

Mathematical modelling of unsteady flows during ureteroscopy



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Statement of Originality

This thesis contains work carried out by myself, Harry Reynolds, at the Mathematical Institute at the University of Oxford from May 2018 to May 2022, under supervision of Professor Derek E. Moulton and Professor Sarah L. Waters. Clinical insight and motivation was provided by Professor Ben W. Turney. Collaboration with Boston Scientific Corporation provided industrial insight and overview via supervision by Niraj Rauniyar and Aditi Ray, as well as access to clinical equipment and experimental data to support my work. I would like to thank all my supervisors for motivation and support throughout my thesis, and I am grateful for funding provided by Boston Scientific Corporation in partnership with the EPSRC Centre For Doctoral Training in Industrially Focused Mathematical Modelling (InFoMM). No part of this thesis has been submitted for any qualification other than the degree of Doctor of Philosophy at the University of Oxford.

Abstract

Ureteroscopy is a procedure used to treat kidney stones, which consists of passing a flexible tool known as a ureteroscope through the urinary system to access the kidney. The scope is hollow along its length, creating a working channel through which working tools can be passed. Once a stone is located, a laser fibre is inserted down the working channel to deliver a high-power laser which reduces the kidney stone to dust. Throughout the procedure constant fluid irrigation is delivered through the scope working channel, which fills and dilates the kidney cavity, and then exits via the space in between the ureteroscope and ureter. Often an access sheath is inserted within the ureter to provide better dilation of the channel. The fluid washes the dust created during lasering out of the kidney, providing the clinician with a clear field of view of the kidney interior via a camera at the tip of the scope. Traditionally this fluid irrigation is provided by a hanging saline bag at a level above the patient, producing a constant, gravity driven flow. Instead, we consider the case where the fluid is driven via a peristaltic pump, providing the clinician with better control over the irrigation delivery. Due to the nature of how these pumps operate, previously unseen time dependence is introduced to the flow, causing it to oscillate in time. Understanding the prevalence of these oscillations throughout the system is key to determining how this new method differs from the traditional set-up, and what benefits the improved delivery method can bring.

This thesis is concerned with the mathematical modelling of oscillatory fluid irrigation throughout the ureteroscopy system, with an aim to determine the prevalence of the fluid oscillations throughout the system and how the time dependence can be harnessed to improve the procedure by decreasing dust wash-out times.

We begin by modelling the flow throughout the cylindrical geometries of the system, namely the tubing connecting the pump and scope, the scope working channel with working tool inserted, and the access sheath with scope inserted. The flow is driven by an upstream oscillatory pressure, with atmospheric pressure at the outlet of the access sheath, where at the junctions between each section we assume continuity of flux and pressure. The model is solved analytically, and it is found that the flow and pressure oscillations are significantly dampened by the time they reach the kidney cavity due to the small radii of the ureteroscope working channel and working tool. The assumptions made at the junctions between each section are then addressed by numerically simulating the flow within these more complicated geometries. Alongside this, we present a model of the peristaltic pump used to drive the fluid, and show

how the characteristics of the flow mean, frequency, and amplitude are related to pump rotations per minute. Both models are validated by comparing the theoretical predictions with experimental data sets and good agreement is shown.

Next we turn to the modelling of fluid flow within the kidney cavity, and the transport of kidney stone dust within said fluid. To represent the cavity, we consider an idealised, two-dimensional, rectangular domain, with one inlet and two outlets on the same side of the domain. A parabolic, oscillatory inlet flow introduces the time-dependence to the system. The problem is solved numerically via finite element formulation. Dust wash-out time is used as a metric, and the parameters of the inflow are varied with an objective to reduce the wash-out time. It is found that an inlet flow which oscillates with a low frequency and high amplitude aids in disturbing vortical structures within the flow, leading to a significantly reduced dust wash-out time due to the resulting mixing effect. We go on to compare the wash-out times under this oscillatory regime to a simulation of a typical industry method termed flushing, and find that this new method performs favourably. The robustness of these results is then examined by demonstrating the impact of changes to the cavity geometry on dust wash-out times, and qualitatively similar wash-out results are seen for all geometries.

Finally, the previous modelling efforts are combined together to create a model of the full ureteroscopy system. Fluid irrigation is modelled through tubing, to scope working channel, and into the kidney, before exiting via the access sheath, all driven by a flux which is determined from the pump model. In addition to this, we introduce the idea of an air dampener upstream of the ureteroscope to the model and consider the impact it has on the oscillatory flows downstream of the scope. The model shows that the air dampener effectively reduces the amplitude of the oscillations as intended. The predictions from the modelling are again validated by experimental data sets and good agreement is found by accounting for a scaling factor on the working tool resistance.

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1 Introduction

1.1 Motivation

Approximately one in every eleven people are affected by kidney stone disease (Scales Jr et al., 2012). The disease is typically more common in males, with around 12% of men and 5% of women developing a kidney stone at some point during their lifetime (Coe et al., 2005). Whilst a healthy lifestyle and balanced diet can reduce the chances of developing kidney stones (Semins et al., 2010; Heilberg and Goldfarb, 2013), genetic factors can also increase the likelihood of suffering from the disease (Howles and Thakker, 2020; Devuyst and Pirson, 2007). If stones are left untreated they can result in infection of the kidney, which may lead to reduced functionality and, in some extreme cases, death (Kum et al., 2016; Whitehurst et al., 2019). The incidence of kidney stones has been on the rise in Europe, North America, and other industrialised countries over the last 100 years (Trinchieri, 2006), and more recently, increases in the prevalence of the disease have been seen worldwide, with similar rates found in developed and developing countries alike (Daudon et al., 2004). This has resulted in an increasing number of patients needing hospital treatment, with over 86,000 stone disease related hospital episodes in the UK in 2015 (Turney et al., 2011; Heers and Turney, 2016). The prevalence of this disease puts a great burden on national health resources around the world, with the financial cost in the USA estimated to exceed five billion dollars by 2030 (Antonelli et al., 2014). Even once the disease is treated, the risk of recurrence is around 40% after 5 years, rising to 75% after 20 years (Worcester and Coe, 2010). Therefore, the need for efficient and effective treatment of kidney stones is more vital than ever.

Healthy kidneys are vital for removing waste from the human body by filtering it out of the blood (Stevens et al., 2006), via a network blood vessel within the tissue of the renal cortex and medulla. The waste is converted into urine which passes from the medulla into the cavity of the renal pelvis, before passing through the ureter towards the bladder, where it is periodically removed from the body (Elbadawi, 1996; Hickling et al., 2017). It is within the smaller calyces of the renal pelvis that kidney stones form (Figure 1). Kidney stones are caused by an imbalance in the concentration of crystal-forming substances within the urine. The composition of the stones varies, leading to many different classifications (Wesson and Ward, 2007; Pawar et al., 2018; Cui et al., 2018), but typically stones comprise of calcium oxalate ($\sim 65\%$), uric acid ($\sim 15\%$), and calcium phosphate ($\sim 10\%$), as well as smaller quantities of other substances (Goldfarb, 2012; Coe et al., 2016; Ma et al., 2018). Stones vary in size from a couple of millimetres up to multiple centimetres (Khan et al., 2016), however some extreme cases display stones measuring up to 10cm at its widest point (Gavande and Gavande, 2019). Whilst smaller stones are usually left to pass naturally out of the body in the urine, larger ones can either remain stuck within the kidney calyx or become trapped within the ureter (Figure 1(a)). When this happens, treatment is typically required.

There are multiple options currently available to treat kidney stones. Shock wave lithotripsy involves targeting the stones with shock waves sent from outside the body, causing them to fragment into smaller pieces which can then pass naturally in the urine (Lingeman et al., 2009). The procedure benefits from being non-invasive to the patient, making it a popular choice for treatment (Heers and Turney, 2016). However, it has been

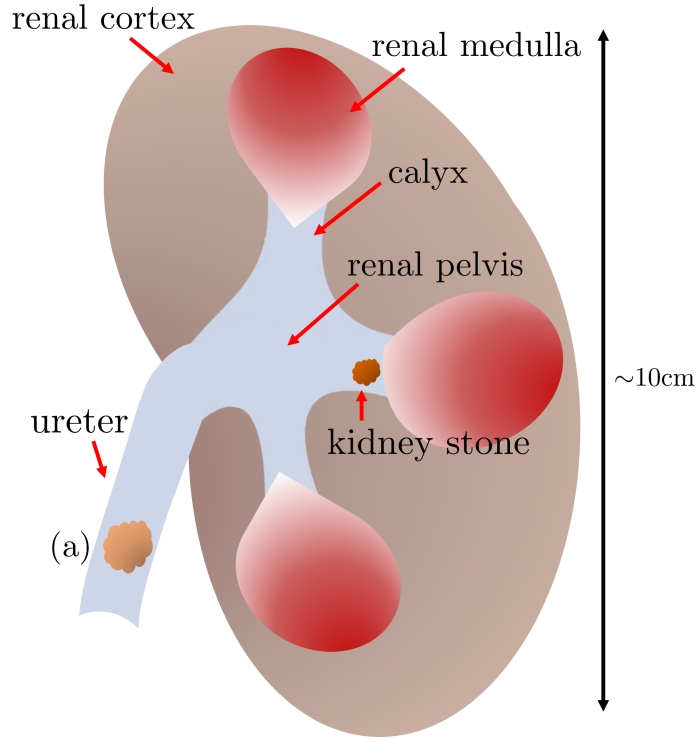


Figure 1: Diagram of a single kidney leading to ureter with a kidney stone located within one of the calyces. (a) A larger stone stuck within the ureter.

found that certain stone compositions are resistant to this treatment and cannot be broken up using shock waves (Zhong and Preminger, 1994). Furthermore, due to its imprecise nature, the treatment can lead to acute injury of the surrounding kidney tissue (Matlaga et al., 2008). A less preferred option for treatment is percutaneous nephrolithotomy, a procedure in which a small incision is made in the back of the patient through which the stones can be located and removed (Wickham and Kellet, 1981). The largest drawback of this procedure is that it requires surgery, leading to complications such as extravasation of blood into the surrounding tissue, transfusion, and fever occurring in around 83% of cases (Michel et al., 2007). It is thus reserved for when other methods have failed.

One approach which is quickly becoming the preferred method for treating kidney stones is ureteroscopy (Heers and Turney, 2016). The procedure involves passing a flexible medical instrument, known as a ureteroscope, through the urinary system to gain access to the kidney (Figure 2). The ureteroscope's hollow centre provides a working channel through which working tools such as baskets and laser fibres can be passed, thus providing access to the renal pelvis. Once a clinician has access, laser lithotripsy is performed to destroy the stone, in which the laser fibre delivers pulses of high-powered lasers at the stone to break it up via shockwaves. High frequency laser pulses are used to reduce the stone to dust, a process known as dusting, whilst lower frequency laser pulses cause the stone to fragment into larger pieces, which are then manually removed by grabbing them using a wire basket. These larger fragments are typically around 1-2mm in size, with no standard definition currently existing for the size of the dust, except that it is free to pass spontaneously in the flow (Doizi et al., 2018). Modern ureteroscopes are fitted with a light

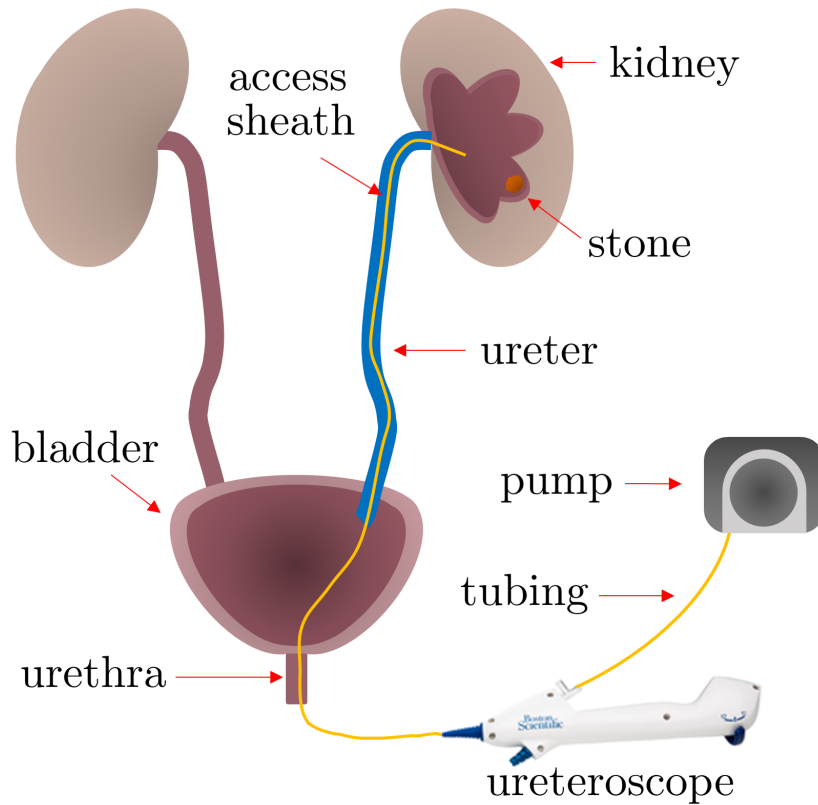


Figure 2: Diagram of the urinary system, showing a ureteroscope (shown in yellow) inserted through the urethra, bladder, and ureter, to reach the kidney, driven by an electronic pump.

source and a camera at their tip, allowing urologists to visualise the urinary system and locate the stones with greater ease (Butticè et al., 2016). Ureteroscopy benefits from the fact that it can precisely locate and remove stones from the kidney through a minimally invasive procedure without the need for surgery.

As well as being used to pass working tools into the kidney, the ureteroscope working channel allows saline irrigation fluid to be delivered into the renal pelvis, which serves to flush stone dust created during the laser treatment out of the patient, while also giving a clear field of view for the camera on the tip of the scope. Fluid flows from a saline reservoir to the ureteroscope through tubing which is connected to the working channel inlet, then down the length of the working channel before exiting out of the tip of the scope into the renal pelvis. The fluid then flows out of the kidney in the space between the outside of the scope and the ureter wall. Often an access sheath is inserted within the ureter, a cylindrical tube which sits tightly against the ureter wall, providing greater ureteral dilation (Kaplan et al., 2016) and creating a channel for the scope to pass through (Figure 3). This provides a larger, more rigid channel, making ureteroscope insertion and removal more simple, whilst also allowing the fluid to exit the body more easily (Kourambas et al., 2001). In addition, the sheath protects the ureter wall from possible abrasions when inserting and removing tools. The fluid irrigation also keeps the renal pelvis dilated, providing better scope manoeuvrability during the procedure.

Traditionally, fluid irrigation is achieved by hanging a saline bag at a height of around

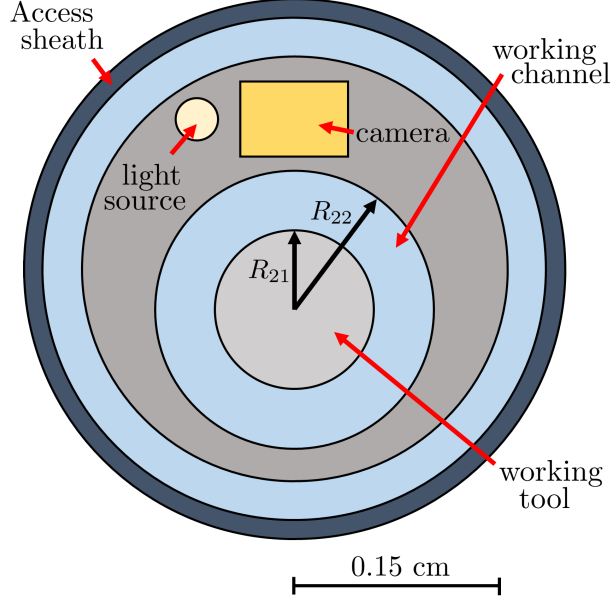


Figure 3: Cross section of the ureterscope within the access sheath, with a working tool inserted within the working channel. Position of the light and camera at the tip of the scope indicated. Irrigation fluid shown in light blue.

60cm above the ureterscope, connected to the working channel inlet via tubing. This creates a hydrostatic pressure at the inlet to the working channel, given by $\rho g H$ for fluid density ρ , gravitational acceleration g , and bag height H , thus providing a constant flow of irrigation fluid to the system. The irrigation fluid flow rate is set by the height of the bag, and can be increased or decreased by raising or lowering this height. Often during a procedure, if the field of view of the camera becomes obscured by a large amount of kidney stone dust, then a temporary increase of flow rate is needed. To achieve this, in a process known as flushing, the bag can be manually squeezed by the clinician or via a saline bag cuff to send an additional bolus of fluid through the system. Using a hanging saline bag to drive the fluid is somewhat crude as it does not allow for precise control over the amount of fluid flow delivered to the system. In an ideal world the relationship between bag height and flow rate would be linear, but it has been shown experimentally that in practice this is not observed (Williams, 2019), making it difficult to predict precise flow rates each time a procedure is performed. Additionally, the increase in flow rate provided by the flushing procedure is very imprecise due to the manual nature of applying additional pressure via ad hoc bag squeezing.

To combat these issues, recent developments in the field involve using an electronic pump to drive the irrigation fluid, allowing for greater control over the fluid flow. Using these pump allows for a system in which a flow rate can simply be set to a constant, or even time varying, value, removing the need for manual adjustment of bag height or squeezing. A common type of electronic pump used to drive fluids for biomedical applications, as well as many others (Klespitz and Kovács, 2014), are peristaltic pumps. Peristaltic pumps are beneficial since the self-contained fluid does not come into contact with exposed pump components, preventing possible contamination, and their design prevents backflow without the use of valves (Esser et al., 2019). These pumps comprise

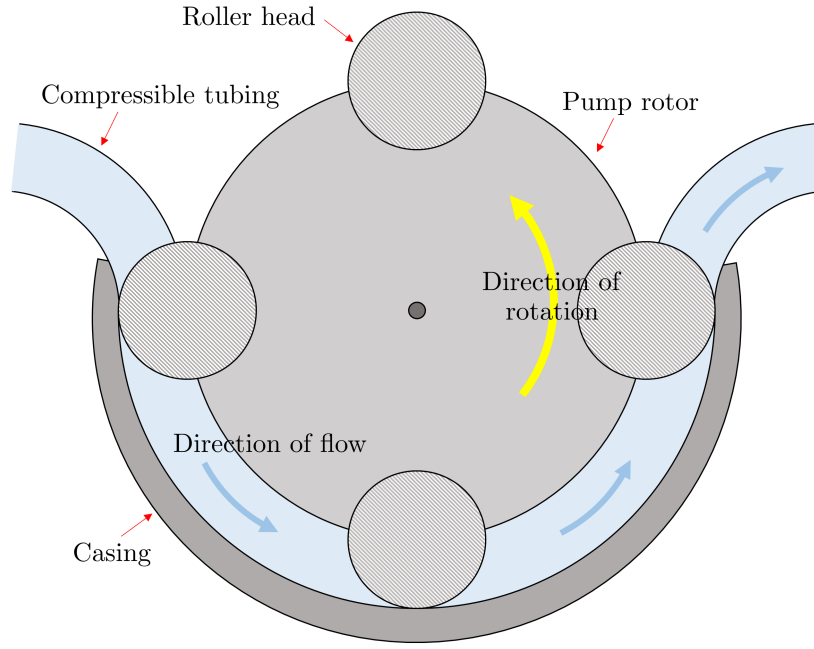


Figure 4: Diagram of a peristaltic pump with four roller heads compressing a section of flexible tubing containing fluid. The direction of rotation is given by the yellow arrow and the resulting direction of fluid flow is given by the blue arrows.

of a large central rotor with several small roller heads arranged around its circumference (Figure 4). Fluid is contained within compressible tubing that is passed through the outer casing of the pump, wrapping partially around the central rotor. As the pump rotor rotates around its centre point, the roller heads compress the tubing, causing a flux of fluid to move ahead of the roller. Typically, two or more roller heads are in contact with the tubing at any one time, causing a body of fluid to be trapped between them. When the leading roller head reaches the other side of the pump the compression is released, and the tubing returns to its original circumference, drawing more fluid into the tube. As the main rotor continues, this process is repeated, thus driving fluid flow (Berg and Dallas, 2015). We anticipate that the repetitive nature of these pumps will introduce natural oscillations into the flow (Latham, 1966).

Natural oscillations in the flow of irrigation fluid would result in a ureteroscopy procedure which differs from the traditional steady flow method. If these natural oscillations are undesired and a system of irrigation fluid which is akin to the traditional method is the objective, then one way to reduce them would be the inclusion an air dampener into the system. An air dampener consists of a chamber of air connected to the fluid network after the pump that acts to absorb oscillations in the fluid. However, introducing an air dampener into the system may hinder potentially desired oscillatory effects. The advancement of driving irrigation fluid with a peristaltic pump could provide additional, previously unexplored, benefits to ureteroscopy, by enabling the possibility for time-dependent flow throughout the system. It is possible that vortical fluid structures in the kidney cavity could trap dust and debris, leading to a less optimised procedure with increased duration and decreased clearance. Ideally, we would aim to disturb these fluid vortices by causing a large variation in the area, shape, and centre point of the vortical

structures over time. The introduction of oscillations in the flow may achieve this, thus disturbing the vortices and enhancing debris removal.

A successful ureteroscopy procedure is one in which the entirety of the kidney stone is removed, as quickly and safely as possible. Therefore, an ideal performance can be indicated by the maximum removal of kidney stone dust in as short a time as possible, whilst also ensuring the pressures inside the kidney are at a safe level, as high kidney pressures during ureteroscopy have been associated to post procedure infections such as sepsis (Wilson and Preminger, 1990).

1.2 Fluid irrigation system and ureteroscopy set-up

The components of the ureteroscopy fluid irrigation system are summarised as follows: (1) a peristaltic pump which generates the driving flow, (2) tubing which leads from the pump to the ureteroscope working channel, (3) the working channel, possibly with a working tool inserted within it, leading into the kidney, (4) the cavity of the kidney calyx, and (5) the exit channel provided by the access sheath with ureteroscope inserted within it. In addition to this, an air dampener may also be introduced to the fluid network at some point between the pump and ureterscope.

Mathematical modelling is vital to gaining a fundamental understanding of the physical processes at play in the fluid irrigation system. Modelling gives insight into how variables such as flow rate and pressure throughout the system are affected as different parameters of the system are varied, removing the need for multiple experiments. The theoretical approach also allows for the inspection of system aspects which may not be practically possible, such as varying the diffusivity of a passive tracer in the flow or changing the position of a working tool within the working channel.

From a fluid mechanics perspective, the key ingredients of the irrigation fluid system are sections of connected pipe flow leading into a cavity, the flow within said cavity, and outlet pipe flow leading to open atmosphere. Within the cavity, we have a diffusive tracer that advects with the flow representing the kidney stone dust which diffuses and is advected by the irrigation fluid. The driving flow is generated by the mechanics of the peristaltic pump. A chamber of pressurised air connected to the fluid network represents the air dampener. Modelling of the flow within each of these individual elements must be considered, as well as modelling the system as a whole. Where possible, such as within the long-thin pipe geometries, simplifications can be made to the fluid system to allow for reduced order modelling. For the more complicated aspects of the system, such as the flow and transport of a tracer within the kidney cavity, numerical techniques will need to be employed.

We proceed by examining the literature of experimental work and mathematical models for such systems, for both ureteroscopic applications and other relevant fields. We begin by discussing the surrounding literature of each individual element of the system, before considering them as a combined network.

1.3 Literature review

1.3.1 Irrigation fluid modelling

The saline irrigation fluid is well-modelled as an incompressible fluid with constant density ρ and dynamic viscosity μ . The velocity and pressure of this fluid is then described by the Navier-Stokes equations and incompressibility condition (Batchelor, 1967, p.147) given by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} , \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 . \quad (2)$$

respectively, where $\mathbf{u} = (u, v, w)$ is the fluid velocity in corresponding coordinate directions ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$), p is pressure, and t is time. The corresponding non-dimensional form of these equations yields the dimensionless Reynolds number $Re = \rho Ua/\mu$, giving the ratio between the inertial and viscous forces, where U is the typical fluid velocity and a is the typical lengthscale (Batchelor, 1967, p.211). We note that the gravitational term has been neglected from equation (1) as these forces are ignored for the purposes of ureteroscopy fluid irrigation due to the patient and fluid system lying horizontal. A ureteroscopy procedure typically has fluid velocities of around 1 cm/s (Williams et al., 2019). When considering flows within the kidney calyx which has a lengthscale of ~ 0.6 cm, this yields a Reynolds number of $O(10^2)$. Within the cylindrical geometries of the system, the long-thin nature of these geometries (radius $\sim 10^{-1}$ cm, length $\sim 10^2$ cm) is exploited and the key dimensionless parameter becomes the reduced Reynolds number, which is $O(10^{-1})$ in tubing, working channel, and access sheath alike.

1.3.2 Peristaltic pump modelling

The design of a peristaltic pump is comparable to the natural peristaltic pumping mechanics displayed in human physiology, with analogies often made between the mechanics of the pump and the contraction of muscular walls, where successive compression of the tubing caused by the rotating pump heads can be viewed as changes to the boundaries of the pipe. Latham (1966) considered an analytical model of peristaltic driving motion in the context of the urinary system, with the peristaltic motion caused by the successive contraction of the muscular walls of the ureter, used to naturally send urine from the kidneys to the bladder. The theory presented a model for two-dimensional channel flow and represented the peristaltic motion as changes in the boundaries of the pipe in the form of a sinusoidal wave. Through reduction of the Navier-Stokes equations, exploiting that the wavelength of the wall motion is long compared to the width of the channel, the authors made theoretical predictions for the fluid motion within the channel which showed good agreement with experimental results. A similar approach was taken by Jaffrin and Shapiro (1971), who developed analytical solutions for short wavelengths motions via the use of hyperbolic and Bessel functions (Bowman, 1958). More recent methods consider a more direct model of the pump mechanics, and as such rely on more complex numerical and computational techniques. In Elabbasi et al. (2011), the successive compression of flexible tubing containing fluid was modelled as a fluid-structure interaction problem via COMSOL Multiphysics (Multiphysics, 1998), and it was shown that the flow rates

displayed sharp changes in magnitude as the roller engaged and disengaged the tube. A similar modelling approach was taken by Formato et al. (2019) and it was indeed found that the flow displayed pulsatile behaviour.

1.3.3 Steady and oscillatory pipe flow

The cylindrical geometries present throughout the ureteroscopy system can each be modelled via laminar flow through a uniform pipe, leading to the classic Poiseuille equation (Batchelor, 1967, p.180). When considering a working tool within the working channel, an annular region is formed for the fluid to flow through, which leads to considerations of tool position and impact of resistance to the flow.

There is a broad background of experimental work which explores the impact working tools have on irrigation flow rates (Pasqui et al., 2004; Nagele et al., 2006; Bach et al., 2008; Ng et al., 2010; Bach et al., 2011; Kruck et al., 2011). The presence of a working tool within the working channel was shown to cause a significant decrease in flow rates, with this effect worsening for larger working tool diameters. This resulted in a decrease of stone dust clearance leading to the undesired effect of reduced visibility in the cavity (Bach et al., 2011). One such way of combating this loss of flow rate is by inserting an access sheath within the ureter, which has been shown to keep renal pressure low and increase irrigation fluid flow (Rehman et al., 2003). To gain a better understanding of the fluid mechanics at play, Williams et al. (2019) considered the mathematical modelling of ureteroscope irrigation for different pipe geometries, with a focus on working tool position and the effect this has on fluid flow rates and pressure. It was found that when a working tool is present in the working channel the flow rates are significantly decreased, due to the decreased cross sectional area offering more resistance to the flow. Furthermore, the position of the tool was found to have a large impact on flow rates, with tools placed at the edge of the channel allowing for higher flow rates compared to those placed centrally, since they provide less resistance to the flow. The work goes on to suggest some other potential geometries that the working channel could take, and for example, shows that modifying the cross sectional shape from a circle to an ellipse can increase flow rates. The theoretical modelling was supported by complementary bench-top experimental work and excellent agreement was found. Further considerations of working channel/working tool geometry and orientation were explored in Williams et al. (2020c). These results highlight the importance of working tool position in the working channel and the impact it has on flow resistance.

When considering oscillatory flow within the cylindrical geometries of the ureteroscopy system, we can again exploit the long-thin geometry of the system to reduce the Navier-Stokes equations and neglect the non-linear terms from the material derivative in equation (1), whilst keeping the $\partial \mathbf{u} / \partial t$ term. Insights into these types of oscillatory pipe flow can be gained from fundamental work by Womersley (1955), who identified the key dimensionless parameter called the Womersley number $\alpha = a\sqrt{\rho\omega/\mu}$, which gives the ratio of timescales between the pulsatile flow frequency ω and viscous effects. In the context of ureteroscopy, the frequency of the oscillatory flow produced by a peristaltic pump can be estimated by considering the number of roller heads n coming into contact with the tubing per second. Whilst designs and models of pump vary, most display a maximum rotations per minute (RPM) of around 400-600 with typically 2-6 rollers. This gives an estimate for upper limit

of the Womersley number to be $\alpha \sim O(10^1)$, with $\alpha \sim O(10^0)$ being a more reasonable choice for the rotation rates needed for the flow rates we anticipate in the system.

Experiments performed by Hino et al. (1976) looked at the transition of turbulent regimes in oscillatory pipe flow for increasing Reynolds numbers, ranging from 100 up to around 5000. They detected three distinct regimes; weakly turbulent, conditionally turbulent, and fully turbulent. It was found that the conditionally turbulent regimes begin to occur at Reynolds numbers of 850. This value is beyond what we would expect to see during ureteroscopy, suggesting laminar flows are appropriate for our modelling purposes. Further to this, Kurzweg et al. (1989) experimentally explored the onset of turbulence through varying the Womersley number. It was shown that for higher Womersley numbers the onset of turbulence is more likely. Again, since we are considering $\alpha \sim O(1)$ we are unlikely to see turbulent regimes in practice.

1.3.4 Fluid flow within the kidney cavity

It is within the kidney calyx where the flow becomes more complicated. From a fluid mechanics perspective, the cavity is described by one inlet and two outlets located on the same side, with pipe flow leading both into and out of the cavity, along with a diffusive dust that advects with the flow. A related but perhaps simpler system is pipe flow with a sudden expansion in the radius of the pipe, which provides one of the classic examples of separated flows. Durst et al. (1993) studied flows in such geometries, and via a numerical finite volume method, they found the flow was symmetric at low Reynolds number and that recirculation zones formed close to the expansion region. As the Reynolds number increased, the zones increased in size and asymmetry in the flow occurred. The flow patterns showed strong agreement with experimental results. A numerical study by Drikakis (1997) explored the asymmetry in the flow further, finding a critical Reynolds number at which a symmetry-breaking bifurcation occurs. Mizushima et al. (1996) showed that the addition of a downstream contraction after the expansion zone acts to re-stabilise the symmetry in the flow, whilst Mizushima and Takahashi (1999) introduced two outlets in the cavity to show that a symmetry-breaking bifurcation can still occur under these conditions.

More relevant to the present research are the papers of Williams et al. (2020a, 2021), which studied steady fluid flow and the advection and diffusion of a passive tracer, representing stone dust, within a two-dimensional representation of the renal cavity. The cavity was described by a rectangular domain with one inlet and two outlets on the same left-hand side, with a variable cavity length. A steady, fully-developed parabolic flow profile was prescribed at the inlet to the cavity, with zero-stress conditions at both outlets. Modelling the fluid via the steady incompressible Navier-Stokes equations, solutions for velocity and pressure in the cavity were obtained via a finite-element formulation using the solver of Farrell et al. (2019). Multiple steady solutions were found, where the solutions exhibited a change from a symmetrical to asymmetrical fluid structure as the inlet velocity was increased. These theoretical results were confirmed by particle image velocimetry (PIV) experiments where the flow displayed the same change in fluid structure. A conjecture was made on the stability of the solutions, and a bifurcation diagram as a function of the Reynolds number was posed. A critical Reynolds number was determined, where the symmetrical flow pattern changed from being stable to unstable, and

two new stable, asymmetrical flow patterns, each a reflection of the other, were seen. It was shown that for fixed Reynolds number a bifurcation in the solution could also occur when varying the cavity length.

1.3.5 Kidney stone dust transport

The studies on fluid flow within the kidney cavity above highlight the presence of recirculation zones, features that play a strong role when it comes to the transport of debris in the fluid. Efficient debris removal is important in many fields, such as in cleaning and decontamination. Min et al. (2020) considered mass transfer in a laminar flow downstream of a backward facing step, for varying Reynolds numbers and Schmidt numbers – the ratio of viscous to molecular diffusion rates. They found that mass accumulates in recirculation zones located near the step. The interplay between advective and diffusive effects was considered, and it was shown that an increase in Schmidt number, corresponding to smaller diffusivity of passive solute, strongly decreased the rate of contaminant removal. Within the biomedical field, accumulation of debris is often undesirable and so techniques to improve mass transfer are well studied. For example, Clavica et al. (2014) characterised the flow patterns in the neighbourhood of side holes in a stented ureter and found that recirculation zones often occur. Jiménez and Davies (2009) similarly modelled blood flow through stented arteries and found that the recirculation zones act as sites for the accumulation of inflammatory elements. Possible redesigns of the stents were shown to reduce recirculation zones and thus improve wash-out. Coppola and Caro (2009) conducted a numerical study of steady and unsteady flow in stented arteries. By prescribing a sinusoidal inlet flow, they found that flow pulsatility generated a sweeping motion in flow vortices. This resulted in a reduction in build-ups of oxygen near the vessel walls and lower wall shear stresses.

Returning to the papers by Williams et al. (2020a, 2021) discussed previously, the transport of a passive tracer in the cavity flow was considered. A cloud of stone dust was introduced into the flow by considering an initial, uniform amount of tracer modelled as a circle of fixed radius, and its position within the cavity was varied for both flow solutions. An objective metric was defined for the time it takes 90% of the tracer to exit the cavity, referred to as T_{90} . Vortical structures seen in the asymmetrical fluid solution were shown to have a negative impact on debris clear out, as dust relied on slow diffusive processes to exit the vortices. Williams et al. (2021) extended this work by allowing the shape of the inlet channel to vary, and found an optimal shape for the rapid wash-out of the passive tracer. Following work from Jeong and Hussain (1995), a metric for the vortical structures in the flow was used as the optimisation object, where vortices were characterised by regions where $\det \nabla \mathbf{u} > 0$.

1.3.6 Urinary and ureteroscopy network system modelling

We have seen that modelling the individual elements of the ureteroscopy system yields complex and varied challenges, but combining each of these individual parts into a full model of the fluid irrigation system is also an involved task. One such way to view the full system is as a network of nodes and edges, with each node representing a pressure at a certain position in the system and each edge representing the flux of fluid travelling between them, where the resistance to the flow in each channel can be thought of as the

weight which that edge holds; channels with a lower resistance offer an easier route for the flux of fluid. Such an imagining is analogous to the modelling of electronic circuits (Abdolrazaghi et al., 2010; Oh et al., 2012). These techniques have been applied in considerations of flow through porous media (Blunt, 2001), where the individual pores make up a network of channels, with common examples being the modelling of oil through porous rock networks (Ryazanov et al., 2009; Yang et al., 2019) and the modelling of filtration systems (Sharma and Yortsos, 1987; Lee and Koplik, 2001; Griffiths et al., 2014). Literature surrounding the flow of blood through the vasculature networks of the body also takes advantage of such techniques (Sherwin et al., 2003; Pozrikidis and Farrow, 2003).

When considering urinary applications, we first turn to literature on modelling the urinary system without a ureteroscope device present. One such application is the modelling of urinary stents – small, flexible tubes that are inserted into the ureter to provide greater dilation and drain the urinary tract (Haleblian et al., 2008). Stents are used when there is a need to restore flow to the ureter due to blockages caused by stones, allowing flow to go around the blockage through holes in the stent wall. They are often inserted after procedures such as ureteroscopy and shock wave lithotripsy to aid in preventing ureteral stricture and facilitating stone fragment passage (Auge and Preminger, 2002). Mathematical models of flow through a stented ureter typically use lubrication theory to take advantage of the long-thin aspect ratio of the geometry and reduce the governing equations for flow (Lykoudis and Roos, 1970; Cummings et al., 2004). The rigidity of the urinary system is explored in Waters et al. (2008), where the ureter wall was modelled as an elastic membrane which expands in the radial direction, along with a renal pelvis modelled as an elastic bag with constant bulk stiffness. Similar fluid-structure interaction models are seen throughout the literature concerning the modelling of physiological processes, such as blood flow within vessels or vocal chord oscillations during phonation, since fluid flow within cylindrical, flexible geometries is common within the fluid carrying vessels throughout the human body (Grotberg and Jensen, 2004; Heil and Hazel, 2011). Computational methods have also been used to model stented ureters, removing the need to make reductions to the fluid mechanics and allowing for more complex flows within more complex geometries to be considered (Tong et al., 2007; Carugo et al., 2014). A comprehensive review of the modelling of the kidney and ureter is given by Zheng et al. (2021).

For a ureteroscopy system that more closely resembles the problem we consider, we turn to work by Williams et al. (2020b), which considered flow through the scope working channel, into the kidney, and back through an access sheath with scope inserted, with flow provided by a hanging saline bag. Fluid was driven by a prescribed steady inlet pressure and flux was conserved in the scope and sheath, with the kidney modelled as a compliant body, resulting in a lump-parameter model for the system. The solutions for unidirectional flow within the pipes were presented in terms of the viscous resistance to the cylindrical and annular geometries. The scope was assumed to be coaxial but not necessarily concentric to the working channel by employing a scaling factor on the flow resistance, based on an analytical solution for flow through an offset annular domain (Heyda, 1959). The difference between a centred and fully offset scope (i.e. up against the sheath wall) gave an upper and lower bound for resistance to the flow. Building upon a linear consideration by Oratis et al. (2018), this work employed an exponential

law for kidney compliance, which was shown to be a better prediction compared to the linear method via the use of experimental data. Ex-vivo, bench-top experiments were performed using porcine kidneys, and excellent agreement between the model and data was found for the flow rates of the system. Some disagreement was seen for predictions of kidney pressure, which was found to be likely due to spatial variations in kidney pressure not captured by the lumped model, supported by numerical cavity flow simulations in a simplified kidney geometry.

1.3.7 Air dampener modelling

An air dampener may be included into such networks as an equation describing one of the nodal pressures where the air chamber is located. By modelling the air as an ideal gas, the pressure P and volume of air V in the dampener satisfies the ideal gas law given by $PV = nRT$ where nRT is a constant for moles of gas n , ideal gas constant R , and temperature T . Such an approach was taken by Deng et al. (2012), where the incoming flow rate fluctuations are expressed by a sinusoidal wave in time. The temporal evolution of the pressure and air volume in the dampener was shown to reach a steady state after an initial period of adjustment from the initial conditions. The variables then fluctuate around this constant value in a sinusoidal form. The theoretical model was validated with experimental results of pressure and flow rate downstream of the dampener, and a significant reduction in oscillations was found. Including such a model into the wider fluid system introduces a time-dependence to the flow as the air in the dampener adjusts to changes in upstream flux. The system no longer responds instantaneously to changes, introducing a delay in reactions to changes throughout the fluid system.

1.4 Thesis aims

The technological advancement of using an electronic pump to deliver irrigation fluid to the ureteroscopy system opens up a gap in the literature when it comes to considering the modelling of unsteady fluid flows under this application. In this thesis, we aim to build upon the current literature by changing the irrigation fluid delivery method from the traditional hanging saline bag to an electronic pump, and thus consider the unsteady fluid system. We focus on oscillatory flows motivated by the type of motion the peristaltic pumps produce.

