence properties of $\hat{\psi}^{m,n}$. Since M^m is Lipschitz continuous, we have by Hölder's inequality and (3.57) that

$$
\int_0^T \|\varrho^{m,n}\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq C(m,\ell). \tag{3.63}
$$

Using the strong convergence (3.62), we obtain that

$$
\begin{aligned}
\int_0^T\int_\Omega (\varrho^{m,n}-\varrho^m)^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \left(\int_D M^m(\hat{\psi}^{m,n}-\hat{\psi}^m)\,\mathrm{d}\boldsymbol{q}\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\leq C(m)\int_0^T\int_\Omega\int_D (\hat{\psi}^{m,n}-\hat{\psi}^m)^2\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&\to 0.
\end{aligned}
\tag{3.64}
$$

Hence, (3.63) and (3.64) imply that, there exists a subsequence such that, as $n \to \infty$,

$$\varrho^{m,n} \rightharpoonup \varrho^m \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.65}$$

$$\varrho^{m,n} \to \varrho^m \qquad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{3.66}$$

where

$$\varrho^m(x,t) = \int_D M^m(\boldsymbol{q})\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{3.67}$$

From the strong convergence (3.66), we deduce that

$$\varrho^{m,n} \to \varrho^m \qquad \text{a.e. in } Q. \tag{3.68}$$

Next, we set $\bar{\varphi} := \varrho^{m,n}$ in (3.16). Note that such a choice is legitimate. We obtain the following identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho^{m,n}\|^2_{L^2(\Omega)} - (\boldsymbol{v}^{m,n}\varrho^{m,n}, \nabla_x\varrho^{m,n}) + \|\nabla_x\varrho^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)} = 0. \tag{3.69}$$

We begin by noting that the second term in the above equality vanishes thanks to the fact that $\operatorname{div}_x \boldsymbol{v}^{m,n} = 0$. Next, integrating (3.69) with respect to time over $(0,t)$, for $t \in (0,T]$, we get

$$\|\varrho^{m,n}(\cdot,t)\|^2_{L^2(\Omega)} + 2\int_0^t\int_\Omega |\nabla_x\varrho^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}\tau = \|\varrho_0^{m,n}\|^2_{L^2(\Omega)}. \tag{3.70}$$

By the definition of $\varrho_0^{m,n}$, we get the following uniform estimate:

$$\sup_{t\in(0,T)}\|\varrho^{m,n}\|^2_{L^2(\Omega)} + 2\int_0^T\int_\Omega |\nabla_x\varrho^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}t \le C(m,\ell). \tag{3.71}$$

Next, we we shall focus on the bounds on the time derivative of $\varrho^{m,n}$. Using (3.41), the uniform bounds (3.52) and (3.71), the Sobolev embedding and Hölder's inequality, we deduce that,

$$\begin{aligned}
\int_0^T \|\partial_t\varrho^{m,n}\|^2_{(W^{1,2}(\Omega))'}\,\mathrm{d}t &= \int_0^T\left(\sup_{\bar{\varphi}\in W^{1,2}(\Omega)}\frac{|(\partial_t\varrho^{m,n},\bar{\varphi})|}{\|\nabla_x\bar{\varphi}\|_{L^2(\Omega;\mathbb{R}^d)}}\right)^2\mathrm{d}t \\
&\le \int_0^T\left(\sup_{\bar{\varphi}\in W^{1,2}(\Omega)}\frac{|(\boldsymbol{v}^{m,n}\varrho^{m,n},\nabla_x\bar{\varphi})| + |(\nabla_x\varrho^{m,n},\nabla_x\bar{\varphi})|}{\|\nabla_x\bar{\varphi}\|_{L^2(\Omega;\mathbb{R}^d)}}\right)^2\mathrm{d}t \\
&\le 2\int_0^T\left(\|\boldsymbol{v}^{m,n}\|^2_{L^\infty(\Omega;\mathbb{R}^d)}\|\varrho^{m,n}\|^2_{L^2(\Omega)} + \|\nabla_x\varrho^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)\mathrm{d}t \\
&\le 2\|\boldsymbol{v}^{m,n}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))}\|\varrho^{m,n}\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\|\nabla_x\varrho^{m,n}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \\
&\le C(m,\ell).
\end{aligned} \tag{3.72}$$

Therefore, we deduce that

$$\partial_t \varrho^{m,n} \rightharpoonup \partial_t \varrho^m \qquad \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'). \tag{3.73}$$

For the convergence properties of the viscous part of the Cauchy stress $S_v^{m,n}$, we first note that since μ is a continuous function and also the function $[\cdot]_+$ is Lipschitz continuous, it follows from (3.68) and the Dominated Convergence Theorem that

$$\mu([\varrho^{m,n}]_+) \to \mu([\varrho^m]_+) \qquad \text{strongly in } L^1(Q). \tag{3.74}$$

By the assumption (3.24), we deduce that

$$\mu([\varrho^{m,n}]_+) \to \mu([\varrho^m]_+) \qquad \text{strongly in } L^p(Q), \tag{3.75}$$

for all $p \in [1,\infty)$. Using the strong convergence results (3.53) and (3.75), and the uniform bound (3.52), we consider the following

$$\begin{aligned}
&\int_0^T \int_\Omega |S_v^{m,n} - S_v^m|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\int_0^T \int_\Omega |\mu([\varrho^{m,n}]_+)D(\boldsymbol{v}^{m,n}) - \mu([\varrho^{m,n}]_+)D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^T \int_\Omega |\mu([\varrho^{m,n}]_+)D(\boldsymbol{v}^m) - \mu([\varrho^m]_+)D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2c_2^2 \int_0^T \int_\Omega |D(\boldsymbol{v}^{m,n}) - D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x\,\mathrm{d}t + C(m,\ell)\int_0^T \int_\Omega |\mu([\varrho^{m,n}]_+) - \mu([\varrho^m]_+)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\to 0.
\end{aligned}$$

Hence, we deduce that

$$S_v^{m,n} \to S_v^m \qquad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{3.76}$$

where

$$S_v^m := \mu([\varrho^m]_+)D(\boldsymbol{v}^m). \tag{3.77}$$

Conclusion. Using the convergence results (3.48), (3.49) for $c_i^{m,n}$, (3.60)–(3.62) for $\hat{\psi}^{m,n}$, (3.53) for the velocity field $\boldsymbol{v}^{m,n}$, (3.65), (3.66) and (3.73) for $\varrho^{m,n}$, (3.76) for $S_v^{m,n}$ and (3.44) for $S_e^{m,n}$, we can pass to the limit $n \to \infty$ in (3.39)–(3.41) to deduce that

$$\begin{aligned}&(\partial_t \boldsymbol{v}^m, \boldsymbol{w}_i) - (\Gamma_\ell(|\boldsymbol{v}^m|)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla_x \boldsymbol{w}_i) + (S_v^m, \nabla_x \boldsymbol{w}_i)\\ &\quad = -(S_e^m, \nabla_x \boldsymbol{w}_i) + (\boldsymbol{f}, \boldsymbol{w}_i) \quad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0,T),\end{aligned} \tag{3.78}$$

$$\begin{aligned}&\langle \partial_t (M^m \hat{\psi}^m), \varphi \rangle_{\mathcal{O}} - (M^m \boldsymbol{v}^m \hat{\psi}^m, \nabla_x \varphi)_{\mathcal{O}} - (M \Lambda_\ell(\hat{\psi}^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi)_{\mathcal{O}}\\ &\quad + (M^m \nabla_x \hat{\psi}^m, \nabla_x \varphi)_{\mathcal{O}} + (M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^m), \nabla_{\boldsymbol{q}} \varphi)_{\mathcal{O}} = 0\\ &\qquad \text{for all } \varphi \in W^{1,2}(\mathcal{O}) \text{ and a.e. } t \in (0,T),\end{aligned} \tag{3.79}$$

$$\begin{aligned}&\langle \partial_t \varrho^m, \bar{\varphi} \rangle - (\boldsymbol{v}^m \varrho^m, \nabla_x \bar{\varphi}) + (\nabla_x \varrho^m, \nabla_x \bar{\varphi}) = 0\\ &\qquad \text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T).\end{aligned} \tag{3.80}$$

It is obvious that $\boldsymbol{v}^m(x,0) = \boldsymbol{v}_0^m(x)$. Moreover, it is standard to show that

$$\lim_{t \to 0+} \|\hat{\psi}^m(\cdot,t) - T_\ell(\hat{\psi}_0^m(\cdot))\|_{L^2(\mathcal{O})} = 0, \quad \lim_{t \to 0+} \|\varrho^m(\cdot,t) - \varrho_0^m(\cdot)\|_{L^2(\Omega)} = 0,$$

where

$$\varrho_0^m(x) = \int_D M^m(\boldsymbol{q}) T_\ell(\hat{\psi}_0^m(x,\boldsymbol{q}))\, \mathrm{d}\boldsymbol{q}.$$

We have shown that

$$\varrho^m(x,t) = \int_D M^m(\boldsymbol{q}) \hat{\psi}^m(x,\boldsymbol{q},t)\, \mathrm{d}\boldsymbol{q},$$

and

$$S_v^m = \mu([\varrho^m]_+) D(\boldsymbol{v}^m). \tag{3.81}$$

Finally, using (3.62) and the Dominated Convergence Theorem, we can pass to the limit in (3.43) to deduce that

$$S_e^m = -k \int_D \left[K M T_\ell(\hat{\psi}^m) I + \sum_{j=1}^K T_\ell(\hat{\psi}^m) \nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \quad \text{a.e. in } Q.$$

3.3.4 Passage to the limit with m

This section is devoted to deriving a priori estimates that are independent of m and from the uniform estimates, we establish the existence of weak solutions to the truncated Navier–Stokes–Fokker–Planck system.

Minimum principle for $\hat{\psi}^m$. Before deriving uniform estimates on $\hat{\psi}^m$, we first show that $\hat{\psi}^m$ is nonnegative a.e. in $\mathcal{O} \times (0,T)$. Setting $\varphi := [\hat{\psi}^m]_- := \min(0, \hat{\psi}^{m,n})$ in (3.79) and following a similar procedure as in (3.54)–(3.56), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M^m \left([\hat{\psi}^m]_-\right)^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}[\hat{\psi}^m]_-\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}& \\ \le C(m,\ell)\int_{\mathcal{O}} M^m \left([\hat{\psi}^m]_-\right)^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}&.\end{aligned} \tag{3.82}$$

Since $\hat{\psi}^m(0) = T_\ell(\hat{\psi}_0^m) \geq 0$, the application of Gronwall's inequality implies that $[\hat{\psi}^m]_- \equiv 0$ in $\mathcal{O} \times (0,T)$, which means that

$$\hat{\psi}^m \geq 0 \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{3.83}$$

By the definition of ϱ^m, it then automatically follows that $\varrho^m \geq 0$ a.e. in Q. Now that we have shown that ϱ^m is nonnegative a.e. in Q, we can replace the $[\varrho^m]_+$ by ϱ^m in (3.81).

Maximum principle for ϱ^m. Setting $\varphi \equiv 1$ in (3.79) and integrating over $(0,t)$, we deduce that

$$\begin{aligned}\int_{\mathcal{O}} M^m(\boldsymbol{q})\,\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x &= \int_{\mathcal{O}} M^m(\boldsymbol{q})\,T_\ell(\hat{\psi}_0^m(x,\boldsymbol{q}))\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \\ \leq \int_{\mathcal{O}} M^m(\boldsymbol{q})\hat{\psi}_0^m(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x &\leq \int_{\mathcal{O}} M(\boldsymbol{q})\,\hat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x \leq C.\end{aligned}$$

Therefore,

$$0 \leq \varrho_0^m(x) = \int_D M^m(\boldsymbol{q}) T_\ell(\hat{\psi}_0^m(x,\boldsymbol{q}))\,\mathrm{d}\boldsymbol{q} \leq \int_D M(\boldsymbol{q})\hat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} = \varrho_0(x) \leq C.$$

Let $\bar{P} = \sup_{x\in\Omega} \varrho_0^m(x)$, then we have

$$\langle \partial_t(\varrho^m - \bar{P}), \bar{\varphi}\rangle - (\boldsymbol{v}^m(\varrho^m - \bar{P}), \nabla_x\bar{\varphi}) + (\nabla_x(\varrho^m - \bar{P}), \nabla_x\bar{\varphi}) = 0,$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$. Setting $\bar{\varphi} = [\varrho^m - \bar{P}]_+$ in the above equation gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ([\varrho^m - \bar{P}]_+)^2\,\mathrm{d}x + \int_\Omega |\nabla_x[\varrho^m - \bar{P}]_+|^2\,\mathrm{d}x = 0.$$

Since the second term on the left-hand side of the above equation is always nonnegative, we then have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ([\varrho^m - \bar{P}]_+)^2\,\mathrm{d}x \leq 0,$$

which then gives

$$\int_\Omega ([\varrho^m - \bar{P}]_+(x,t))^2 \,\mathrm{d}x \le \int_\Omega ([\varrho^m - \bar{P}]_+(x,0))^2 \,\mathrm{d}x = \int_\Omega ([\varrho_0^m - \bar{P}]_+)^2 \,\mathrm{d}x.$$

By the definition of $\bar{P}$, $[\varrho_0^m - \bar{P}]_+ = 0$ a.e. in Ω. Thus, $[\varrho^m - \bar{P}]_+ = 0$ a.e. in Ω. Hence,

$$\|\varrho^m\|_{L^\infty(Q)} \le \|\varrho_0^m\|_{L^\infty(Q)} \le \|\varrho_0\|_{L^\infty(Q)} \le C. \tag{3.84}$$

Next, we derive uniform bounds for the velocity field $\boldsymbol{v}^m$ and for the probability density function $\hat{\psi}^m$. To begin with, we set $\varphi := \log(\hat{\psi}^m + \delta) + 1$ in (3.79), where $\delta > 0$ is arbitrary. We define the following functions:

$$\mathcal{F}_\delta(s) := (s+\delta)\log(s+\delta) + 1, \qquad \mathcal{F}(s) := s\log s + 1,$$
$$T_{\delta,\ell}(s) := \int_0^s \frac{\Lambda_\ell(t)}{t+\delta}\,\mathrm{d}t = \int_0^s \frac{t\Gamma_\ell(t)}{t+\delta}\,\mathrm{d}t.$$

Note that $\mathcal{F}_\delta(s) \ge 0$ for all $s \ge 0$. Then we obtain from (3.79) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\mathcal{O} M^m \mathcal{F}_\delta(\hat{\psi}^m)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \Big(M^m\boldsymbol{v}^m, \nabla_x \mathcal{F}_\delta(\hat{\psi}^m)\Big)_\mathcal{O} + \left(\frac{M^m}{\hat{\psi}^m+\delta}\nabla_x\hat{\psi}^m, \nabla_x\hat{\psi}^m\right)_\mathcal{O}$$
$$+ \left(\frac{M^m}{\hat{\psi}^m+\delta}\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^m), \nabla_{\boldsymbol{q}}\hat{\psi}^m\right)_\mathcal{O} = \Big(M(\nabla_x\boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_\mathcal{O}. \tag{3.85}$$

Next, we integrate (3.85) with respect to time over $(0,t)$. Using the assumption (3.10) and noting that the second term vanishes since $\mathrm{div}_x\,\boldsymbol{v}^m = 0$, we obtain

$$\int_\mathcal{O} M^m \mathcal{F}_\delta(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + C\int_0^t\int_\mathcal{O} \frac{M^m}{\hat{\psi}^m+\delta}|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau$$
$$\le \int_\mathcal{O} M^m\mathcal{F}_\delta(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_0^t \Big(M(\nabla_x\boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}}T_{\delta,\ell}(\hat{\psi}^m)\Big)_\mathcal{O}\,\mathrm{d}\tau. \tag{3.86}$$

Now we consider the limit of (3.86) as $\delta \to 0+$. The following two limits can be easily identified:

$$\lim_{\delta\to 0+}\int_\mathcal{O} M^m\mathcal{F}_\delta(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_\mathcal{O} M^m\mathcal{F}(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q},$$
$$\lim_{\delta\to 0+}\int_\mathcal{O} M^m\mathcal{F}_\delta(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_\mathcal{O} M^m\mathcal{F}(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Also, by the Monotone Convergence Theorem, we have

$$\lim_{\delta\to 0+}\int_0^t\int_\mathcal{O}\frac{M^m}{\hat{\psi}^m+\delta}|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau = \int_0^t\int_\mathcal{O}\frac{M^m}{\hat{\psi}^m}|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau$$
$$= 4\int_0^t\int_\mathcal{O} M^m\left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau.$$

Thus, we obtain

$$\int_{\mathcal{O}} M^m \mathcal{F}(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C\int_0^t \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau$$
$$\leq \int_{\mathcal{O}} M^m \mathcal{F}(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \limsup_{\delta\to 0+}\int_0^t \Big(M(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau. \quad (3.87)$$

By integration by parts, noting the divergence-free property of $\boldsymbol{v}^m$, we have

$$\int_0^t \Big(M(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau = -\int_0^t \Big(\mathrm{div}_{\boldsymbol{q}}(M(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}), T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau$$
$$= -\sum_{j=1}^K \int_0^t \Big(\nabla_x \boldsymbol{v}^m, T_{\delta,\ell}(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j\Big)_{\mathcal{O}}\,\mathrm{d}\tau.$$

Since $|T_{\delta,\ell}(s) - T_\ell(s)| \leq \delta\log(1+\frac{\ell}{\delta})$, we can identify the limit in the last term in (3.87) and we arrive at

$$\int_{\mathcal{O}} M^m \mathcal{F}(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C\int_0^t \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau$$
$$\leq \int_{\mathcal{O}} M^m \mathcal{F}(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \sum_{j=1}^K \int_0^t \Big(\nabla_x \boldsymbol{v}^m, T_\ell(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j\Big)_{\mathcal{O}}\,\mathrm{d}\tau. \quad (3.88)$$

By setting $\bar{\varphi} = \varrho^m$ in (3.80), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\varrho^m)^2\,\mathrm{d}x + \int_\Omega |\nabla_x \varrho^m|^2\,\mathrm{d}x = 0. \quad (3.89)$$

Direct integration with respect to t and the application of (3.84) give also

$$\int_0^T \int_\Omega |\nabla_x \varrho^m|^2\,\mathrm{d}x\,\mathrm{d}t \leq C. \quad (3.90)$$

Next we multiply the i-th equation of (3.78) by $c_i^m(t)$ to deduce the following energy identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m\|^2_{L^2(\Omega;\mathbb{R}^d)} + (S_v^m, D(\boldsymbol{v}^m)) = -(S_e^m, D(\boldsymbol{v}^m)) + (\boldsymbol{f}, \boldsymbol{v}^m). \quad (3.91)$$

Using $\mathrm{div}_x\, \boldsymbol{v}^m = 0$ and the definition of S_e^m we deduce that

$$(S_e^m, \nabla_x \boldsymbol{v}^m) = -k\sum_{j=1}^K \Big(\nabla_x \boldsymbol{v}^m, T_\ell(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j\Big)_{\mathcal{O}}.$$

Integrating (3.89) and (3.91) over $(0,t)$, combining the result with (3.88) and noting the boundedness (3.24) of $\mu(\cdot)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{v}^m(\cdot,t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho^m(\cdot,t))^2 \,\mathrm{d}x + c_1 \int_0^t \int_\Omega |D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega |\nabla_x \varrho^m|^2 \,\mathrm{d}x \,\mathrm{d}\tau + k \int_{\mathcal{O}} M^m \mathcal{F}(\hat{\psi}^m(\cdot,t)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + 4kC \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m} \right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}\tau \\
&\le k \int_{\mathcal{O}} M^m \mathcal{F}(T_\ell(\hat{\psi}_0^m)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0^m(\cdot)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho_0^m(\cdot))^2 \,\mathrm{d}x + \int_0^t (\boldsymbol{f}, \boldsymbol{v}^m) \,\mathrm{d}\tau.
\end{aligned} \tag{3.92}$$

From the assumption on the initial data, we have the following estimates that are uniform with respect to m:

$$\begin{aligned}
&\sup_{t\in(0,T)} \Big(\|\boldsymbol{v}^m(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\varrho^m(\cdot,t)\|_{L^\infty(\Omega)} + \|M^m \mathcal{F}(\hat{\psi}^m(\cdot,t))\|_{L^1(\mathcal{O})} \Big) \\
&\quad + \int_0^T \int_\Omega |\nabla_x \varrho^m|^2 \,\mathrm{d}x \,\mathrm{d}t + c_1 \int_0^T \|\boldsymbol{v}^m\|^2_{W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\
&\quad + \int_0^T \|\sqrt{M^m} \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m}\|^2_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})} \,\mathrm{d}t \\
&\le C(1 + \|M^m \mathcal{F}(T_\ell(\hat{\psi}_0^m))\|_{L^1(\mathcal{O})}) \le C(\ell).
\end{aligned} \tag{3.93}$$

Also, the definition of S_e^m and the assumption on M give

$$|S_e^m| \le C(\ell). \tag{3.94}$$

Interpolating between the uniform estimates for $\boldsymbol{v}^m$ in (3.93), we deduce that

$$\|\boldsymbol{v}^m\|_{L^{\frac{2(d+2)}{d}}(\Omega;\mathbb{R}^d)} \le C(\ell). \tag{3.95}$$

By using (3.78), the uniform estimates (3.93), (3.94) and (3.95), and the boundedness

(3.24) of $\mu(\cdot)$, we deduce by using Hölder's inequality that, for $p \in \big(1, \frac{d+2}{d}\big]$,

$$
\begin{aligned}
&\int_0^T \|\partial_t \boldsymbol{v}^m\|^p_{W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t = \int_0^T \left(\sup_{\boldsymbol{w}\in W^{1,p'}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \frac{(\partial_t \boldsymbol{v}^m, \boldsymbol{w})}{\|\boldsymbol{w}\|_{W^{1,p'}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)}} \right)^p \mathrm{d}t \\
&\le C \int_0^T \left(\sup_{\boldsymbol{w}\in W^{1,p'}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \frac{|(\Gamma_\ell(|\boldsymbol{v}^m|)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla_x \boldsymbol{w})| + |(S^m_v, \nabla_x \boldsymbol{w})|}{\|\nabla_x \boldsymbol{w}\|_{L^{p'}(\Omega;\mathbb{R}^{d\times d})}} \right)^p \mathrm{d}t \\
&\quad + C \int_0^T \left(\sup_{\boldsymbol{w}\in W^{1,p'}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \frac{|(S^m_e, \nabla_x \boldsymbol{w})|}{\|\nabla_x \boldsymbol{w}\|_{L^{p'}(\Omega;\mathbb{R}^{d\times d})}}\,\mathrm{d}t \right)^p \mathrm{d}t \\
&\quad + C \int_0^T \left(\sup_{\boldsymbol{w}\in W^{1,p'}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \frac{|(\boldsymbol{f}, \boldsymbol{w})|}{\|\boldsymbol{w}\|_{L^{p'}(\Omega;\mathbb{R}^d)}} \right)^p \mathrm{d}t \\
&\le C \int_0^T \|\Gamma_\ell(|\boldsymbol{v}^m|)\boldsymbol{v}^m \otimes \boldsymbol{v}^m\|^p_{L^p(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t + C \int_0^T \|S^m_v\|^p_{L^p(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t \\
&\quad + C \int_0^T \|S^m_e\|^p_{L^p(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t + C \int_0^T \|\boldsymbol{f}\|^p_{L^p(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \\
&\le C(\ell)\|\boldsymbol{v}^m\|^{\frac{2(d+2)}{d}}_{L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)} + C\|D(\boldsymbol{v}^m)\|^{\frac{p}{2}}_{L^2(Q;\mathbb{R}^{d\times d})} + C\|S^m_e\|^p_{L^\infty(Q;\mathbb{R}^{d\times d})} + C\|\boldsymbol{f}\|^{\frac{p}{2}}_{L^2(Q;\mathbb{R}^d)} \\
&\le C(\ell).
\end{aligned}
\tag{3.96}
$$

Next we derive bounds on the time derivative of ϱ^m. Similarly as in (3.72), by using (3.80) and the uniform estimates (3.93), we deduce that

$$
\begin{aligned}
&\int_0^T \|\partial_t \varrho^m\|^2_{(W^{1,2}(\Omega))'}\,\mathrm{d}t = \int_0^T \left(\sup_{\bar{\varphi}\in W^{1,2}(\Omega)} \frac{|(\partial_t \varrho^m, \bar{\varphi})|}{\|\nabla_x \bar{\varphi}\|_{L^2(\Omega;\mathbb{R}^d)}} \right)^2 \mathrm{d}t \\
&\le \int_0^T \left(\sup_{\bar{\varphi}\in W^{1,2}(\Omega)} \frac{|(\boldsymbol{v}^m \varrho^m, \nabla_x \bar{\varphi})| + |(\nabla_x \varrho^m, \nabla_x \bar{\varphi})|}{\|\nabla_x \bar{\varphi}\|_{L^2(\Omega;\mathbb{R}^d)}} \right)^2 \mathrm{d}t \\
&\le 2\int_0^T \left(\|\boldsymbol{v}^m\|^2_{L^2(\Omega;\mathbb{R}^d)} \|\varrho^m\|^2_{L^\infty(\Omega)} + \|\nabla_x \varrho^m\|^2_{L^2(\Omega;\mathbb{R}^d)} \right) \mathrm{d}t \\
&\le 2\|\boldsymbol{v}^m\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \|\varrho^m\|^2_{L^\infty(Q)} + 2\|\nabla_x \varrho^m\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \\
&\le C(\ell).
\end{aligned}
\tag{3.97}
$$

Using the uniform estimates (3.93), (3.94), (3.95), (3.96) and (3.97), we deduce the existence of subsequences (not relabelled), such that

$$\varrho^m \rightharpoonup^* \varrho \qquad \text{weak* in } L^\infty(Q), \tag{3.98}$$

$$\varrho^m \rightharpoonup \varrho \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.99}$$

$$\partial_t\varrho^m \rightharpoonup \partial_t\varrho \qquad \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'), \tag{3.100}$$

$$\boldsymbol{v}^m \rightharpoonup^* \boldsymbol{v} \qquad \text{weak* in } L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{3.101}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{3.102}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d), \tag{3.103}$$

$$\partial_t\boldsymbol{v}^m \rightharpoonup \partial_t\boldsymbol{v} \qquad \text{weakly in } L^p(0,T;W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \text{ for any } p \in \left(1, \frac{d+2}{d}\right], \tag{3.104}$$

$$S_e^m \rightharpoonup^* S_e \qquad \text{weak* in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d})). \tag{3.105}$$

Then, by using the Aubin–Lions Lemma, we deduce that

$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{3.106}$$

$$\varrho^m \to \varrho \qquad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.107}$$

The above strong convergence (3.107) can be interpolated with (3.84) to get

$$\varrho^m \to \varrho \qquad \text{strongly in } L^p(0,T;L^p(\Omega)) \text{ for any } p \in [1,\infty). \tag{3.108}$$

Convergence for S_v^m. Now we are in a good position to deduce the convergence of the viscous part of the Cauchy stress S_v^m. It follows from the strong convergence (3.108) that

$$\varrho^m \to \varrho \qquad \text{a.e. in } Q. \tag{3.109}$$

Since $\mu(\cdot)$ is a continuous function, we have that

$$\mu(\varrho^m) \to \mu(\varrho) \qquad \text{a.e. in } Q. \tag{3.110}$$

From the boundedness (3.24) of $\mu(\cdot)$, we deduce that

$$\mu(\varrho^m) \to \mu(\varrho) \qquad \text{strongly in } L^p(Q) \text{ for any } p \in [1,\infty). \tag{3.111}$$

Using (3.102), we deduce that

$$S_v^m = \mu(\varrho^m)D(\boldsymbol{v}^m) \rightharpoonup \mu(\varrho)D(\boldsymbol{v}) =: S_v \qquad \text{weakly in } L^q(Q;\mathbb{R}^{d\times d}) \text{ for any } q \in [1,2). \tag{3.112}$$

To show this, we let $\boldsymbol{w} \in L^{\frac{2p}{p-2}}(Q;\mathbb{R}^d)$ with $p \in (2,\infty)$ to be an arbitrary test function. We consider

$$\begin{aligned}
&\left|\int_0^T \int_\Omega (\mu(\varrho^m)D(\boldsymbol{v}^m) - \mu(\varrho)D(\boldsymbol{v})) : \boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \le \left|\int_0^T \int_\Omega (\mu(\varrho^m)D(\boldsymbol{v}^m) - \mu(\varrho)D(\boldsymbol{v}^m)) : \boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad + \left|\int_0^T \int_\Omega (\mu(\varrho)D(\boldsymbol{v}^m) - \mu(\varrho)D(\boldsymbol{v})) : \boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \le \int_0^T \int_\Omega |\mu(\varrho^m) - \mu(\varrho)||D(\boldsymbol{v}^m)||\boldsymbol{w}|\,\mathrm{d}x\,\mathrm{d}t + \left|\int_0^T \int_\Omega \mu(\varrho)(D(\boldsymbol{v}^m) - D(\boldsymbol{v})) : \boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad =: I_1^m + I_2^m.
\end{aligned}$$

For I_1^m, we deduce from the (3.93), (3.111) and Hölder's inequality that

$$I_1^m \le \|\mu(\varrho^m) - \mu(\varrho)\|_{L^p(Q)} \|D(\boldsymbol{v}^m)\|_{L^2(Q;\mathbb{R}^{d\times d})} \|\boldsymbol{w}\|_{L^{\frac{2p}{p-2}}(Q;\mathbb{R}^{d\times d})} \to 0 \qquad \text{as } m \to \infty.$$

For I_2^m, thanks to the weak convergence result (3.102), we only need to show that $\boldsymbol{w} \in L^2(Q;\mathbb{R}^{d\times d})$, which is given by

$$\int_0^T \int_\Omega |\boldsymbol{w}|^2\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_0^T \int_\Omega |\boldsymbol{w}|^{\frac{2p}{p-2}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p-2}{p}} \left(\int_0^T \int_\Omega 1^{\frac{p}{2}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} \le C.$$

Then $I_2^m \to 0$ as $m \to \infty$. Therefore, we obtain the weak convergence result (3.112).

Convergence for $\hat{\psi}^m$. We focus on the convergence properties of $\hat{\psi}^m$. First we define $\psi^m = M^m \hat{\psi}^m$. From (3.93), the definition of $\mathcal{F}$ and the fact that $M^m \le C$, we have

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} \psi^m(x,\boldsymbol{q},t)\log(1+\psi^m(x,\boldsymbol{q},t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le C(\ell),$$

which implies that ψ^m is bounded in $L^\infty(0,T;L^1(\mathcal{O}))$ and is uniformly integrable in $L^1(\mathcal{O}\times(0,T))$. Then by De la Vallée Poussin's Theorem (c.f. Theorem 2.4.2), there exists a subsequence, which we do not relabel, such that

$$\psi^m \rightharpoonup \psi \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T)). \tag{3.113}$$

From (3.34), we deduce that $1/M^m$ converges to $1/M$ in $C(\varkappa)$ where $\varkappa \subset D$ is compact. Therefore, it follows that

$$\hat{\psi}^m \rightharpoonup \hat{\psi} \qquad \text{weakly in } L^1_{loc}(\mathcal{O}\times(0,T)). \tag{3.114}$$

Next, we will show that

$$\hat{\psi}^m \rightharpoonup \hat{\psi} \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{3.115}$$

Let $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$ be an arbitrary Lipschitz domain. From (3.93) and the properties of M^m, we have that

$$\sup_{t\in(0,T)} \|\sqrt{\hat{\psi}^m(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\hat{\psi}^m}\|^2_{W^{1,2}(\mathcal{O}_0)} \,\mathrm{d}t \le C(\mathcal{O}_0). \tag{3.116}$$

By the interpolation inequality (2.3), we obtain

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^m|^{\frac{(K+1)d+2}{d(K+1)}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = \int_0^T \int_{\mathcal{O}_0} \left|\sqrt{\hat{\psi}^m}\right|^{\frac{2((K+1)d+2)}{d(K+1)}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0). \tag{3.117}$$

The application of Hölder's inequality together with (3.116) gives, for any $\beta \in [1,2)$, that

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^m|^\beta \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = 2^\beta \int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m}\right|^\beta \left|\sqrt{\hat{\psi}^m}\right|^\beta \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t$$
$$\le C \left(\int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{\beta}{2}} \left(\int_0^T \int_{\mathcal{O}_0} \left|\sqrt{\hat{\psi}^m}\right|^{\frac{2\beta}{2-\beta}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{2-\beta}{\beta}} \le C(\mathcal{O}_0).$$

The inequality holds provided that

$$\frac{2\beta}{2-\beta} \le \frac{2((K+1)d+2)}{d(K+1)}.$$

By choosing $\beta = \frac{(K+1)d+2}{(K+1)d+1}$, we get

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^m|^{\frac{(K+1)d+2}{(K+1)d+1}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0). \tag{3.118}$$

Since we have proved that $\|\varrho^m\|_{L^\infty(Q)} \le C$ and $\psi^m \ge 0$, we can improve the integrability of ψ^m as follows:

$$\|\psi^m\|_{L^\infty(Q;L^1(D))} \le C.$$

Parabolic interpolation between the above equation and (3.117) and the property of M^m implies that for any $q_1 \in (1,\infty)$ there exists a $q_2 > 1$ such that

$$\|\hat{\psi}^m\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \le C(\mathcal{O}_0).$$

Recalling (3.93), we deduce using Hölder's inequality that there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^m \hat{\psi}^m\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \le C(\mathcal{O}_0). \tag{3.119}$$

Next, we shall prove the pointwise convergence of $\hat{\psi}^m$ using the Div-Curl Lemma. We set

$$
\begin{aligned}
H^m &:= (M^m\hat{\psi}^m, M^m\hat{\psi}^m\boldsymbol{v}^m - M^m\nabla_x\hat{\psi}^m, M\Lambda_\ell(\hat{\psi}^m)(\nabla_x\boldsymbol{v}^m)\boldsymbol{q} - M^m\nabla_{\boldsymbol{q}}\hat{\psi}^m),\\
Q^m &:= ((1+\hat{\psi}^m)^\gamma, \underbrace{0,\dots,0}_{(d+Kd)\text{-times}}),
\end{aligned}
$$

for some $\gamma \in (0, 1/2)$. Recalling the convergence results (3.101), (3.102) and (3.106) for $\boldsymbol{v}^m$ together with the uniform bounds (3.118) and (3.119), we see that there exist subsequences, such that

$$
\begin{aligned}
H^m &\rightharpoonup H \qquad \text{weakly in } L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^{1+d+Kd}),\\
Q^m &\rightharpoonup Q \qquad \text{weakly in } L^{\frac{1}{\gamma}}(\mathcal{O}_0\times(0,T);\mathbb{R}^{1+d+Kd}),
\end{aligned}
$$

where,

$$
\begin{aligned}
H &:= (M\hat{\psi}, M\hat{\psi}\boldsymbol{v} - M\nabla_x\hat{\psi}, M\overline{\Lambda_\ell(\hat{\psi})}(\nabla_x\boldsymbol{v})\boldsymbol{q} - M\nabla_{\boldsymbol{q}}\hat{\psi}),\\
Q &:= (\overline{(1+\hat{\psi})^\gamma}, 0,\dots,0).
\end{aligned}
$$

Now we need to verify the remaining conditions of the Div-Curl Lemma. First, it follows from (3.79) that

$$
\operatorname{div}_{t,x,\boldsymbol{q}} H^m = 0 \qquad \text{in } \mathcal{O}_0\times(0,T),
$$

which implies that $(\operatorname{div}_{t,x,\boldsymbol{q}} H^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0\times(0,T))$. Moreover, we have the following

$$
\begin{aligned}
\int_0^T\int_{\mathcal{O}_0} |\operatorname{curl} Q^m|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &= \int_0^T\int_{\mathcal{O}_0} |\nabla_{t,x,\boldsymbol{q}}Q^m - (\nabla_{t,x,\boldsymbol{q}}Q^m)^{\mathrm{T}}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C\int_0^T\int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}(1+\hat{\psi}^m)^\gamma|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C\int_0^T\int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C(\mathcal{O}_0).
\end{aligned}
$$

Therefore, $(\operatorname{curl}_{t,x,\boldsymbol{q}} Q^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0\times(0,T))$. By choosing $\gamma < \frac{\delta}{1+\delta}$, the application of the Div-Curl Lemma implies that

$$
H^m\cdot Q^m \rightharpoonup H\cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0\times(0,T)).
$$

In particular, we have

$$
M^m\hat{\psi}^m(1+\hat{\psi}^m)^\gamma \rightharpoonup M\hat{\psi}\overline{(1+\hat{\psi})^\gamma} \qquad \text{weakly in } L^1(\mathcal{O}_0\times(0,T)).
$$

Since M^m converges to M uniformly, we obtain

$$(1+\hat{\psi}^m)^{\gamma+1} \rightharpoonup (1+\hat{\psi})\overline{(1+\hat{\psi})^\gamma} \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).$$

Then by Theorem 2.4.6, since the function $s \in [0,\infty) \mapsto s^{\gamma+1} \in [0,\infty)$ is convex, we can deduce that

$$(1+\hat{\psi})^{\gamma+1} \leq (1+\hat{\psi})\overline{(1+\hat{\psi})^\gamma} \quad \Longrightarrow \quad (1+\hat{\psi})^\gamma \leq \overline{(1+\hat{\psi})^\gamma}.$$

Since the function $s \in [0,\infty) \mapsto s^\gamma \in [0,\infty)$ is concave, we arrive at

$$(1+\hat{\psi})^\gamma = \overline{(1+\hat{\psi})^\gamma}.$$

Finally, an application of Theorem 2.4.6 again shows that there exists a subsequence, such that

$$\hat{\psi}^m \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Since M^m converges to M uniformly, we get

$$\psi^m \to \psi \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Now we want to extend the pointwise convergence result of ψ^m to the whole of our domain $\mathcal{O} \times (0,T)$. For this purpose, we choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k\in\mathbb{N}}$, i.e. $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \cdots \subset \mathcal{O}_0^k \subset \cdots$, satisfying $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of ψ^m that is pointwise convergent to ψ in $\mathcal{O}_0^k \times (0,T)$. Arguing by a diagonal procedure, we deduce that there exists a subsequence, such that

$$\psi^m \to \psi \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{3.120}$$

By uniform integrability combined with (3.120) and using Vitali's Convergence Theorem, we obtain that

$$\psi^m \to \psi \qquad \text{strongly in } L^1(0,T;L^1(\mathcal{O})). \tag{3.121}$$

Therefore, thanks to the presence of the truncation T_ℓ, we apply the Dominated Convergence Theorem to deduce that

$$S_e^m \to S_e \qquad \text{strongly in } L^1(0,T;L^1(\Omega;\mathbb{R}^{d\times d})), \tag{3.122}$$

where

$$S_e = -k\int_D \left[KMT_\ell(\hat{\psi})I + \sum_{j=1}^K T_\ell(\hat{\psi})\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q}, \quad \text{a.e. in } Q. \tag{3.123}$$

From (3.121), we see that

$$\varrho^m = \int_D \psi^m \,\mathrm{d}\boldsymbol{q} \to \int_D \psi \,\mathrm{d}\boldsymbol{q} \qquad \text{strongly in } L^1(Q). \tag{3.124}$$

By uniqueness of the limit, recalling the strong convergence (3.108), the function ϱ is then given by

$$\varrho(x,t) = \int_D \psi(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q} = \int_D M(\boldsymbol{q})\hat{\psi}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{3.125}$$

Passage to the limit in the governing equations. The rest of this section is devoted to identifying all limits in (3.79) as $m \to \infty$. Recall that we have $\|\varrho^m\|_{L^\infty(Q)} \leq C$. Interpolating with the strong convergence result (3.121) for ψ^m, we get that

$$\psi^m \to \psi \quad \text{strongly in } L^p(Q; L^1(D)) \text{ for any } p \in [1,\infty). \tag{3.126}$$

For any measurable $U \subset (Q \times D)$ with $|U| \leq \delta$, we can apply Hölder's inequality to deduce that

$$\begin{aligned}
\int_U M^m |\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &= 2\int_U M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right| \sqrt{\hat{\psi}^m}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\leq 2\left(\int_U M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_U \psi^m\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{\frac{1}{2}}
\end{aligned}$$

thanks to the a priori estimate (3.93) and the uniform equi-integrability of ψ^m. Then there exists a subsequence such that

$$M^m \nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup \overline{M\nabla_{x,\boldsymbol{q}}\hat{\psi}} \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}). \tag{3.127}$$

Recalling the estimate (3.118) we have

$$\int_0^T\int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \leq \int_0^T\int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|^{\frac{(K+1)d+2}{(K+1)d+1}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \leq C(\mathcal{O}_0).$$

Therefore $\nabla_{x,\boldsymbol{q}}\hat{\psi}^m$ converges weakly to $\nabla_{x,\boldsymbol{q}}\hat{\psi}$ in $L^1(\mathcal{O}_0\times(0,T))$ for any Lipschitz domain $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since M^m converges to M uniformly, we can identify the weak limit $\overline{M\nabla_{x,\boldsymbol{q}}\hat{\psi}}$ as $M\nabla_{x,\boldsymbol{q}}\hat{\psi}$. From the a priori estimate (3.93), we also have

$$\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m} \rightharpoonup \sqrt{M}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}} \qquad \text{weakly in } L^2(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}). \tag{3.128}$$

Now we consider the following

$$\begin{aligned}
&\int_Q \left(\int_D M^m \left|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m\right| \,\mathrm{d}\boldsymbol{q}\right)^2 \mathrm{d}x\,\mathrm{d}t = 4\int_Q \left(\int_D M^m \sqrt{\hat{\psi}^m} \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right| \,\mathrm{d}\boldsymbol{q}\right)^2 \mathrm{d}x\,\mathrm{d}t\\
&\le 4\int_Q \left(\int_D M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \mathrm{d}\boldsymbol{q}\right)\left(\int_D M^m\hat{\psi}^m \,\mathrm{d}\boldsymbol{q}\right) \mathrm{d}x\,\mathrm{d}t\\
&\le C\sup_Q \|\psi^m\|_{L^1(D)} \int_Q \|\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\|^2_{L^2(D;\mathbb{R}^{d(K+1)})}\,\mathrm{d}x\,\mathrm{d}t\\
&\le C.
\end{aligned}$$

The above inequality follows from (3.84) and (3.93). Therefore, we deduce that

$$M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup M\nabla_{x,\boldsymbol{q}}\hat{\psi} \qquad \text{weakly in } L^2(Q;L^1(D;\mathbb{R}^{d(K+1)})). \tag{3.129}$$

Combining (3.106) and (3.126), we get that, for any $p \in [1,2)$,

$$M^m\hat{\psi}^m\boldsymbol{v}^m \to \psi\boldsymbol{v} \qquad \text{strongly in } L^p(Q;L^1(D;\mathbb{R}^d)), \tag{3.130}$$

Also, using (3.102) and (3.126), we can deduce that, for any $p \in [1,2)$,

$$M\Lambda_\ell(\hat{\psi}^m)\nabla_x\boldsymbol{v}^m \rightharpoonup M\Lambda_\ell(\hat{\psi})\nabla_x\boldsymbol{v} \qquad \text{weakly in } L^p(Q\times D;\mathbb{R}^{d\times d}). \tag{3.131}$$

Using (3.79), we rewrite $\partial_t(M^m\hat{\psi}^m)$ as

$$\begin{aligned}
\partial_t(M^m\hat{\psi}^m) &= \operatorname{div}_{x,\boldsymbol{q}}\left(M^m\boldsymbol{v}^m\hat{\psi}^m - M^m\nabla_x\hat{\psi}^m, M\Lambda_\ell(\hat{\psi}^m)(\nabla_x\boldsymbol{v}^m)\boldsymbol{q} - M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^m)\right)\\
&=: \operatorname{div}_{x,\boldsymbol{q}}\mathcal{H}^m.
\end{aligned}$$

From the convergence results (3.129), (3.130) and (3.131), we deduce that, for any $p \in [1,2)$,

$$\mathcal{H}^m \rightharpoonup \mathcal{H} \qquad \text{weakly in } L^p(0,T;L^1(\mathcal{O};\mathbb{R}^{d(K+1)})), \tag{3.132}$$

where

$$\mathcal{H} = \left(M\boldsymbol{v}\hat{\psi} - M\nabla_x\hat{\psi}, M\Lambda_\ell(\hat{\psi})(\nabla_x\boldsymbol{v})\boldsymbol{q} - M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi})\right).$$

Therefore, for any $p \in [1,2)$,

$$\partial_t(M^m\hat{\psi}^m) \rightharpoonup \partial_t\psi \qquad \text{weakly in } L^p(0,T;W^{-1,1}(\mathcal{O})). \tag{3.133}$$

Conclusion. Using the convergence results (3.101), (3.102), (3.106) and (3.104) for $\boldsymbol{v}^m$, (3.98),(3.99), (3.100) and (3.107) for ϱ^m, (3.126), (3.129), (3.130), (3.131) and (3.133) for $\hat{\psi}^m$, (3.122) for S_e^m and (3.3.5) for S_v^m and the linearity of the map $B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$, we can pass to the limit as $m \to \infty$ in (3.78)–(3.79) to deduce the following (here we reinstate the superscript ℓ) :

$$\begin{aligned} &\langle \partial_t \boldsymbol{v}^\ell, \boldsymbol{w}\rangle - (\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{w}) + (S_v^\ell, \nabla_x \boldsymbol{w}) \\ &\quad = -(S_e^\ell, \nabla_x \boldsymbol{w}) + (\boldsymbol{f}, \boldsymbol{w}) \quad \text{for all } \boldsymbol{w} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d) \text{ and a.e. } t \in (0,T), \end{aligned} \tag{3.134}$$

$$\begin{aligned} &\langle \partial_t (M\hat{\psi}^\ell), \varphi\rangle_{\mathcal{O}} - (M\boldsymbol{v}^\ell \hat{\psi}^\ell, \nabla\varphi)_{\mathcal{O}} - (M\Lambda_\ell(\hat{\psi}^\ell)(\nabla \boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}} \\ &\quad + (M\nabla\hat{\psi}^\ell, \nabla\varphi)_{\mathcal{O}} + (M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^\ell), \nabla_{\boldsymbol{q}}\varphi)_{\mathcal{O}} = 0 \\ &\quad \text{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})) \text{ and a.e. } t \in (0,T), \end{aligned} \tag{3.135}$$

$$\begin{aligned} &\langle \partial_t \varrho^\ell, \bar{\varphi}\rangle - (\boldsymbol{v}^\ell \varrho^\ell, \nabla_x \bar{\varphi}) + (\nabla_x \varrho^\ell, \nabla_x \bar{\varphi}) = 0 \\ &\quad \text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T). \end{aligned} \tag{3.136}$$

The initial conditions are attained in the following sense:

$$\begin{aligned} \lim_{t\to 0+} \|\boldsymbol{v}^\ell(\cdot,t) - \boldsymbol{v}_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^d)} &= 0, \\ \lim_{t\to 0+} \|\hat{\psi}^\ell(\cdot,t) - T_\ell(\hat{\psi}_0(\cdot))\|_{L^1_M(\mathcal{O})} &= 0, \\ \lim_{t\to 0+} \|\varrho^\ell(\cdot,t) - \varrho_0^\ell(\cdot)\|_{L^1(\Omega)} &= 0, \end{aligned}$$

where

$$\varrho_0^\ell(x,t) := \int_D M(\boldsymbol{q}) T_\ell(\hat{\psi}_0(x,\boldsymbol{q},t))\,\mathrm{d}\boldsymbol{q}.$$

The polymer number density ϱ^ℓ is given by

$$\varrho^\ell(x,t) = \int_D M(\boldsymbol{q})\hat{\psi}^\ell(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

Now we focus on the energy inequality and a priori estimates. We let $m \to \infty$ in (3.92) to deduce by weak lower semi-continuity that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\boldsymbol{v}^\ell(\cdot,t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho^\ell(\cdot,t))^2\,\mathrm{d}x + c_1\int_0^t\int_\Omega |D(\boldsymbol{v}^\ell)|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \int_0^t\int_\Omega |\nabla_x \varrho^\ell|^2\,\mathrm{d}x\,\mathrm{d}\tau + k\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}^\ell(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad + 4kC\int_0^t\int_{\mathcal{O}} M\left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^\ell}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ &\leq k\int_{\mathcal{O}} M\mathcal{F}(T_\ell(\hat{\psi}_0^\ell))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho_0(\cdot))^2\,\mathrm{d}x + \int_0^t (\boldsymbol{f},\boldsymbol{v}^\ell)\,\mathrm{d}\tau. \end{aligned} \tag{3.137}$$

where $\mathcal{F}(s) := s \log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s \to 0+} \mathcal{F}(s) = 1$. Also, letting $m \to \infty$ in (3.93), we can deduce the following a priori estimate

$$\begin{aligned}
&\sup_{t \in (0,T)} \Big(\|\boldsymbol{v}^\ell(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\hat{\psi}^\ell(\cdot,t) \log \hat{\psi}^\ell(\cdot,t)\|_{L^1_M(\mathcal{O})} + \|\varrho^\ell(\cdot,t)\|_{L^\infty(\Omega)} \Big) \\
&\quad + c_1 \int_0^T \|\boldsymbol{v}^m\|^2_{W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\
&\quad + \int_0^T \|\sqrt{\hat{\psi}^\ell}\|^2_{W^{1,2}_M(\mathcal{O})} + \|S^\ell_e\|_{L^1(\Omega;\mathbb{R}^{d\times d})} + \|\nabla_x \varrho^\ell\|^2_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\
&\le C(1 + \|M\mathcal{F}(T_\ell(\hat{\psi}_0))\|_{L^1(\mathcal{O})}) \le C
\end{aligned} \tag{3.138}$$

by the assumption (3.22). Indeed, passage to the limits in all terms on the left-hand side of (3.92) and (3.93) can be justified using Fatou's Lemma.

In the next section, we shall pass to the limit as $\ell \to \infty$ which then completes the proof of Theorem 3.2.2.

3.3.5 Passage to the limit with ℓ

This section is devoted to proving the existence of a global weak solution from the sequence of approximate solutions $(\boldsymbol{v}^\ell, \hat{\psi}^\ell, \varrho^\ell)$ constructed above.

Similarly as in the previous section, from (3.138) we deduce the existence of subsequences, which we do not relabel, such that

$$\varrho^\ell \rightharpoonup^* \varrho \qquad \text{weak* in } L^\infty(Q), \tag{3.139}$$

$$\varrho^\ell \rightharpoonup \varrho \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.140}$$

$$\boldsymbol{v}^\ell \rightharpoonup^* \boldsymbol{v} \qquad \text{weak* in } L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{3.141}$$

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)). \tag{3.142}$$

By standard interpolation, we obtain that

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d). \tag{3.143}$$

From the definition (3.123) for S^ℓ_e, we apply integration by parts noting the fact that $M = 0$ on ∂D to get

$$\begin{aligned}
S^\ell_e &= -k \int_D \left[K M T_\ell(\hat{\psi}^\ell) I + \sum_{j=1}^K T_\ell(\hat{\psi}^\ell) \nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \\
&= k \sum_{j=1}^K \int_D M \nabla_{\boldsymbol{q}^j} T_\ell(\hat{\psi}^\ell) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}.
\end{aligned} \tag{3.144}$$

The integration by parts can be justified by Lemma 3.1 in [7]. Then by Fundamental Theorem of Calculus, we deduce that

$$\begin{aligned}\|S_e^\ell\|_{L^2(Q;\mathbb{R}^{d\times d})} &\leq C(M)\int_0^T\int_\Omega\int_D |\nabla_{\boldsymbol{q}^j} T_\ell(\hat{\psi}^\ell)|^2 \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(M)\int_0^T\int_\Omega\int_D (\Gamma_\ell(\hat{\psi}^\ell))^2 \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(M).\end{aligned} \tag{3.145}$$

Therefore, we deduce the existence of a subsequence, which we do not relabel, such that

$$S_e^\ell \rightharpoonup S_e \qquad \text{weakly in } L^2(Q;\mathbb{R}^{d\times d}). \tag{3.146}$$

Similarly as in (3.96), by using (3.134) and the uniform bounds (3.138), (3.143) and (3.145), we deduce that

$$\int_0^T \|\partial_t \boldsymbol{v}^\ell\|^p_{W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \leq C \tag{3.147}$$

for any $p \in \left(1, \frac{d+2}{d}\right]$. Therefore,

$$\partial_t \boldsymbol{v}^\ell \rightharpoonup \partial_t \boldsymbol{v} \qquad \text{weakly in } L^p(0,T;W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \text{ for any } p \in \left(1, \frac{d+2}{d}\right]. \tag{3.148}$$

It then follows from the Aubin–Lions Lemma that

$$\boldsymbol{v}^\ell \to \boldsymbol{v} \qquad \text{strongly in } L^2(Q;\mathbb{R}^d). \tag{3.149}$$

By using (3.136) and the uniform estimates (3.138), we deduce that

$$\int_0^T \|\partial_t \varrho^\ell\|^2_{(W^{1,2}(\Omega))'}\,\mathrm{d}t \leq C(\ell), \tag{3.150}$$

which then gives

$$\partial_t \varrho^\ell \rightharpoonup \partial_t \varrho \qquad \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'). \tag{3.151}$$

Again by Aubin–Lions Lemma and interpolation with the uniform estimates (3.138), we deduce that

$$\varrho^\ell \to \varrho \qquad \text{strongly in } L^p(Q) \text{ for any } p \in [1,\infty). \tag{3.152}$$

Using (3.142) and and (3.152), we deduce that

$$\begin{aligned}S_v^\ell = \mu(\varrho^\ell) D(\boldsymbol{v}^\ell) \rightharpoonup \mu(\varrho) D(\boldsymbol{v}) =: S_v& \\ \text{weakly in } L^q(Q;\mathbb{R}^{d\times d}) \text{ for any } q \in [1,2).&\end{aligned} \tag{3.153}$$

Convergence for $\hat{\psi}^\ell$. We can mimic the procedure in Section 3.3.4 and deduce the convergence results for $\hat{\psi}^\ell$. Here we omit the details and we only present some significant results, which will be useful for identifying S_e. First we note from (3.138) that

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} M(\boldsymbol{q})\hat{\psi}^\ell(x,\boldsymbol{q},t)\log(1+\hat{\psi}^\ell(x,\boldsymbol{q},t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le C.$$

Then we can deduce the uniform equi-integrability of the sequence $\hat{\psi}^\ell$, and the existence of a subsequence such that

$$\hat{\psi}^\ell \rightharpoonup \hat{\psi} \qquad \text{weakly in } L^1_M(\mathcal{O}\times(0,T)). \tag{3.154}$$

Next, we will show that there exists a subsequence, such that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T). \tag{3.155}$$

Similarly as in Section 3.3.4, we let $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$ be an arbitrary Lipschitz domain. Since M is bounded below, from the uniform estimates (3.138), we have that

$$\sup_{t\in(0,T)} \|\sqrt{\hat{\psi}^\ell(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\hat{\psi}^\ell}\|^2_{W^{1,2}(\mathcal{O}_0)}\,\mathrm{d}t \le C(\mathcal{O}_0).$$

Interpolation then gives

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^\ell|^{\frac{(K+1)d+2}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0).$$

By similar arguments as in Section 3.3.4, we arrive at

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\hat{\psi}^\ell|^{\frac{(K+1)d+2}{(K+1)d+1}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0). \tag{3.156}$$

Since we have $\|\varrho^\ell\|_{L^\infty(Q)} \le C$, we have

$$\|\psi^\ell\|_{L^\infty(Q;L^1(D))} \le C.$$

Again by parabolic interpolation and the properties of M, we see that for any $q_1 \in (1,\infty)$ there exists a $q_2 > 1$ such that

$$\|\hat{\psi}^\ell\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \le C(\mathcal{O}_0).$$

Consequently, from the convergence properties of $\boldsymbol{v}^\ell$, the definition of Λ_ℓ, the fact that D_0 is bounded and Hölder's inequality, we deduce that there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^\ell\hat{\psi}^\ell\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} + \|\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x\boldsymbol{v}^\ell)\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \le C(\mathcal{O}_0). \tag{3.157}$$

Then, we define

$$
\begin{aligned}
H^\ell &\coloneqq (M\hat{\psi}^\ell, M\hat{\psi}^\ell \boldsymbol{v}^\ell - M\nabla_x \hat{\psi}^\ell, M\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q} - M\nabla_{\boldsymbol{q}}\hat{\psi}^\ell),\\
Q^\ell &\coloneqq ((1+\hat{\psi}^\ell)^\gamma, \underbrace{0,\dots,0}_{(d+Kd)\text{-times}}),
\end{aligned}
$$

for some $\gamma > 0$ with $\gamma < \frac{\delta}{1+\delta}$. Using the convergence properties (3.141), (3.142) and (3.149) for $\boldsymbol{v}^\ell$, (3.156) and (3.157), we deduce that there exist subsequences such that

$$
\begin{aligned}
H^\ell &\rightharpoonup H \qquad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),\\
Q^\ell &\rightharpoonup Q \qquad \text{weakly in } L^{\frac{1}{\gamma}}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),
\end{aligned}
$$

where,

$$
\begin{aligned}
H &\coloneqq (M\hat{\psi}, M\hat{\psi}\boldsymbol{v} - M\nabla_x\hat{\psi}, M\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q} - M\nabla_{\boldsymbol{q}}\hat{\psi}),\\
Q &\coloneqq (\overline{(1+\hat{\psi})^\gamma}, 0, \dots, 0).
\end{aligned}
$$

Thus, by the application of the Div-Curl Lemma, we deduce that

$$
H^\ell \cdot Q^\ell \rightharpoonup H \cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).
$$

Note that

$$
\hat{\psi}^\ell(1+\hat{\psi}^\ell)^\gamma \rightharpoonup \overline{\hat{\psi}(1+\hat{\psi})^\gamma} \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)),
$$

which by an analogous argument as previously in Section 3.3.4 gives

$$
\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).
$$

We choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k\in\mathbb{N}}$, i.e. $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \cdots \subset \mathcal{O}_0^k \subset \cdots$, satisfying $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of ψ^ℓ that is pointwise convergent to ψ in $\mathcal{O}_0^k \times (0,T)$. Arguing by a diagonal procedure, we deduce that there exists a subsequence such that (3.155) holds. Hence, by Vitali's Convergence Theorem, we obtain that

$$
\hat{\psi}^\ell \to \hat{\psi} \qquad \text{strongly in } L^1(0,T; L^1_M(\mathcal{O})). \tag{3.158}
$$

Moreover, using the fact that $\|\varrho^\ell\|_{L^\infty(Q)} \le C$, interpolation shows that

$$
\psi^\ell \to \psi \qquad \text{strongly in } L^p(Q; L^1(D)) \text{ for any } p \in [1,\infty). \tag{3.159}
$$

Then we can follow the arguments in Section 3.3.4 to obtain the following convergence results:

$$\nabla_{x,\boldsymbol{q}}\hat{\psi}^\ell \rightharpoonup \nabla_{x,\boldsymbol{q}}\hat{\psi} \qquad \text{weakly in } L^1(0,T;L^1_M(\mathcal{O};\mathbb{R}^{d(K+1)})), \tag{3.160}$$

$$\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^\ell} \rightharpoonup \nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}} \qquad \text{weakly in } L^2(0,T;L^2_M(\mathcal{O};\mathbb{R}^{d(K+1)})), \tag{3.161}$$

$$\nabla_{x,\boldsymbol{q}}\hat{\psi}^\ell \rightharpoonup \nabla_{x,\boldsymbol{q}}\hat{\psi} \qquad \text{weakly in } L^2(Q;L^1_M(D;\mathbb{R}^{d(K+1)})). \tag{3.162}$$

Moreover, by similar arguments as in Section 3.3.4, using the convergence results (3.141), (3.142) and (3.149) for $\boldsymbol{v}^\ell$, we deduce that

$$\Lambda_\ell(\hat{\psi}^\ell)\nabla_x\boldsymbol{v}^\ell \rightharpoonup \hat{\psi}\nabla_x\boldsymbol{v} \qquad \text{weakly in } L^p(Q;L^1_M(D;\mathbb{R}^{d\times d})) \text{ for any } p\in[1,2), \tag{3.163}$$

$$M\hat{\psi}^\ell\boldsymbol{v}^\ell \to M\hat{\psi}\boldsymbol{v} \qquad \text{strongly in } L^p(Q;L^1(D;\mathbb{R}^d)) \text{ for any } p\in[1,2). \tag{3.164}$$

Hence, using (3.135) and the convergence results (3.160), (3.163) and (3.164), we deduce using similar arguments as in Section 3.3.4 that

$$\partial_t(M\hat{\psi}^\ell) \rightharpoonup \partial_t(M\hat{\psi}) \qquad \text{weakly in } L^p(0,T;W^{-1,1}(\mathcal{O})) \text{ for any } p\in[1,2). \tag{3.165}$$

Therefore, using the convergence results derived in this section and the linearity of the map $B\in\mathbb{R}^{d(K+1)} \mapsto \mathbb{A}(B)\in\mathbb{R}^{d(K+1)}$, we can let $\ell\to\infty$ in (3.134)–(3.136) to deduce that

$$\begin{aligned}&\int_0^T \langle\partial_t\boldsymbol{v},\boldsymbol{w}\rangle\,\mathrm{d}t + \int_0^T [-(\boldsymbol{v}\otimes\boldsymbol{v},\nabla_x\boldsymbol{w}) + (S_v,\nabla_x\boldsymbol{w})]\,\mathrm{d}t \\ &= \int_0^T [-(S_e,\nabla_x\boldsymbol{w}) + (\boldsymbol{f},\boldsymbol{w})]\,\mathrm{d}t, \quad \text{for all } \boldsymbol{w}\in L^\infty(0,T;W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\end{aligned} \tag{3.166}$$

$$\begin{aligned}&\int_0^T \left[\left\langle\partial_t(M\hat{\psi}),\varphi\right\rangle_{\mathcal{O}} - \left(M\boldsymbol{v}\hat{\psi},\nabla_x\varphi\right)_{\mathcal{O}} - \left(M\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q},\nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\right]\mathrm{d}t \\ &\quad + \int_0^T \left[\left(M\nabla\hat{\psi},\nabla_x\varphi\right)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}),\nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\right]\mathrm{d}t = 0, \\ &\quad \text{for all } \varphi\in L^\infty(0,T;W^{1,\infty}(\mathcal{O})).\end{aligned} \tag{3.167}$$

$$\begin{aligned}&\int_0^T [\langle\partial_t\varrho,\bar{\varphi}\rangle - (\boldsymbol{v}\varrho,\nabla_x\bar{\varphi}) + (\nabla_x\varrho,\nabla_x\bar{\varphi})]\,\mathrm{d}t = 0, \\ &\text{for all } \bar{\varphi}\in L^2(0,T;W^{1,2}(\Omega)).\end{aligned} \tag{3.168}$$

Also it follows from (3.159) that

$$\varrho(x,t) = \int_D M(\boldsymbol{q})\hat{\psi}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

By the convergence results (3.159) and (3.160), we deduce from (3.144) that

$$S_e(x,t) = k\sum_{j=1}^K \int_D M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j}\hat{\psi}(x,\boldsymbol{q},t)\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}.$$

For the energy identity, letting $\ell \to \infty$ in (3.137), we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{v}(\cdot,t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho(\cdot,t))^2 \,\mathrm{d}x + c_1 \int_0^t \int_\Omega |D(\boldsymbol{v})|^2 \,\mathrm{d}x \,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega |\nabla_x \varrho|^2 \,\mathrm{d}x \,\mathrm{d}\tau + k \int_{\mathcal{O}} M \mathcal{F}(\hat{\psi}(\cdot,t)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + 4kC \int_0^t \int_{\mathcal{O}} M \left| \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}} \right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}\tau \\
&\leq k \int_{\mathcal{O}} M \mathcal{F}(\hat{\psi}_0) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\varrho_0)^2 \,\mathrm{d}x + \int_0^t (\boldsymbol{f}, \boldsymbol{v}) \,\mathrm{d}\tau.
\end{aligned} \tag{3.169}$$

Next, we show the attainment of initial conditions $(\boldsymbol{v}_0, \hat{\psi}_0, \varrho_0)$. First, we set $\boldsymbol{w}(x,t) := \chi_{[0,t]}\boldsymbol{u}(x)$ in (3.134), where $\boldsymbol{u} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ is arbitrary. Then we have

$$\begin{aligned}
(\boldsymbol{v}^\ell(t), \boldsymbol{u}) + \int_0^t [-(\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{u}) + (S^\ell_v, \nabla_x \boldsymbol{u})] \,\mathrm{d}\tau \\
= \int_0^t [-(S^\ell_e, \nabla_x \boldsymbol{u}) + (\boldsymbol{f}, \boldsymbol{u})] \,\mathrm{d}\tau + (\boldsymbol{v}_0, \boldsymbol{u}).
\end{aligned}$$

Letting $\ell \to \infty$ in the above equation and using the convergence results from preceding sections, we get for a.e. $t \in (0,T)$,

$$\begin{aligned}
(\boldsymbol{v}(t), \boldsymbol{u}) + \int_0^t [-(\boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{u}) + (S_v, \nabla_x \boldsymbol{u})] \,\mathrm{d}\tau \\
= \int_0^t [-(S_e, \nabla_x \boldsymbol{u}) + (\boldsymbol{f}, \boldsymbol{u})] \,\mathrm{d}\tau + (\boldsymbol{v}_0, \boldsymbol{u}).
\end{aligned}$$

We note that after a possible redefinition of $\boldsymbol{v}$ on a set of measure zero, the above equation holds for all $t \in (0,T)$. Then taking the limit as $t \to 0+$, we see that for all $\boldsymbol{u} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$,

$$\lim_{t \to 0+} (\boldsymbol{v}(t), \boldsymbol{u}) = (\boldsymbol{v}_0, \boldsymbol{u}).$$

Therefore, we have $\boldsymbol{v}(t)$ converges to $\boldsymbol{v}_0$ weakly in $L^2(\Omega;\mathbb{R}^d)$ as $t \to 0+$.

