Analysis of Gradient Flows Relating to Minimal Surfaces

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Statement of Originality

I declare that all the work presented in this thesis is my own except where explicitly stated otherwise. Chapter 3 and associated Appendix B contains the results of joint work with Michael Struwe and my supervisor Melanie Rupflin. All other chapters are my own work. I have not submitted this thesis in whole or part for examination at another university.

Abstract

In this thesis, I study certain geometric gradient flow equations, some old, some new, which all seek to construct minimal surfaces.

The first equation I study is the half-harmonic map flow, also known as the Plateau flow, which is an evolution equation for maps from the circle $\mathcal{S}^1$ into a smooth closed submanifold N of $\mathbb{R}^n$. This is a gradient flow for the half-energy functional, introduced by Da Lio and Rivière, the critical points of which are half-harmonic maps. These are of geometric interest since the harmonic extension of a half-harmonic map to the unit disc parametrises a free boundary minimal surface. This flow equation has been studied by several authors, in particular by Wettstein and Struwe and it is two questions of Struwe which I will answer in this part of the thesis. The first of these is to prove uniqueness of weak solutions along which the energy is non-increasing, improving upon the existing uniqueness result proved by Struwe and bringing the theory in line with what is known about the harmonic map flow. The second question concerns the relation of half-harmonic maps with the classical Plateau problem when the target N is a closed curve Γ. In particular, I prove the monotonicity of half-harmonic maps, meaning that half-harmonic maps give rise to a solution of the Plateau problem, but perhaps with multiplicity.

The second part of this thesis presents joint work with Melanie Rupflin and Michael Struwe which introduces a new system of equations. This flow is designed to produce free boundary minimal surfaces with topology of any fixed compact surface, with boundary supported on a smooth closed submanifold $N \hookrightarrow \mathbb{R}^n$. This is also a gradient flow for the half energy, but now in its generalised form as introduced by Da Lio and Pigati. We couple the equation for the map evolution with an equation to evolve the metric on the domain, which is inspired by the Teichmüller harmonic map flow introduced by Rupflin and Topping. For this system, we establish key results on existence and regularity of solutions, along with an analysis of singularity formation and asymptotic convergence.

In the final part of this thesis, I study a question about solutions of the Teichmüller harmonic map flow, which is a gradient flow of the Dirichlet energy with respect to both the map and domain metric which was introduced to produce closed minimal surfaces in a smooth closed target manifold (N, h). In particular, I study the fine structure of the limit object at infinite time in the setting where the domain metric degenerates and so part of the limiting map collapses to a curve. I study sufficient conditions to ensure that this curve is a geodesic in (N, h) for two slightly modified versions of the flow.

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Chapter 1

Introduction

In this introduction, I will aim to give a background on the area of geometric analysis I have been studying for the last four years and for which this thesis is my contribution. I will begin with providing some history and broader background on the areas of minimal surfaces, harmonic maps and gradient flows before specialising to the theory which directly informs the questions I have tackled, namely the Teichmüller harmonic map flow and half-harmonic maps. The particular work which I have done fits very much in the paradigm of geometric flow equations, with key analytic questions relating to existence, uniqueness, singularity formation and asymptotic convergence being the focus of my efforts.

1.1 Historical and General Background

1.1.1 Minimal Surfaces and Harmonic Maps

Minimal surfaces have been studied in one way or another for hundreds of years, and yet they remain at the heart of the field of geometric analysis to this day. I will not pretend to give a full history of their study here, but I will aim to recall the rich variety of methods which have been developed for the study of minimal surfaces, and which in many cases have found extensive applications in other areas of mathematics.

The story of minimal surfaces traces back to the origins of the calculus of variations, since the area of a surface is a very natural and physically relevant quantity to wish to minimise. Indeed, Lagrange computed the variation of surface area for a graph, arriving at the *minimal surface equation*

$$(1+(\partial_x u)^2)\partial_{yy}u - 2\partial_x u\partial_y u\partial_{xy}u + (1+(\partial_y u)^2)\partial_{xx}u = 0 \tag{1.1.1}$$

where $u = u(x, y)$ defines a graphical surface $\{(x, y, u(x, y))\}$ in $\mathbb{R}^3$. This is a quasi-linear elliptic PDE, and Lagrange was unable to find any examples beyond the trivial one of a plane. One thing to note here is that despite the name minimal, this equation is satisfied by all critical points of the area functional, and it is customary to call any critical point of area a *minimal surface*, whether minimising or not.

It was subsequently realised, by Meusnier, that this equation corresponds exactly to the geometric condition of vanishing mean curvature, and hence he showed that the catenoid and helicoid are examples of minimal surfaces. Meusnier thus produced the first non-trivial examples of minimal surfaces, but coming up with further explicit solutions of this equation turns out to be very hard, and it required decades of work and new techniques to expand beyond these simple examples.

A major leap forward came about in the nineteenth century, when new methods using complex analysis were developed. In particular, Scherk constructed some new explicit examples, now known as Scherk surfaces, which are built periodically from solutions on quadrilaterals. More systematically, Weierstrass and Enneper introduced a general representation formula using complex functions. Specifically, they proved that given a simply-connected domain $0 \in \Omega \subseteq \mathbb{C}$, a non-constant holomorphic function $f : \Omega \to \mathbb{C}$, a meromorphic $g : \Omega \to \mathbb{C}$ such that fg^2 is holomorphic, and constants x_0, y_0, z_0, the mapping

$$w = u + iv \mapsto \begin{pmatrix} x_0 + \operatorname{Re}\left(\int_0^w \frac{1}{2}f(z)(1-g(z)^2)\mathrm{d}z\right) \\ y_0 + \operatorname{Re}\left(\int_0^w \frac{1}{2}if(z)(1+g(z)^2)\mathrm{d}z\right) \\ z_0 + \operatorname{Re}\left(\int_0^w f(z)g(z)\mathrm{d}z\right) \end{pmatrix} \tag{1.1.2}$$

parametrises a minimal surface, and conversely *any* simply-connected minimal surface can be parametrised in this way. The works coming from this period made many advances in our understanding of minimal surfaces, for example allowing for a systematic treatment of branch points, and they cemented the link between minimal surfaces, complex analysis and harmonic functions. However, this is not the end of the story, since this representation is not well defined in general when we consider a multiply connected domain (the integral becomes potentially path dependent), and even when restricted to the simply connected case, it has drawbacks. For example, it does not provide any means to either solve boundary value problems such as the Plateau problem discussed below or to study the embeddedness of the resulting surface. A particular result coming from this time that we will see many iterations of in this thesis can be stated, slightly informally, as follows: a parametrised surface is minimal if and only if its parametrisation is *harmonic* and *conformal*.

One specific question which has been the focus of intensive study is the so called *Plateau problem*, originally raised by Lagrange and named after the physicist Joseph Plateau, who conducted extensive experiments with soap films and derived a number of empirical laws for their behaviour. There are several different mathematical formulations of the Plateau problem which have been studied, but all of them ask the following basic question: given a closed Jordan curve (or sometimes curves) Γ, does there exist a surface Σ of least area which has boundary spanning Γ, and if so what are its properties? For a curve Γ sitting in $\mathbb{R}^3$, intuitively this surface will be the shape formed by dipping a wire in the shape of Γ into a soap solution and lifting it out, yielding a film which due to the elastic energy of the liquid will seek to minimise surface area. However, to tackle this problem in a mathematically rigorous way took centuries of work and the development of many new ideas and methods.

The major breakthroughs came around 1930 with the independent works of Douglas [Dou31] and Radó [Rad30]. Both authors worked with parametrised surfaces with parameter space $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and used the aforementioned result that $u : \mathbb{D} \to \mathbb{R}^n$ represents a minimal surface if it is *harmonic* and *conformal*. However, beyond this their approaches were quite different. Radó used an approximation scheme, based upon polygonal approximation and conformal mapping theory, which finds a sequence of harmonic and *nearly* conformal maps, which map $\partial\mathbb{D}$ *close* to Γ, and then took a limit to solve the true Plateau problem for any rectifiable curve $\Gamma \subseteq \mathbb{R}^3$. Douglas instead introduced a new functional which depends just on the boundary values of the parametrisation. This functional is defined by

$$A(g) := \frac{1}{4\pi} \int_{\partial\mathbb{D}\times\partial\mathbb{D}} \frac{|g(\theta) - g(\phi)|^2}{4\sin^2\left(\frac{\theta-\phi}{2}\right)} \mathrm{d}\theta \mathrm{d}\phi \tag{1.1.3}$$

for a vector valued function $g = (g_1, \ldots, g_n) : \partial\mathbb{D} \to \Gamma \subseteq \mathbb{R}^n$. Douglas provided the nice geometric interpretation of the integrand as being the length of the chord connecting $g(\theta)$ to $g(\phi)$ divided by the distance between θ and ϕ as measured in the disc, all squared, and the reader familiar with fractional Sobolev spaces will see that this is in fact the $H^{\frac{1}{2}}(\mathcal{S}^1)$ semi-norm, a fact we return to later on. However, the key property of this functional is that if g minimises A (or more generally is a critical point), among all functions which parametrise Γ in a suitable sense, then the harmonic extension of g to the unit disc will parametrise a minimal surface spanning the curve Γ. One analytic benefit of this functional to Douglas was that it contains no derivatives, and hence is more amenable to study with the more limited functional analytic theory available at the time.[1] Modern expositions of the Plateau problem for minimal discs typically use the approach of directly minimising the Dirichlet energy of maps from the disc. This idea had been around since before Radó's and Douglas' works, but it took until Courant in [Cou37] for the technical issues to be overcome. A modern proof for the existence of a solution to the Plateau problem using this method can be found in [CM11, Chapter 4].

The above approaches to the Plateau problem all worked with parametrised surfaces, and hence fixed in advance the topology of the resulting minimal surface. If instead the problem is posed for the existence of an area minimising surface among *all* possible topologies, then a fundamentally different approach is needed. This came with the development of *geometric measure theory* around the year 1960 with key contributions by Federer and Fleming, and many others. In addition to not prescribing the topology, this method also

[1]For some context, Sobolev's fundamental contributions were yet to come later that decade.

allows for more singular minimisers such as triple junctions, which are seen in physical soap film configurations.

The Plateau problem is probably the most extensively studied minimal surface problem, but it is not the only one. An interesting variant, which provides the motivation for much of this thesis, is to find *free boundary minimal surfaces*. There are several different versions of this problem which have been studied, tracing back to at least 1940 with the paper [Cou40] of Courant. In that work, Courant studied the problem of finding a minimal surface which is topologically an annulus where one boundary curve has the Plateau boundary condition, i.e. maps monotonically onto a given simple Jordan curve $\Gamma \subseteq \mathbb{R}^n$, whereas the other boundary curve is constrained to lie on a given submanifold $N_0 \subseteq \mathbb{R}^n$, giving the so called *free boundary*. More generally, Courant studied the case when the area of a surface is minimised subject to the whole boundary of the surface being restricted to lie on a submanifold, and using a *linking condition* was able to perform a direct minimisation of the Dirichlet energy to produce some existence results, see for example [Cou50, Chapter 6]. Courant's linking condition is not always available, for example when the support manifold $N_0 \subseteq \mathbb{R}^3$ is diffeomorphic to a sphere. This case is treated by Struwe in [Str84], who used a minmax approach combined with an approximation scheme modelled on that of Sacks and Uhlenbeck to prove the existence of a free boundary minimal disc. One feature to emphasise in the above works is that the resulting minimal surface is *not* required to lie on one side of the support manifold N_0, indeed the setup does not impose that N_0 is a hypersurface, and so it may not even make sense to discuss which side the minimal surface lies on. This contrasts with a different formulation of the free boundary minimal surface problem that instead of working with a submanifold $N_0 \subseteq \mathbb{R}^n$, works with a manifold N with boundary, and looks for critical points for area among surfaces contained in N whose boundaries lie in ∂N. This is the setup used for example by Jost and Gruter in [GJ86]. In this thesis, I will exclusively work with the former setup used by Courant and Struwe.

A special case of the free boundary minimal surface problem where $N_0 = \mathcal{S}^{n-1}$ has attracted particular attention. Despite the apparent simplicity of this geometric setup, the resulting theory of free boundary minimal surfaces is very rich. For example, it was not known until 2024 that there exists an oriented free boundary minimal surface of every topological type when $n = 3$, see [KKMS24], and compare Lawson's constructions of closed minimal surfaces in $\mathcal{S}^3$ referenced below. Another reason this particular geometry is interesting is that there are strong links to the Steklov eigenvalue problem. Specifically, if a metric g on Σ, a surface with boundary, maximises the first Steklov eigenvalue, then the corresponding eigenfunctions yield an immersion as a free boundary minimal surface into a ball B^n for some dimension n, and indeed it is using this eigenvalue approach, with several clever modifications, that yielded the above existence theorem in [KKMS24].

The final existence problem for minimal surfaces that I mention here is to find a *closed* minimal surface of a given topology, and sometimes also homotopy class, as a submanifold of a fixed ambient manifold N. The case when $N = \mathbb{R}^n$ turns out to be trivial, as there are *no* closed minimal submanifolds of $\mathbb{R}^n$. However, the case of the three dimensional sphere $\mathcal{S}^3$ has a much richer theory, with Lawson in [Law70] constructing examples of immersed minimal surfaces in $\mathcal{S}^3$ of every topological type except the real projective plane, which is known not to exist. Moreover the orientable examples were embedded. For general ambient manifolds, key progress on the existence of closed minimal surfaces was obtained by Sacks and Uhlenbeck, and independently Schoen and Yau, see [SU81], [SU82], [SY79]. These authors proved that if Σ is a closed surface, N is a smooth manifold of dimension at least 3 and $f : \Sigma \to N$ is a homotopically non-trivial continuous map which induces an injection on the fundamental groups, i.e. $f_* : \pi_1(\Sigma) \to \pi_1(N)$ is injective, then there exists a branched minimal immersion $h : \Sigma \to N$ which induces the same map of fundamental groups and can be selected to have least area among such maps. Moreover, if $\pi_2(N) = 0$, then h can further be taken to be homotopic to f. A map f satisfying the above conditions is called *incompressible.*

A key idea in both proofs of the above result is another instance of the recurring idea that a parametrisation is minimal if it is conformal and harmonic. However, now that we are working with Riemannian manifolds instead of domains in Euclidean space, we need to replace the notion of harmonic functions with that of *harmonic maps.* Note that whilst our focus is on two-dimensional domains, harmonic maps are defined and studied in all dimensions. To introduce harmonic maps, we first need to recall the *Dirichlet energy*, a functional defined for maps $f : (M, g) \to (N, h)$ with regularity H^1 by the formula

$$E(f, g) := \frac{1}{2} \int_M |\mathrm{d}f|^2 \, \mathrm{d}v_g \tag{1.1.4}$$

where $\mathrm{d}v_g$ is the Riemannian volume form on M. A function $f \in H^1(M; N)$ is then called a *harmonic map* if it is a critical point of the Dirichlet energy. Computing the Euler-Lagrange equation yields the PDE

$$\tau_g(f) := \mathrm{tr}\nabla \mathrm{d}f = 0 \tag{1.1.5}$$

where $\tau_g(f)$ is called the *tension field*. Often, it is convenient to embed the target manifold $N \hookrightarrow \mathbb{R}^n$ isometrically, which is always possible by Nash's embedding theorem, [Nas56], as this allows us to write the tension field extrinsically as

$$\tau_g(f) = P_f(\Delta_g f) = \Delta_g f + A(f)(\nabla f, \nabla f) \tag{1.1.6}$$

where $\Delta_g f$ is the usual Laplace-Beltrami operator, applied coordinate-wise to f, P_x is the orthogonal projection of $\mathbb{R}^n$ onto $T_x N$ at a point $x \in N$, and A is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^n$. Apart from simplifying the analysis in some instances, this also makes clear the quadratic nature of the non-linearity of the harmonic map equation which becomes

$$\Delta_g f = A(f)(\nabla f, \nabla f) \tag{1.1.7}$$

and $|A(f)(\nabla f, \nabla f)| \leqslant C |\nabla f|^2$ for a constant C depending only on N. This is behind why in dimension two, the harmonic map equation is critical in a PDE sense and so of particular interest. For one-dimensional domains the equation is sub-critical and for dimensions higher than 2, it is super-critical.

Two key examples to keep in mind are the cases where $N = \mathbb{R}$, which recovers the classical notion of harmonic functions, and the case where $M = \mathcal{S}^1$, where harmonic maps are readily seen to be closed geodesics. The study of harmonic maps goes back to at least 1940 by Bochner in [Boc40], who considered the special case where the domain is a subset of the Euclidean plane. In an influential paper [ES64], Eells and Sampson studied harmonic maps in the general case, and in the setting where the target manifold (N, h) has non-positive sectional curvature obtained powerful existence results. I will return to this work later on in the discussion of gradient flows.

With an appropriate understanding of what harmonic means in the setting of maps between manifolds in hand, let us now turn to the second condition needed to ensure f parametrises a minimal surface, namely that f is conformal. What is interesting here is that for two dimensional domains, the conformality of the map also has a variational interpretation, and moreover this is in terms of exactly the Dirichlet energy used to define harmonic maps, only now considered as a function of the domain metric g. This follows by a standard computation of the variation of the Dirichlet energy, see for example [BW03, Chapter 3], which yields for a fixed map $f : M \to N$, fixed metric h on N and a one parameter family of metrics g_ε on M with $\partial_\varepsilon g_\varepsilon|_{\varepsilon=0} = \delta g$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E(f, g_\varepsilon) = -\frac{1}{2}\int_M \langle k(f, g_0), \delta g\rangle \mathrm{d}v_{g_0} \tag{1.1.8}$$

where $k(f, g) := f^*h - \frac{1}{2}|\mathrm{d}f|^2 g$ is the *stress-energy tensor*. This computation is valid for all dimensions, but is most useful in dimension two. In this setting, a common alternative viewpoint is to write the stress-energy tensor in terms of the *Hopf differential*, which in local complex coordinates $z = x + iy$ is

$$\Phi(f, g) := \varphi(f, g)\mathrm{d}z^2 = \left(|\partial_x f|^2 - |\partial_y f|^2 - 2i\langle\partial_x f, \partial_y f\rangle\right)\mathrm{d}z^2 \tag{1.1.9}$$

Then the relation is $k(f, g) = \frac{1}{2}\mathrm{Re}(\Phi(f, g))$, and the vanishing of this corresponds exactly to $|\partial_x f|^2 = |\partial_y f|^2$ and $\langle\partial_x f, \partial_y f\rangle = 0$, i.e. that f is weakly conformal[2]. Note that in this thesis, I will always allow for branch points in the definition of conformal, which is what is meant here by weakly conformal. One final point to note here is that, again just for two dimensional domains, the Dirichlet energy is invariant under a conformal change of metric, i.e. replacing g by $\rho^2 g$ for a function $\rho > 0$, and hence E in fact does not depend fully on the metric g, only on its conformal class.

[2]In dimensions other than two, the vanishing of the stress-energy tensor instead implies that the map is constant, see e.g. [BW03, Chapter 3]

The above discussion relates back to minimal surfaces through the following key result of Sacks and Uhlenbeck.

Theorem 1.1.1 (Theorem 1.8, [SU81]). *Let Σ be a closed surface, (N, h) a Riemannian manifold and suppose that (u, g) is a critical point of the Dirichlet energy E with respect to both the function and the conformal class of the metric. Then $u : (\Sigma, g) \to (N, h)$ is a branched minimal immersion or is constant.*

1.1.2 Gradient Flows and Their Applications to Geometry

In this section, I will explain the other major topic of this thesis, namely *gradient flows*. Following a short general introduction, I will be paying special attention to the case of harmonic map flow, which is not only historically significant, but further is of particular relevance to the work I present in the remainder of this thesis.

When studying the existence and properties of objects which arise as critical points of a functional, it is very natural to consider the gradient flow associated to this functional. In a simple abstract setting, suppose we have a functional $F : V \to \mathbb{R}$, for a vector space V equipped with an inner product.[3] Then we can compute the gradient of the functional F, denoted ∇F and defined by $\langle \nabla F, w \rangle = \mathrm{d}F(w)$ for all $w \in V$, which of course depends on the choice of inner product[4]. From this, it follows that critical points are those $v \in V$ with $\nabla F(v) = 0$ and the gradient flow is the evolution equation

$$\partial_t v = -\nabla F(v). \tag{1.1.10}$$

The hope is then that we can find solutions $v(t)$ to this equation for a given initial data $v_0 \in V$ and which exist for all time, describing a curve in V, and that in a suitable sense, $v(t)$ should converge to a critical point v_∞ as $t \to \infty$. The key property of this solution

[3]This is not the most general setting for defining gradient flows, but is convenient to get across the key ideas. I will later work in Sobolev spaces taking values in a manifold, which are not vector spaces but Banach manifolds, but the discussion here applies equally well in this setting making only the natural adjustments.

[4]Often this inner product will end up being L^2, but in Chapter 4, the equation I study arises from a weighted inner product.

which justifies this hope is the following equation governing the functional value along the solution $v(t)$. A formal computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}F(v(t)) = \mathrm{d}F(v(t))(\partial_t v) = \langle \nabla F(v(t)), \partial_t v\rangle = -\|\nabla F(v(t))\|^2. \tag{1.1.11}$$

This not only provides global in time bounds on the quantity $F(v(t))$, which are often very useful for the analysis of (1.1.10), but importantly also implies the existence of a sequence of times $t_j \to \infty$ along which $\|\nabla F(v(t_j))\| \to 0$. In the presence of a suitable compactness theory, it is then expected that a subsequence of the $v(t_j)$ would converge to a critical point of F.

Of course, there is a lot to be made precise in this approach, and in practice there are many difficulties to be overcome, but in important cases, this idea has bourne fruit.

To give a simple concrete example, consider a domain $U \subseteq \mathbb{R}^n$ and the functional

$$F(u) := \frac{1}{2}\int_U |\nabla u|^2 \, \mathrm{d}x$$

defined for functions $u \in H^1(U; \mathbb{R})$. Then of course critical points of F are the familiar harmonic functions, and the associated L^2-gradient flow is nothing more than the classical heat equation.

The first time that a gradient flow approach has been used to study a geometric problem, and indeed the most relevant example for us, was the introduction of the *harmonic map flow* by Eells and Sampson in [ES64]. The setting there is for maps $u : (M, g) \to (N, h)$ between Riemannian manifolds of *any* dimension, with M closed, and the functional is the Dirichlet energy (1.1.4) introduced above. As already discussed, critical points of this functional are harmonic maps, and the question these authors sought to answer is known as the homotopy problem for harmonic maps. This asks if it is possible to find a harmonic map which is homotopic to a given map. To study this, they introduced the L^2-gradient flow of the Dirichlet energy, which is given by the PDE

$$\partial_t u = \tau_g(u). \tag{1.1.12}$$

Specifically, they showed that under the assumption that the target manifold (N, h) is compact and has non-positive sectional curvature, any continuously differentiable initial map $u_0 : M \to N$ yields a global solution to the flow (1.1.12), which moreover converges to a harmonic limit along some sequence of times $t_j \to \infty$. Since their solution of the flow is continuous, in fact even smooth, this solved very neatly the homotopy problem, under this restriction on the curvature of the target manifold. The theory of harmonic map flow for non-positively curved targets was further developed by Hartman in [Har67], who proved among other results that the convergence is independent of the sequence of times $t_j \to \infty$, and that the set of harmonic maps homotopic to a given map is connected.

It turns out that the homotopy problem for harmonic maps is not solvable in all cases, as for example there is no harmonic degree one map from a torus to a sphere, see [EW76]. Hence, in some settings it is necessary that a solution of the harmonic map flow forms some kind of singularity either in finite time or at infinite time.

In the eighties, Struwe made a major contribution to the understanding of the harmonic map flow in [Str85]. Here, instead of imposing a curvature condition on the target manifold (N, h), he considered the case where the domain (M, g) is two dimensional, i.e. a surface. Without the curvature condition used by Eells and Sampson, it is not possible to rule out singularities in solutions of the harmonic map flow, and so instead Struwe analysed potential singularities and constructed a weak solution of the flow. Note that the example above of degree one maps from a torus to a sphere shows that singularities are a real feature of the flow, and so this analysis is necessary.

In more detail, Struwe showed that for $u_0 \in H^1(M; N)$, with $\dim M = 2$, there exists a solution $u \in H^1_{\text{loc}}([0, \infty) \times M; N)$ to the harmonic map flow with initial data u_0. Moreover, the Dirichlet energy is non-increasing along this solution and u is smooth away from finitely many space-time points. In addition, Struwe showed the uniqueness of solutions with this regularity. Key to building this global in time solution is an analysis of the singularities that can arise for classical solutions of the flow. In this case, Struwe showed that any breakdown of a classical solution occurs due to a concentration of the

Dirichlet energy around finitely many points in M. Around each of these singular points, after a suitable rescaling the map converges to a non-constant harmonic map $v : \mathcal{S}^2 \to N$, which is usually called a *bubble*. This is reminiscent of the bubbling phenomena first studied by Sacks and Uhlenbeck in [SU81] in their construction of minimal spheres.

Struwe's paper led to many further works which improved our understanding of this weak solution. In one direction, the uniqueness theory has been improved. In particular, Freire in [Fre95] showed that uniqueness holds for solutions in $H^1_{\text{loc}}([0, \infty) \times M; N)$ subject only to the condition that energy is non-increasing. It is perhaps surprising that energy monotonicity needs to be assumed at all for a gradient flow, but the existence of backwards bubbles due to Topping, [Top02], shows that both energy monotonicity and uniqueness can fail if we look in the class of weak solutions with no additional assumptions. Following this, Rupflin strengthened Freire's result in [Rup08] to show that uniqueness holds even when we allow for small increases in energy, with the smallness needed to exclude backwards bubbles.

Another direction of research has been to develop a more refined understanding of the singularities of the flow. For simplicity, as is common in the literature since this is a local phenomenon, consider maps $u_i : \mathbb{D} \to N$, for $\mathbb{D} \subseteq \mathbb{R}^2$ the unit disc, where energy is concentrating at the origin and u_i will be taken to be a solution to harmonic map flow evaluated at a suitably chosen t_i. The initial analysis done by Struwe finds scales $\lambda_i \searrow 0$ and points $a_i \to 0$ such that $u_i(\lambda_i(x - a_i))$ converges strongly in $H^1_{\text{loc}}(\mathbb{R}^2)$ to a non-constant map $v : \mathbb{R}^2 \to N$ which is harmonic. Typically, by a conformal change and an application of the removable singularity theorem of Sacks-Uhlenbeck, this limit is instead viewed as a map from the sphere $\mathcal{S}^2$, explaining why it is called a bubble. Intuitively, the scales λ_i are taken small enough so that the Dirichlet energy of the rescaled maps is no longer concentrating around 0, but large enough so that the limit has positive energy. However, this only extracts one bubble from the energy concentration, and there could be multiple bubbles forming at potentially different scales. Several authors subsequently refined this analysis to account for all these bubbles by an iterative rescaling procedure that captures each suitably distinct scaling sequence $\{\lambda_i^1\}, \dots \{\lambda_i^m\}$ and approach sequence

$\{a_i^1\}, \dots \{a_i^m\}$ to construct a more detailed limit object called a *bubble tree*. Here, suitably distinct means that for $k \neq j$,

$$\frac{\lambda_i^k}{\lambda_i^j} + \frac{\lambda_i^j}{\lambda_i^k} + \frac{\left|a_i^j - a_i^k\right|^2}{\lambda_i^j \lambda_i^k} \to \infty \text{ as } i \to \infty. \tag{1.1.13}$$

The limiting bubble tree consists of a *base map*, $u_b : M \to N$, which is simply the weak H^1 limit of the original sequence u_i, together with a collection of non-constant harmonic maps, the bubbles, $\omega_1, \dots, \omega_m : \mathcal{S}^2 \to N$ which are the weak H^1_{loc} limits of the rescaled maps $u_i(\lambda_i^k(x - a_i^k))$. These bubbles are connected together to the base map or another bubble by curves called *necks*, i.e. the curve γ_k associated to ω_k joins to either the base map u_b or a bubble ω_j which is extracted at a larger scale. Two key questions arising in this picture are one: whether the total energy of the bubble tree equals the limit of the energy of the sequence u_j,

$$\lim_{i \to \infty} E(u_j) = E(u_b) + \sum_{k=1}^{m} E(\omega_k), \tag{1.1.14}$$

called the no loss of energy property and two: do all the necks have zero length, which is called the no-neck property. Constructing this bubble tree limit and answering these two questions when the maps u_i are taken to be approaching either finite time or infinite time singularities from the harmonic map flow have been investigated by many authors, with key works including [Qin95], [DT95], [QT97a], [Top04]. In particular, these works have shown that for infinite time singularities of the harmonic map flow, both the no loss of energy and no-neck properties hold, whilst for finite time singularities, the no loss of energy property remains true but non-trivial necks can form. The key difference between these situations is that for infinite time singularities the energy decay formula allows us to find a time sequence along which the tension field is bounded in L^2 (in fact even tending to zero), but this can fail at finite time singularities. A key topic which remains not well understood is that of uniqueness of the bubble tree limit, i.e. whether the limit depends on the choice of time sequence. There are examples where uniqueness can fail, see e.g. [Top04] for the finite time case, but it is conjectured that uniqueness must hold when the target is real analytic. This remains very much open in general, though the recent paper

[Rup25] showed that for the simplest limit bubble tree at infinite time, where the base map is constant and exactly one bubble forms, the location of the bubble is independent of the time sequence.

The harmonic map flow has been studied further in the higher dimensional case for general target manifolds, for instance by Struwe in [Str88] and Struwe and Chen in [CS89]. Here, different techniques are used, and the results are necessarily weaker than the two-dimensional case as the equation is super-critical.

The harmonic map flow is not the only flow equation which has interested geometers, with the key innovation of Eells and Sampson inspiring a substantial body of research. Two of the most studied geometric flow equations are the *mean curvature flow* and the *Ricci flow*, the first of which is the L^2 gradient flow for the area functional and the second, whilst not introduced as a gradient flow, is behind the famous resolution of the Poincaré conjecture by Perelman. One interesting equation which does not have a gradient flow structure is the *inverse mean curvature flow*, which has found applications in general relativity, see [HI01].

1.2 Review of Relevant Existing Theory

With an idea of the history of and general background behind minimal surfaces and gradient flows, I now turn to giving a more detailed exposition of the more closely relevant existing work, which I shall reference, make use of and build upon throughout the rest of this thesis.

1.2.1 Teichmüller Harmonic Map Flow

One key body of work which relates to the rest of the thesis is the study of the *Teichmüller harmonic map flow*. On one hand, the final chapter of this thesis, Chapter 4, concerns some results I obtained relating to this equation, and on the other hand, the new flow studied in Chapter 3 builds heavily upon this theory. Throughout this section, (M, g) and (N, h) are both closed Riemannian manifolds, with M being two-dimensional.

The Teichmüller harmonic map flow is a geometric gradient flow of the Dirichlet energy, introduced by Rupflin and Topping in [RT16], which couples the harmonic map flow seen above with an evolution equation for the domain metric, with the aim of constructing critical points of the Dirichlet energy with respect to both the map and the metric. Recalling Theorem 1.1.1, such critical points are branched minimal immersions.

To get a well behaved evolution equation for the metric requires a bit more care, and in particular to take advantage of the symmetries of the problem. The first symmetry to take note of is the conformal invariance of the energy. By the classical uniformisation theorem, every closed orientable surface (M, g) is conformal to a unique surface with constant curvature $\kappa = 1, 0, -1$. Moreover, $\kappa = 1$ if M is a sphere, $\kappa = 0$ if M is a torus and $\kappa = -1$ if M has higher genus. It is helpful then to introduce the notation $\mathcal{M}_\kappa$ for the space of smooth metrics on M with constant curvature κ, with the additional constraint that the total area is 1 if $\kappa = 0$, and then to restrict the metric g to the set $\mathcal{M}_\kappa$. The other symmetry present in the energy is that of simultaneous pullback of the map and metric by a diffeomorphism. It is a bit more subtle to account for this, but the main idea is to constrain the metric to flow in directions orthogonal to the action of diffeomorphisms. This same idea is needed for the new flow we introduce in Chapter 3, and so the full details will be explained there. The resulting system of equations, called Teichmüller harmonic map flow, was introduced by Rupflin and Topping in [RT16] and is given by

$$\partial_t u = \tau_g(u) \tag{1.2.1}$$

$$\partial_t g = \frac{\eta^2}{4} Re(P_g^H(\Phi(u, g))) \tag{1.2.2}$$

where $\eta > 0$ is a constant we are free to choose, Φ is the Hopf differential defined in (1.1.9) and P_g^H is the L^2-orthogonal projection of quadratic differentials onto the finite dimensional subspace $H(g)$ of holomorphic quadratic differentials. When the domain is a sphere, there are no non-trivial holomorphic quadratic differentials, so the metric is stationary and this is the same as the classical harmonic map flow. If the domain is a torus, then $H(g)$ is an integrable distribution on $\mathcal{M}_0$ and so the equation reduces to two ODEs. Moreover, as a consequence of the completeness of the Teichmüller space, no singularities can form in the metric in finite time. In fact, the flow in this case is equivalent to a flow introduced earlier by Ding, Li and Liu in [DLL06]. In light of these observations, the focus will be on the case when the domain has genus at least 2.

In two initial papers first appearing in 2012, [RT16] and [Rup14], Rupflin and Topping established the basic theory for this system, demonstrating the following properties of solutions in the setting where the metric is well behaved:

1. For any initial data $(u_0, g_0) \in H^1(M; N) \times \mathcal{M}_\kappa(M)$, there exists a solution (u, g) to (1.2.1) and (1.2.2) which is smooth for $t > 0$ away from finitely many space-time points and exists until a maximal time $T \in (0, \infty]$, which is infinite unless the metric degenerates, i.e. $\liminf_{t \nearrow T} \operatorname{inj}(g(t)) = 0$.

2. At each singular time $T_s < T$, energy concentrates at a finite set of points $x_1, \ldots, x_k$,
$$\inf_{r \searrow 0} \limsup_{t \nearrow T_s} \int_{B_r^{g(t)}(x_i)} |\nabla u(t)|^2 \, \mathrm{d}v_g > 0$$
and after a suitable rescaling about x_i, the map converges to a non-constant harmonic map $v_i : \mathcal{S}^2 \to N$. Moreover, the metric $g(t)$ is Lipschitz in time across T_s (using any C^k metric on $\mathcal{M}_\kappa$).

3. The energy $E(u(t), g(t))$ is non-increasing in time.

4. The solution is uniquely determined by the initial data within the class of weak solutions with non-increasing energy.

5. If $T = \infty$ and further $\inf_{t \geqslant 0} \operatorname{inj}(g(t)) > 0$, i.e. the metric does not degenerate even at infinity, then there is a sequence of times $t_j \to \infty$ and orientation preserving diffeomorphisms $f_j : \Sigma \to \Sigma$ such that $(f_j^* u(t_j), f_j^* g(t_j))$ converges to a pair $(u_\infty, g_\infty) \in C^\infty(M; N) \times \mathcal{M}_\kappa$ which is a critical point of the Dirichlet energy. The

convergence of the metrics is smooth, and the convergence of $f_j^* u(t_j)$ is smooth away from finitely many points, where energy concentrates and at least one bubble forms as above.

A further refinement of the asymptotic convergence was also obtained in [KRT22], namely that if the target manifold is real analytic and no singularities form as $t \to \infty$, then the convergence is smooth, independent of the time subsequence $t_j \to \infty$ and there is no need to pull back by diffeomorphisms.

So far, this mirrors very closely results for the harmonic map flow in two dimensions. What is new for this flow is the possibility that the metric g can degenerate, leading to a new type of singularity formation, either in finite or infinite time. Key to understanding the behaviour of these new potential singularities in the higher genus case is the description of how hyperbolic surfaces degenerate. More details can be found in Appendix C, but for now note that given a sequence of hyperbolic metrics g_n on a surface M satisfying $\mathrm{inj} g_n \to 0$, then there is a finite and disjoint collection of simple closed geodesics $\sigma_1^n, \ldots, \sigma_k^n$ in (M, g_n) which have length going to zero. It is then possible to understand the limit of these surfaces as a punctured hyperbolic surface $\tilde{M}$, with the punctures corresponding to the collapsed closed geodesics. This limit may be disconnected. Using the Keen-Randol collar lemma, C.0.1, each closed geodesic has an explicit special cylindrical neighbourhood, called a collar, and as the length of the closed geodesic goes to 0, the conformal length of the collar goes to infinity.

The analysis of possible finite time metric singularities was done in [RT19]. The authors proved that there is a canonical means to continue the flow past a finite time metric degeneration as a flow on a collection of lower genus surfaces. In particular this proves the existence of a global weak solution for any initial data. The authors further establish a no loss of energy result at each finite time singularity. When the metric is not degenerating, this follows closely from the analogous statements for the harmonic map flow, but the authors were able to extend this to also cover the case where the metric degenerates. A key observation is that there can be at most finitely many such singularities, and so when studying the asymptotic behaviour of solutions of the flow and the fine structure of any singularity, it suffices to move past the last finite time singularity to where the solution is globally smooth.

The case where the metric degenerates, but only as $t \to \infty$, is studied first in [RTZ13] and later refined in [HRT16]. The full details of this convergence are more technical, so I defer them until Chapter 4, and simply record that away from the collapsing parts of the surface, the solution of the flow converges to a branched minimal immersion on a possibly disconnected limit surface. Furthermore, the degenerating collar neighbourhoods are mapped close to curves. A key open question raised in [HRT16] is whether this curve is a geodesic in (N, h). This is known to be true in one case, as shown by Ding, Li and Liu in [DLL06], specifically when the domain is a torus and the energy converges to zero at infinite time, but in general this remains open. It is this question which I investigate in Chapter 4.

1.2.2 Half-Harmonic Maps

We have seen above that there is a strong relationship between harmonic maps and minimal surfaces, which has been exploited both in connection to the Plateau problem and the construction of closed minimal surfaces in Riemannian manifolds. This has more recently been further extended to apply to the free boundary problem using *half-harmonic maps*. These were first explicitly introduced by Da Lio and Rivière in [LR11]. In this paper, they introduced a functional they called the *line energy* for maps $u : \mathbb{R} \to N_0 \subseteq \mathbb{R}^n$ as

$$L(u) := \int_{\mathbb{R}} \left| (-\Delta)^{\frac{1}{4}} u(x) \right|^2 \mathrm{d}x \tag{1.2.3}$$

where $N_0 \subseteq \mathbb{R}^n$ is a sufficiently smooth submanifold. As the authors point out, a simple computation shows that this energy is, up to a constant, exactly the $H^{\frac{1}{2}}(\mathbb{R})$ semi-norm. Since we will repeatedly see fractional Sobolev spaces in this thesis, a short introduction and summary is provided in Appendix A.1.1. In [LR11], the authors explicitly state they introduce this energy as a one-dimensional analogue of the Dirichlet energy in two-dimensions, noting both the quadratic integrand and the invariance of L under the trace of a conformal diffeomorphism which preserves the upper half-plane. A key difference compared with the Dirichlet energy is that this line energy is a *non-local* functional.

There are several equivalent ways this energy can be expressed. It was observed in [LR11] that L is related to the Dirichlet energy on the upper half-plane $\mathcal{H} = \mathbb{R} \times [0, \infty)$ by

$$L(u) = \inf \left\{ \int_{\mathcal{H}} |\nabla w|^2 \, \mathrm{d}v : w|_{\partial \mathcal{H}} = u \right\}, \tag{1.2.4}$$

with this infimum realised when w is the unique finite energy harmonic extension of u. Consequently, introducing the notation $u_{\mathcal{H}}$ for this unique finite energy solution to

$$\begin{cases} \Delta u_{\mathcal{H}} = 0 & \text{in } \mathcal{H} \\ u_{\mathcal{H}} = u & \text{on } \partial \mathcal{H} \end{cases} \tag{1.2.5}$$

allows L to be rewritten as

$$L(u) = \int_{\mathcal{H}} |\nabla u_{\mathcal{H}}|^2 \, \mathrm{d}v. \tag{1.2.6}$$

Using the conformal invariance of the Dirichlet energy, it is possible to change the domain to the unit disc $\mathbb{D}$, which has the benefit of being compact. Moreover, as above the energy can be written in terms of the boundary values using the fractional Laplacian, via

$$\int_{\partial \mathbb{D}} \left| (-\Delta)^{\frac{1}{4}} u \right|^2 \mathrm{d}\theta = \int_{\mathbb{D}} |\nabla u_{\mathbb{D}}|^2 \, \mathrm{d}v. \tag{1.2.7}$$

Working on the unit circle, it is interesting to note that in fact, the line energy introduced by Da Lio and Rivière is up to a constant exactly Douglas' functional A, see (1.1.3), introduced to solve the Plateau problem, since this can be seen to be the $H^{\frac{1}{2}}(\mathcal{S}^1)$ semi-norm. To clarify terminology, I will call this functional in all its forms the *half-energy*, and denote it by $E_{\frac{1}{2}}$.

This notion of half-energy has been extended by Da Lio and Pigati in [PL17] to more general domains. To do this, they start from the harmonic extension viewpoint and for a compact Riemannian surface with boundary (Σ, g), an embedded submanifold $N_0 \subseteq \mathbb{R}^n$

and a map $u \in H^{\frac{1}{2}}(\partial\Sigma; N_0)$, they define the *half-energy* as

$$E_{\frac{1}{2}}(u, g) := \frac{1}{2}\int_\Sigma |\nabla u_g|^2 \, \mathrm{d}v_g. \tag{1.2.8}$$

Here, u_g denotes the unique solution of the problem

$$\begin{cases} \Delta u_g = 0 & \text{in } \Sigma \\ u_g = u & \text{on } \partial\Sigma \end{cases} \tag{1.2.9}$$

for a given $u \in H^{\frac{1}{2}}(\partial\Sigma; N_0)$ and metric g on Σ.

Let us now turn to the critical points of the half-energy, which following Da Lio and Rivière are called *half-harmonic maps*. More precisely, these are critical points of the half-energy among maps taking values in the support manifold $N_0 \hookrightarrow \mathbb{R}^n$. When the domain is $\mathcal{S}^1 = \partial\mathbb{D}$, equivalently $\mathbb{R}$, then half-harmonic maps are characterised in [LR11] as solutions of the equation

$$P_u(-\Delta)^{\frac{1}{2}} u = 0 \tag{1.2.10}$$

recalling that P_x is the orthogonal projection of $\mathbb{R}^n$ onto $T_x N_0$.

In the case considered by Da Lio and Pigati where the domain is the boundary of a compact surface Σ, instead of the first variation being written in terms of the half-Laplacian, it now involves the *Dirichlet-to-Neumann* operator, specifically the Euler-Lagrange equation obtained in [PL17] is

$$P_u \partial_\nu u_g = 0 \tag{1.2.11}$$

where ∂_ν is the outward unit normal derivative. Note that in the case where $\Sigma = \mathbb{D}$ with Euclidean metric, the equality $\partial_\nu u_g = (-\Delta)^{\frac{1}{2}} u$ holds, which is a simple case of the theory of representing fractional operators developed by Caffarelli and Silvestre in [CS06].

One key property that has been established for half-harmonic maps is their regularity. In particular, if the target manifold N_0 is smooth, then any half-harmonic map into N_0 is also smooth, see [DR11] for the case when the domain is $\mathbb{R}$ and [PL17] for the case of a compact surface with boundary.

Finally, as promised, there is a strong connection between half-harmonic maps and free boundary minimal surfaces. First, for half-harmonic maps from the circle, it was observed in [MS15, Remark 4.28] and [DLMR15] that $u : \mathcal{S}^1 \to N_0$ is half-harmonic if and only if its harmonic extension $u_{\mathbb{D}}$ is conformal. From this it follows, as we have now seen several times, that $u_{\mathbb{D}}$ is a branched minimal immersion since it is harmonic and conformal. Moreover, the Euler-Lagrange equation for half-harmonic maps implies $\partial_r u_{\mathbb{D}} \perp T_u N_0$, i.e. that the surface meets N_0 orthogonally, and hence $u_{\mathbb{D}}$ parametrises a free boundary minimal disc. When the domain is no longer $\mathcal{S}^1$ but the boundary of a compact surface (Σ, g), then it is clear that a half-harmonic map u has extension u_g which is harmonic, by definition, and which meets the support manifold N_0 orthogonally, by the Euler-Lagrange equation (1.2.11). However, it is no longer true that u_g is automatically a parametrisation of a free boundary minimal surface, since it may fail to be conformal. For this to hold, we also require that the metric g on Σ is a critical point of the energy, very much in analogy with Theorem 1.1.1 above. This is shown by Da Lio and Pigati.

Theorem 1.2.1 (Theorems 1.6 and 3.4 from [PL17])**.** *Let (Σ, g) be a surface with non-empty boundary, $N_0 \hookrightarrow \mathbb{R}^n$ a smooth closed submanifold and $u \in H^{\frac{1}{2}}(\partial\Sigma; N_0)$ a half-harmonic map. Then the harmonic extension of u is a free boundary branched minimal immersion if and only if g is a critical point of the energy $E_{\frac{1}{2}}(u, g)$ with respect to variations of the metric.*

Remark 1.2.2. Recall that we take the definition of free boundary minimal surfaces in line with Courant and Struwe, in particular the support set for the boundary need not be a hypersurface and the minimal surface need not lie exclusively on one side even when it is.

1.2.3 Half-Harmonic Gradient Flow (Plateau Flow)

To date, there have been several works studying gradient flow equations for the half-energy. The paper of Sire, Wei and Zheng, [SWZ21], considered the special case when the domain is $\mathbb{R}$ and the target manifold is the circle $\mathcal{S}^1$. In this case, the L^2-gradient flow takes the form

$$\partial_t u = -(-\Delta)^{\frac{1}{2}} u + \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|u(x) - u(s)|^2}{|x - s|^2} \mathrm{d}s \right) u. \tag{1.2.12}$$

The authors use an inner-outer gluing scheme, a technique which has found applications in constructing singular solutions of several non-linear equations, see for example [DdPW20] in the case of harmonic map flow, to find solutions with a prescribed singularity formation at infinite time. They in fact conjecture that any solution of this flow cannot have a finite time singularity. Proving or disproving this conjecture remains an interesting open problem.

In a series of papers [Wet22, Wet21, Wet23] as part of his PhD, Wettstein studied the same flow with domain $\mathcal{S}^1$ and a more general target, either a sphere $\mathcal{S}^{n-1} \subseteq \mathbb{R}^n$ or a general smooth submanifold $N_0 \subseteq \mathbb{R}^n$ for any dimension n. In this more general setting, the gradient flow equation is then given by

$$\partial_t u + P_u((-\Delta)^{\frac{1}{2}} u) = 0 \text{ on } [0, T) \times \partial\mathbb{D}. \tag{1.2.13}$$

Wettstein established many key properties of solutions of this flow. In particular, he showed that for any given initial data $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D}; N_0)$, there exists a solution to (1.2.13) $u \in H^1_{\text{loc}}([0, \infty); L^2(\partial\mathbb{D}; N_0)) \cap L^\infty([0, \infty); H^{\frac{1}{2}}(\partial\mathbb{D}; N_0))$ with non-increasing energy and which is smooth away from finitely many singular times T_i. Moreover, he obtained results on the uniqueness and asymptotic convergence of solutions which have *small* energy.

Wettstein worked extensively with tools and ideas from fractional calculus for his results. Subsequently, Struwe in [Str24] investigated the same flow equation, but now emphasising the equivalent formulation using the Dirichlet-to-Neumann operator, changing the equation to

$$\partial_t u = -P_u \partial_\nu u_{\mathbb{D}}. \tag{1.2.14}$$

This allows much of the analysis to be carried out with classical differential operators acting on the harmonic extension of the map. One restriction that Struwe introduced compared with Wettstein's work is that the target manifold N_0 is assumed to be embedded in $\mathbb{R}^n$ in such a way that the normal bundle $T^\perp N_0$ is parallelisable. This holds for example when N_0 is either a hypersurface or a collection of closed curves, but can fail such as when N_0 is not orientable[5]. On the other hand, Struwe was able to remove the restriction that the initial energy be small that Wettstein needed for his uniqueness and asymptotic convergence results, and was able to prove the smoothness of solutions at the singular times T_i away from finitely many points in $\mathcal{S}^1$. Note that both Wettstein and Struwe performed an analysis of the singularities of the flow, and showed that these are caused by energy concentration and the formation of a half-harmonic map from the circle, closely matching Struwe's analysis of the harmonic map flow seen above. Finally, a consequence of the uniqueness result of Struwe is that his solution coincides with Wettstein's, and in the following I will call this solution the *almost smooth solution*.

In [Str24], Struwe raised several open questions. One relates to the uniqueness of solutions. Struwe proved the uniqueness of solutions which have additional regularity compared with what is assumed for a weak solution, but asks if in fact uniqueness holds in the more general class of weak solutions along which the half-energy is non-increasing, see Section 1.3.1 for the precise statement. This is in direct analogy with what is known for the harmonic map flow. In Chapter 2, I answer this question positively.

Another question raised by Struwe relates to its possible applications to the Plateau problem when the support manifold is a curve $\Gamma \subseteq \mathbb{R}^n$, a link evidenced by his terming the flow equation the *Plateau flow*. We have seen above that the stationary solutions of the equation are free boundary minimal discs whose boundary lies on Γ, and Struwe showed convergence of solutions (perhaps with bubbling) along a sequence of times $t_j \to \infty$ to a half-harmonic limit. The question arises as to whether a half-harmonic map parametrises in any suitable sense the curve Γ. In his original paper on the Plateau problem, Douglas overcame this issue by minimising the half-energy within the closed class of weakly monotone maps from $\mathcal{S}^1$ to Γ. However, it is not clear that even if the initial map u_0 is monotone that this would be preserved by the flow. In Chapter 2, I show that a non-constant half-harmonic map into a curve Γ must indeed be monotone.

[5]This follows since if there is a global basis of $T^{\perp}N_0$, this can be used to construct a volume form on N_0, hence implying N_0 is orientable.

Finally, it is raised again by Struwe, but also already by Da Lio and Pigati in [PL17], that the half-harmonic map flow could be extended to the case of maps from the boundary of a compact surfaces. They ask specifically if by combining the above gradient flow for the map with a suitable equation to flow the metric on the domain surface, as is done for Teichmüller harmonic map flow, it is possible to define a coupled flow equation to produce conformal half-harmonic maps, which we have seen above are branched free boundary minimal surfaces. This is what I, in joint work with Melanie Rupflin and Michael Struwe, do in Chapter 3, where we introduce such a flow and prove key results about its solutions, outlined in more detail in Section 1.3.2.

1.3 My Contribution

To finish this introduction, I will outline the new results which make up this thesis, which is organised into three main chapters. Note that in the above I used the notation N_0 for the submanifold of $\mathbb{R}^n$ when talking about half-harmonic maps and (N, h) for the target when discussing harmonic map flow. From now, I use N in both cases to be consistent with the existing literature. I never discuss these two settings together so it should be clear which is meant.

1.3.1 Results for Plateau Flow on $\mathcal{S}^1$

The first part of this thesis, contained in Chapter 2 and comprising of work published in [Wri24] along with a short unpublished final section, is devoted to answering two open questions raised by Struwe regarding the Plateau flow in the setting of maps from the circle $\mathcal{S}^1$. The first of these relates to the uniqueness of weak solutions, and so I begin by defining two notions of solution for the Plateau flow.

Definition 1.3.1. A function $u : [0, T) \times \partial\mathbb{D} \to N$ is called a *weak solution* to the Plateau flow (1.2.14) on the time interval $[0, T)$ and with initial data $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$ if it satisfies

1. $u \in H^1_{\mathrm{loc}}([0,T); L^2(\partial\mathbb{D};\mathbb{R}^n)) \cap L^\infty_{\mathrm{loc}}([0,T); H^{\frac{1}{2}}(\partial\mathbb{D}; N))$.

2. u solves the equation weakly, in the sense that for all $\varphi \in L^2((0,T); H^1(\mathbb{D};\mathbb{R}^n))$,

$$\int_0^T \int_{\partial\mathbb{D}} \partial_t u \cdot \varphi \mathrm{d}s\mathrm{d}t + \int_0^T \int_{\mathbb{D}} \langle \nabla u_{\mathbb{D}}, \nabla(P_u(\varphi))\rangle \mathrm{d}x\mathrm{d}t = 0.$$

3. The trace of u on $\{0\} \times \partial\mathbb{D}$ equals u_0.

A weak solution which has the additional regularity $u \in H^1_{\mathrm{loc}}([0,T) \times \partial\mathbb{D}; N)$ and satisfies the energy inequality

$$E_{\frac{1}{2}}(u(t_2)) + \int_{t_1}^{t_2} \int_{\partial\mathbb{D}} |\partial_t u|^2 \,\mathrm{d}s\mathrm{d}t \leqslant E_{\frac{1}{2}}(u(t_1))$$

for each $0 < t_1 < t_2 < T$ is called an *energy-class solution.*

In [Str24], Struwe then proved the following uniqueness result, which in particular proves uniqueness of solutions of the regularity he constructed for the flow.

Theorem 1.3.2 (Theorem 7.1 in [Str24])**.** *Let u, v be energy class solutions to* (1.2.14) *on an interval* $[0,T)$ *whose traces agree at time $t = 0$, and suppose further that u, v are smooth for $t > 0$. Then $u = v$.*

The focus of this part of my thesis has been on extending this uniqueness result to a more general class of functions. This question arises naturally, and indeed is asked in [Str24], when we compare with the corresponding theory of harmonic map flow. In that setting, Struwe constructed a global weak solution in [Str85] with a uniqueness result similar to Theorem 1.3.2, in that uniqueness is obtained for solutions under additional regularity assumptions. However, it was subsequently shown by Freire in [Fre95] that uniqueness holds in the class of weak solutions subject only to the condition that energy is non-increasing. It is perhaps surprising that energy monotonicity needs to be assumed at all for a gradient flow, but the existence of backwards bubbles due to Topping, [Top02], shows that energy monotonicity can fail in the class of weak solutions with no additional assumptions. Following this, Rupflin strengthened Freire's result in [Rup08] to show that uniqueness holds even when we allow for small positive jumps in the energy, with the smallness needed to exclude backwards bubbles. What I will show is that for the Plateau flow on the circle, uniqueness of solutions holds for weak solutions which do not have any large positive jumps in energy, which improves Struwe's uniqueness result in this setting and mirrors Rupflin's result for harmonic map flow. To state these, define the quantity $\varepsilon^* = \varepsilon^*(N) > 0$ by

$$\varepsilon^* := \inf\{E_{\frac{1}{2}}(u) : u : \partial\mathbb{D} \to N \text{ non-constant and solves } P_u(\partial_\nu u_{\mathbb{D}}) = 0\}. \tag{1.3.1}$$

The first version of the uniqueness result I show is the following.

Theorem 1.3.3 (W. [Wri24, Theorem 1.2])**.** *Let $N \hookrightarrow \mathbb{R}^n$ be a smooth closed manifold with parallelisable normal bundle. Then there exists $\varepsilon_0 \in (0, \varepsilon^*]$, depending only on $N \hookrightarrow \mathbb{R}^n$, such that if u is a weak solution to the Plateau flow* (1.2.14) *on $[0, T)$ satisfying*

$$\limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) < E_{\frac{1}{2}}(u(t)) + \varepsilon_0 \tag{1.3.2}$$

for all $t \in [0, T)$, then u is equal to the almost smooth solution constructed by Wettstein/Struwe with the same initial data.

Remark 1.3.4. In particular, this applies to weak solutions with non-increasing energy.

In addition, I prove the following slightly improved result.

Theorem 1.3.5 (W. [Wri24, Theorem 1.3])**.** *Let $N \hookrightarrow \mathbb{R}^n$ be a smooth closed manifold with parallelisable normal bundle. Suppose u is a weak solution to the Plateau flow* (1.2.14) *on $[0, T)$ satisfying the two conditions*

$$\limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) < E_{\frac{1}{2}}(u(t)) + \varepsilon^* \tag{1.3.3}$$

for all $t \in [0, T)$ and that the set

$$S := \left\{ t \in [0, T) : \limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) \geqslant E_{\frac{1}{2}}(u(t)) + \varepsilon_0 \right\} \tag{1.3.4}$$

has no accumulation points, where ε_0 is the constant from Theorem 1.3.3. Then u is equal to the almost smooth solution with the same initial data.

Remark 1.3.6. Note that $\varepsilon^* \geqslant \varepsilon_0$ and so the assumptions here really are weaker, and further that the condition (1.3.4) is satisfied for example if the map $t \mapsto E_{\frac{1}{2}}(u(t))$ has locally finite total variation.

The second question related to the Plateau flow on $\mathcal{S}^1$ that I investigate comes from another question raised by Struwe in the case that the support manifold N is a closed curve Γ. Given that the harmonic extension of a half-harmonic map from $\mathcal{S}^1$ into Γ parametrises a minimal disc, with boundary contained in Γ, it is very natural to compare this to the classical solutions of the Plateau problem. The difference is that for the Plateau problem, it is usually required that $\mathcal{S}^1$ is mapped *monotonically* and *injectively* onto Γ.

Since a half-harmonic map is not in general degree one, we need to make precise what notion of monotonicity we are using, for which we give two equivalent options, one global and one local. First, let $\gamma : \mathcal{S}^1(L) \to \Gamma$ be an arc-length parametrisation of Γ. Then for a given smooth map $u : \mathcal{S}^1 \to \Gamma$, we can find a smooth map $s : \mathcal{S}^1 \to \mathcal{S}^1(L)$ such that $u(\theta) = \gamma(s(\theta))$, which is unique up to rotation. We then say u is monotonic if the lift of s to a k-fold cover of $\mathcal{S}^1(L)$ by a degree k covering map, is injective, where k is the topological degree of s. Note that this is a strict notion of monotonicity, we do not allow the map to be constant on any open set. The local equivalent notion which does not consider the lifted map instead insists that at each point $x \in \mathcal{S}^1$, there is a small neighbourhood of x on which u is injective.

If we establish the monotonicity of a half-harmonic map u, then the injectivity reduces to asking that the degree of u is ± 1, and so it is the monotonicity that is the principle obstacle. Indeed, if u is a monotone half-harmonic map with degree greater than 1, then its extension is a Plateau solution with multiplicity. Struwe asks if the Plateau flow should be considered as producing generalised solutions of the Plateau problem, since there is no immediate reason to believe that the flow would preserve monotonicity.

Considering the links to the original work of Douglas on the Plateau problem as the half-energy coincides with Douglas' functional, it seems reasonable to ask if in fact we can prove some monotonicity properties of half-harmonic maps. One form of non-monotonicity has already been ruled out by a lemma of Douglas, which shows that a half-harmonic map cannot be constant on any open subset of the domain. Douglas needed this since he worked in a class of functions which included non-proper parametrisations of the curve Γ, and so had to show that the energy minimising map was a proper parametrisation. Using Douglas' form of the energy functional and using the regularity of half-harmonic maps, I prove the following result.

Theorem 1.3.7. *Let $\Gamma \subseteq \mathbb{R}^n$ be a simple smooth closed curve and $u : \mathcal{S}^1 \to \Gamma$ be a non-constant half-harmonic map. Then u is monotone (in the sense described above).*

This provides some immediate consequences. First, we see that there can be no non-trivial degree zero half-harmonic maps into Γ, and so any non-constant half harmonic map is in fact surjective, which in particular applies to the bubbles which form along solutions of the flow. Hence by starting with an initial map which has non-zero degree, we know the flow will produce some monotone half-harmonic map, either as a smooth limit at infinite time or as a bubble.

What remains unclear is whether we can ensure the existence of a degree one half-harmonic map using the flow. This is due to the fact that in the setting of the half-energy, we do not have the detailed singularity analysis such as that described above for the harmonic map flow, and so cannot yet answer questions about how singularities affect the homotopy type of the solution. There is a paper by Da Lio which proved more refined results for singularities forming in sequences of half-harmonic maps into spheres, [DL15], but there remain many questions about bubbling for sequences of *almost* half-harmonic maps, as would be needed to analyse solutions of the gradient flow.

1.3.2 A Gradient Flow for Constructing Free Boundary Minimal Surfaces

The second main chapter of my thesis contains work on generalising the half-harmonic map flow from the circle, considered by Wettstein, Struwe and others, with the aim of constructing free boundary minimal surfaces with other topologies, in the sense of Courant, Struwe, Da Lio and Pigati etc, see earlier Remark 1.2.2. This is the contents of Chapter 3 and this is all work done jointly with Michael Struwe and Melanie Rupflin for a paper in preparation. In this, we combine the map evolution equation introduced by Struwe [Str24], but now posed on the boundary of a compact Riemannian surface (Σ, g), with a metric evolution equation closely related to the Teichmüller harmonic map flow. Our results apply to any compact surface with boundary, and we can treat *any* closed Riemannian submanifold target $N \subseteq \mathbb{R}^n$, as we have succeeded in removing the restriction on the normal bundle required by Struwe in [Str24]. Note that in the arguments we will assume that Σ is orientable, but the results also apply in the non-orientable case by passing to the orientable double cover and working with metrics and function spaces which are invariant under the non-trivial covering space transformation.

We turn now to providing a brief motivation behind the definition of this system of equations. For this we first introduce the notation $\pi : N_\eta \to N$ for the nearest point projection onto N, which is well-defined and smooth on the tubular neighbourhood

$$N_\eta := \{y \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{R}^n}(y, N) < \eta\} \tag{1.3.5}$$

of N if $\eta > 0$ is chosen small enough. Recall that for any $p \in N$, the orthogonal projection $P_p : \mathbb{R}^n \to T_pN$ of the ambient $\mathbb{R}^n$ onto the tangent space T_pN to N at p can be equivalently written as $P_p = \mathrm{d}\pi(p)$.