We aim to build a theoretical model which describes the fluid irrigation system present for ureteroscopy. The system is characterised by cylindrical and annular pipe geometries, created by the tubing, the working tool within the working channel, and the scope within the access sheath, as well as a kidney cavity which offers up more complicated fluid dynamics. We will model time-dependent flows to mathematically describe how the fluid velocity and pressure throughout the system depend upon the system geometries and driving forces. The focus will be on oscillatory flows, and the magnitude, amplitude, and frequency of these oscillations will be explored with an aim to determine their prevalence throughout the system. A model of the peristaltic pump which generates these flows will give insight into how the fluid delivery can be controlled. A more detailed look at the complicated fluid flows within the kidney cavity will be considered, and we will determine how these flows impact the transport and clearance of kidney stone dust throughout the

cavity. The oscillation parameters will be varied to determine whether dust clearance time can be decreased. We aim to combine all these elements together and create a model of the full ureteroscopy system which can be used to predict downstream factors such as kidney pressure and dust wash-out time based on the upstream fluid delivery coming from the peristaltic pump parameters. Where possible, the models will be compared against experiments to validate their real world applications. Through this modelling, we seek to answer two key questions described as follows.

Firstly, does the new method of irrigation fluid delivery differ significantly from the traditional hanging saline bag set-up? We anticipate that the peristaltic pump will introduce oscillations into the fluid, and so aim to determine the prevalence of these oscillations by exploring how the amplitude of the pressures and fluxes vary throughout the system. We are most interested in these amplitudes in the key area downstream of the scope in the kidney cavity, where the dusting and removal of the kidney stones occurs. Fluid irrigation here is vital for removing debris, and so large variations in the fluid flux or pressure may result in dust removal that cannot be carefully controlled or in pressures that cannot be maintained at a safe patient level. A system in which the oscillations are significantly dampened is akin to the traditional set-up, and so we determine which parameters of the system have the greatest impact on reducing oscillations amplitudes to more closely reflect the traditional steady system. We consider how an air dampener can be included into the system to reduce natural oscillations created by the pump, and explore the impact it has on the rest of the system.

Secondly, we ask whether the new irrigation fluid delivery method can be used to improve the effectiveness and efficiency of the ureteroscopy procedure. The new delivery method gives much finer control over the delivery of irrigation fluid, with the ability for the flow to vary over time. We have seen in the previous literature how kidney stone dust can become trapped within fluid vortices in the kidney cavity (Williams, 2019). Motivated by this, we aim to determine whether the time it takes for irrigation fluid to wash dust out of the cavity can be shortened by exploiting oscillatory flows. We anticipate that by introducing oscillations in the cavity flow, we can disturb clouds of dust which have accumulated in fluid vortices, as has been seen in the literature (Coppola and Caro, 2009). Additionally, we must ensure that the oscillations the upstream pump is imposing into the fluid are still significant downstream of the scope, and so we require a full model of the ureteroscopy fluid irrigation system. We consider whether the air dampener may hinder the potential upsides of time-dependent flows by determining the impact it has on dampening the imposed oscillations.

1.5 Thesis structure

In Chapter 2, we begin by presenting a mathematical model for the flow of irrigation fluid through the cylindrical geometries of the ureteroscopy system, namely the tubing, ureteroscope working channel with working tool inserted, and access sheath with ureteroscope inserted. These regions are connected to one another by assuming continuity of flux and pressure at the junctions between each region. The flow within the more complicated kidney calyx is not considered in this Chapter, where instead we match pressures at the scope tip and sheath entrance where the cavity is located. The fluid is driven by an inlet pressure which oscillates in time, representing the peristaltic pump driving the fluid. We

determine the prevalence of pressure and flux oscillations throughout the system, and determine which parameters of the system have the greatest impact on dampening them. We test the validity of the continuity assumptions made at the junctions between each region by computing numerical solutions of the flow at said junctions. Alongside this, we present a model of a peristaltic pump to provide insight into how the upstream oscillatory flows are generated, and how this flow depends on the features of the pump such as the number of roller heads and rotation rate. We combine the pump model and cylindrical geometry model together and compare the theory against experimental results for varying pump rotation rates.

Having previously neglected the kidney cavity, in Chapter 3, we explore cavity flow and the associated dust transport. We consider a single calyx of the kidney and model this cavity as a two-dimensional rectangular domain. We impose a sinusoidal parabolic flow profile at the cavity inlet, and model kidney stone dust as a passive tracer transported within and eventually out of the cavity by the combined effects of diffusion and advection. The model is solved numerically via a finite element formulation approach. We characterise the presence and size of fluid vortices, their disturbance due to the oscillatory component of the inlet flow profile, and the impact this disturbance has on debris wash-out. We determine how the flow and transport properties depend on the inlet flow mean value, amplitude, and frequency, as well as the diffusivity of the tracer. The results of the oscillatory cases are compared to a simulation of a typical industry method termed flushing, in which there is an instantaneous jump in flow rate from one steady value to another for a brief period of time. We go on to test the robustness of our model by comparing the wash-out times in the idealised geometry against varying two-dimensional cavity shapes.

Motivated by the modelling efforts of Chapters 2 and 3, in Chapter 4 we aim to combine all elements together into a full ureteroscopy fluid irrigation system model. The system is characterised by a series of pressure-flux relations drawn from the cylindrical geometries throughout the system, together with a pressure-flux relation for the cavity extracted from the cavity simulations. Additionally, we model the inclusion of an air dampener into the fluid network which serves to reduce natural oscillations created by the peristaltic pump. We exploit a small parameter, representing the ratio between the time it takes for the dampener to fill due to the initial flux of fluid and the frequency of oscillations, and decompose the system into its quasi-steady and oscillatory parts. Analysing each of these parts separately, we first solve the quasi-steady system, and determine the time it takes the system to adjust to steady state from its initial conditions, as well as how it adjusts as a working tool is inserted and removed from the working channel. Parameters of the system are varied to explore how the adjustment times are impacted. These quasi-steady results are supported by experimental data of tool insertion and removal. We then go on to solve the oscillatory system, where we analyse the impact the air dampener has on reducing oscillations created by the pump.

In Chapter 5, we summarise the content of each thesis chapter, recap the conclusions drawn, and suggest possible avenues for future research.

2 Systems model of oscillatory fluid flows throughout the ureteroscopy system

2.1 Introduction

We begin by considering oscillatory fluid flow throughout the cylindrical geometries of the ureteroscopy system. The path the fluid takes through the system is as follows; an upstream peristaltic pump draws fluid from a reservoir, which is then driven through a section of tubing to a ureteroscope which is inserted into the patient. The fluid travels down the length of the ureteroscope working channel before reaching the kidney. It exits the kidney via the access sheath, where it travels down the length of the sheath and exits into the atmosphere. The working channel of the ureteroscope may also have a working tool inserted within it during the procedure.

The pressures within the kidney must be maintained at a safe level during the procedure, as increasing the kidney pressure may cause damage or distress to the patient (Zhong et al., 2015), and so the upstream pump which drives the irrigation fluid must be carefully regulated. We will determine how the kidney pressure depends on the upstream driving pressure, various system radii, and other parameters of the system in order to ascertain which of these parameters has the largest impact on kidney pressure. This understanding will then enable us to say how the pressure imposed by the pump should be set to ensure safe levels of kidney pressure.

The use of a peristaltic pump to drive irrigation fluid through the ureteroscopy system introduces time-dependent oscillations into the flow. Whilst a steady flow and the associated constant system pressures are easy to regulate, a flow which oscillates in time will cause time-dependent variations in the pressures throughout the system. However, oscillations within the flow could also provide benefits. As we have seen from previous literature, oscillating flows lead to the disturbance of vortical structure within the flow (Coppola and Caro, 2009), which could be beneficial for the removal of kidney stone debris from the kidney cavity. Therefore, we will establish the prevalence of the oscillations throughout the system and determine how the magnitude, amplitude, and frequency of these oscillations are impacted by the parameters of the system. The insight gained from this information will lead to greater control over the oscillations in the flow in areas where it is important, such as within the kidney cavity.

The cylindrical sections are connected via complicated geometries, such as within the renal pelvis, the flow through which would be equally complicated. As we are focusing on the pressures and velocities throughout the system as a whole, in this chapter we neglect the modelling of flow within these junctions, before returning to them in Chapter 3. Due to the long, thin nature of the geometries we are considering, we assume unidirectional, axisymmetric flow and at the junctions between these section we assume continuity of pressures and fluxes. We test the validity of this assumption later through numerical simulations of steady fluid flow through said junctions, where these assumptions are anticipated to break down.

2.1.1 Chapter outline

We consider fluid flow throughout the cylindrical geometries of the ureteroscopy system, namely the tubing, ureteroscope working channel, and access sheath, each referred to as a region of the system. The flow is driven by an upstream oscillatory pressure, with atmospheric pressure at the outlet of the access sheath. At the junctions between each region we assume continuity of flux and pressure. The model is outlined in Section 2.2 with the theoretical formulation given in Section 2.3. We solve the model analytically to determine the pressure and fluid velocity throughout the system in Section 2.4, and in Section 2.5, we explore how varying the model parameters impacts the flow. In Section 2.6, we address the assumptions made in the model by considering the flow characteristics at the junctions between regions. In Section 2.7, we present a model of a peristaltic pump and determine how the magnitude, amplitude, and frequency of the oscillatory flow rate of the system are related to the rotation rate and geometry of the pump. Finally, in Section 2.8, the predictions of the combined system model and pump model are compared against experimental data, and good agreement between the two is found.

2.2 Problem description

We consider the flow in three distinct regions representing the tubing, the ureteroscope working channel with working tool inserted, and the access sheath with ureteroscope inserted respectively. Region one is modelled as a rigid cylinder with circular cross section of radius R_1 and length L_1 (Figure 5). Regions two and three are both modelled as annular regions formed from two concentric rigid cylinders with circular cross section, with fluid flowing between the inner and outer cylinder. In region two, we define R_{21} as the radius of the inner cylinder, R_{22} as the radius of the outer cylinder, and L_2 as the length of both cylinders. In region three, we define R_{31} as the radius of the inner cylinder, R_{32} as the radius of the outer cylinder, and L_3 as the length of both cylinders. We define $P_0^*(t)$ as the imposed upstream pressure, $P_1^*(t)$ as the pressure at the junction between regions one and region two, $P_2^*(t)$ as the pressure at the junction between regions two and three, and $P_3^*(t)$ as the downstream pressure. The kidney pressure is represented by $P_2^*(t)$, the pressure at the junction between regions two and three. We assume continuity of both fluid flux and pressure at each junction and assume no-slip of fluid at all rigid boundaries. To mimic the oscillations produced by the pump, we assume the upstream pressure $P_0^*(t)$ oscillates sinusoidally in time with mean A_0^* , frequency ω^* , and amplitude B_0^* . The fluid is modelled as a viscous, incompressible, Newtonian fluid of density ρ^* and dynamic viscosity μ^* .

2.3 Model set-up

We consider a cylindrical polar coordinate system (r^*, θ, z^*) with corresponding coordinate directions $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$. We assume a unidirectional, axisymmetric flow such that the fluid velocity is $\mathbf{u}_j^* = (0, 0, w_j^*)$ and the pressure is $p_j^* = p_j^*(r^*, z^*, t^*)$, with $w_j^* = w_j^*(r^*, z^*, t^*)$ denoting fluid velocity in the z -direction, for $j = 1, 2, 3$ in each region

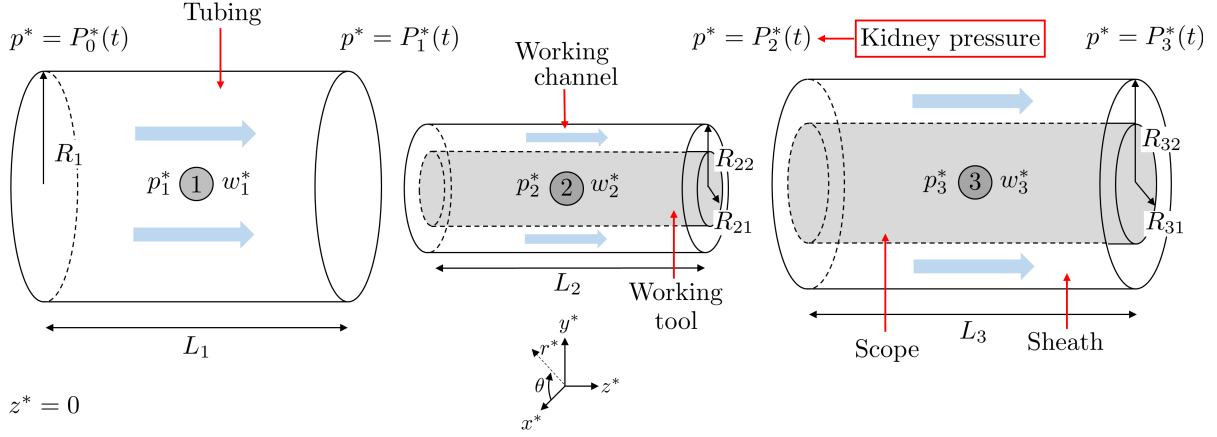


Figure 5: Model sketch of the three cylindrical geometries, region one representing tubing, region two representing the working channel with working tool inserted, and region three representing access sheath with scope inserted. The fluid flows through the regions marked by blue arrows.

respectively, where t^* denotes time. The continuity and Navier-Stokes equations are then

$$\frac{\partial w_j^*}{\partial z^*} = 0, \quad (3)$$

$$0 = -\frac{\partial p_j^*}{\partial r^*}, \quad (4)$$

$$\rho^* \left(\frac{\partial w_j^*}{\partial t^*} + w_j^* \frac{\partial w_j^*}{\partial z^*} \right) = -\frac{\partial p_j^*}{\partial z^*} + \mu^* \left(\frac{\partial^2 w_j^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial w_j^*}{\partial r^*} + \frac{\partial^2 w_j^*}{\partial z^{*2}} \right). \quad (5)$$

for $j = 1, 2, 3$. From equation (3), w_j^* is independent of z^* , and from equation (4), we find that $p_j^* = p_j^*(z^*, t^*)$. Equation (5) then simplifies to

$$\mu^* \left(\frac{\partial^2 w_j^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial w_j^*}{\partial r^*} \right) - \rho^* \frac{\partial w_j^*}{\partial t^*} = \frac{\partial p_j^*}{\partial z^*}, \quad (6)$$

for $j = 1, 2, 3$. Since the LHS of equation (6) is independent of z^* and the RHS is independent of r^* , each side is a function of time only.

To solve for the velocity and pressure in each region, we impose pressure conditions at the upstream and downstream ends, no-slip conditions at the region boundaries, and a symmetry condition at the centreline of region 1. This gives conditions in region 1 of

$$w_1^*(R_1, t^*) = 0, \quad (7)$$

$$\frac{\partial w_1^*}{\partial r^*}(0, t^*) = 0, \quad (8)$$

$$p_1^*(0, t^*) = P_0^*(t^*) = A_0^* + B_0^* e^{i\omega^* t^*}, \quad (9)$$

$$p_1^*(L_1, t^*) = P_1^*(t^*), \quad (10)$$

in region 2 of

$$w_2^*(R_{21}, t^*) = 0 , \quad (11)$$

$$w_2^*(R_{22}, t^*) = 0 , \quad (12)$$

$$p_2^*(L_1, t^*) = P_1^*(t^*) , \quad (13)$$

$$p_2^*(L_1 + L_2, t^*) = P_2^*(t^*) , \quad (14)$$

and in region 3 of

$$w_3^*(R_{31}, t^*) = 0 , \quad (15)$$

$$w_3^*(R_{32}, t^*) = 0 , \quad (16)$$

$$p_3^*(L_1 + L_2, t^*) = P_2^*(t^*) , \quad (17)$$

$$p_3^*(L_1 + L_2 + L_3, t^*) = 0 , \quad (18)$$

where $P_1^*(t^*)$ and $P_2^*(t^*)$ are unknown functions to be found. Equation (9) has real part understood. To close the system, we impose continuity of flux at each junction to give

$$Q_1^* = Q_2^* \text{ on } z^* = L_1 , \quad (19)$$

$$Q_2^* = Q_3^* \text{ on } z^* = L_1 + L_2 , \quad (20)$$

where

$$Q_1^* = 2\pi \int_0^{R_1} w_1^* r^* dr^* , \quad (21)$$

$$Q_2^* = 2\pi \int_{R_{21}}^{R_{22}} w_2^* r^* dr^* , \quad (22)$$

$$Q_3^* = 2\pi \int_{R_{31}}^{R_{32}} w_3^* r^* dr^* . \quad (23)$$

2.3.1 System decomposition

Motivated by the form of the imposed upstream pressure, and utilising the linearity of equation (6), we consider an ansatz of

$$w_j^*(r^*, t^*) = \bar{w}_j^*(r^*) + \hat{w}_j^*(r^*) e^{i\omega^* t^*} , \quad (24)$$

$$p_j^*(z^*, t^*) = \bar{p}_j^*(z^*) + \hat{p}_j^*(z^*) e^{i\omega^* t^*} , \quad (25)$$

$$Q_j^*(t^*) = \bar{Q}_j^* + \hat{Q}_j^* e^{i\omega^* t^*} , \quad (26)$$

for $j = 1, 2, 3$, real part understood, and

$$P_j^*(t^*) = A_j^* + B_j^* e^{i\omega^* t^*} , \quad (27)$$

for $j = 1, 2$, where we have decomposed the depend variables into a time-independent and time-dependent part represented by bars and hats respectively. Substituting equations (24) to (27) into equation (6) gives the time-independent system as

$$\mu^* \left(\frac{d^2 \bar{w}_j^*}{dr^{*2}} + \frac{1}{r^*} \frac{d\bar{w}_j^*}{dr^*} \right) = \frac{d\bar{p}_j^*}{dz^*} , \quad (28)$$

and the time-dependent system as

$$\mu^* \left(\frac{d^2 \hat{w}_j^*}{dr^{*2}} + \frac{1}{r^*} \frac{d \hat{w}_j^*}{dr^*} \right) - \rho^* i \omega^* \hat{w}_j^* = \frac{d \hat{p}_j^*}{dz^*} , \quad (29)$$

for $j = 1, 2, 3$, with appropriate boundary conditions for each region. We see that the time-independent and time-dependent problems are independent and thus we opt to treat them separately.

2.3.2 Dimensionless governing equations and boundary conditions

We non-dimensionalise as follows; the geometry variables of the system are scaled on the radius and length of region one, such that

$$r^* = \delta L_1 r , \quad z^* = L_1 z , \quad (30)$$

where $\delta = R_1/L_1$. The time-independent and time-dependent variables are scaled as follows

$$\bar{w}_j^* = \frac{\delta^2 L_1 A_0^*}{\mu^*} \bar{w}_j , \quad \bar{p}_j^* = A_0^* \bar{p}_j , \quad \bar{Q}_j^* = \frac{\delta^4 L_1^3 A_0^*}{\mu^*} \bar{Q}_j , \quad (31)$$

$$\hat{w}_j^* = \omega^* L_1 \hat{w}_j , \quad \hat{p}_j^* = \frac{\mu^* \omega^*}{\delta^2} \hat{p}_j , \quad \hat{Q}_j^* = \delta^2 \omega^* L_1^3 \hat{Q}_j , \quad (32)$$

for $j = 1, 2, 3$. We have scaled the time-independent pressure on the upstream mean pressure, resulting in the associated time-independent velocity scale. The time-dependent pressure is scaled on a viscous pressure scale resulting from the velocity scale set by frequency of the oscillations. Finally time is non-dimensionalised on the oscillatory timescale, such that

$$t^* = \frac{1}{\omega^*} t . \quad (33)$$

Applying these scalings to equation (28) gives

$$\frac{d^2 \bar{w}_j}{dr^2} + \frac{1}{r} \frac{d \bar{w}_j}{dr} = \frac{d \bar{p}_j}{dz} , \quad (34)$$

for $j = 1, 2, 3$, with boundary conditions in region 1 given by

$$\bar{w}_1(1) = 0 , \quad \frac{d \bar{w}_1}{dr}(0) = 0 , \quad \bar{p}_1(0) = 1 , \quad \bar{p}_1(1) = A_1 , \quad (35)$$

where $A_1 = A_1^*/A_0^*$, in region 2 given by

$$\bar{w}_2(r_1) = 0 , \quad \bar{w}_2(r_2) = 0 , \quad \bar{p}_2(1) = A_1 , \quad \bar{p}_2(1 + l_1) = A_2 , \quad (36)$$

where $r_1 = R_{21}/R_1$, $r_2 = R_{22}/R_1$, $l_1 = L_2/L_1$, and $A_2 = A_2^*/A_0^*$, and in region 3 given by

$$\bar{w}_3(r_3) = 0 , \quad \bar{w}_3(r_4) = 0 , \quad \bar{p}_3(1 + l_1) = A_2 , \quad \bar{p}_3(1 + l_1 + l_2) = A_3 = 0 , \quad (37)$$

where $r_3 = R_{31}/R_1$, $r_4 = R_{32}/R_1$, $l_2 = L_3/L_1$, and $A_3 = A_3^*/A_0^*$. Finally, the flux matching conditions are given by

$$\bar{Q}_1 = \bar{Q}_2 \quad \text{on } z = 1 , \quad (38)$$

$$\bar{Q}_2 = \bar{Q}_3 \quad \text{on } z = 1 + l_1 . \quad (39)$$

where

$$\bar{Q}_1 = 2\pi \int_0^1 \bar{w}_1 r \, dr , \quad (40)$$

$$\bar{Q}_2 = 2\pi \int_{r_1}^{r_2} \bar{w}_2 r \, dr , \quad (41)$$

$$\bar{Q}_3 = 2\pi \int_{r_3}^{r_4} \bar{w}_3 r \, dr . \quad (42)$$

The dimensionless form of equation (29) is

$$\frac{d^2 \hat{w}_j}{dr^2} + \frac{1}{r} \frac{d\hat{w}_j}{dr} - i\alpha^2 \hat{w}_j = \frac{d\hat{p}_j}{dz} , \quad (43)$$

for $j = 1, 2, 3$, where $\alpha = R_1 \sqrt{\rho^* \omega^* / \mu^*}$ is the Womersley number and α^2 is the ratio of the timescale for the diffusion of vorticity relative to the oscillation timescale. The boundary conditions in region 1 are

$$\hat{w}_1(1) = 0 , \quad \frac{d\hat{w}_1}{dr}(0) = 0 , \quad \hat{p}_1(0) = B_0 , \quad \hat{p}_1(1) = B_1 , \quad (44)$$

where $B_0 = \delta^2 B_0^* / \mu^* \omega^*$ and $B_1 = \delta^2 B_1^* / \mu^* \omega^*$, in region 2 are

$$\hat{w}_2(r_1) = 0 , \quad \hat{w}_2(r_2) = 0 , \quad \hat{p}_2(1) = B_1 , \quad \hat{p}_2(1 + l_1) = B_2 , \quad (45)$$

where $B_2 = \delta^2 B_2^* / \mu^* \omega^*$, and in region 3 are

$$\hat{w}_3(r_3) = 0 , \quad \hat{w}_3(r_4) = 0 , \quad \hat{p}_3(1 + l_1) = B_2 , \quad \hat{p}_3(1 + l_1 + l_2) = B_3 = 0 , \quad (46)$$

where $B_3 = \delta^2 B_3^* / \mu^* \omega^*$. Finally, we have the flux matching conditions for the time-dependent system, given by

$$\hat{Q}_1 = \hat{Q}_2 \quad \text{on } z = 1 , \quad (47)$$

$$\hat{Q}_2 = \hat{Q}_3 \quad \text{on } z = 1 + l_1 . \quad (48)$$

where

$$\hat{Q}_1 = 2\pi \int_0^1 \hat{w}_1 r \, dr , \quad (49)$$

$$\hat{Q}_2 = 2\pi \int_{r_1}^{r_2} \hat{w}_2 r \, dr , \quad (50)$$

$$\hat{Q}_3 = 2\pi \int_{r_3}^{r_4} \hat{w}_3 r \, dr . \quad (51)$$

2.4 Fluid solution

Beginning with the time-independent system, the pressure gradient is constant, and is equal to the difference between the upstream and downstream time-independent pressures divided by the length of the region, giving

$$\frac{d\bar{p}_j}{dz} = \frac{L_1(A_j - A_{j-1})}{L_j}, \quad (52)$$

for $j = 1, 2, 3$, noting $A_0 = 1$ and $A_3 = 0$. We then solve equation (34), subject to the no-slip boundary conditions and symmetry condition from equations (35)-(37), for each of the three regions, giving the solution for the velocity in each region as follows

$$\bar{w}_1 = \frac{1}{4}(A_1 - 1)(r^2 - 1), \quad (53)$$

$$\bar{w}_2 = \frac{(A_2 - A_1)((r_2^2 - r_1^2)\ln(r) + (r - r_2)(r + r_2)\ln(r_1) + (r_1^2 - r^2)\ln(r_2))}{4l_1(\ln(r_1) - \ln(r_2))}, \quad (54)$$

$$\bar{w}_3 = \frac{-A_2((r_3^2 - r_4^2)\ln(r) + (r - r_4)(r + r_4)\ln(r_3) + (r_3^2 - r^2)\ln(r_4))}{4l_2(\ln(r_3) - \ln(r_4))}, \quad (55)$$

respectively, with A_1 and A_2 to be determine. To do this, the time-independent fluxes $\bar{Q}_1$, $\bar{Q}_2$, and $\bar{Q}_3$ are found by inserting equations (53)-(55) into the time-independent flux definition from equations (40)-(42). The flux matching conditions from equations (38) and (39) are then applied, and the values of A_1 and A_2 are then found by solving the resulting system. Integrating equation (52) with respect to z and applying the pressure boundary conditions from equations (35)-(37) then gives the time-independent pressure in each region.

We now consider the time-dependent system, with a solution method following similarly to before. The pressure gradient is given by

$$\frac{d\hat{p}_j}{dz} = \frac{L_1(B_j - B_{j-1})}{L_j}, \quad (56)$$

for $j = 1, 2, 3$, noting $B_3 = 0$. From equation (43), we find a general solution for the velocity as

$$\hat{w}_{jGS} = \frac{i}{\alpha^2} \frac{L_1(B_j - B_{j-1})}{L_j} r^2 + C_{j3} J_0(i^{3/2}\alpha r) + C_{j4} Y_0(-i^{3/2}\alpha r), \quad (57)$$

for $j = 1, 2, 3$, where C_{j3} and C_{j4} are unknown constants for each region, $J_0(r)$ is a Bessel function of the first kind, and $Y_0(r)$ is a Bessel function of the second kind. The unknown constants of C_{j3} and C_{j4} for each $j = 1, 2, 3$ are resolved by applying the no-slip and symmetry boundary conditions given by equations (44)-(46) in each region, to give the solution for time-dependent part of the velocity in regions one, two, and three, as given in Appendix B.1, with B_1 and B_2 to be determined. Similarly to before, this is done by first determining the time-dependent fluxes $\hat{Q}_1$, $\hat{Q}_2$, and $\hat{Q}_3$ by inserting equations (205)-(207) into the time-independent flux definition from equations (49)-(51). The flux matching conditions from equations (47) and (48) are then applied, and the values of B_1 and B_2 are then found by solving the resulting system. We then find the time-dependent pressure

Table 1: Table of parameters and dimensionless quantities.

Parameter	Description	Value	Units
R_1	Radius of tubing	0.2	(cm)
R_{21}	Radius of working tool	0.045	(cm)
R_{22}	Radius of working channel	0.06	(cm)
R_{31}	Radius of ureteroscope	0.15675	(cm)
R_{32}	Radius of access sheath	0.2145	(cm)
L_1	Length of tubing	30	(cm)
L_2	Length of ureteroscope	68	(cm)
L_3	Length of sheath	36	(cm)
A_0^*	Upstream constant pressure	6×10^4	(g/cm·s ²)
B_0^*	Upstream amplitude of pressure	6×10^4	(g/cm·s ²)
ω^*	Frequency of pressure oscillations	1	(1/s)
ρ^*	Density of irrigation fluid	0.997	(g/cm ³)
μ^*	Dynamic viscosity of irrigation fluid	8.9×10^{-3}	(g/cm·s)
W	Typical velocity scale	1 - 10	(cm/s)
Re	Reynolds number: $WR_1\rho^*/\mu^*$	10 - 100	-
α	Womersley number: $R_1\sqrt{\rho^*\omega^*/\mu^*}$	1 - 10	-

amplitude in each region by integrating equation (56) with respect to z and applying the pressure boundary conditions from equations (44)-(46).

Combining the two solutions back together, the dimensionless velocity and pressure throughout the system are given by

$$w_j = \sigma \bar{w}_j + \text{Re}(\hat{w}_j e^{it}) , \quad (58)$$

$$p_j = \sigma \bar{p}_j + \text{Re}(\hat{p}_j e^{it}) , \quad (59)$$

$$Q_j = \sigma \bar{Q}_j + \text{Re}(\hat{Q}_j e^{it}) , \quad (60)$$

for $j = 1, 2, 3$, where we have taken the real part of the time-dependent part of the solution and the scaling factor $\sigma = \mu^*\omega^*/A_0^*\delta^2$ reflects the different non-dimensional schemes for the steady and time-dependent quantities.

2.5 Fluid results

Unless stated otherwise, the default parameters are chosen to reflect a realistic ureteroscopy system, as given in Table 1. The radii and length of the ureteroscope are taken from a Boston Scientific LithovueTM Single-Use Digital Flexible Ureteroscope. The radius of the working tool is taken from a Boston Scientific FlexivaTM Laser Fibre. The radius and length of the access sheath are taken from a Boston Scientific 13F diameter NavigatorTM Ureteral Access Sheath. The radius and length of the tubing are taken from flexible medical tubing, the same kind used during any experiments performed for use in this thesis. The values of upstream pressure mean, amplitude, and frequency are expected values during a ureteroscopy procedure using a four head peristaltic pump at a low rotation rate.

2.5.1 Pressure throughout the system

We begin by considering how the pressure varies throughout the system. Focusing on the time-independent system, in Figure 6, we plot the pressure $\bar{p}_j$ as a function of z , for region one, two, and three in purple, blue, and green respectively. We see that the majority of the mean pressure drop occurs in region two, within the working channel of the ureterscope, with regions one and three experiencing much smaller reductions, as highlighted by the red boxes in the figure. The larger reduction in mean pressure throughout region two is due to the extremely small channel through which the fluid must flow, offering a higher resistance compared to the other two regions. The mean pressure within the kidney, located between regions two and three, will be significantly smaller than that imposed by the upstream pump, suggesting that the upstream pressure needs to be substantial in order to maintain an adequate level of fluid irrigation within the kidney.

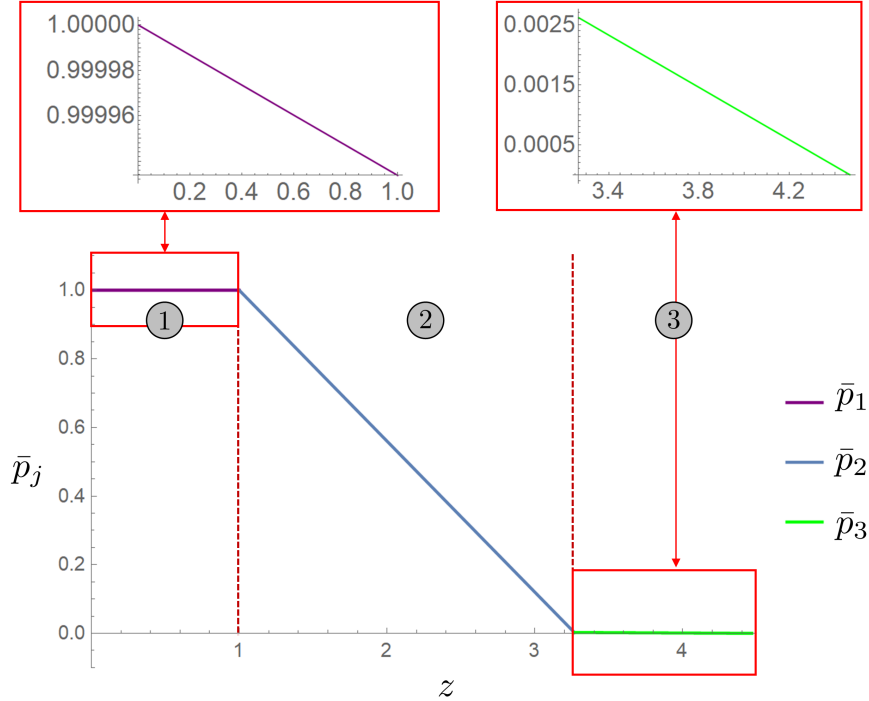


Figure 6: Time-independent mean pressure $\bar{p}_j$ as a function of z , for region one, two, and three in purple, blue, and green respectively, with parameters from Table 1. Dotted, red lines represent the junctions between each region, with the kidney located between regions 2 and 3. The two red boxes show a close-up of the mean pressure drop in regions one and two.

Turning to the time-dependent system, we explore the prevalence of the oscillations imposed by the upstream pump throughout the system. In Figure 7, we give the time-dependent amplitude of pressure $\hat{p}_j$ as a function of z , for region one, two, and three in purple, blue, and green respectively. Similarly to the mean pressure, we see that the amplitude of pressure is reduced most significantly throughout region two, the working channel of the ureterscope, again due to the small radii of this channel. This result suggests that any oscillations created in the flow by the upstream pump will be significantly

dampened by the time they reach the kidney, downstream of the ureterscope working channel.

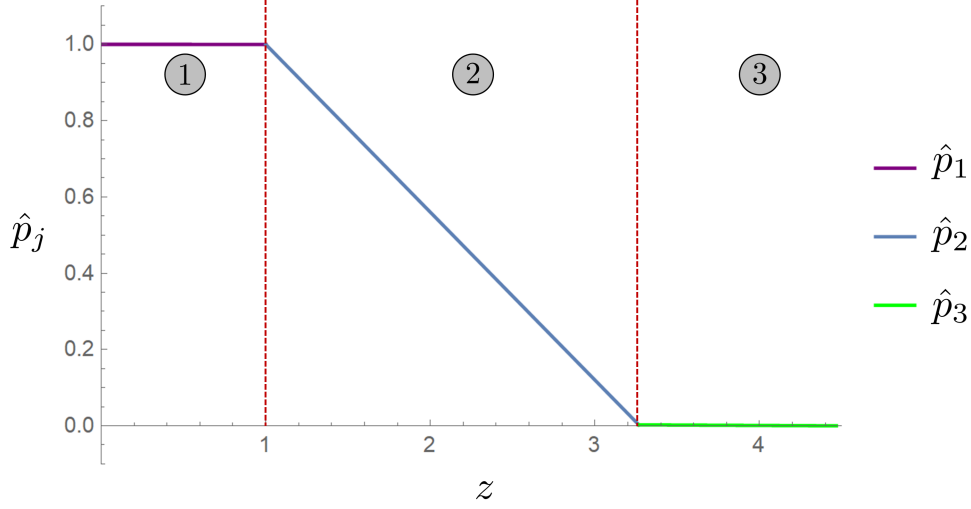


Figure 7: Time-dependent amplitude of pressure $\hat{p}_j$ as a function of z , for region one, two, and three in purple, blue, and green respectively, with parameters from Table 1. Dotted, red lines represent the junctions between each region, with the kidney located between regions 2 and 3.

2.5.2 The effect of a working tool

In practice, a working tool may not always be present within the working channel, as tools are inserted and removed during the procedure. As we have seen, the small cross sectional area created by the tool within the channel leads to a large drop in the mean pressure, as well as significant dampening of the oscillations within the flow. By removing the working tool, the pressure mean and amplitude within the kidney will increase due to the larger working channel cross sectional area. To represent this scenario, we adapt the model by changing region two to a single cylinder of radius R_{22} , similar to region one. In Figure 8, we give the amplitude of pressure $\hat{p}_j$ as a function of z in regions one, two, and three, for a working channel both with and without a working tool inserted, given in purple and blue respectively. With the working tool removed, there is a larger channel for the fluid to flow through, resulting in the amplitude of oscillations being less reduced at the junction between regions two and three. This suggests any oscillations created by the upstream pump could still have a significant impact downstream within the kidney cavity.

2.5.3 The effect of varying the Womersley number

Having seen that the amplitude of the pressure can be significant downstream of the ureterscope when a working tool is removed, we now explore how this amplitude varies with α , again considering a working channel with no working tool. In Figure 9(d), we show the amplitude of pressure $\hat{p}_j$ as a function of z in regions one, two, and three, for $\alpha = 1$, $\alpha = 8$, and $\alpha = 15$ in purple, blue, and green respectively. Alongside this, in

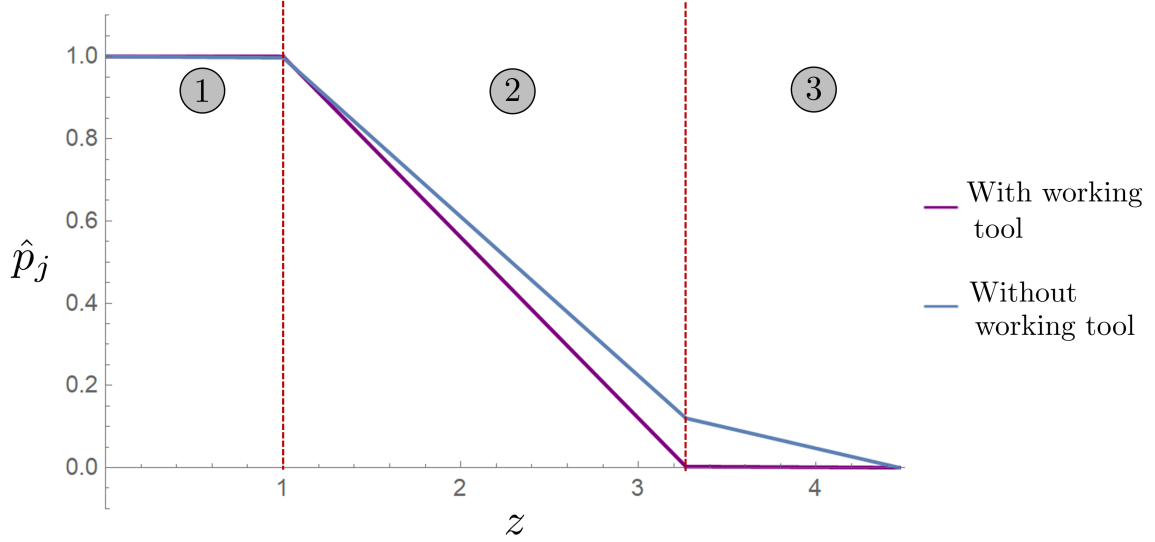


Figure 8: Amplitude of pressure $\hat{p}_j$ as a function of z in regions one, two, and three, for a working channel both with and without a working tool inserted, given in purple and blue respectively, with parameters from Table 1. Dotted, red lines represent the junctions between each region.

Figures 9(a), (b), and (c), we plot the gradient of the amplitude of pressure $d\hat{p}_j/dz$ as a function of α in regions one, two, and three respectively. We see that as α increases, the gradient of pressure amplitude decreases in regions one and two, while in region three it increases. This behaviour is displayed across the entire system in Figure 9(d). Focusing on the junction between region two and three, representing the kidney, we see that as α increases, the amplitude of pressure decreases. An increase in α is representative of an increase in frequency, suggesting that the amplitude of kidney pressure can be reduced by increasing the frequency of the oscillations.

