Combinatorics, Probability and Computing (2026), 1–12
doi:10.1017/S0963548326100431

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ARTICLE

Defect and transference versions of the Alon–Frankl–Lovász theorem

Lior Gishboliner[1], Stefan Glock[2], Peleg Michaeli[3] and Amedeo Sgueglia[2]

[1]University of Toronto, Canada, [2]Universität Passau, Germany, and [3]University of Oxford, UK
Corresponding author: Amedeo Sgueglia; Email: amedeo.sgueglia@uni-passau.de

(Received 9 September 2025; revised 18 February 2026; accepted 18 February 2026)

Abstract

Confirming a conjecture of Erdős on the chromatic number of Kneser hypergraphs, Alon, Frankl and Lovász proved that in any q-colouring of the edges of the complete r-uniform hypergraph, there exists a monochromatic matching of size $\lfloor \frac{n+q-1}{r+q-1} \rfloor$. In this paper, we prove a transference version of this theorem. More precisely, for fixed q and r, we show that with high probability, a monochromatic matching of approximately the same size exists in any q-colouring of a random hypergraph, already when the average degree is a sufficiently large constant. In fact, our main new result is a defect version of the Alon–Frankl–Lovász theorem for almost complete hypergraphs. From this, the transference version is obtained via a variant of the weak hypergraph regularity lemma. The proof of the defect version uses tools from extremal set theory developed in the study of the Erdős matching conjecture.

Keywords: Alon–Frankl–Lovász Theorem; Matchings; Transference and defect versions

2020 MSC Codes: Primary: 05C70; Secondary: 05C65, 05C80, 05D15

1. Introduction

A flourishing trend in combinatorics has been showing that classical theorems concerning dense graphs (or hypergraphs) have corresponding analogues in the setting of (sparse) random graphs. Such results are usually known as *transference* theorems and include, among many others, the breakthroughs of Rödl and Ruciński [31] on the transference of Ramsey's theorem, and of Conlon and Gowers [7] and Schacht [32] on the transference of Turán's theorem. Moreover, the study of combinatorial theorems for random graphs has generated several exciting recent developments in probabilistic and extremal combinatorics, including the sparse regularity method, hypergraph containers, the KŁR conjecture and the absorption method. We refer to the surveys of Conlon [6] and Böttcher [4] (and the references therein) for more details.

Here we are interested in a transference version of the celebrated result of Alon, Frankl and Lovász [1] concerning the chromatic number of Kneser hypergraphs. In 1955, Kneser [25] conjectured that if the r-subsets of a $(2r+q-1)$-element set are split into q classes, then one of the classes will contain two disjoint r-subsets. The conjecture remained open for 23 years, until Lovász [27] gave a topological proof using the Borsuk–Ulam theorem. His contribution is often considered to be the start of the field of topological combinatorics, and we refer to the book of Matoušek [29] for more examples. More generally, given $n, r, k \in \mathbb{N}$ with $n \geq kr$, the *Kneser hypergraph* $KG^k(n, r)$ is the k-uniform hypergraph (short: *k-graph*) where the vertices are all the r-subsets of $[n]$ and a collection of k vertices forms an edge if the corresponding r-sets are pairwise disjoint. Then, Kneser's conjecture is equivalent to the statement that $KG^2(n, r)$ is not q-colourable if $n \geq 2r+q-1$. As a generalisation of this, Erdős [10] conjectured even further that $KG^k(n,r)$ is not q-colourable if $n \geq (q-1)(k-1)+kr$. The case $k=2$ corresponds to Kneser's original conjecture. Moreover, the validity of the case $r=2$ is a classical result by Cockayne and Lorimer [5]. The conjecture of Erdős was finally resolved by Alon, Frankl and Lovász [1], also using topological methods. We state their result in an equivalent form concerning the size of a monochromatic matching that we can guarantee in any colouring of the edges of a complete hypergraph.

Theorem 1.1 (Alon, Frankl and Lovász [1]). *Let $n, r, q \in \mathbb{N}$ with $r, q \geq 2$. Then any q-colouring of the edges of the complete n-vertex r-graph $K_n^{(r)}$ contains a monochromatic matching of size at least* $\lfloor \frac{n+q-1}{r+q-1} \rfloor$.

The bound $k := \lfloor \frac{n+q-1}{r+q-1} \rfloor$ on the size of the matching is best possible as shown by the following construction. Partition $[n]$ into q sets $V_1, \ldots, V_q$ such that V_i has size at most k for each $1 \leq i \leq q-1$ and V_q has size at most $rk+r-1$. Given an edge $e \in K_n^{(r)}$, for $1 \leq i \leq q-1$, we assign colour i to e if and only if e intersects V_i, and we assign colour q to e if and only if e is completely contained in V_q. Note that an edge might be assigned several colours $1 \leq i \leq q-1$, and in this case we choose one of these colours arbitrarily. Then this yields an extremal q-colouring with respect to Theorem 1.1. Indeed, for $1 \leq i \leq q-1$, every edge with colour i has to intersect V_i and hence any matching of colour i has size at most $|V_i| \leq k$. Moreover, every edge with colour q is completely contained in V_q and hence the size of any matching of colour q is at most $\lfloor |V_q|/r \rfloor \leq k$.

Here, we prove a transference version of Theorem 1.1. The random hypergraph model we consider is the binomial random r-graph $\mathbb{G}^{(r)}(n,p)$, which has n vertices, and where each r-set of vertices forms an edge independently with probability p. We show that if $p \gg n^{-r+1}$, then $\mathbb{G}^{(r)}(n,p)$ typically contains a monochromatic matching of asymptotically the same size as what is guaranteed to exist in $K_n^{(r)}$ by Theorem 1.1.

Theorem 1.2 (Transference version of the AFL Theorem). *For all $r, q \in \mathbb{N}$ with $r, q \geq 2$, and all $\mu > 0$, there exists $C > 0$ such that, provided $p \geq Cn^{-r+1}$, w.h.p. the following holds for $G \sim \mathbb{G}^{(r)}(n,p)$: For any q-colouring of the edges of G, there exists a monochromatic matching of size at least* $(1-\mu)\frac{n}{r+q-1}$.