Similarly, setting $\varphi := \chi_{[0,t]}\phi(x,\boldsymbol{q})$ in (3.135), where $\phi \in W^{1,\infty}(\mathcal{O})$ is arbitrary, we deduce that

$$\begin{aligned}
(M\hat{\psi}^\ell(t), \phi)_{\mathcal{O}} - \int_0^t \Big[(M\boldsymbol{v}^\ell \hat{\psi}^\ell, \nabla_x \phi)_{\mathcal{O}} - (M\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}} \phi)_{\mathcal{O}} \Big] \,\mathrm{d}\tau \\
+ \int_0^t \Big[(M\nabla_x \hat{\psi}^\ell, \nabla_x \phi)_{\mathcal{O}} + (M\mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^\ell), \nabla_{\boldsymbol{q}} \phi)_{\mathcal{O}} \Big] \,\mathrm{d}\tau = (MT_\ell(\hat{\psi}_0), \phi)_{\mathcal{O}}.
\end{aligned}$$

We let $\ell \to \infty$ in the above equation. By applying the above convergence results, we see that for all $t \in (0,T)$ (after a possible redefinition of $\hat{\psi}$ on a set of measure zero), we have

$$(M\hat{\psi}(t), \phi)_{\mathcal{O}} - \int_0^t \Big[(M\boldsymbol{v}\hat{\psi}, \nabla_x \phi)_{\mathcal{O}} - (M\hat{\psi}(\nabla_x \boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\phi)_{\mathcal{O}}\Big]\,\mathrm{d}\tau$$
$$+ \int_0^t \Big[(M\nabla_x\hat{\psi}, \nabla_x\phi)_{\mathcal{O}} + (M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}), \nabla_{\boldsymbol{q}}\phi)_{\mathcal{O}}\Big]\,\mathrm{d}\tau = (M\hat{\psi}_0, \phi)_{\mathcal{O}}.$$

Therefore,

$$\lim_{t\to 0+}(M\hat{\psi}(t), \phi)_{\mathcal{O}} = (M\hat{\psi}_0, \phi)_{\mathcal{O}} \qquad \text{for all } \phi \in W^{1,\infty}(\mathcal{O}). \tag{3.170}$$

In addition, we note from the energy identity (3.169) that $(M\mathcal{F}(\hat{\psi}(t)))_{t\geq 0}$ is bounded in $L^1(\mathcal{O})$, and the De la Vallée Poussin's Theorem (c.f. Theorem 2.4.2) implies that that $(M\hat{\psi}(t))_{t\geq 0}$ is equi-integrable. By the Dunford–Pettis Theorem (c.f. Theorem 2.4.3), $(M\hat{\psi}(t))_{t\geq 0}$ has a weakly convergence subsequence in $L^1(\mathcal{O})$. Since we already have the weak limit (3.170), by uniqueness of the limit, we deduce that $\hat{\psi}(t)$ converges to $\hat{\psi}_0$ weakly in $L^1_M(\mathcal{O})$ as $t \to 0+$.

Now, we are in a good position to deduce the strong attainment of the initial condition. Letting $\ell \to \infty$ in the energy inequality and using the convergence results above, we obtain, for all $t \in (0, T)$, that

$$\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot, t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega |\boldsymbol{v}(\cdot, t)|^2\,\mathrm{d}x$$
$$\leq \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x + \int_0^t (\boldsymbol{f}, \boldsymbol{v})\,\mathrm{d}\tau.$$

First, we note that

$$\limsup_{t\to 0+}\left[\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot, t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\right] \leq \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Since $\mathcal{F}(s) = s\log s + 1$ is a nonnegative strictly convex function on $(0, \infty)$, we use the convergence results of $\boldsymbol{v}(t)$ and $\hat{\psi}(t)$ to deduce that

$$\liminf_{t\to 0+}\left[\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot, t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\right] \geq \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Consequently, we have

$$\lim_{t\to 0+}\left[\int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot, t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\right] = \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}. \tag{3.171}$$

Next, we shall separately discuss the initial conditions for $\hat{\psi}$ and $\boldsymbol{v}$ using the above equality; we shall proceed by contradiction. We first assume

$$\limsup_{t\to 0+}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} > \|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Then from (3.171), we get that

$$\liminf_{t\to 0+} \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} < \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

However, using the weak lower semi-continuity of the convex function $\mathcal{F}$, we also have

$$\liminf_{t\to 0+} \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \geq \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q},$$

which is a contradiction. Hence, we have

$$\begin{gathered}\lim_{t\to 0+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} = \|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)},\\ \lim_{t\to 0+} \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_{\mathcal{O}} M\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.\end{gathered} \tag{3.172}$$

Since we have already proved that $\boldsymbol{v}(t)$ converges to $\boldsymbol{v}_0$ weakly in $L^2(\Omega;\mathbb{R}^d)$ as $t \to 0+$, we obtain

$$\lim_{t\to 0+} \|\boldsymbol{v}(\cdot,t) - \boldsymbol{v}_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^d)} = 0. \tag{3.173}$$

Since $\mathcal{F}$ is strictly convex and we have that $\hat{\psi}(t)$ converges to $\hat{\psi}_0$ weakly in $L^1_M(\mathcal{O})$ as $t \to 0+$, it follows from (3.172) that as $t \to 0+$,

$$M\hat{\psi}(\cdot,t) \to M\hat{\psi}_0(\cdot) \quad \text{a.e. in } \mathcal{O}. \tag{3.174}$$

Therefore, we get

$$\lim_{t\to 0+} \|\hat{\psi}(\cdot,t) - \hat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})} = 0. \tag{3.175}$$

Then it is easy to see that

$$\lim_{t\to 0+} \|\varrho(\cdot,t) - \varrho_0(\cdot)\|_{L^1(\Omega)} = 0. \tag{3.176}$$

Therefore, we have completed the proof of Theorem 3.2.2.

In this chapter, we have shown the existence of large-data global-in-time weak solutions to a general class of kinetic models for the incompressible homogeneous dilute polymeric fluids, featuring the presence of a polymer-number-density-dependent viscosity coefficient in the viscous part of the Cauchy stress. In the next chapter, we consider a more complicated model, which pertains to incompressible nonhomogeneous dilute polymeric fluids and we shall prove the existence of weak solutions to the coupled Navier–Stokes–Fokker–Planck system that arises in models of nonhomogeneous dilute polymeric fluids with variable density.

Chapter 4

Existence of weak solutions to a model of a nonhomogeneous dilute polymeric fluid

This chapter, which is based on our paper [37], is aimed to prove the existence of global-in-time large-data weak solutions to a Navier–Stokes–Fokker–Planck system that arises in models of nonhomogeneous dilute polymeric fluids. The key difference of the nonhomogeneous model, compared to the homogeneous version, is that now the density ρ varies in the domain. The evolution of the density ρ is governed by the transport equation. Another main feature of this model is that there is a density-dependent drag coefficient $\zeta(\rho)$ in the Fokker–Planck equation. Since ρ satisfies the transport equation, then under certain assumptions, ρ is also a solution to the renormalized equation, i.e. ρ also satisfies the following equation

$$\frac{\partial \beta(\rho)}{\partial t} + \mathrm{div}_x(\beta(\rho)\boldsymbol{v}) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d \times (0,T)),$$

provided that $\beta \in C^1([0,\infty))$. This is a consequence of the Friedrichs Commutator Lemma (c.f., Lemma 6.7 in [53]). Therefore, the introduction of $\zeta(\rho)$ in the system is always possible since $\boldsymbol{v}$ is divergence-free and $\zeta(\cdot)$ is a C^1 function. We consider the full model described in Section 1.1 in Chapter 1, and we apply similar techniques as in Chapter 3. However, the introduction of variable density and density-dependent drag coefficient significantly complicates the proof. Compared to the homogeneous case, where the Aubin–Lions Lemma can be directly applied, in the nonhomogeneous case, one can only derive fractional time-derivative bounds on the velocity field $\boldsymbol{v}$ and on the probability density function $\hat{\psi}$, expressed in terms of Nikolskiĭ norm estimates, which is motivated by [14].

In Section 4.1, we first state the governing equations of the Navier–Stokes–Fokker–Planck system under consideration and state some assumptions on the data. In Section 4.2, we formulate our main result to be proved in this chapter. In Section 4.3, we begin our proof by introducing a truncated Navier–Stokes–Fokker–Planck system with truncation parameter ℓ in Subsection 4.3.1. In Subsection 4.3.2, we perform a spatial Galerkin semi-discretization of the velocity field and the probability density function with parameters n and m. Given a sufficiently smooth velocity field, i.e., $\boldsymbol{v} \in L^1(0,T;W^{1,1}(\Omega;\mathbb{R}^d))$, the proofs of existence and uniqueness of the weak solution to the transport equation can be found in [14], for example. We then use Schauder's Fixed-Point Theorem to prove the existence of solutions to our partially Galerkin discretized system in Subsection 4.3.3. In Subsection 4.3.4, we derive n-independent a priori estimates, which allow us to pass to the limit as $n \to \infty$. In Subsection 4.3.5, we first prove the boundedness of the sequence of approximate densities ρ^m and the nonnegativity of the Galerkin approximations $\hat{\psi}^m$. Then we derive an m-independent a priori estimate. Finally, in Subsection 4.3.6, we derive ℓ-independent estimates and apply similar techniques as in Subsection 4.3.5 to pass to the limit as $\ell \to \infty$. We also show the attainment of the initial conditions.

4.1 Statement of the problem

We consider the following system of partial differential equations for the density $\rho(x,t)$, the velocity field $\boldsymbol{v}(x,t)$ and the probability density function $\psi(x,\boldsymbol{q},t)$:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}_x(\boldsymbol{v}\rho) = 0 \qquad \text{in } Q, \tag{4.1}$$

$$\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \operatorname{div}_x(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}_x \mathcal{T} = \rho\boldsymbol{f} \qquad \text{in } Q, \tag{4.2}$$

$$\operatorname{div}_x \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{4.3}$$

subject to the initial conditions

$$\begin{aligned} \rho(\cdot,0) &= \rho_0(\cdot) && \text{in } \Omega,\\ (\rho\boldsymbol{v})(\cdot,0) &= (\rho_0\boldsymbol{v}_0)(\cdot) && \text{in } \Omega, \end{aligned} \tag{4.4}$$

and the no-slip boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega \times (0,T). \tag{4.5}$$

The Cauchy stress $\mathcal{T}$ is decomposed as

$$\mathcal{T} := -pI + S_v + S_e. \tag{4.6}$$

The viscous part S_v of the Cauchy stress $\mathcal{T}$ is defined by

$$S_v := \mu(\rho, \varrho) D(\boldsymbol{v}), \tag{4.7}$$

where $D(\boldsymbol{v}) := \frac{1}{2}(\nabla_x \boldsymbol{v} + (\nabla_x \boldsymbol{v})^{\mathrm{T}})$ denotes the symmetric part of the velocity gradient and ϱ is the polymer number density defined by

$$\varrho(x,t) := \int_D \psi(x, \boldsymbol{q}, t) \, \mathrm{d}\boldsymbol{q}. \tag{4.8}$$

The extra-stress tensor S_e is given by the Kramers expression:

$$S_e(x,t) := k \left(\sum_{j=1}^{K} \int_D \psi(x, \boldsymbol{q}, t) \boldsymbol{q}^j {\boldsymbol{q}^j}^{\mathrm{T}} (U^j)' \left(\frac{1}{2} |\boldsymbol{q}^j|^2 \right) \mathrm{d}\boldsymbol{q} - K \varrho(x,t) I \right), \tag{4.9}$$

where ψ is the probability density function satisfying the Fokker–Planck equation

$$\begin{aligned} \frac{\partial \psi}{\partial t} &+ \mathrm{div}_x(\boldsymbol{v}\psi) + \sum_{j=1}^{K} \mathrm{div}_{\boldsymbol{q}^j}((\nabla_x \boldsymbol{v}) \boldsymbol{q}^j \psi) \\ &= \Delta_x \left(\frac{\psi}{\zeta(\rho)} \right) + \frac{1}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \, \mathrm{div}_{\boldsymbol{q}^j} \left(M \nabla_{\boldsymbol{q}^i} \left(\frac{\psi}{\zeta(\rho) M} \right) \right), \end{aligned} \tag{4.10}$$

in $\mathcal{O} \times (0,T)$, with $\mathcal{O} := \Omega \times D$. In the above equation $\zeta(\cdot) \in \mathbb{R}_{>0}$ is a density-dependent scaled drag coefficient. We impose the following boundary conditions, for all $j = 1, \dots, K$:

$$\left[\frac{1}{4\lambda} \sum_{i=1}^{K} A_{ij} M \nabla_{\boldsymbol{q}^i} \left(\frac{\psi}{\zeta(\rho) M} \right) - (\nabla_x \boldsymbol{v}) \boldsymbol{q}^j \psi \right] \cdot \boldsymbol{n}^j = 0 \quad \text{on } \Omega \times \partial \bar{D}^j \times (0,T), \tag{4.11}$$

$$\nabla_x \left(\frac{\psi}{\zeta(\rho)} \right) \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times D \times (0,T), \tag{4.12}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, and we impose the following initial condition:

$$\psi(x, \boldsymbol{q}, 0) = \psi_0(x, \boldsymbol{q}) \qquad \text{in } \mathcal{O}. \tag{4.13}$$

To simplify the presentation of the Fokker–Planck equation (4.10), we define

$$\hat{\psi} := \frac{\psi}{\zeta(\rho) M}, \quad \hat{\psi}_0 = \frac{\psi_0}{\zeta(\rho_0) M}.$$

Furthermore, we associate with A the linear mapping $\mathbb{A} : \mathbb{R}^{d \times K} \to \mathbb{R}^{d \times K}$ defined, for any $B = (B_i^j)_{i=1,\dots,d}^{j=1,\dots,K} \in \mathbb{R}^{d \times K}$, by $(\mathbb{A}(B))_i^j := \sum_{k=1}^{K} B_i^k A_{kj}$, and let $\mathbb{A}^j : \mathbb{R}^{d \times K} \to \mathbb{R}^d$ be the linear mapping defined by $(\mathbb{A}^j(B))_i := (\mathbb{A}(B))_i^j$, for $i = 1, \dots, d$ and $j = 1, \dots, K$. By the positive definiteness of the Rouse matrix $A \in \mathbb{R}^{K\times K}_{sym}$, there exist positive constants C_1 and C_2 such that

$$C_1|B|^2 \leq \mathbb{A}(B) : B \leq C_2|B|^2 \qquad \forall\, B \in \mathbb{R}^{d\times K}. \tag{4.14}$$

The extra-stress tensor S_e can be rewritten as

$$S_e(x,t) = k\sum_{j=1}^{K}\int_D M\zeta(\rho)\nabla_{\boldsymbol{q}^j}\hat{\psi}(x,\boldsymbol{q},t)\otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}. \tag{4.15}$$

The Fokker–Planck equation (4.10) can be rewritten as

$$\begin{aligned}\partial_t(M\zeta(\rho)\hat{\psi}) + \operatorname{div}_x(M\zeta(\rho)\hat{\psi}\boldsymbol{v}) + \operatorname{div}_{\boldsymbol{q}}\Big(M\zeta(\rho)\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q}\Big)&\\ - \Delta_x(M\hat{\psi}) - \operatorname{div}_{\boldsymbol{q}}\mathbb{A}(M\nabla_{\boldsymbol{q}}\hat{\psi}) = 0.&\end{aligned} \tag{4.16}$$

The above equation is supplemented by the following initial and boundary conditions:

$$\hat{\psi}(x,\boldsymbol{q},0) = \hat{\psi}_0(x,\boldsymbol{q}) \qquad \text{in } \mathcal{O}, \tag{4.17}$$

$$M\nabla_x\hat{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\times D\times(0,T), \tag{4.18}$$

$$\Big(M\zeta(\rho)\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q}^j - \mathbb{A}^j(M\nabla_{\boldsymbol{q}}\hat{\psi})\Big)\cdot\boldsymbol{n}^j = 0 \qquad \text{on } \Omega\times\partial\bar{D}^j\times(0,T), \tag{4.19}$$

for all $j = 1, \dots, K$ and $\boldsymbol{n}^j = (n_1^j, \dots, n_d^j)^{\mathrm{T}}$ is the unit outward normal vector to $\partial\bar{D}^j$.

Next, we need to make a few assumptions. We assume that $\partial\Omega \in C^{0,1}$. For the Maxwellian M we assume that

$$M \in C_0(\overline{D}) \cap C^{0,1}_{loc}(D) \cap W^{1,1}_0(D). \tag{4.20}$$

For the initial density ρ_0 we assume that

$$\rho_0 \in [\rho_{\min}, \rho_{\max}], \text{ with } \rho_{\min} > 0. \tag{4.21}$$

For the initial velocity $\boldsymbol{v}_0$ we assume that

$$\boldsymbol{v}_0 \in L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d). \tag{4.22}$$

For the initial value ψ_0 of the probability density function we assume that

$$\begin{gathered}\psi_0 \geq 0 \quad \text{a.e. in } \mathcal{O}, \quad \hat{\psi}_0\log\hat{\psi}_0 \in L^1_M(\mathcal{O}),\\ 0 \leq \varrho_0(x) := \int_D \psi_0(\cdot,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} \leq \varrho_{\max} \quad \text{a.e. in } \Omega, \quad \int_{\mathcal{O}}\psi_0(x,\boldsymbol{q})\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 1.\end{gathered} \tag{4.23}$$

We shall further assume that

$$\mu \in C^1([\rho_{\min},\rho_{\max}]\times[0,\infty),[\mu_{\min},\mu_{\max}]), \quad \zeta \in C^1([\rho_{\min},\rho_{\max}],[\zeta_{\min},\zeta_{\max}]), \tag{4.24}$$

with $\mu_{\min}, \zeta_{\min} > 0$. Finally, we assume that

$$\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{4.25}$$

In the next section, we state the main result to be proved in this chapter.

4.2 The main result

In this section, we first formulate the notion of weak solution to the coupled Navier–Stokes–Fokker–Planck system introduced in Section 4.1.

Definition 4.2.1. *We say that the triple* $(\rho, \boldsymbol{v}, \psi)$, *with*

$$\psi(x, \boldsymbol{q}, t) = \zeta(\rho(x,t))M(\boldsymbol{q})\hat{\psi}(x, \boldsymbol{q}, t),$$

is a weak solution to the system of nonlinear partial differential equations (4.1)–(4.13), *if the following conditions hold:*

(i) The functions $(\rho, \boldsymbol{v}, \hat{\psi})$ *belong to the following function spaces:*

$$\begin{aligned} \rho &\in L^\infty(\Omega \times (0,T)) \cap C([0,T]; L^p(\Omega)), \text{ where } p \in [1,\infty), \\ \boldsymbol{v} &\in L^\infty(0,T; L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)) \cap L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \\ \hat{\psi} &\in L^\infty(\Omega \times (0,T); L^1_M(D)) \cap L^2(0,T; W^{1,1}_M(\mathcal{O})), \\ &\hat{\psi} \geq 0 \text{ a.e. in } \mathcal{O} \times (0,T), \\ \partial_t \rho &\in L^2(0,T; W^{-1,p}(\Omega)), \text{ where } p \in [1,\infty) \text{ when } d = 2 \\ &\quad \text{and } p \in [1,6] \text{ when } d = 3, \\ \partial_t(\rho \boldsymbol{v}) &\in L^p(0,T; W^{-1,p}(\Omega;\mathbb{R}^d)), \text{ for any } p \in \left(1, \frac{d+2}{d}\right], \\ \partial_t(M\zeta(\rho)\hat{\psi}) &\in L^p(0,T; W^{-1,1}(\mathcal{O})), \text{ for any } p \in [1,2), \end{aligned}$$

where $\mathcal{O} := \Omega \times D$.

(ii) The system (4.1)–(4.13) *is satisfied in the following sense:*

$$\int_0^T [\langle \partial_t \rho, \eta \rangle - (\boldsymbol{v}\rho, \nabla_x \eta)] \,\mathrm{d}t = 0, \quad \text{for all } \eta \in L^1(0,T; W^{1,\frac{q}{q-1}}(\Omega)), \tag{4.26}$$

where $q \in (2,\infty)$ *when* $d = 2$ *and* $q \in [3,6]$ *when* $d = 3$,

$$\begin{aligned} &\int_0^T \langle \partial_t(\rho \boldsymbol{v}), \boldsymbol{w} \rangle \,\mathrm{d}t + \int_0^T [-(\rho \boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{w}) + (S_v, \nabla_x \boldsymbol{w})] \,\mathrm{d}t \\ &= \int_0^T [-(S_e, \nabla_x \boldsymbol{w}) + (\rho \boldsymbol{f}, \boldsymbol{w})] \,\mathrm{d}t, \quad \text{for all } \boldsymbol{w} \in L^s(0,T; W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \end{aligned} \tag{4.27}$$

with $s > d$, and

$$\begin{aligned}
&\int_0^T \Big\langle \partial_t(M\zeta(\rho)\hat{\psi}), \varphi \Big\rangle_{\mathcal{O}} - \Big(M\zeta(\rho)\boldsymbol{v}\hat{\psi}, \nabla_x \varphi \Big)_{\mathcal{O}} \,\mathrm{d}t \\
&\quad - \int_0^T \Big(M\zeta(\rho)\hat{\psi}(\nabla_x \boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi \Big)_{\mathcal{O}} \,\mathrm{d}t \\
&\quad + \int_0^T (M\nabla_x \hat{\psi}, \nabla_x \varphi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}), \nabla_{\boldsymbol{q}} \varphi \Big)_{\mathcal{O}} \,\mathrm{d}t = 0, \\
&\quad \textit{for all } \varphi \in L^\infty(0,T; W^{1,\infty}(\mathcal{O})),
\end{aligned} \tag{4.28}$$

where the polymer number density ϱ is given by

$$\varrho(x,t) := \zeta(\rho) \int_D M(\boldsymbol{q})\hat{\psi}(x,\boldsymbol{q},t) \,\mathrm{d}\boldsymbol{q} \quad \textit{for a.e. } (x,t) \in \Omega \times (0,T), \tag{4.29}$$

the viscous part S_v of the Cauchy stress is given by

$$S_v(x,t) = \mu(\rho,\varrho) D(\boldsymbol{v}) \quad \textit{for a.e. } (x,t) \in \Omega \times (0,T), \tag{4.30}$$

and the extra-stress tensor S_e is given by

$$S_e(x,t) := k \sum_{j=1}^K \int_D M\zeta(\rho) \nabla_{\boldsymbol{q}^j} \hat{\psi}(x,\boldsymbol{q},t) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q} \quad \textit{for a.e. } (x,t) \in \Omega \times (0,T). \tag{4.31}$$

(iii) The following weak continuity properties hold:

$$\begin{aligned}
&t \mapsto \int_\Omega \rho(x,t)\boldsymbol{v}(x,t) \cdot \boldsymbol{u} \,\mathrm{d}x \in C([0,T]) \quad \textit{for any } \boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d), \\
&t \mapsto \int_{\mathcal{O}} M(\boldsymbol{q})\zeta(\rho(x,t))(t)\hat{\psi}(x,\boldsymbol{q},t)\, \phi(x,\boldsymbol{q}) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \in C([0,T]) \\
&\quad \textit{for any } \phi \in W^{1,\infty}(\mathcal{O}),
\end{aligned} \tag{4.32}$$

with $s > 2$, and the initial data are attained in the following sense:

$$\begin{aligned}
&\lim_{t \to 0+} \|\boldsymbol{v}(\cdot,t) - \boldsymbol{v}_0(\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} + \|\rho(\cdot,t) - \rho_0(\cdot)\|_{L^p(\Omega)} = 0, \ \textit{for all } p \in [1,\infty). \\
&\lim_{t \to 0+} \int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} = \int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q}.
\end{aligned} \tag{4.33}$$

Then we state our main result, which we shall prove in the next sections.

Theorem 4.2.2. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded open Lipschitz domain. Let $K \in \mathbb{N}$ be arbitrary and let $D^i \subset \mathbb{R}^d$, $i = 1,\dots,K$, be bounded open balls centred at the origin. Suppose that $\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. Assume that the map $\mathbb{A} : B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$ is linear and satisfies (4.14), the Maxwellian $M : D \to \mathbb{R}$*

satisfies (4.20), $\mu(\cdot,\cdot)$ and $\zeta(\cdot)$ satisfy (4.24), and the initial data $(\rho_0, \boldsymbol{v}_0, \psi_0)$ satisfy (4.21)–(4.23). Then, there exists a triple $(\rho, \boldsymbol{v}, \psi)$ which is a weak solution to the system (4.1)–(4.13) *in the sense of Definition 4.2.1. Moreover, for a.e. $t \in (0,T)$, the following energy inequality holds:*

$$
\begin{aligned}
&k \int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_{\Omega} \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x \\
&\quad + \int_0^t \int_{\Omega} \mu(\rho,\varrho)|D(\boldsymbol{v})|^2\,\mathrm{d}x\,\mathrm{d}s + 4kC_1 \int_0^t \int_{\mathcal{O}} M \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}}\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \qquad (4.34)\\
&\leq k \int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_{\Omega} \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x + \int_0^t (\rho\boldsymbol{f}, \boldsymbol{v})\,\mathrm{d}s,
\end{aligned}
$$

where $\mathcal{F}(s) = s\log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s\to 0+}\mathcal{F}(s) = 1$.

We will prove this result by constructing a sequence of approximations to the problem under consideration. We shall then pass to the limits in the various approximation parameters — first in the dimensions n and m of the Galerkin subspaces, and then in the parameter ℓ in the truncation process that we shall next introduce, to deduce the convergence of the sequence of approximations to a global-in-time weak solution of the problem.

4.3 Existence proof

4.3.1 The first level of approximation: truncation

To approximate our original Navier–Stokes–Fokker–Planck system, we begin by introducing a smooth nonnegative function $\Gamma \in C_0^\infty((-2,2))$, such that $\Gamma(s) = 1$ for all $s \in [-1,1]$ and for an arbitrary $\ell \in \mathbb{N}$ we define $\Gamma_\ell(s) := \Gamma(\frac{s}{\ell})$. The primitive function to Γ_ℓ is the truncation function defined by

$$T_\ell(s) := \int_0^s \Gamma_\ell(r)\,\mathrm{d}r.$$

We define the ℓ-th approximation of (4.1) as follows:

$$\frac{\partial \rho^\ell}{\partial t} + \mathrm{div}_x(\boldsymbol{v}^\ell \rho^\ell) = 0 \qquad \text{in } Q, \tag{4.35}$$

subject to the following initial condition:

$$\rho^\ell(\cdot,0) = \rho_0(\cdot) \qquad \text{in } \Omega. \tag{4.36}$$

We define the ℓ-th approximation of (4.2) and (4.3) as follows:

$$\begin{aligned} \frac{\partial(\rho^\ell \boldsymbol{v}^\ell)}{\partial t} + \operatorname{div}_x(\rho^\ell \boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell) - \operatorname{div}_x S_v^\ell + \nabla_x p^\ell = \rho^\ell \boldsymbol{f} + \operatorname{div}_x S_e^\ell \qquad & \text{in } Q, \\ \operatorname{div}_x \boldsymbol{v}^\ell = 0 \qquad & \text{in } Q, \end{aligned} \tag{4.37}$$

with initial and boundary conditions given by

$$\begin{aligned} \boldsymbol{v}^\ell(\cdot, 0) &= \boldsymbol{v}_0(\cdot) \qquad && \text{in } \Omega, \\ \boldsymbol{v}^\ell &= \boldsymbol{0} \qquad && \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{4.38}$$

The ℓ-th approximation ϱ^ℓ of the polymer number density ϱ is defined by

$$\varrho^\ell(x,t) := \zeta(\rho^\ell(x,t)) \int_D M(\boldsymbol{q}) \hat{\psi}^\ell(x, \boldsymbol{q}, t) \, \mathrm{d}\boldsymbol{q}. \tag{4.39}$$

The ℓ-th approximation S_v^ℓ of the Newtonian stress tensor S_v is given by

$$S_v^\ell(x,t) := \mu(\rho^\ell, \varrho^\ell) D(\boldsymbol{v}^\ell), \tag{4.40}$$

and the ℓ-th approximation S_e^ℓ of the polymeric extra stress tensor S_e is given by

$$S_e^\ell(x,t) := k \sum_{j=1}^K \int_D M(\boldsymbol{q}) \zeta(\rho^\ell(x,t)) \nabla_{\boldsymbol{q}^j} T_\ell(\hat{\psi}^\ell(x, \boldsymbol{q}, t)) \otimes \boldsymbol{q}^j \, \mathrm{d}\boldsymbol{q}, \tag{4.41}$$

where $\hat{\psi}^\ell$ is the solution of the initial-boundary-value problem (4.43)–(4.46) stated below.

By partial integration we have that

$$\begin{aligned} S_e^\ell(x,t) = -k \int_D K M(\boldsymbol{q}) \zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell(x, \boldsymbol{q}, t)) I \, \mathrm{d}\boldsymbol{q} \\ - k \sum_{j=1}^K \int_D \zeta(\rho^\ell) T_\ell(\hat{\psi}^\ell(x, \boldsymbol{q}, t)) \nabla_{\boldsymbol{q}^j} M(\boldsymbol{q}) \otimes \boldsymbol{q}^j \, \mathrm{d}\boldsymbol{q} \end{aligned} \tag{4.42}$$

on observing that the boundary term on ∂D vanishes since $M = 0$ on ∂D. We shall also modify the Fokker–Planck equation (4.10). We first set

$$\Lambda_\ell(s) := s \Gamma_\ell(s).$$

The ℓ-th approximation of (4.10) is given by

$$\begin{aligned} \frac{\partial(M \zeta(\rho^\ell) \hat{\psi}^\ell)}{\partial t} + \operatorname{div}_x(M \zeta(\rho^\ell) \hat{\psi}^\ell \boldsymbol{v}^\ell) + \operatorname{div}_{\boldsymbol{q}}(M \zeta(\rho^\ell) \Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell) \boldsymbol{q}) \\ - \Delta_x(M \hat{\psi}^\ell) - \operatorname{div}_{\boldsymbol{q}} \mathbb{A} \left(M \nabla_{\boldsymbol{q}} \hat{\psi}^\ell \right) = 0 \end{aligned} \tag{4.43}$$

on $\mathcal{O} \times (0,T)$, where $\mathcal{O} := \Omega \times D$, and is supplemented by the following boundary conditions:

$$\Big[\mathbb{A}^j(M\nabla_{\boldsymbol{q}}\hat{\psi}^\ell) - M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q}^j\Big] \cdot \boldsymbol{n}^j = 0 \qquad \text{on } \Omega \times \partial\bar{D}^j \times (0,T), \tag{4.44}$$

$$M\nabla_x\hat{\psi}^\ell \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T), \tag{4.45}$$

for all $j = 1, \dots, K$. We also truncate the initial condition for $\hat{\psi}^\ell$ as follows:

$$\hat{\psi}^\ell(x,\boldsymbol{q},0) = T_\ell(\hat{\psi}_0(x,\boldsymbol{q})) \qquad \text{for } (x,\boldsymbol{q}) \in \Omega \times D. \tag{4.46}$$

For simplicity, we shall omit the superscript ℓ temporarily in the following discussions; we shall reinstate it later and will then pass to the limit $\ell \to \infty$. We need to show first, however, that this approximating problem has a solution $(\rho^\ell, \boldsymbol{v}^\ell, \hat{\psi}^\ell, \varrho^\ell)$ for each $\ell \geq 1$; we shall do so by constructing a two-level Galerkin approximation to it and passing to the limits in the sequences of Galerkin approximations.

4.3.2 A two-stage Galerkin approximation

First, we define an approximate Maxwellian M^m by fixing a sequence of positive functions $(\overline{M}^m)_{m\in\mathbb{N}} \subset C_0^{0,1}(\overline{D})$ such that for each compact set $\varkappa \subset D$ the following holds:

$$\lim_{m\to\infty} \|\overline{M}^m - M\|_{C(\overline{D})\cap W_0^{1,1}(D)} + \|(\overline{M}^m)^{-1} - M^{-1}\|_{C(\varkappa)} = 0. \tag{4.47}$$

Then, the approximate Maxwellian M^m is defined by

$$M^m := \overline{M}^m + \frac{1}{m}, \quad \text{for } m = 1, 2, \dots.$$

Let $(\boldsymbol{f}^m)_{m=1}^\infty$ be a sequence of functions in $C([0,T]; L^2(\Omega;\mathbb{R}^d))$ converging to $\boldsymbol{f}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. Let further $(\rho_0^m)_{m=1}^\infty$ be a sequence of functions in $C^1(\overline{\Omega})$ such that $\rho_0^m \in [\rho_{\min}, \rho_{\max}]$, with $\rho_{\min} > 0$, which converges to ρ_0 in $L^1(\Omega)$; such a sequence can be constructed by extending ρ_0 from Ω to $\mathbb{R}^d$ by 0 and convolving the resulting function, still denoted by ρ_0, with θ^m, where $\theta^m(x) := m^d\theta(mx)$, $\theta \in C_0^\infty(\mathbb{R}^d)$, $\theta \geq 0$, and $\int_{\mathbb{R}^d} \theta(x)\,\mathrm{d}x = 1$, and observing that

$$\rho_0^m(x) - \rho_{\min} = \int_{\mathbb{R}^d} (\rho_0(x-y) - \rho_{\min})\,\theta^m(y)\,\mathrm{d}y \geq 0,$$

and

$$\rho_0^m(x) - \rho_{\max} = \int_{\mathbb{R}^d} (\rho_0(x-y) - \rho_{\max})\,\theta^m(y)\,\mathrm{d}y \leq 0.$$

Next, we shall introduce the Galerkin basis functions for the velocity field and the probability density function respectively. Similarly as in Chapter 3, by the

Hilbert–Schmidt Theorem, there exists a sequence $(\boldsymbol{w}_i)_{i=1}^{\infty}$ of eigenfunctions in $W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)$ whose linear span is dense in $L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$, such that the $\boldsymbol{w}_i$, $i = 1, 2, \ldots$, are orthogonal in the inner product of $W^{d+1,2}(\Omega;\mathbb{R}^d)$ and orthonormal in the inner product of $L^2(\Omega;\mathbb{R}^d)$. By Sobolev embedding, it follows that $\boldsymbol{w}_i \in C^1(\overline{\Omega};\mathbb{R}^d)$ for all $i = 1, 2, \ldots$. Similarly, for each $m \in \mathbb{N}$ we find a sequence $(\varphi_i^m)_{i=1}^{\infty}$ of eigenfunctions in $W^{(K+1)d+1,2}(\mathcal{O})$ that are orthogonal in $W^{1,2}_{M^m}(\mathcal{O})$ and orthonormal in $L^2_{M^m}(\mathcal{O})$. As $\mathcal{O} = \Omega \times D \subset \mathbb{R}^{(K+1)d}$, by Sobolev embedding we deduce that $\varphi_i^m \in C^1(\mathcal{O})$.

Finally, we fix $m, n \in \mathbb{N}$ and look for $(\rho^{m,n}, \boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$, where $\boldsymbol{v}^{m,n}$ and $\hat{\psi}^{m,n}$ are of the form

$$\boldsymbol{v}^{m,n}(x,t) \coloneqq \sum_{i=1}^{m} c_i^{m,n}(t)\boldsymbol{w}_i(x), \tag{4.48}$$

$$\hat{\psi}^{m,n}(x,\boldsymbol{q},t) \coloneqq \sum_{i=1}^{n} d_i^{m,n}(t)\varphi_i^m(x,\boldsymbol{q}), \tag{4.49}$$

that solve

$$(\partial_t \rho^{m,n}, \eta) - (\boldsymbol{v}^{m,n}\rho^{m,n}, \nabla_x \eta) = 0, \tag{4.50}$$

for all $\eta \in C^{0,1}(\overline{\Omega})$ and a.e. $t \in (0,T)$,

$$\begin{aligned} &(\partial_t(\rho^{m,n}\boldsymbol{v}^{m,n}), \boldsymbol{w}_i) - (\rho^{m,n}\boldsymbol{v}^{m,n} \otimes \boldsymbol{v}^{m,n}, \nabla_x \boldsymbol{w}_i) + (S_v^{m,n}, \nabla_x \boldsymbol{w}_i) \\ &\quad = -(S_e^{m,n}, \nabla_x \boldsymbol{w}_i) + (\rho^{m,n}\boldsymbol{f}^m, \boldsymbol{w}_i) \end{aligned} \tag{4.51}$$

for all $i = 1, \ldots, m$ and a.e. $t \in (0,T)$,

$$\begin{aligned} &\Big(\partial_t(M^m\zeta(\rho^{m,n})\hat{\psi}^{m,n}), \varphi_i^m\Big)_{\mathcal{O}} - \Big(M^m\zeta(\rho^{m,n})\boldsymbol{v}^{m,n}\hat{\psi}^{m,n}, \nabla_x\varphi_i^m\Big)_{\mathcal{O}} \\ &\quad - \Big(M\zeta(\rho^{m,n})\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x \boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}} + (M^m\nabla_x\hat{\psi}^{m,n}, \nabla_x\varphi_i^m)_{\mathcal{O}} \\ &\quad + \Big(M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}), \nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}} = 0, \end{aligned} \tag{4.52}$$

for all $i = 1, \ldots, n$ and a.e. $t \in (0,T)$. Here $S_v^{m,n}$ is defined by

$$S_v^{m,n} \coloneqq \mu(\rho^{m,n}, \varrho^{m,n})D(\boldsymbol{v}^{m,n}), \tag{4.53}$$

where $\varrho^{m,n}$ is defined by

$$\varrho^{m,n}(x,t) := \zeta(\rho^{m,n}(x,t)) \int_D M^m(\boldsymbol{q})\, [\hat{\psi}^{m,n}(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q} \quad \text{for a.e. } (x,t) \in Q, \tag{4.54}$$

where, for a real number s, $[s]_+ := \max(0,s)$, and the expression $S_e^{m,n}$ is defined as follows:

$$S_e^{m,n} := -k \int_D \Bigg[KM\zeta(\rho^{m,n}) T_\ell(\hat{\psi}^{m,n}) I \\ + \sum_{j=1}^{K} \zeta(\rho^{m,n}) T_\ell(\hat{\psi}^{m,n}) \nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \Bigg] \,\mathrm{d}\boldsymbol{q} \qquad \text{a.e. in } Q. \quad (4.55)$$

The initial data are given by

$$\begin{aligned} \rho^{m,n}(x,0) &= \rho_0^m(x) && \text{on } \Omega, \\ \boldsymbol{v}^{m,n}(x,0) &= \boldsymbol{v}_0^m(x) := \sum_{i=1}^{m} (\boldsymbol{v}_0, \boldsymbol{w}_i)\boldsymbol{w}_i(x) && \text{on } \Omega, \\ \hat{\psi}^{m,n}(x,\boldsymbol{q},0) &= \hat{\psi}_0^{m,n}(x,\boldsymbol{q}) := \sum_{i=1}^{n} (T_\ell(\hat{\psi}_0^m), \varphi_i^m)_{\mathcal{O}}\, \varphi_i^m(x,\boldsymbol{q}) && \text{on } \mathcal{O}, \end{aligned} \qquad (4.56)$$

where

$$\hat{\psi}_0^m := \hat{\psi}_0 \frac{M}{M^m}. \qquad (4.57)$$

Note that in the third term on the left-hand side of (4.52) and in (4.55) the Maxwellian M has, intentionally, not been replaced by the approximate Maxwellian M^m. Note also that we do not perform Galerkin discretizations of the density ρ and of the polymer number density ϱ in the above system (4.50)–(4.57), so at this point we have no guarantee that solutions to this system exist. Our aim in the next section is therefore to show that solutions to this partially Galerkin-discretized system do in fact exist. Having done so, we shall pass to the limit $n \to \infty$, then to the limit $m \to \infty$, and finally we shall let $\ell \to \infty$.

4.3.3 Existence of solutions to the partially Galerkin-discretized system

In this subsection, we will first show that solutions $(\rho^{m,n}, \boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$ exist for the system (4.50)–(4.57) using Schauder's Fixed-Point Theorem.

For any integers $m, n \geq 1$, let $V^m = \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ and $X^n = \operatorname{span}\{\varphi_1^m, \ldots, \varphi_n^m\}$ be the finite-dimensional Galerkin approximation spaces under consideration. Let $\boldsymbol{u}^{m,n} \in C([0,T]; V^m)$ and $\xi^{m,n} \in C([0,T]; X^n)$ be given. First let us consider the following transport problem:

$$\frac{\partial \rho^{m,n}}{\partial t} + \operatorname{div}_x(\rho^{m,n}\boldsymbol{u}^{m,n}) = 0, \qquad (4.58)$$

subject to the initial condition

$$\rho^{m,n}(0) = \rho_0^m. \tag{4.59}$$

Since $\boldsymbol{u}^{m,n} \in L^1(0,T;W^{1,1}(\Omega;\mathbb{R}^d))$ and $\operatorname{div}_x \boldsymbol{u}^{m,n} = 0$, one can show, by following the arguments in Chapter VI in [14], that there exists a unique solution $\rho^{m,n}$ to the transport problem which satisfies

$$0 < \rho_{\min} \leq \rho^{m,n} \leq \rho_{\max} \tag{4.60}$$

and, in addition, $\rho^{m,n} \in C([0,T];L^p(\Omega))$ for $1 \leq p < \infty$. We also define

$$\lambda^{m,n}(x,t) := \zeta(\rho^{m,n}(x,t)) \int_D M^m(\boldsymbol{q})\, [\xi^{m,n}(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q}.$$

Now that we have built $\rho^{m,n}$ and $\lambda^{m,n}$, we seek $\boldsymbol{v}^{m,n} \in C^1([0,T];V^m)$ and $\hat{\psi}^{m,n} \in C^1([0,T];X^n)$ satisfying

$$\begin{aligned}
&\int_\Omega \rho^{m,n} \left(\frac{\partial \boldsymbol{v}^{m,n}}{\partial t} + (\boldsymbol{u}^{m,n} \cdot \nabla_x)\boldsymbol{v}^{m,n} \right) \cdot \boldsymbol{w} \,\mathrm{d}x + \int_\Omega S_v^{m,n} : D(\boldsymbol{w}) \,\mathrm{d}x \\
&= \int_\Omega \rho^{m,n} \boldsymbol{f}^m \cdot \boldsymbol{w} \,\mathrm{d}x - \int_\Omega S_e^{m,n} : \nabla_x \boldsymbol{w} \,\mathrm{d}x,
\end{aligned} \tag{4.61}$$

$$\begin{aligned}
&\int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \frac{\partial \hat{\psi}^{m,n}}{\partial t} \varphi \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) (\nabla_x \hat{\psi}^{m,n} \cdot \boldsymbol{u}^{m,n}) \varphi \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + \int_{\mathcal{O}} M^m \nabla_x \hat{\psi}^{m,n} \cdot \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}) : \nabla_{\boldsymbol{q}} \varphi \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&= \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_\ell(\xi^{m,n}) (\nabla_x \boldsymbol{u}^{m,n}) \boldsymbol{q} : \nabla_{\boldsymbol{q}} \varphi \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q},
\end{aligned} \tag{4.62}$$

for any $\boldsymbol{w} \in V^m$ and $\varphi \in X^n$, and $S_v^{m,n}$ is given by

$$S_v^{m,n} = \mu(\rho^{m,n}, \lambda^{m,n}) D(\boldsymbol{v}^{m,n}), \tag{4.63}$$

and $S_e^{m,n}$ is given by

$$S_e^{m,n} = -k \int_D \left[K M \zeta(\rho^{m,n}) T_\ell(\xi^{m,n}) I + \sum_{j=1}^K \zeta(\rho^{m,n}) T_\ell(\xi^{m,n}) \nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q}. \tag{4.64}$$

First we note that since by hypothesis ζ is a continuous function of its argument and $\rho^{m,n}$ is bounded, $\zeta(\rho^{m,n})$ is bounded above. Thanks to the presence of the truncation function T_ℓ we deduce that

$$|S_e^{m,n}| \leq C(\ell, M, \zeta_{\max}). \tag{4.65}$$

Also note that since we have replaced the unknown advection vector field $\boldsymbol{v}^{m,n}$ in the convective term by the known vector field $\boldsymbol{u}^{m,n}$, (4.61), (4.62) is now a system of linear ordinary differential equations.

Claim 4.3.1. *Problem (4.61) has a unique solution* $\boldsymbol{v}^{m,n} \in C^1([0,T];V^m)$ *subject to the initial condition*

$$\boldsymbol{v}^{m,n}(\cdot,0) = \boldsymbol{v}_{0,m} := \sum_{i=1}^{m} (\boldsymbol{v}_0, \boldsymbol{w}_i)\boldsymbol{w}_i.$$

Proof. We shall seek the solution in the form $\boldsymbol{v}^{m,n} = \sum_{i=1}^{m} \alpha_i^{m,n}(t)\boldsymbol{w}_i$. Thereby the equation (4.61) can be re-written as

$$M(t)\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}(t) = A(t)\boldsymbol{\alpha}(t) + B(t), \qquad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \tag{4.66}$$

where $\boldsymbol{\alpha}(t) = (\alpha_1^{m,n}(t), \dots, \alpha_m^{m,n}(t))^{\mathrm{T}} \in \mathbb{R}^m$ is the unknown m-component coefficient vector of $\boldsymbol{v}^{m,n}$ at time t, with associated initial datum $\boldsymbol{\alpha}_0 = ((\boldsymbol{v}_0, \boldsymbol{w}_0), \dots (\boldsymbol{v}_0, \boldsymbol{w}_m))^{\mathrm{T}} \in \mathbb{R}^m$. In the above equation

$$\begin{aligned}
(M(t))_{ij} &:= \int_\Omega \rho^{m,n} \boldsymbol{w}_i \cdot \boldsymbol{w}_j \,\mathrm{d}x, \\
(A(t))_{ij} &:= -\left(\int_\Omega \rho^{m,n} (\boldsymbol{u}^{m,n} \cdot \nabla_x)\boldsymbol{w}_i \cdot \boldsymbol{w}_j \,\mathrm{d}x + \int_\Omega \mu(\rho^{m,n}, \lambda^{m,n}) D(\boldsymbol{w}_i) : D(\boldsymbol{w}_j)\,\mathrm{d}x \right), \\
(B(t))_j &:= \int_\Omega \rho^{m,n} \boldsymbol{f}^m \cdot \boldsymbol{w}_j \,\mathrm{d}x - \int_\Omega S_e^{m,n} : \nabla_x \boldsymbol{w}_j \,\mathrm{d}x,
\end{aligned}$$

where $i, j \in \{1, \dots, m\}$ and $S_e^{m,n}$ is a function of $\xi^{m,n}$, as given in (4.64). Note that $M(t)$, $A(t)$ and $B(t)$ are continuous with respect to t. Since $M(t)$ is the Gram matrix associated with the basis $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ with respect to the inner product (note the two-sided bound (4.60) on $\rho^{m,n}$) defined by

$$[\boldsymbol{v}, \boldsymbol{w}]_{\rho^{m,n}(t)} := \int_\Omega \rho^{m,n}(t,x)\boldsymbol{v}(x) \cdot \boldsymbol{w}(x)\,\mathrm{d}x,$$

it follows that $M(t)$ is invertible for all $t \in [0,T]$. By the Cauchy–Lipschitz Theorem the initial-value problem (4.66) has a unique global solution. As a consequence, (4.61) has a unique solution $\boldsymbol{v}^{m,n} \in C^1([0,T];V^m)$ subject to the initial condition $\boldsymbol{v}^{m,n}(0) = \boldsymbol{v}_{0,m}$. ☐

Claim 4.3.2. *Problem (4.62) has a unique solution* $\hat{\psi}^{m,n} \in C^1([0,T];X^n)$ *subject to the initial condition*

$$\hat{\psi}^{m,n}(\cdot,\cdot,0) = \hat{\psi}_{0,n} := \sum_{i=1}^{n} (T_\ell(\hat{\psi}_0^m), \varphi_i^m)_{\mathcal{O}}\, \varphi_i^m.$$

Proof. Note that (4.62) is a system of linear ordinary differential equations. Thus, by writing $\hat{\psi}^{m,n} = \sum_{i=1}^{n} \beta_i^{m,n}(t)\varphi_i^m$, we deduce that (4.62) has a unique solution $\hat{\psi}^{m,n} \in C^1([0,T];X^n)$ subject to the initial condition $\hat{\psi}^{m,n}(0) = \hat{\psi}_{0,n}$. The detailed argument is similar to the proof of Claim 4.3.1. ☐

Let $\|\cdot\|_{V^m}$ and $\|\cdot\|_{X^n}$ be norms on V^m and X^n, respectively. Since V^m and X^n are finite-dimensional linear spaces, and all norms on finite-dimensional linear spaces are equivalent, the precise choice of these norms is of no relevance in the discussion that follows.

Claim 4.3.3. *Let*

$$\mathcal{K} := \left\{ \boldsymbol{v} \in C^1([0,T];V^m);\ \sup_{t\in[0,T]} \|\boldsymbol{v}(t)\|_{L^2(\Omega;\mathbb{R}^d)} \le C, \quad \sup_{t\in[0,T]} \left\| \frac{\partial \boldsymbol{v}}{\partial t} \right\|_{V^m} \le C(m,n) \right\}$$

be a subset of $C([0,T];V^m)$ *and*

$$\mathcal{S} := \left\{ \hat{\psi} \in C^1([0,T];X^n);\ \sup_{t\in[0,T]} \|\hat{\psi}(t)\|_{L^2_{M^m}(\mathcal{O})} \le C, \quad \sup_{t\in[0,T]} \left\| \frac{\partial \hat{\psi}}{\partial t} \right\|_{X^n} \le C(m,n) \right\}$$

be a subset of $C([0,T];X^n)$. *Let* $\Theta : \overline{\mathcal{K}\times\mathcal{S}} \to \overline{\mathcal{K}\times\mathcal{S}}$ *denote the map that takes the pair* $(\boldsymbol{u}^{m,n}, \xi^{m,n})$ *to* $(\boldsymbol{v}^{m,n}, \hat{\psi}^{m,n}) =: \Theta(\boldsymbol{u}^{m,n}, \xi^{m,n})$ *via the procedure (4.61) and (4.62); then, for* C *and* $C(m,n)$ *sufficiently large, the mapping* Θ *has a fixed point in* $\overline{\mathcal{K}\times\mathcal{S}}$.

Proof. To show that Θ has a fixed point we apply Schauder's Fixed-Point Theorem. First we note that $\overline{\mathcal{K}\times\mathcal{S}}$ is obviously nonempty and it is easy to show that $\overline{\mathcal{K}\times\mathcal{S}}$ is convex. Then, it remains to show that: (i) Θ maps $\mathcal{K}\times\mathcal{S}$ into itself; (ii) $\mathcal{K}\times\mathcal{S}$ is relatively compact in $C([0,T];V^m)\times C([0,T];X^n)$; then $\overline{\mathcal{K}\times\mathcal{S}}$ is compact in $C([0,T];V^m)\times C([0,T];X^n)$; (iii) $\Theta : \overline{\mathcal{K}\times\mathcal{S}} \to \overline{\mathcal{K}\times\mathcal{S}}$ is continuous.