2.5.4 Fluid velocity flow profiles

To further analyse the impact of varying α , we explore the flow patterns in each region of the system and how they depend on the Womersley number. In Figure 10(a), (b), and (c), we plot the fluid velocity w_j as a function of r over half a period, in regions one, two, and three respectively, for $\alpha = 1$. We note that the velocity remains positive over the period due to the choice of values for the upstream oscillatory pressure, where we have ensured no backflow. All three flow profiles display a familiar parabolic profile, which varies in magnitude over the period. We see that the velocity in region two is an order of magnitude larger than the other two regions, as well as having the largest variation in velocity over time.

To determine how varying α can impact these flow profiles, in Figure 11, we give the fluid velocity in the tubing w_1 as a function of r , for increasing α , at $t = 0$. We see that as α increases we start to see more plug-like flow profiles, with a decreasing maximum velocity at $r = 0$.

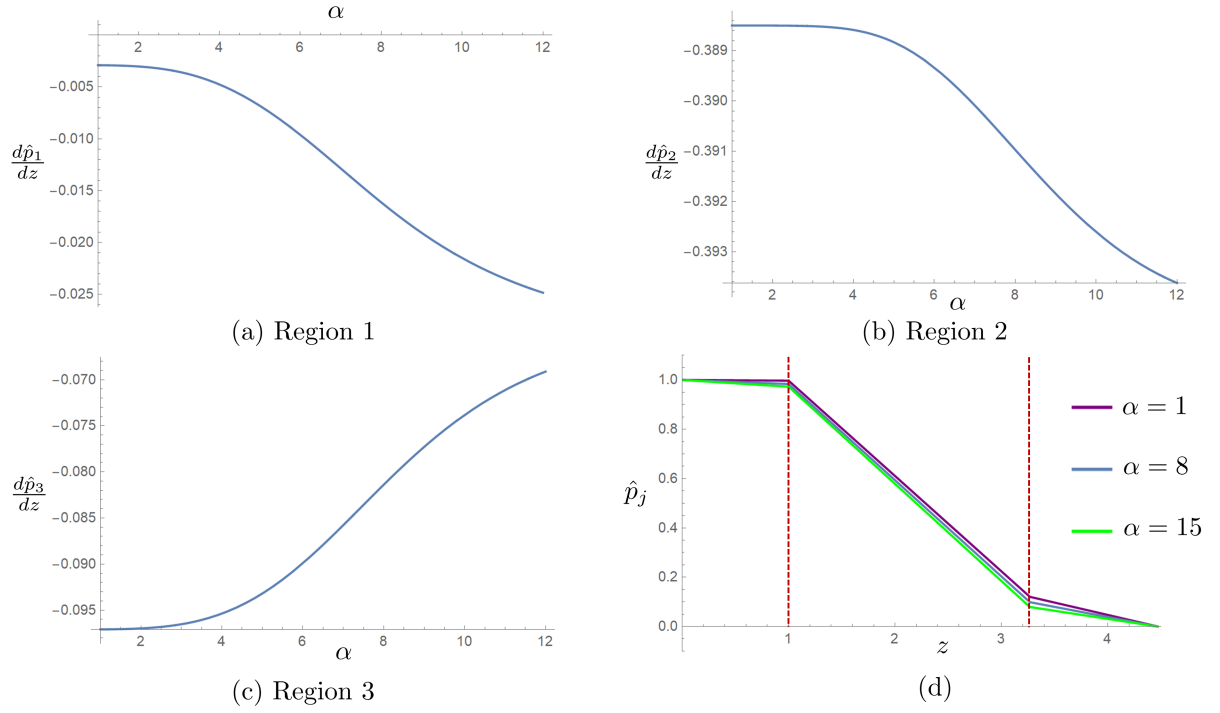


Figure 9: (d) Amplitude of pressure $\hat{p}_j$ as a function of z in regions one, two, and three, for $\alpha = 1$, $\alpha = 8$, and $\alpha = 15$ in purple, blue, and green respectively, with dotted, red lines representing the junctions between each region. Alongside this, we plot the gradient of the amplitude of pressure $d\hat{p}_j/dz$ as a function of α in regions (a) one, (b) two, and (c) three respectively.

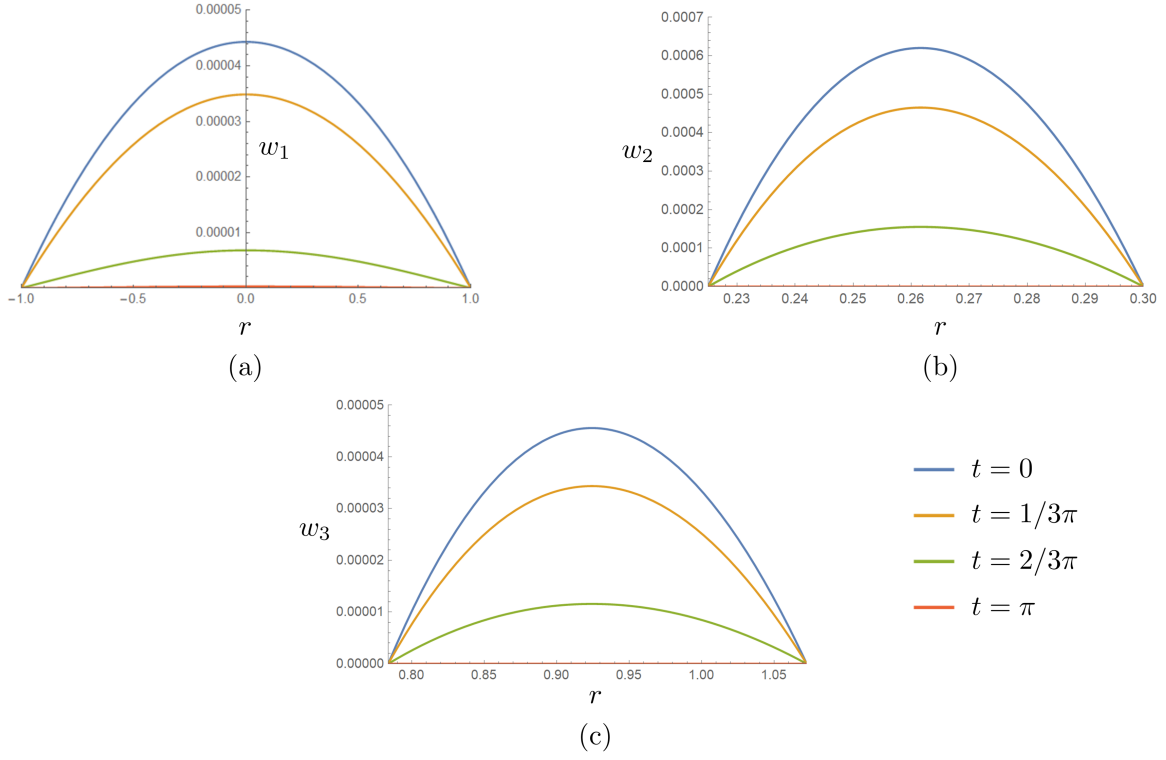


Figure 10: Fluid velocity w_j as a function of r over half a period, in regions (a) one, (b) two, and (c) three, for $\alpha = 1$.

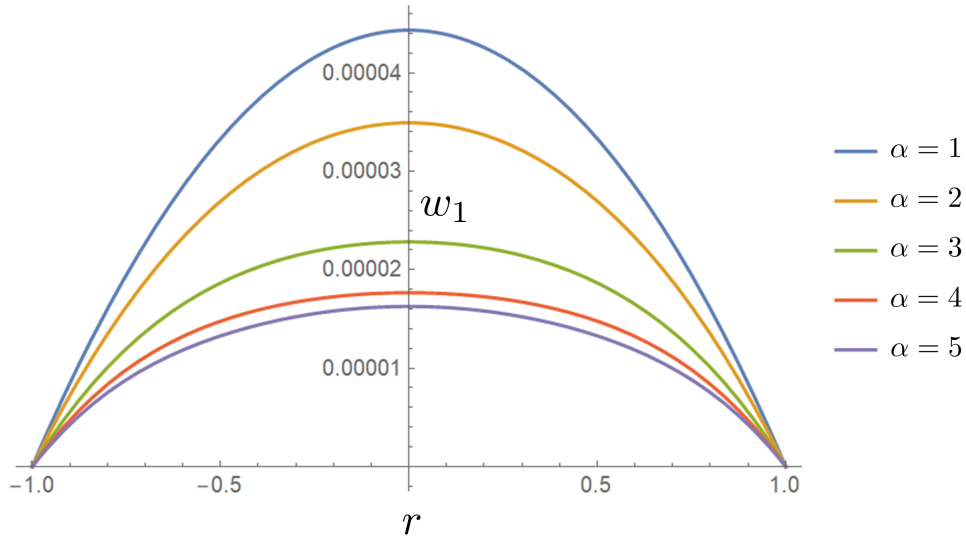


Figure 11: Fluid velocity in the tubing w_1 as a function of r , for increasing α , at $t = 0$.

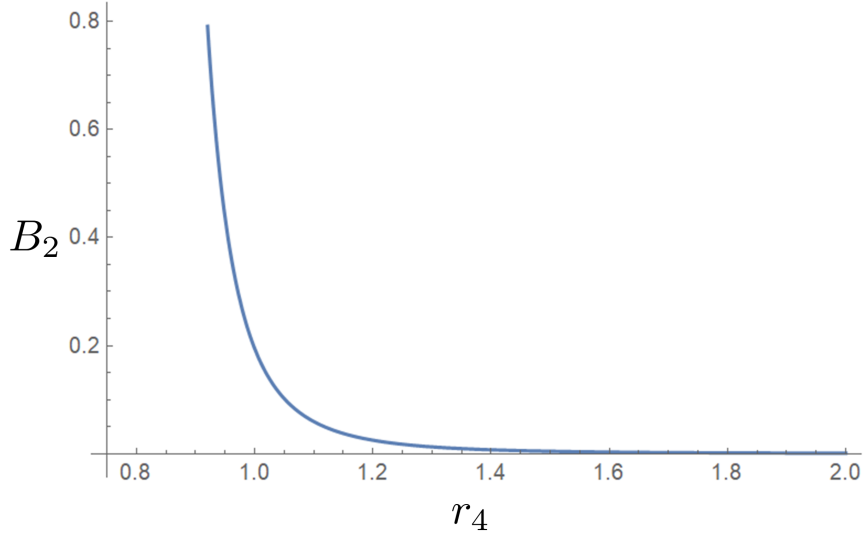


Figure 12: Amplitude of the oscillatory component of kidney pressure B_2 as a function of access sheath radius r_4 .

2.5.5 The effect of varying access sheath radius

The pressure within the kidney depends largely upon the radii within the system, as we have already seen when comparing the pressures with and without a working tool. In Figure 12, we give the amplitude of the oscillatory component of kidney pressure B_2 as a function of access sheath radius r_4 . We see that the amplitude of pressure in the kidney quickly decreases as we increase the radius of the access sheath, due to the exiting fluid having a wider channel to travel through, offering less resistance. In practice, there is little flexibility to have with increasing this radius, since the access sheath must be small enough to fit comfortably within the ureter of the patient.

2.6 Addressing model assumptions

We now explore the accuracy of the assumptions made in the model. As the fluid exits one region and enters another, the flow will no longer be unidirectional. We now aim to justify the assumptions by considering the flow between regions. We focus on the junction between regions two and three, where the kidney cavity is located, which can be thought of as a cavity or pipe expansion. Specifically, we consider flow through a cavity, similar to that given in Mizushima and Takahashi (1999).

Rather than consider axisymmetric pipe flows entering and exiting a three-dimensional cavity, we make two significant model reductions for simplicity. Firstly, we reduce the problem to a 2D Cartesian system x, y with corresponding coordinate directions $\mathbf{i}, \mathbf{j}$. Secondly, since we have seen that the amplitude of oscillations can be significantly dampened downstream of the scope within the kidney, we consider only steady flow, and solve the steady, two-dimensional Navier-Stokes equations. We solve the governing equations via a numerical finite element implementation.

For the domain, we model flow through a channel, representing the ureterscope working channel, which then flows into an expansion zone, representing the kidney cavity.

The fluid then exits out of the cavity via two channels, representing the access sheath with ureterscope inserted. These components together define the flow domain Ω (Figure 13). We define the entrance to the domain to be at Γ_{in} and the two exits of the domain to be at Γ_{out} , with the rest of the domain bounded by rigid, impermeable boundaries Γ_{walls} . The entrance channel has width $2R_{22}$ and length L_2 , the exit channels both have width $R_{32} - R_{31}$ and length L_3 , and the cavity has width $2R_{cav}$ and length L_{cav} . The flow is driven by a prescribed upstream parabolic profile at the channel entrance, with zero stress conditions on the exits. The fluid is modelled as a viscous, incompressible, Newtonian fluid of density ρ^* and dynamic viscosity μ^* .

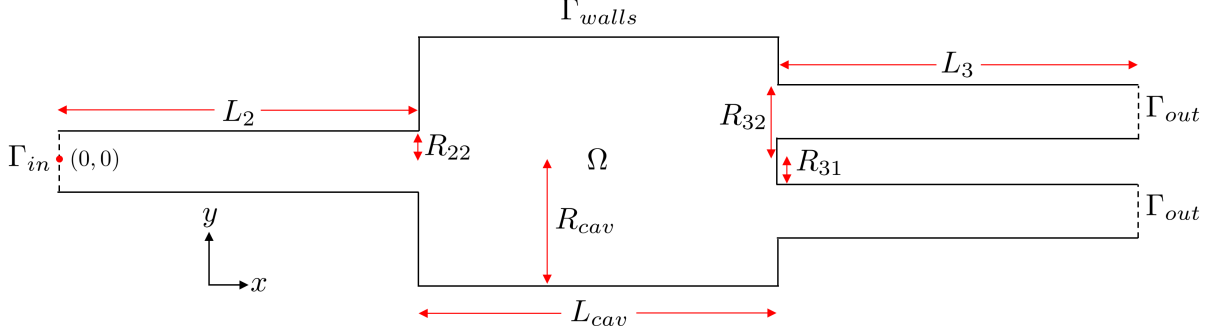


Figure 13: Sketch of the flow domain Ω , for the cavity flow problem, with inlet Γ_{in} , outlets Γ_{out} , and solid black lines representing the boundaries Γ_{walls} .

The flow is governed by the two-dimensional Navier-Stokes equations, where the fluid velocity is given by $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ and the pressure is given by p . Scaling on the entrance channel half width, we non-dimensionalise the lengths with R_{22} , with velocity scaling U and pressure scaling μ^*U/R_{22} . The dimensionless, steady, two-dimensional Navier-Stokes and continuity equations are given by

$$\nabla \cdot \mathbf{u} = 0 , \quad (61)$$

$$Re(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} , \quad (62)$$

where $Re = \rho^*UR_{22}/\mu^*$. We prescribe a parabolic flow profile at the inlet to the channel, given as

$$u = 1 - y^2 , v = 0 \quad \text{on} \quad \Gamma_{in} , \quad (63)$$

as well as a no-stress condition at both outlets of the channel, given as

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{out} , \quad (64)$$

where $\mathbf{n}$ is the outward normal to the boundary. We prescribe no-slip conditions along all walls of the channel, given as

$$u = v = 0 \quad \text{on} \quad \Gamma_{walls} . \quad (65)$$

We solve the problem numerically via a finite element formulation. The problem is implemented using the open source finite element library Firedrake (Rathgeber et al.,

2017). To solve, we use a stationary, incompressible Navier-Stokes equations solver with a Reynolds-robust pre-conditioner, which was provided by the authors of Farrell et al. (2019). The domain is discretised and a mesh structure is generated using the open source finite element mesh generator software Gmsh (Geuzaine and Remacle, 2009). We use the default triangular meshing settings with an target element size at each node of $dx = 0.25$. Visualisation of the finite element results is produced using software package Paraview (Ahrens et al., 2005), with instantaneous streamlines being generating by the software's default streamline plotter.

In Figure 14(a) and (b), we give the solution for fluid velocity u and v respectively. Alongside this, in Figure 14(c), we give the solution for pressure p throughout the domain. Finally, in Figure 14(d), we give the fluid velocity magnitude with streamlines. These results are generated through a cavity with $L_{\text{cav}} = 0.6$ and a Reynolds number of $Re = 1$. We see little variation in pressure at the inlet and outlets of the cavity, and that the flow returns to be unidirectional upon re-entering the outlet channels.

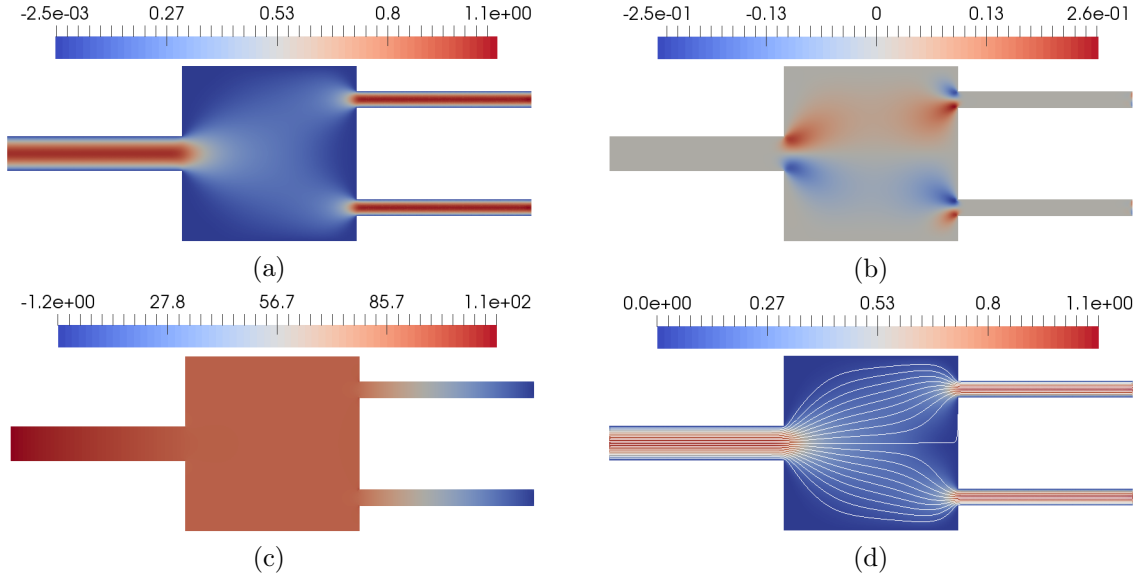


Figure 14: Flow through cavity solution plots of: (a) fluid velocity u , (b) fluid velocity v , (c) pressure p , and (d) fluid velocity magnitude with streamlines, for $L_{\text{cav}} = 0.6$ and $Re = 1$.

To test these assumptions further, we explore how the pressure changes across the cavity region for varying cavity lengths and Reynolds numbers. In Figure 15(a), (b), and (c), we give pressure profiles across the inlet and outlets of the cavity, for varying Reynolds number, for a cavity length of $L_{\text{cav}} = 0.6$. From Figure 15(a), we see that as the Reynolds number increases, the pressure at the inlet also increases, but remains fairly constant, suggesting that the flow will remain unidirectional here. From Figures 15(b) and (c), we see that an increasing Reynolds number causes the pressure across the outlets to become less constant. This suggests that the earlier assumption of unidirectional flow at junctions begins to break-down as the Reynolds number increases. In Figure 15(d), we give the error between the mean pressure across the inlet and the mean pressure across

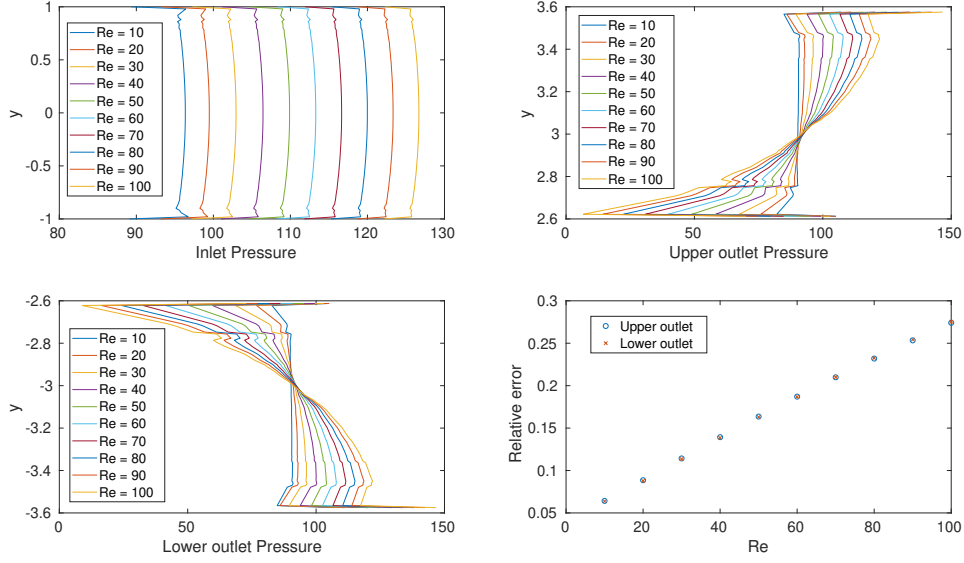


Figure 15: (a) Pressure across the cavity inlet, (b) pressure across the upper cavity outlet, (c) pressure across the lower cavity outlet, and (d) relative error between the mean inlet and outlet pressure, for varying Reynolds number, with $L_{cav} = 0.6$.

each outlet, defined as

$$\varepsilon_{rel} = \left| \frac{mean(p_{in}) - mean(p_{outlow})}{mean(p_{in})} \right|, \quad (66)$$

and similarly for the upper outlet, and see that the error linearly increases as the Reynolds number increases. This result suggests that the assumption of the pressures matching between each junction is valid for low Reynolds numbers, but becomes less valid as the Reynolds number grows.

Since we are extracting data from the inlet and outlets of the pipe expansion cavity, the sharp corners of the domain lead to numerical discontinuities, as seen in the results. The value used for the target element size of the mesh can be reduced to help smooth out these results, but at the expense of computational run time. An example of the same results at $Re = 10$ and $L_{cav} = 0.6$, but with a mesh which is refined more at these domain corners is presented in Appendix B.2.

In Figure 16, we give the same pressure plots as presented Figure 16, but now for varying cavity length, with a fixed width of 0.6 and a Reynolds number of $Re = 10$. We see from the results that the pressure at the inlet and outlets remains fairly constant and the error between the mean pressure at each end of the cavity remains low as the length of the cavity varies. These results suggest the earlier assumptions are valid and not impacted by cavity length.

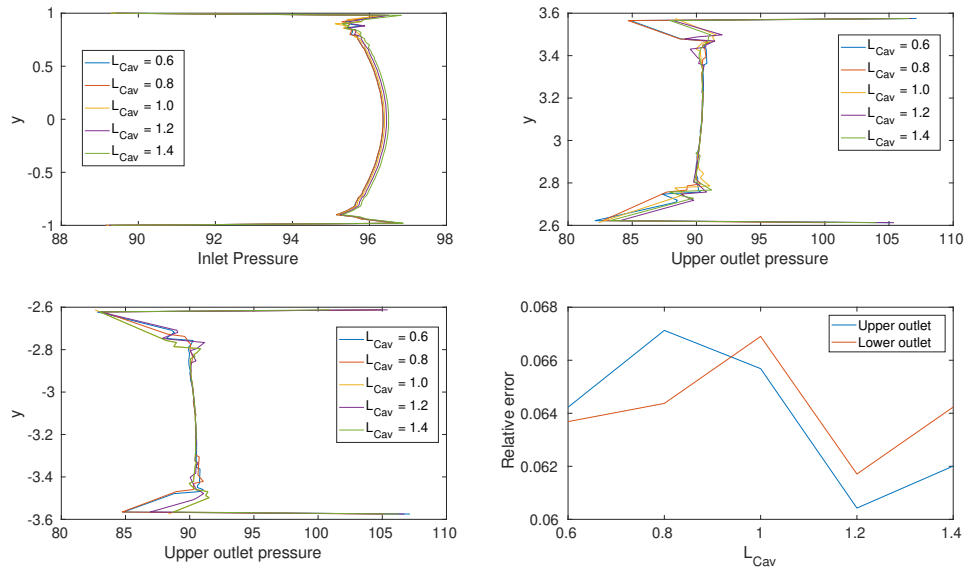


Figure 16: (a) Pressure across the cavity inlet, (b) pressure across the upper cavity outlet, (c) pressure across the lower cavity outlet, and (d) relative error between the mean inlet and outlet pressure, for varying L_{cav} , with $Re = 10$.

2.7 Peristaltic pump model

Throughout the analysis of the systems model, we have assumed that the upstream pressure parameters A_0^* , B_0^* , and ω^* can be varied independently of one another. However, we anticipate that these parameters may not be independent. Therefore, we ask what is an accurate choice for the upstream pressure that is produced by the peristaltic pump? To determine this we present a model of a peristaltic pump.

We consider a pump with radius r_p and n roller heads, rotating at an angular velocity Ω , surrounded by tubing of radius r_t (Figure 17). We assume the heads are spaced evenly around the pump, and that their radius is equal to twice that of the tubing. We define the angle between each roller head as ψ and the angle the roller head occupies as ϕ . These angles satisfy $\phi = 4 \arcsin(r_p/r_t)$ and $\psi = 2\pi/n - \phi$.

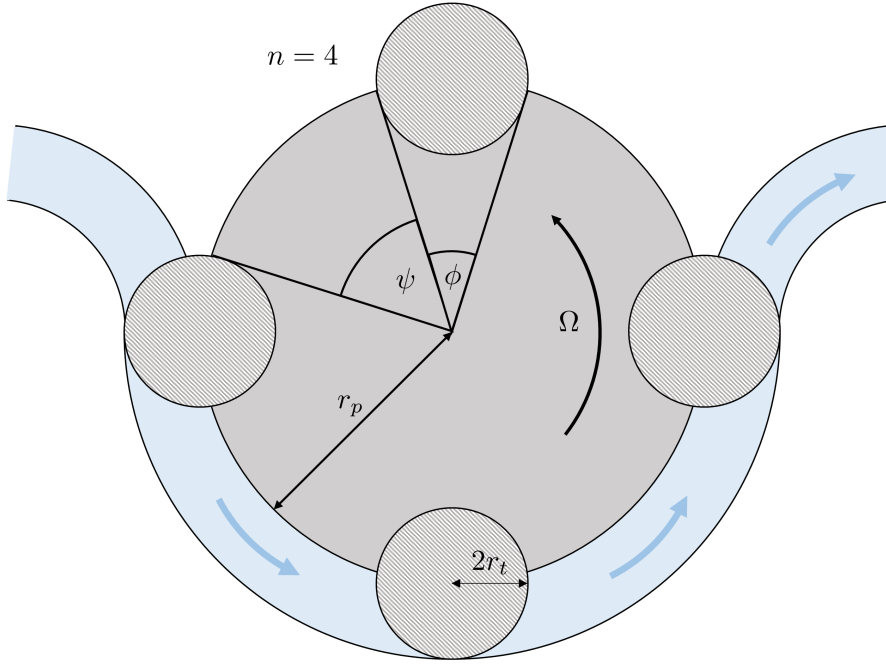


Figure 17: Model sketch of the peristaltic pump for $n = 4$ roller heads.

We consider the flux produced by the pump, which once found is used to determine the upstream pressure for use in the system model. Firstly, we consider the maximum flux being produced by the pump. To determine this, we take the cross sectional area of the tubing $A = \pi r_t^2$ and multiply it by the velocity at which the pump is rotating, giving $Q_{\max} = \pi r_t^2 \Omega R$. As the pump rotates, the roller head lifts away from the tubing creating additional space which must be occupied. This space can only be filled by the fluid downstream of the roller head, which produces a reduction from the maximum flux (Figure 18).

Therefore, the flux for the period of time during which the roller is lifting away from the tubing is given by $Q_{\max} - \dot{V}_{\text{roller}}$, where V_{roller} is the volume within the tube the head occupies. We linearly approximate this change in volume over time as $\dot{V}_{\text{roller}} \sim V_{\text{roller}}/(\phi/\Omega)$, where ϕ/Ω is the time it takes for the roller head to completely lift away from the tubing. Combining this information we have the upstream flux for one period

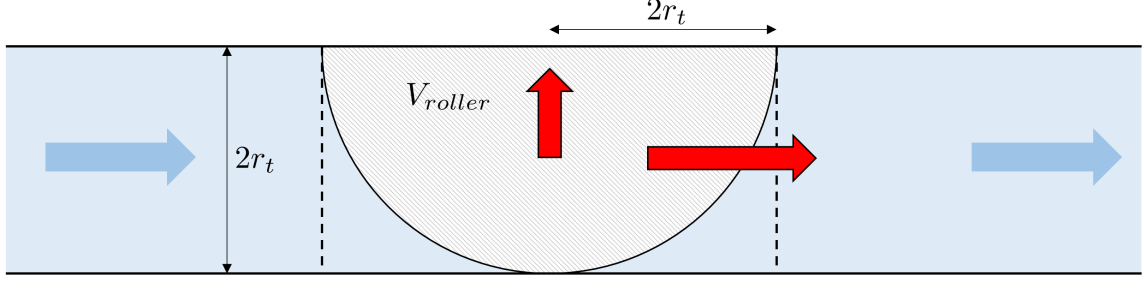


Figure 18: Sketch of the movement of a roller head, with fluid given in blue, roller head given in grey, and red arrows indicating movement of the roller head as the pump rotates. The volume of the roller head is given by V_{roller} .

of roller rotations as

$$Q_0 = \begin{cases} Q_{\text{max}} & 0 \leq t \leq \frac{\psi}{\Omega} \\ Q_{\text{max}} - \frac{V_{\text{roller}}}{\phi/\Omega} & \frac{\psi}{\Omega} \leq t \leq \frac{\psi}{\Omega} + \frac{\phi}{\Omega} \end{cases} . \quad (67)$$

Following from the ansatz made for the form of the flux in the systems model, we fit a curve of the form $Q_0 = \bar{Q}_0 + \hat{Q}_0 e^{i\omega^* t}$ to this piecewise function. The frequency is simply given by the number of times we see a roller head per rotation of the pump, giving $\omega^* = n\Omega$. The mean flux $\bar{Q}_0$ and the amplitude of flux $\hat{Q}_0$ are found by solving

$$\bar{Q}_0 + \hat{Q}_0 e^{i\omega^* t_1} = Q_{\text{max}} \quad (68)$$

$$\bar{Q}_0 + \hat{Q}_0 e^{i\omega^* t_2} = Q_{\text{max}} - V_{\text{roller}}/(\phi/\Omega) , \quad (69)$$

where $t_1 = \psi/\Omega$ and $t_2 = \phi/\Omega$. Figure 19 shows the fitted curve against the piecewise flux, for some realistic parameter values. The accuracy of this fit is improved when t_1 and t_2 are similar. Notably, the fit captures the important features such as frequency, maximum and minimum flux, and mean flux over time.

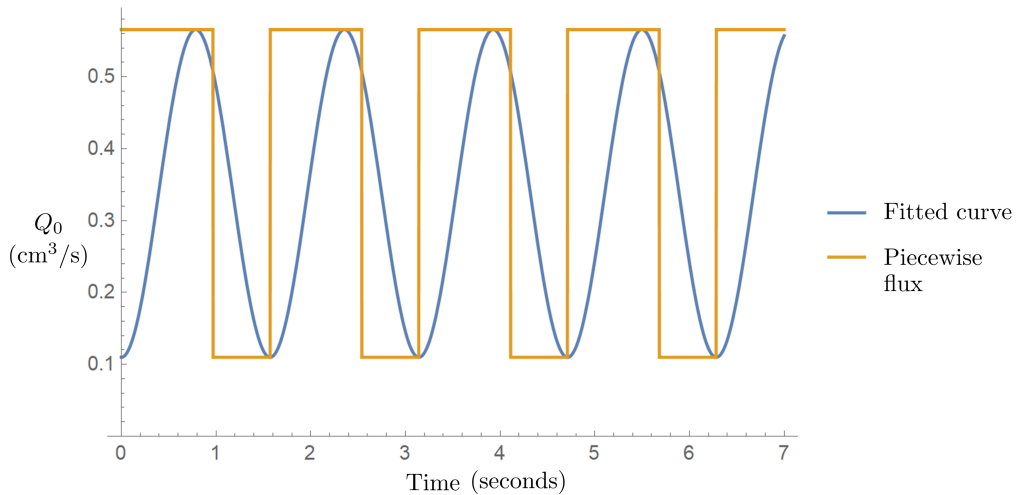


Figure 19: Curve fitted to piecewise flux, for $\Omega = 1$ rotations per second, $r_t = 0.2\text{cm}$, $r_p = 2\text{cm}$, and $n = 4$ rollers heads.

We now return to the analysis of the systems model, where now we drive the fluid using the predicted flow rate from the peristaltic pump model, to explore the effect that varying the pump parameters has on the system. Since we have presented a pump model which produces an upstream flux and not pressure, the systems model must be adapted to accommodate this. The model set-up and solution method proceeds as before, however, the values of A_0^* and B_0^* in the boundary condition given by equation (9) are no longer given. Instead, the form of this equation becomes an ansatz of the system, and the known values are now Q_0^* (derived from the pump model) and P_3^* . Since the flux throughout the system is matched at each junction, the values of Q_j^* for $j = 1, 2, 3$ are now known, and from here the unknowns of the system are found.

During a ureteroscopy procedure, the only parameter which the clinician can control is the rotation speed of the pump, given by the angular velocity Ω . Focusing on this parameter, Figure 20 shows the kidney pressure over time for increasing values of Ω . As the angular velocity of the pump increases, the mean kidney pressure, the amplitude of kidney pressure, and the frequency all increase. As expected, these three parameters cannot be varied independently, and are all related by the angular velocity of the pump. These results suggest that if larger mean pressures are required to drive the irrigation fluid then similarly large amplitudes may be unavoidable when using a peristaltic pump.

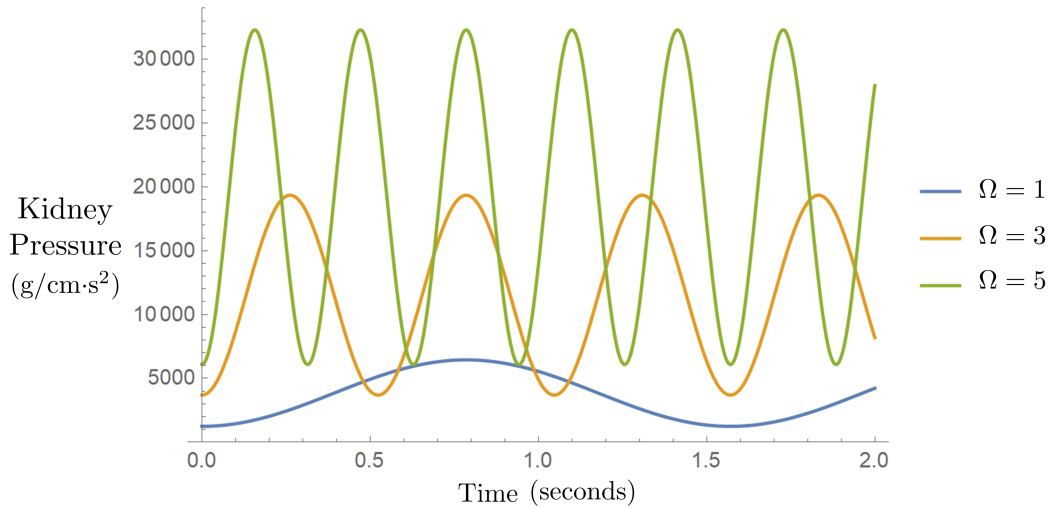


Figure 20: Kidney pressure over time for increasing angular velocity Ω , with $r_t = 0.2\text{cm}$, $r_p = 3\text{cm}$, and $n = 4$ rollers heads.

2.8 Comparison to experimental data

In order to validate the system and pump modelling in this chapter, we compare the results against data obtained from experiments.

The experimental set-up is as follows; a peristaltic pump draws water from a reservoir. Tubing of length 98cm connects the pump to a flow meter, followed by a pressure sensor 8cm after. The pressure sensor is then connected to a ureteroscope working channel with another section of tubing of length 20cm. The water travels down the length of the ureteroscope working channel and exits the tip into open air (Figure 21). All tubing in

the set-up has radius $R_1 = 0.3175\text{cm}$, with total length $L_1 = 126\text{cm}$. The pump has a radius of $r_p = 2.55\text{cm}$ with six pump heads ($n = 6$). The scope working channel has radius $R_{22} = 0.06\text{cm}$ and length $L_2 = 79\text{cm}$. We note that a working tool is not inserted within the scope.

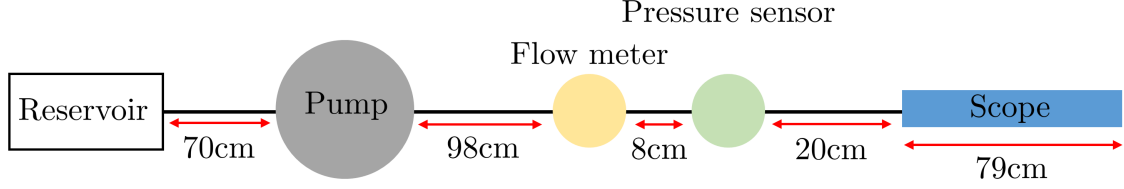


Figure 21: Schematic of the experimental set-up, with peristaltic pump in grey, flow meter in yellow, pressure sensor in green, and ureterscope in blue.

The experiment was performed at three different pump speeds, measured in rotations per minute (RPM), of $\text{RPM} = 50, 289$, and 595 . For each speed, the pressure sensor recorded the pressure in mmHg over time for a ten second period. Prior to this, the pump was allowed to run for one to two minutes, to ensure the system had adjusted to the initial delivery of fluid and reached a quasi-steady state. The first two pump speeds are typical of what would be used during a ureteroscopy procedure, with the third speed being an upper bound on the capabilities of the peristaltic pump.

In order to replicate the experimental set-up, the system model presented in Section 2.3 is adjusted. Since the ureterscope exits into open air, the access sheath represented by region three of the systems model is removed, and the pressure at the exit to the ureterscope is set to $P_2 = 0$. As a working tool is not present during the experiments, this is also removed from the model, and the boundary condition in region two of $w_2(r_1) = 0$ is replaced by $w_2'(0) = 0$. We note that the results are redimensionalised for comparison with the data, where pressure is converted from cgs units to mmHg by dividing the pressure value by 1333.22. To compare the model against the experimental data, we evaluate the pressure p_1 at $z = 106$, for $t = 0$ to $t = 10$. Since the data takes a ten second snapshot from a longer time frame, we allow the time window of t to be shifted to match with the period of the data. The systems model is adapted to be driven by an upstream flux, as described in Section 2.7, with said flux coming from the predictions of the pump model. Since the pump model uses angular velocity Ω , we convert from RPM via

$$\Omega = \frac{2\pi\text{RPM}}{60 \times 15}, \quad (70)$$

where the 15 in the denominator represents the 15:1 gear ratio for the type of pump used. The system can then take RPM as an input, and give the solution of pressure throughout the system as an output.