As seen above, we recall that a map $u \in H^{\frac{1}{2}}(\partial\Sigma; N)$ is half-harmonic if it is a critical point of $E_{\frac{1}{2}}(u, g)$, i.e. $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} E_{\frac{1}{2}}(\pi(u+\varepsilon v), g) = 0$ for every $v \in H^{\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$. As computed by Da Lio and Pigati in [PL17],

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E_{\frac{1}{2}}(\pi(u+\varepsilon v), g) = \int_{\partial\Sigma} \partial_\varepsilon \pi(u+\varepsilon v)|_{\varepsilon=0} \cdot \partial_{\nu_g} u_g \mathrm{d}s_g = \int_{\partial\Sigma} v \cdot P_u(\partial_{\nu_g} u_g) \mathrm{d}s_g, \tag{1.3.6}$$

where ν_g is the outwards unit normal of (Σ, g) along $\partial\Sigma$, and so the $L^2(\partial\Sigma)$-gradient of the half energy with respect to the map is $P_u(\partial_{\nu_g} u_g)$. Hence u being half-harmonic is equivalent to asking that the harmonic extension u_g of u satisfies the equation

$$P_u(\partial_{\nu_g} u_g) = 0 \quad \text{on } \partial\Sigma. \tag{1.3.7}$$

Next, in order for the harmonic extension of a half-harmonic map to parametrise a free boundary minimal surface, it must also be conformal, which we have seen above by Theorem 1.2.1 is equivalent to the pair (u, g) being a critical point of the half-energy. Since we will use the corresponding properties of the variation of the half-energy in our construction of the flow, we briefly recall how this key property can be obtained.

To this end, we first note that the variation of the Dirichlet energy for a fixed metric g along maps $v_\varepsilon = u_{g_\varepsilon}$ that are obtained as harmonic extensions of a fixed map $u : \partial\Sigma \to \mathbb{R}^n$ with respect to a family of metrics g_ε with $g_{\varepsilon=0} = g$ vanishes. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(v_\varepsilon, g) = \int_\Sigma \langle \nabla_g(\partial_\varepsilon v_\varepsilon), \nabla_g u_g \rangle \mathrm{d}v_g = -\int_\Sigma \partial_\varepsilon v_\varepsilon \cdot \Delta_g u_g \mathrm{d}v_g + \int_{\partial\Sigma} \partial_\varepsilon v_\varepsilon \cdot \partial_{\nu_g} u_g \mathrm{d}s_g = 0$$

as u_g is harmonic and $\partial_\varepsilon v_\varepsilon|_{\partial\Sigma} = 0$, where the derivatives with respect to ε are evaluated at $\varepsilon = 0$. Hence, the variation of the half-energy with respect to the metric reduces to the variation of the Dirichlet energy of the fixed map u_g with respect to a varying metric g_ε, which we recall is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E_{\frac{1}{2}}(u, g_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(u_g, g_\varepsilon) = -\frac{1}{2} \int_\Sigma \langle \partial_\varepsilon g_\varepsilon, k(u_g, g) \rangle_g \mathrm{d}v_g, \tag{1.3.8}$$

with $k(v, g) = v^* g_{\mathbb{R}^n} - \frac{1}{2} |\mathrm{d}v|_g^2 g$ the stress-energy tensor introduced earlier. We recall that $k(v, g)$ vanishes if and only if v is conformal, and for later reference, we note that k is always *trace free* and for a harmonic map is also *divergence free*.

Since the energy is conformally invariant, we can restrict the class of admissible metrics to consist of a unique representative for each conformal class of metrics. For this we make use of standard uniformisation results, see e.g. [OPS88]. For Σ with negative Euler characteristic we will work with the unique representative g of each conformal class for which the boundary curves are geodesics in (Σ, g) and which is hyperbolic in the interior. When Σ is a cylinder we instead choose as our unique representative the flat metric g for which (Σ, g) has unit area and the boundary curves are geodesic. We denote by $\mathcal{M}(\Sigma)$

the space of such constant curvature metrics and note that our definition of the flow will ensure that for an initial metric $g_0 \in \mathcal{M}(\Sigma)$, the evolving metrics always will be in $\mathcal{M}(\Sigma)$.

Remark 1.3.8. In contrast with the uniformisation of closed surfaces, there are many choices of constant curvature uniformisations available for compact surfaces with boundary. Two standard ones are described in [OPS88], and we shall make use of the type I choice from this paper, described above. We make this choice for convenience, so as to be able to use a doubling argument which allows us to easily make use of results from the Teichmüller harmonic map flow. However, choosing the type II uniformisation or indeed some other choice would be perfectly valid, just requiring more work to obtain the results in Section 3.3. For the scope of results we consider, namely excluding the possibility of the metric degenerating, it is not expected to make a substantive difference which uniformisation is chosen, however when studying solutions where the metric is degenerating, it is possible that there could be genuinely different behaviour. It would make for interesting future work to compare and contrast different uniformisations for such solutions.

We recall that in addition to the conformal invariance of the Dirichlet energy, there is also the invariance under the simultaneous pull-back of the map and metric by a diffeomorphism. To account also for this, as is necessary to obtain an equation with good properties, we follow the ideas of [RT16] and note that at any $g \in \mathcal{M}(\Sigma)$, the tangent space $T_g\mathcal{M}(\Sigma)$ splits $L^2(\Sigma, g)$-orthogonally as

$$T_g\mathcal{M}(\Sigma) = \{L_X g\} \oplus H(g). \tag{1.3.9}$$

Here, the space $\{L_X g\}$ is the infinite dimensional space of Lie-derivatives generated by vector fields that are parallel to $\partial\Sigma$ on $\partial\Sigma$, giving rise to 1-parameter families of diffeomorphisms of Σ, and $H(g)$ is the finite dimensional *horizontal space* which consists of all symmetric $(0,2)$-tensors h which are trace-free, divergence-free and which satisfy $h(\nu_g, \tau_g) = 0$ on $\partial\Sigma$, τ_g a unit tangent vector field along $(\partial\Sigma, \mathrm{d}s_g)$, c.f. [Tro92]. We introduce the notation P_g^H for the orthogonal projection of symmetric $(0,2)$-tensors onto $H(g)$.

We can compare this final condition defining $H(g)$ with the equation for half-harmonic maps to see that $k(u_g, g)(\nu_g, \tau_g) = \langle \partial_{\nu_g} u_g, \partial_{\tau_g} u_g \rangle = 0$ on $\partial\Sigma$ when u is half-harmonic. Hence, as k is also trace and divergence free when the map is harmonic, we have that $k(u_g, g) \in H(g)$ for *any* half-harmonic map. Consequently, we can conclude that if u is half-harmonic, then its harmonic extension u_g is conformal if and only if $P_g^H(k(u_g, g)) = 0$.

Finally, since any variation of the metric in a direction $L_X g$ corresponds to the action of a diffeomorphism, we can exploit the aforementioned invariance of the energy under simultaneous pullback of map and metric by a diffeomorphism to restrict the metric to flow in horizontal directions. Intuitively, this is helpful as it reduces an infinite dimensional flow to a finite dimensional one, and so we expect the behaviour to be closer to that of an ODE system, but we stress that the flow is still posed on the infinite dimensional space $\mathcal{M}(\Sigma)$.

The system we study is therefore the following.

$$\partial_t u = -\nabla_u^{L^2} E_{\frac{1}{2}}(u, g) = -P_u(\partial_{\nu_g} u_g) \tag{1.3.10}$$
$$\partial_t g = -P_g^H\big(\nabla_g^{L^2} E_{\frac{1}{2}}(u, g)\big) = \frac{1}{2} P_g^H(k(u_g, g)). \tag{1.3.11}$$

The evolution of the map component $u = u(t)$ is described by a variant of the Plateau flow studied by Struwe in [Str24], albeit now considered on a general surface Σ with a time-dependent metric $g(t)$ rather than on a disc with fixed metric. The evolution of the metric $g = g(t)$ on the other hand can be viewed as an evolution equation in the infinite dimensional manifold $\mathcal{M}(\Sigma)$ of constant curvature metrics with geodesic boundary curves described above. Note that whilst it is tempting in light of the symmetries present to consider (1.3.11) on the finite dimensional Teichmüller space, this is not possible since the map equation (1.3.10) depends on the metric itself, not just its Teichmüller class.[6]

Remark 1.3.9. We observe that in the case when $\Sigma = D$ we have $H(g) = \{0\}$ for any metric g on D, and so our coupled gradient flow (3.1.7) reduces to the Plateau flow for maps $u(t)$ on the disc with fixed metric, already considered in [Str24]. Accordingly, we

[6]When the domain is a cylinder, in fact we *can* exploit the integrability of the horizontal space to reduce the metric evolution to a single ODE. This is explained further in Section 3.3.3.

will focus our attention on the novel case where the metric evolution is non-trivial.

In Chapter 3, we provide the fundamental results for this new system, particularly obtaining the existence and regularity of solutions, establishing a basic analysis of the singularities that might arise, and proving the asymptotic convergence in the non-singular setting. Furthermore, as mentioned above, we were able to do this without making the restriction on the normal bundle assumed by Struwe in [Str24].

The first result that we obtain is the following, which establishes the existence of an almost smooth solution for as long as the metric remains non-degenerate.

Theorem 1.3.10. *Let $\Sigma \neq D$ be any orientable surface with boundary and $N \hookrightarrow \mathbb{R}^n$ any compact smooth embedded submanifold. Then to any initial metric $g_0 \in \mathcal{M}(\Sigma)$ and any initial map $u_0 \in H^{\frac{1}{2}}((\partial\Sigma, g_0); N)$, there exists a weak solution (u, g) of the coupled flow*

$$\partial_t u = -P_u(\partial_{\nu_g} u_g), \qquad \partial_t g = \frac{1}{2} P_g^H(k(u_g, g)) \tag{1.3.12}$$

which has non-increasing energy and is defined on a maximal interval $[0, T_\infty)$ where $T_\infty = \infty$ unless the domain metrics degenerate in finite time, i.e. unless

$$\mathrm{inj}(\Sigma, g(t)) \to 0 \text{ as } t \nearrow T_\infty \text{ for a finite } T_\infty.$$

Furthermore,

1. *Away from a finite number of singular times $0 < T_i^s < T_\infty$, both the map and metric component of the flow are smooth and the energy decays according to*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\frac{1}{2}}(u, g) = -\|P_u(\partial_{\nu_g} u_g)\|^2_{L^2(\partial\Sigma, g)} - \frac{1}{4}\|P_g^H(k(u_g, g))\|^2_{L^2(\Sigma, g)}. \tag{1.3.13}$$

2. *Across each T_i^s the flow of metrics remains regular in the sense that $g(t)$ is Lipschitz continuous in time with respect to any C^k-metric in space.*

3. *Any such singular time T_i^s is characterised by the* bubbling-off *of a finite number of minimal discs of the map component, exactly as in [Str24], see also Section 3.5.6.*

Here and in the following, we say that (u, g) is a weak solution of the flow to initial data (u_0, g_0) on an interval $[0, T)$ if:

1. The metric component is a continuous curve $g : [0, T) \to (\mathcal{M}^3(\Sigma), \text{dist}_{H^3})$ with $g(0) = g_0$ which is differentiable at a.e. $t \in [0, T)$ and so that the second equation in (1.3.12) is satisfied at a.e. such t. Here $\mathcal{M}^3$ denotes the set of constant curvature metrics with coefficients in H^3 as discussed in Section 3.2 below.

2. The map component is given by a $u \in L^\infty([0, T); H^{\frac{1}{2}}((\partial\Sigma, \mathrm{d}s_g); N)$ which is so that $\partial_t u \in L^2_{loc}([0, T); L^2(\partial\Sigma, g))$ and so that the first equation of (1.3.12) is satisfied in the sense of distributions.

The second theorem we obtain provides a result on the asymptotic behaviour of the flow in the non-singular case.

Theorem 1.3.11. *Suppose that (u, g) is a global weak solution of* (1.3.12) *for which* $\text{inj}(\Sigma, g)$ *remains bounded away from zero and for which energy does not concentrate as $t \to \infty$, i.e.* $\limsup_{t\to\infty} \sup_{x\in\partial\Sigma} E(u_g(t), g(t); B_r^{g(t)}(x)) \to 0$ *as $r \to 0$.*

Then there exists a sequence $t_j \to \infty$ so that, after pull-back by diffeomorphisms, the pairs $(u(t_j), g(t_j))$ converge smoothly to a limiting pair $(u^*, g^*) \in C^\infty(\partial\Sigma; N) \times \mathcal{M}(\Sigma)$ which is so that $u^*_{g^*} : (\Sigma, g^*) \to \mathbb{R}^n$ is conformal and harmonic, and so that $u^*_{g^*}(\Sigma)$ meets N orthogonally; that is, $u^*_{g^*}(\Sigma)$ represents a (possibly branched) free boundary minimal surface supported by N.[7]

This work opens up many new avenues of study. Certainly, there is a lot still to learn about the singularity formation in this flow. It is not known if finite time singularities in the map are possible: recall that Sire, Wei and Zheng conjectured that for domain $\mathbb{R}$ and target $\mathcal{S}^1$ that no finite time singularities are possible, but even if this is true, it is not clear that the flow with general domains and targets should behave in the same way. Furthermore, when the domain is not $\mathcal{S}^1$, there is a whole new set of singularities that

[7]Recall that our notion of free boundary minimal surface is coming from Theorem 1.2.1.

can form from the metric degenerating. This can occur either from an interior geodesic collapsing, for which we would expect the behaviour to be reminiscent of the metric singularities in Teichmüller harmonic map flow[8], or from a boundary curve collapsing, which is a new feature of this flow. The case of a cylinder is substantially simpler as the metric can only degenerate at infinite time and is governed by a single ODE. It also remains to undertake a more refined analysis of map singularities even in the case of $\mathcal{S}^1$, and so to investigate if the no loss of energy and no neck properties for the harmonic map flow also hold in the half-harmonic case. In fact, it is this gap which currently stands in the way of obtaining improved results regarding the asymptotic convergence of the flow when the metric remains non-degenerate but the map forms at least one bubble.

1.3.3 Analysis of Metric Singularities in the Teichmüller Harmonic Map Flow

In the final chapter of my thesis, I will present some results concerning the asymptotic behaviour of the Teichmüller harmonic map flow, in particular investigating the fine structure of the limit in the case where the metric degenerates at infinite time.

To provide context for my work, I first recall, informally for now (see Chapter 4 for the detailed statements) what is already known about the asymptotic convergence of this flow. In particular, in the paper [RTZ13], the authors establish that away from the collapsing geodesics and along a sequence of times, the solution (u, g) converges, perhaps with bubbling, to a conformal harmonic map on a new, possibly disconnected, punctured hyperbolic surface, with the punctures arising from the collapsing geodesics. Subsequently, the paper [HRT16] studied the behaviour in a neighbourhood of the collapsing geodesics. Specifically, they look at following the *collar neighbourhoods*. We recall that by the Keen-Randol collar lemma, C.0.1, surrounding each closed geodesic $\sigma \subseteq \Sigma$ of length ℓ, there is a neighbourhood which is isometric to the cylinder $C_\ell = [-X(\ell), X(\ell)] \times \mathcal{S}^1$ with metric

[8] For Teichmüller harmonic map flow, if the target manifold does not admit any bubbles then no finite time metric singularities can occur, see [RT18]. Since this is true for $\mathbb{R}^n$, it is plausible that the same holds for our new flow.

$g_\ell = \rho_\ell(s)^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)$, where

$$X(\ell) := \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\frac{\ell}{2}\right)\right) \tag{1.3.14}$$

$$\rho_\ell(s) := \frac{\ell}{2\pi\cos\left(\frac{\ell s}{2\pi}\right)} \tag{1.3.15}$$

A key observation is that as $\ell \searrow 0$, then $X(\ell) \nearrow \infty$. In [HRT16], they show that the along a time sequence, the restrictions of the maps $u(t_i)$ to the degenerating collar neighbourhood converge to a *full bubble branch*, see Definition 4.1.2, which essentially means that $u(t_i)$ converges to a sequence of bubbles connected by curves.

What remains open from these results is the nature of the connecting curves. In [HRT16], the authors ask the following.

Question 1.3.12. *For a solution of the Teichmüller harmonic map flow where the metric degenerates at infinite time, are the connecting curves converging to geodesics in (N, h)?*

There is one, and only one, setting where this is known to be true, which is the result of Ding, Li and Liu from [DLL06], where they show that if the domain is a torus, the energy converges to zero and $\liminf_{t\to\infty} \operatorname{inj} g(t) = 0$, then along a sequence of times $t_i \to \infty$ the map converges to either a point or a closed geodesic. The aim of this part of my thesis is to investigate these connecting curves in the higher genus setting.

The key result which guides the approaches I follow below is a result of Rupflin which provides a sharp sufficient condition on the decay of the tension which ensures the convergence to a geodesic.

Theorem 1.3.13 (Theorem 1.3, [Rup22]). *Let $\ell_i \to 0$ and $u_i : \mathcal{C}_{\ell_i} \to N$ be a sequence of maps from hyperbolic cylinders $(\mathcal{C}_{\ell_i}, g_{\ell_i})$ which have uniformly bounded energy, satisfy*

$$\ell_i^{-\frac{1}{2}}\|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}_{\ell_i}, g_{\ell_i})} \to 0$$

and which converge to a full bubble branch as in Definition 4.1.2, with associated sequences s_i^k. Then there exist sequences a_i^k and b_i^k, which split the cylinder into extended bubble regions $[a_i^k, b_i^k] \times \mathcal{S}^1$ and connecting cylinders $[b_i^k, a_i^{k+1}] \times \mathcal{S}^1$, such that $s_i^{k-1} < b_i^{k-1} \leqslant a_i^k <$

s_i^k and both $b_i^k - s_i^k \to \infty$ and $s_i^k - a_i^k \to \infty$ as $i \to \infty$, and so that

1. *No energy is lost on the extended bubble regions.*
2. *No necks form in the extended bubble regions.*
3. *The images of the connecting cylinders subconverge to geodesics in the following sense. The curves $v_i^k : [-c_i^k, c_i^k] \to N$, which are the maps*
$$s \mapsto \Pi\left(\frac{1}{2\pi}\int_{\{s\}\times\mathcal{S}^1} u_i(s,\theta)\mathrm{d}\theta\right)$$
reparametrised by arc length, where Π is the nearest point projection of $\mathbb{R}^n$ onto N, satisfy
$$\|\tau(v_i^k)\|_{L^p([-c_i^k,c_i^k])} \to 0$$
for each $p \in [1,2]$. Hence on passing to a subsequence, v_i^k converges to a geodesic locally in H^2, either of trivial length, finite length or infinite length.

This result translates answering Question 1.3.12 into proving that along some sequence of times, the tension is decaying sufficiently fast with respect to the rate of decay of the metric. The two results I present below provide some instances where the necessary tension decay estimate can be achieved for two *close variants* of the original flow. In particular, I will replace the constant coupling factor η with a variable factor which depends on the injectivity radius. Note that whilst this has not specifically been studied before, Huxol in [Hux17] studied the limits of solutions of the flow where the coupling constant is taken to 0. Furthermore, we note that the short time existence, uniqueness and regularity theory of this adjusted flow goes through without major alterations. It is only in the asymptotic behaviour that qualitatively different behaviour is potentially present, and we are making these changes precisely to try and arrange the correct asymptotics of the solutions.

In the nearly twenty years since Ding, Li and Liu's result, there has been no further progress even in the simplified setting of the torus, and so instead of trying to treat the full flow equation, I first considered a model problem. This aims to capture the key features of the original flow, whilst making a number of helpful simplifications and adjustments.

First of all, since it is the behaviour on the collar neighbourhood that we are interested in, I switch to considering the Teichmüller harmonic map flow on cylinders introduced by Rupflin in [Rup17]. This works on a cylinder

$$\mathcal{C}_\ell := [-Y(\ell), Y(\ell)] \times \mathcal{S}^1 \tag{1.3.16}$$

where for $\ell > 0$,

$$Y(\ell) := \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\frac{\ell}{2}\right)\right). \tag{1.3.17}$$

Let (s, θ) be the coordinates on $\mathcal{C}_\ell$, then we equip $\mathcal{C}_\ell$ with the metric

$$g_\ell := \rho_\ell(s)^2\left(\mathrm{d}s^2 + \mathrm{d}\theta^2\right) \tag{1.3.18}$$

where the conformal factor ρ_ℓ is defined by

$$\rho_\ell(s) := \frac{\ell}{2\pi\cos\left(\frac{\ell s}{2\pi}\right)}. \tag{1.3.19}$$

One point to notice is that the length of this cylinder is slightly different to the one guaranteed by the collar lemma, compare $Y(\ell)$ to $X(\ell)$ defined in (1.3.14), but these are very similar for small ℓ. The reason for this choice of length is that it allows us to reduce the metric equation to a single ODE for the length of the central geodesic, ℓ, see [Rup17] for further details of this. The equations are then

$$\partial_t u = \tau_{g_\ell}(u) \tag{1.3.20}$$

$$\partial_t \ell = -\frac{\ell}{4\pi}\eta^2\lambda \tag{1.3.21}$$

where λ is the quantity

$$\lambda := 8\pi^3\ell^{-2}\|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)}\int_{\mathcal{C}_\ell}\left(|u_s|^2 - |u_\theta|^2\right)\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta. \tag{1.3.22}$$

Note that the equation here for $\frac{\mathrm{d}\ell}{\mathrm{d}t}$ agrees to leading order with the expression for the true evolution of the length of a closed geodesic under Teichmüller harmonic map flow on a hyperbolic surface computed in [RT18]. We will use Dirichlet boundary conditions for the map component in our analysis.

We also assume now that the initial map u does not depend on the angular variable θ, i.e. that the map is already starting out as a curve. On first glance, this may seem quite a restrictive assumption, however firstly we know we are converging to a one-dimensional limit and we are mostly interested in the properties of this, not in how the second dimension is collapsing and moreover there are strong estimates which show the exponential decay of the energy in the angular direction under the assumption that energy is not concentrated, see for example Lemma 3.1 in [RT18], and so the contribution to the dynamics in the θ-direction are not expected to affect the dynamics in the s-direction to leading order. Hence, in this first investigation to understand the leading order dynamics, we restrict to this θ-independent setting. Note further that by removing the θ-dependence, we have also prevented any bubble formation.

The final point of departure from the usual Teichmüller harmonic map flow is that we exploit the flexibility in the choice of inner product we use to define the gradient flow to allow the coupling constant η to depend on the parameter ℓ, which more geometrically we can think of as the injectivity radius of the metric. We do this in a way that $\eta \searrow 0$ as $\ell \searrow 0$, effectively slowing down the metric degeneration compared with the standard flow. Intuitively, this adjusts the balance between the decay of the tension and the degeneration of the metric, and so in light of Theorem 4.1.5 it is reasonable to expect that the asymptotic solution of this flow more easily converges to a geodesic.

The result I obtain about this model problem is the following.

Proposition 1.3.14. *Let* $0 < \underline{L} < \overline{L}$ *and* $\beta > \frac{1}{2}$ *be given constants. Then there exists a constant* $0 < \ell_1 < \operatorname{arsinh}(1)$*, depending only on* $\underline{L}$*,* $\overline{L}$ *and* β*, and a universal constant* $M_0 > 0$ *satisfying the following. Let* (u, ℓ) *be any* θ*-independent smooth solution to the flow* (4.2.6),(4.2.7) *on* $[0, \infty)$ *with coupling factor* $\eta(\ell) = |\log(\ell)|^{-\beta}$ *and so that the length of the curve* $u(t)$ *satisfies* $\underline{L} \leqslant L(u(t)) \leqslant \overline{L}$ *for all time. Suppose additionally that the initial data satisfies*

1. $\ell(0) \leqslant \ell_1$

2. $E(u(0), \ell(0)) \leqslant M_0 \left|\log(\ell(0))\right|^{2\beta-1} \ell(0)$.

Then there exists a subsequence of times $t_n \to \infty$ along which the map u converges to a geodesic.

Let us make a couple of remarks about this result. First, since this result concerns sufficient conditions on the initial data, it is important to ask if these are ever satisfied in a non-trivial way. To begin, the condition that the length is bounded below by some $\underline{L}$ can be easily satisfied by taking $\underline{L}$ to be the length of the shortest geodesic in N joining the endpoints of u, since we impose Dirichlet boundary conditions. The upper bound $\overline{L}$ is not easy to explicitly impose, but it is still reasonable for us to first restrict our attention to convergence to finite length geodesics in this initial result. The condition that the width of the collar be sufficiently small is quite natural since we are interested in solutions for which this goes to zero anyway. The final condition, that the energy be sufficiently small in terms of the parameter ℓ, is more artificial. It is coming out of particular estimates that we make, in particular in equations (4.2.23) and (4.2.31). Having said that, we will show in a key step of the proof that this condition is in fact preserved under the flow, and that whilst we *cannot* at present ensure that any solution will eventually satisfy the condition (the energy could be decaying too slowly compared with the parameter ℓ), for any given map $u(0)$, by reducing ℓ sufficiently and linearly rescaling $u(0)$ to fit the new larger cylinder, it is possible to satisfy the bound $E(u(0), l(0)) \leqslant M_0 \left|\log(\ell(0))\right|^{2\beta-1} \ell(0)$, by noting that energy scales with ℓ, yet the factor $M_0 \left|\log(\ell(0))\right|^{2\beta-1}$ blows up as $\ell \searrow 0$. Hence, it is easy to find initial data for which the theorem applies, and moreover it applies to *any* geometric curve in N, just needing perhaps re-parametrising over a longer cylinder. Finally, the upper bound arsinh(1) for ℓ_1 could be replaced by any number less than 1 for this result, but the choice of arsinh(1) is geometrically significant as the collar neighbourhoods of closed geodesics of length less than arsinh(1) on a hyperbolic surface are disjoint, see [RTZ13, Appendix A].

My second investigation of the question of the convergence to a geodesic limit of the Teichmüller harmonic map flow took a different approach. Instead of considering directly the tension on the whole cylinder, which stood in for the degenerating collar, I instead worked with a weighted tension quantity defined on the true collar neighbourhood C_ℓ of a degenerating geodesic in a hyperbolic surface (Σ, g). Essentially, this approach splits the collar into a central region and an outer region, and studies the behaviour in the central one. The quantity in question is

$$\mathcal{T}_w(u,\ell)^2 := \int_{C_\ell} |\tau_{g_\ell}(u)|^2 \rho_\ell(s)^{-2}\varphi(\rho_\ell(s))^2 \mathrm{d}v_{g_\ell} \tag{1.3.23}$$

where $\varphi \in C_c^\infty([0,\frac{1}{\pi});[0,1])$ is a cut-off function satisfying $\varphi \equiv 1$ on $[0,\frac{1}{2\pi}]$ and $|\varphi'| \leqslant 4\pi$.

To understand this quantity better, fix any $\Lambda > 1$ and consider the region

$$C_\ell^\Lambda := \left[-\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right), \frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)\right] \subseteq [-X(\ell), X(\ell)] \tag{1.3.24}$$

for small enough ℓ, noting that $\rho_\ell(\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)) = \Lambda\ell$. Then on C_ℓ^Λ, $\Lambda^{-2}\ell^{-2} \leqslant \rho_\ell(s)^{-2} \leqslant \ell^{-2}$ and so control of $\mathcal{T}_w(u,\ell)$ implies control of $\ell^{-2}\mathcal{T}(u,\ell)$ on C_ℓ^Λ. Note then that

$$\frac{\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)}{X(\ell)} \to \frac{2}{\pi}\arccos\left(\frac{1}{2\pi\Lambda}\right) \text{ as } \ell \searrow 0 \tag{1.3.25}$$

and

$$\frac{2}{\pi}\arccos\left(\frac{1}{2\pi\Lambda}\right) \to 1 \text{ as } \Lambda \nearrow \infty \tag{1.3.26}$$

so the central region C_ℓ^Λ covers a fixed portion of the collar in the limit as $\ell \searrow 0$, and that this proportion can be taken arbitrarily close to 1.

This weighted quantity $\mathcal{T}_w$ was introduced in [RT18], which studied the behaviour of Teichmüller harmonic map flow when the target manifold has non-positive sectional curvature, or more generally admits no bubbles. I will also be considering this setting so as to make use of the estimates derived there.

As above, I will also be assuming that the coupling factor η depends on the injectivity radius in a particular way. This time, it is assumed that $\eta \nearrow \infty$ as $\ell \searrow 0$, though not too fast. Specifically, I assume

$$\eta \to \infty \text{ as } \ell \to 0 \tag{1.3.27}$$

$$\eta^2\ell \to 0 \text{ as } \ell \to 0. \tag{1.3.28}$$

Interestingly, this contrasts with the previous result in that I am speeding up the degeneration of the metric which intuitively makes it harder for the map to reach a geodesic.

In this setting, I obtained the following theorem.

Proposition 1.3.15. *Suppose* $(u(t), g(t))$ *is a smooth solution to the flow* (1.2.1),(1.2.2) *on* $[0, \infty)$ *with domain* Σ *being a closed surface with genus at least* 2 *and target* (N, h) *not admitting any bubbles. Let* $\mathcal{C}$ *be a collar on which* $0 < \ell < 2\,\mathrm{arsinh}(1)$ *for all time and* $\ell \to 0$ *as* $t \to \infty$. *Suppose further that* η *satisfies* (1.3.27) *and* (1.3.28). *Then there exists a sequence of times* $t_j \to \infty$ *along which* $\mathcal{T}_w^2\ell(t_j) \to 0$.

Consequently, for any fixed $\Lambda > 1$, $\mathcal{T}^2\ell^{-1}(t_j) \to 0$ *on* C_ℓ^Λ *and so the curve restricted to the central region* C_ℓ^Λ *sub-converges to a geodesic.*

This result gives a positive answer to the question of convergence to a geodesic for this slightly modified flow on a large part of the collar neighbourhood. It remains to be seen what can be said about the more delicate in between region where the collar has width of order strictly between ℓ and 1.

Chapter 2

Plateau Flow: Results for Domain $\mathcal{S}^1$

In this chapter, I will present two results about the Plateau flow in the case where the domain is $\mathcal{S}^1$. In particular, I give answers to two questions raised by Struwe in [Str24]. The first and more substantial question relates to the uniqueness of solutions, and is based on the published paper [Wri24]. The second smaller result, presented in Section 2.5, relates to the application of this flow to the Plateau problem.

2.1 Uniqueness of Solutions of the Plateau Flow

In this section, I consider the uniqueness of solutions to a gradient flow of maps from the unit circle $\mathcal{S}^1$ into a smoothly embedded closed submanifold $N \hookrightarrow \mathbb{R}^n$. This flow was introduced and studied by Wettstein in [Wet22, Wet21, Wet23] as the equation

$$\partial_t u + P_u \left((-\Delta)^{\frac{1}{2}} u \right) = 0 \text{ on } [0, T) \times \mathcal{S}^1 \tag{2.1.1}$$

where P_x is the orthogonal projection of $\mathbb{R}^n$ onto the tangent space of N at x. This is the L^2-gradient flow of the *half-energy*

$$E_{\frac{1}{2}}(u) := \frac{1}{2} \int_{\mathcal{S}^1} \left| (-\Delta)^{\frac{1}{4}} u \right|^2 \mathrm{d}s \tag{2.1.2}$$

within the function class $u \in H^{\frac{1}{2}}(\mathcal{S}^1; N)$. Critical points of this energy are called *half-harmonic maps*, and are characterised as solutions to

$$P_u\left((-\Delta)^{\frac{1}{2}}u\right) = 0 \text{ on } \mathcal{S}^1. \tag{2.1.3}$$

The half-energy and half-harmonic maps were first introduced by Da Lio and Rivière in [LR11], and a special case of the gradient flow is studied by Sire, Wei and Zheng in [SWZ21]. The geometric motivation behind studying half-harmonic maps comes from their connection with free boundary minimal surfaces. It is observed in [MS15] and in [LMR15] that if $u : \partial\mathbb{D} \to N \subseteq \mathbb{R}^n$ is a non-constant half-harmonic map, where $\mathbb{D} \subseteq \mathbb{R}^2$ is the closed unit disc, then the harmonic extension $u : \mathbb{D} \to \mathbb{R}^n$ parametrises a free boundary branched minimal immersion.

In [Wet21, Wet23], Wettstein established the existence of a weak solution with regularity $u \in H^1_{\text{loc}}([0,\infty); L^2(\mathcal{S}^1; N)) \cap L^\infty([0,\infty); H^{\frac{1}{2}}(\mathcal{S}^1; N))$ to (2.1.1) for two cases. First, in the case of an arbitrary target N, existence of weak solutions is established for initial data with small energy. Second, if the target manifold is a sphere, then existence of a weak solution is established for arbitrary initial data. Moreover, these solutions have non-increasing energy and are smooth away from finitely many singular times. Wettstein also obtained a uniqueness result in [Wet21], but this applies only to solutions with small energy throughout, leaving open the question of uniqueness of general weak solutions.

In [Str24], Struwe studied this geometric flow using very different techniques. Whereas Wettstein used the methods of fractional calculus, Struwe used the alternative characterisation of the half-Laplacian as

$$(-\Delta)^{\frac{1}{2}}u = \partial_\nu u \tag{2.1.4}$$

where ∂_ν is the Dirichlet-to-Neumann operator associated to the Laplacian on $\mathbb{D}$. This has previously been used for example by Millot and Sire in [MS15] and Moser in [Mos11] and is a special case of a more general theory of representing fractional operators, developed by Caffarelli and Silvestre in [CS06]. The relation (2.1.4) transforms the gradient flow to the following,

$$\partial_t u + P_u(\partial_\nu u) = 0 \text{ on } [0,T) \times \partial\mathbb{D} \tag{2.1.5}$$

which is what Struwe defined as the Plateau flow. One advantage of this approach is that it allows for the localisation of many of the arguments by studying the harmonic extension of a map defined on $\partial\mathbb{D}$ to all of $\mathbb{D}$. For this reason, we do not distinguish in notation between a function defined on $\partial\mathbb{D}$ and its harmonic extension defined on $\mathbb{D}$. This also allows the half-energy to be replaced by the better understood Dirichlet energy

$$E(u) := \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2 \, \mathrm{d}x \tag{2.1.6}$$

where equation (1.5) in [Str24] shows that this is equal to the half energy in this setting. We refer to [Str24] for further details behind the flow and the motivation for using Struwe's approach. One detail we note is that Struwe assumed that N is embedded into $\mathbb{R}^n$ in such a way that the normal bundle is parallelisable. Struwe made this assumption so as to make use of a globally defined function $\mathrm{dist}_N : \mathbb{R}^n \to \mathbb{R}^m$ (defined in Section 1.9 of [Str24]). This assumption is satisfied in particular for hypersurfaces and curves in $\mathbb{R}^n$. Since we build upon Struwe's results, we also make this restriction.

Struwe obtained the existence of weak solutions to (2.1.5) with non-increasing energy for arbitrary initial data $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$. Moreover, Struwe gave a more detailed description of the regularity at singular times, ensuring that his solution is smooth for $t > 0$ except at finitely many spacetime points. Further, Struwe obtained a uniqueness result for arbitrary initial data, however with extra regularity of the flow assumed. To state this, let us recall the following definitions of solution of the equation.

Definition 2.1.1. A function $u : [0,T) \times \partial\mathbb{D} \to N$ is called a *weak solution* to the Plateau flow (2.1.5) on the time interval $[0,T)$ and with initial data $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$ if it satisfies

1. $u \in H^1_{\mathrm{loc}}([0,T); L^2(\partial\mathbb{D};\mathbb{R}^n)) \cap L^\infty_{\mathrm{loc}}([0,T); H^{\frac{1}{2}}(\partial\mathbb{D}; N))$.

2. u solves the equation weakly, in the sense that for all $\varphi \in L^2((0,T); H^1(\mathbb{D};\mathbb{R}^n))$,

$$\int_0^T \int_{\partial\mathbb{D}} \partial_t u \cdot \varphi \mathrm{d}s\mathrm{d}t + \int_0^T \int_{\mathbb{D}} \langle \nabla u, \nabla(P_u(\varphi))\rangle \mathrm{d}x\mathrm{d}t = 0.$$

3. The trace of u on $\{0\} \times \partial\mathbb{D}$ equals u_0.

A weak solution which has the additional regularity $u \in H^1_{\mathrm{loc}}([0,T) \times \partial\mathbb{D}; N)$ and satisfies the energy inequality

$$E_{\frac{1}{2}}(u(t_2)) + \int_{t_1}^{t_2} \int_{\partial\mathbb{D}} |\partial_t u|^2 \,\mathrm{d}s\mathrm{d}t \leqslant E_{\frac{1}{2}}(u(t_1))$$

for each $t_1 < t_2 < T$ is called an *energy-class solution*.

In his paper, Struwe states the following result for solutions which are smooth for positive time.

Theorem 2.1.2 ([Str24, Theorem 7.1])**.** *Let u, v be energy class solutions to the Plateau Flow* (2.1.5) *on the time interval* $[0,T)$*,* $T \leqslant \infty$*, with the same initial data* $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$*. Suppose that additionally u and v are smooth for all $t \in (0,T)$. Then $u = v$.*

We will see later in Proposition 2.2.3 that the same proof works under weaker regularity assumptions.

From this local uniqueness result, combined with the fact that at a singular time T_s, the weak $H^{\frac{1}{2}}(\partial\mathbb{D})$ limit as $t \nearrow T_s$ is unique (i.e. independent of choice to time sequence $t_n \nearrow T_s$), Struwe deduces a global uniqueness result which can be stated in the following way. Let $u, v : [0,\infty) \times \partial\mathbb{D} \to N$ be weak solutions to the flow which have the same initial data and are such that there exist a finite number of times $0 = T_0 < T_1 < \cdots < T_k = \infty$ such that for each $i = 1, \ldots, k$, u and v are energy class solutions on $[T_{i-1}, T_i)$ which are moreover smooth on (T_{i-1}, T_i). Then $u = v$.

Remark 2.1.3. If at a time T_s a weak solution of the flow is forming a bubble as described in Section 8 of [Str24], see also Theorem 2.3.1, then $u \notin H^1([0,T_s] \times \partial\mathbb{D}; N)$, which is why it is necessary to consider the solution on the intervals $[T_{i-1}, T_i)$. To see this, note that the regularity $H^1([0,T_s] \times \partial\mathbb{D}; N)$ is sufficient to ensure the energy identity

$$E(u(t_2)) - E(u(t_1)) = \int_{t_1}^{t_2} \int_{\partial\mathbb{D}} |\partial_t u|^2 \,\mathrm{d}s\mathrm{d}t$$

and thus the convergence $E(u(t)) \to E(u(T_s))$ as $t \nearrow T_s$. Hence, no energy is lost at the limit T_s and so no bubbling can have occurred as $t \nearrow T_s$. Similarly, if there exists a *reverse bubble* solution similar to Topping's construction for harmonic map flow in [Top02], say occurring at T_s, then necessarily $u \notin H^1([T_s, T_s + \varepsilon] \times \partial\mathbb{D}; N)$ for any $\varepsilon > 0$.

Following this, and by comparison with the theory of harmonic map flow, it is natural to consider if we can obtain uniqueness of solutions within a more general class of functions. Indeed, Struwe directly raised the question of whether uniqueness holds in the larger space of weak solutions with non-increasing energy, and so in particular removing the regularity requirements imposed by being an energy class solution. We can answer this question positively in the following theorem, which in fact also allows for small increases in energy. As in the thesis introduction, we call the solution obtained by Wettstein/Struwe the *almost smooth solution* associated to a particular initial map and define the quantity

$$\varepsilon^* := \inf\{E_{\frac{1}{2}}(u) : u : \partial\mathbb{D} \to N \text{ non-constant and solves } P_u(\partial_\nu u_{\mathbb{D}}) = 0\}. \tag{2.1.7}$$

Theorem 2.1.4. *Let $N \hookrightarrow \mathbb{R}^n$ be a smooth manifold which is smoothly embedded into $\mathbb{R}^n$ with parallelisable normal bundle. There exists $\varepsilon_0 \in (0, \varepsilon^*]$, depending only on N, such that if u is a weak solution to the Plateau flow* (2.1.5) *defined on the time interval $[0, T)$, $T \leqslant \infty$, satisfying*

$$\limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) < E_{\frac{1}{2}}(u(t)) + \varepsilon_0 \tag{2.1.8}$$

for all $t \in [0, T)$, then u is equal to the almost smooth solution with the same initial data.

This result provides a parallel with the theory of harmonic map flow. In that setting, Struwe constructed a global weak solution in [Str85] with a uniqueness result similar to Theorem 2.1.2, in that uniqueness is obtained for solutions under additional regularity assumptions. However, it was subsequently shown by Freire in [Fre95] that uniqueness holds in a weaker class of solutions subject only to the condition that the energy is non-increasing. It is perhaps surprising that energy monotonicity needs to be assumed at all for a gradient flow, but the existence of backwards bubbles due to Topping, [Top02], shows that energy monotonicity can fail if we look in the class of weak solutions with no additional assumptions. Following this, Rupflin strengthened Freire's result in [Rup08] to show that uniqueness holds even when we allow for small increases in energy, with the smallness needed to exclude backwards bubbles. We adapt the methods of Rupflin for our proofs below, and note that our results provide analogues of Theorems 1.1 and 1.2 from [Rup08] in the Plateau flow setting.

Our Theorem 2.1.4 is enough to cover the case of weak solutions with non-increasing energy, but since we do not have any lower bound on the value of ε_0, we cannot use this result to conclude that non-uniqueness must be caused by backwards bubbling. To get closer to this, we prove the following slightly strengthened theorem. Note that since $\varepsilon^* \geqslant \varepsilon_0$, the assumptions are indeed weaker, than in Theorem 2.1.4, or at worst equivalent.

Theorem 2.1.5. *Let $N \hookrightarrow \mathbb{R}^n$ be a smooth manifold which is smoothly embedded into $\mathbb{R}^n$ with parallelisable normal bundle. Suppose u is a weak solution to the Plateau flow* (2.1.5) *on the time interval $[0,T)$, $T \leqslant \infty$, satisfying the two conditions*

$$\limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) < E_{\frac{1}{2}}(u(t)) + \varepsilon^* \tag{2.1.9}$$

for all $t \in [0,T)$ and that the set

$$S := \{t \in [0,T) : \limsup_{s \searrow t} E_{\frac{1}{2}}(u(s)) \geqslant E_{\frac{1}{2}}(u(t)) + \varepsilon_0\} \tag{2.1.10}$$

has no accumulation points, where ε_0 is the constant from Theorem 2.1.4. Then u is equal to the almost smooth solution with the same initial data.

Note that the second condition (2.1.10) is satisfied in particular if $t \mapsto E_{\frac{1}{2}}(t)$ has locally finite total variation.

2.2 Proof of Theorem 2.1.4

In this section I present the first improved uniqueness result for solutions of the Plateau flow.

It will be important for later arguments that we can study weak solutions at fixed times, and so we give the following elementary lemma.

Lemma 2.2.1. *Let u be a weak solution to the Plateau flow (2.1.5) on $[0,T]$. Then for almost all $t \in [0,T]$, $u(t) \in H^{\frac{1}{2}}(\partial\mathbb{D};N)$ solves the stationary equation, namely for all $\varphi_0 \in H^1(\mathbb{D};\mathbb{R}^n)$*

$$\int_{\partial\mathbb{D}} \partial_t u \cdot \varphi_0 \mathrm{d}s + \int_{\mathbb{D}} \nabla u \cdot \nabla(P_u^{TN}(\varphi_0))\mathrm{d}x = 0. \tag{2.2.1}$$

Proof. Let $\{\varphi_k : k \in \mathbb{N}\}$ be a countable dense subset of $H^1(\mathbb{D};N)$. Define the function $f_k \in L^1([0,T])$ by

$$f_k(t) = \int_{\partial\mathbb{D}} \partial_t u \cdot \varphi_k \mathrm{d}s + \int_{\mathbb{D}} \nabla u \cdot \nabla(P_u^{TN}(\varphi_k))\mathrm{d}x.$$

Then we use the test function $\varphi_{k,a,b} \in L^2([0,T];H^1(\mathbb{D};\mathbb{R}^n))$, defined by

$$\varphi_{k,a,b}(x,t) = \begin{cases} \varphi_k(x) & t \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

to see that for all $0 \leqslant a < b \leqslant T$

$$\int_a^b f_k(t)\mathrm{d}t = 0$$

and hence $f_k = 0$ except on a null set $N_k \subseteq [0,T]$. Therefore on the set $[0,T] \setminus \bigcup_{k\in\mathbb{N}} N_k$, (2.2.1) is satisfied with $\varphi_0 = \varphi_k$ for each k. The conclusion follows be density of the φ_k in $H^1(\mathbb{D};N)$. □

We additionally recall the following standard result on Sobolev spaces, see for example the book of Evans [Eva21].

Lemma 2.2.2 (Theorem 2, Section 5.9.2 in [Eva21])**.** *Let $u \in H^1([0,T];X)$ where $(X, \|\cdot\|)$ is a Banach space. Then for all $0 \leqslant s < t \leqslant T$,*

$$u(t) = u(s) + \int_s^t \partial_\tau u(\tau)\mathrm{d}\tau$$

and so $t \mapsto u(t)$ is continuous as a map $[0,T] \to X$.

In particular, this applies to weak solutions of the Plateau flow with $X = L^2(\partial\mathbb{D})$.

The proof of Theorem 2.1.4 now comes in three steps. The first is to go through Struwe's proof of Theorem 2.1.2 and weaken the regularity assumptions obtaining Proposition 2.2.3. Secondly, we show that under the conditions of Theorem 2.1.4, there is a limit on how much energy can concentrate over short time intervals. Finally, using some regularity theory of the inhomogeneous half-harmonic equation, we prove that weak solutions satisfying (2.1.8) meet the regularity requirements of Proposition 2.2.3.

2.2.1 Struwe's Uniqueness Proof

In order to give Struwe's original proof, we need to recall the definition of the function dist_N from [Str24]. To define dist_N, we first take $\nu_1, ..., \nu_m$ to be an orthonormal frame of the normal bundle defined on N. We then use that $T : (p, y_1, ..., y_m) \mapsto p + \sum_{i=1}^m y_i \nu_i(p)$ is a diffeomorphism from $N \times B_\rho(0, \mathbb{R}^m)$ to a tubular neighbourhood of N for sufficiently small ρ. Letting π be the nearest point projection onto N, we can define $h_i(q) := \nu_i(q) \cdot (q - \pi(q))$. Finally, fixing a smooth cut-off function $\eta : \mathbb{R} \to [0, 1]$ which is 1 for $|s| \leqslant \frac{1}{2}\rho$ and 0 for $|s| > \frac{3}{4}\rho$, we set

$$\mathrm{dist}_N^i(q) := \eta(h_i(q)) \tag{2.2.2}$$

and

$$\mathrm{dist}_N(q) := (\mathrm{dist}_n^1(q), ..., \mathrm{dist}_N^m(q)). \tag{2.2.3}$$

We moreover extend the vector fields ν_i to all of $\mathbb{R}^n$ by setting $\nu_i := \nabla \mathrm{dist}_N^i$. The function dist_N should be thought of as measuring the signed distance of a point in $\mathbb{R}^n$ from N which has been smoothly cut-off so as to be defined on all of $\mathbb{R}^n$ and have compact support.

Now we summarise Struwe's original proof of Theorem 2.1.2, after which we will explain the very minor adjustments needed to prove Proposition 2.2.3.

Proof of Theorem 2.1.2, [Str24]. Let $w = u - v$. Then on $\partial\mathbb{D}$ and for $t > 0$, we have the following, which is equation (7.1) in [Str24]

$$\partial_\nu w + \partial_t w = \nu(u)\partial_\nu(\mathrm{dist}_N(u)) - \nu(v)\partial_\nu(\mathrm{dist}_N(v)) \tag{2.2.4}$$

$$= (\nu(u) - \nu(v))\partial_\nu(\mathrm{dist}_N(u)) + \nu(v)\partial_\nu(\mathrm{dist}_N(u) - \mathrm{dist}_N(v))$$

where we have written $\nu(u)\partial_\nu \mathrm{dist}_N(u)$ as an abbreviation for

$$\sum_{i=1}^{m} \nu_i(u)\nu_i(u) \cdot \partial_\nu u = \partial_\nu u - P_u^{TN}(\partial_\nu u)$$

Testing the left hand side of this against w and integrating by parts yields

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}}{\mathrm{d}t} |w|^2 \,\mathrm{d}s + 2\int_{\mathbb{D}} |\nabla w|^2 \,\mathrm{d}x = 2\int_{\partial\mathbb{D}} (\partial_t w + \partial_\nu w) \cdot w \mathrm{d}s.$$

Taking absolute values, integrating from t_0 to T_0, where $0 < t_0 < T_0 \leqslant T$, and taking the limit $t_0 \to 0$, we obtain

$$\sup_{t\in(0,T_0)} \|w(t)\|_{L^2(\partial\mathbb{D})}^2 + \int_0^{T_0} \int_{\mathbb{D}} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}t \leqslant 2\int_0^{T_0} \left| \int_{\partial\mathbb{D}} (\partial_t w + \partial_\nu w) \cdot w \mathrm{d}s \right| \mathrm{d}t.$$

Hence, writing

$$\begin{aligned} I &:= \int_{\partial\mathbb{D}} (w(\nu(u) - \nu(v))) \cdot \partial_\nu(\mathrm{dist}_N(u))\mathrm{d}s \\ II &:= \int_{\partial\mathbb{D}} w\nu(v) \cdot (\partial_\nu(\mathrm{dist}_N(u) - \mathrm{dist}_N(v)))\,\mathrm{d}s \end{aligned}$$

we obtain from (2.2.4)

$$\sup_{t\in(0,T_0)} \|w(t)\|_{L^2(\partial\mathbb{D})}^2 + \int_0^{T_0} \int_{\mathbb{D}} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}t \leqslant 2\int_0^{T_0} |I + II| \,\mathrm{d}t \tag{2.2.5}$$

for any $T_0 \leqslant T$. The next step is to estimate I and II. The key to this is the following formula, which is (3.5) from [Str24] and holds for smooth harmonic functions $u : \mathbb{D} \to \mathbb{R}^n$

$$\Delta(\mathrm{dist}_N^i(u)) = \nabla u \cdot (\mathrm{d}\nu_i)_u(\nabla u) := (\partial_1 u) \cdot (\mathrm{d}\nu_i)_u(\partial_1 u) + (\partial_2 u) \cdot (\mathrm{d}\nu_i)_u(\partial_2 u) \tag{2.2.6}$$

For I, we have for each $\varepsilon > 0$

$$\begin{aligned} |I| &= \left| \sum_{i=1}^{m} \int_{\partial\mathbb{D}} [w(\nu(u) - \nu(v))]_i \partial_\nu(\mathrm{dist}_N^i(u))\mathrm{d}s \right| \\ &\leqslant \sum_{i=1}^{m} \int_{\mathbb{D}} |[w(\nu(u) - \nu(v))]_i| \,|\nabla u \cdot (\mathrm{d}\nu_i)_u(\nabla u)| \,\mathrm{d}x \\ &\quad + \sum_{i=1}^{m} \int_{\mathbb{D}} |\nabla([w(\nu(u) - \nu(v))]_i)| \left|\nabla(\mathrm{dist}_N^i(u))\right| \mathrm{d}x \end{aligned}$$

$$\leqslant C \int_{\mathbb{D}} (|\nabla w|\,|w| + |w|^2\,|\nabla u|)\,|\nabla u|\,\mathrm{d}x$$
$$\leqslant \varepsilon \|\nabla w\|^2_{L^2(\mathbb{D})} + C(\varepsilon)\|w\|^2_{L^4(\mathbb{D})}\|\nabla u\|^2_{L^4(\mathbb{D})} \tag{2.2.7}$$

Here we have integrated by parts, used (2.2.6) and that dist_N and ν are smooth with compact support. The constant $C(\varepsilon)$ depends on $N \hookrightarrow \mathbb{R}^n$ and ε.

Similarly, we estimate II as

$$\begin{aligned}
|II| &= \left|\sum_{i=1}^m \left[\int_{\partial\mathbb{D}} [w\nu(v)]_i \partial_\nu(\mathrm{dist}^i_N(u))\mathrm{d}s - \int_{\partial\mathbb{D}} [w\nu(v)]_i \partial_\nu(\mathrm{dist}^i_N(v))\mathrm{d}s\right]\right| \\
&\leqslant \sum_{i=1}^m \int_{\mathbb{D}} |[w\nu(v)]_i|\,|\nabla u \cdot (\mathrm{d}\nu_i)_u(\nabla u) - \nabla v \cdot (\mathrm{d}\nu_i)_v(\nabla v)|\,\mathrm{d}x \\
&+ \sum_{i=1}^m \int_{\mathbb{D}} \left|\nabla([w\nu(v)]_i) \cdot \nabla(\mathrm{dist}^i_N(u) - \mathrm{dist}^i_N(v))\right|\mathrm{d}x \\
&\leqslant C \int_{\mathbb{D}} |w| \left(|w|\,|\nabla u|^2 + (|\nabla u| + |\nabla v|)\,|\nabla w|\right)\mathrm{d}x \\
&+ C\sum_{i=1}^m \int_{\mathbb{D}} (|\nabla w| + |w|\,|\nabla v|)\left|\nabla(\mathrm{dist}^i_N(u) - \mathrm{dist}^i_N(v))\right|\mathrm{d}x \\
&\leqslant \frac{1}{2}\varepsilon\|\nabla w\|^2_{L^2(\mathbb{D})} + C(\varepsilon)\|w\|^2_{L^4(\mathbb{D})}(\|\nabla u\|^2_{L^4(\mathbb{D})} + \|\nabla v\|^2_{L^4(\mathbb{D})}) \\
&+ C\left(\|\nabla w\|_{L^2(\mathbb{D})} + \||w||\nabla v|\|_{L^2(\mathbb{D})}\right)\|\nabla(\mathrm{dist}_N(u) - \mathrm{dist}_N(v))\|_{L^2(\mathbb{D})} \\
&\leqslant \varepsilon\|\nabla w\|^2_{L^2(\mathbb{D})} + C(\varepsilon)\|w\|^2_{L^4(\mathbb{D})}(\|\nabla u\|^2_{L^4(\mathbb{D})} + \|\nabla v\|^2_{L^4(\mathbb{D})})
\end{aligned} \tag{2.2.8}$$

Here, we have additionally used the estimates

$$|(\mathrm{dist}_N(u) - \mathrm{dist}_N(v))| \leqslant C\,|w|$$
$$|\nabla u \cdot (\mathrm{d}\nu_i)_u(\nabla u) - \nabla v \cdot (\mathrm{d}\nu_i)_v(\nabla v)| \leqslant C\left(|w|\,|\nabla u|^2 + (|\nabla u| + |\nabla v|)\,|\nabla w|\right)$$

which come from dist_N and ν being $C^\infty_c(\mathbb{R}^n)$ and

$$\|\nabla(\mathrm{dist}_N(u) - \mathrm{dist}_N(v))\|^2_{L^2(\mathbb{D})} \leqslant C\int_{\mathbb{D}} |w|\left(|w|\,|\nabla u|^2 + (|\nabla u| + |\nabla v|)\,|\nabla w|\right)\mathrm{d}x \tag{2.2.9}$$

which is obtained from (2.2.6) by multiplying with $\mathrm{dist}^i_N(u) - \mathrm{dist}^i_N(v)$ and integrating by parts.

Integrating the above estimates with $\varepsilon = \frac{1}{8}$, we then obtain for sufficiently small T_0

$$\begin{aligned}\sup_{t\in(0,T_0)} \|w(t)\|^2_{L^2(\partial\mathbb{D})} + \|\nabla w\|^2_{L^2(\mathbb{D}\times[0,T_0])} &\leqslant \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{D}\times[0,T_0])} \\ &\quad + C\int_0^{T_0} \|w\|^2_{L^4(\mathbb{D})}(\|\nabla u\|^2_{L^4(\mathbb{D})} + \|\nabla v\|^2_{L^4(\mathbb{D})})\mathrm{d}t \\ &\leqslant \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{D}\times[0,T_0])} \\ &\quad + C\sup_{t\in(0,T_0)} \|w\|^2_{L^2(\partial\mathbb{D})}\int_0^{T_0} (\|\nabla u\|^2_{L^4(\mathbb{D})} + \|\nabla v\|^2_{L^4(\mathbb{D})})\mathrm{d}t \\ &\leqslant \frac{1}{2}\left(\sup_{t\in(0,T_0)} \|w(t)\|^2_{L^2(\partial\mathbb{D})} + \|\nabla w\|^2_{L^2(\mathbb{D}\times[0,T_0])}\right)\end{aligned}$$

We have used that $\|u\|_{L^4(\mathbb{D})} \leqslant C\|u\|_{L^2(\partial\mathbb{D})}$ which follows from Theorem A.1.4 and the Sobolev embeddings. Also we can find a suitably small T_0 for the last step to hold by absolute continuity of the Lebesgue integral. For this to hold, we need to know that $\|\nabla u\|_{L^4(\mathbb{D})}, \|\nabla v\|_{L^4(\mathbb{D})} \in L^2([0,T_0])$. This follows from u, v being energy class solutions, and hence in $H^1_{\text{loc}}([0,T)\times\partial\mathbb{D};N)$, since $\|\nabla u\|_{L^4(\mathbb{D})} \leqslant C\|u\|_{H^1(\partial\mathbb{D})}$.

Therefore we have uniqueness on $[0,T_0]$ for a small time T_0. Using this proof starting again at T_0, we can see that $u = v$ on a set of the form $[0,T_1)$. Theorem 2.2.2 tells us that if $T_1 < T$, then $u = v$ also at $t = T_1$ and so we must have $u = v$ on all of $[0,T)$. □

Having recalled Struwe's method, we now state a modified version which requires only minor adjustments to the proof.

Proposition 2.2.3. *Let u, v be weak solutions to the Plateau Flow* (2.1.5) *with the same initial data $u_0 \in H^{\frac{1}{2}}(\partial\mathbb{D};N)$ on the time interval* $[0,T)$. *Suppose that additionally $u, v \in H^1_{\text{loc}}(\partial\mathbb{D}\times[0,T);N)$ (i.e. we also have $\partial_s u, \partial_s v \in L^2_{\text{loc}}(\partial\mathbb{D}\times[0,T);\mathbb{R}^n)$). Then $u = v$.*

Proof. First, we note that (2.2.4) still holds provided we interpret it in a distributional sense, since $\partial_\nu(\mathrm{dist}_N(u))$ is now an element of $H^{-\frac{1}{2}}(\partial\mathbb{D})$. In particular we have the following lemma which gives meaning to $\partial_\nu(\mathrm{dist}_N(u))$.

Lemma 2.2.4. *Let $u \in H^{\frac{3}{2}}(\mathbb{D};\mathbb{R}^n)$ be harmonic. Then for all $\varphi \in H^1(\mathbb{D};\mathbb{R})$*

$$\int_{\partial\mathbb{D}} \varphi\partial_\nu(\mathrm{dist}^i_N(u))\mathrm{d}s = \int_{\mathbb{D}} \varphi\nabla u\cdot(\mathrm{d}\nu_i)_u(\nabla u)\mathrm{d}x + \int_{\mathbb{D}} \nabla\varphi\cdot\nabla(\mathrm{dist}^i_N(u))\mathrm{d}x. \qquad (2.2.10)$$

Proof. First note that all the integrals exist using the Sobolev embeddings and that ν_i is smooth with compact support. Next, for u smooth, we can multiply (2.2.6) by φ and integrate by parts to obtain (2.2.10). To obtain the result in general, let $u_k \in H^1(\partial\mathbb{D};\mathbb{R}^n)$ be smooth and $u_k \to u$ in $H^1(\partial\mathbb{D};\mathbb{R}^n)$. Then by Theorem A.1.4, the harmonic extensions of u_k converge to u in $H^{\frac{3}{2}}(\mathbb{D};\mathbb{R}^n)$. Since dist_N^i is smooth and compactly supported and using the Sobolev embeddings, we can take the limit of (2.2.10) with u_k to get the result for u. $\square$

Next, we see that the steps to obtain (2.2.5) still work without any trouble using the weak formulation of the Dirichlet-to-Neumann operator.

The estimates of I and II are still true, however the proofs require us to replace the integration by parts with Lemma 2.2.4. We also note that these estimates hold for almost all $t \in [0,T]$, since the assumed regularity only gives us $u, v \in H^{\frac{3}{2}}(\mathbb{D};\mathbb{R}^n)$ for almost all t. Also, the estimate (2.2.9) is instead proved by using $\varphi = \text{dist}_N^i(u) - \text{dist}_N^i(v)$ in (2.2.10).

The last part of the argument needs no modification. $\square$

2.2.2 Regularity and Energy Concentration Results

Here we prove an energy concentration lemma and state some existing regularity results which we shall need in the proof of the main result. First of all, we have the following, which is based upon the ideas from Lemma 3.3 in [Rup08].

Lemma 2.2.5. *Let u be a weak solution to the Plateau flow (2.1.5) on $[0,T)$ and suppose that at some time t_0, u satisfies*

$$\limsup_{s \searrow t_0} E(u(s)) < E(u(t_0)) + \varepsilon$$

for some $\varepsilon > 0$. Then there exists $t_1 > t_0$ and $r > 0$ such that

$$\sup_{x\in\mathbb{D}, t\in[t_0,t_1]} E(u(t); B_r(x)) \leqslant \varepsilon.$$

Proof. Let $\rho > 0$ satisfy

$$\limsup_{s \searrow t_0} E(u(s)) \leqslant E(u(t_0)) + \varepsilon - \rho. \tag{2.2.11}$$

Since u is an weak solution of the flow, $E(u(t_0)) < \infty$. Therefore there exists a covering $\{B_{r_i}(x_i) : 1 \leqslant i \leqslant m\}$ of $\mathbb{D}$ satisfying

$$E(u(t_0); B_{2r_i}(x_i)) \leqslant \frac{1}{2}\rho \tag{2.2.12}$$

for each i.