We start by showing suitable energy estimates. Since $\boldsymbol{v}^{m,n} \in C^1([0,T];V^m)$, we can take the test function $\boldsymbol{w} = \boldsymbol{v}^{m,n}(s)$ in (4.61) and integrate with respect to time over $(0,t)$, where $t \in (0,T]$. Then we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho^{m,n}(t)|\boldsymbol{v}^{m,n}(t)|^2 \,\mathrm{d}x - \frac{1}{2}\int_0^t \int_\Omega \frac{\partial \rho^{m,n}}{\partial t} |\boldsymbol{v}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_\Omega \rho^{m,n}\boldsymbol{u}^{m,n} \cdot \nabla_x(|\boldsymbol{v}^{m,n}|^2) \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \mu(\rho^{m,n},\lambda^{m,n})|D(\boldsymbol{v}^{m,n})|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{2}\int_\Omega \rho_0^m |\boldsymbol{v}_{0,m}|^2 \,\mathrm{d}x + \int_0^t \int_\Omega \rho^{m,n} \boldsymbol{f}^m \cdot \boldsymbol{v}^{m,n} \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_\Omega S_e^{m,n} : \nabla_x \boldsymbol{v}^{m,n} \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{4.67}$$

Since $\boldsymbol{v}^{m,n} \in C^1([0,T];V^m)$, we can test the transport equation (4.58) with $|\boldsymbol{v}^{m,n}|^2$ and we see that the second and third term in the above identity add up to 0. From the bounds (4.60) and (4.65), the assumption (4.24), Young's inequality and Korn's inequality, we deduce from (4.67) that

$$\begin{aligned} &\int_\Omega \rho^{m,n}(t)|\boldsymbol{v}^{m,n}(t)|^2\,\mathrm{d}x + \mu_{\min}\int_0^t\int_\Omega |D(\boldsymbol{v}^{m,n})|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \rho_{\max}\int_\Omega |\boldsymbol{v}_{0,m}|^2\,\mathrm{d}x + \rho_{\max}\int_0^t\int_\Omega |\boldsymbol{f}^m|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \rho_{\max}\int_0^t\int_\Omega |\boldsymbol{v}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \frac{1}{\mu_{\min}c_0}\int_0^t\int_\Omega |S_e^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le C + C\int_0^t\int_\Omega |\boldsymbol{v}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{4.68}$$

where C is a constant depending on the data $\ell, \boldsymbol{f}, \boldsymbol{v}_0, \mu_{\min}, M, \zeta_{\max}$. On noting that $\rho^{m,n} \ge \rho_{\min} > 0$, we have in particular that

$$\int_\Omega |\boldsymbol{v}^{m,n}(t)|^2\,\mathrm{d}x \le C + C\int_0^t\int_\Omega |\boldsymbol{v}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}s.$$

By Gronwall's inequality we have that

$$\sup_{t\in[0,T]} \|\boldsymbol{v}^{m,n}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \mu_{\min}\int_0^T\int_\Omega |D(\boldsymbol{v}^{m,n})|^2\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{4.69}$$

Similarly, by taking the test function $\varphi = \hat{\psi}^{m,n}(s)$ in (4.62) and integrating with respect to time over $(0,t)$, where $t \in (0,T]$, we get

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})\left(\frac{\partial(\hat{\psi}^{m,n})^2}{\partial t} + \boldsymbol{u}^{m,n}\cdot\nabla_x(\hat{\psi}^{m,n})^2\right)\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathcal{O}} M^m|\nabla_x\hat{\psi}^{m,n}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathcal{O}} M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}) : \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\ &= \int_0^t\int_{\mathcal{O}} M\zeta(\rho^{m,n})\Lambda_\ell(\xi^{m,n})(\nabla_x\boldsymbol{u}^{m,n})\boldsymbol{q} : \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s. \end{aligned} \tag{4.70}$$

Since ζ is a C^1 function of the density, by the renormalization property we have that

$$\begin{aligned} \int_0^t\int_{\mathcal{O}} \zeta(\rho^{m,n})\left(\frac{\partial\phi}{\partial t} + \boldsymbol{u}^{m,n}\cdot\nabla_x\phi\right)\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s - \int_{\mathcal{O}}\zeta(\rho^{m,n}(t))\phi(t)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ + \int_{\mathcal{O}}\zeta(\rho_0^m)\phi(0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 0 \end{aligned} \tag{4.71}$$

for any $\phi \in C^{0,1}([0,T]\times\overline{\mathcal{O}})$. As $\hat{\phi}^{m,n} \in C^1([0,T]; W^{(K+1)d+1,2}(\mathcal{O})) \hookrightarrow C^1([0,T]; C^1(\overline{\mathcal{O}}))$, it follows that $|\hat{\psi}^{m,n}|^2 \in C^1([0,T]; C^1(\overline{\mathcal{O}}))$. Thanks to the assumed smoothness of M

(and thereby also of M^m), we can take the test function $\phi = M^m|\hat{\psi}^{m,n}|^2$ in (4.71) to get that

$$\int_0^t \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \left(\frac{\partial(\hat{\psi}^{m,n})^2}{\partial t} + \boldsymbol{u}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n})^2 \right) \mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s$$
$$= \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}(t)) |\hat{\psi}^{m,n}(t)|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} - \int_{\mathcal{O}} M^m \zeta(\rho_0^m) |\hat{\psi}_{0,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q}. \quad (4.72)$$

By multiplying (4.72) by $1/2$ and subtracting the resulting equation from (4.70) we obtain

$$\begin{aligned}
&\frac{1}{2} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}(t)) |\hat{\psi}^{m,n}(t)|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_0^t \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}) : \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&= \frac{1}{2} \int_{\mathcal{O}} M^m \zeta(\rho_0^m) |\hat{\psi}_{0,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + \int_0^t \int_{\mathcal{O}} M \zeta(\rho^{m,n}) \Lambda_\ell(\xi^{m,n}) (\nabla_x \boldsymbol{u}^{m,n}) \boldsymbol{q} : \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s.
\end{aligned}$$

On noting (4.14), (4.24) and the presence of the truncation function $\Lambda_\ell(\cdot)$ we can apply Hölder's inequality and Young's inequality to get that

$$\begin{aligned}
&\zeta_{\min} \int_{\mathcal{O}} M^m |\hat{\psi}^{m,n}(t)|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + 2 \int_0^t \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&\quad + C_1 \int_0^t \int_{\mathcal{O}} M^m |\nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&\leq \zeta_{\max} \int_{\mathcal{O}} M^m |\hat{\psi}_{0,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + \frac{1}{C_1} \int_0^t \int_{\mathcal{O}} M |\zeta(\rho^{m,n})|^2 |\Lambda_\ell(\xi^{m,n})|^2 |\nabla_x \boldsymbol{u}^{m,n}|^2 |\boldsymbol{q}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}s \\
&\leq C(\ell, M, \zeta_{\max}, \hat{\psi}_0).
\end{aligned} \quad (4.73)$$

From the estimates (4.69) and (4.73) we deduce that

$$\sup_{t \in [0,T]} \|\boldsymbol{v}^{m,n}(t)\|_{L^2(\Omega;\mathbb{R}^d)} \leq C, \quad (4.74)$$

$$\sup_{t \in [0,T]} \|\hat{\psi}^{m,n}(t)\|_{L^2_{M^m}(\mathcal{O})} \leq C. \quad (4.75)$$

Next, we shall derive estimates for the norms of $\partial_t \boldsymbol{v}^{m,n}$ and $\partial_t \hat{\psi}^{m,n}$. We take the test function $\boldsymbol{w} = \partial_t \boldsymbol{v}^{m,n}$ in (4.61). Since V^m is finite-dimensional, all norms on V^m are equivalent (with constants depending on m). We obtain, using Hölder's inequality, that

$$\begin{aligned}
&\int_\Omega \rho^{m,n}\left|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right|^2 \mathrm{d}x \\
&\quad\leq C\|\boldsymbol{f}^m\|_{L^2(\Omega;\mathbb{R}^d)}\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{L^2(\Omega;\mathbb{R}^d)} + C\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{W^{1,2}(\Omega;\mathbb{R}^d)} \\
&\qquad + C\|\boldsymbol{u}^{m,n}\|_{L^3(\Omega;\mathbb{R}^d)}\|\nabla_x \boldsymbol{v}^{m,n}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{L^6(\Omega;\mathbb{R}^d)} \\
&\quad\leq C(m,n)\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{L^2(\Omega;\mathbb{R}^d)}.
\end{aligned} \tag{4.76}$$

On noting that $\rho^{m,n} \geq \rho_{\min}$ it follows that

$$\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{L^2(\Omega;\mathbb{R}^d)} \leq \frac{C(m,n)}{\rho_{\min}}.$$

Since all norms on V^m are equivalent, we have that

$$\sup_{t\in[0,T]}\left\|\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}\right\|_{V^m} \leq C(m,n). \tag{4.77}$$

Now let us take the test function $\varphi = \partial_t\hat{\psi}^{m,n}$ in (4.62). Since X^n is finite-dimensional, all norms on X^n are equivalent (with constants depending on n). We obtain, using Hölder's inequality, that

$$\begin{aligned}
&\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})\left|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&\quad\leq C(M,\zeta_{\max})\|\nabla_x\hat{\psi}^{m,n}\|_{L^2(\mathcal{O};\mathbb{R}^d)}\|\boldsymbol{v}^{m,n}\|_{L^3(\Omega;\mathbb{R}^d)}\left\|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right\|_{L^6(\mathcal{O})} \\
&\qquad + C(M)\|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}\left\|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right\|_{W^{1,2}(\mathcal{O})} \\
&\qquad + C(\ell,M,\zeta_{\max})\|\nabla_x\boldsymbol{u}^{m,n}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\left\|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right\|_{W^{1,2}(\mathcal{O})} \\
&\quad\leq C(n)\left\|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right\|_{L^2(\mathcal{O})}.
\end{aligned} \tag{4.78}$$

Since $\zeta(\rho^{m,n}) \geq \zeta_{\min}$ and $M^m \geq 1/m$, we have

$$\left\|\frac{\partial\hat{\psi}^{m,n}}{\partial t}\right\|_{L^2(\mathcal{O})} \leq \frac{mC(n)}{\zeta_{\min}}.$$

Since all norms on X^n are equivalent, we have that

$$\sup_{t\in[0,T]} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{X^n} \leq C(m,n). \tag{4.79}$$

From the estimates (4.77) and (4.79) we obtain that

$$\sup_{t\in[0,T]} \left\| \frac{\partial \boldsymbol{v}^{m,n}}{\partial t} \right\|_{V^m} \leq C(m,n), \tag{4.80}$$

$$\sup_{t\in[0,T]} \left\| \frac{\partial \hat{\psi}^{m,n}}{\partial t} \right\|_{X^n} \leq C(m,n). \tag{4.81}$$

From (4.74), (4.75), (4.80) and (4.81) we see that if we take $(\boldsymbol{u}^{m,n}, \xi^{m,n}) \in \mathcal{K} \times \mathcal{S}$, then $\Theta(\boldsymbol{u}^{m,n}, \xi^{m,n}) = (\boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$ still belongs to $\mathcal{K} \times \mathcal{S}$. Therefore, we have shown that (i) Θ maps $\mathcal{K} \times \mathcal{S}$ into itself.

(ii) From (4.74) we obtain that, for any $t \in [0,T]$, $\|\boldsymbol{v}^{m,n}(t)\|_{V^m} \leq C(m)$, thanks to the fact that all norms are equivalent in finite-dimensional spaces. Then, the subset $\mathcal{K}(t) := \{\boldsymbol{v}(t); \boldsymbol{v} \in \mathcal{K}\}$ is relatively compact in V^m. Also, since $\boldsymbol{v} \in C^1([0,T]; V^m)$ and $\boldsymbol{v}$ satisfies (4.80), we deduce that

$$\|\boldsymbol{v}(t_1) - \boldsymbol{v}(t_2)\|_{V^m} \leq C(m,n)|t_1 - t_2|.$$

Then, for all $t_1 \in [0,T]$ and for all $\varepsilon > 0$, there exists a $\delta = \varepsilon/2C(m,n) > 0$ such that

$$\|\boldsymbol{v}(t_1) - \boldsymbol{v}(t_2)\|_{V^m} < \varepsilon$$

for all $t_2 \in [0,T]$ with $|t_1 - t_2| < \delta$ and for all $\boldsymbol{v} \in \mathcal{K}$. Therefore, by the Arzelà–Ascoli Theorem, we deduce that $\mathcal{K}$ is relatively compact in $C([0,T]; V^m)$.

Since $M^m \geq 1/m$, we have from (4.75) that, for any $t \in [0,T]$, $\|\hat{\psi}^{m,n}(t)\|_{L^2(\mathcal{O})} \leq C(m)$. This then gives $\|\hat{\psi}^{m,n}(t)\|_{X^n} \leq C(m,n)$, since all norms are equivalent in finite-dimensional spaces. Thus the subset $\mathcal{S}(t) := \{\hat{\psi}(t) : \hat{\psi} \in \mathcal{S}\}$ is relatively compact in X^n. Also, since $\hat{\psi} \in C^1([0,T]; X^n)$ and $\hat{\psi}$ satisfies (4.81), we deduce that

$$\|\hat{\psi}(t_3) - \hat{\psi}(t_4)\|_{X^n} \leq C(m,n)|t_3 - t_4|.$$

Then, for all $t_3 \in [0,T]$ and for all $\varepsilon' > 0$, there exists a $\delta' = \varepsilon'/2C(m,n) > 0$ such that

$$\|\hat{\psi}(t_3) - \hat{\psi}(t_4)\|_{X^n} < \varepsilon'$$

for all $t_4 \in [0,T]$ with $|t_3 - t_4| < \delta'$ and for all $\hat{\psi} \in \mathcal{S}$. Therefore, by the Arzelá–Ascoli Theorem, $\mathcal{S}$ is relatively compact in $C([0,T]; X^n)$. Hence, we have shown that $\mathcal{K} \times \mathcal{S}$ is relatively compact in $C([0,T]; V^m) \times C([0,T]; X^n)$.

(iii) Next we will show that Θ is continuous; to this end it suffices to show that Θ is sequentially continuous. Let $(\boldsymbol{u}^{m,n}_{(r)})_{r=1}^{\infty}$ be a sequence in $C([0,T];V^m)$ which converges to some $\boldsymbol{u}^{m,n}$ in $C([0,T];V^m)$ as $r \to \infty$ and let $(\xi^{m,n}_{(r)})_{r=1}^{\infty}$ be a sequence in $C([0,T];X^n)$ which converges to some $\xi^{m,n}$ in $C([0,T];X^n)$ as $r \to \infty$. To show that Θ is sequentially continuous we shall show that $(\boldsymbol{v}^{m,n}_{(r)}, \hat{\psi}^{m,n}_{(r)}) = \Theta(\boldsymbol{u}^{m,n}_{(r)}, \xi^{m,n}_{(r)})$ converges to $\Theta(\boldsymbol{u}^{m,n}, \xi^{m,n})$ as $r \to \infty$ and $\Theta(\boldsymbol{u}^{m,n}, \xi^{m,n}) = (\boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$.

For any r, let $\rho^{m,n}_{(r)}$ be the unique solution to the transport equation (4.58) corresponding to the velocity field $\boldsymbol{u}^{m,n}_{(r)}$. Let $\rho^{m,n}$ be the unique solution to (4.58) corresponding to the velocity field $\boldsymbol{u}^{m,n}$. All of these transport problems are associated with the same initial datum $\rho_0^m \in W^{d+1,2}(\Omega)$ satisfying (4.21) (and converging to ρ_0 in $L^1(\Omega)$ as $m \to \infty$; here though, $m \geq 1$ and $n \geq 1$ are fixed, and we are interested, instead, in the limit $r \to \infty$).

We further define $\lambda^{m,n}_{(r)}$ and $\lambda^{m,n}$ by

$$\lambda^{m,n}_{(r)}(x,t) := \zeta(\rho^{m,n}_{(r)}(x,t)) \int_D M^m(\boldsymbol{q})\, [\xi^{m,n}_{(r)}(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q},$$

$$\lambda^{m,n}(x,t) := \zeta(\rho^{m,n}(x,t)) \int_D M^m(\boldsymbol{q})\, [\xi^{m,n}(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q}.$$

By Theorem VI.1.9 in [14] we deduce that, as $r \to \infty$,

$$\rho^{m,n}_{(r)} \longrightarrow \rho^{m,n} \qquad \text{strongly in } C([0,T];L^p(\Omega)) \text{ for any } p \in [1,\infty). \tag{4.82}$$

From the assumptions (4.24) on ζ we then deduce that, as $r \to \infty$,

$$\zeta(\rho^{m,n}_{(r)}) \longrightarrow \zeta(\rho^{m,n}) \qquad \text{strongly in } C([0,T];L^p(\Omega)) \text{ for any } p \in [1,\infty). \tag{4.83}$$

Hence, and thanks to the global Lipschitz continuity of the mapping $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_{\geq 0}$, we have that

$$\lambda^{m,n}_{(r)} \longrightarrow \lambda^{m,n} \qquad \text{strongly in } C([0,T];L^p(\Omega)) \text{ for any } p \in [1,\infty). \tag{4.84}$$

From the assumptions (4.24) on μ we then deduce that, as $r \to \infty$,

$$\mu(\rho^{m,n}_{(r)}, \lambda^{m,n}_{(r)}) \longrightarrow \mu(\rho^{m,n}, \lambda^{m,n}) \qquad \text{strongly in } L^\infty(0,T;L^p(\Omega)) \text{ for any } p \in [1,\infty). \tag{4.85}$$

For any r, we take the test function in (4.61) and (4.62) to be $\boldsymbol{v}_{(r)}^{m,n}$ and $\hat{\psi}_{(r)}^{m,n}$ respectively and perform a similar procedure as in (4.67)–(4.73) to deduce that

$$\sup_r \|\boldsymbol{v}_{(r)}^{m,n}\|_{C([0,T];L^2(\Omega;\mathbb{R}^d))} \leq C, \tag{4.86}$$

$$\sup_r \|\nabla_x \boldsymbol{v}_{(r)}^{m,n}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \leq C, \tag{4.87}$$

$$\sup_r \|\hat{\psi}_{(r)}^{m,n}\|_{C([0,T];L^2_{M^m}(\mathcal{O}))} \leq C, \tag{4.88}$$

$$\sup_r \|\nabla_{x,\boldsymbol{q}} \hat{\psi}_{(r)}^{m,n}\|_{L^2(0,T;L^2_{M^m}(\mathcal{O};\mathbb{R}^{d(K+1)}))} \leq C. \tag{4.89}$$

Taking the test function in (4.61) to be $\partial_t \boldsymbol{v}_{(r)}^{m,n}$ we perform a similar argument as in (4.76), (4.77) to deduce that

$$\sup_r \left\| \frac{\partial \boldsymbol{v}_{(r)}^{m,n}}{\partial t} \right\|_{C([0,T];V^m)} \leq C(m,n).$$

Similarly, taking the test function in (4.62) to be $\partial_t \hat{\psi}_{(r)}^{m,n}$, as in (4.78), (4.79), we get

$$\sup_r \left\| \frac{\partial \hat{\psi}_{(r)}^{m,n}}{\partial t} \right\|_{C([0,T];X^n)} < C(m,n).$$

Now since $\boldsymbol{v}_{(r)}^{m,n} \in \mathcal{K}$ for all r and $\mathcal{K}$ is relatively compact in $C([0,T];V^m)$, there exists a subsequence (not relabelled) such that, as $r \to \infty$,

$$\boldsymbol{v}_{(r)}^{m,n} \longrightarrow \boldsymbol{v}' \qquad \text{strongly in } C([0,T];V^m). \tag{4.90}$$

From the bounds (4.86) and (4.87) we deduce the following weak convergence:

$$\boldsymbol{v}_{(r)}^{m,n} \rightharpoonup \boldsymbol{v}' \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d)). \tag{4.91}$$

With the convergence results (4.82), (4.84), (4.85), (4.83), (4.90) and (4.91) we can pass to the limit as $r \to \infty$ in the equation satisfied by $\boldsymbol{v}_{(r)}^{m,n}$ and the limit $\boldsymbol{v}'$ satisfies (4.61). However, with $\rho^{m,n}$, $\boldsymbol{u}^{m,n}$ and $\xi^{m,n}$ given, (4.61) can be solved with a unique solution $\boldsymbol{v}^{m,n}$ as proved in Claim 4.3.1, which then implies that $\boldsymbol{v}' \equiv \boldsymbol{v}^{m,n}$.

Meanwhile, since $\hat{\psi}_{(r)}^{m,n} \in \mathcal{S}$ for all r and $\mathcal{S}$ is relatively compact in $C([0,T];X^n)$, there exists a subsequence (not relabelled) such that, as $r \to \infty$,

$$\hat{\psi}_{(r)}^{m,n} \longrightarrow \hat{\psi}' \qquad \text{strongly in } C([0,T];X^n). \tag{4.92}$$

From the bounds (4.88) and (4.89) we deduce the following weak convergence:

$$\sqrt{M^m}\nabla_{x,\boldsymbol{q}} \hat{\psi}_{(r)}^{m,n} \rightharpoonup \sqrt{M^m}\nabla_{x,\boldsymbol{q}} \hat{\psi}' \qquad \text{weakly in } L^2(0,T;L^2(\mathcal{O};\mathbb{R}^{d(K+1)})). \tag{4.93}$$

With the convergence results (4.83), (4.92) and (4.93) we can pass to the limit as $r \to \infty$ in the equation satisfied by $\hat{\psi}^{m,n}_{(r)}$ and the limit $\hat{\psi}'$ satisfies (4.62). However, with $\rho^{m,n}$, $\boldsymbol{u}^{m,n}$ and $\xi^{m,n}$ given, (4.62) can be solved uniquely with $\hat{\psi}^{m,n}$ as proved in Claim 4.3.2, which implies that $\hat{\psi}' \equiv \hat{\psi}^{m,n}$. Hence, we have shown that the mapping Θ is continuous.

Finally, by Schauder's Fixed-Point Theorem, we deduce that $\Theta : \overline{\mathcal{K} \times \mathcal{S}} \to \overline{\mathcal{K} \times \mathcal{S}}$ has a fixed point $(\boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$ in $\overline{\mathcal{K} \times \mathcal{S}}$. Thus, also,

$$\varrho^{m,n}(x,t) = \zeta(\rho^{m,n}(x,t)) \int_D M^m(\boldsymbol{q})\, [\hat{\psi}^{m,n}(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q},$$

and

$$S_v^{m,n} = \mu(\rho^{m,n}, \varrho^{m,n}) D(\boldsymbol{v}^{m,n}),$$

and

$$\begin{aligned} S_e^{m,n}(x,t) = -k \int_D \Bigg[& KM(\boldsymbol{q})\zeta(\rho^{m,n}(x,t)) T_\ell(\hat{\psi}^{m,n}(x,\boldsymbol{q},t)) I \\ & + \sum_{j=1}^{K} \zeta(\rho^{m,n}(x,t)) T_\ell(\hat{\psi}^{m,n}(x,\boldsymbol{q},t) \nabla_{\boldsymbol{q}^j} M(\boldsymbol{q}) \otimes \boldsymbol{q}^j \Bigg] \,\mathrm{d}\boldsymbol{q}. \end{aligned}$$

That completes the proof of the existence of a solution $(\rho^{m,n}, \boldsymbol{v}^{m,n}, \hat{\psi}^{m,n})$ to the partially Galerkin discretized system (4.50)–(4.57). $\square$

Remark 4.3.4. *Although it seems tempting to add a parabolic regularization term* $-\alpha\Delta\rho^{m,n}$ *in the transport equation and then take the limit as* $\alpha \to 0+$ *as in [9], there is a problem with the presence of the density-dependent drag coefficient* $\zeta(\rho^{m,n})$*. The problem lies in the lack of knowledge showing the evolution of* $\zeta(\rho^{m,n})$ *under the parabolic regularization. Even if we assume that*

$$\frac{\partial \zeta(\rho^{m,n})}{\partial t} + \mathrm{div}_x(\zeta(\rho^{m,n})\boldsymbol{v}) - \alpha\Delta_x\zeta(\rho^{m,n}) = 0$$

holds for the moment, the presence of the term $-\alpha\Delta_x\zeta(\rho^{m,n})$ *makes it difficult to derive a reasonable energy estimates for* $\hat{\psi}$*. Therefore, we have taken the approach using Schauder's Fixed-Point Theorem motivated by [14] and [63].*

In the following sections we shall derive uniform bounds independent of the parameters n, m and ℓ, and then use those to successively pass to the limits with $n, m, \ell \to \infty$.

4.3.4 Passage to the limit with n

The goal of this section is to pass to the limit as $n \to \infty$. To achieve this we shall first derive uniform estimates independent of n. We note that from the definition (4.55) of $S_e^{m,n}$ we deduce that

$$|S_e^{m,n}| \leq C(\ell, \zeta_{\max}, M), \tag{4.94}$$

which implies that there exists a subsequence (not relabelled) such that

$$S_e^{m,n} \rightharpoonup S_e^m \qquad \text{weak* in } L^\infty(Q; \mathbb{R}^{d\times d}). \tag{4.95}$$

By Theorem VI.1.6 in [14] we find that

$$\sup_{t\in(0,T)} \|\rho^{m,n}(t)\|_{L^\infty(\Omega)} \leq \|\rho_0^m\|_{L^\infty(\Omega)} \leq \rho_{\max}.$$

Moreover, Proposition VI.1.8 in [14] gives

$$\rho^{m,n} \geq \rho_{\min}. \tag{4.96}$$

We deduce the existence of subsequences (not relabelled) such that, as $n \to \infty$,

$$\rho^{m,n} \rightharpoonup \rho^m \qquad \text{weak* in } L^\infty(Q), \tag{4.97}$$

$$(\rho^{m,n})^2 \rightharpoonup \vartheta^m \qquad \text{weak* in } L^\infty(Q). \tag{4.98}$$

By the renormalization property, for any $\beta \in C^1(\mathbb{R})$, $\beta(\rho^{m,n})$ satisfies

$$\frac{\partial \beta(\rho^{m,n})}{\partial t} + \operatorname{div}_x(\beta(\rho^{m,n})\boldsymbol{v}^{m,n}) = 0 \tag{4.99}$$

in the distributional sense.

We multiply the i-th equation in (4.51) by $c_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, m$ to deduce the following identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{m,n}|\boldsymbol{v}^{m,n}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{\partial \rho^{m,n}}{\partial t}|\boldsymbol{v}^{m,n}|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega \rho^{m,n}\boldsymbol{v}^{m,n}\cdot\nabla_x(|\boldsymbol{v}^{m,n}|^2)\,\mathrm{d}x$$
$$+ \int_\Omega \mu(\rho^{m,n}, \varrho^{m,n})|D(\boldsymbol{v}^{m,n})|^2 \,\mathrm{d}x = -\int_\Omega S_e^{m,n} : D(\boldsymbol{v}^{m,n})\,\mathrm{d}x + (\rho^{m,n}\boldsymbol{f}^m, \boldsymbol{v}^{m,n}).$$

The second term and the third term in the above equality add up to 0, since we can take the test function $\eta = |\boldsymbol{v}^{m,n}|^2$ in (4.50). Using (4.24), (4.94), Young's inequality and Gronwall's inequality, we find that

$$\sup_{t\in(0,T)} \|\sqrt{\rho^{m,n}(t)}\boldsymbol{v}^{m,n}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \mu_{\min}\int_0^T\int_\Omega |D(\boldsymbol{v}^{m,n})|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C(\ell, M, \boldsymbol{v}_0, \boldsymbol{f}, \mu_{\min}, \rho_{\max}).$$

Using (4.96) and Korn's inequality we deduce that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c_0\mu_{\min}\int_0^T \|\boldsymbol{v}^{m,n}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \le C(\ell), \tag{4.100}$$

which implies, by the orthogonality of the basis, that

$$\sup_{t\in(0,T);\, i=1,\dots,m} |c_i^{m,n}(t)| \le C(m,\ell). \tag{4.101}$$

From the definition of $\boldsymbol{v}^{m,n}$ we see that

$$\frac{\partial \boldsymbol{v}^{m,n}}{\partial t}(x,t) = \sum_{j=1}^m \frac{\mathrm{d}c_j^{m,n}(t)}{\mathrm{d}t}\boldsymbol{w}_j(x). \tag{4.102}$$

Substituting (4.102) into (4.51) we have that, for a fixed $i \in \{1,\dots,m\}$,

$$\begin{aligned}
\rho_{\min}\left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| &\le |(\partial_t\rho^{m,n}\boldsymbol{v}^{m,n},\boldsymbol{w}_i)| + |(\rho^{m,n}\boldsymbol{v}^{m,n}\otimes\boldsymbol{v}^{m,n},\nabla_x\boldsymbol{w}_i)| \\
&\quad + |(\mu(\rho^{m,n},\varrho^{m,n})D(\boldsymbol{v}^{m,n}),\nabla_x\boldsymbol{w}_i)| \\
&\quad + |(S_e^{m,n},\nabla_x\boldsymbol{w}_i)| + |(\rho^{m,n}\boldsymbol{f}^m,\boldsymbol{w}_i)| \\
&\le \|\partial_t\rho^{m,n}\|_{W^{-1,2}(\Omega)}\|\boldsymbol{v}^{m,n}\|_{W^{1,2}(\Omega;\mathbb{R}^d)}\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \\
&\quad + \rho_{\max}\|\boldsymbol{v}^{m,n}\|^2_{L^2(\Omega;\mathbb{R}^d)}\|\nabla_x\boldsymbol{w}_i\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \\
&\quad + \mu_{\max}\|D(\boldsymbol{v}^{m,n})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\nabla_x\boldsymbol{w}_i\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \\
&\quad + C(\ell)\|\nabla_x\boldsymbol{w}_i\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} + C(\boldsymbol{f},\rho_{\max})\|\nabla_x\boldsymbol{w}_i\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \\
&\le C(m,\ell)\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}.
\end{aligned}$$

Since $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we have that $\|\boldsymbol{w}_i\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(m)$. Therefore,

$$\sup_{t\in(0,T);\, i=1,\dots,m}\left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| \le C(m,\ell). \tag{4.103}$$

By the definition of $\boldsymbol{v}^{m,n}$ it is easy to see that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}(t)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(m,\ell). \tag{4.104}$$

By using (4.100) and the Sobolev Embedding Theorem it then follows from (4.99) that

$$\begin{aligned}
\left\|\frac{\partial\beta(\rho^{m,n})}{\partial t}\right\|_{L^2(0,T;(W^{1,p'}(\Omega))')} &\le \|\beta(\rho^{m,n})\boldsymbol{v}^{m,n}\|_{L^2(0,T;L^p(\Omega;\mathbb{R}^d))} \\
&\le C\|\boldsymbol{v}^{m,n}\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} \le C
\end{aligned} \tag{4.105}$$

for any $p \in (1,\infty)$ when $d = 2$ and $p \in (1,6]$ when $d = 3$; the constant C is independent of m and n.

The uniform bounds (4.101) and (4.103) imply the following convergence results:

$$c_i^{m,n} \rightharpoonup c_i^m \qquad \text{weak* in } W^{1,\infty}((0,T)), \tag{4.106}$$

$$c_i^{m,n} \to c_i^m \qquad \text{strongly in } C([0,T]), \tag{4.107}$$

which then imply that

$$\boldsymbol{v}^{m,n} \to \boldsymbol{v}^m \qquad \text{strongly in } C([0,T]; W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)). \tag{4.108}$$

Noting (4.97) and (4.98) we deduce that

$$\rho^{m,n}\boldsymbol{v}^{m,n} \rightharpoonup \rho^m\boldsymbol{v}^m \qquad \text{weakly in } L^p(Q;\mathbb{R}^d), \tag{4.109}$$

$$(\rho^{m,n})^2\boldsymbol{v}^{m,n} \rightharpoonup \vartheta^m\boldsymbol{v}^m \qquad \text{weakly in } L^p(Q;\mathbb{R}^d), \tag{4.110}$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,6]$ when $d = 3$. From the estimate (4.105) we deduce that

$$\frac{\partial \rho^{m,n}}{\partial t} \rightharpoonup \frac{\partial \rho^m}{\partial t} \qquad \text{weakly in } L^2(0,T;W^{-1,p}(\Omega)), \tag{4.111}$$

$$\frac{\partial (\rho^{m,n})^2}{\partial t} \rightharpoonup \frac{\partial \vartheta^m}{\partial t} \qquad \text{weakly in } L^2(0,T;W^{-1,p}(\Omega)), \tag{4.112}$$

where $p \in (1,\infty)$ when $d = 2$ and $p \in (1,6]$ when $d = 3$. With the convergence results (4.111) and (4.109) we pass to the limit as $n \to \infty$ and deduce that ρ^m is the (unique) weak solution of

$$\frac{\partial \rho^m}{\partial t} + \mathrm{div}_x(\rho^m\boldsymbol{v}^m) = 0 \tag{4.113}$$

with the initial condition

$$\rho^m(0) = \rho_0^m. \tag{4.114}$$

By the renormalization property, if we define $P^{m,n} = (\rho^{m,n})^2$, then $P^{m,n}$ solves the following initial-value problem:

$$\frac{\partial P^{m,n}}{\partial t} + \mathrm{div}_x(P^{m,n}\boldsymbol{v}^{m,n}) = 0,$$
$$P^{m,n}(0) = (\rho_0^m)^2.$$

Similarly, $P^m = (\rho^m)^2$ solves the following problem:

$$\frac{\partial P^m}{\partial t} + \mathrm{div}_x(P^m\boldsymbol{v}^m) = 0,$$
$$P^m(0) = (\rho_0^m)^2.$$

With the convergence results (4.112) and (4.110) we deduce that ϑ^m also solves the following initial-value problem

$$\begin{aligned} \frac{\partial \vartheta^m}{\partial t} + \mathrm{div}_x(\vartheta^m \boldsymbol{v}^m) &= 0, \\ \vartheta^m(0) &= (\rho_0^m)^2. \end{aligned} \tag{4.115}$$

However, since (4.115) is of the same form as (4.113), the solution to (4.115) is unique. Hence, $\vartheta^m = (\rho^m)^2$. Then, the convergence result (4.98) gives

$$\int_0^T \int_\Omega |\rho^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega |\rho^m|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

which then implies that

$$\rho^{m,n} \to \rho^m \qquad \text{strongly in } L^2(Q).$$

It then follows from (4.97) that

$$\rho^{m,n} \to \rho^m \qquad \text{strongly in } L^p(Q), \tag{4.116}$$

for any $p \in [1, \infty)$. With the convergence result (4.108) for $\boldsymbol{v}^{m,n}$ we can perform a similar argument as in Theorem VI.1.9 in [14] and strengthen the above convergence to get

$$\rho^{m,n} \to \rho^m \qquad \text{strongly in } C([0,T]; L^p(\Omega)), \tag{4.117}$$

for any $p \in [1, \infty)$. Thanks to the assumption (4.24) on ζ we then have that

$$\zeta(\rho^{m,n}) \to \zeta(\rho^m) \qquad \text{strongly in } C([0,T]; L^p(\Omega)). \tag{4.118}$$

We multiply the i-th equation in (4.52) by $d_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, n$ to deduce the following identity

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M^m \zeta(\rho^{m,n})(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_{\mathcal{O}} M^m \frac{\partial \zeta(\rho^{m,n})}{\partial t}(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad - \frac{1}{2}\int_{\mathcal{O}} M^m \zeta(\rho^{m,n})\boldsymbol{v}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad + \int_{\mathcal{O}} M^m |\nabla_x \hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}) : \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &= \Big(M^m \zeta(\rho^{m,n}) \Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x \boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}\Big)_{\mathcal{O}}. \end{aligned} \tag{4.119}$$

Note that $\zeta(\rho^{m,n})$ is a renormalized solution in the sense that

$$\langle \partial_t \zeta(\rho^{m,n}), \eta \rangle - (\boldsymbol{v}^{m,n}\zeta(\rho^{m,n}), \nabla_x \eta) = 0 \qquad \forall\, \eta \in C^{0,1}([0,T]; C^{0,1}(\overline{\Omega})).$$

Taking the test function $\eta = \int_D M^m(\hat{\psi}^{m,n})^2 \,\mathrm{d}\boldsymbol{q}$ we see that the second term and the third term in (4.119) add up to 0. By Young's inequality, the definition of Λ_ℓ, (4.24) and (4.104), we deduce that

$$\begin{aligned}
&\Big(M^m\zeta(\rho^{m,n})\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x \boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}\Big)_{\mathcal{O}} \\
&\le \frac{C_1}{2}\int_{\mathcal{O}} M^m|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&\quad + \frac{C(\ell,\zeta_{\max})}{2}\|\nabla_x \boldsymbol{v}^{m,n}\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&\le \frac{C_1}{2}\int_{\mathcal{O}} M^m|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&\quad + C(m,\ell)\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.
\end{aligned} \tag{4.120}$$

Inserting (4.120) into (4.119) and using (4.14) we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}& \\
\le C(m,\ell)\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})(\hat{\psi}^{m,n})^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.&
\end{aligned} \tag{4.121}$$

Gronwall's inequality gives that

$$\sup_{t\in(0,T)} \|\sqrt{\zeta(\rho^{m,n}(t))}\hat{\psi}^{m,n}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + \int_0^T\int_{\mathcal{O}} M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(m,\ell).$$

By noting that $M^m \ge 1/m$ and (4.24) we deduce that

$$\sup_{t\in(0,T)} \|\hat{\psi}^{m,n}(t)\|^2_{L^2(\mathcal{O})} + \int_0^T\int_{\mathcal{O}} |\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(m,\ell). \tag{4.122}$$

Since M^m is Lipschitz continuous, we can further deduce that

$$\int_0^T \|M^m\hat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \le C(m,\ell).$$

Then, using (4.52) and a standard calculation, we obtain that

$$\int_0^T \|\partial_t(M^m\zeta(\rho^{m,n})\hat{\psi}^{m,n})\|^2_{(W^{1,2}(\mathcal{O}))'}\,\mathrm{d}t \le C(m,\ell). \tag{4.123}$$

Next, we shall focus on the fractional time derivative of $\hat{\psi}^{m,n}$. Integrating (4.52) with respect to time over $(s, s+h)$, with $s < T-h$, we have, noting that in fact φ_i^m

can be replaced by any test function $\varphi^m \in \operatorname{span}\{\varphi_1^m, \dots, \varphi_n^m\}$,

$$\begin{aligned}
&\int_{\mathcal{O}} M^m [(\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s)]\varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&= \int_s^{s+h} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})\hat{\psi}^{m,n}\boldsymbol{v}^{m,n} \cdot \nabla_x \varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\quad - \int_s^{s+h} \int_{\mathcal{O}} M^m \nabla_x \hat{\psi}^{m,n} : \nabla_x \varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\quad - \int_s^{s+h} \int_{\mathcal{O}} M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^{m,n}) : \nabla_{\boldsymbol{q}} \varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\quad + \int_s^{s+h} \int_{\mathcal{O}} M \zeta(\rho^{m,n})\Lambda_\ell(\hat{\psi}^{m,n})(\nabla_x \boldsymbol{v}^{m,n})\boldsymbol{q} : \nabla_{\boldsymbol{q}} \varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&=: U_1 + U_2 + U_3 + U_4.
\end{aligned}$$

For U_1, we have by Hölder's inequality and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
|U_1| &\leq C(M, \zeta_{\max})\|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega;L^2(D))} \|\boldsymbol{v}^{m,n}\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\
&\leq C(M, \zeta_{\max})\|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \\
&\quad \times \int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega;L^2(D))} \|\boldsymbol{v}^{m,n}\|_{L^2(\Omega;\mathbb{R}^d)}^{1-\frac{\mathrm{d}}{q}} \|\boldsymbol{v}^{m,n}\|_{W^{1,2}(\Omega;\mathbb{R}^d)}^{\frac{\mathrm{d}}{q}} \,\mathrm{d}t \\
&\leq C(M, \zeta_{\max}) h^{\frac{q-d}{2q}} \|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \|\boldsymbol{v}^{m,n}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}^{1-\frac{\mathrm{d}}{q}} \\
&\quad \times \|\boldsymbol{v}^{m,n}\|_{L^2(s,s+h;W^{1,2}(\Omega;\mathbb{R}^d))}^{\frac{\mathrm{d}}{q}} \left(\int_s^{s+h} \|\hat{\psi}^{m,n}\|_{L^q(\Omega;L^2(D))}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq C h^{\frac{q-d}{2q}} \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})},
\end{aligned}$$

where $q \in (2,\infty)$ when $d = 2$ and $q \in (3,6]$ when $d = 3$. For U_2, we have that

$$|U_2| \leq C(M) h^{\frac{1}{2}} \|\nabla_x \hat{\psi}^{m,n}\|_{L^2(s,s+h;L^2(\mathcal{O};\mathbb{R}^d))} \|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq C h^{\frac{1}{2}} \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}.$$

Similarly as above, for U_3 we have that

$$|U_3| \leq C h^{\frac{1}{2}} \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}.$$

For U_4, thanks to the presence of the truncation function Λ_ℓ, we obtain that

$$|U_4| \leq C h^{\frac{1}{2}} \|\nabla_{\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{K+1})} \|\nabla_x \boldsymbol{v}^{m,n}\|_{L^2(s,s+h;L^2(\Omega;\mathbb{R}^d))} \leq C h^{\frac{1}{2}} \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}.$$

Hence,

$$\begin{aligned}
&\left| \int_{\mathcal{O}} M^m [(\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s)]\varphi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \right| \\
&\qquad \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}, \quad (4.124)
\end{aligned}$$

where $q \in (2,\infty)$ when $d = 2$ and $q \in (3,6]$ when $d = 3$. By the renormalization property $\zeta(\rho^{m,n})$ satisfies

$$\int_0^T \int_\Omega \left(\frac{\partial \zeta(\rho^{m,n})}{\partial t} \eta - \zeta(\rho^{m,n}) \boldsymbol{v}^{m,n} \cdot \nabla_x \eta \right) \mathrm{d}x \, \mathrm{d}t = 0.$$

Taking $\eta = \chi_{[s,s+h]} \int_D M^m \hat{\psi}^{m,n}(s) \varphi^m \, \mathrm{d}\boldsymbol{q}$ in the above equation we obtain that

$$\begin{aligned} &\int_{\mathcal{O}} M^m [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)] \hat{\psi}^{m,n} \varphi^m \, \mathrm{d}x \, \mathrm{d}\boldsymbol{q} \\ &\qquad = \int_s^{s+h} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n}) \boldsymbol{v}^{m,n} \cdot \nabla_x (\hat{\psi}^{m,n}(s) \varphi^m) \, \mathrm{d}x \, \mathrm{d}\boldsymbol{q} \, \mathrm{d}t. \end{aligned} \tag{4.125}$$

By Hölder's inequality we obtain that

$$\begin{aligned} &\left| \int_{\mathcal{O}} M^m [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)] \hat{\psi}^{m,n} \varphi^m \, \mathrm{d}x \, \mathrm{d}\boldsymbol{q} \right| \\ &\leq C(M, \zeta_{\max}) h^{\frac{1}{2}} \|\boldsymbol{v}^{m,n}\|_{L^2(s,s+h;L^q(\Omega;\mathbb{R}^d))} \|\nabla_x (\hat{\psi}^{m,n}(s) \varphi^m)\|_{L^{\frac{q}{q-1}}(\mathcal{O};\mathbb{R}^d)} \\ &\leq C(M, \zeta_{\max}) h^{\frac{1}{2}} \|\boldsymbol{v}^{m,n}\|_{L^2(s,s+h;L^q(\Omega;\mathbb{R}^d))} \Bigg(\|\nabla_x \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O};\mathbb{R}^d)} \|\varphi^m\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \\ &\quad + \|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \|\hat{\psi}^{m,n}(s)\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \Bigg), \end{aligned} \tag{4.126}$$

where $q \in (2, 2(K+1)]$ when $d = 2$ and $q \in (3,6]$ when $d = 3$. Since

$$\begin{aligned} &(\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s+h) - (\zeta(\rho^{m,n})\hat{\psi}^{m,n})(s) \\ &= \zeta(\rho^{m,n})(s+h)[\hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)] + [\zeta(\rho^{m,n})(s+h) - \zeta(\rho^{m,n})(s)]\hat{\psi}^{m,n}(s), \end{aligned}$$

it follows from (4.124) and (4.126) that

$$\begin{aligned} &\left| \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})(s+h) [\hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)] \varphi^m \, \mathrm{d}x \, \mathrm{d}\boldsymbol{q} \right| \\ &\quad \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \|\nabla_{x,\boldsymbol{q}} \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})} \\ &\qquad + C h^{\frac{1}{2}} \Bigg(\|\nabla_x \hat{\psi}^{m,n}(s)\|_{L^2(\mathcal{O};\mathbb{R}^d)} \|\varphi^m\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} + \|\nabla_x \varphi^m\|_{L^2(\mathcal{O};\mathbb{R}^d)} \|\hat{\psi}^{m,n}(s)\|_{L^{\frac{2q}{q-2}}(\mathcal{O})} \Bigg), \end{aligned}$$

where $q \in (2, 2(K+1)]$ when $d = 2$ and $q \in (3,6]$ when $d = 3$. Taking $\varphi^m = \hat{\psi}^{m,n}(s+h) - \hat{\psi}^{m,n}(s)$ in the above inequality and integrating with respect to s we obtain that

$$
\begin{aligned}
&\int_0^{T-h}\int_{\mathcal{O}} M^m\zeta(\rho^{m,n})(s+h)[\hat{\psi}^{m,n}(s+h)-\hat{\psi}^{m,n}(s)]^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s\\
&\leq 2C(h^{\frac{1}{2}}+h^{\frac{q-d}{2q}})\left(\int_0^T\|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n}(s)\|^2_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+2Ch^{\frac{1}{2}}\left(\int_0^T\|\nabla_x\hat{\psi}^{m,n}(s)\|^2_{L^2(\mathcal{O};\mathbb{R}^d)}\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^T\|\hat{\psi}^{m,n}(s)\|^2_{L^{\frac{2q}{q-2}}(\mathcal{O})}\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+2Ch^{\frac{1}{2}}\left(\int_0^T\|\nabla_x\hat{\psi}^{m,n}(s)\|^2_{L^2(\mathcal{O};\mathbb{R}^d)}\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^T\|\hat{\psi}^{m,n}(s)\|^2_{L^{\frac{2q}{q-2}}(\mathcal{O})}\,\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leq Ch^{\frac{q-d}{2q}},
\end{aligned}
$$

where $q\in(2,2(K+1)]$ when $d=2$ and $q\in(3,6]$ when $d=3$. The above inequality follows from Hölder's inequality, the uniform estimate (4.122) and the Sobolev embedding $W^{1,2}(\mathcal{O})\hookrightarrow L^p(\mathcal{O})$, $p\in\left[1,\frac{2(K+1)d}{(K+1)d-2}\right]$. Since $\zeta(\rho^{m,n})\geq\zeta_{\min}$ and $M^m\geq 1/m$, we have that

$$\|\hat{\psi}^{m,n}(\cdot+h)-\hat{\psi}^{m,n}(\cdot)\|_{L^2(0,T-h;L^2(\mathcal{O}))}\leq Ch^{\gamma},$$

where $0<\gamma\leq\frac{K}{4(K+1)}$ when $d=2$ and $0<\gamma\leq 1/8$ when $d=3$. We have therefore shown the following Nikolskiĭ norm estimate:

$$\|\hat{\psi}^{m,n}\|_{N_2^{\gamma}(0,T;L^2(\mathcal{O}))}:=\sup_{0<h<T}h^{-\gamma}\|\hat{\psi}^{m,n}(\cdot+h)-\hat{\psi}^{m,n}(\cdot)\|_{L^2(0,T-h;L^2(\mathcal{O}))}\leq C,\quad(4.127)$$

where $0<\gamma\leq\frac{K}{4(K+1)}$ when $d=2$ and $0<\gamma\leq 1/8$ when $d=3$.

From (4.122), (4.123) and (4.127) and by using the Aubin–Lions Lemma we deduce that there exists a subsequence (not relabelled) such that

$$
\begin{aligned}
\hat{\psi}^{m,n}&\rightharpoonup\hat{\psi}^m &&\text{weakly in } L^2(0,T;W^{1,2}(\mathcal{O})), &(4.128)\\
\partial_t(M^m\zeta(\rho^{m,n})\hat{\psi}^{m,n})&\rightharpoonup\partial_t(M^m\zeta(\rho^m)\hat{\psi}^m) &&\text{weakly in } L^2(0,T;(W^{1,2}(\mathcal{O}))'), &(4.129)\\
\hat{\psi}^{m,n}&\to\hat{\psi}^m &&\text{strongly in } L^2(0,T;L^2(\mathcal{O})). &(4.130)
\end{aligned}
$$

It then follows from (4.118), (4.130) and the global Lipschitz continuity of the mapping $s\in\mathbb{R}\mapsto[s]_+\in\mathbb{R}_{\geq0}$ that

$$\varrho^{m,n}\to\varrho^m\quad\text{strongly in } L^2(0,T;L^2(\Omega)),$$

where

$$\varrho^m=\zeta(\rho^m)\int_D M^m\,[\hat{\psi}^m]_+\,\mathrm{d}\boldsymbol{q},$$

and therefore, thanks to (4.117) and the assumption (4.24) on μ, the Dominated Convergence Theorem implies that

$$\mu(\rho^{m,n}, \varrho^{m,n}) \to \mu(\rho^m, \varrho^m) \qquad \text{strongly in } L^1(Q).$$

Hence, because $0 \leq \mu_{\min} \leq \mu(\cdot,\cdot) \leq \mu_{\max} < \infty$, it follows that

$$\mu(\rho^{m,n}, \varrho^{m,n}) \to \mu(\rho^m, \varrho^m) \qquad \text{strongly in } L^p(Q), \tag{4.131}$$

for any $p \in [1,\infty)$.

Now the above convergence results (4.109), (4.111) and (4.116) for $\rho^{m,n}$, (4.131) for $\mu(\rho^{m,n}, \varrho^{m,n})$, (4.118) for $\zeta(\rho^{m,n})$, (4.128), (4.129) and (4.130) for $\hat{\psi}^{m,n}$, (4.106) and (4.107) for $c^{m,n}$, (4.108) for $\boldsymbol{v}^{m,n}$ and (4.95) for $S_e^{m,n}$ enable us to pass to the limit $n \to \infty$ in (4.50)–(4.52) to obtain the following:

$$\langle \partial_t \rho^m, \eta\rangle - (\boldsymbol{v}^m \rho^m, \nabla_x \eta) = 0, \quad \text{for all } \eta \in C^{0,1}(\overline{\Omega}) \text{ and a.e. } t \in (0,T), \tag{4.132}$$

$$\begin{aligned} &\langle \partial_t(\rho^m \boldsymbol{v}^m), \boldsymbol{w}_i\rangle - (\rho^m \boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla_x \boldsymbol{w}_i) + (\mu(\rho^m, \varrho^m) D(\boldsymbol{v}^m), \nabla_x \boldsymbol{w}_i) \\ &\quad = -(S_e^m, \nabla_x \boldsymbol{w}_i) + (\rho^m \boldsymbol{f}^m, \boldsymbol{w}_i) \quad \text{for all } i = 1,\dots,m \text{ and a.e. } t \in (0,T), \end{aligned} \tag{4.133}$$

and

$$\begin{aligned} &\Big\langle \partial_t (M^m \zeta(\rho^m)\hat{\psi}^m), \varphi\Big\rangle_{\mathcal{O}} - \Big(M^m \zeta(\rho^m)\boldsymbol{v}^m \hat{\psi}^m, \nabla_x \varphi\Big)_{\mathcal{O}} \\ &\quad - \Big(M\zeta(\rho^m)\Lambda_\ell(\hat{\psi}^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}} + (M^m \nabla_x \hat{\psi}^m, \nabla_x \varphi)_{\mathcal{O}} \\ &\quad + \Big(M^m \mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^m), \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}} = 0 \quad \text{for all } \varphi \in W^{1,2}(\mathcal{O}) \text{ and a.e. } t \in (0,T). \end{aligned} \tag{4.134}$$

It is obvious that $\boldsymbol{v}^m(x,0) = \boldsymbol{v}_0^m(x)$. We can deduce using standard calculations that

$$\lim_{t\to 0+} \|\hat{\psi}^m(\cdot,t) - T_\ell(\hat{\psi}_0^m(\cdot))\|_{L^2(\mathcal{O})} = 0.$$

Also, from (4.118) we deduce that

$$\zeta(\rho^{m,n}) \to \zeta(\rho^m) \qquad \text{a.e. in } Q. \tag{4.135}$$

Similarly, from (4.130) we deduce that

$$\hat{\psi}^{m,n} \to \hat{\psi}^m \qquad \text{a.e. in } \mathcal{O} \times (0,T).$$

Therefore,

$$\zeta(\rho^{m,n})\hat{\psi}^{m,n} \to \zeta(\rho^m)\hat{\psi}^m \qquad \text{a.e. in } \mathcal{O}\times(0,T).$$

Finally, the Dominated Convergence Theorem gives that

$$S_e^m = -k\int_D \left[KM\zeta(\rho^m)T_\ell(\hat{\psi}^m)I + \sum_{j=1}^K \zeta(\rho^m)T_\ell(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \quad \text{a.e. in } Q. \tag{4.136}$$

4.3.5 Passage to the limit with m

Since $\boldsymbol{v}^m \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d)) \hookrightarrow L^1(0,T;W^{1,1}(\Omega;\mathbb{R}^d))$ and the initial data $\rho^m(0) = \rho_0^m \in C^1(\overline{\Omega}) \subset L^\infty(\Omega)$, we deduce from Theorem VI.1.6 in [14] that there exists a unique weak solution $\rho^m \in L^\infty(\Omega \times (0,T))$ that solves (4.132). Moreover, by Theorem VI.1.3 in [14] we have that $\rho^m \in C([0,T];L^p(\Omega))$, where $1 \leq p < \infty$. The solution satisfies

$$\begin{gathered}\sup_{t\in(0,T)} \|\rho^m(t)\|_{L^\infty(\Omega)} \leq \|\rho_0^m\|_{L^\infty(\Omega)} \leq \rho_{\max}, \\ \rho^m \geq \rho_{\min} \quad \text{a.e. in } Q.\end{gathered} \tag{4.137}$$

By the renormalization property, for any $\beta \in C^1(\mathbb{R})$, $\beta(\rho^m)$ satisfies

$$\frac{\partial \beta(\rho^m)}{\partial t} + \mathrm{div}_x(\beta(\rho^m)\boldsymbol{v}^m) = 0 \tag{4.138}$$

in the distributional sense. We note that $\zeta(\rho^m)$ is a renormalized solution in the sense that (4.138) is satisfied with $\beta = \zeta$.

We note further that thanks to (4.104) it follows by weak lower semi-continuity and passage to the limit as $n \to \infty$ that

$$\|\boldsymbol{v}^m\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))} \leq C(m,\ell). \tag{4.139}$$

We would now like to show that $\hat{\psi}^m \geq 0$ a.e. in $\mathcal{O}\times(0,T)$. Before doing so, we shall revisit the question of regularity of ρ^m and will show that, thanks to the regularity of $\boldsymbol{v}^m$ and ρ_0^m, in fact $\partial_t\rho^m \in L^\infty(Q)$ and $\nabla_x\rho^m \in L^\infty(Q;\mathbb{R}^d)$, and therefore also $\partial_t\zeta(\rho^m) \in L^\infty(Q)$ and $\nabla_x\zeta(\rho^m) \in L^\infty(Q;\mathbb{R}^d)$, whereby the renormalized equation (4.138) does in fact hold in the sense that

$$\partial_t\zeta(\rho^m) + \nabla_x \cdot (\boldsymbol{v}^m\zeta(\rho^m)) = 0 \qquad \text{a.e. in } Q. \tag{4.140}$$

To this end we note that after passing to the limit $n \to \infty$ in (4.48),

$$\boldsymbol{v}^m(x,t) \coloneqq \sum_{i=1}^m c_i^m(t)\, \boldsymbol{w}_i(x), \tag{4.141}$$

where $c_i^{m,n} \rightharpoonup c_i^m \in W^{1,\infty}((0,T))$ for each $m \geq 1$ as $n \to \infty$ by (4.106) and $\boldsymbol{w}_i \in C^1(\overline{\Omega};\mathbb{R}^d)$ for each $i = 1,\dots,m$. Thanks to the orthonormality of $(\boldsymbol{w}_i)_{i=1}^m$ in $L^2(\Omega;\mathbb{R}^d)$ it therefore follows that

$$\|\boldsymbol{v}^m(\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} = \left(\sum_{i=1}^m |c_i^m(\cdot)|^2\right)^{\frac{1}{2}} \in W^{1,\infty}((0,T)).$$

For $x \in \overline{\Omega}$ and $t \in [0,T]$, consider the following initial-value problem for a system of ordinary differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}X^m}{\mathrm{d}s}(x,t;s) &= \boldsymbol{v}^m(X^m(x,t;s),s), \qquad s \in [0,T], \\
X^m(x,t;t) &= x.
\end{aligned}
\tag{4.142}
$$

By the Cauchy–Lipschitz theorem, whose assumptions are satisfied thanks to the regularity properties of c_i^m and $\boldsymbol{w}_i$, $i = 1,\dots,m$, we deduce that, for each $(x,t) \in \overline{\Omega} \times [0,T]$, the system (4.142) has a unique solution $s \in [0,T] \mapsto X^m(x,t;s) \in C^1([0,T];\mathbb{R}^d)$. Furthermore, thanks to the specific form of (4.141) and the fact that $\boldsymbol{w}_i \in C^1(\overline{\Omega};\mathbb{R}^d)$ for $i = 1,\dots,m$, it follows that X^m is a differentiable function of x, and, for every $x \in \overline{\Omega}$ and $t \in [0,T]$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(\nabla_x X^m(x,t;s)) &= \nabla_x \boldsymbol{v}^m(X^m(x,t;s),s)\, \nabla_x X^m(x,t;s), \qquad s \in [0,T], \\
\nabla_x X^m(x,t;t) &= I,
\end{aligned}
$$

where I is the $d \times d$ identity matrix. Thus, upon integration and recalling the form of $\boldsymbol{v}^m$ from (4.141), we deduce that

$$
\begin{aligned}
\nabla_x X^m(x,t;s) &= \nabla_x X^m(x,t;t) \\
&\qquad + \int_t^s \sum_{i=1}^m c_i^m(\sigma)\, (\nabla_x \boldsymbol{w}_i)(X^m(x,t;\sigma))\, \nabla_x X^m(x,t;\sigma)\, \mathrm{d}\sigma.
\end{aligned}
$$

By Liouville's formula for matrix-differential equations, and recalling that the functions $\boldsymbol{w}_i$, $i = 1,2,\dots$ are divergence-free, whereby the matrix $\nabla_x \boldsymbol{v}^m$ is trace-free, we have that

$$
\begin{aligned}
\det\left(\nabla_x X^m(x,t;s)\right) &= \det\left(\nabla_x X^m(x,t;t)\right) \exp\left(\int_t^s \mathrm{tr}[(\nabla_x \boldsymbol{v}^m)(X^m(x,t;\sigma),\sigma)]\, \mathrm{d}\sigma\right) \\
&= \det(I)\exp(0) \\
&= 1
\end{aligned}
$$

for all $x \in \overline{\Omega}$ and all $t,s \in [0,T]$. Thus, the mapping $x \in \overline{\Omega} \mapsto X^m(x,t;s) \in \overline{\Omega}$ has a nonvanishing Jacobian for all $x \in \overline{\Omega}$ and all $t,s \in [0,T]$ and is therefore an invertible C^1 bijection from $\overline{\Omega}$ onto $\overline{\Omega}$, with C^1 inverse $y \in \overline{\Omega} \mapsto X^m(y;s;t)$ for all $t,s \in [0,T]$. In what follows we require a bound on $\nabla_x X^m$.