In Figure 22(a), (b), and (c), we give the pressure readings from the experiments over a period of ten seconds in orange, for the three rotation speeds of 50, 289, and 595 respectively. On the same plot, in blue, we give the predicted pressure in the system from the model. Mean pressure values for both the experimental readings and model predictions are given by dotted blue and orange lines respectively. Looking at the experimental data for the lowest RPM of 50, in Figure 22(a), the trough of each curve appears to be negative

for some time, being -18.4 at its minimum. This behaviour is unexpected, as the flow is assumed to travel downstream from pump to scope. The model prediction fails to capture this since the pump model is bound to not give negative flow rates, only predicting a minimum pressure of 5.6 . The under-prediction in amplitude leads to an over-prediction in mean pressure, with the model giving a mean pressure of 18.4 while the data showed a mean pressure of 8.3 . Aside from this, the model still accurately captures the aspects of flow frequency. At the higher RPM of 289 , in Figure 22(b), we see that the model accurately captures all three aspects of pressure frequency, amplitude, and mean rate, with the mean pressure having a relative difference of only 0.074 . Increasing to the maximum RPM of 595 , in Figure 22(c), we see the model over-predicts the amplitude of the pressure with the minimum pressure in the model dropping to 56.1 whilst the data only shows a minimum of 215.4 . This in turn results in a significant difference between the mean pressures. However, the relative difference of the maximum pressure is very low at only 0.014 , suggesting that if the over-prediction of the minimum pressure can be corrected for then the model accuracy will be significantly improved, which will be discussed in Section 2.9. Despite some discrepancies, we conclude that the system and pump model manages to accurately capture the data, with this accuracy decreasing for higher RPMs.

2.9 Summary and discussion

To summarise, we have developed a mathematical model of fluid flow throughout the cylindrical geometries of the ureteroscopy system, namely the tubing, ureteroscope working channel with working tool inserted, and access sheath with ureteroscope inserted. Each of these regions is connected by assuming continuity of pressure and flux at the junctions between them. The flow is driven by a pressure drop across the system, with an upstream pressure which oscillates in time and a downstream pressure set to atmospheric. The model is solved analytically to determine the pressure and fluid velocity throughout the system. We have determined the magnitude, amplitude, and frequency of the pressure within the kidney cavity, and explored how these values are varied by changing the parameters of the system. We conclude that due to the small radii of the working channel, oscillations are significantly dampened by the time they reach the kidney downstream of the ureteroscope. It is found that the oscillations become more significant if a working tool is not present within the working channel, due to the increased cross sectional area for the fluid to flow through. We go on to test the continuity assumptions made at the junctions between each region by numerically solving for the flow within the more complicated geometry of a two-dimensional cavity. The assumptions made in the system modelling are justified as we find that the pressure drop across the cavity is small, and that the flow returns to being unidirectional shortly after exiting the cavity. Finally, we present a model of a peristaltic pump to determine how the upstream pressure parameters of magnitude, amplitude, and frequency are related to one another via the geometry and rotation rate of the pump driving the fluid throughout the system. The systems model and pump model are then compared against data obtained from an experimental set-up of flow through tubing and scope, driven by a peristaltic pump. Good agreement between model and data is found, with the accuracy decreasing as pump rotation rate increases.

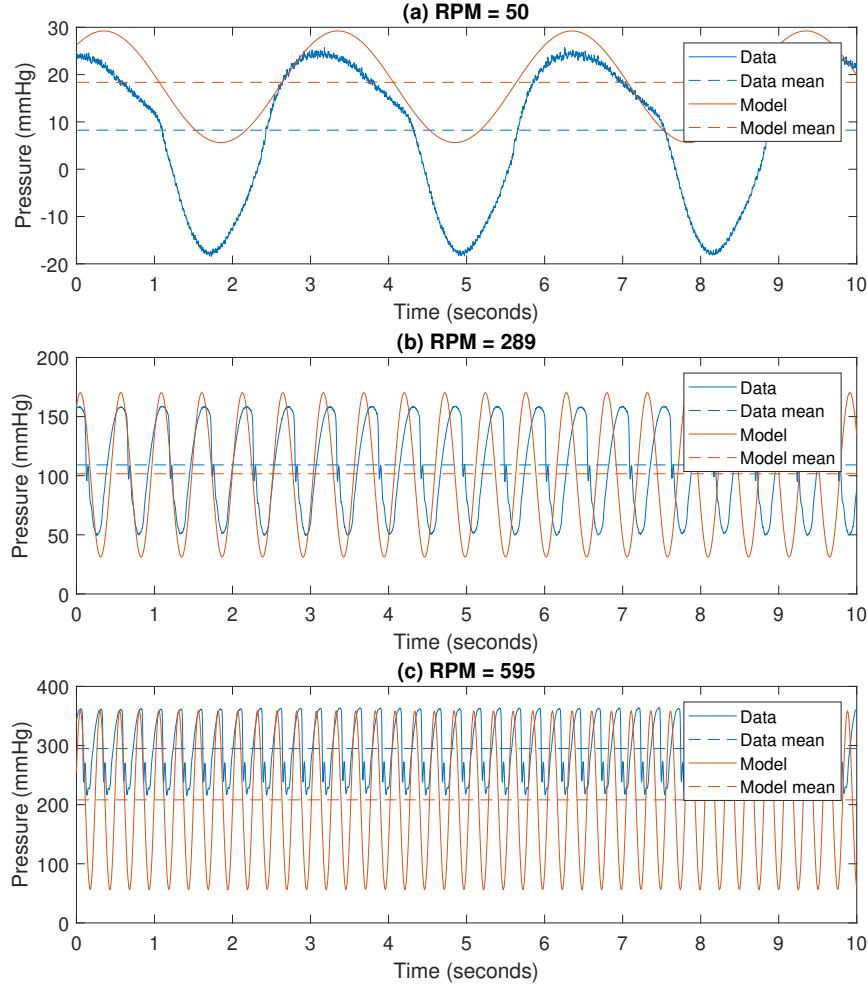


Figure 22: Pressure readings from the experiments in orange and pressure predictions from the model in blue, over a ten second period, for RPMs of (a) 50, (b) 289, and (c) 595. Mean pressure values for each are given by dotted lines.

While the continuity assumptions made at the junctions have been examined, reductions had to be made in order to simplify the problem. We have considered a cavity which has inlet channels and outlet channels on opposite edges. A more realistic set-up would be to have the inlet and outlets on the same edge, representing the ureterscope tip within the access sheath, pointing towards the cavity. Further to this, we have only considered steady fluid flow, however we have seen that oscillations within the cavity can become significant under some circumstances. These two aspects are explored further in the next chapter, where a more in depth look at modelling oscillatory fluid flow within a kidney cavity is explored.

The discrepancies seen between the model and associated experimental data at the higher rotation rate of 595 suggests that the pump model is failing to capture the amplitude of the flow rates created by the pump. To explore this error, we turn to the

assumptions made in the modelling of the peristaltic pump. We have assumed that the tubing containing the fluid will be fully compressed by the pump head on every rotation. In reality, it is likely that the tubing will have some compliance, and not become fully compressed. On top of this, we assume the tubing will instantaneously return to its normal state in between each compression, while it is likely that this process would take some amount of time. As the rotation rate increases, the time in between each compression is reduced, giving the tubing less time to readjust to its resting state. This could account for the over-prediction of amplitude within the model for higher rotation rates. Modelling such a scenario would require more complicated techniques, such as the numerical simulations discussed in the literature (Elabbasi et al., 2011; Formato et al., 2019). However, we may estimate this behaviour by reducing the volume within the tubing which the pump head occupies, introducing a scaling factor to the parameter V_{roller} . We can see from the form of equation (67), this would result in an increase in the minimum flux produced by the pump, while leaving Q_{max} unchanged, which matches up with the discrepancy seen in Figure 22(c). In Figure 23, we give the pressure readings from the experiments in orange and pressure predictions from the adjusted model, where a scaling factor of $SF = 0.44$ has been introduced to the pump model, in blue, over a ten second period, for $\text{RPM} = 595$. We see that this scaling results in a more accurate prediction of the pressure.

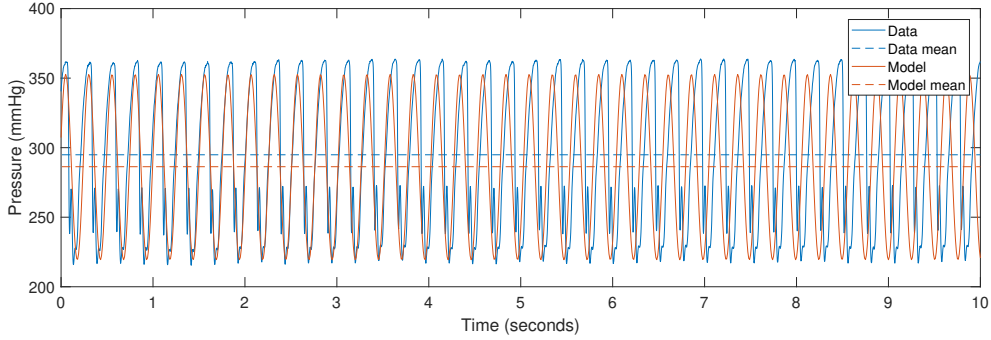


Figure 23: Pressure readings from the experiments in orange and pressure predictions from the adjusted model, where a scaling factor of $SF = 0.44$ has been introduced, in blue, over a ten second period, for $\text{RPM} = 595$. Mean pressure values for each are given by dotted lines.

In conclusion, the results of the system model suggest that the oscillations in pressure and velocity are significantly dampened once they reach the kidney, even more so when a working tool is present, due to the radii and lengths of the pipes. The oscillations of the velocity and pressure are more sensitive to the geometric parameters, however they can also be reduced by increasing the frequency of the imposed oscillations. If we desire to harness the potential of oscillatory flow within the kidney, lower upstream frequencies are required to observe this behaviour downstream of the pulsatile pump.

3 Kidney cavity model

3.1 Introduction

The ultimate goal of a ureteroscopy procedure is to locate and remove kidney stones from the kidney cavity. There are multiple ways this is achieved, one of which being the aforementioned lasering process, in which the stone is reduced to dust which is then washed away by the irrigation fluid. Efficient wash-out of dust by the irrigation fluid is important to maintain a clear field of view for the camera at the tip of the scope, whilst also ensuring maximum stone dust removal. It is therefore of interest to understand the flow of irrigation fluid within the cavity in order to explore the role it plays in dust removal.

Key to previous work on debris wash-out is the conceptual point that wash-out time is strongly linked to the presence of vortical structures in which debris may become trapped. It is worth noting that the studies by Williams et al. discussed in Chapter 1 utilised a fixed upstream pressure at the entrance to the ureteroscope, such that the fluid boundary condition at the inlet to the cavity was constant in time. In this chapter, we relax this assumption and consider a time varying boundary condition at the inlet to the cavity. A strong motivating factor for the present research is a recent advancement in the field of ureteroscopy: the traditional fluid delivery method of an upstream hydrostatic pressure generated by a hanging bag of saline is replaced by a fluid management system that drives fluid flow throughout the system via a peristaltic pump. In contrast to gravity-driven flow, this delivery method enables a much finer control over the flow rate, with the ability to modulate the flow rate as a function of time. This technology, combined with previous studies such as Coppola and Caro (2009) that have demonstrated the potential to disturb vortices using flow pulsatility, motivates the key question we seek to answer in this chapter: can a time-varying inlet flow be utilised to reduce the wash-out time of debris?

In Chapter 2, we have seen that the peristaltic pump creates natural oscillations in the flow. However, due to the small radii of the working channel and working tool, these natural oscillations are significantly dampened within the kidney cavity. However, varying the rotation speed of the pump offers the potential to deliver irrigation fluid to the ureteroscopy system in such a way that the flow profile at the cavity inlet oscillates. By considering fluid flow and dust transport within a cavity with an inlet flow profile which oscillates in time, we aim to determine how we can adjust the magnitude and frequency of the oscillations to maximise dust wash-out.

Motivated by the previous work on steady cavity flow (Williams et al., 2019), we consider a system which is initially fully irrigated by a steady upstream flow, such that the inlet flow to the cavity is a steady parabolic profile. Then at a certain time, the proposed oscillatory irrigation technique is administered so that the inlet flow profile begins to oscillate. We consider a cavity which has a uniform concentration of kidney stone dust at this time. The model we propose describes the fluid flow and dust transport beginning at this time, that is from the time at which the oscillations are administered into the flow.

3.1.1 Chapter outline

Following Williams et al. (2019, 2020b, 2021), we consider a single calyx of the kidney and model this cavity as a two-dimensional rectangular domain. We impose a sinusoidal parabolic flow profile at the cavity inlet, and we model kidney stone dust as a passive tracer transported within and eventually out of the cavity by the combined effects of diffusion and advection. The model is outlined in Section 3.2, with theoretical formulation for the fluid and dust given in Sections 3.3 and 3.6 respectively. Within this setup, we characterise in Sections 3.5 and 3.7 the presence and size of fluid vortices, their disturbance due to the oscillatory component of the inlet flow profile, and the impact this disturbance has on debris wash-out. We characterise how the flow and transport properties depend on the nature of the inlet flow (base value, amplitude, and frequency) and the diffusivity of the tracer. Within this 4 dimensional space we demonstrate that oscillations can indeed be harnessed to reduce the wash-out time. In Section 3.9, we then compare the reduction in wash-out time under the oscillatory regime to a simulation of a typical industry method termed flushing – an instantaneous jump in flow rate from one steady value to another for a brief period of time. In Section 3.10, we demonstrate that our conclusions are robust to changes in the cavity geometry. Conclusions and future work are given in Section 3.11.

3.2 Problem description

To represent the kidney calyx, we consider an idealised, two-dimensional, rectangular cavity, as shown in Figure 24. The domain Ω , given in white, has length l_c and width $2(a + b + d)$, with boundary Γ^* comprising of inlet, outlets, and rigid, impermeable boundaries. The fluid enters the cavity through a single inlet of width $2a$ denoted Γ_{in}^* on the left-hand side of the domain, representing the working channel of the ureteroscope entering the cavity. As a two-dimensional representation of a scope sitting centrally within an access sheath, we position the outlets, denoted by Γ_{out}^* , on the same side of the cavity above and below the inlet. Each outlet is of width d . The remaining boundaries are denoted by Γ_{wall}^* , representing the tip of the scope on the left-hand side, as well as the walls of the cavity on the top, bottom, and right-hand side, which are all impermeable to both fluid and dust. The values of a , b , d , and l_c are determined based on typical working channel, ureteroscope, and access sheath dimensions ($a = 0.06\text{cm}$, $b = 0.09675\text{cm}$, $d = 0.05775\text{cm}$, $l_c = 0.6\text{cm}$). The fluid has constant density ρ^* and dynamic viscosity μ^* , and the kidney stone dust has constant diffusion coefficient D^* .

To mimic oscillatory flow imposed by the upstream pump, at the inlet we prescribe a parabolic flow profile with the associated flux consisting of a mean $\bar{Q}$ and with a sinusoidal component of amplitude $\hat{Q}$ and frequency ω^* . To prevent back-flow we ensure that $\hat{Q} \leq \bar{Q}$. At both outlets we apply a stress free condition on the fluid. At the impermeable, solid walls we impose no-flux and no-slip of the fluid.

We assume that the kidney stone has been completely reduced to dust and no larger fragments remain, and that the dust is small enough such that it does not influence the fluid flow. The dust satisfies an advection-diffusion equation, with no-flux boundary conditions at both the impermeable, solid walls and inlet. At the outlet, we impose a no-diffusion condition, and say that the dust is free to advect out of the cavity.

To represent a system which is fully irrigated, in which the upstream pump is delivering a steady flow but not administering oscillations, we assume that initially the fluid

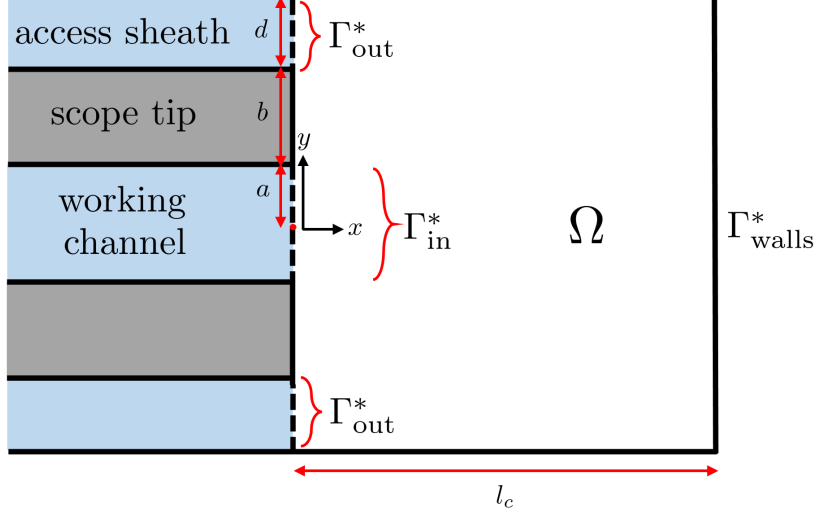


Figure 24: Kidney cavity domain sketch with domain Ω of length l_c and width $2(a+b+d)$, with boundary Γ comprising of inlet Γ_{in} , outlets Γ_{out} , and rigid, impermeable boundaries Γ_{wall} .

in the cavity has the solution of the steady problem. We assume that the dust initially occupies the entire cavity domain with a constant uniform concentration C^* .

3.3 Fluid flow

3.3.1 Model set-up

We begin by describing the fluid flow within the cavity. Once the solution for the flow is determined, we then substitute this into the advection-diffusion equation to determine dust transport.

We consider a two-dimensional, Cartesian coordinate system $\mathbf{x}^* = (x^*, y^*)$ with corresponding coordinate directions $(\mathbf{e}_x, \mathbf{e}_y)$. We assume the fluid is incompressible, with velocity $\mathbf{u}^* = (u^*(x^*, y^*, t^*), v^*(x^*, y^*, t^*))$ and pressure $p^*(x^*, y^*, t^*)$, where t^* denotes time. The time-dependent Navier-Stokes equations are given by

$$\rho^* \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) = -\nabla^* p^* + \mu^* \nabla^{*2} \mathbf{u}^* , \quad (71)$$

$$\nabla^* \cdot \mathbf{u}^* = 0 . \quad (72)$$

At the inlet to the cavity we prescribe a fully-developed parabolic flow which oscillates in time such that

$$\mathbf{u}^* = \left(\frac{1}{a^2} \left(\bar{U}^* + \hat{U}^* \sin(\omega^* t) \right) (a^2 - y^{*2}), 0 \right) \quad \text{on} \quad \Gamma_{\text{in}}^* , \quad (73)$$

where $\bar{U}^*$ is the mean maximum velocity and $\hat{U}^*$ is the amplitude of oscillations, such

that

$$\bar{Q} = \frac{4}{3}a\bar{U} , \quad (74)$$

$$\hat{Q} = \frac{4}{3}a\hat{U} \sin(\omega^*t) , \quad (75)$$

are the associated fluxes. To prevent backflow at the inlet we require that $\hat{U} \sin(\omega^*t) \leq \bar{U}$ for all time. For the outlet boundaries we prescribe zero stress such that

$$\boldsymbol{\sigma}^* \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{out}}^* , \quad (76)$$

where $\mathbf{n}$ is the outward normal to the boundary. Finally, we impose no-slip on all impermeable walls, given by

$$\mathbf{u}^* = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{walls}}^* . \quad (77)$$

We assume that initially the cavity is fully irrigated by a steady flow, so that

$$\mathbf{u}^*(\mathbf{x}^*, 0) = \mathbf{u}_{\text{steady}}^* . \quad (78)$$

We non-dimensionalise the problem via the scalings

$$\mathbf{x}^* = a\mathbf{x} , \quad \mathbf{u}^* = \bar{U}^*\mathbf{u} , \quad p^* = \frac{\mu^*\bar{U}^*}{a}p , \quad t^* = \frac{a}{\bar{U}^*}t , \quad (79)$$

where the length scale corresponds to the width of the inlet channel, and the velocity scale is the maximum mean inlet velocity. The dimensionless Navier-Stokes equations are given by

$$Re \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \nabla^2 \mathbf{u} , \quad (80)$$

$$\nabla \cdot \mathbf{u} = 0 , \quad (81)$$

which must be solved subject to boundary conditions

$$\mathbf{u} = \left(\left(1 + \frac{\hat{U}}{Re} \sin\left(\frac{\omega}{Re}t\right) \right) (1 - y^2) , 0 \right) \quad \text{on} \quad \Gamma_{\text{in}} , \quad (82)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{out}} , \quad (83)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{walls}} , \quad (84)$$

where $Re = \rho^*\bar{U}^*a/\mu^*$ is the Reynolds number, $\hat{U} = \rho^*\hat{U}^*a/\mu^*$, and $\omega = \rho^*\omega^*a^2/\mu^*$. The dimensionless initial condition is given by

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_{\text{steady}}(Re) , \quad (85)$$

where the steady solution used as an initial condition and the unsteady problem being considering have the same Reynolds number. The dimensionless steady system and its numerical solution method are described in Appendix C.1.

The fluid problem is characterised by three non-dimensional parameters: Re , ω , and $\hat{U}$. The Reynolds number Re is the ratio of inertial to viscous forces. Typically the Reynolds number takes values within the range $10 - 100$, corresponding to irrigation fluid with dynamic viscosity $\mu = 8.9 \times 10^{-3} \text{g/cm}\cdot\text{s}$ and density $\rho = 0.997 \text{g/cm}^3$, a working channel radius $a = 0.06 \text{cm}$, and mean fluid velocity $\bar{U}$ within the range of $1 - 15 \text{cm/s}$. The parameter ω is the dimensionless inlet frequency corresponding to the ratio of the viscous timescale to the timescale of oscillations, also called the Womersely number. Finally, $\hat{U}$ is the dimensionless amplitude of inlet oscillations. We note that to prevent backflow, we must have that $\hat{U} \leq Re$.

3.3.2 Numerical solution method

We solve the system numerically via a finite element formulation implemented using the open source finite element library Firedrake (Rathgeber et al., 2017). We begin by transforming equations (80) and (81) into weak form by multiplying each equation by test functions $\mathbf{v}$ and q in some function spaces V and Q respectively, and integrating over the domain Ω , to give

$$Re \int_{\Omega} \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial t} d\Omega + Re \int_{\Omega} \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} d\Omega + \int_{\Omega} \mathbf{v} \cdot \nabla p d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla^2 \mathbf{u} d\Omega = 0 , \quad (86)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u} d\Omega = 0 . \quad (87)$$

To reduce the order of the problem and relax the differentiability requirement on the pressure space, we use integration by parts on the third and fourth term in equation (86), to give

$$Re \int_{\Omega} \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial t} d\Omega + Re \int_{\Omega} \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} d\Omega - \int_{\Omega} p \nabla \cdot \mathbf{v} d\Omega + \int_{\Omega} \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} d\Omega = 0 , \quad (88)$$

where the surface terms have been eliminated via the boundary conditions of the problem given by equations (83) and (84) and via choosing that $\mathbf{v} = 0$ on the boundary.

Having determined the weak form of the governing fluid equations, we go on to solve using an iterative time solver. Our approach to solving is the following; we discretise the space by generating a meshed structure of the cavity geometry. The cavity mesh structure is generated on the dimensionless domain via the open source finite element mesh generator software Gmsh (Geuzaine and Remacle, 2009), using the default triangular meshing settings with a target element size at each node of $dx = 0.25$ (Figure 25). We then begin the iteration over time, starting from $t = 0$, up to $t = T$, with a step size of $dt_{\text{fluid}} = 0.1$. At each time-step the problem to solve is then: find $(\mathbf{u}, p) \in V \times Q$ such that

$$Re \int_{\Omega} \mathbf{v} \cdot \frac{u - u_{\text{prev}}}{dt_{\text{fluid}}} d\Omega + Re \int_{\Omega} \mathbf{v} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} d\Omega + \int_{\Omega} \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} d\Omega - \int_{\Omega} p \nabla \cdot \mathbf{v} d\Omega - \int_{\Omega} q \nabla \cdot \mathbf{u} d\Omega = 0 , \quad (89)$$

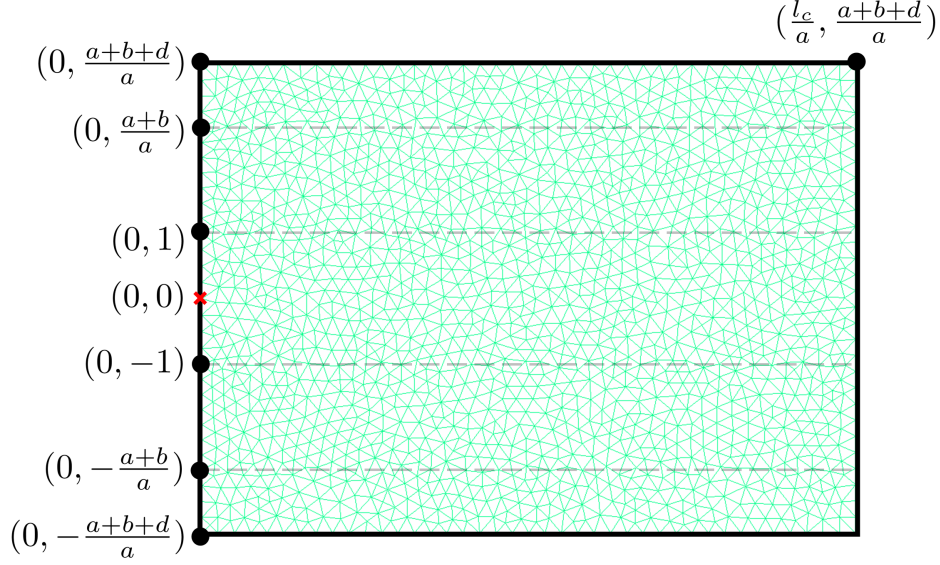


Figure 25: The structured mesh of triangular elements with target element size $dx = 0.25$, constructed in Gmsh.

for all $(\mathbf{v}, q) \in V \times Q$ where V and Q are appropriate function spaces, where we have discretised the time derivative such that

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{u - u_{\text{prev}}}{dt_{\text{fluid}}} , \quad (90)$$

where dt_{fluid} is the time-step of the fluid problem and u_{prev} is the solution to the fluid velocity at the previous time-step. Since u_{prev} is not known for the initial step at $t = 0$, the problem is initialised using the solution to the steady problem, as given by equation (85). At each time-step the problem is solved by employing a stationary incompressible Navier-Stokes solver with Reynolds robust pre-conditioner (Farrell et al., 2019), where we use the previous time-step solution as the value of u_{prev} . The solver handles the choice of functions spaces V and Q as discussed in Farrell et al. (2019).

Further numerical detail on the convergence of the code in support of the values used for time-stepping and mesh size is outlined in Appendix C.4. Visualisation of the finite element results is produced using software package Paraview (Ahrens et al., 2005), with instantaneous streamlines being generating by the software’s default streamline plotter.

3.3.3 Quantifying vortical fluid structures

Previous research indicates that fluid recirculation zones are likely to be problematic when it comes to dust wash-out (Clavica et al., 2014; Coppola and Caro, 2009; Jiménez and Davies, 2009; Min et al., 2020). To quantify such fluid structures, we turn to Jeong and Hussain (1995) who define vortices within a flow field as regions where $\det \nabla \mathbf{u} > 0$. Williams et al. (2021) used this definition to form an objective which is used to minimise recirculation zones. Following this approach, we define the metric

$$\theta(x, y, t) = \max(0, \det \nabla \mathbf{u}) . \quad (91)$$

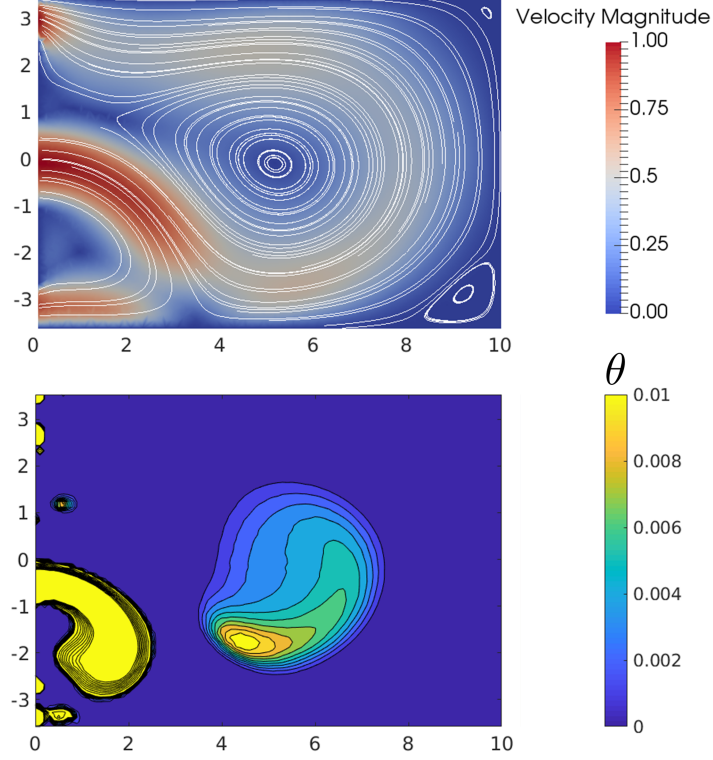


Figure 26: Fluid velocity magnitude with instantaneous streamlines compared with a contour map of θ , for $Re = 50$, $\hat{U} = 0$, and $t = 1000$.

To test this metric, in Figure 26, we give an example which compares the fluid velocity magnitude with instantaneous streamlines in the cavity with the corresponding contours of $\theta(x, y, t)$, for $Re = 50$ and $\hat{U} = 0$, at a time of $t = 1000$. We see strong correlation between the areas of closed streamlines and increased values of θ , with θ capturing the vortices near the inlet as well as the large central vortex in the cavity.

By integrating over the domain we obtain a single measure of vortical structures over time:

$$\Theta(t) = \iint_{\Omega} \theta \, d\Omega \quad (92)$$

A larger value of Θ corresponds to a higher area of the fluid domain containing vortical structures, while variation in Θ over time provides a measure of the disturbance to the vortical structures as their position and magnitude vary. To allow comparison across varying Reynolds numbers, we normalise with respect to the mean velocity scale, and present ΘRe as the metrics of interest.

3.4 Steady flow solution

3.4.1 Evolution of the unsteady problem to steady state

To justify the use of the steady flow solution as an initial condition for the unsteady problem, we investigate the evolution of the unsteady problem from a zero initial condition,

such that $\mathbf{u}^*(\mathbf{x}^*, 0) = \mathbf{0}$. We consider flows driven by a steady parabolic inlet flow profile by setting $\hat{U} = 0$ and examine the evolution of the system to steady state. In addition to the parameters describing the cavity geometry, the system is characterised by the parameter Re . In Figure 27, we present the fluid velocity magnitude with instantaneous streamlines for various times, at $t = 50, 150, 250$, stopping at $T = 1000$, with $Re = 50$. Early on we see a symmetrical flow pattern, with a structure akin to an impinging jet. The symmetry is then broken, as the fluid settles into an asymmetrical state. The flow pattern now displays a single, large vortex within the centre of the cavity. We verify that the solution has indeed settled to its steady state solution by comparing it to the solution of the steady system with the same Reynolds number, and find that the two are the same.

3.4.2 Dependence on Reynolds number

Previous work on steady cavity flow by Williams et al. (2020a) found that there existed a critical Reynolds number beyond which the steady system solution would undergo a saddle bifurcation and display multiple solutions. Below the critical Reynolds number the system had one, stable, symmetrical solution, and above it, this solution became unstable, and two new asymmetrical solutions were found. Since we are using a similar cavity to the previous work, we anticipate a similar solution structure will persist, with a different critical Reynolds number due to the cavity dimensions slightly differing to those used by Williams et al. (2020a). Motivated by this, we now turn to solving the steady fluid problem (as outlined in Appendix C.1) for varying Reynolds number, with the aim of determining the critical Reynolds number.

Having defined a metric to capture vortical structures, we analyse the impact of the Reynolds number on the fluid structure by calculating Θ as the Reynolds number varies. In Figure 28, we give ΘRe for increasing Reynolds number with the fluid velocity magnitude with streamlines for $Re = 15, 40, 70$. We see that as the Reynolds number increases, the value of ΘRe also increases, reflected by the display of a larger central vortex as Re increases. The fluid solutions below $Re = 20$ (circled in red) display the symmetrical fluid solution, and solutions above $Re = 20$ display the asymmetrical fluid solution, highlighted by a significant change in gradient after this Reynolds number.

It appears that the critical Reynolds number lies somewhere around $Re = 20$. To verify this, we aim to determine the solution structure of the steady system. We require a measure to distinguish between different solutions, and so following similar techniques used by Williams et al. (2020a), we take a slice of the fluid velocity u in the y -direction, determine the fluid flux in the upper and lower half, and calculate the difference. We define this parameter as

$$\tilde{u} = \int_0^{a+b+d} u \, dy - \int_{-a-b-d}^0 u \, dy \quad \text{at} \quad x = x_{\text{slice}} . \quad (93)$$

where a value of zero indicates a symmetrical solution and any other value indicates that the solution does not have symmetry in y .

Multiple solutions at a fixed Reynolds number are found by following the techniques used in Williams et al. (2020a), where the initial guess used by the solver is varied to find different solutions. An initial guess of $\mathbf{u} = \mathbf{0}$ returns the symmetrical fluid solution. To

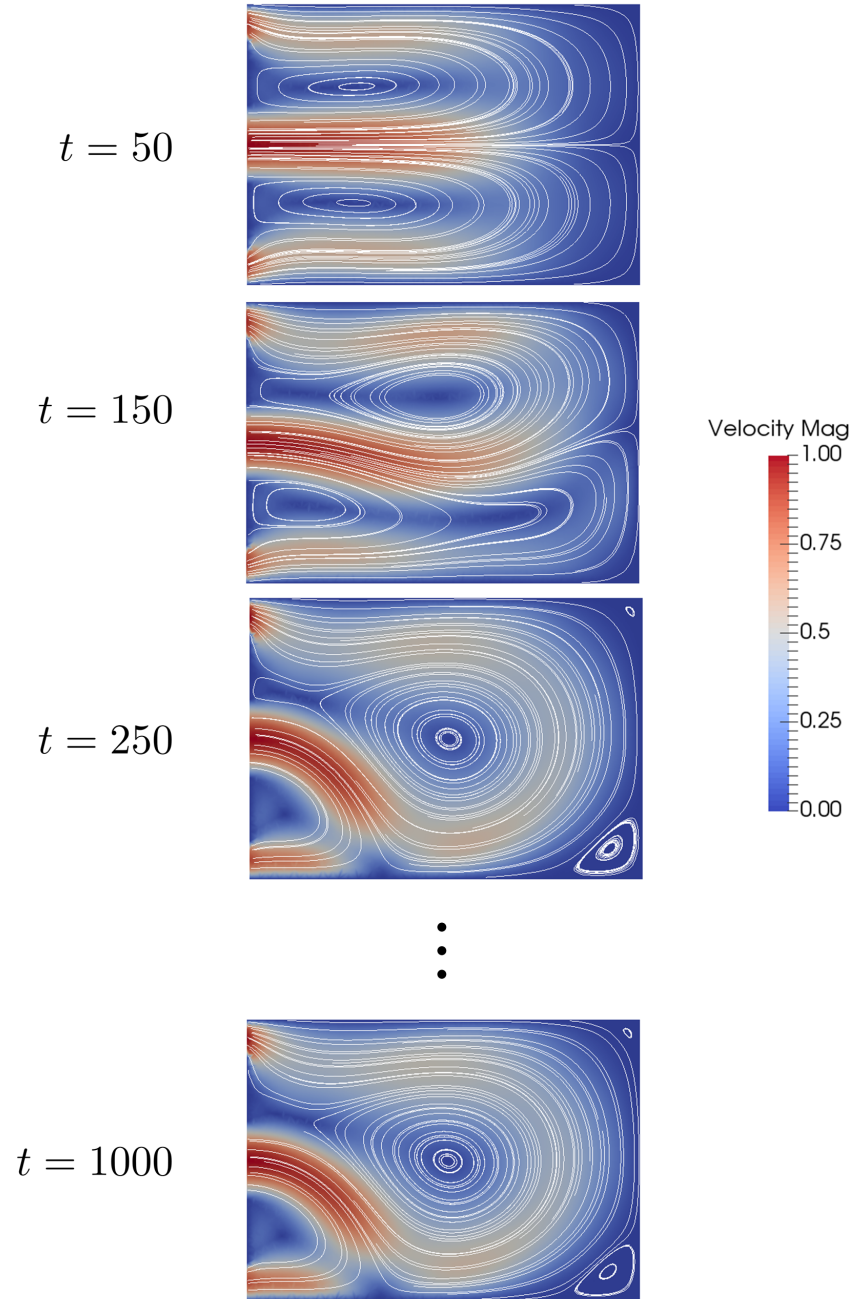


Figure 27: Plots of fluid velocity magnitude with instantaneous streamlines for $t = 50, 150, 250, 1000$, with $Re = 50$

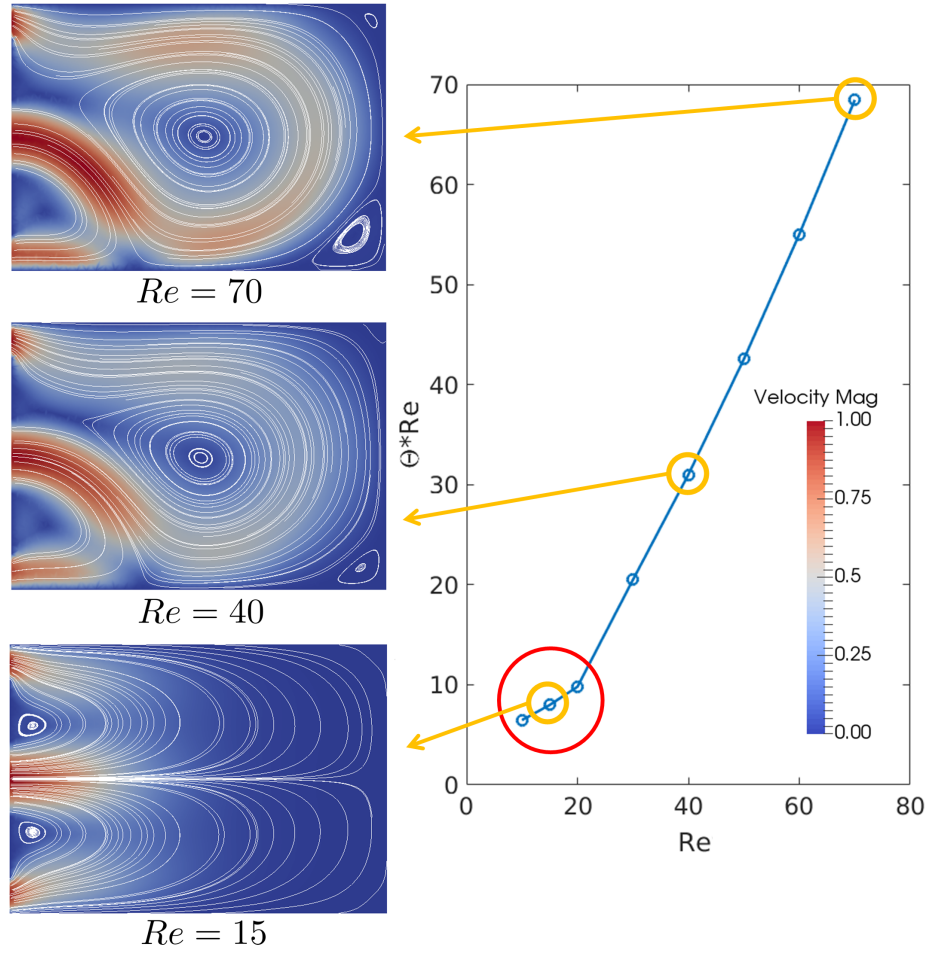


Figure 28: ΘRe for varying Reynolds number, with solutions of fluid velocity magnitude with streamlines for $Re = 15, 40, 70$. Circled in red are the solutions which display a symmetrical fluid structure.