We claim then that there exists $t_1 > t_0$ such that for all $t \in [t_0, t_1]$ and for all indices i,

$$E(u(t); B_{2r_i}(x_i)) \leqslant \varepsilon$$

Arguing by contradiction, suppose that there is a sequence of times $t_k \searrow t_0$ and a sequence of indices i_k such that

$$E(u(t_k); B_{2r_{i_k}}(x_{i_k})) > \varepsilon \tag{2.2.13}$$

By passing to a subsequence, we assume without loss of generality that $i_k = 1$ and let $r = r_1$. Taking the $\limsup$, we estimate

$$\begin{aligned}
\varepsilon &\leqslant \limsup_{k\to\infty} E(u(t_k); B_{2r}(x_1)) \\
&= \limsup_{k\to\infty} \left(E(u(t_k)) - E(u(t_k); \mathbb{D} \setminus B_{2r}(x_1))\right) \\
&\leqslant \limsup_{k\to\infty} E(u(t_k)) - \liminf_{k\to\infty} E(u(t_k); \mathbb{D} \setminus B_{2r}(x_1)) \\
&\leqslant E(u(t_0)) + \varepsilon - \rho - \liminf_{k\to\infty} E(u(t_k); \mathbb{D} \setminus B_{2r}(x_1))
\end{aligned}$$

with the final line following from the choice of ρ. Next we estimate the $\liminf$ term. For this, we note that as u is a weak solution, $u(t_k)$ is a bounded sequence in $H^{\frac{1}{2}}(\partial\mathbb{D}; N)$ and hence the harmonic extensions form a bounded sequence in $H^1(\mathbb{D}; \mathbb{R}^n)$. Therefore, on passing to a subsequence we can assume that

$$u(t_k) \to u_\infty \text{ strongly in } L^2(\mathbb{D}; \mathbb{R}^n)$$

$$\nabla u(t_k) \rightharpoonup \nabla u_\infty \text{ weakly in } L^2(\mathbb{D}; \mathbb{R}^n)$$

This implies the same convergence in $L^2(\mathbb{D} \setminus B_{2r}(x_1))$. The weak convergence of the gradients implies

$$\|\nabla u_\infty\|_{L^2(\mathbb{D}\setminus B_{2r}(x_1);\mathbb{R}^n)} \leqslant \liminf_{k\to\infty} \|\nabla u(t_k)\|_{L^2(\mathbb{D}\setminus B_{2r}(x_1);\mathbb{R}^n)}$$

and Lemma 2.2.2 implies that $u(t_k) \to u(t_0)$ in $L^2(\mathbb{D} \setminus B_{2r}(x_1); \mathbb{R}^n)$. Hence $u_\infty = u(t_0)$ and therefore

$$\liminf_{k\to\infty} E(u(t_k); \mathbb{D} \setminus B_{2r}(x_1)) \geqslant E(u(t_0); \mathbb{D} \setminus B_{2r}(x_1))$$

Hence, we get

$$\begin{aligned}
\varepsilon &\leqslant E(u(t_0)) + \varepsilon - \rho - \liminf_{k\to\infty} E(u(t_k); \mathbb{D} \setminus B_{2r}(x_1)) \\
&\leqslant E(u(t_0)) + \varepsilon - \rho - E(u(t_0); \mathbb{D} \setminus B_{2r}(x_1)) \\
&\leqslant E(u(t_0); B_{2r}(x_1)) + \varepsilon - \rho \\
&\leqslant \varepsilon - \frac{1}{2}\rho
\end{aligned}$$

which gives the desired contradiction. To conclude, we can select $r = \min\{r_1, ..., r_m\}$ as this ensures that any ball of radius r lies entirely within $B_{2r_i}(x_i)$ for some i. □

Next we state some regularity results. First of all, we have the following qualitative H^1 regularity result due to Wettstein. Note that this result is originally stated with the fractional Laplacian, but as remarked in the beginning of this section, this is equivalent to the Dirichlet-to-Neumann operator that we use.

Proposition 2.2.6 (Lemma 3.4 [Wet22],Proposition 4.1 [Wet21])**.** *Let $u \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$ and suppose u satisfies*

$$P_u^{TN}(\partial_\nu u) = f \tag{2.2.14}$$

for some $f \in L^2(\partial\mathbb{D}; \mathbb{R}^n)$. Then $u \in H^1(\partial\mathbb{D}; N)$

We note that this result is exactly found as Lemma 3.4 in [Wet22] in the case where N is a sphere, and the proof in the general case can be extracted from the proof of

Proposition 4.1 by replacing $\partial_t u$ with f.

Next we need some local quantitative H^1 estimates for the same equation. The following, due to Struwe from [Str24], are stated and proved for smooth functions u, but as remarked in the paper, the proofs work just the same for the following statements.

Proposition 2.2.7 (Proposition 3.3 from [Str24])**.** *There exists a constant $\delta > 0$, depending only on $N \hookrightarrow \mathbb{R}^n$, such that if $u \in H^1(\partial\mathbb{D}; N)$ solves (2.2.14) with $f \in L^2(\partial\mathbb{D}; \mathbb{R}^n)$ and $x_0 \in \partial\mathbb{D}$, $0 < r < \frac{1}{2}$ satisfy*

$$E(u; B_r(x_0)) \leqslant \delta^2$$

then

$$\int_{B_{r^2}(x_0)\cap\partial\mathbb{D}} |\partial_s u|^2 \, \mathrm{d}s \leqslant C\left(\|f\|^2_{L^2(B_r(x_0)\cap\partial\mathbb{D};\mathbb{R}^n)} + E(u)\right) \tag{2.2.15}$$

where C depends only on r.

This then yields by a covering argument the following global version.

Proposition 2.2.8 (Proposition 3.4 from [Str24])**.** *There exists a constant $\delta > 0$, depending only on $N \hookrightarrow \mathbb{R}^n$, such that if $u \in H^1(\partial\mathbb{D}; N)$ solves (2.2.14) with $f \in L^2(\partial\mathbb{D}; \mathbb{R}^n)$ and $0 < r < \frac{1}{2}$ satisfies*

$$\sup_{x\in\mathbb{D}} E(u; B_r(x)) \leqslant \delta^2 \tag{2.2.16}$$

then

$$\|\partial_s u\|^2_{L^2(\partial\mathbb{D};\mathbb{R}^n)} \leqslant C\left(\|f\|^2_{L^2(\partial\mathbb{D};\mathbb{R}^n)} + E(u)\right) \tag{2.2.17}$$

where C depends only on r.

Remark 2.2.9. The constant δ^2 must satisfy $\delta^2 \leqslant \varepsilon^*$, since it is coming from a global estimate Proposition 3.1 of [Str24] which in turn is used to deduce the positivity of ε^* in Corollary 3.2 of that paper.

2.2.3 Proof of Theorem 2.1.4

Finally, using the above results we can prove our first uniqueness theorem.

Proof of Theorem 2.1.4. Using Lemma 2.2.1, we have that for almost all times, $u(t)$ is a weak solution to the stationary equation. Note also that $\partial_t u(t) \in L^2(\partial\mathbb{D};\mathbb{R}^n)$ for almost all $t \in [0,T)$. Hence, we can apply Proposition 2.2.6 to obtain that for almost all times, $u(t) \in H^1(\partial\mathbb{D};N)$.

Then we let $\varepsilon_0 = \delta^2$ from Proposition 2.2.8, which as noted in Remark 2.2.9 satisfies $0 < \delta^2 \leqslant \varepsilon^*$. For each $t_0 \in [0,T)$, by Lemma 2.2.5, there exists $t_1 > t_0$ and $r > 0$ such that

$$\sup_{x\in\mathbb{D},t\in[t_0,t_1]} E(u(t);B_r(x)) \leqslant \varepsilon_0$$

Hence by Proposition 2.2.8, for almost every $t \in [t_0,t_1]$

$$\|\partial_s u(t)\|^2_{L^2(\partial\mathbb{D};\mathbb{R}^n)} \leqslant C\left(\|\partial_t u(t)\|^2_{L^2(\partial\mathbb{D};\mathbb{R}^n)} + E(u(t))\right)$$

Since u is a weak solution to the flow, we can integrate over $[t_0,t_1]$ and get $u \in H^1(\partial\mathbb{D} \times [t_0,t_1];\mathbb{R}^n)$. Hence we can apply Proposition 2.2.3 to get uniqueness of solutions on each $[t_0,t_1]$. Let v be the almost smooth solution with initial data u_0. Then by the above, the set of times on which $u = v$ is right-open, and so contains a set $[0,T_1)$. Suppose then, for a contradiction, that the maximal such $T_1 < T$. Then by Lemma 2.2.2, $u(t) \to u(T_1)$ as $t \nearrow T_1$ and $v(t) \to v(T_1)$ as $t \nearrow T_1$ in $L^2(\partial\mathbb{D};\mathbb{R}^n)$. Since $u(t) = v(t)$ for $t < T_1$, we have that $u(T_1) = v(T_1)$. But then there exists some $T_2 > T_1$ such that we have uniqueness on $[T_1,T_2]$, contradicting maximality of T_1.

Hence, we have that u is the almost smooth solution with initial data u_0 on all of $[0,T)$. □

2.3 Bubble Formation in Sequences of Almost Critical Maps

In this section, we investigate the process of bubble formation in the solution, continuing work of Struwe from Section 8 in [Str24] where the process of forwards in time bubble formation is studied. We begin by giving an overview of the analogous picture for harmonic map flow. Then we use the arguments of Struwe from [Str24] to extract a simple result proving that energy concentration forces the formation of a bubble for suitable sequences of maps. This is only the first step in trying to understand singularity formation, but it is enough for us to later obtain a slight upgrade to our uniqueness result.

Before we go any further, we want to clarify what we mean by bubbles. This term has been used to describe the maps which can be extracted by rescaling at singularities of harmonic map flow, the name coming from the fact these can be seen as spheres in the image. We shall use this terminology for maps arising out of singularities of the Plateau flow.

2.3.1 Brief Overview of Results in the Almost Harmonic Setting

This phenomena is seen in the work of Sacks and Uhlenbeck in [SU81], but the first major result on bubble formation relating to harmonic map flow was contained in [Str85]. This has several important consequences, but perhaps most importantly it proves that there can be only be finitely many singular times, since at each such time, there is a strictly positive lower bound on the amount of energy that must be lost.

In subsequent works, many authors have obtained results which improve our understanding of the bubble formation process. In particular, in [DT95], Ding and Tian show that the energy lost to bubbles accounts for all of the energy lost at a singular time. Also we mention the result of Tian and Qing in [QT97b], which shows that the limiting bubble tree of maps has connected image and that the images converge pointwise. A nice summary of these and other results is contained in the paper of Topping [Top04], which itself contains several further results on bubble formation.

Another import question to investigate is whether a bubble can form at all in finite time. It is quite straightforward to construct examples where a bubble has to form at either finite or *infinite* time, by starting with a map which has no harmonic map in its homotopy class, e.g. a degree 1 map from a torus to a sphere (see [EW76] for more details on this). However, it was not until the paper [CDY92] by Chang, Ding and Ye that an example of a finite time bubble was known to exist.

Finally, it is interesting to ask if the bubble tree limit is unique, since the arguments in the construction require passing to a subsequence using some kind of compactness. A result from [Top04] shows that the bubble tree limit can in fact be non-unique due to a winding behaviour. However, in light of the work of Simon on Łojasiewicz-type inequalities, it seems possible that with the right restriction on the target, uniqueness results may be possible. In this direction, we note the recent preprint of Rupflin, [Rup25], which obtains a Łojasiewicz-Simon inequality in the case of a single bubble.

2.3.2 Result on Bubble Formation

In this section, we use the analysis from Section 8 of [Str24] to give a result on bubble formation for a sequence of maps defined on $\mathbb{D}$. Since we will later need to analyse bubble formation backwards along solutions, we have extracted the following version of Struwe's results, and for completeness provide the proof, though this follows exactly the work of [Str24].

Theorem 2.3.1 (Contents of Section 8, [Str24]). *Let (N, h) be a closed Riemannian manifold smoothly embedded in $\mathbb{R}^n$. Suppose that $u_k : \partial\mathbb{D} \to N$ are smooth maps which are extended harmonically to the interior of $\mathbb{D}$ and satisfy $E(u_k) \leqslant \bar{E}$ for some $\bar{E} < \infty$. Suppose that there are sequences $x_k \in \mathbb{D}$ and $r_k \searrow 0$ and a $\delta > 0$ such that*

$$E(u_k; B_{r_k}(x_k)) \geqslant \delta \text{ for all } k \tag{2.3.1}$$

$$r_k^{\frac{1}{2}} \|P_{u_k}^{TN}(\partial_\nu u_k)\|_{L^2(\partial\mathbb{D})} \to 0 \text{ as } k \to 0 \tag{2.3.2}$$

Then there exists a subsequence of the u_k, a map $\bar{u} \in H^{\frac{1}{2}}(\partial\mathbb{D}; N)$, a point $x_0 \in \partial\mathbb{D}$ and a sequence of smooth conformal bijections $\Phi_k : \mathbb{D} \to \mathbb{D}$ such that

- $\Phi_k \rightharpoonup x_0$ *in* $H^1(\mathbb{D})$.

- *$\bar{u}$ is non-constant and $\frac{1}{2}$-harmonic.*

- *$u_k \circ \Phi_k \rightharpoonup \bar{u}$ in $H^1(\mathbb{D})$.*

Proof. First of all, by adjusting x_k and r_k, we assume that for some fixed $c_0 > 0$,

$$\delta = E(u_k; B_{r_k}(x_k)) \geqslant E(u_k; B_{r_k}(x)) \tag{2.3.3}$$

for all $x \in B_{c_0}(x_k)$. Additionally, we assume that δ is sufficiently small, in a way to be chosen later (though still independent of u_k). Further, using compactness of $\mathbb{D}$ and passing to a subsequence, we assume that $x_k \to x_0 \in \mathbb{D}$. We deduce later that x_0 in fact lies on the boundary.

We claim now that $r_k^{-1}\text{dist}(x_k, \partial\mathbb{D})$ is bounded. So suppose, for a contradiction, that $r_k^{-1}\text{dist}(x_k, \partial\mathbb{D})$ is unbounded. We pass to a subsequence and assume then that $r_k^{-1}\text{dist}(x_k, \partial\mathbb{D}) \nearrow \infty$. Then consider the rescaled maps $v_k(x) := u_k(x_k + r_k x)$ defined on the domains $\Omega_k := \{x : x_k + r_k x \in \mathbb{D}\}$. Then we have that

$$B_{r_k^{-1}\text{dist}(x_k, \partial\mathbb{D})}(0) \subseteq \Omega_k$$

and so the Ω_k exhaust $\mathbb{R}^2$.

Since v_k are harmonic, by the maximum principle they achieve their maximum values on $\partial\Omega_k$ and so the v_k are uniformly bounded in L^∞ by compactness of N. Further, by conformal invariance of the Dirichlet energy, $E(v_k) = E(u_k) \leqslant \bar{E}$. Therefore the sequence v_k is locally bounded in $H^1(\mathbb{R}^2)$. Hence, we can pass to a subsequence which converges weakly locally in $H^1(\mathbb{R}^2)$ to some $v_\infty \in H^1(\mathbb{R}^2)$. This weak convergence ensures that v_∞ is bounded and harmonic. Since $B_1(0)$ is compactly contained in all Ω_k for k sufficiently large, then by the usual uniform C^l estimates ($l \in \mathbb{N}$) for harmonic functions, we have

$$E(v_\infty; B_1(0)) = \lim_{k\to\infty} E(v_k; B_1(0)) = \delta.$$

Therefore v_∞ is a non-constant, bounded and harmonic function on $\mathbb{R}^2$. But this is a contradiction and so $r_k^{-1}\text{dist}(x_k, \partial\mathbb{D})$ is bounded.

Note that this in particular implies that $\text{dist}(x_k, \partial\mathbb{D}) \to 0$ and since $x_k \to x_0$, then $x_0 \in \partial\mathbb{D}$. By applying a rotation, we then assume that $x_k = (0, -y_k)$ with $1 - y_k \leqslant Mr_k$. We then define a new sequence of points $\tilde{x}_k := (0, -1)$ and radii $\tilde{r}_k := (M+1)r_k$. Since $B_{r_k}(x_k) \subseteq B_{\tilde{r}_k}(\tilde{x}_k)$, we have

$$E(u_k; B_{\tilde{r}_k}(\tilde{x}_k)) \geqslant E(u_k; B_{r_k}(x_k)) = \delta$$

Moreover, by (2.3.3) and by covering $B_{\tilde{r}_k}(\tilde{x}_k)$ by balls of radius r_k, we have

$$E(u_k; B_{\tilde{r}_k}(\tilde{x}_k)) \leqslant L\delta$$

for a constant L which does not depend on k. We then define the rescaled functions $v_k(x) := u_k(\tilde{x}_k + \tilde{r}_k x)$ which are defined on the domains $\Omega_k = \{x : \tilde{x}_k + \tilde{r}_k x \in \mathbb{D}\}$. Because of our choice of $\tilde{x}_k$ and $\tilde{r}_k$, we know that Ω_k is $B_{\tilde{r}_k^{-1}}((0, \tilde{r}_k^{-1}))$. We also note here that the rescaled maps satisfy

$$\|P^{TN}_{v_k}(\partial_\nu v_k)\|_{L^2(\partial\Omega_k)} \to 0$$

Let $\Phi_k : \bar{\mathbb{H}} \to \bar{B}_{\tilde{r}_k^{-1}}((0, \tilde{r}_k^{-1}))$ be the smooth conformal bijections, which in complex coordinates are defined by

$$\Phi_k(z) := \frac{2z}{2 - i\tilde{r}_k z}$$

Note that $\Phi_k \to \text{id}$ smoothly on each compact set in $\bar{\mathbb{H}}$. Then we define the maps $w_k := v_k \circ \Phi_k$.

By conformal invariance of the Dirichlet energy, we have $E(w_k) = E(v_k) \leqslant \bar{E}$. Then by the maximum principle we have w_k uniformly bounded in L^∞ as well. Hence by passing to a subsequence, we can assume $w_k \rightharpoonup w_\infty$ weakly locally in $H^1(\bar{\mathbb{H}})$. This weak convergence ensures that w_∞ is harmonic in $\mathbb{H}$ and maps $\partial\mathbb{H}$ into N almost everywhere.

Next, for each fixed $K \subseteq \partial\mathbb{H}$ compact, we estimate

$$\begin{aligned}
\int_K \left|P^{TN}_{w_k}(\partial_\nu w_k)\right|^2 \mathrm{d}s &\leqslant \int_{\Phi_k(K)} |\nabla\Phi_k|^2 \left|P^{TN}_{v_k}(\partial_\nu v_k)\right|^2 \left|\nabla(\Phi_k^{-1})\right| \mathrm{d}s \\
&\leqslant \sup_{k\in\mathbb{N}, x\in K} |\nabla\Phi_k(x)| \int_{\partial\Omega_k} |P_{v_k} TTN(\partial_\nu v_k)|^2 \mathrm{d}s
\end{aligned}$$

$$\to 0 \text{ as } k \to 0$$

and so $P^{TN}_{w_k}(\partial_\nu w_k) \to 0$ in $L^2_{\text{loc}}(\partial\mathbb{H})$. Using (2.3.3) and by assuming δ small enough, we can apply Proposition 2.2.7 to $w_k \circ \Psi$ for some fixed conformal $\Psi : \bar{B} \to \bar{\mathbb{H}}$ to obtain

$$\int_I |\partial_x w_k|^2 \, \mathrm{d}s \leqslant C(I)$$

for each compact interval $I \subseteq \partial\mathbb{H}$, where crucially the constant is independent of k. Next, we apply Lemma A.1.5 to $w_k \circ \Psi$ to get that for each $K \subseteq \overline{\mathbb{H}}$ compact, w_k is a bounded sequence $H^{\frac{3}{2}}(K)$. Using the compactness of $H^{\frac{3}{2}} \hookrightarrow H^1$ and the local L^2 convergence, we obtain local strong convergence $w_k \to w_\infty$ in $H^1(\mathbb{H})$. This gives us convergence of the energy, and so

$$\begin{aligned} E(w_\infty; B_2(0) \cap \mathbb{H}) &= \lim_{k\to\infty} E(w_k; B_2(0) \cap \mathbb{H}) \\ &\geqslant \liminf_{k\to\infty} E(u_k; B_{\tilde{r}_k}(\tilde{x}_k)) \geqslant \delta \end{aligned}$$

which tells us that w_∞ is non-constant.

Finally, using the weak convergence of $\nabla w_k \rightharpoonup \nabla w_\infty$ in $L^2_{\text{loc}}(\partial\mathbb{H})$ and the local uniform convergence $w_k \to w_\infty$ on $\partial\mathbb{H}$, we have $P^{TN}_{w_k}(\partial_\nu w_k) \rightharpoonup P^{TN}_{w_\infty}(\partial_\nu w_\infty)$ in $L^2_{\text{loc}}(\partial\mathbb{H})$, and hence $P^{TN}_{w_\infty}(\partial_\nu w_\infty) = 0$.

We can then conclude the proof by setting $\bar{u} = w_\infty \circ \Psi^{-1}$.

□

2.4 Proof of Theorem 2.1.5

In this section, we obtain an improvement on our earlier uniqueness result, Theorem 2.1.4. This gives us a better geometric understanding on the allowable energy jumps. The ideas of this result are based upon Theorem 1.2 in [Rup08].

The proof follows in two steps. First, we obtain a result, Proposition 2.4.2, which provides conditions for a bubble to form backwards in time. We can then use a similar argument to the proof of Theorem 2.1.4.

2.4.1 Extraction of a Backwards Bubble

Here we provide criterion for the formation of a bubble backwards in time. To prove this, we shall need the following simple estimate for harmonic functions, which can be found in [Str24].

Lemma 2.4.1 (Formula 2.2, Lemma 2.2 [Str24]). *Let $f : \mathbb{D} \to \mathbb{R}$ be a smooth function which is harmonic on $\mathbb{D}$. Then for any $x_0 \in \mathbb{D}$ and $0 < r < 1$*

$$\int_{\mathbb{D}\cap B_r(x_0)} |f|^2 \, \mathrm{d}x \leqslant 2r \int_{\partial\mathbb{D}} |f|^2 \, \mathrm{d}s.$$

With this, we can state and prove the following. This is based on the analysis in Section 8 of [Str24].

Proposition 2.4.2. *Let u be a weak solution to the Plateau flow (2.1.5) on $[0,T)$. Suppose that there exists $T_0 \in [0,T)$, $\delta > 0$, $x_k \to x_0 \in \bar{\mathbb{D}}$, $r_k \searrow 0$, $t_k \searrow T_0$ such that u is smooth on $(T_0, T_0+\varepsilon]$ for some $\varepsilon > 0$, $E(u(T_0)) < \infty$ and*

$$E(u(t_k); B_{r_k}(x_k)) \geqslant \delta \text{ for all } k. \tag{2.4.1}$$

Then there exist times $\tilde{t}_k \searrow T_0$ such that $u_k := u(\tilde{t}_k)$ satisfy the hypothesis for Theorem 2.3.1, with perhaps a smaller δ.

Proof. First, we note that any choice of $\tilde{t}_k \in [t_k, t_k + r_k]$ will satisfy all the conditions of Theorem 2.3.1, except perhaps for (2.3.1) and (2.3.2).

For the energy concentration, let $\varphi_k \in C^\infty(\mathbb{D};[0,1])$ be a cut-off function satisfying $\varphi_k(x) \equiv 0$ outside $B_{2r_k}(x_0)$, $\varphi_k(x) \equiv 1$ on $B_{r_k}(x_0)$ and $r_k \sup_{x\in\mathbb{D}} |\nabla\varphi_k(x)| \leqslant C$, for a constant C not depending on k. Then we note by a direct computation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{D}} |\nabla u|^2 \varphi_k^2 \mathrm{d}x = -\int_{\partial\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}s - 2\int_{\mathbb{D}} (\partial_t u)(\nabla u)\varphi_k \nabla\varphi_k \mathrm{d}x$$

Integrating this from t_k to $t_k + r_k t$ for any $t \in (0,1]$ and applying Cauchy-Schwarz, we obtain

$$\begin{aligned}\int_{\mathbb{D}} |\nabla u(t_k + r_k t)|^2 \varphi_k^2 \mathrm{d}x \geqslant \int_{\mathbb{D}} |\nabla u(t_k)|^2 \varphi_k^2 \mathrm{d}x - \int_{t_k}^{t_k + r_k t} \int_{\partial\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}s\mathrm{d}t \\ - \left(\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\nabla u|^2 |\nabla \varphi_k|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}\end{aligned} \tag{2.4.2}$$

We treat the first negative term by

$$\int_{t_k}^{t_k + r_k t} \int_{\partial\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}s\mathrm{d}t \leqslant \int_{T_0}^{t_k + r_k t} \int_{\partial\mathbb{D}} |\partial_t u|^2 \mathrm{d}s\mathrm{d}t \to 0$$

as $k \to \infty$ since $\partial_t u \in L^2(\partial\mathbb{D} \times [0,T])$ and $t_k + r_k t \to T_0$. For the product term, we estimate

$$\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\nabla u|^2 |\nabla \varphi_k|^2 \mathrm{d}x\mathrm{d}t \leqslant C r_k^{-2} \int_{t_k}^{t_k + r_k t} E(u(t))\mathrm{d}t \leqslant C r_k^{-1} \bar{E}$$

and using Lemma 2.4.1

$$\begin{aligned}\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}x\mathrm{d}t &\leqslant \int_{t_k}^{t_k + r_k t} \int_{\mathbb{D} \cap B_{2r}(y)} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \\ &\leqslant 4 r_k \int_{t_k}^{t_k + r_k t} \int_{\partial\mathbb{D}} |\partial_t u|^2 \mathrm{d}s\mathrm{d}t\end{aligned}$$

which combine to yield

$$\left(\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\nabla u|^2 |\nabla \varphi_k|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{t_k}^{t_k + r_k t} \int_{\mathbb{D}} |\partial_t u|^2 \varphi_k^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \to 0$$

as $k \to \infty$, and hence

$$\int_{\mathbb{D}} |\nabla u(t_k + r_k t)|^2 \varphi_k^2 \mathrm{d}x \geqslant \int_{\mathbb{D}} |\nabla u(t_k)|^2 \varphi_k^2 \mathrm{d}x - o(1) \tag{2.4.3}$$

as $k \to \infty$. Hence replacing δ by $\frac{1}{2}\delta$, we can satisfy (2.3.1) with any $\tilde{t}_k \in [t_k, t_k + r_k]$.

Finally, we use that $\partial_t u \in L^2(\partial\mathbb{D} \times [0,T])$ to get

$$\int_{t_k}^{t_k + r_k} \int_{\partial\mathbb{D}} \left| P_u^{TN}(\partial_\nu u) \right|^2 \mathrm{d}s\mathrm{d}t \to 0$$

as $k \to \infty$. Hence we can choose any $\tilde{t}_k \in [t_k, t_k + r_k]$ satisfying

$$\int_{\partial \mathbb{D}} \left| P_u^{TN}(\partial_\nu u(\tilde{t}_k)) \right|^2 \mathrm{d}s \leqslant \frac{1}{r_k} \int_{t_k}^{t_k + r_k} \int_{\partial \mathbb{D}} \left| P_u^{TN}(\partial_\nu u) \right|^2 \mathrm{d}s\mathrm{d}t$$

to obtain (2.3.2) □

2.4.2 Proof of Result

We can now give the proof of main result of this section.

Proof of Theorem 2.1.5. First, we note that on $[0,T) \setminus S$, the conditions for Theorem 2.1.4 hold. Hence if $s_0 = \min S$, we have uniqueness on $[0, s_0)$, so we assume without loss of generality that $s_0 = 0$. As S has no accumulation points, there is then some time $T_1 > 0$ such that $(0, T_1] \subseteq [0,T) \setminus S$. Then we know that for any $0 < t_0 < T_1$, u must equal the almost smooth solution with initial data $u(t_0)$ on $[t_0, T_1]$. By reducing T_1 to before the first bubble, we then additionally assume that u is smooth on $[t_0, T_1]$.

Now, we claim that there exists some $r > 0$ and $t_1 \in (0, T_1]$ such that for all $t \in [0, t_1]$ and for all $x \in \mathbb{D}$

$$E(u(t); B_r(x)) \leqslant \varepsilon_0$$

To prove this claim, we argue by contradiction. So suppose that there exist sequences $t_k \searrow 0$, $r_k \to 0$ such that

$$\sup_{x \in \mathbb{D}} E(u(t_k); B_{r_k}(x)) > \varepsilon_0$$

We can then find a sequence x_k such that for each k,

$$E(u(t_k); B_{r_k}(x_k)) > \varepsilon_0$$

By passing to a subsequence, we can assume the $x_k \to x_0$. Then we have the conditions to apply Proposition 2.4.2 and subsequently Theorem 2.3.1, and so let $\bar{u}, \Phi_k, \bar{t}_k$ be the resulting functions and time sequence.

We then fix $\rho > 0$ which satisfies

$$\limsup_{s \searrow 0} E(u(s)) \leqslant E(u(0)) + \varepsilon^* - \rho$$

and select $r_0 > 0$ such that

$$E(u(0); B_{r_0}(x_0)) \leqslant \frac{1}{2}\rho$$

By Theorem 2.3.1, we have

$$\liminf_{k \to \infty} E(u(\bar{t}_k); B_{r_0}(x_0)) \geqslant E(\bar{u}) \geqslant \varepsilon^*$$

Combining this with the argument from Lemma 2.2.5 to estimate the lim inf term, we obtain

$$\begin{aligned}
E(u(0)) &\leqslant E(u(0); \mathbb{D} \setminus B_{r_0}(x_0)) + \frac{1}{2}\rho \\
&\leqslant \liminf_{k \to \infty} E(u(\bar{t}_k); \mathbb{D} \setminus B_{r_0}(x_0)) + \frac{1}{2}\rho \\
&\leqslant \limsup_{k \to \infty} E(u(\bar{t}_k)) - \liminf_{k \to \infty} E(u(\bar{t}_k); B_{r_0}(x_0)) + \frac{1}{2}\rho \\
&< E(u(0)) + \varepsilon^* - \rho - \varepsilon^* + \frac{1}{2}\rho
\end{aligned}$$

which gives the required contradiction.

From this, we argue exactly as in the proof of Theorem 2.1.4 to conclude that u must in fact be equal to the almost smooth solution down to and including time $t = 0$. This argument can be repeated for all the times $t \in S$ to obtain the result. □

2.5 Connections with the Classical Plateau Problem

In this short section, I will present a result which strengthens the links between half-harmonic maps with the classical Plateau problem. As mentioned in the introduction to this chapter, Struwe raised the question in [Str24] of whether the Plateau flow produces a solution to the Plateau problem when the target manifold N is a closed curve Γ in some suitable sense. What I will show is the result that a non-constant half-harmonic map from $\mathcal{S}^1$ to a curve Γ is monotone.

The difference between half-harmonic maps and solutions of the Plateau problem is that usually the Plateau problem requires that $\mathcal{S}^1$ is mapped *monotonically* and *injectively* onto Γ. Since a half-harmonic map is not in general degree one, we need to make precise what notion of monotonicity we are using, for which we give two equivalent options, one global and one local. First, let $\gamma : \mathcal{S}^1(L) \to \Gamma$ be an arc-length parametrisation of Γ. Then for a given smooth map $u : \mathcal{S}^1 \to \Gamma$, we can find a smooth map $s : \mathcal{S}^1 \to \mathcal{S}^1(L)$ such that $u(\theta) = \gamma(s(\theta))$, which is unique up to rotation. We then say u is monotonic if the lift of s to a k-fold cover of $\mathcal{S}^1(L)$ by a degree k covering map[1], is injective, where k is the topological degree of s. Note that this is a strict notion of monotonicity, we do not allow the map to be constant on any open set. The local equivalent notion which does not consider the lifted map instead insists that at each point $x \in \mathcal{S}^1$, there is a small neighbourhood of x on which u is injective.

If we establish the monotonicity of a half-harmonic map u, then the injectivity reduces to asking that the degree of u is ± 1, and so it is the monotonicity that is the principle obstacle. Indeed, if u is a monotone half-harmonic map with degree greater than 1, then its extension is a Plateau solution with multiplicity. Struwe asks if the Plateau flow should be considered as producing generalised solutions of the Plateau problem, since there is no immediate reason to believe that the flow would preserve monotonicity.

I briefly introduced in Section 1.1.1 the work of Douglas, and in particular mentioned that the functional he introduced to solve the Plateau problem is precisely the half-energy defining half-harmonic maps. Although he principally studied the minimisers of this energy over monotone maps satisfying a three point condition, we can extract some useful ideas from his work which apply to all half-harmonic maps from the circle. In particular, in the course of showing that the minimising parametrisation is proper, he showed the following result, which I have restated in the language of half-harmonic maps.

Lemma 2.5.1 (Section 18, [Dou31])**.** *Let $u : \mathcal{S}^1 \to \Gamma \subseteq \mathbb{R}^n$ be a half-harmonic map into*

[1]Such a lifted map always exists as if $p : \mathcal{S}^1 \to \mathcal{S}^1$ is a degree k covering map, then $p_*\pi_1(\mathcal{S}^1) = s_*\pi_1(\mathcal{S}^1) = k\mathbb{Z}$

a smooth simple curve. Suppose that u is constant on some non-empty open set $\Omega \subseteq \mathcal{S}^1$. Then u is constant on all of $\mathcal{S}^1$.

Douglas' proof is an application of the Riemann-Schwarz reflection principle and the identity theorem for a suitable holomorphic function.

This result rules out one form of failure of monotonicity for non-constant half-harmonic maps, and in fact applies also for general target manifolds N. It is possible to use this result to go further and show the full monotonicity of half-harmonic maps. It will be most convenient to work below with the formula for the half-energy used by Douglas, which we recall is, up to a constant,

$$E_{\frac{1}{2}}(u) = \frac{1}{2}\int_{\mathcal{S}^1\times\mathcal{S}^1} \frac{|u(\theta)-u(\varphi)|^2}{4\sin^2\left(\frac{\theta-\varphi}{2}\right)}\mathrm{d}\theta\mathrm{d}\varphi.$$

Theorem 2.5.2. *Let $u : \mathcal{S}^1 \to \Gamma$ be a non-constant half-harmonic map. Then u is monotone (in the sense described above).*

This result provides some immediate consequences. First, we see that there can be no non-trivial degree zero half-harmonic maps into Γ, and so any non-constant half-harmonic map is in fact surjective and monotone, importantly applying to the bubbles which can form along solutions of the flow. Hence by starting with an initial map which has non-zero degree, we know the flow will produce some monotone half-harmonic map, either as a smooth limit at infinite time or as a bubble. This provides a solution to the Plateau problem up to having potentially *multiplicity*.

It remains unclear whether we can ensure the existence of a degree one half-harmonic map, and thus a true Plateau solution, using the flow. This is due to the fact that in the setting of the half-energy, we do not have the detailed singularity analysis such as that described above for the harmonic map flow, and so cannot yet answer questions about how singularities affect the homotopy type of the solution. There is a paper by Da Lio which proves more refined results for singularities forming in sequences of half-harmonic maps into spheres, [DL15], but there remain many questions about bubbling for sequences of *almost* half-harmonic maps, as would be needed to analyse solutions of the gradient flow.

Proof of Theorem 2.5.2. Let $\delta = \delta(\Gamma) > 0$ be a small constant to be fixed later in the proof, and take the function s as above which is associated to u, so that $u(\theta) = \gamma(s(\theta))$. Since half-harmonic maps into Γ are smooth, as shown in [DR11], then also the function s is smooth.

Assume then, for a contradiction, that u, and hence s, is not monotone, and replace s by its lift by the k-fold covering map, where k is the topological degree of u. Then we can find $\theta_1 < \theta_2$ such that $s(\theta_1) = s(\theta_2) =: s_0$ and that for all $\theta \in (\theta_1, \theta_2)$, $s_0 - \delta \leqslant s(\theta) \leqslant s_0 + \delta$. Moreover, thanks to Douglas' result we know that s is not constant on this interval. Then for each $\varepsilon \in [-1, 1]$, define the following functions

$$s_\varepsilon(\theta) := \begin{cases} s_0 - \varepsilon(s(\theta) - s_0) & \text{for } \theta \in [\theta_1, \theta_2] \\ s(\theta) & \text{otherwise} \end{cases} \tag{2.5.1}$$

This defines a variation $u_\varepsilon(\theta) = \gamma(s_\varepsilon(\theta))$ of u. The claim is that the half-energy $E_{\frac{1}{2}}$ is *not* stationary with respect to this variation. Indeed, a simple computation gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E_{\frac{1}{2}}(u_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2} \int_{\mathcal{S}^1 \times \mathcal{S}^1} \frac{|u_\varepsilon(\theta) - u_\varepsilon(\varphi)|^2}{4\sin^2(\frac{\theta-\varphi}{2})} \mathrm{d}\theta \mathrm{d}\varphi \\ &= \int_{\mathcal{S}^1 \times \mathcal{S}^1} \frac{(\gamma(s(\theta)) - \gamma(s(\varphi))) \cdot \left(\gamma'(s(\theta)) \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} s_\varepsilon(\theta) - \gamma'(s(\varphi)) \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} s_\varepsilon(\varphi)\right)}{4\sin^2(\frac{\theta-\varphi}{2})} \mathrm{d}\theta \mathrm{d}\varphi \\ &= \int_{[\theta_1, \theta_2]^2} \frac{-(\gamma(s(\theta)) - \gamma(s(\varphi))) \cdot (\gamma'(s(\theta))(s(\theta) - s_0) - \gamma'(s(\varphi))(s(\varphi) - s_0))}{4\sin^2(\frac{\theta-\varphi}{2})} \mathrm{d}\theta \mathrm{d}\varphi. \end{aligned}$$

To estimate the integrand, let $s_1 = s(\theta)$ and $s_2 = s(\varphi)$. We then use Taylor's theorem, noting that γ is smooth, to get for $i = 1, 2$

$$\gamma(s_i) = \gamma(s_0) + \gamma'(s_0)(s_i - s_0) + O(\delta^2)$$
$$\gamma'(s_i) = \gamma'(s_0) + \gamma''(s_0)(s_i - s_0) + O(\delta^2)$$

and hence the left hand term in the inner product is

$$\gamma(s_1) - \gamma(s_2) = \gamma'(s_0)(s_1 - s_2) + O(\delta^2).$$

and the right hand term is

$$\gamma'(s_1)(s_1 - s_0) - \gamma'(s_2)(s_2 - s_0) = \gamma'(s_0)(s_1 - s_2) + O(\delta^2).$$

Combined, this shows that the inner product is bounded below by $|\gamma'(s_0)|^2 (s_1 - s_2)^2 - C\delta^3$ for a constant C depending only on the curve Γ, so for sufficiently small $\delta \leqslant \frac{1}{2C}$, we have the estimate

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E_{\frac{1}{2}}(u_\varepsilon) \leqslant \int_{[\theta_1,\theta_2]^2} \frac{-\frac{1}{2}(s(\theta) - s(\varphi))^2}{4\sin^2(\frac{\theta-\varphi}{2})} \mathrm{d}\theta \mathrm{d}\varphi < 0.$$

This shows that u is not a stationary point of the half-energy, and hence not half-harmonic.

□

2.6 Future Outlook

The results of this chapter provide answers to some of the questions raised in the literature about this equation, particularly by Struwe in [Str24], but there remains a lot to be understood about both the half-energy functional, half-harmonic maps and the gradient flow studied here.

Following on from the uniqueness results I obtained here, it would be very natural to ask if the reverse bubble solution found by Topping in [Top02] from the harmonic map flow case could be adapted to this setting, demonstrating the need carefully consider energy jumps, or if instead such solutions cannot occur and perhaps uniqueness holds in an even wider class of functions.

Another natural question, which has already been raised by Struwe and Wettstein, is to find out if finite time singularities can actually occur in solutions. The original method of Chang, Ding and Ye [CDY92] seems not to apply, at least not in a straightforward way, but there are by now many different constructions for forcing finite time blow-ups in such geometric PDEs, such as the gluing methods in [DdPW20], or the construction of a specialised target by Topping in [Top04]. On the other hand finding methods of ruling out bubble formation is likely to be key in some applications.

Certainly, whether occurring at finite or infinite time, in studying applications of this flow, it will be necessary to develop a more refined analysis of bubble formation, specifically relating to the no-neck and no loss of energy type results and to study the change in topology caused by bubbling. It is also likely of interest to understand better the rate at which energy concentrates.

In a more general sense, it will be interesting to see what applications of this flow can be developed. I discussed in the previous section the possible applications to the classical Plateau problem, and more generally in finding settings where the flow can be shown to produce non-trivial free boundary minimal surfaces in the limit. With regards the Plateau application, it remains an open question if the monotonicity of the parametrisation is preserved along solutions of the flow, and not as I showed just obtained in the limit. Also, there have been recent applications of such flows to study rigidity estimates, such as Topping's paper [Top23] on harmonic maps between spheres. In the half-harmonic setting, there has been some work on such estimates, such as the paper [DScW23]. It is noted there that such estimates have applications to the study of the gradient flow, and so it seems of interest to further explore the interplay of these two topics.

Chapter 3

Plateau Flow: Extension to General Domains

In this chapter, I will present work from a joint project I did in collaboration with Melanie Rupflin and Michael Struwe. The aim of this work was to bring together the results of Struwe's previous paper, [Str24], with the Teichmüller harmonic map flow introduced in [RT16], to build a theory of a gradient flow for the half-energy on any compact surface with boundary. This chapter thus introduces a system of equations which generalises the Plateau flow equation studied in Chapter 2 and develops the theory of this new system. Specifically, we have obtained key results on the existence and regularity of solutions, given in Theorem 3.1.4, together with a result on the asymptotic behaviour of the solution, given in 3.1.5. What I present below is taken from our paper in preparation.

3.1 Introduction

3.1.1 Half-Harmonic Maps

We begin by briefly recalling the notion of the half-energy and half-harmonic maps introduced in Section 1.2.2. Let $N \hookrightarrow \mathbb{R}^n$ be a smooth, not necessarily connected, closed submanifold of any dimension and let Σ be a closed surface with boundary $\partial\Sigma \neq \emptyset$. Given a metric g on Σ, following [LR11] and [PL17], we define the half-energy $E_{\frac{1}{2}}(u,g)$ of a function $u \in H^{\frac{1}{2}}(\partial\Sigma; N)$ as the Dirichlet energy of its harmonic extension, i.e.

$$E_{\frac{1}{2}}(u,g) := E(u_g, g) := \frac{1}{2}\int_\Sigma |\mathrm{d}u_g|_g^2 \,\mathrm{d}v_g. \tag{3.1.1}$$

Here and in the following $E(\cdot, g)$ is the standard Dirichlet energy with respect to the metric g and $u_g \colon \Sigma \to \mathbb{R}^n$ denotes the harmonic extension with respect to the metric g of a given map $u \in H^{\frac{1}{2}}(\partial\Sigma; N) = H^{\frac{1}{2}}((\partial\Sigma, \mathrm{d}s_g); N)$, i.e. the unique function $u_g : \Sigma \to \mathbb{R}^n$ with $\Delta_g u_g = 0$ and trace $u|_{\partial\Sigma} = u$.

We furthermore denote by $\pi : N_\eta \to N$ the nearest point projection onto N, which is well-defined and smooth on the tubular neighbourhood

$$N_\eta := \{y \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{R}^n}(y, N) < \eta\}$$

of N if $\eta > 0$ is chosen small enough, and recall that for any $p \in N$ the orthogonal projection $P_p : \mathbb{R}^n \to T_pN$ of the ambient $\mathbb{R}^n$ onto the tangent space T_pN to N at p can be equivalently written as $P_p = \mathrm{d}\pi(p)$.

As in [LR11] and [PL17], we call a map $u \in H^{\frac{1}{2}}(\partial\Sigma; N)$ half-harmonic if it is a critical point of $E_{\frac{1}{2}}(u,g)$ in the sense that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} E_{\frac{1}{2}}(\pi(u+\varepsilon v), g) = 0$ for every $v \in H^{\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$. As

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E_{\frac{1}{2}}(\pi(u+\varepsilon v), g) = \int_{\partial\Sigma} \partial_\varepsilon \pi(u+\varepsilon v)|_{\varepsilon=0} \cdot \partial_{\nu_g} u_g \mathrm{d}s_g = \int_{\partial\Sigma} v \cdot P_u(\partial_{\nu_g} u_g) \mathrm{d}s_g, \tag{3.1.2}$$

where ν_g is the outwards unit normal of (Σ, g) along $\partial\Sigma$, this is equivalent to asking that the harmonic extension u_g of u satisfies the equation

$$P_u(\partial_{\nu_g} u_g) = 0 \quad \text{on } \partial\Sigma. \tag{3.1.3}$$

As explained in more detail in the introduction, half-harmonic maps are closely related to free boundary minimal surfaces. In particular, it was shown by Da Lio and Pigati [PL17] that the harmonic extension of a map $u : (\partial\Sigma, g) \to N \subseteq \mathbb{R}^n$ parametrises a free boundary minimal surface supported on N if and only if u is half-harmonic and conformal. Note that we use throughout the definition of conformal $v^*g_{\mathbb{R}^N} = \rho^2 g$ for a function $\rho \geqslant 0$, sometimes called weakly conformal as we allow for branch points.

We note that if $\Sigma = D$, then as observed in [LMR15] and [MS15], the equation (3.1.3) is in fact a sufficient condition for weak conformality of the harmonic function u_g, compare also Remark 3.1.2 below. While for surfaces Σ of higher genus or higher connectivity this condition is not sufficient, we can exploit that maps $v\colon \Sigma \to \mathbb{R}^n$ are conformal if and only if the half-energy is critical with respect to variations of the metric. This was shown by Da Lio and Pigati [PL17] in the setting of the half-energy and of half-harmonic maps considered here. Since we will use the corresponding properties of the variation of the half-energy in our construction of the flow, we briefly recall how this key property can be obtained.

To this end, we note that the first variation of the Dirichlet energy E (for fixed metric g) along maps $v_\varepsilon = u_{g_\varepsilon}$ that are obtained as harmonic extensions of a fixed map $u : \partial\Sigma \to \mathbb{R}^n$ with respect to a family of metrics g_ε with $g_{\varepsilon=0} = g$ is always so that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(v_\varepsilon, g) = \int_\Sigma \langle \nabla_g(\partial_\varepsilon v_\varepsilon), \nabla_g u_g \rangle \mathrm{d}v_g = -\int_\Sigma \partial_\varepsilon v_\varepsilon \cdot \Delta_g u_g \mathrm{d}v_g + \int_{\partial\Sigma} \partial_\varepsilon v_\varepsilon \cdot \partial_{\nu_g} u_g \mathrm{d}s_g = 0$$

as u_g is harmonic and $\partial_\varepsilon v_\varepsilon|_{\partial\Sigma} = 0$. The variation of the half-energy with respect to the metric hence reduces to the variation of the Dirichlet energy of the fixed map u_g with respect to a varying metric g_ε, which we recall is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E_{\frac{1}{2}}(u, g_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(u_g, g_\varepsilon) = -\frac{1}{2}\int_\Sigma \langle \partial_\varepsilon g_\varepsilon, k(u_g, g)\rangle_g \mathrm{d}v_g, \tag{3.1.4}$$

see for instance [BW03, Lemma 3.4.1]. Here and in the following

$$k(v, g) = v^* g_{\mathbb{R}^n} - \frac{1}{2}\,|\mathrm{d}v|_g^2\, g \tag{3.1.5}$$

denotes the stress-energy tensor of maps $v : \Sigma \to \mathbb{R}^n$, which we note is always trace-free and of course vanishes if and only if v is conformal.

3.1.2 Gradient Flow

In view of the above, in order to define a flow that will evolve an initial surface to a free boundary minimal surface it seems natural to consider a gradient flow of the half-energy.

In the case where Σ is a disc, we have seen in the previous chapter that this has been studied already. We now consider the corresponding problem for surfaces Σ of higher genus and/or connectivity. We introduce a coupled flow which evolves both an initial map $u_0 \colon \partial\Sigma \to N$ and an initial domain metric g_0 so as to yield a critical point of the half-energy with respect to both u and g, hence inducing a free boundary minimal surface.

Remark 3.1.1. In the following, we will only consider domain surfaces Σ which are orientable, without further comment, as for non-orientable Σ all results can be recovered by passing to the orientable double cover and working with metrics and function spaces which are invariant under the non-trivial covering space transformation.

Since the energy is conformally invariant, we can restrict the class of admissible metrics to consist of a unique representative for each conformal class of metric. For this we make use of standard uniformisation results[1], see e.g. [OPS88]. For $\Sigma \neq D$ with negative Euler characteristic, we will work with the unique representative g of each conformal class for which the boundary curves are geodesics in (Σ, g) and which is hyperbolic. When Σ is a cylinder we instead choose as our unique representative the flat metric g for which (Σ, g) has unit area. We denote by $\mathcal{M}(\Sigma)$ the space of such constant curvature metrics and note that our definition of the flow will ensure that for an initial metric $g_0 \in \mathcal{M}(\Sigma)$, the evolving metrics always will be in $\mathcal{M}(\Sigma)$.

We recall that at any $g \in \mathcal{M}(\Sigma)$ the tangent space splits $L^2(\Sigma, g)$-orthogonally as

$$T_g\mathcal{M}(\Sigma) = \{L_X g\} \oplus H(g), \tag{3.1.6}$$

where $\{L_X g\}$ is the set of Lie-derivatives generated by vector fields that are parallel to $\partial\Sigma$ on $\partial\Sigma$, giving rise to 1-parameter families of diffeomorphisms of Σ, and where the *horizontal space* $H(g)$ consists of all symmetric $(0,2)$-tensors h which are trace-free and

[1] See Remark 1.3.8 from the introduction for more discussion on this.

divergence-free and which satisfy $h(\nu_g, \tau_g) = 0$ on $\partial\Sigma$, τ_g a unit tangent vector field along $(\partial\Sigma, \mathrm{d}s_g)$, c.f. [Tro92].

Remark 3.1.2. We note that for a half-harmonic map $u : (\partial\Sigma, g) \to N$ the tensor $k(u_g, g)$ is always an element of $H(g)$. Indeed since u_g is harmonic k is divergence free, while (3.1.3) implies that $k(u_g, g)(\nu_g, \tau_g) = \langle \partial_{\nu_g} u_g, \partial_{\tau_g} u_g \rangle$ vanishes on the boundary since u maps into N. Hence the harmonic extension of a half-harmonic map is conformal if and only if $P_g^H(k(u_g, g)) = 0$, where we define P_g^H as the $L^2(\Sigma, g)$-orthogonal projection from the space of symmetric $(0, 2)$-tensors to $H(g)$.

Following the construction of the Teichmüller harmonic map flow in [RT16], we now exploit the fact that the energy is invariant under simultaneous pull-back of both the map and the metric (by the same diffeomorphism) to restrict the movement of the metric component to be L^2-orthogonal to the space of tensors $\{L_X g\}$ that is generated by the action of diffeomorphisms on the metric. We hence define our flow as

$$\partial_t u = -\nabla_u^{L^2} E_{\frac{1}{2}}(u, g) = -P_u(\partial_{\nu_g} u_g) \tag{3.1.7}$$

$$\partial_t g = -P_g^H\big(\nabla_g^{L^2} E_{\frac{1}{2}}(u, g)\big) = \frac{1}{2} P_g^H(k(u_g, g)). \tag{3.1.8}$$

The evolution of the map component $u = u(t)$ is hence described by a variant of the Plateau flow studied by Struwe in [Str24], albeit now considered on a general surface Σ with a time-dependent metric $g(t)$ rather than on a disc with fixed metric; the evolution of the metric $g = g(t)$ on the other hand can be viewed as an evolution equation in the infinite dimensional manifold $\mathcal{M}(\Sigma)$ of constant curvature metrics with geodesic boundary curves described above.

Remark 3.1.3. We observe that in the case when $\Sigma = D$ we have $H(g) = \{0\}$ for any metric g on D, and so our coupled gradient flow (3.1.7) reduces to the Plateau flow for maps $u(t)$ on the disc with fixed metric considered in [Str24]. Because of this, we will henceforth not consider the case of the disc as results we obtain are simply those of [Str24] and the methods developed below are unnecessary in this instance. One possible small exception to this is that we have been able to remove the restriction on N having a parallelisable normal bundle present in [Str24], and so one or two of our proofs could be of interest to the disc case.

3.1.3 Main Results

Our first main result establishes the existence of solutions to this new geometric flow and gives a description of the potential singularities that the components of this system of equations might form.

Theorem 3.1.4. *Let $\Sigma \neq D$ be any orientable surface with boundary and let $\mathcal{M}(\Sigma)$ be the set of metrics with constant curvature and geodesic boundary considered above. Then to any initial metric $g_0 \in \mathcal{M}(\Sigma)$ and any initial map $u_0 \in H^{\frac{1}{2}}((\partial\Sigma, g_0); N)$ there exists a weak solution (u, g) of the coupled flow*

$$\partial_t u = -P_u(\partial_{\nu_g} u_g), \qquad \partial_t g = \frac{1}{2} P_g^H(k(u_g, g)) \tag{3.1.9}$$

which has non-increasing energy and is defined on a maximal interval $[0, T_\infty)$ where $T_\infty = \infty$ unless the domain metrics degenerate in finite time, i.e. unless

$$\mathrm{inj}(\Sigma, g(t)) \to 0 \text{ as } t \nearrow T_\infty \text{ for a finite } T_\infty.$$

Furthermore,

1. *Away from a finite number of singular times $0 < T_i^s < T_\infty$, both the map and metric component of the flow are smooth and the energy decays according to*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\frac{1}{2}}(u, g) = -\|P_u(\partial_{\nu_g} u_g)\|^2_{L^2(\partial\Sigma, g)} - \frac{1}{4}\|P_g^H(k(u_g, g))\|^2_{L^2(\Sigma, g)}. \tag{3.1.10}$$

2. *Across each T_i^s the flow of metrics remains regular in the sense that $g(t)$ is Lipschitz continuous in time with respect to any C^k-metric in space.*

3. *Any such singular time T_i^s is characterised by the* bubbling-off *of a finite number of minimal discs of the map component, exactly as in [Str24], see also Section 3.5.6.*

Here and in the following we say that (u, g) is a weak solution of the flow to initial data (u_0, g_0) on an interval $[0, T)$ if:

1. the metric component is a continuous curve $g : [0, T) \to (\mathcal{M}^3(\Sigma), \mathrm{dist}_{H^3})$ with $g(0) = g_0$ which is differentiable at a.e. $t \in [0, T)$ and so that the second equation in (3.1.9) is satisfied at a.e. such t. Here $\mathcal{M}^3$ denotes the set of constant curvature metrics with coefficients in H^3 as discussed in Section 3.2 below.

2. the map component is given by a $u \in L^\infty([0, T); H^{\frac{1}{2}}((\partial\Sigma, \mathrm{d}s_g); N)$ which is so that $\partial_t u \in L^2_{loc}([0, T); L^2(\partial\Sigma, g))$ and so that the first equation of (3.1.9) is satisfied in the sense of distributions.

In the absence of singularities at infinity, the following result ensures that the flow deforms the given initial map into a free boundary minimal surface as desired.

Theorem 3.1.5. *Suppose that (u, g) is a global weak solution of (3.1.9) for which $\mathrm{inj}(\Sigma, g)$ remains bounded away from zero and for which energy does not concentrate as $t \to \infty$, i.e. so that $\limsup_{t\to\infty} \sup_{x\in\partial\Sigma} E(u_g(t), g(t); B_r^{g(t)}(x)) \to 0$ as $r \to 0$.*

Then there exist $t_j \to \infty$ so that after pull-back by diffeomorphisms the pairs $(u(t_j), g(t_j))$ converge smoothly to a limiting pair $(u^*, g^*) \in C^\infty(\partial\Sigma; N) \times \mathcal{M}(\Sigma)$ which is so that $u^*_{g^*} : (\Sigma, g^*) \to \mathbb{R}^n$ is conformal and harmonic, and so that $u^*_{g^*}(\Sigma)$ meets N orthogonally; that is, $u^*_{g^*}(\Sigma)$ represents a (possibly branched) free boundary minimal surface supported by N.

Remark 3.1.6. We note that in the special case where N is given by a collection of k disjoint closed curves $\Gamma_1, \ldots, \Gamma_k$, k the number of boundary components σ_i of Σ, and where each $u_0|_{\sigma_i}$ is a map into Γ_i with non-zero degree, this property is preserved along regular solutions of the flow. In this case the minimal surface obtained in the above theorem can be thought of as a solution of a generalised version of the Douglas-Plateau problem in which multiple coverings and non-monotone parametrisations of the prescribed curves Γ_i are allowed, compare also the discussion in Sections 1.5 and 1.6 of [Str24]. In light of the monotonicity result from the previous chapter, Theorem 2.5.2, it might be hoped that the parametrisation is in fact monotone, but this is yet to be seen, and the functional lacks the convenient representation solely in terms of the function on the boundary available in the case of a circle.[2]

We structure the proofs of these two theorems in the following way. We discuss the key steps of the proof of short time existence of solutions to the flow in Section 3.2 and carry out the required detailed analysis of the equations satisfied by the metric and map components in the subsequent Sections 3.3 and 3.4. In Section 3.5, we prove higher regularity estimates which enable us to deduce the smoothness of the solution until either the metric degenerates or energy concentrates, which combined with the analysis of finite time singularities allows us to complete the proof of our first main result, Theorem 3.1.4. Finally, in Section 3.6 we study the asymptotic behaviour, proving Theorem 3.1.5.

3.2 Short Time Existence of Solutions

In this section, we explain the key steps needed to prove local existence of solutions of (3.1.9) for given smooth initial data (u_0, g_0). This proof will be based on an iteration argument for a regularised system of equations that is carried out on a sufficiently small interval $[0, T]$ and in suitable Sobolev spaces $X_m(T)$ of maps of regularity H^m and curves of metrics in $\mathcal{M}^{m+1}$ of regularity $C_t^1 H^{m+1}$ as made precise below. This is then combined with a compactness argument to obtain our solution.

In the following, we will focus mainly on the case of domains with negative Euler characteristic as the analysis of the metric component simplifies very significantly if Σ is a cylinder, as in this case the equation for the metric reduces to a well-controlled ODE. We will discuss this special case in Section 3.3.3 and for now simply note that all results stated below are applicable also for Σ a cylinder and the corresponding space $\mathcal{M}(\Sigma)$ of flat unit area metrics.

[2] In [PL17], they do obtain a representation of the half-energy in terms of the boundary functions, but this is more complicated and involves interactions between the different boundary components.

3.2.1 Set-up

Given a surface Σ with boundary, we recall that we may think of this domain as a fixed subset of the Schottky double $\hat{\Sigma}$, which is the closed surface that we obtain as quotient of $\Sigma \times \{-1, 1\}$ under the identification $(p, 1) \sim (p, -1)$ for $p \in \partial\Sigma$. Moreover, we note that if g is any hyperbolic metric on Σ for which the boundary curves are geodesics, then g can be extended by even symmetry to a hyperbolic metric $\hat{g}$ on $\hat{\Sigma}$. One way to see this is to note that a neighbourhood of each boundary curve can be described by an collar neighbourhood, see (3.3.3) and (3.3.3) and surrounding discussion below. The even extension of these collars can then be described on $[-X(\ell), X(\ell)] \times \mathcal{S}^1$ to give a smooth hyperbolic metric, and hence the reflected metric $\hat{g}$ is smooth across each of these geodesics, and thus on all of $\hat{\Sigma}$. It is precisely here and for this reason that we are exploiting our choice of uniformisation, see Remark 1.3.8.

Following the approach of Tromba [Tro92], given any $k \in \mathbb{N}_{\geq 3}$ and any fixed finite (and symmetric) set of (smooth) coordinate charts on $\hat{\Sigma}$ we can consider the set $\mathcal{M}^k = \mathcal{M}^k(\Sigma)$ of metrics g on Σ which we obtain as restrictions of hyperbolic metrics $\hat{g}$ on $\hat{\Sigma}$ which have the above even symmetry across $\partial\Sigma$ and whose coefficients are of class H^k in these charts. We measure the distance between two elements of $\mathcal{M}^k$ by letting

$$\mathrm{dist}_{H^k}(g_1, g_2) := \inf \int_1^2 \|\partial_s g(s)\|_{H^k(\Sigma, g(s))} \mathrm{d}s,$$

where the infimum is taken over the set of all C^1-paths $g = g(s) \in \mathcal{M}^k(\Sigma)$, $1 \leq s \leq 2$, that connect $g_1 = g(1)$ and $g_2 = g(2)$. Here and in the following all Sobolev norms are computed using the Levi-Cività connection with respect to the indicated metric.