To this end, let $\|\cdot\|$ be a matrix norm on $\mathbb{R}^{d\times d}$ induced by a vector norm on $\mathbb{R}^d$.

Then, $\|I\| = 1$ and therefore, because $X^m(x,t;t) = I$, also

$$\begin{aligned}
&\|\nabla_x X^m(x,t;s)\| \\
&\leq 1 + \left| \int_t^s \sum_{i=1}^m |c_i^m(\sigma)| \, \|(\nabla_x \boldsymbol{w}_i)(X^m(x,t;\sigma))\| \, \|\nabla_x X^m(x,t;\sigma)\| \, \mathrm{d}\sigma \right| \\
&\leq 1 + \left| \int_t^s \sum_{i=1}^m |c_i^m(\sigma)| \, \|\nabla_x \boldsymbol{w}_i\|_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \, \|\nabla_x X^m(x,t;\sigma)\| \, \mathrm{d}\sigma \right| \\
&\leq 1 + \left| \int_t^s \left(\sum_{i=1}^m |c_i^m(\sigma)|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|\nabla_x \boldsymbol{w}_i\|^2_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \right)^{\frac{1}{2}} \|\nabla_x X^m(x,t;\sigma)\| \, \mathrm{d}\sigma \right| \\
&= 1 + \left(\sum_{i=1}^m \|\nabla_x \boldsymbol{w}_i\|^2_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \right)^{\frac{1}{2}} \left| \int_t^s \|\boldsymbol{v}^m(\sigma)\|_{L^2(\Omega;\mathbb{R}^d)} \, \|\nabla_x X^m(x,t;\sigma)\| \, \mathrm{d}\sigma \right| \\
&\leq 1 + \left(\sum_{i=1}^m \|\nabla_x \boldsymbol{w}_i\|^2_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \right)^{\frac{1}{2}} \|\boldsymbol{v}^m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \left| \int_t^s \|\nabla_x X^m(x,t;\sigma)\| \, \mathrm{d}\sigma \right|.
\end{aligned}$$

Thus, by Gronwall's inequality, we have that

$$\|\nabla_x X^m(x,t;s)\| \leq \exp\left(|s-t| \left(\sum_{i=1}^m \|\nabla_x \boldsymbol{w}_i\|^2_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \right)^{\frac{1}{2}} \|\boldsymbol{v}^m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \right).$$

Consequently,

$$\max_{x\in\overline{\Omega},\, t,s\in[0,T]} \|\nabla_x X^m(x,t;s)\| \leq C_m := \exp\left(T \left(\sum_{i=1}^m \|\nabla_x \boldsymbol{w}_i\|^2_{C(\overline{\Omega};\mathbb{R}^{d\times d})} \right)^{\frac{1}{2}} \|\boldsymbol{v}^m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \right).$$

We shall show that the unique weak solution $\rho^m \in L^\infty(Q)$ of (4.132) is given by the expression

$$\rho^m(x,t) = \rho_0^m(X^m(x,t;0)), \qquad x \in \overline{\Omega},\, t \in [0,T].$$

Indeed, for any $\eta \in \mathcal{D}(\Omega) := C_0^\infty(\Omega)$, we have as equalities in $\mathcal{D}'(0,T)$, that

$$\begin{aligned}
\langle \partial_t \rho^m(\cdot,t), \eta \rangle_{\mathcal{D}(\Omega)} &= \frac{\mathrm{d}}{\mathrm{d}t} \langle \rho^m(\cdot,t), \eta \rangle_{\mathcal{D}(\Omega)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho^m(x,t)\, \eta(x) \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho_0^m(X^m(x,t;0))\, \eta(x) \, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho_0^m(y)\, \eta(X^m(y,0;t)) \det[(\nabla_y X^m)(y,0;t)] \, \mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho_0^m(y)\, \eta(X^m(y,0;t)) \, \mathrm{d}y \\
&= \int_\Omega \rho_0^m(y)\, (\nabla_x \eta)(X^m(y,0;t)) \cdot \frac{\mathrm{d}X^m}{\mathrm{d}t}(y,0;t) \, \mathrm{d}y.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \partial_t \rho^m, \eta \rangle_{\mathcal{D}(\Omega)} &= \int_\Omega \rho_0^m(y)\, (\nabla_x \eta)(X^m(y,0;t)) \cdot \boldsymbol{v}^m(X^m(y,0;t))\, \mathrm{d}y \\
&= \int_\Omega \rho_0^m(X^m(x,t,0))\, \nabla_x \eta(x) \cdot \boldsymbol{v}^m(x) \det[(\nabla_x X^m)(x,t;0)]\, \mathrm{d}x \\
&= \int_\Omega \rho_0^m(X^m(x,t,0))\, \nabla_x \eta(x) \cdot \boldsymbol{v}^m(x)\, \mathrm{d}x \\
&= \int_\Omega \rho^m(x,t)\, \nabla_x \eta(x) \cdot \boldsymbol{v}^m(x)\, \mathrm{d}x \\
&= \int_\Omega \rho^m(x,t)\, \boldsymbol{v}^m(x) \cdot \nabla_x \eta(x)\, \mathrm{d}x \\
&= \langle \rho^m(\cdot,t)\, \boldsymbol{v}^m, \nabla_x \eta \rangle_{\mathcal{D}(\Omega)} .
\end{aligned}$$

Thus,

$$\langle \partial_t \rho^m, \eta \rangle_{\mathcal{D}(\Omega)} = \langle \rho^m(\cdot,t)\, \boldsymbol{v}^m, \nabla_x \eta \rangle_{\mathcal{D}(\Omega)} = \langle -\nabla_x \cdot (\rho^m(\cdot,t)\, \boldsymbol{v}^m), \eta \rangle_{\mathcal{D}(\Omega)} ,$$

as equalities in $\mathcal{D}'(0,T)$. We have shown that ρ^m defined by $\rho^m(x,t) = \rho_0^m(X^m(x,t;0))$ for $x \in \overline{\Omega}$ and $t \in [0,T]$ satisfies the equation

$$\frac{\partial \rho^m}{\partial t} + \mathrm{div}_x(\boldsymbol{v}^m \rho^m) = 0$$

in $\mathcal{D}'(Q)$; in addition $\rho^m(x,0) = \rho_0^m(x)$ for all $x \in \Omega$. By the uniqueness of the weak solution to the problem (4.132) asserted by Theorem VI.1.3 in [14], it follows that the weak solution in question is given by $\rho^m(x,t) = \rho_0^m(X^m(x,t;0))$ for $x \in \overline{\Omega}$ and $t \in [0,T]$.

Using this representation of ρ^m we are now in a position to show that, in addition to the regularity $\rho^m \in C([0,T]; L^p(\Omega))$, where $1 \leq p < \infty$, guaranteed by Theorem VI.1.3 in [14], in our case, thanks to the regularity of $\boldsymbol{v}^m$ and ρ_0^m, ρ^m has additional regularity.

Indeed, because $\rho_0^m \in C^1(\overline{\Omega})$ and the mapping $x \in \overline{\Omega} \mapsto X^m(x,t;0) \in \overline{\Omega}$ is a C^1 mapping, it follows by the chain rule that

$$\nabla_x \rho^m(x,t) = \nabla_x \left[\rho_0^m(X^m(x,t;0))\right] = \left[(\nabla_x X^m)(x,t;0)\right]^{\mathrm{T}} (\nabla \rho_0^m)(X^m(x,t;0))$$

for all $x \in \overline{\Omega}$ and all $t \in [0,T]$, whereby

$$\|\nabla_x \rho^m\|_{L^\infty(Q;\mathbb{R}^d)} \leq C_m \|\nabla \rho_0^m\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq C(m).$$

As $\boldsymbol{v}^m \in W^{1,\infty}(Q;\mathbb{R}^d)$, it then also follows that

$$\begin{aligned}
\left\| \frac{\partial \rho^m}{\partial t} \right\|_{L^\infty(Q)} &\leq \|\boldsymbol{v}^m\|_{L^\infty(Q;\mathbb{R}^d)} \|\nabla_x \rho^m\|_{L^\infty(Q;\mathbb{R}^d)} \\
&\leq C_m \|\nabla_x \rho_0^m\|_{L^\infty(\Omega;\mathbb{R}^d)} \|\boldsymbol{v}^m\|_{L^\infty(Q;\mathbb{R}^d)} \leq C(m).
\end{aligned}$$

With these bounds it follows that

$$\frac{\partial \rho^m}{\partial t} + \mathrm{div}_x(\boldsymbol{v}^m \rho^m) = \frac{\partial \rho^m}{\partial t} + \boldsymbol{v}^m \cdot \nabla_x \rho^m = 0 \qquad \text{a.e. in } Q,$$

and $\rho^m(x,0) = \rho_0^m(x)$ for all $x \in \Omega$. Hence, the renormalized equation also holds in the sense that

$$\frac{\partial \beta(\rho^m)}{\partial t} + \mathrm{div}_x(\boldsymbol{v}^m \beta(\rho^m)) = \frac{\partial \beta(\rho^m)}{\partial t} + \boldsymbol{v}^m \cdot \nabla_x \beta(\rho^m) = 0 \qquad \text{a.e. in } Q, \tag{4.143}$$

for any $\beta \in C^1(\mathbb{R})$, and $\beta(\rho^m(x,0)) = \beta(\rho_0^m(x))$ for all $x \in \Omega$.

We are now ready to return to the proof of the nonnegativity of $\hat{\psi}^m$. By following a similar procedure as in (4.119)–(4.121), but now using the renomalization property in the form (4.143), and bearing in mind that

$$\nabla_x \boldsymbol{v}^m \in L^\infty(Q;\mathbb{R}^{d\times d}), \quad \nabla_x \rho^m \in L^\infty(Q;\mathbb{R}^d) \quad \text{and} \quad \frac{\partial \rho^m}{\partial t} \in L^\infty(Q), \tag{4.144}$$

we obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})((\hat{\psi}^m)_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m |\nabla_{x,\boldsymbol{q}}(\hat{\psi}^m)_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ \leq C(m,\ell) \int_{\mathcal{O}} M^m \zeta(\rho^{m,n})((\hat{\psi}^m)_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}. \end{aligned}$$

Since $\hat{\psi}^m(0) = T_\ell(\hat{\psi}_0^m) \geq 0$ and $\zeta(\cdot) \geq \zeta_{\min} > 0$, Gronwall's inequality implies that $(\hat{\psi}^m)_- \equiv 0$ in $\mathcal{O} \times (0,T)$. Thus we deduce that

$$\hat{\psi}^m \geq 0 \quad \text{a.e. in } \mathcal{O} \times (0,T).$$

Next, we will derive m-independent estimates for $\boldsymbol{v}^m$ and $\hat{\psi}^m$. We define

$$\begin{aligned} &\mathcal{F}_\delta(s) \coloneqq (s+\delta)\log(s+\delta) + 1, \quad \mathcal{F}(s) \coloneqq s\log s + 1, \\ &T_{\delta,\ell}(s) \coloneqq \int_0^s \frac{\Lambda_\ell(t)}{t+\delta}\,\mathrm{d}t = \int_0^s \frac{t\Gamma_\ell(t)}{t+\delta}\,\mathrm{d}t. \end{aligned}$$

We set $\varphi \coloneqq \log(\hat{\psi}^m + \delta) + 1$ in (4.134), where $\delta > 0$ is arbitrary, to obtain the following identity:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} M^m \zeta(\rho^m) \mathcal{F}_\delta(\hat{\psi}^m) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \int_{\mathcal{O}} M^m \partial_t \zeta(\rho^m)(\delta \log(\hat{\psi}^m + \delta) + 1 - \hat{\psi}^m) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad - \int_{\mathcal{O}} M^m \zeta(\rho^m) \frac{\hat{\psi}^m}{\hat{\psi}^m + \delta} (\boldsymbol{v}^m \cdot \nabla_x \hat{\psi}^m) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} \nabla_x \hat{\psi}^m : \nabla_x \hat{\psi}^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad + \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} \mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^m) : \nabla_{\boldsymbol{q}} \hat{\psi}^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \Big(M\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m) \Big)_{\mathcal{O}}. \end{aligned} \tag{4.145}$$

Since $\hat{\psi}^m \in W^{1,2}(\mathcal{O})$, we can test the equation (4.140) with the function

$$\eta = \int_D M^m(\delta \log(\hat{\psi}^m + \delta) + 1 - \hat{\psi}^m)\,\mathrm{d}\boldsymbol{q}, \quad \nabla_x \eta = -\int_D \frac{M^m \hat{\psi}^m \nabla_x \hat{\psi}^m}{\hat{\psi}^m + \delta}\,\mathrm{d}\boldsymbol{q}$$

to deduce that the second term and the third term in (4.145) add up to 0. Next, we integrate the resulting equation with respect to time over $(0,t)$ and use the assumption (4.14), to obtain that

$$\begin{aligned}
&\int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot,t))\mathcal{F}_\delta(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + C_1 \int_0^t \int_{\mathcal{O}} \frac{M^m}{\hat{\psi}^m + \delta} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^m|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\
&\quad \leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m)\mathcal{F}_\delta(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_0^t \Big(M\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}s.
\end{aligned}$$

Taking the limit $\delta \to 0_+$ in the above inequality, the first term on the left-hand side and the first-term on the right-hand side can be easily dealt with. For the second term on the left-hand side we apply the Monotone Convergence Theorem. Therefore we get

$$\begin{aligned}
&\int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot,t))\mathcal{F}(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m} \right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\
&\quad \leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m)\mathcal{F}(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \limsup_{\delta \to 0+} \int_0^t \Big(M\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}s.
\end{aligned} \tag{4.146}$$

For the last term on the right-hand side we use integration by parts (the boundary term vanishes since $M = 0$ on ∂D) and note that $\mathrm{div}_x\, \boldsymbol{v}^m = 0$. We obtain

$$\begin{aligned}
&\int_0^t \Big(M\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}s \\
&\quad = -\int_0^t \Big(\mathrm{div}_{\boldsymbol{q}} \left(M\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}\right), T_{\delta,\ell}(\hat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}s \\
&\quad = -\sum_{j=1}^K \int_0^t \Big(\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m), T_{\delta,\ell}(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \Big)_{\mathcal{O}}\,\mathrm{d}s.
\end{aligned} \tag{4.147}$$

Since $|T_{\delta,\ell}(s) - T_\ell(s)| \leq \delta \log\left(1 + \frac{\ell}{\delta}\right)$ for all $s \in [0,\infty)$, using (4.147) in (4.146) we pass to the limit in (4.146) to obtain

$$\begin{aligned}
&\int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot,t))\mathcal{F}(\hat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left| \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m} \right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\
&\quad \leq \int_{\mathcal{O}} M^m \zeta(\rho_0^m)\mathcal{F}(T_\ell(\hat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \sum_{j=1}^K \int_0^t \Big(\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m), T_\ell(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \Big)_{\mathcal{O}}\,\mathrm{d}s.
\end{aligned} \tag{4.148}$$

We multiply the i-th equation in (4.133) by $c_i^m(t)$ and sum with respect to $i = 1, \dots, m$ to deduce the following energy identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^m |\boldsymbol{v}^m|^2 \,\mathrm{d}x + \int_\Omega \mu(\rho^m, \varrho^m)|D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x \\ = -\int_\Omega S_e^m : \nabla_x \boldsymbol{v}^m \,\mathrm{d}x + (\rho^m \boldsymbol{f}^m, \boldsymbol{v}^m). \quad (4.149)$$

Using $\mathrm{div}_x\, \boldsymbol{v}^m = 0$ and (4.136) we have that

$$\int_\Omega S_e^m : \nabla_x \boldsymbol{v}^m \,\mathrm{d}x = -k \sum_{j=1}^{K} \Big(\zeta(\rho^m)(\nabla_x \boldsymbol{v}^m), T_\ell(\hat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j\Big)_{\mathcal{O}}.$$

Integrating (4.149) with respect to time over $(0, t)$ and multiplying (4.148) by k and adding the results we get

$$k\int_{\mathcal{O}} M^m \zeta(\rho^m(\cdot,t))\mathcal{F}(\hat{\psi}^m(\cdot,t)) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho^m(\cdot,t)|\boldsymbol{v}^m(\cdot,t)|^2 \,\mathrm{d}x \\ + \int_0^t \int_\Omega \mu(\rho^m, \varrho^m)|D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x\,\mathrm{d}s + 4kC_1 \int_0^t \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s \\ \leq k\int_{\mathcal{O}} M^m \zeta(\rho_0^m)\mathcal{F}(T_\ell(\hat{\psi}_0^m)) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho_0^m |\boldsymbol{v}_0|^2 \,\mathrm{d}x + \int_0^t \int_\Omega \rho^m \boldsymbol{f}^m \cdot \boldsymbol{v}^m \,\mathrm{d}x\,\mathrm{d}s. \\ (4.150)$$

Using the assumption (4.24), Korn's inequality and Gronwall's inequality we arrive at the following uniform estimate:

$$k\zeta_{\min} \sup_{t\in(0,T)} \|M^m \mathcal{F}(\hat{\psi}^m(\cdot,t))\|_{L^1(\mathcal{O})} + \frac{\rho_{\min}}{2} \sup_{t\in(0,T)} \|\boldsymbol{v}^m(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)} \\ + \frac{\mu_{\min} c_0}{2}\int_0^T \|\boldsymbol{v}^m\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t + 4kC_1 \int_0^T \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ \leq k\zeta_{\max}\int_{\mathcal{O}} M^m \mathcal{F}(T_\ell(\hat{\psi}_0^m)) \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{\rho_{\max}}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)} + \frac{\rho_{\max}^2}{2\mu_{\min} c_0}\int_0^T \|\boldsymbol{f}^m\|^2_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\ \leq C(\ell, \zeta_{\max}, \rho_{\max}, \hat{\psi}_0, \boldsymbol{v}_0, \boldsymbol{f}). \\ (4.151)$$

By interpolation we have the following estimate for $\boldsymbol{v}^m$:

$$\|\boldsymbol{v}^m\|_{L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)} \leq C(\ell). \quad (4.152)$$

It directly follows from (4.24), (4.133), (4.151) and (4.152) that

$$\int_0^T \|\partial_t(\rho^m \boldsymbol{v}^m)\|^q_{W^{-1,q}_{\mathrm{div}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \leq C(\ell), \quad \text{where } q \in \left(1, \frac{d+2}{d}\right]. \quad (4.153)$$

By a similar calculation as in (4.105) we have that

$$\|\partial_t \rho^m\|_{L^2(0,T;W^{-1,p}(\Omega))} \leq C(\ell), \tag{4.154}$$

where $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 6]$ when $d = 3$. Thanks to the presence of the truncation function $T_\ell(\cdot)$ we have that

$$|S_e^m| \leq C(\ell). \tag{4.155}$$

Next, we shall focus on the fractional time derivative of $\boldsymbol{v}^m$. Integrating (4.133) with respect to time over $(s, s+h)$, with $s < T - h$, we have that, noting here that $\boldsymbol{w}_i$ can be replaced by any test function $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$,

$$\begin{aligned}
&\int_\Omega [(\rho^m \boldsymbol{v}^m)(s+h) - (\rho^m \boldsymbol{v}^m)(s)] \cdot \boldsymbol{w} \,\mathrm{d}x \\
&\quad = \int_s^{s+h} \int_\Omega \rho^m \boldsymbol{v}^m \otimes \boldsymbol{v}^m : \nabla_x \boldsymbol{w} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad\quad - \int_s^{s+h} \int_\Omega \mu(\rho^m, \varrho^m) D(\boldsymbol{v}^m) : D(\boldsymbol{w}) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad\quad - \int_s^{s+h} \int_\Omega S_e^m : \nabla_x \boldsymbol{w} \,\mathrm{d}x \,\mathrm{d}t + \int_s^{s+h} \rho^m \boldsymbol{f}^m \cdot \boldsymbol{w} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.156}$$

For I_1, we have by Hölder's inequality, the Gagliardo–Nirenberg inequality and Korn's inequality that

$$\begin{aligned}
|I_1| &\leq \rho_{\max} \left(\int_s^{s+h} \|\boldsymbol{v}^m\|_{L^q(\Omega;\mathbb{R}^d)} \|\boldsymbol{v}^m\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \right) \|\nabla_x \boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
&\leq C\rho_{\max} \left(\int_s^{s+h} \|\boldsymbol{v}^m\|_{L^q(\Omega;\mathbb{R}^d)} \|\boldsymbol{v}^m\|_{L^2(\Omega;\mathbb{R}^d)}^{1-\frac{\mathrm{d}}{q}} \|\boldsymbol{v}^m\|_{W^{1,2}(\Omega;\mathbb{R}^d)}^{\frac{\mathrm{d}}{q}} \,\mathrm{d}t \right) \|\nabla_x \boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
&\leq C\rho_{\max} \|\nabla_x \boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \|\boldsymbol{v}^m\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}^{1-\frac{\mathrm{d}}{q}} \left(\int_s^{s+h} \|\boldsymbol{v}^m\|_{W^{1,2}(\Omega;\mathbb{R}^d)}^2 \,\mathrm{d}t \right)^{\frac{\mathrm{d}}{2q}} \\
&\quad \times \left(\int_s^{s+h} \|\boldsymbol{v}^m\|_{L^q(\Omega;\mathbb{R}^d)}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_s^{s+h} 1^{\frac{2q}{q-d}} \,\mathrm{d}t \right)^{\frac{q-d}{2q}} \\
&\leq C h^{\frac{q-d}{2q}} \|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})},
\end{aligned}$$

where $q \in (2, \infty)$ when $d = 2$ and $q \in (3, 6]$ when $d = 3$. For I_2, we have by (4.151) that

$$|I_2| \leq \mu_{\max} h^{\frac{1}{2}} \|D(\boldsymbol{v}^m)\|_{L^2(s,s+h;L^2(\Omega;\mathbb{R}^{d\times d}))} \|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq C h^{\frac{1}{2}} \|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}.$$

For I_3, we have by (4.155) that

$$|I_3| \leq Ch^{\frac{1}{2}}\|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}.$$

For I_4, we have by Hölder's inequality and Korn's inequality that

$$|I_4| \leq \rho_{\max}h^{\frac{1}{2}}\|\boldsymbol{f}^m\|_{L^2(s,s+h;L^2(\Omega;\mathbb{R}^d))}\|\boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq Ch^{\frac{1}{2}}\|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}.$$

Hence,

$$\left|\int_\Omega [(\rho^m\boldsymbol{v}^m)(s+h) - (\rho^m\boldsymbol{v}^m)(s)]\cdot\boldsymbol{w}\,\mathrm{d}x\right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}})\|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}, \quad (4.157)$$

where $q \in (2,\infty)$ when $d = 2$ and $q \in (3,6]$ when $d = 3$. Next, we take the test function $\eta = \chi_{[s,s+h]}(\boldsymbol{v}^m(s)\cdot\boldsymbol{w})$ in

$$\int_0^T\int_\Omega\left(\frac{\partial\rho^m}{\partial t}\eta - \rho^m\boldsymbol{v}^m\cdot\nabla_x\eta\right)\mathrm{d}x\,\mathrm{d}t = 0,$$

to deduce that

$$\int_\Omega[\rho^m(s+h) - \rho^m(s)](\boldsymbol{v}^m(s)\cdot\boldsymbol{w})\,\mathrm{d}x = \int_s^{s+h}\int_\Omega\rho^m\boldsymbol{v}^m\cdot\nabla_x(\boldsymbol{v}^m(s)\cdot\boldsymbol{w})\,\mathrm{d}x\,\mathrm{d}t.$$

By Hölder's inequality we have that

$$\begin{aligned}
&\left|\int_\Omega[\rho^m(s+h) - \rho^m(s)](\boldsymbol{v}^m(s)\cdot\boldsymbol{w})\,\mathrm{d}x\right| \\
&\leq \rho_{\max}h^{\frac{1}{2}}\|\boldsymbol{v}^m\|_{L^2(s,s+h;L^q(\Omega;\mathbb{R}^d))}\|\nabla_x(\boldsymbol{v}^m(s)\cdot\boldsymbol{w})\|_{L^{\frac{q}{q-1}}(\Omega;\mathbb{R}^d)} \\
&\leq \rho_{\max}h^{\frac{1}{2}}\|\boldsymbol{v}^m\|_{L^2(s,s+h;L^q(\Omega;\mathbb{R}^d))}\times\bigg(\|\nabla_x\boldsymbol{v}^m(s)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\boldsymbol{w}\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} \\
&\qquad + \|\nabla_x\boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\boldsymbol{v}^m(s)\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)}\bigg).
\end{aligned} \quad (4.158)$$

Since

$$(\rho^m\boldsymbol{v}^m)(s+h) - (\rho^m\boldsymbol{v}^m)(s) = \rho^m(s+h)[\boldsymbol{v}^m(s+h) - \boldsymbol{v}^m(s)] + [\rho^m(s+h) - \rho^m(s)]\boldsymbol{v}^m(s),$$

it follows from (4.157) and (4.158) that

$$\begin{aligned}
&\left|\int_\Omega\rho^m(s+h)[\boldsymbol{v}^m(s+h) - \boldsymbol{v}^m(s)]\cdot\boldsymbol{w}\,\mathrm{d}x\right| \leq C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}})\|D(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\
&\quad + Ch^{\frac{1}{2}}\bigg(\|\nabla_x\boldsymbol{v}^m(s)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\boldsymbol{w}\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} + \|\nabla_x\boldsymbol{w}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\|\boldsymbol{v}^m(s)\|_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)}\bigg),
\end{aligned}$$

where $q \in (2,\infty)$ when $d=2$ and $q \in (3,6]$ when $d=3$. Taking $\boldsymbol{w} = \boldsymbol{v}^m(s+h) - \boldsymbol{v}^m(s)$ in the above inequality and integrating with respect to s we obtain that

$$\begin{aligned}
&\left| \int_0^{T-h} \int_\Omega \rho^m(s+h)[\boldsymbol{v}^m(s+h) - \boldsymbol{v}^m(s)]^2 \,\mathrm{d}x \,\mathrm{d}s \right| \\
&\leq 2C(h^{\frac{1}{2}} + h^{\frac{q-d}{2q}}) \left(\int_0^T \|D(\boldsymbol{v}^m(s))\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \,\mathrm{d}s \right)^{\frac{1}{2}} \\
&\quad + 2Ch^{\frac{1}{2}} \left(\int_0^T \|\nabla_x \boldsymbol{v}^m(s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \,\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_0^T \|\boldsymbol{v}^m(s)\|^2_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \right)^{\frac{1}{2}} \\
&\quad + 2Ch^{\frac{1}{2}} \left(\int_0^T \|\nabla_x \boldsymbol{v}^m(s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \,\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_0^T \|\boldsymbol{v}^m(s)\|^2_{L^{\frac{2q}{q-2}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq Ch^{\frac{q-d}{2q}},
\end{aligned}$$

where $q \in (2,\infty)$ when $d=2$ and $q \in (3,6]$ when $d=3$. The above inequality follows from Hölder's inequality, the uniform estimate (4.151) and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, $p \in [1,\infty)$ when $d=2$ and $p \in [1,6]$ when $d=3$. Since $\rho^m \geq \rho_{\min}$, we have

$$\|\boldsymbol{v}^m(\cdot + h) - \boldsymbol{v}^m(\cdot)\|_{L^2(0,T-h;L^2(\Omega;\mathbb{R}^d))} \leq Ch^\gamma,$$

where $0 < \gamma < 1/4$ when $d=2$ and $0 < \gamma \leq 1/8$ when $d=3$. Therefore, we obtain the following Nikolskiĭ norm estimate:

$$\|\boldsymbol{v}^m\|_{N_2^\gamma(0,T;L^2(\Omega;\mathbb{R}^d))} := \sup_{0<h<T} h^{-\gamma} \|\boldsymbol{v}^m(\cdot + h) - \boldsymbol{v}^m(\cdot)\|_{L^2(0,T-h;L^2(\Omega;\mathbb{R}^d))} \leq C, \quad (4.159)$$

where $0 < \gamma < 1/4$ when $d=2$ and $0 < \gamma \leq 1/8$ when $d=3$. Using the m-independent estimates (4.137), (4.151), (4.159), (4.152), (4.153), (4.154) and (4.155) and the Aubin–Lions Lemma we deduce the existence of subsequences (not relabelled) such that, as $m \to \infty$,

$$\begin{aligned}
\rho^m &\rightharpoonup \rho && \text{weak* in } L^\infty(Q), && (4.160)\\
\partial_t \rho^m &\rightharpoonup \partial_t \rho && \text{weakly in } L^2(0,T;W^{-1,p}(\Omega)), && (4.161)\\
S_e^m &\rightharpoonup S_e && \text{weak* in } L^\infty(Q;\mathbb{R}^{d\times d}), && (4.162)\\
\boldsymbol{v}^m &\rightharpoonup \boldsymbol{v} && \text{weak* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)), && (4.163)\\
\boldsymbol{v}^m &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d)), && (4.164)\\
\partial_t(\rho^m \boldsymbol{v}^m) &\rightharpoonup \partial_t(\rho \boldsymbol{v}) && \text{weakly in } L^q(0,T;W^{-1,q}_{\mathrm{div}}(\Omega;\mathbb{R}^d)), && (4.165)\\
\boldsymbol{v}^m &\to \boldsymbol{v} && \text{strongly in } L^2(0,T;L^p(\Omega;\mathbb{R}^d)), && (4.166)
\end{aligned}$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,6)$ when $d = 3$, and $q \in \left(1, \frac{d+2}{d}\right]$. By the strong convergence of $\boldsymbol{v}^m$, we can perform a similar argument as in (4.113)–(4.116) to deduce that,

$$\rho^m \to \rho \qquad \text{strongly in } L^p(Q), \tag{4.167}$$

for any $p \in [1,\infty)$. With the convergence result (4.166) for $\boldsymbol{v}^m$ we can perform a similar argument as in Theorem VI.1.9 in [14] and strengthen the convergence above to get

$$\rho^m \to \rho \qquad \text{strongly in } C([0,T]; L^p(\Omega)), \tag{4.168}$$

for any $p \in [1,\infty)$. Therefore,

$$\zeta(\rho^m) \to \zeta(\rho) \qquad \text{strongly in } C([0,T]; L^p(\Omega)), \tag{4.169}$$

for any $p \in [1,\infty)$.

Next, we shall focus on the convergence properties of $\hat{\psi}^m$. First, we set $\varphi \equiv 1$ in (4.134) and deduce that

$$\begin{aligned} 0 &\leq \int_{\mathcal{O}} M^m(\boldsymbol{q})\zeta(\rho^m(x,t))\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_{\mathcal{O}} M^m \zeta(\rho_0^m) T_\ell(\hat{\psi}_0^m)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\leq \zeta_{\max} \int_{\mathcal{O}} M^m \hat{\psi}_0^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \zeta_{\max} \int_{\mathcal{O}} M\hat{\psi}_0\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C. \end{aligned}$$

As $\hat{\psi}^m \geq 0$ a.e. in $\mathcal{O} \times (0,T)$, it follows that

$$\begin{aligned} \varrho^m(x,t) &= \zeta(\rho^m(x,t)) \int_D M^m(\boldsymbol{q})\, [\hat{\psi}^m(x,\boldsymbol{q},t)]_+ \,\mathrm{d}\boldsymbol{q} \\ &= \zeta(\rho^m(x,t)) \int_D M^m(\boldsymbol{q})\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q} \geq 0. \end{aligned} \tag{4.170}$$

By defining

$$\lambda^m(x,t) := \int_D M^m(\boldsymbol{q})\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q} \tag{4.171}$$

and setting $\varphi(x,\boldsymbol{q},t) := \bar{\varphi}(x,t)$ in (4.134), we have the following equation satisfied by λ^m:

$$\begin{aligned} &\langle \partial_t(\zeta(\rho^m)\lambda^m), \bar{\varphi}\rangle - (\zeta(\rho^m)\lambda^m \boldsymbol{v}^m, \nabla_x\bar{\varphi}) + (\nabla_x\lambda^m, \nabla_x\bar{\varphi}) = 0 \\ &\qquad\qquad\qquad\qquad \text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T), \end{aligned} \tag{4.172}$$

supplemented by the initial condition $\lambda^m(0) := \lambda_0^m$, where, thanks to (4.23),

$$0 \leq \lambda_0^m(x) := \int_D M^m T_\ell(\hat{\psi}_0^m)\,\mathrm{d}\boldsymbol{q} \leq \int_D M\hat{\psi}_0\,\mathrm{d}\boldsymbol{q} = \frac{1}{\zeta(\rho_0)}\int_D \psi_0\,\mathrm{d}\boldsymbol{q} \leq \frac{\varrho_{\max}}{\zeta_{\min}}.$$

Let $\omega := \sup_{x\in\Omega}\lambda_0^m(x)$, test (4.140) with the function $\eta = \omega\bar{\varphi}$, and subtract (4.172) from the resulting equation; this gives that

$$\langle \partial_t(\zeta(\rho^m)(\omega-\lambda^m)), \bar{\varphi}\rangle - (\zeta(\rho^m)(\omega-\lambda^m)\boldsymbol{v}^m, \nabla_x\bar{\varphi}) + (\nabla_x(\omega-\lambda^m), \nabla_x\bar{\varphi}) = 0, \quad (4.173)$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and for a.e. $t \in (0,T)$. Note that $\hat{\psi}^m \in L^2(0,T;W^{1,2}(\mathcal{O}))$ by (4.128). It therefore follows from (4.171) that $\lambda^m \in L^2(0,T;W^{1,2}(\Omega))$. Hence, because $\zeta(\rho^m) \in W^{1,\infty}(Q)$, which follows from (4.144), we have that

$$\zeta(\rho^m)(\omega-\lambda^m) \in L^2(0,T;W^{1,2}(\Omega)).$$

Furthermore, since $\zeta(\rho^m)(\omega-\lambda^m)\boldsymbol{v}^m \in L^2(0,T;L^2(\Omega))$ and $\nabla_x(\omega-\lambda^m) \in L^2(0,T;L^2(\Omega))$, it follows from (4.173) that

$$\partial_t(\zeta(\rho^m)(\omega-\lambda^m)) \in L^2(0,T;(W^{1,2}(\Omega))').$$

where $(W^{1,2}(\Omega))'$ is the dual space of $W^{1,2}(\Omega)$. In addition, since $\zeta(\rho^m) \in W^{1,\infty}(Q)$ and $\lambda^m \in L^2(0,T;W^{1,2}(\Omega))$, we have that $\varrho^m \in L^2(0,T;W^{1,2}(\Omega))$. By (4.129) and (4.144) we also have that $\partial_t(M^m\zeta(\rho^m)\hat{\psi}^m) \in L^2(0,T;(W^{1,2}(\mathcal{O}))')$, and therefore $\partial_t\varrho^m \in L^2(0,T;(W^{1,2}(\Omega))')$. Thus, in summary,

$$\varrho^m \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t\varrho^m \in L^2(0,T;(W^{1,2}(\Omega))'). \qquad (4.174)$$

We shall now show that $\partial_t\lambda^m \in L^2(0,T;(W^{1,2}(\Omega))')$. For any $\phi \in W^{1,2}(\Omega)$ and $\theta \in C_0^\infty((0,T))$, we have

$$\begin{aligned}
\int_0^T \langle \partial_t\lambda^m, \phi\rangle\,\theta\,\mathrm{d}t &= -\int_0^T\int_\Omega \lambda^m\,\phi\,\partial_t\theta\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \varrho^m\,\frac{\phi}{\zeta(\rho^m)}\,\partial_t\theta\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\int_\Omega \varrho^m\,\phi\left[\partial_t\left(\frac{\theta}{\zeta(\rho^m)}\right) - \theta\,\partial_t\left(\frac{1}{\zeta(\rho^m)}\right)\right]\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \left\langle \partial_t\varrho^m, \frac{\phi}{\zeta(\rho^m)}\right\rangle\theta\,\mathrm{d}t - \int_0^T\left(\int_\Omega \varrho^m\,\phi\,\frac{\zeta'(\rho^m)\,\partial_t\rho^m}{\zeta(\rho^m)^2}\,\mathrm{d}x\right)\theta\,\mathrm{d}t.
\end{aligned}$$

Note that $\partial_t\rho^m$ is integrable since $\rho^m \in W^{1,\infty}(Q)$ by (4.144). From the last equality, we deduce that

$$\langle \partial_t\lambda^m, \phi\rangle = \left\langle \partial_t\varrho^m, \frac{\phi}{\zeta(\rho^m)}\right\rangle - \int_\Omega \varrho^m\,\phi\,\frac{\zeta'(\rho^m)\,\partial_t\rho^m}{\zeta(\rho^m)^2}\,\mathrm{d}x \qquad \text{a.e. in } \Omega$$

for all $\phi \in W^{1,2}(\Omega)$. It follows that, for all $\phi \in W^{1,2}(\Omega)$,

$$|\langle \partial_t\lambda^m, \phi\rangle| \leq \|\partial_t\varrho^m\|_{(W^{1,2}(\Omega))'}\left\|\frac{\phi}{\zeta(\rho^m)}\right\|_{W^{1,2}(\Omega)} + \|\varrho^m\|_{L^2(\Omega)}\left\|\phi\,\frac{\zeta'(\rho^m)\,\partial_t\rho^m}{\zeta(\rho^m)^2}\right\|_{L^2(\Omega)}.$$

Note that by (4.174) we have that $\|\partial_t \varrho^m\|_{(W^{1,2}(\Omega))'} \leq C(m)$ and $\|\varrho^m\|_{L^2(\Omega)} \leq C(m)$. Also, by using $\|\nabla_x \rho^m\|_{L^\infty(Q;\mathbb{R}^d)} \leq C(m)$ and $\|\partial_t \rho^m\|_{L^\infty(Q)} \leq C(m)$, we obtain the following bounds

$$\left\| \frac{\phi}{\zeta(\rho^m)} \right\|_{W^{1,2}(\Omega)} \leq C(m)\|\phi\|_{W^{1,2}(\Omega)},$$

$$\left\| \phi \, \frac{\zeta'(\rho^m)\, \partial_t \rho^m}{\zeta(\rho^m)^2} \right\|_{L^2(\Omega)} \leq C(m)\|\phi\|_{W^{1,2}(\Omega)}.$$

Hence,

$$|\langle \partial_t \lambda^m, \phi \rangle| \leq C(m)\|\phi\|_{W^{1,2}(\Omega)},$$

which then implies that

$$\|\partial_t \lambda^m\|^2_{L^2(0,T;(W^{1,2}(\Omega))')} = \int_0^T \|\partial_t \lambda^m\|^2_{(W^{1,2}(\Omega))'} \,\mathrm{d}t \leq C(m).$$

We have shown that

$$\lambda^m \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \lambda^m \in L^2(0,T;(W^{1,2}(\Omega))'),$$

whereby, upon defining $\alpha^m := \omega - \lambda^m$, also

$$\alpha^m \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \alpha^m \in L^2(0,T;(W^{1,2}(\Omega))'). \tag{4.175}$$

It then follows from (4.173) that, for any $\overline{\varphi} \in W^{1,2}(\Omega)$ and any $\theta \in C_0^\infty((0,T))$, we have

$$\begin{aligned}
0 &= -\int_0^T \int_\Omega \zeta(\rho^m)\, \alpha^m\, \overline{\varphi}\, \partial_t \theta \,\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \zeta(\rho^m)\, \alpha^m\, \boldsymbol{v}^m \cdot (\nabla_x \overline{\varphi})\, \theta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \overline{\varphi})\, \theta \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_\Omega \alpha^m \left[\partial_t(\zeta(\rho^m)\theta) - (\partial_t \zeta(\rho^m))\, \theta\right] \overline{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \alpha^m \left[\nabla_x \cdot (\zeta(\rho^m)\, \boldsymbol{v}^m\, \overline{\varphi}) - (\nabla_x \cdot (\zeta(\rho^m)\, \boldsymbol{v}^m))\, \overline{\varphi}\right] \theta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \overline{\varphi})\, \theta \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_\Omega \alpha^m \left[\partial_t(\zeta(\rho^m)\, \theta)\, \overline{\varphi} + \nabla_x \cdot (\zeta(\rho^m)\, \boldsymbol{v}^m\, \overline{\varphi})\, \theta\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \alpha^m \left[\partial_t \zeta(\rho^m) + \nabla_x \cdot (\zeta(\rho^m)\, \boldsymbol{v}^m)\right] \overline{\varphi}\, \theta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \overline{\varphi})\, \theta \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_\Omega \alpha^m \left[\partial_t(\zeta(\rho^m)\, \theta)\, \overline{\varphi} + \nabla_x \cdot (\zeta(\rho^m)\, \boldsymbol{v}^m\, \overline{\varphi})\, \theta\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \overline{\varphi})\, \theta \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where in the transition to the last line we made use the fact that the renormalized equation (4.143) satisfied by ρ^m, with $\beta = \zeta$, holds almost everywhere on $Q = \Omega \times (0,T)$. We then deduce from the last equality that

$$\int_0^T \langle \partial_t \alpha^m, \zeta(\rho^m)\,\overline{\varphi}\rangle\,\theta\,\mathrm{d}t + \int_0^T \left(\int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m)\,\boldsymbol{v}^m\,\overline{\varphi}\,\mathrm{d}x\right)\theta\,\mathrm{d}t + \int_0^T \left(\int_\Omega \nabla_x \alpha^m \cdot (\nabla_x \overline{\varphi})\mathrm{d}x\right)\theta\,\mathrm{d}t = 0,$$

for all $\overline{\varphi} \in W^{1,2}(\Omega)$ and all $\theta \in C_0^\infty((0,T))$. Hence, also

$$\langle \partial_t \alpha^m, \zeta(\rho^m)\,\overline{\varphi}\rangle + \int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m)\,\boldsymbol{v}^m\,\overline{\varphi}\,\mathrm{d}x + \int_\Omega \nabla_x \alpha^m \cdot \nabla_x \overline{\varphi}\,\mathrm{d}x = 0 \qquad (4.176)$$

for all $\overline{\varphi} \in W^{1,2}(\Omega)$ and for a.e. $t \in (0,T)$. Our objective is to show that $\alpha^m = \omega - \lambda^m \geq 0$ a.e. in Q. To this end, it seems tempting to take $\overline{\varphi} = [\alpha^m]_-$ in (4.176); however, the calculation that would by use of the renormalized equation (4.143) satisfied by ρ^m with $\beta = \zeta$ result in the desired inequality

$$\int_\Omega \zeta(\rho)\,([\alpha^m(x,t)]_-)^2\,\mathrm{d}x \leq \int_\Omega \zeta(\rho)\,([\alpha^m(x,0)]_-)^2\,\mathrm{d}x = \int_\Omega \zeta(\rho)\,([\omega - \lambda_0^m(x)]_-)^2\,\mathrm{d}x = 0,$$

which would then ultimately imply that $[\omega - \lambda^m(x,t)]_- = 0$, a.e. in Q, is difficult to justify rigorously. The main obstacle in the approach is the limited regularity of α^m in conjunction with the fact that the function $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_+$ is only Lipschitz continuous. An equivalent way of phrasing this would be to say that we define $G(s) := \frac{1}{2}([s]_+)^2$ for $s \in \mathbb{R}$, and we take as our test function in (4.176) the function $\overline{\varphi} = G'(\alpha^m)$. We shall overcome this difficulty by making a different choice of the function G.

For $\delta \in (0,1)$, let

$$G_\delta(s) := \begin{cases} \frac{s^2-\delta^2}{2\delta} + s(\log\delta - 1) + 1 & \text{for } s \leq \delta, \\ s(\log s - 1) + 1 & \text{for } s \geq \delta. \end{cases}$$

It then follows that

$$G_\delta'(s) = \begin{cases} \frac{s}{\delta} + \log\delta - 1 & \text{for } s \leq \delta, \\ \log s & \text{for } s \geq \delta, \end{cases}$$

and

$$G_\delta''(s) = \begin{cases} 1/\delta & \text{for } s \leq \delta, \\ 1/s & \text{for } s \geq \delta. \end{cases}$$

Clearly $G_\delta \in C^{2,1}(\mathbb{R})$, $G_\delta(s) \geq 0$ for all $s \in \mathbb{R}$, $G_\delta(1) = 0$, G_δ is strictly convex, and in addition

$$G_\delta(s) \geq \begin{cases} \frac{s^2}{2\delta} & \text{for } s \leq 0, \\ 0 & \text{for } s \geq 0. \end{cases}$$

We shall choose $\overline{\varphi} = G'_\delta(\alpha^m)$ in (4.176), and will at the end of the calculation pass to the limit $\delta \to 0_+$. Hence, a.e. in $(0,T)$,

$$\langle \partial_t \alpha^m, \zeta(\rho^m)\, G'_\delta(\alpha^m)\rangle + \int_\Omega \nabla_x \alpha^m \cdot \zeta(\rho^m)\, \boldsymbol{v}^m\, G'_\delta(\alpha^m)\,\mathrm{d}x + \int_\Omega \nabla_x \alpha^m \cdot \nabla_x G'_\delta(\alpha^m)\,\mathrm{d}x = 0.$$

Equivalently,

$$\langle \partial_t \alpha^m,\, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle + \int_\Omega \zeta(\rho^m)\,\boldsymbol{v}^m \cdot \nabla_x G_\delta(\alpha^m)\,\mathrm{d}x + \int_\Omega G''_\delta(\alpha^m)\,|\nabla_x \alpha^m|^2\,\mathrm{d}x = 0$$

a.e. in $(0,T)$. Thus, after partial integration in the second term on the left-hand side noting that $\boldsymbol{v}^m|_{\partial\Omega\times(0,T)} = 0$ and $\operatorname{div}\boldsymbol{v}^m = 0$ on Q we have that, a.e. in $(0,T)$,

$$\langle \partial_t \alpha^m,\, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle - \int_\Omega (\boldsymbol{v}^m \cdot \nabla_x \zeta(\rho^m))\, G_\delta(\alpha^m)\,\mathrm{d}x + \int_\Omega G''_\delta(\alpha^m)\,|\nabla_x \alpha^m|^2\,\mathrm{d}x = 0.$$

Next, we invoke the renormalized equation (4.143) to transform the second integral on the left-hand side further, resulting in

$$\langle \partial_t \alpha^m,\, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle + \int_\Omega (\partial_t \zeta(\rho^m))\, G_\delta(\alpha^m)\,\mathrm{d}x + \int_\Omega G''_\delta(\alpha^m)\,|\nabla_x \alpha^m|^2\,\mathrm{d}x = 0 \quad (4.177)$$

a.e. in $(0,T)$. Our objective is now to show that the first term of the left-hand side can be rewritten as follows:

$$\langle \partial_t \alpha^m, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle = \langle \partial_t G_\delta(\alpha^m), \zeta(\rho^m)\rangle \qquad \text{a.e. in } (0,T).$$

To this end, we extend α^m by 0 from $\Omega \times [0,T]$ to $\Omega \times \mathbb{R}$, and we mollify the resulting function, still denoted by α^m, with respect to t; to be more specific, for a nonnegative function $\chi \in C_0^\infty((0,T))$ such that $\int_{\mathbb{R}} \chi(t)\,\mathrm{d}t = 1$ and $\varepsilon \in (0,1)$, we let $\chi_\varepsilon(t) = \frac{1}{\varepsilon}\chi(\frac{t}{\varepsilon})$, and define

$$\alpha^m_\varepsilon := \alpha^m *_t \chi_\varepsilon \in C_0^\infty(\mathbb{R}; W^{1,2}(\Omega)),$$

where $*_t$ denotes convolution with respect to t. If then follows from (4.175) that

$$\|\alpha^m_\varepsilon - \alpha^m\|_{L^2(0,T;W^{1,2}(\Omega))} \to 0 \qquad \text{as } \varepsilon \to 0_+, \tag{4.178}$$

and, by Young's inequality for convolutions,

$$\|\partial_t \alpha^m_\varepsilon\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leq \|\partial_t \alpha^m\|_{L^2(0,T;(W^{1,2}(\Omega))')}. \tag{4.179}$$

Now, because $\alpha_\varepsilon^m \in C_0^\infty(\mathbb{R}; W^{1,2}(\Omega))$, it follows by the chain rule and an application of Hölder's inequality that $\partial_t G_\delta(\alpha_\varepsilon^m) = G'_\delta(\alpha_\varepsilon^m)\,\partial_t\alpha_\varepsilon^m \in C([0,T]; L^2(\Omega))$, and therefore, for any $\theta \in C_0^\infty((0,T))$,

$$\begin{aligned}
&\int_0^T \langle \partial_t G_\delta(\alpha_\varepsilon^m), \zeta(\rho^m)\rangle\,\theta(t)\,\mathrm{d}t \\
&= \int_0^T\int_\Omega \partial_t G_\delta(\alpha_\varepsilon^m)\,\zeta(\rho^m)\,\theta(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_\Omega G'_\delta(\alpha_\varepsilon^m)\,\partial_t\alpha_\varepsilon^m\,\zeta(\rho^m)\,\theta(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_\Omega G'_\delta(\alpha^m)\,\partial_t\alpha_\varepsilon^m\,\zeta(\rho^m)\,\theta(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega (G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m))\,\partial_t\alpha_\varepsilon^m\,\zeta(\rho^m)\,\theta(t)\,\mathrm{d}x\,\mathrm{d}t \\
&=: \mathrm{T}_{1,\varepsilon} + \mathrm{T}_{2,\varepsilon}.
\end{aligned} \tag{4.180}$$

We shall now pass to the limit $\varepsilon \to 0_+$ in this equality. First, because, by (4.179), $\partial_t\alpha_\varepsilon^m$ is, for each $m \geq 1$, uniformly bounded in $L^2(0,T;(W^{1,2}(\Omega))')$ and because, by definition, $L^2(0,T;(W^{1,2}(\Omega))')$ is the dual space of the normed linear space $L^2(0,T;W^{1,2}(\Omega))$, thanks to the Banach–Alaoglu theorem we can extract a weak* convergent subsequence, for which (without indicating the subsequence in our notation) we have by the uniqueness of the weak* limit that

$$\partial_t\alpha_\varepsilon^m \rightharpoonup^* \partial_t\alpha^m \qquad \text{in } L^2(0,T;(W^{1,2}(\Omega))').$$

Because $G'_\delta(\alpha^m)\,\zeta(\rho^m)\,\theta \in L^2(0,T;W^{1,2}(\Omega))$, the predual of $L^2(0,T;(W^{1,2}(\Omega))')$ it therefore follows that

$$\lim_{\varepsilon\to 0_+} \mathrm{T}_{1,\varepsilon} = \int_0^T \langle \partial_t\alpha^m, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle\,\theta(t)\,\mathrm{d}t.$$

Next, we will show that $\mathrm{T}_{2,\varepsilon} \to 0$ as $\varepsilon \to 0_+$. We begin by noting that

$$\begin{aligned}
|\mathrm{T}_{2,\varepsilon}| &\leq \|\partial_t\alpha_\varepsilon^m\|_{L^2(0,T;(W^{1,2}(\Omega))')}\,\|\zeta(\rho^m)\,\theta\,(G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m))\|_{L^2(0,T;W^{1,2}(\Omega))} \\
&\leq \sqrt{2}\left(\|\zeta(\rho^m)\,\theta\|^2_{L^\infty(Q)} + \|\nabla_x(\zeta(\rho^m))\,\theta\|^2_{L^\infty(Q;\mathbb{R}^d)}\right)^{\frac{1}{2}} \\
&\quad\times \|G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)\|_{L^2(0,T;W^{1,2}(\Omega))}.
\end{aligned} \tag{4.181}$$

It therefore remains to show that

$$\|G'_\delta(\alpha_\varepsilon^m) - G'_\delta(\alpha^m)\|_{L^2(0,T;W^{1,2}(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0_+. \tag{4.182}$$

First, observe that

$$\begin{aligned} \|G'_\delta(\alpha^m_\varepsilon) - G'_\delta(\alpha^m)\|_{L^2(0,T;L^2(\Omega))} &\leq \|G''_\delta\|_{L^\infty(\mathbb{R})}\|\alpha^m_\varepsilon - \alpha^m\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \frac{1}{\delta}\|\alpha^m_\varepsilon - \alpha^m\|_{L^2(0,T;L^2(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0_+. \end{aligned} \tag{4.183}$$

Next we shall show that $\|\nabla_x(G'_\delta(\alpha^m_\varepsilon)) - \nabla_x(G'_\delta(\alpha^m))\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \to 0$. We begin by observing that

$$\nabla_x(G'_\delta(\alpha^m_\varepsilon)) = G''_\delta(\alpha^m_\varepsilon)\,\nabla_x\alpha^m_\varepsilon.$$

Thanks to the strong conference (4.178), for m fixed, there exists a subsequence (with respect to ε, not indicated) such that $\alpha^m_\varepsilon \to \alpha^m$ a.e. in Q. Because $G''_\delta \in C^{0,1}(\mathbb{R})$, it then follows that $G''_\delta(\alpha^m_\varepsilon) \to G''_\delta(\alpha^m)$ a.e. in Q. Furthermore, $0 \leq G''_\delta(\alpha^m_\varepsilon) \leq 1/\delta$ a.e. in Q. In addition, thanks to a 'converse' of the Dominated Convergence Theorem, which asserts that each subsequence of a strongly convergent sequence in $L^1(Q)$ contains a dominated sub-subsequence (c.f., for example, Theorem 1 in [59]), because by (4.178) $|\alpha^m_\varepsilon - \alpha^m|^2 + |\nabla_x\alpha^m_\varepsilon - \nabla_x\alpha^m|^2 \to 0$ in $L^1(Q)$ as $\varepsilon \to 0_+$, there exists a nonnegative function $g \in L^1(Q)$ such that for a particular sub-subsequence (not indicated)

$$|\alpha^m_\varepsilon(x,t) - \alpha^m(x,t)|^2 + |\nabla_x\alpha^m_\varepsilon(x,t) - \nabla_x\alpha^m(x,t)|^2 \leq g(x,t) \tag{4.184}$$

for a.e. $(x,t) \in Q$. For this same sub-subsequence,

$$G''_\delta(\alpha^m_\varepsilon)\,\nabla_x\alpha^m_\varepsilon \to G''_\delta(\alpha^m)\,\nabla_x\alpha^m \qquad \text{a.e. in } Q$$

and

$$|G''_\delta(\alpha^m_\varepsilon(x,t))\,\nabla_x\alpha^m_\varepsilon(x,t) - G''_\delta(\alpha^m(x,t))\,\nabla_x\alpha^m(x,t)|^2 \leq \frac{2}{\delta^2}\,g(x,t) + \frac{4}{\delta^2}\,|\nabla_x\alpha^m(x,t)|^2$$

a.e. in Q. Thus, we can pass to the limit over this sub-subsequence to deduce by the Dominated Convergence Theorem that, as $\varepsilon \to 0_+$,

$$\begin{aligned} &\|\nabla_x(G'_\delta(\alpha^m_\varepsilon)) - \nabla_x(G'_\delta(\alpha^m))\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad = \int_Q |G''_\delta(\alpha^m_\varepsilon(x,t))\,\nabla_x\alpha^m_\varepsilon(x,t) - G''_\delta(\alpha^m(x,t))\,\nabla_x\alpha^m(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \to 0. \end{aligned} \tag{4.185}$$

By passing to the limit over this sub-subsequence, (4.183) and (4.185) imply (4.182), and then using (4.182) in (4.181), again by passage to the limit with $\varepsilon \to 0_+$ over this sub-subsequence implies that $\lim_{\varepsilon\to 0_+} |\mathrm{T}_{2,\varepsilon}| = 0$. Hence, from (4.180) we have, by passage to the limit over this sub-subsequence,

$$\lim_{\varepsilon\to 0_+} \int_0^T \langle \partial_t G_\delta(\alpha^m_\varepsilon), \zeta(\rho^m)\rangle\,\theta(t)\,\mathrm{d}t = \int_0^T \langle \partial_t\alpha^m, G'_\delta(\alpha^m)\,\zeta(\rho^m)\rangle\,\theta(t)\,\mathrm{d}t. \tag{4.186}$$

On the other hand, thanks to the smoothness of α_ε^m with respect to t, use of the chain rule and since $\partial_t G_\delta(\alpha_\varepsilon^m) \in C([0,T]; L^2(\Omega))$, the expression on the left-hand side of (4.186) can be rewritten as follows:

$$\begin{aligned}\int_0^T \langle \partial_t G_\delta(\alpha_\varepsilon^m), \zeta(\rho^m)\rangle\, \theta(t)\, \mathrm{d}t &= \int_0^T \int_\Omega \partial_t G_\delta(\alpha_\varepsilon^m)\, \zeta(\rho^m)\, \theta(t)\, \mathrm{d}x\, \mathrm{d}t \\ &= -\int_0^T \int_\Omega G_\delta(\alpha_\varepsilon^m)\, \partial_t(\zeta(\rho^m)\, \theta(t))\, \mathrm{d}x\, \mathrm{d}t.\end{aligned} \tag{4.187}$$

Now, with the sub-subsequence under consideration $G_\delta(\alpha_\varepsilon^m) \to G_\delta(\alpha^m)$ as $\varepsilon \to 0_+$ a.e. in Q. Also, because $0 \le G_\delta(s) \le C_\delta(s^2+1)$, it follows from (4.184) that

$$0 \le G_\delta(\alpha_\varepsilon^m(x,t)) \le C_\delta(|\alpha_\varepsilon^m(x,t)|^2 + 1) \le C_\delta + 2C_\delta\, g(x,t) + 2C_\delta\, |\alpha^m(x,t)|^2.$$

As the right-hand side of this is a function in $L^1(Q)$, by the Dominated Convergence Theorem we can pass to the limit over the sub-subsequence under consideration to deduce strong convergence of $G_\delta(\alpha_\varepsilon^m)$ to $G_\delta(\alpha^m)$ in $L^1(Q)$ as $\varepsilon \to 0_+$. It then follows from (4.186) and (4.187) that

$$-\int_0^T \int_\Omega G_\delta(\alpha^m)\, \partial_t(\zeta(\rho^m)\, \theta(t))\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \langle \partial_t \alpha^m, G_\delta'(\alpha^m)\, \zeta(\rho^m)\rangle\, \theta(t)\, \mathrm{d}t. \tag{4.188}$$

Next, we multiply (4.177) by θ, integrate this over $(0,T)$, and use (4.188) to rewrite the resulting first term on the left-hand side; hence.

$$\begin{aligned}-\int_0^T \int_\Omega G_\delta(\alpha^m)\, \partial_t(\zeta(\rho^m)\, \theta(t))\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega (\partial_t \zeta(\rho^m))\, G_\delta(\alpha^m)\, \theta(t)\, \mathrm{d}x\, \mathrm{d}t \\ + \int_0^T \int_\Omega G_\delta''(\alpha^m)\, |\nabla_x \alpha^m|^2\, \theta(t)\, \mathrm{d}x\, \mathrm{d}t = 0.\end{aligned} \tag{4.189}$$