The size of the matching is asymptotically best possible since one cannot do better even in the complete r-graph, as explained above. In fact, it is also necessary that C is sufficiently large given μ, since G must contain at most $O(\mu n)$ isolated vertices. Indeed, by the optimality of Theorem 1.1, if G is an n-vertex r-graph with n' isolated vertices, then there is a q-colouring of its edges whose largest monochromatic matching has size at most $\lfloor \frac{(n-n')+q-1}{r+q-1} \rfloor$.

The graph case $r=2$ (i.e. a transference version of the Cockayne–Lorimer theorem) was already proved by Gishboliner, Krivelevich and Michaeli [18], with earlier results of Letzter [26] and Dudek and Prałat [8] implying the cases $(r,q)=(2,2)$ and $(r,q)=(2,3)$, respectively.

We will prove Theorem 1.2 by combining the sparse hypergraph regularity method with the following 'defect' version of the Alon–Frankl–Lovász theorem, which shows that, for large n, the conclusion of Theorem 1.1 approximately holds even for edge-colourings of *almost* complete r-uniform hypergraphs.

Theorem 1.3 (Defect version of the AFL Theorem). *For all $r, q \in \mathbb{N}$ with $r, q \geq 2$, and all $\mu > 0$, there exists $\varepsilon > 0$ such that the following holds for all sufficiently large n. Let G be an n-vertex r-graph whose edges are q-coloured and assume $e(G) \geq (1-\varepsilon)\binom{n}{r}$. Then G contains a monochromatic matching of size at least* $(1-\mu)\frac{n}{r+q-1}$.

The deduction of Theorem 1.2 from Theorem 1.3 via the regularity method works in a similar way as in the graph case. However, finding a strategy to prove Theorem 1.3 presents several challenges. First, the topological proof of the Alon–Frankl–Lovász theorem does not seem to be robust enough to generalise to hypergraphs that miss a significant proportion of the edges. Moreover, the proof of the graph case ($r = 2$) from [18] relies on a good understanding of matchings in graphs in the form of the Tutte–Berge formula, no analogue of which is available for hypergraphs. Finally, one could try to prove the existence of certain small coloured configurations ('gadgets') that can be repeatedly removed until the remaining hypergraph has a very specific structure. For instance, in the case of two colours, in [17] it is implicitly proved that either there exist two edges of different colour that share $r-1$ vertices, or the colouring is almost monochromatic. For general q, one possible gadget would be a set of $r+q-1$ vertices that contains an edge of each colour. However, for $r \geq 3$, there are constructions where all colour classes are large yet there is no such gadget. For instance, consider the complete r-graph on the disjoint union of $q-1$ sets $V_1, \ldots, V_{q-1}$ of the same size, where each edge completely contained in some V_i is coloured with colour i and all other edges are coloured with colour q. Thus this approach seems infeasible as well. We instead follow a new approach and make use of tools from extremal set theory developed in the study of the Erdős matching conjecture (see the overview in Section 2 for more details).

Theorem 1.2 also has implications to the discrepancy of perfect matchings in random hypergraphs. The general question in discrepancy theory is whether, given a ground set Ω, a family $\mathcal{P} \subseteq 2^{\Omega}$ and a positive integer $q \geq 2$, there exists a q-colouring of Ω such that each set in $\mathcal{P}$ contains roughly the same number of elements of each colour. In the setting of (hyper)graphs, the main interest is finding conditions on a hypergraph G under which any q-edge-colouring of G contains a particular substructure with high discrepancy, meaning that significantly more than a $1/q$-proportion of the substructure's edges are in the same colour. The question goes back to works of Erdős and Spencer [12], and of Erdős, Füredi, Loebl and Sós [11]. Recently, this has been extensively studied for minimum degree conditions forcing perfect matchings with high discrepancy in hypergraphs (see [3, 17, 20, 28]), with the graph case having been considered earlier (see [2, 15, 19]). Hàn, Lang, Marciano, Pavez-Signé, Sanhueza-Matamala, Treglown and Zárate-Guerén [20, Section 5.3] showed that there exists a constant $C > 0$ such that if $p \geq C\sqrt{n^{2-r}}$, then w.h.p. in every q-colouring of the edges of $\mathbb{G}^{(r)}(n, p)$ there is a perfect matching with high discrepancy. However, due to the celebrated result of Johansson, Kahn, and Vu [22], we know that a perfect matching already typically exists for $p \gg n^{-r+1} \log n$ (cf. Theorem 3.5). A natural question, which was asked in [20], is to determine the correct threshold (depending on r, q) for the property of having a perfect matching with high discrepancy in every q-colouring. Motivated by this, we prove the following result, which starts working at the threshold for the existence of a perfect matching, and even yields an asymptotically optimal bound on the discrepancy. However, note that we assume the colouring is known a priori, that is, the statement we prove to hold w.h.p. is for a fixed colouring.

Theorem 1.4. *For all $r, q \in \mathbb{N}$ with $r, q \geq 2$, and all $\mu > 0$, there exists $C > 0$ such that, provided $p \geq Cn^{-r+1} \log n$, the following holds. For any q-colouring of the r-subsets of $[n]$, the random hypergraph $G \sim \mathbb{G}^{(r)}(n, p)$ contains w.h.p. a perfect matching with at least $(1-\mu)\frac{n}{r+q-1}$ edges of the same colour.*

Organisation. The rest of the paper is organised as follows. The proof of the defect theorem (Theorem 1.3) is discussed in Section 2 and the proof of the transference theorem (Theorem 1.2) in Section 3. The proof of Theorem 1.4 is also discussed there. Finally, we give concluding remarks in Section 4. In the arXiv version of this paper, we provided the proof of the multicolour weak sparse hypergraph regularity lemma in the appendix for the sake of completeness.

Notation. We use standard graph theory notation. In particular, we let $K_n^{(r)}$ denote the complete r-graph on n vertices, and for a graph G and a subset $W \subseteq V(G)$ we let $G[W]$ denote the subgraph of G induced by W.

We let $[n]$ denote the set $\{1, \ldots, n\}$ and, given a set X and an integer $i \geq 0$, we write $\binom{X}{i}$ for the collection of all subsets of X of size i.

For $a, b, c \in (0, 1]$, we write $a \ll b \ll c$ in our statements to mean that there are increasing functions $f, g:(0, 1] \to (0, 1]$ such that whenever $a \le f(b)$ and $b \le g(c)$, then the subsequent result holds.