Given any $m \geqslant 2$ and a curve of metrics $g \in C^1([0, T]; \mathcal{M}^{m+1})$ we will work with maps $u : \partial\Sigma \times [0, T] \to \mathbb{R}^n$ that are contained in the space

$$X_m(T, g) = \{u : u_g \in L^\infty([0, T];\ H^m(\Sigma, g)) \text{ and } \partial_t(u_g) \in L^\infty([0, T]; H^1(\Sigma, g))\}, \quad (3.2.1)$$

which we equip with the corresponding norm

$$\|u\|^2_{X_m(T,g)} := \|u_g\|^2_{L^\infty([0,T];H^m(\Sigma,g))} + \|\partial_t(u_g)\|^2_{L^\infty([0,T];H^1(\Sigma,g))}. \tag{3.2.2}$$

While these norms depend on the specific choice of g, we will see that the spaces $X_m(T,g)$ themselves are indeed independent of the choice of g as the change of harmonic extensions and Sobolev norms along such curves of metrics is well controlled, compare Lemma 3.3.8. We hence often drop the reference to g in the notation for $X_m(T,g)$ and simply write $X_m(T)$, and will at times also use the shorthand $X_{m,loc}(T) := \bigcup_{T'<T} X_m(T')$. We furthermore note that in the proof of short-time existence we will for the most part only need to consider curves of metrics which are contained in a small neighbourhood $\mathcal{U}$ of the initial metric g_0 and whose velocity is uniformly bounded and we note that for such curves of metrics these norms are uniformly equivalent, compare Lemma 3.3.3 below, which will allow us to work with respect to the fixed norm $\|\cdot\|_{X_m(T,g_0)}$ in the relevant fixed point argument.

We furthermore note that we can always work with the spaces $\mathcal{M}^{m+1}$ which we obtain by fixing coordinate charts in which the initial metric is smooth, and will see that this property is preserved along the flow. We can hence in the following always assume that the initial metric is an element of $\mathcal{M}^{m+1}$ for any $m \in \mathbb{N}$.

In the following we also use the convention that all constants are allowed to depend on the setting we consider, i.e. the fixed topological type of the surface Σ, the fixed manifold $N \hookrightarrow \mathbb{R}^n$ and later on also on the fixed extensions $P_\cdot$ and $P_\cdot^\perp$ of the orthogonal projections onto $T_\cdot N$ respectively $T_\cdot^\perp N$ chosen in Remark 3.2.4, without further mentioning this.

3.2.2 Evolution of the Metric Component

We first consider the problem of solving the equation

$$\partial_t g = \frac{1}{2} P_g^H(k(v_g, g)) \text{ with } g(0) = g_0 \tag{3.2.3}$$

for a given map $v \in X_2(T, g_0)$, k the stress-energy tensor defined in (3.1.5). In Section 3.3 we will prove

Proposition 3.2.1. *For any $m \geq 2$ and any $\iota_0 > 0$ there exist constants $C, \delta > 0$ so that the following holds true for any $T > 0$, any $g_0 \in \mathcal{M}^{m+1}$ with $\mathrm{inj}(\Sigma, g_0) \geqslant 2\iota_0$ and the neighbourhood $\mathcal{U}$ of g_0 in $\mathcal{M}^{m+1}$ described in Lemma 3.3.3.*

For any $v \in X_2(T)$ there exists a unique solution $g_v \in C^1([0,T_0]; \mathcal{M}^{m+1})$ of (3.2.3) which is defined and so that $g_v(t) \in \mathcal{U}$, and hence in particular

$$\mathrm{inj}(\Sigma, g_v(t)) \geqslant \frac{1}{2}\mathrm{inj}(\Sigma, g_0) \geqslant \iota_0, \tag{3.2.4}$$

at least on the interval $I = [0, T_0]$ for $T_0 := \min(T, \delta/\hat{E})$, $\hat{E}$ chosen so that

$$\sup_{t\in[0,T]} E_{\frac{1}{2}}(v(t), g_0) \leqslant \hat{E}. \tag{3.2.5}$$

Moreover, for every $t, \tilde{t} \in [0, T_0]$ we have

$$\|\partial_t g_v(t)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C E_{\frac{1}{2}}(v(t), g_0), \tag{3.2.6}$$

and

$$\|\partial_t g_v(t) - \partial_t g_v(\tilde{t})\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\left|t-\tilde{t}\right| \left[\hat{E}^{\frac{1}{2}}\|\partial_t v_{g_0}\|_{L^\infty(I;H^1(\Sigma,g_0))} + \hat{E}\right]. \tag{3.2.7}$$

Moreover, the map $v \mapsto g_v$ is Lipschitz the sense that the estimates

$$\mathrm{dist}_{H^{m+1}}(g_v(t), g_{\tilde{v}}(t)) \leqslant C\hat{E}^{\frac{1}{2}} t\|(v-\tilde{v})_{g_0}\|_{L^\infty(I;H^1(\Sigma,g_0))} \tag{3.2.8}$$

and

$$\|\partial_t(g_v - g_{\tilde{v}})(t)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\hat{E}^{\frac{1}{2}}\left[\|(v-\tilde{v})_{g_0}(t)\|_{H^1(\Sigma,g_0)} + \hat{E}t\|(v-\tilde{v})_{g_0}\|_{L^\infty(I;H^1(\Sigma,g_0))}\right] \tag{3.2.9}$$

hold for all $t \in [0, T_0]$ and for all v and $\tilde{v}$ which satisfy (3.2.5) for a given $\hat{E}$.

Remark 3.2.2. The neighbourhood $\mathcal{U} = \mathcal{U}^{m+1}(g_0)$ in Lemma 3.3.3 is given by a ball in $\mathcal{M}^{m+1}$ around g_0 whose radius only depends on ι_0 and which is chosen so that metrics in $\mathcal{U}$ are uniformly equivalent and induce uniformly equivalent H^{m+1}-norms. Hence the above estimates all remain valid also if some or all of the norms are computed with respect to another metric of $\mathcal{U}$, such as $g(t)$, instead of g_0.

We can apply the above lemma iteratively to see that unless the injectivity radius tends to zero as t approaches some time $T_1 < T$, the solution of (3.2.3) is guaranteed to exist on the whole interval $[0, T]$ where v is defined. Moreover, the above lemma shows that the estimate

$$\|\partial_t g_v(t)\|_{H^{m+1}(\Sigma, g(t))} \leqslant C E_{\frac{1}{2}}(v(t), g(t)) \tag{3.2.10}$$

holds true with a constant C that only depends on a lower bound on the injectivity radius of $(\Sigma, g(t))$ at that specific time t and the exponent m.

We note that for the proof of short-time existence of a solution of our coupled flow (3.1.9), a weaker version of the above lemma, in which the constants and the size of the neighbourhood are allowed to depend on the initial metric g_0 itself, instead of just on ι_0, would suffice, but that we choose to formulate Proposition 3.2.1 and (3.2.10) in the above more precise form to make them applicable also for the long-term analysis of the flow. Namely, in Section 3.5.6 we will use that the above lemma immediately implies the following.

Remark 3.2.3. Let $(u, g)(t)$, $t \in [0, T]$, be any weak solution of our flow (3.1.9) with non-increasing energy. Then the metric component remains in the neighbourhood $\mathcal{U}(g(0))$ described in Lemma 3.3.3 at least on a time interval of the form $[0, \min(T, T_0)]$, for a number $T_0 = T_0(\hat{E}, \iota_0) > 0$ that only depends on an upper bound on the initial energy and a lower bound $2\iota_0$ on the injectivity radius of the initial metric.

3.2.3 Regularized Equation for the Map Component

Instead of directly proving a result similar to Proposition 3.2.1 also for the map component, we follow the approach of the second author from [Str24, Lemma 5.2], and first consider the simpler, and in particular linear, problem of determining a solution $u = u_{\varepsilon,g,v}$

of the regularized equation

$$\partial_t u = -(\varepsilon + P_v)\partial_{\nu_g} u_g \quad \text{on } \partial\Sigma \tag{3.2.11}$$

for given $\varepsilon > 0$ and given curves of maps $v : \partial\Sigma \to N_\eta$ and metrics g.

Remark 3.2.4. Here and in the following we use fixed extensions $P_\cdot, P_\cdot^\perp \in C_b^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ of the orthogonal projections $P_p : \mathbb{R}^n \to T_pN$, $P_p^\perp : \mathbb{R}^n \to T_p^\perp N$ which are chosen so that

$$P_p = P_{\pi(p)} \text{ and } P_p^\perp = P_{\pi(p)}^\perp \text{ for all } p \in N_\eta \tag{3.2.12}$$

on a small tubular neighbourhood N_η where $\eta > 0$ is chosen so that the nearest point projection is well defined and smooth on the closure $\bar{N}_\eta$ of N_η. We note that this choice of extension in particular ensures that P_v and $P_v^\perp$ are orthogonal projections with $P_v + P_v^\perp =$ Id for maps v whose image is contained in this tubular neighbourhood N_η.

We note that we should not expect solutions of the above equation (3.2.11) to remain in N even if v maps into N, since we no longer evaluate the projection P at the points $u(x,t)$ and since the term $\varepsilon\partial_{\nu_g} u_g$ in general will not be tangential to N. Thus, in order to later be able to apply fixed point arguments, we consider (3.2.11) not only for maps whose image is contained in N, but allow the maps v to take values in this tubular neighbourhood N_η.

Since the term $-\varepsilon\partial_{\nu_g} u_g$ compensates for the degeneracy introduced by the projection operator, we can use a Galerkin approximation to establish local existence for (3.2.11). While this proof of existence of solutions to the regularised equation (3.2.11) crucially uses that $\varepsilon > 0$, we will be able to obtain H^m-bounds for these solutions which are independent of ε. This will be essential in later arguments where we will send ε to zero in order to obtain a solution of our original equation (3.1.7).

The relevant results on the regularised equation (3.2.11), which will be proven in Section 3.4.1, can be summarised by the following proposition.

Proposition 3.2.5. *Let* $\varepsilon \in (0, \frac{1}{2}]$, $m \geq 4$ *and* $T > 0$. *Then for any given* $u_0 \in H^{m-\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$, *any* $g_0 \in \mathcal{M}^{m+1}$, *any* $g \in C^1([0,T]; \mathcal{M}^{m+1})$ *with* $g(0) = g_0$, *and any*

$v : [0,T] \times \partial\Sigma \to N_\eta$ with $v \in X_m(T)$, there exists a unique solution $u = u_{\varepsilon,g,v} \in X_m(T)$ of (3.2.11) with $u(0) = u_0$ and this solution is furthermore so that $u \in L^2([0,T]; H^m(\partial\Sigma, g))$ and so that the estimates

$$\|u_g\|^2_{L^\infty([0,T];H^m(\Sigma,g)))} \leqslant e^{CT}\|(u_0)_{g_0}\|^2_{H^m(\Sigma,g_0)} \tag{3.2.13}$$

and

$$\varepsilon\|\partial_\nu \nabla^{m-1} u_g\|^2_{L^2([0,T];L^2(\partial\Sigma,g))} \leqslant 2e^{CT}\|(u_0)_{g_0}\|^2_{H^m(\Sigma,g_0)} \tag{3.2.14}$$

hold for a constant C that only depends on m and numbers $0 < \iota_0, M, \Lambda < \infty$, which are chosen so that for all $t \in [0,T]$,

$$\mathrm{inj}(g(t)) \geqslant \iota_0, \quad \|\partial_t g(t)\|_{H^m(\Sigma,g(t))} \leqslant M, \quad \|(v(t))_{g(t)}\|_{H^m(\Sigma,g(t))} \leqslant \Lambda. \tag{3.2.15}$$

We stress in particular that C is independent *of ε.*

In addition to this result, we will also show that for every $\varepsilon \in (0, \frac{1}{2}]$, the mapping $(v,g) \mapsto u_{\varepsilon,v,g}$ is Lipschitz in the sense described in Lemma 3.4.4.

Combining these results on the map and metric component will then allow us to prove that for suitably small $T = T(\varepsilon, u_0, g_0, m) > 0$, the map

$$\Psi = \Psi_{\varepsilon,u_0,g_0} : v \mapsto u_{\varepsilon,v,g_v}$$

is a contraction on a suitably chosen subset of $X_m(T)$, see Lemma 3.4.5 for details. We can hence deduce that for each $\varepsilon \in (0, \frac{1}{2}]$, the regularised system

$$\partial_t u_\varepsilon = -(\varepsilon + P_{u_\varepsilon})\partial_{\nu_{g_\varepsilon}} u_{g_\varepsilon}, \quad \partial_t g_\varepsilon = \frac{1}{2} P^H_{g_\varepsilon}(k(u_{g_\varepsilon}, g_\varepsilon)) \tag{3.2.16}$$

can be solved at least on a small, a priori *ε-dependent*, time interval interval $[0, T_\varepsilon]$.

The fact that the estimate (3.2.13) on the evolution of the H^m norm of the map component is ε-independent, combined with the uniform control on the metric component obtained in Proposition 3.2.1, will then allow us to establish that these solutions of the regularised system (3.2.16) indeed exist, and remain well controlled, on an *ε-independent* time interval. To be more precise, in Section 3.4.2 we will show the following Proposition.

Proposition 3.2.6. *For any $m \geq 4$, any $\iota_0 > 0$ and any $\Lambda_0 < \infty$ there exists a time $T > 0$ and a constant C so that for any initial metric $g_0 \in \mathcal{M}^{m+1}$ with $\mathrm{inj}(\Sigma, g_0) \geqslant 2\iota_0$ and any initial map $u_0 \in H^{m-\frac{1}{2}}((\partial\Sigma, g_0); N)$ with $\|(u_0)_{g_0}\|_{H^m(\Sigma,g_0)} \leqslant \Lambda_0$ the following holds true.*

For every $0 < \varepsilon \leq \frac{1}{2}$ there exists a unique solution $(u_\varepsilon, g_\varepsilon) \in X_m(T) \times C^1([0,T]; \mathcal{M}^{m+1})$ of the regularised system (3.2.16) with $u_\varepsilon(0) = u_0$, $g_\varepsilon(0) = g_0$ and on this ε-independent interval $[0,T]$ the map component remains bounded by

$$\|(u_\varepsilon)_{g_\varepsilon}\|_{H^m(\Sigma,g_\varepsilon)} \leqslant 2\Lambda_0 \tag{3.2.17}$$

while the metric component g_ε remains in the neighbourhood $\mathcal{U} \subset \mathcal{M}^{m+1}$ of g_0 obtained in Lemma 3.3.3 and is so that

$$\|g_\varepsilon\|_{C^{1,1}([0,T];\mathcal{M}^{m+1})} \leqslant C. \tag{3.2.18}$$

As neither $T > 0$ nor C depend on $\varepsilon > 0$, we can exploit the above uniform bounds on the solutions $(u_\varepsilon, g_\varepsilon)$, $\varepsilon \in (0, \frac{1}{2}]$ of the regularised flow (3.2.16) and pass to the limit along suitable $\varepsilon \downarrow 0$ to obtain a solution of our original problem on the interval $[0,T]$ for $T = T(\Lambda_0, \iota_0, m) > 0$ as in the above Proposition 3.2.6. Iteratively applying this short-time existence result, which yields a solution for an additional time interval whose length is bounded away from zero for as long as both the injectivity radius and the H^m-norm remain controlled, we can hence deduce

Proposition 3.2.7. *For every initial data $(u_0, g_0) \in H^{m-\frac{1}{2}}(\partial\Sigma; N) \times \mathcal{M}^{m+1}$, $m \geqslant 4$, there exists a solution $(u,g) \in X_{loc}(T_m^*) \times C^1_{loc}([0,T_m^*); \mathcal{M}^{m+1})$ to the coupled flow (3.1.9) on a maximal time interval $[0, T_m^*)$, where $T_m^* = \infty$ unless either*

$$\lim_{t \uparrow T_m^*} \mathrm{inj}(g(t)) = 0 \text{ or } \lim_{t \uparrow T_m^*} \|u(t)\|_{H^m(\Sigma, g(t))} = \infty. \tag{3.2.19}$$

In Section 3.5 we will establish that the H^m norm remains indeed controlled for as long as there is no concentration of energy. Combined with the analysis of potential finite time singularities of the map component that is carried out in Section 3.5.6, following [Str24], this will complete the proof of our main Theorem 3.1.4.

3.3 Analysis of the Metric Component

We will first carry out the analysis of the metric component in detail in the more difficult case where our domain surface Σ is neither a cylinder nor a disc. We will discuss the adaptions, and the significant simplifications, of this analysis for the case where Σ is cylinder later in Section 3.3.3 and again recall that in the case where Σ is a disc, the flow reduces to the Plateau flow studied by Struwe in [Str24].

3.3.1 Basic Properties of the Set $\mathcal{M}(\Sigma)$ of Hyperbolic Metrics

Let Σ be an orientable surface of general type and let $\mathcal{M}(\Sigma)$ be the set of hyperbolic metrics for which the boundary curves are geodesics as considered above. In this section we collect some useful properties of these metrics which will be used not only in the analysis of horizontal curves as considered in Proposition 3.2.1, but also later on in the paper. To lighten the notation we will often use the short-hand

$$\mathcal{M}_{\iota_0} := \{g \in \mathcal{M} : \mathrm{inj}(\Sigma, g) \geqslant \iota_0\} \text{ for } \iota_0 > 0 \tag{3.3.1}$$

as well as $\mathcal{M}_{\iota_0}^{m+1} := \{g \in \mathcal{M}^{m+1} : \mathrm{inj}(\Sigma, g) \geqslant \iota_0\}$ and note the following.

Remark 3.3.1. It is easy to see that the subset of moduli space which corresponds to metrics with $\mathrm{inj}(g) \geqslant \iota_0$, $\iota_0 > 0$ any fixed number, is compact. Hence the usual Sobolev embedding theorems (for both functions and tensors) are all applicable on surfaces (Σ, g), $g \in \mathcal{M}_{\iota_0}$, with constants that do not depend on the specific choice of metric $g \in \mathcal{M}_{\iota_0}$, but only on $\iota_0 > 0$ (and of course the exponents of the involved Sobolev spaces and as usual the topology of Σ). We also stress that all Sobolev spaces and norms are to be understood with respect to the corresponding metric g (at the relevant time if g is time dependent) unless specified otherwise. We also note that while the Sobolev spaces $H^s(\partial\Sigma, g)$ and

$H^s(\Sigma, g)$ of any order s are defined for all metrics $g \in \mathcal{M}^{m+1}$, we will only ever consider such spaces for exponents $s \leqslant m$ to ensure that the spaces that we obtain are independent of the choice of metric $g \in \mathcal{M}^{m+1}$ and note that the dependence on g of the corresponding norms can be controlled as described in Lemma 3.3.8.

We furthermore recall for such hyperbolic metrics on Σ, there is a neighbourhood $\mathcal{C}(\sigma_i, g)$ of each boundary curve σ_i of Σ which is isometric to the hyperbolic cylinder

$$\big([0, X(\ell)) \times \mathcal{S}^1, \rho_\ell^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)\big), \tag{3.3.2}$$

for $\ell = L_g(\sigma_i)$ and with σ_i corresponding to $\{0\} \times \mathcal{S}^1$, where

$$X(\ell) = \tfrac{2\pi}{\ell}[\tfrac{\pi}{2} - \arctan(\sinh\tfrac{\ell}{2})], \qquad \rho_\ell(s) = \tfrac{\ell}{2\pi}[\cos(\tfrac{\ell s}{2\pi})]^{-1}. \tag{3.3.3}$$

By passing to the double $\hat{\Sigma}$ and using the corresponding version of the collar lemma for all simple closed geodesics, or exploiting the above remark, it is furthermore easy to see that the lengths $L_g(\sigma_i)$ of the boundary curves are bounded both away from zero and from above uniformly for all metrics in $g \in \mathcal{M}_{\iota_0}$.

As a result, we obtain the following uniform control on the behaviour of the metric on the neighbourhood $\mathcal{C}(\partial\Sigma, g) := \bigcup_i \mathcal{C}(\sigma_i, g)$ of the boundary of such surfaces (Σ, g), $g \in \mathcal{M}_{\iota_0}$.

Remark 3.3.2. For any $g \in \mathcal{M}_{\iota_0}$, $\iota_0 > 0$ any given number, estimates of the form

$$0 < c \leqslant X(\ell_i) \leqslant C \text{ and } 0 < c \leqslant \rho_{\ell_i} \leqslant C \text{ on } [0, X(\ell_i)], \quad \ell_i := L_g(\sigma_i) \tag{3.3.4}$$

are valid for constants $c > 0$ and C that only depend on ι_0 (and as usual the topology of Σ), while for any $k \in \mathbb{N}$ the estimate $\|\rho_{\ell_i)}\|_{C^k(\mathcal{C}(\sigma_i,g))} \leqslant C$ holds true for a constant that additionally depends on k.

For each such surface we can hence in particular choose a cut-off function $\varphi \in C_c^\infty(\mathcal{C}(\partial\Sigma, g))$ with $\varphi \equiv 1$ in a neighbourhood of $\partial\Sigma$ which is so that $\|\varphi\|_{C^k(\mathcal{C}(\partial\Sigma,g))} \leqslant C = C(\iota_0, k)$ and so that $\mathrm{dist}_g(\mathrm{supp}(\nabla\varphi), \partial\Sigma) \geqslant c(\iota_0) > 0$.

On occasion we will also want to extend the unit tangent τ_g and the unit normal

ν_g from $(\partial\Sigma, g)$ to this collar neighbourhood and will always do this by choosing the vector fields which are given in collar coordinates by the fixed rescalings of the standard coordinate vector fields

$$\tau := \rho_{\ell_i}(0)^{-1}\frac{\partial}{\partial\theta} \text{ and } \nu := -\rho_{\ell_i}(0)^{-1}\frac{\partial}{\partial s}. \tag{3.3.5}$$

We note that while this does not yield vector fields with unit length, this choice of extension is convenient since it yields $\partial_\nu^2 U + \partial_\tau^2 U = 0$ for any function U which is harmonic with respect to g.

We note that for (orientable) surfaces Σ different from the disc or the cylinder, it has already been pointed out by Tromba in [Tro92] that the map $g \mapsto H(g)$ is not integrable in the sense of Frobenius's theorem; that is, there exists no submanifold $\mathcal{M}_H$ of $\mathcal{M}(\Sigma)$ (or of $\mathcal{M}_{\iota_0}^{m+1}(\Sigma)$ for any finite $m \geq 2$) whose tangent spaces $T_g\mathcal{M}_H$ coincide with the corresponding horizontal spaces

$$H(g) := \{h \in \Gamma^{sym}(T^*\Sigma \otimes T^*\Sigma) : \mathrm{tr}_g(h) = 0, \quad \mathrm{div}_g(k) = 0, \quad h(\nu_g, \tau_g)|_{\partial\Sigma} = 0\}. \tag{3.3.6}$$

In practice this means that curves which are horizontal, i.e. whose velocity $\partial_t g$ at each time is given by an element of the corresponding horizontal space $H(g(t))$ and which start at a given initial metric g_0 will not be constrained to a finite dimensional manifold. Conversely, in the case where Σ is a cylinder such horizontal curves are constrained to an explicit 1-dimensional submanifold of $\mathcal{M}$, compare Section 3.3.3 below.

We also stress that the equation for the metric component of our coupled flow (3.1.9) cannot be viewed as an equation on Teichmüller space but that it is important to keep track of the metric $g(t)$ itself as different representatives $\tilde{g}(t) = f_t^* g(t)$ of the same curve in Teichmüller space lead to different PDEs for the map component.

For the proof of short-time existence we will be working with metrics which are contained in a small neighbourhood of a given $g_0 \in \mathcal{M}^{m+1}$ which is chosen as in the following lemma.

Lemma 3.3.3. *For every $\iota_0 > 0$ and $m \geqslant 2$ there exists $r_0 = r_0(\iota_0, m) > 0$ so that for* *each* $g_0 \in \mathcal{M}^{m+1}_{2\iota_0}$ *all metrics in the neighbourhood*

$$\mathcal{U} = \mathcal{U}^{m+1}_{g_0,r_0} := \{g \in \mathcal{M}^{m+1} : \mathrm{dist}_{H^{m+1}}(g_0, g) \leqslant r_0\} \tag{3.3.7}$$

of g_0 *are uniformly equivalent in the sense that*

$$\frac{1}{4}g \leqslant \tilde{g} \leqslant 4g \text{ for all } g, \tilde{g} \in \mathcal{U} \tag{3.3.8}$$

and hence in particular so that

$$\mathrm{inj}(\Sigma, g) \geqslant \frac{1}{2}\mathrm{inj}(\Sigma, g_0) \geqslant \iota_0 \qquad \text{for all } g \in \mathcal{U}. \tag{3.3.9}$$

Furthermore, all metrics in $\mathcal{U}$ *induce uniformly equivalent Sobolev norms in the sense that there exists a constant* $C = C(m, \iota_0)$ *so that the estimates*

$$\|v\|_{H^k(\Sigma,g)} \leqslant C\|v\|_{H^k(\Sigma,\tilde{g})} \qquad \text{and} \qquad \|h\|_{H^k(\Sigma,g)} \leqslant C\|h\|_{H^k(\Sigma,\tilde{g})} \tag{3.3.10}$$

hold true for all $g, \tilde{g} \in \mathcal{U}$*, all* $0 \leqslant k \leqslant m+1$*, all functions* $v \in H^k(\Sigma; \mathbb{R}^n)$ *and all* $(0, 2)$ *tensors* h *on* Σ *whose coefficients are in* H^k.

Remark 3.3.4. As explained in Section 3.3.4, such metrics also yield uniformly equivalent half-energies, namely are so that the estimate

$$E_{\frac{1}{2}}(v, g) \leqslant C E_{\frac{1}{2}}(v, \tilde{g}) \text{ for all } g, \tilde{g} \in \mathcal{U}^{m+1}_{g_0,r_0} \tag{3.3.11}$$

holds true for all $v \in H^{\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$ for a constant $C = C(\iota_0)$.

Proof of Lemma 3.3.3. Let $\mathcal{U}$ be as defined above for a number $r_0 > 0$ that we determine below. Given any $g_1 \in \mathcal{U}$ we can consider a curve of metrics $g(t), t \in [0, 1]$, with $g(0) = g_0$ and $g(1) = g_1$ so that $\int_0^1 \|\partial_t g(t)\|_{H^{m+1}(\Sigma,g(t))}\mathrm{d}t < 2r_0$ and we let $t_0 \in (0, 1]$ be the maximal time so that (3.3.9) holds for all metrics $g(t)$ with $t \in [0, t_0]$.

As we have pointwise bounds of

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|X|^2_g\right| = |(\partial_t g)(X, X)| \leqslant |\partial_t g|_g |X|^2_g \leqslant \|\partial_t g\|_{L^\infty(\Sigma,g)} |X|^2_g \tag{3.3.12}$$

for any vector field X on Σ and as the Sobolev-embedding theorem $H^2(\Sigma, g) \hookrightarrow L^\infty(\Sigma, g)$

is valid with the same constant $C_0 = C_0(\iota_0)$ for all metrics $g(t)$ with $t \in [0, t_0]$, we can integrate the resulting estimate of $\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(|X|_g^2)\right| \leqslant \|\partial_t g\|_{L^\infty(\Sigma,g)} \leqslant C_0\|\partial_t g\|_{H^{m+1}(\Sigma,g)}$ over any subinterval of $[0, t_0]$ and use that $\int_0^1 \|\partial_t g\|_{H^{m+1}(\Sigma,g)}\mathrm{d}t < 2r_0$ to deduce that

$$g(t) \leqslant e^{2C_0 r_0} g(\tilde{t}) \leqslant 2g(\tilde{t}) \text{ for all } t, \tilde{t} \in [0, t_0], \tag{3.3.13}$$

where the last inequality holds provided $r_0 > 0$ is small enough.

We conclude that the lengths of any curve σ in Σ with respect to two such metrics are related by $L_{g(t)}(\sigma) \leqslant \sqrt{2}L_{g(\tilde{t})}(\sigma)$. Since the injectivity radius of hyperbolic surfaces (Σ, g) is given by half of the length of the shortest curve which is either closed and non-contractible or so that it connects two points of $\partial\Sigma$, we can hence deduce that $\mathrm{inj}(\Sigma, g(t)) \leqslant \sqrt{2}\mathrm{inj}(\Sigma, g(\tilde{t}))$ for all $t, \tilde{t} \in [0, t_0]$. In particular,

$$\mathrm{inj}(\Sigma, g(t_0)) \geqslant \tfrac{1}{\sqrt{2}}\mathrm{inj}(\Sigma, g_0) > \tfrac{1}{2}\mathrm{inj}(\Sigma, g_0),$$

so as t_0 was chosen as the maximal number $t_0 \leqslant 1$ so that (3.3.9) holds on $[0, t_0]$ we must have $t_0 = 1$.

The estimates on $g(t)$ and $\mathrm{inj}(\Sigma, g(t))$ obtained above are hence in particular applicable for $g_1 = g(1)$ which immediately yields the first two claims (3.3.8) and (3.3.9) of the lemma.

We can then use that pointwise estimates of the form

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\nabla^k_{g(t)} v\right|_{g(t)} \leqslant C \sum_{i+j\leqslant k} \left|\nabla^i_{g(t)}\partial_t g(t)\right|_{g(t)} \cdot \left|\nabla^j_{g(t)} v\right|_{g(t)}, \qquad C = C(k) \tag{3.3.14}$$

hold true for every $k \in \{0, \ldots, m\}$, for all sufficiently smooth $v : \Sigma \to \mathbb{R}^n$ and every curve of hyperbolic metrics. As (3.3.9) and Remark 3.3.1 ensure that the standard Sobolev embeddings $H^2(\Sigma, g) \hookrightarrow L^\infty(\Sigma, g)$ and $H^1(\Sigma, g) \hookrightarrow L^4(\Sigma, g)$ are applicable with the same constant for all $g \in \mathcal{U}$, and as the change in the volume element is controlled by $\|\partial_t g\|_{L^\infty(\Sigma,g)}$, we can hence bound

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^k(\Sigma,g))}\right| \leqslant C\|\partial_t g\|^2_{H^{m+1}(\Sigma,g)}\|v\|^2_{H^k(\Sigma,g)} \text{ for some } C = C(m, \iota_0, \Sigma) \tag{3.3.15}$$

for all $k \in \{0, \ldots, m+1\}$ and all functions $v \in H^k(\Sigma; \mathbb{R}^n)$ as claimed. Integrated over t

this immediately yields the claimed equivalence of norms of functions, and we note that the same argument also applies for tensors. □

3.3.2 Analysis of Horizontal Curves of Hyperbolic Metrics

The key tool needed for the proof of Proposition 3.2.1 is the following Lipschitz property of the projection P^H.

Lemma 3.3.5. *For any $\iota_0 > 0$ and $m \geqslant 2$, there exist constants $C < \infty$ and $r_1 \in (0, r_0)$, $r_0 > 0$ as in the above Lemma 3.3.3, so that for any $g_0 \in \mathcal{M}^{m+1}_{2\iota_0}$ the following assertions hold true for every metric g in the neighbourhood $\mathcal{U} = \mathcal{U}^{m+1}_{g_0,r_1} := \{g \in \mathcal{M}^{m+1} : \mathrm{dist}_{H^{m+1}}(g_0, g) \leqslant r_1\}$ of g_0 in $\mathcal{M}^{m+1}$.*

Let P^H_g be the $L^2(\Sigma, g)$-orthogonal projection from the space $\Gamma^{sym}_{L^2(\Sigma,g)}(T^*\Sigma \otimes T^*\Sigma)$ of symmetric $(0,2)$-tensors on Σ with finite $L^2(\Sigma, g)$-norm to the horizontal space $H(g)$. Then for every $h \in \Gamma^{sym}_{L^2(\Sigma,g)}(T^*\Sigma \otimes T^*\Sigma)$ and all $g, \tilde{g} \in \mathcal{U}$ we can bound

$$\|P^H_g(h)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\|h\|_{L^1(\Sigma,g_0)} \tag{3.3.16}$$

as well as

$$\|P^H_g(h) - P^H_{\tilde{g}}(h)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\mathrm{dist}_{H^{m+1}}(g, \tilde{g})\|h\|_{L^1(\Sigma,g_0)}, \tag{3.3.17}$$

and for any C^1-curve of metrics g_ε in $\mathcal{U}$, we have

$$\|\partial_\varepsilon P^H_{g_\varepsilon}(h)\|_{H^{m+1}(\Sigma,g_0)} \leq C\|\partial_\varepsilon g_\varepsilon\|_{H^{m+1}(\Sigma,g_0)}\|h\|_{L^1(\Sigma,g_0)}. \tag{3.3.18}$$

We note that (3.3.16) in particular ensures that the projection has a unique continuous extension to the space $\Gamma^{sym}_{L^1(\Sigma,g)}(T^*\Sigma \otimes T^*\Sigma)$ of tensors with finite L^1 norm, and remark that the above estimates remain valid if we compute some or all of the norms with respect to g since Lemma 3.3.3 ensures that the metrics in $\mathcal{U}$ and the induced H^{m+1} norms are uniformly equivalent.

Proof of Lemma 3.3.5. As in the introduction we let $\hat{\Sigma}$ be the Schottky double of Σ and let $\hat{\mathcal{M}}^{m+1}$ be the manifold of all hyperbolic metrics on $\hat{\Sigma}$ whose coefficients are in the

Sobolev space H^{m+1} with respect to the fixed coordinate charts on $\hat{\Sigma}$. Given $\hat{g} \in \hat{\mathcal{M}}^{m+1}$ we let $\hat{H}(\hat{g}) = \hat{H}(\hat{\Sigma}, \hat{g})$ be the space of all symmetric (0,2)-tensors on $\hat{\Sigma}$ which are trace free and divergence free and let $P_{\hat{g}}^{\hat{H}}$ be the $L^2(\hat{\Sigma}, \hat{g})$-orthogonal projection from the space of $(0,2)$-tensors to $\hat{H}$.

We now make use of Lemma 2.9 from [Rup14], which ensures that for every $\hat{g} \in \hat{\mathcal{M}}^{m+1}$ there exists a neighbourhood $\hat{\mathcal{U}}$ of $\hat{g}$ in $\hat{\mathcal{M}}^{m+1}$ and a constant $C > 0$ so that $P_{\hat{g}}^{\hat{H}}$ satisfies the analogues of the claims on P_g^H made in Lemma 3.3.5. While the results in [Rup14] are stated using a different notion of H^{m+1} norm, there computed using a fixed set of coordinate chart, we note that for each fixed g_0 this norm is equivalent to the $H^{m+1}(\Sigma, g_0)$ norm which is computed using the Levi-Cività connection we use in the present paper. The results of [Rup14] hence guarantee that there exists a neighbourhood $\hat{\mathcal{U}}$ of $\hat{g}_0$ and a constant $C = C(\hat{g}_0, m)$ so that for any L^2-tensor $\hat{h}$ on $\hat{\Sigma}$, any $\hat{g}_1, \hat{g}_2 \in \hat{\mathcal{U}}$ and any C^1-curve of metrics $\hat{g}_\varepsilon$ in $\hat{\mathcal{U}}$ we have

$$\|P_{\hat{g}_1}^{\hat{H}}(\hat{h})\|_{H^{m+1}(\hat{\Sigma}, \hat{g}_0)} \leqslant C\|\hat{h}\|_{L^1(\hat{\Sigma}, \hat{g}_0)}, \tag{3.3.19}$$

$$\|P_{\hat{g}_1}^{\hat{H}}(\hat{h}) - P_{\hat{g}_2}^{\hat{H}}(\hat{h})\|_{H^{m+1}(\hat{\Sigma}, \hat{g}_0)} \leqslant C \mathrm{dist}_{H^{m+1}}(\hat{g}_1, \hat{g}_2)\|\hat{h}\|_{L^1(\hat{\Sigma}, \hat{g}_0)}, \tag{3.3.20}$$

as well as

$$\|\partial_\varepsilon P_{\hat{g}_\varepsilon}^{\hat{H}}(\hat{h})\|_{H^{m+1}(\hat{\Sigma}, g_0)} \leq C\|\partial_\varepsilon \hat{g}_\varepsilon\|_{H^{m+1}(\hat{\Sigma}, g_0)}\|\hat{h}\|_{L^1(\hat{\Sigma}, \hat{g}_0)}; \tag{3.3.21}$$

compare also Proposition 2.1 of [Rup14]. We now observe that since these claims are invariant under pull-back by diffeomorphism, and since the subset of moduli space that corresponds to metrics with $\mathrm{inj}(\hat{\Sigma}, g) \geqslant 2\hat{\iota}_0 > 0$ is compact, we indeed obtain that the above estimates (3.3.19)-(3.3.21) holds true on a ball $\hat{\mathcal{U}}_{\hat{g}_0, \hat{r}_1} := \{\hat{g} \in \hat{\mathcal{M}}^m : \mathrm{dist}_{H^{m+1}}(\hat{g}, \hat{g}_0) \leqslant \hat{r}_1\}$ whose radius $\hat{r}_1 > 0$ only depends on m and a lower bound $2\hat{\iota}_0$ on $\mathrm{inj}(\hat{\Sigma}, \hat{g}_0)$ and with a constant $C = C(\hat{\iota}_0, m)$.

To deduce the claims of Lemma 3.3.5 from these facts we can now argue as follows: Given any $h \in \Gamma^{sym}(T^*\Sigma \otimes T^*\Sigma)$ we extend h to the disjoint union $\hat{\Sigma} = \Sigma \times \{-1, 1\}$ by simply setting $\hat{h}(p, \pm 1) = h(p)$ for each $p \in \Sigma$. Away from $\partial\Sigma$, and hence almost everywhere, this yields a well defined tensor $\hat{h}$ on the double $\hat{\Sigma}$ which is even with respect to $\partial\Sigma$ in the sense that

$$\hat{h}(p,1)(w_1,w_2) = \hat{h}(p,-1)(w_1,w_2) \text{ for } p \in \Sigma \setminus \partial\Sigma \text{ and } w_{1,2} \in T_p\Sigma \hat{=} T_{(p,\pm 1)}\hat{\Sigma}. \quad (3.3.22)$$

In this way we can extend any L^2-tensor field on Σ to an L^2-tensor field on the double with even symmetry. We do not claim that for smooth h this extension always yields a smooth, or even just continuous, tensor field $\hat{h}$ as some of the components of $\hat{h}$ can jump across $\partial\Sigma$ in $\hat{\Sigma}$.

To see this we can use the description and coordinates of the collar neighbourhoods $\mathcal{C}(\sigma_i)$ of the boundary curves σ_i which we recalled in Section 3.3.1. On the corresponding neighbourhood $\mathcal{C}(\sigma_i) \times \{-1,+1\}/\sim$ of σ_i in the double $(\hat{\Sigma}, \hat{g})$ we can then work in the coordinates that we obtain by identifying $((s,\theta),\pm 1)$ with $(\pm s,\theta)$. The components of the extended tensor $\hat{h}$ in these coordinates at $(\pm s,\theta)$ are so that $\hat{h}_{ss}(\pm s,\theta) = h_{ss}(s,\theta)$ and $\hat{h}_{\theta\theta}(\pm s,\theta) = h_{\theta\theta}(s,\theta)$ are continuous across $\partial\Sigma$ whenever h is continuous, but so that $\hat{h}_{s\theta}(\pm s,\theta) = \pm h_{s\theta}(s,\theta)$ jumps unless $h_{s\theta}$ vanishes on $\partial\Sigma$, that is, unless $h(\nu_g,\tau_g)|_{\partial\Sigma} \equiv 0$.

In the special case where we extend the metric tensor g itself in this way, the resulting tensor $\hat{g}$ is not only continuous but indeed smooth across $\partial\Sigma$ since $\rho_\ell(s) = \rho_\ell(-s)$.

Similarly, the extension of any tensor $h \in H(g)$ yields an element of $\hat{H}(\hat{g})$, so in particular a smooth tensor. To see this, we first recall that such a tensor can be written as $h = \mathrm{Re}(\Psi)$ for a holomorphic quadratic differential Ψ on Σ which is real on the boundary in the sense that in collar coordinates (s,θ) near any σ_i we have $\Psi = \psi(\mathrm{d}s + i\mathrm{d}\theta)^2$ for a holomorphic $\psi : [0, X(\ell)) \times \mathcal{S}^1$ which is real on the circle $\{0\} \times \mathcal{S}^1$. The extended tensor $\hat{h}$ is then given by $\mathrm{Re}(\hat{\psi}(\mathrm{d}s + i\mathrm{d}\theta)^2)$ for $\hat{\psi}$ defined by $\mathrm{Re}(\hat{\psi}(\pm s,\theta)) = \mathrm{Re}(\psi(s,\theta))$ and $\mathrm{Im}(\hat{\psi}(\pm s,\theta)) = \pm\mathrm{Im}(\psi(s,\theta))$. As $\mathrm{Im}(\psi(0,\theta)) = 0$ on $\partial\Sigma$ this function $\hat{\psi}$ is holomorphic on $\hat{\Sigma}$, and hence in particular smooth across $\partial\Sigma$, and the extended tensor $\hat{h} = \mathrm{Re}(\hat{\psi}dz^2)$ is an element of $\hat{H}(\hat{g})$ as elements of this space can be equivalently characterised as real parts of holomorphic quadratic differentials.

We now claim that to project a given $h \in \Gamma_{L^2}^{sym}(T^*\Sigma \otimes T^*\Sigma)$ onto $H(g)$ we can equivalently first extend h to the even L^2-tensor $\hat{h}$ on the double as described above, then project this tensor onto $\hat{H}$, and, finally, again restrict $\hat{h}$ to Σ. That is, we claim that

$$P_g^H(h) = P_{\hat{g}}^{\hat{H}}(\hat{h})|_\Sigma. \tag{3.3.23}$$

To see that this holds true, we split $\hat{H} = \hat{H}_{even} \oplus \hat{H}_{odd}$ into the tensors with even or odd symmetry with respect to $\partial\Sigma$, respectively, that is, into tensors on $\hat{\Sigma}$ corresponding to tensors h on $\Sigma \times \{-1, 1\}$ with $h(p, 1) = h(p, -1)$ for all p or $h(p, 1) = -h(p, -1)$ for all p, respectively. As discussed above, the extension of any element of $H(g)$ yields an element of $\hat{H}_{even}$ and conversely the restriction of elements of $\hat{H}_{even}$ to Σ is contained in $H(g)$ since the (s, θ) component of such smooth even tensors on $\hat{\Sigma}$ must vanish on $\partial\Sigma$. As the extension $\hat{h}$ of any $h \in \Gamma_{L^2}^{sym}(T^*\Sigma \otimes T^*\Sigma)$ is even we hence deduce that $P_g^H(h) = P_{\hat{g}}^{\hat{H}_{even}}(\hat{h})|_\Sigma$. Combined with the fact that being even forces the extended tensor $\hat{h}$ to be $L^2(\hat{\Sigma}, \hat{g})$-orthogonal to $\hat{H}_{odd}$, this gives (3.3.23).

As choosing $r_1 := \frac{1}{\sqrt{2}}\hat{r}_1$ ensures that the extension of metrics $g \in \mathcal{U}_{g_0,r_1}$ is contained in the neighbourhood $\hat{\mathcal{U}}_{\hat{g}_0,\hat{r}_1}$ of $\hat{g}_0$ on which (3.3.19), (3.3.20) and (3.3.21) hold, we can hence immediately deduce the claims of Lemma 3.3.5 from these properties of the projection $P^{\hat{H}}$ that were proven in [Rup14]. □

Based on these Lipschitz estimates we can now deduce

Lemma 3.3.6. *For any $m \geq 2$ and $\iota_0 > 0$ there exist constants $C, \delta_1 > 0$ so that the following holds true. Let $g_0 \in \mathcal{M}^{m+1}$ be any metric with $\operatorname{inj}(\Sigma, g_0) \geqslant 2\iota_0$, let $h \in C^1([0, T]; \Gamma_{L^1(\Sigma,g_0)}^{sym}(T^*\Sigma \otimes T^*\Sigma))$ for some $T > 0$. Then there is a unique solution $g \in C^1([0, T_1]; \mathcal{M}^{m+1})$ of*

$$\partial_t g = P_g^H(h) \text{ with } g(0) = g_0 \tag{3.3.24}$$

which is defined and remains in the neighbourhood $\mathcal{U} = \mathcal{U}_{g_0,r_1}$ of g_0 where both Lemmas 3.3.3 and 3.3.5 are applicable at least until $T_1 := \min(T, \delta_1/M_1)$ for M_1 chosen so that $\sup_{[0,T]} \|h\|_{L^1(\Sigma,g_0)} \leqslant M_1$. Furthermore this solution satisfies

$$\|\partial_t g(t)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\|h(t)\|_{L^1(\Sigma,g_0)} \text{ for all } t \in [0, T_1] \tag{3.3.25}$$

and the map $h \mapsto g$ is Lipschitz, in the sense that the solutions $g_{1,2}$ of (3.3.24) to tensors

$h_{1,2}$ *with* $\sup_{[0,T]} \|h_{1,2}\|_{L^1(\Sigma,g_0)} \leqslant M_1$ *for some given* M_1 *satisfy*

$$\mathrm{dist}_{H^{m+1}}(g_1(t), g_2(t)) \leqslant Ct\|h_1 - h_2\|_{L^\infty([0,T_1];L^1(\Sigma,g_0))} \text{ for all } 0 \leqslant t \leqslant T_1. \tag{3.3.26}$$

Proof of Lemma 3.3.6. Let g_0, $\mathcal{U} = \mathcal{U}^{m+1}_{g_0,r_1}$, h and M_1 be as in the lemma and let $T_1 := \min(T, \delta_1/M_1)$ for $\delta_1 > 0$ to be determined below.

Given any curve of metrics $g \in C^0([0,T_1];\mathcal{U})$ we define

$$G_g(t) = g_0 + \int_0^t P_g^H(h(t'))\mathrm{d}t' \tag{3.3.27}$$

and note that a sufficiently small choice of $\delta_1 = \delta_1(\iota_0, m) > 0$ ensures that $G_g(t) \in \mathcal{U}$ for all $t \in [0, T_1]$ as the estimate (3.3.16) of Lemma 3.3.5 allows us to bound

$$\mathrm{dist}_{H^{m+1}}(g_0, G_g(t)) \leqslant C\int_0^t \|h(t')\|_{L^1(\Sigma,g_0)}\mathrm{d}t' \leqslant CM_1 t \tag{3.3.28}$$

for as long as $G_g(t)$ is in $\mathcal{U}$, hence ensuring that $\mathrm{dist}_{H^{m+1}}(g_0, G_g(t))$ remains strictly less than r_1 on all of $[0,T_1]$ if $\delta_1 < C^{-1}r_1$.

To obtain the desired solution of (3.3.24) we now want to argue that a sufficiently small choice of δ_1 ensures that this map $g \mapsto G_g$, which we have just established is mapping $C^0([0,T_1];\mathcal{U})$ to itself, is a contraction.

Given $g_{1,2} \in C^0([0,T_1];\mathcal{U})$ we can always choose metrics $g_s(t) \in \mathcal{U}, t \in [0,T_1], s \in [1,2]$, so that for each t the curve $s \mapsto g_s(t)$ is continuously differentiable, interpolates between $g_1(t)$ and $g_2(t)$ and is so that $\int_1^2 \|\partial_s g_s(t)\|_{H^{m+1}(\Sigma,g_s(t))}\mathrm{d}s \leqslant 2\sup_{[0,T_1]}\mathrm{dist}_{H^{m+1}}(g_1, g_2)$, while for each s the function $t \mapsto g_s(t)$ is continuous away from a finite set of (s independent) times t_i.

As the estimate (3.3.28) is applicable also for such piecewise continuous curves of metrics $t \mapsto g(t) \in \mathcal{U}$, the curves of metrics $t \mapsto G_s(t) := G_{g_s}(t)$, $s \in [1,2]$, which satisfy (3.3.27) for g_s instead of g, are again contained in $\mathcal{U}$ for all $t \in [0,T_1]$. We can hence exploit the equivalence of the induced H^{m+1} and L^1-norms and apply (3.3.18) to bound

$$\|\partial_s G_s(t)\|_{H^{m+1}(\Sigma,G_s(t))} \leqslant C\|\partial_s G_s(t)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\int_0^t \|\partial_s\partial_t G_s(t')\|_{H^{m+1}(\Sigma,g_0)}\mathrm{d}t'$$

$$
\begin{aligned}
&= C\int_0^t \|\partial_s P_{g_s(t')}(h(t'))\|_{H^{m+1}(\Sigma,g_0)}\mathrm{d}t' \\
&\leqslant CM_1\int_0^t \|\partial_s g_s(t')\|_{H^{m+1}(\Sigma,g_0)}\mathrm{d}t' \\
&\leqslant C\delta_1 \sup_{[0,T_1]} \mathrm{dist}_{H^{m+1}}(g_1,g_2)
\end{aligned}
$$

for every $t \in [0,T_1]$ and $s \in [1,2]$ and for a constant $C = C(\iota_0, m)$. After reducing $\delta_1 = \delta_1(\iota_0,m) > 0$ if necessary, we hence deduce that

$$
\sup_{[0,T_1]} \mathrm{dist}_{H^{m+1}}(G_{g_1},G_{g_2}) \leq \frac{1}{2}\sup_{[0,T_1]} \mathrm{dist}_{H^{m+1}}(g_1,g_2),
$$

i.e. that $g \mapsto G_g$ is indeed a contraction from $C^0([0,T_1];\mathcal{U})$ to itself. As $\mathcal{U}$ is defined as a closed ball of a complete metric space, we hence obtain the existence of a unique solution $g \in C^1([0,T_1];\mathcal{U})$ of (3.3.24) from Banach's fixed point theorem. We furthermore observe that the claimed estimate (3.3.25) is an immediate consequence of (3.3.16).

To prove the second part of the theorem we can argue similarly, except that we now consider the family g_s of curves that solve (3.3.24) for the family $h_s := (2-s)h_1+(s-1)h_2$, $s \in [1,2]$, of tensors that interpolates between the two given tensors $h_{1,2}$. The estimates (3.3.16) and (3.3.18) from Lemma 3.3.5 are again applicable and now allow us to bound

$$
\|\partial_t\partial_s g_s\|_{H^{m+1}(\Sigma,g_0)} = \|\partial_s P_{g_s}(h_s)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\|\partial_s g_s(t)\|_{H^{m+1}(\Sigma,g_0)}M_1 + C\|h_1-h_2\|_{L^1(\Sigma,g_0)}.
$$

This allows us to deduce that

$$
\sup_{[0,t]} \|\partial_s g_s\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\delta_1 \sup_{[0,t]} \|\partial_s g_s\|_{H^{m+1}(\Sigma,g_0)} + Ct\|h_1-h_2\|_{L^\infty([0,T];L^1(\Sigma,g_0))},
$$

for a constant $C = C(\iota_0,m)$. After reducing $\delta_1 = \delta_1(\iota_0,m) > 0$ further if necessary, we can absorb the first term of the right hand side into the left hand side and integrate the resulting estimate over $s \in [1,2]$ to obtain the final claim (3.3.26) of the lemma. □

For the proof of Proposition 3.2.1, we can additionally use that the harmonic extension depends continuously on the domain metric as described in detail in Lemma 3.3.8. This immediately implies that the stress-energy tensors $k(v_g,g)$ satisfy the following Lipschitz estimates.

Lemma 3.3.7. *Let $g_0 \in \mathcal{M}^{m+1}_{2\iota_0}$ for some $\iota_0 > 0$ and let $\mathcal{U} = \mathcal{U}^m_{g_0,r_0} \subset \mathcal{M}^{m+1}$ be the neighbourhood of g_0 considered in Lemma 3.3.3 above. Then for any $v, \tilde{v} \in H^{\frac{1}{2}}(\partial\Sigma, g_0)$ and any $g, \tilde{g} \in \mathcal{U}$, we can bound*

$$\|k(v_g, g)\|_{L^1(\Sigma,g_0)} \leqslant CE_{\frac{1}{2}}(v, g_0) \tag{3.3.29}$$

and control the difference between the corresponding stress-energy tensors by

$$\|k(v_g, g) - k(\tilde{v}_{\tilde{g}}, \tilde{g})\|_{L^1(\Sigma,g_0)} \leqslant C\|(v - \tilde{v})_{g_0}\|_{H^1(\Sigma,g_0)}\big(E_{\frac{1}{2}}(v, g_0) + E_{\frac{1}{2}}(\tilde{v}, g_0)\big)^{\frac{1}{2}} \tag{3.3.30}$$

$$+ C\mathrm{dist}_{H^{m+1}}(g, \tilde{g}) \cdot E_{\frac{1}{2}}(v, g_0) \tag{3.3.31}$$

for a constant C that only depends on ι_0.

We are now finally in a position to complete the proof of Proposition 3.2.1 for surfaces Σ of general type.

Proof of Proposition 3.2.1. Let $\iota_0 > 0$ and $m \geqslant 2$ be any fixed numbers, let $g_0 \in \mathcal{M}^{m+1}_{2\iota_0}$ and let $\mathcal{U} = U^m_{g_0,r_1}$ be the neighbourhood of g_0 obtained in Lemma 3.3.5 which we recall is so that also Lemmas 3.3.3 and 3.3.7 are applicable.

Given $v \in X_2(T)$, for $X_2(T)$ as defined in (3.2.1), we let $\hat{E}$ be so that (3.2.5) holds and note that (3.3.29) ensures that

$$\|k(v(t)_g, g)\|_{L^1(\Sigma,g_0)} \leqslant C\hat{E} \text{ for every } t \in [0, T] \text{ and every } g \in \mathcal{U},$$

where here and in the following constants C only depend on ι_0 and m.

Setting $\delta := \delta_1/C$ for this constant C and the number $\delta_1 > 0$ obtained in Lemma 3.3.6, we deduce from this lemma that for every $g \in C^0([0, T_0]; \mathcal{U})$, $T_0 := \min(T, \delta/\hat{E})$, the solution $G_{v,g}$ of

$$\partial_t G_{v,g} = P_{G_{v,g}}(k(v_g, g)) \text{ with } \tilde{g}(0) = g_0 \tag{3.3.32}$$

is defined and remains in $\mathcal{U}$ on the whole interval $[0, T_0]$. The Lipschitz estimate (3.3.26) obtained in this lemma, combined with (3.3.30), furthermore allows us to see that curves of metrics $G_{1,2} = G_{v,g_{1,2}}$ obtained from (3.3.32) for the fixed map v and for curves of metrics $g_{1,2} \in C^0([0,T_0];\mathcal{U})$ satisfy

$$\begin{aligned}\sup_{[0,T_0]} \mathrm{dist}_{H^{m+1}}(G_1,G_2) &\leqslant CT_0\|k(v_{g_1},g_1)-k(v_{g_2},g_2)\|_{L^\infty([0,T_0];L^1(\Sigma,g_0))}\\ &\leqslant CT_0\hat{E}\sup_{[0,T_0]}\mathrm{dist}_{H^{m+1}}(G_1,G_2)\leqslant C\delta\sup_{[0,T_0]}\mathrm{dist}_{H^{m+1}}(G_1,G_2)\\ &\leqslant \frac{1}{2}\sup_{[0,T_0]}\mathrm{dist}_{H^{m+1}}(G_1,G_2)\end{aligned}$$

where the last estimate holds after reducing $\delta=\delta(\iota_0)>0$ if necessary. Banach's Fixed point theorem hence yields the existence of a unique solution $g_v \in C^1([0,T_0];\mathcal{U})$ of (3.2.3).

Thanks to (3.3.16) and (3.3.29), we can furthermore bound

$$\|\partial_t g_v(t)\|_{H^{m+1}(\Sigma,g_0)} \leqslant C\|k(v_{g(t)},g(t))\|_{L^1(\Sigma,g_0)} \leqslant CE_{\frac{1}{2}}(v(t),g_0)$$

as claimed in (3.2.6), while a combination of (3.3.16), (3.3.18), (3.3.29), (3.3.30) and (3.2.5) yields

$$\begin{aligned}\|\partial_t g_v(t)-\partial_t g_v(\tilde{t})\|_{H^{m+1}(\Sigma,g_0)} &\leqslant \|\big(P_{g_v(t)}-P_{g_v(\tilde{t})}\big)(k(v_{g_v},g_v)(t))\|_{H^{m+1}(\Sigma,g_0)}\\ &\quad+\|P_{g_v(\tilde{t})}\big((k(v_{g_v},g_v)(t)-k(v_{g_v},g_v)(\tilde{t})\big)\|_{H^{m+1}(\Sigma,g_0)}\\ &\leqslant C\mathrm{dist}_{H^{m+1}}(g_v(t),g_v(\tilde{t}))\hat{E}+C\hat{E}^{\frac{1}{2}}\|(v(t)-v(\tilde{t}))_{g_0}\|_{H^1(\Sigma,g_0)}\end{aligned}$$

for all $t,\tilde{t}\in[0,T_0]$. As the estimate (3.2.6), which we have proven above, ensures that

$$\mathrm{dist}_{H^{m+1}}(g_v(t),g_v(\tilde{t})) \leqslant C\hat{E}\left|t-\tilde{t}\right|,$$

this immediately yields the second a priori bound (3.2.7) claimed in the lemma.

Given v and $\tilde{v}$ which satisfy (3.2.5) for some given $\hat{E}$, we can then combine (3.3.26) with (3.3.30) to see that for any $t\in[0,T_0]$,

$$\begin{aligned}\sup_{[0,t]}\mathrm{dist}(g_v,g_{\tilde{v}}) &\leqslant Ct\|k(v_{g_v},g_v)-k(\tilde{v}_{g_{\tilde{v}}},g_{\tilde{v}})\|_{L^\infty([0,t];L^1(\Sigma,g_0))}\\ &\leqslant Ct\hat{E}^{\frac{1}{2}}\|(v-\tilde{v})_{g_0}\|_{L^\infty([0,T];H^1(\Sigma,g_0))}+Ct\hat{E}\sup_{[0,t]}\mathrm{dist}_{H^{m+1}}(g_v,g_{\tilde{v}}).\end{aligned}$$

After reducing $\delta>0$ if necessary to ensure that $CT_0\hat{E}\leqslant C\delta\leqslant\frac{1}{2}$, we can absorb the last term into the left hand side resulting in the claimed Lipschitz estimate (3.2.8).

Similarly, (3.3.17), (3.3.16) and (3.3.30) allow us to bound

$$\begin{aligned}\|\partial_t(g_v - g_{\tilde{v}})\|_{H^{m+1}(\Sigma,g_0)} &\leqslant \|(P_{g_v} - P_{g_{\tilde{v}}})(k(v_{g_v}, g_v))\|_{H^{m+1}(\Sigma,g_0)} \\ &\quad + \|P_{g_{\tilde{v}}}(k(v_{g_v}, g_v) - k(\tilde{v}_{g_{\tilde{v}}}, g_{\tilde{v}}))\|_{H^{m+1}(\Sigma,g_0)} \\ &\leqslant C\mathrm{dist}_{H^{m+1}}(g_v, g_{\tilde{v}})\hat{E} + C\hat{E}^{\frac{1}{2}}\|(v-\tilde{v})_{g_0}\|_{H^1(\Sigma,g_0)},\end{aligned}$$

which, when combined with (3.2.8), yields the final claim (3.2.9) of the lemma. □

3.3.3 Simplified Analysis of the Metric Components for the Cylinder

We finally turn to the case where Σ is a cylinder where the above arguments simplfy significantly since the horizontal space $H(g)$ is not only 1-dimensional, but also so that $g \mapsto H(g)$ is an *integrable* distribution on the space $\mathcal{M}(\Sigma)$ of flat unit area metrics with geodesic boundary curves. This ensures that every horizontal curve that starts at an initial metric $g_0 \in \mathcal{M}$ evolves in a 1-dimensional integral manifold $\mathcal{M}_H(g_0) \subseteq \mathcal{M}(\Sigma)$. We can hence either prove Proposition 3.2.1 directly, using that the evolution of the metric reduces to an ODE, or simply observe that the above proof still applies but that it can be simplified in the following way.

Given a flat unit cylinder (Σ, g_0) with geodesic boundary curves, we can always introduce coordinates (x, θ) in which

$$\Sigma = [-\pi, \pi] \times \mathcal{S}^1 \text{ and } g_0 = g_{a_0} := (2\pi)^{-2}(a_0^{-1}\mathrm{d}x^2 + a_0\mathrm{d}\theta^2), \tag{3.3.33}$$

where a_0 is determined by the length $\ell_0 = L_{g_0}(\sigma_1) = L_{g_0}(\sigma_2)$ of the boundary curves via $a_0 = \ell_0^2$ and related to the injectivity radius by $\mathrm{inj}(g_0) = \frac{1}{2}\min(a_0, a_0^{-1})^{1/2}$. It can then be easily checked that the one-dimensional submanifold

$$\mathcal{M}_H = \{g_a := (2\pi)^{-2}(a^{-1}\mathrm{d}x^2 + a\mathrm{d}\theta^2) : a > 0\}$$

of $\mathcal{M}$ is an integral manifold of $g \mapsto H(g)$, i.e. that $T_g\mathcal{M}_H = H(g)$ for all $g \in \mathcal{M}_H$. One way to see this is to set $s := a^{-1}x$ to obtain coordinates (s, θ) in which g_a is conformal to the standard cylindrical metric, namely given by $a(2\pi)^{-2}(\mathrm{d}s^2+\mathrm{d}\theta^2)$, and use that in these coordinates each horizontal tensor can be written as $h=\mathrm{Re}(\psi(s+i\theta)(\mathrm{d}s+i\mathrm{d}\theta)^2)$ for a function ψ which is holomorphic and real on the boundary curves and hence given by $\psi\equiv c$ for some $c\in\mathbb{R}$. Thus, each such h is given by $h=c(\mathrm{d}s^2-\mathrm{d}\theta^2)=c(a^{-2}\mathrm{d}x^2-\mathrm{d}\theta^2)=-c(2\pi)^2\partial_a g_a$.