As $\partial_t(\zeta(\rho^m)\,\theta) = \partial_t(\zeta(\rho^m))\,\theta + \zeta(\rho^m)\,\partial_t\theta$, the first integral on the left-hand side of (4.189) can be written as a sum of two integrals, the first of which cancels with the penultimate integral on the left-hand side of (4.189); hence,

$$-\int_0^T \int_\Omega G_\delta(\alpha^m)\, \zeta(\rho^m)\, \partial_t\theta\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega G_\delta''(\alpha^m)\, |\nabla_x \alpha^m|^2\, \theta\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Equivalently, since θ is independent of x,

$$-\int_0^T \left(\int_\Omega G_\delta(\alpha^m)\, \zeta(\rho^m)\, \mathrm{d}x\right) \partial_t\theta\, \mathrm{d}t + \int_0^T \left(\int_\Omega G_\delta''(\alpha^m)\, |\nabla_x \alpha^m|^2\, \mathrm{d}x\right) \theta\, \mathrm{d}t = 0 \tag{4.190}$$

for all $\theta \in C_0^\infty((0,T))$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G_\delta(\alpha^m)\, \zeta(\rho^m)\, \mathrm{d}x + \int_\Omega G_\delta''(\alpha^m)\, |\nabla_x \alpha^m|^2\, \mathrm{d}x = 0 \qquad \text{a.e. in } (0,T).$$

This implies, upon integration with respect to t and thanks to the properties of G_δ that, for all $t \in (0,T)$,

$$\begin{aligned}\int_\Omega G_\delta(\alpha^m(x,t))\,\zeta(\rho^m(x,t))\,\mathrm{d}x + \frac{1}{\delta}\int_0^t\int_\Omega |\nabla_x \alpha^m(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_\Omega G_\delta(\alpha^m(x,0))\,\zeta(\rho^m(x,0))\,\mathrm{d}x = \int_\Omega G_\delta(\omega - \lambda_0^m(x))\,\zeta(\rho_0^m(x))\,\mathrm{d}x.\end{aligned}$$

Thus, for all $t \in (0,T)$,

$$\int_\Omega G_\delta(\alpha^m(x,t))\,\zeta(\rho^m(x,t))\,\mathrm{d}x \leq \int_\Omega G_\delta(\omega - \lambda_0^m(x))\,\zeta(\rho_0^m(x))\,\mathrm{d}x.$$

Let us now denote by $\Omega_-(t)$, for $t \in (0,T)$, the set of all $x \in \Omega$ such that $\alpha^m(x,t) < 0$. Once again, appealing to the properties of G_δ, we have that $G_\delta(\alpha^m(s,t)) \geq |\alpha^m(s,t)|^2/(2\delta)$ for all $x \in \Omega_-(t)$. Therefore, for all $t \in (0,T)$,

$$\zeta_{\min}\int_{\Omega_-(t)} |\alpha^m(x,t)|^2\,\mathrm{d}x \leq 2\delta\int_\Omega G_\delta(\omega - \lambda_0^m(x))\,\zeta(\rho_0^m(x))\,\mathrm{d}x. \tag{4.191}$$

On the other hand, because $\omega - \lambda_0^m(x) \geq 0$ on Ω, by passing to the limit $\delta \to 0_+$ it follows that

$$\begin{aligned}\lim_{\delta\to 0_+}\int_\Omega G_\delta(\omega - \lambda_0^m(x))\,\zeta(\rho_0^m(x))\,\mathrm{d}x \\ = \int_\Omega [(\omega - \lambda_0^m(x))\,(\log(\omega - \lambda_0^m(x)) - 1) + 1]\,\zeta(\rho_0^m(x))\,\mathrm{d}x.\end{aligned}$$

Hence, by passing to the limit $\delta \to 0_+$ in (4.191) it follows that

$$\int_{\Omega_-(t)} |\alpha^m(x,t)|^2\,\mathrm{d}x \leq 0 \qquad \text{for all } t \in (0,T).$$

Therefore $\mathrm{meas}(\Omega_-(t)) = 0$ for all $t \in (0,T)$. In other words, $0 \leq \lambda^m(x,t) \leq \omega$ for a.e. $(x,t) \in Q$. Consequently,

$$\|\lambda^m\|_{L^\infty(Q)} \leq \|\lambda_0^m\|_{L^\infty(Q)} \leq \frac{\varrho_{\max}}{\zeta_{\min}} \leq C. \tag{4.192}$$

Noting that $\zeta(\cdot) \leq \zeta_{\max}$, we obtain from (4.170) and (4.192) that

$$\|\varrho^m\|_{L^\infty(Q)} \leq \zeta_{\max}\|\lambda^m\|_{L^\infty(Q)} \leq C. \tag{4.193}$$

By setting $\bar{\varphi} = \lambda^m$ in (4.172) and using that $\zeta(\rho^m)$ satisfies (4.140) we further deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \zeta(\rho^m(x,t))[\lambda^m(x,t)]^2\,\mathrm{d}x + \int_\Omega |\nabla_x \lambda^m(x,t)|^2\,\mathrm{d}x = 0,$$

and therefore, upon integration with respect to t, also

$$\int_0^T \int_\Omega |\nabla_x \lambda^m|^2 \,\mathrm{d}x \,\mathrm{d}t \le C.$$

From the uniform estimate (4.151) we deduce that

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} \tilde{\psi}^m(x,\boldsymbol{q},t) \log(1+\tilde{\psi}^m(x,\boldsymbol{q},t)) \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \le C(\ell),$$

where $\tilde{\psi}^m := M^m \hat{\psi}^m$. By De la Vallée Poussin's Theorem (c.f. Theorem 2.4.2) the sequence $\tilde{\psi}^m$ is uniformly equi-integrable. Hence, by the Dunford–Pettis Theorem (c.f. Theorem 2.4.3), the sequence $(\tilde{\psi}^m)_{m\ge1}$ is weakly relatively compact in $L^1(\mathcal{O}\times(0,T))$, which implies the existence of a subsequence (not relabelled) such that

$$\tilde{\psi}^m \rightharpoonup \tilde{\psi} \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T)).$$

Since M^m converges to M uniformly in $C(\overline{D})$, we deduce that

$$\hat{\psi}^m \rightharpoonup \frac{\tilde{\psi}}{M} =: \hat{\psi} \qquad \text{weakly in } L^1_{loc}(\mathcal{O}\times(0,T)).$$

Next, we shall show that

$$\hat{\psi}^m \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T).$$

Let $\mathcal{O}_0$ be a Lipschitz subdomain of $\mathcal{O}$ such that $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since $\mathcal{F}(s) = s\log s + 1 \ge s$ for all $s \in \mathbb{R}_{\ge0}$ and M^m is bounded below on $\mathcal{O}_0$ by a positive constant (which may depend on $\mathcal{O}_0$), we have from (4.151) that

$$\sup_{t\in(0,T)} \|\sqrt{\hat{\psi}^m(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\hat{\psi}^m}\|^2_{W^{1,2}(\mathcal{O}_0)} \,\mathrm{d}t \le C(\mathcal{O}_0). \tag{4.194}$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{(K+1)d}$, standard interpolation gives that

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^m|^{\frac{(K+1)d+2}{d(K+1)}} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t = \int_0^T \int_{\mathcal{O}_0} \left|\sqrt{\hat{\psi}^m}\right|^{\frac{2((K+1)d+2)}{d(K+1)}} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \le C(\mathcal{O}_0). \tag{4.195}$$

The application of Hölder's inequality gives that

$$\begin{aligned}\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^m|^p \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t &= 2^p \int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^p \left|\sqrt{\hat{\psi}^m}\right|^p \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \\ &\le 2^p \left(\int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t\right)^{\frac{p}{2}} \\ &\quad \times \left(\int_0^T \int_{\mathcal{O}_0} \left|\sqrt{\hat{\psi}^m}\right|^{\frac{2p}{2-p}} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t\right)^{\frac{2-p}{2}} \\ &\le C(\mathcal{O}_0),\end{aligned}$$

provided that

$$\frac{2p}{2-p} \leq \frac{2((K+1)d+2)}{d(K+1)}. \tag{4.196}$$

By selecting $p = \frac{d(K+1)+2}{d(K+1)+1}$, which is the largest value satisfying (4.196), we obtain

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^m|^{\frac{d(K+1)+2}{d(K+1)+1}} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \leq C(\mathcal{O}_0). \tag{4.197}$$

It directly follows from (4.192) that

$$\|\tilde{\psi}^m\|_{L^\infty(Q;L^1(D))} \leq C, \tag{4.198}$$

and therefore,

$$\|\hat{\psi}^m\|_{L^\infty(\Omega_0\times(0,T);L^1(D_0))} \leq C, \tag{4.199}$$

where $\mathcal{O}_0 = \Omega_0 \times D_0$. Interpolating between (4.195), (4.197) and (4.199) we see that for any two real numbers q_1 and q_2, with

$$\frac{(K+1)d+2}{(K+1)d} \leq q_1 < \infty, \qquad 1 < q_2 \leq \frac{(K+1)d+2}{(K+1)d},$$

and satisfying the relation

$$q_1\left(1 - \frac{1}{q_2}\right) = \frac{2}{(K+1)d},$$

we have that

$$\|\hat{\psi}^m\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \leq C(\mathcal{O}_0).$$

Since $\zeta(\cdot) \leq \zeta_{\max}$, using (4.152) and Hölder's inequality we deduce that there exists a $\delta > 0$ such that

$$\|\zeta(\rho^m)\boldsymbol{v}^m\hat{\psi}^m\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \leq C(\mathcal{O}_0). \tag{4.200}$$

Similarly, from (4.151), we deduce that

$$\|\zeta(\rho^m)\Lambda_\ell(\hat{\psi}^m)(\nabla_x \boldsymbol{v}^m)\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \leq C(\mathcal{O}_0). \tag{4.201}$$

To apply the Div-Curl Lemma, let us first define the following sequences of $(1+d+Kd)$-component vector fields:

$$\begin{aligned} H^m &:= (M^m\zeta(\rho^m)\hat{\psi}^m, M^m\zeta(\rho^m)\hat{\psi}^m\boldsymbol{v}^m - M^m\nabla_x\hat{\psi}^m, \\ &\qquad M\zeta(\rho^m)\Lambda_\ell(\hat{\psi}^m)(\nabla_x\boldsymbol{v}^m)\boldsymbol{q} - M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^m)), \\ Q^m &:= ((1+\hat{\psi}^m)^\alpha, \underbrace{0,\ldots,0}_{(d+Kd)\text{-times}}), \end{aligned}$$

for some $\alpha \in (0, 1/2)$. Consequently, using (4.151), (4.200) and (4.201), we deduce the existence of subsequences (not relabelled) such that

$$\begin{aligned} H^m &\rightharpoonup H \qquad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}), \\ Q^m &\rightharpoonup Q \qquad \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}), \end{aligned}$$

where, noting the uniform convergence of M^m and the strong convergences of $\zeta(\rho^m)$ and $\boldsymbol{v}^m$,

$$\begin{aligned} H &:= (M\zeta(\rho)\hat{\psi}, M\zeta(\rho)\hat{\psi}\boldsymbol{v} - M\nabla_x\hat{\psi}, M\zeta(\rho)\overline{\Lambda_\ell(\hat{\psi})}(\nabla_x \boldsymbol{v})\boldsymbol{q} - M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi})), \\ Q &:= (\overline{(1+\hat{\psi})^\alpha}, 0, \dots, 0). \end{aligned}$$

It follows from (4.134) that

$$\operatorname{div}_{t,x,\boldsymbol{q}} H^m = 0 \qquad \text{in } \mathcal{O}_0 \times (0,T).$$

Since $\alpha \in (0, 1/2)$, we obtain by using (4.194) that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}_0} |\operatorname{curl}_{t,x,\boldsymbol{q}} Q^m|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &= \int_0^T \int_{\mathcal{O}_0} |\nabla_{t,x,\boldsymbol{q}} Q^m - (\nabla_{t,x,\boldsymbol{q}} Q^m)^{\mathrm{T}}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ &\leq C \int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} (1+\hat{\psi}^m)^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ &\leq C \int_0^T \int_{\mathcal{O}_0} \left| \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^m} \right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ &\leq C(\mathcal{O}_0). \end{aligned}$$

Hence, $(\operatorname{div}_{t,x,\boldsymbol{q}} H^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$ and $(\operatorname{curl}_{t,x,\boldsymbol{q}} Q^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$. By choosing $\alpha < \frac{\delta}{1+\delta}$ we deduce using the Div-Curl Lemma that

$$H^m \cdot Q^m \rightharpoonup H \cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).$$

In particular, we have that

$$M^m \zeta(\rho^m)\hat{\psi}^m (1+\hat{\psi}^m)^\alpha \rightharpoonup M\zeta(\rho)\hat{\psi}\overline{(1+\hat{\psi})^\alpha}.$$

Because M^m converges to M uniformly and $\zeta(\rho^m)$ converges to $\zeta(\rho)$ strongly in $L^\infty(0,T; L^p(\Omega))$, $1 < p < \infty$, the above implies that

$$\hat{\psi}^m (1+\hat{\psi}^m)^\alpha \rightharpoonup \hat{\psi}\overline{(1+\hat{\psi})^\alpha}. \tag{4.202}$$

Since $(1+\hat{\psi}^m)^\alpha \rightharpoonup \overline{(1+\hat{\psi})^\alpha}$ in $L^1(\mathcal{O}_0 \times (0,T))$, we can add this to (4.202), which gives

$$(1+\hat{\psi}^m)^{\alpha+1} = (1+\hat{\psi}^m)(1+\hat{\psi}^m)^\alpha \rightharpoonup (1+\hat{\psi})\overline{(1+\hat{\psi})^\alpha}.$$

Thanks to the weak lower semi-continuity of the continuous convex function $s \in [0, \infty) \mapsto s^{\alpha+1} \in [0, \infty)$ we have that

$$(1 + \hat{\psi})^{\alpha+1} \leq (1 + \hat{\psi})\overline{(1 + \hat{\psi})^{\alpha}},$$

which is equivalent to

$$(1 + \hat{\psi})^{\alpha} \leq \overline{(1 + \hat{\psi})^{\alpha}}.$$

On the other hand, the function $s \in [0, \infty) \mapsto s^{\alpha} \in [0, \infty)$ is concave. Again, by the weak lower semi-continuity of the continuous convex function $s \in [0, \infty) \mapsto -s^{\alpha} \in [0, \infty)$, we deduce that

$$-(1 + \hat{\psi})^{\alpha} \leq -\overline{(1 + \hat{\psi})^{\alpha}}.$$

Therefore,

$$(1 + \hat{\psi})^{\alpha} = \overline{(1 + \hat{\psi})^{\alpha}}.$$

Since $s \in [0, \infty) \mapsto s^{\alpha} \in [0, \infty)$ is strictly concave, thanks to Theorem 2.4.6, there exists a subsequence (not relabelled), such that

$$\hat{\psi}^m \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0, T).$$

Since M^m converges uniformly to M, we have that

$$\tilde{\psi}^m \to \tilde{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0, T).$$

Now we want to extend the pointwise convergence result of $\tilde{\psi}^m$ to the whole of our domain $\mathcal{O} \times (0, T)$. For this purpose we choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k=1}^{\infty}$, i.e., $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \cdots \subset \mathcal{O}_0^k \subset \cdots$, satisfying $\cup_{k=1}^{\infty} \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of $(\tilde{\psi}^m)_{m=1}^{\infty}$ that is pointwise convergent to $\tilde{\psi}$ a.e. in $\mathcal{O}_0^k \times (0, T)$. Arguing by a diagonal procedure we deduce that there exists a subsequence such that

$$\tilde{\psi}^m \to \tilde{\psi} \qquad \text{a.e. in } \mathcal{O} \times (0, T). \tag{4.203}$$

Since $\tilde{\psi}^m$ is uniformly equi-integrable, using Vitali's Convergence Theorem, we obtain that

$$\tilde{\psi}^m \to \tilde{\psi} \qquad \text{strongly in } L^1(0, T; L^1(\mathcal{O})). \tag{4.204}$$

Interpolating between (4.198) and (4.204) gives that

$$M^m \hat{\psi}^m = \tilde{\psi}^m \to \tilde{\psi} = M\hat{\psi} \qquad \text{strongly in } L^p(Q; L^1(D)), \text{ for all } p \in [1, \infty). \tag{4.205}$$

Thus, by recalling (4.169) and (4.171) it follows that

$$\varrho^m \to \varrho \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty),$$

where

$$\varrho(x,t) = \zeta(\rho) \int_D M(\boldsymbol{q})\hat{\psi}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

Hence, and by recalling (4.168), we have that

$$\mu(\rho^m,\varrho^m) \to \mu(\rho,\varrho) \qquad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty). \tag{4.206}$$

For any measurable $U \subset (\mathcal{O} \times (0,T))$, with $|U| \leq \delta$, we use Hölder's inequality to deduce that

$$\begin{aligned}
&\int_U M^m |\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&= 2\int_U M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|\left|\sqrt{\hat{\psi}^m}\right|\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\leq 2\left(\int_U M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_U \tilde{\psi}^m\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{\frac{1}{2}},
\end{aligned} \tag{4.207}$$

which follows from the uniform estimate (4.151) and the uniform equi-integrability of $\tilde{\psi}^m$. By the Dunford–Pettis Theorem (c.f. Theorem 2.4.3) we can extract a subsequence such that

$$M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup \overline{M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m} \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}). \tag{4.208}$$

From (4.197) we deduce that $\nabla_{x,\boldsymbol{q}}\hat{\psi}^m$ weakly converges to $\nabla_{x,\boldsymbol{q}}\hat{\psi}$ locally in L^1. Since M^m converges uniformly to M, we can identify the weak limit $\overline{M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m}$ as $M\nabla_{x,\boldsymbol{q}}\hat{\psi}$. Analogously as in (4.208) it also follows from (4.197) that

$$\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m} \rightharpoonup \sqrt{M}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}} \qquad \text{weakly in } L^2(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}). \tag{4.209}$$

Since $M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m = 2\sqrt{M^m}\sqrt{\hat{\psi}^m}\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m} = 2\sqrt{\tilde{\psi}^m}\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}$, by using a similar calculation as in (4.207) we also see that

$$\begin{aligned}
\int_Q \left(\int_D M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|\,\mathrm{d}\boldsymbol{q}\right)^2\mathrm{d}x\,\mathrm{d}t &= 4\int_Q\left(\int_D \sqrt{\tilde{\psi}^m}\left|\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\right|\,\mathrm{d}\boldsymbol{q}\right)^2\mathrm{d}x\,\mathrm{d}t \\
&\leq 4\int_Q \|\tilde{\psi}^m\|^2_{L^1(D)}\|\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\|^2_{L^2(D)}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq 4\|\lambda^m\|^2_{L^\infty(Q)}\int_Q \|\sqrt{M^m}\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^m}\|^2_{L^2(D)}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C,
\end{aligned}$$

where we have used the estimates (4.151) and (4.192). Therefore, we can strengthen (4.208) as follows:

$$M^m \nabla_{x,\boldsymbol{q}} \hat{\psi}^m \rightharpoonup M \nabla_{x,\boldsymbol{q}} \hat{\psi} \qquad \text{weakly in } L^2(Q; L^1(D; \mathbb{R}^{d(K+1)})). \tag{4.210}$$

With the strong convergence results (4.166), (4.169) and (4.205) for $\boldsymbol{v}^m$, $\zeta(\rho^m)$ and $\tilde{\psi}^m$ we deduce that

$$M^m \zeta(\rho^m) \boldsymbol{v}^m \hat{\psi}^m \to M \zeta(\rho) \boldsymbol{v} \hat{\psi} \qquad \text{strongly in } L^p(Q; L^1(D; \mathbb{R}^d)), \tag{4.211}$$

where $p \in [1, \frac{2(d+2)}{d})$. It follows from (4.203) and the fact that $\Gamma_\ell \in C_0^\infty((-2\ell, 2\ell))$ that

$$\hat{\psi}^m \Gamma_\ell(\hat{\psi}^m) = \Lambda_\ell(\hat{\psi}^m) \to \Lambda_\ell(\hat{\psi}) = \hat{\psi} \Gamma_\ell(\hat{\psi}) \qquad \text{a.e. in } \mathcal{O} \times (0, T).$$

Hence, using (4.169) and the boundedness of the function $\Lambda_\ell(\cdot)$,

$$\zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m) \to \zeta(\rho) \Lambda_\ell(\hat{\psi}) \qquad \text{strongly in } L^p(\mathcal{O} \times (0, T)), \text{ for all } p \in [1, \infty).$$

It then follows from (4.164) that

$$M \zeta(\rho^m) \Lambda_\ell(\hat{\psi}^m)(\nabla_x \boldsymbol{v}^m) \rightharpoonup M \zeta(\rho) \Lambda_\ell(\hat{\psi})(\nabla_x \boldsymbol{v}) \quad \text{weakly in } L^p(Q \times D; \mathbb{R}^{d \times d}), \tag{4.212}$$

for all $p \in [1, 2)$. Using (4.134) and the convergence results (4.210), (4.211) and (4.212) we obtain that

$$\partial_t (M^m \zeta(\rho^m) \hat{\psi}^m) \rightharpoonup \partial_t (M \zeta(\rho) \hat{\psi}) \qquad \text{weakly in } L^p(0, T; W^{-1,1}(\mathcal{O})), \tag{4.213}$$

for all $p \in [1, 2)$. From (4.169) we deduce that

$$\zeta(\rho^m) \to \zeta(\rho) \qquad \text{a.e. in } Q.$$

Therefore, using (4.203) and the Dominated Convergence Theorem (thanks to the presence of the truncation T_ℓ), we can let $m \to \infty$ in (4.136) to deduce that

$$S_e^m \to S_e \qquad \text{strongly in } L^1(Q; \mathbb{R}^{d \times d}), \tag{4.214}$$

where

$$S_e = -k \int_D \left[K M \zeta(\rho) T_\ell(\hat{\psi}) I + \sum_{j=1}^{K} \zeta(\rho) T_\ell(\hat{\psi}) \nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \quad \text{a.e. in } Q. \tag{4.215}$$

Now let us reinstate the superscript ℓ. Collecting the convergence results (4.161), (4.164), (4.166), (4.167), (4.206), (4.169), (4.204), (4.210), (4.211), (4.212), (4.213) and (4.214), and using the fact that $\boldsymbol{f}^m$ converges to $\boldsymbol{f}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, we can pass to the limit as $m\to\infty$ in (4.132)–(4.134) to obtain the following:

$$\int_0^T [\langle \partial_t \rho^\ell, \eta\rangle - (\boldsymbol{v}^\ell \rho^\ell, \nabla_x \eta)]\,\mathrm{d}t = 0, \quad \text{for all } \eta \in L^1(0,T;W^{1,\frac{q}{q-1}}(\Omega)), \tag{4.216}$$

where $q\in(2,\infty)$ when $d=2$ and $q\in[3,6]$ when $d=3$,

$$\begin{aligned}
&\int_0^T \langle \partial_t(\rho^\ell \boldsymbol{v}^\ell), \boldsymbol{w}\rangle\,\mathrm{d}t + \int_0^T [-(\rho^\ell \boldsymbol{v}^\ell\otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{w}) + (\mu(\rho^\ell,\varrho^\ell)D(\boldsymbol{v}^\ell), \nabla_x \boldsymbol{w})]\,\mathrm{d}t\\
&\quad = \int_0^T [-(S_e^\ell, \nabla_x \boldsymbol{w}) + (\rho^\ell \boldsymbol{f}, \boldsymbol{w})]\,\mathrm{d}t\\
&\quad \text{for all } \boldsymbol{w}\in L^s(0,T;W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)) \quad \text{with } s>2,
\end{aligned}\tag{4.217}$$

and

$$\begin{aligned}
&\int_0^T \Big\langle \partial_t(M\zeta(\rho^\ell)\hat{\psi}^\ell), \varphi\Big\rangle_{\mathcal{O}} - \Big(M\zeta(\rho^\ell)\boldsymbol{v}^\ell\hat{\psi}^\ell, \nabla_x\varphi\Big)_{\mathcal{O}} - \Big(M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t\\
&\quad + \int_0^T (M\nabla_x\hat{\psi}^\ell, \nabla_x\varphi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^\ell), \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t = 0\\
&\quad \text{for all } \varphi\in L^\infty(0,T;W^{1,\infty}(\mathcal{O})).
\end{aligned}\tag{4.218}$$

Letting $m\to\infty$ in (4.150), we deduce the following energy inequality:

$$\begin{aligned}
&k\int_{\mathcal{O}} M\zeta(\rho^\ell(\cdot,t))\mathcal{F}(\hat{\psi}^\ell(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho^\ell(\cdot,t)|\boldsymbol{v}^\ell(\cdot,t)|^2\,\mathrm{d}x\\
&\quad + \int_0^t\int_\Omega \mu(\rho^\ell)|D(\boldsymbol{v}^\ell)|^2\,\mathrm{d}x\,\mathrm{d}s + 4kC_1\int_0^t\int_{\mathcal{O}} M\left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}^\ell}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s\\
&\quad \le k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(T_\ell(\hat{\psi}_0))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x + \int_0^t (\rho^\ell\boldsymbol{f}, \boldsymbol{v}^\ell)\,\mathrm{d}s.
\end{aligned}\tag{4.219}$$

Letting $m\to\infty$ in (4.137), (4.193) and (4.151), by the weak lower semi-continuity of norms, we deduce the following a priori estimate:

$$\begin{aligned}
&\sup_{t\in(0,T)}\big(\|\rho^\ell(\cdot,t)\|_{L^\infty(\Omega)} + \|\varrho^\ell(\cdot,t)\|_{L^\infty(\Omega)}\big)\\
&\quad + 2k\zeta_{\min}\sup_{t\in(0,T)}\|\mathcal{F}(\hat{\psi}^\ell(\cdot,t))\|_{L^1_M(\mathcal{O})} + \rho_{\min}\sup_{t\in(0,T)}\|\boldsymbol{v}^\ell(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad + \mu_{\min}c_0\int_0^T \|\boldsymbol{v}^\ell\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t + 8kC_1\int_0^T \left\|\sqrt{\hat{\psi}^\ell}\right\|^2_{W^{1,2}_M(\mathcal{O})}\,\mathrm{d}t\\
&\le 2k\zeta_{\max}\int_{\mathcal{O}} M\mathcal{F}(T_\ell(\hat{\psi}_0))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \rho_{\max}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad + \frac{\rho^2_{\max}}{\mu_{\min}c_0}\int_0^T \|\boldsymbol{f}\|^2_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}t\\
&\le C,
\end{aligned}\tag{4.220}$$

where C is a positive constant independent of ℓ and

$$\varrho^\ell(x,t) = \zeta(\rho^\ell(x,t)) \int_D M(\boldsymbol{q})\hat{\psi}^\ell(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{4.221}$$

We are now ready to pass to the final limit, $\ell \to \infty$.

4.3.6 Passage to the limit with ℓ

In this section, we focus on deriving uniform estimates independent of ℓ and passing to the limit as $\ell \to \infty$. We shall also show the weak attainment of the initial conditions.

From the uniform estimate (4.220) (noting that the constant C does not depend on ℓ) we deduce the existence of a subsequence (not relabelled) such that, as $\ell \to \infty$,

$$\begin{aligned} \boldsymbol{v}^\ell &\rightharpoonup \boldsymbol{v} && \text{weak* in } L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\\ \boldsymbol{v}^\ell &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)). \end{aligned} \tag{4.222}$$

By standard interpolation

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d).$$

From the definition (4.215) for S_e^ℓ, we apply integration by parts noting the fact that $M = 0$ on ∂D to get

$$\begin{aligned} S_e^\ell &= -k\int_D \left[KM\zeta(\rho)T_\ell(\hat{\psi})I + \sum_{j=1}^K \zeta(\rho)T_\ell(\hat{\psi})\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q}\\ &= k\sum_{j=1}^K \int_D M\zeta(\rho^\ell)\nabla_{\boldsymbol{q}^j} T_\ell(\hat{\psi}^\ell) \otimes \boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}. \end{aligned} \tag{4.223}$$

The integration by parts can be justified by Lemma 3.1 in [7]. Then by Fundamental Theorem of Calculus, we deduce that

$$\begin{aligned} \|S_e^\ell\|_{L^2(Q;\mathbb{R}^{d\times d})} &\le C(M,\zeta_{\max}) \int_0^T \int_\Omega \int_D |\nabla_{\boldsymbol{q}^j} T_\ell(\hat{\psi}^\ell)|^2\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(M,\zeta_{\max}) \int_0^T \int_\Omega \int_D (\Gamma_\ell(\hat{\psi}^\ell))^2\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(M,\zeta_{\max}), \end{aligned} \tag{4.224}$$

which then implies the existence of a subsequence (not relabelled) such that

$$S_e^\ell \rightharpoonup S_e \qquad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})). \tag{4.225}$$

We can perform a similar argument as in (4.156)–(4.159) to deduce that

$$\|\boldsymbol{v}^\ell\|_{N_2^\gamma(0,T;L^2(\Omega;\mathbb{R}^d))} \le C,$$

where $0 < \gamma < 1/4$ when $d = 2$ and $0 < \gamma \leq 1/8$ when $d = 3$. By the Aubin–Lions Lemma it follows that

$$\boldsymbol{v}^{\ell} \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^p(\Omega;\mathbb{R}^d)), \tag{4.226}$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,6)$ when $d = 3$.

From the uniform estimate (4.220) we deduce the existence of a subsequence (not relabelled) such that, as $\ell \to \infty$,

$$\rho^{\ell} \rightharpoonup \rho \qquad \text{weak* in } L^{\infty}(Q).$$

Using (4.216), (4.220) and Sobolev embedding we have that

$$\|\partial_t \rho^{\ell}\|_{L^2(0,T;W^{-1,p}(\Omega))} \leq C,$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,6]$ when $d = 3$. Therefore, we have that

$$\partial_t \rho^{\ell} \rightharpoonup \partial_t \rho \qquad \text{weakly in } L^2(0,T;W^{-1,p}(\Omega)), \tag{4.227}$$

where $p \in (1,\infty)$ when $d = 2$ and $p \in (1,6]$ when $d = 3$. We can also show that

$$\rho^{\ell} \to \rho \qquad \text{strongly in } L^p(Q), \text{ for any } p \in [1,\infty); \tag{4.228}$$

the proof proceeds similarly as in (4.113)–(4.116). With the convergence result (4.226) for $\boldsymbol{v}^{\ell}$ we can perform a similar argument as in Theorem VI.1.9 in [14] and strengthen the above convergence to get

$$\rho^{\ell} \to \rho \qquad \text{strongly in } C([0,T];L^p(\Omega)), \text{ for any } p \in [1,\infty). \tag{4.229}$$

From the assumption (4.24) we further deduce that

$$\zeta(\rho^{\ell}) \to \zeta(\rho) \qquad \text{strongly in } C([0,T];L^p(\Omega)), \text{ for any } p \in [1,\infty). \tag{4.230}$$

Using (4.217), (4.220) and (4.224) we have that

$$\int_0^T \|\partial_t(\rho^{\ell}\boldsymbol{v}^{\ell})\|^p_{W^{-1,p}_{\mathrm{div}}(\Omega)} \,\mathrm{d}t \leq C,$$

where $p \in \left(1, \frac{d+2}{d}\right]$ and C is a positive constant independent of ℓ, which then implies that

$$\partial_t(\rho^{\ell}\boldsymbol{v}^{\ell}) \rightharpoonup \partial_t(\rho\boldsymbol{v}) \qquad \text{weakly in } L^p(0,T;W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)). \tag{4.231}$$

Next, we will show the strong convergence of $\hat{\psi}^{\ell}$. From (4.220) we have that

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} M(\boldsymbol{q})\hat{\psi}^{\ell}(x,\boldsymbol{q},t)\log(1+\hat{\psi}^{\ell}(x,\boldsymbol{q},t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C,$$

where C is a positive constant independent of ℓ, which implies the uniform equi-integrability of the sequence $(\hat{\psi}^{\ell})_{\ell\geq 0}$. By the Dunford–Pettis Theorem (c.f. Theorem 2.4.3) the sequence $(\hat{\psi}^{\ell})_{\ell\geq 0}$ is weakly relatively compact in $L^1_M(\mathcal{O}\times(0,T))$. Hence, there exists a $\hat{\psi} \in L^1_M(\mathcal{O}\times(0,T))$ and a subsequence (not relabelled) such that

$$\hat{\psi}^{\ell} \rightharpoonup \hat{\psi} \qquad \text{weakly in } L^1_M(\mathcal{O}\times(0,T)).$$

The next step is to show that

$$\hat{\psi}^{\ell} \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T).$$

Let $\mathcal{O}_0$ be a Lipschitz subdomain of $\mathcal{O}$ such that $\mathcal{O}_0 \subset \overline{\mathcal{O}} \subset \mathcal{O}$. Since M is bounded below in $\mathcal{O}_0$, we have that

$$\sup_{t\in(0,T)} \|\sqrt{\hat{\psi}^{\ell}(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\hat{\psi}^{\ell}}\|^2_{W^{1,2}(\mathcal{O}_0)}\,\mathrm{d}t \leq C(\mathcal{O}_0).$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{d(K+1)}$, standard interpolation gives that

$$\int_0^T\int_{\mathcal{O}_0} |\hat{\psi}^{\ell}|^{\frac{(K+1)d+2}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = \int_0^T\int_{\mathcal{O}_0} \left|\sqrt{\hat{\psi}^{\ell}}\right|^{\frac{2((K+1)d+2)}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \leq C(\mathcal{O}_0).$$

The application of Hölder's inequality gives that

$$\int_0^T\int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\hat{\psi}^{\ell}|^{\frac{d(K+1)+2}{d(K+1)+1}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \leq C(\mathcal{O}_0). \tag{4.232}$$

Thanks to (4.220) we have that

$$\|\hat{\psi}^{\ell}\|_{L^\infty(Q;L^1_M(D))} \leq C,$$

where C is a positive constant independent of ℓ. It then follows that for any $q_1 \in (1,\infty)$ there exists a $q_2 > 1$ such that

$$\|\hat{\psi}^{\ell}\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \leq C(\mathcal{O}_0). \tag{4.233}$$

Hence, using (4.220), (4.233), the fact that $\zeta(\cdot) \leq \zeta_{\max}$ and Hölder's inequality, there exists a $\delta > 0$ such that

$$\|\zeta(\rho^{\ell})\boldsymbol{v}^{\ell}\hat{\psi}^{\ell}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} + \|\zeta(\rho^{\ell})\Lambda_{\ell}(\hat{\psi}^{\ell})(\nabla_x\boldsymbol{v}^{\ell})\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \leq C(\mathcal{O}_0). \tag{4.234}$$

To apply the Div-Curl Lemma we define

$$
\begin{aligned}
&H^\ell := (M\zeta(\rho^\ell)\hat{\psi}^\ell, M\zeta(\rho^\ell)\hat{\psi}^\ell \boldsymbol{v}^\ell - M\nabla_x\hat{\psi}^\ell, M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q} - M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}^\ell)),\\
&Q^\ell := ((1+\hat{\psi}^\ell)^\alpha, \underbrace{0,\dots,0}_{(d+Kd)\text{-times}}),
\end{aligned}
$$

for some $\alpha \in (0, \frac{\delta}{1+\delta})$. Consequently, using (4.220), (4.232) and (4.234), we deduce the existence of subsequences (not relabelled) such that

$$
\begin{aligned}
H^\ell &\rightharpoonup H \qquad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),\\
Q^\ell &\rightharpoonup Q \qquad \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),
\end{aligned}
$$

where, noting the strong convergences of $\zeta(\rho^\ell)$ and $\boldsymbol{v}^\ell$,

$$
\begin{aligned}
&H := (M\zeta(\rho)\hat{\psi}, M\zeta(\rho)\hat{\psi}\boldsymbol{v} - M\nabla_x\hat{\psi}, M\zeta(\rho)\overline{\Lambda_\ell(\hat{\psi})}(\nabla_x \boldsymbol{v})\boldsymbol{q} - M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi})),\\
&Q := (\overline{(1+\hat{\psi})^\alpha}, 0, \dots, 0).
\end{aligned}
$$

Similarly as above we can show that $\mathrm{div}_{t,x,\boldsymbol{q}}\, H^\ell$ and $\mathrm{curl}_{t,x,\boldsymbol{q}}\, Q^\ell$ are precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$. Thanks to our choice of α we can apply the Div-Curl Lemma to deduce that

$$
H^\ell \cdot Q^\ell \rightharpoonup H \cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).
$$

In particular, we have that

$$
M\zeta(\rho^\ell)\hat{\psi}^\ell(1+\hat{\psi}^\ell)^\alpha \rightharpoonup M\zeta(\rho)\overline{\hat{\psi}(1+\hat{\psi})^\alpha}.
$$

Thanks to the strong convergence of $\zeta(\rho^\ell)$ we deduce that

$$
\hat{\psi}^\ell(1+\hat{\psi}^\ell)^\alpha \rightharpoonup \overline{\hat{\psi}(1+\hat{\psi})^\alpha},
$$

which then implies that

$$
\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).
$$

By selecting a nondecreasing sequence of nested set $(\mathcal{O}_0^k)_{k\geq 1}$ such that $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$ and deducing pointwise convergence on each $\mathcal{O}_0^k$, we can argue following a diagonal procedure to deduce that there exists a subsequence such that

$$
\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O} \times (0,T).
$$

By Vitali's Convergence Theorem we obtain that

$$
\hat{\psi}^\ell \to \hat{\psi} \qquad \text{strongly in } L^1(0,T; L^1_M(\mathcal{O})).
$$

By standard interpolation with (4.220) we deduce that

$$\hat{\psi}^{\ell} \to \hat{\psi} \qquad \text{strongly in } L^p(Q; L^1_M(D)) \text{ for all } p \in [1,\infty). \tag{4.235}$$

We can now use (4.230) and (4.235) to pass to the limit $\ell \to \infty$ in (4.221) to deduce that

$$\varrho^{\ell} \to \varrho \qquad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty),$$

where

$$\varrho(x,t) = \zeta(\rho(x,t)) \int_D M(\boldsymbol{q}) \hat{\psi}(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}.$$

Thus, by noting (4.24) and (4.229), we have that

$$\mu(\rho^{\ell},\varrho^{\ell}) \to \mu(\rho,\varrho) \qquad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty). \tag{4.236}$$

Following a similar argument as in (4.207)–(4.210) we deduce the following convergence results:

$$\begin{aligned} \nabla_{x,\boldsymbol{q}} \hat{\psi}^{\ell} &\rightharpoonup \nabla_{x,\boldsymbol{q}} \hat{\psi} && \text{weakly in } L^1(0,T; L^1_M(\mathcal{O}; \mathbb{R}^{d(K+1)})), \\ \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}^{\ell}} &\rightharpoonup \nabla_{x,\boldsymbol{q}} \sqrt{\hat{\psi}} && \text{weakly in } L^2(0,T; L^2(\mathcal{O}; \mathbb{R}^{d(K+1)})), \\ \nabla_{x,\boldsymbol{q}} \hat{\psi}^{\ell} &\rightharpoonup \nabla_{x,\boldsymbol{q}} \hat{\psi} && \text{weakly in } L^2(Q; L^1_M(D; \mathbb{R}^{d(K+1)})). \end{aligned} \tag{4.237}$$

From the Lipschitz continuity of Γ, and therefore the Lipschitz continuity of Λ_ℓ, we obtain for any $p \in [1,\infty)$ that

$$\begin{aligned} \|\Lambda_\ell(\hat{\psi}^{\ell}) - \hat{\psi}\|_{L^p(Q;L^1_M(D))} &\leq \|\Lambda_\ell(\hat{\psi}^{\ell}) - \Lambda_\ell(\hat{\psi})\|_{L^p(Q;L^1_M(D))} + \|\Lambda_\ell(\hat{\psi}) - \hat{\psi}\|_{L^p(Q;L^1_M(D))} \\ &\leq C\|\hat{\psi}^{\ell} - \hat{\psi}\|_{L^p(Q;L^1_M(D))} + \|\Lambda_\ell(\hat{\psi}) - \hat{\psi}\|_{L^p(Q;L^1_M(D))}. \end{aligned}$$

The first term in the above inequality converges to 0 as $\ell \to \infty$ on noting (4.235). The second term in the above inequality also converges to 0 as $\ell \to \infty$ on noting that $\Lambda_\ell(\hat{\psi})$ converges to $\hat{\psi}$ almost everywhere in $\mathcal{O} \times (0,T)$ and applying the Dominated Convergence Theorem. Therefore,

$$\Lambda_\ell(\hat{\psi}^{\ell}) \to \hat{\psi} \qquad \text{strongly in } L^p(Q; L^1_M(D)) \text{ for all } p \in [1,\infty).$$

Moreover, using (4.222) and (4.230) we deduce that

$$\zeta(\rho^{\ell})\Lambda_\ell(\hat{\psi}^{\ell})(\nabla_x \boldsymbol{v}^{\ell}) \rightharpoonup \zeta(\rho)\hat{\psi}(\nabla_x \boldsymbol{v}) \quad \text{weakly in } L^p(Q; L^1_M(D; \mathbb{R}^{d\times d})), \tag{4.238}$$

for all $p \in [1,2)$. With the convergence results (4.222), (4.230) and (4.235) we deduce that

$$\zeta(\rho^{\ell})\boldsymbol{v}^{\ell}\hat{\psi}^{\ell} \to \zeta(\rho)\boldsymbol{v}\hat{\psi} \quad \text{strongly in } L^p(Q; L^1_M(D; \mathbb{R}^d)), \tag{4.239}$$

for any $p \in \left[1, \frac{2(d+2)}{d}\right)$. Consequently, using the identity (4.218) and the convergence results (4.237), (4.238) and (4.239), it follows that

$$\partial_t(M\zeta(\rho^\ell)\hat{\psi}^\ell) \rightharpoonup \partial_t(M\zeta(\rho)\hat{\psi}) \qquad \text{weakly in } L^p(0,T;W^{-1,1}(\mathcal{O})), \tag{4.240}$$

for any $p \in [1,2)$.

Next, we shall deduce the expression for the extra-stress tensor S_e. Using (4.230), (4.235) and (4.237) we pass to the limit as $\ell \to \infty$ in (4.223) to identify the weak limit S_e in (4.225) as

$$S_e = k\sum_{j=1}^{K}\int_D M\zeta(\rho)\nabla_{\boldsymbol{q}^j}\hat{\psi}\otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}.$$

With the convergence results (4.222), (4.225), (4.226), (4.227), (4.228), (4.231), (4.235), (4.236), (4.237), (4.238), (4.239) and (4.240), we pass to the limit as $\ell \to \infty$ in (4.216)–(4.218) to deduce that

$$\int_0^T [\langle \partial_t\rho, \eta\rangle - (\boldsymbol{v}\rho, \nabla_x\eta)]\,\mathrm{d}t = 0, \quad \text{for all } \eta \in L^1(0,T;W^{1,\frac{q}{q-1}}(\Omega)),$$

where $q \in (2,\infty)$ when $d=2$ and $q \in [3,6]$ when $d=3$,

$$\begin{aligned}&\int_0^T \langle \partial_t(\rho\boldsymbol{v}), \boldsymbol{w}\rangle\,\mathrm{d}t + \int_0^T [-(\rho\boldsymbol{v}\otimes\boldsymbol{v}, \nabla_x\boldsymbol{w}) + (\mu(\rho,\varrho)D(\boldsymbol{v}), \nabla_x\boldsymbol{w})]\,\mathrm{d}t\\ &\quad = \int_0^T [-(S_e, \nabla_x\boldsymbol{w}) + (\rho\boldsymbol{f}, \boldsymbol{w})]\,\mathrm{d}t \quad \text{for all } \boldsymbol{w} \in L^s(0,T;W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)) \qquad \text{where } s>2,\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \Big\langle \partial_t(M\zeta(\rho)\hat{\psi}), \varphi\Big\rangle_{\mathcal{O}} - \Big(M\zeta(\rho)\boldsymbol{v}\hat{\psi}, \nabla_x\varphi\Big)_{\mathcal{O}} - \Big(M\zeta(\rho)\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t\\ &\quad + \int_0^T (M\nabla_x\hat{\psi}, \nabla_x\varphi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}), \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t = 0\\ &\quad \text{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})).\end{aligned}$$

Letting $\ell \to \infty$ in (4.219) we deduce the following energy inequality, stated in (4.34):

$$\begin{aligned}&k\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\\ &\quad + \int_0^t\int_\Omega \mu(\rho,\varrho)|D(\boldsymbol{v})|^2\,\mathrm{d}x\,\mathrm{d}s + 4kC_1\int_0^t\int_{\mathcal{O}} M\left|\nabla_{x,\boldsymbol{q}}\sqrt{\hat{\psi}}\right|^2\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}s\\ &\le k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x + \int_0^t(\rho\boldsymbol{f},\boldsymbol{v})\,\mathrm{d}s,\end{aligned} \tag{4.241}$$

where $\mathcal{F}(s) = s \log s + 1$ for $s > 0$ and $\mathcal{F}(0) := \lim_{s\to 0+} \mathcal{F}(s) = 1$.

It remains to prove the weak continuity properties stated in (4.32) and the attainment of the initial data asserted in (4.33). First, we set $\boldsymbol{w}(x,t) := \chi_{[0,t]}\boldsymbol{u}(x)$ in (4.217), where $\boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ is arbitrary, with $s > 2$, to deduce that

$$\begin{aligned}((\rho^\ell \boldsymbol{v}^\ell)(t), \boldsymbol{u}) &+ \int_0^t [-(\rho^\ell \boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{u}) + (\mu(\rho^\ell, \varrho^\ell) D(\boldsymbol{v}^\ell), \nabla_x \boldsymbol{u})]\, \mathrm{d}\tau \\ &= \int_0^t [-(S_e^\ell, \nabla_x \boldsymbol{u}) + (\rho^\ell \boldsymbol{f}, \boldsymbol{w})]\, \mathrm{d}\tau + (\rho_0 \boldsymbol{v}_0, \boldsymbol{u}).\end{aligned} \tag{4.242}$$

Letting $\ell \to \infty$ in (4.242) and using the convergence results above we deduce, for almost all $t \in (0,T)$, that, for each $\boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ with $s > 2$,

$$\begin{aligned}((\rho \boldsymbol{v})(t), \boldsymbol{u}) &+ \int_0^t [-(\rho \boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{u}) + (\mu(\rho, \varrho) D(\boldsymbol{v}), \nabla_x \boldsymbol{u})]\, \mathrm{d}\tau \\ &= \int_0^t [-(S_e, \nabla_x \boldsymbol{u}) + \langle \rho \boldsymbol{f}, \boldsymbol{w}\rangle]\, \mathrm{d}\tau + (\rho_0 \boldsymbol{v}_0, \boldsymbol{u}).\end{aligned} \tag{4.243}$$

After a possible redefinition of $\rho\boldsymbol{v}$ on a set of measure zero, the above equation holds for all $t \in (0,T)$. Thus, by letting $t \to 0_+$, we deduce that

$$\lim_{t\to 0+} ((\rho \boldsymbol{v})(t), \boldsymbol{u}) = (\rho_0 \boldsymbol{v}_0, \boldsymbol{u}) \qquad \text{for all } \boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d) \text{ with } s > 2. \tag{4.244}$$

Replacing t with t' in (4.243) and subtracting the resulting equality from (4.243) we deduce, for almost every $t, t' \in (0,T)$ that, for each $\boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ with $s > 2$,

$$\begin{aligned}((\rho \boldsymbol{v})(t), \boldsymbol{u}) &+ \int_{t'}^t [-(\rho \boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{u}) + (\mu(\rho, \varrho) D(\boldsymbol{v}), \nabla_x \boldsymbol{u})]\, \mathrm{d}\tau \\ &= \int_{t'}^t [-(S_e, \nabla_x \boldsymbol{u}) + \langle \rho \boldsymbol{f}, \boldsymbol{w}\rangle]\, \mathrm{d}\tau + ((\rho \boldsymbol{v})(t'), \boldsymbol{u}).\end{aligned} \tag{4.245}$$

As the integrands in the integrals appearing on the left-hand side and right-hand side of (4.245) belong to $L^1(0,T)$, it follows by the Fundamental Theorem of Calculus for Lebesgue Integral (c.f., Theorem 7.18 in [61]), again, after a possible of redefinition of $\rho\boldsymbol{v}$ on a set of measure zero, that $t \mapsto ((\rho\boldsymbol{v})(t), \boldsymbol{u})$ is, for each $\boldsymbol{u} \in W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ with $s > 2$, absolutely continuous on $[0,T]$.

Similarly, setting $\varphi := \chi_{[0,t]}\phi(x,\boldsymbol{q})$ in (4.218), where $\phi \in W^{1,\infty}(\mathcal{O})$ is arbitrary, we deduce that

$$\begin{aligned}(M(\zeta(\rho^\ell)\hat{\psi}^\ell)(t), \phi)_{\mathcal{O}} &- \int_0^t \Big(M\zeta(\rho^\ell)\boldsymbol{v}^\ell \hat{\psi}^\ell, \nabla_x \phi\Big)_{\mathcal{O}}\, \mathrm{d}\tau \\ &- \int_0^t \Big(M\zeta(\rho^\ell)\Lambda_\ell(\hat{\psi}^\ell)(\nabla_x \boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}} \phi\Big)_{\mathcal{O}}\, \mathrm{d}\tau \\ &+ \int_0^t \Big[(M\nabla_x \hat{\psi}^\ell, \nabla_x \phi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}} \hat{\psi}^\ell), \nabla_{\boldsymbol{q}} \phi\Big)_{\mathcal{O}}\Big]\, \mathrm{d}\tau = (M\zeta(\rho_0)\hat{\psi}_0, \phi)_{\mathcal{O}}.\end{aligned} \tag{4.246}$$

Letting $\ell \to \infty$ in (4.246) and using the convergence results above we deduce, for almost all $t \in (0,T)$, that, for each $\phi \in W^{1,\infty}(\mathcal{O})$,

$$(M(\zeta(\rho)\hat{\psi})(t),\phi)_{\mathcal{O}} - \int_0^t \left[\left(M\zeta(\rho)\boldsymbol{v}\hat{\psi},\nabla_x\phi\right)_{\mathcal{O}} \mathrm{d}\tau + \left(M\zeta(\rho)\hat{\psi}(\nabla_x\boldsymbol{v})\boldsymbol{q},\nabla_{\boldsymbol{q}}\phi\right)_{\mathcal{O}}\right] \mathrm{d}\tau$$
$$+ \int_0^t \left[(M\nabla_x\hat{\psi},\nabla_x\phi)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\boldsymbol{q}}\hat{\psi}),\nabla_{\boldsymbol{q}}\phi\right)_{\mathcal{O}}\right] \mathrm{d}\tau = (M\zeta(\rho_0)\hat{\psi}_0,\phi)_{\mathcal{O}}.$$

After a possible redefinition of $\zeta(\rho)\hat{\psi}$ on a set of measure zero, the above equation holds for all $t \in (0,T)$. Letting $t \to 0_+$, we deduce that

$$\lim_{t\to 0+} (M(\zeta(\rho)\hat{\psi})(t),\phi)_{\mathcal{O}} = (M\zeta(\rho_0)\hat{\psi}_0,\phi)_{\mathcal{O}} \qquad \text{for all } \phi \in W^{1,\infty}(\mathcal{O}). \tag{4.247}$$

Similarly as in the case of $\rho\boldsymbol{v}$ above, after a possible redefinition on a set of measure zero, the function $t \mapsto (M(\zeta(\rho)\hat{\psi})(t),\phi)_{\mathcal{O}}$ is absolutely continuous on $[0,T]$ for each $\phi \in W^{1,\infty}(\mathcal{O})$. This completes the proofs of the assertion (4.32).

Next, we shall strengthen the attainment of the initial conditions by the following arguments. Since we have assumed that $\rho_0 \geq \rho_{\min} > 0$, i.e. there is no vacuum region in the fluid domain, we can strengthen the attainment of the initial conditions for ρ_0 and $\boldsymbol{v}_0$. Following a similar argument as in Theorem 2.2 in [46], we can show that

$$\rho\boldsymbol{v} \in C([0,T];L^2(\Omega;\mathbb{R}^d)),\ \boldsymbol{v} \in C([0,T];L^2(\Omega;\mathbb{R}^d)),$$

and also

$$\lim_{t\to 0+} \|\boldsymbol{v}(\cdot,t) - \boldsymbol{v}_0(\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} = 0, \tag{4.248}$$

provided that $S_e \in L^2(Q;\mathbb{R}^{d\times d})$, which holds by (4.225) and $\operatorname{div}_x \boldsymbol{v}_0 = 0$, which has been assumed. Furthermore, since $\rho \in C([0,T];L^p(\Omega))$ for all $p \in [1,\infty)$, then we deduce that

$$\lim_{t\to 0+} \|\rho(\cdot,t) - \rho_0(\cdot)\|_{L^p(\Omega)} = 0, \tag{4.249}$$

for all $p \in [1,\infty)$. Next, we shall focus on the attainment of the initial condition for $\hat{\psi}$. From the energy inequality (4.241), it is easy to see that

$$\begin{aligned}&\limsup_{t\to 0+} \left[k\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_{\Omega} \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\right]\\ &\quad\leq k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \frac{1}{2}\int_{\Omega}\rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x.\end{aligned} \tag{4.250}$$

Next, we will show that

$$\begin{aligned}&\liminf_{t\to 0+}\left[k\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\frac{1}{2}\int_\Omega \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\right]\\ &\quad\geq k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\frac{1}{2}\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x.\end{aligned} \tag{4.251}$$

First we note that by weak-strong lower semicontinuity (c.f., Theorem 13.1.1 in [2]),

$$\liminf_{t\to 0+} k\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\geq k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}, \tag{4.252}$$

since $\mathcal{F}(\cdot)$ is convex and we have shown that $\rho(\cdot,t)$ converges to $\rho_0(\cdot)$ strongly in $C([0,T];L^p(\Omega))$ as $t\to 0+$ by (4.249). Similarly, by weak-strong lower semicontinuity and the convexity of the squared norm $|\cdot|^2$, we deduce that

$$\liminf_{t\to 0+}\frac{1}{2}\int_\Omega \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\geq\frac{1}{2}\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x. \tag{4.253}$$

Combining (4.252) and (4.253), we obtain that

$$\begin{aligned}&\lim_{t\to 0+}\left[k\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\frac{1}{2}\int_\Omega \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\right]\\ &\quad= k\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}+\frac{1}{2}\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x.\end{aligned} \tag{4.254}$$

Since we have shown that the initial conditions for $\boldsymbol{v}$ and ρ are attained strongly, we deduce that

$$\lim_{t\to 0+}\int_\Omega \rho(\cdot,t)|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x=\int_\Omega \rho_0|\boldsymbol{v}_0|^2\,\mathrm{d}x. \tag{4.255}$$

Therefore, we have strengthened the attainment of the initial condition for $\hat{\psi}$ to

$$\lim_{t\to 0+}\int_{\mathcal{O}} M\zeta(\rho(\cdot,t))\mathcal{F}(\hat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}=\int_{\mathcal{O}} M\zeta(\rho_0)\mathcal{F}(\hat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}. \tag{4.256}$$

Thereby the proof of Theorem 4.2.2 is complete.

Remark 4.3.5. *However, since we do not have sufficient information about the structure of the function $\zeta(\cdot)$, it is hard to exploit the convexity of the function $\mathcal{F}(\cdot)$ to deduce the strong attainment of the initial condition for $\hat{\psi}$ in $L^1_M(\mathcal{O})$ in the sense that:*

$$\lim_{t\to 0+}\|\hat{\psi}(\cdot,t)-\hat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})}=0.$$

We have presented the existence proof to the Naiver–Stokes–Fokker–Planck system modelling incompressible nonhomogeneous dilute polymeric fluids with variable density in this chapter. The main feature is that the system involves a density-dependent and polymer-number-density-dependent viscosity coefficient and a density-dependent drag coefficient. In the next chapter, we shall take temperature effect into account. In addition to the Navier–Stokes equation and the Fokker–Planck equation, the system also includes a temperature evolution equation so that the whole system is thermodynamically consistent.

Chapter 5

Existence of weak solutions to a model of a nonisothermal homogeneous dilute polymeric fluid

So far, we have been considering isothermal models of incompressible, viscous Newtonian fluids. In this chapter, we focus on incompressible, viscous, nonisothermal dilute polymeric fluids. In particular, we show the existence of global-in-time weak solutions to the corotational Navier–Stokes–Fokker–Planck system. The proof is based on the Galerkin method as in the previous chapters. We also apply similar techniques as in [16] and [19].

In Section 5.1, we formulate the system of equations under consideration in detail, which includes an evolution equation for the temperature θ. We also introduce an additional cut-off function $\beta^L(\theta) := \min(\theta, L)$, where $L \gg 1$, in the coefficients for second order terms in the Fokker–Planck equation. In Section 5.2, we derive a formal energy identity and show the nonnegativity of the temperature θ and the probability density function ψ. In Section 5.3, we state the main result to be proved in this chapter, followed by the proof of the main result presented in Section 5.4. In addition to the truncation and Galerkin parameters introduced in similar ways as in the previous chapters, we also introduce a parameter δ in Subsection 5.4.2 to guarantee the positiveness of the coefficients in the Fokker–Planck equation. In Subsection 5.4.4–5.4.8, we derive uniform estimates independent of each parameter and then pass to the relevant limits. However, as we progress to deducing the limits for the Galerkin parameter m in Subsection 5.4.7, the presence of the term $2\nu(\theta^m)D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m)$ in the temperature equation requires strong convergence of the velocity gradient $D(\boldsymbol{v}^m)$. This is achieved by manipulating the convergence results for $\boldsymbol{v}^m$ and using the weak lower semi-continuity of norms. Also, since $D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m)$ only belongs to $L^1(Q)$, to deduce uniform bounds for θ^m, we take a special test function $(\theta^m)^\lambda$ with $\lambda \in (-1, 0)$ for the temperature equation, which is motivated by [16]. The same problem persists when we investigate the limits as the truncation parameter $\ell \to \infty$ in Subsection 5.4.8. To circumvent this, we consider the total energy given by $E^\ell := \frac{1}{2}|\boldsymbol{v}^\ell| + \theta^\ell$. Then the velocity gradient term in the temperature equation cancels with the viscous part of the Cauchy stress in the Navier–Stokes equation.