We say that an event holds with high probability (w.h.p.) if the probability that it holds tends to 1 as the number of vertices n tends to infinity.

2. Proof of the defect theorem

We start by describing our strategy for the defect theorem (Theorem 1.3). Let G be an n-vertex q-edge-coloured r-graph G with $e(G) \ge (1 - o(1))\binom{n}{r}$, and recall that we want to show that G contains a monochromatic matching of size $(1 - o(1))\frac{n}{r+q-1}$. Our first idea is to fix a large $k \in \mathbb{N}$ (which does not depend on n), and consider the family $\mathcal{F}$ of the k-subsets F of $V(G)$ for which $G[F]$ is a complete r-graph. Observe that since G is almost-complete, almost all k-subsets of $V(G)$ belong to $\mathcal{F}$, i.e. $|\mathcal{F}| \ge (1 - o(1))\binom{n}{k}$. It may seem at this stage that we have not really made any progress as, thinking of $\mathcal{F}$ as a k-uniform hypergraph on $V(G)$, we still have that $\mathcal{F}$ is only almost-complete. However, we have gained the flexibility of choosing k, and our argument relies on choosing k to be large enough in terms of r, q.

For each $F \in \mathcal{F}$, we can apply the Alon–Frankl–Lovász theorem as a black box to $G[F]$ and get a monochromatic matching M_F of $G[F]$ of size roughly $\frac{k}{r+q-1}$. Assign to F the colour of the monochromatic matching M_F, giving a q-colouring of $\mathcal{F}$. Let blue be the most popular colour in this colouring of $\mathcal{F}$, and let $\mathcal{F}' := \{F \in \mathcal{F} : M_F \text{ is blue}\}$; so $|\mathcal{F}'| \ge (1 - o(1))\binom{n}{k}/q$. Our goal is now to find a certain structure in $\mathcal{F}'$ which would translate to a large blue matching in the original hypergraph G.

A naive approach would be to look for a large blue matching in $\mathcal{F}'$. More precisely, we need a matching of size $(1 - o(1))\frac{n}{k}$ in $\mathcal{F}'$ to obtain a matching of size $(1 - o(1))\frac{n}{r+q-1}$ in G. With this approach, the relevant question is what size of a matching is guaranteed to exist in an n-vertex k-graph with a given number of edges. This is a classical problem of Erdős [9], known as the Erdős matching conjecture, asking, given $n, k, t \in \mathbb{N}$, to determine the maximum number of edges of an n-vertex k-graph which does not contain a matching of size $t + 1$. If $n < k(t + 1)$, the problem is trivial as the family $\binom{[n]}{k}$ itself does not contain a matching of size $t + 1$. When $n \ge k(t + 1)$, Erdős conjectured that the answer is $\max\{\binom{n}{k} - \binom{n-t}{k}, \binom{k(t+1)-1}{k}\}$. The two bounds correspond to two natural extremal constructions: a star-like construction, obtained by taking all edges intersecting $[t]$ in at least one vertex, and a clique-like construction, obtained by taking all edges completely contained in $[k(t + 1) - 1]$. However, even if the conjecture was known to be true, it would not imply the desired bound on the size of the matching. Indeed, for $t = (1 - \varepsilon)\frac{n}{k}$, the clique-like construction has density roughly $(1 - \varepsilon)^k$, and, for large k, the star-like construction has density roughly $1 - e^{-(1-\varepsilon)}$. Hence, the density of $\mathcal{F}'$, which is about $1/q$, is not enough to guarantee the existence of an almost perfect matching.

Our second idea is that instead of a matching, we look in $\mathcal{F}'$ for a collection of edges which are allowed to overlap (only) mildly. More precisely, our goal is to find $F_1, \ldots, F_s \in \mathcal{F}'$ with $s = (1 - o(1))\frac{n}{k}$ such that the set W of vertices appearing in more than one of the sets $F_1, \ldots, F_s$ is small. Perhaps surprisingly, such $F_1, \ldots, F_s$ always exist provided that k is large enough as a function of $\binom{n}{k}/|\mathcal{F}'|$. Then, removing all edges in $\bigcup_{i=1}^{s} M_{F_i}$ which intersect W, we obtain a blue matching in G of the desired size $(1 - o(1))\frac{n}{r+q-1}$.

The key step is hence to find the desired sets $F_1, \ldots, F_s \in \mathcal{F}'$. We now state the result which allows one to do this. Roughly speaking, it says that in a hypergraph with large uniformity a constant density suffices to guarantee an 'almost-perfect almost-cover'.

Theorem 2.1. *Let* $1/n \ll 1/k \ll 1/C \ll \alpha, \beta$ *and* $\mathcal{F} \subseteq \binom{[n]}{k}$ *with* $|\mathcal{F}| \ge \beta\binom{n}{k}$. *Then there exist* $s := \lceil (1 - \alpha)\frac{n}{k} \rceil$ *sets* $F_1, \ldots, F_s \in \mathcal{F}$ *with* $|F_1 \cup F_2 \cup \cdots \cup F_s| \ge (k - C)s$.

We remark that, for the proof of Theorem 2.1 to work, we only need that $(1-\alpha)^{C+1} < \beta/2$, $k \geq C+1$ and $n \geq \max\{2k(k+1), 2k/(1-\alpha)\}$. The proof of Theorem 2.1 uses tools from extremal set theory and we start by introducing the relevant definitions. For a k-graph $\mathcal{F} \subseteq \binom{[n]}{k}$ and $0 \leq \ell < k$, we let $\sigma_\ell(\mathcal{F})$ denote the ℓ-*shadow* of $\mathcal{F}$. Namely,

$$\sigma_\ell(\mathcal{F}) := \left\{ G \in \binom{[n]}{\ell} : \exists F \in \mathcal{F} \text{ with } G \subseteq F \right\},$$

where we observe in particular that if $\mathcal{F} \neq \emptyset$, then $\sigma_0(\mathcal{F}) = \{\emptyset\}$ and $|\sigma_0(\mathcal{F})| = 1$. In his work on the Erdős matching conjecture, Frankl [14] proved the following fundamental inequality which generalises Katona's intersection theorem [24]: For any integer $s \geq 2$ and any $\mathcal{F} \subseteq \binom{[n]}{k}$ which does not contain s pairwise disjoint sets, it holds that

$$|\sigma_{k-1}(\mathcal{F})| \geq |\mathcal{F}|/(s-1). \tag{2.1}$$

Note that $\mathcal{F}$ is a family as above if and only if $|F_1 \cup F_2 \cup \cdots \cup F_s| \leq ks - 1$ for each $F_1, F_2, \ldots, F_s \in \mathcal{F}$, as otherwise the sets $\{F_i : i \in [s]\}$ would be pairwise disjoint. We can generalise (2.1) as follows.