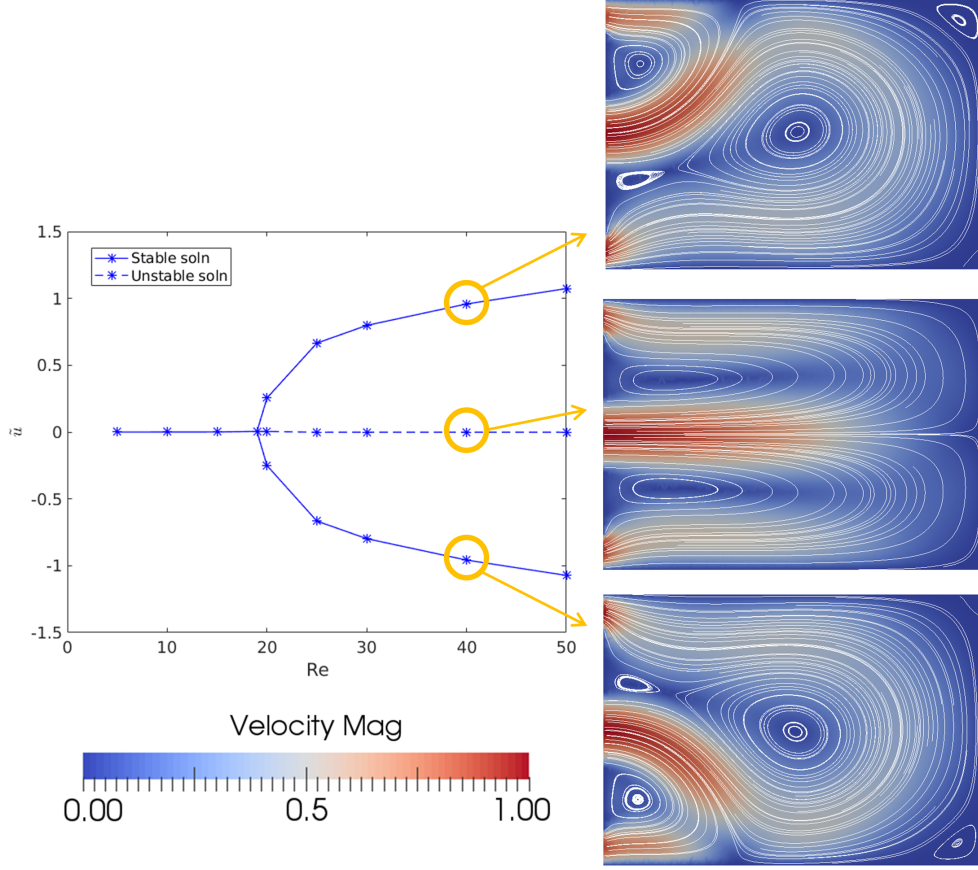


Figure 29: Solution structure of the steady system showing $\tilde{u}$ as Reynolds number increases, along with plots of the fluid velocity magnitude with streamlines for the three solutions at $Re = 40$. Unstable solutions are represented by a dashed line.

seek other solutions, we solve a problem where we block either the upper or lower outlet, by setting $\mathbf{u} = \mathbf{0}$ on the outlet boundary, and then use this solution as the initial guess for the two-outlet problem. This yields one of the two asymmetrical solutions, depending on which outlet is blocked. Performing this technique below the critical Reynolds number always returns the symmetrical fluid solution, regardless on the initial guess used. We anticipate the stability of each solution branch will be the same as displayed in Williams et al. (2020a).

In Figure 29, we present $\tilde{u}$ for increasing Reynolds number, along with fluid velocity with streamlines for the three solutions at $Re = 40$. As expected, we see a similar solution structure as presented in Williams et al. (2020a). We find that the critical Reynolds number is indeed at $Re = 20$. Past this value, the lower branch represents the asymmetrical solution which we have already seen as the steady state solution to the time-dependent problem when using a zero fluid velocity initial condition.

3.5 Oscillatory inlet flow profile results

Returning to the time-dependent problem, we now consider an oscillatory fluid inlet. We are most interested in the impact the oscillations have on disturbing vortical structures

within the cavity, such as the large central vortex displayed in the steady solutions above the critical Reynolds number. Therefore, we choose to focus on Reynolds numbers above the solution bifurcation point of $Re = 20$.

Turning to the two parameters which govern the oscillations of the inlet flow profile, namely ω and $\hat{U}$, we aim to evaluate how effective the proposed oscillatory fluid irrigation delivery method is at reducing vortices within the cavity.

3.5.1 The effect of inlet flow frequency

We begin by varying the inlet frequency ω , fixing the value of Reynolds number at $Re = 50$ and inlet amplitude at $\hat{U} = 20$, representing a 40% variation from the mean flow. Figure 30(b) gives ΘRe over time for inlet flow oscillations with a low frequency of $\omega = 10$ in red and a high frequency of $\omega = 100$ in blue, with $Re = 50$ and $T = 100$. The average value of each case is given as dashed lines, where Θ_{avg} is defined as the average value of Θ over all time. The ΘRe value of a steady inlet flow is also given in green for comparison purposes. Comparing the $\Theta_{\text{avg}} Re$ values of both time-dependent solutions to the steady case, we find a larger value for both a low and high frequencies compared to the steady cases. This implies that on average an oscillating inlet will lead to a system which displays a larger area of vortical fluid structures compared to the steady system. Alongside this, in Figures 30(a) and 30(c) we present contour plots of the cavity circulation metric θ for half a period, for a high and low frequency respectively. The yellow areas within the plots identify zones in which the highest values of θ are present, where the metric has identified vortical structures in the flow. Focusing on the large central vortex being captured by θ , for the high frequency solution in Figure 30(a), the snapshots reveal that its size and location within the cavity does not vary greatly over its half period, with qualitative similarity between each snapshot. In the low frequency case in Figure 30(c) however, we find that θ captures a significant difference in the size and location of the central vortex over the half period. This indicates that, unlike in the high frequency or steady cases, a low frequency inlet flow causes disturbances to the vortical structures within the cavity, most prominently to the central vortex. These vortex disturbances suggest that a low frequency inlet may perform favourably when it comes to dust removal.

It is clear that Θ is periodic in time, thus we define an additional metric Θ_{amp} as the amplitude of Θ over all time, where the larger the value, the larger the variation in vortex structures.

3.5.2 The effect of inlet flow amplitude

Motivated by the result that a low frequency inlet flow causes greater variations in vortical structures over time when compared to a high frequency, we now determine if this behaviour persists for varying values of oscillation amplitude. In Figure 31, we present Θ_{avg} for increasing amplitude $\hat{U}$, with bars for Θ_{amp} , for both a low and high frequency of $\omega = 10$ and $\omega = 100$ given in red and blue respectively, with $Re = 50$ and $T = 100$. We again give the ΘRe value of the steady system for comparison. Note that we do not consider $\hat{U}$ above 50 as values larger than this would cause backflow at the inlet. We see that as the amplitude increases, the average measure of the vortices Θ_{avg} increases, at a similar rate for both frequencies. Compared to the steady system, we see that large amplitude oscillations lead to a much larger measure of vortical structures within the

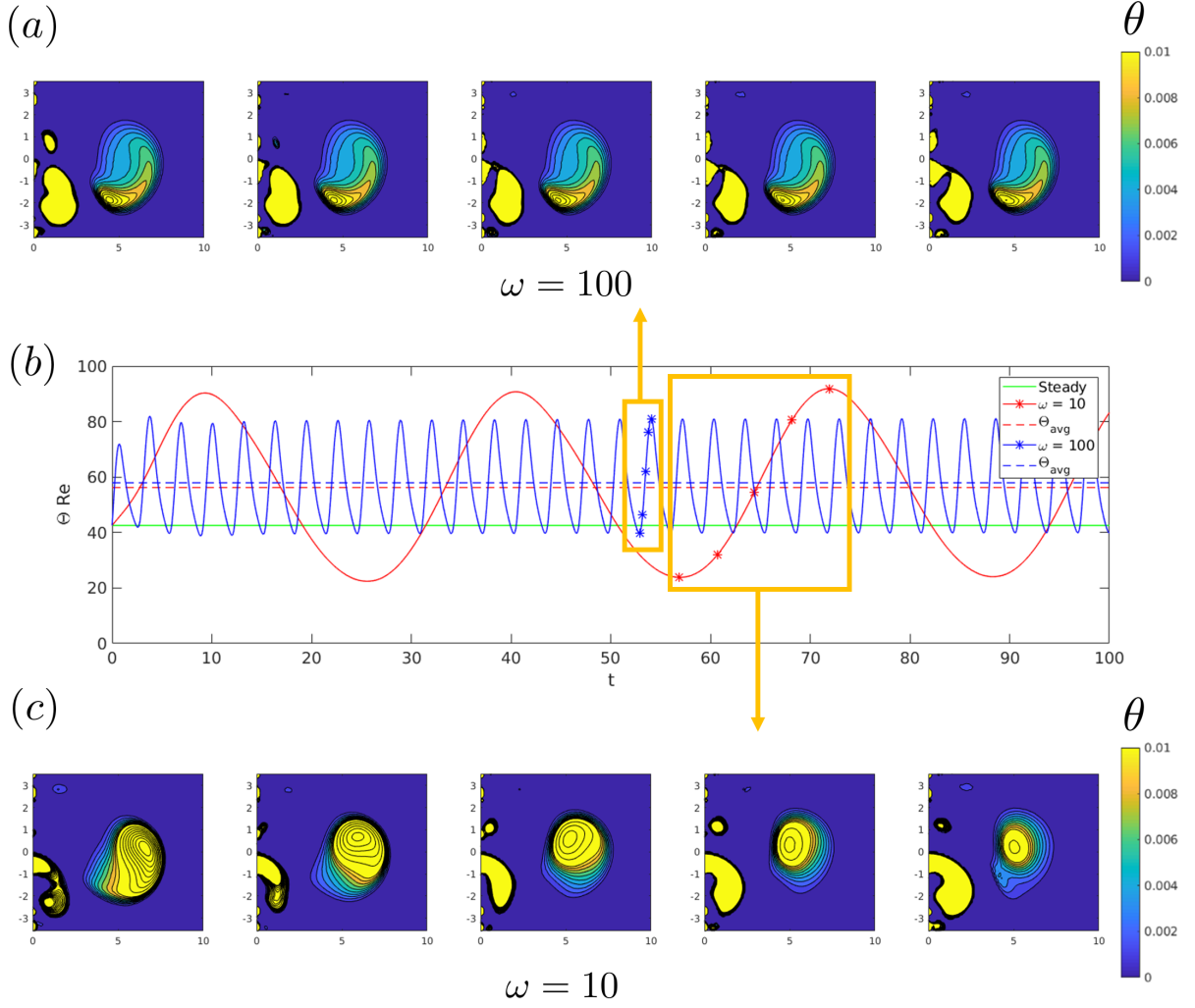


Figure 30: (b) ΘRe over time for $\omega = 10$ in red and $\omega = 100$ in blue, with the average value of each case given as dashed lines, along with the ΘRe value of the steady system in green for comparison purposes. Alongside this, we show contour plots of θ for half a period, with each snapshot time marked by a star in (b), for both (a) $\omega = 100$ and (c) $\omega = 10$, with $Re = 50$ and $\hat{U} = 20$ for each.

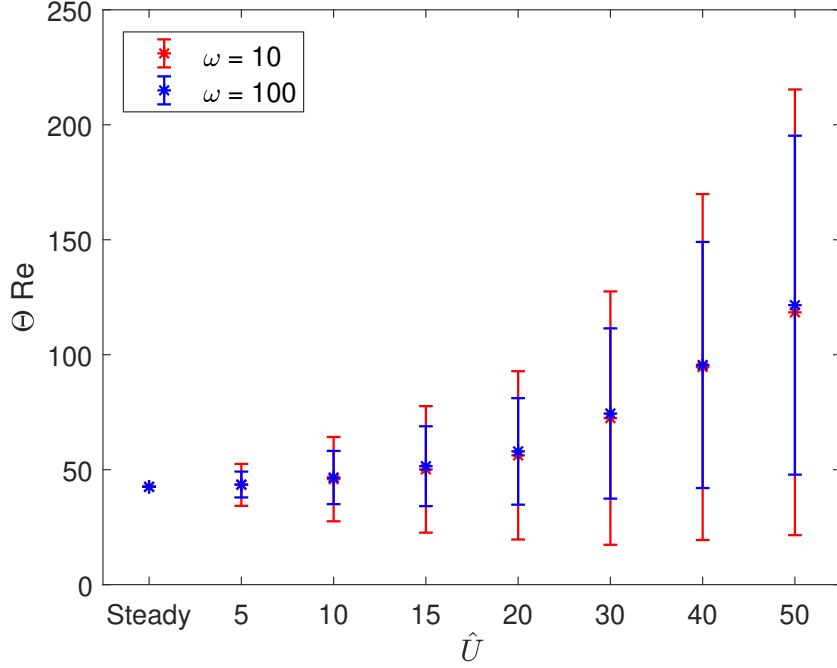


Figure 31: Θ_{avg} for increasing $\hat{U}$ with bars for Θ_{amp} , for $Re = 50$ and $\omega = 10, 100$.

cavity. We find that a lower frequency causes a larger variation in vortices than a higher frequency, and this difference increases as the amplitude increases.

These results suggest that vortical flow structures within the cavity experience the greatest disturbance when the frequency is low and the amplitude is large. In the next section we determine whether disturbing the vortical flow structures will lead to a more efficient removal of kidney stone dust.

3.6 Dust transport model

We consider a dust concentration $c^*(x^*, y^*, t^*)$, which has diffusivity D^* and is advected by the fluid flow. Via conservation of mass we have the advection-diffusion equations given by

$$\frac{\partial c^*}{\partial t^*} + \nabla \cdot \mathbf{J}^* = 0 \quad (94)$$

where

$$\mathbf{J}^* = \mathbf{u}^* c^* - D^* \nabla c^* \quad (95)$$

is the flux of dust where the first term describes the advective transport of dust due to the background fluid flow and the second term describes the diffusion of the dust. Since the flow is incompressible, we can rewrite the advection-diffusion equation as

$$\frac{\partial c^*}{\partial t^*} = D^* \nabla^2 c^* - \mathbf{u}^* \cdot \nabla c^* . \quad (96)$$

We impose that there is no flux of dust through the impermeable walls of the cavity, so that

$$\mathbf{J}^* \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{walls}}^* . \quad (97)$$

Since $\mathbf{u}^* = \mathbf{0}$ on Γ_{walls}^* (See equation (77)), we automatically have no advective flux and to satisfy the condition of no total flux it is sufficient to impose

$$\nabla c^* \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{walls}}^* . \quad (98)$$

To model a cavity which is being irrigated by clean water, at the inlet to the cavity we impose that there is no-flux of dust, such that

$$\mathbf{J}^* \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{in}}^* . \quad (99)$$

We consider a dust concentration that can exit the cavity via advection only, and so at the outlet we impose that the dust cannot diffuse through the boundaries via

$$\nabla c^* \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{out}}^* . \quad (100)$$

Finally, to close the problem, we assume that initially throughout the entire cavity domain there is a uniform concentration of dust, giving the initial condition

$$c^*(\mathbf{x}^*, 0) = C^* \quad \text{on} \quad \Omega^* , \quad (101)$$

where C^* is a constant.

We non-dimensionalise as before, via equation (79), together with

$$c^* = C^* c . \quad (102)$$

The non-dimensional advection-diffusion equation is then

$$\frac{\partial c}{\partial t} = \frac{1}{ReSc} \nabla^2 c - \mathbf{u} \cdot \nabla c , \quad (103)$$

where $Sc = \mu^*/\rho^*D^*$ is the Schmidt number. Equation (103) is then solved subject to the boundary conditions

$$\nabla c \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{walls}}, \Gamma_{\text{out}} , \quad (104)$$

$$\left(\mathbf{u}c - \frac{1}{ReSc} \nabla c \right) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{in}} , \quad (105)$$

and initial condition

$$c(\mathbf{x}, 0) = 1 \quad \text{on} \quad \Omega . \quad (106)$$

The dust transport problem introduces an additional parameter into the system, the Schmidt number Sc , which is the ratio of viscosity and mass diffusivity and characterises the diffusivity of the dust.

3.7 Dust transport results

3.7.1 Metrics

To characterise dust concentration and the efficiency of wash-out, we define two metrics. To analyse the change in concentration across the entire cavity over time, we evaluate the net concentration at any given time and compare it with the net concentration at time zero, to give the concentration percentage metric γ , defined by

$$\gamma = 100 \frac{\iint_{\Omega} (c(\mathbf{x}, t)) \, d\Omega}{\iint_{\Omega} c(\mathbf{x}, 0) \, d\Omega} . \quad (107)$$

We also follow Williams et al. (2021) and define the metric T_{90} as the time taken for 90% of dust to leave the cavity via

$$\frac{\iint_{\Omega} (c(\mathbf{x}, 0) - c(\mathbf{x}, T_{90})) \, d\Omega}{\iint_{\Omega} c(\mathbf{x}, 0) \, d\Omega} = 0.9 . \quad (108)$$

The smaller the value of this metric the more efficient the fluid irrigation is at dust removal. As before, to allow comparison across varying Reynolds numbers, we normalise T_{90} with respect to the mean velocity scale, and present T_{90}/Re as the metric of interest.

3.7.2 Numerical details

The system is solved numerically via a finite element formulation, implemented using the open source finite element library Firedrake (Rathgeber et al., 2017). We transform equation (103) into weak form by multiplying by an arbitrary test function v in a space of continuous, Galerkin functions V and integrating over the domain Ω , to give

$$\int_{\Omega} v \frac{\partial c}{\partial t} \, d\Omega = \frac{1}{ReSc} \int_{\Omega} v \nabla^2 c \, d\Omega - \int_{\Omega} v (\mathbf{u} \cdot \nabla) c \, d\Omega , \quad (109)$$

Performing integration by parts on the second and third terms in equation (109) and applying boundary conditions from equations (104) gives the weak form of the advection-diffusion equation

$$\begin{aligned} \int_{\Omega} v \frac{\partial c}{\partial t} \, d\Omega = & -\frac{1}{ReSc} \int_{\Omega} \nabla v \cdot \nabla c \, d\Omega \\ & + \int_{\Omega} c \nabla \cdot (v \mathbf{u}) \, d\Omega - \int_{\Gamma_{\text{out}}} v c \mathbf{u} \cdot \mathbf{n} \, d\Gamma_{\text{out}} . \end{aligned} \quad (110)$$

The fluid solution is taken as an input to equation (110) which we then solve numerically via the inbuilt Firedrake solver. We use a midpoint method for the time derivative and a Runge-Kutta scheme for time-stepping. The cavity mesh structure is unchanged from before. We take the time-step of the dust simulation to be $dt_{\text{dust}} = 0.01$. Since this is an order of magnitude smaller than the fluid time-step, the fluid solution used as input into the dust transport equation at each dust time-step is the solution at the next available fluid time-step. For example, from $t = 0.1, 0.11, 0.12 \dots$ up to $t = 0.19$, the dust transport equation uses the fluid solution from $t = 0.2$. We opt to change the end-time condition so that both the fluid and dust transport solvers now terminate when $t = T_{90}$. Appendix C.5 gives further numerical details on convergence of the dust transport solver in support of the values used for time-stepping and mesh size.

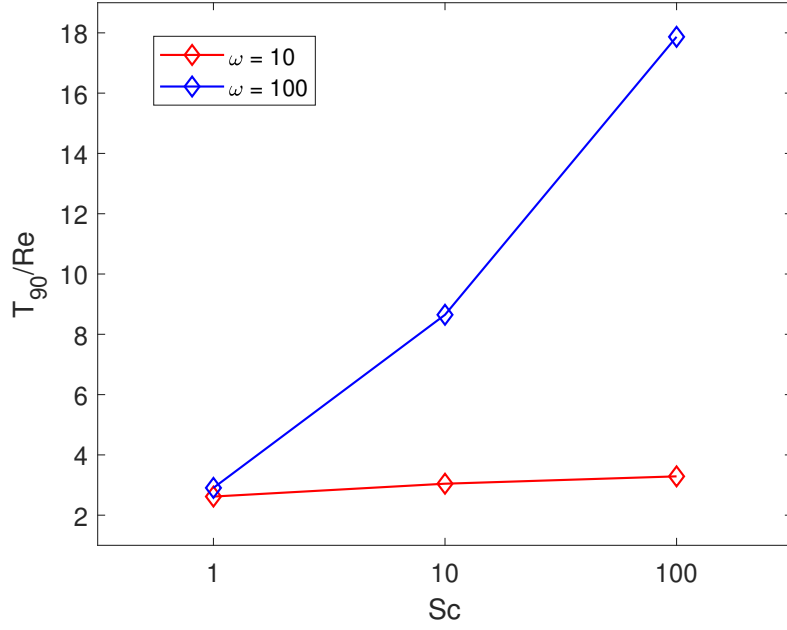


Figure 32: T_{90}/Re as a function of Schmidt number for $\omega = 10, 100$ in red and blue respectively, with $Re = 50$ and $\hat{U} = 20$.

3.7.3 The effect of the Schmidt number

We now examine the role of the Schmidt number on the dust transport characteristics. In Figure 32, we plot T_{90}/Re as a function of Schmidt number, for both low and high frequency inlet flow given in red and blue respectively, with $Re = 50$ and $\hat{U} = 20$. As the Schmidt number increases, T_{90}/Re increases, and the increase with Sc is greater for the higher frequencies. At lower Schmidt numbers, corresponding to dust with a high diffusivity, dust is able to diffuse across closed streamlines within vortices more rapidly than higher Schmidt number dust. Therefore, we predict that disturbing the vortices will have greater impact on dust wash-out at higher Sc , and this is indeed seen in Figure 32.

We do not have an exact value for the diffusivity of kidney stone dust, but we assume it to be within the range of similar substances, such as small spherical particles in water, which closely resembles the saline solution used during ureteroscopy (Williams, 2019). We therefore opt to select a Schmidt number based on the need for a sufficiently low value such that dust will become trapped within vortices, whilst also reducing computation time. Computationally, due to the iterative nature of the solver, the higher the value of T_{90} the longer the computation, so that simulation time increases significantly with Sc . To draw focus on the advective effects of the fluid, going forward we fix the Schmidt number to $Sc = 10$. This value provides slower diffusivity, for which dust clouds will persist in cavity vortices for longer time, while still granting computational ease by reducing the overall solution time. When analysing the system with $Sc = 10$, we keep in mind that larger Schmidt numbers would lead to a slower diffusion process, so any scenarios which rely heavily on diffusive effects would have increased wash-out times. We anticipate that although the quantitative characteristics, such as wash-out time, may change, the qualitative conclusions drawn will still apply for higher Schmidt numbers.

3.7.4 The effect of inlet flow frequency

Having determined the effect low and high frequency fluid inlet oscillations have on the vortical structures in the flow, we are motivated to discover how this impacts dust wash-out. In Figure 33(b), we give concentration percentage γ over time for inlet flows with frequency $\omega = 10$ and $\omega = 100$, in blue and orange respectively, with $Re = 50$, $Sc = 10$, and $\hat{U} = 20$. The red star indicates the time at which T_{90} has been reached, when 90% of the dust has left the cavity. Initially both frequency cases remove dust at a similar rate. However, when around 40% of the dust remains, the high frequency case dramatically slows down, taking a significant amount of time to reach the T_{90} value. In Figures 33(a) and 33(c), we give snapshots of the dust concentration in the cavity for the high and low frequency cases respectively, at $t = 25, 75, 125$. For both cases, initially the dust is washed out of the cavity through both the upper and lower outlets. In the high frequency case, dust starts to accumulate in the centre of the cavity as it becomes trapped within the large central vortex. As time continues, the dust cloud persists as the dust relies on the slow diffusive effects to exit the vortex. This behaviour explains the high T_{90} value seen for a high frequency inlet flow. In the low frequency case however, we see that the dust does not accumulate within the centre of the cavity due to the mixing effect arising from the disturbance of the vortical structures caused by the low frequency inlet flow, as shown earlier (Figure 30). This explains the much quicker wash-out time compared to the former case, as the dust transport does not need to rely on diffusive effects.

Taking the frequency analysis further, we explore a wider range of frequencies, and in Figure 34, we present T_{90}/Re for increasing frequency ω , for two amplitudes of $\hat{U} = 5, 20$ in blue and orange respectively, with $Re = 50$ and $Sc = 10$. The T_{90}/Re value for a steady inlet flow is also shown in red for comparison purposes. We see that optimal wash-out times occur for a frequency around $\omega = 10$. As we decrease the frequency from this value towards zero, wash-out times tend towards the steady value. This agrees with the theoretical formulation where setting ω to zero recovers the steady system. As the frequency increases from $\omega = 10$, the wash-out times increase, with the high frequency wash-out times again becoming more similar to the steady system. This result is explained by the observation that at a high frequency the large, central fluid vortex in the cavity remains undisturbed, much like in the steady scenario. We conclude that lower frequencies around $\omega = 10$ produce the best wash-out times due to their ability to disturb vortices in the flow.

3.7.5 The effect of inlet flow amplitude

We now turn to investigating a wider range of amplitudes and the impact on dust transport. In Figure 35, we give T_{90}/Re for increasing amplitude $\hat{U}$, for inlet flow frequencies of $\omega = 10$ and $\omega = 100$, in red and blue respectively, with $Re = 50$ and $Sc = 10$. For the high frequency case, a larger amplitude causes an increase in wash-out times. This is due to the central vortex becoming larger as the amplitude increases, as shown in Figure 31, causing more dust to be trapped within it. For the lower frequency case however, the opposite behaviour occurs, with the wash-out time decreasing as the amplitude increases. The mixing effect we have seen at lower frequencies (Figure 33) intensifies as amplitude increases, leading to faster wash-out times.

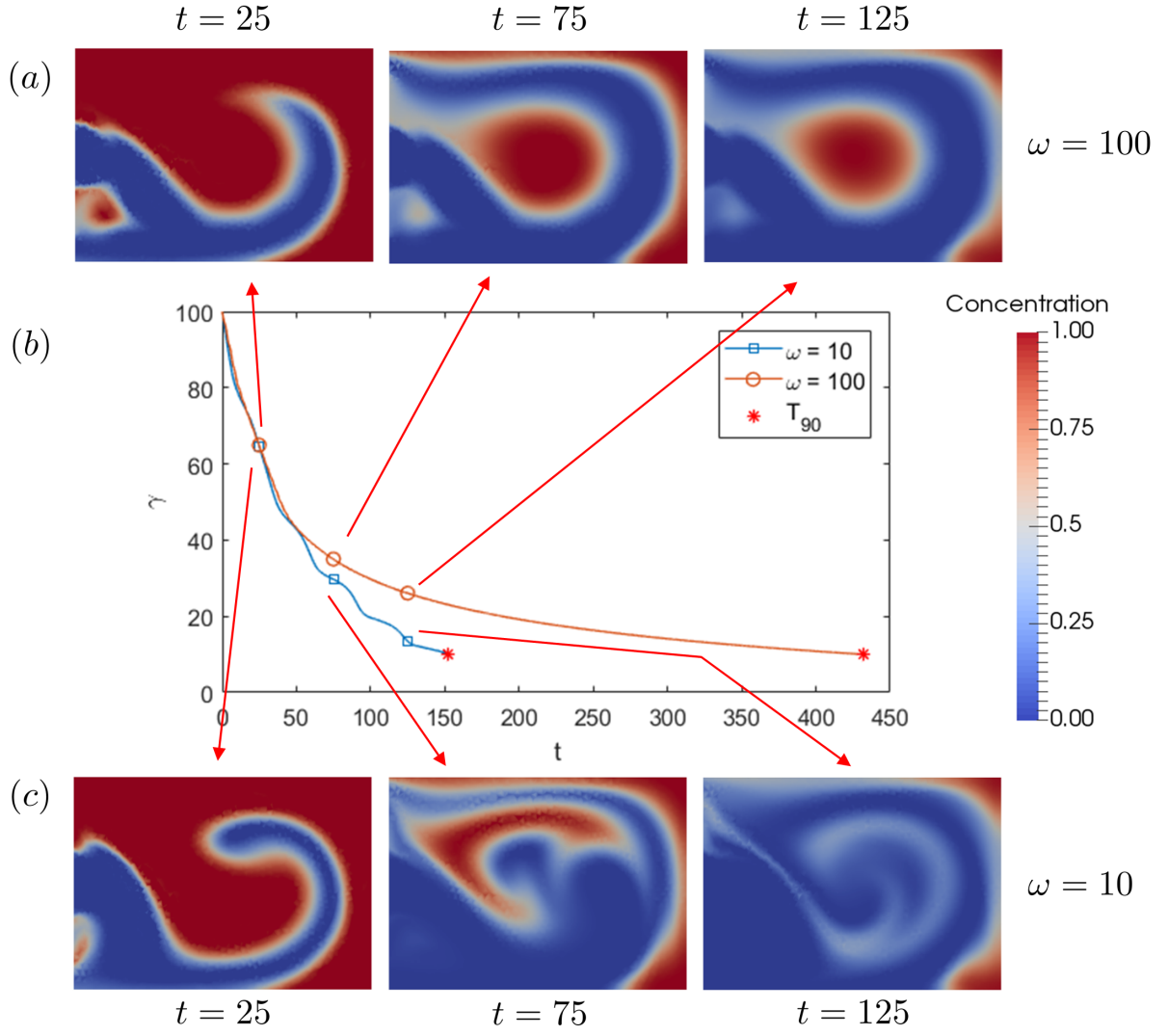


Figure 33: (b) Concentration percentage γ over time for $\omega = 10$ and $\omega = 100$ in blue and orange respectively, along with snapshots of dust concentration c in the cavity for (a) $\omega = 100$ and (c) $\omega = 10$, for $Re = 50$, $Sc = 10$ and $\hat{U} = 20$. The red stars in (b) indicate when the value of T_{90} has been reached.

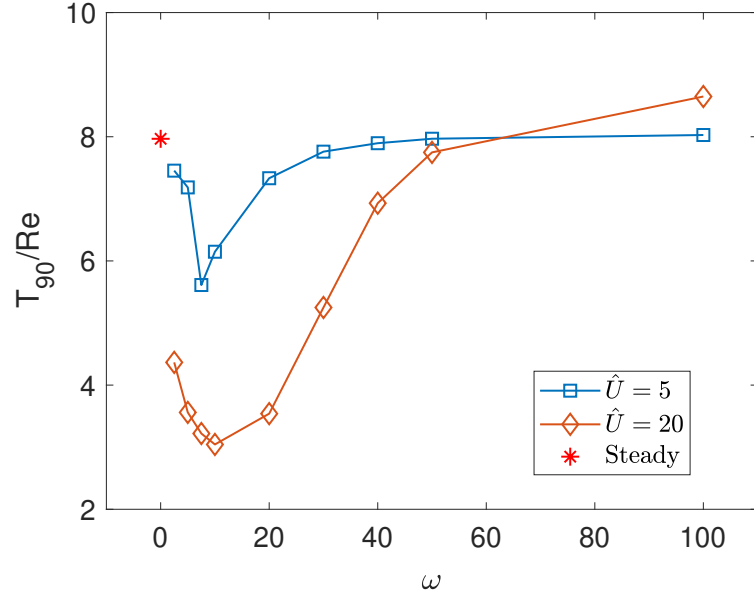


Figure 34: T_{90}/Re as a function of ω for $\hat{U} = 5$ and $\hat{U} = 20$ in blue and orange respectively, with $Re = 50$ and $Sc = 10$. The red star indicates the value of T_{90}/Re for a steady parabolic inlet flow.

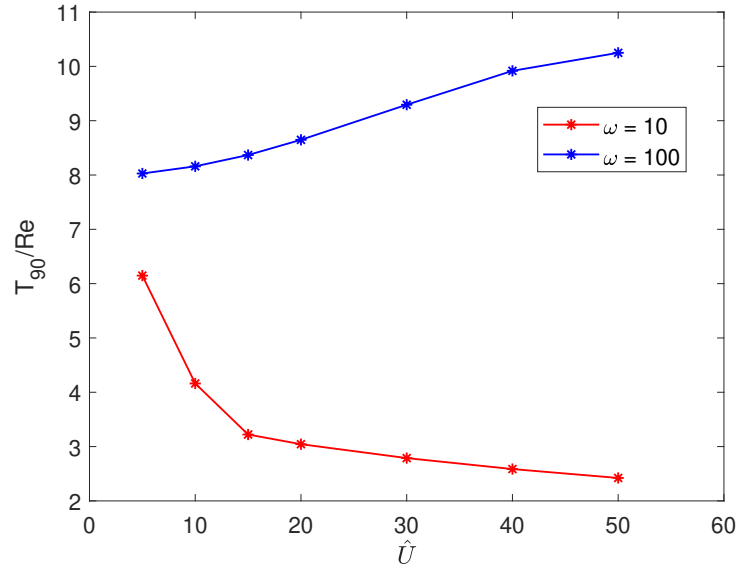


Figure 35: T_{90}/Re for increasing $\hat{U}$ for $\omega = 10$ and $\omega = 100$ in red and blue respectively, for $Re = 50$ and $Sc = 10$.

3.7.6 The effect of the Reynolds number

These results from Sections 3.7.4 and 3.7.5 lead to the conclusion that in the regime of fixed Reynolds number $Re = 50$, an inlet flow which is oscillating at a low frequency of $\omega \sim 10$ and a high amplitude of $\hat{U} \sim 20 - 50$ produces minimal wash-out times, due to the disturbance of vortical structures within the flow.

Having established how the inlet frequency and amplitude impact wash-out times, we now vary the Reynolds number to determine whether the features we have found for fixed $Re = 50$ still hold. In Figure 36, we show T_{90}/Re for increasing Reynolds number with $Sc = 10$, for a low and high frequency of $\omega = 10, 100$ represented by red and blue lines respectively, and for a low and high amplitude of $\hat{U} = 5, 20$ represented by square and diamond marks respectively. Alongside this we give the wash-out times for the steady fluid solution represented by pink stars. We focus on the asymmetrical fluid solution and hence start at $Re = 30$. For all values of Reynolds number, the previous results hold; a low frequency and high amplitude inlet flow produces the best wash-out times. For the high frequency inlet flows given in blue, an increase in Reynolds number leads to an increase in wash-out times, due to the central vortex growing in size as the Reynolds number increases, as previously shown in Figure 28. These results also highlight the similarity between the steady fluid solution (pink stars) and an inlet flow profile with a high frequency and low amplitude (blue squares), with the two yielding comparable wash-out times. Turning to the low frequency inlet flows given in red, we see that for a low amplitude (red squares) the wash-out times stay fairly consistent for increasing Reynolds number, until $Re = 70$ when an increase is seen. For inlet flows with a high amplitude (red diamonds), there is an optimal Reynolds number, somewhere around $Re = 50$. Both these cases highlight the competition between the mixing effect created by a low frequency inlet and the size of the central vortex. Where the mixing effect is dominant, a larger Reynolds number can actually be beneficial, since when dust is disturbed out of the central vortex it is advected away faster, as shown by the improving wash-out times from $Re = 30$ to $Re = 50$ in the high frequency, low amplitude case.

3.8 Relating fluid characteristics to wash-out times

From all of the above results, we conclude that to optimise wash-out times, we require a fluid inlet velocity profile that is oscillating at high amplitude and low frequency. Such oscillating conditions give rise to the largest disturbance of the central fluid vortex. We hypothesise that wash-out times are directly related to vortex properties, such that wash-out times will decrease for flows which display less vortices and have a greater variation in vortices over time. To test this hypothesis, we explore a wider range of system parameters Re , ω , and $\hat{U}$, the full list of which is given in Appendix C.2. In all cases the Schmidt number is $Sc = 10$. For each case we calculate the Θ_{avg} , Θ_{amp} , and T_{90} metrics, and create a linear polynomial fit to determine whether a relationship exists between them, such that

$$T_{90} = p_{00} + p_{10}\Theta_{\text{avg}} + p_{01}\Theta_{\text{amp}} , \quad (111)$$

The values of p_{00} , p_{10} , and p_{01} are determined using the inbuilt curve fitting tool in Matlab (The MathWorks, 2020). In Figure 37, we present the linear polynomial fit of

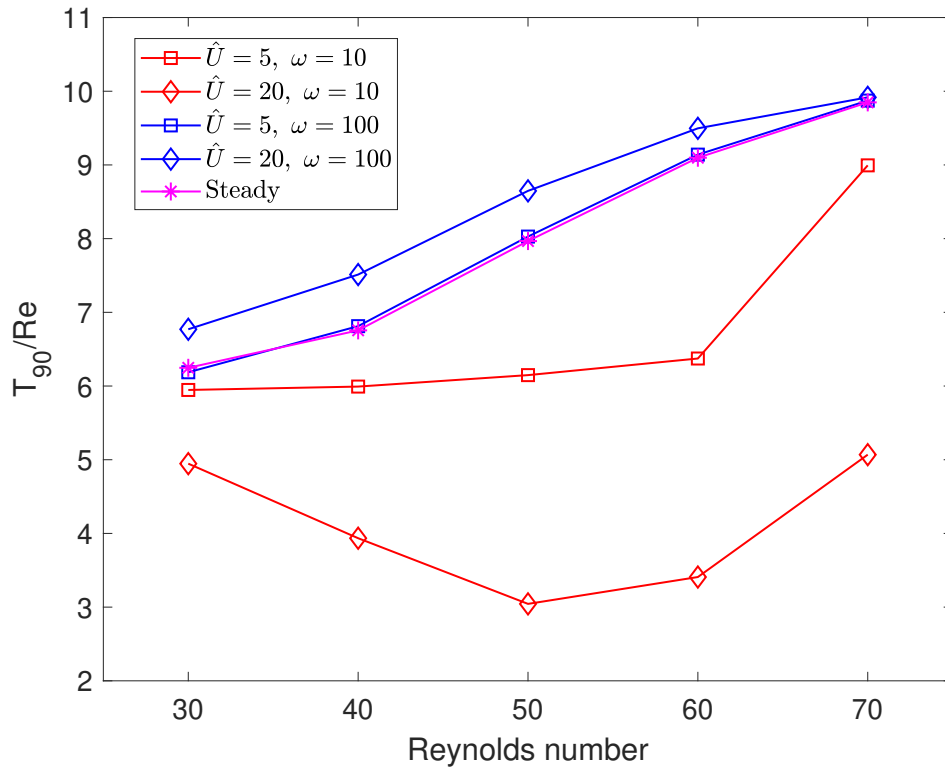


Figure 36: T_{90} for increasing Reynolds number, for $Sc = 10$, $\hat{U} = 5, 20$, and $\omega = 10, 100$, as well as the steady results.