5.1 Statement of the problem

Based on the construction in [27], we identify the system of governing equations for the velocity $\boldsymbol{v}(x,t)$, the temperature $\theta(x,t)$ and the probability density function $\psi(x, \boldsymbol{q}, t)$ as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{div}_x(\boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div}_x \mathcal{T} + \boldsymbol{f} \qquad \text{in } Q, \tag{5.1}$$

$$\operatorname{div}_x \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{5.2}$$

and

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \operatorname{div}_x \left(\boldsymbol{v}\psi - \frac{k_B \theta}{2\zeta} \nabla_x \psi \right) \\ &\quad + \operatorname{div}_{\boldsymbol{q}} \left(\omega(\boldsymbol{v}) \boldsymbol{q} \psi - \frac{2\boldsymbol{F}}{\zeta}\psi - \frac{2k_B\theta}{\zeta} \nabla_{\boldsymbol{q}} \psi \right) = 0 \qquad \text{in } \mathcal{O} \times (0,T), \end{aligned} \tag{5.3}$$

where $\omega(\boldsymbol{v}) := \frac{1}{2}(\nabla_x \boldsymbol{v} - (\nabla_x \boldsymbol{v})^{\mathrm{T}})$. In the above system, $\boldsymbol{f}$ denotes the external body force, ζ denotes the hydrodynamic drag coefficient, which we take to be 1/2, and k_B denotes the Boltzmann constant, which we take to be 1. The spring force $\boldsymbol{F}$ is given via the potentials U_e and U_η by the formula

$$\begin{aligned} \boldsymbol{F} &= \nabla_{\boldsymbol{q}} \left[U_e \left(\frac{1}{2} \left| \frac{\boldsymbol{q}}{q_{ref}} \right|^2 \right) + \frac{\theta}{\theta_{ref}} U_\eta \left(\frac{1}{2} \left| \frac{\boldsymbol{q}}{q_{ref}} \right|^2 \right) \right] \\ &= \left(\frac{\mathrm{d}U_e}{\mathrm{d}s} \bigg|_{s = \frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2} + \frac{\theta}{\theta_{ref}} \frac{\mathrm{d}U_\eta}{\mathrm{d}s} \bigg|_{s = \frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2} \right) \frac{\boldsymbol{q}}{q_{ref}^2}, \end{aligned}$$

and the Cauchy stress tensor is given by the formula

$$\mathcal{T} = -pI + 2\nu(\theta) D(\boldsymbol{v}),$$

where $p = -\frac{1}{3} \operatorname{tr} \mathcal{T}$ is the mean normal stress and $D(\boldsymbol{v}) := (\nabla_x \boldsymbol{v} + (\nabla_x \boldsymbol{v})^{\mathrm{T}})/2$ is the symmetric part of the velocity gradient.

Remark 5.1.1. *Note that unlike Chapters 3 and 4, the Cauchy stress tensor here does not contain the polymeric part S_e, which is given by the Kramers expression. This is because the expression of the Cauchy stress tensor $\mathcal{T}$ is a consequence of the energy storage and entropy production mechanisms. With the corotational Fokker–Planck equation, the contribution resulting from the presence of polymer molecules in the solvent to the Cauchy stress tensor vanishes.*

Compared to the isothermal Fokker–Planck equation, the nonisothermal version (5.3) has an extra coefficient θ for second order terms. With the coefficient θ present, the terms $\theta \nabla_x \psi$ and $\theta \nabla_{\boldsymbol{q}} \psi$ lack integrability even in the weak formulation. The details will be discussed at the end of the proof. Motivated by [4], we introduce a cut-off function $\beta^L \in C([0,\infty))$ with

$$\beta^L(\theta) := \begin{cases} \theta & \text{if } \theta \leq L, \\ L & \text{if } \theta > L. \end{cases} \tag{5.4}$$

Then, for $L \gg 1$, (5.3) becomes

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \operatorname{div}_x \left(\boldsymbol{v}\psi - \frac{k_B \beta^L(\theta)}{2\zeta} \nabla_x \psi \right) \\ &\qquad + \operatorname{div}_{\boldsymbol{q}} \left(\omega(\boldsymbol{v}) \boldsymbol{q} \psi - \frac{2\boldsymbol{F}}{\zeta} \psi - \frac{2 k_B \beta^L(\theta)}{\zeta} \nabla_{\boldsymbol{q}} \psi \right) = 0 \qquad \text{in } \mathcal{O} \times (0,T). \end{aligned} \tag{5.5}$$

Remark 5.1.2. *This is achieved by replacing θ by the cut-off function $\beta^L(\theta)$ in the energy fluxes $\boldsymbol{j}_{\varphi,x}$ and $\boldsymbol{j}_{\varphi,\boldsymbol{q}}$ in Section 5.2 in [27] and then the derivation follows. This change is physically reasonable because the temperature cannot go off to infinity in an isolated container.*

The temperature evolution equation is given as follows:

$$\begin{aligned} c_V \left(\frac{\partial \theta}{\partial t} + \operatorname{div}_x(\boldsymbol{v}\theta) \right) &= \operatorname{div}_x(\kappa(\theta) \nabla_x \theta) + 2\nu(\theta) D(\boldsymbol{v}) : D(\boldsymbol{v}) \\ &\quad + \frac{2}{\zeta} \int_D (\nabla_{\boldsymbol{q}} U_e) \cdot \nabla_{\boldsymbol{q}} \left(U_e + \frac{\theta}{\theta_{ref}} U_\eta \right) \psi \, \mathrm{d}\boldsymbol{q} \\ &\quad - \frac{2 k_B \theta}{\zeta} \int_D \left[\Delta_{\boldsymbol{q}} U_e \left(\frac{1}{2} \left| \frac{\boldsymbol{q}}{q_{ref}} \right|^2 \right) \right] \psi \, \mathrm{d}\boldsymbol{q} \qquad \text{in } Q, \end{aligned} \tag{5.6}$$

where $c_V > 0$ denotes the specific heat at constant volume. For simplicity, we take c_V to be 1, q_{ref} to be 1, and θ_{ref} to be 1.

We shall also specify the initial and boundary conditions. The Navier–Stokes equation (5.1) is supplemented by the following no-slip boundary condition:

$$\boldsymbol{v} = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{5.7}$$

The Fokker–Planck equation (5.5) is supplemented by the following boundary conditions:

$$\left[\omega(\boldsymbol{v})\boldsymbol{q}\psi - \frac{2\boldsymbol{F}}{\zeta}\psi - \frac{2k_B\beta^L(\theta)}{\zeta}\nabla_{\boldsymbol{q}}\psi\right]\cdot \boldsymbol{n}_{\boldsymbol{q}} = 0 \qquad \text{on } \Omega\times\partial D\times(0,T), \tag{5.8}$$

$$\nabla_x\psi\cdot\boldsymbol{n}_x = 0 \qquad \text{on } \partial\Omega\times D\times(0,T), \tag{5.9}$$

where $\boldsymbol{n}_{\boldsymbol{q}}$ is the unit outward normal vector to ∂D and $\boldsymbol{n}_x$ is the unit outward normal vector to $\partial\Omega$. The boundary condition for the temperature θ is given as follows:

$$\nabla_x\theta\cdot\boldsymbol{n}_x = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{5.10}$$

where $\boldsymbol{n}_x$ is the unit outward normal vector to $\partial\Omega$. We impose the following as initial conditions:

$$\begin{aligned} \boldsymbol{v}(x,0) &= \boldsymbol{v}_0(x) && \text{in } \Omega,\\ \theta(x,0) &= \theta_0(x) && \text{in } \Omega,\\ \psi(x,\boldsymbol{q},0) &= \psi_0(x,\boldsymbol{q}) && \text{in } \mathcal{O}. \end{aligned} \tag{5.11}$$

By introducing the Maxwellian

$$M(x,\boldsymbol{q},t) := \frac{\mathrm{e}^{-\frac{U\left(\frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2\right)}{k_B\,\theta(x,t)}}}{\int_D \mathrm{e}^{-\frac{U\left(\frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2\right)}{k_B\,\theta(x,t)}}\,\mathrm{d}\boldsymbol{q}},$$

we define $\hat{\psi} := \frac{\psi}{M}$ and rewrite the Fokker–Planck equation as

$$\begin{aligned}\frac{\partial(M\hat{\psi})}{\partial t} &+ \mathrm{div}_x\left(\boldsymbol{v}M\hat{\psi} - \beta^L(\theta)\nabla_x(M\hat{\psi})\right)\\ &+ \mathrm{div}_{\boldsymbol{q}}\left(\omega(\boldsymbol{v})\boldsymbol{q}M\hat{\psi} - 4\beta^L(\theta)M\nabla_{\boldsymbol{q}}\hat{\psi}\right) = 0. \end{aligned} \tag{5.12}$$

On assuming that $U_e = 0$, the temperature evolution equation becomes

$$\frac{\partial\theta}{\partial t} + \mathrm{div}_x(\boldsymbol{v}\theta) = \mathrm{div}_x(\kappa(\theta)\nabla_x\theta) + 2\nu(\theta)D(\boldsymbol{v}) : D(\boldsymbol{v}). \tag{5.13}$$

The Maxwellian is then also simplified and becomes

$$M(\boldsymbol{q}) = \frac{\mathrm{e}^{-U_\eta\left(\frac{1}{2}|\boldsymbol{q}|^2\right)}}{\int_D \mathrm{e}^{-U_\eta\left(\frac{1}{2}|\boldsymbol{q}|^2\right)}\,\mathrm{d}\boldsymbol{q}},$$

which is independent of x and t.

Next, we shall introduce a few assumptions on the data. We assume that $\partial\Omega \in C^{0,1}$. For the Maxwellian M, we assume that

$$M \in C(\overline{D}) \cap C^{0,1}_{loc}(D) \cap W^{1,1}_0(D). \tag{5.14}$$

We impose the following assumptions on the function $\nu(\cdot)$ and $\kappa(\cdot)$. We assume that ν and κ are Lipschitz functions of θ on $[0,\infty)$ and satisfy

$$0 < \nu_{\min} \leq \nu(\theta) \leq \nu_{\max}, \tag{5.15}$$

$$0 < \kappa_{\min} \leq \kappa(\theta) \leq \kappa_{\max}, \tag{5.16}$$

for all $\theta \geq 0$. For the initial velocity, we assume that

$$\boldsymbol{v}_0 \in L^2_{0,\mathrm{div}}(\Omega; \mathbb{R}^d). \tag{5.17}$$

For the initial probability density, we define $\hat{\psi}_0 := \frac{\psi_0}{M}$. We assume that

$$\hat{\psi}_0 \geq 0 \qquad \text{a.e. in } \mathcal{O}, \qquad \hat{\psi}_0 \in L^2(0,T; L^2_M(\mathcal{O})). \tag{5.18}$$

For the initial temperature, we assume that

$$\theta_0 \geq \theta_{\min} > 0, \tag{5.19}$$

for a.e. $x \in \Omega$.

In the next section, we shall establish a formal energy identity, which will guide the subsequent rigorous proof of existence of global-in-time weak solutions to the system under consideration.

5.2 Energy identity

In this section, we shall derive a formal energy identity assuming that $\boldsymbol{v}$, $\hat{\psi}$ and θ exist and are sufficiently smooth. Taking the $L^2(\Omega; \mathbb{R}^d)$ inner product of equation (5.1) with $\boldsymbol{v}$, we deduce upon partial integration and noting the boundary condition (5.7) on $\boldsymbol{v}$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}|\boldsymbol{v}|^2 \,\mathrm{d}x\right) + \int_\Omega 2\nu(\theta)|D(\boldsymbol{v})|^2 \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x. \tag{5.20}$$

Multiplying (5.12) by $\hat{\psi}$, integrating over $\mathcal{O}$ and using integration by parts together with the boundary conditions (5.8) and (5.9), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{O}} M|\hat{\psi}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q}\right) + \int_{\mathcal{O}} \beta^L(\theta) M \left|\nabla_x \hat{\psi}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\ + 4\int_{\mathcal{O}} \beta^L(\theta) M \left|\nabla_{\boldsymbol{q}} \hat{\psi}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} = 0.\end{aligned} \tag{5.21}$$

Integrating (5.13) over Ω and using the Divergence Theorem give that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \theta \,\mathrm{d}x\right) = \int_\Omega 2\nu(\theta)|D(\boldsymbol{v})|^2 \,\mathrm{d}x. \tag{5.22}$$

Adding (5.20), (5.21) and (5.22), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}|\boldsymbol{v}|^2 + \theta + \frac{1}{2}\int_D M|\hat{\psi}|^2 \,\mathrm{d}\boldsymbol{q}\right] \mathrm{d}x + \int_{\mathcal{O}} \beta^L(\theta) M \left|\nabla_x \hat{\psi}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\ + 4\int_{\mathcal{O}} \beta^L(\theta) M \left|\nabla_{\boldsymbol{q}} \hat{\psi}\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x.\end{aligned}$$

For the above identity to make sense, we show in the next section that θ and $\hat{\psi}$ are nonnegative.

5.2.1 Nonnegativity

In this section, we will show the nonnegativity of the temperature θ and the probability density function $\hat{\psi}$. We multiply (5.13) by $[\theta]_- := \min(0,\theta)$ and integrate over Ω to get, using the fact that $\mathrm{div}_x\, \boldsymbol{v} = 0$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|[\theta]_-\|^2_{L^2(\Omega)} + \int_\Omega \kappa(\theta)|\nabla_x[\theta]_-|^2 \,\mathrm{d}x = \int_\Omega 2\nu(\theta)\, |D(\boldsymbol{v})|^2\, [\theta]_- \,\mathrm{d}x.$$

Integrating the above identity with respect to time over $(0,t)$ and noting that $[\theta]_-$ is nonpositive and ν is nonnegative, we obtain

$$\|[\theta]_-(t)\|^2_{L^2(\Omega)} + 2\kappa_{\min}\int_0^t\int_\Omega |\nabla_x[\theta]_-|^2 \,\mathrm{d}x\,\mathrm{d}\tau \le \|[\theta]_-(0)\|^2_{L^2(\Omega)} = 0. \tag{5.23}$$

We have $[\theta]_-(0) \equiv 0$ since the initial temperature θ_0 is nonnegative. The second term on the left-hand side of (5.23) is nonnegative. Therefore, we have

$$\|[\theta]_-(t)\|^2_{L^2(\Omega)} \le 0,$$

which implies that

$$\theta \ge 0 \quad \text{a.e. in } Q.$$

Next, multiplying (5.12) by $[\hat{\psi}]_- := \min(0,\hat{\psi})$ and integrating over $\mathcal{O}$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M\,([\hat{\psi}]_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} \beta^L(\theta) M\, |\nabla_x[\hat{\psi}]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ + 4\int_{\mathcal{O}} \beta^L(\theta) M\, |\nabla_{\boldsymbol{q}}[\hat{\psi}]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 0.\end{aligned}$$

Integrating the above identity with respect to time over $(0,t)$, we get

$$\begin{aligned}\|[\hat{\psi}]_-(t)\|^2_{L^2_M(\mathcal{O})} &+ 2\int_0^t \int_{\mathcal{O}} \beta^L(\theta) M\, |\nabla_x [\hat{\psi}]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ &+ 8\int_0^t \int_{\mathcal{O}} \beta^L(\theta) M\, |\nabla_{\boldsymbol{q}} [\hat{\psi}]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau = \|[\hat{\psi}]_-(0)\|^2_{L^2_M(\mathcal{O})} = 0,\end{aligned}$$

since $\hat{\psi}_0$ is nonnegative. Since both integrals on the left-hand side are nonnegative, we have

$$\|[\hat{\psi}]_-(t)\|^2_{L^2_M(\mathcal{O})} \leq 0.$$

Therefore,

$$\hat{\psi} \geq 0 \quad \text{a.e. in } \mathcal{O} \times (0,T).$$

Now we have completed the set-up of the problem and derived a formal energy identity. In the next section, we shall state the main result.

5.3 The main result

In this section, we formulate our main result, which will be proved in the following sections.

Definition 5.3.1. *We say that the tuple* $(\boldsymbol{v}^L, \psi^L, \theta^L, E^L)$*, with*

$$\psi^L(x,\boldsymbol{q},t) = M(\boldsymbol{q})\hat{\psi}^L(x,\boldsymbol{q},t),$$

is a weak solution to the system of nonlinear partial differential equations (5.1)–(5.13)*, if the following conditions hold:*

(i) The functions $(\boldsymbol{v}^L, \hat{\psi}^L, \theta^L, E^L)$ *belong to the following function spaces:*

$$\begin{aligned}
\boldsymbol{v}^L &\in L^\infty(0,T; L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)) \cap L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\\
\hat{\psi}^L &\in L^\infty(0,T; L^2_M(\mathcal{O})) \cap L^2(0,T; W^{1,2}_M(\mathcal{O})),\ \hat{\psi}^L \geq 0 \textit{ a.e. in } \mathcal{O}\times(0,T),\\
\theta^L &\in L^p(Q) \cap L^s(0,T; W^{1,s}(\Omega)),\\
&\qquad \textit{for all } p \in \left[1, \frac{d+2}{d}\right) \textit{ and } s \in \left[1, \frac{d+2}{d+1}\right),\ \theta^L \geq \theta_{\min} > 0 \textit{ a.e. in } Q,\\
E^L &\in L^\infty(0,T; L^1(\Omega)),\ \textit{where } E^L = \frac{1}{2}|\boldsymbol{v}^L|^2 + \theta^L,\\
\partial_t \boldsymbol{v}^L &\in L^p(0,T; W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)),\ \textit{for all } p \in \left(1, \frac{d+2}{d}\right],\\
\partial_t(M\hat{\psi}^L) &\in L^p(0,T; W^{-1,1}(\mathcal{O})),\ \textit{for all } p \in [1,2),\\
\partial_t E^L &\in L^p(0,T; (W^{1,p'}(\Omega))'),\ \textit{for all } p \in \left[1, \frac{2(d+2)}{3d}\right),
\end{aligned}$$

where $Q := \Omega \times (0,T)$ *and* $\mathcal{O} := \Omega \times D$.

(ii) The system (5.1)–(5.13) *is satisfied in the weak form as follows:*

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{v}^L, \boldsymbol{w}\rangle \,\mathrm{d}t + \int_0^T [-(\boldsymbol{v}^L \otimes \boldsymbol{v}^L, \nabla_x \boldsymbol{w}) + (2\nu(\theta^L) D(\boldsymbol{v}^L), \nabla_x \boldsymbol{w})]\,\mathrm{d}t \\ = \int_0^T (\boldsymbol{f}, \boldsymbol{w})\,\mathrm{d}t, \quad \textit{for all } \boldsymbol{w} \in L^s(0,T; W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)),\end{aligned} \tag{5.24}$$

with $s > d$,

$$\begin{aligned}\int_0^T \Big\langle \partial_t(M\hat{\psi}^L), \varphi\Big\rangle_{\mathcal{O}} - \Big(M\boldsymbol{v}^L\hat{\psi}^L, \nabla_x\varphi\Big)_{\mathcal{O}} - \Big(M\hat{\psi}^L\,\omega(\boldsymbol{v}^L)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t \\ + \int_0^T \Big(M\beta^L(\theta^L)\nabla_x\hat{\psi}^L, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(4M\beta^L(\theta^L)\nabla_{\boldsymbol{q}}\hat{\psi}^L, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\,\mathrm{d}t = 0, \\ \textit{for all } \varphi \in L^\infty(0,T; W^{1,\infty}(\mathcal{O})),\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\int_0^T \langle \partial_t E^L, u\rangle\,\mathrm{d}t + \int_0^T [(\kappa(\theta^L)\nabla_x\theta^L, \nabla_x u) + (2\nu(\theta^L)D(\boldsymbol{v}^L)\boldsymbol{v}^L, \nabla_x u) - (E^L\boldsymbol{v}^L, \nabla_x u)]\,\mathrm{d}t \\ = \int_0^T (\boldsymbol{f}, \boldsymbol{v}^L u)\,\mathrm{d}t, \quad \textit{for all } u \in L^\infty(0,T; W^{1,\infty}(\Omega)).\end{aligned} \tag{5.26}$$

(iii) The initial conditions are satisfied in the following sense:

$$\lim_{t\to 0_+} \|\boldsymbol{v}^L(\cdot,t) - \boldsymbol{v}_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\hat{\psi}^L(\cdot,t) - \hat{\psi}_0(\cdot)\|^2_{L^2_M(\mathcal{O})} + \|\theta^L(\cdot,t) - \theta_0\|_{L^1(\Omega)} = 0. \tag{5.27}$$

Next we state the main theorem to be proved in this chapter.

Theorem 5.3.2. *Let* $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, *be a bounded open Lipschitz domain. Let* $K \in \mathbb{N}$ *be arbitrary and let* $D^i \subset \mathbb{R}^d$, $i = 1,\dots,K$, *be bounded open balls centred at the origin. Suppose that* $\boldsymbol{f} \in L^2(0,T; L^2(\Omega;\mathbb{R}^d))$. *Assume that the Maxwellian* $M : D \to \mathbb{R}$ *satisfies* (5.14), $\nu(\cdot)$ *and* $\kappa(\cdot)$ *satisfy* (5.15) *and* (5.16), *and the initial data* $(\boldsymbol{v}_0, \hat{\psi}_0, \theta_0)$ *satisfy* (5.17)–(5.19). *Then, there exists a tuple* $(\boldsymbol{v}^L, \hat{\psi}^L, \theta^L, E^L)$ *which is a weak solution to the system* (5.1)–(5.13) *in the sense of Definition 5.3.1. Moreover, for a.e.* $t \in (0,T)$, *the following energy inequality holds:*

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}|\boldsymbol{v}^L(\cdot,t)|^2 + \theta^L(\cdot,t)\right)\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}^L(\cdot,t))^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad + \int_0^t\int_{\mathcal{O}} M\beta^L(\theta^L)|\nabla_x\hat{\psi}^L|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau + 4\int_0^t\int_{\mathcal{O}} M\beta^L(\theta^L)|\nabla_{\boldsymbol{q}}\hat{\psi}^L|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ &\le \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}^L\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \left(\frac{1}{2}|\boldsymbol{v}_0(\cdot)|^2 + \theta_0(\cdot)\right)\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}_0(\cdot))^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.\end{aligned}$$

We have introduced the superscript L in the functions $(\boldsymbol{v}^L, \hat{\psi}^L, \theta^L, E^L)$ to remind the readers that the system is dependent on the cut-off parameter L, but later in the analysis, we shall omit the superscript L for the simplicity of presentation.

In the next section, we shall start the proof of Theorem 5.3.2 by constructing a sequence of approximations. Then, we shall pass to the limits in the approximation parameters by deriving uniform estimates.

5.4 Existence proof

This section is devoted to proving Theorem 5.3.2. Before embarking on the proof, we shall first apply three levels of approximations:

1. the first level of approximation is to truncate the convective term by a truncation parameter ℓ in Subsection 5.4.1, and we also truncate the initial condition of $\hat{\psi}$ to avoid technical difficulties;

2. since now the second derivative terms in the Fokker–Planck equation have additional coefficients $\beta^L(\theta)$, we need to ensure that the coefficients are strictly positive by introducing a parameter $\delta > 0$ in Subsection 5.4.2;

3. finally, we apply Galerkin semi-discretization in the spatial domains of $\boldsymbol{v}$, $\hat{\psi}$ and θ with parameters m, n and k in Subsection 5.4.3.

In Subsections 5.4.4 – 5.4.8, we derive uniform estimates independent of the parameters k, δ, n, m and ℓ and pass to the limit as $k, n, m, \ell \to \infty$ and $\delta \to 0_+$.

5.4.1 The first level of approximation: truncation

In order to preserve the energy identity, we truncate the convective term in the Navier–Stokes equation. We introduce the following smooth nonnegative function $\Gamma \in C_0^\infty((-2,2))$, such that $\Gamma(s) = 1$ for all $s \in [-1,1]$. For an arbitrary $\ell \in \mathbb{N}$, we define $\Gamma_\ell(s) := \Gamma(\frac{s}{\ell})$. The primitive function to Γ_ℓ is given by

$$T_\ell(s) := \int_0^s \Gamma_\ell(r)\,\mathrm{d}r.$$

Then, the ℓ-approximation of (5.1) and (5.2) are defined by

$$\frac{\partial \boldsymbol{v}^\ell}{\partial t} + \mathrm{div}_x(\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell) - \mathrm{div}_x(2\nu([\theta^\ell]_+)D(\boldsymbol{v}^\ell)) = \boldsymbol{f} \qquad \text{in } Q, \tag{5.28}$$

$$\mathrm{div}_x\, \boldsymbol{v}^\ell = 0 \qquad \text{in } Q, \tag{5.29}$$

where $[\theta^\ell]_+ \coloneqq \max(\theta^\ell, 0)$, with the following initial and boundary conditions:

$$\boldsymbol{v}^\ell(\cdot, 0) = \boldsymbol{v}_0(\cdot) \qquad \text{in } \Omega, \tag{5.30}$$

$$\boldsymbol{v}^\ell = 0 \qquad \text{on } \partial\Omega \times (0, T). \tag{5.31}$$

The ℓ-th approximation of the temperature evolution equation (5.13) is given by

$$\frac{\partial \theta^\ell}{\partial t} + \operatorname{div}_x(\theta^\ell \boldsymbol{v}^\ell) + \operatorname{div}_x(\kappa([\theta^\ell]_+)\nabla_x \theta^\ell) = 2\nu([\theta^\ell]_+) D(\boldsymbol{v}^\ell) : D(\boldsymbol{v}^\ell) \qquad \text{in } Q, \tag{5.32}$$

which is supplemented by the following initial and boundary conditions:

$$\theta^\ell(\cdot, x) = \theta_0(\cdot) \qquad \text{in } \Omega, \tag{5.33}$$

$$\nabla_x \theta^\ell \cdot \boldsymbol{n}_x = 0 \qquad \text{on } \partial\Omega \times (0, T). \tag{5.34}$$

Finally, we define the ℓ-th approximation of the Fokker–Planck equation by

$$\frac{\partial (M\hat{\psi}^\ell)}{\partial t} + \operatorname{div}_x \left(\boldsymbol{v}^\ell M \hat{\psi}^\ell - \beta^L(\theta^\ell) \nabla_x (M\hat{\psi}^\ell) \right) + \operatorname{div}_{\boldsymbol{q}} \left(\omega(\boldsymbol{v}^\ell) \boldsymbol{q} M \hat{\psi}^\ell - 4M\beta^L(\theta^\ell) \nabla_{\boldsymbol{q}} \hat{\psi}^\ell \right) = 0 \tag{5.35}$$

in $\mathcal{O} \times (0, T)$. The above equation is supplemented by the following boundary conditions:

$$\left[\omega(\boldsymbol{v}^\ell) \boldsymbol{q} M \hat{\psi}^\ell - 4M\beta^L(\theta^\ell) \nabla_{\boldsymbol{q}} \hat{\psi}^\ell \right] \cdot \boldsymbol{n}_{\boldsymbol{q}} = 0 \quad \text{on } \Omega \times \partial D \times (0, T), \tag{5.36}$$

$$M \nabla_x \hat{\psi}^\ell \cdot \boldsymbol{n}_x = 0 \quad \text{on } \partial\Omega \times D \times (0, T), \tag{5.37}$$

and the following initial condition:

$$\hat{\psi}^\ell(x, \boldsymbol{q}, 0) = T_\ell(\hat{\psi}_0(x, \boldsymbol{q})) \qquad \text{for } (x, \boldsymbol{q}) \in \mathcal{O}. \tag{5.38}$$

Note that we do not truncate the Fokker–Planck equation since in the case of the corotational model, the drag term will vanish after testing the equation (5.35) with $\hat{\psi}^\ell$ since $\omega(\boldsymbol{v})$ is skew-symmetric. However, we truncate the initial condition for $\hat{\psi}^\ell$ since at a certain point in the proof we shall need to use the fact that $T_\ell(\hat{\psi}_0)$ is bounded above by $C(\ell)$. In the following sections, we shall temporarily omit the superscript ℓ, and then reinstate ℓ in the last step as $\ell \to \infty$.

5.4.2 Positivity of the coefficients in the Fokker–Planck equation

Since we need to guarantee the positivity of the coefficients of the second derivative terms in the Fokker–Planck equation, we shall introduce an additional parameter $\delta > 0$ in the early stages of our proof and we will pass to the limit as $\delta \to 0+$ later. We shall modify the approximate Fokker–Planck equation (3.3) as:

$$\begin{aligned}\frac{\partial(M\hat{\psi}^\delta)}{\partial t} &+ \mathrm{div}_x\Big(\boldsymbol{v}^\delta M\hat{\psi}^\delta - (\beta^L([\theta^\delta]_+) + \delta)\nabla_x(M\hat{\psi}^\delta)\Big)\\ &+ \mathrm{div}_{\boldsymbol{q}}\Big(\omega(\boldsymbol{v}^\delta)\boldsymbol{q}M\hat{\psi}^\delta - 4M(\beta^L([\theta^\delta]_+) + \delta)\nabla_{\boldsymbol{q}}\hat{\psi}^\delta\Big) = 0 \quad \text{in } \mathcal{O} \times (0,T),\end{aligned} \tag{5.39}$$

with the following boundary conditions:

$$\Big[\omega(\boldsymbol{v}^\delta)\boldsymbol{q}M\hat{\psi}^\delta - 4M(\beta^L([\theta^\delta]_+) + \delta)\nabla_{\boldsymbol{q}}\hat{\psi}^\delta\Big] \cdot \boldsymbol{n}_{\boldsymbol{q}} = 0 \quad \text{on } \Omega \times \partial D \times (0,T), \tag{5.40}$$

$$M\nabla_x\hat{\psi}^\delta \cdot \boldsymbol{n}_x = 0 \quad \text{on } \partial\Omega \times D \times (0,T), \tag{5.41}$$

and the following initial condition:

$$\hat{\psi}^\delta(x,\boldsymbol{q},0) = T_\ell(\hat{\psi}_0(x,\boldsymbol{q})) \qquad \text{for } (x,\boldsymbol{q}) \in \mathcal{O}. \tag{5.42}$$

The velocity $\boldsymbol{v}^\delta$ satisfies the following equations:

$$\frac{\partial \boldsymbol{v}^\delta}{\partial t} + \mathrm{div}_x(\Gamma_\ell(|\boldsymbol{v}^\delta|)\boldsymbol{v}^\delta \otimes \boldsymbol{v}^\delta) - \mathrm{div}_x(2\nu([\theta^\delta]_+)D(\boldsymbol{v}^\delta)) = \boldsymbol{f} \qquad \text{in } Q, \tag{5.43}$$

$$\mathrm{div}_x\, \boldsymbol{v}^\delta = 0 \qquad \text{in } Q, \tag{5.44}$$

with the following initial and boundary conditions:

$$\boldsymbol{v}^\delta(\cdot,0) = \boldsymbol{v}_0(\cdot) \qquad \text{in } \Omega, \tag{5.45}$$

$$\boldsymbol{v}^\delta = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{5.46}$$

The temperature θ^δ satisfies the following equation:

$$\frac{\partial \theta^\delta}{\partial t} + \mathrm{div}_x(\theta^\delta \boldsymbol{v}^\delta) + \mathrm{div}_x(\kappa([\theta^\delta]_+)\nabla_x\theta^\delta) = 2\nu([\theta^\delta]_+)D(\boldsymbol{v}^\delta) : D(\boldsymbol{v}^\delta) \quad \text{in } Q, \tag{5.47}$$

which is supplemented by the following initial and boundary conditions:

$$\theta^\delta(\cdot,x) = \theta_0(\cdot) \qquad \text{in } \Omega, \tag{5.48}$$

$$\nabla_x\theta^\delta \cdot \boldsymbol{n}_x = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{5.49}$$

In the next section, we shall formulate the Galerkin approximation of the system which is the starting point of our proof.

5.4.3 Galerkin approximation

Before we introduce the Galerkin approximation, we first define an approximate Maxwellian M^m by fixing a sequence of positive functions $(\overline{M}^m)_{m\in\mathbb{N}} \subset C^{0,1}_0(\overline{D})$ such that for each compact set $\varkappa \subset D$ the following holds:

$$\lim_{m\to\infty} \|\overline{M}^m - M\|_{C(\overline{D})\cap W^{1,1}_0(D)} + \|(\overline{M}^m)^{-1} - M^{-1}\|_{C(\varkappa)} = 0. \tag{5.50}$$

Then, the approximate Maxwellian M^m is defined by

$$M^m := \overline{M}^m + \frac{1}{m}, \quad \text{for } m = 1, 2, \dots.$$

Now we introduce the Galerkin basis functions. Similarly as in Chapter 3, by the Hilbert–Schmidt Theorem, there exists a sequence $(\boldsymbol{w}_i)_{i=1}^\infty$ of eigenfunctions in $W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)$ whose linear span is dense in $L^2_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$, such that the $\boldsymbol{w}_i$, $i = 1, 2, \dots$, are orthogonal in the inner product of $W^{d+1,2}(\Omega;\mathbb{R}^d)$ and orthonormal in the inner product of $L^2(\Omega;\mathbb{R}^d)$. To introduce the basis functions for the probability density function, we find a sequence $(\varphi_i)_{i=1}^\infty$ of eigenfunctions in $W^{(K+1)d+1,2}(\mathcal{O})$ that are orthogonal in $W^{(K+1)d+1,2}_{M^m}(\mathcal{O})$ and orthonormal in $L^2_{M^m}(\mathcal{O})$. For the basis functions for the temperature, we find a sequence $(u_i)_{i=1}^\infty$ of eigenfunctions in $W^{1,2}(\Omega)$ that are orthogonal in $W^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$. For $\delta > 0$ and the parameters $m, n, k \in \mathbb{N}$ fixed, we seek $(\boldsymbol{v}^{m,n,k,\delta}, \hat{\psi}^{m,n,k,\delta}, \theta^{m,n,k,\delta})$ given by

$$\boldsymbol{v}^{m,n,k,\delta}(x,t) := \sum_{i=1}^{m} c_i^{m,n,k,\delta}(t)\boldsymbol{w}_i(x), \tag{5.51}$$

$$\hat{\psi}^{m,n,k,\delta}(x,\boldsymbol{q},t) := \sum_{i=1}^{n} d_i^{m,n,k,\delta}(t)\varphi_i(x,\boldsymbol{q}), \tag{5.52}$$

$$\theta^{m,n,k,\delta}(x,t) := \sum_{i=1}^{k} e_i^{m,n,k,\delta}(t)u_i(x), \tag{5.53}$$

which solve

$$\begin{aligned}&\big(\partial_t \boldsymbol{v}^{m,n,k,\delta}, \boldsymbol{w}_i\big) - \big(\Gamma_\ell(|\boldsymbol{v}^{m,n,k,\delta}|)\boldsymbol{v}^{m,n,k,\delta} \otimes \boldsymbol{v}^{m,n,k,\delta}, \nabla_x \boldsymbol{w}_i\big)\\ &\quad + \big(2\nu([\theta^{m,n,k,\delta}]_+)\, D(\boldsymbol{v}^{m,n,k,\delta}), \nabla_x \boldsymbol{w}_i\big) = (\boldsymbol{f}, \boldsymbol{w}_i)\end{aligned} \tag{5.54}$$

for all $i = 1, \dots, m$ and a.e. $t \in (0,T)$,

$$\begin{aligned}&\Big(\partial_t(M^m\, \hat{\psi}^{m,n,k,\delta}), \varphi_i\Big)_{\mathcal{O}} - \Big(M^m\, \boldsymbol{v}^{m,n,k,\delta}\, \hat{\psi}^{m,n,k,\delta}, \nabla_x \varphi_i\Big)_{\mathcal{O}}\\ &\quad - \Big(M\, \hat{\psi}^{m,n,k,\delta}\, \omega(\boldsymbol{v}^{m,n,k,\delta})\, \boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi_i\Big)_{\mathcal{O}}\\ &\quad + \Big(M^m\, (\beta^L([\theta^{m,n,k,\delta}]_+) + \delta)\, \nabla_x \hat{\psi}^{m,n,k,\delta}, \nabla_x \varphi_i\Big)_{\mathcal{O}}\\ &\quad + \Big(4M^m\, (\beta^L([\theta^{m,n,k,\delta}]_+) + \delta)\, \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n,k,\delta}, \nabla_{\boldsymbol{q}} \varphi_i\Big)_{\mathcal{O}} = 0\end{aligned} \tag{5.55}$$

for all $i = 1, \ldots, n$ and a.e. $t \in (0,T)$,

$$\begin{aligned}&\big(\partial_t \theta^{m,n,k,\delta}, u_i\big) - \big(\boldsymbol{v}^{m,n,k,\delta}\,\theta^{m,n,k,\delta}, \nabla_x u_i\big) + \big(\kappa([\theta^{m,n,k,\delta}]_+)\,\nabla_x \theta^{m,n,k,\delta}, \nabla_x u_i\big)\\ &\quad = \big(2\nu([\theta^{m,n,k,\delta}]_+)\,D(\boldsymbol{v}^{m,n,k,\delta}) : D(\boldsymbol{v}^{m,n,k,\delta}), u_i\big)\end{aligned} \tag{5.56}$$

for all $i = 1, \ldots, k$ and a.e. $t \in (0,T)$. The initial data are given by

$$\begin{aligned}\boldsymbol{v}^{m,n,k,\delta}(x,0) &= \boldsymbol{v}_0^m(x) := \sum_{i=1}^m (\boldsymbol{v}_0, \boldsymbol{w}_i)\boldsymbol{w}_i(x),\\ \hat{\psi}^{m,n,k,\delta}(x,\boldsymbol{q},0) &= \hat{\psi}_0^{m,n}(x,\boldsymbol{q}) := \sum_{i=1}^n (T_\ell(\hat{\psi}_0^m), \varphi_i)_{\mathcal{O}}\varphi_i(x,\boldsymbol{q}),\\ \theta^{m,n,k,\delta}(x,0) &= \theta_0^k(x) := \sum_{i=1}^k (\theta_0, u_i)u_i(x),\end{aligned} \tag{5.57}$$

where

$$\hat{\psi}_0^m := \hat{\psi}_0 \frac{M}{M^m}. \tag{5.58}$$

The local-in-time existence of solutions to the system (5.54)–(5.56) follows from Carathéodory's Existence Theorem, since (5.54)–(5.56) are now ordinary differential equations. Then in the following sections, we shall derive uniform bounds independent of the parameters k, δ, n, m, ℓ, pass to the limits with $k, n, m, \ell \to \infty$ and $\delta \to 0_+$, which then gives global-in-time existence of solutions to the problem stated in Section 5.1.

5.4.4 Passage to the limit as $k \to \infty$

In this section, we derive uniform estimates independent of the parameter k and pass to the limit as $k \to \infty$. First, we multiply the i-th equation in (5.54) by $c_i^{m,n,k,\delta}(t)$ and sum with respect to $i = 1, \ldots, m$ to deduce the following identity,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^{m,n,k,\delta}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_\Omega 2\nu([\theta^{m,n,k,\delta}]_+)\,|D(\boldsymbol{v}^{m,n,k,\delta})|^2\,\mathrm{d}x = (\boldsymbol{f}, \boldsymbol{v}^{m,n,k,\delta}),$$

noting that the convective term vanishes since $\mathrm{div}_x\, \boldsymbol{v}^{m,n,k,\delta} = 0$. Using (5.15), Korn's inequality, Young's inequality and Gronwall's inequality, we obtain that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n,k,\delta}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c_0\nu_{\min}\int_0^T \|\boldsymbol{v}^{m,n,k,\delta}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{v}_0). \tag{5.59}$$

From the above bounds and using the orthogonality of the basis and the embedding $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we have

$$\sup_{t\in(0,T);i=1,\ldots,m} |c_i^{m,n,k,\delta}(t)| \le C(m), \qquad \sup_{t\in(0,T);i=1,\ldots,m} \left|\frac{\mathrm{d}c_i^{m,n,k,\delta}(t)}{\mathrm{d}t}\right| \le C(m). \tag{5.60}$$

Hence, by the definition of $\boldsymbol{v}^{m,n,k,\delta}$, we deduce that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n,k,\delta}(t)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \leq C(m). \tag{5.61}$$

Next, we shall derive uniform bounds on $\theta^{m,n,k,\delta}$. Similarly, we multiply the i-th equation of (5.56) by $e_i^{m,n,k,\delta}(t)$ and sum with respect to $i = 1, \dots, k$ to deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{m,n,k,\delta}\|^2_{L^2(\Omega)} + \int_\Omega \kappa([\theta^{m,n,k,\delta}]_+)\,|\nabla_x \theta^{m,n,k,\delta}|^2 \,\mathrm{d}x \\ = \int_\Omega 2\nu([\theta^{m,n,k,\delta}]_+)\,|D(\boldsymbol{v}^{m,n,k,\delta})|^2\,\theta^{m,n,k,\delta}\,\mathrm{d}x,\end{aligned}$$

where we have used the property $\mathrm{div}_x\, \boldsymbol{v}^{m,n,k,\delta} = 0$ again. Using the assumptions (5.15) and (5.16) and the bound (5.61), we deduce from the above identity that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{m,n,k,\delta}\|^2_{L^2(\Omega)} + 2\kappa_{\min}\int_\Omega |\nabla_x\theta^{m,n,k,\delta}|^2\,\mathrm{d}x \leq C(m,\nu_{\max})\|\theta^{m,n,k,\delta}\|^2_{L^2(\Omega)}.$$

Then, by Gronwall's inequality, we obtain that

$$\sup_{t\in(0,T)} \|\theta^{m,n,k,\delta}(t)\|^2_{L^2(\Omega)} + 2\kappa_{\min}\int_0^T \|\theta^{m,n,k,\delta}\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq C(m,\nu_{\max},\theta_0). \tag{5.62}$$

By using (5.56), it follows from (5.59), (5.61) and (5.62) that

$$\int_0^T \|\partial_t\theta^{m,n,k,\delta}\|^2_{(W^{1,2}(\Omega))'}\,\mathrm{d}t \leq C(m). \tag{5.63}$$

We shall follow a similar argument to derive uniform bounds on $\hat{\psi}^{m,n,k,\delta}$. We multiply the i-th equation of (5.55) by $d_i^{m,n,k,\delta}(t)$ and sum with respect to $i = 1, \dots, n$ to deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\psi}^{m,n,k,\delta}\|^2_{L^2_{M^m}(\mathcal{O})} + \int_{\mathcal{O}} M^m\,(\beta^L([\theta^{m,n,k,\delta}]_+)+\delta)\,|\nabla_x\hat{\psi}^{m,n,k,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ + 4\int_{\mathcal{O}} M^m\,(\beta^L([\theta^{m,n,k,\delta}]_+)+\delta)\,|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n,k,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 0,\end{aligned}$$

where we have used the incompressibility of $\boldsymbol{v}^{m,n,k,\delta}$ and the fact that $\omega(\boldsymbol{v}^{m,n,k,\delta})$ is skew-symmetric. Then, by Gronwall's inequality, we get

$$\begin{aligned}\sup_{t\in(0,T)} \|\hat{\psi}^{m,n,k,\delta}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + 2\delta\int_0^T\int_{\mathcal{O}} M^m\,|\nabla_x\hat{\psi}^{m,n,k,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ + 8\delta\int_0^T\int_{\mathcal{O}} M^m\,|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n,k,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \leq C(m,\ell,M,\hat{\psi}_0).\end{aligned} \tag{5.64}$$

By using the fact that $M^m \geq \frac{1}{m}$, we also deduce from (5.64) that

$$\int_0^T \|\hat{\psi}^{m,n,k,\delta}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \leq C(m,\delta,\ell). \tag{5.65}$$

By orthonormality of the basis $(\varphi_i)_{i=1}^\infty$ in $L^2_{M^m}(\mathcal{O})$, we have from (5.64) that

$$\sup_{t\in(0,T);i=1,\dots,n} |d_i^{m,n,k,\delta}(t)| \leq C(m,n,\ell). \tag{5.66}$$

Also note that since $\mathcal{O} \subset \mathbb{R}^{d(K+1)}$, the Sobolev embedding and the orthogonality of the basis give that $\|\nabla_{x,\boldsymbol{q}}\varphi_i\|_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})} \leq C(m,n)$ for all $i = 1,\dots,n$. By substituting

$$\frac{\partial\hat{\psi}^{m,n,k,\delta}}{\partial t}(x,\boldsymbol{q},t) = \sum_{i=1}^n \frac{\mathrm{d}d_i^{m,n,k,\delta}(t)}{\mathrm{d}t}\varphi_i(x,\boldsymbol{q}),$$

in (5.55), we get by using Hölder's inequality and the uniform bounds (5.59), (5.62) and (5.65) that

$$\left|\frac{\mathrm{d}d_i^{m,n,k,\delta}(t)}{\mathrm{d}t}\right| \leq C(m,\delta)\|\nabla_{x,\boldsymbol{q}}\varphi_i\|_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})} \leq C(m,n,\delta,\ell). \tag{5.67}$$

The definition of $\hat{\psi}^{m,n,k,\delta}$ gives that

$$\sup_{t\in(0,T)} \|\hat{\psi}^{m,n,k,\delta}(t)\|_{W^{1,\infty}(\mathcal{O})} \leq C(m,n,\ell). \tag{5.68}$$

Since M^m is Lipschitz continuous, we also have

$$\int_0^T \|M^m\hat{\psi}^{m,n,k,\delta}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \leq C(m,\delta,\ell).$$

We use the k-independent estimates (5.59), (5.60), (5.62), (5.63), (5.64), (5.65), (5.66) and (5.67) and the Aubin–Lions Lemma to deduce that there exist subsequences, which we do not relabel, such that

$$c_i^{m,n,k,\delta} \rightharpoonup^* c_i^{m,n,\delta} \qquad \text{weak* in } W^{1,\infty}(0,T), \tag{5.69}$$

$$c_i^{m,n,k,\delta} \to c_i^{m,n,\delta} \qquad \text{strongly in } C([0,T]), \tag{5.70}$$

$$\boldsymbol{v}^{m,n,k,\delta} \to \boldsymbol{v}^{m,n,\delta} \qquad \text{strongly in } C([0,T]; W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)), \tag{5.71}$$

$$d_i^{m,n,k,\delta} \rightharpoonup^* d_i^{m,n,\delta} \qquad \text{weak* in } W^{1,\infty}(0,T), \tag{5.72}$$

$$d_i^{m,n,k,\delta} \to d_i^{m,n,\delta} \qquad \text{strongly in } C([0,T]), \tag{5.73}$$

$$\hat{\psi}^{m,n,k,\delta} \rightharpoonup \hat{\psi}^{m,n,\delta} \qquad \text{strongly in } C([0,T]; W^{(K+1)d+1,2}(\mathcal{O})), \tag{5.74}$$

$$\theta^{m,n,k,\delta} \rightharpoonup^* \theta^{m,n,\delta} \qquad \text{weak* in } L^\infty(0,T;L^2(\Omega)), \tag{5.75}$$

$$\theta^{m,n,k,\delta} \rightharpoonup \theta^{m,n,\delta} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.76}$$

$$\partial_t\theta^{m,n,k,\delta} \rightharpoonup \partial_t\theta^{m,n,\delta} \qquad \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'), \tag{5.77}$$

$$\theta^{m,n,k,\delta} \to \theta^{m,n,\delta} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{5.78}$$

Since $\nu(\cdot)$, $\kappa(\cdot)$ and the function $s \in \mathbb{R} \mapsto [s]_+ \in \mathbb{R}_+$ are Lipschitz continuous and $\beta^L(\cdot)$ is continuous, we can pass to the limit as $k \to \infty$ in (5.54)–(5.56) to deduce that

$$\begin{aligned}
&\big(\partial_t \boldsymbol{v}^{m,n,\delta}, \boldsymbol{w}_i\big) - \big(\Gamma_\ell(|\boldsymbol{v}^{m,n,\delta}|)\,\boldsymbol{v}^{m,n,\delta} \otimes \boldsymbol{v}^{m,n,\delta}, \nabla_x \boldsymbol{w}_i\big) \\
&\quad + \big(2\nu([\theta^{m,n,\delta}]_+)\, D(\boldsymbol{v}^{m,n,\delta}), \nabla_x \boldsymbol{w}_i\big) = (\boldsymbol{f}, \boldsymbol{w}_i) \qquad (5.79)\\
&\quad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0,T),\\
&\Big(\partial_t(M^m\, \hat{\psi}^{m,n,\delta}), \varphi_i\Big)_{\mathcal{O}} - \Big(M^m\, \boldsymbol{v}^{m,n,\delta}\, \hat{\psi}^{m,n,\delta}, \nabla_x \varphi_i\Big)_{\mathcal{O}}\\
&\quad - \Big(M\, \hat{\psi}^{m,n,\delta}\, \omega(\boldsymbol{v}^{m,n,\delta})\, \boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi_i\Big)_{\mathcal{O}}\\
&\quad + \Big(M^m\, (\beta^L([\theta^{m,n,\delta}]_+) + \delta)\, \nabla_x \hat{\psi}^{m,n,\delta}, \nabla_x \varphi_i\Big)_{\mathcal{O}} \qquad (5.80)\\
&\quad + \Big(4M^m\, (\beta^L([\theta^{m,n,\delta}]_+) + \delta)\, \nabla_{\boldsymbol{q}} \hat{\psi}^{m,n,\delta}, \nabla_{\boldsymbol{q}} \varphi_i\Big)_{\mathcal{O}} = 0\\
&\quad \text{for all } i = 1, \dots, n \text{ and a.e. } t \in (0,T),\\
&\big\langle \partial_t \theta^{m,n,\delta}, u\big\rangle - \big(\boldsymbol{v}^{m,n,\delta}\, \theta^{m,n,\delta}, \nabla_x u\big) + \big(\kappa([\theta^{m,n,\delta}]_+)\, \nabla_x \theta^{m,n,\delta}, \nabla_x u\big)\\
&\quad = \big(2\nu([\theta^{m,n,\delta}]_+)\, D(\boldsymbol{v}^{m,n,\delta}) : D(\boldsymbol{v}^{m,n,\delta}), u\big) \qquad (5.81)\\
&\quad \text{for all } u \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T).
\end{aligned}$$

It is also straightforward to show that the above system is complemented by the following initial conditions:

$$\boldsymbol{v}^{m,n,\delta}(x,0) = \boldsymbol{v}_0^m(x), \qquad (5.82)$$

$$\hat{\psi}^{m,n,\delta}(x,\boldsymbol{q},0) = \varphi_0^{m,n}(x,\boldsymbol{q}), \qquad (5.83)$$

and

$$\lim_{t \to 0_+} \|\theta^{m,n,\delta}(\cdot,t) - \theta_0(\cdot)\|^2_{L^2(\Omega)} = 0. \qquad (5.84)$$

Now that we have successfully passed to the limit as $k \to \infty$, we can show that $\theta^{m,n,\delta}$ is positive a.e. in Q, which enables us to get rid of the parameter δ by sending $\delta \to 0_+$ in the next section.

5.4.5 Passage to the limit as $\delta \to 0_+$

In this section, we first show that $\theta^{m,n,\delta}$ is bounded below. Then, we can derive δ-independent a priori estimates and pass to the limit as $\delta \to 0_+$.

Minimum principle for $\theta^{m,n,\delta}$. Recall that we assume the initial temperature θ_0 satisfies $0 < \theta_{\min} \le \theta_0$ for a.e. $x \in \Omega$. To show that $\theta^{m,n,\delta}$ is bounded below, we set $u = (\theta^{m,n,\delta} - \theta_{\min})_-$ in (5.81) and integrate the resulting identity with respect to time over $(0, t)$; we get that

$$\begin{aligned} &\|(\theta^{m,n,\delta}(t) - \theta_{\min})_-\|^2_{L^2(\Omega)} + 2\int_0^t \int_\Omega \kappa([\theta^{m,n,\delta}]_+)\, |\nabla_x(\theta^{m,n,\delta} - \theta_{\min})_-|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &= \|(\theta_0 - \theta_{\min})_-\|^2_{L^2(\Omega)} + 4\int_0^t \int_\Omega \nu([\theta^{m,n,\delta}]_+)\, |D(\boldsymbol{v}^{m,n,\delta})|^2\, (\theta^{m,n,\delta} - \theta_{\min})_- \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{5.85}$$

Since $\theta_0 \ge \theta_{\min}$ and $\nu(\cdot)$ is nonnegative, the right-hand side of (5.85) is nonpositive. Also since $\kappa(\cdot)$ is nonnegative, the second integral on the left-hand side of (5.85) is nonnegative. Therefore

$$\|(\theta^{m,n,\delta}(t) - \theta_{\min})_-\|^2_{L^2(\Omega)} \le 0,$$

which then implies

$$\theta^{m,n,\delta}(x, t) \ge \theta_{\min} > 0 \qquad \text{for a.e. } (x, t) \in Q. \tag{5.86}$$

The limit $\delta \to 0_+$. Since we have proved that $\theta^{m,n,\delta} \ge \theta_{\min} > 0$, we can replace all $[\theta^{m,n,\delta}]_+$ in (5.79)–(5.81) by $\theta^{m,n,\delta}$. Similarly as in Section 5.4.4, we multiply the i-th equation in (5.79) by $c_i^{m,n,\delta}(t)$ and sum with respect to $i = 1, \dots, m$ to obtain, noting that $\mathrm{div}_x\, \boldsymbol{v}^{m,n,\delta} = 0$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^{m,n,\delta}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_\Omega 2\nu(\theta^{m,n,\delta})\, |D(\boldsymbol{v}^{m,n,\delta})|^2 \,\mathrm{d}x = (\boldsymbol{f}, \boldsymbol{v}^{m,n,\delta}).$$

Using (5.15), Korn's inequality, Young's inequality and Gronwall's inequality, we obtain that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n,\delta}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c_0\nu_{\min}\int_0^T \|\boldsymbol{v}^{m,n,\delta}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \le C(\boldsymbol{f}, \boldsymbol{v}_0). \tag{5.87}$$

By orthogonality of the basis and the embedding $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we have

$$\sup_{t\in(0,T);i=1,\dots,m} |c_i^{m,n,\delta}(t)| \le C(m), \qquad \sup_{t\in(0,T);i=1,\dots,m} \left|\frac{\mathrm{d}c_i^{m,n,\delta}(t)}{\mathrm{d}t}\right| \le C(m). \tag{5.88}$$

From the above bounds, we can also strengthen the norm as follows:

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n,\delta}(t)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le C(m). \tag{5.89}$$

Next, we shall derive uniform bounds for $\theta^{m,n,\delta}$ independent of δ. To achieve this, we set $u = \theta^{m,n,\delta}$ in (5.81) to deduce the following identity, noting that $\mathrm{div}_x\, \boldsymbol{v}^{m,n,\delta} = 0$,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{m,n,\delta}\|^2_{L^2(\Omega)} &+ \int_\Omega \kappa(\theta^{m,n,\delta})\,|\nabla_x \theta^{m,n,\delta}|^2 \,\mathrm{d}x \\
&= \int_\Omega 2\nu(\theta^{m,n,\delta})\,|D(\boldsymbol{v}^{m,n,\delta})|^2\,\theta^{m,n,\delta}\,\mathrm{d}x \\
&\leq 2\nu_{\max}\left(\int_\Omega |D(\boldsymbol{v}^{m,n,\delta})|^4\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |\theta^{m,n,\delta}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq C(\nu_{\max})\int_\Omega |D(\boldsymbol{v}^{m,n,\delta})|^4\,\mathrm{d}x + C(\nu_{\max})\int_\Omega |\theta^{m,n,\delta}|^2\,\mathrm{d}x \\
&\leq C(m,\nu_{\max}) + C(\nu_{\max})\|\theta^{m,n,\delta}\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have applied Hölder's inequality and Young's inequality, the assumption (5.15) and the bound (5.89). Using the assumption (5.16), we apply Gronwall's inequality to deduce that

$$\sup_{t\in(0,T)}\|\theta^{m,n,\delta}(t)\|^2_{L^2(\Omega)} + \kappa_{\min}\int_0^T \|\theta^{m,n,\delta}\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq C(m,\nu_{\max},\theta_0). \tag{5.90}$$

Using (5.81) and the uniform estimates (5.89) and (5.90), we deduce that

$$\int_0^T \|\partial_t\theta^{m,n,\delta}\|^2_{(W^{1,2}(\Omega))'}\,\mathrm{d}t \leq C(m). \tag{5.91}$$

By multiplying the i-th equation of (5.80) by $d_i^{m,n,\delta}(t)$ and summing with respect to $i = 1,\dots,n$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\psi}^{m,n,\delta}\|^2_{L^2_{M^m}(\mathcal{O})} &+ \int_{\mathcal{O}} M^m\,(\beta^L(\theta^{m,n,\delta})+\delta)\,|\nabla_x\hat{\psi}^{m,n,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\
&+ 4\int_{\mathcal{O}} M^m\,(\beta^L(\theta^{m,n,\delta})+\delta)\,|\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 0,
\end{aligned}$$

where we have used the fact that $\mathrm{div}_x\,\boldsymbol{v}^{m,n,\delta} = 0$ and that $\omega(\boldsymbol{v}^{m,n,\delta})$ is skew-symmetric. Since we have shown that $\theta^{m,n,\delta} \geq \theta_{\min}$, we apply Gronwall's inequality to deduce that

$$\sup_{t\in(0,T)}\|\hat{\psi}^{m,n,\delta}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + \theta_{\min}\int_0^T\int_{\mathcal{O}} M^m\,|\nabla_{x,\boldsymbol{q}}\hat{\psi}^{m,n,\delta}|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C(m,\ell,M,\hat{\psi}_0). \tag{5.92}$$

Since we still have the bound $\|\nabla_{x,\boldsymbol{q}}\varphi_i\|_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})} \leq C(m,n)$ for all $i = 1,\dots,n$ by Sobolev embedding and the orthogonality of the basis, we can deduce the following bounds similarly as in Section 5.4.4:

$$\sup_{t\in(0,T);i=1,\dots,n}|d_i^{m,n,\delta}(t)| \leq C(m,n,\ell), \qquad \sup_{t\in(0,T);i=1,\dots,n}\left|\frac{\mathrm{d}d_i^{m,n,\delta}(t)}{\mathrm{d}t}\right| \leq C(m,n,\ell). \tag{5.93}$$

Therefore,

$$\sup_{t\in(0,T)} \|\hat{\psi}^{m,n,\delta}(t)\|_{W^{1,\infty}(\mathcal{O})} \leq C(m,n,\ell). \tag{5.94}$$

By using the fact that $M^m \geq \frac{1}{m}$, we also deduce from (5.92) that

$$\int_0^T \|\hat{\psi}^{m,n,\delta}\|^2_{W^{1,2}(\mathcal{O})}\,\mathrm{d}t \leq C(m,\ell). \tag{5.95}$$

Now we have δ-independent estimates (5.87), (5.88), (5.89), (5.90), (5.91), (5.92), (5.93) and (5.95). Then, there exist subsequences (not relabelled), such that as $\delta \to 0_+$,

$$\begin{aligned}
c_i^{m,n,\delta} &\rightharpoonup^* c_i^{m,n} && \text{weak* in } W^{1,\infty}(0,T), && (5.96)\\
c_i^{m,n,\delta} &\to c_i^{m,n} && \text{strongly in } C([0,T]), && (5.97)\\
\boldsymbol{v}^{m,n,\delta} &\to \boldsymbol{v}^{m,n} && \text{strongly in } C([0,T]; W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)), && (5.98)\\
d_i^{m,n,\delta} &\rightharpoonup^* d_i^{m,n} && \text{weak* in } W^{1,\infty}(0,T), && (5.99)\\
d_i^{m,n,\delta} &\to d_i^{m,n} && \text{strongly in } C([0,T]), && (5.100)\\
\hat{\psi}^{m,n,\delta} &\rightharpoonup \hat{\psi}^{m,n} && \text{strongly in } C([0,T]; W^{(K+1)d+1,2}(\mathcal{O})), && (5.101)\\
\theta^{m,n,\delta} &\rightharpoonup^* \theta^{m,n} && \text{weak* in } L^\infty(0,T;L^2(\Omega)), && (5.102)\\
\theta^{m,n,\delta} &\rightharpoonup \theta^{m,n} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), && (5.103)\\
\partial_t\theta^{m,n,\delta} &\rightharpoonup \partial_t\theta^{m,n} && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'), && (5.104)\\
\theta^{m,n,\delta} &\to \theta^{m,n} && \text{strongly in } L^2(0,T;L^2(\Omega)), && (5.105)
\end{aligned}$$

where we have used the Aubin–Lions Lemma to deduce strong convergence for $\theta^{m,n,\delta}$. Also, since $\nu(\cdot)$ and $\kappa(\cdot)$ are Lipschitz continuous and $\beta^L(\cdot)$ is continuous, we can pass to the limit as $\delta \to 0_+$ in (5.79)–(5.81) to deduce that

$$\begin{aligned}
&(\partial_t\boldsymbol{v}^{m,n},\boldsymbol{w}_i) - (\Gamma_\ell(|\boldsymbol{v}^{m,n}|)\,\boldsymbol{v}^{m,n}\otimes\boldsymbol{v}^{m,n},\nabla_x\boldsymbol{w}_i) + (2\nu(\theta^{m,n})\,D(\boldsymbol{v}^{m,n}),\nabla_x\boldsymbol{w}_i)\\
&\quad = (\boldsymbol{f},\boldsymbol{w}_i) \quad \text{for all } i=1,\dots,m \text{ and a.e. } t\in(0,T),
\end{aligned} \tag{5.106}$$

$$\begin{aligned}
&\Big(\partial_t(M^m\,\hat{\psi}^{m,n}),\varphi_i\Big)_{\mathcal{O}} - \Big(M^m\,\boldsymbol{v}^{m,n}\,\hat{\psi}^{m,n},\nabla_x\varphi_i\Big)_{\mathcal{O}}\\
&\quad - \Big(M\,\hat{\psi}^{m,n}\,\omega(\boldsymbol{v}^{m,n})\,\boldsymbol{q},\nabla_{\boldsymbol{q}}\varphi_i\Big)_{\mathcal{O}} + \Big(M^m\,\beta^L(\theta^{m,n})\,\nabla_x\hat{\psi}^{m,n},\nabla_x\varphi_i\Big)_{\mathcal{O}}\\
&\quad + \Big(4M^m\,\beta^L(\theta^{m,n})\,\nabla_{\boldsymbol{q}}\hat{\psi}^{m,n},\nabla_{\boldsymbol{q}}\varphi_i\Big)_{\mathcal{O}} = 0\\
&\quad \text{for all } i=1,\dots,n \text{ and a.e. } t\in(0,T),
\end{aligned} \tag{5.107}$$

$$\begin{aligned}
&\langle\partial_t\theta^{m,n},u\rangle - (\boldsymbol{v}^{m,n}\,\theta^{m,n},\nabla_x u) + (\kappa(\theta^{m,n})\,\nabla_x\theta^{m,n},\nabla_x u)\\
&\quad = (2\nu(\theta^{m,n})\,D(\boldsymbol{v}^{m,n}) : D(\boldsymbol{v}^{m,n}),u)\\
&\quad \text{for all } u\in W^{1,2}(\Omega) \text{ and a.e. } t\in(0,T).
\end{aligned} \tag{5.108}$$

The above system is complemented by the following initial conditions:

$$\boldsymbol{v}^{m,n}(x,0) = \boldsymbol{v}_0^m(x), \tag{5.109}$$

$$\hat{\psi}^{m,n}(x,\boldsymbol{q},0) = \hat{\psi}_0^{m,n}(x,\boldsymbol{q}), \tag{5.110}$$

and

$$\lim_{t\to 0_+} \|\theta^{m,n}(\cdot,t) - \theta_0(\cdot)\|^2_{L^2(\Omega)} = 0. \tag{5.111}$$

In the next section, we shall study n-independent estimates and pass to the limit as $n \to \infty$.