Theorem 2.2. *Let $k \geq 1$, $s \geq 2$ and $1 \leq b \leq k$. Assume that $\mathcal{F} \subseteq \binom{[n]}{k}$ is a family such that*

$$|F_1 \cup F_2 \cup \cdots \cup F_s| \leq ks - (b-1)(s-1) - 1 \tag{2.2}$$

for all $F_1, \ldots, F_s \in \mathcal{F}$. Then

$$|\sigma_{k-b}(\mathcal{F})| \geq \frac{|\mathcal{F}|}{(s-1)^b}\,. \tag{2.3}$$

We remark that Theorem 2.2 can be deduced from a technical result of Frankl (c.f. [13, Corollary 4]). Here, for the sake of completeness, we give a direct proof, which proceeds analogously to Frankl's proof [14] of (2.1).

Proof of Theorem 2.2. It is well known (see, for example, [13, Proposition 1]) that we can assume that $\mathcal{F}$ is a *shifted* family, that is, for all $1 \leq i < j \leq n$ and $F \in \mathcal{F}$, the conditions $i \notin F$, $j \in F$ imply that $F \cup \{i\} \setminus \{j\} \in \mathcal{F}$ as well.

We first prove the statement for $k = 1$ (and thus $b = 1$) and any $s \geq 2$. Observe that if $F_1, \ldots, F_s \in \binom{[n]}{1}$ and $|F_1 \cup F_2 \cup \cdots \cup F_s| < s$, then the F_i cannot be all distinct. Therefore if $\mathcal{F} \subseteq \binom{[n]}{1}$ satisfies (2.2), then $|\mathcal{F}| \leq s-1$. Since $|\sigma_0(\mathcal{F})| = 1$, (2.3) follows.

Assume now that $k \geq 2$. We first prove the statement for all $n \leq ks - (b-1)(s-1) - 1$. Consider the bipartite graph with partite sets $\mathcal{F}$ and $\sigma_{k-b}(\mathcal{F})$, where there is an edge connecting $F \in \mathcal{F}$ and $G \in \sigma_{k-b}(\mathcal{F})$ if and only if $G \subseteq F$. Each $F \in \mathcal{F}$ has degree $\binom{k}{k-b} = \binom{k}{b}$ and each $G \in \sigma_{k-b}(\mathcal{F})$ has degree at most $\binom{n-|G|}{b} = \binom{n-k+b}{b}$. Therefore, by double counting the edges, $\binom{k}{b} \cdot |\mathcal{F}| \leq \binom{n-k+b}{b} \cdot |\sigma_{k-b}(\mathcal{F})|$, implying

$$\frac{|\mathcal{F}|}{|\sigma_{k-b}(\mathcal{F})|} \leq \frac{\binom{n-k+b}{b}}{\binom{k}{b}} = \prod_{j=0}^{b-1} \frac{n-k+b-j}{k-j} \leq (s-1)^b\,,$$

where the last inequality can be justified as follows. Since $n \leq ks - (b-1)(s-1) - 1$, it is enough to show that $\frac{ks-(b-1)(s-1)-1-k+b-j}{k-j} \leq s-1$, which is equivalent to $(b-j-1)(s-2) \geq 0$. This is clearly true as $s \geq 2$ and $0 \leq j \leq b-1$. We obtain (2.3) by rearranging.

We now suppose that $n \geq ks - (b-1)(s-1)$ and proceed by induction on n. So assume that the statement holds for $n-1$ for both k and $k-1$. Define

$$\mathcal{F}(\overline{n}) := \{F \in \mathcal{F} : n \notin F\} \quad \text{and} \quad \mathcal{F}(n) := \{F \setminus \{n\} : F \in \mathcal{F}, n \in F\}\,,$$

and observe that $\mathcal{F}(\overline{n}) \subseteq \binom{[n-1]}{k}$ and $\mathcal{F}(n) \subseteq \binom{[n-1]}{k-1}$. Since $\mathcal{F}(\overline{n}) \subseteq \mathcal{F}$, for any $F_1, \ldots, F_s \in \mathcal{F}(\overline{n})$ we have

$$|F_1 \cup F_2 \cup \cdots \cup F_s| \leq ks - (b-1)(s-1) - 1\,.$$

Therefore, by induction, $|\mathcal{F}(\overline{n})| \leq (s-1)^b \cdot |\sigma_{k-b}(\mathcal{F}(\overline{n}))|$. Moreover, we claim that for any $G_1, \ldots, G_s \in \mathcal{F}(n)$ we must have

$$|G_1 \cup G_2 \cup \cdots \cup G_s| \leq (k-1)s - (b-1)(s-1) - 1\,. \tag{2.4}$$

Indeed, suppose this is not the case and let $c \geq 0$ be such that

$$|G_1 \cup G_2 \cup \cdots \cup G_s| = (k-1)s - (b-1)(s-1) + c\,.$$

By definition of $\mathcal{F}(n)$, we have $F_i := G_i \cup \{n\} \in \mathcal{F}$ and thus

$$ks - (b-1)(s-1) - 1 \geq |F_1 \cup F_2 \cup \cdots \cup F_s| = |G_1 \cup G_2 \cup \cdots \cup G_s| + 1\,,$$

from which it follows that $c \leq s-2$. Moreover, by our assumption on n, we have

$$n - 1 \geq ks - (b-1)(s-1) - 1 = |G_1 \cup G_2 \cup \cdots \cup G_s| + s - c - 1,$$

and $s - c - 1 \geq 1$. Therefore there exist distinct $a_1, \ldots, a_{s-c-1} \in [n-1]$ which do not belong to any of $G_1, \ldots, G_s$. Let $F_i' := F_i \cup \{a_i\} \setminus \{n\}$ for $i \leq s-c-1$, and $F_i' := F_i$ for $s - c \leq i \leq s$. Then, since $\mathcal{F}$ is shifted, $F_i' \in \mathcal{F}$ for each $i \in [s]$. However,

$$|F_1' \cup F_2' \cup \cdots \cup F_s'| = |G_1 \cup G_2 \cup \cdots \cup G_s| + s - c - 1 + 1 = ks - (b-1)(s-1)\,,$$

which contradicts our assumption. Hence, (2.4) holds.