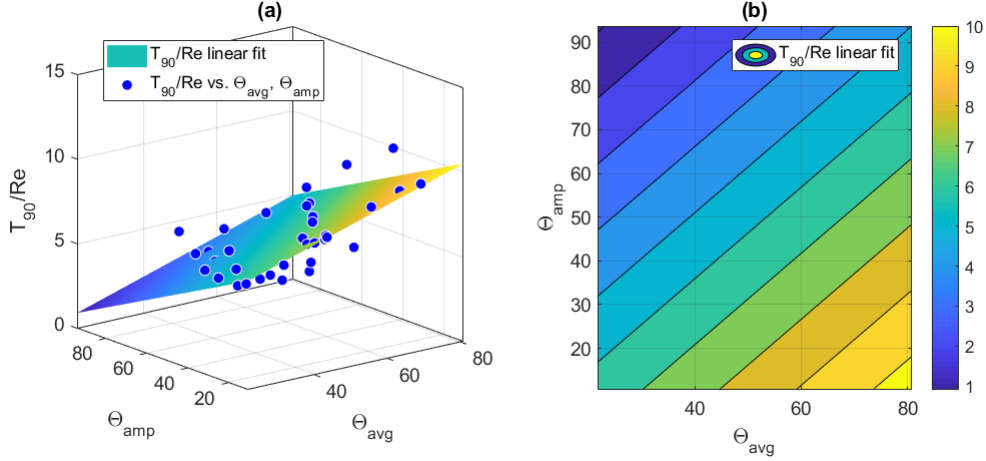


Figure 37: (a) Data points of T_{90} against Θ_{avg} and Θ_{amp} , given by blue dots, along with a linear polynomial fit of the data, with a contour plot of this fit in (b).

T_{90}/Re vs. Θ_{avg} and Θ_{amp} for 42 different parameter variations. In Figure 37(a), we give the produced fit compared against the values of the data points, with this fit given as a contour plot in Figure 37(b). The fit has an R squared value of $R^2 = 0.62$. From the contour plot, we have agreement with the previous conclusion; the smallest T_{90}/Re values are given when Θ_{avg} is smallest and Θ_{amp} is largest; when the system displays a smaller area of vortical structures and the variation in these structures over time is large.

3.9 Comparison to the leading industry method

We have shown that under the right conditions an oscillatory cavity inlet flow can disturb vortices within the cavity and reduce wash-out times. In practice, a clinician has a similar technique to disturb accumulations of kidney stone dust. The process involves delivering an extra bolus of fluid into the cavity to wash away any excess dust that may be obscuring the view of the camera. In the steady, hanging saline bag set-up, a bolus of fluid is delivered by squeezing the bag to introduce more upstream pressure and hence a brief increase in flow rate. This technique is known as flushing.

An electronic pump may replicate the flushing process by increasing its rotation rate for a short period of time, thus increasing the system flow rate. Due to the nature of how the flush is administered, the magnitude of the flush can be quite variable. Instead, using an electronic pump to control the delivery of a flush opens the possibility to have finer control over the magnitude of the flush, as well as the time at which it is administered. Therefore, it is of interest to determine how varying the magnitude, start time, and duration of the flow rate increase impacts wash-out times. To further analyse the conclusions on oscillatory flow, we also compare the wash-out times in a simulation of the flushing procedure to oscillatory inlet flow scenarios. By computing the T_{90} times, we determine how flushing parameters impact wash-out and moreover how an oscillatory inlet flow compares to the current leading industry method for disturbing cavity debris.

To simulate the flushing process, we consider an initially steady inflow of fluid, i.e. a fixed parabolic inlet flow profile. Then, at a given time, the inlet flow rate is increased, either for a fixed duration or until the end of the simulation, which is defined as before by

90% of the dust being advected out of the cavity. The model is the same as constructed in Section 3.3, consisting of the time-dependent Navier-Stokes equations (71) and (72) and outlet and wall boundary conditions (76) and (77); the only change when simulating flushing being that we replace the oscillatory inlet condition (73) by the piece-wise step function

$$\mathbf{u}^* = \left(\frac{1}{a^2} (\bar{U}^* + k^*) (a^2 - y^{*2}), 0 \right) \quad \text{on} \quad \Gamma_{\text{in}}^*, \quad (112)$$

where

$$k^* = \begin{cases} 0 & t^* < F_{\text{start}}^* \\ \kappa^* & F_{\text{start}}^* \leq t^* < F_{\text{stop}}^* \\ 0 & F_{\text{stop}}^* \leq t^* \end{cases}, \quad (113)$$

where κ^* is the increase in mean velocity during flushing, F_{start}^* is the time at which flushing starts, and F_{stop}^* is the time at which the flushing stops.

The model for the transport of dust remains unchanged, given by equations (103) and (104). The dust initial condition also remains the same, such that dust fills the entire domain at $t = 0$.

We non-dimensionalise via equation (79) to obtain the non-dimensional, Navier-Stokes equations as given by equations (80) and (81) with boundary conditions given by equations (83), (84), and

$$\mathbf{u} = \left(\left(1 + \frac{k}{Re} \right) (1 - y^2), 0 \right) \quad \text{on} \quad \Gamma_{\text{in}}, \quad (114)$$

where

$$k = \begin{cases} 0 & t < F_{\text{start}} Re \\ \kappa & F_{\text{start}} Re \leq t < F_{\text{stop}} Re \\ 0 & F_{\text{stop}} Re \leq t \end{cases}, \quad (115)$$

where $\kappa = \rho^* \kappa^* a / \mu^*$ and $F_i = \mu^* F_i^* / \rho^* a^2$ for $i = \text{start}, \text{stop}$. Here κ/Re characterises the fractional percentage increase from the initial maximum inlet flow velocity, the larger the increase the larger the flush magnitude has been applied. The numerical solver terminates once the value of T_{90} has been reached.

As defined here, flushing consists of 3 independent parameters; κ , F_{start} , and F_{stop} . Conceptually, we anticipate that both the sudden increase in flow rate at $t = F_{\text{start}} Re$ and subsequent decrease at $t = F_{\text{stop}} Re$ creates a vortex disturbance in a similar manner to the oscillatory inlet, as shown in Section 3.5. To investigate the significance of the start and stop times on wash-out reduction, we begin by fixing the flushing magnitude and varying the time at which the flush is administered for three scenarios, outlined below and illustrated in Figure 38. The light blue points correspond to the case of a continuous flush which once started does not stop. The second and third cases, given in orange and yellow respectively, are flushes of a specified duration of $t = 25$, a short flush, and $t = 75$, a long flush, i.e. $F_{\text{stop}} Re = F_{\text{start}} Re + 25$ and $F_{\text{stop}} Re = F_{\text{start}} Re + 75$. In all of these cases the magnitude of the flush is 80%, with $\kappa = 40$, to allow comparison with an inlet

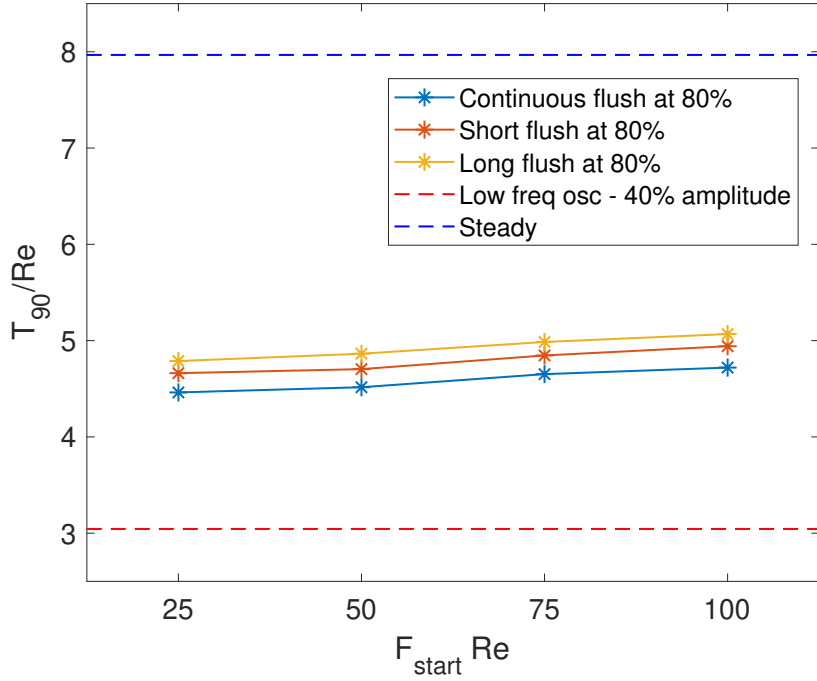


Figure 38: T_{90}/Re for varying flush start times, for both a continuous, short, and long flush, compared with a steady inlet and an oscillating inlet with $\omega = 10$ and $\hat{U} = 20$, for $Re = 50$, $Sc = 10$, and $\kappa = 40$.

flow profile oscillating with an amplitude of 40%. In Figure 38, we also give the wash-out times for both the steady inlet flow and oscillatory inlet flow with $\omega = 10$ and $\hat{U} = 20$, in dark blue and red dashed, horizontal lines respectively. We see that all cases of flushing provide much better wash-out times compared to the steady scenario. We find that the later the flush is administered, the less impact it has on washing away dust, although the difference from 25 to 100 is small compared to the steady scenario. A short flush performs better than a long flush, suggesting that the closer the start and end times are together the greater the impact of dust wash-out. However, the benefit of increased magnitude gained from continually flushing seems to out-perform the added benefit of a second mixing effect generated from stopping the flush, since better wash-out times are seen if the flush is not turned off once administered. This suggests that administering a flush aids in improving wash-out times by disturbing the central vortex in the cavity, however the time at which the flush starts and stops has little impact on the degree to which the vortex is disturbed. Comparing the flushing cases to a low frequency, high amplitude oscillatory inlet flow, we see that the oscillating inlet performs favourably to all cases.

Having found that an oscillating inlet flow profile produces a significantly lower T_{90}/Re compared to flushing at 80% magnitude, it is of interest to determine what level of flush magnitude would be required in order to perform comparably with the oscillatory case. In Figure 39, we plot the maximum inlet velocity over time for increasing flush magnitudes from 0 – 200%, along with an oscillatory inlet flow with $\omega = 10$ and $\hat{U} = 20$. The time at which each case stops is indicated with a yellow marker, when 90% of the dust has been

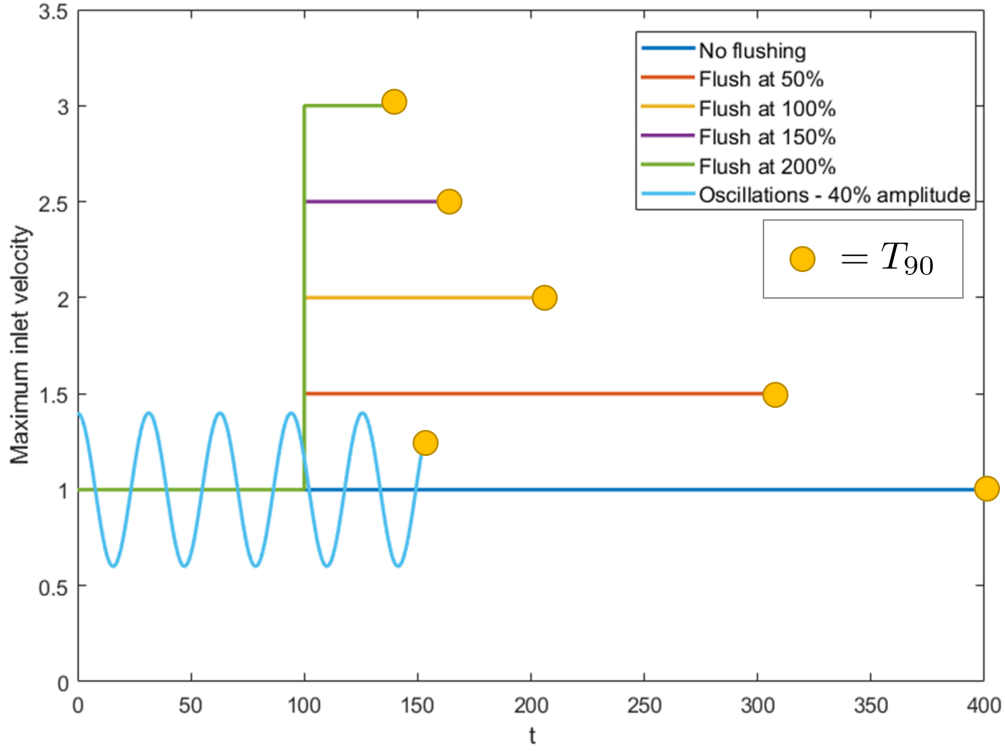


Figure 39: Maximum inlet velocity over time for increasing flush magnitude with $F_{\text{start}}Re = 100$, compared with an oscillating inlet flow profile with $\omega = 10$ and $\hat{U} = 20$, for $Re = 50$ and $Sc = 10$. Each case stops once the T_{90} value has been reached, indicated by the yellow markers.

cleared from the cavity. We consider a continuous flush which is not stopped, and the time at which the flush is introduced is the same for all cases, with $F_{\text{start}}Re = 100$. We see that as we increase the flush magnitude so that the maximum inlet velocity increases, the dust is washed away faster. The relationship between T_{90} and κ/Re appears to be non-linear, suggesting less of an improvement for ever increasing flush magnitudes. While these results show increasing the magnitude of the flush improves dust removal time, it may have additional negative effects within the cavity such as introducing higher shear stresses on the kidney walls and increasing cavity pressures. Comparing these results to the best case oscillatory scenario we see similar wash-out times to the 200% flush. However, the oscillatory case only increases to a maximum of 40% larger than the base maximum inlet flow velocity. We also see the maximum inlet velocity dropping below the base maximum inlet velocity for a significant period of time. We conclude that oscillating the incoming fluid achieves similar wash-out times at a drastically reduced flow. The oscillating inlet technique benefits from being passive throughout the ureteroscopy procedure, reducing the likelihood of stagnant vortices forming within the cavity.

3.10 Extension to different domains

Up to now we have considered a single, idealised geometry of the kidney cavity, which we now refer to as *ideal*. This domain is both two-dimensional and rectangular to simplify the problem. Relaxing the assumption of a two-dimensional cavity and exploring a three-dimensional geometry is beyond the scope of this work. The question remains as to whether the results that an oscillatory inlet improving wash-out times is robust to changes in the size and shape of the domain? To investigate, we consider six different cavity geometries, five of which are extensions of the previous rectangular domain, and a sixth based on more complex geometries seen in situ. We determine whether the same qualitative fluid flow features persist. Further to this, we determine if the dust wash-out results from the original domain still hold by evaluating the wash-out time in two scenarios for each domain: the first with a steady parabolic inlet and the second with an unsteady, oscillatory inlet flow profile. We choose to set the parameters based on the previous analysis, and focus on a single oscillatory case of low frequency and high amplitude inlet flow for each domain. The steady and oscillatory parameters considered are the same across all domains. We aim to test whether the conclusion of oscillatory flow improving wash-out times holds in other domains.

In Figure 40, we present the seven domains considered, along with their respective steady fluid velocity magnitude solutions with streamlines. On each domain sketch (apart from *kidney*) the original geometry *ideal* is shown in grey for comparison purposes, with the inlet shown in blue and the outlets shown in red. The dimensions of each domain are given in Appendix C.3. In all cases, we see similar qualitative fluid structure characterised by a large central vortex.

We now turn to the wash-out times for each scenario. The steady and oscillatory T_{90} value for each case is shown in Figure 41, along with the percentage difference from the steady case. The results are generated for $Re = 50$, $Sc = 10$, $\omega = 10$, and $\hat{U} = 20$.

In all cases an oscillatory inlet improves wash-out times, with most times being at least halved. The best improvement is seen in *large*. A much larger central vortex can be seen in this case, which leads to much higher T_{90} times, due to more dust becoming trapped. However, because of this, a mixing effect introduced by an oscillatory inlet flow is extremely beneficial, giving a 74.81% improvement.

For *scopein* we consider the notion of a ureteroscope that has been pushed slightly further towards the back of the cavity to aid in washing away excess debris. We find that since the scope is inserted further in, the cavity length is reduced. Wash-out times are slightly quicker here due to dust contained within the arms either side of the scope being easily advected out of the cavity. Mixing created by an oscillating inlet flow is still beneficial in this slightly shorter cavity.

Turning to the opposite scenario, in which the scope tip is not insert very far inside the cavity, thus resulting in a longer cavity, in *long* we find that wash-out times are much longer than those of *ideal*. This is due to the presence of a second vortex forming at the back of the cavity. In Figure 42, we show the dust concentration for $t = 50, 100, 1000$. We see that a large amount of the dust becomes trapped within this secondary vortex, which after a long time still remains, hence the much longer wash-out times. In reality, this situation would be avoided by inserting the scope tip further within the cavity, suggesting there may be an optimal minimum distance for which the scope should be inserted. An

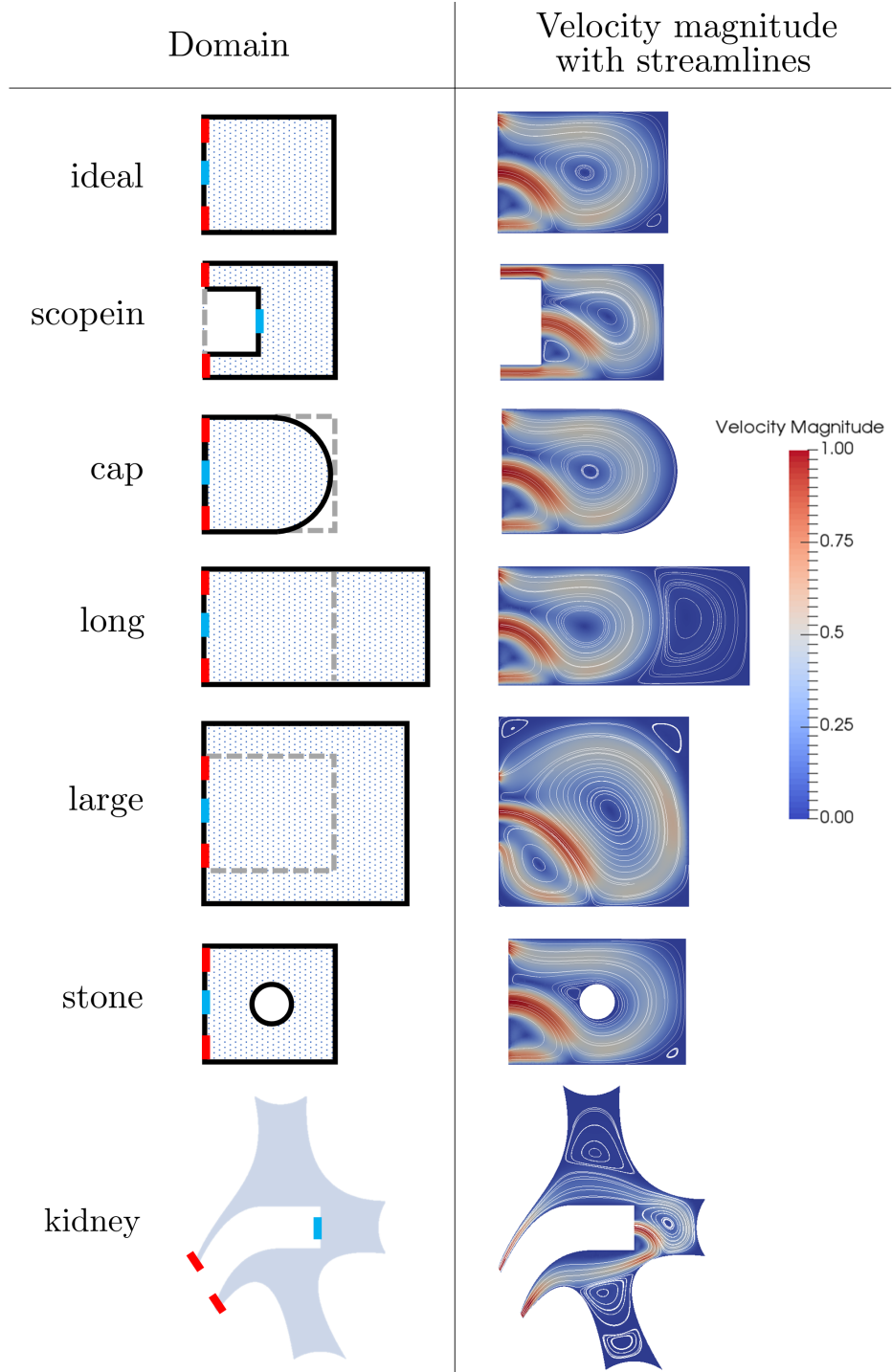


Figure 40: The seven cavity geometries domains along with their steady solutions for fluid velocity magnitude with streamlines. The domain inlets and outlets are marked as blue and red respectively.

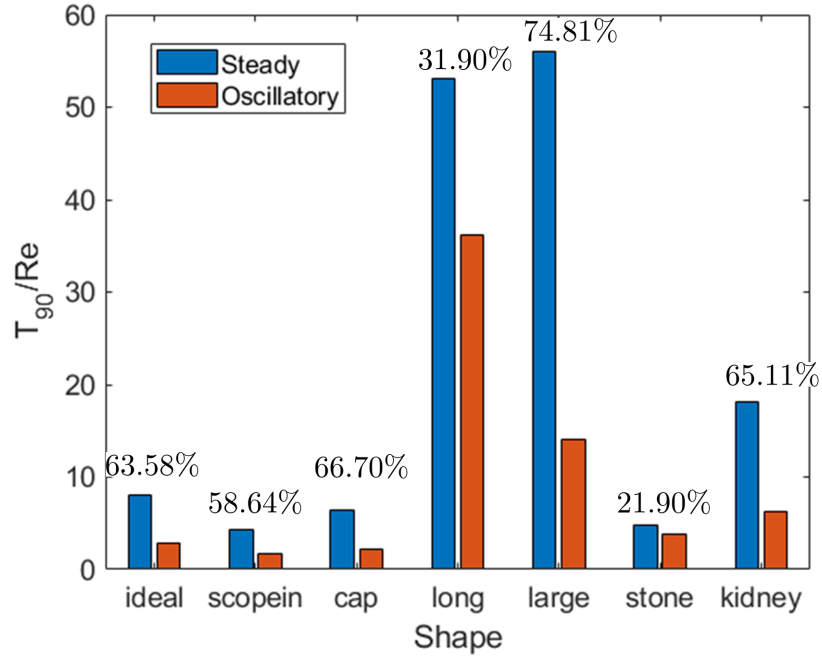


Figure 41: The steady and oscillatory T_{90} results for each shape considered, along with a percentage difference from the steady case.

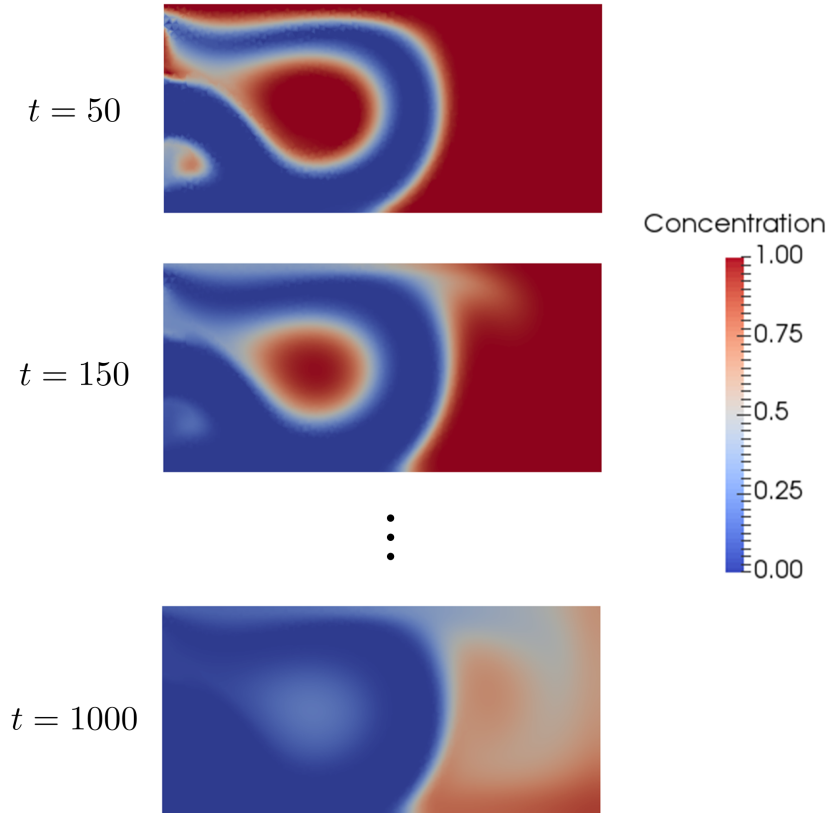


Figure 42: Dust concentration in *long* for $t = 50, 100, 1000$.

oscillatory inlet flow is less impactful here. While the oscillating inlet flow is still able to disturb the primary vortex, it fails to aid in disturbing the additional vortex, and so any dust trapped there is not mixed.

We explore whether the corners present in the ideal domain have a significant impact on the dust wash-out by comparing the results against those of the domain *cap*, in which the back wall of the cavity has been rounded to remove said corners. Very similar results are seen between the two cases, suggesting the corners of the ideal domain have minimal impact on wash-out times.

The domain *stone* considers a circular cut-out being removed from the original domain, representing a larger stone fragment being present in the flow, notably one that has become stuck within the central vortex. Although this fragment is stationary, it still provides insight into how the fluid structure and dust wash-out changes in this scenario. We see the smallest improvement here, at only 21.90%, due to the fact that there is less area within the central fluid vortex to trap dust since it is occupied by the stone.

The illustrative example *kidney* aims to give a better representation of the kidney cavity in situ. In Figure 43, we present a sketch of the kidney interior with ureterscope inserted. The domain has three cavities modelling kidney calyces. A ureterscope is inserted inside of the kidney, pointing towards one of the cavities where the stone dust is contained. In this case, the dust does not occupy the entire domain, instead only occupying the right-most cavity to which the scope is pointed. We find that the dust is washed away alongside the scope and out through the base outlets. The flow pattern seen in the right-most cavity has strong similarities to the original domain, supporting the idealised rectangular domain assumption. Again, introducing oscillations to this domain gives a significant improvement of 65% to wash-out time.

We conclude that the vortex characteristics seen in the idealised domain extend to the other six geometries, and that oscillating the inlet flow at a low frequency and high amplitude improves wash-out times when comparing to a steady inlet for all geometries.

3.11 Summary and discussion

Efficient kidney stone dust removal is vital for an effective ureteroscopy procedure. The clearance of stone dust via fluid irrigation is essential for providing a clear field of view for the clinician. By modelling the irrigation fluid flow and kidney stone dust transport within the cavity we can gain an understanding of the effectiveness of different approaches to debris clearance. The ability to have greater control over irrigation fluid delivery opens up the possibility to harness fluid behaviours to improve dust wash-out times.

We have modelled the fluid flow and dust transport in a rectangular, two-dimensional cavity, with an oscillatory, parabolic inlet flow. We have found that a high frequency inlet oscillation results in flow patterns which are similar to the steady system, where a large central vortex is seen. Dust can become trapped in said vortex, leading to long wash-out times reliant on slow diffusive processes for transport. Reducing the frequency of the inlet flow oscillations caused disturbances to the central vortex. This created a mixing effect which resulted in freeing debris trapped within the vortex and a reduction in wash-out times. Larger amplitudes were shown to further amplify the mixing effect. We concluded that an inlet flow with a low frequency and high amplitude significantly improved wash-out times when compared to those of a high frequency or steady inlet

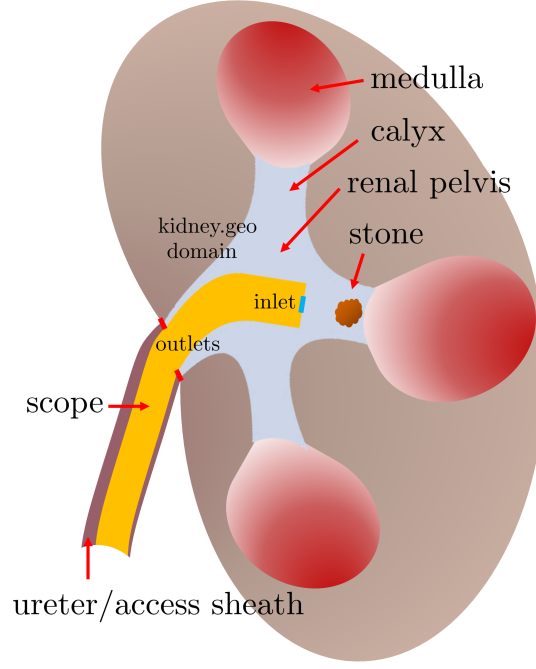


Figure 43: A sketch of the kidney interior containing a kidney stone, with ureterscope (yellow) inserted. The domain *kidney.geo* (grey) with inlet (blue) and outlets (red) is shown.

flow. By exploring a wider range of parameters and calculating the wash-out times and vortical structure metric for each, we created a linear fit which supported the conclusions. We compared the suggested oscillatory mixing technique to the current leading industry standard for excess debris removal, known as flushing. We found an oscillatory inlet to perform favourably over flushing, achieving the same wash-out times at a much lower flow rate. To test the robustness of results to changes in the idealised cavity geometry, we went on to compare wash-out times in six new domains. We found similar qualitative flow patterns in each, and that low frequency, high amplitude oscillations improve wash-out times in all scenarios, further supporting the previous results.

The simplifying assumption of massless dust particles allowed us to model the dust transport via an advection-diffusion model. Relaxing this assumption would open up the consideration of particle size, which may be of industrial interest as this feature can be somewhat controlled via different lasering techniques. One could consider a single particle of a given radius within the cavity flow and calculate the forces acting upon it, such as fluid drag and gravity, to determine its trajectory. If this particle was small enough, the effect it has on the flow could be ignored and a one-way coupled system could be considered. If a larger particle were to be considered, such as those larger than 1mm which are collected via a wire basket working tool, the system would need to be fully coupled, with a fluid-structure interaction problem being considered. A time discontinuous approach could be taken to tackle this scenario, where the fluid domain would contain a cut-out to represent the large stone fragment, such as in the example *stone* in Section 3.10, and the fluid problem would then be solved in this domain for a given time. The forces acting upon the boundaries of this cut-out could then be calculated, and the domain could be altered such that the cut-out is moved a small amount in the appropriate direction. The fluid problem would

then be re-solved at the next time step, and this process would continue to determine the trajectory of the large particle over time. A similar technique is employed in work by Williams (2019) when considering working tool position. The problem could be taken even further by modelling multiple particles in the flow and considering the particle-particle interaction problem, which would considerably increase the complexity of the work. The advection-diffusion modelling performed in this thesis gives the initial insight into how low frequency, high amplitude flows can benefit dust wash-out; qualitative behaviour which may carry over into these more complicated modelling approaches.

Although we have tested the idealised domain by considering various shapes, we have remained in two dimensions. A natural extension to the study would be exploring a three-dimensional cavity. Doing this would introduce a lot more complexity into the problem, with the need to solve the unsteady Navier-Stokes and advection diffusion equations in a full three-dimensional domain. Such an undertaking would be beneficial in creating a system which is more realistic. However, there is a question to whether the symmetry breaking behaviour we see in the two-dimensional case would carry over to the full three-dimensional cavity. If the same qualitative fluid behaviour pertains then oscillatory fluid flow may still be beneficial in decreasing wash-out times.

Another change to the domain to consider would be relaxing the assumption of a rigid boundary, as in reality, the kidney is likely to have some amount of deformability. Once such way to do this would be to adapt the boundary on the right hand side of the domain to be an elastic membrane of some prescribed elasticity representing the mechanical properties of the kidney. This would open up the ability to explore kidney compliance and how it would adjust to increases in flow rate and pressure. Additionally, the membrane would adjust in response to the oscillating inlet flow, which may lead to the oscillations of the flow being naturally dampened by the cavities deformability. Modelling this more realistic kidney structure would give insight into how the kidney copes with increases in kidney pressure, whilst also providing understanding into another contribution to oscillation dampening in the system.

In the analysis we have focused on the effect of an oscillating inlet flow. By varying the frequency and amplitude of said inflow, we have been able to consider a wide range of different inlet conditions. However, with the ability to solve the full unsteady system, different inlet conditions could be considered, with the aim to reduce wash-out times further. Oscillating the inlet flow at multiple different modes may combine to provide an overall better wash-out time.

4 Full ureteroscopy system model

4.1 Introduction

Historically, fluid irrigation during ureteroscopy is delivered by a hanging bag of saline above the level of the patient, providing a constant fluid flow throughout the ureteroscopy system. This technique provides the clinician very little control over the fluid flow, with no direct knowledge of the resulting pressures within the kidney. A key improvement in ureteroscopic devices will be integrated fluid systems to manage the delivery of irrigation fluid. Such a system would provide simple and fine control over the flow via a delivery system which can be easily adjusted, such as a peristaltic pump. If flow is driven by a peristaltic pump, high frequency oscillations are naturally induced, which raises the question: are these detrimental? Should something be added to the ureteroscopy irrigation system to try to remove them?

A design feature which has been considered within the industry is the inclusion of an air dampener within the fluid irrigation system, with a view to remove the natural oscillations created by the peristaltic pump. Located somewhere between the pump and the ureteroscope, an air dampener consists of a chamber of air connected to the fluid network. From the conclusions drawn in this thesis so far, it is not clear whether the inclusion of a dampener to reduce oscillations is either necessary or useful. Chapter 2 provided some evidence that oscillations are largely removed by the time the irrigation fluid reaches the kidney, due to the natural dampening arising from the small radii of the working channel, thus questioning the need for an air dampener. However, Chapter 2 presents the simplest model of the system due to the assumptions made to simplify the problem. Therefore, updating the model to include an air dampener opens up the possibility to explore further scenarios whilst also representing a system more accurate to reality. As we have shown in Chapter 3, there is a potential benefit to not dampening the low frequency oscillations within the cavity, due to their ability to reduce kidney stone dust wash-out times. Determining how an air dampener works by including it into the model allows us to explore how different ranges of frequencies are affected, and whether it is possible to filter out certain undesired frequencies while maintaining the desired ones.

Translating all of these insights for clinical applications requires a more comprehensive description of the fluid flow, not just in the cavity but throughout the network. Our aim in this chapter is to combine the previous modelling efforts into one full systems model. Having established models for the peristaltic pump and oscillatory fluid flow throughout the cylindrical geometries of the system in Chapter 2, and oscillatory fluid flow within the kidney cavity in Chapter 3, we set out to create a model which describes the ureteroscopy system as a whole by combining these individual elements together, along with a new model for the air dampener.

Our objective is to obtain a description of the fluid irrigation network consisting of a set of algebraic pressure-flux relations for the kidney cavity and the cylindrical geometries of the network, combined with flux continuity and a gas law applied to the air chamber, resulting in a model which is easily solved to yield the properties of the system. Such a tool would provide access to pressure and flow conditions throughout the network, in particular within the kidney, as a function of the upstream conditions that drive the flow. It would allow us to determine the consequences of varying the resistance within

the network, for example by inserting a working tool into the working channel. We can directly test the impact and utility of the air dampener to determine its effectiveness in reducing oscillations, while also connecting more directly the results of chapter 3 to the operation of the system in order to enhance stone dust washout.

The full ureteroscopy system that we model is as follows: a peristaltic pump draws fluid from a reservoir and sends it through a section of tubing. The tubing then splits at a junction into two paths. One path leads to an air dampener, which is a rigid, impermeable container that contains a volume of air. The other path leads through more tubing, which then connects to a ureteroscope. The fluid flows down the working channel of the scope, before exiting into a cavity, representing a kidney calyx, modelled as a two-dimensional, rigid domain, with one inlet and two outlets on the same side, as described in Chapter 3. The fluid then exits this cavity and travels down an access sheath before exiting into open air (Figure 44). The sections of tubing, working channel, and access sheath are all modelled as rigid, impermeable cylinders, as described in Chapter 2.

4.1.1 Chapter outline

We consider fluid flow throughout the full ureteroscopy system. The model is outlined in Section 4.2 with the theoretical formulation given in Section 4.3. The problem is then non-dimensionalised and a small parameter representing the ratio between the oscillatory and dampener timescales, is exploited to expand the system and decompose it into quasi-steady and oscillatory parts in Sections 4.4 and 4.5 respectively. In Section 4.6, we solve the quasi-steady system, by considering the fluid flow through the cylindrical geometries of the system, as well as the flow in the cavity, from which we determine a series of pressure-flux relations for the network. These results are then analysed and compared against data in Section 4.7. We then go on to solving the oscillatory system, where we reduce the problem by removing the cavity, and examine the results in Section 4.8 and 4.9 respectively. Finally, in Section 4.10, we discuss the results and suggest reasons for discrepancies between model and data.

4.2 Problem description

We split up the ureteroscopy system into six distinct regions, labelled 0 to 4 and D for the dampener, which are connected to one another via a series of nodes, labelled 0 to 5 and D. At each node we denote the pressure, P_i^* , for $i = 0, 1, D, 2, 3, 4, 5$ (Figure 44). We consider the fluid flow in each region, with a flux and resistance given by Q_i^* and R_i^* respectively, for $i = 0, 1, D, 2, 3, 4$. Starting upstream, node 0 represents a peristaltic pump that drives the fluid throughout the system. Region 0 represents tubing that connects the pump to a junction at node 1, where the fluid splits into two regions. The first split at this junction leads to region D, which represents tubing that connects the junction to an air dampener. The second split leads to region 1, representing more tubing which connects the junction to a ureteroscope working channel. The working channel is represented by region 2, which leads into the kidney cavity, represented by region 3. The fluid then exits the cavity into an access sheath with ureterscope inserted, represented by region 4. The fluid exits the access sheath into open air, represented by pressure P_5^* , which is known. The peristaltic pump produces a flux $q_{in}^*(t^*)$, which is assumed to be known and can be

determined from the peristaltic pump model. The tubing in regions 0,1, and D, and the working channel in region 2, are all modelled as rigid cylinders of radius r_i^* and length L_i^* for $i = 0, 1, D, 2$ respectively. The air dampener is a solid, impermeable container containing air of volume V^* . The initial pressure and air volume in the dampener are at P_{D0}^* and V_0^* respectively. The kidney cavity is modelled the same as in Chapter 3, as a two-dimensional, rectangular domain of length l_c and width $2(a + b + d)$, with one inlet of width $2a$ and two outlets of width d on the left hand side of the domain. The access sheath with ureterscope inserted is modelled as two concentric cylinders of length L_4^* , with inner cylinder radius r_{41}^* and outer cylinder radius r_{42}^* . During the procedure, a working tool may be inserted into the working channel. To represent this scenario, we change region two to be modelled as two concentric cylinders, with the outer cylinder unchanged from before, and the inner cylinder with radius r_{21}^* . The fluid is modelled as a viscous, incompressible, Newtonian fluid of density ρ^* and dynamic viscosity μ^* .

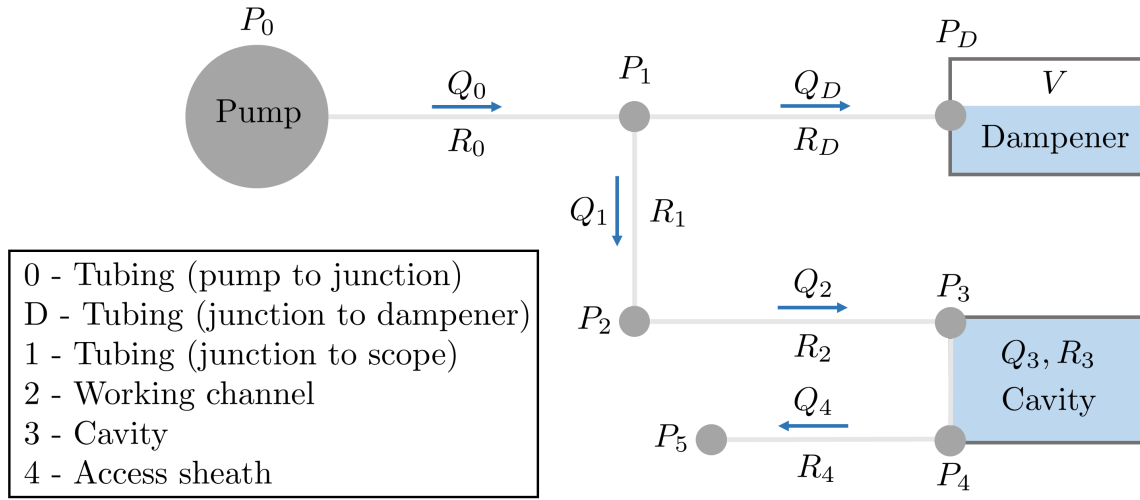


Figure 44: Network model diagram of the full ureteroscopy system: (0) Tubing connects the peristaltic pump to a junction. (D) Tubing connects the junction to the air dampener. (1) Tubing connects the junction to ureterscope working channel. (2) Working channel leads to kidney cavity. (3) Fluid enters the kidney cavity from the working channel and exits into the access sheath. (4) Access sheath with working tool inserted connects the cavity to the open air.