Instead of working with metrics g in a full H^{m+1} neighbourhood of g_0 in $\mathcal{M}^{m+1}$, the existence of such an integral manifold means that it suffices to consider metrics in a neighbourhood $\mathcal{U}(g_{a_0})=\{g_a:\frac{1}{2}a_0\leqslant a\leqslant 2a_0\}$ of $g_0=g_{a_0}$ in this explicit 1-dimensional set. These metrics trivially satisfy the equivalence properties stated in Lemma 3.3.3 while the properties of the projection onto $H(g)$ for metrics $g\in\mathcal{U}(g_0)$ are an immediate consequence of the explicit formula

$$\begin{aligned}P^H_{g_a}(h)&=\|\partial_a g_a\|^{-2}_{L^2(\Sigma,g_a)}\langle h,\partial_a g_a\rangle_{L^2(\Sigma,g_a)}\partial_a g_a\\&=\frac{a^2}{2(2\pi)^4}\langle h,a^{-2}\mathrm{d}x^2-\mathrm{d}\theta^2\rangle_{L^2(\Sigma,g_a)}\left(a^{-2}\mathrm{d}x^2-\mathrm{d}\theta^2\right).\end{aligned}$$

The proof of Proposition 3.2.1 in this simpler case, can hence again be obtained based on a fixed point argument as carried out above.

3.3.4 Dependence of Key Quantities on the Metric

In the next section we will often need to switch back and forth between norms computed with respect to different metrics. For this it is helpful to record that while a change in metric g leads to a change of the harmonic extensions and of the Sobolev norms of maps, these changes are well controlled along horizontal curves of metrics as considered here, as we can combine the estimates on the H^{m+1} norm for the velocity of such curves obtained above with the following basic lemma.

Lemma 3.3.8. *For any integer $m\geqslant 2$ and any $\iota_0>0$ there exists a constant C so that the following holds true for any curve $g\in C^1([0,T];\mathcal{M}^{m+1}_{\iota_0})$.*

For any $j \in \{0, \dots, m\}$ and any $v \in H^j(\Sigma)$ we can bound

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla_g^j v\|^2_{L^2(\Sigma,g(t))}\right| \leqslant C\|\partial_t g\|_{H^m(\Sigma,g(t))}\|v\|^2_{H^j(\Sigma,g(t))} \tag{3.3.34}$$

while for maps in $v \in H^{j+\frac{1}{2}}(\Sigma)$ we can also bound

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla_g^j v\|^2_{L^2(\partial\Sigma,g(t))}\right| \leqslant C\|\partial_t g\|_{H^{m+1}(\Sigma,g(t))}\sum_{i=0}^{j}\|\nabla_g^i v\|^2_{L^2(\partial\Sigma,g(t))}. \tag{3.3.35}$$

Furthermore, the change of the harmonic extension of any $f \in H^{m-\frac{1}{2}}(\partial\Sigma)$ is bounded by

$$\|\partial_t[(f)_{g(t)}]\|_{H^m(\Sigma,g(t))} \leqslant C\|\partial_t g(t)\|_{H^m(\Sigma,g(t))}\|(f)_{g(t)}\|_{H^m(\Sigma,g(t))}, \tag{3.3.36}$$

and the change of the half-energy by

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}E_{\frac{1}{2}}(f,g(t))\right| \leqslant C\|\partial_t g\|_{L^\infty(\Sigma,g(t))}E_{\frac{1}{2}}(f,g(t)). \tag{3.3.37}$$

In particular, if M is chosen so that $\|\partial_t g\|_{H^m(\Sigma,g)} \leqslant M$ on $[0,T]$ then we can bound

$$\|(f)_{g(t_0)}\|^2_{H^m(\Sigma,g(t_0))} \leq e^{C|t_1-t_0|M}\|(f)_{g(t_1)}\|^2_{H^m(\Sigma,g(t_1))} \text{ for all } t_{0,1} \in [0,T] \tag{3.3.38}$$

as well as

$$\|v\|^2_{H^m(\Sigma,g(t_0))} \leq e^{C|t_0-t_1|M}\|v\|^2_{H^m(\Sigma,g(t_1))} \text{ for all } t_{0,1} \in [0,T]. \tag{3.3.39}$$

Proof of Lemma 3.3.8. We first note that the estimates (3.3.34) and (3.3.35) follow from the pointwise estimate (3.3.14), the fact that the change of the volume form $\partial_t \mathrm{d}v_{g(t)} = \frac{1}{2}\mathrm{tr}(\partial_t g)\mathrm{d}v_g$ is controlled by $\|\partial_t g\|_{L^\infty(\Sigma)}$ and that $H^m(\Sigma, g)$ embeds into both $W^{m-1,4}(\Sigma, g)$ and $L^\infty(\Sigma, g)$.

The estimate (3.3.37) for the half-energy is a direct consequence of the formula (3.1.4) for the variation of the half-energy and the estimate (3.3.29) on the stress-energy tensor. It thus remains to control the evolution of the harmonic extension. For this, we can use that $\partial_t(f_{g(t)}) = 0$ on $\partial\Sigma$ to apply standard L^2 elliptic estimates to bound

$$\|\partial_t(f_{g(t)})\|_{H^m(\Sigma,g(t))} \leqslant C\|\Delta_{g(t)}\partial_t(f_{g(t)})\|_{H^{m-2}(\Sigma,g(t))} \tag{3.3.40}$$

$$= C\| - \partial_\varepsilon \Delta_{g(t+\varepsilon)} f_{g(t)}\big|_{\varepsilon=0}\|_{H^{m-2}(\Sigma,g(t))} \tag{3.3.41}$$

$$\leqslant C\|\partial_t g\|_{H^m(\Sigma)}\|f_{g(t)}\|_{H^m(\Sigma,g(t))} \tag{3.3.42}$$

where the second step follows since $\Delta_{g(t)}f_{g(t)} = 0$ for all t. □

3.4 Analysis of the Map Component

In this section we complete the analysis of the map component that is required to establish the existence and properties of solutions of our coupled flow (3.1.7) for initial maps $u_0 \in H^{m-\frac{1}{2}}(\partial\Sigma; N)$, $m \geqslant 4$, claimed in Proposition 3.2.6. As outlined in Section 3.2 we will proceed in several steps, first considering the linearised and regularised equation (3.2.11) for given curves of maps v and metrics g, see Section 3.4.1, then the non-linear, but still regularised, coupled system of equations (3.2.16), see Section 3.4.2, before using the obtained uniform control on the solutions of these auxiliary problems to deduce the desired short-time existence results for our flow (3.1.9), in Section 3.4.3.

3.4.1 Analysis of the Linearised and Regularised Equation (3.2.11) for u

The main goal of this section is to establish the claims made in Proposition 3.2.5 about the existence and properties of solutions of (3.2.11) for given curves of maps $v : [0,T] \times \partial\Sigma \to N_\eta$ and metrics g as described in this proposition. For this it will be convenient to write this equation (3.2.11) for short as

$$\partial_t u = Q_{\varepsilon,v}\partial_{\nu_g}u_g \text{ for } Q_{\varepsilon,v} = -(\varepsilon\mathrm{Id} + P_v),$$

where we recall that the chosen extension of the projection is so that $P_\cdot = P_{\pi(\cdot)}$ on the closure of N_η. This in particular ensures that $P_\cdot$ is an orthogonal projection on $\bar{N}_\eta$ and hence the matrices $(\varepsilon\mathrm{Id}+P_x)$ are invertible for $x \in \bar{N}_\eta$ and further the maps $x \mapsto (\varepsilon\mathrm{Id}+P_x)$ and $x \mapsto (\varepsilon\mathrm{Id} + P_x)^{-1}$ are smooth on $\bar{N}_\eta$.

If it is clear from the context what ε and v are we will drop the indices and simply write

$Q = Q_{\varepsilon,v}$ to lighten the notation and similarly, we will often drop references to the metric for geometric quantities and operators, such as volume forms $\mathrm{d}v_g = \mathrm{d}v$, $\mathrm{d}s_g = \mathrm{d}s$ and gradients $\nabla_g = \nabla$, in situations where we only work with one, possibly time dependent, curve of metrics and continue to use the convention that all Sobolev spaces of maps from Σ or $\partial\Sigma$ and their norms are to considered with respect to g at the relevant time unless indicated otherwise.

As in [Str24], we want to use Galerkin's method with Steklov eigenfunctions to establish the existence of solutions to this linear equation (3.2.11). To use this method, we first consider the problem for a time-independent metric g, to allow us to work with time-independent finite dimensional spaces of functions. To simplify the proof of the regularity of weak solutions, compare Lemma 3.4.2, we will furthermore initially also only consider maps v which are independent of time, and will later remove both of these restrictions by using a time-discretisation argument.

Lemma 3.4.1. *Let $\varepsilon \in (0, \frac{1}{2}]$, $T > 0$, $g \in \mathcal{M}^3$ and $v \in H^{1/2}(\partial\Sigma; N_\eta)$. Then for any $u_0 \in H^{\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$ there exists a unique weak solution $u : [0,T] \times \partial\Sigma \to \mathbb{R}^n$ of (3.2.11) with regularity*

$$u \in H^1([0,T]; L^2(\partial\Sigma)) \cap L^2([0,T]; H^1(\partial\Sigma)) \text{ and } u_g \in L^\infty([0,T]; H^1(\Sigma))$$

whose $L^2(\partial\Sigma)$ trace at $t = 0$ is given by u_0. Furthermore, along weak solution with this regularity we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\frac{1}{2}}(u, g) \leqslant -\frac{1}{2}\|\partial_t u\|^2_{L^2(\partial\Sigma)}. \tag{3.4.1}$$

Proof of Lemma 3.4.1. As g is independent of time, we can consider a fixed orthonormal basis $\varphi_0, \varphi_1, \ldots$ of $L^2(\partial\Sigma)$ which consists of Steklov eigenfunctions corresponding to non-decreasing eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, i.e. functions $\varphi_j : \partial\Sigma \to \mathbb{R}$ with

$$\partial_\nu\big[(\varphi_j)_g\big] = \lambda_j \varphi_j \text{ on } \partial\Sigma,$$

and work with the $\mathbb{R}^n$-valued span

$$V_\ell := \left\{ u : \partial\Sigma \to \mathbb{R}^n : u = \sum_{j=0}^{\ell} \varphi_j b_j \text{ for some } b_j \in \mathbb{R}^n \right\}.$$

For any $\ell \in \mathbb{N}$, we consider the functions $u_\ell(t,x) = \sum_{j=0}^{\ell} a_{j,\ell}(t)\varphi_j(x) \in C^\infty([0,T]; V_\ell)$, and their harmonic extensions $U_\ell = (u_\ell)_g$, which are chosen so that $u_\ell(0)$ is given as the $L^2(\partial\Sigma)$-orthonormal projection of u_0 onto V_ℓ and which are so that

$$\int_{\partial\Sigma} \partial_t u_\ell \cdot v \mathrm{d}s = \int_{\partial\Sigma} Q(\partial_\nu U_\ell) v \mathrm{d}s \quad \text{for all } v \in V_\ell \tag{3.4.2}$$

holds at each time in $[0,T]$. As $\partial_\nu U_\ell(t)|_{\partial\Sigma} = \sum_{j=0}^{\ell} a_{j,\ell}(t)\lambda_j\varphi_j \in V_\ell$ at each such time, and as the φ_j are orthonormal, we can obtain these functions u_ℓ simply by solving the corresponding system of ODEs

$$\partial_t a_{j,\ell} = \sum_{i=0}^{\ell} \lambda_i \int_{\partial\Sigma} Q(a_{i,\ell})\varphi_i\varphi_j \mathrm{d}s, \ 0 \le j \le \ell,$$

with the initial condition

$$a_{j,\ell}(0) = a_j(u_0) := \int_{\partial\Sigma} u_0 \varphi_j \mathrm{d}s \in \mathbb{R}^n, \ 0 \le j \le \ell.$$

As (3.4.2) is in particular applicable for $v = \partial_\nu u_\ell(t) \in V_\ell$ and as $\Delta U_\ell = 0$ we note that the energy of these Galerkin approximations decays according to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla U_\ell\|^2_{L^2(\Sigma)} &= \int_\Sigma \langle \nabla\partial_t U_\ell, \nabla U_\ell\rangle \mathrm{d}v = \int_{\partial\Sigma} \partial_t u_\ell \cdot \partial_\nu U_\ell \mathrm{d}s \\ &= \int_{\partial\Sigma} Q(\partial_\nu U_\ell)\cdot \partial_\nu U_\ell \mathrm{d}s = -\varepsilon\|\partial_\nu U_\ell\|^2_{L^2(\partial\Sigma)} - \|P_v\partial_\nu U_\ell\|^2_{L^2(\partial\Sigma)} \\ &\le -\frac{1}{2}\|\partial_t u_\ell\|^2_{L^2(\partial\Sigma)},\end{aligned} \tag{3.4.3}$$

where the last step follows since $P_\cdot$ is an orthogonal projection on N_η, and hence

$$\|\partial_t u_\ell\|^2_{L^2(\partial\Sigma)} = \|(\varepsilon + P_v)\partial_\nu U_\ell\|^2_{L^2(\partial\Sigma)} = \varepsilon^2\|\partial_\nu U_\ell\|^2_{L^2(\partial\Sigma)} + (1+2\varepsilon)\|P_v\partial_\nu U_\ell\|^2_{L^2(\partial\Sigma)},$$

and since $\varepsilon \le \frac{1}{2}$. We can hence deduce that

$$\sup_{0\le t\le T} \|\nabla U_\ell(t)\|^2_{L^2(\Sigma)} + \|\partial_t u_\ell\|^2_{L^2([0,T]\times\partial\Sigma)} + \varepsilon\|\partial_\nu U_\ell\|^2_{L^2([0,T]\times\partial\Sigma)} \leqslant 3\|\nabla U_\ell(0)\|^2_{L^2(\Sigma)} \tag{3.4.4}$$

which also ensures that

$$\|U_\ell(t)\|_{L^2(\Sigma)} \le C\|u_\ell(t)\|_{L^2(\partial\Sigma)} \le C\big(\|u_\ell(0)\|_{L^2(\partial\Sigma)} + T^{\frac{1}{2}}\|\partial_t u_\ell\|_{L^2([0,T]\times\partial\Sigma)}\big)$$
$$\le C\|U_\ell(0)\|^2_{H^1(\Sigma)}$$

for every $t \in [0,T]$, for a constant that is allowed to depend on T and a lower bound ι_0 on $\mathrm{inj}(g)$, but that is independent of ℓ. Hence we have that

$$\|U_\ell\|^2_{L^\infty([0,T];H^1(\Sigma))} + \|\partial_t u_\ell\|^2_{L^2([0,T]\times\partial\Sigma)} + \varepsilon\|\partial_\nu U_\ell\|^2_{L^2([0,T]\times\partial\Sigma)} \le C\|u_\ell(0)\|^2_{H^1(\Sigma)}, \tag{3.4.5}$$

again with a constant $C = C(T,\iota_0)$.

By construction, $\|u_\ell(0)\|_{L^2(\partial\Sigma)} \le \|u_0\|_{L^2(\partial\Sigma)}$ since $u_\ell(0)$ is the L^2-orthogonal projection of u_0 onto V_ℓ. Importantly, the analogous $\dot{H}^1(\Sigma)$-estimate $\|\nabla U_\ell(0)\|_{L^2(\Sigma)} \leqslant \|\nabla(u_0)_g\|_{L^2(\Sigma)}$ also holds, since the relation

$$\int_\Sigma \langle\nabla(\varphi_j)_g, \nabla w_g\rangle \mathrm{d}v = \int_{\partial\Sigma} \partial_\nu(\varphi_j)_g \cdot w\,\mathrm{d}s = \lambda_j \int_{\partial\Sigma} \varphi_j w\,\mathrm{d}s \text{ for every } w \in H^{\frac{1}{2}}(\partial\Sigma)$$

ensures that the $\lambda_j^{-\frac{1}{2}}(\varphi_j)_g, j \geqslant 1$, are $\dot{H}^1(\Sigma)$-orthonormal and that $U_\ell(0) - a_0(u_0)$ can alternatively be obtained as $\dot{H}^1(\Sigma)$-orthogonal projection of $(u_0)_g$ onto the span of $(\phi_1)_g, \ldots, (\phi_\ell)_g$.

Combining the resulting uniform bounds

$$\|U_\ell(0)\|_{H^1(\Sigma)} \leqslant C(\|U_\ell(0)\|_{L^2(\partial\Sigma)} + \|\nabla U_\ell(0)\|_{L^2(\Sigma)})$$
$$\leqslant C(\|u_0\|_{L^2(\partial\Sigma)} + \|\nabla(u_0)_g\|_{L^2(\Sigma)}) \leqslant C\|u_0\|_{H^1(\Sigma)}$$

on the initial maps $U_\ell(0)$ with the ℓ-independent estimate (3.4.5) thus allows us to conclude that the functions $U_\ell = (u_\ell)_g$ are uniformly bounded in $L^\infty([0,T];H^1(\Sigma))$ and, as $\varepsilon > 0$ is fixed, that the functions $\partial_t u_\ell$ and $\partial_{\nu_g} U_\ell$ are uniformly bounded in $L^2([0,T]\times\partial\Sigma)$. This allows us to pass to a subsequence, still indexed by ℓ, along which U_ℓ converges weak-$*$ in $L^\infty([0,T];H^1(\Sigma))$ to a function $U = (u)_g$ and along which $\partial_t u_\ell$ as well as $\partial_{\nu_g} U_\ell$ converge weakly in $L^2([0,T]\times\partial\Sigma)$ to $\partial_t u$ and $\partial_{\nu_g} U$, respectively.

This function hence has the regularity asked for in the lemma and since u_ℓ satisfies

$$\int_0^T \int_{\partial\Sigma} \partial_t u_\ell \cdot w \mathrm{d}s_g \mathrm{d}t = \int_0^T \int_{\partial\Sigma} Q(\partial_\nu(u_\ell)_g) \cdot w \mathrm{d}s_g \mathrm{d}t \tag{3.4.6}$$

for all $w \in C^\infty([0,T];V_\ell)$, we find that u satisfies (3.4.6) for all $w \in \bigcup_{j=1}^\infty C^\infty([0,T];V_j)$ and is hence a weak solution of (3.2.11) as required.

We furthermore observe that for solutions u of this regularity, the computation carried out in (3.4.3) yields the claimed estimate (3.4.1) on the decay of the energy. Finally we note that as our equation is *linear* this in turn ensures the uniqueness of the solution in the considered class of functions. □

We recall that in the case of $\Sigma = D$ discussed in [Str24], improved regularity of these solutions for initial maps satisfying stronger regularity assumptions was obtained by establishing uniform estimates on the Galerkin approximates in higher order Sobolev spaces. In the present setting this argument is no longer applicable since for general surfaces we cannot expect derivatives of φ_k to be contained in V_k and hence cannot use these functions as test-functions in (3.4.2) above.

However, we can easily circumvent this difficulty by differentiating (3.2.11) in time. Since v and g are time independent, this leaves the equation unchanged allowing us to use a simple iterative argument that combines Lemma 3.4.1 with the fundamental theorem of calculus.

Lemma 3.4.2. *Let $\varepsilon \in (0, \frac{1}{2}]$, $T > 0$ and $m \geqslant 2$. Then for any $g \in \mathcal{M}^{m+1}$, any $v \in H^{m-\frac{1}{2}}((\partial\Sigma, g); N_\eta)$ and any $u_0 \in H^{m-\frac{1}{2}}((\partial\Sigma, g); \mathbb{R}^n)$, the corresponding solution u of* (3.2.11) *obtained in Lemma 3.4.1 has regularity*

$$u \in L^2([0,T]; H^m(\partial\Sigma)) \text{ and } u_g \in L^\infty([0,T]; H^m(\Sigma)).$$

We note that this claim on the regularity could be equivalently stated as

$$u \in L^2([0,T]; H^m(\partial\Sigma)) \cap X_m(T, g)$$

since $\partial_t u_g = (Q\partial_\nu u)_g \in L^\infty([0,T]; H^1(\Sigma, g))$ if $u_g \in L^\infty([0,T]; H^2(\Sigma))$.

Proof of Lemma 3.4.2. We first consider the case $m = 2$. We let w be the solution to (3.2.11) with initial data $w_0 := Q_{v,\varepsilon}\partial_\nu(u_0)_g \in H^{\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n)$ obtained in Lemma 3.4.1. We then note that

$$\tilde{u}(x,t) := u_0(x) + \int_0^t w(x,t')\mathrm{d}t'$$

not only satisfies the initial condition $\tilde{u}(0) = u_0$ but also solves (3.2.11). Indeed, since v and g, and hence also Q and ν, are independent of time, the harmonic extension $\tilde{u}_g$ satisfies

$$Q\partial_\nu \tilde{u}_g(t) = Q\partial_\nu (u_0)_g + \int_0^t Q\partial_\nu w_g(t')\mathrm{d}t' = Q\partial_\nu (u_0)_g + \int_0^t \partial_t w(t')\mathrm{d}t' = w(t) = \partial_t \tilde{u}(t).$$

By the uniqueness of solutions of (3.2.11) established in Lemma 3.4.1, we thus have $u = \tilde{u}$ which implies that $Q\partial_\nu u_g = \partial_t u = w \in L^\infty([0,T]; H^{\frac{1}{2}}(\partial\Sigma)) \cap L^2([0,T]; H^1(\partial\Sigma))$. As $x \mapsto (\varepsilon\mathrm{Id} + P_\cdot)^{-1}$ is smooth and as $v \in H^{3/2}(\partial\Sigma)$ we thus find that both $\partial_\nu u_g$ and $\partial_t u$ are in $L^\infty([0,T]; H^{\frac{1}{2}}(\partial\Sigma)) \cap L^2([0,T]; H^1(\partial\Sigma))$. Elliptic regularity hence allows us to deduce that $u_g \in L^\infty([0,T]; H^2(\Sigma)) \cap L^2([0,T]; H^2(\partial\Sigma))$, and that $\partial_t u_g \in L^\infty([0,T]; H^1(\Sigma; \mathbb{R}^n))$, which gives that $u \in X_2(T,g) \cap L^2([0,T]; H^2(\partial\Sigma))$ as claimed.

The claim for general $m \geqslant 3$ can now be obtained by iterating this argument. □

We next establish the following a priori estimates on the $H^m(\Sigma)$-norm of solutions of equation (3.2.11), where we note that we now also allow g and v to depend on time.

Lemma 3.4.3. *Let $m \geqslant 3$, $T > 0$, $g \in C^1([0,T]; \mathcal{M}^{m+1})$ and $v : [0,T] \times \partial\Sigma \to N_\eta$ with $v \in X_m(T,g)$. Then for any $0 \leqslant \varepsilon \leqslant \frac{1}{2}$ and any solution u of (3.2.11) with*

$$u \in L^2([0,T]; H^m(\partial\Sigma)) \text{ and } u_g \in L^\infty([0,T]; H^m(\Sigma)) \tag{3.4.7}$$

we can bound

$$\|u_g(t)\|^2_{H^m(\Sigma,g(t))} + \tfrac{\varepsilon}{2}\|\partial_\nu \nabla^{m-1} u_g\|^2_{L^2([0,t];L^2(\partial\Sigma,g))} \leqslant e^{C_1 t}\|(u(0))_{g(0)}\|^2_{H^m(\Sigma,g(0))} \tag{3.4.8}$$

for every $t \in [0,T]$ and for a constant $C_1 = C(\iota_0, m, M)\Lambda^m$ where $\Lambda \geqslant 1$ and ι_0 are as in (3.2.15) *while M is chosen so that $\|\partial_t g\|_{H^m(\Sigma,g)} \leqslant M$ on $[0,T]$.*

We stress that the constant C_1 is *independent* of ε, and it is this key feature of the above lemma which will allow us to obtain a solution of the original problem from solutions of the regularised equation as $\varepsilon \to 0$ later in Section 3.4.3.

We note that as our domain surface is in general not flat, we cannot expect higher derivatives to commute, which results in many additional error terms when compared with the case of $\Sigma = D$ treated in [Str24], though these will all be of lower order.

Throughout this proof and also later on in Section 3.5 we will also often want to exchange the order in which operations such as derivatives, projections and harmonic extensions are applied. To do this efficiently, we have collected the relevant estimates on the resulting commutator terms in Appendix B, see in particular Lemma B.0.1 and Remark B.0.2.

Proof of Lemma 3.4.3. As (3.4.7) ensures that $\partial_\nu u_g \in L^2([0,T]; H^{m-1}(\partial\Sigma, g))$ we have that also $\partial_t u = Q_{\varepsilon,v}\partial_\nu u_g \in L^2([0,T]; H^{m-1}(\partial\Sigma, g))$, which gives us sufficient regularity to compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_g\|^2_{H^m(\Sigma,g)} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\|u_{g(\cdot+\varepsilon)}\|^2_{H^m(\Sigma,g(\cdot+\varepsilon))} + 2\sum_{j=0}^{m}\int_\Sigma \langle \nabla^j u_g, \nabla^j(\partial_t u)_g\rangle \mathrm{d}v \\ &\leqslant C\|\partial_t g\|_{H^m(\Sigma,g)}\|u_g\|^2_{H^m(\Sigma,g)} + 2\sum_{j=0}^{m}\int_\Sigma \langle \nabla^j u_g, \nabla^j(\partial_t u)_g\rangle \mathrm{d}v,\end{aligned}$$

where we use of Lemma 3.3.8 in the second step and recall that $\|\partial_t g\|_{H^m(\Sigma,g)} \leqslant M$. The claimed estimate (3.4.8) hence follows once we establish that $J_j := \int_\Sigma \langle \nabla^j u_g, \nabla^j(\partial_t u_g)\rangle \mathrm{d}v$ satisfies

$$J_j \leqslant C\big(1 + \|v_g\|^m_{H^m(\Sigma)}\big)\|u_g\|^2_{H^m(\Sigma)} \text{ for } j = 0, 1, \ldots, m-1 \tag{3.4.9}$$

while

$$J_m \leqslant -\tfrac{\varepsilon}{2}\|P_v\partial_\nu\nabla^{m-1}u_g\|^2_{L^2(\partial\Sigma)} + C\big(1 + \|v_g\|^m_{H^m(\Sigma)}\big)\|u_g\|^2_{H^m(\Sigma)} \tag{3.4.10}$$

where here and in the following C denotes a constant that is allowed to depend on ι_0, m and as usual our setting, but that is *independent* of ε.

The required bound (3.4.9) on the lower order terms immediately follows from the estimate

$$\|\nabla^j(\partial_t u)_g\|_{L^2(\Sigma)} = \|\nabla^j(\varepsilon\partial_\nu u_g + P_v\partial_\nu u_g)_g\|_{L^2(\Sigma)} \tag{3.4.11}$$

$$\leqslant C(1+\|v_g\|_{H^{m-1}(\Sigma)}^{m-1})\|u\|_{H^{m-1}(\Sigma)}+C\|u_g\|_{H^m(\Sigma)} \tag{3.4.12}$$

which we can e.g. obtain by observing that $\|P_v\nabla^j\partial_\nu u\|_{L^2(\Sigma)}\leqslant\|\nabla^j\partial_\nu u\|_{L^2(\Sigma)}\leqslant\|u_g\|_{H^m(\Sigma)}$, compare (B.0.3), and that each $P_v\nabla^j\partial_\nu u-\nabla^j(P_v\partial_\nu u_g)_g$ can be thought of as a commutator term $C_j(v_g,u_g)$ of the form (B.0.19) for which Lemma B.0.1 is applicable.

To estimate the leading order term J_m, we integrate by parts to rewrite

$$\begin{aligned}J_m&=\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,\nabla^{m-1}(\partial_t u)_g\rangle\mathrm{d}s-\int_\Sigma\langle\Delta\nabla^{m-1}u_g,\nabla^{m-1}(\partial_t u)_g\rangle\mathrm{d}v\\&=-\varepsilon\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,\nabla^{m-1}(\partial_\nu u_g)_g\rangle\mathrm{d}s-\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,\nabla^{m-1}(P_v\partial_\nu u_g)_g\rangle\mathrm{d}s\\&\quad+\mathrm{err}_1\end{aligned}$$

and exploit that u_g is harmonic, and hence $\|\Delta\nabla^{m-1}u_g\|_{L^2(\Sigma)}\leqslant C\|u_g\|_{H^m(\Sigma)}$, as well as (3.4.11) to bound the resulting error $\mathrm{err}_1:=\int_\Sigma\langle\Delta\nabla^{m-1}u_g,\nabla^{m-1}(\partial_t u)_g\rangle\mathrm{d}v$ by

$$\mathrm{err}_1\leqslant C\|u_g\|_{H^m(\Sigma)}\|\nabla^{m-1}(\partial_t u)_g\|_{L^2(\Sigma)}\leqslant C(1+\|v_g\|_{H^{m-1}(\Sigma)}^{m-1})\|u_g\|_{H^m(\Sigma)}^2. \tag{3.4.13}$$

We now further rewrite

$$J_m=-\varepsilon\|\partial_\nu\nabla^{m-1}u_g\|_{L^2(\partial\Sigma)}^2-\|P_v\partial_\nu\nabla^{m-1}u_g\|_{L^2(\partial\Sigma)}^2+\mathrm{err}_1+\mathrm{err}_2+\mathrm{err}_3 \tag{3.4.14}$$

for

$$\mathrm{err}_2=\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,P_v\nabla^{m-1}(\partial_\nu u_g)_g-\nabla^{m-1}(P_v\partial_\nu u_g)_g\rangle\mathrm{d}s$$

and

$$\begin{aligned}\mathrm{err}_3&:=\varepsilon\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,\partial_\nu\nabla^{m-1}u_g-\nabla^{m-1}(\partial_\nu u_g)_g\rangle\mathrm{d}s\\&\quad+\int_{\partial\Sigma}\langle\partial_\nu\nabla^{m-1}u_g,P_v\left(\partial_\nu\nabla^{m-1}u_g-\nabla^{m-1}(\partial_\nu u_g)_g\right)\rangle\mathrm{d}s\\&=\int_{\partial\Sigma}\langle\varepsilon\partial_\nu\nabla^{m-1}u_g+P_v\partial_\nu\nabla^{m-1}u_g,\partial_\nu\nabla^{m-1}u_g-\nabla^{m-1}(\partial_\nu u_g)_g\rangle\mathrm{d}s.\end{aligned}$$

Thanks to (B.0.3) we can bound

$$\mathrm{err}_3\leqslant C\varepsilon\|\partial_\nu\nabla^{m-1}u_g\|_{L^2(\partial\Sigma)}\|u_g\|_{H^m(\Sigma)}+C\|P_v\partial_\nu\nabla^{m-1}u_g\|_{L^2(\partial\Sigma)}\|u_g\|_{H^m(\Sigma)}$$

$$\leqslant \varepsilon^2 \|\partial_\nu \nabla^{m-1} u_g\|^2_{L^2(\partial\Sigma)} + \frac{1}{2}\|P_v \partial_\nu \nabla^{m-1} u_g\|^2_{L^2(\partial\Sigma)} + C\|u_g\|^2_{H^m(\Sigma)}$$

where we stress that C is independent of $\varepsilon \in [0, \frac{1}{2}]$. To bound err_2, we note that $C_{m-1}(v_g, u_g) = P_{v_g}\nabla^{m-1}(\partial_\nu u_g)_g - \nabla^{m-1}(P_v \partial_\nu u_g)_g$ and $\nabla C_{m-1}(v_g, u_g)$ are commutator terms of the form (B.0.19) (for $j = m-1$ and $j = m$, respectively) and that Lemma B.0.1 hence in particular implies that

$$\|C_{m-1}(v_g, u_g)\|_{H^1(\Sigma)} \leqslant C(1 + \|v_g\|^m_{H^m(\Sigma)})\|u_g\|_{H^m(\Sigma)}. \tag{3.4.15}$$

Rewriting err_2 as an integral over the surface and using again that $\Delta\nabla^{m-1}u_g$ is of lower order we hence obtain that

$$\begin{aligned}\mathrm{err}_2 &= \int_{\partial\Sigma} \langle \partial_\nu \nabla^{m-1} u_g, C_{m-1}(v_g, u_g)\rangle \mathrm{d}s \\ &\leqslant \int_\Sigma \langle \nabla^m u_g, \nabla C_{m-1}(v_g, u_g)\rangle \mathrm{d}v + C\|\Delta\nabla^{m-1}u_g\|_{L^2(\Sigma)}\|C_{m-1}(v_g, u_g)\|_{L^2(\Sigma)} \\ &\leqslant C\left(1 + \|v_g\|^m_{H^m(\Sigma)}\right)\|u_g\|^2_{H^m(\Sigma)}.\end{aligned}$$

Inserting these error estimates into (3.4.14) yields the claimed bound (3.4.10) on J_m and hence completes the proof of the lemma. □

We can now combine the above lemmas with a time-discretisation argument to establish the existence of solutions of (3.2.11) of this regularity for time-dependent g and v as claimed in Proposition 3.2.5.

Proof of Proposition 3.2.5. As we have already established the claimed a priori estimates in Lemma 3.4.3 above, it remains to show that to any $\varepsilon > 0$, $T > 0$, g, v and u_0 as in the proposition there exists a unique solution $u \in X_m(T) \cap L^2([0,T]; H^m(\partial\Sigma))$ of (3.2.11) with $u(0) = u_0$.

To prove this we want to apply a time-discretisation argument in which we solve (3.2.11) for maps and metrics $(\tilde{v}, \tilde{g})$ which are piecewise constant (in time) in place of (v, g). To this end, given any partition $t_0 = 0 < t_1 < \ldots < t_K < t_{K+1} = T$ of the time interval, we can first use that $u_0 \in H^{m-\frac{1}{2}}(\partial\Sigma, g(t_0) = g_0)$ to apply Lemmas 3.4.1 and 3.4.2

on $[0,t_1]$ to obtain $\tilde u \in X_m([0,t_1],g(t_0)) \cap L^2([0,t_1];H^m(\partial\Sigma,g(t_0)))$ which solves (3.2.11) with $v(t_0)$ and $g(t_0)$ in place of the time dependent v and g. As the spaces $H^{m-\frac{1}{2}}(\partial\Sigma,g(t))$ all coincide, compare Lemma 3.3.8, we obtain that $\tilde u(t_1) \in H^{m-\frac{1}{2}}(\partial\Sigma,g(t_1))$, so can iterate this argument to obtain $\tilde u \in X_m(T,\tilde g) \cap L^2([0,T];H^m(\partial\Sigma,\tilde g))$ which solves (3.2.11) for the piecewise constant in time metrics and maps $(\tilde g,\tilde v)|_{[t_{k-1},t_k)} = (g,v)(t_{k-1})$.

On each individual interval $[t_{k-1},t_k)$ the assumptions of Lemma 3.4.3 are satisfied so we can control the evolution of $\|\tilde u(t)\|_{H^m(\Sigma,\tilde g)}$ by the estimate (3.4.8) obtained in this Lemma 3.4.3. Since the bounds on the H^m-norm and injectivity radius in (3.2.15) hold for the same numbers Λ and ι_0 also for $\tilde v$ and $\tilde g$, we hence obtain that

$$\begin{aligned}\|(\tilde u(t))_{\tilde g(t)}\|^2_{H^m(\Sigma,\tilde g(t))} + \tfrac{\varepsilon}{2}\|\partial_\nu \nabla^{m-1} u_g\|^2_{L^2([t_{k-1},t];L^2(\partial\Sigma,\tilde g(t_{k-1})))}&\\ \leq e^{C_1(t-t_{k-1})}\|(\tilde u(t_{k-1}))_{\tilde g(t_{k-1})}\|^2_{H^m(\Sigma,\tilde g(t_{k-1}))}&\end{aligned}$$

for every $t \in [t_{k-1},t_k)$ and for a constant $C_1 = C(\iota_0,m,\Lambda)$.

We note that while the map $\tilde u(t): \partial\Sigma \to \mathbb{R}^n$ is continuous across the times t_j, since the metric jumps at t_j, its harmonic extensions $(\tilde u(t))_{\tilde g(t)}$ and the corresponding $H^m(\Sigma,\tilde g(t))$-norms may jump. This jump is however controlled by the estimate (3.3.38) from Lemma 3.3.8 which ensures that

$$\|(\tilde u(t_k))_{\tilde g(t_k)}\|^2_{H^m(\Sigma,\tilde g(t_k))} \leq e^{CM|t_k-t_{k-1}|}\lim_{t\nearrow t_k}\|(\tilde u(t))_{\tilde g(t)}\|^2_{H^m(\Sigma,\tilde g(t))} \tag{3.4.16}$$

for a constant $C = C(\iota_0,m)$.

Combined with (3.4.8) we hence deduce that the harmonic extension $\tilde U = (\tilde u)_{\tilde g}$ obtained from any such $\tilde u$ is so that

$$\|\tilde U\|^2_{H^m(\Sigma,\tilde g(t))} + \tfrac{\varepsilon}{2}\|\partial_\nu \nabla^{m-1}_{\tilde g}\tilde U\|^2_{L^2([0,t];L^2(\partial\Sigma,\tilde g))} \leq e^{Ct}\|(u_0)_{g_0}\|^2_{H^m(\Sigma,g_0)} \tag{3.4.17}$$

for every $t \in [0,T]$ and a constant $C = C(\iota_0,m,\Lambda,M)$. Combined with standard elliptic estimates and Lemma 3.3.8 this hence yields uniform bounds of

$$\|\tilde U\|^2_{L^\infty([0,T];H^m(\Sigma,g_0))} + \varepsilon\|\nabla^m \tilde U\|^2_{L^2([0,T];L^2(\partial\Sigma,g_0))} \leq C\|(u_0)_{g_0}\|^2_{H^m(\Sigma,g_0)}, \tag{3.4.18}$$

$C = C(\iota_0, T, m)$ for all $\tilde{U} = (\tilde{u})_{\tilde{g}}$ obtained by such a time discretisation.

Applying this argument for a sequence $(\tilde{g}_\ell, \tilde{v}_\ell)$ which corresponds to partitions whose mesh-size tends to zero, we can hence pass to a subsequence so that the resulting maps $\tilde{U}_\ell = (\tilde{u}_\ell)_{\tilde{g}_\ell}$ converge weak-$*$ in $L^\infty([0,T]; H^m(\Sigma, g_0))$ and weakly in $L^2([0,T]; H^{m+\frac{1}{2}}(\Sigma, g_0))$ to a limiting map $U \in L^\infty([0,T]; H^m(\Sigma, g_0)) \cap L^2([0,T]; H^{m+\frac{1}{2}}(\Sigma, g_0))$. Since $\Delta_{\tilde{g}_\ell} U_\ell = 0$ for every ℓ and since $\tilde{g}_\ell \to g$ in $L^\infty([0,T]; \mathcal{M}^{m+1}(\Sigma))$ we must have $\Delta_g U = 0$, and can hence view $U = u_g$ as the harmonic extension of the weak limit u of the traces $\tilde{u}_\ell$.

As maps in $X_m(T)$ are continuous on $\partial\Sigma \times [0,T]$, compare Lemma 3.4.6 below, we can additionally use that $\tilde{v}_\ell$ converge to v uniformly on $[0,T] \times \Sigma$ to pass to the limit in $\partial_t u_\ell = -(\varepsilon + P_{\tilde{v}_\ell})\partial_{\nu_{\tilde{g}_\ell}} U_\ell$ to obtain that u is the desired solution of

$$\partial_t u = -(\varepsilon + P_v)\partial_{\nu_g} U = -(\varepsilon + P_v)\partial_{\nu_g} u_g \text{ with } u(0) = u_0.$$

Finally, to establish the uniqueness of solutions of the given regularity to this linear equation it suffices to exclude the possibility that there is a non-trivial solution that evolves from $u_0 = 0$ and this follows from Gronwall's inequality as any solution of (3.2.11) satisfies

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} E_{\frac{1}{2}}(u,g) &= \tfrac{\mathrm{d}}{\mathrm{d}\delta} E_{\frac{1}{2}}(u, g(\cdot + \delta)) - \varepsilon \|\partial_\nu u_g\|^2_{L^2(\partial\Sigma)} - \|P_v \partial_\nu u_g\|^2_{L^2(\partial\Sigma)} \\ &\leqslant C(\iota_0) M E_{\frac{1}{2}}(u,g). \end{aligned} \tag{3.4.19}$$

□

In addition to the properties of the solutions $u = u_{\varepsilon,v,g}$ of (3.2.11) that we have just established, we also show that the dependence of this solution on curves of maps v and metrics g is Lipschitz in the following sense.

Lemma 3.4.4. *For any $\iota_0 > 0$ and $M, \Lambda < \infty$, there exists a constant C so that the following holds true for every $0 < \varepsilon \leq \frac{1}{2}$ and any $0 < T \leq 1$. Let $m = 3$, let $(u_0, g_0) \in H^{m-\frac{1}{2}}(\partial\Sigma; \mathbb{R}^n) \times \mathcal{M}^{m+1}_{2\iota_0}$, let $\mathcal{U} = \mathcal{U}^{m+1}$ be the neighbourhood of g_0 from Proposition 3.2.1, and let $v_{1,2} \in X_m(T)$ and $g_{1,2} \in C^1([0,T]; \mathcal{U})$ be any pair of maps and metrics such that $v_{1,2}(\partial\Sigma \times [0,T]) \subset N_\eta$, $(v_{1,2}(0), g_{1,2}(0)) = (u_0, g_0)$ and for which (3.2.15) holds for this*

$m=3$, M and Λ. Then the corresponding solutions $u_i = u_{\varepsilon,v_i,g_i}$, $i=1,2$ of (3.2.11) *with initial conditions $u_i(0)=u_0$ satisfy*

$$\sup_{[0,t]} \|\nabla_{g_1}(u_2-u_1)_{g_1}\|^2_{L^2(\Sigma,g_1)} + \int_0^t \|\partial_t(u_2-u_1)\|^2_{L^2(\partial\Sigma,g_1)}\mathrm{d}t$$
$$\leq C\varepsilon^{-1} t \sup_{[0,t]} \left(\mathrm{dist}^2_{H^3}(g_1,g_2) + \|(v_2-v_1)_{g_1}\|^2_{H^1(\Sigma,g_1)}\right)$$

for all $0\leq t\leq \min\{T,\delta/M\}$, where $\delta>0$ is as in Proposition 3.2.1.

We note that we can of course apply this lemma also for any other choice of $m\geqslant 3$ since we can assume that the neighbourhoods $\mathcal{U}^k = \mathcal{U}^k(g_0)$ are chosen so that $\mathcal{U}^{k+1}\subseteq\mathcal{U}^k$ and since increasing the exponent m simply strengthens the assumptions while leaving the conclusions invariant.

Proof of Lemma 3.4.4. We first note that $w=u_2-u_1$ satisfies an inhomogenous version of equation (3.2.11), namely

$$\partial_t w = -(\varepsilon + P_{v_1})\partial_{\nu_1} w_{g_1} + f, \tag{3.4.20}$$

where $\nu_1=\nu_{g_1}$ and where $f=f_1+f_2+f_3$ for

$$f_1 = (\varepsilon+P_{v_1})\partial_{\nu_1}\left((u_2)_{g_1}-(u_2)_{g_2}\right),$$
$$f_2 = (\varepsilon+P_{v_1})\left(\partial_{\nu_1}-\partial_{\nu_2}\right)(u_2)_{g_2},$$
$$f_3 = \left(P_{v_1}-P_{v_2}\right)\partial_{\nu_2}(u_2)_{g_2}.$$

We furthermore note that $(u_1)_{g_1}$ and $(u_2)_{g_1}$ are uniformly bounded in $H^m(\Sigma,g_1)$ since Proposition 3.2.5 yields uniform $H^m(\Sigma,g_i)$ bounds on $(u_i)_{g_i}$ and since the dependence of the norms and the harmonic extension on the metric are controlled by Lemma 3.3.8. We can thus apply this Lemma 3.3.8 to bound the error terms by

$$\begin{aligned}
\|f_1\|_{L^2(\partial\Sigma,g_1)} &\leq 2\|\partial_{\nu_1}\left((u_2)_{g_1}-(u_2)_{g_2}\right)\|_{L^2(\partial\Sigma,g_1)} \leq C\mathrm{dist}_{H^m}(g_1,g_2)\\
\|f_2\|_{L^2(\partial\Sigma,g_1)} &\leq 2\|\left(\partial_{\nu_1}-\partial_{\nu_2}\right)(u_2)_{g_2}\|_{L^2(\partial\Sigma,g_1)} \leq C\mathrm{dist}_{H^m}(g_1,g_2)\\
\|f_3\|_{L^2(\partial\Sigma,g_1)} &= \|\left(P_{v_1}-P_{v_2}\right)\partial_{\nu_2}(u_2)_{g_2}\|_{L^2(\partial\Sigma,g_1)} \leq C\|v_2-v_1\|_{L^2(\partial\Sigma,g_1)}
\end{aligned} \tag{3.4.21}$$

at any time in $[0,T]$ and for $C = C(\iota_0, \Lambda)$. We can furthermore compute

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E_{\frac{1}{2}}(w, g_1) = \int_\Sigma \langle \nabla_{g_1} w_{g_1}, \nabla_{g_1}\partial_t w_{g_1}\rangle_{g_1} \mathrm{d}v_{g_1} + \tfrac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0} E_{\frac{1}{2}}(w, g_1(t+\delta)) \tag{3.4.22}$$

$$= -\varepsilon \|\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} - \|P_{v_1}\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} \tag{3.4.23}$$

$$+ \int_{\partial\Sigma} \partial_{\nu_1} w_{g_1} \cdot f \mathrm{d}s_{g_1} + \tfrac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0} E_{\frac{1}{2}}(w, g_1(t+\delta)) \tag{3.4.24}$$

and note that (3.3.37) allows us to bound $\tfrac{\mathrm{d}}{\mathrm{d}\delta}\big|_{\delta=0} E_{\frac{1}{2}}(w, g_1(t+\delta)) \leq CE_{\frac{1}{2}}(w, g_1)$ for $C = C(M, \iota_0)$. We can hence apply Young's inequality and use the above estimates (3.4.21) on $f = f_1 + f_2 + f_3$ to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\frac{1}{2}}(w, g_1) \leqslant -\frac{1}{2}\varepsilon \|\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} - \|P_{v_1}\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} + CE_{\frac{1}{2}}(w, g_1)$$
$$+ C\varepsilon^{-1}\left(\mathrm{dist}^2_{C^3}(g_1, g_2) + \|v_2 - v_1\|^2_{L^2(\partial\Sigma, g_1)}\right),$$

so as $w(0) = 0$ and $T \leqslant 1$, that

$$\|\nabla_{g_1} w_{g_1}(t)\|^2_{L^2(\Sigma, g_1(t))} + \int_0^t \varepsilon \|\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} + \|P_{v_1}\partial_{\nu_1} w_{g_1}\|^2_{L^2(\partial\Sigma, g_1)} \mathrm{d}t'$$
$$\leq C\varepsilon^{-1} t \sup_{[0,t]} \left(\mathrm{dist}_{C^3}(g_1, g_2)^2 + \|v_2 - v_1\|^2_{L^2(\partial\Sigma, g_1)}\right).$$

This immediately implies the claim of the lemma since $|\partial_t w| \leqslant \varepsilon\, |\partial_{\nu_1} w_{g_1}| + |P_{v_1}\partial_{\nu_1} w_{g_1}| + |f|$, for f controlled by (3.4.21), and since $\|v_2 - v_1\|_{L^2(\partial\Sigma, g_1)} \leqslant C\|(v_2 - v_1)_{g_1}\|_{H^1(\Sigma, g_1)}$. □

3.4.2 Analysis of the Regularised Coupled Flow

We now want to establish the existence of a solution of the non-linear, but still regularised, system (3.2.16) based on the results on the metric obtained in Section 3.3 and the properties of the solutions $u_{\varepsilon,v,g}$ of the regularised and linearised equation (3.2.11). To this end we show the following.

Lemma 3.4.5. *For any $\varepsilon \in (0, \frac{1}{2}]$, $m \geqslant 3$, $\iota_0, d_0 > 0$ and $E_0, \Lambda_0 < \infty$ there exists $T > 0$ and $C_0 \geqslant 2$ so that the following holds true for any $g_0 \in \mathcal{M}^{m+1}_{2\iota_0}$, the corresponding neighbourhood $\mathcal{U}$ from Proposition 3.2.1 and any $u_0 \in H^{m-\frac{1}{2}}(\partial\Sigma; N_\eta)$ with $\|(u_0)_{g_0}\|_{H^m(\Sigma, g_0)} \leqslant \Lambda_0$, $E_{\frac{1}{2}}(v_0, g_0) \leqslant E_0$ and $\mathrm{dist}(u_0(\partial\Sigma), \partial N_\eta) > d_0$.*

For any v in the set

$$S = \{v \in X_m(T) : v(0) = u_0, \|v_{g_0}\|_{X_m(T,g_0)} \le \Lambda := C_0\Lambda_0, E_{\frac{1}{2}}(v(t), g_0) \le 2E_0\}$$

the unique solution g_v of (3.2.3) *with $g_v(0) = g_0$ and the unique solution $u_{\varepsilon,g_v,v}$ of* (3.2.11) *with $u(0) = u_0$ are defined on all of $[0, T]$ with $g_v(t) \in \mathcal{U}$ and $u(t)(\partial\Sigma) \subset N_\eta$ for $0 < t \le T$, and the map*

$$\Psi = \Psi_{\varepsilon,u_0,g_0} : v \mapsto u_{\varepsilon,g_v,v}$$

maps the set S into itself and is a contraction with respect to the norm

$$\|v\|_S^2 := \sup_{0\le t\le T} \|(v(t))_{g_0}\|^2_{H^1(\Sigma,g_0)} + \int_0^T \int_{\partial\Sigma} |\partial_t v|^2 \, \mathrm{d}s_{g_0}\mathrm{d}t.$$

For the proof of this lemma, and in the next section, we will use that the spaces X_m have the following useful properties.

Lemma 3.4.6. *Let $g_0 \in \mathcal{M}^{m+1}_{\iota_0}$ for some $m \geqslant 2$ and $\iota_0 > 0$, and let $X_m(T, g_0)$ be as defined in* (3.2.1)*. Then the harmonic extension $v \mapsto v_{g_0}$ is a compact operator from $(X_m(T), \|\cdot\|_{(X_m(T),g_0)})$ to $C^0([0, T]; H^{m-1}(\Sigma, g_0))$ and there exists a constant $C = C(\iota_0)$ so that*

$$\|v(t) - v(s)\|_{C^0(\partial\Sigma)} \leqslant C\|v\|_{X_m} |t - s|^{\frac{1}{2}} \tag{3.4.25}$$

for any $v \in X_m(T, g_0)$ and for all $t, s \in [0, T]$.

Remark 3.4.7. We note in particular that if $v \in X_m(T, g_0)$ is so that $v_0 := v(t = 0)$ maps $\partial\Sigma$ into N_η and if Λ is so that $\|v\|_{X_m} \leqslant \Lambda$ then we are guaranteed that $v(t)(\partial\Sigma) \subset N_\eta$ at least for times $t \in [0, T_2]$, where $T_2 := c_0\Lambda^{-2}\mathrm{dist}^2_{\mathbb{R}^n}(v_0(\partial\Sigma), \partial N_\eta)$ for a constant $c_0 = c_0(\iota_0) > 0$.

Proof of Lemma 3.4.6. The first claim follows by combining the compact embedding of $W^{1,\infty}([0, T]; H^1(\Sigma, g_0)) \hookrightarrow C^0([0, T]; L^2(\Sigma, g_0))$ and the interpolation inequality

$$\|f\|^m_{H^{m-1}(\Sigma,g_0)} \leqslant C\|f\|_{L^2(\Sigma,g_0)}\|f\|^{m-1}_{H^m(\Sigma,g_0)}, \quad C = C(\iota_0, m)$$

which follows inductively from $\|f\|^2_{H^k} \leqslant C\|f\|_{H^{k-1}}\|f\|_{H^{k+1}}$, compare also Theorem (A.2.3).

To obtain the second claim it suffices to note that the trace operator is continuous from $H^{\frac{3}{2}}(\Sigma)$ into $C^0(\partial\Sigma)$ and that for any $0 \leqslant s \leqslant t \leqslant T$, we can bound

$$\begin{aligned}\|v(s)-v(t)\|^2_{H^{\frac{3}{2}}(\Sigma,g_0)} &\leqslant C\|v(s)-v(t)\|_{H^2(\Sigma,g_0)}\|v(t)-v(s)\|_{H^1(\Sigma,g_0)}\\ &\leqslant C\|v\|_{X_2}\int_s^t \|\partial_t v(t')\|_{H^1(\Sigma,g_0)}\mathrm{d}t' \leqslant C\|v\|^2_{X_m}|s-t|\,.\end{aligned}$$

□

For the proof of Lemma 3.4.5 we will furthermore use that

$$\|P_v\partial_\nu u_g\|_{H^{1/2}(\partial\Sigma,g)} \leqslant C(1+E_{\frac{1}{2}}(v,g)^{\frac{1}{2}})\|u_g\|_{H^3(\Sigma)}, \tag{3.4.26}$$

and hence that

$$\begin{aligned}\|(\partial_t u)_g\|_{H^1(\Sigma,g)} &\leqslant C\|\partial_t u\|_{H^{1/2}(\partial\Sigma,g)}\\ &\leqslant C\varepsilon\|\partial_\nu u_g\|_{H^{1/2}(\partial\Sigma,g)} + C\|P_v\partial_\nu u_g\|_{H^{1/2}(\partial\Sigma,g)}\\ &\leqslant C(1+E_{\frac{1}{2}}(v,g)^{\frac{1}{2}})\|u_g\|_{H^3(\Sigma)}\end{aligned} \tag{3.4.27}$$

along any solution $u = \Psi(v)$ of (3.2.11) as considered in Lemma 3.4.5.

We note that to obtain (3.4.26) we can use that if $f_1 \in H^{\frac{1}{2}}(\partial\Sigma)$ and $f_2 \in H^{3/2}(\partial\Sigma)$ then the product $f_1 f_2$ is in $H^{1/2}(\partial\Sigma)$ with

$$\|f_1 f_2\|_{H^{1/2}(\partial\Sigma)} \leqslant C\|f_1\|_{H^{1/2}(\partial\Sigma)}\|f_2\|_{H^{3/2}(\partial\Sigma)}$$

and that $\|P_v\|_{H^{1/2}(\partial\Sigma)} \leqslant C(1+\|v\|_{H^{1/2}(\partial\Sigma)})$ since $x \mapsto P_x$ is smooth, hence allowing us to bound

$$\begin{aligned}\|P_v\partial_\nu u_g\|_{H^{1/2}(\partial\Sigma)} &\leqslant C(1+\|v\|_{H^{1/2}(\partial\Sigma)})\|\partial_\nu u_g\|_{H^{3/2}(\partial\Sigma)}\\ &\leqslant C(1+E_{\frac{1}{2}}(v,g)^{\frac{1}{2}})\|u_g\|_{H^3(\Sigma)}.\end{aligned}$$

Proof of Lemma 3.4.5. We first recall from Remark 3.4.7 that if we choose

$$T \leq c_0\Lambda^{-2}d_0^2,$$

then $v([0,T]\times\partial\Sigma) \subset N_\eta$ for all $v \in S$.

We then note that Proposition 3.2.1 ensures that if $T > 0$ is chosen to be no more than $\delta/(4E_0)$, for δ as in that proposition, then the curve of metrics $g = g_v$ exists and satisfies $g_v(t) \in \mathcal{U}$ for all $t \in [0, T]$ as well as

$$\|\partial_t g\|_{H^{m+1}(\Sigma,g)} \leq CE_0 =: M \tag{3.4.28}$$

for a constant M that only depends on E_0 and ι_0.

We furthermore recall that Lemma 3.3.8 ensures that for any $f \in H^{m-\frac{1}{2}}(\partial\Sigma)$,

$$\|f_{g(t_0)}\|^2_{H^j(\Sigma,g(t_0))} \leq e^{C|t_1-t_0|M}\|f_{g(t_1)}\|^2_{H^j(\Sigma,g(t_1))} \leq 2\|f_{g(t_1)}\|^2_{H^j(\Sigma,g(t_1))} \tag{3.4.29}$$

for all $0 \leq t_{0,1} \leq T$ and all $j = 1, \ldots, m$ where the last inequality holds after possibly reducing $\delta > 0$ and ensuring that $T \leq \delta/M$. In particular, we have

$$\|(v(t))_{g(t)}\|_{H^m(\Sigma,g(t))} \leq 2\Lambda \text{ for all } v \in S \text{ and all } t \in [0, T]. \tag{3.4.30}$$

Hence, for such $0 < T < \delta/M$ and any $v \in S$ we can apply Proposition 3.2.5, with Λ replaced by 2Λ, to obtain a unique weak solution $u = u_{\varepsilon,g,v}$ of the linear equation (3.2.11) which, after possibly further reducing δ, satisfies

$$\begin{aligned}\|(u(t))_{g_0}\|_{H^m(\Sigma,g_0)} &\leqslant 2\|(u(t))_{g(t)}\|_{H^m(\Sigma,g(t))} \leq 2e^{CT}\|(u_0)_{g_0}\|_{H^m(\Sigma,g_0)} \\ &\leq 4\|(u_0)_{g_0}\|_{H^m(\Sigma,g_0)} \leq 4\Lambda_0\end{aligned} \tag{3.4.31}$$

for all $0 < t \leq T$.

From (3.4.29), (3.4.27) and (3.4.31) we furthermore obtain

$$\begin{aligned}\|\partial_t(u(t))_{g_0}\|_{H^1(\Sigma,g_0)} &= \|(\partial_t u(t))_{g_0}\|_{H^1(\Sigma,g_0)} \leq 2\|(\partial_t u(t))_{g(t)}\|_{H^1(\Sigma,g(t))} \\ &\leq C(1 + E_{\frac{1}{2}}(v(t), g(t))^{\frac{1}{2}})\|u_{g(t)}\|_{H^3(\Sigma,g(t))} \\ &\leq C(1 + E_0^{\frac{1}{2}})\Lambda_0.\end{aligned} \tag{3.4.32}$$

Combined with (3.4.31), we thus obtain

$$\|u\|_{X_m(T,g_0)} \leq C(1 + E_0^{\frac{1}{2}})\Lambda_0 \leqslant C_0\Lambda_0 = \Lambda,$$

if $C_0 = C_0(E_0, \iota_0) > 0$ is chosen sufficiently large.

To obtain the desired bound on the energy we can then apply the estimate (3.4.19) on the evolution of the energy obtained above which, for sufficiently small $\delta > 0$ and $T \le \delta/M$, implies the bound

$$E_{\frac{1}{2}}(u(t), g_0) \leqslant e^{CMT} E_{\frac{1}{2}}(u(t), g(t)) \le e^{2CMT} E_{\frac{1}{2}}(u_0, g_0) \le 2E_0.$$

This completes the proof that $u \in S$, which as already observed also ensures that $u([0,T] \times \partial\Sigma) \subset N_\eta$.

It remains to show that $\Psi\colon S \to S$ is a contraction with respect to the norm $\|\cdot\|_S$. So we let $v_{1,2} \in S$, and let $g_{1,2} = g_{v_{1,2}}$ as well as $u_{1,2} = \Psi(v_{1,2})$ be the corresponding solutions of (3.2.3) and (3.2.11).

Since $w := u_2 - u_1$ is so that $w(0) = 0$, and since we can assume that $T \leqslant 1$, we can bound

$$\|w_{g_1}\|^2_{L^2(\Sigma,g_1)} \le C\|w\|^2_{L^2(\partial\Sigma,g_1)} \le C\bigg(\int_0^t \|\partial_t w\|_{L^2(\partial\Sigma,g_1)}\mathrm{d}t\bigg)^2 \le C\|\partial_t w\|^2_{L^2([0,T];L^2(\partial\Sigma,g_1))}$$

at any $0 \le t \le T$. As the curves of metrics $g_{1,2}$ induce uniformly equivalent Sobolev norms and harmonic extensions as recalled above we can hence bound

$$\begin{aligned}
\|w\|^2_S &= \sup_{t\in[0,T]} \|w_{g_0}\|^2_{H^1(\Sigma,g_0)} + \|\partial_t w\|^2_{L^2([0,T]\times\partial\Sigma,g_0)} \\
&\le 2\sup_{t\in[0,T]} \|w_{g_1}\|^2_{H^1(\Sigma,g_1)} + 2\|\partial_t w\|^2_{L^2([0,T]\times\partial\Sigma,g_1)} \\
&\leqslant 2\sup_{t\in[0,T]} \|\nabla_{g_1}(u_2 - u_1)_{g_1}\|^2_{L^2(\Sigma,g_1)} + C\|\partial_t w\|^2_{L^2([0,T]\times\partial\Sigma,g_1)} \\
&\leqslant C\varepsilon^{-1}T\sup_{t\in[0,T]}\Big(\mathrm{dist}^2_{H^3}(g_1,g_2) + \|(v_2 - v_1)_{g_1}\|^2_{H^1(\Sigma,g_1)}\Big)
\end{aligned}$$

where the last step follows from the Lipschitz estimate proven in Lemma 3.4.4.

We note that (3.2.8) implies

$$\mathrm{dist}_{H^3}(g_1(t), g_2(t)) \leqslant CT\sup_{t\in[0,T]} \|(v_2(t) - v_1(t))_{g_0}\|_{H^1(\Sigma,g_0)} \leqslant CT\|v_2 - v_1\|_S \tag{3.4.33}$$

while (3.4.29) ensures that $\sup_{t\in[0,T]} \|(v_2 - v_1)_{g_1}\|^2_{H^1(\Sigma,g_1)} \leqslant 2\|v_2 - v_1\|^2_S$. Combined we hence deduce that

$$\|u_2 - u_1\|^2_S \leqslant C\varepsilon^{-1}T\|v_2 - v_1\|^2_S \leqslant \frac{1}{2}\|v_2 - v_1\|^2_s$$

where the last inequality holds after possibly further reducing $T > 0$ to ensure that $C\varepsilon^{-1}T < \frac{1}{2}$ for the constant $C = C(\iota_0, E_0, \Lambda_0)$ obtained in the above argument. □

Noting the completeness of the metric space $(S, \|\cdot\|_S)$ and by iteration, this immediately provides the following existence result.