If $b \leq k-1$, then we can apply induction and get that $|\mathcal{F}(n)| \leq (s-1)^b \cdot |\sigma_{k-1-b}(\mathcal{F}(n))|$. Note that $|\mathcal{F}| = |\mathcal{F}(\overline{n})| + |\mathcal{F}(n)|$. Moreover, if $A \in \sigma_{k-b}(\mathcal{F}(\overline{n}))$, then $n \notin A$ and $A \in \sigma_{k-b}(\mathcal{F})$. Similarly, if $B \in \sigma_{k-1-b}(\mathcal{F}(n))$, then $n \notin B$ and $B \cup \{n\} \in \sigma_{k-b}(\mathcal{F})$. Therefore

$$|\sigma_{k-b}(\mathcal{F})| \geq |\sigma_{k-b}(\mathcal{F}(\overline{n}))| + |\sigma_{k-1-b}(\mathcal{F}(n))|\,,$$

and

$$|\mathcal{F}| = |\mathcal{F}(\overline{n})| + |\mathcal{F}(n)| \leq (s-1)^b \cdot \Big(|\sigma_{k-b}(\mathcal{F}(\overline{n}))| + |\sigma_{k-1-b}(\mathcal{F}(n))| \Big) \leq (s-1)^b \cdot |\sigma_{k-b}(\mathcal{F})|\,,$$

which proves (2.3) when $b \leq k-1$.

When $b = k$, (2.4) reads as $|G_1 \cup \cdots \cup G_s| \leq (k-1)s - (k-1)(s-1) - 1 = k-2$, which cannot be satisfied as $|G_1 \cup \cdots \cup G_s| \geq |G_1| = k-1$. Therefore $\mathcal{F}(n) = \emptyset$ and

$$|\mathcal{F}| = |\mathcal{F}(\overline{n})| \leq (s-1)^b \cdot |\sigma_0(\mathcal{F}(\overline{n}))| = (s-1)^b \cdot |\sigma_0(\mathcal{F})|\,,$$

which proves (2.3) in the remaining case $b = k$. □

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Suppose the statement is false. Then for any $F_1, \ldots, F_s \in \mathcal{F}$, we have $|F_1 \cup F_2 \cup \cdots \cup F_s| < (k-C)s$ and, choosing $b := C+1$, we have $(k-C)s \leq ks - (b-1)(s-1) - 1$. Moreover, since $1/n \ll 1/k, \alpha$, it holds that $s \geq 2$ and thus we can use Theorem 2.2 (applied with the above choice of b) and we have

$$|\sigma_{k-b}(\mathcal{F})| \geq \frac{|\mathcal{F}|}{(s-1)^b}.$$

In particular,

$$|\sigma_{k-b}(\mathcal{F})| \geq \frac{\frac{\beta}{2}\frac{n^k}{k!}}{(1-\alpha)^b (n/k)^b} \geq \frac{\beta}{2(1-\alpha)^b}\binom{n}{k-b} > \binom{n}{k-b}.$$

where the first inequality follows from $s-1 \leq (1-\alpha)\frac{n}{k}$ and $|\mathcal{F}| \geq \beta\binom{n}{k} \geq \frac{\beta}{2}\frac{n^k}{k!}$, the second inequality from $\frac{k^b}{k!} \geq \frac{1}{(k-b)!}$ and the last one uses that $b = C+1$ and $1/C \ll \alpha, \beta$. This is a contradiction to the trivial upper bound $|\sigma_{k-b}(\mathcal{F})| \leq \binom{n}{k-b}$. □

We conclude this section by proving Theorem 1.3, the defect version of the Alon–Frankl–Lovász theorem.

Proof of Theorem 1.3. First observe that we can assume that $\mu \ll 1/r, 1/q$. Then choose new constants k, C such that $k \in \mathbb{N}$ and $1/n \ll \varepsilon \ll 1/k \ll 1/C \ll \mu \ll 1/r, 1/q$. Let $\mathcal{F} \subseteq \binom{V(G)}{k}$ be the family of k-subsets $F \subseteq V(G)$ for which $G[F]$ is the complete r-graph on F.

Claim 1. *We have* $|\mathcal{F}| \geq \frac{1}{2}\binom{n}{k}$.

Proof of claim: Let F be a uniformly chosen subset of $V(G)$ of size k and observe that it is enough to show that $\mathbb{P}[F \notin \mathcal{F}] \leq 1/2$, where we recall that $F \notin \mathcal{F}$ is equivalent to the condition that $G[F]$ contains a non-edge. Given r vertices $v_1, \ldots, v_r$, we have $\mathbb{P}[v_1, \ldots, v_r \in F] = \binom{n-r}{k-r} \cdot \binom{n}{k}^{-1}$ and thus the expected number of non-edges in $G[F]$ is at most $\varepsilon \cdot \binom{n}{r} \cdot \binom{n-r}{k-r} \cdot \binom{n}{k}^{-1} = \varepsilon\binom{k}{r} \leq 1/2$, where the inequality uses $\varepsilon \ll 1/k, 1/r$. By Markov's inequality, the probability that $G[F]$ contains a non-edge is at most $1/2$, and we are done. □

Using the Alon–Frankl–Lovász theorem (Theorem 1.1), for each $F \in \mathcal{F}$, we find in $G[F]$ a monochromatic matching, which we denote by M_F, of size at least $\lfloor \frac{k}{r+q-1} \rfloor \geq (1-\mu^2)\frac{k}{r+q-1}$, using $1/k \ll \mu, 1/r, 1/q$. By averaging, there exists a colour c and a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at least $\frac{|\mathcal{F}|}{q} \geq \frac{1}{2q}\binom{n}{k}$ such that for each $F \in \mathcal{F}'$, the monochromatic matching M_F has colour c.