4.3 Theoretical formulation

We aim to represent the system as a series of pressure-flux relations for each region, such that

$$Q_i^* = f_i^*(P_i^*, P_{i+1}^*) , \quad (116)$$

$$Q_D^* = f_D^*(P_1^*, P_D^*) , \quad (117)$$

for $i = 0, 1, 2, 3, 4$, where f_i^* are unknown functions to be determined, describing the relationship between the flux and the two pressures for each region. These relations will be determined by linking pipe flow throughout the network to a gas law in the chamber,

and an extracted pressure-flux relation for the cavity. We assume continuity of flux throughout the system, such that

$$Q_0^* = Q_D^* + Q_1^* , \quad (118)$$

$$Q_1^* = Q_2^* = Q_3^* = Q_4^* . \quad (119)$$

To drive the flow, the upstream flux is given by

$$Q_0^*(t^*) = q_{\text{in}}^* . \quad (120)$$

We follow from the pump model in Section 2.7 of Chapter 2, and assume that the upstream flux is of the form

$$q_{\text{in}}^* = \bar{q}_{\text{in}}^* + \text{Re}(\hat{q}_{\text{in}}^* e^{i\omega^* t^*}) , \quad (121)$$

where $\bar{q}_{\text{in}}^*$ is the steady upstream flux, and $\hat{q}_{\text{in}}^*$ and ω^* are the amplitude and frequency of the oscillatory component of the upstream flux respectively. The values of $\bar{q}_{\text{in}}^*$, $\hat{q}_{\text{in}}^*$, and ω^* are determined from the peristaltic pump model for given pump parameters, as outlined in Section 2.7. The system is closed by setting atmospheric pressure at the outlet, given by

$$P_5^* = 0 . \quad (122)$$

To model the fluid flow in the cylindrical geometries throughout the system, represented by regions 0,1,2,4, and D, we follow Chapter 2; we consider a cylindrical polar coordinate system (r^*, θ, z^*) with corresponding coordinate directions $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$. We assume a unidirectional, axisymmetric flow such that the fluid velocity is $\mathbf{u}_i^* = (0, 0, w_i^*)$ and the pressure is $p_i^* = p_i^*(z^*, t^*)$, with $w_i^* = w_i^*(r^*, t^*)$ denoting fluid velocity in the axial z -direction, for $i = 0, 1, 2, 4, D$ in each region respectively, where t^* denotes time. The continuity and Navier-Stokes equations governing flow through these regions are then

$$\frac{\partial w_i^*}{\partial z^*} = 0 , \quad (123)$$

$$0 = -\frac{\partial p_i^*}{\partial r^*} , \quad (124)$$

$$\mu^* \left(\frac{\partial^2 w_i^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial w_i^*}{\partial r^*} \right) - \rho^* \frac{\partial w_i^*}{\partial t^*} = \frac{\partial p_i^*}{\partial z^*} , \quad (125)$$

for $i = 0, 1, 2, 4, D$. Since the LHS of equation (125) is independent of z^* and the RHS is independent of r^* , each side is a function of time only. Therefore, the pressure gradients are defined in terms of the pressure difference along the length of each cylinder, giving

$$\frac{\partial p_i^*}{\partial z^*} = \frac{(P_{i+1}^* - P_i^*)}{L_i^*} , \quad (126)$$

$$\frac{\partial p_D^*}{\partial z^*} = \frac{(P_1^* - P_D^*)}{L_D^*} , \quad (127)$$

for $i = 0, 1, 2, 4$. We prescribe typical no-slip conditions on the boundaries of each region, giving

$$w_i^*(r_i^*, t^*) = 0 , \quad (128)$$

for $i = 0, 1, 2, 41, 42, D$, along with a symmetry condition for the regions with fluid at $r = 0$, giving

$$\frac{\partial w_i^*}{\partial r^*}(0, t^*) = 0 . \quad (129)$$

for $i = 0, 1, 2, D$. The fluid flux in regions 0,1,2,4, and D, from equation (116) is then determined from the axial flow via

$$Q_i^*(t^*) = 2\pi \int_0^{r_i^*} w_i^* r^* dr^* , \quad (130)$$

$$Q_4^*(t^*) = 2\pi \int_{r_{41}^*}^{r_{42}^*} w_i^* r^* dr^* , \quad (131)$$

for $i = 0, 1, 2, D$.

To determine P_D^* we consider the air dampener, where we model the air as an ideal gas so that

$$P_D^* V^* = c^* , \quad (132)$$

where $c^* = nRT$ is a constant, for moles of gas n , ideal gas constant R , and temperature T . The volume of air in the chamber satisfies

$$V^* = V_0^* - \int_0^{t^*} Q_D^*(t^*) dt^* , \quad (133)$$

where V_0^* is the initial air volume and Q_D^* is the flux of fluid entering the dampener. Differentiating equation (133) with respect to time gives

$$\frac{dV^*}{dt^*} = -Q_D^* . \quad (134)$$

The initial condition for the volume of air the dampener is given by

$$V^*(0) = V_0^* . \quad (135)$$

For region 3, representing the kidney cavity, we follow from Chapter 3, and determine the fluid flow for a given pressure drop via numerical cavity flow simulations. Computing the fluid velocities and pressure within the cavity for a varying range of parameters, we aim to find a relationship between the inlet flux and the inlet and outlet pressures of the form

$$Q_3^* = f_3^*(P_3^*, P_4^*) . \quad (136)$$

The results of the simulations, drawing a relationship from them, and relating it back to the system are described in more detail in section 4.6.1. Once such a relationship is determined, it is then coupled with the systems model, as described by equation (116).

4.4 Non-dimensionalisation

We non-dimensionalise as follows,

$$\begin{aligned} r^* &= r_2^* r, \quad z^* = r_2^* z, \quad w_i^* = \frac{\bar{q}_{\text{in}}^*}{r_2^{*2}} w_i, \quad Q_i^* = \bar{q}_{\text{in}}^* Q_i, \\ p_i^* &= \frac{\mu^* \bar{q}_{\text{in}}^*}{r_2^{*3}} p_i, \quad P_j^* = \frac{\mu^* \bar{q}_{\text{in}}^*}{r_2^{*3}} P_j, \quad V^* = V_0^* V, \quad t^* = \frac{1}{\omega^*} t, \end{aligned} \quad (137)$$

for $i = 0, 1, 2, 3, 4, D$ and $j = 0, 1, 2, 3, 4, 5, D$, where we have scaled the velocity on the mean upstream flux and time on the frequency of oscillations. The dimensionless governing equations for the air dampener and fluid flow through the cylindrical geometries are then

$$\frac{\partial^2 w_i}{\partial r^2} + \frac{1}{r} \frac{\partial w_i}{\partial r} - \alpha^2 \frac{\partial w_i}{\partial t} = \frac{\partial p_i}{\partial z}, \quad (138)$$

$$\frac{\partial V}{\partial t} = -\delta Q_D, \quad (139)$$

$$P_D V = c, \quad (140)$$

for $i = 0, 1, 2, 4, D$, where $\alpha = r_2^* \sqrt{\rho^* \omega^* / \mu^*}$ is the Womersley number, $c = r_2^{*3} c^* / \bar{q}_{\text{in}}^* \mu^* V_0^*$ is a dimensionless constant, and $\delta = \bar{q}_{\text{in}}^* / V_0^* \omega^*$ is the ratio between the time taken for the air in the dampener to adjust from its initial volume due to the upstream steady flux and the frequency of oscillations. The pressure gradients from equation (138) are given by

$$\frac{\partial p_i}{\partial z} = \frac{(P_{i+1} - P_i)}{L_i}, \quad (141)$$

$$\frac{\partial p_D}{\partial z} = \frac{(P_D - P_1)}{L_D}, \quad (142)$$

where $L_i = L_i^* / r_2^*$, for $i = 0, 1, 2, 4$, and continuity of flux is given by

$$Q_0 = Q_D + Q_1, \quad (143)$$

$$Q_1 = Q_2 = Q_3 = Q_4, \quad (144)$$

where the flux in regions 0,1,2,4, and D, is given by

$$Q_i(t) = 2\pi \int_0^{r_i} w_i r \, dr, \quad (145)$$

$$Q_4(t) = 2\pi \int_{r_{41}}^{r_{42}} w_i r \, dr, \quad (146)$$

for $i = 0, 1, 2, D$, where $r_i = r_i^* / r_2^*$. The boundary and initial conditions are given by

$$w_i(r_i, t) = 0 \quad \text{for } i = 0, 1, 2, D, \quad (147)$$

$$\frac{\partial w_i}{\partial r}(0, t) = 0 \quad \text{for } i = 0, 1, 2, D, \quad (148)$$

$$w_4(r_j, t) = 0 \quad \text{for } j = 41, 42, \quad (149)$$

$$Q_0(t) = 1 + \hat{q}_{\text{in}} e^{it}, \quad (150)$$

$$P_5 = 0, \quad (151)$$

$$V(0) = 1, \quad (152)$$

where $\hat{q}_{\text{in}} = \hat{q}_{\text{in}}^* / \bar{q}_{\text{in}}^*$.

4.5 System expansions

Using the parameter values from Table 3 in Appendix D.1, we can estimate the size of the dimensionless parameters in the system. We find that $\alpha \sim 1$, $\delta \sim 0.001$, and $c \gg 1$. Motivated by the fact that δ is small, to proceed with solving the problem, we introduce a multi-scale assumption on the variables of the system of such that

$$w_i = w_i(r, t, \tau) , \quad (153)$$

where $\tau = \delta t$ with $\delta \ll 1$, and similarly for p_i , V , Q_i , and P_i . This assumption is motivated by the form of equation (139), where the change in volume of the dampener over time is scaled by δ . We pose that two distinct timescales exist in the problem: a short timescale of the frequency of oscillations, represented by t , and a long timescale of the time it takes the dampener to fill due to the mean flux, represented by τ . Since we anticipate that the changes in dampener volume are happening slowly compared to the oscillations in the system, we refer to the short and long timescales as the oscillatory and quasi-steady timescales respectively.

We expand all time derivatives such that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau} , \quad (154)$$

so that equations (138)-(140) are now

$$\frac{\partial^2 w_i}{\partial r^2} + \frac{1}{r} \frac{\partial w_i}{\partial r} - \alpha^2 \frac{\partial w_i}{\partial t} - \delta \alpha^2 \frac{\partial w_i}{\partial \tau} = \frac{\partial p_i}{\partial z} , \quad (155)$$

$$\frac{\partial V}{\partial t} + \delta \frac{\partial V}{\partial \tau} = -\delta Q_D , \quad (156)$$

$$P_D V = c , \quad (157)$$

for $i = 0, 1, 2, 4, D$, with the remaining equations unchanged.

Motivated by the form of the upstream flux, we expand all variables into their mean and oscillatory components, such that

$$w_i = \bar{w}_i(r, \tau) + \hat{w}_i(r, \tau)e^{it} , \quad (158)$$

for $i = 0, 1, 2, 4, D$, with bars and hats respectively, and similarly for p_i , V , Q_i , and P_i . Substituting these expansions into equations (155)-(157) gives

$$\begin{aligned} \frac{\partial^2}{\partial r^2} (\bar{w}_i + \hat{w}_i e^{it}) + \frac{1}{r} \frac{\partial}{\partial r} (\bar{w}_i + \hat{w}_i e^{it}) - \alpha^2 i \hat{w}_i e^{it} - \delta \alpha^2 \frac{\partial}{\partial \tau} (\bar{w}_i + \hat{w}_i e^{it}) \\ = \frac{\partial}{\partial z} (\bar{p}_i + \hat{p}_i e^{it}) , \end{aligned} \quad (159)$$

$$i \hat{V} e^{it} + \delta \frac{\partial}{\partial \tau} (\bar{V} + \hat{V} e^{it}) = -\delta \bar{Q}_D - \delta \hat{Q}_D e^{it} , \quad (160)$$

$$(\bar{P}_D + \hat{P}_D e^{it}) (\bar{V} + \hat{V} e^{it}) = c , \quad (161)$$

for $i = 0, 1, 2, 4, D$, with similar expansions for the boundary and initial conditions.

We expand all variables as follows

$$\bar{w}_i = \bar{w}_i^{(0)} + \delta \bar{w}_i^{(1)} + \delta^2 \bar{w}_i^{(2)} + O(\delta^3) , \quad (162)$$

for $i = 0, 1, 2, 4, D$, and similarly for $\hat{w}_i$, $\bar{p}_i$, $\hat{p}_i$, $\bar{V}$, $\hat{V}$, $\bar{Q}_i$, $\hat{Q}_i$, $\bar{P}_i$, and $\hat{P}_i$. Applying these expansions in δ to the system, we next proceed to solve the system at each order.

4.5.1 Leading order system

Exploring the system at leading order, we find that equation (159) has quasi-steady and oscillatory parts of

$$\frac{\partial^2 \bar{w}_i^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_i^{(0)}}{\partial r} = \frac{\partial \bar{p}_i^{(0)}}{\partial z} , \quad (163)$$

$$\frac{\partial^2 \hat{w}_i^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{w}_i^{(0)}}{\partial r} - \alpha^2 i \hat{w}_i^{(0)} = \frac{\partial \hat{p}_i^{(0)}}{\partial z} , \quad (164)$$

respectively, for $i = 0, 1, 2, 4, D$, where

$$\frac{\partial \bar{p}_i^{(0)}}{\partial z} = \frac{(\bar{P}_{i+1}^{(0)} - \bar{P}_i^{(0)})}{L_i} , \quad \frac{\partial \bar{p}_D^{(0)}}{\partial z} = \frac{(\bar{P}_D^{(0)} - \bar{P}_1^{(0)})}{L_D} \quad (165)$$

$$\frac{\partial \hat{p}_i^{(0)}}{\partial z} = \frac{(\hat{P}_{i+1}^{(0)} - \hat{P}_i^{(0)})}{L_i} , \quad \frac{\partial \hat{p}_D^{(0)}}{\partial z} = \frac{(\hat{P}_D^{(0)} - \hat{P}_1^{(0)})}{L_D} , \quad (166)$$

for $i = 0, 1, 2, 4$. From equation (160), we find that at leading order

$$\hat{V}^{(0)} = 0 , \quad (167)$$

which then gives the quasi-steady and oscillatory parts of equation (161) as

$$\bar{P}_D^{(0)} \bar{V}^{(0)} = c , \quad (168)$$

$$\hat{P}_D^{(0)} = 0 , \quad (169)$$

respectively. We see that equation (160) at leading order does not contain any quasi-steady variables, and so we proceed to order δ in order to extract a relationship for the change in volume over time. The quasi-steady part of equation (160) at order δ gives

$$\frac{\partial \bar{V}^{(0)}}{\partial \tau} = -\bar{Q}_D^{(0)} . \quad (170)$$

From the continuity of flux equations we have

$$\bar{Q}_0^{(0)} = \bar{Q}_D^{(0)} + \bar{Q}_1^{(0)} , \quad (171)$$

$$\hat{Q}_0^{(0)} = \hat{Q}_D^{(0)} + \hat{Q}_1^{(0)} , \quad (172)$$

$$\bar{Q}_1^{(0)} = \bar{Q}_2^{(0)} = \bar{Q}_3^{(0)} = \bar{Q}_4^{(0)} , \quad (173)$$

$$\hat{Q}_1^{(0)} = \hat{Q}_2^{(0)} = \hat{Q}_3^{(0)} = \hat{Q}_4^{(0)} . \quad (174)$$

The boundary and initial conditions of the leading order quasi-steady system are given by

$$\bar{w}_i^{(0)}(r_i, t) = 0 \quad \text{for } i = 0, 1, 2, D , \quad (175)$$

$$\frac{\partial \bar{w}_i^{(0)}}{\partial r}(0, t) = 0 \quad \text{for } i = 0, 1, 2, D , \quad (176)$$

$$\bar{w}_4^{(0)}(r_j, t) = 0 \quad \text{for } j = 41, 42 , \quad (177)$$

$$\bar{Q}_0^{(0)} = 1 , \quad (178)$$

$$\bar{P}_5^{(0)} = 0 , \quad (179)$$

$$\bar{V}^{(0)}(0) = 1 , \quad (180)$$

and for the oscillatory system are given by

$$\hat{w}_i^{(0)}(r_i, t) = 0 \quad \text{for } i = 0, 1, 2, D, \quad (181)$$

$$\frac{\partial \hat{w}_i^{(0)}}{\partial r}(0, t) = 0 \quad \text{for } i = 0, 1, 2, D, \quad (182)$$

$$\hat{w}_4^{(0)}(r_j, t) = 0 \quad \text{for } j = 41, 42, \quad (183)$$

$$\hat{Q}_0^{(0)} = \hat{q}_{\text{in}}, \quad (184)$$

$$\hat{P}_5^{(0)} = 0. \quad (185)$$

We have now defined a system of independent quasi-steady and oscillatory leading order variables given by equations (163) to (185), which when paired with the solution to the cavity relationship given by equation (136), can be solved to give the leading order system solution. We do not consider the solution at higher orders, and so moving forward we drop the superscript (0) notation from the variables for ease.

4.6 Quasi-steady system solution

We begin by solving the quasi-steady system. We focus on the system adjustment to steady state from its initial conditions and its response to changes in resistance via the introduction of a working tool. The quasi-steady time dependence of the problem arises due to the air dampener adjusting to said changes over time.

To solve, we take the following approach: we first consider the fluid flow in the cylindrical geometries of the system. For known flux and pressure gradient, we can represent the pressure-flux relations in terms of the resistance that each tube provides, such that

$$\bar{Q}_0 = \frac{\bar{P}_0 - \bar{P}_1}{\bar{R}_0}, \quad \bar{Q}_1 = \frac{\bar{P}_1 - \bar{P}_2}{\bar{R}_1}, \quad \bar{Q}_2 = \frac{\bar{P}_2 - \bar{P}_3}{\bar{R}_2}, \quad \bar{Q}_4 = \frac{\bar{P}_4 - \bar{P}_5}{\bar{R}_5}, \quad \bar{Q}_D = \frac{\bar{P}_1 - \bar{P}_D}{\bar{R}_D}. \quad (186)$$

We then move onto obtaining a pressure-flux relationship from the cavity, which we determine via numerical simulations. This results in an ODE for the change in the volume of air in the dampener over time, as given by equation (170), in terms of the known variables of $\bar{Q}_0$ and $\bar{P}_5$, which can easily be numerically solved.

The resistances in the cylindrical geometries of the network model are determined through solving equation (163) along with the form of pressure gradients from equation (165), subject to boundary conditions from equations (175)-(177). Recalling the time-independent system solution given in Chapter 2, the resistances for unidirectional fluid flow through the cylindrical geometries of the network are given by

$$\bar{R}_0 = \frac{8L_0}{\pi r_0^4}, \quad \bar{R}_1 = \frac{8L_1}{\pi r_1^4}, \quad \bar{R}_2 = \frac{8L_2}{\pi}, \quad \bar{R}_D = \frac{8L_D}{\pi r_D^4}, \quad (187)$$

and the resistance for unidirectional flow through two concentric cylinders (i.e. the access sheath with ureterscope inserted) is given by

$$\bar{R}_4 = \frac{8L_4 (\ln(r_{42}) - \ln(r_{41}))}{\pi(r_{41} - r_{42})(r_{41} + r_{42})((r_{41}^2 + r_{42}^2)(\ln(r_{41}) - \ln(r_{42})) - r_{41}^2 + r_{42}^2)}. \quad (188)$$

4.6.1 Pressure-flux relation from modelling steady flow within a cavity

To determine the pressure-flux relationship for the cavity, we return to cavity flow simulations, as outlined in Chapter 3. As we are considering the quasi-steady system, we opt to numerically solve a series of steady cavity flow problems, and vary the velocity of the incoming fluid to build a relationship between pressure and flux. We recall that the cavity problem is solved in two dimensions, with the two outlets representing the three-dimensional annular geometry of the access sheath with scope inserted. Since the full system model presented in this chapter is three-dimensional, we must be careful about how we relate the two. Therefore, we begin by solving and analysing the results of the steady flow cavity simulations in order to determine how to connect them back to the network.

Following from Chapter 3, we consider a two-dimensional, Cartesian coordinate system $\mathbf{x}^* = (x^*, y^*)$ with corresponding coordinate directions $(\mathbf{e}_x, \mathbf{e}_y)$. We assume the fluid is incompressible, with velocity $\mathbf{u}^* = (u^*(x^*, y^*), v^*(x^*, y^*))$ and pressure $p^*(x^*, y^*, t^*)$. We opt to non-dimensionalise the problem via the scalings presented in Chapter 3, as given by equation (79), and later re-scale the resulting pressure-flux relationship to align with the full system model non-dimensionalisation scheme. The dimensionless, steady cavity flow problem is then defined as follows: the steady Navier-Stokes equations which govern the fluid are given by

$$Re((\mathbf{u} \cdot \nabla) \mathbf{u}) = -\nabla p + \nabla^2 \mathbf{u} , \quad (189)$$

$$\nabla \cdot \mathbf{u} = 0 , \quad (190)$$

where $Re = \rho^* U^* a / \mu^*$ is the Reynolds number. At the inlet to the cavity we prescribe fully-developed parabolic flow, given by

$$\mathbf{u} = ((1 - y^2), 0) \quad \text{on} \quad \Gamma_{\text{in}} , \quad (191)$$

At the outlet boundaries we prescribe zero stress such that

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{out}} . \quad (192)$$

We impose no-slip on all impermeable walls, given by

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{walls}} . \quad (193)$$

Since the full system model assumes unidirectional flow throughout the working channel and access sheath, we adapt the cavity domain to ensure unidirectional flow at the inlet and outlets of the cavity. To achieve this, we include additional channels of length L_{channel} to the original cavity domain (Figure 45(a)), with the original cavity dimensions unchanged from before ($a = 0.06\text{cm}$, $b = 0.09675\text{cm}$, $d = 0.05775\text{cm}$, $l_c = 0.6\text{cm}$). As the fluid exits the cavity and enters the outlet channels, we anticipate the flow will become unidirectional, provided the outlet channels are sufficiently long. To increase the accuracy of the results extracted from the inlet and outlet locations a finer meshing structure is considered in those locations. A target element size of $dx = 0.15$ is used for the nodes at the ends of these channels, as shown in red on Figure 45(b), with the remaining points outlining the cavity staying unchanged from before at $dx = 0.25$.

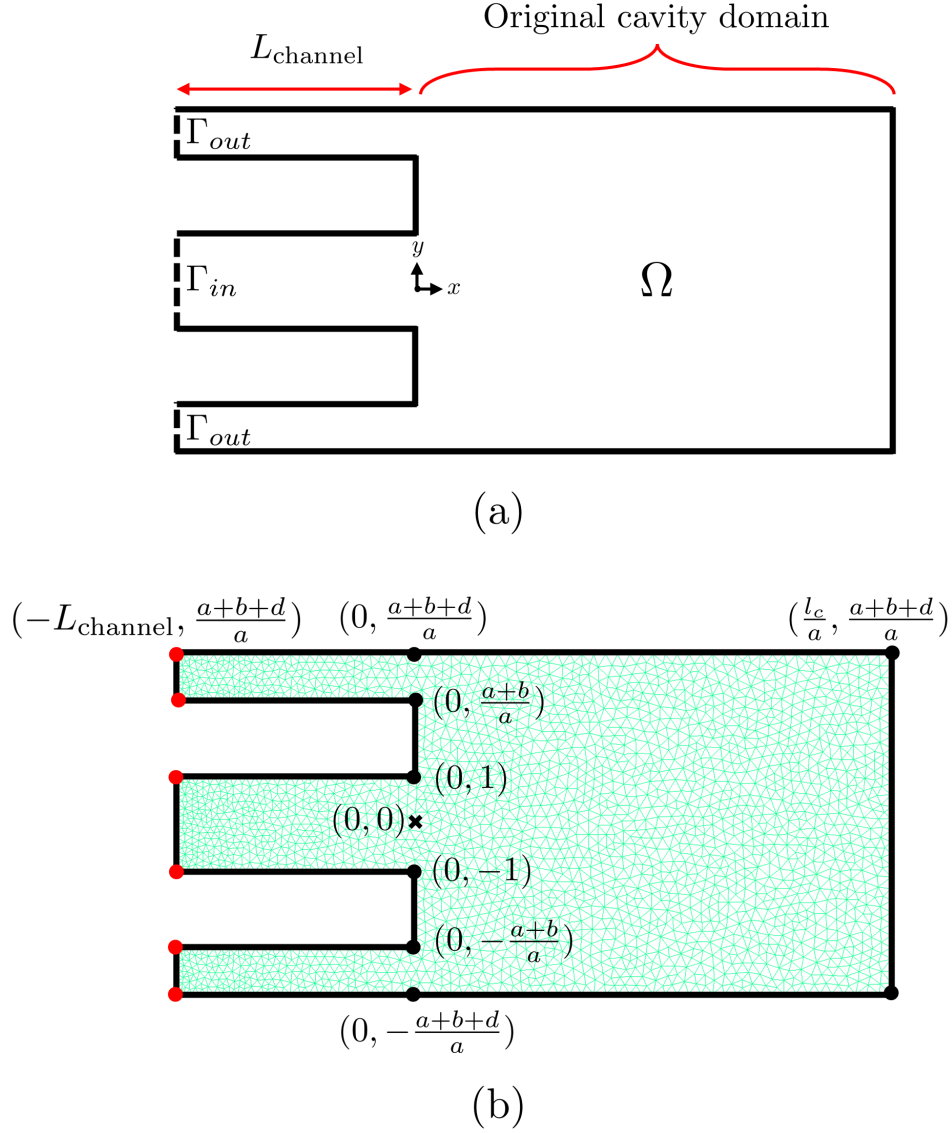


Figure 45: (a) Adapted original cavity domain with channels of length L_{channel} included. (b) Structured mesh of triangular elements with target element size $dx = 0.25$ for black nodes and $dx = 0.15$ for red nodes, constructed in Gmsh. ($a = 0.06\text{cm}$, $b = 0.09675\text{cm}$, $d = 0.05775\text{cm}$, $l_c = 0.6\text{cm}$).

The solution method then follows as described in Chapter 3, in which we numerically solve the problem via a finite element formulation, where the steady fluid problem being solved is as given by equations (189) to (193), using the updated domain given in Figure 45. To allow comparison across varying Reynolds numbers, we normalise the velocities, pressure, and wash-out times with respect to the mean velocity scale, and present uRe , vRe , pRe , and T_{90}/Re as the variables of interest.

To determine if the flow becomes unidirectional within the outlet channels, we examine the fluid velocity in the y -direction, v , and see how this changes as the flow rate increases, represented by an increase in Reynolds number. For the unidirectionality assumption to hold, we expect this value to be zero, with only fluid velocity in the x -direction, u , to be present. In Figure 46(a), we plot vRe across a cross section of the upper outlet channel at $x = -4$, for increasing Reynolds number, with $L_{\text{channel}} = 5$. We see that as the Reynolds number increases we begin to see flow in the y -direction, with the weak velocity increasing as Reynolds number increases. To ensure unidirectional outlet flow a longer outlet channel may be considered, at the expense of computational run time. However, we see that the v velocities remain small compared to the size of the u velocities (Figure 46(b)), and so the assumption of unidirectional flow at the outlets is justified.

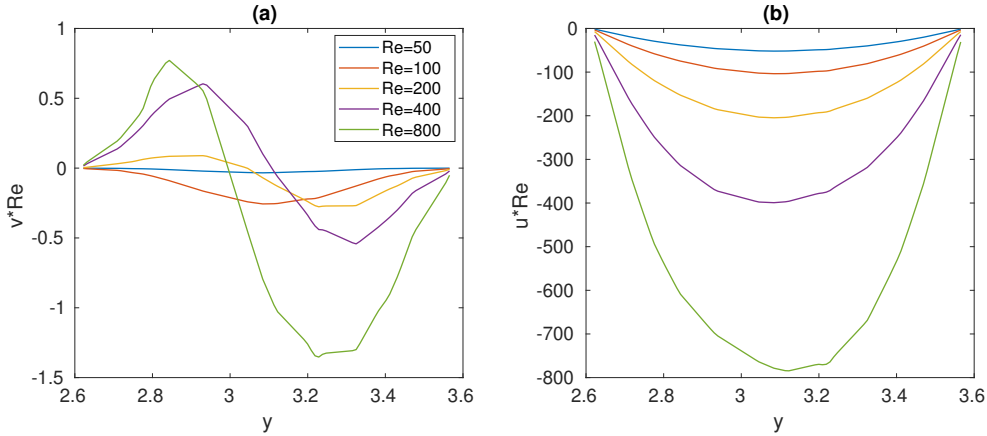


Figure 46: (a) Velocity in the y -direction, vRe and (b) velocity in the x -direction, uRe , across a cross section of the upper outlet channel at $x = -4$, for increasing Reynolds number, with $L_{\text{channel}} = 5$.

From the analysis performed in Chapter 3 we know that multiple fluid solutions exist. In Figure 47, we show an example of velocity magnitudes with streamlines for two different solutions at $Re = 50$. We refer to the solution in Figure 47(a) as the symmetrical solution and in Figure 47(b) as the asymmetrical solution. We note that a final third solution exists, a reflection of the solution in Figure 47(b) in $y = 0$. The previous cavity flow analysis revealed that at $Re = 20$ a pitchfork bifurcation occurs. Below this threshold the cavity flow has a single, stable solution, which is the symmetrical solution. Above this threshold, the symmetrical solution becomes unstable, and the two asymmetrical solutions are stable solutions of the system.

From the asymmetrical solution it is clear that the flux of fluid exiting each outlet is likely to be different. To investigate, we determine the maximum velocity at the outlet to each channel. Since we assume the flow to be fully-developed at the outlets, we define

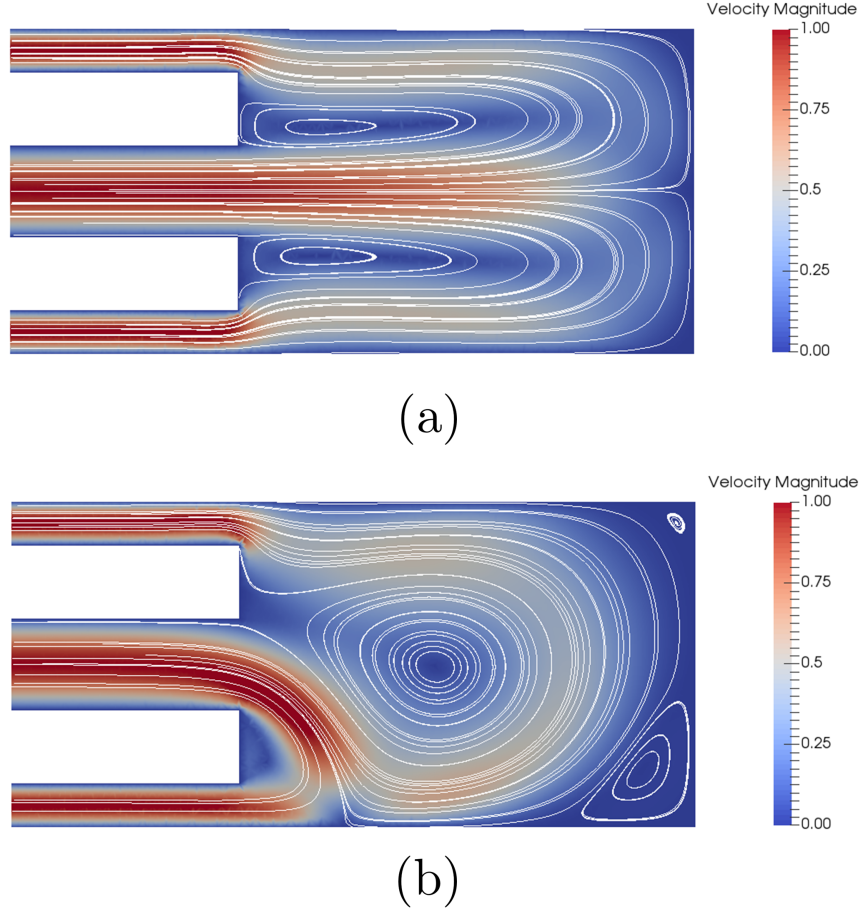


Figure 47: Fluid velocity magnitude with streamlines for the (a) symmetrical and (b) asymmetrical solutions, with $Re = 50$.

the maximum velocity $u_{\max}$ as the velocity at the centre of the channel, given by u at $x = -5$ and $y = \pm 3.09375$ for the upper and lower outlet respectively. In Figure 48, we present the maximum velocity $u_{\max}$ for the symmetrical solution at the upper and lower outlet in blue and orange respectively, alongside the maximum velocity for the asymmetrical solution at the upper and lower outlet in yellow and purple respectively, all for increasing Reynolds number. We note that the asymmetrical solution does not exist as a solution to the system at and below $Re = 20$. Naturally, the symmetrical solution has the same maximum velocity in both outlets, and this is unaffected by Reynolds number. In the asymmetrical case however, as expected, there is a difference in fluid velocities between the two outlets, with the upper outlet experiencing a higher velocity of fluid. This difference grows larger as Reynolds number increases.

This difference in outlet flux becomes problematic when relating the two-dimensional cavity simulations back to the three-dimensional system model. Since the two outlets are representing the same axisymmetric, three-dimensional geometry, the flux of fluid exiting each outlet must be equal. We see an equal outlet flux in the symmetrical solution, however, as discussed previously, we know that the symmetrical solution is unstable above $Re = 20$, making it an unsuitable choice for use in the model. Therefore, we proceed by analysing the results of the asymmetrical solution and comparing them against the

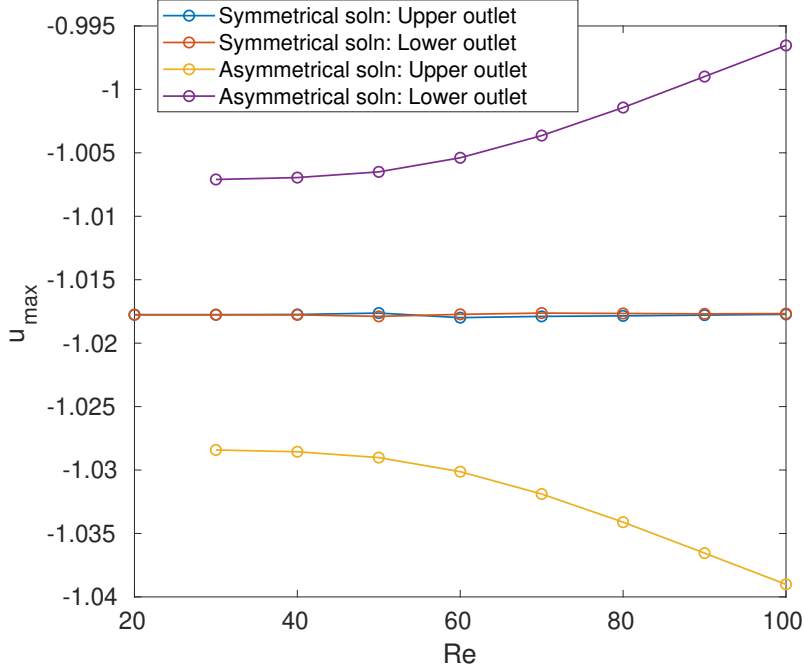


Figure 48: Maximum velocity $u_{\max}$ for the symmetrical solution at the upper and lower outlet in blue and orange respectively, alongside the maximum velocity for the asymmetrical solution at the upper and lower outlet in yellow and purple respectively, all for increasing Reynolds number.

symmetrical solution, before making conclusions on how to relate them back to the three-dimensional network.

As mentioned, the desired output from the cavity simulations is a relationship between the inlet flux Q_2 and the inlet and outlet pressures $\bar{P}_3$ and $\bar{P}_4$. We note that since we have prescribed zero stress at the outlet to the cavity, and also included additional channels to ensure unidirectional flow at the outlets, this results in the pressure $\bar{P}_4$ being equal to zero at the outlets. We can account for this by simply including a pressure shift in the system to ensure that the network pressure at the end of the access sheath $\bar{P}_5$ is equal to zero, which is possible since we are relating flux to a pressure drop throughout the network. Therefore, the equivalent relationship to draw from the cavity results is how the pressure at the inlet to the cavity, p_{in} , varies with Reynolds number. We note that the pressure at the inlet to the cavity does not vary in y , and so is presented as a single value.

In Figure 49, we give the pressure at the inlet to the cavity $p_{\text{in}} Re$ for both the symmetrical and asymmetrical solution in blue and orange respectively, for increasing Reynolds number. We see that as Reynolds number increases, inlet pressure also increases, for both solutions. For lower Reynolds number, we see that the two solutions yield similar results. However, for the larger Reynolds numbers, we start to see a difference, as the asymmetrical solution grows larger than the symmetrical.

As well as extracting the pressure-flux relation needed to solve the network model, we may also use the results from the cavity simulations to compute metrics that are of interest during ureteroscopy, namely the dust wash-out time T_{90} and the maximum

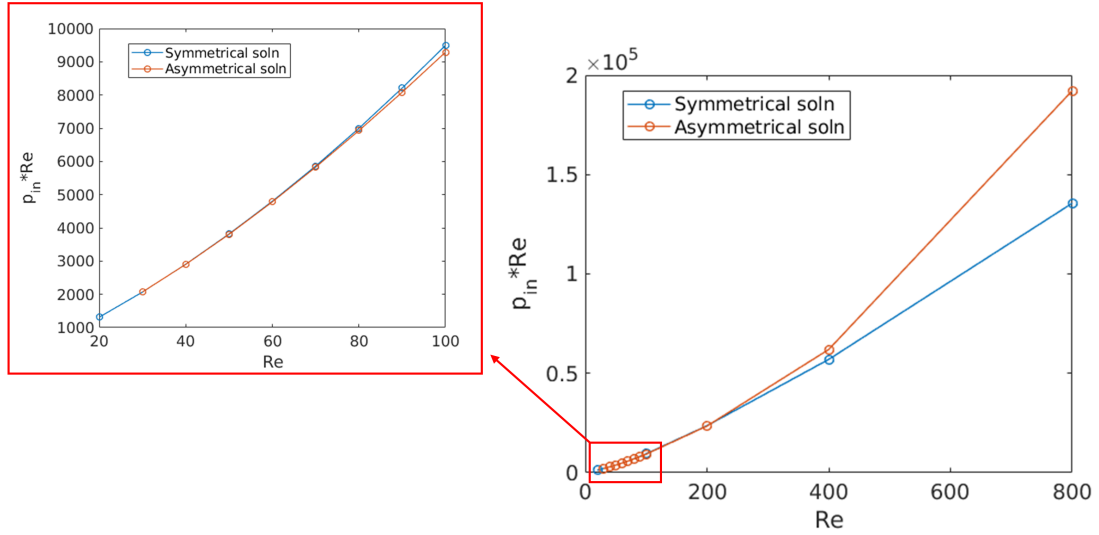


Figure 49: Pressure at the inlet to the cavity $p_{in}Re$ for both the symmetrical and asymmetrical solution in blue and orange respectively, for increasing Reynolds number.

pressure within the cavity. By using the cavity results to determine a relationship between these metrics and the cavity inlet flux, we can then use the network model to predict how these metrics vary as we vary the upstream flux imposed by the pump.