5.4.6 Passage to the limit as $n \to \infty$

In this section, we derive n-independent bounds on $\boldsymbol{v}^{m,n}$, $\hat{\psi}^{m,n}$ and $\theta^{m,n}$, which then allow us to pass to the limit in (5.106)–(5.108) and identify the system of equations satisfied by the triplet $(\boldsymbol{v}^m, \hat{\psi}^m, \theta^m)$.

By using the strong convergence (5.105) in (5.86), we deduce that

$$\theta^{m,n}(x,t) \geq \theta_{\min} > 0 \qquad \text{for a.e. } (x,t) \in Q. \tag{5.112}$$

Similarly as above, we multiply the i-th equation in (5.106) by $c_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, m$, by using Korn's inequality, Young's inequality and Gronwall's inequality, we deduce that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c_0\nu_{\min}\int_0^T \|\boldsymbol{v}^{m,n}\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \leq C(\boldsymbol{f},\boldsymbol{v}_0). \tag{5.113}$$

By orthogonality of the basis and the embedding $W^{d+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we also have

$$\sup_{t\in(0,T);i=1,\dots,m} |c_i^{m,n}(t)| \leq C(m), \qquad \sup_{t\in(0,T);i=1,\dots,m} \left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| \leq C(m). \tag{5.114}$$

From the above bounds, we can also strengthen the norm as follows:

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}(t)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \leq C(m). \tag{5.115}$$

By the uniform estimates (5.113), (5.114) and (5.115) and using the fact that all norms are equivalent in finite-dimensional spaces, we deduce that there exist subsequences (not relabelled), such that as $n \to \infty$,

$$c_i^{m,n} \rightharpoonup^* c_i^m \qquad \text{weak* in } W^{1,\infty}(0,T), \tag{5.116}$$

$$c_i^{m,n} \to c_i^m \qquad \text{strongly in } C([0,T]), \tag{5.117}$$

$$\boldsymbol{v}^{m,n} \to \boldsymbol{v}^m \qquad \text{strongly in } C([0,T]; W^{1,2}_{0,\mathrm{div}} \cap W^{d+1,2}(\Omega;\mathbb{R}^d)). \tag{5.118}$$

For the approximate probability density function $\hat{\psi}^{m,n}$, we multiply the i-th equation in (5.107) by $d_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, n$, by using the bound (5.112) and Gronwall's inequality, we deduce by similar arguments as above that

$$\sup_{t\in(0,T)} \|\hat{\psi}^{m,n}(t)\|^2_{L^2_{M^m}(\mathcal{O})} + \theta_{\min} \int_0^T \int_{\mathcal{O}} M^m \, |\nabla_{x,\boldsymbol{q}} \hat{\psi}^{m,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \le C(m, \ell, M, \hat{\psi}_0). \quad (5.119)$$

Note that $M^m \ge \frac{1}{m}$, therefore we have

$$\sup_{t\in(0,T)} \|\hat{\psi}^{m,n}(t)\|^2_{L^2(\mathcal{O})} + \theta_{\min} \int_0^T \int_{\mathcal{O}} |\nabla_{x,\boldsymbol{q}} \hat{\psi}^{m,n}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \le C(m, \ell, M, \hat{\psi}_0). \quad (5.120)$$

Also since M^m is Lipschitz continuous, we deduce from above that

$$\int_0^T \|M^m \hat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})} \,\mathrm{d}t \le C(m, \ell). \quad (5.121)$$

Thanks to the presence of the cut-off function $\beta^L(\cdot)$, by using (5.107), the bounds (5.115) and (5.120), and Hölder's inequality, we deduce that

$$\int_0^T \|\partial_t (M^m \hat{\psi}^{m,n})\|^2_{(W^{1,2}(\mathcal{O}))'} \,\mathrm{d}t \le C(m, \ell, L, M, \hat{\psi}_0). \quad (5.122)$$

By the n-independent estimates (5.120) and (5.122), we deduce that there exists a subsequence (not relabelled), such that as $n \to \infty$,

$$\begin{aligned}
\hat{\psi}^{m,n} &\rightharpoonup \hat{\psi}^m && \text{weakly in } L^2(0,T; W^{1,2}(\mathcal{O})), && (5.123)\\
\partial_t (M^m \hat{\psi}^{m,n}) &\rightharpoonup \partial_t (M^m \hat{\psi}^m) && \text{weakly in } L^2(0,T; (W^{1,2}(\mathcal{O}))'), && (5.124)\\
\hat{\psi}^{m,n} &\rightharpoonup \hat{\psi}^m && \text{strongly in } L^2(0,T; L^2(\mathcal{O})), && (5.125)
\end{aligned}$$

where we have applied the Aubin–Lions Lemma for the strong convergence of $\hat{\psi}^m$.

To derive uniform bounds on the approximate temperature $\theta^{m,n}$, we set the test function $u = \theta^{m,n}$ in (5.108). Noting that $\mathrm{div}_x \, \boldsymbol{v}^{m,n} = 0$, we deduce that

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\theta^{m,n}\|^2_{L^2(\Omega)} + \int_\Omega \kappa(\theta^{m,n}) |\nabla_x \theta^{m,n}|^2 \,\mathrm{d}x &= \int_\Omega 2\nu(\theta^{m,n}) |D(\boldsymbol{v}^{m,n})|^2 \theta^{m,n} \,\mathrm{d}x \\
&\le C(m, \nu_{\max}) \|\theta^{m,n}\|^2_{L^2(\Omega)},
\end{aligned} \quad (5.126)$$

where we have made use of the assumption (5.15) and the uniform bound (5.115). Using Gronwall's inequality, and the assumption (5.16), we obtain that

$$\sup_{t\in(0,T)} \|\theta^{m,n}\|^2_{L^2(\Omega)} + \kappa_{\min} \int_0^T \|\theta^{m,n}\|^2_{W^{1,2}(\Omega)} \,\mathrm{d}t \le C(m, \nu_{\max}, \theta_0). \quad (5.127)$$

Using (5.108), the assumptions (5.15) and (5.16) and the uniform bounds (5.115) and (5.127), we apply Hölder's inequality to obtain that

$$\int_0^T \|\partial_t \theta^{m,n}\|^2_{(W^{1,2}(\Omega))'} \,\mathrm{d}t \le C(m, \kappa_{\max}, \nu_{\max}, \theta_0). \tag{5.128}$$

By the n-independent estimates (5.127) and (5.128), we deduce that there exists a subsequence (not relabelled), such that as $n \to \infty$,

$$\theta^{m,n} \rightharpoonup^* \theta^m \qquad \text{weak* in } L^\infty(0,T;L^2(\Omega)), \tag{5.129}$$

$$\theta^{m,n} \rightharpoonup \theta^m \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.130}$$

$$\partial_t\theta^{m,n} \rightharpoonup \partial_t\theta^m \qquad \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'), \tag{5.131}$$

$$\theta^{m,n} \to \theta^m \qquad \text{strongly in } L^2(Q), \tag{5.132}$$

where we have applied the Aubin–Lions Lemma to get the strong convergence of $\theta^{m,n}$.

By the convergence results (5.116)–(5.118), (5.123)–(5.125) and (5.129)–(5.132), and the fact that $\nu(\cdot)$ and $\kappa(\cdot)$ are Lipschitz continuous and $\beta^L(\cdot)$ is continuous, we can pass to the limit as $n \to \infty$ in (5.106)–(5.108) to deduce that

$$\begin{aligned}&(\partial_t \boldsymbol{v}^m, \boldsymbol{w}_i) - (\Gamma_\ell(|\boldsymbol{v}^m|)\, \boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla_x \boldsymbol{w}_i) + (2\nu(\theta^m)\, D(\boldsymbol{v}^m), \nabla_x \boldsymbol{w}_i)\\ &\quad = (\boldsymbol{f}, \boldsymbol{w}_i) \quad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0,T),\end{aligned} \tag{5.133}$$

$$\begin{aligned}&\langle \partial_t (M^m \hat{\psi}^m), \varphi\rangle_{\mathcal{O}} - \Big(M^m \boldsymbol{v}^m \hat{\psi}^m, \nabla_x \varphi\Big)_{\mathcal{O}} - \Big(M\, \hat{\psi}^m\, \omega(\boldsymbol{v}^m)\, \boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\\ &\quad + \Big(M^m \beta^L(\theta^m)\, \nabla_x \hat{\psi}^m, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(4M^m \beta^L(\theta^m)\, \nabla_{\boldsymbol{q}} \hat{\psi}^m, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}} = 0\\ &\quad \text{for all } \varphi \in W^{1,2}(\mathcal{O}) \text{ and a.e. } t \in (0,T),\end{aligned} \tag{5.134}$$

$$\begin{aligned}&\langle \partial_t \theta^m, u\rangle - (\boldsymbol{v}^m\, \theta^m, \nabla_x u) + (\kappa(\theta^m)\, \nabla_x \theta^m, \nabla_x u)\\ &\quad = (2\nu(\theta^m)\, D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m), u) \quad \text{for all } u \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T).\end{aligned} \tag{5.135}$$

Obviously, we have $\boldsymbol{v}^m(x,0) = \boldsymbol{v}_0^m(x)$. For the initial conditions for $\hat{\psi}^m$ and θ^m, it is straightforward to show that

$$\lim_{t\to 0_+} \|\hat{\psi}^m(\cdot,t) - T_\ell(\hat{\psi}_0^m(\cdot))\|^2_{L^2_{M^m}(\mathcal{O})} = 0, \tag{5.136}$$

$$\lim_{t\to 0_+} \|\theta^m(\cdot,t) - \theta_0(\cdot)\|^2_{L^2(\Omega)} = 0. \tag{5.137}$$

In the next section, we shall derive m-independent estimates and pass to the limit as $m \to \infty$.

5.4.7 Passage to the limit as $m \to \infty$

In this section, we first show that $\hat{\psi}^m \ge 0$ a.e. in $\mathcal{O} \times (0,T)$. Then we derive uniform estimates independent of m and use these uniform estimates to deduce relevant convergence results.

Minimum principle for $\hat{\psi}^m$. First we show that $\hat{\psi} \geq 0$ a.e. in $\mathcal{O} \times (0,T)$ using standard arguments. By setting $\varphi = [\hat{\psi}^m]_- := \min(0, \hat{\psi}^m)$ in (5.134), we deduce by using the fact that $\operatorname{div}_x \boldsymbol{v}^m = 0$ and that $\omega(\boldsymbol{v}^m)$ is skew-symmetric, that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} M^m ([\hat{\psi}^m]_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \,\beta^L(\theta^m)\, |\nabla_x [\hat{\psi}^m]_-|^2 \,\mathrm{d}x\mathrm{d}\boldsymbol{q} \\ + 4 \int_{\mathcal{O}} M^m \,\beta^L(\theta^m)\, |\nabla_{\boldsymbol{q}} [\hat{\psi}^m]_-|^2 \,\mathrm{d}x\mathrm{d}\boldsymbol{q} = 0.$$

Direct integration with respect to time over $(0,t)$ gives that

$$\int_{\mathcal{O}} M^m ([\hat{\psi}^m]_-(t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_0^t \int_{\mathcal{O}} M^m \,\beta^L(\theta^m)\, |\nabla_x [\hat{\psi}^m]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ + 4 \int_0^t \int_{\mathcal{O}} M^m \,\beta^L(\theta^m)\, |\nabla_{\boldsymbol{q}} [\hat{\psi}^m]_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau = \int_{\mathcal{O}} M^m ([\hat{\psi}^m]_-(0))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Since the second and third integrals on the left-hand side are nonnegative, we have

$$\int_{\mathcal{O}} M^m ([\hat{\psi}^m]_-(t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq \int_{\mathcal{O}} M^m ([\hat{\psi}^m]_-(0))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ = \int_{\mathcal{O}} M^m ([T_\ell(\hat{\psi}_0^m)]_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = 0,$$

where we have used that $T_\ell(\hat{\psi}_0^m) \geq 0$. Thus, we have shown that

$$\hat{\psi}^m \geq 0 \quad \text{a.e. in } \mathcal{O} \times (0,T).$$

Passage to the limit as $m \to \infty$. Our goal is to pass to the limit as $m \to \infty$ in (5.133)–(5.135). The main difficulty lies in passing to the limit in the temperature equation where we shall need strong convergence of the velocity gradient $D(\boldsymbol{v}^m)$. In the preceding sections, the strong convergence of $D(\boldsymbol{v}^m)$ naturally held since we have been working in a finite-dimensional space for $\boldsymbol{v}^m$. However, as $m \to \infty$, we are no longer in finite dimensions. Therefore, we shall apply a different technique to deduce the strong convergence of $D(\boldsymbol{v}^m)$. Also, thanks to the presence of the velocity gradient term in the temperature equation, we cannot set the test function $u = \theta^m$ in (5.135). To solve this, we instead take the test function $u = (\theta^m)^\lambda$ with $\lambda \in (-1, 0)$ in (5.135) which is motivated by [16]. In this section, we first derive uniform estimates as in the previous sections.

By using the strong convergence (5.132) in (5.112), we deduce that

$$\theta^m(x,t) \geq \theta_{\min} > 0 \qquad \text{for a.e. } (x,t) \in Q. \tag{5.138}$$

Similarly as above, we multiply the i-th equation in (5.133) by $c_i^m(t)$ and sum with respect to $i = 1, \dots, m$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_\Omega 2\nu(\theta^m)D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m)\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^m\,\mathrm{d}x. \tag{5.139}$$

Then, by using Korn's inequality, Young's inequality and Gronwall's inequality, we deduce that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^m(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c_0\nu_{\min}\int_0^T \|\boldsymbol{v}^m\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \leq C(\boldsymbol{f}, \boldsymbol{v}_0). \tag{5.140}$$

By standard interpolation, we have

$$\|\boldsymbol{v}^m\|_{L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)} \leq C(\boldsymbol{f}, \boldsymbol{v}_0). \tag{5.141}$$

To deduce the strong convergence of $\boldsymbol{v}^m$, we shall derive a uniform bound on the time derivative of $\boldsymbol{v}^m$. Thanks to the presence of the truncation $\Gamma_\ell(\cdot)$ in the convective term, we deduce by using (5.140) that

$$\int_0^T \|\partial_t \boldsymbol{v}^m\|^2_{W^{-1,2}_{\mathrm{div}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \leq C(\ell). \tag{5.142}$$

From the uniform bounds (5.140) and (5.141), we deduce the existence of a subsequence (not relabelled) such that, as $m \to \infty$,

$$\boldsymbol{v}^m \rightharpoonup^* \boldsymbol{v} \qquad \text{weak* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{5.143}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{5.144}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d). \tag{5.145}$$

By the uniform estimate (5.142), we apply the Aubin–Lions Lemma to deduce the following strong convergence:

$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^p(\Omega;\mathbb{R}^d)), \tag{5.146}$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,6)$ when $d = 3$.

For the approximate temperature θ^m, we start off by setting the test function $u = 1$ in (5.135) to get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^m\|_{L^1(\Omega)} = \int_\Omega 2\nu(\theta^m)D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m)\,\mathrm{d}x. \tag{5.147}$$

Integrating (5.139) and (5.147) with respect to time over $(0,t)$ and adding the results together, we obtain

$$\int_\Omega \left(\frac{1}{2}|\boldsymbol{v}^m(\cdot,t)|^2 + \theta^m(\cdot,t)\right) \mathrm{d}x = \int_0^t \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^m \,\mathrm{d}x \,\mathrm{d}\tau + \int_\Omega \left(\frac{1}{2}|\boldsymbol{v}_0^m(\cdot)|^2 + \theta_0(\cdot)\right) \mathrm{d}x. \tag{5.148}$$

Using the uniform bounds (5.15) and (5.140), we deduce from (5.147) that

$$\sup_{t\in(0,T)} \|\theta^m\|_{L^1(\Omega)} \le C(\nu_{\max}, \boldsymbol{f}, \boldsymbol{v}_0, \theta_0). \tag{5.149}$$

Since neither the temperature θ^m nor the velocity gradient $D(\boldsymbol{v}^m)$ has enough intergrability, it would be difficult to obtain uniform bounds for θ^m if we took the test function $u = \theta^m$ in (5.135). Motivated by the approach in [16], we set the test function $u = (\theta^m)^\lambda$ with $\lambda \in (-1,0)$ in (5.135). It follows that $u = (\theta^m)^\lambda \le (\theta_{\min})^\lambda \le C$ for some constant C. Then, we obtain, using the fact that $\operatorname{div}_x \boldsymbol{v}^m = 0$,

$$\begin{aligned}\frac{1}{\lambda+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\theta^m)^{\lambda+1}\,\mathrm{d}x &= \int_\Omega 2\nu(\theta^m)|D(\boldsymbol{v}^m)|^2(\theta^m)^\lambda\,\mathrm{d}x \\ &\quad - \lambda\int_\Omega \kappa(\theta^m)(\theta^m)^{\lambda-1}|\nabla_x\theta^m|^2\,\mathrm{d}x.\end{aligned} \tag{5.150}$$

We note here that the second term in the right-hand side of (5.150) can be rewritten as

$$-\lambda\int_\Omega \kappa(\theta^m)(\theta^m)^{\lambda-1}|\nabla_x\theta^m|^2\,\mathrm{d}x = -\frac{4\lambda}{(\lambda+1)^2}\int_\Omega \kappa(\theta^m)\left|\nabla_x(\theta^m)^{\frac{\lambda+1}{2}}\right|^2\mathrm{d}x,$$

and is nonnegative since $\lambda < 0$. Therefore, integrating (5.150) with respect to time over $(0,T)$, we obtain that

$$\begin{aligned}&\int_0^T\int_\Omega 2\nu(\theta^m)|D(\boldsymbol{v}^m)|^2(\theta^m)^\lambda\,\mathrm{d}x\,\mathrm{d}t - \frac{4\lambda}{(\lambda+1)^2}\int_0^T\int_\Omega \kappa(\theta^m)\left|\nabla_x(\theta^m)^{\frac{\lambda+1}{2}}\right|^2\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{\lambda+1}(\|(\theta^m)^{\lambda+1}(T)\|_{L^1(\Omega)} - \|(\theta^m)^{\lambda+1}(0)\|_{L^1(\Omega)}) \\ &\quad \le \frac{(\theta_{\min})^\lambda}{\lambda+1}\|\theta^m(T)\|_{L^1(\Omega)} - \frac{1}{\lambda+1}\|(\theta_0)^{\lambda+1}\|_{L^1(\Omega)} \\ &\quad \le C(\nu_{\max}, \boldsymbol{f}, \boldsymbol{v}_0, \theta_0),\end{aligned}$$

where we have made use of (5.149). Therefore, we obtain that

$$\int_0^T\int_\Omega \left|\nabla_x(\theta^m)^{\frac{\lambda+1}{2}}\right|^2\mathrm{d}x\,\mathrm{d}t \le C(\nu_{\max}, \kappa_{\min}, \boldsymbol{f}, \boldsymbol{v}_0, \theta_0). \tag{5.151}$$

From (5.149) and (5.151), we deduce that

$$(\theta^m)^{\frac{\lambda+1}{2}} \in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^{\frac{2}{\lambda+1}}(\Omega)).$$

Applying parabolic interpolation (c.f. Lemma 2.3), we deduce that

$$\|(\theta^m)^{\frac{\lambda+1}{2}}\|_{L^r(Q)} \leq C \quad \text{for any } r \in \left[1, \frac{2(2+d(\lambda+1))}{d(\lambda+1)}\right),$$

which in turn gives

$$\|\theta^m\|_{L^p(Q)} \leq C \quad \text{for any } p \in \left[1, \frac{d+2}{d}\right). \tag{5.152}$$

By applying Hölder's inequality and the uniform bounds (5.151) and (5.152), we see that

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla_x \theta^m|^s \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega |\nabla_x \theta^m|^s (\theta^m)^{(\lambda-1)\frac{s}{2}} (\theta^m)^{(1-\lambda)\frac{s}{2}} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \left(\int_0^T \int_\Omega |\nabla_x \theta^m|^2 (\theta^m)^{(\lambda-1)} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{s}{2}} \left(\int_0^T \int_\Omega (\theta^m)^{(1-\lambda)\frac{s}{2-s}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2-s}{2}} \\
&\leq \left(\int_0^T \int_\Omega \frac{4}{(\lambda+1)^2} \left|\nabla_x (\theta^m)^{\frac{\lambda+1}{2}}\right|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{s}{2}} \left(\int_0^T \int_\Omega (\theta^m)^{\frac{s(1-\lambda)}{2-s}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2-s}{2}} \\
&\leq C,
\end{aligned}$$

provided that $s \in \left[1, \frac{d+2}{d+1}\right)$. Therefore,

$$\|\theta^m\|_{L^s(0,T;W^{1,s}(\Omega))} \leq C \quad \text{for any } s \in \left[1, \frac{d+2}{d+1}\right). \tag{5.153}$$

For the strong convergence of θ^m, we shall need to control the time derivative of $\partial_t \theta^m$. However, because of the presence of $|D(\boldsymbol{v}^m)|^2$ on the right-hand side of (5.135), we have to consider test functions in the space $W^{1,q}(\Omega)$ with q sufficiently large. Using

(5.140), (5.141), (5.152) and (5.153), we deduce by applying Hölder's inequality that

$$\begin{aligned}
&\int_0^T \|\partial_t\theta^m\|_{(W^{1,q}(\Omega))'}\,\mathrm{d}t = \int_0^T \left(\sup_{u\in W^{1,q}(\Omega)} \frac{|\langle \partial_t\theta^m, u\rangle|}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t \\
&\quad\le \int_0^T \left(\sup_{u\in W^{1,q}(\Omega)} \frac{|(\boldsymbol{v}^m\theta^m, \nabla_x u)|}{\|\nabla_x u\|_{L^q(\Omega;\mathbb{R}^d)}} + \sup_{u\in W^{1,q}(\Omega)} \frac{|(\kappa(\theta^m)\nabla_x\theta^m, \nabla_x u)|}{\|\nabla_x u\|_{L^q(\Omega;\mathbb{R}^d)}}\right. \\
&\qquad\quad \left. + \sup_{u\in W^{1,q}(\Omega)} \frac{|(2\nu(\theta^m)D(\boldsymbol{v}^m) : D(\boldsymbol{v}^m), u)|}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t \\
&\le \int_0^T \left(\|\boldsymbol{v}^m\theta^m\|_{L^{q'}(\Omega;\mathbb{R}^d)} + \kappa_{\max}\|\nabla_x\theta^m\|_{L^{q'}(\Omega;\mathbb{R}^d)}\right. \\
&\qquad\quad \left. + \sup_{u\in W^{1,q}(\Omega)} \frac{2\nu_{\max}\|u\|_{L^\infty(\Omega)}\|D(\boldsymbol{v}^m)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t \\
&\le C\|\theta^m\|_{L^p(Q)}\|\boldsymbol{v}^m\|_{L^{\frac{pq'}{p-q'}}(Q;\mathbb{R}^d)} + C\|\nabla_x\theta^m\|_{L^s(Q;\mathbb{R}^d)} \\
&\quad + \int_0^T \left(\sup_{u\in W^{1,q}(\Omega)} \frac{2\nu_{\max}C\|u\|_{W^{1,q}(\Omega)}\|D(\boldsymbol{v}^m)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t \\
&\le C\|\theta^m\|_{L^p(Q)}\|\boldsymbol{v}^m\|_{L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)} + C\|\nabla_x\theta^m\|_{L^s(Q;\mathbb{R}^d)} + C\|D(\boldsymbol{v}^m)\|_{L^2(Q;\mathbb{R}^{d\times d})} \\
&\le C,
\end{aligned} \tag{5.154}$$

provided that $q \in \left(\frac{2(d+2)}{4-d}, \infty\right)$. Note that in the above inequality, we have used the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$.

From the uniform estimates (5.152), (5.153) and (5.154), we deduce the existence of a subsequence (not relabelled), such that, as $m \to \infty$,

$$\theta^m \rightharpoonup \theta \qquad \text{weakly in } L^p(Q) \text{ for any } p \in \left[1, \frac{d+2}{d}\right), \tag{5.155}$$

$$\theta^m \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \text{ for any } s \in \left[1, \frac{d+2}{d+1}\right), \tag{5.156}$$

$$\partial_t\theta^m \rightharpoonup \partial_t\theta \qquad \text{weakly in } \mathcal{M}(0,T;(W^{1,q}(\Omega))'), \text{ where } p > \frac{2(d+2)}{4-d}. \tag{5.157}$$

Using the generalized version of the Aubin–Lions Lemma (c.f., Corollary 7.9 in [60]), we deduce the following strong convergence:

$$\theta^m \to \theta \qquad \text{strongly in } L^p(Q) \text{ for any } p \in \left[1, \frac{d+2}{d}\right). \tag{5.158}$$

Next, we shall derive uniform bounds on the approximate probability density function $\hat{\psi}^m$. Setting $\varphi = 1$ in (5.134) and integrating over $(0,t)$, we obtain

$$\int_{\mathcal{O}} M^m(\boldsymbol{q})\,\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_{\mathcal{O}} M^m(\boldsymbol{q})\,T_\ell(\hat{\psi}^m_0(x,\boldsymbol{q}))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}$$
$$\leq \int_{\mathcal{O}} M^m(\boldsymbol{q})\hat{\psi}^m_0(x,\boldsymbol{q})\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq \int_{\mathcal{O}} M(\boldsymbol{q})\,\hat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C.$$

We define the approximate polymer number density by

$$\varrho^m(x,t) := \int_D M^m(\boldsymbol{q})\,[\hat{\psi}^m(x,\boldsymbol{q},t)]_+\,\mathrm{d}\boldsymbol{q} = \int_D M^m(\boldsymbol{q})\,\hat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q} \geq 0,$$

since we have shown that $\hat{\psi}^m \geq 0$ a.e. in $\mathcal{O} \times (0,T)$. By setting the test function $\varphi(x,\boldsymbol{q},t) := \bar{\varphi}(x,t)$ in (5.134), we deduce the equation satisfied by ϱ^m:

$$\langle \partial_t \varrho^m, \bar{\varphi}\rangle - (\boldsymbol{v}^m \varrho^m, \nabla_x \bar{\varphi}) + (\beta^L(\theta^m)\nabla_x \varrho^m, \nabla_x \bar{\varphi}) = 0 \tag{5.159}$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$, supplemented by the initial condition $\varrho^m(0) := \varrho^m_0$, where

$$0 \leq \varrho^m_0(x) := \int_D M^m(\boldsymbol{q}) T_\ell(\hat{\psi}^m_0(x,\boldsymbol{q}))\,\mathrm{d}\boldsymbol{q} \leq \int_D M(\boldsymbol{q})\hat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} = \varrho_0(x).$$

Let $\omega = \sup_{x\in\Omega} \varrho^m_0(x)$; then we have

$$\langle \partial_t(\varrho^m - \omega), \bar{\varphi}\rangle - (\boldsymbol{v}^m(\varrho^m - \omega), \nabla_x\bar{\varphi}) + (\beta^L(\theta^m)\nabla_x(\varrho^m - \omega), \nabla_x\bar{\varphi}) = 0,$$

for all $\bar{\varphi} \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$. Setting $\bar{\varphi} = [\varrho^m - \omega]_+$ in the above equation gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ([\varrho^m - \omega]_+)^2\,\mathrm{d}x + \int_\Omega \beta^L(\theta^m)|\nabla_x[\varrho^m - \omega]_+|^2\,\mathrm{d}x = 0.$$

Since $\theta^m \geq \theta_{\min} > 0$, the second term on the left-hand side of the above equation is always nonnegative. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ([\varrho^m - \omega]_+)^2\,\mathrm{d}x \leq 0,$$

which then gives

$$\int_\Omega ([\varrho^m - \omega]_+(x,t))^2\,\mathrm{d}x \leq \int_\Omega ([\varrho^m - \omega]_+(x,0))^2\,\mathrm{d}x = \int_\Omega ([\varrho^m_0 - \omega]_+)^2\,\mathrm{d}x.$$

By the definition of ω, $[\varrho^m_0 - \omega]_+ = 0$ a.e. in Ω. Thus, $[\varrho^m - \omega]_+ = 0$ a.e. in Ω. Hence,

$$\|\varrho^m\|_{L^\infty(Q)} \leq \|\varrho^m_0\|_{L^\infty(\Omega)} = \|\varrho_0\|_{L^\infty(\Omega)} \leq C. \tag{5.160}$$

By setting $\bar{\varphi} = \varrho^m$ in (5.159), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\varrho^m)^2 \,\mathrm{d}x + \int_\Omega \beta^L(\theta^m)|\nabla_x \varrho^m|^2 \,\mathrm{d}x = 0.$$

Direct integration with respect to t and the application of (5.138) and (5.160) give also

$$\int_0^T \int_\Omega |\nabla_x \varrho^m|^2 \,\mathrm{d}x \,\mathrm{d}t \leq C. \tag{5.161}$$

By setting $\varphi = \hat{\psi}^m$ in (5.134), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M^m(\hat{\psi}^m)^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} M^m \beta^L(\theta^m)|\nabla_x \hat{\psi}^m|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\ + 4\int_{\mathcal{O}} M^m \beta^L(\theta^m)|\nabla_{\boldsymbol{q}} \hat{\psi}^m|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} = 0.$$

Integrating the above identity with respect to time over $(0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\mathcal{O}} M^m(\hat{\psi}^m(\cdot,t))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} + \int_0^t \int_{\mathcal{O}} M^m \beta^L(\theta^m)|\nabla_x \hat{\psi}^m|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}\tau \\ &\quad + 4\int_0^t \int_{\mathcal{O}} M^m \beta^L(\theta^m)|\nabla_{\boldsymbol{q}} \hat{\psi}^m|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}\tau = \frac{1}{2}\int_{\mathcal{O}} M^m (T_\ell(\hat{\psi}_0^m(\cdot)))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q}. \end{aligned} \tag{5.162}$$

Using Gronwall's inequality and the bound (5.138), we obtain that

$$\begin{aligned} &\sup_{t\in(0,T)} \int_{\mathcal{O}} M^m \left(\hat{\psi}^m(x,\boldsymbol{q},t)\right)^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\ &\quad + \theta_{\min} \int_0^T \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}} \hat{\psi}^m\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \leq C(\ell, M, \hat{\psi}_0). \end{aligned} \tag{5.163}$$

Since now we do not have a positive lower bound on M^m that is uniform in m, we set $\psi^m := M^m \hat{\psi}^m$, use the above inequality and deduce that

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} (\psi^m(x,\boldsymbol{q},t))^2 \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}x \leq C \sup_{t\in(0,T)} \int_{\mathcal{O}} M^m \left(\hat{\psi}^m(x,\boldsymbol{q},t)\right)^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \leq C(\ell),$$

noting that we have used the fact that M^m is bounded above. Then the sequence $(\psi^m)_{m=1}^\infty$ is uniformly equi-integrable on $\mathcal{O} \times (0,T)$, thanks to the De la Vallée Poussin's Theorem (c.f. Theorem 2.4.2). Hence, by the Dunford–Pettis Theorem (c.f. Theorem 2.4.3), the sequence $(\psi^m)_{m=1}^\infty$ is weakly relatively compact in $L^1(\mathcal{O}\times(0,T))$, which then implies the existence of a subsequence (not relabelled) that

$$\psi^m \rightharpoonup \psi \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T)). \tag{5.164}$$

Since M^m converges to M uniformly in $C(\overline{D})$, we have that

$$\hat{\psi}^m \rightharpoonup \frac{\psi}{M} =: \hat{\psi} \qquad \text{weakly in } L^1_{loc}(\mathcal{O} \times (0,T)). \tag{5.165}$$

Next, we shall show that

$$\hat{\psi}^m \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{5.166}$$

Let $\mathcal{O}_0$ be a Lipschitz subdomain of $\mathcal{O}$ such that $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since M^m is bounded below on $\mathcal{O}_0$ by a positive constant (which may depend on $\mathcal{O}_0$), we deduce from (5.163) that

$$\sup_{t\in(0,T)} \|\hat{\psi}^m(\cdot,t)\|^2_{L^2(\mathcal{O}_0)} + \theta_{\min} \int_0^T \|\hat{\psi}^m\|^2_{W^{1,2}(\mathcal{O}_0)} \,\mathrm{d}t \leq C(\mathcal{O}_0). \tag{5.167}$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{d(K+1)}$, by standard function space interpolation, we get

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^m|^{\frac{2((K+1)d+2)}{d(K+1)}} \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \leq C(\mathcal{O}_0). \tag{5.168}$$

From (5.168) and (5.140), we deduce by using Hölder's inequality that there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^m \hat{\psi}^m\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \leq C(\mathcal{O}_0). \tag{5.169}$$

Similarly, we have

$$\|\hat{\psi}^m \omega(\boldsymbol{v}^m)\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \leq C(\mathcal{O}_0). \tag{5.170}$$

To be more specific, we have $\delta \in \left(0, \frac{1}{(K+1)d+1}\right]$. Then we define the following sequence of $(1+d+Kd)$-component vector fields:

$$\begin{aligned}
&H^m := (M^m\,\hat{\psi}^m, M^m\,\hat{\psi}^m\,\boldsymbol{v}^m - M^m\,\beta^L(\theta^m)\,\nabla_x\hat{\psi}^m, M\,\hat{\psi}^m\,\omega(\boldsymbol{v}^m)\,\boldsymbol{q} - M^m\,\beta^L(\theta^m)\,\nabla_{\boldsymbol{q}}\hat{\psi}^m),\\
&Q^m := ((1+\hat{\psi}^m)^\alpha, \underbrace{0,\dots,0}_{(d+Kd)\text{-times}}),
\end{aligned}$$

for some $\alpha \in (0, 1/2)$. Therefore, using (5.158), (5.167), (5.169) and (5.170), we deduce the existence of subsequences (not relabelled) such that

$$\begin{aligned}
H^m &\rightharpoonup H \qquad \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),\\
Q^m &\rightharpoonup Q \qquad \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}),
\end{aligned}$$

where, noting the uniform convergence of M^m, the strong convergence (5.146) of $\boldsymbol{v}^m$ and the strong convergence (5.158) of θ^m and the fact that $\beta^L(\cdot)$ is continuous,

$$\begin{aligned}
&H := (M\hat{\psi}, M\hat{\psi}\boldsymbol{v} - M\beta^L(\theta)\nabla_x\hat{\psi}, M\hat{\psi}\,\omega(\boldsymbol{v})\boldsymbol{q} - M\beta^L(\theta)\nabla_{\boldsymbol{q}}\hat{\psi}),\\
&Q := (\overline{(1+\hat{\psi})^\alpha}, 0, \dots, 0).
\end{aligned}$$

It follows from (5.134) that

$$\operatorname{div}_{t,x,\boldsymbol{q}} H^m = 0 \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Also, we obtain from (5.167) that

$$\begin{aligned}
\int_0^T \int_{\mathcal{O}_0} |\operatorname{curl}_{t,x,\boldsymbol{q}} Q^m|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &= \int_0^T \int_{\mathcal{O}_0} |\nabla_{t,x,\boldsymbol{q}} Q^m - (\nabla_{t,x,\boldsymbol{q}} Q^m)^{\mathrm{T}}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C \int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}(1+\hat{\psi}^m)^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C \int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m\right|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\\
&\le C(\mathcal{O}_0).
\end{aligned}$$

Hence, $(\operatorname{div}_{t,x,\boldsymbol{q}} H^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$ and $(\operatorname{curl}_{t,x,\boldsymbol{q}} Q^m)_{m=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$. By choosing $\alpha < \frac{\delta}{1+\delta}$, we apply the Div-Curl Lemma to deduce that

$$H^m \cdot Q^m \rightharpoonup H \cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).$$

In particular, we have that

$$M^m\hat{\psi}^m(1+\hat{\psi}^m)^\alpha \rightharpoonup M\hat{\psi}\overline{(1+\hat{\psi})^\alpha}.$$

Since M^m converges to M uniformly, the above gives that

$$\hat{\psi}^m(1+\hat{\psi}^m)^\alpha \rightharpoonup \hat{\psi}\overline{(1+\hat{\psi})^\alpha}. \tag{5.171}$$

Since $(1+\hat{\psi}^m)^\alpha \rightharpoonup \overline{(1+\hat{\psi})^\alpha}$ in $L^1(\mathcal{O}_0 \times (0,T))$, we can add this to (5.171), which gives

$$(1+\hat{\psi}^m)^{\alpha+1} = (1+\hat{\psi}^m)(1+\hat{\psi}^m)^\alpha \rightharpoonup (1+\hat{\psi})\overline{(1+\hat{\psi})^\alpha}.$$

Thanks to the weak lower semi-continuity of the continuous convex function $s \in [0,\infty) \mapsto s^{\alpha+1} \in [0,\infty)$ we have that

$$(1+\hat{\psi})^{\alpha+1} \le (1+\hat{\psi})\overline{(1+\hat{\psi})^\alpha},$$

which is equivalent to

$$(1+\hat{\psi})^\alpha \leq \overline{(1+\hat{\psi})^\alpha}.$$

On the other hand, the function $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is concave. Again, by the weak lower semi-continuity of the continuous convex function $s \in [0,\infty) \mapsto -s^\alpha \in [0,\infty)$, we deduce that

$$-(1+\hat{\psi})^\alpha \leq \overline{-(1+\hat{\psi})^\alpha}.$$

Therefore,

$$(1+\hat{\psi})^\alpha = \overline{(1+\hat{\psi})^\alpha}.$$

Since $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is strictly concave, thanks to Theorem 2.4.6, there exists a subsequence (not relabelled), such that

$$\hat{\psi}^m \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Since M^m converges uniformly to M, we have that

$$\psi^m \to \psi \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Now we want to extend the pointwise convergence result of ψ^m to the whole of our domain $\mathcal{O}\times(0,T)$. For this purpose we choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k=1}^\infty$, i.e., $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \cdots \subset \mathcal{O}_0^k \subset \cdots$, satisfying $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of $(\psi^m)_{m=1}^\infty$ that is pointwise convergent to ψ a.e. in $\mathcal{O}_0^k \times (0,T)$. Arguing by a diagonal procedure we deduce that there exists a subsequence such that

$$\psi^m \to \psi \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{5.172}$$

Since ψ^m is uniformly equi-integrable on $\mathcal{O} \times (0,T)$, using Vitali's Convergence Theorem (c.f., Theorem 2.24 in [34]), we obtain that

$$M^m\hat{\psi}^m = \psi^m \to \psi = M\hat{\psi} \qquad \text{strongly in } L^1(0,T;L^1(\mathcal{O})). \tag{5.173}$$

Interpolating between (5.160) and (5.173), we obtain that

$$\psi^m \to \psi \qquad \text{strongly in } L^p(Q;L^1(D)), \text{ for all } p \in [1,\infty). \tag{5.174}$$

With the strong convergence result (5.146) for $\boldsymbol{v}^m$, we deduce that

$$M^m\boldsymbol{v}^m\hat{\psi}^m \to M\boldsymbol{v}\hat{\psi} \quad \text{strongly in } L^p(Q;L^1(D;\mathbb{R}^d)), \text{ where } p \in [1,2). \tag{5.175}$$

Also, using (5.144), we obtain that

$$\hat{\psi}^m \omega(\boldsymbol{v}^m) \rightharpoonup \hat{\psi}\omega(\boldsymbol{v}) \quad \text{weakly in } L^p(Q; L^1(D; \mathbb{R}^{d\times d})), \text{ where } p \in [1,2). \tag{5.176}$$

Next, we shall derive convergence results for $\nabla_{x,\boldsymbol{q}}\hat{\psi}^m$. First, we note that for any measurable $U \subset (\mathcal{O} \times (0,T))$ with $|U| < \varepsilon$, we can apply Hölder's inequality to deduce that

$$\begin{aligned}\int_U M^m |\nabla_{x,\boldsymbol{q}}\hat{\psi}^m| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &\le C\int_U \sqrt{M^m}\left|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m\right| \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ &\le C\left(\int_U M^m \left|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_U 1\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\overset{(5.163)}{\le} C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

It follows from the Dunford–Pettis Theorem that there exists a subsequence (not relabelled) such that

$$M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup \overline{M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m} \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}). \tag{5.177}$$

We identify the weak limit $\overline{M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m}$ as $M\nabla_{x,\boldsymbol{q}}\hat{\psi}$ since we have that $\nabla_{x,\boldsymbol{q}}\hat{\psi}^m$ weakly converges to $\nabla_{x,\boldsymbol{q}}\hat{\psi}$ locally in $L^1(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)})$ and M^m converges to M uniformly in $C(\overline{D})$. Again, we use Hölder's inequality and the uniform estimate (5.163) to deduce the following bound:

$$\int_0^T\int_\Omega\left(\int_D M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|\,\mathrm{d}\boldsymbol{q}\right)^2 \mathrm{d}x\,\mathrm{d}t \le C\int_0^T\int_\Omega\left(\int_D M^m|\nabla_{x,\boldsymbol{q}}\hat{\psi}^m|^2\,\mathrm{d}\boldsymbol{q}\right)\mathrm{d}x\,\mathrm{d}t \le C(\ell),$$

which then allows us to strengthen (5.177) to

$$M^m\nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup M\nabla_{x,\boldsymbol{q}}\hat{\psi} \qquad \text{weakly in } L^2(Q;L^1(D;\mathbb{R}^{d(K+1)})). \tag{5.178}$$

From the strong convergence result (5.158) for θ^m, we deduce that

$$\theta^m \to \theta \qquad \text{a.e. in } \mathcal{O}\times(0,T). \tag{5.179}$$

Since the cut-off function $\beta^L(\cdot)$ is continuous and bounded above, we deduce that

$$\beta^L(\theta^m) \to \beta^L(\theta) \qquad \text{strongly in } L^p(Q) \text{ for any } p\in[1,\infty). \tag{5.180}$$

The strong convergence result (5.180) combined with (5.178) then gives

$$M^m\beta^L(\theta^m)\nabla_{x,\boldsymbol{q}}\hat{\psi}^m \rightharpoonup M\beta^L(\theta)\nabla_{x,\boldsymbol{q}}\hat{\psi} \quad \text{weakly in } L^p(Q;L^1(D;\mathbb{R}^{d(K+1)})), \tag{5.181}$$

for all $p \in [1,2)$. Finally, using (5.134) and the convergence results (5.175), (5.176) and (5.181), we obtain that

$$\partial_t(M^m \hat{\psi}^m) \rightharpoonup \partial_t(M\hat{\psi}) \qquad \text{weakly in } L^p(0,T;W^{-1,1}(\mathcal{O})), \tag{5.182}$$

for all $p \in [1,2)$. With the convergence results (5.144), (5.146), (5.158), (5.173), (5.175), (5.176), (5.181) and (5.182) we can pass to the limit as $m \to \infty$ in (5.133) and (5.134) to get

$$\big\langle \partial_t \boldsymbol{v}^\ell, \boldsymbol{w} \big\rangle - \big(\Gamma_\ell(|\boldsymbol{v}^\ell|)\, \boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{w}\big) + \big(2\nu(\theta^\ell) D(\boldsymbol{v}^\ell), \nabla_x \boldsymbol{w}\big) = (\boldsymbol{f}, \boldsymbol{w}), \tag{5.183}$$

for all $\boldsymbol{w} \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ and a.e. $t \in (0,T)$, and

$$\begin{aligned} \langle \partial_t(M\hat{\psi}^\ell), \varphi\rangle_{\mathcal{O}} &- \Big(M\boldsymbol{v}^\ell \hat{\psi}^\ell, \nabla_x \varphi\Big)_{\mathcal{O}} - \Big(M\hat{\psi}^\ell \omega(\boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi\Big)_{\mathcal{O}} \\ &+ \Big(M\beta^L(\theta^\ell)\nabla_x \hat{\psi}^\ell, \nabla_x \varphi\Big)_{\mathcal{O}} + \Big(4M\beta^L(\theta^\ell)\nabla_{\boldsymbol{q}} \hat{\psi}^\ell, \nabla_{\boldsymbol{q}} \varphi\Big)_{\mathcal{O}} = 0, \end{aligned} \tag{5.184}$$

for all $\varphi \in W^{1,\infty}(\mathcal{O})$ and a.e. $t \in (0,T)$. The initial conditions are satisfied in the following sense:

$$\lim_{t\to 0_+} \|\boldsymbol{v}^\ell(\cdot,t) - \boldsymbol{v}_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^d)} = 0, \tag{5.185}$$

$$\lim_{t\to 0_+} \|\hat{\psi}^\ell(\cdot,t) - T_\ell(\hat{\psi}_0(\cdot))\|_{L^1_M(\mathcal{O})} = 0. \tag{5.186}$$

Before passing to the limit as $m \to \infty$ in the temperature equation, we shall first show the strong convergence of the velocity gradient. We multiply (5.133) by $c_i^m(t)$ and sum over $i = 1, \dots, m$, then integrate the resulting identity over $(0,T)$ to get

$$\int_0^T \int_\Omega 2\nu(\theta^m)|D(\boldsymbol{v}^m)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{v}^m)\,\mathrm{d}t + \frac{1}{2}\Big(\|\boldsymbol{v}_0^m\|^2_{L^2(\Omega;\mathbb{R}^d)} - \|\boldsymbol{v}^m(T)\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big). \tag{5.187}$$

By setting the test function $\boldsymbol{w} \coloneqq \boldsymbol{v}$ in (5.183) and integrating the resulting identity over $(0,T)$, we obtain the following equality (omitting the superscript ℓ):

$$\int_0^T \int_\Omega 2\nu(\theta)|D(\boldsymbol{v})|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{v})\,\mathrm{d}t + \frac{1}{2}\Big(\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)} - \|\boldsymbol{v}(T)\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big). \tag{5.188}$$

By the weak lower semi-continuity and the weak convergence of $\boldsymbol{v}^m$, we get from

(5.187) and (5.188) that

$$\begin{aligned}
&\limsup_{m\to\infty}\int_0^T\int_\Omega \nu(\theta^m)|D(\boldsymbol{v}^m)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\limsup_{m\to\infty}\left(\int_0^T(\boldsymbol{f},\boldsymbol{v}^m)\,\mathrm{d}t+\frac{1}{2}\|\boldsymbol{v}_0^m\|^2_{L^2(\Omega;\mathbb{R}^d)}-\frac{1}{2}\|\boldsymbol{v}^m(T)\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)\\
&\quad=\int_0^T(\boldsymbol{f},\boldsymbol{v})\,\mathrm{d}t+\frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}-\frac{1}{2}\liminf_{m\to\infty}\|\boldsymbol{v}^m(T)\|^2_{L^2(\Omega;\mathbb{R}^d)} \qquad (5.189)\\
&\quad\le\int_0^T(\boldsymbol{f},\boldsymbol{v})\,\mathrm{d}t+\frac{1}{2}\|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}-\frac{1}{2}\|\boldsymbol{v}(T)\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad=\int_0^T\int_\Omega \nu(\theta)|D(\boldsymbol{v})|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Recall (5.158) which shows that θ^m converges to θ a.e. in Q. Since $\nu(\cdot)$ is Lipschitz continuous, we have

$$\nu(\theta^m)\to\nu(\theta) \qquad \text{a.e. in } Q. \tag{5.190}$$

From the weak convergence (5.144) of $\boldsymbol{v}^m$, we deduce that

$$\lim_{m\to\infty}\int_0^T\int_\Omega (D(\boldsymbol{v}^m)-D(\boldsymbol{v})):\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t=0,$$

for any $\boldsymbol{u}\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$. Setting $\boldsymbol{u}\coloneqq D(\boldsymbol{v})w$, where $w\in L^\infty(Q)$, we obtain from the above identity that

$$\lim_{m\to\infty}\int_0^T\int_\Omega (D(\boldsymbol{v}^m):D(\boldsymbol{v})-D(\boldsymbol{v}):D(\boldsymbol{v}))w\,\mathrm{d}x\,\mathrm{d}t=0,$$

which then implies

$$D(\boldsymbol{v}^m):D(\boldsymbol{v})\rightharpoonup D(\boldsymbol{v}):D(\boldsymbol{v}) \qquad \text{weakly in } L^1(Q). \tag{5.191}$$

Thanks to the convergence result (5.190), we further deduce that

$$\nu(\theta^m)D(\boldsymbol{v}^m):D(\boldsymbol{v})\rightharpoonup \nu(\theta)D(\boldsymbol{v}):D(\boldsymbol{v}) \qquad \text{weakly in } L^1(Q), \tag{5.192}$$

as a product of a weakly convergent sequence in L^1 and a uniformly bounded sequence that converges almost everywhere (c.f., Proposition 2.61 in [34]). Thus, by the lower semi-continuity of the norm function, we have

$$\int_0^T\int_\Omega \nu(\theta)D(\boldsymbol{v}):D(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\le\liminf_{m\to\infty}\int_0^T\int_\Omega \nu(\theta^m)D(\boldsymbol{v}^m):D(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t. \tag{5.193}$$

Therefore, from the assumption (5.15), we deduce the following inequality:

$$\begin{aligned}
&\lim_{m\to\infty} \nu_{\min} \int_0^T \int_\Omega |D(\boldsymbol{v}^m) - D(\boldsymbol{v})|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \lim_{m\to\infty} \int_0^T \int_\Omega \nu(\theta^m) |D(\boldsymbol{v}^m) - D(\boldsymbol{v})|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \limsup_{m\to\infty} \int_0^T \int_\Omega \nu(\theta^m) \big(|D(\boldsymbol{v}^m)|^2 + |D(\boldsymbol{v})|^2 - 2D(\boldsymbol{v}^m) : D(\boldsymbol{v})\big) \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \int_\Omega \nu(\theta) \big(|D(\boldsymbol{v})|^2 + |D(\boldsymbol{v})|^2 - 2D(\boldsymbol{v}) : D(\boldsymbol{v})\big) \,\mathrm{d}x\,\mathrm{d}t \\
&= 0,
\end{aligned}$$

where we have used (5.189), (5.190) and (5.193). In this way, we have strengthened the convergence property (5.144) for $\boldsymbol{v}^m$ to

$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)). \tag{5.194}$$

Therefore, we also have

$$|D(\boldsymbol{v}^m)|^2 \to |D(\boldsymbol{v})|^2 \qquad \text{strongly in } L^1(0,T; L^1(\Omega)). \tag{5.195}$$

We consider

$$\begin{aligned}
&\left| \int_0^T \int_\Omega (\nu(\theta^m)|D(\boldsymbol{v}^m)|^2 - \nu(\theta)|D(\boldsymbol{v})|^2) \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq \int_0^T \int_\Omega \nu(\theta^m) \left| |D(\boldsymbol{v}^m)|^2 - |D(\boldsymbol{v})|^2 \right| \,\mathrm{d}x\,\mathrm{d}t + \left| \int_0^T \int_\Omega (\nu(\theta^m) - \nu(\theta))|D(\boldsymbol{v})|^2 \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq \nu_{\max} \int_0^T \int_\Omega \left| |D(\boldsymbol{v}^m)|^2 - |D(\boldsymbol{v})|^2 \right| \,\mathrm{d}x\,\mathrm{d}t + \left| \int_0^T \int_\Omega (\nu(\theta^m) - \nu(\theta))|D(\boldsymbol{v})|^2 \,\mathrm{d}x\,\mathrm{d}t \right|.
\end{aligned}$$

The first integral in the last line converges to 0 thanks to (5.195). For the second integral in the last line, we have

$$|\nu(\theta^m) - \nu(\theta)| \leq |\nu'(\xi^m)||\theta^m - \theta| \leq \|\nu'\|_{L^\infty(\Omega)}|\theta^m - \theta|,$$

where ξ^m is some value in between θ^m and θ. It then follows from Hölder' s inequality and the strong convergence (5.158) of θ^m that

$$\int_0^T \int_\Omega |\nu(\theta^m) - \nu(\theta)| \,\mathrm{d}x\,\mathrm{d}t \leq \|\nu'\|_{L^\infty(\Omega)} \left(\int_0^T \int_\Omega |\theta^m - \theta| \,\mathrm{d}x\,\mathrm{d}t \right),$$

which converges to 0. Hence,

$$\nu(\theta^m) \to \nu(\theta) \quad \text{strongly in } L^1(Q).$$

As $\nu(\theta^m) \leq \nu_{\max}$, $\nu(\theta^m)$ converges weakly* in $L^\infty(Q)$. By uniqueness of the weak* limit, it follows that

$$\nu(\theta^m) \rightharpoonup^* \nu(\theta) \quad \text{weak* in } L^\infty(Q).$$

As $|D(\boldsymbol{v})|^2 \in L^1(Q)$, it is a legitimate test function for weak* convergence in $L^\infty(Q)$. Hence, as $m \to \infty$,

$$\left| \int_0^T \int_\Omega (\nu(\theta^m) - \nu(\theta)) |D(\boldsymbol{v})|^2 \,\mathrm{d}x \,\mathrm{d}t \right| \to 0.$$

Therefore, we deduce that

$$\nu(\theta^m)|D(\boldsymbol{v}^m)|^2 \to \nu(\theta)|D(\boldsymbol{v})|^2 \qquad \text{strongly in } L^1(0,T;L^1(\Omega)). \tag{5.196}$$

Using (5.135), and the convergence results (5.156), (5.158) and (5.196), we deduce that

$$\partial_t \theta^m \rightharpoonup \partial_t \theta \qquad \text{weakly in } L^1(0,T;(W^{1,q}(\Omega))'), \text{ where } q > \frac{2(d+2)}{4-d}. \tag{5.197}$$

Finally, we pass to the limit as $m \to \infty$ in (5.135) to deduce that

$$\left\langle \partial_t \theta^\ell, u \right\rangle - \left(\boldsymbol{v}^\ell\, \theta^\ell, \nabla_x u \right) + \left(\kappa(\theta^\ell)\, \nabla_x \theta^\ell, \nabla_x u \right) = \left(2\nu(\theta^\ell)\, D(\boldsymbol{v}^\ell) : D(\boldsymbol{v}^\ell), u \right), \tag{5.198}$$

for all $u \in W^{1,q}(\Omega)$, where $q > \frac{2(d+2)}{4-d}$ and a.e. $t \in (0,T)$. The initial condition is satisfied in the following sense:

$$\lim_{t \to 0_+} \|\theta^\ell(\cdot,t) - \theta_0(\cdot)\|_{L^1(\Omega)} = 0. \tag{5.199}$$

With the convergence results derived in this section, we pass to the limit as $m \to \infty$ in (5.148) and (5.162) to obtain the following energy inequality:

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}|\boldsymbol{v}^\ell(\cdot,t)|^2 + \theta^\ell(\cdot,t) \right) \mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}^\ell(\cdot,t))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \\
&\quad + \int_0^t \int_{\mathcal{O}} M\beta^L(\theta^\ell)|\nabla_x \hat{\psi}^\ell|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}\tau + 4\int_0^t \int_{\mathcal{O}} M\beta^L(\theta^\ell)|\nabla_{\boldsymbol{q}} \hat{\psi}^\ell|^2 \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}x \,\mathrm{d}\tau \\
&\leq \int_0^t \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^\ell \,\mathrm{d}x \,\mathrm{d}\tau + \int_\Omega \left(\frac{1}{2}|\boldsymbol{v}_0(\cdot)|^2 + \theta_0(\cdot) \right) \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathcal{O}} M(T_\ell(\hat{\psi}_0(\cdot)))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q},
\end{aligned} \tag{5.200}$$

where we have used the weak lower semi-continuity of norms to pass to the limit in all terms on the left-hand side. Collecting the uniform estimates (5.138), (5.140), (5.149), (5.151), (5.152), (5.153), (5.160), (5.161) and (5.163), we apply the weak lower semicontinuity of norms or Fatou's Lemma to derive the following ℓ-independent estimates:

$$\begin{aligned}
&\sup_{t\in(0,T)} \Big(\|\boldsymbol{v}^\ell(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\theta^\ell(t)\|_{L^1(\Omega)} + \|\hat{\psi}^\ell(t)\|^2_{L^2_M(\mathcal{O})} + \|\varrho^\ell(t)\|_{L^\infty(\Omega)} \Big) \\
&\quad + c_0\nu_{\min} \int_0^T \|\boldsymbol{v}^\ell\|^2_{W^{1,2}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t + \int_0^T \|\theta^\ell\|^p_{L^p(\Omega)} \,\mathrm{d}t + \int_0^T \|\theta^\ell\|^s_{W^{1,s}(\Omega)} \,\mathrm{d}t \\
&\quad \int_0^T \|(\theta^\ell)^{\frac{\lambda+1}{2}}\|^2_{W^{1,2}(\Omega)} \,\mathrm{d}t + \int_0^T \|\varrho^\ell\|^2_{W^{1,2}(\Omega)} \,\mathrm{d}t + \theta_{\min} \int_0^T \|\hat{\psi}^\ell\|^2_{W^{1,2}_M(\mathcal{O})} \,\mathrm{d}t \\
&\quad \le C(\boldsymbol{f}, M, \nu_{\max}, \boldsymbol{v}_0, \theta_0, \varphi_0),
\end{aligned} \tag{5.201}$$

for any $p \in \big[1, \frac{d+2}{d}\big)$, $s \in \big[1, \frac{d+2}{d+1}\big)$ and $\lambda \in (-1, 0)$. Note that the above bound does not depend on the parameter ℓ. This is because as $m \to \infty$, we have that

$$\int_{\mathcal{O}} M^m (T_\ell(\hat{\psi}_0^m(\cdot)))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \to \int_{\mathcal{O}} M (T_\ell(\hat{\psi}_0(\cdot)))^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \le C.$$

Thanks to the strong convergence in (5.158), we also have

$$\theta^\ell(x,t) \ge \theta_{\min} > 0 \qquad \text{for a.e. } (x,t) \in Q. \tag{5.202}$$

Therefore, we are ready to pass to the limit as $\ell \to \infty$ in the next section.