Applying Theorem 2.1 to $\mathcal{F}'$ gives that there exist $F_1, \ldots, F_s \in \mathcal{F}'$ with $|F_1 \cup F_2 \cup \cdots \cup F_s| \geq (k-C)s$, where $s := \lceil (1-\mu^2)\frac{n}{k} \rceil$. We would like to estimate the number of edges appearing in more than one of the matchings M_{F_i} and, for that, we first estimate the number of vertices appearing in more than one of the sets F_i.

Let $U := F_1 \cup F_2 \cup \cdots \cup F_s$ and note that $|U| \geq (1-2\mu^2)n$. For a vertex $u \in U$, let $d(u) := |\{i \in [s] : u \in F_i\}|$ and set $W := \{u \in U : d(u) \geq 2\}$, i.e. W is the set of vertices which belong to at least two of the sets F_i. By double-counting, we have

$$|U| + \sum_{u \in W}(d(u)-1) = \sum_{u \in U} d(u) = \sum_{i \in [s]} |F_i| = ks \leq n,$$

so $\sum_{u \in W}(d(u)-1) \leq n - |U| \leq 2\mu^2 n$. Since $d(u) \leq 2(d(u)-1)$ for all $u \in W$, we then deduce that $\sum_{u \in W} d(u) \leq 4\mu^2 n$. Let M be the multiset consisting of all the edges in M_{F_i} for $i \in [s]$ and delete all the edges which contain a vertex from W. The remaining edges then form a matching. Note that, for each vertex $u \in W$, we delete at most $d(u)$ edges since for each $i \in [s]$, at most one edge of M_{F_i} contains u. Hence, the final matching has size at least

$$s \cdot (1-\mu^2)\frac{k}{r+q-1} - \sum_{u \in W} d(u) \geq (1-2\mu^2)\frac{n}{k} \cdot \frac{k}{r+q-1} - 4\mu^2 n \geq (1-\mu)\frac{n}{r+q-1},$$

and it is monochromatic (of colour c) by construction. □

3. Proof of the transference and the discrepancy theorems

The transference theorem (Theorem 1.2) follows from our defect theorem via the multicolour weak sparse hypergraph regularity lemma, which we now state after introducing a few definitions.

Let H be an r-graph. For disjoint vertex sets $X_1, \ldots, X_r$ denote by $E(X_1, \ldots, X_r)$ the set of edges having exactly one vertex in each X_i, $i = 1, \ldots, r$. The *density* between $X_1, \ldots, X_r$ is

$$d(X_1, \ldots, X_r) := \frac{|E(X_1, \ldots, X_r)|}{|X_1| \cdot \ldots \cdot |X_r|}.$$

For $\varepsilon > 0$ and $p \in [0, 1]$, an r-partite r-uniform hypergraph with parts $V_1, \ldots, V_r$ is *(ε, p)-regular* if for every $X_i \subseteq V_i$, $i = 1, \ldots, r$, with $|X_i| \geq \varepsilon |V_i|$, one has

$$|d(X_1, \ldots, X_r) - d(V_1, \ldots, V_r)| \leq \varepsilon p.$$

A partition $\mathcal{P} := (V_1, \ldots, V_t)$ of H is called *(ε, p)-regular* if it is an *equipartition* (i.e. the sizes of the parts differ by at most 1) and for all but at most $\varepsilon\binom{t}{r}$ of the r-sets $(V_{i_1}, \ldots, V_{i_r})$, the induced r-partite r-uniform hypergraph with parts $V_{i_1}, \ldots, V_{i_r}$ is (ε, p)-regular. Moreover we say that t is the *order* of $\mathcal{P}$. For $\eta > 0$ and $D > 1$, we say that H is *(η, p, D)-upper-uniform* if for any disjoint vertex sets $X_1, \ldots, X_r$ with $|X_i| \geq \eta |V(H)|$ one has $d(X_1, \ldots, X_r) \leq Dp$.

We can now state the lemma we need. The proof is a straightforward adaptation of a proof of the sparse regularity lemma for graphs (see, for example, the survey of Gerke and Steger [16]). For completeness, we provide a proof in the appendix of the arXiv version of this paper.

Lemma 3.1. *Let $\eta \ll 1/T \ll 1/t_0 \ll \varepsilon, 1/D, 1/q, 1/r$, with $T, D, q, t_0, r \in \mathbb{N}$ and $D > 1$. Then for every $p \in [0, 1]$, if $H_1, \ldots, H_q$ are (η, p, D)-upper-uniform r-graphs on the same vertex set V with $|V| \geq t_0$ then there exists a partition $V_1, \ldots, V_t$ of V with $t_0 \leq t \leq T$ which is (ε, p)-regular with respect to H_j for all $j = 1, \ldots, q$.*

Given an (ε, p)-regular partition $V_1, \ldots, V_t$ of V as in Lemma 3.1, we define the *cluster hypergraph* with respect to $\{V_1, \ldots, V_t\}$ as the r-graph on $[t]$ with an edge $e := \{i_1, i_2, \ldots, i_r\}$ if and only if there exists $i := i(e) \in [q]$ such that the induced r-partite r-subgraph of H_i with parts $V_{i_1}, \ldots, V_{i_r}$ is (ε, p)-regular and $d_{H_i}(V_{i_1}, \ldots, V_{i_r}) > \varepsilon p$. The cluster hypergraph inherits a q-edge-colouring from H by colouring the edge e in colour $i(e)$ (if there is more than one choice for $i(e)$, we pick one arbitrarily).

We apply Lemma 3.1 to $\mathbb{G}^{(r)}(n, p)$ and thus we need to show that $\mathbb{G}^{(r)}(n, p)$ is upper-uniform.

Lemma 3.2. *Let $1/C \ll \eta, 1/r$ with $r \in \mathbb{N}$ and $r \geq 2$. Provided that $p \geq Cn^{-r+1}$, the random hypergraph $G \sim \mathbb{G}^{(r)}(n, p)$ has w.h.p. the following property: for any r pairwise disjoint vertex sets $X_1, \ldots, X_r$, each having size at least ηn, it holds that $p/2 \leq d(X_1, \ldots, X_r) \leq 3p/2$.*

Lemma 3.2 follows from a simple application of the following version of Chernoff's bound (see e.g. [21, Corollary 2.3]).