Corollary 3.4.8. *For given constants $\iota_0, \Lambda_0 > 0$ and $\varepsilon \in (0, \frac{1}{2}]$ and initial data $(u_0, g_0) \in H^{m-\frac{1}{2}}(\partial\Sigma; N) \times \mathcal{M}_{\iota_0}$ satisfying $\|(u_0)_{g_0}\|_{H^m(\Sigma,g_0)} \leqslant \Lambda_0$, there exists a unique solution of the flow* (3.2.16) *on a time interval* $[0, T]$*, where we take $T = 1$ unless one of the following occurs for $T < 1$:*

$$\text{dist}(u(T, \partial\Sigma), \partial N_\eta) = \frac{1}{2}\eta, \quad \|u(T)_{g(T)}\|_{H^m(\Sigma,g(T))} = 2\Lambda_0 \text{ or } \text{inj}(g(T)) = \frac{1}{2}\iota_0 \qquad (3.4.34)$$

Comparing with the claim in Proposition 3.2.6, it remains only to show that there is an ε-independent lower bound for the time $T > 0$ such that one of the three conditions of Corollary 3.4.8 can hold.

Proof of Proposition 3.2.6. Let u_0 and g_0 be as in the Proposition and let $T > 0$ be an ε-independent number that is fixed below.

Given any $\varepsilon > 0$, we can use that Corollary 3.4.8 provides the existence of a solution $(u_\varepsilon, g_\varepsilon)$ on $[0, T]$ until one of the three conditions (3.4.34) is met, or $T = 1$ in which case this evidently doesn't depend on ε. Note that on the interval $[0, T]$, the estimate $\|u_\varepsilon(T)_{g(t)}\|_{H^m(\Sigma,g_\varepsilon(t))} \leqslant 2\Lambda_0$ holds, $g_\varepsilon(t) \in \mathcal{M}_{\frac{1}{2}\iota_0}$ and $E_{\frac{1}{2}}(u_\varepsilon(t), g_\varepsilon(t)) \leqslant E_0$.

As in Remark 3.4.7, we obtain a time $T_1 = T_1(\Lambda_0, \iota_0, E_0, \eta, m)$ such that
$\text{dist}(u_\varepsilon(t, \partial\Sigma), \partial N_\eta) \geqslant \frac{1}{2}\eta$ on $[0, \min(T, T_1)]$.

Next, we recall the estimates from Lemma 3.3.3 show that first for the injectivity radius to half, the metric g_ε must have reached the boundary of the neighbourhood $\mathcal{U}$. Then by Remark 3.2.3, there is a time $T_2 = T_2(E_0, \iota_0)) > 0$ such that $\text{inj}(g_\varepsilon(t)) \geqslant \frac{1}{2}\iota_0$ on $[0, \min(T, T_2)]$.

Finally, noting the bound on $\|\partial_t g_\varepsilon\|_{H^{m+1}(\Sigma, g_\varepsilon)} \leqslant CE_0$ for a constant $C = C(\iota_0)$ from (3.2.10) and the uniform H^m bound from Proposition 3.2.5, we obtain a third constant $T_3 = T_3(m, \iota_0, E_0, \Lambda_0) > 0$ so that $\|u_\varepsilon(T)_{g(t)}\|_{H^m(\Sigma, g_\varepsilon(t))} \leqslant 2\Lambda_0$ on $[0, \min(T, T_3)]$.

Hence we see $T \geqslant \min(T_1, T_2, T_3, 1) > 0$ and this lower bound is independent of ε.

We finally remark that the claimed $C^{1,1}$ estimate for the metric component is an immediate consequence of the a priori estimates (3.2.6) and (3.2.7) obtained in Proposition 3.2.1 as the estimate $E_{\frac{1}{2}}(u_\varepsilon(t), g_0) \leqslant CE_{\frac{1}{2}}(u_\varepsilon(t), g_\varepsilon(t)) \leqslant CE_0$, $C = C(\iota_0)$ remains valid for as long as g_ε remains in $\mathcal{U}$. □

3.4.3 Local Existence of Solutions of the Original Flow (3.1.9)

We are now finally in a position to establish the existence of solutions to our original flow (3.1.9) for sufficiently regular initial data based on the ε-independent control on the solutions $(u_\varepsilon, g_\varepsilon)$ of the regularised flow (3.2.16) that we have just established.

Proof of Proposition 3.2.7. Let u_0, g_0 be as in the proposition and let $(u_\varepsilon, g_\varepsilon)$, $\varepsilon \in (0, \frac{1}{2}]$ be the corresponding solutions of the regularised system (3.2.16). Proposition 3.2.6 ensures that these solutions all exist at least on the ε-independent interval $[0, T]$, for $T > 0$ as in that proposition, that the metric components are uniformly bounded in $C^{1,1}([0, T]; \mathcal{M}^{m+1})$ and contained in $\mathcal{U}$, which, combined with (3.2.17), ensures that the maps $U_\varepsilon := (u_\varepsilon)_{g_\varepsilon}$ are uniformly bounded in $L^\infty([0, T]; H^m(\Sigma, g_0))$. As u_ε solves (3.2.16), these estimates of course also provide uniform bounds on $\partial_t(u_\varepsilon)_{g_0}$ in $L^\infty([0, T]; H^1(\Sigma, g_0))$, so u_ε is bounded uniformly in the space $(X_m(T), \|\cdot\|_{X_m(T, g_0)})$, which embeds compactly into $C^0([0, T]; H^{m-1}(\Sigma, g_0))$, compare Lemma 3.4.6.

We can hence choose $\varepsilon_n \to 0$ so that g_{ε_n} converges strongly in $C^1([0, T]; \mathcal{M}^m)$ to a limiting curve of metrics $g : [0,T] \to \mathcal{U}$ while the maps u_{ε_n} converge weak-$*$ in $X_m(T, g_0)$ to a limit $u \in X_m(T, g_0)$ which is furthermore so that harmonic extensions $U_{\varepsilon_n} = (u_{\varepsilon_n})_{g_{\varepsilon_n}}$ converge strongly in $C^0([0,T]; H^{m-1}(\Sigma))$ to u_g.

This allows us to pass to the limit in (3.2.16) to conclude that (u, g) solves the original equation (3.1.9) on this interval $[0, T]$.

We stress that the length of the interval on which this argument applies does not depend on the specific choice of u_0 and g_0, but only on upper bounds the initial energy and the initial $H^m(\Sigma)$-norm, and on a lower bound on the injectivity radius. As the energy is decreasing along the flow, we can hence apply this argument iteratively to establish that the solution indeed exists for as long as $\mathrm{inj}(\Sigma, g) \not\to 0$ and $\|u_g\|_{H^m(\Sigma,g)} \not\to \infty$. $\square$

3.5 Proof of Theorem 3.1.4

In the previous section, we constructed a solution (u, g) to the flow (3.1.9) which remains well controlled until either $\mathrm{inj}(\Sigma, g) \to 0$ or $\|u_g\|_{H^m(\Sigma,g)} \to \infty$.

In this section, we will complete the proof of our first main theorem, Theorem 3.1.4. In pursuit of this, we establish bounds on the $H^m(\Sigma, g)$-norm of u_g which remain valid for as long both $\mathrm{inj}(g(t))$ and the radius of balls on which a certain (small) amount of energy is concentrated remains bounded away from zero. We will then prove some local energy estimates and analyse the behaviour around points where energy concentrates.

The first aim is to show the following smooth existence result.

Proposition 3.5.1. *To every initial data* $(u_0, g_0) \in H^{\frac{1}{2}}(\partial\Sigma) \times \mathcal{M}(\Sigma)$*, there exists a weak solution* (u, g) *of the coupled flow* (3.1.9) *which is smooth on an interval* $(0, T_*)$ *for a maximal time* $T_* > 0$*. Furthermore, if* T_* *is finite, then we either have that* $\mathrm{inj}(g(t)) \to 0$ *as* $t \nearrow T^*$ *or that* $\iota_0 := \inf_{[0,T_*)} \mathrm{inj}(g) > 0$ *and* $\liminf_{t \nearrow T_*} \sup_{x \in \partial\Sigma} E(u(t)_{g(t)}, g(t); B_r^{g(t)}(x)) \geqslant \frac{1}{2}\delta_0$ *for every* $r > 0$*,* $\delta_0 = \delta_0(\iota_0, E_0) > 0$ *as in Proposition 3.5.4.*

To prove this we require a number of auxiliary results. First, the following lemma, which extends Proposition 3.3 from [Str24] to maps from general surfaces.

Lemma 3.5.2. *For any $\iota_0 > 0$, there are constants $\delta_1 > 0$, $c_1 \in (0, \frac{1}{2}\iota_0)$ and $C > 0$ such that the following holds true for any metric $g \in \mathcal{M}_{\iota_0}$ and any map $u \in H^1(\partial\Sigma; N)$. If $x_0 \in \partial\Sigma$ and $0 < r < c_1$ are so that*

$$E(u_g, g; B^g_{2r}(x_0)) \leqslant \delta_1 \tag{3.5.1}$$

then we can bound

$$\begin{aligned}\|\nabla u_g\|_{L^2(\partial\Sigma\cap B^g_r(x_0),g)} &\leqslant C\|P_u\partial_\nu u_g\|_{L^2(\partial\Sigma\cap B^g_{2r}(x_0),g)} \\ &\quad + Cr^{-\frac{1}{2}}\|\nabla u_g\|_{L^2(\Sigma\cap B^g_{2r}(x_0),g)} + Cr^{-\frac{3}{2}}\|u_g\|_{L^2(\Sigma\cap B^g_{2r}(x_0),g)}.\end{aligned} \tag{3.5.2}$$

In addition, we will use that solutions of the regularity constructed in the previous section in fact have a little bit more regularity.

Lemma 3.5.3. *Let $m \geqslant 3$ and let $(u, g) \in X_m(T) \times C^1_{loc}([0, T]; \mathcal{M}^{m+1})$ be any solution of the coupled flow* (3.1.9)*. Then $u_g \in L^2([0, T]; H^{m+\frac{1}{2}}(\Sigma))$.*

For such solutions, we consider the evolution of the quantities

$$I_k(u, g) := 1 + \frac{1}{2}\int_\Sigma \left|\nabla^k u_g\right|^2 \mathrm{d}v, \quad I_{k+\frac{1}{2}}(u, g) := 1 + \frac{1}{2}\int_{\partial\Sigma} \left|\nabla^k u_g\right|^2 \mathrm{d}s \tag{3.5.3}$$

for $k = 1, \dots, m$.

In Section 3.5.2 we will prove the following proposition, which provides the necessary generalisations of Propositions 4.6 and 4.11 of [Str24] to the present setting of maps from general surfaces.

Proposition 3.5.4. *For any $\iota_0 > 0$ and any $\bar{E} > 0$, there exists $\delta_0 = \delta_0(\iota_0, \bar{E}) \in (0, \delta_1]$, $\delta_1 = \delta_1(\iota_0)$ as in Lemma 3.5.2, so that the following holds true. Let (u, g) be a solution to the flow* (3.1.9) *which satisfies $g(t) \in \mathcal{M}_{\iota_0}$, $E_{\frac{1}{2}}(u, g) \leqslant \bar{E}$ on a time interval $[t_1, t_2]$ and is so that $u_g \in L^\infty([t_1, t_2]; H^s(\Sigma)) \cap L^2([t_1, t_2]; H^{s+\frac{1}{2}}(\Sigma))$ for some $s \in \frac{1}{2}\mathbb{N}_{\geqslant 3}$. Let $\Lambda \geqslant 1$ be so that*

$$\|u_g(t)\|^2_{H^{s-\frac{1}{2}}(\Sigma,g(t))} \leqslant \Lambda \text{ for } t \in [t_1, t_2]. \tag{3.5.4}$$

Then the estimate

$$\sup_{t\in[t_1,t_2]} I_s(t) + \int_{t_1}^{t_2} I_{s+\frac{1}{2}}(t)\mathrm{d}t \leqslant e^{C|t_2-t_1|\Lambda^{\lceil s\rceil}} I_s(t_1) \tag{3.5.5}$$

holds true for a constant C which for $s > \frac{3}{2}$ only depends on ι_0, s, while for $s = \frac{3}{2}$ we have $C = C_0(\iota_0, \bar{E})r_0^{-2}$ for $r_0 \in (0, \iota_0)$ chosen so that

$$E(u(t), g(t); B_{r_0}^{g(t)}(x)) \leqslant \delta_0 \text{ for all } x \in \partial\Sigma \text{ and all } t \in [t_1, t_2]. \tag{3.5.6}$$

Combined with a standard iteration argument, these a priori estimates allow us to obtain the desired H^m control on the map component as described in the following proposition.

Proposition 3.5.5. *For any solution $(u, g) \in X_m(T) \times C^1([0, T]; \mathcal{M}(\Sigma))$, $m \geqslant 4$, we can bound*

$$\sup_{t\in[0,T]} \|u_g(t)\|_{H^m(\Sigma,g)} \leqslant C(m, \|(u_0)_{g_0}\|_{H^m(\Sigma,g_0)}, \iota_0, r_0, \bar{E}), \tag{3.5.7}$$

and

$$\sup_{t\in[\tau,T]} \|u_g(t)\|_{H^m(\Sigma,g)} \leqslant C(m, \tau, \iota_0, r_0, \bar{E}) \text{ for any } 0 < \tau < T, \tag{3.5.8}$$

for $\iota_0 > 0$ and $r_0 > 0$ chosen so that $\mathrm{inj}_g \geqslant \iota_0$ and (3.5.6) hold on $[0, T]$ and where $\bar{E}$ is an upper bound on $E_{\frac{1}{2}}(u_0, g_0)$.

Combined with local energy estimates which are stated and proven in Section 3.5.5, this proposition then allows us to deduce Proposition 3.5.1.

The results of Section 3.3 guarantee that if $\mathrm{inj}(g(t)) \nrightarrow 0$ as $t \to T_*$ then the metrics $g(t)$ converge smoothly to a limiting metric $g(T_*) \in \mathcal{M}(\Sigma)$ as $t \nearrow T_*$. As the energy is non-increasing and as $\int_0^{T_*} \|\partial_t u_g\|^2_{L^2(\Sigma,g)}\mathrm{d}t$ is finite, we hence deduce that the maps $u_{g(t)}(t)$ converge weakly in $H^1(\Sigma, g(T_*))$ to a limiting map which is harmonic with respect to $g(T_*)$ and can hence be written as $u(T_*)_{g(T_*)}$ for some $u(T_*) \in H^{\frac{1}{2}}(\partial\Sigma)$. We can thus apply the above proposition for this new initial data $(u(T_*), g(T_*))$ to restart the flow and hence continue the flow past any such finite time singularities at which the injectivity radius remains bounded away from zero.

Furthermore, we shall see later in this section that any such singularity must be caused by the bubbling off of a non-constant half-harmonic map from the disc. Since the energy of such maps is bounded from below by a constant $\varepsilon_0 > 0$ which only depends on N, compare Remark 3.5.9, the number of such singular times is a priori bounded in terms of the initial energy. Repeating the above argument a finite number of times, we hence obtain the existence of a weak solution (u, g) for as long as the metric does not degenerate.

This completes the proof of our main Theorem 3.1.4 up to the proof of the auxiliary results that we stated above, which will be carried out in the subsequent Sections 3.5.1-3.5.5, and the characterisation of finite-time singularities of the map component, which is discussed in Section 3.5.6.

In these proofs we will always work with metrics $g \in \mathcal{M}_{\iota_0}$ for a fixed $\iota_0 > 0$ and with a fixed number $m \geqslant 3$, so will at times use the following convention to lighten the notation.

Remark 3.5.6. We write for short $a \lesssim b$ if $a \leqslant Cb$ for a number C that only depends on m, ι_0 and as always the topology of the surface Σ and the given manifold N.

3.5.1 Analysis of Terms Involving $P_\cdot^\perp$

For the proofs of both Lemma 3.5.3 and Proposition 3.5.4, we will use that terms that involve the projection onto the normal space have improved regularity properties, as summarized in the following lemma.

Lemma 3.5.7. *Let $m \geqslant 3$ and $\iota_0 > 0$ be any fixed numbers and let $g \in \mathcal{M}_{\iota_0}^{m+1}$ and $u \in H^{m-\frac{1}{2}}(\partial\Sigma; N)$. Then both $P_u^\perp \partial_\nu \nabla^{m-1} u_g$ and $\nabla^{m-1} P_u^\perp \partial_\nu u_g$ are in $L^2(\partial\Sigma)$ and we can bound*

$$\|P_u^\perp \partial_\nu \nabla u_g\|_{L^2(\partial\Sigma)}^2 + \|\nabla(P_u^\perp \partial_\nu u_g)_g\|_{L^2(\partial\Sigma)}^2 \lesssim \|\nabla^2 u\|_{L^2(\Sigma)}^2 (1 + \|\nabla u\|_{L^4(\Sigma)}^2) + (1 + \|\nabla u\|_{L^4(\Sigma)}^2)^3, \qquad (3.5.9)$$

and, for $k \in \{3, \ldots, m\}$,

$$\|P_u^\perp \partial_\nu \nabla^{k-1} u_g\|_{L^2(\partial\Sigma)}^2 + \|\nabla^{k-1}(P_u^\perp \partial_\nu u_g)_g\|_{L^2(\partial\Sigma)}^2$$

$$\lesssim I_{3/2}(u)I_k(u) + I_{5/2}(u)S_{k-1}(u) + S_{k-1}(u)^{k+1}. \qquad (3.5.10)$$

Here and in the following, we use the shorthand $\leqslant$ introduced in Remark 3.5.6 and write for short

$$S_s(u,g) := \sum_{\{\tilde{s}\in\frac{1}{2}\mathbb{N}_{\geqslant 2}:\tilde{s}\leqslant s\}} I_{\tilde{s}}(u,g), \quad \text{for } I_{\tilde{s}}(u,g) \text{ defined in (3.5.3)}, \qquad (3.5.11)$$

where we drop the reference to the metric if we work with respect to fixed (curves of) metrics g.

Proof of Lemma 3.5.7. The main reason for the improved regularity of the above terms is that the projection of $\partial_\tau u$ onto $T_u^\perp N$ vanishes identically on $\partial\Sigma$ since u maps $\partial\Sigma$ into N. This will allow us to obtain the claims made in the lemma by writing all relevant expressions in terms of derivatives of the corresponding harmonic extension $(P_u\partial_\tau u)_g \equiv 0$ and commutator terms, whose improved regularity properties are discussed in the appendix.

To make this precise, we recall that Lemma B.0.1 and Remark B.0.2 ensure that commutator terms $C_{k-1}(u_g) = C_{k-1}(u_g,u_g)$, $2 \leqslant k \leqslant m$, as considered in (B.0.4), (B.0.18) and (B.0.19) are in $L^2(\partial\Sigma)$ and that their norms $\|C_{k-1}(u_g)\|^2_{L^2(\partial\Sigma)} + \|C_k(u_g)\|^2_{L^{4/3}(\Sigma)}$ are bounded by the right hand side of the claimed estimates (3.5.9) (for $k=2$) respectively (3.5.10) (for $k \geqslant 3$).

As $\varphi\big[P_u^\perp\partial_\nu\nabla^{k-1}u_g - \nabla^{k-1}(P_u^\perp\partial_\nu u_g)_g\big]$, φ a cut-off function as in Remark 3.3.2, is given by such a commutator term $C_{k-1}(u_g,u_g)$ of the form (B.0.18), it hence suffices to establish the claims for $P_u^\perp\partial_\nu\nabla^{k-1}u_g$.

In collar coordinates (s,θ) on a neighbourhood of the boundary component σ_i, this term can be written as a linear combination of terms of the form $P_u^\perp\partial_s^i\partial_\theta^j u_g$ with $1 \leqslant i \leqslant k-j$. As the claimed estimate is trivially true if $i < k-j$, it suffices to establish that all terms of the form $P_u^\perp\partial_s^{k-j}\partial_\theta^j u_g$, $j = 0,\dots,k-1$ are controlled in $L^2(\sigma_i, \mathrm{d}\theta)$ by the right-hand sides of (3.5.9) respectively (3.5.10).

For $j \geqslant 1$ we use that $(P_u^\perp\partial_\theta u)_g$ vanishes identically since it is the harmonic extension of $P_u^\perp \partial_\theta u|_{\partial\Sigma} \equiv 0$. This allows us to rewrite

$$P_u^\perp \partial_s^{k-j}\partial_\theta^j u_g = \partial_s^{k-j}\partial_\theta^{j-1}\big[P_u^\perp \partial_\theta u_g - (P_u^\perp \partial_\theta u)_g\big] - \big[\partial_s^{k-j}\partial_\theta^{j-1} P_u^\perp \partial_\theta u_g - P_u^\perp \partial_s^{k-j}\partial_\theta^j u_g\big]$$

on the collar neighbourhood. As these are again commutator terms of the form (B.0.18) respectively (B.0.4), we immediately obtain the required estimate on $\|P_u^\perp \partial_s^{k-j}\partial_\theta^j u_g\|_{L^2(\partial\Sigma)}$ from Lemma B.0.1 for any $j \geqslant 1$.

Finally, for $j = 0$ we can use $(\partial_s^2 + \partial_\theta^2)u_g = 0$ to rewrite the corresponding term as $P_u^\perp \partial_s^k u_g = -P_u^\perp \partial_s^{k-2}\partial_\theta^2 u_g$ reducing this final case to the case where $j = 2$ discussed above. □

Since $H^{\frac{1}{2}}(\Sigma) \hookrightarrow L^4(\Sigma)$, the above lemma in particular ensures that for any $k \in \{2, \ldots, m\}$

$$\begin{aligned} &\|P_u^\perp \partial_\nu \nabla^{k-1} u_g\|_{L^2(\partial\Sigma)}^2 + \|\nabla^{k-1}(P_u^\perp \partial_\nu u_g)_g\|_{L^2(\partial\Sigma)}^2 \\ &\qquad \lesssim \|u_g\|_{H^{3/2}(\Sigma)}^2 \|u_g\|_{H^k(\Sigma)}^2 + (1 + \|u_g\|_{H^{k-1/2}(\Sigma)}^2)^{k+1}. \end{aligned} \tag{3.5.12}$$

Writing the equation for the map component as

$$\partial_t u + \partial_\nu u_g = P_u^\perp(\partial_\nu u_g) \tag{3.5.13}$$

hence allows to derive the extra regularity claimed in Lemma 3.5.3 using the following simple argument.

Proof of Lemma 3.5.3. A short calculation shows that if a function u satisfies $\partial_t u + \partial_\nu u = f$ for some $f : [0, T] \times \partial\Sigma \to \mathbb{R}^n$ with $\nabla^{m-1} f_g \in L^2([0, T]; L^2(\partial\Sigma, g))$, then

$$\begin{aligned} \|u_g(t)\|_{H^m(\Sigma)}^2 + \int_0^t \|\partial_\nu \nabla^{m-1} u_g\|_{L^2(\partial\Sigma)}^2 \mathrm{d}t' &\lesssim \|u_g(0)\|_{H^m(\Sigma)}^2 + \int_0^t \|u_g(t)\|_{H^m(\Sigma)}^2 \mathrm{d}t' \\ &\quad + \int_0^t \|\nabla^{m-1} f_g\|_{L^2(\partial\Sigma)}^2 + \|\nabla^{m-1} f_g\|_{L^2(\Sigma)}^2 \mathrm{d}t'. \end{aligned}$$

As (3.5.12) ensures that $\nabla_g^{m-1} P_u^\perp(\partial_\nu u_g) \in L^2([0, T]; L^2(\partial\Sigma, g))$ for solutions (u, g) of (3.1.9) as considered in the lemma, we can hence deduce that $u_g \in L^2([0, T]; H^{m+\frac{1}{2}}(\Sigma, g))$. □

3.5.2 Proof of Proposition 3.5.4

We now want to use the above improved estimates on terms involving $P_u^\perp$ to derive the a priori estimates on the evolution of the quantities I_s, $s \in \frac{1}{2}\mathbb{N}_{\geqslant 3}$ defined in (3.5.3) that we claimed in Proposition 3.5.4.

Proof of Proposition 3.5.4. Given (u, g) as in the proposition we will write for short $I_s(t) = I_s(u(t), g(t))$ for $t \in [t_1, t_2]$ as well as $S_s(t) := S_s(u(t), g(t))$ and will often drop the reference to t if there is no room for confusion. As $S_{s-\frac{1}{2}} \leqslant C(1 + \|u_g\|^2_{H^{s-1/2}(\Sigma,g)}) \leqslant C\Lambda$ on $[t_1, t_2]$ for Λ as in (3.5.4), it suffices to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_s + \frac{1}{4} I_{s+\frac{1}{2}} \leqslant C S_{s-\frac{1}{2}}^{\lceil s \rceil} S_s \text{ on } [t_1, t_2]$$

for a constant C as considered in the proposition. To carry out this proof simultaneously for all $s \in \frac{1}{2}\mathbb{N}_{\geqslant 3}$, we write all such exponents as $s = k - \frac{1}{2}\beta$ for $k = \lceil s \rceil \in \mathbb{N}_{\geqslant 2}$ and $\beta = 2(\lceil s \rceil - s) \in \{0, 1\}$.

We first note that the control on the evolution of the metric component established in Lemma 3.3.8 and (3.2.10) allows us to bound the contribution of the change of the metric to the evolution of I_s by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} I_s(u, g(\cdot + \varepsilon)) \leqslant C\|\partial_t g\|_{H^k(\Sigma)} S_s \leqslant C S_s, \quad C = C(\iota_0, \bar{E}).$$

Writing for short

$$\hat{I}_s := \int_{\partial\Sigma} \langle \partial_\nu^{1-\beta} \nabla^{k-1} u_g, \nabla^{k-1}(P_u \partial_\nu u)_g \rangle \mathrm{d}s \tag{3.5.14}$$

and recalling that $\partial_t u = -P_u \partial_\nu u_g$, we then note that for $s = k - \frac{1}{2}$ (and hence $\beta = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} I_s(u(\cdot + \varepsilon), g) = \int_{\partial\Sigma} \langle \nabla^{k-1} u_g, \nabla^{k-1}(\partial_t u)_g \rangle \mathrm{d}s = -\hat{I}_s,$$

while for $s = k \in \mathbb{N}_{\geqslant 2}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} I_s(u(\cdot + \varepsilon), g) &= \int_\Sigma \langle \nabla^k u_g, \nabla^k(\partial_t u)_g \rangle \mathrm{d}v = \int_{\partial\Sigma} \langle \partial_\nu \nabla^{k-1} u_g, \nabla^{k-1}(\partial_t u)_g \rangle \mathrm{d}s + \mathrm{err}_1 \\ &= -\hat{I}_s + \mathrm{err}_1\end{aligned}$$

for $\mathrm{err}_1 = -\int_\Sigma \langle \Delta(\nabla^{k-1}u_g), \nabla^{k-1}(P_u(\partial_\nu u_g))_g\rangle \mathrm{d}v$. As $\|\Delta\nabla^{k-1}u\|_{L^2(\Sigma)} \lesssim \|u_g\|_{H^k(\Sigma)}$ and as we can view $\nabla^{k-1}(P_u(\partial_\nu u_g))_g - P_u(\partial_\nu \nabla^{k-1}u_g)$ as a commutator term $C_{k-1}(u_g, u_g)$ of the form (B.0.4), we can bound this error term by

$$|\mathrm{err}_1| \lesssim \|u_g\|^2_{H^k(\Sigma)} + \|C_{k-1}(u_g, u_g)\|^2_{L^2(\Sigma)} \lesssim I_k + S^k_{k-\frac{1}{2}} \lesssim S^{k-1}_{k-\frac{1}{2}} S_k, \tag{3.5.15}$$

compare (B.0.6) and (B.0.12).

In both cases we hence obtain that $\frac{\mathrm{d}}{\mathrm{d}t} I_s \leq -\hat{I}_s + C S^{k-1}_{s-\frac{1}{2}} I_s$, reducing the proof of (3.5.2) to the proof that

$$-\hat{I}_s \leqslant -\frac{1}{4} I_{s+\frac{1}{2}} + C S^k_{s-\frac{1}{2}} S_s. \tag{3.5.16}$$

To show this we write $P_u = \mathrm{Id} - P_u^\perp$ and split $\hat{I}_s = T_s - T_s^\perp$ for

$$T_s := \int_{\partial\Sigma} \langle \partial_\nu^{1-\beta} \nabla^{k-1} u_g, \nabla^{k-1}(\partial_\nu u_g)_g \rangle \mathrm{d}s \tag{3.5.17}$$

$$T_s^\perp := \int_{\partial\Sigma} \langle \partial_\nu^{1-\beta} \nabla^{k-1} u_g, \nabla^{k-1} \left(P_u^\perp(\partial_\nu u_g)\right)_g \rangle \mathrm{d}s \tag{3.5.18}$$

To bound the first term for $s = k \in \mathbb{N}_{\geqslant 2}$ (corresponding to $\beta = 0$), we can use that $\|\partial_\nu \nabla^{k-1} u_g - \nabla^{k-1}(\partial_\nu u_g)_g\|^2_{L^2(\partial\Sigma)} \lesssim S_{k-\frac{1}{2}}$, compare (B.0.3) and (B.0.1), as well as that

$$\frac{1}{2}\int_{\partial\Sigma} \left|\nabla^k u_g\right|^2 \mathrm{d}s \leqslant \int_{\partial\Sigma} \left|\partial_\nu \nabla^{k-1} u_g\right|^2 \mathrm{d}s + C S_{k-\frac{1}{2}},$$

which can e.g. be seen by viewing these expressions in collar coordinates and exploiting that $\partial_s^2 u = -\partial_\theta^2 u$. Combined, this allows us to bound

$$-T_k \leqslant -\int_{\partial\Sigma} \left|\partial_\nu \nabla^{k-1} u_g\right|^2 \mathrm{d}s + C\|\partial_\nu \nabla^{k-1} u_g\|_{L^2(\partial\Sigma)} S^{\frac{1}{2}}_{k-\frac{1}{2}} \leqslant -I_{k+\frac{1}{2}} + C S_{k-\frac{1}{2}}$$

for every $k \in \mathbb{N}_{\geqslant 2}$. Similarly, we have

$$\begin{aligned}
-T_{k-\frac{1}{2}} &\leqslant -\int_{\partial\Sigma} \langle \nabla^{k-1} u_g, \partial_\nu \nabla^{k-1} u_g \rangle \mathrm{d}s + C\|\nabla^{k-1} u_g\|_{L^2(\partial\Sigma)} S^{\frac{1}{2}}_{k-\frac{1}{2}} \\
&\leqslant -\frac{1}{2}\int_{\partial\Sigma} \partial_\nu \left|\nabla^{k-1} u_g\right|^2 \mathrm{d}s + C S_{k-\frac{1}{2}} = -\frac{1}{2}\int_\Sigma \Delta \left|\nabla^{k-1} u_g\right|^2 \mathrm{d}v + C S_{k-\frac{1}{2}} \\
&\leqslant -\int_\Sigma \left|\nabla^k u_g\right|^2 \mathrm{d}v + C S_k^{\frac{1}{2}} S_{k-1}^{\frac{1}{2}} + C S_{k-\frac{1}{2}} \\
&\leq -2I_k + C I_k^{\frac{1}{2}} S_{k-1}^{\frac{1}{2}} + C S_{k-\frac{1}{2}} \leqslant -I_k + C S_{k-\frac{1}{2}}.
\end{aligned}$$

We hence conclude that

$$-T_s \leqslant -I_{s+\frac{1}{2}} + CS_s \text{ for all } s \geqslant \tfrac{3}{2}. \tag{3.5.19}$$

To bound $T_s^\perp$ for $s \geqslant \frac{5}{2}$ we use that (3.5.10) ensures that for $k \geqslant 3$

$$\|\nabla^{k-1}\left(P_u^\perp \partial_\nu u_g\right)_g\|_{L^2(\partial\Sigma)}^2 \lesssim S_{3/2}I_k + S_{5/2}S_{k-1} + S_{k-1}^{k+1}.$$

For $s = k \geqslant 3$ this term is in particular bounded by $CS_sS_{s-1}^k$ allowing us to estimate

$$\left|T_s^\perp\right| \leqslant I_{k+\frac{1}{2}}^{\frac{1}{2}}\|\nabla^{k-1}\left(P_u^\perp \partial_\nu u_g\right)_g\|_{L^2(\partial\Sigma)} \leqslant \frac{1}{4}I_{s+\frac{1}{2}} + CS_sS_{s-1}^s$$

while for $s = k - \frac{1}{2}$, $k \geqslant 3$, we can use that $\frac{5}{2} \leqslant s$ and $\frac{3}{2} \leqslant s-1$ to bound

$$\begin{aligned}\left|T_s^\perp\right| &\leqslant I_{k-\frac{1}{2}}^{\frac{1}{2}}\|\nabla^{k-1}\left(P_u^\perp \partial_\nu u_g\right)_g\|_{L^2(\partial\Sigma)} \leqslant CI_s^{\frac{1}{2}}(S_{s-1}I_{s+\frac{1}{2}} + S_sS_{s-\frac{1}{2}} + S_{s-\frac{1}{2}}^{k+1})^{\frac{1}{2}} &(3.5.20)\\ &\leqslant \frac{1}{4}I_{s+\frac{1}{2}} + CS_{s-\frac{1}{2}}^kS_s. &(3.5.21)\end{aligned}$$

Similarly, applying (3.5.12) for $k = 2$ allows us to bound

$$\begin{aligned}\left|T_2^\perp\right| &\leqslant \|\partial_\nu\nabla u_g\|_{L^2(\partial\Sigma)}\|\nabla\left(P_u^\perp \partial_\nu u_g\right)_g\|_{L^2(\partial\Sigma)} \leqslant CI_{5/2}^{1/2}\left[I_{3/2}I_2 + S_{3/2}^3\right]^{\frac{1}{2}} &(3.5.22)\\ &\leqslant \frac{1}{4}I_{5/2} + CS_{3/2}^2S_2. &(3.5.23)\end{aligned}$$

Combined, these estimates on T_s and $T_s^\perp$, $s \geqslant 2$, imply that

$$-\hat{I}_s = -T_s + T_s^\perp \leqslant -\frac{1}{4}I_{s+\frac{1}{2}} + CS_{s-\frac{1}{2}}^kS_s$$

for all such $s \geqslant 2$ and a constant $C = C(\iota_0, s)$, i.e. establish the estimate (3.5.16) that was required to prove the claim of the proposition for these s.

It hence remains to prove this estimate (3.5.16) for $s = \frac{3}{2}$, where we recall that in this one case we have claimed that such an estimate holds for a constant of the form $C = C_0(\iota_0)r_0^{-2}$, r_0 so that (3.5.6) holds, rather than for a constant $C = C(\iota_0, s)$.

While we can continue to use (3.5.19) to bound $-T_s$ if $s = \frac{3}{2}$, in this case we need to analyse the term $T_s^\perp$ more carefully. For this we use the well known fact that

$$\int_\Sigma |\nabla U|^4 \,\mathrm{d}v \leqslant C \sup_{x\in\Sigma} E(U,g;B_r^g(x))\|U\|_{H^2(\Sigma)}^2 + Cr^{-2}E(U,g), \quad C = C(\iota_0) \tag{3.5.24}$$

for any $U \in H^2(\Sigma)$ and any $r \in (0, \mathrm{inj}(g))$, which can e.g. be obtained by applying the bound

$$\|\phi\,|\nabla u|\,\|_{L^4(\Sigma)}^4 = \|\phi^2\,|\nabla u|^2\,\|_{L^2(\Sigma)}^2 \leqslant C\|\phi^2\,|\nabla u|^2\,\|_{W^{1,1}(\Sigma)}^2$$

for a suitable cover of balls and corresponding cut-off functions ϕ.

Choosing $r = r_0 \in (0, \frac{1}{2}c_1)$, $c_1 > 0$ as in Lemma 3.5.2 so that (3.5.6) holds for $\delta_0 = \delta_0(\iota_0, \bar{E}) \in (0, \min(1, \delta_1))$ chosen below, this allows us to estimate

$$\|\nabla u_g\|_{L^4(\Sigma)} \leqslant C\delta_0^{1/4} I_2^{1/4} + Cr_0^{-1/2} I_1^{1/4}$$

for a constant C which only depends on ι_0. Combined with (3.5.9), and recalling that $I_1 \leqslant 1 + \bar{E}$, this allows us to bound

$$\begin{aligned} \|\nabla \left(P_u^\perp(\partial_\nu u_g)\right)_g\|_{L^2(\partial\Sigma)} &\leqslant C(1 + \|\nabla u_g\|_{L^4(\Sigma)})\|\nabla^2 u\|_{L^2(\Sigma)} + C(1 + \|\nabla u_g\|_{L^4(\Sigma)})^3 \\ &\leqslant C\delta_0^{1/4} I_2^{3/4} + Cr_0^{-1/2} I_2^{\frac{1}{2}} + Cr_0^{-3/2} \end{aligned}$$

for $C = C(\iota_0, \bar{E})$. As the standard interpolation inequality from Theorem A.2.3 ensures that

$$\|\nabla u_g\|_{L^2(\partial\Sigma)} \leqslant C\|u_g\|_{H^{3/2}(\Sigma)} \leqslant C\|u_g\|_{H^1(\Sigma)}^{\frac{1}{2}} \|u_g\|_{H^2(\Sigma)}^{\frac{1}{2}} \leqslant CS_2^{\frac{1}{4}}, \quad C = C(\iota_0, \bar{E})$$

we hence obtain that $T_{\frac{3}{2}}^\perp = \int_{\partial\Sigma} \langle \nabla u_g, \nabla(P_u^\perp \partial_\nu u_g)_g \rangle \mathrm{d}s$ is bounded by

$$\left|T_{3/2}^\perp\right| \leqslant C\delta_0^{1/4} I_2 + Cr_0^{-1/2} I_2^{3/4} + Cr_0^{-3/2} I_2^{1/4} \tag{3.5.25}$$

$$\leqslant (C\delta_0^{1/4} + \frac{1}{8}) I_2 + Cr_0^{-2} \leqslant \frac{1}{4} I_2 + Cr_0^{-2} \tag{3.5.26}$$

for a constant $C = C(\iota_0, \bar{E})$ and where the last estimate holds provided $\delta_0 = \delta_0(\iota_0, \bar{E}) > 0$ is chosen small enough.

This completes the proof of the bound (3.5.16) in this final case where $s = \frac{3}{2}$ and hence completes the proof of the key Proposition 3.5.4. □

3.5.3 Proof of Lemma 3.5.2

We will make use of the fact that for small enough r, the domain $B^g_{2r}(x_0)$ lies within a collar neighbourhood of a boundary curve $\sigma \subseteq \partial\Sigma$, which we can further view as a subset of the Euclidean half-plane $(\mathcal{H}, g_0) = ([0,\infty) \times \mathbb{R}, \mathrm{d}x^2 + \mathrm{d}y^2)$, with x_0 given coordinates $(0,0) \in \mathcal{H}$. We hence extract the main estimate in the following lemma.

Lemma 3.5.8. *Let $0 < c_0 < C_0 < \infty$. Then there exist constants $C, \delta_1 > 0$, depending only on c_0, C_0, such that for any radius $0 < r < \frac{\pi}{C_0}$, any non-empty subsets $\Omega_1 \subseteq \Omega_0 \subseteq B^{g_0}_{C_0 r}(0)$ which satisfy $\mathrm{dist}_{g_0}(\partial\Omega_1 \setminus \partial\mathcal{H}, \partial\Omega_0 \setminus \partial\mathcal{H}) \geqslant c_0 r$, and any harmonic function $w : B^{g_0}_{C_0 r}(0) \to \mathbb{R}^n$ with $w(\{0\} \times [-C_0 r, C_0 r]) \subseteq N$ and $E(w, g_0; \Omega_0) \leqslant \delta_1$, then*

$$\|\nabla_{g_0} w\|^2_{L^2(\Omega_1 \cap \partial\mathcal{H})} \leqslant C(\|P_w \partial_x w\|^2_{L^2(\Omega_0 \cap \partial\mathcal{H})} + r^{-1}\|\nabla_{g_0} w\|^2_{L^2(\Omega_0)} + r^{-3}\|w\|^2_{L^2(\Omega_0)}) \quad (3.5.27)$$

Once we have proved this lemma, we can deduce Lemma 3.5.2 by working in the corresponding collar coordinates and using the uniform control on the conformal factor ρ and the size of the corresponding cylinder $[0, X(\ell_i)] \times \mathcal{S}^1$ obtained in Remark 3.3.2. In particular, we can find a constant $C_0 = C_0(\iota_0) > 0$ such that $B^g_{2r}(x_0) \subseteq B^{g_0}_{C_0 r}(0)$, and hence choose $c_1 = \min(\frac{\pi}{C_0}, \frac{1}{2}\iota_0)$. We then set $\Omega_1 = B^g_r$ and $\Omega_0 = B^g_{2r}$ and use that $\mathrm{dist}_{g_0}(\cdot,\cdot) \geqslant c_0 \mathrm{dist}_g(\cdot,\cdot)$ for a constant c_0 given by a lower bound on ρ, and hence depending only on ι_0. Lemma 3.5.8 then provides the constant $\delta_1 = \delta_1(C_0, c_0) = \delta_1(\iota_0)$. We then apply Lemma 3.5.8 with $w = u_g$, noting that by conformal invariance of the energy, $\|\nabla_g u_g\|_{L^2(\Omega_0, g)} = \|\nabla_{g_0} u_g\|_{L^2(\Omega_0, g_0)}$, and so assumption (3.5.1) implies the small energy assumption of Lemma 3.5.8, and hence we conclude the proof of Lemma 3.5.2.

Proof of Lemma 3.5.8. We first without loss of generality take $r = \frac{\pi}{2C_0}$, since the lemma is scale invariant under dilations of $\mathcal{H}$. Next, we recall that we can find a cut-off function $\varphi : B^{g_0}_{C_0 r} = B^{g_0}_{\pi/2} \to \mathbb{R}$ such that $\varphi \equiv 1$ on Ω_1, $\mathrm{supp}(\varphi) \subseteq \Omega_0$, and with $\|\nabla^k_{g_0}\varphi\|_{L^\infty} \leqslant C = C(k, c_0, C_0)$. For example, this can be obtained by taking the convolution of the indicator function on the subset $\{x : \mathrm{dist}_{g_0}(x, \Omega_1) \leqslant \frac{\pi c_0}{4C_0}\}$ with a mollifying kernel with radius $\frac{\pi c_0}{8C_0}$, noting that the rescalled subsets satisfy $\mathrm{dist}_{g_0}(\partial\Omega_1 \setminus \partial\mathcal{H}, \partial\Omega_0 \setminus \partial\mathcal{H}) \geqslant \frac{\pi c_0}{2C_0}$.

We then note that $B^{g_0}_{\pi/2} \subseteq [0,2] \times [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and hence we can view this as a subset of the fixed Euclidean cylinder $\Sigma_0 := [0,2] \times \mathcal{S}^1$ with coordinates $(s,\theta) = (x,y)$ and metric $g_0 = \mathrm{d}s^2 + \mathrm{d}\theta^2$, and in the rest of this proof we will always use this metric for computations.

We now begin the proof of (3.5.27) by recalling the estimate

$$\|\nabla v\|_{L^2(\partial\Sigma_0)} \leqslant C\|\partial_s v\|_{L^2(\partial\Sigma_0)} \tag{3.5.28}$$

for all harmonic functions $v : \Sigma_0 \to \mathbb{R}^n$. We then proceed to localise this estimate as follows, where we write $\sigma_0 = \{0\} \times \mathcal{S}^1$ for the left boundary curve of Σ_0 and allow the constant C to depend only on c_0 and C_0. We will write in this proof v_{Σ_0} for the harmonic extension of a map v defined on $\partial\Sigma_0$ to all of Σ_0 with respect to the Euclidean metric.

$$\begin{aligned}
\|\varphi\nabla w\|_{L^2(\partial\Sigma_0)} &\leqslant \|\nabla(\varphi w)\|_{L^2(\partial\Sigma_0)} + \|\nabla(\varphi)w\|_{L^2(\partial\Sigma_0)} \\
&\leqslant \|\nabla(\varphi w)_{\Sigma_0}\|_{L^2(\partial\Sigma_0)} + \|\nabla(\varphi w - (\varphi w)_{\Sigma_0})\|_{L^2(\partial\Sigma_0)} + \|\nabla(\varphi)w\|_{L^2(\partial\Sigma_0)} \\
&\leqslant C\|\partial_s(\varphi w)_{\Sigma_0}\|_{L^2(\partial\Sigma_0)} + \|\nabla(\varphi w - (\varphi w)_{\Sigma_0})\|_{L^2(\partial\Sigma_0)} + \|\nabla(\varphi)w\|_{L^2(\partial\Sigma_0)} \\
&\leqslant C\|\varphi\partial_s w\|_{L^2(\partial\Sigma_0)} + C\|\nabla(\varphi w - (\varphi w)_{\Sigma_0})\|_{L^2(\partial\Sigma_0)} + C\|\nabla(\varphi)w\|_{L^2(\partial\Sigma_0)}
\end{aligned}$$

Next, using the bounds $\|\nabla(\varphi w - (\varphi w)_{\Sigma_0})\|_{L^2(\partial\Sigma_0)} \leqslant C\|w\|_{H^1(\Omega_0)}$ and $\|\nabla(\varphi)w\|_{L^2(\partial\Sigma_0)} \leqslant C\|w\|_{H^1(\Omega_0)}$, together with the fact $\varphi \equiv 0$ on $\partial\Sigma_0 \setminus \sigma_0$, we obtain

$$\|\varphi\nabla w\|_{L^2(\partial\Sigma_0)} \leqslant C\|\varphi\partial_s w\|_{L^2(\sigma_0)} + C\|w\|_{H^1(\Omega_0)}.$$

Since $w(\sigma_0) \subseteq N$, we can express $\|\varphi\partial_s w\|^2_{L^2(\sigma_0)} = \|\varphi P_w\partial_s w\|^2_{L^2(\sigma_0)} + \|\varphi P^\perp_w\partial_s w\|^2_{L^2(\sigma_0)}$ and hence it remains only to estimate $\|\varphi P^\perp_w \partial_s w\|^2_{L^2(\sigma_0)}$, which we will show is bounded by

$$\|\varphi P^\perp_w\partial_s w\|^2_{L^2(\sigma_0)} \leqslant C\delta_1^{\frac{1}{2}}\|\varphi\nabla w\|^2_{L^2(\partial\Sigma_0)} + C\|w\|^2_{H^1(\Omega_0)}. \tag{3.5.29}$$

We first prove

$$\|\varphi P^\perp_w\partial_s w\|^2_{L^2(\sigma_0)} \leqslant C\delta_1^{\frac{1}{2}}\|\varphi\nabla w\|^2_{L^4(\Sigma_0)} + C\|w\|^2_{H^1(\Omega_0)}. \tag{3.5.30}$$

We let err be an error term bounded by $C\int_{\Sigma_0}\varphi^2\,|\nabla w|^3\,\mathrm{d}v + C\int_{\Sigma_0}\varphi\,|\nabla\varphi|\,|\nabla w|^2\,\mathrm{d}v$, and compute as follows using the divergence theorem, the fact $P^\perp_w$ is a projection matrix on

σ_0, the fact $\varphi \equiv 0$ on $\partial\Sigma_0 \setminus \sigma_0$ and finally that $\partial_{ss}w + \partial_{\theta\theta}w = 0$ to obtain

$$\begin{aligned}
\|\varphi P_w^\perp \partial_s w\|_{L^2(\sigma_0)}^2 &= \int_{\partial\Sigma_0} \partial_s w \cdot \varphi^2 P_w^\perp \partial_s w \mathrm{d}\theta = \int_{\Sigma_0} \nabla w \cdot \nabla(\varphi^2 P_w^\perp \partial_s w)\mathrm{d}v \\
&= \int_{\Sigma_0} \partial_s w \cdot \partial_s(\varphi^2 P_w^\perp \partial_s w) + \partial_\theta w \cdot \partial_\theta(\varphi^2 P_w^\perp \partial_s w)\mathrm{d}v \\
&= \int_{\Sigma_0} \partial_s w \cdot (\varphi^2 P_w^\perp \partial_{ss} w) + \partial_\theta w \cdot (\varphi^2 P_w^\perp \partial_{s\theta} w)\mathrm{d}v + \mathrm{err} \\
&= \int_{\Sigma_0} \partial_s w \cdot (-\varphi^2 P_w^\perp \partial_{\theta\theta} w) + \partial_\theta w \cdot \partial_s(\varphi^2 P_w^\perp \partial_\theta w)\mathrm{d}v + \mathrm{err} \\
&= \int_{\Sigma_0} -\partial_s w \cdot \partial_\theta(\varphi^2 P_w^\perp \partial_\theta w) + \partial_\theta w \cdot \partial_s(\varphi^2 P_w^\perp \partial_\theta w)\mathrm{d}v + \mathrm{err} \\
&= \int_{\sigma_0} \partial_\theta w \cdot \varphi^2 P_w^\perp \partial_\theta w \mathrm{d}\theta - \int_{\partial\Sigma_0\setminus\sigma_0} \partial_\theta w \cdot \varphi^2 P_w^\perp \partial_\theta w \mathrm{d}\theta + \mathrm{err}
\end{aligned}$$

where the last line follows from integrating by parts the first term with respect to θ and the second term with respect to s, noting that the surface integrals exactly cancel. Then we observe that the boundary integrals vanish as $\varphi^2 P_w^\perp \partial_\theta w$ is zero on $\partial\Sigma_0$, due to the fact $\partial_\theta w$ is tangent to N on σ_0 and φ vanishes on the other boundary curve. An application of Hölder's inequality gives the bound

$$\mathrm{err} \leqslant C\|\nabla w\|_{L^2(\Omega_0)} \|\varphi \nabla w\|_{L^4(\Sigma_0)}^2 + C\|w\|_{H^1(\Omega_0)}^2$$

from which we deduce the estimate (3.5.30). Finally, we bound $\|\varphi\nabla w\|_{L^4(\Sigma_0)}^2$ using the embedding $H^{\frac{1}{2}}(\Sigma_0) \hookrightarrow L^4(\Sigma_0)$ and the elliptic estimate $\|f_{\Sigma_0}\|_{H^{\frac{1}{2}}(\Sigma_0)} \leqslant C\|f\|_{L^2(\partial\Sigma_0)}$, which combined with a commutator estimate gives

$$\begin{aligned}
\|\varphi\nabla u\|_{L^4(\Sigma_0)} &\leqslant \|\nabla(\varphi u)\|_{L^4(\Sigma_0)} + \|\nabla(\varphi)u\|_{L^4(\Sigma_0)} \\
&\leqslant C\|\nabla(\varphi u)\|_{H^{\frac{1}{2}}(\Sigma_0)} + C\|u\|_{H^1(\Omega_0)} \\
&\leqslant C\|(\nabla(\varphi u))_{\Sigma_0}\|_{H^{\frac{1}{2}}(\Sigma_0)} + C\|\nabla(\varphi u) - (\nabla(\varphi u))_{\Sigma_0}\|_{H^{\frac{1}{2}}(\Sigma_0)} + C\|u\|_{H^1(\Omega_0)} \\
&\leqslant C\|\nabla(\varphi u)\|_{L^2(\partial\Sigma_0)} + C\|u\|_{H^1(\Omega_0)} \\
&\leqslant C\|\varphi\nabla u\|_{L^2(\partial\Sigma_0)} + C\|u\|_{H^1(\Omega_0)}
\end{aligned}$$

where we use the simple commutator estimate

$$\|\nabla(\varphi u) - (\nabla(\varphi u))_{\Sigma_0}\|_{H^{\frac{1}{2}}(\Sigma_0)} \leqslant \|\nabla(\varphi u) - (\nabla(\varphi u))_{\Sigma_0}\|_{H^1(\Sigma_0)}$$

$$\leqslant C\|\Delta\nabla(\varphi u)\|_{H^{-1}(\Sigma_0)}$$
$$\leqslant C\|\Delta(\varphi u)\|_{L^2(\Sigma_0)} \leqslant C\|u\|_{H^1(\Omega_0)}$$

and so deduce (3.5.29). Upon taking δ_1 small enough, we can absorb the term $\|\varphi\nabla u\|_{L^2(\partial\Sigma_0)}$ and hence conclude the proof. □

Remark 3.5.9. We note here that this same argument can be carried out on the semi-infinite cylinder $[0,\infty)\times\mathcal{S}^1$, which is conformal to the punctured disc, and without the cut-off function φ, i.e. taking $\Omega_1 = [0,\infty)\times\mathcal{S}^1$, to obtain an estimate for harmonic functions with small total energy of the form

$$\|\nabla w\|_{L^2(\mathcal{S}^1)} \leqslant C\|P_w\partial_r w\|_{L^2(\mathcal{S}^1)}.$$

This in particular can be used to find a positive lower bound for any half-harmonic disc. This reproves Corollary 3.2 by the second author in [Str24] in a way which avoids using the assumption made there on the normal bundle being parallelisable.

3.5.4 Proof of Proposition 3.5.5

Proof of Proposition 3.5.5. We note that it suffices to prove this proposition for $T \leqslant 1$ as the desired estimates at times $t_0 > 1$ can be obtained by considering the solution $(u,g)(t+t_0-1)$ on $[0,1]$ and using that the initial energy of this flow is also bounded by $\bar{E}$ since the energy is non-increasing along the flow.

As Lemma 3.5.3 ensures that the map component of the solution has the regularity required in Proposition 3.5.4 for any $s \leqslant m$, we can obtain the first estimate directly by applying this proposition iteratively on $[0,T]$, starting with $s=\frac{3}{2}$ for which (3.5.4) is valid for $\Lambda = C(1+\bar{E})$ as the energy is non-increasing along the flow.

To prove the second estimate (3.5.8) we first apply Lemma 3.5.2 to conclude that on $[0,T]$

$$\|u_g\|^2_{H^{3/2}(\Sigma)} + \|\nabla u_g\|^2_{L^2(\partial\Sigma)} \leqslant C(r_T,\iota_T)\left(\|P_u\partial_\nu u_g\|^2_{L^2(\partial\Sigma)} + \|u_g\|^2_{H^1(\Sigma)}\right),$$

where we note that this lemma is applicable as $\delta_0 \leqslant \delta_1$. As the energy decays according to (3.1.10) along the flow, we hence deduce that

$$\int_0^T \int_{\partial\Sigma} |\nabla u_g|^2 \,\mathrm{d}s\mathrm{d}t \leqslant C(1+T) \leqslant C \text{ for } C = C(\iota_0, r_0, \bar{E}). \tag{3.5.31}$$

We can now use this bound to select a time $t_1 \in [0, \tau/2]$ with $I_{\frac{3}{2}}(t_1) \leqslant C_1 = \frac{2}{\tau}C(\iota_0, r_0, \bar{E})$. Proposition 3.5.4, applied for $s = \frac{3}{2}$ on $[t_1, T]$, then ensures that

$$\sup_{t\in[t_1,T]} I_{\frac{3}{2}} + \int_{t_1}^T I_2 \mathrm{d}t \leqslant C I_{\frac{3}{2}}(t_1) \leqslant C_2(\iota_0, r_0, \tau, \bar{E}),$$

This not only provides a uniform estimate for $I_{\frac{3}{2}}$ on $[\tau, T] \subseteq [t_1, T]$, but also allows us to select the next $t_2 \in (\frac{1}{2}\tau, \frac{3}{4}\tau)$ with which we can continue to iteratively apply Proposition 3.5.4 to establish the claimed bound (3.5.8). □

3.5.5 Local Energy Estimates and Proof of Proposition 3.5.1

Both in the proof of the existence of solutions for general initial data as well as in the analysis of finite time singularities, we will make use of the following local energy estimates.

Lemma 3.5.10. *Along any solution (u, g) of the flow* (3.1.9) *and for any cut-off function $\varphi \in C_c^1(\Sigma; [0,1])$, the evolution of the local energy*

$$E_\varphi := \frac{1}{2}\int_\Sigma \varphi^2 |\nabla u|^2 \,\mathrm{d}v \tag{3.5.32}$$

is bounded by

$$\left| \tfrac{\mathrm{d}}{\mathrm{d}t} E_\varphi + \int_{\partial\Sigma} \varphi^2 |\partial_t u|^2 \,\mathrm{d}s \right| \leqslant C E_\varphi + C(1 + \|\partial_t u\|_{L^2(\partial\Sigma)}) \|\nabla\varphi\|_{L^4(\mathrm{supp}(\varphi))} E_\varphi^{\frac{1}{2}} \tag{3.5.33}$$

for a constant $C > 0$ depending only on an upper bound $\bar{E}$ on the initial energy $E_{\frac{1}{2}}(u(0), g(0))$ and a lower bound $\iota_0 > 0$ on the injectivity radius $\mathrm{inj}(g(t))$.

Proof of Lemma 3.5.10. From the equation of the flow and upon integrating by parts, we get that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E_\varphi + \int_{\partial\Sigma} \varphi^2 |\partial_t u|^2 \,\mathrm{d}s = \frac{1}{2}\int_\Sigma \langle \partial_t g, k(u_g, g)\rangle \varphi^2 \mathrm{d}v - 2\int_\Sigma \varphi \partial_t u_g \cdot \langle \nabla u_g, \nabla\varphi\rangle \mathrm{d}v.$$

As $k(u_g, g) \leqslant C\,|\mathrm{d}u_g|_g^2$ pointwise, the first term is bounded by $\|\partial_t g\|_{L^\infty(\Sigma)} \int_\Sigma \varphi^2\,|k(u_g,g)|\,\mathrm{d}v \leqslant CE_\varphi$, we can hence bound

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}E_\varphi + \int_{\partial\Sigma} \varphi^2\,|\partial_t u|^2\,\mathrm{d}s\right| \leqslant CE_\varphi + CE_\varphi^{\frac{1}{2}}\|\nabla\varphi\|_{L^4(\Sigma)}\|\partial_t u_g\|_{L^4(\Sigma)}$$

and it remains to check that $\|\partial_t u\|_{L^4(\Sigma)} \leqslant C(\|\partial_t u\|_{L^2(\partial\Sigma)} + 1)$ for some $C = C(\iota_0, \bar{E})$.

To show this, we split

$$\partial_t u_g = (\partial_t u)_g + \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}(u)_{g(\cdot+\varepsilon)} \tag{3.5.34}$$

and use the estimates of Lemma 3.3.8, the Sobolev embedding $H^{\frac{1}{2}}(\Sigma) \hookrightarrow L^4(\Sigma)$ and the elliptic regularity estimate $\|f_g\|_{H^{\frac{1}{2}}(\Sigma)} \leqslant C\|f\|_{L^2(\partial\Sigma)}$, to estimate

$$\begin{aligned}
\|\partial_t u_g\|_{L^4(\Sigma)} &\leqslant \|(\partial_t u)_g\|_{L^4(\Sigma)} + \|\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}(u)_{g(\cdot+\varepsilon)}\|_{L^4(\Sigma)} \\
&\leqslant C\|(\partial_t u)_g\|_{H^{\frac{1}{2}}(\Sigma)} + C\|\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}(u)_{g(\cdot+\varepsilon)}\|_{H^{\frac{1}{2}}(\Sigma)} \\
&\leqslant C\left(\|\partial_t u\|_{L^2(\partial\Sigma)} + \|u_g\|_{H^1(\Sigma)}\right) \\
&\leqslant C\left(\|\partial_t u\|_{L^2(\partial\Sigma)} + 1\right)
\end{aligned}$$

as required. □

We can now combine these local energy estimates with the control on the evolution of the metric obtained in Section 3.3 and with Propositions 3.2.7 and 3.5.5, to deduce a lower bound on the time it takes for energy to concentrate.

Lemma 3.5.11. *For any $\bar{E}, \iota_0, r_0 > 0$ there exists a time $T_{\min} = T_{\min}(\bar{E}, \iota_0, r_0) > 0$ so that for any initial data $(u_0, g_0) \in C^\infty(\partial\Sigma, N) \times \mathcal{M}(\partial\Sigma)$ satisfying*

$$\mathrm{inj}(g_0) \geqslant 2\iota_0 \text{ and } \sup_{x\in\partial\Sigma} E((u_0)_{g_0}, g_0; B_{4r_0}^{g_0}(x)) \leqslant \frac{1}{2}\delta_0, \tag{3.5.35}$$

there exists a smooth solution (u, g) of the flow (3.1.9) on $[0, T_{\min}]$ which furthermore satisfies at each time $t \in [0, T_{\min}]$

$$\mathrm{inj}(g(t)) \geqslant \iota_0 \text{ and } \sup_{x\in\partial\Sigma} E((u(t))_{g(t)}, g(t); B_{r_0}^{g(t)}(x)) \leqslant \delta_0 \tag{3.5.36}$$

for $\delta_0 = \delta_0(\iota_0, \bar{E})$ is as in Proposition 3.5.4.

Proof of Lemma 3.5.11. Given any such initial data, we can use that Propositions 3.2.7 and 3.5.5 yield the existence of a smooth solution (u, g) to the flow (3.1.9) on a maximal time interval $[0, T_{\max}(u_0, g_0))$ which can only be finite if $\mathrm{inj}(g(t)) \to 0$ or if energy concentrates as $t \nearrow T_{\max}$.