5.4.8 Passage to the limit as $\ell \to \infty$

This section is devoted to deriving the final limit as $\ell \to \infty$, which will then allow us to remove the truncation in the convective term in the Navier–Stokes equation. However, a problem arises in the justification of strong convergence of the velocity gradient since the velocity field will not be an admissible test function in the limit equation. To solve this problem, we consider the global energy $E^\ell := \frac{1}{2}|\boldsymbol{v}^\ell|^2 + \theta^\ell$. Recall that the equation satisfied by $\boldsymbol{v}^\ell$ is given by

$$\big\langle \partial_t \boldsymbol{v}^\ell, \boldsymbol{w} \big\rangle - \big(\Gamma_\ell(|\boldsymbol{v}^\ell|)\, \boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{w} \big) + \big(2\nu(\theta^\ell) D(\boldsymbol{v}^\ell), \nabla_x \boldsymbol{w} \big) = (\boldsymbol{f}, \boldsymbol{w}), \tag{5.203}$$

for all $\boldsymbol{w} \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. Setting $\boldsymbol{w} := \boldsymbol{v}^\ell u$, where $u \in W^{1,\infty}(\Omega)$, and adding the resulting identity to (5.198), we obtain

$$\begin{aligned}
&\langle \partial_t E^\ell, u \rangle - (\boldsymbol{v}^\ell \theta^\ell, \nabla_x u) - (\Gamma_\ell(|\boldsymbol{v}^\ell|) \boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell : \nabla_x \boldsymbol{v}^\ell, u) \\
&\quad - (\Gamma_\ell(|\boldsymbol{v}^\ell|)(\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell)\boldsymbol{v}^\ell, \nabla_x u) + (\kappa(\theta^\ell)\nabla_x \theta^\ell, \nabla_x u) \\
&\quad + (2\nu(\theta^\ell) D(\boldsymbol{v}^\ell)\boldsymbol{v}^\ell, \nabla_x u) = (\boldsymbol{f}, \boldsymbol{v}^\ell u).
\end{aligned} \tag{5.204}$$

First, we focus on the convergence properties of $\boldsymbol{v}^\ell$. From (5.201), we have by standard function space interpolation that

$$\|\boldsymbol{v}^\ell\|_{L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)} \leq C. \tag{5.205}$$

To prove a uniform bound on the time derivative of $\boldsymbol{v}^\ell$, we use (5.203) and the uniform bounds (5.201) and (5.205) to deduce that

$$\int_0^T \|\partial_t \boldsymbol{v}^\ell\|^p_{W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \leq C \tag{5.206}$$

for all $p \in \left(1, \frac{d+2}{d}\right]$.

Also, using (5.198) and the uniform bounds (5.201) and (5.205), we can follow a similar argument as (5.154) to deduce that

$$\int_0^T \|\partial_t \theta^\ell\|_{(W^{1,q}(\Omega))'} \,\mathrm{d}t \leq C, \tag{5.207}$$

where $q > \frac{2(d+2)}{4-d}$. Therefore, from the ℓ-independent estimates (5.201), (5.202) (5.205), (5.206) and (5.207), we deduce the existence of subsequences (not relabelled), such that, as $\ell \to \infty$,

$$\boldsymbol{v}^\ell \rightharpoonup^* \boldsymbol{v} \qquad \text{weak* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{5.208}$$

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)), \tag{5.209}$$

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d), \tag{5.210}$$

$$\partial_t\boldsymbol{v}^\ell \rightharpoonup \partial_t\boldsymbol{v} \qquad \text{weakly in } L^p(0,T;W^{-1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \text{ for any } p \in \left(1, \frac{d+2}{d}\right], \tag{5.211}$$

$$\theta^\ell \rightharpoonup \theta \qquad \text{weakly in } L^p(Q) \text{ for any } p \in \left[1, \frac{d+2}{d}\right), \tag{5.212}$$

$$\theta^\ell \rightharpoonup \theta \qquad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \text{ for any } s \in \left[1, \frac{d+2}{d+1}\right), \tag{5.213}$$

$$\partial_t\theta^\ell \rightharpoonup \partial_t\theta \qquad \text{weakly in } \mathcal{M}(0,T;(W^{1,q}(\Omega))'), \text{ where } p > \frac{2(d+2)}{4-d}. \tag{5.214}$$

Then, by (the generalized version of) the Aubin–Lions Lemma, we deduce that

$$\begin{aligned} \boldsymbol{v}^\ell \to \boldsymbol{v} \qquad & \text{strongly in } L^2(0,T;L^p(\Omega;\mathbb{R}^d)), \\ & \quad \text{where } p \in [1,\infty) \text{ when } d=2 \text{ and } p \in [1,6) \text{ when } d=3, \end{aligned} \tag{5.215}$$

$$\theta^\ell \to \theta \qquad \text{strongly in } L^p(Q) \text{ for any } p \in \left[1, \frac{d+2}{d}\right). \tag{5.216}$$

Next, we show the convergence properties of E^ℓ. From the strong convergence (5.215) of $\boldsymbol{v}^\ell$, we deduce that there exists a subsequence (not relabelled), such that

$$\boldsymbol{v}^\ell \to \boldsymbol{v} \qquad \text{a.e. in } Q. \tag{5.217}$$

Also, since the sequence $\boldsymbol{v}^\ell$ is bounded in $L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)$, then we have

$$\boldsymbol{v}^\ell \to \boldsymbol{v} \qquad \text{strongly in } L^p(Q;\mathbb{R}^d), \text{ for all } p \in \left[1, \frac{2(d+2)}{d}\right). \tag{5.218}$$

Therefore, combining (5.216) and (5.218), we have by Hölder's inequality that

$$\boldsymbol{v}^\ell\theta^\ell \to \boldsymbol{v}\theta \qquad \text{strongly in } L^p(Q;\mathbb{R}^d), \text{ for all } p \in \left[1, \frac{2(d+2)}{3d}\right). \tag{5.219}$$

It follows from (5.208) and (5.215) and function space interpolation (2.6) that

$$\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell \to \boldsymbol{v} \otimes \boldsymbol{v} \qquad \text{strongly in } L^p(Q;\mathbb{R}^{d\times d}), \tag{5.220}$$

where $p \in [1,\infty)$ when $d = 2$ and $p \in [1,3)$ when $d = 3$. Since $\Gamma_\ell(\cdot)$ is Lipschitz continuous, it follows from (5.220) and (5.209) that

$$\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell : \nabla_x \boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} : \nabla_x \boldsymbol{v} \qquad \text{weakly in } L^p(Q), \tag{5.221}$$

where $p \in [1,2)$ when $d = 2$ and $p \in \left[1, \frac{6}{5}\right)$ when $d = 3$. Also, it follows from (5.218) that

$$\Gamma_\ell(|\boldsymbol{v}^\ell|)(\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell)\boldsymbol{v}^\ell \to (\boldsymbol{v} \otimes \boldsymbol{v})\boldsymbol{v}$$
$$\text{strongly in } L^p(Q;\mathbb{R}^d), \text{ for all } p \in \left[1, \frac{2(d+2)}{3d}\right). \tag{5.222}$$

Since $\nu(\cdot)$ is Lipschitz continuous, it follows from (5.218), (5.216) and (5.209) that

$$\nu(\theta^\ell)D(\boldsymbol{v}^\ell)\boldsymbol{v}^\ell \rightharpoonup \nu(\theta)D(\boldsymbol{v})\boldsymbol{v} \qquad \text{weakly in } L^p(Q;\mathbb{R}^d), \text{ for all } p \in \left[1, \frac{d+2}{d+1}\right). \tag{5.223}$$

Using (5.204) and the convergence results above, we deduce that

$$\partial_t E^\ell \rightharpoonup \partial_t E \qquad \text{weakly in } L^p(0,T;(W^{1,p'}(\Omega))'), \text{ for all } p \in \left[1, \frac{2(d+2)}{3d}\right). \tag{5.224}$$

With the convergence results (5.209), (5.215), (5.216), (5.218) and (5.224) we can pass to the limit as $\ell \to \infty$ in (5.203) and (5.204) to deduce that

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle \,\mathrm{d}t + \int_0^T [-(\boldsymbol{v} \otimes \boldsymbol{v}, \nabla_x \boldsymbol{w}) + (2\nu(\theta)D(\boldsymbol{v}), \nabla_x \boldsymbol{w})] \,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{w}) \,\mathrm{d}t, \tag{5.225}$$

for all $\boldsymbol{w} \in L^s(0,T;W^{1,s}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d))$, where $s > d$, and

$$\int_0^T \langle \partial_t E, u\rangle \,\mathrm{d}t + \int_0^T [(\kappa(\theta)\nabla_x\theta, \nabla_x u) + (2\nu(\theta)D(\boldsymbol{v})\boldsymbol{v}, \nabla_x u) - (E\boldsymbol{v}, \nabla_x u)]\,\mathrm{d}t \\ = \int_0^T (\boldsymbol{f}, \boldsymbol{v}u)\,\mathrm{d}t, \quad (5.226)$$

for all $u \in L^\infty(0,T;W^{1,\infty}(\Omega))$.

Next, we show the convergence properties of $\hat{\psi}^\ell$. From the uniform bound (5.201), we have

$$\hat{\psi}^\ell \rightharpoonup \hat{\psi} \qquad \text{weakly in } L^2_M(\mathcal{O}\times(0,T)). \quad (5.227)$$

Our goal is to deduce the strong convergence of $\hat{\psi}^\ell$. To achieve this, we first show that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T). \quad (5.228)$$

The proof is similar to that in Section 5.4.7. Let $\mathcal{O}_0$ be a Lipschitz subdomain of $\mathcal{O}$ such that $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$. Since M is bounded below on $\mathcal{O}_0$ by a positive constant (which may depend on $\mathcal{O}_0$), we deduce from (5.201) that

$$\sup_{t\in(0,T)} \|\hat{\psi}^\ell(\cdot,t)\|^2_{L^2(\mathcal{O}_0)} + \theta_{\min}\int_0^T \|\hat{\psi}^\ell\|^2_{W^{1,2}(\mathcal{O}_0)}\,\mathrm{d}t \le C(\mathcal{O}_0). \quad (5.229)$$

Since $\mathcal{O}_0 \subset \mathcal{O} \subset \mathbb{R}^{d(K+1)}$, by standard function space interpolation, we get

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^\ell|^{\frac{2((K+1)d+2)}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0). \quad (5.230)$$

From (5.230) and (5.201), we deduce by using Hölder's inequality that there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^\ell\hat{\psi}^\ell\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \le C(\mathcal{O}_0). \quad (5.231)$$

Similarly, we have

$$\|\hat{\psi}^\ell\omega(\boldsymbol{v}^\ell)\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T);\mathbb{R}^d)} \le C(\mathcal{O}_0). \quad (5.232)$$

To be more specific, we have $\delta \in \left(0, \frac{1}{(K+1)d+1}\right]$. Then we define the following sequence of $(1+d+Kd)$-component vector fields:

$$H^\ell := (M\hat{\psi}^\ell, M\hat{\psi}^\ell\,\boldsymbol{v}^\ell - M\beta^L(\theta^\ell)\nabla_x\hat{\psi}^\ell, M\hat{\psi}^\ell\omega(\boldsymbol{v}^\ell)\,\boldsymbol{q} - M\beta^L(\theta^\ell)\nabla_{\boldsymbol{q}}\hat{\psi}^\ell),$$
$$Q^\ell := ((1+\hat{\psi}^\ell)^\alpha, \underbrace{0,\dots,0}_{(d+Kd)\text{-times}}),$$

for some $\alpha \in (0, 1/2)$. Therefore, using (5.229), (5.231) and (5.232), we deduce the existence of subsequences (not relabelled) such that

$$\begin{aligned} H^\ell \rightharpoonup H \qquad & \text{weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}), \\ Q^\ell \rightharpoonup Q \qquad & \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0,T); \mathbb{R}^{1+d+Kd}), \end{aligned}$$

where, noting the strong convergence (5.215) of $\boldsymbol{v}^\ell$ and the strong convergence (5.216) of θ^ℓ and the fact that $\beta^L(\cdot)$ is continuous,

$$\begin{aligned} H &:= (M\hat{\psi}, M\hat{\psi}\boldsymbol{v} - M\beta^L(\theta)\nabla_x\hat{\psi}, M\hat{\psi}\,\omega(\boldsymbol{v})\boldsymbol{q} - M\beta^L(\theta)\nabla_{\boldsymbol{q}}\hat{\psi}), \\ Q &:= (\overline{(1+\hat{\psi})^\alpha}, 0, \dots, 0). \end{aligned}$$

It follows from (5.184) that

$$\operatorname{div}_{t,x,\boldsymbol{q}} H^\ell = 0 \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Also, we obtain from (5.229) that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}_0} |\operatorname{curl}_{t,x,\boldsymbol{q}} Q^\ell|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t &= \int_0^T \int_{\mathcal{O}_0} |\nabla_{t,x,\boldsymbol{q}} Q^\ell - (\nabla_{t,x,\boldsymbol{q}} Q^\ell)^{\mathrm{T}}|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \\ &\leq C \int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} (1+\hat{\psi}^\ell)^\alpha|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \\ &\leq C \int_0^T \int_{\mathcal{O}_0} \left|\nabla_{x,\boldsymbol{q}} \hat{\psi}^\ell\right|^2 \,\mathrm{d}x \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}t \\ &\leq C(\mathcal{O}_0). \end{aligned}$$

Hence, $(\operatorname{div}_{t,x,\boldsymbol{q}} H^\ell)_{\ell=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$ and $(\operatorname{curl}_{t,x,\boldsymbol{q}} Q^\ell)_{\ell=1}^\infty$ is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$. By choosing $\alpha < \frac{\delta}{1+\delta}$, we apply the Div-Curl Lemma to deduce that

$$H^\ell \cdot Q^\ell \rightharpoonup H \cdot Q \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)).$$

In particular, we have that

$$\hat{\psi}^\ell (1+\hat{\psi}^\ell)^\alpha \rightharpoonup \hat{\psi}\overline{(1+\hat{\psi})^\alpha}. \tag{5.233}$$

Since $(1+\hat{\psi}^\ell)^\alpha \rightharpoonup \overline{(1+\hat{\psi})^\alpha}$ in $L^1(\mathcal{O}_0 \times (0,T))$, we can add this to (5.233), which gives

$$(1+\hat{\psi}^\ell)^{\alpha+1} = (1+\hat{\psi}^\ell)(1+\hat{\psi}^\ell)^\alpha \rightharpoonup (1+\hat{\psi})\overline{(1+\hat{\psi})^\alpha}.$$

Thanks to the weak lower semi-continuity of the continuous convex function $s \in [0,\infty) \mapsto s^{\alpha+1} \in [0,\infty)$ we have that

$$(1+\hat{\psi})^{\alpha+1} \leq (1+\hat{\psi})\overline{(1+\hat{\psi})^\alpha},$$

which is equivalent to

$$(1+\hat{\psi})^\alpha \leq \overline{(1+\hat{\psi})^\alpha}.$$

On the other hand, the function $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is concave. Again, by the weak lower semi-continuity of the continuous convex function $s \in [0,\infty) \mapsto -s^\alpha \in [0,\infty)$, we deduce that

$$-(1+\hat{\psi})^\alpha \leq -\overline{(1+\hat{\psi})^\alpha}.$$

Therefore,

$$(1+\hat{\psi})^\alpha = \overline{(1+\hat{\psi})^\alpha}.$$

Since $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is strictly concave, thanks to Theorem 2.4.6, there exists a subsequence (not relabelled), such that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T).$$

Now we want to extend the pointwise convergence result of $\hat{\psi}^\ell$ to the whole of our domain $\mathcal{O} \times (0,T)$. For this purpose we choose a nondecreasing sequence of nested sets $(\mathcal{O}_0^k)_{k=1}^\infty$, i.e., $\mathcal{O}_0^1 \subset \mathcal{O}_0^2 \subset \cdots \subset \mathcal{O}_0^k \subset \cdots$, satisfying $\cup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$. For each $k \in \mathbb{N}$, we deduce the existence of a subsequence of $(\hat{\psi}^\ell)_{\ell=1}^\infty$ that is pointwise convergent to $\hat{\psi}$ a.e. in $\mathcal{O}_0^k \times (0,T)$. Arguing by a diagonal procedure we deduce that there exists a subsequence such that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{5.234}$$

Combining (5.227) and (5.234), we obtain that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{strongly in } L^p(0,T;L^p_M(\mathcal{O})), \text{ for all } p \in [1,2). \tag{5.235}$$

Interpolating with (5.201), we deduce that

$$\hat{\psi}^\ell \to \hat{\psi} \qquad \text{strongly in } L^p(Q;L^1_M(D)), \text{ for all } p \in [1,\infty). \tag{5.236}$$

Also, it follows from (5.201) that

$$M^{\frac{1}{2}}\nabla_{x,\boldsymbol{q}}\hat{\psi}^\ell \rightharpoonup M^{\frac{1}{2}}\nabla_{x,\boldsymbol{q}}\hat{\psi} \qquad \text{weakly in } L^2(0,T;L^2(\mathcal{O})). \tag{5.237}$$

Combining (5.218) and (5.235), we obtain that

$$M\boldsymbol{v}^\ell\hat{\psi}^\ell \to M\boldsymbol{v}\hat{\psi} \qquad \text{strongly in } L^p(Q;\mathbb{R}^d), \text{ for all } p \in \left[1, \frac{d+2}{d+1}\right). \tag{5.238}$$

Combining (5.209) and (5.236), we have that

$$\hat{\psi}^\ell \omega(\boldsymbol{v}^\ell) \rightharpoonup \hat{\psi}\omega(\boldsymbol{v}) \quad \text{weakly in } L^p(Q; L^1_M(D; \mathbb{R}^{d\times d})), \text{ for all } p \in [1,2). \tag{5.239}$$

To show this, we take the test function $\boldsymbol{u} \in L^{p'}(Q; L^\infty(D; \mathbb{R}^{d\times d}))$, where $p' > 2$, and consider

$$\begin{aligned}
&\int_0^T \int_\Omega \int_D (M\hat{\psi}^\ell \omega(\boldsymbol{v}^\ell) - M\hat{\psi}\omega(\boldsymbol{v})) : \boldsymbol{u} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \int_D (M\hat{\psi}^\ell \omega(\boldsymbol{v}^\ell) - M\hat{\psi}\omega(\boldsymbol{v}^\ell)) : \boldsymbol{u} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \int_D (M\hat{\psi} \omega(\boldsymbol{v}^\ell) - M\hat{\psi}\omega(\boldsymbol{v})) : \boldsymbol{u} \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&=: I_1 + I_2.
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 &= \int_0^T \int_\Omega \omega(\boldsymbol{v}^\ell) : \left(\int_D (M\hat{\psi}^\ell - M\hat{\psi})\boldsymbol{u} \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x\,\mathrm{d}t \\
&\leq \left(\int_0^T \int_\Omega |\omega(\boldsymbol{v}^\ell)|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^T \int_\Omega |\operatorname{ess\,sup}_{\boldsymbol{q}\in D} \boldsymbol{u}(x,\boldsymbol{q},t)|^2 \left(\int_D (M\hat{\psi}^\ell - M\hat{\psi}) \,\mathrm{d}\boldsymbol{q} \right)^2 \mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq \|\omega(\boldsymbol{v}^\ell)\|_{L^2(Q;\mathbb{R}^{d\times d})} \left(\int_0^T \int_\Omega |\operatorname{ess\,sup}_{\boldsymbol{q}\in D} \boldsymbol{u}(x,\boldsymbol{q},t)|^{p'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p'}} \\
&\quad \times \left(\int_0^T \int_\Omega \left(\int_D (M\hat{\psi}^\ell - M\hat{\psi}) \,\mathrm{d}\boldsymbol{q} \right)^{\frac{2p'}{p'-2}} \mathrm{d}x\,\mathrm{d}t \right)^{\frac{p'-2}{2p'}} \\
&\leq \|\omega(\boldsymbol{v}^\ell)\|_{L^2(Q;\mathbb{R}^{d\times d})} \|\boldsymbol{u}\|_{L^{p'}(Q;L^\infty(D;\mathbb{R}^{d\times d}))} \|M\hat{\psi}^\ell - M\hat{\psi}\|_{L^{\frac{2p'}{p'-2}}(Q;L^1(D))}.
\end{aligned}$$

The second norm on the last line of the above inequality is bounded since $\boldsymbol{u} \in L^{p'}(Q; L^\infty(D; \mathbb{R}^{d\times d}))$. It follows from (5.201) that $\|\omega(\boldsymbol{v}^\ell)\|_{L^2(Q;\mathbb{R}^{d\times d})} \leq C$. Since $p' > 2$, it follows from (5.235) that $\|M\hat{\psi}^\ell - M\hat{\psi}\|_{L^{\frac{2p'}{p'-2}}(Q;L^1(D))} \to 0$ as $\ell \to \infty$. Therefore, $I_1 \to 0$.

For I_2, we have

$$I_2 = \int_0^T \int_\Omega (\omega(\boldsymbol{v}^\ell) - \omega(\boldsymbol{v})) : \left(\int_D M\hat{\psi}\boldsymbol{u} \,\mathrm{d}\boldsymbol{q} \right) \mathrm{d}x\,\mathrm{d}t.$$

Next, we show that $\int_D M\hat{\psi}\boldsymbol{u}\,\mathrm{d}\boldsymbol{q} \in L^2(Q;\mathbb{R}^d)$, which makes it a valid test function for the weak convergence of $\omega(\boldsymbol{v}^\ell)$ in $L^2(Q;\mathbb{R}^{d\times d})$ that follows from (5.209).

$$\begin{aligned}
&\int_0^T \int_\Omega \left| \int_D M\hat{\psi}\boldsymbol{u}\,\mathrm{d}\boldsymbol{q} \right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \int_\Omega |\operatorname{ess\,sup}_{\boldsymbol{q}\in D} \boldsymbol{u}(x,\boldsymbol{q},t)|^2 \left(\int_D M\hat{\psi}\,\mathrm{d}\boldsymbol{q} \right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\leq \|\boldsymbol{u}\|^2_{L^{p'}(Q;L^\infty(D;\mathbb{R}^{d\times d}))} \|M\hat{\psi}\|^2_{L^{\frac{2p'}{p'-2}}(Q;L^1(D))} \\
&\leq \|\boldsymbol{u}\|^2_{L^{p'}(Q;L^\infty(D;\mathbb{R}^{d\times d}))} \|\varrho\|^2_{L^\infty(Q)} \\
&\leq C.
\end{aligned}$$

Therefore, $I_2 \to 0$. We have shown that

$$\int_0^T \int_\Omega \int_D (M\hat{\psi}^\ell \omega(\boldsymbol{v}^\ell) - M\hat{\psi}\omega(\boldsymbol{v})) : \boldsymbol{u}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \to 0,$$

for all $\boldsymbol{u} \in L^{p'}(Q;L^\infty(D;\mathbb{R}^{d\times d}))$, where $p' > 2$, which implies the convergence result (5.239).

It follows from (5.216) that

$$\theta^\ell \to \theta \qquad \text{a.e. in } Q. \tag{5.240}$$

Since $\beta^L(\cdot)$ is continuous and bounded above, then we obtain that

$$\beta^L(\theta^\ell) \to \beta^L(\theta) \qquad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty). \tag{5.241}$$

Also, combining (5.241) and (5.237), we obtain that

$$\begin{aligned}
&M\beta^L(\theta^\ell)\nabla_{x,\boldsymbol{q}}\hat{\psi}^\ell \rightharpoonup M\beta^L(\theta)\nabla_{x,\boldsymbol{q}}\hat{\psi} \\
&\qquad\qquad\text{weakly in } L^p(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)}), \text{ for all } p \in [1,2).
\end{aligned} \tag{5.242}$$

Using (5.184) and the convergence results above, we deduce that

$$\partial_t(M\hat{\psi}^\ell) \rightharpoonup \partial_t(M\hat{\psi}) \qquad \text{weakly in } L^p(0,T;W^{-1,1}(\mathcal{O})), \text{ for all } p \in [1,2). \tag{5.243}$$

Finally, we pass to the limit as $\ell \to \infty$ in (5.184) to get

$$\begin{aligned}
&\int_0^T \Big\langle \partial_t(M\hat{\psi}), \varphi \Big\rangle_{\mathcal{O}} - \Big(M\boldsymbol{v}\hat{\psi}, \nabla_x\varphi \Big)_{\mathcal{O}} - \Big(M\hat{\psi}\,\omega(\boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi \Big)_{\mathcal{O}}\,\mathrm{d}t \\
&\quad + \int_0^T \Big(M\beta^L(\theta)\nabla_x\hat{\psi}, \nabla_x\varphi \Big)_{\mathcal{O}} + \Big(4M\beta^L(\theta)\nabla_{\boldsymbol{q}}\hat{\psi}, \nabla_{\boldsymbol{q}}\varphi \Big)_{\mathcal{O}}\,\mathrm{d}t = 0,
\end{aligned} \tag{5.244}$$

for all $\varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O}))$.

For the last part of the proof, we shall focus on the attainment of initial conditions. Choosing $\boldsymbol{w} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$ in (5.203) and integrating with respect to time over $(0,t)$, we get

$$(\boldsymbol{v}^\ell(t),\boldsymbol{w}) + \int_0^t [-\big(\Gamma_\ell(|\boldsymbol{v}^\ell|)\,\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla_x \boldsymbol{w}\big) + \big(2\nu(\theta^\ell)D(\boldsymbol{v}^\ell), \nabla_x \boldsymbol{w}\big)]\,\mathrm{d}\tau \\ = \int_0^t (\boldsymbol{f},\boldsymbol{w})\,\mathrm{d}\tau + (\boldsymbol{v}_0,\boldsymbol{w}).$$

Letting $\ell \to \infty$ in the above equation, and using the convergence results (5.209), (5.215) and (5.216), we deduce that, for a.e. $t \in (0,T)$,

$$(\boldsymbol{v}(t),\boldsymbol{w}) + \int_0^t [-(\boldsymbol{v}\otimes\boldsymbol{v}, \nabla_x \boldsymbol{w}) + (2\nu(\theta)D(\boldsymbol{v}), \nabla_x \boldsymbol{w})]\,\mathrm{d}\tau = \int_0^t (\boldsymbol{f},\boldsymbol{w})\,\mathrm{d}\tau + (\boldsymbol{v}_0,\boldsymbol{w}).$$

After a possible redefinition of $\boldsymbol{v}$ on a set of measure zero, the above identity holds for all $t \in (0,T)$. Therefore,

$$\lim_{t\to 0_+} (\boldsymbol{v}(t),\boldsymbol{w}) = (\boldsymbol{v}_0,\boldsymbol{w}) \qquad \text{for all } \boldsymbol{w} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d).$$

Since $\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d))$, we deduce that, as $t \to 0_+$,

$$\boldsymbol{v}(t) \rightharpoonup \boldsymbol{v}_0 \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^d). \tag{5.245}$$

Note that for $\boldsymbol{v}^\ell$, we have the following energy inequality:

$$\frac{1}{2}\int_\Omega |\boldsymbol{v}^\ell(\cdot,t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega 2\nu(\theta^\ell)|D(\boldsymbol{v}^\ell)|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}^\ell\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x.$$

Letting $\ell \to \infty$ and neglecting the nonnegative term on the left-hand side, we deduce by weak lower semi-continuity that

$$\frac{1}{2}\int_\Omega |\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x \le \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x.$$

Thus, we have that

$$\limsup_{t\to 0_+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} \le \|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Using the weak convergence result (5.245), we obtain by weak lower semi-continuity that

$$\liminf_{t\to 0_+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} \ge \|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Therefore,

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} = \|\boldsymbol{v}_0\|^2_{L^2(\Omega;\mathbb{R}^d)}. \tag{5.246}$$

Following a similar argument, we integrate (5.184) with respect to time over $(0,t)$ to get

$$\begin{aligned}(M\hat{\psi}^\ell(t),\varphi)_{\mathcal{O}} &- \int_0^t \left[\Big(M\boldsymbol{v}^\ell\hat{\psi}^\ell, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(M\hat{\psi}^\ell\omega(\boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\right] \mathrm{d}\tau \\ &+ \int_0^t \left[\Big(M\beta^L(\theta^\ell)\nabla_x\hat{\psi}^\ell, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(4M\beta^L(\theta^\ell)\nabla_{\boldsymbol{q}}\hat{\psi}^\ell, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\right] \mathrm{d}\tau = (M\hat{\psi}_0,\varphi),\end{aligned}$$

for all $\varphi \in W^{1,\infty}(\mathcal{O})$ and a.e. $t \in (0,T)$. Letting $\ell \to \infty$, and using the convergence results (5.236), (5.237), (5.238), (5.239) and (5.242), we deduce that, for a.e. $t \in (0,T)$,

$$\begin{aligned}(M\hat{\psi}(t),\varphi)_{\mathcal{O}} &- \int_0^t \left[\Big(M\boldsymbol{v}\hat{\psi}, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(M\hat{\psi}\omega(\boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\right] \mathrm{d}\tau \\ &+ \int_0^t \left[\Big(M\beta^L(\theta)\nabla_x\hat{\psi}, \nabla_x\varphi\Big)_{\mathcal{O}} + \Big(4M\beta^L(\theta)\nabla_{\boldsymbol{q}}\hat{\psi}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\right] \mathrm{d}\tau = (M\hat{\psi}_0,\varphi).\end{aligned}$$

After a possible redefinition of $\hat{\psi}$ on a set of measure zero, the above identity holds for all $t \in (0,T)$. Therefore,

$$\lim_{t\to 0_+} (M\hat{\psi}(t),\varphi)_{\mathcal{O}} = (M\hat{\psi}_0,\varphi) \qquad \text{for all } \varphi \in W^{1,\infty}(\mathcal{O}).$$

Since $\hat{\psi} \in L^2(0,T;L^2_M(\mathcal{O}))$, we deduce that, as $t \to 0_+$,

$$\hat{\psi}(t) \rightharpoonup \hat{\psi}_0 \qquad \text{weakly in } L^2_M(\mathcal{O}). \tag{5.247}$$

Recall that for $\hat{\psi}^\ell$, we have the following energy inequality:

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}^\ell(\cdot,t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} &+ \int_0^t \int_{\mathcal{O}} M\beta^L(\theta^\ell)|\nabla_x\hat{\psi}^\ell|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ &+ 4\int_0^t \int_{\mathcal{O}} M\beta^L(\theta^\ell)|\nabla_{\boldsymbol{q}}\hat{\psi}^\ell|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \le \frac{1}{2}\int_{\mathcal{O}} M(T_\ell(\hat{\psi}_0(\cdot)))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.\end{aligned}$$

Letting $\ell \to \infty$ neglecting the nonnegative term on the left-hand side, we deduce that

$$\frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}(\cdot,t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le \frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}_0(\cdot))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Thus, we have that

$$\limsup_{t\to 0_+} \int_{\mathcal{O}} M(\hat{\psi}(\cdot,t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le \int_{\mathcal{O}} M(\hat{\psi}_0(\cdot))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Using the weak convergence result (5.247), we obtain by weak lower semi-continuity that

$$\liminf_{t\to 0_+} \int_{\mathcal{O}} M(\hat{\psi}(\cdot,t))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \geq \int_{\mathcal{O}} M(\hat{\psi}_0(\cdot))^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Therefore,

$$\lim_{t\to 0_+} \|\hat{\psi}(t)\|^2_{L^2_M(\mathcal{O})} = \|\hat{\psi}_0\|^2_{L^2_M(\mathcal{O})}. \tag{5.248}$$

Similarly as above, integrating (5.204) with respect to time over $(0,t)$, we obtain

$$\begin{aligned}(E^\ell(t),u) &+ \int_0^t \left[-(\boldsymbol{v}^\ell\theta^\ell,\nabla_x u) - (\Gamma_\ell(|\boldsymbol{v}^\ell|)\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell : \nabla_x\boldsymbol{v}^\ell,u)\right]\mathrm{d}\tau\\ &+ \int_0^t \left[-(\Gamma_\ell(|\boldsymbol{v}^\ell|)(\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell)\boldsymbol{v}^\ell,\nabla_x u) + (\kappa(\theta^\ell)\nabla_x\theta^\ell,\nabla_x u)\right]\mathrm{d}\tau\\ &+ \int_0^t \left[(2\nu(\theta^\ell)D(\boldsymbol{v}^\ell)\boldsymbol{v}^\ell,\nabla_x u)\right]\mathrm{d}\tau = \int_0^t (\boldsymbol{f},\boldsymbol{v}^\ell u)\,\mathrm{d}\tau + (E^\ell(0),u),\end{aligned}$$

for all $u \in W^{1,\infty}(\Omega)$ and a.e. $t \in (0,T)$. By the definition of E, we deduce from (5.185) and (5.199) that $E^\ell(0) = E_0 := \frac{1}{2}|\boldsymbol{v}_0|^2 + \theta_0$. Letting $\ell \to \infty$, and using the convergence results (5.209), (5.213), (5.215) and (5.216), we deduce that, for a.e. $t \in (0,T)$,

$$\begin{aligned}(E(t),u) + \int_0^t [(\kappa(\theta)\nabla_x\theta,\nabla_x u) + (2\nu(\theta)D(\boldsymbol{v})\boldsymbol{v},\nabla_x u) - (E\boldsymbol{v},\nabla_x u)]\,\mathrm{d}\tau&\\ = \int_0^t (\boldsymbol{f},\boldsymbol{v}u)\,\mathrm{d}t + (E_0,u).&\end{aligned}$$

After a possible redefinition of E on a set of measure zero, the above identity holds for all $t \in (0,T)$. Therefore,

$$\lim_{t\to 0_+} (E(t),u) = (E_0,u) \qquad \text{for all } u \in W^{1,\infty}(\Omega).$$

Since $\boldsymbol{v} \in L^{\frac{2(d+2)}{d}}(Q;\mathbb{R}^d)$ and $\theta \in L^p(Q)$ for $p \in \left[1,\frac{d+2}{d}\right)$, we deduce that, as $t \to 0_+$,

$$\left(\frac{1}{2}|\boldsymbol{v}(t)|^2 + \theta(t)\right) = E(t) \rightharpoonup E_0 = \left(\frac{1}{2}|\boldsymbol{v}_0|^2 + \theta_0\right) \qquad \text{weakly in } L^p(\Omega),$$

for $p \in \left[1,\frac{d+2}{d}\right)$. Letting $\ell \to \infty$ in (5.200), neglecting the nonnegative integrals on the left-hand side and using the weak convergence results deduced in this section, we obtain that

$$\begin{aligned}\int_\Omega \left(\frac{1}{2}|\boldsymbol{v}(\cdot,t)|^2 + \theta(\cdot,t)\right)\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} M(\hat{\psi}(\cdot,t))^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}&\\ \leq \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega\left(\frac{1}{2}|\boldsymbol{v}_0|^2 + \theta_0\right)\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} M\hat{\psi}_0^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.&\end{aligned}$$

First we note that

$$\limsup_{t\to 0_+}\left[\int_\Omega\left(\frac{1}{2}|\boldsymbol{v}(\cdot,t)|^2+\theta(\cdot,t)\right)\,\mathrm{d}x+\frac{1}{2}\int_{\mathcal{O}}M(\hat{\psi}(\cdot,t))^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\right]$$
$$\leq\int_\Omega\left(\frac{1}{2}|\boldsymbol{v}_0|^2+\theta_0\right)\,\mathrm{d}x+\frac{1}{2}\int_{\mathcal{O}}M\hat{\psi}_0^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}.$$

Since we already have strong attainment of initial conditions for $\boldsymbol{v}$ and $\hat{\psi}$, we deduce from the above inequality that

$$\limsup_{t\to 0_+}\int_\Omega\theta(\cdot,t)\,\mathrm{d}x\leq\int_\Omega\theta_0\,\mathrm{d}x. \tag{5.249}$$

To show the strong attainment of the initial condition for θ, we let $u=(1+\theta)^{-\frac{1}{2}}\tilde{u}$ in (5.198), where $\tilde{u}\in W^{1,\infty}(\Omega)$. Note that u is an admissible test function. Then we have

$$\begin{aligned}2\langle\partial_t\sqrt{1+\theta^\ell},\tilde{u}\rangle+\int_\Omega\left(\frac{\kappa(\theta^\ell)\nabla_x\theta^\ell}{\sqrt{1+\theta^\ell}}-2\sqrt{1+\theta^\ell}\,\boldsymbol{v}^\ell\right)\cdot\nabla_x\tilde{u}\,\mathrm{d}x\\ -\frac{1}{2}\int_\Omega\frac{\kappa(\theta^\ell)\tilde{u}}{(1+\theta^\ell)^{\frac{3}{2}}}\nabla_x\theta^\ell\cdot\nabla_x\theta^\ell\,\mathrm{d}x=\int_\Omega\frac{2\nu(\theta^\ell)|D(\boldsymbol{v}^\ell)|^2\tilde{u}}{\sqrt{1+\theta^\ell}}\,\mathrm{d}x.\end{aligned} \tag{5.250}$$

Using (5.250), we deduce that, for $q>d+2$,

$$\begin{aligned}\int_0^T\|\partial_t\sqrt{1+\theta^\ell}\|_{(W^{1,q}(\Omega))'}\,\mathrm{d}t&=\int_0^T\left(\sup_{u\in W^{1,q}(\Omega)}\frac{|\langle\partial_t\sqrt{1+\theta^\ell},u\rangle|}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t\\ &\leq\frac{1}{2}\int_0^T\left(\sup_{u\in W^{1,q}(\Omega)}\frac{\left|\left(\frac{\kappa(\theta^\ell)\nabla_x\theta^\ell}{\sqrt{1+\theta^\ell}},\nabla_x u\right)\right|}{\|\nabla_x u\|_{L^q(\Omega;\mathbb{R}^d)}}\right)\mathrm{d}t\\ &\quad+\int_0^T\left(\sup_{u\in W^{1,q}(\Omega)}\frac{|(\sqrt{1+\theta^\ell}\boldsymbol{v}^\ell,\nabla_x u)|}{\|\nabla_x u\|_{L^q(\Omega;\mathbb{R}^d)}}\right)\mathrm{d}t\\ &\quad+\frac{1}{4}\int_0^T\left(\sup_{u\in W^{1,q}(\Omega)}\frac{\left|\left(\frac{\kappa(\theta^\ell)\nabla_x\theta^\ell\cdot\nabla_x\theta^\ell}{(1+\theta^\ell)^{\frac{3}{2}}},u\right)\right|}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t\\ &\quad+\int_0^T\left(\sup_{u\in W^{1,q}(\Omega)}\frac{\left|\left(\frac{\nu(\theta^\ell)|D(\boldsymbol{v}^\ell)|^2}{\sqrt{1+\theta^\ell}},u\right)\right|}{\|u\|_{W^{1,q}(\Omega)}}\right)\mathrm{d}t\\ &=:\frac{1}{2}I_1+I_2+\frac{1}{4}I_3+I_4.\end{aligned}$$

For I_1, we have by Hölder's inequality and (5.16),(5.202) and (5.201) that

$$I_1 \le \int_0^T \left\| \frac{\kappa(\theta^\ell)\nabla_x \theta^\ell}{\sqrt{1+\theta^\ell}} \right\|_{L^{q'}(\Omega;\mathbb{R}^d)} \mathrm{d}t \le C(\kappa_{\max}, \theta_{\min}) \|\nabla_x \theta^\ell\|_{L^s(Q;\mathbb{R}^d)} \le C,$$

provided that $q' < \frac{d+2}{d+1}$, i.e. $q > d+2$. For I_2, by applying Hölder's inequality, we deduce from the uniform bound (5.201) that

$$\begin{aligned} I_2 &\le \int_0^T \|\sqrt{1+\theta^\ell}\boldsymbol{v}^\ell\|_{L^{q'}(\Omega;\mathbb{R}^d)} \,\mathrm{d}t \\ &\le \int_0^T \left(\int_\Omega |1+\theta^\ell|^p \,\mathrm{d}x\right)^{\frac{1}{2p}} \left(\int_\Omega |\boldsymbol{v}^\ell|^{\frac{2pq'}{2p-q'}} \,\mathrm{d}x\right)^{\frac{2p-q'}{2pq'}} \mathrm{d}t \\ &\le \left(\int_0^T \int_\Omega |1+\theta^\ell|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2p}} \left(\int_0^T \int_\Omega |\boldsymbol{v}^\ell|^{\frac{2pq'}{2p-q'}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2p-q'}{2pq'}} \left(\int_0^T 1^{\frac{q'}{q'-1}} \,\mathrm{d}t\right)^{\frac{q'-1}{q'}} \\ &\le C\|\theta^\ell\|^{\frac{1}{2}}_{L^p(Q)} \|\boldsymbol{v}^\ell\|_{L^{\frac{2(d+2)}{d}}(Q;\mathrm{R}^d)} \\ &\le C, \end{aligned}$$

provided that $q' < \frac{d+2}{d}$, which naturally holds since $q > d+2$. For I_3, we have

$$\begin{aligned} I_3 &\le \kappa_{\max} \left(\sup_{u \in W^{1,q}(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{W^{1,q}(\Omega)}} \right) \int_0^T \int_\Omega (\theta^\ell)^{-\frac{3}{2}} |\nabla_x \theta^\ell|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \kappa_{\max} \left(\sup_{u \in W^{1,q}(\Omega)} \frac{C\|u\|_{W^{1,q}(\Omega)}}{\|u\|_{W^{1,q}(\Omega)}} \right) \int_0^T \int_\Omega \left|\nabla_x (\theta^\ell)^{\frac{1}{4}}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C, \end{aligned}$$

which follows from the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ when $q > d+2$ and the uniform bound (5.201) with $\lambda = \frac{1}{2}$. Similarly, for I_4, we have

$$I_4 \le C(\nu_{\max}, \theta_{\min}) \left(\sup_{u \in W^{1,q}(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{W^{1,q}(\Omega)}} \right) \int_0^T \int_\Omega |D(\boldsymbol{v}^\ell)|^2 \,\mathrm{d}x\,\mathrm{d}t \le C,$$

where we again use the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ when $q > d+2$ and the uniform estimates (5.15), (5.202) and (5.201). Finally, we obtain that

$$\int_0^T \|\partial_t \sqrt{1+\theta^\ell}\|_{(W^{1,q}(\Omega))'} \,\mathrm{d}t \le C, \tag{5.251}$$

where $q > d+2$.

It follows from (5.215) that

$$\sqrt{1+\theta^\ell} \to \sqrt{1+\theta} \qquad \text{strongly in } L^2(Q).$$

From the uniform bound (5.251), we deduce that

$$\partial_t\sqrt{1+\theta^\ell} \rightharpoonup^* \partial_t\sqrt{1+\theta} \qquad \text{weakly* in } \mathcal{M}(0,T;(W^{1,q}(\Omega))'), \text{ where } q > d+2.$$

Since the time derivative is a measure, it makes sense to define

$$\begin{aligned}\sqrt{1+\theta(\tau_+)} &\coloneqq \lim_{t\to\tau_+}\sqrt{1+\theta(t)},\\ \sqrt{1+\theta(\tau_-)} &\coloneqq \lim_{t\to\tau_-}\sqrt{1+\theta(t)},\end{aligned}$$

for all $\tau \in (0,T)$, where both limits are considered in the space $(W^{1,p}(\Omega))'$. Also, it follows from (5.201) and (5.216) that $\sqrt{1+\theta} \in L^\infty(0,T;L^2(\Omega))$. Then, since $W^{1,p}(\Omega)$ is dense in $L^2(\Omega)$, it follows from above that

$$\sqrt{1+\theta(t)} \rightharpoonup \sqrt{1+\theta(\tau_+)} \qquad \text{weakly in } L^2(\Omega) \text{ as } t \to \tau_+, \tag{5.252}$$

$$\sqrt{1+\theta(t)} \rightharpoonup \sqrt{1+\theta(\tau_-)} \qquad \text{weakly in } L^2(\Omega) \text{ as } t \to \tau_-. \tag{5.253}$$

Since θ^ℓ is positive, then we consider $\tilde{u} \in W^{1,\infty}(\Omega)$, with $\tilde{u} \geq 0$ and we obtain that

$$\begin{aligned}&2\langle\partial_t\sqrt{1+\theta^\ell},\tilde{u}\rangle + \int_\Omega\left(\frac{\kappa(\theta^\ell)\nabla_x\theta^\ell}{\sqrt{1+\theta^\ell}} - 2\sqrt{1+\theta^\ell}\,\boldsymbol{v}^\ell\right)\cdot\nabla_x\tilde{u}\,\mathrm{d}x\\ &\quad - \frac{1}{2}\int_\Omega\frac{\kappa(\theta^\ell)\tilde{u}}{(1+\theta^\ell)^{\frac{3}{2}}}\nabla_x\theta^\ell\cdot\nabla_x\theta^\ell\,\mathrm{d}x \geq 0.\end{aligned}$$

Integrating over $(0,\tau)$ gives that

$$\begin{aligned}&2\int_\Omega\sqrt{1+\theta^\ell(\tau)}\,\tilde{u}\,\mathrm{d}x + \int_0^\tau\int_\Omega\left(\frac{\kappa(\theta^\ell)\nabla_x\theta^\ell}{\sqrt{1+\theta^\ell}} - 2\sqrt{1+\theta^\ell}\,\boldsymbol{v}^\ell\right)\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \frac{1}{2}\int_0^\tau\int_\Omega\frac{\kappa(\theta^\ell)\tilde{u}}{(1+\theta^\ell)^{\frac{3}{2}}}\nabla_x\theta^\ell\cdot\nabla_x\theta^\ell\,\mathrm{d}x\,\mathrm{d}t \geq 2\int_\Omega\sqrt{1+\theta_0}\,\tilde{u}\,\mathrm{d}x.\end{aligned}$$

Then integrating over $\tau \in (t_0, t_0+h)$ gives that

$$\begin{aligned}&2\int_{t_0}^{t_0+h}\int_\Omega\sqrt{1+\theta^\ell(\tau)}\,\tilde{u}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega\frac{\kappa(\theta^\ell)}{\sqrt{1+\theta^\ell}}\nabla_x\theta^\ell\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau\\ &\quad - 2\int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega\sqrt{1+\theta^\ell}\,\boldsymbol{v}^\ell\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau\\ &\quad - \frac{1}{2}\int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega\frac{\kappa(\theta^\ell)\tilde{u}}{(1+\theta^\ell)^{\frac{3}{2}}}\nabla_x\theta^\ell\cdot\nabla_x\theta^\ell\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \geq 2\int_{t_0}^{t_0+h}\int_\Omega\sqrt{1+\theta_0}\,\tilde{u}\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

With the convergence results derived in this section, we pass to the limit as $\ell \to \infty$ in the above inequality to obtain that

$$\begin{aligned}
2\int_{t_0}^{t_0+h}&\int_\Omega \sqrt{1+\theta(\tau)}\,\tilde{u}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega \frac{\kappa(\theta)}{\sqrt{1+\theta}}\nabla_x\theta\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\
&- 2\int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega \sqrt{1+\theta}\,\boldsymbol{v}\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\
&- \frac{1}{2}\int_{t_0}^{t_0+h}\int_0^\tau\int_\Omega \frac{\kappa(\theta)\tilde{u}}{(1+\theta)^{\frac{3}{2}}}\nabla_x\theta\cdot\nabla_x\theta\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \geq 2\int_{t_0}^{t_0+h}\int_\Omega \sqrt{1+\theta_0}\,\tilde{u}\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Dividing by h and letting $h \to 0_+$ and using (5.252) and (5.253), we deduce that, for all $\tau \in (0, T)$ and for all nonnegative $\tilde{u} \in W^{1,\infty}(\Omega)$, we have

$$\begin{aligned}
2\int_\Omega \sqrt{1+\theta(\tau_+)}\,\tilde{u}\,\mathrm{d}x &+ \int_0^\tau\int_\Omega \left(\frac{\kappa(\theta)\nabla_x\theta}{\sqrt{1+\theta}} - 2\sqrt{1+\theta}\,\boldsymbol{v}\right)\cdot\nabla_x\tilde{u}\,\mathrm{d}x\,\mathrm{d}t \\
&- \frac{1}{2}\int_0^\tau\int_\Omega \frac{\kappa(\theta)\tilde{u}}{(1+\theta)^{\frac{3}{2}}}\nabla_x\theta\cdot\nabla_x\theta\,\mathrm{d}x\,\mathrm{d}t \geq 2\int_\Omega \sqrt{1+\theta_0}\,\tilde{u}\,\mathrm{d}x.
\end{aligned}$$

Letting $\tau \to 0_+$ and using (5.252) and (5.253), we deduce that, for all nonnegative $\tilde{u} \in L^2(\Omega)$,

$$\liminf_{t\to 0_+}\int_\Omega \sqrt{1+\theta(t)}\,\tilde{u}\,\mathrm{d}x \geq \int_\Omega \sqrt{1+\theta_0}\,\tilde{u}\,\mathrm{d}x. \tag{5.254}$$

It follows from (5.249) that

$$\limsup_{t\to 0_+}\|\sqrt{1+\theta(t)}\|^2_{L^2(\Omega)} \leq \|\sqrt{1+\theta_0}\|^2_{L^2(\Omega)}. \tag{5.255}$$

We take $\tilde{u} = 1$ in (5.254) and consider the resulting inequality together with (5.255) to deduce that

$$\lim_{t\to 0_+}\|\sqrt{1+\theta(t)}\|^2_{L^2(\Omega)} = \|\sqrt{1+\theta_0}\|^2_{L^2(\Omega)},$$

which then implies

$$\lim_{t\to 0_+}\|\theta(\cdot, t) - \theta_0(\cdot)\|_{L^1(\Omega)} = 0. \tag{5.256}$$

This completes the proof of Theorem 5.3.2.

In this chapter, we have proved the existence of global-in-time weak solutions to the corotational Navier–Stokes–Fokker–Planck system with a cut-off function in the Fokker–Planck equation coupled with a temperature evolution equation.

Remark 5.4.1. *We note that there is an additional cut-off parameter in the Fokker–Planck equation. The reason is that θ and $\nabla_{x,\boldsymbol{q}}\hat{\psi}$ have limited integrability which causes lack of integrability of their product $\theta\nabla_{x,\boldsymbol{q}}\hat{\psi}$. We have introduced the cut-off function β^L so that the second order terms in the Fokker–Planck equation can remain bounded in spaces with sufficient integrability. To remove the cut-off parameter L, we make the following observation:*

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} M\theta\nabla_{x,\boldsymbol{q}}\hat{\psi}\cdot\nabla_{x,\boldsymbol{q}}\varphi \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = \int_0^T \int_{\mathcal{O}} \sqrt{M}\sqrt{\theta}\sqrt{M\theta}\nabla_{x,\boldsymbol{q}}\hat{\psi}\cdot\nabla_{x,\boldsymbol{q}}\varphi \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\quad\le \int_0^T \|\sqrt{M}\sqrt{\theta}\|_{L^2(\mathcal{O})}\|\sqrt{M\theta}\nabla_{x,\boldsymbol{q}}\hat{\psi}\|_{L^2(\mathcal{O};\mathbb{R}^{d(K+1)})}\|\nabla_{x,\boldsymbol{q}}\varphi\|_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})}\,\mathrm{d}t \\
&\quad\le C(M)\|\sqrt{M\theta}\nabla_{x,\boldsymbol{q}}\hat{\psi}\|_{L^2(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)})}\left(\int_0^T \|\theta\|_{L^1(\Omega)}\|\nabla_{x,\boldsymbol{q}}\varphi\|^2_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})}\,\mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}$$

From the energy identity, we deduce that

$$\|\sqrt{M\theta}\nabla_{x,\boldsymbol{q}}\hat{\psi}\|_{L^2(\mathcal{O}\times(0,T);\mathbb{R}^{d(K+1)})} \le C,$$

for some constant C. Then, we apply Hölder's inequality to deduce that

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} M\theta\nabla_{x,\boldsymbol{q}}\hat{\psi}\cdot\nabla_{x,\boldsymbol{q}}\varphi \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\quad\le C(M)\left(\int_0^T \|\theta\|^p_{L^1(\Omega)}\,\mathrm{d}t\right)^{\frac{1}{2p}}\left(\int_0^T \|\nabla_{x,\boldsymbol{q}}\varphi\|^{\frac{2p}{p-1}}_{L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)})}\,\mathrm{d}t\right)^{\frac{p-1}{2p}} \\
&\quad\le C(M)\|\theta\|^{\frac{1}{2}}_{L^p(Q)}\|\nabla_{x,\boldsymbol{q}}\varphi\|_{L^{\frac{2p}{p-1}}(0,T;L^\infty(\mathcal{O};\mathbb{R}^{d(K+1)}))},
\end{aligned}$$

where $p\in\left(1,\frac{d+2}{d}\right)$. Since we have proved that $\theta\in L^p(Q)$ for $p\in\left[1,\frac{d+2}{d}\right)$, then the integral

$$\int_0^T \int_{\mathcal{O}} M\theta\nabla_{x,\boldsymbol{q}}\hat{\psi}\cdot\nabla_{x,\boldsymbol{q}}\varphi \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t$$

is finite. The above arguments indicate that removing the cut-off function β^L in the Fokker–Planck equation may be possible, but this will not be pursued further in this thesis.

Chapter 6

Conclusions

In this thesis, we showed the existence of global-in-time weak solutions to several classes of coupled bead-spring chain models for incompressible dilute polymeric fluids. Our proofs, based on the Galerkin method, have been applicable to models of homogeneous, nonhomogeneous and nonisothermal dilute polymeric fluids. Compared to earlier contributions [6, 5, 8] based on time discretization, our approach is considerably simpler.

Our approach can also be applied to other models. One direct extension of Chapter 5 is to consider the non-corotational Fokker–Planck equation. For the non-corotational case, the Cauchy stress tensor will include the polymeric contribution

$$\begin{aligned}\mathcal{T}_{polymer} &= \int_D \left[\left(\left.\frac{\mathrm{d}U_e}{\mathrm{d}s}\right|_{s=\frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2} + \frac{\theta}{\theta_{ref}} \left.\frac{\mathrm{d}U_\eta}{\mathrm{d}s}\right|_{s=\frac{1}{2}\left|\frac{\boldsymbol{q}}{q_{ref}}\right|^2} \right) \frac{\boldsymbol{q}}{q_{ref}} \otimes \frac{\boldsymbol{q}}{q_{ref}} \psi \right] \mathrm{d}\boldsymbol{q} \\ &\quad - 2k_B\theta \left(\int_D \psi \,\mathrm{d}\boldsymbol{q} \right) I \\ &= \int_D \boldsymbol{F} \otimes \boldsymbol{q} \psi \,\mathrm{d}\boldsymbol{q} - 2k_B\theta \left(\int_D \psi \,\mathrm{d}\boldsymbol{q} \right) I,\end{aligned}$$

which was rigorously derived in [27]. Note that in Chapter 5, we also made the simplification that $U_e = 0$. One can also consider the more general model with U_e being an arbitrary smooth function. The main difficulty lies in deriving a reasonable energy identity. If one manages to derive a suitable energy identity, then we hypothesise that the existence of weak solutions to the nonisothermal Navier–Stokes–Fokker–Planck system could perhaps also be proved in that more general case using similar arguments to those presented in this thesis. Note that we have introduced a cut-off parameter L in Chapter 5 and how to remove this cut-off parameter remains an open problem. Remark 5.4.1 provides a possible direction of solving this problem and is certainly worth looking into.

We have assumed here that the fluids under consideration are incompressible. To consider more general situations, one can investigate compressible polymeric fluids. In [9, 10, 11], the existence of global-in-time weak solutions to an isothermal, compressible, viscous Navier–Stokes–Fokker–Planck system was proved. One can presumably extend the results to nonisothermal compressible polymeric fluids, or more generally, nonisothermal implicitly constituted models as described in [21, 64]. For nonisothermal compressible fluids, there are a wide range of open problems. Fully macroscopic models of viscoelastic fluids, the existence and uniqueness of local strong solutions and the existence of global solutions near equilibrium was proved in [38, 39, 40, 41, 56, 57]. The existence of global-in-time weak solutions to fully macroscopic models of viscoelastic rate-type fluids was shown in [17]. For compressible nonisothermal Navier–Stokes–Fokker–Planck system, the existence of weak solutions is not yet available. It would be interesting to investigate further if our approach can be applied to this case.

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