Lemma 3.3 (Chernoff's inequality). *Let X be the sum of independent Bernoulli random variables. Then for all $0 < \beta < 1$, we have*

$$\mathbb{P}\Big[|X - \mathbb{E}[X]| \geq \beta \mathbb{E}[X]\Big] \leq 2 \exp\left(-\frac{\beta^2}{3}\mathbb{E}[X]\right).$$

Proof of Lemma 3.2 Consider any r pairwise disjoint vertex sets $X_1, \ldots, X_r$ each having size at least ηn. Then $\mathbb{E}[e(X_1, \ldots, X_r)] = p|X_1| \cdot \ldots \cdot |X_r| \geq (\eta n)^r p$. Therefore, by Chernoff's bound, we have

$$\mathbb{P}\left[\Big|e(X_1, \ldots, X_r) - p|X_1| \cdot \ldots \cdot |X_r|\Big| \geq \frac{1}{2} \cdot p|X_1| \cdot \ldots \cdot |X_r|\right] \leq 2 \exp\left(-\frac{(\eta n)^r}{12} p\right) \leq 2 \exp\left(-\frac{C\eta^r}{12} n\right)$$

which, by choosing C large enough, beats the union bound over the at most 2^{rn} choices of $X_1, \ldots, X_r$. □

After applying the regularity lemma to $G \sim \mathbb{G}^{(r)}(n, p)$, we consider the corresponding reduced hypergraph R. Such a graph is almost complete and inherits an edge-colouring from G, by colouring an edge of R in colour c if the r-partite subgraph of G induced by the corresponding clusters is dense in colour c. In particular, R is suitable for an application of Theorem 1.3 and contains a monochromatic matching of size roughly $\frac{|V(R)|}{r+q-1}$. By applying the following standard technique, this monochromatic matching translates into a monochromatic matching of G of size roughly $\frac{|V(G)|}{r+q-1}$, as desired.

Lemma 3.4. *Let $\varepsilon > 0$, $p \in [0, 1]$ and G be an (ε, p)-regular r-partite r-graph with parts $V_1, \ldots, V_r$, each of size m or $m + 1$, and with $d(V_1, \ldots, V_r) > \varepsilon p$. Then G has a matching of size at least $(1 - \varepsilon)m$.*

Proof. Let M be a maximum matching of G and suppose that $|M| < (1 - \varepsilon)m$. Let $X_i := V_i \setminus V(M)$ be the set of uncovered vertices in V_i and observe that $|X_i| \geq \varepsilon m$ for each $i \in [r]$. Then, by the definition of (ε, p)-regularity, $d(X_1, \ldots, X_r) \geq d(V_1, \ldots, V_r) - \varepsilon p > 0$ and thus the sets $X_1, \ldots, X_r$ span at least one edge, contradicting the maximality of M. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\varepsilon, \eta > 0$ and $t_0, T \in \mathbb{N}$ be new constants such that

$$1/C \ll \eta \ll 1/T \ll 1/t_0 \ll \varepsilon \ll \mu, 1/r, 1/q.$$

Then let $p \geq Cn^{-r+1}$ and let $G \sim \mathbb{G}^{(r)}(n, p)$. By Lemma 3.2, w.h.p. G has the property that for any r pairwise disjoint vertex sets $X_1, \ldots, X_r$, each having size at least ηn, it holds that

$$p/2 \leq d_G(X_1, \ldots, X_r) \leq 3p/2\,. \tag{3.1}$$

Fix now such a G and consider a q-colouring of the edges of G with colours in $[q]$. For each $i \in [q]$, let G_i be the spanning subgraph of G consisting of the edges of colour i. Observe that by (3.1), G is $(\eta, p, 3/2)$-upper-uniform and thus each G_i is $(\eta, p, 3/2)$-upper-uniform as well.

By the weak sparse hypergraph regularity lemma (Lemma 3.1), there exists a partition $V_1, \ldots, V_t$ of $V(G)$ with $t_0 \leq t \leq T$ which is (ε, p)-regular with respect to G_i for each $i \in [q]$. Let R be the (q-edge-coloured) cluster hypergraph associated with this partition. Observe that, by the definition of an (ε, p)-regular partition, all but at most $q \cdot \varepsilon\binom{t}{r}$ of the sets $\{i_1, i_2, \ldots, i_r\} \subseteq [t]$ are such that the induced r-partite r-subgraph of H_i with parts $V_{i_1}, \ldots, V_{i_r}$ is (ε, p)-regular for each $i \in [q]$. Moreover, by (3.1), for any such r-set $\{i_1, i_2, \ldots, i_r\}$ it holds that $d_G(V_{i_1}, \ldots, V_{i_r}) \geq p/2$ and thus, by averaging, there exists $i \in [q]$ such that $d_{G_i}(V_{i_1}, \ldots, V_{i_r}) \geq p/(2q) > \varepsilon p$. Therefore $e(R) \geq (1 - q \cdot \varepsilon)\binom{t}{r}$.

By Theorem 1.3 (applied with $\mu/2$ playing the role of μ), R contains a monochromatic matching of size at least $(1 - \frac{\mu}{2})\frac{t}{r+q-1}$, say of colour 1. By Lemma 3.4, each edge $\{i_1, \ldots, i_r\}$ of M gives rise (in G) to a monochromatic matching (of colour 1) of size at least $(1 - \varepsilon)\lfloor n/t \rfloor$ between the sets $V_{i_1}, \ldots, V_{i_r}$. Moreover, these matchings (for different choices of $\{i_1, \ldots, i_r\}$) are pairwise vertex-disjoint. Therefore the union of these matchings is a monochromatic matching (of colour 1) in G of size at least $(1 - \frac{\mu}{2})\frac{t}{r+q-1} \cdot (1 - \varepsilon)\lfloor \frac{n}{t} \rfloor \geq (1 - \mu)\frac{n}{r+q-1}$, as wanted. □

Finally we prove Theorem 1.4, which follows by combining our Theorem 1.2 and (a special case of) a result due to Johansson, Kahn, and Vu [22], which we state below.