As Remark 3.2.3 ensures that $g(t)$ remains in the neighbourhood $\mathcal{U}(g_0)$ of the metric g_0 considered in Lemma 3.3.3 at least until $\min(T_0, T_{\max})$, $T_0 = T_0(\bar{E}, \iota_0) > 0$ as in that remark, we can bound

$$\mathrm{inj}(g(t)) \geqslant \frac{1}{2}\mathrm{inj}(g_0) \geqslant \iota_0 \text{ and } \frac{1}{4}g(t) \leqslant g_0 \leqslant 4g(t) \text{ for all } t \in [0, \min(T_0, T_{\max})).$$

It remains to bound the energy concentration. For any given fixed $x_0 \in \partial\Sigma$, we then take $\varphi : \Sigma \to \mathbb{R}$ to be a smooth cut-off function satisfying $\varphi(x) \equiv 1$ on $B_{2r}^{g_0}(x_0)$, $\mathrm{supp}(\varphi) \subseteq B_{4r_0}^{g_0}(x_0)$ and $|\nabla_{g_0}\varphi| \leqslant \frac{C}{r}$ for a universal constant C. Noting the bounds on the metric $g(t)$ above coming from $g(t) \in \mathcal{U}(g_0)$, we have that for all $t \leqslant \min(T_0, T_{\max})$, $\|\nabla_{g(t)}\varphi\|_{L^4(\mathrm{supp}(\varphi),g)} \leqslant Cr^{-\frac{1}{2}}$ and

$$B_{r_0}^{g(t)}(x_0) \subseteq B_{2r_0}^{g_0}(x_0)$$

which serves to control the size of the domain $\{\varphi(x) = 1\}$ in the evolving metric $g(t)$.

With this choice of φ, we introduce E_φ as the local energy of the solution (u, g). Noting that $E_\varphi \leqslant \bar{E}$, we can apply the estimate (3.5.33) to give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\varphi &\leqslant CE_\varphi + C(1 + \|\partial_t u\|_{L^2(\partial\Sigma)})\|\nabla_{g(t)}\varphi\|_{L^4(\mathrm{supp}(\varphi),g)}E_\varphi^{\frac{1}{2}} \\ &\leqslant C\bar{E} + C(1 + \|\partial_t u\|_{L^2(\partial\Sigma)})r^{-\frac{1}{2}}\bar{E}^{\frac{1}{2}} \\ &\leqslant C\left(1 + \|\partial_t u\|_{L^2(\partial\Sigma)}\right)\end{aligned}$$

where the constant $C > 0$ depends only $r, \bar{E}, \iota_0$, and crucially *not* on x_0. We obtain from this

$$\begin{aligned}E_\varphi(t) &\leqslant E_\varphi(0) + C\int_0^t (1 + \|\partial_t u_k(t')\|_{L^2(\partial\Sigma)})\mathrm{d}t' \\ &\leqslant \frac{1}{2}\delta_0 + C\left(t + t^{\frac{1}{2}}\left(\int_0^t \|\partial_t u(t')\|_{L^2(\partial\Sigma)}^2 \mathrm{d}t'\right)^{\frac{1}{2}}\right)\end{aligned}$$

$$\leqslant \frac{1}{2}\delta_0 + C(t + t^{\frac{1}{2}}\bar{E}^{\frac{1}{2}}).$$

Hence, for $t \leqslant \min(T_0, T_{\max}, \hat{T})$ where $C(\hat{T} + \hat{T}^{\frac{1}{2}}\bar{E}^{\frac{1}{2}}) = \frac{1}{2}\delta_0$ we have the estimate

$$\int_{B_r^{g(t)}(x_0)} \left|\nabla(u(t))_{g(t)}\right|^2 \mathrm{d}v \leqslant E_\varphi(t) \leqslant \delta_0$$

and so conclude by taking $T_{\min} := \min(T_0, T_{\max}, \hat{T})$. □

We are now finally in a position to complete the proof of Proposition 3.5.1.

Proof of Proposition 3.5.1. Given $(u_0, g_0) \in H^{\frac{1}{2}}(\partial\Sigma; N) \times \mathcal{M}(\Sigma)$, we set $\bar{E} := 2E_{\frac{1}{2}}(u_0, g_0)$, $\iota_0 := \frac{1}{2}\mathrm{inj}(g_0)$ and fix $r_0 > 0$ so that $\sup_{x\in\partial\Sigma} E((u_0)_{g_0}, g_0; B^{g_0}_{4r_0}(x)) \leqslant \frac{1}{4}\delta_0$ for $\delta_0 = \delta_0(\iota_0, \bar{E})$ as in Proposition 3.5.4.

Letting $u_0^k \in C^\infty(\partial\Sigma; N)$ be a sequence of maps which converges to u_0 in $H^{\frac{1}{2}}(\partial\Sigma)$, we can then use that the conditions of Lemma 3.5.11 are satisfied for all sufficiently large k. After dropping the first few u_0^k if necessary, we hence deduce that the corresponding solutions (u_k, g_k) of (3.1.9) all exist and are so that (3.5.36) is satisfied on the k independent interval $[0, T_{\min}]$. Proposition 3.5.5 hence ensures that

$$\sup_k \|(u_k(t))_{g_k(t)}\|_{L^\infty([\tau,T_{\min}];H^m(\Sigma,g_k(t)))} < \infty \text{ for every } \tau > 0 \text{ and } m \in \mathbb{N}.$$

Combined with the control on the metric established in Section 3.3 and the fact that (u, g) satisfy the evolution equations (3.1.9), this yields k-independent bounds on all space-time derivatives of u_k and g_k on $[\tau, T_{\min}] \times \Sigma$, $\tau > 0$. Hence the (u_k, g_k) subconverge smoothly locally on $(0, T_{\min}] \times \Sigma$ and weakly in $H^1([0, T_{\min}] \times \Sigma)$ to a solution (u, g) of (3.1.9) to initial data (u_0, g_0) which is smooth away from time $t = 0$. This completes the proof of the proposition since the characterisation of the maximal time T_* until which this solution remains smooth immediately follows from Lemma 3.5.11. □

3.5.6 Analysis of Finite Time Singularities

In this section, we generalise the analysis of finite time singularities from the disc case treated in [Str24] to the general setting.

We first note that away from the boundary curves, the maps are uniformly bounded in C^k in terms of just the energy and the distance $\mathrm{dist}(x, \partial\Sigma)$, thanks to standard C^k estimates for harmonic functions and noting that we assume a uniform lower bound $\iota_0 > 0$ on the injectivity radius of the metrics. Therefore, potential singularities can only form near the boundary curves σ_i, and so we will work on collar neighbourhoods $\mathcal{C}(\sigma_i)$ as described at the start of Section 3.3. In particular, recall that $\mathcal{C}(\sigma_i)$ is isometric to $([0, c_1] \times \mathcal{S}^1, \rho(s)^2(\mathrm{d}s^2 + \mathrm{d}\theta^2))$ where c_1 and ρ are bounded uniformly above and below in terms of ι_0.

In Section 8 of [Str24], Struwe carried out a rescaling analysis of the singularities directly for solutions of the flow on the disc. The local nature of this analysis allows it to be easily transferred to the case of a flat cylinder, which by the uniform estimates on the hyperbolic metric on collar neighbourhoods will be applicable to our setting. Furthermore, the same analysis applies to the more general setting of suitable sequences of maps, and hence we extract from [Str24] the following result.

Lemma 3.5.12. *Let $c_1 > 0$ be any fixed number, $\mathcal{C}(c_1) := [0, c_1] \times \mathcal{S}^1$ and $g_0 := \mathrm{d}s^2 + \mathrm{d}\theta^2$ the flat metric on $\mathcal{C}(c_1)$. Suppose that $v_k : \mathcal{C}(c_1) \to \mathbb{R}^n$ is a sequence of harmonic functions which is bounded in $H^1(\mathcal{C}(c_1)) \cap L^\infty(\mathcal{C}(c_1))$ and for which there exist radii $r_k \searrow 0$, points $x_k = (s_k, \theta_k) \in \mathcal{C}(c_1)$ with $s_k \leqslant \frac{1}{2}c_1$ and a constant $\delta > 0$ such that*

$$E(v_k, g_0; B^{g_0}_{r_k}(x_k)) \geqslant \delta.$$

Then $r_k^{-1} s_k$ is bounded.

If furthermore $v_k(\{0\} \times \mathcal{S}^1) \subset N$ and

$$r_k \| P_{v_k} \partial_s v_k \|^2_{L^2(\{0\}\times\mathcal{S}^1, \mathrm{d}\theta)} \to 0 \tag{3.5.37}$$

then there exist $\hat{\theta}_k \in \mathcal{S}^1$ *and scales* $\hat{r}_k \searrow 0$ *so that the rescaled maps*

$$\hat{v}_k(s,\theta) := v_k(\hat{r}_k(s,\theta - \hat{\theta}_k))$$

converge weakly in $H^{3/2}_{loc}(\mathcal{H})$*, where* $\mathcal{H} := \{(s,\theta), s \geqslant 0, \theta \in \mathbb{R}\}$*, to a finite energy limiting harmonic function* v_∞ *which maps* $\partial\mathcal{H}$ *into* N *and with*

$$P_{v_\infty}\partial_s v_\infty \equiv 0 \text{ on } \partial\mathcal{H} = \{0\} \times \mathbb{R},$$

and hence upon composing with a conformal bijection $\psi : \mathbb{D} \to \mathcal{H}$*, we obtain a half-harmonic disc* $v_\infty \circ \psi$*.*

As solutions of our flow for which the injectivity radius is bounded from below remain well controlled for as long as the condition (3.5.6) on the smallness of energy remains valid on fixed size balls, we obtain the desired claim on the behaviour of our flow at finite singular times by combining the above result from [Str24] with the following lemma.

Lemma 3.5.13. *Let* (u,g) *be a smooth solution of the flow* (3.1.9) *on an interval* $[0,T_*)$ *with* $\iota_0 := \inf_{[0,T_*)} \text{inj}(g) > 0$*. Suppose that energy concentrates near at least one of the boundary curves* $\sigma_i \subset \partial\Sigma$ *as* $t \nearrow T_*$ *in the sense that*

$$\sup_{t\in[0,T_*)} \sup_{x\in\sigma_i} E(u_{g(t)}(t), g(t); B_r^{g(t)}(x)) \geqslant \frac{3}{4}\delta_0 \text{ for every } r > 0, \tag{3.5.38}$$

for the constant $\delta_0 = \delta_0(\bar{E}, \iota_0) > 0$*,* $\bar{E} := E_{\frac{1}{2}}(u(0), g(0))$*, obtained in Proposition 3.5.4. Then there are times* $t_k \nearrow T_*$ *and a number* $c_1 = c_1(\iota_0) > 0$ *so that the maps* $u_k : \mathcal{C}(c_1) \subset \mathcal{C}(X(L_{g(t_k)}(\sigma_i))) \to \mathbb{R}^n$ *which represent* $u(t_k)$ *in the collar coordinates of* $(\Sigma, g(t_k))$ *around* σ_i *satisfy the assumptions of Lemma 3.5.12 and hence undergo bubbling as described in this lemma.*

Proof. We first recall from Remark 3.2.3 that the metrics are uniformly equivalent on the interval $I_0 := [\max(0, T_* - T_0), T_*)]$, $T_0 = T_0(\bar{E}, \frac{1}{2}\iota_0) > 0$ as in this remark, namely that $g(t) \leqslant 4g(t')$ for all $t, t' \in I_0$.

Given any $r \in (0, \iota_0)$ and $x \in \partial\Sigma$, we can hence obtain a function $\phi_{x,r}$ which satisfies

$$\phi_{x,r} \equiv 1 \text{ on } B^{g(t)}_{\frac{1}{4}r}(x), \quad \operatorname{supp}(\phi_{x,r}) \subseteq B^{g(t)}_{2r}(x) \text{ and } \|\nabla_{g(t)}\phi_{x,r}\|_{L^\infty(\Sigma,g(t))} \leqslant Cr^{-1}$$

for all $t \in I_0$ by fixing any $t_0 \in I_0$ and taking a standard cut-off function on $(\Sigma, g(t_0))$ which vanishes outside of the corresponding ball with radius r and is equal to 1 on $B^{g(t_0)}_{\frac{1}{2}r}(x)$.

For any such x and r, we can hence apply the estimates on the evolution of the localised energy $E_{\phi_{x,r}}$ obtained in Lemma 3.5.10 to deduce that

$$E(u(t'); B^{g(t')}_{4r}(x)) \geqslant E(u(t); B^{g(t)}_{\frac{1}{4}r}(x)) - C\left(t - t' + \|\partial_t u\|^2_{L^2([t',t]\times\partial\Sigma)}\right) - Cr^{-\frac{1}{2}}(t - t') \tag{3.5.39}$$

for any $t' < t$ with $t, t' \in I_0$ and for a constant $C = C(\iota_0, \bar{E})$.

As the assumption (3.5.38), allows us to choose times $\tilde{t}_k \nearrow T_*$, points $\tilde{x}_k \in \sigma_i$ and radii $\tilde{r}_k \to 0$ with $E(u(\tilde{t}_k)_{g(\tilde{t}_k)}, g(\tilde{t}_k); B^{g(\tilde{t}_k)}_{\frac{1}{4}\tilde{r}_k}(x_k)) \geqslant \frac{1}{2}\delta_1$ we hence deduce that

$$\inf_{t\in I_k} E(u(t)_{g(t)}, g(t); B^{g(t)}_{4\tilde{r}_k}(x_k)) \geqslant \frac{1}{2}\delta_1 - C\left(t - t' + \|\partial_t u\|^2_{L^2(I_k\times\partial\Sigma)}\right) - Cc_0 \geqslant \frac{1}{4}\delta_1,$$

on $I_k := [\tilde{t}_k - c_0 r_k, \tilde{t}_k]$, $c_0 := \frac{1}{8}C^{-1}\delta_1$, where the last estimate holds for all sufficiently large k since $\|\partial_t(u, g)\|^2_{L^2(I_k\times\partial\Sigma)} \to 0$ as $k \to \infty$.

Choosing $t_k \in I_k$ so that $\|P_{u(t_k)}\partial_\nu(u(t_k))_{g(t_k)}\|^2_{L^2(\partial\Sigma)} \leqslant 2\,|I_k|^{-1}\int_{I_k}\|P_u\partial_\nu u_g\|^2_{L^2(\partial\Sigma)}dt$ and using that $\int_{I_k}\|P_u\partial_\nu u_g\|^2_{L^2(\partial\Sigma)}dt = \|\partial_t u\|^2_{L^2(I_k;L^2(\partial\Sigma))} \to 0$, we hence deduce that

$$E(u(t_k)_{g(t_k)}, g(t_k); B^{g(t_k)}_{4\tilde{r}_k}(x_k)) \geqslant \frac{1}{4}\delta_1 > 0 \text{ and } r_k\|P_{u(t_k)}\partial_\nu(u(t_k))_{g(t_k)}\|^2_{L^2(\partial\Sigma)} \to 0.$$

As the conformal factors ρ_k of the metrics which represent $g(t_k)$ in the collar coordinates around σ_i are bounded below by some constant $c = c(\iota_0) > 0$, and as $X(L_{g(t_k)}(\sigma_i)) \geqslant c_1 = c_1(\iota_0)$, compare Remark 3.3.2, we hence deduce that the maps $u_k : \mathcal{C}(c_1) \to \mathbb{R}^n$ which represent $u(t_k)_{g(t_k)}$ in these coordinates are indeed so that

$$E(u_k, g_0; B^{g_0}_{r_k}(x_k)) \geqslant \frac{1}{4}\delta_1 \text{ and } r_k\|P_{u_k}\partial_s u_k\|^2_{L^2(\{0\}\times\mathcal{S}^1,d\theta)} \to 0 \text{ for } r_k = 4c^{-1}\tilde{r}_k \to 0.$$

□

3.6 Asymptotics

In this final section, we complete the proof of Theorem 3.1.5, which gives a result on the asymptotic convergence of the flow under the assumption that neither the metric, nor the map, develop singularities as $t \to \infty$.

For this, we prove the following sequential compactness result, which can be readily applied to the solution of the flow studied above.

Proposition 3.6.1. *Let $(u_n, g_n) \in C^\infty(\partial\Sigma; N) \times \mathcal{M}(\Sigma)$ be a sequence with half-energy uniformly bounded from above, $E_{\frac{1}{2}}(u_n, g_n) \leqslant \bar{E} < \infty$, and injectivity radius uniformly bounded from below, $\operatorname{inj}(g_n) \geqslant \iota_0 > 0$. Suppose further that energy does not concentrate, so there is an $r > 0$ such that*

$$\sup_n \sup_{x_0 \in \Sigma} \int_{B_r^{g_n}(x_0)} |\nabla (u_n)_{g_n}|^2 \,\mathrm{d}v < \delta_1 = \delta_1(\iota_0), \tag{3.6.1}$$

where δ_1 is the constant from Lemma 3.5.2. Suppose further that (u_n, g_n) are an almost critical sequence for the half-energy functional, in the sense that

$$\|P_{u_n} \partial_\nu (u_n)_{g_n}\|_{L^2(\partial\Sigma, g_n)} \to 0 \tag{3.6.2}$$

$$\|P^H_{g_n}(k((u_n)_{g_n}, g_n)\|_{L^2(\Sigma, g_n)} \to 0. \tag{3.6.3}$$

Then there exist diffeomorphisms $f_n : \Sigma \to \Sigma$ such that along a subsequence,

$$f_n^* g_n \to g_\infty \text{ smoothly} \tag{3.6.4}$$

$$u_n \circ f_n \to u_\infty \text{ strongly in } H^{\frac{1}{2}}(\partial\Sigma, g_\infty), \text{ weakly in } H^1(\partial\Sigma, g_\infty) \tag{3.6.5}$$

where (u_∞, g_∞) is a critical point of the half-energy $E_{\frac{1}{2}}$, i.e. $g_\infty \in \mathcal{M}(\Sigma)$, u_∞ is a g_∞ half-harmonic map and $(u_\infty)_{g_\infty}$ is conformal.

Proof. Firstly, since $\operatorname{inj}(g_n) \geqslant \iota_0 > 0$, we can apply Mumford's compactness theorem to immediately obtain the diffeomorphisms f_n and the convergence of the metrics (3.6.4). We then define $\tilde{u}_n := u_n \circ f_n$ and $\tilde{g}_n := f_n^* g_n$, and note that $P_{\tilde{u}_n} \partial_{\nu_{\tilde{g}_n}} (\tilde{u}_n)_{\tilde{g}_n} = f_n^* \left(P_{u_n} \partial_{\nu_{g_n}} (u_n)_{g_n} \right)$

and hence the assumed convergence (3.6.2) implies that

$$\|P_{\tilde{u}_n}\partial_{\nu_{\tilde{g}_n}}(\tilde{u}_n)_{\tilde{g}_n}\|_{L^2(\partial\Sigma,\tilde{g}_n)}\to 0. \tag{3.6.6}$$

Similarly, we see that the local energy bound (3.6.1) holds for the sequence $(\tilde{u}_n,\tilde{g}_n)$, and so combining the estimate (3.5.2) of Lemma 3.5.2 together with the convergence (3.6.4), we have that $\tilde{u}_n$ is a bounded sequence in $H^1(\partial\Sigma,g_\infty)$ and that $\partial_{\nu_{g_\infty}}(\tilde{u}_n)_{g_\infty}$ is a bounded sequence in $L^2(\partial\Sigma,g_\infty)$. Hence, upon passing to a subsequence, we can assume that $\tilde{u}_n\rightharpoonup u_\infty$ weakly in $H^1(\partial\Sigma,g_\infty)$ and $\partial_{\nu_{g_\infty}}(\tilde{u}_n)_{g_\infty}\rightharpoonup\partial_{\nu_\infty}(u_\infty)_{g_\infty}$ weakly in $L^2(\partial\Sigma,g_\infty)$. By the compactness of the embedding $H^1(\partial\Sigma)\hookrightarrow C(\partial\Sigma)$, we further have uniform convergence of $\tilde{u}_n$ and hence also the projections $P_{\tilde{u}_n}$ converge uniformly to P_{u_∞}. We thus conclude that

$$P_{u_\infty}\partial_{\nu_{g_\infty}}(u_\infty)_{g_\infty}=0,$$

i.e. that u_∞ is a half-harmonic map (with respect to g_∞).

It remains to show that $(u_\infty)_{g_\infty}$ is conformal. For this, we first note that pulling back by diffeomorphisms does not affect (3.6.3), and so

$$\|P^H_{\tilde{g}_n}(k((\tilde{u}_n)_{\tilde{g}_n},\tilde{g}_n)\|_{L^2(\Sigma)}\to 0.$$

We can now use the estimates from Section 3.3 on the stress-energy tensor and horizontal projection to show that $P^H_{g_\infty}k((u_\infty)_{g_\infty},g_\infty)=0$.

Specifically, we use (3.3.16) and (3.3.30) to get the convergence

$$\|P^H_{g_\infty}\left(k((u_\infty)_{g_\infty},g_\infty)-k((\tilde{u}_n)_{\tilde{g}_n},\tilde{g}_n)\right)\|_{L^2(\Sigma)}\to 0$$

and the bounds (3.3.17) and (3.3.29) to show

$$\|(P^H_{g_\infty}-P^H_{\tilde{g}_n})k((\tilde{u}_n)_{\tilde{g}_n},\tilde{g}_n)\|_{L^2(\Sigma)}\to 0$$

which all combine to show $P^H_{g_\infty}k((u_\infty)_{g_\infty},g_\infty)=0$.

Finally, we recall Remark 3.1.2, which tells us that since u_∞ is g_∞ half-harmonic, having $P^H_{g_\infty}(k((u_\infty)_{g_\infty},g_\infty)=0$ is sufficient for the full stress-energy tensor to vanish.

Hence $(u_\infty)_{g_\infty}$ is conformal, and the proof is complete. □

Proof of Theorem 3.1.5. If (u, g) is a solution to the flow (3.1.9) with $\mathrm{inj}(g(t)) \geqslant \iota_0$ and no energy concentration, then from the energy decay formula (3.1.10), there is a sequence of times t_j such that for $(u_j, g_j) = (u(t_j), g(t_j))$ the estimates (3.6.2) and (3.6.3) hold. Hence applying Proposition 3.6.1 gives the required limiting pair (u^*, g^*), the smooth convergence $g(t_j) \to g^*$ and weak $H^1(\partial\Sigma, g^*)$ convergence of the map component $u(t_j) \to u^*$. We then note the estimate (3.5.8) provides uniform H^k bounds on $u(t_j)$, which allows us to upgrade to smooth convergence and hence conclude the proof. □

3.7 Future Outlook

The theory we have developed here is in an early stage, and a lot of work remains to be done. Certain fundamental results should be established, particularly the uniqueness of solutions and the local regularity estimates needed to ensure the smoothness of solutions away from the finite spacetime points where bubbling of the map occurs, but these are not likely to be of great interest in their proof. More interesting will be the study of the asymptotic behaviour of this flow, especially when the metric is degenerating, which is what sets this apart from the simpler case when the domain is $\mathcal{S}^1$. In general, the ways in which the metric can degenerate are quite complicated, and we have seen both in the introduction and below in Chapter 4 that it took quite some work to develop this theory in the arguably simpler case of Teichmüller harmonic map flow where metric degeneration happens in essentially one way. It is likely that studying the simpler case of a cylinder will serve as a stepping stone to the more general case, much like Ding, Li and Liu studied the torus years before the general Teichmüller harmonic map flow theory was developed.

As for the case of the flow from the circle, it would be interesting to study this flow in the setting of the Plateau problem, choosing N to be a collection of disjoint simple closed curves. It would be important to see if similar results on the monotonicity of critical points holds in this general setting. Further, it may be interesting to see what singular solutions come out of the flow. For example, taking the case of a cylinder, if the two boundary curves are sufficiently far apart then there is no minimal cylinder surface connecting them, but it is tempting to conjecture that the flow would in this case converge to two minimal discs joined be a straight line, which we can interpret as a valid singular solution to the problem. Of course, much work remains to be done to realise such a program, but it gives a motivating case for developing a more complete theory of the flow.

Chapter 4

Investigation of the Infinite Time Limit of Teichmüller Harmonic Map Flow

In this chapter, I present some investigations into a question about the solutions of the Teichmüller harmonic map flow, in particular the fine structure of the limit at infinite time.

4.1 Background on Teichmüller Harmonic Map Flow

We recall that this is a coupled system of equations for pairs (u, g) where u is a map $u : (\Sigma, g) \to (N, h)$ for a closed surface (Σ, g) and target (N, h) a fixed closed manifold of any dimension, given by

$$\partial_t u = \tau_g(u) \tag{4.1.1}$$

$$\partial_t g = \frac{\eta^2}{4} Re(P_g^H(\Phi(u, g))) \tag{4.1.2}$$

Before presenting my results, I recall again the main existing convergence results from

[RTZ13] and [HRT16] for this flow, where I refer to Appendix C for a more detailed description of the punctured surfaces and notion of convergence. The first result, due to Rupflin, Topping and Zhu, establishes the asymptotic behaviour of solutions away from the collapsing geodesics.

Theorem 4.1.1 (Theorem 1.1, [RTZ13])**.** *Let $(u(t), g(t))$ be a solution to* (4.1.1),(4.1.2) *which is defined for all time $t > 0$, satisfies $\liminf_{t\to\infty} \operatorname{inj} g(t) = 0$ and where the domain surface Σ has genus $\gamma \geqslant 2$. Then there exists a sequence of times $t_i \to \infty$, an integer $1 \leqslant k \leqslant 3(\gamma - 1)$ and a (possibly disconnected) punctured hyperbolic surface $(\tilde{\Sigma}, \tilde{g}, \tilde{c})$ with $2k$ punctures such that*

1. *the surfaces $(\Sigma, g(t_i), c_i)$ converge to $(\tilde{\Sigma}, \tilde{g}, \tilde{c})$ in the sense of Proposition C.0.2 by collapsing k simple closed geodesics σ_i^j, and hence there are diffeomorphisms $f_i : \tilde{\Sigma} \to \Sigma \setminus \bigcup_{j=1}^k \sigma_i^j$ such that*

$$f_i^* g(t_i) \to \tilde{g} \text{ and } f_i^* c(t_i) \to \tilde{c} \text{ smoothly locally}$$

   where $c(t)$ denotes the complex structure of $(\Sigma, g(t))$.

2. *the maps $u(t_i) \circ f_i$ converge weakly locally in $H^1(\tilde{\Sigma})$ and weakly locally in $H^2(\tilde{\Sigma} \setminus S)$ to a map $u_\infty : \tilde{\Sigma} \to N$, where S is a finite set where energy concentrates.*

3. *u_∞ extends to a branched minimal immersion or constant map on each connected component of the compactification of $(\tilde{\Sigma}, \tilde{g})$.*

Next, we recall that by the Keen-Randol collar lemma, C.0.1, surrounding each closed geodesic $\sigma \subseteq \Sigma$ of length ℓ, there is a neighbourhood which is isometric to $C_\ell = [-X(\ell), X(\ell)] \times \mathcal{S}^1$ with metric $g_\ell = \rho_\ell(s)^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)$, where

$$X(\ell) := \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\frac{\ell}{2}\right)\right) \tag{4.1.3}$$

$$\rho_\ell(s) := \frac{\ell}{2\pi\cos\left(\frac{\ell s}{2\pi}\right)} \tag{4.1.4}$$

A key observation is that as $\ell \searrow 0$, then $X(\ell) \nearrow \infty$.

The second result recalled here provides information about the flow on the degenerating parts of the surface, namely the collar neighbourhoods of the geodesics which collapse as $t_j \to \infty$. To state this requires us recall the following definition from [HRT16].

Definition 4.1.2. Let $X_i \to \infty$ and $u_i : [-X_i, X_i] \times \mathcal{S}^1 \to N$ be a sequence of smooth maps. We say that u_i converge to a *full bubble branch* if all of the following hold.

1. There exists an integer m and sequences $s_i^0, ..., s_i^m$ satisfying $-X_i = s_i^0 < s_i^1 < \cdots < s_i^m = X_i$ for each i and such that for each $k \in \{0, 1, ..., m-1\}$, $s_i^{k+1} - s_i^k \to \infty$ as $i \to \infty$.

2. For each $k \in \{0, ..., m-1\}$, the maps $u_i^k(s, \theta) := u_i(s + s_i^m, \theta)$ converge to a non trivial bubble branch, meaning that there is some harmonic $u_\infty^k : \mathbb{R} \times \mathcal{S}^1 \to N$ and a finite set S such that $u_i^k \rightharpoonup u_\infty^k$ weakly in $H^1_{\text{loc}}(\mathbb{R} \times \mathcal{S}^1)$ and $u_i^k \to u_\infty^k$ strongly in $H^2_{\text{loc}}(\mathbb{R} \times \mathcal{S}^1 \setminus S)$.

3. The connecting cylinders $[s_i^k + \lambda, s_i^{k+1} - \lambda] \times \mathcal{S}^1$ are mapped near curves, in the sense

$$\lim_{\lambda \to \infty} \limsup_{i \to \infty} \sup_{s \in [s_i^k + \lambda, s_i^{k+1} - \lambda]} \operatorname{osc}(u_i(s, \theta); \{s\} \times \mathcal{S}^1) = 0.$$

With this, Huxol, Rupflin and Topping showed the following.

Theorem 4.1.3 (Theorem 1.11, [HRT16])**.** *Suppose $(u(t), g(t))$ satisfy the conditions of Theorem 4.1.1. Then there is a sequence of times $t_i \to \infty$, a subsequence of the time sequence from Theorem 4.1.1, such that the restrictions of the maps $u(t_i)$ to each degenerating collar converge to a full bubble branch.*

Collectively, these existing results tell us that away from the collapsing geodesics in Σ, the flow converges to a branched minimal immersion as desired, and that in a collar neighbourhood of the collapsing geodesic the map is essentially given by a bubble tree consisting of minimal spheres connected by curves. However, these results tell us nothing about the nature of the connecting curves. In [HRT16], the authors ask the following.

Question 4.1.4. *For a solution of the Teichmüller harmonic map flow where the metric degenerates at infinite time, are the connecting curves converging to geodesics in (N, h)?*

There is one, and only one, setting where this is known to be true, which is the result of Ding, Li and Liu from [DLL06], where they show that if the domain is a torus, the energy converges to zero and $\liminf_{t\to\infty} \operatorname{inj} g(t) = 0$, then along a sequence of times $t_i \to \infty$ the map converges to either a point or a closed geodesic. The aim of this part of my thesis is to investigate these connecting curves in the higher genus setting.

The key result which guides the approaches I follow below is a result of Rupflin which provides a sharp sufficient condition on the decay of the tension which ensures the convergence to a geodesic.

Theorem 4.1.5 (Theorem 1.3, [Rup22]). *Let $\ell_i \to 0$ and $u_i : \mathcal{C}_{\ell_i} \to N$ be a sequence of maps from hyperbolic cylinders $(\mathcal{C}_{\ell_i}, g_{\ell_i})$ which have uniformly bounded energy, satisfy*

$$\ell_i^{-\frac{1}{2}} \|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}_{\ell_i}, g_{\ell_i})} \to 0$$

and which converge to a full bubble branch as in Definition 4.1.2, with associated sequences s_i^k. Then there exist sequences a_i^k and b_i^k, which split the cylinder into extended bubble regions $[a_i^k, b_i^k] \times \mathcal{S}^1$ and connecting cylinders $[b_i^k, a_i^{k+1}] \times \mathcal{S}^1$, such that $s_i^{k-1} < b_i^{k-1} \leqslant a_i^k < s_i^k$ and both $b_i^k - s_i^k \to \infty$ and $s_i^k - a_i^k \to \infty$ as $i \to \infty$, and so that

1. *No energy is lost on the extended bubble regions.*
2. *No necks form in the extended bubble regions.*
3. *The images of the connecting cylinders subconverge to geodesics in the following sense. The curves $v_i^k : [-c_i^k, c_i^k] \to N$, which are the maps*

$$s \mapsto \Pi\left(\frac{1}{2\pi}\int_{\{s\}\times\mathcal{S}^1} u_i(s,\theta)\mathrm{d}\theta\right)$$

reparametrised by arc length, where Π is the nearest point projection of $\mathbb{R}^n$ onto N, satisfy

$$\|\tau(v_i^k)\|_{L^p([-c_i^k, c_i^k])} \to 0$$

for each $p \in [1,2]$. Hence on passing to a subsequence, v_i^k converges to a geodesic locally in H^2, either of trivial length, finite length or infinite length.

This is sharp in the sense that a solution for which $0 < c \leqslant \ell_i^{-\frac{1}{2}} \|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}_{\ell_i}, g_{\ell_i})} \leqslant C < \infty$, is *not* guaranteed to converge to a geodesic. The counterexample is simple and given in [HRT16, Proposition 1.14]: take a constant speed fixed curve $\gamma : [-1, 1] \to N$ which is not a geodesic, take the map $u_i : \mathcal{C}_{\ell_i} \to N$ as $u_i(s, \theta) := \gamma(\frac{s}{X_i})$ and take $\ell_i \searrow 0$. Then the tension satisfies $\|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}_{\ell_i}, g_{\ell_i})} \leqslant C\ell_i^{\frac{1}{2}}$. Furthermore, note that in the setting where the maps u_i are *harmonic*, then it is known that the limit curve is geodesic, see [CLW12] and [CT99].

This result translates answering Question 4.1.4 into proving that along some sequence of times, the tension is decaying sufficiently fast with respect to the rate of decay of the metric. The two results I present below provide some instances where the necessary tension decay estimate can be achieved for two *close variants* of the original flow. In particular, I will replace the constant coupling factor η with a variable factor which depends on the injectivity radius. Note that whilst this has not specifically been studied before, Huxol in [Hux17] studied the limits of solutions of the flow where the coupling constant is taken to 0. Furthermore, we note that the short time existence, uniqueness and regularity theory of this adjusted flow goes through without major alterations. It is only in the asymptotic behaviour that qualitatively different behaviour is potentially present, and we are making these changes precisely to try and arrange the correct asymptotics of the solutions.

4.2 First Approach

In the nearly twenty years since Ding, Li and Liu's result, there has been no further progress even in the simplified setting of the torus, and so instead of trying to treat the full flow equation, I first considered a model problem. This aims to capture the key features of the original flow, whilst making a number of helpful simplifications and adjustments.

First of all, since it is the behaviour on the collar neighbourhood that we are interested in, I switch to considering the Teichmüller harmonic map flow on cylinders introduced by Rupflin in [Rup17]. This works on a cylinder

$$\mathcal{C}_\ell := [-Y(\ell), Y(\ell)] \times \mathcal{S}^1 \tag{4.2.1}$$

where for $\ell > 0$,

$$Y(\ell) := \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\frac{\ell}{2}\right)\right). \tag{4.2.2}$$

Note that

$$Y(\ell) = \pi^2\ell^{-1} + O(1). \tag{4.2.3}$$

Let (s, θ) be the coordinates on $\mathcal{C}_\ell$, then we equip $\mathcal{C}_\ell$ with the metric

$$g_\ell := \rho_\ell(s)^2\left(\mathrm{d}s^2 + \mathrm{d}\theta^2\right) \tag{4.2.4}$$

where the conformal factor ρ_ℓ is defined by

$$\rho_\ell(s) := \frac{\ell}{2\pi\cos\left(\frac{\ell s}{2\pi}\right)}. \tag{4.2.5}$$

One point to notice is that the length of this cylinder is slightly different to the one guaranteed by the collar lemma, compare $Y(\ell)$ to $X(\ell)$ defined in (4.1.3), but these are very similar for small ℓ. The reason for this choice of length is that it allows us to reduce the metric equation to a single ODE for the length of the central geodesic, ℓ, see [Rup17] for further details of this. The equations are then

$$\partial_t u = \tau_{g_\ell}(u) \tag{4.2.6}$$

$$\partial_t \ell = -\frac{\ell}{4\pi}\eta^2\lambda \tag{4.2.7}$$

where λ is the quantity

$$\lambda := 8\pi^3\ell^{-2}\|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)}\int_{\mathcal{C}_\ell}\left(|u_s|^2 - |u_\theta|^2\right)\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta. \tag{4.2.8}$$

Note that the equation here for $\frac{\mathrm{d}\ell}{\mathrm{d}t}$ agrees to leading order with the expression for the true evolution of the length of a closed geodesic under Teichmüller harmonic map flow on a hyperbolic surface computed in [RT18]. We will use Dirichlet boundary conditions for the map component in our analysis.

We also assume now that the initial map u does not depend on the angular variable θ, i.e. that the map is already starting out as a curve. On first glance, this may seem quite a restrictive assumption, however firstly we know we are converging to a one-dimensional limit and we are mostly interested in the properties of this, not in how the second dimension is collapsing and moreover there are strong estimates which show the exponential decay of the energy in the angular direction under the assumption that energy is not concentrated, see for example Lemma 3.1 in [RT18], and so the contribution to the dynamics in the θ-direction are not expected to affect the dynamics in the s-direction to leading order. Hence, in this first investigation to understand the leading order dynamics, we restrict to this θ-independent setting. Note further that by removing the θ-dependence, we have also prevented any bubble formation.

The final point of departure from the usual Teichmüller harmonic map flow is that we exploit the flexibility in the choice of inner product we use to define the gradient flow to allow the coupling constant η to depend on the parameter ℓ, which more geometrically we can think of as the injectivity radius of the metric. We do this in a way that $\eta \searrow 0$ as $\ell \searrow 0$, effectively slowing down the metric degeneration compared with the standard flow. Intuitively, this adjusts the balance between the decay of the tension and the degeneration of the metric, and so in light of Theorem 4.1.5 it is reasonable to expect that the asymptotic solution of this flow more easily converges to a geodesic.

The result I obtain about this model problem is the following.

Proposition 4.2.1. *Let $0 < \underline{L} < \overline{L}$ and $\beta > \frac{1}{2}$ be given constants. Then there exists a constant $0 < \ell_1 < \operatorname{arsinh}(1)$, depending only on $\underline{L}$, $\overline{L}$ and β, and a universal constant $M_0 > 0$ satisfying the following. Let (u, ℓ) be any θ-independent smooth solution to the flow (4.2.6),(4.2.7) on $[0, \infty)$ with coupling factor $\eta(\ell) = |\log(\ell)|^{-\beta}$ and so that the length of the curve $u(t)$ satisfies $\underline{L} \leqslant L(u(t)) \leqslant \overline{L}$ for all time. Suppose additionally that the initial data satisfies*

1. $\ell(0) \leqslant \ell_1$

2. $E(u(0), \ell(0)) \leqslant M_0 \left|\log(\ell(0))\right|^{2\beta-1} \ell(0)$.

Then there exists a subsequence of times $t_n \to \infty$ along which the map u converges to a geodesic.

Let us make a couple of remarks about this result. First, since this result concerns sufficient conditions on the initial data, it is important to ask if these are ever satisfied in a non-trivial way. To begin, the condition that the length is bounded below by some $\underline{L}$ can be easily satisfied by taking $\underline{L}$ to be the length of the shortest geodesic in N joining the endpoints of u, since we impose Dirichlet boundary conditions. The upper bound $\overline{L}$ is not easy to explicitly impose, but it is still reasonable for us to first restrict our attention to convergence to finite length geodesics in this initial result. The condition that the width of the collar be sufficiently small is quite natural since we are interested in solutions for which this goes to zero anyway. The final condition, that the energy be sufficiently small in terms of the parameter ℓ, is more artificial. It is coming out of particular estimates that we make, in particular in equations (4.2.23) and (4.2.31). Having said that, we will show in a key step of the proof that this condition is in fact preserved under the flow, and that whilst we *cannot* at present ensure that any solution will eventually satisfy the condition (the energy could be decaying too slowly compared with the parameter ℓ), for any given map $u(0)$, by reducing ℓ sufficiently and linearly rescaling $u(0)$ to fit the new larger cylinder, it is possible to satisfy the bound $E(u(0), l(0)) \leqslant M_0 \left|\log(\ell(0))\right|^{2\beta-1} \ell(0)$, by noting that energy scales with ℓ, yet the factor $M_0 \left|\log(\ell(0))\right|^{2\beta-1}$ blows up as $\ell \searrow 0$. Hence, it is easy to find initial data for which the theorem applies, and moreover it applies to *any* geometric curve in N, just needing perhaps re-parametrising over a longer cylinder. Finally, the upper bound arsinh(1) for ℓ_1 could be replaced by any number less than 1 for this result, but the choice of arsinh(1) is geometrically significant as the collar neighbourhoods of closed geodesics of length less than arsinh(1) on a hyperbolic surface are disjoint, see [RTZ13, Appendix A].

4.2.1 Static Estimates

The first ingredient in the proof of Proposition 4.2.1 is a collection of pointwise estimates on the velocity of the curve. These are similar to estimates from [Rup22].

Lemma 4.2.2. *Let* $u : [-Y(\ell), Y(\ell)] \to (N, h)$ *be smooth. Then for all* $s \in [-Y(\ell), Y(\ell)]$,

$$||u_s(s)| - |u_s(0)|| \leqslant \mathcal{T} \rho_\ell(s)^{\frac{1}{2}}, \tag{4.2.9}$$

where we let $\mathcal{T} := \|\tau_{g_\ell}(u)\|_{L^2(\mathcal{C}_\ell)}$. *Consequently we get the following estimates*

$$\left||u_s(s)|^2 - |u_s(0)|^2\right| \leqslant \mathcal{T}^2 \rho_\ell(s) + 2\,|u_s(0)|\,\mathcal{T}\rho_\ell(s)^{\frac{1}{2}} \tag{4.2.10}$$

$$|u_s(s)|^2 \geqslant \frac{1}{2}\,|u_s(0)|^2 - \mathcal{T}^2 \rho_\ell(s) \tag{4.2.11}$$

Proof. We compute first of all

$$\partial_s |u_s|^2 = 2\langle u_{ss}, u_s\rangle = 2\langle \tau_0(u), u_s\rangle$$

where τ_0 is the tension computed with respect to the euclidean metric $g_0 = \mathrm{d}s^2 + \mathrm{d}\theta^2$ on $\mathcal{C}_\ell$. This follows since the tension is the projection of u_{ss} onto the tangent space of (N, h). Hence we have

$$\partial_s |u_s| = \frac{\langle \tau_0(u), u_s\rangle}{|u_s|}$$

almost everywhere. Using Cauchy-Schwarz and integrating this up yields

$$\begin{aligned}
||u_s(s)| - |u_s(0)||^2 &\leqslant \left(\int_0^s |\tau_0(u)| \mathrm{d}s\right)^2 \\
&\leqslant \left(\int_0^s \rho_\ell(s)^2 \mathrm{d}s\right)\left(\int_0^s |\tau_0(u)|^2 \rho_\ell(s)^{-2} \mathrm{d}s\right) \\
&\leqslant \rho_\ell(s) \int_{\mathcal{C}_\ell} |\tau_0(u)|^2 \rho_\ell(s)^{-4} \mathrm{d}v_{g_\ell} \\
&= \rho_\ell(s) \int_{\mathcal{C}_\ell} |\tau_{g_\ell}(u)|^2 \mathrm{d}v_{g_\ell} \\
&= \mathcal{T}^2 \rho_\ell(s)
\end{aligned}$$

where we have used the relation $\tau_{g_\ell} = \rho_\ell^{-2}\tau_0$ and the estimate

$$\int_0^s \rho_\ell(x)^2 \mathrm{d}x \leqslant \rho_\ell(s).$$

This gives (4.2.9). Then (4.2.10) is obtained from (4.2.9) via

$$\begin{aligned}\left||u_s(s)|^2 - |u_s(0)|^2\right| &= ||u_s(s)| - |u_s(0)|| \, ||u_s(0)| - |u_s(0)| + 2\,|u_s(0)|| \\ &\leqslant ||u_s(s)| - |u_s(0)||^2 + 2\,|u_s(0)|\, ||u_s(s)| - |u_s(0)|| \\ &\leqslant \mathcal{T}^2\rho_\ell(s) + 2\,|u_s(0)|\,\mathcal{T}\rho_\ell(s)^{\frac{1}{2}}\end{aligned}$$

Finally (4.2.11) is obtained from (4.2.9) using Young's inequality on the cross term. □

In order to make use of the above estimate, we need to also obtain estimates on $|u_s(0)|$. We do this in Lemma 4.2.4, but first we state the following estimates which we make several appeals to later on.

Lemma 4.2.3. *Let $\alpha \in \mathbb{R}$. Then there is a constant C_α only depending on α such that for any $0 < \ell < \operatorname{arsinh}(1)$*

$$\int_{-Y(\ell)}^{Y(\ell)} \rho_\ell(s)^\alpha \mathrm{d}s \leqslant C_\alpha \ell^{\alpha-1} \qquad \text{for } \alpha < 1 \tag{4.2.12}$$

$$\int_{-Y(\ell)}^{Y(\ell)} \rho_\ell(s) \mathrm{d}s \leqslant C_1 \left|\log(\ell)\right| \tag{4.2.13}$$

$$\int_{-Y(\ell)}^{Y(\ell)} \rho_\ell(s)^\alpha \mathrm{d}s \leqslant C_\alpha \qquad \text{for } \alpha > 1 \tag{4.2.14}$$

The proof of this lemma is just an explicit calculation.

Lemma 4.2.4. *There exist universal constants $C, \underline{c}, \overline{C}$, with $0 < \underline{c} < \overline{C}$, such that for any $0 < \ell < \operatorname{arsinh}(1)$ and any smooth map $u : [-Y(\ell), Y(\ell)] \to (N, h)$*

$$\underline{c}\ell L(u) - C\ell^{\frac{1}{2}}\mathcal{T} \leqslant |u_s(0)| \leqslant \overline{C}\ell L(u) + C\ell^{\frac{1}{2}}\mathcal{T} \tag{4.2.15}$$

where $L(u)$ denotes the length of the curve given by u.

Proof. We can write the length of u as

$$L(u) = \int_{-Y(\ell)}^{Y(\ell)} (|u_s(s)| - |u_s(0)| + |u_s(0)|)\, \mathrm{d}s$$

Rearranging this and using (4.2.9), (4.2.3) and (4.2.12) gives

$$\begin{aligned} \left| |u_s(0)| - \frac{L(u)}{2Y(\ell)} \right| &\leqslant \frac{1}{2Y(\ell)} \int_{-Y(\ell)}^{Y(\ell)} ||u_s(s)| - |u_s(0)||\, \mathrm{d}s \\ &\leqslant C\ell \int_{-Y(\ell)}^{Y(\ell)} \mathcal{T} \rho_\ell(s)^{\frac{1}{2}} \mathrm{d}s \\ &\leqslant C\mathcal{T}\ell^{\frac{1}{2}} \end{aligned}$$

Using (4.2.3) again on the remaining $Y(\ell)$ gives the constants $\underline{c}$ and $\overline{C}$ and the result. □

Next we deduce several bounds on the quantity λ defined in (4.2.8). The first of these estimates λ by replacing the velocity by $|u_s(0)|$ and then using Lemma 4.2.2.

Lemma 4.2.5. *There exists a universal constant $C > 0$ such that for any $0 < \ell <$ $\operatorname{arsinh}(1)$ and any smooth map $u : [-Y(\ell), Y(\ell)] \to (N, h)$*

$$|\lambda(u, \ell) - \lambda_0(u, \ell)| \leqslant C \left(\mathcal{T}^2 \ell^{-1} + \mathcal{T} |u_s(0)| \, \ell^{-\frac{3}{2}} \right) \tag{4.2.16}$$

where we define

$$\lambda_0(u, \ell) := 8\pi^3 \ell^{-2} \|\mathrm{d}z^2\|_{L^2(\mathcal{C}_\ell)}^{-2} \int_{\mathcal{C}_\ell} |u_s(0)|^2 \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \tag{4.2.17}$$

Consequently we get the upper bound

$$\lambda(u, \ell) \leqslant C \left(L(u)^2 + \ell^{-1} \mathcal{T}^2 \right) \tag{4.2.18}$$

where C is a universal constant and $L(u)$ is the length of the curve.

We also have the corresponding lower bound

$$\lambda(u, \ell) \geqslant cL(u)^2 - C\ell^{-1}\mathcal{T}^2 \tag{4.2.19}$$

Proof. Noting the simple estimate

$$\|\mathrm{d}z^2\|_{L^2(\mathcal{C}_\ell)}^2 = 32\pi^5 \ell^{-3} + O(1), \tag{4.2.20}$$

combined with (4.2.12) and (4.2.10), we get

$$\begin{aligned}|\lambda(u,\ell)-\lambda_0(u,\ell)| &\leqslant \left(\frac{1}{4\pi^2}\ell+O(\ell^4)\right)\int_{\mathcal{C}_\ell}\left||u_s(s)|^2-|u_s(0)|^2\right|\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta\\ &\leqslant C\ell\left(\int_{\mathcal{C}_\ell}\mathcal{T}^2\rho_\ell(s)^{-1}\mathrm{d}s\mathrm{d}\theta+\int_{\mathcal{C}_\ell}2\mathcal{T}\left|u_s(0)\right|\rho_\ell(s)^{-\frac{3}{2}}\mathrm{d}s\mathrm{d}\theta\right)\\ &\leqslant C\ell\left(\mathcal{T}^2\ell^{-2}+\mathcal{T}\left|u_s(0)\right|\ell^{-\frac{5}{2}}\right)\\ &\leqslant C\left(\mathcal{T}^2\ell^{-1}+\mathcal{T}\left|u_s(0)\right|\ell^{-\frac{3}{2}}\right)\end{aligned}$$

which gives the first estimate. To get (4.2.18), we first compute

$$\lambda_0=2\pi^3\ell^{-2}\left|u_s(0)\right|^2. \tag{4.2.21}$$

Then we estimate

$$\begin{aligned}\lambda &\leqslant \lambda_0+|\lambda-\lambda_0|\\ &\leqslant 2\pi^3\ell^{-2}|u_s(0)|^2+C\left(\ell^{-1}\mathcal{T}^2+\ell^{-\frac{3}{2}}\mathcal{T}|u_s(0)|\right)\\ &\leqslant CL(u)^2+C\ell^{-1}\mathcal{T}^2+C\ell^{-\frac{1}{2}}\mathcal{T}L(u)\\ &\leqslant C\left(L(u)^2+\ell^{-1}\mathcal{T}^2\right)\end{aligned}$$

where we have used (4.2.16) and (4.2.15).

For the lower bound, we proceed similarly, only using the estimate (4.2.11) instead of (4.2.10) to get

$$\begin{aligned}\lambda &= 8\pi^3\ell^{-2}\|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)}\int_{\mathcal{C}_\ell}|u_s|^2\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta\\ &\geqslant \left(\frac{1}{4\pi^2}l+O(\ell^4)\right)\int_{\mathcal{C}_\ell}\left(\frac{1}{2}\left|u_s(0)\right|^2-\mathcal{T}^2\rho_\ell(s)\right)\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta\\ &\geqslant c\ell^{-2}\left|u_s(0)\right|^2-C\ell^{-1}\mathcal{T}^2\\ &\geqslant cL(u)^2-C\ell^{-1}\mathcal{T}^2\end{aligned}$$

□

The next estimate relates λ to the energy after a suitable rescaling.

Lemma 4.2.6. *There exists a universal constant $C>0$ such that for any $0<\ell<$*

arsinh(1) *and any smooth map* $u : [-Y(\ell), Y(\ell)] \to (N, h)$

$$|\ell\lambda(u,\ell) - E(u,\ell)| \leqslant C\left(|\log(\ell)|\,\mathcal{T}^2 + \ell^{-\frac{1}{2}}\mathcal{T}|u_s(0)| + |u_s(0)|^2\right) \tag{4.2.22}$$

from which we get the estimate

$$|\ell\lambda(u,\ell) - E(u,\ell)| \leqslant C\left(|\log(\ell)|\,\mathcal{T}^2 + L(u)^2\ell\,|\log(\ell)|^{-1}\right) \tag{4.2.23}$$

where $L(u)$ *is the length of the curve.*

Proof. The main idea is to once again replace the true velocity by the proxy $|u_s(0)|$. We do this using quantities λ_0 defined in (4.2.17) and E_0 defined as

$$E_0 := \frac{1}{2}\int_{\mathcal{C}_\ell} |u_s(0)|^2 \mathrm{d}s\mathrm{d}\theta \tag{4.2.24}$$

to approximate E and λ respectively. Hence we shall use the decomposition

$$|\ell\lambda - E| \leqslant |\ell\lambda - \ell\lambda_0| + |\ell\lambda_0 - E_0| + |E_0 - E| \tag{4.2.25}$$

This gives us three sub-estimates to work on. First, we recall (4.2.21) and similarly compute

$$E_0 = 2\pi Y(\ell)\,|u_s(0)|^2 \tag{4.2.26}$$

Combining this with (4.2.3) gives

$$\begin{aligned}
|\ell\lambda_0 - E_0| &= \left|2\pi^3\ell^{-1}|u_s(0)|^2 - 2\pi Y(\ell)|u_s(0)|^2\right| \\
&= 2\pi\,|u_s(0)|^2\left|\pi^2\ell^{-1} - Y(\ell)\right| \\
&\leqslant C\,|u_s(0)|^2
\end{aligned} \tag{4.2.27}$$

For the final estimate, we use (4.2.10), (4.2.12) and (4.2.13) to get

$$\begin{aligned}
|E_0 - E| &\leqslant \frac{1}{2}\int_{\mathcal{C}_\ell} \left||u_s(0)|^2 - |u_s(s)|^2\right| \mathrm{d}s\mathrm{d}\theta \\
&\leqslant \frac{1}{2}\int_{\mathcal{C}_\ell} \left(\mathcal{T}^2\rho_\ell(s) + 2\mathcal{T}\rho_\ell(s)^{\frac{1}{2}}|u_s(0)|\right)\mathrm{d}s\mathrm{d}\theta \\
&\leqslant C\left(|\log(\ell)|\,\mathcal{T}^2 + \ell^{-\frac{1}{2}}\mathcal{T}|u_s(0)|\right)
\end{aligned} \tag{4.2.28}$$

Plugging (4.2.27), (4.2.28) and (4.2.16) from Lemma 4.2.5 into (4.2.25) then yields

$$\begin{aligned}|\ell\lambda - E| &\leqslant C\left(\mathcal{T}^2 + \mathcal{T}\,|u_s(0)|\,\ell^{-\frac{1}{2}} + |u_s(0)|^2 + |\log(\ell)|\,\mathcal{T}^2 + \mathcal{T}\,|u_s(0)|\,\ell^{-\frac{1}{2}}\right)\\ &\leqslant C\left(\mathcal{T}\,|u_s(0)|\,\ell^{-\frac{1}{2}} + |u_s(0)|^2 + |\log(\ell)|\,\mathcal{T}^2\right)\end{aligned}$$

This gives us (4.2.22). To get (4.2.23), we apply (4.2.15) and use Young's inequality to get

$$2L(u)\ell^{\frac{1}{2}}\mathcal{T} \leqslant |\log(\ell)|\,\mathcal{T}^2 + L(u)^2\ell\,|\log(\ell)|^{-1}$$

□

The final λ estimate gives an alternative upper bound using the energy.

Lemma 4.2.7. *Let $\ell > 0$ and $u : [-Y(\ell), Y(\ell)] \to (N, h)$ be smooth. Then there is a universal constant $C > 0$ such that*

$$\lambda(u, \ell) \leqslant CE\ell^{-1}. \tag{4.2.29}$$

Proof. This follows from (4.2.20) and the inequality $\rho_\ell(s)^{-2} \leqslant \frac{4\pi^2}{\ell^2}$, giving

$$\begin{aligned}\lambda &= 8\pi^3\ell^{-2}\|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)}\int_{\mathcal{C}_\ell}|u_s|^2\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta\\ &\leqslant \left(\frac{1}{4\pi^2}\ell + O(\ell^4)\right)\int_{\mathcal{C}_\ell}|u_s|^2\frac{4\pi^2}{\ell^2}\mathrm{d}s\mathrm{d}\theta\\ &\leqslant \left(\ell^{-1} + O(\ell^2)\right)E\\ &\leqslant CE\ell^{-1}\end{aligned}$$

□

4.2.2 Dynamic Estimates

Now we use the above static estimates and apply them to solutions to the flow (4.2.6),(4.2.7). The first thing to note is that energy decays via the following formula

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\|\tau_{g_\ell}(u)\|^2_{L^2(\mathcal{C}_\ell)} - \frac{\eta^2}{16}\int_{\mathcal{C}_\ell}|Re(P^H_{g_\ell}(\Phi(u, g_\ell)))|^2\mathrm{d}A_{g_\ell} \tag{4.2.30}$$

$$= -\mathcal{T}^2 - \frac{1}{2}\eta^2 \|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)} \left(\int_{\mathcal{C}_\ell} \left(|u_s|^2 - |u_\theta|^2\right) \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \right)^2$$
$$- 2\eta^2 \|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)} \left(\int_{\mathcal{C}_\ell} \langle u_s, u_\theta \rangle \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \right)^2 .$$

The key idea in the rest of this section is to study the evolution of $E\ell^{-1}$. In particular, we have the following.

Lemma 4.2.8. *There exists a constant $C > 0$ such that for any θ-independent solution (u, ℓ) to the flow* (4.2.6),(4.2.7) *satisfying $\ell(0) < \operatorname{arsinh}(1)$, the following holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant -\mathcal{T}^2\ell^{-1} + C\eta^2\lambda \left(\left(|\log(\ell)| + \ell^3(1 + L^2)\right) \ell^{-1}\mathcal{T}^2 + L^2 \left|\log(\ell)\right|^{-1} \right) \qquad (4.2.31)$$

where $L = L(u)$ is the length of the curve.

Proof. First of all, we note that when $|u_\theta| = 0$, the energy decay formula (4.2.30) simplifies to

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= -\|\tau_{g_\ell}(u)\|^2_{L^2(\mathcal{C}_\ell)} - \frac{1}{2}\eta^2\|\mathrm{d}z^2\|^{-2}_{L^2(\mathcal{C}_\ell)} \left(\int_{\mathcal{C}_\ell} |u_s|^2 \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \right)^2 \\
&= -\mathcal{T}^2 - \frac{1}{128\pi^6}\eta^2 \|\mathrm{d}z^2\|^2_{L^2(\mathcal{C}_\ell)} \ell^4 \lambda^2 \\
&= -\mathcal{T}^2 - \frac{1}{4\pi}\eta^2(\ell + O(\ell^4))\lambda^2 \qquad (4.2.32)
\end{aligned}$$

where we use (4.2.20) in the last step. Using (4.2.32) and (4.2.7), we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) &= \frac{\mathrm{d}E}{\mathrm{d}t}\ell^{-1} - E\ell^{-2}\frac{\mathrm{d}\ell}{\mathrm{d}t} \\
&= -\mathcal{T}^2\ell^{-1} - \frac{1}{4\pi}\eta^2 \left(1 + O(\ell^3)\right) \lambda^2 + \frac{1}{4\pi}\eta^2\ell^{-1}\lambda E \\
&= -\mathcal{T}^2\ell^{-1} - \frac{1}{4\pi}\eta^2 \lambda \left(\lambda - \ell^{-1}E + O(\ell^3)\lambda\right)
\end{aligned}$$

Using the assumed upper bound on the length of the curve and (4.2.23), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant -\mathcal{T}^2\ell^{-1} + C\eta^2\lambda \left(|\log(\ell)|\, \ell^{-1}\mathcal{T}^2 + L(u)^2 \left|\log(\ell)\right|^{-1} + \ell^3\lambda\right)$$

Finally, using (4.2.18) on the final term, gives the desired estimate. □

At this point we specialise to a specific functional form for the coupling factor η. The following result gives the key estimate we need to prove the main result.

Lemma 4.2.9. *Let* $0 < \underline{L} < \overline{L}$ *and* $\beta > \frac{1}{2}$ *be given constants. Then there exist constants* $\delta > 0$, $C_0 > 0$ *and* $0 < \ell_0 < \operatorname{arsinh}(1)$ *depending only on* $\underline{L}$, $\overline{L}$ *and* β *and a universal constant* $M_0 > 0$ *such that for any smooth solution* (u, ℓ) *to the flow* (4.2.6),(4.2.7) *on an interval* $[t_0, T)$ *(allowing* $T = \infty$*) with coupling factor* $\eta(\ell) = |\log(\ell)|^{-\beta}$*, if it holds that for all* $t \in [t_0, T)$

1. *the length of the curve satisfies* $\underline{L} \leqslant L(u(t)) \leqslant \overline{L}$
2. $\ell(t) \leqslant \ell_0$
3. $E(u(t), \ell(t)) \leqslant 2M_0 \left|\log(\ell(t)\right|^{2\beta-1} \ell(t)$

then the following estimate holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant \left(C_0 \left|\log(\ell(t))\right|^{-2\beta-1} - \frac{1}{2}\mathcal{T}^2\ell^{-1}\right)\chi_{R_1}(t) - \frac{1}{4}\delta\chi_{R_2}(t) \tag{4.2.33}$$

where

$$R_1 := \{t \in [t_0, T) : \ell^{-1}\mathcal{T}^2 < \delta\}$$
$$R_2 := \{t \in [t_0, T) : \ell^{-1}\mathcal{T}^2 \geqslant \delta\}$$

Proof. The first step is to find δ. For this, we note that the estimates (4.2.18), (4.2.19) imply that there exist constants $\delta, c_0 > 0$, depending only on $\underline{L}, \overline{L}$, such that

$$\ell^{-1}\mathcal{T}^2 \leqslant \delta \Rightarrow c_0 \leqslant \lambda \leqslant c_0^{-1} \tag{4.2.34}$$

Using this δ we define R_1 and R_2 as in the statement. We say the flow is in regime 1 at time t if $t \in R_1$ and similarly regime 2 if $t \in R_2$.

For notational convenience, we introduce the quantity $\varphi(t) := |\log(\ell(t))|$. It is clear that an upper bound on ℓ then corresponds to a lower bound on φ.

Next we estimate $\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1})$ in both regimes. First let us suppose we are in regime 1.

Then starting from Lemma 4.2.8 and (4.2.34), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) &\leqslant -\mathcal{T}^2\ell^{-1} + C\varphi^{-2\beta}\lambda\left(\varphi\ell^{-1}\mathcal{T}^2 + \varphi^{-1}\right)\\ &\leqslant \mathcal{T}^2\ell^{-1}\left(C\varphi^{1-2\beta} - \frac{1}{2}\right) + \left(C\varphi^{-2\beta-1} - \frac{1}{2}\mathcal{T}^2\ell^{-1}\right)\end{aligned}$$

If we assume $\varphi > \varphi_1$, where φ_1 is chosen such that $C\varphi^{1-2\beta} - \frac{1}{2} < 0$ (and so φ_1 depends only on β, $\underline{L}$ and $\overline{L}$), then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant C\varphi^{-2\beta-1} - \frac{1}{2}\mathcal{T}^2\ell^{-1} \tag{4.2.35}$$

for a constant C depending only on β, $\underline{L}$ and $\overline{L}$. We then set C_0 from the statement to be this C and note that φ_1 gives an upper bound on ℓ_0.

Now suppose we are in regime 2, then from Lemma 4.2.8, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) &\leqslant -\mathcal{T}^2\ell^{-1} + C\varphi^{-2\beta}\lambda\left(\varphi\ell^{-1}\mathcal{T}^2 + L^2\varphi^{-1}\right)\\ &\leqslant \left(C\varphi^{-2\beta+1}\lambda - \frac{1}{2}\right)\mathcal{T}^2\ell^{-1} + \left(C_L\varphi^{-2\beta-1}\lambda - \frac{1}{2}\mathcal{T}^2\ell^{-1}\right)\\ &\leqslant \left(C\varphi^{-2\beta+1}\lambda - \frac{1}{2}\right)\mathcal{T}^2\ell^{-1} + \left(C_L\varphi^{-2\beta-1} - \frac{1}{4}\mathcal{T}^2\ell^{-1}\right) + \left(C_L\varphi^{-2\beta-1} - \frac{1}{4}\right)\mathcal{T}^2\ell^{-1}\\ &\leqslant \left(C\varphi^{-2\beta+1}\lambda - \frac{1}{2}\right)\mathcal{T}^2\ell^{-1} + \left(C_L\varphi^{-2\beta-1} - \frac{1}{4}\delta\right) + \left(C_L\varphi^{-2\beta-1} - \frac{1}{4}\right)\mathcal{T}^2\ell^{-1}\end{aligned}$$

where we have used the bound (4.2.18) and $-\mathcal{T}^2\ell^{-1} \leqslant -\delta$ and written C_L for a constant depending additionally on $\overline{L}$ (and later also $\underline{L}$). Then we can find a φ_2 depending only on β, $\underline{L}$, $\overline{L}$ and δ such that provided $\varphi > \varphi_2$, the second and third terms satisfy

$$\begin{aligned}C_L\varphi^{-2\beta-1} - \frac{1}{4}\delta &\leqslant -\frac{1}{8}\delta\\ \left(C_L\varphi^{-2\beta-1} - \frac{1}{4}\right)\mathcal{T}^2\ell^{-1} &\leqslant -\frac{1}{8}\delta\end{aligned}$$

Taking $\max\{\varphi_1, \varphi_2\}$ and translating back to ℓ gives the constant ℓ_0 from the statement. To treat the first term, we use (4.2.29). Then we can see that provided $E\ell^{-1} \leqslant 2M_0\varphi^{2\beta-1}$, where M_0 is a constant depending on β, $\underline{L}$ and $\overline{L}$, the first term is negative, which gives us our constant M_0.

Combining the estimates from both regimes gives the result. □

4.2.3 Proof of Proposition 4.2.1

Using the estimates of the previous two sections and Rupflin's theorem, Theorem 4.1.5, we can provide the following proof.

Proof of Proposition 4.2.1. The key in this proof is to first note that we have given ourselves extra room to work with in the third condition compared with Lemma 4.2.9. We then use estimates on the decay of ℓ to show that we can never reach the equality case $E\ell^{-1} = 2M_0 \left|\log(\ell)\right|^{2\beta-1}$ meaning (4.2.33) holds for all time.

We make use of $\varphi, \delta, c_0, C_0, R_1, R_2$ from Lemma 4.2.9 (and it's proof). Additionally we assume that $\ell_1 \leqslant \ell_0$ where ℓ_0 is from Lemma 4.2.9. We may need to decrease this later on.

First we derive a bound on the behaviour of φ. By (4.2.7), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)^{2\beta+1} &= (2\beta+1)\varphi^{2\beta}\left(-\ell^{-1}\right)\frac{\mathrm{d}\ell}{\mathrm{d}t} \\ &= (2\beta+1)\varphi^{2\beta}\ell^{-1}\frac{\ell}{4\pi}\eta^2\lambda \\ &= \frac{2\beta+1}{4\pi}\lambda \end{aligned}$$

If the flow is in regime 1, then using (4.2.34) we have the bound

$$c_0 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{2\beta+1}$$

where c_0 is a constant depending only on $\underline{L}$ and β. If instead we are in regime 2, then as $\lambda \geqslant 0$, we estimate by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)^{2\beta+1} \geqslant 0$$

Combining these and integrating from t_0 to t, we obtain

$$\varphi(t)^{2\beta+1} \geqslant \varphi(t_0)^{2\beta+1} + c_0 m_1(t) \tag{4.2.36}$$

where $m_1(t) := \int_{t_0}^{t} \chi_{R_1}(t')\mathrm{d}t'$

Now we suppose that $t_1 > t_0$ is the first time that $E\ell^{-1} = 2M_0 \left|\log(\ell(t_0)\right|^{2\beta-1}$, if it exists. If no such time exists then it is clear that the flow will satisfy the conditions of Lemma 4.2.9 for all time since ℓ is decreasing. We now claim that $\varphi(t_1) \geqslant 2^{\frac{1}{2\beta-1}}\varphi(t_0)$. Suppose this is not the case. Then since $\varphi(t)$ is increasing, we have for all $t \in [t_0, t_1]$ that $\varphi(t) < 2^{\frac{1}{2\beta-1}}\varphi(t_0)$. Combining this with (4.2.36) gives

$$m_1(t_1) \leqslant C\left(2^{\frac{2\beta+1}{2\beta-1}} - 1\right)\varphi(t_0)^{2\beta+1} \tag{4.2.37}$$

where C depends only on β and $\underline{L}$. Since the conditions of Lemma 4.2.9 hold on $[t_0, t_1]$, we can use (4.2.33) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant C_0\varphi^{-2\beta-1}\chi_{R_1}(t)$$

Since $\varphi(t_0) \leqslant \varphi(t) \leqslant 2^{\frac{1}{2\beta-1}}\varphi(t_0)$, we can then integrate the above between t_0 and t_1 to get

$$M_0\varphi(t_0)^{2\beta-1} \leqslant C_0 m_1(t_1)\varphi(t_0)^{-2\beta-1}$$

since $E\ell^{-1}$ has increased by at least $M_0\varphi(t_0)^{2\beta-1}$. Rearranging and combining with (4.2.37) gives

$$c_1\varphi(t_0)^{2\beta+1}M_0\varphi(t_0)^{2\beta-1} \leqslant m_1(t_1) \leqslant C\left(2^{\frac{2\beta+1}{2\beta-1}} - 1\right)\varphi(t_0)^{2\beta+1}$$

This leads to

$$M_0\varphi(t_0)^{2\beta-1} \leqslant C$$

for a constant C depending only on $\underline{L}$, $\overline{L}$ and β. By decreasing ℓ_1 if necessary, this gives a contradiction.

Therefore we have $\varphi(t_1) \geqslant 2^{\frac{1}{2\beta-1}}\varphi(t_0)$ and hence

$$E(u(t_1))\ell(t_1)^{-1} \leqslant M_0 \left|\log(\ell(t_1)\right|^{2\beta-1}$$

This implies that the flow satisfies the same conditions at time t_1 as it did at t_0. So by repeating this argument inductively we get that the conditions of Lemma 4.2.9 hold with $T = \infty$. Now we use (4.2.33) to find a subsequence $t_j \to \infty$ along which $\mathcal{T}^2\ell^{-1} \to 0$.

Suppose first that there is some $T > t_0$ such that for all $t > T$, whenever the flow is in regime 1, it holds that

$$C_0\varphi^{-2\beta-1} < \frac{1}{4}\mathcal{T}^2\ell^{-1}$$

where C_0 is the constant from (4.2.33). Then for all $t > T$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant -\frac{1}{4}\mathcal{T}^2\ell^{-1}\chi_{R_1}(t) - \frac{1}{4}\delta\chi_{R_2}(t)$$

Since $E\ell^{-1}$ is decreasing (at least after T) and bounded below (certainly by 0 and in fact by some positive constant proportional to $\underline{L}^2$), we can integrate this formula from T to ∞ and get a finite result. This gives us that the measure of $R_2 \cap [T, \infty)$ is finite and so $R_1 \cap [T, \infty)$ has infinite measure. Since $\int_{R_1\cap[T,\infty)} \mathcal{T}^2\ell^{-1}\mathrm{d}t < \infty$, we can extract the desired subsequence.

Now suppose there is no such time T. We claim that there is a sequence of times $t_j \to \infty$ such that $t_j \in R_1$ and $\mathcal{T}^2\ell^{-1} \leqslant 4C_0\varphi^{-2\beta-1}$ along t_j. To prove this, suppose, for a contradiction, that there is some $T_0 > T$ such that the flow remains in regime 2 for all $t > T_0$. Then from (4.2.33) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\ell^{-1}) \leqslant -\frac{1}{4}\delta$$

But $E\ell^{-1} \geqslant 0$ so integrating from T_0 to ∞ gives the contradiction. This provides the required sequence of times. Since $\varphi(t) \to \infty$ as $t \to \infty$, this is a suitable sequence. $\square$

4.3 Second Approach

My second investigation of the question of the convergence to a geodesic limit of the Teichmüller harmonic map flow took a different approach. Instead of considering directly the tension on the whole cylinder, which stood in for the degenerating collar, I instead worked with a weighted tension quantity defined on the true collar neighbourhood $\mathcal{C}_\ell$ of a degenerating geodesic in a hyperbolic surface (Σ, g). Essentially, this approach splits the collar into a central region and an outer region, and studies the behaviour in the central one. The quantity in question is

$$\mathcal{T}_w(u,\ell)^2 := \int_{C_\ell} |\tau_{g_\ell}(u)|^2 \rho_\ell(s)^{-2}\varphi(\rho_\ell(s))^2 \mathrm{d}v_{g_\ell} \tag{4.3.1}$$

where $\varphi \in C_c^\infty([0,\frac{1}{\pi}),[0,1])$ is a cut-off function satisfying $\varphi \equiv 1$ on $[0,\frac{1}{2\pi}]$ and $|\varphi'| \leqslant 4\pi$.

To understand better this quantity, fix any $\Lambda > 1$ and consider the region

$$C_\ell^\Lambda := \left[-\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right), \frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)\right] \subseteq [-X(\ell), X(\ell)]$$

for small enough ℓ, noting that $\rho_\ell(\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)) = \Lambda\ell$. Then on C_ℓ^Λ, $\Lambda^{-2}\ell^{-2} \leqslant \rho_\ell(s)^{-2} \leqslant \ell^{-2}$ and so control of $\mathcal{T}_w(u,\ell)$ implies control of $\ell^{-2}\mathcal{T}(u,\ell)$ on C_ℓ^Λ. Note that

$$\frac{\frac{2\pi}{\ell}\arccos\left(\frac{1}{2\pi\Lambda}\right)}{X(\ell)} \to \frac{2}{\pi}\arccos\left(\frac{1}{2\pi\Lambda}\right) \quad \text{as } \ell \searrow 0$$

and

$$\frac{2}{\pi}\arccos\left(\frac{1}{2\pi\Lambda}\right) \to 1 \text{ as } \Lambda \nearrow \infty$$

so the central region C_ℓ^Λ covers a fixed portion of the collar in the limit as $\ell \searrow 0$, and that this proportion can be taken arbitrarily close to 1.

This weighted quantity $\mathcal{T}_w$ was introduced in [RT18], which studied the behaviour of Teichmüller harmonic map flow when the target manifold has non-positive sectional curvature, or more generally admits no bubbles. I will also be considering this setting so as to make use of the estimates derived there.

As above, I will also be assuming that the coupling factor η depends on the injectivity radius in a particular way. This time, it is assumed that $\eta \nearrow \infty$ as $\ell \searrow 0$, though not too fast. Specifically, assume

$$\eta \to \infty \text{ as } \ell \to 0 \tag{4.3.2}$$

$$\eta^2\ell \to 0 \text{ as } \ell \to 0 \tag{4.3.3}$$

Interestingly, this contrasts with the previous result in that I am speeding up the degeneration of the metric which intuitively makes it harder for the map to reach a geodesic.

In this setting, I obtained the following theorem.

Proposition 4.3.1. *Suppose* $(u(t), g(t))$ *is a smooth solution to the flow* (4.1.1),(4.1.2) *on* $[0, \infty)$ *with domain* Σ *being a closed surface with genus at least* 2 *and target* (N, h) *not admitting any bubbles. Let* $\mathcal{C}$ *be a collar on which* $0 < \ell < 2\,\mathrm{arsinh}(1)$ *for all time and* $\ell \to 0$ *as* $t \to \infty$*. Suppose further that* η *satisfies* (4.3.2) *and* (4.3.3)*. Then there exists a sequence of times* $t_j \to \infty$ *along which* $\mathcal{T}_w^2 \ell(t_j) \to 0$.

Consequently, for any fixed $\Lambda > 1$, $\mathcal{T}^2 \ell^{-1}(t_j) \to 0$ *on* C_ℓ^Λ *and so the curve restricted to the central region* C_ℓ^Λ *sub-converges to a geodesic.*

This result gives a positive answer to the question of convergence to a geodesic for this slightly modified flow on a large part of the collar neighbourhood. It remains to be seen what can be said about the more delicate in between region where the collar has width of order strictly between ℓ and 1.

4.3.1 Required Estimates From Existing Literature

We quote here three key estimates which we need for the main result, all coming from [RT18]. The first gives a bound on the evolution of ℓ.

Lemma 4.3.2 (Part of Lemma 2.3, [RT18]). *Let* $(u(t), g(t))$ *be a smooth solution to the flow* (4.1.1),(4.1.2) *on* $[0, T)$ *and* $\mathcal{C}$ *be a collar with* $\ell < 2\,\mathrm{arsinh}(1)$ *for all* $t \in [0, T)$*. Then there is a constant* $C > 0$ *depending only on the genus of the surface such that*

$$\left| \frac{\mathrm{d}\ell}{\mathrm{d}t} + \frac{\ell^2\eta^2}{16\pi^3} \int_{\mathcal{C}_\ell} \left(|u_s|^2 - |u_\theta|^2 \right) \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \right| \leqslant C\ell^2\eta^2 E_0 \tag{4.3.4}$$

where E_0 *is an upper bound on the initial energy.*

The second is an estimate on the weighted angular energy, defined

$$I^{(\theta)}(u, \ell) := \int_{\mathcal{C}_\ell} |u_\theta|^2 \rho_\ell(s)^{-2} \mathrm{d}s\mathrm{d}\theta \tag{4.3.5}$$

Note there is no cut-off function here, but this quantity bounds the cut-off version which appears later on.

Lemma 4.3.3 (Lemma 3.1, [RT18]). *Let $E_0 < \infty$ be given and assume (N, h) supports no bubbles. Then there is a constant C depending only on E_0 and (N, h) such that for any $0 < \ell < 2\operatorname{arsinh}(1)$ and smooth map $u : (\mathcal{C}_\ell, g_\ell) \to (N, h)$ with $E(u) \leqslant E_0$*

$$I^{(\theta)}(u, \ell) \leqslant C(1 + \mathcal{T}^2) \tag{4.3.6}$$

Finally, we define a weighted energy

$$\mathcal{I}(u, \ell) := \frac{1}{2} \int_{\mathcal{C}_\ell} \left(|u_s|^2 + |u_\theta|^2 \right) \rho_\ell(s)^{-2} \varphi(\rho_\ell(s))^2 \mathrm{d}s\mathrm{d}\theta \tag{4.3.7}$$

Note that we have not lost much using the cut-off function, since we have the estimate (which is (5.2) from [RT18])

$$\frac{1}{2} \int_{\mathcal{C}_\ell} \left(|u_s|^2 + |u_\theta|^2 \right) \rho_\ell(s)^{-2} (1 - \varphi(\rho_\ell(s))^2) \mathrm{d}s\mathrm{d}\theta \leqslant 4\pi^2 E \tag{4.3.8}$$

which shows the error compared to the full weighted energy is controllable. Also note that since $\rho_\ell(Y(\ell)) = \frac{1}{\pi} + \frac{\ell^2}{8\pi} + O(\ell^4)$, the cut-off function is 0 at the ends and only cuts off a small bit of the cylinder.

We need the following estimate on the evolution of this weighted energy (4.3.7). This is contained in Lemma 5.1 from [RT18] with two small adjustments. First we keep track of the η dependence since we are considering η blowing up as opposed to η constant. Second, we keep half of the weighted tension term which appears but is then estimated away.

Lemma 4.3.4 (Essentially Lemma 5.1, [RT18]). *Assume (N, h) supports no bubbles and let (u, g) be a smooth solution to the flow on $[0, T)$, $\mathcal{C}$ be a collar with $\ell < 2\operatorname{arsinh}(1)$ for all $t \in [0, T)$ and suppose η satisfies (4.3.2) and (4.3.3). Then there is a constant $C > 0$ depending only on the genus of the surface, (N, h) and E_0 (a bound in initial energy) such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I} \leqslant C \left(\eta^2 + \mathcal{I} + \eta^2 \ell \mathcal{I} \mathcal{T}^2 \right) - \frac{1}{2} \mathcal{T}_w^2 \tag{4.3.9}$$

4.3.2 Proof of Proposition 4.3.1

With these results, we can prove Proposition 4.3.1.

Proof. The key quantity we study here is $\mathcal{I}\ell$. We want to use the evolution of this for which we need the following estimate, a consequence of Lemma 4.3.2, (4.3.8) and Lemma 4.3.3. Note the constants C, c depend only on the genus of the domain surface, (N, h) and E_0.

$$\begin{aligned}\frac{\mathrm{d}\ell}{\mathrm{d}t} &\leqslant -\frac{\ell^2}{16\pi^3}\eta^2\int_{\mathcal{C}_\ell}\left(|u_s|^2-|u_\theta|^2\right)\rho_\ell(s)^{-2}\mathrm{d}s\mathrm{d}\theta + C\eta^2\ell^2\\ &\leqslant -\frac{\ell^2}{16\pi^3}\eta^2\left(2\mathcal{I}-C-I^{(\theta)}\right)+C\ell^2\eta^2\\ &\leqslant -c\ell^2\eta^2\mathcal{I}+C\ell^2\eta^2 I^{(\theta)}+C\ell^2\eta^2\\ &\leqslant -c\ell^2\eta^2\mathcal{I}+C\ell^2\eta^2(1+\mathcal{T}^2)\end{aligned}$$

Hence using Lemma 4.3.4 we have the following bound on the evolution of $\mathcal{I}\ell$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{I}\ell) &= \ell\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}+\mathcal{I}\frac{\mathrm{d}\ell}{\mathrm{d}t}\\ &\leqslant C\left(\eta^2\ell+\mathcal{I}\ell+\mathcal{I}\eta^2\mathcal{T}^2\ell^2\right)-\frac{1}{2}\mathcal{T}_w^2\ell+\mathcal{I}\left(-c\ell^2\eta^2\mathcal{I}+C\ell^2\eta^2(1+\mathcal{T}^2)\right)\\ &\leqslant C\left(\eta^2\ell+(\mathcal{I}\ell)+\eta^2\ell\mathcal{T}^2(\mathcal{I}\ell)\right)-c\eta^2(\mathcal{I}\ell)^2-\frac{1}{2}\mathcal{T}_w^2\ell\end{aligned}$$

Now we note that by the formula for the decay of energy, $\int_0^\infty \mathcal{T}^2\mathrm{d}t < E_0$. So combined with the assumption (4.3.3), $\int_0^\infty \eta^2\ell\mathcal{T}^2\mathrm{d}t<\infty$. This enables us to proceed as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{I}\ell e^{-\int_0^t C\eta^2\ell\mathcal{T}^2\mathrm{d}t'}\right) &= \left(e^{-\int_0^t C\eta^2\ell\mathcal{T}^2\mathrm{d}t'}\right)\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{I}\ell)-C\eta^2\ell\mathcal{T}^2(\mathcal{I}\ell)e^{-\int_0^t C\eta^2\ell\mathcal{T}^2\mathrm{d}t'}\\ &\leqslant C\left(\eta^2\ell+(\mathcal{I}\ell)+\eta^2\ell\mathcal{T}^2(\mathcal{I}\ell)\right)-c\eta^2(\mathcal{I}\ell)^2-c\mathcal{T}_w^2\ell-C\eta^2\ell\mathcal{T}^2(\mathcal{I}l)\\ &= C\left(\eta^2\ell+(\mathcal{I}\ell)\right)-c\eta^2(\mathcal{I}\ell)^2-c\mathcal{T}_w^2\ell\\ &\leqslant C(\eta^2\ell+\eta^{-2})-c\mathcal{T}_w^2\ell\end{aligned}$$

where we have used that $\delta \leqslant e^{-\int_0^t C\eta^2\ell\mathcal{T}^2\mathrm{d}t'}\leqslant 1$ for some $\delta>0$ independent of t. We can now consider two cases to extract the required subsequence.

First suppose that there is some time t_0 such that for all $t \geqslant t_0$, it holds that

$$\mathcal{T}_w^2 \ell > 2Cc^{-1}(\eta^{-2} + \eta^2 \ell)$$

Then for all $t \geqslant t_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathcal{I} \ell e^{-\int_0^t C\eta^2 \ell \mathcal{T}^2 \mathrm{d}t'} \right) \leqslant -\frac{1}{2} c \mathcal{T}_w^2 \ell$$

Since $\mathcal{I} \ell e^{-\int_0^t C\eta^2 \ell \mathcal{T}^2 \mathrm{d}t'}$ is non-negative, it follows that $\int_{t_0}^{\infty} \mathcal{T}_w^2 \ell \mathrm{d}t < \infty$ which allows us to extract the desired subsequence.

Now suppose that no such time t_0 exists. Then we can find a sequence of times $t_j \to \infty$ along which $\mathcal{T}_w^2 \ell \leqslant 2Cc^{-1}(\eta^{-2} + \eta^2 \ell)$. By (4.3.2) and (4.3.3), we therefore have that $\mathcal{T}_w^2 \ell \to 0$ along this sequence. □

4.4 Future Outlook

Answering the main question of this chapter, Question 4.1.4 remains a hard task. One way to interpret the results I have presented above is that I have approached the true flow equation from both sides, in the sense that I have both sped up and slowed down the metric degeneration, and found in both cases partial positive answers showing convergence to a geodesic. Whilst this does not directly apply to the original equations, it at least provides evidence supporting Question 4.1.4. Moreover, many of the estimates used above apply for any η, and so perhaps with some refinements or in combination with further techniques, the idea of proving convergence to a geodesic using Theorem 4.1.5 could be achieved.

Were this question to be satisfactorily answered, it would show that the Teichmüller harmonic map flow despite its singularity formation provides a complete decomposition of any initial map into a collection of harmonic objects.

Appendix A

Fractional Sobolev Spaces

In this appendix, we recall the definitions and key properties of the fractional Sobolev spaces and operators that we use in this thesis.

A.1 Fractional Sobolev Spaces

First of all, we give the definition of fractional Sobolev spaces $W^{s,p}(\Omega)$. There are several equivalent ways of introducing these. In the case when $p = 2$ and the domain is $\mathbb{R}^n$, perhaps the simplest is via the Fourier transform. This has the advantage of very neatly dealing will all real values of s, however it requires more work to subsequently generalise to other domains. This approach is followed in Section 25 of [Tre75]. Another way of introducing these spaces, which works in a very broad context, is to use interpolation theory for Banach spaces. This has the advantage of being very general, for example also be able to define Besov spaces with the same procedure, and it provides access to some nice machinery, e.g. the ability to generalise certain results for integer order Sobolev spaces immediately to fractional Sobolev spaces, however it is more abstract and so does not provide a concrete idea of what properties functions in $W^{s,p}(\Omega)$ have. Chapter 4 in [Tay10] covers this method. The way that we shall introduce fractional Sobolev spaces is via the so called Sobolev–Slobodeckij semi norms, which give a very explicit description of the spaces $W^{s,p}(\Omega)$. The paper [DPV12] gives a nice presentation of this approach. We focus on the case $1 \leqslant p < \infty$ since we do not make use of the spaces $W^{s,\infty}(\Omega)$.

Definition A.1.1. Let Ω be an n-dimensional subset of a Riemannian manifold (M, g), $1 \leqslant p < \infty$ and $s > 0$. Then we define the Sobolev–Slobodeckij semi-norm $[\cdot]_{W^{t,p}(\Omega)}$ for functions $u : \Omega \to \mathbb{R}$ by

$$[u]_{W^{t,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+tp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}$$

for $0 < t < 1$. Now let $s = k + t$, where $k \in \mathbb{N}$ and $0 < t < 1$. Then we define the space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \{u \in W^{k,p}(\Omega) : [\partial^\alpha u]_{W^{t,p}(\Omega)} < \infty \text{ for each } k\text{'th order multi-index } \alpha\}$$

where we take $W^{0,p}(\Omega) = L^p(\Omega)$. We equip this space with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha u]^p_{W^{t,p}(\Omega)} \right)^{\frac{1}{p}}$$

When $p = 2$ we write $W^{s,2}(\Omega)$ as $H^s(\Omega)$. As for the usual Sobolev spaces, we also introduce the spaces $W^{s,p}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $W^{s,p}(\Omega)$. Finally, for $s < 0$ and $1 < p < \infty$, we define the space $W^{s,p}(\Omega)$ to be the dual space of $W^{-s,q}_0(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Next, we state the embedding theorems for fractional Sobolev spaces. For this, we define the Sobolev exponent

$$p^* := \frac{np}{n - sp}$$

where $n \in \mathbb{N}$, $0 < s < 1$ and $1 \leqslant p < \infty$.

Theorem A.1.2 (See results of Sections 6,7 in [DPV12])**.** *Let Ω be a bounded Lipschitz n-dimensional subset of a Riemannian manifold (M, g), $1 \leqslant p < \infty$ and $0 < s < 1$ and suppose $sp < n$. Then for each $q \in [1, p^*]$, there is a constant $C > 0$ depending only on n,p,s,Ω such that for each $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)}.$$

Moreover the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for each $q \in [1, p^*)$.

Next we give the trace theorem for the spaces $W^{s,p}$. For simplicity, we only state this for $p = 2$.

Theorem A.1.3 (See Proposition 4.5, Chapter 4 in [Tay10])**.** *Let Ω be a smooth n-dimensional subset of a Riemannian manifold (M, g) with $\partial\Omega \neq \emptyset$ and $s > \frac{1}{2}$. Then the trace map*

$$\tau : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

is a well defined bounded linear operator.

We note here that as mentioned in [Tar07] in Chapter 16, we cannot extend the trace mapping to $s \leqslant \frac{1}{2}$ since smooth functions which vanish in a neighbourhood of the boundary are dense in $H^s(\Omega)$ for $s \leqslant \frac{1}{2}$.

Next we need a result on the operator which maps a function on the boundary to its harmonic extension.

Theorem A.1.4 (Proposition 1.8, [Tay10])**.** *Let Ω be a smooth compact n-dimensional subset of a Riemannian manifold (M, g) with $\partial\Omega \neq \emptyset$ and $s \geqslant -\frac{1}{2}$. Then the map*

$$h : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega)$$

which maps $u \in H^s(\partial\Omega)$ to its unique harmonic extension is a bounded operator.

In the case when $s = 1$, we shall also need a localised version of this estimate, which we state and prove below. We work with $\Omega = \mathbb{D}$ as this is the case we need.

Lemma A.1.5. *Let $u : \bar{B}_{2r}(x_0) \cap \mathbb{D} \to \mathbb{R}^n$ with $0 < r < \frac{1}{2}$ be harmonic in $B_{2r}(x_0) \cap \mathbb{D}$ and have boundary regularity $u \in H^1(\bar{B}_{2r}(x_0) \cap \partial\mathbb{D})$. Then $u \in H^{\frac{3}{2}}(\bar{B}_r(x_0) \cap \bar{\mathbb{D}})$ with the estimate*

$$\|u\|_{H^{\frac{3}{2}}(\bar{B}_r(x_0)\cap\bar{\mathbb{D}})} \leqslant C\left(r^{-2}\|u\|_{L^2(\bar{B}_{2r}(x_0)\cap\mathbb{D})} + r^{-1}\|\nabla u\|_{L^2(\bar{B}_{2r}(x_0)\cap\mathbb{D})} + \|u\|_{H^1(\bar{B}_{2r}(x_0)\cap\partial\mathbb{D})}\right) \tag{A.1.1}$$

for a universal constant C.

Proof. Let φ be a smooth cut-off function supported on $B_{2r}(x_0)$ satisfying $\varphi(x) \equiv 1$ on $B_r(x_0)$. Let $f \in H^1(\bar{B}_{2r}(x_0) \cap \partial\mathbb{D})$ be the trace of u. Then set $\tilde{u} = \varphi u$ and extend $\tilde{u}$ by 0 to be defined on all of $\bar{\mathbb{D}}$. Then $\tilde{u}$ satisfies the BVP

$$\begin{cases} \Delta\tilde{u} = 2\nabla\varphi \cdot \nabla u + u\Delta\varphi & \text{in } \mathbb{D} \\ \tilde{u} = \varphi f & \text{on } \partial\mathbb{D} \end{cases}$$

Using linearity, we can write $\tilde{u} = u_1 + u_2$ where u_1, u_2 respectively solve

$$\begin{cases} \Delta u_1 = 2\nabla\varphi \cdot \nabla u + u\Delta\varphi & \text{in } \mathbb{D} \\ u_1 = 0 & \text{on } \partial\mathbb{D} \end{cases}$$

and

$$\begin{cases} \Delta u_2 = 0 & \text{in } \mathbb{D} \\ u_2 = \varphi f & \text{on } \partial\mathbb{D} \end{cases}$$

Now we can use the usual H^2 global regularity estimate for Poisson's equation to obtain,

$$\begin{aligned} \|u_1\|_{H^2(\mathbb{D})} &\leqslant C\|2\nabla\varphi \cdot \nabla u + u\Delta\varphi\|_{L^2(\mathbb{D})} \\ &\leqslant C\left(r^{-1}\|\nabla u\|_{L^2(\mathbb{D}} + r^{-2}\|u\|_{L^2(\mathbb{D})}\right) \end{aligned}$$

Next, we can use the global result, Theorem A.1.4, to obtain

$$\|u_2\|_{H^{\frac{3}{2}}(\mathbb{D})} \leqslant C\|\varphi f\|_{H^1(\partial\mathbb{D})} \leqslant C\|f\|_{H^1(\bar{B}_{2r}(x_0)\cap\partial\mathbb{D})}$$

Combining the estimates for u_1 and u_2 gives the result. □

A.2 Key Properties of Operators

Here we provide the definitions for the operators discussed in Chapters 2 and 3. We begin with the fractional Laplacian. There are several ways of defining this operator, which all agree when the domain is $\mathbb{R}^n$ or T^n, but which can give rise to different operators for more general domains. See for example [LPG+20] for an overview of these.

Definition A.2.1. Let $f : \partial\mathbb{D} \to \mathbb{R}$ and $0 < s < 1$. Then define $(-\Delta)^s f$ as

$$(-\Delta)^s f(x) := C(s) \lim_{\varepsilon \to 0} \int_{\partial B \setminus B_\varepsilon(x)} \frac{f(x) - f(y)}{|x-y|^{1+2s}} \mathrm{d}y$$

provided this exists, where $C(s)$ is a normalisation constant, depending only on s.

Since we do not use the fractional Laplacian, we do not pursue the function space on which this is well defined.

Next, we define the Dirichlet-to-Neumann operator associated to the Laplacian.

Definition A.2.2. Let (M, g) be a smooth compact n-dimensional Riemannian manifold with $\partial M \neq \emptyset$. Then the Dirichlet-to-Neumann operator $\partial_\nu : H^{\frac{1}{2}}(\partial M) \to H^{-\frac{1}{2}}(\partial M)$ is defined by asking that

$$\int_M \langle \nabla u_g, \nabla v_g \rangle \mathrm{d}v_g = \int_{\partial M} \partial_\nu u \cdot v \mathrm{d}s_g$$

for all $v \in H^{\frac{1}{2}}(\partial M)$, where u_g, v_g denote the harmonic extensions of u, v respectively and $\mathrm{d}v_g, \mathrm{d}s_g$ the volume forms.

Finally we recall a key Sobolev inequality we use in Chapter 3.

Theorem A.2.3 ([BM18, Special Case of Theorem 1])**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Suppose that s, s_1, s_2, θ are constants satisfying $s, s_1, s_2 \geqslant 0$, $0 < \theta < 1$, $s = \theta s_1 + (1-\theta)s_2$. Then there is a constant $C > 0$ depending only on the domain and the above constants such that for all $f \in H^{s_1}(\Omega) \cap H^{s_2}(\Omega)$*

$$\|f\|_{H^s(\Omega)} \leqslant C \|f\|_{H^{s_1}(\Omega)}^{\theta} \|f\|_{H^{s_2}(\Omega)}^{1-\theta}.$$

Appendix B

Commutator Estimates

While many of the operations we use repeatedly throughout Chapter 3, such as harmonic extensions, derivatives and projections, do not commute, the error terms resulting from changing the order of operation will in general be of lower order. The purpose of this appendix is to collect and prove the relevant estimates of such commutators which will be used throughout the chapter.

To begin with we note that terms which are obtained from commuting two differential operators $L_{1,2}$ of total order $j_1 + j_2 = j \in \mathbb{N}$ can be bounded by

$$\|L_1L_2w - L_2L_1w\|_{L^2(\Sigma)} \leqslant C\|w\|_{H^{j-1}(\Sigma)} \text{ and } \|L_1L_2w - L_2L_1w\|_{L^2(\partial\Sigma)} \leqslant C \sum_{k\leqslant j-1} \|\nabla^k w\|_{L^2(\partial\Sigma)} \tag{B.0.1}$$

We also recall that the extensions τ and ν of the unit tangent and normal vector fields on $\partial\Sigma$ are chosen so that $\partial_\eta u_g$, $\eta \in \{\tau, \nu\}$ are again harmonic on the corresponding neighbourhood $\mathcal{U}(\partial\Sigma)$ of the boundary. As the C^k norm of harmonic functions can be controlled just in terms of the energy away from the boundary, we can hence bound the difference between $\partial_\eta u_g$ and its harmonic extension by

$$\|\varphi\big[\partial_\eta u_g - (\partial_\eta u_g)_g\big)\big]\|_{C^k(\mathcal{U})} \leqslant CE(u_g, g)^{\frac{1}{2}} \tag{B.0.2}$$

for any $k \in \mathbb{N}$, any $g \in \mathcal{M}_\iota$ and a constant C that only depends on k and ι_0.

Combined, these estimates in particular allow us to bound

$$\|\nabla^j(\partial_\eta u_g)_g - \partial_\eta \nabla^j u_g\|_{L^2(\partial\Sigma)} \leqslant C\sum_{k\leqslant j}\|\nabla^k u_g\|_{L^2(\partial\Sigma)} \leqslant C\|w_g\|_{H^{j+1}(\Sigma)} \tag{B.0.3}$$

where the last estimate is not optimal, but will often be sufficient in applications.

We will need commutator estimates mostly for derivatives and harmonic extensions of functions of the form $w = P_{v_g}(\partial_\eta u_g)$ and $w = P^\perp_{v_g}(\partial_\eta u_g)$, where $P_p, P_p^\perp \in C_b^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ denote the (fixed) extensions of the projections onto the tangent and normal space of our support manifold.

To state the relevant estimates in a form that makes them applicable to treat both types of terms, we let $A \in C_b^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ be any bounded matrix valued function with bounded derivatives and consider commutator terms of the form

$$C_j(U,V) := L_1L_2\big(\varphi A(V)(\partial_\eta U)\big) - L_1\big(\varphi A(V)(\partial_\eta L_2 U)\big) \tag{B.0.4}$$

for linear differential operators $L_{1,2}$ of order $j_1 \geqslant 0$, $j_2 \geqslant 1$ with $j_1 + j_2 \leqslant j$ and a cut-off function $\varphi \in C_c^\infty(\mathcal{C}(\partial\Sigma))$ which is chosen as in Remark 3.3.2 and is in particular so that $\varphi \equiv 1$ near $\partial\Sigma$. A short argument, which is carried out below allows us to prove the following basic, but useful, estimates.

Lemma B.0.1. *Let $\iota_0 > 0, g \in \mathcal{M}_{\iota_0}$, $A \in C_b^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ and let $U, V \in H^j(\mathcal{C}(\partial\Sigma))$, for some $j \geqslant 2$, $\mathcal{C}(\partial\Sigma) = \bigcup_i \mathcal{C}(\sigma_i, g)$ the union of the neighbourhood of the boundary curves σ described in Section 3.3.1. Then the following holds true for commutator terms $C_j(U,V)$ and $C_{j-1}(U,V)$ of the form* (B.0.4) *obtained from differential operators $L_{1,2}$ of order $j_{1,2}$ with $j_2 \geqslant 1$ and $j_1 + j_2 \leqslant j$ respectively $j_1 + j_2 \leqslant j - 1$.*

If $j = 2$ then $C_2(U,V) \in L^{4/3}(\Sigma)$ and $C_1(U,V) \in L^2(\partial\Sigma)$ and we can bound

$$\begin{aligned} &\|C_2(U,V)\|_{L^{4/3}(\Sigma)} + \|C_1(U,V)\|_{L^2(\partial\Sigma)} \\ &\qquad \leqslant C(1 + \|\nabla V\|_{L^4(\Sigma)})\|U\|_{H^2(\Sigma)} + C\|V\|_{H^2(\Sigma)}\|\nabla U\|_{L^4(\Sigma)} \\ &\qquad + C(1 + \|\nabla V\|^2_{L^4(\Sigma)})\|\nabla U\|_{L^4(\Sigma)} \end{aligned} \tag{B.0.5}$$

while for functions $U, V \in H^{5/2}(\Sigma)$ *we furthermore have that* $C_2(U,V) \in L^2(\Sigma)$ *and*

$$\|C_2(U,V)\|_{L^2(\Sigma)} \leqslant C(1+\|V\|_{H^{3/2}(\Sigma)})\|U\|_{H^{5/2}(\Sigma)} + C\|V\|_{H^{5/2}(\Sigma)}\|U\|_{H^{3/2}(\Sigma)} \tag{B.0.6}$$

$$+ C(1+\|V\|^2_{H^2(\Sigma)})\|U\|_{H^{3/2}(\Sigma)}. \tag{B.0.7}$$

If $j \geqslant 3$ *then* $C_j(U,V) \in L^2(\Sigma)$ *and* $C_{j-1}(U,V) \in L^2(\partial\Sigma)$ *and we can bound*

$$\|C_j(U,V)\|_{L^{4/3}(\Sigma)} + \|C_{j-1}(U,V)\|_{L^2(\partial\Sigma)} \tag{B.0.8}$$

$$\leqslant C(1+\|V\|_{H^{3/2}(\Sigma)})\|U\|_{H^j(\Sigma)} + C\|U\|_{H^{3/2}(\Sigma)}\|V\|_{H^j(\Sigma)} \tag{B.0.9}$$

$$+ C\|V\|_{H^{5/2}(\Sigma)}\|U\|_{H^{j-1}(\Sigma)} + C\|U\|_{H^{5/2}(\Sigma)}\|V\|_{H^{j-1}(\Sigma)} \tag{B.0.10}$$

$$+ C(1+\|V\|^j_{H^{j-1}(\Sigma)})\|U\|_{H^{j-1}(\Sigma)} \tag{B.0.11}$$

and

$$\|C_j(U,V)\|_{L^2(\Sigma)} \leqslant C(1+\|V\|_{H^{5/2}(\Sigma)})\|U\|_{H^j(\Sigma)} + C\|U\|_{H^{5/2}(\Sigma)}\|V\|_{H^j(\Sigma)} \tag{B.0.12}$$

$$+ C(\|V\|_{H^{5/2}} + \|V\|^2_{H^2(\Sigma)})\|U\|_{H^{j-\frac{1}{2}}(\Sigma)} \tag{B.0.13}$$

$$+ C(\|U\|_{H^{5/2}(\Sigma)} + \|V\|_{H^2(\Sigma)}\|U\|_{H^2(\Sigma)})\|V\|_{H^{j-1/2}(\Sigma)} \tag{B.0.14}$$

$$+ C(1+\|V\|^j_{H^{j-1}(\Sigma)})\|U\|_{H^{j-1}(\Sigma)} \tag{B.0.15}$$

for a constant C *that only depends on the operators* $L_{1,2}$*,* ι_0 *and* A*.*

At times, it will suffice to use that the above estimates in particular imply that

$$\|C_j(u_g,u_g)\|^2_{L^2(\Sigma)} \leqslant S_{j-\frac{1}{2}}(u,g)^j S_j(u,g), \text{ for } S_s(u,g) \text{ as in (3.5.11)} \tag{B.0.16}$$

We can apply the above lemma not only to control error terms that result from commuting derivatives and projections, but also to compare functions of the form $w = P_{v_g}\partial_\eta u_g$ with their harmonic extensions. To this end, we note that (cut-off) versions of derivatives of $\Delta(A(v_g)\partial_\eta u_g)$ can be viewed as commutator terms $\varphi\nabla^{j-2}\Delta(A(v_g)\partial_\eta u_g) = C_j(v_g,u_g)$ of the above form for $L_1 = \nabla_g^{j-2}$ and $L_2 = \Delta_g$ since $\Delta_g u_g \equiv 0$. To bound the difference between such $w = \varphi A(v_g)\partial_\eta u_g$ and their harmonic extension w_g we will hence be able to combine the above commutator estimates with the standard elliptic estimate

$$\|w - w_g\|_{W^{k,p}(\Sigma)} \leqslant C\|\Delta_g w\|_{W^{k-2,p}(\Sigma)}, \quad C = C(k, p, \iota_0), \quad 1 < p < \infty \tag{B.0.17}$$

and the fact that $W^{1,\frac{4}{3}}(\Sigma) \hookrightarrow L^2(\partial\Sigma)$.

This immediately allows us to deduce that the above commutator estimates also apply for terms of the form $\nabla^j(\varphi\big[A(v_g)\partial_\eta u_g - (A(v_g)\partial_\eta u_g)_g\big])$ and, as $(\partial_\eta u_g)_g - \partial_\eta u_g$ is controlled by (B.0.2), also for terms of the form $\nabla^j(\varphi\big[A(v_g)(\partial_\eta u_g)_g - (A(v_g)\partial_\eta u_g)_g\big])$. As harmonic functions are controlled in any C^k in terms of their energy away from the boundary, and hence on the support of $(1-\varphi)$, and as we can view $\nabla^j(\varphi A(v_g)\partial_\eta u_g) - \nabla^{j_1}(\varphi A(v_g)\partial_\eta \nabla^{j_2} u_g)$ as another commutator term of the form (B.0.4), we hence obtain the following useful consequence of the above lemma.

Remark B.0.2. The commutator estimates stated in the above Lemma B.0.1 are all valid also for terms of the form

$$C_j(v_g, u_g) = \varphi\big(\nabla_g^j\big[(A(v_g)\partial_\eta u_g)_g\big] - \nabla_g^{j_1}\big[A(v_g)\partial_\eta \nabla_g^{j_2} u_g\big]\big), \quad j_1 + j_2 = j \tag{B.0.18}$$

and

$$C_j(v_g, u_g) = \nabla_g^j\big[(A(v_g)\partial_\eta u_g)_g\big] - \nabla_g^{j_1}\big[A(v_g)\nabla_g^{j_2}(\partial_\eta u_g)_g\big], \quad j_1 + j_2 = j \tag{B.0.19}$$

Proof of Lemma B.0.1. To obtain the claim for $j = 2$ we can use that ∇C_1 and C_2 satisfy pointwise bounds of

$$|C_2(U,V)| + |\nabla C_1(U,V)| \leqslant C\big[\left|\nabla^2 U\right|(1 + |\nabla V|) + \left|\nabla^2 V\right||\nabla U| + (1 + |\nabla V|^2)\,|\nabla U|\big]$$

C as in the lemma. The claims in this case can hence be obtained by using that $H^{\frac{1}{2}}(\Sigma) \hookrightarrow L^4(\Sigma)$ and $W^{1,\frac{4}{3}}(\Sigma) \hookrightarrow L^2(\partial\Sigma)$ and applying Hölder's inequality.

Similarly, for $j \geqslant 3$ we have pointwise estimates of

$$\begin{aligned} |C_j(U,V)| + |\nabla C_{j-1}(U,V)| \leqslant{}& (1 + |\nabla V|)\left|\nabla^j U\right| + \left|\nabla^j V\right||\nabla U| \\ &+ (|\nabla V|^2 + \left|\nabla^2 V\right| + 1)\left|\nabla^{j-1} U\right| \\ &+ (|\nabla U||\nabla V| + \left|\nabla^2 U\right|)\left|\nabla^{j-1} V\right| \end{aligned}$$

$$+ \sum_{\substack{|\alpha|\leqslant j+1 \\ 1\leqslant\alpha_i\leqslant j-2}} |\nabla^{\alpha_1} V| \cdots |\nabla^{\alpha_{\ell-1}} V| \, |\nabla^{\alpha_\ell} U| \, .$$

The claims in this case can hence be obtained by using the Sobolev embeddings $H^{\frac{1}{2}}(\Sigma) \hookrightarrow L^4(\Sigma)$ and $H^{j-1}(\Sigma) \hookrightarrow W^{j-2,p}(\Sigma)$, $p < \infty$, and $H^{3/2}(\Sigma) \hookrightarrow L^\infty(\Sigma)$ as well as Hölder's inequality. $\square$

Appendix C

Closed Hyperbolic Surfaces

Here we summarise two key results on hyperbolic surfaces which we refer to in Chapter 4. We use the appendix of [RTZ13] as our source as the results have been formulated in a convenient manner, and refer to the references of that paper for the original results.

The first result we need is the following description of hyperbolic surfaces near simple closed geodesics, which is a version of the Keen-Randol collar lemma.

Lemma C.0.1 (Lemma A.4 [RTZ13])**.** *Let (Σ, g) be a closed surface with constant curvature -1 and let σ be a simple closed geodesic in (Σ, g) of length ℓ. Then there exists a neighbourhood U of σ which is isometric to the cylinder $\mathcal{C}_\ell = [-X(\ell), X(\ell)] \times \mathcal{S}^1$ with metric $g_\ell = \rho_\ell(s)^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)$, where*

$$X(\ell) := \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\frac{\ell}{2}\right)\right)$$
$$\rho_\ell(s) := \frac{\ell}{2\pi\cos\left(\frac{\ell s}{2\pi}\right)}$$

and the geodesic σ is mapped to the curve $\{0\} \times \mathcal{S}^1$.

The second key result we need is a description of the limits of degenerating hyperbolic surfaces. This is addressed in the following version of the Deligne-Mumford compactness theorem.

Proposition C.0.2 (Proposition A.2, [RTZ13])**.** *Let (Σ, g_i, c_i) be a sequence of connected* *closed hyperbolic Riemann surfaces of genus* $\gamma \geqslant 2$ *and satisfying* $\liminf_{i\to\infty} \operatorname{inj} g_i = 0$, *where* c_i *denotes the complex structure. Then there is a subsequence such that the surfaces converge (in the sense described below) to a (potentially disconnected) punctured hyperbolic Riemann surface* $(\tilde{\Sigma}, g, c)$. *The limit surface* $\tilde{\Sigma}$ *arises by collapsing a finite collection of pairwise disjoint simple homotopically non-trivial closed curves* $\sigma^1, ..., \sigma^k$ *in* Σ *to points* $q^1, ..., q^k$ *to get a space* M, *and then setting* $\tilde{\Sigma} = M \setminus \bigcup_{j=1}^k q^j$. *We note* $1 \leqslant k \leqslant 3\gamma - 3$.

By the convergence, we mean that for each i, *there are simple closed geodesics* $\sigma_i^1, ..., \sigma_i^k$ *such that* σ^j *and* σ_i^j *are homotopic for each* i, j *and there is a continuous map* $\tau_i : \Sigma \to M$ *which satisfies*

1. *for each* i, j, $\tau_i(\sigma_i^j) = q^j$.

2. *for each* j, *the length of* σ_i^j *converges to* 0 *as* $i \to \infty$.

3. *for each* i, $\tau_i : \Sigma \setminus \bigcup_{j=1}^k \sigma^j \to \tilde{\Sigma}$ *is a diffeomorphism, with inverse denoted* f_i.

4. $f_i^* g_i \to g$ *locally smoothly on* $\tilde{\Sigma}$.

5. $f_i^* c_i \to c$ *locally smoothly on* $\tilde{\Sigma}$.

Bibliography

[BM18] Haïm Brezis and Petru Mironescu. Gagliardo–Nirenberg Inequalities and Non-Inequalities: The Full Story. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(5):1355–1376, 2018.

[Boc40] Salomon Bochner. Harmonic Surfaces in Riemann Metric. *Transactions of the American Mathematical Society*, 47(1):146–154, 1940.

[BW03] Paul Baird and John C. Wood. *Harmonic Morphisms Between Riemannian Manifolds.* Oxford University Press, 03 2003.

[CDY92] Kung-Ching Chang, Wei Yue Ding, and Rugang Ye. Finite-Time Blow-Up of the Heat Flow of Harmonic Maps from Surfaces. *Journal of Differential Geometry*, 36(2):507–515, 1992.

[CLW12] Li Chen, Yuxiang Li, and Youde Wang. The Refined Analysis on the Convergence Behavior of Harmonic Map Sequence from Cylinders. *Journal of Geometric Analysis*, 22(4):942–963, 2012.

[CM11] Tobias Colding and William Minicozzi. *A Course in Minimal Surfaces.* Graduate Studies in Mathematics, American Mathematical Society, 2011.

[Cou37] Richard Courant. Plateau's Problem and Dirichlet's Principle. *Annals of Mathematics*, 38(3):679–724, 1937.

[Cou40] Richard Courant. The Existence of Minimal Surfaces of Given Topological Structure Under Prescribed Boundary Conditions. *Acta Mathematica*, 72:51–98, 1940.

[Cou50] Richard Courant. *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*. Springer New York, NY, 1950.

[CS89] Yunmei Chen and Michael Struwe. Existence and Partial Regularity Results for the Heat Flow for Harmonic Maps. *Mathematische Zeitschrift*, 201(1):83 – 103, 1989.

[CS06] Luis Caffarelli and Luis Silvestre. An Extension Problem Related to the Fractional Laplacian. *Communications in Partial Differential Equations*, 32:1245–1260, 08 2006.

[CT99] Jingyi Chen and Gang Tian. Compactification of Moduli Space of Harmonic Mappings. *Commentarii Mathematici Helvetici*, 74(2):201 – 237, 1999.

[DdPW20] Juan Dávila, Manuel del Pino, and Juncheng Wei. Singularity Formation for the Two-Dimensional Harmonic Map Flow into S^2. *Inventiones mathematicae*, 219(2):345– 466, 2020.

[DL15] Francesca Da Lio. Compactness and Bubble Analysis for 1/2-Harmonic Maps. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 32(1):201–224, 2015.

[DLL06] Weiyue Ding, Jiayu Li, and Qingyue Liu. Evolution of Minimal Torus in Riemannian Manifolds. *Inventiones mathematicae*, 165(2):225–242, 2006.

[DLMR15] Francesca Da Lio, Luca Martinazzi, and Tristan Rivière. Blow-Up Analysis of a Nonlocal Liouville-Type Equation. *Analysis and PDE*, 8, 03 2015.

[Dou31] Jesse Douglas. Solution of the Problem of Plateau. *Transactions of the American Mathematical Society*, 33(1):263–321, 1931.

[DPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bulletin des Sciences Mathématiques*, 136(5):521–573, 2012.

[DR11] Francesca Da Lio and Tristan Rivière. Sub-Criticality of Non-Local Schrödinger Systems with Antisymmetric Potentials and Applications to Half-Harmonic Maps. *Advances in Mathematics*, 227(3):1300–1348, 2011.

[DScW23] Bin Deng, Liming Sun, and Jun cheng Wei. Non-Degeneracy and Quantitative Stability of Half-Harmonic Maps from R to S. *Advances in Mathematics*, 420:108979, 2023.

[DT95] Weiyue Ding and Gang Tian. Energy Identity for a Class of Approximate Harmonic Maps from Surfaces. *Communications in Analysis and Geometry*, 3(4):543–554, 1995.

[ES64] James Eells and Joseph Sampson. Harmonic Mappings of Riemannian Manifolds. *American Journal of Mathematics*, 86(1):109–160, 1964.

[Eva21] Lawrence Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2021.

[EW76] James Eells and John Wood. Restrictions on Harmonic Maps of Surfaces. *Topology*, 15(3):263–266, 1976.

[Fre95] Alexandre Freire. Uniqueness for the Harmonic Map Flow from Surfaces to General Targets. *Commentarii Mathematici Helvetici*, 70:310–338, 1995.

[GJ86] Michael Grüter and Jürgen Jost. On Embedded Minimal Disks in Convex Bodies. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 3(5):345–390, 1986.

[Har67] Philip Hartman. On Homotopic Harmonic Maps. *Canadian Journal of Mathematics*, 19:673–687, 1967.

[HI01] Gerhard Huisken and Tom Ilmanen. The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality. *Journal of Differential Geometry*, 59(3):353 – 437, 2001.

[HRT16] Tobias Huxol, Melanie Rupflin, and Peter Topping. Refined Asymptotics of the Teichmüller Harmonic Map Flow into General Targets. *Calculus of Variations and Partial Differential Equations*, 55(4):85, 2016.

[Hux17] Tobias Huxol. Limiting behaviour of the teichmüller harmonic map flow, 2017. https://arxiv.org/abs/1711.08844.

[KKMS24] Mikhail Karpukhin, Robert Kusner, Peter McGrath, and Daniel Stern. Embedded Minimal Surfaces in $\mathbb{S}^3$ and $\mathbb{B}^3$ via Equivariant Eigenvalue Optimization, 2024. https://arxiv.org/abs/2402.13121.

[KRT22] James Kohout, Melanie Rupflin, and Peter Topping. Uniqueness and Nonuniqueness of Limits of Teichmüller Harmonic Map Flow. *Advances in Calculus of Variations*, 15(3):369–384, 2022.

[Law70] Herbert Lawson. Complete Minimal Surfaces in $\mathcal{S}^3$. *Annals of Mathematics Second Series*, 92(3):335–374, 1970.

[LMR15] Francesca Da Lio, Luca Martinazzi, and Tristan Rivière. Blow-up Analysis of a Nonlocal Liouville-Type Equation. *Analysis and PDE*, 8(7):1757–1805, 2015.

[LPG+20] Anna Lischke, Guofei Pang, Mamikon Gulian, Fangying Song, Christian Glusa, Xiaoning Zheng, Zhiping Mao, Wei Cai, Mark M. Meerschaert, Mark Ainsworth, and George Em Karniadakis. What is the Fractional Laplacian? A Comparative Review with New Results. *Journal of Computational Physics*, 404, 2020.

[LR11] Francesca Da Lio and Tristan Rivière. Three-Term Commutator Estimates and the Regularity of 1/2-Harmonic Maps into Spheres. *Analysis and PDE*, 4(1):149–190, 2011.

[Mos11] Roger Moser. Intrinsic Semiharmonic Maps. *Journal of Geometric Analysis*, 21:588–598, 07 2011.

[MS15] Vincent Millot and Yannick Sire. On a Fractional Ginzburg–Landau Equation and 1/2-Harmonic Maps into Spheres. *Archive for Rational Mechanics and Analysis*, 215(1):125–210, 2015.

[Nas56] John Nash. The Imbedding Problem for Riemannian Manifolds. *Annals of Mathematics*, 63(1):20–63, 1956.

[OPS88] Brad Osgood, Ralph Phillips, and Peter Sarnak. Extremals of Determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148–211, 1988.

[PL17] Alessandro Pigati and Francesca Da Lio. Free Boundary Minimal Surfaces: A Nonlocal Approach. *Annali Scuola Normale Superiore - Classe di Scienze*, 20:437–489, 12 2017.

[Qin95] Jie Qing. On Singularities of the Heat Flow for Harmonic Maps from Surfaces into Spheres. *Communications in Analysis and Geometry*, 3(2):297–315, 1995.

[QT97a] Jie Qing and Gang Tian. Bubbling of the Heat Flows for Harmonic Maps from Surfaces. *Communications on Pure and Applied Mathematics*, 50(4):295–310, 1997.

[QT97b] Jie Qing and Gang Tian. Bubbling of the Heat Flows for Harmonic Maps from Surfaces. *Communications on Pure and Applied Mathematics*, 50(4):295–310, 1997.

[Rad30] Tibor Radó. The Problem of the Least Area and the Problem of Plateau. *Mathematische Zeitschrift*, 32(1):763–796, 1930.

[RT16] Melanie Rupflin and Peter Topping. Flowing Maps to Minimal Surfaces. *American Journal of Mathematics*, 138(4):1095–1115, 2016.

[RT18] Melanie Rupflin and Peter Topping. Teichmüller Harmonic Map Flow into Nonpositively Curved Targets. *Journal of Differential Geometry*, 108(1):135–184, 2018.

[RT19] Melanie Rupflin and Peter Topping. Global Weak Solutions of the Teichmüller Harmonic Map Flow into General Targets. *Analysis and PDE*, 12(3):815–842, 2019.

[RTZ13] Melanie Rupflin, Peter Topping, and Miaomiao Zhu. Asymptotics of the Teichmüller Harmonic Map Flow. *Advances in Mathematics*, 244:874–893, 2013.

[Rup08] Melanie Rupflin. An Improved Uniqueness Result for the Harmonic Map Flow in Two Dimensions. *Calculus of Variations and Partial Differential Equations*, 33:329–341, 2008.

[Rup14] Melanie Rupflin. Flowing Maps to Minimal Surfaces: Existence and Uniqueness of Solutions. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 31(2):349–368, 2014.

[Rup17] Melanie Rupflin. Teichmüller Harmonic Map Flow from Cylinders. *Mathematische Annalen*, 368(3):1227–1276, 2017.

[Rup22] Melanie Rupflin. Convergence of Almost Harmonic Maps to Geodesic Bubble Trees, 2022. https://arxiv.org/abs/2210.13367.

[Rup25] Melanie Rupflin. Lojasiewicz Inequalities for Almost Harmonic Maps near Simple Bubble Trees. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 42(4):807 – 850, 2025.

[Str84] Michael Struwe. On a Free Boundary Problem for Minimal Surfaces. *Inventiones mathematicae*, 75(3):547–560, 1984.

[Str85] Michael Struwe. On the Evolution of Harmonic Mappings of Riemannian Surfaces. *Commentarii Mathematici Helvetici*, 60:558–581, 1985.

[Str88] Michael Struwe. On the Evolution of Harmonic Maps in Higher Dimensions. *Journal of Differential Geometry*, 28(3):485 – 502, 1988.

[Str24] Michael Struwe. Plateau Flow or the Heat Flow for Half-Harmonic Maps. *Analysis and PDE*, 17(4):1397–1438, 2024.

[SU81] Jonathan Sacks and Karen Uhlenbeck. The Existence of Minimal Immersions of 2-Spheres. *Annals of Mathematics*, 113(1):1–24, 1981.

[SU82] Jonathan Sacks and Karen Uhlenbeck. Minimal Immersions of Closed Riemann Surfaces. *Transactions of the American Mathematical Society*, 271(2):639–652, 1982.

[SWZ21] Yannick Sire, Juncheng Wei, and Youquan Zheng. Infinite Time Blow-Up for Half-Harmonic Map Flow from R into S1. *American Journal of Mathematics*, 143:1261–1335, 01 2021.

[SY79] Richard Schoen and Shing-Tung Yau. Existence of Incompressible Minimal Surfaces and the Topology of Three Dimensional Manifolds with Non-Negative Scalar Curvature. *Annals of Mathematics*, 110(1):127–142, 1979.

[Tar07] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*. Springer-Verlag Berlin Heidelberg, 2007.

[Tay10] Michael Taylor. *Partial Differential Equations 1*. Springer New York, NY, 2 edition, 2010.

[Top02] Peter Topping. Reverse Bubbling and Nonuniqueness in the Harmonic Map Flow. *International Mathematics Research Notices*, 2002(10):505–520, 2002.

[Top04] Peter Topping. Winding Behaviour of Finite-Time Singularities of the Harmonic Map Heat Flow. *Mathematische Zeitschrift*, 247(2):279–302, 2004.

[Top23] Peter Topping. A Rigidity Estimate for Maps from S^2 to S^2 via the Harmonic Map Flow. *Bulletin of the London Mathematical Society*, 55(1):338–343, 2023.

[Tre75] François Treves. *Basic Linear Partial Differential Equations.* Academic Press, 1975.

[Tro92] Anthony Tromba. *Teichmüller Theory in Riemannian Geometry.* Lectures in Mathematics, ETH Zürich. Springer, 1992.

[Wet21] Jerome Wettstein. Existence, Uniqueness and Regularity of the Fractional Harmonic Gradient Flow in General Target Manifolds, 2021. https://arxiv.org/abs/2109.11458.

[Wet22] Jerome Wettstein. Uniqueness and Regularity of the Fractional Farmonic Gradient Flow in S^{n-1}. *Nonlinear Analysis*, 214, 2022.

[Wet23] Jerome Wettstein. Half-Harmonic Gradient Flow: Aspects of a Non-Local Geometric PDE. *Mathematics in Engineering*, 5(3):1 – 38, 2023.

[Wri24] Christopher Wright. Uniqueness of Weak Solutions of the Plateau Flow. *Calculus of Variations and Partial Differential Equations*, 63(6), 2024.