Theorem 3.5 (Johansson, Kahn, and Vu [22]). *For all $r \in \mathbb{N}$ with $r \geq 2$, there exists $C > 0$ such that, provided $p \geq Cn^{-r+1} \log n$, w.h.p. $\mathbb{G}^{(r)}(n, p)$ has a perfect matching.*

Proof of Theorem 1.4. Let $C := C(q, r, \mu)$ be a large enough constant. We expose the random hypergraph $G \sim \mathbb{G}^{(r)}(n, Cn^{-r+1} \log n)$ in two stages, and will after each step fix an outcome that holds with high probability. For that, pick C' such that $(1 - Cn^{-r+1})(1 - C'n^{-r+1} \log n)$

$=(1-Cn^{-r+1}\log n)$ (then $C'\geq C/2$), and let $G_1\sim\mathbb{G}^{(r)}(n,Cn^{-r+1})$ and $G_2\sim\mathbb{G}^{(r)}(n,C'n^{-r+1}\log n)$ be independent binomial random r-graphs on $[n]$. Then $G\sim G_1\cup G_2$.

Fix a q-colouring of the r-subsets of $[n]$. Reveal the edges of G_1. Then, by Theorem 1.2, w.h.p. G_1 contains a monochromatic matching of size $\left\lceil(1-\mu)\frac{n}{r+q-1}\right\rceil$. We fix such an outcome, and let M_1 denote this matching and $W:=[n]\setminus V(M_1)$.

Reveal now the edges of $G_2[W]$. Observe $n^{-r+1}\log n=\Omega_{r,q,\mu}(|W|^{-r+1}\log|W|)$ since $|W|=\Theta_{r,q,\mu}(n)$, and thus, by Theorem 3.5, w.h.p. $G_2[W]$ has a perfect matching. We fix such an outcome and let M_2 denote this matching.

This concludes the proof as $M_1\cup M_2$ is a perfect matching of G with at least $|M_1|\geq(1-\mu)\frac{n}{r+q-1}$ edges of the same colour, as desired. □

Recently Kahn [23] determined the sharp threshold for the existence of a perfect matching in hypergraphs, proving that the conclusion of Theorem 3.5 holds with $C=(r-1)!+\varepsilon$ for any $\varepsilon>0$, and that the constant $(r-1)!$ cannot be improved. We conjecture that Theorem 1.4 is true already at the sharp threshold, namely that, for any $r,q\in\mathbb{N}$ and $\varepsilon,\mu>0$, its conclusion holds for $C=(r-1)!+\varepsilon$. Note that, in order to achieve that, only replacing Theorem 3.5 with the sharp version is not enough: indeed, using the same notation as in the proof, since the size of W is much smaller than n, having $p=((r-1)!+\varepsilon)n^{-r+1}\log n$ is not enough to guarantee a perfect matching in $G_2[W]$.

4. Concluding remarks

In this paper, we proved a transference and a defect version of the Alon–Frankl–Lovász theorem. It would be very interesting to characterise the extremal colourings for the AFL Theorem, that is, those q-colourings for which the bound in Theorem 1.1 is tight. In the graph case $r=2$, the extremal colourings were characterised in [33] using the Gallai–Edmonds decomposition theorem. Closely related to this, it would be desirable to have a stability version, which should say that any q-colouring for which the largest monochromatic matching has size at most $(1+\mu)\frac{n}{r+q-1}$ must be ε-close to one of the extremal examples, that is, by recolouring at most εn^r edges, we obtain one of the extremal examples.

We point out that the colouring described after Theorem 1.1 is definitely not the only extremal example. For instance, let $x_1,\ldots,x_q$ be any positive integers such that $x_1+\cdots+x_q=r+q-1$. (The choice $x_1=\cdots=x_{q-1}=1$ and $x_q=r$ corresponds to the construction in Section 1.) Then partition $[n]$ into sets $V_1,\ldots,V_q$ such that $|V_i|=x_i\cdot\frac{n}{r+q-1}$ (we are ignoring rounding issues here). For any edge $e\in K_n^{(r)}$, there must be $i\in[q]$ such that $|e\cap V_i|\geq x_i$. (If not, $|e|\leq\sum_{i=1}^{q}|e\cap V_i|\leq\sum_{i=1}^{q}(x_i-1)=r+q-1-q<r$.) If there are multiple such i, just pick one arbitrarily. Then colour e with colour i. Observe that, for every $i\in[q]$, every edge with colour i intersects V_i in at least x_i vertices and thus any matching in colour i has size at most $|V_i|/x_i=\frac{n}{r+q-1}$.

One of the referees raised the following additional problem, which seems still open: Given $r,q\in\mathbb{N}$ with $r,q\geq 2$, what is the threshold above which w.h.p. every q-colouring of the edges of $\mathbb{G}^{(r)}(n,p)$ contains a monochromatic matching of the same size as that given by the Alon–Frankl–Lovász theorem? That is, which p allows one to remove the error term in Theorem 1.2? We can observe the following for the graph case with two colours (corresponding to $r=q=2$) where, assuming for simplicity that n is divisible by 3, we aim for a monochromatic matching of size $n/3$. Corollary 1.4 in [30] implies that $p>2/3$ suffices. On the other hand, we can easily observe that the threshold has to be at least constant. Indeed, suppose that an n-vertex graph G has two distinct non-adjacent vertices, say v and w, of degree smaller than $n/6$. Then partition the vertex set in two sets A,B with $|A|=n/3$ such that v and its neighbours are in A, $w\in B$ and the neighbours of w are in A. Colour by red all the edges in B and by blue every other edge. Then w is isolated in

$G[B]$ and any blue matching saturating A must use an edge inside A to cover v. Therefore G has no monochromatic matching of size $n/3$.

Acknowledgements

We thank the anonymous referees for their valuable comments.

Funding statement

LG is supported in part by the NSERC Discovery Grant "Problems in Extremal and Probabilistic Combinatorics". SG is funded by the Deutsche Forschungsge-meinschaft (DFG, German Research Foundation) – 542321564. PM is supported by ERC Advanced Grant 883810 and by UKRI under the Horizon Europe Guarantee for MSCA Postdoctoral Fellowships (EP/Z001781/1, IRIS). AS is funded by the Alexander von Humboldt Foundation.

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Cite this article: Gishboliner L, Glock S, Michaeli P, and Sgueglia A (2026). Defect and transference versions of the Alon–Frankl–Lovász theorem. *Combinatorics, Probability and Computing*. https://doi.org/10.1017/S0963548326100431