We define the maximum pressure in the cavity p_{max} as the largest values of p for $x > 0$. In Figure 50(b), we give p_{max} for both the symmetrical and asymmetrical solution in blue and orange respectively, for increasing Reynolds number. Alongside this in Figure 50(a) and (c), we give two examples of pressure profiles within the cavity at $Re = 100$ for the asymmetrical and symmetrical solution respectively, with a yellow point indicating the location of the maximum pressure in each. Despite the two solutions having different locations within the cavity for maximum pressure, we see that the value of p_{max} is similar between the two, with the asymmetrical solution producing slightly larger maximum pressures as the Reynolds number increases.

In Figure 51, we give a log-log plot of T_{90}/Re for increasing Reynolds number, for both the symmetrical and asymmetrical solutions in blue and orange respectively. For the asymmetrical case, we see that at first the wash-out times increase as Reynolds number increases, due to the size of the vortex within the cavity growing larger, trapping more dust within it; a result discussed in Chapter 3. As Reynolds number increases further, dust wash-out times start to decrease, as the benefit of higher fluid velocities starts to become impactful. Turning to the symmetrical case, we see that as Reynolds numbers decrease from $Re = 50$, the wash-out times increase exponentially. This is due to the fluid velocity being too low to reach the dust towards the back of the cavity. At around $Re = 60$, we start to see similar behaviour as displayed by the asymmetrical case as Reynolds number increases, again due to the competition between increasing vortex size and increasing velocities, however for an overall much lower set of wash-out times compared to the asymmetrical results.

These results show us that for the purposes of extracting a pressure-flux relationship from the cavity simulations, as well as maximum pressure relations, the symmetrical and asymmetrical solutions do not differ greatly. Therefore, for the systems model, we opt to

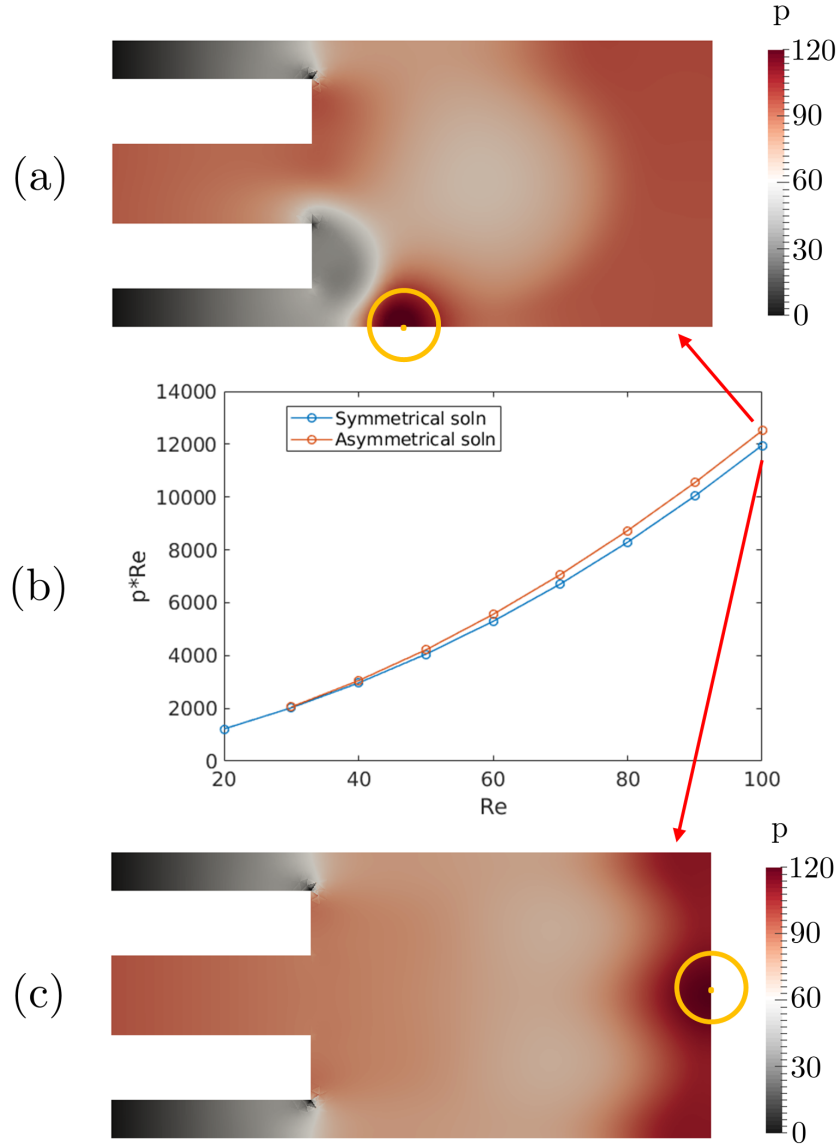


Figure 50: (b) Maximum cavity pressure $p_{\max}$ for both the symmetrical and asymmetrical solution in blue and orange respectively, for increasing Reynolds number. Alongside this in we give two examples of pressure profiles within the cavity at $Re = 100$ for the (a) asymmetrical and (c) symmetrical solution, with a yellow point indicating the location of the maximum pressure in each.

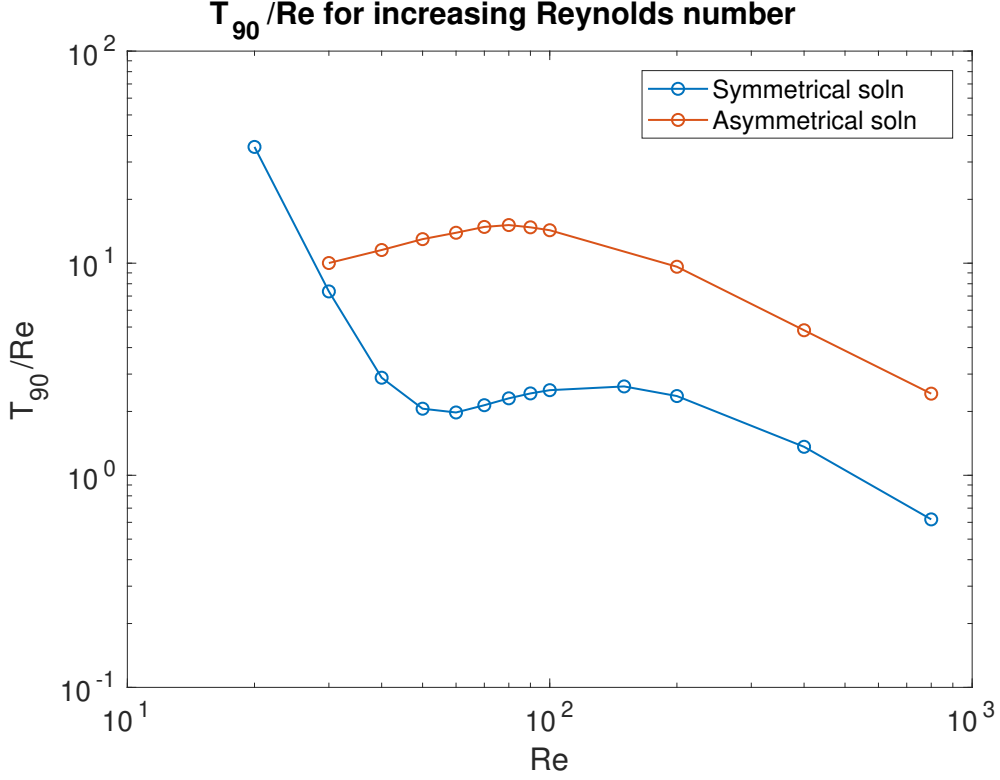


Figure 51: Log-log plot of T_{90}/Re for increasing Reynolds number, for both the symmetrical and asymmetrical solutions in blue and orange respectively.

use the asymmetrical solution in order to extract the relation between flux and pressure, while remaining aware of the difference in wash-out times. While the difference in flux in each outlet channel remains unrealistic for a three-dimensional geometry, it does not have an impact when drawing the relation from the results, since we only consider the total flux in and out.

Since we have non-dimensionalised the network model and cavity model differently, we present dimensional results from the cavity, which we then non-dimensionalise using the network model scalings. Using the inbuilt curve fitting tool in Matlab (The MathWorks, 2020), we determine a quadratic relationship between the dimensional cavity inlet pressure p_{in}^* and the dimensional flux in the inlet channel Q_{2D}^* , defined in terms of the maximum fluid velocity U^* as

$$Q_{2D}^* = \frac{4r_2^*}{3}U^* . \quad (194)$$

where $U^* = \mu^* Re / \rho^* a$. This relationship is given by

$$p_{in}^* = \lambda_0^* Q_{2D}^{*2} + \lambda_1^* Q_{2D}^* + \lambda_2^* . \quad (195)$$

Performing the fit from the asymmetric results for data values of $Re = (0, 30, 40, \dots, 90, 100, 200, 400, 800)$, we find the values of $\lambda_0^* = 32.65$, $\lambda_1^* = 134.9$, and $\lambda_2^* = -4.293$, with a comparison of this fit to the dimensional results given by the log-log plot in Figure 52.

We are now left to convert this two-dimensional fit to the three-dimensional network. We ensure that the cavity inlet pressure p_{in}^* is equivalent to the network pressure P_3^* by

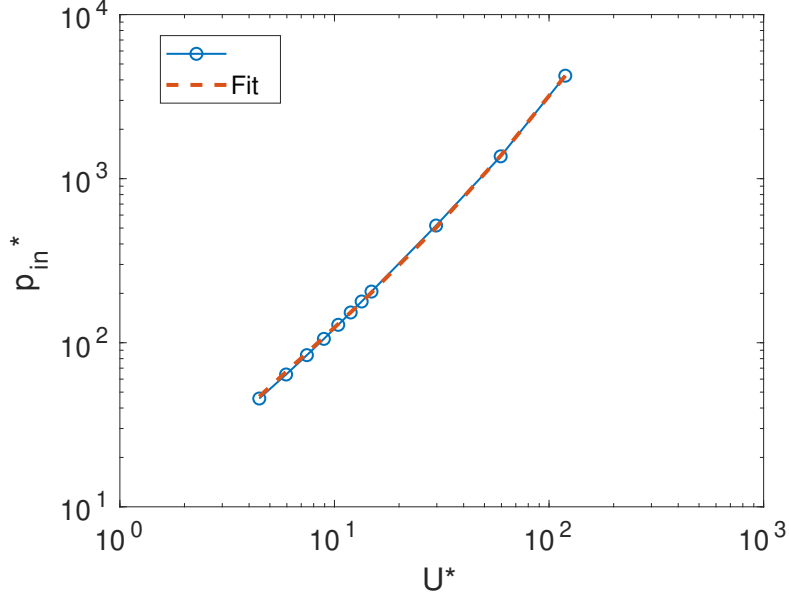


Figure 52: Log-log plot of inlet pressure p_{in}^* for increasing maximum fluid velocity U^* in blue compared to the quadratic fit in orange.

reducing the working channel length L_2 by $L_{channel}$, due to the inclusion of the arms in the cavity modelling. We opt to match the fluxes in the two and three-dimensional cases, such that $Q_{2D}^* = Q_3^*$, giving equation (195) as

$$P_3^* = \lambda_0^* \bar{Q}_3^{*2} + \lambda_1^* \bar{Q}_3^* + \lambda_2^* . \quad (196)$$

We then apply the non-dimensional scalings given by equation (137), to give the pressure-flux relation of

$$P_3 = \lambda_0 \bar{Q}_3^2 + \lambda_1 \bar{Q}_3 + \lambda_2 . \quad (197)$$

where $\lambda_0 = \lambda_0^* r_2^{*2} \bar{q}_{in}^* / \mu^*$, $\lambda_1 = \lambda_1^* r_2^{*2} / \mu^*$, and $\lambda_2 = \lambda_2^* r_2^{*2} / \bar{q}_{in}^* \mu^*$.

The quasi-steady system can now be solved. The pressure-flux relations throughout the system given by equations (186) and (197), the continuity of flux equations given by equations (171) and (173), and the inlet and outlet conditions given by (178) and (179), are solved algebraically to give the solution to pressure and flux throughout the system in terms of two unknown variables of $\bar{P}_D$ and $\bar{V}$. This leaves a simple ODE to be solved, given by equations (170) and (168) subject to initial conditions given by equation (180), which we opt to solve numerically using the inbuilt Mathematica function NDSolve (Inc., 2021).

4.7 Quasi-steady system results

Unless stated otherwise, the parameter values used throughout the analysis are as given in Table 3 in Appendix D.1.

4.7.1 Transient system behaviour

We begin by exploring how the system adjusts from its initial state. In Figure 53(a) and (b), we plot the dampener pressure and volume respectively, up to $t = 300$. We see an initial period of adjustment as the volume of air in the dampener decreases and the pressure increases from their respective initial conditions. The system eventually settles to a steady state after around 100 dimensionless time units, corresponding to around 20 seconds.

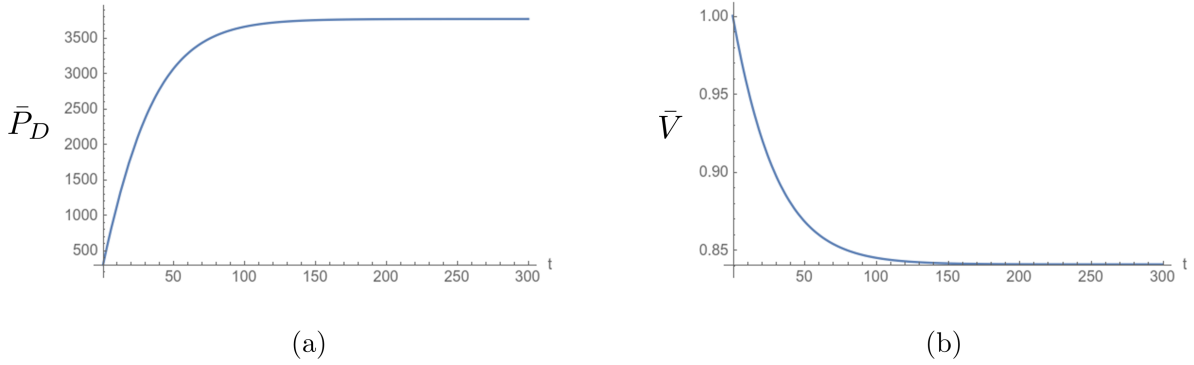


Figure 53: (a) Pressure $\bar{P}_D$ and (b) air volume $\bar{V}$ in the dampener over time.

We see how this adjustment in the dampener volume and pressure impacts the rest of the pressures throughout the system in Figure 54(a), where we plot the pressure values $\bar{P}_i$ for $i = 0, 1, 2, 3, 4$ over time. We categorise the pressures into two distinct groups: pressures before the scope, given by $i = 0, 1, 2$, and pressures after the scope, given by $i = 3, 4$, due to the large differences seen between each group. This is due to the higher resistance within the working channel compared the resistances in the tubing and access sheath, which results in a significant pressure drop as discussed in Chapter 2. By zooming in on the upper and lower groups of pressures, in Figures 54(b) and (c) respectively, the small differences between the pressures within each group are noticeable.

To see how the fluid flux responds to the dampener adjustment, in Figure 55, we plot the fluid flux at the junction in the system, $\bar{Q}_1$ and $\bar{Q}_D$, over time, in orange and green respectively. Initially, as the dampener begins to fill, the majority of the upstream flux is in the tubing leading to the dampener, given by $\bar{Q}_D$, due to the lower resistance of this path. As the system adjusts, the flux towards the dampener reduces to zero, and the flux in tubing leading towards the scope increases, as indicated by $\bar{Q}_1$. We note that due to continuity of flux the remaining fluxes of the system are all equal to $\bar{Q}_1$.

4.7.2 System response for increasing upstream flux

Having seen how the full network adjusts to steady state, we now determine how the adjustment period changes as we vary parameters of the system. In Figure 56(a) and (b), we give the pressures and fluxes throughout the system for increasing upstream flux $\bar{q}_{in}^*$ respectively, with red arrows indicating a change from lower to higher upstream flux, for $\bar{q}_{in}^* = 100/60, 200/60, 300/60$. We see that as the upstream flux increases, the variables of the system reach steady state quicker, reducing the transient time.

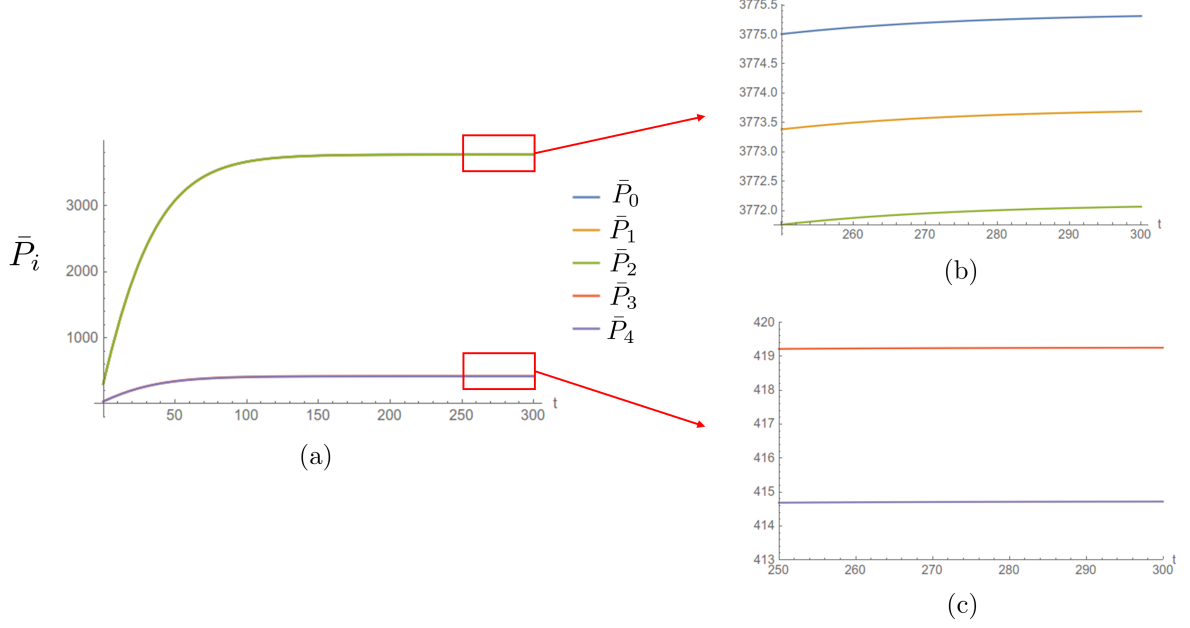


Figure 54: (a) Pressures throughout the system over time, with a close-up of the pressure values categorised into (b) before the scope and (c) after the scope.

4.7.3 The effect of inserting a working tool

A working tool may be inserted into the working channel during a ureteroscopy procedure. To determine the impact this has on the system, we adjust the resistance in the section of the system representing the working channel. We consider how the system responds when a working tool is inserted, and subsequently removed. We model a resistance that varies in time such that

$$R_3(t) = R_{\text{out}} - \frac{1}{2}(R_{\text{out}} - R_{\text{in}}) \left(\tanh\left(\frac{t - t_{\text{in}}}{\sigma}\right) - \tanh\left(\frac{t - t_{\text{out}}}{\sigma}\right) \right), \quad (198)$$

where $R_{\text{out}} = \bar{R}_2$ is the resistance when the tool is not present, R_{in} is the resistance when the tool is present, t_{in} is the time the tool is inserted into the working channel, t_{out} is the time the tool is removed, and σ controls the smoothness of the function, representing the action of tool insertion. The resistance when the tool is present is determined from solving flow in an annulus, as given in Chapter 2, and is given by

$$R_{\text{in}} = \frac{8L_1 \ln(r_{21})}{\pi (1 - r_{21}^2) (1 - r_{21}^2 + (1 + r_{21}^2) \ln(r_{21}))}, \quad (199)$$

where r_{21} is the dimensionless radius of the working tool.

We consider a scenario where a working tool of dimensionless radius $r_{21} = 0.3617$ is inserted into the working channel at time $t_{\text{in}} = 1000$ and removed at time $t_{\text{out}} = 1800$, using a smoothness factor of $\sigma = 5$. The time at which the tool is inserted is chosen sufficiently late to ensure that the system has reached steady state, in order to isolate the impact of the tool insertion. Similarly, the tool removal time is chosen to ensure its effects are isolated.

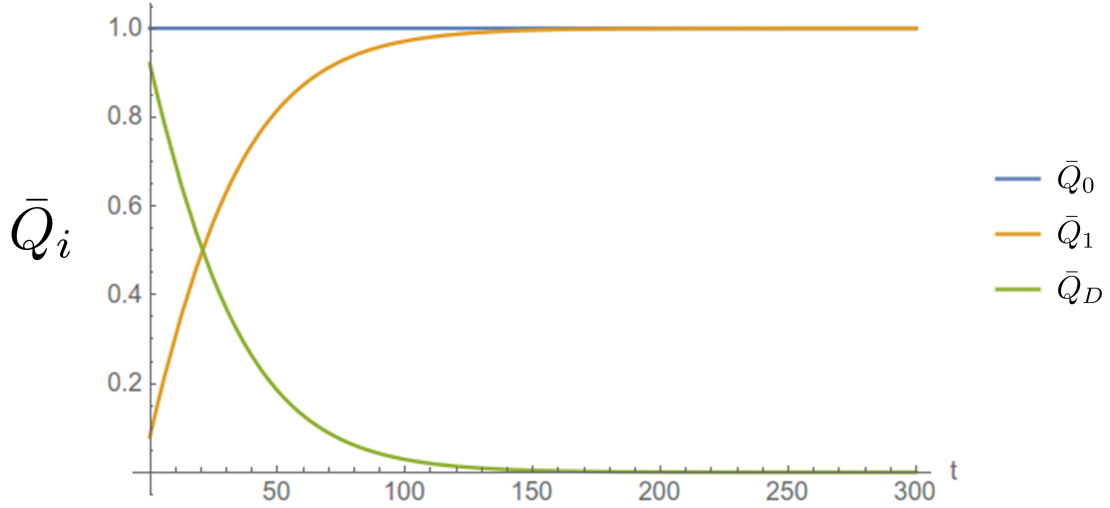


Figure 55: Fluid flux at the junction in the system, $\bar{Q}_D$ and $\bar{Q}_1$, over time, alongside $\bar{Q}_0$ for comparison.

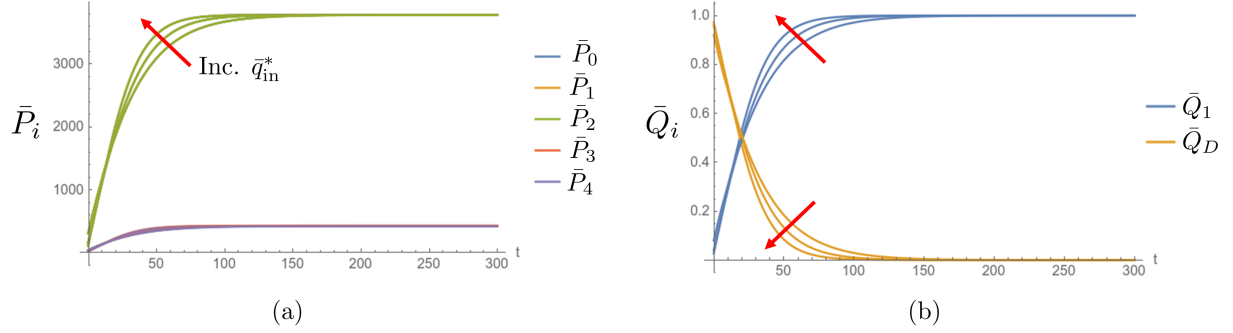


Figure 56: (a) Pressures and (b) fluxes throughout the system for increasing upstream flux $\bar{q}_{in}^*$ respectively, with red arrows indicating a change from lower to higher upstream flux.

In Figure 57, we show how the system variables respond to a working tool being inserted and removed. Figure 57(a) shows how the pressure in the dampener significantly increases when the tool is inserted, with the volume of air dropping to almost half its initial value, as shown in Figure 57(b). This behaviour is reflected by the fluid flux at the junction, where in Figure 57(c) we see an increase in the flux towards the dampener, $\bar{Q}_D$, for a short period, before reducing back down to zero. This period of increased flux towards the dampener also results in a reduction of flux to the rest of the system for a brief period, as shown by $\bar{Q}_1$. The rest of the pressures upstream of the scope also significantly increase due to the presence of the working tool, as shown in Figure 57(d). As the tool is later removed, we see the variables returning to their original steady state values. The drop in pressure within the dampener as the tool is removed causes a large increase in flux throughout the rest of the system, which in turn briefly increases the pressures downstream of the scope, as shown by $\bar{P}_4$ in Figure 57(d). The tool removal sends a larger amount of flux into the system than that being provided by the upstream pump ($\bar{Q}_0 = 1$). From a clinical perspective, this is particularly relevant to note since

the upstream flux is carefully maintained to ensure patient safety, so increasing the flux beyond this level could be problematic.

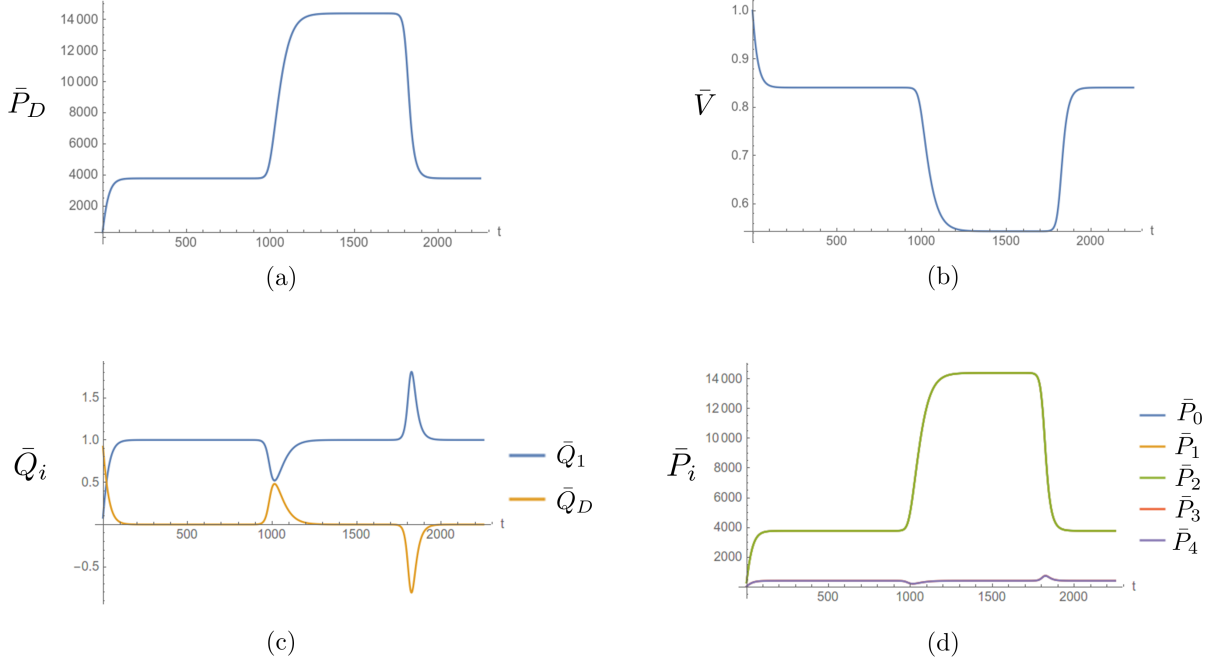


Figure 57: System variables (a) $\bar{P}_D$, (b) $\bar{V}$, (c) $\bar{Q}_1$ and $\bar{Q}_D$, and (d) $\bar{P}_i$ for $i = 0, 1, 2, 3, 4$ responding to a working tool being inserted and removed, for $r_{21} = 0.3617$, $t_{\text{in}} = 1000$, $t_{\text{out}} = 1800$, and $\sigma = 5$.

The large increase in pressures we see upstream of the scope are due to the working tool occupying a significant cross sectional area of the working channel, leaving a small space for the fluid to flow, thus offering a high resistance. We explore how varying the working tool radius can impact the system, and consider three radii of $r_{21} = 1/6, 1/4, 1/3$. In Figure 58, we show how the system variables respond to an increasing working tool radius, with red arrows indicating an increase in radius. Figure 58(a) shows how the flux of fluid into the dampener increases in magnitude as working tool radius increases, causing a decrease in air volume, as shown in Figure 58(b). We see that the pressure upstream of the scope, given by Figure 58(c), increases significantly as the working tool radius increases. When removing the working tool, the pressures downstream of the scope, given by Figure 58(d), see larger increases in pressure as the working tool radius increases.

4.7.4 Comparison to experimental results

To test the theory, we compare the model predictions of working tool insertion and removal against experimental data. The experimental set-up was as described by the model in section 4.2, with the same parameters from Table 3. The data was taken using a flow meter and pressure sensor located just before the ureteroscope, represented by Q_1^* and P_2^* in the model. The system was driven by a constant upstream flux of 100 ml/min, and once it had appeared to reach steady state (i.e. no variations in pressure were seen),

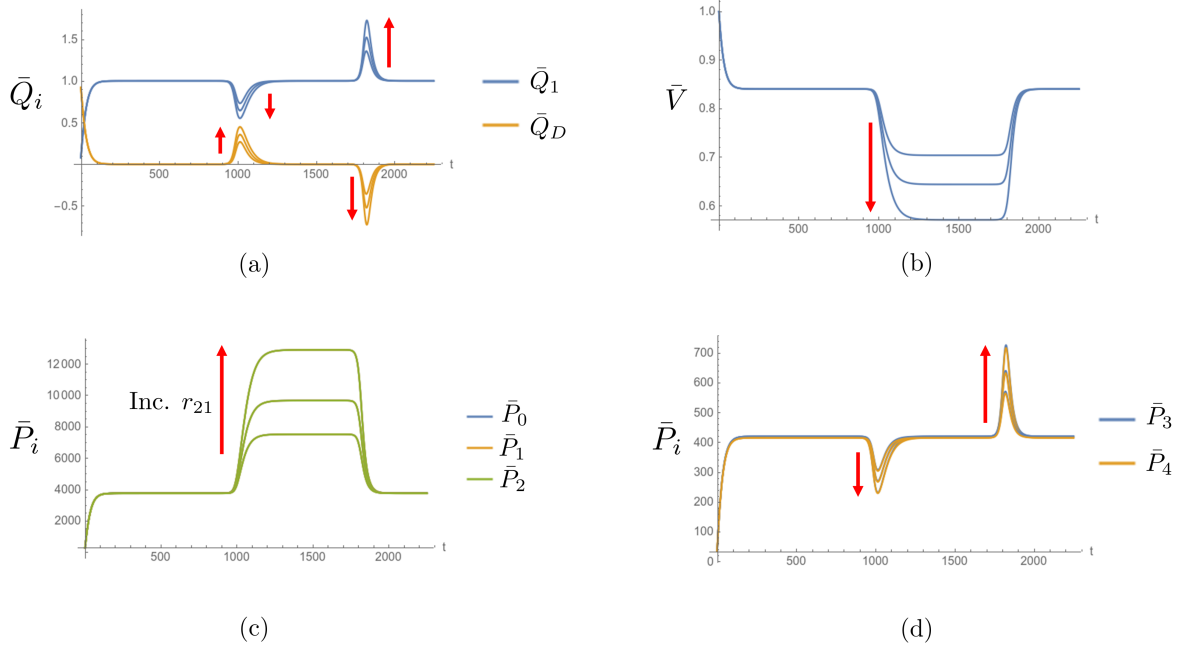


Figure 58: System variables (a) $\bar{P}_D$, (b) $\bar{V}$, (c) $\bar{Q}_1$ and $\bar{Q}_D$, and (d) $\bar{P}_i$ for $i = 0, 1, 2, 3, 4$ responding to a working tool being inserted and removed, with red arrows indicating an increase in working tool radius, for $r_{21} = 1/6, 1/4, 1/3$.

data was recorded as a working tool was inserted into the working channel, being left for 180 seconds, and then was removed. The radius of the working tool was $r_{21}^* = 0.0217\text{cm}$.

In Figure 59(a) and (b), we compare the model predictions for flow rate Q_1^* and pressure P^* respectively against the data obtained from the experiment. The model qualitatively captures the behaviour shown by the data, but is over predicting the impact that inserting a working tool has on the pressure and flow rate of the system. An explanation behind this over-prediction will be discussed later in section 4.10.

4.8 Oscillatory system solution

We now turn to the oscillatory system. As before, we begin by solving the resistances in the cylindrical geometries of the network, which are determined through solving equation (164) along with the form of the pressure gradients from equation (166), subject to boundary conditions from equations (181)-(183). Following from the time-dependent system solution given in Chapter 2, the resistances for unidirectional fluid flow through

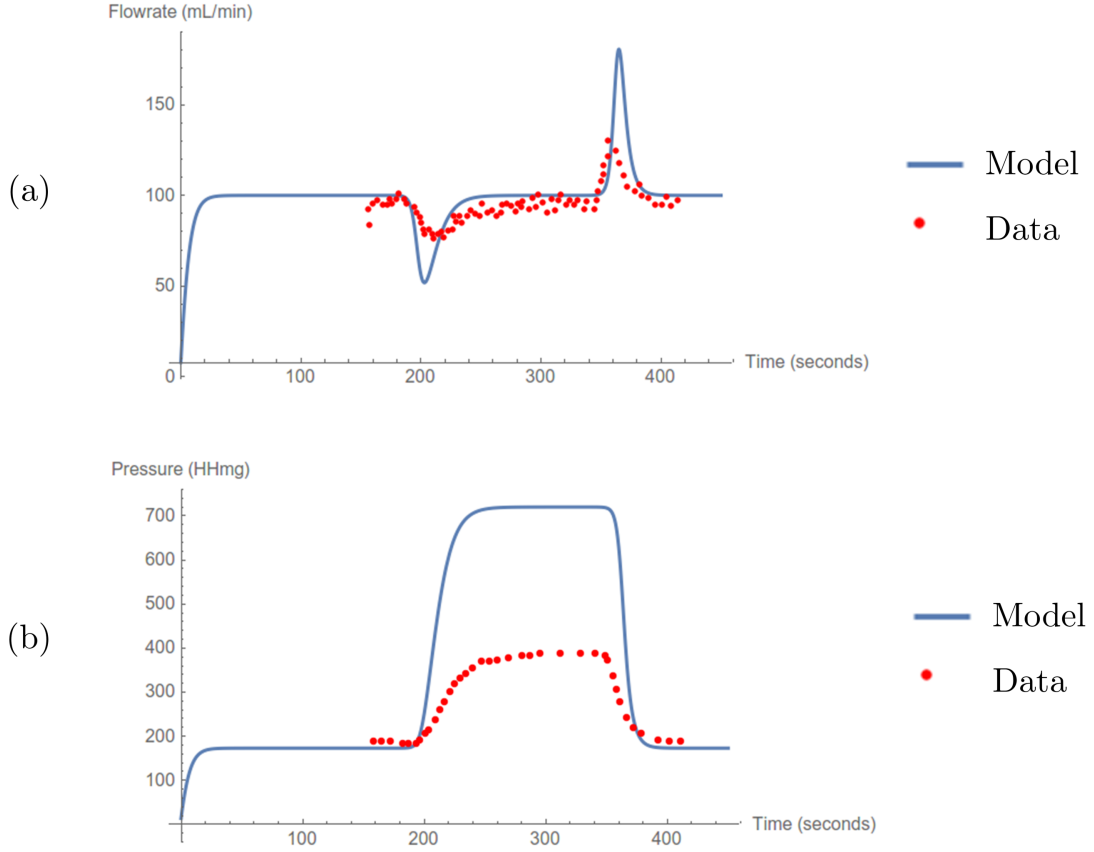


Figure 59: Comparison between the model predictions for (a) flow rate and (b) pressure in blue and the data obtained from experiments in red.

the cylindrical geometries of the network are given by

$$\hat{R}_0 = \frac{i\alpha^3 L_0 J_0(i^{3/2} r_0 \alpha)}{\pi \alpha r_0^2 J_0(i^{3/2} r_0 \alpha) + 2i^{3/2} \pi r_0 I_1(i^{1/2} r_0 \alpha)} , \quad (200)$$

$$\hat{R}_1 = \frac{i\alpha^3 L_1 J_0(i^{3/2} r_1 \alpha)}{\pi \alpha r_1^2 J_0(i^{3/2} r_1 \alpha) + 2i^{3/2} \pi r_1 I_1(i^{1/2} r_1 \alpha)} , \quad (201)$$

$$\hat{R}_2 = \frac{i\alpha^2 L_2 I_0(i^{1/2} \alpha)}{\pi I_2(i^{1/2} \alpha)} , \quad (202)$$

$$\hat{R}_D = \frac{i\alpha^3 L_D J_0(i^{3/2} r_D \alpha)}{\pi \alpha r_D^2 J_0(i^{3/2} r_D \alpha) + 2i^{3/2} \pi r_D I_1(i^{1/2} r_D \alpha)} , \quad (203)$$

where $J_i(r)$ is a Bessel function of the first kind, and $I_i(r)$ is a modified Bessel function of the first kind, and the resistance for unidirectional flow through two concentric cylinders (i.e. the access sheath with ureterscope inserted) is calculated similarly. The oscillatory pressure-flux relations are then given by

$$\hat{Q}_0 = \frac{\hat{P}_0 - \hat{P}_1}{\hat{R}_0} , \quad \hat{Q}_1 = \frac{\hat{P}_1 - \hat{P}_2}{\hat{R}_1} , \quad \hat{Q}_2 = \frac{\hat{P}_2 - \hat{P}_3}{\hat{R}_2} , \quad \hat{Q}_4 = \frac{\hat{P}_4 - \hat{P}_5}{\hat{R}_5} , \quad \hat{Q}_D = \frac{\hat{P}_1 - \hat{P}_D}{\hat{R}_D} . \quad (204)$$

Having determined these relationships, this only leaves the pressure-flux relation from the cavity to be determined. Extracting such a relationship is beyond the scope of the work in this thesis. The complexities of solving time-dependent flow in a cavity in order to extract a pressure-flux relationship are discussed in section 4.10. However, we may still proceed by considering a system which does not include a cavity, and exits into open air at the tip of the ureteroscope, which is a common set-up used for experiments. Due to the modular nature of the model, we can modify the model to represent this scenario.

We change the outlet condition to be at the tip of the scope, replacing equation (185) with $\hat{P}_3 = 0$. The elements of the model beyond this point are removed and not modelled in the system. The system can then be solved algebraically, via the pressure-flux relations given by equations (204), the continuity of flux equations given by equations (172) and (174), and the inlet and outlet conditions given by equations (184) and (185), where we set $\hat{P}_D = 0$ via equation (169).

4.9 Oscillatory system results

We focus on how effective the dampener is in reducing the amplitude of the oscillations in pressure and flux throughout the system. In Figure 60(a), we present the oscillatory pressure over time, defined as $\text{Re}(\hat{P}_j e^{it})$, for $\hat{P}_0$, $\hat{P}_1$, and $\hat{P}_2$ in blue, orange, and green respectively. We see that $\hat{P}_1$ and $\hat{P}_2$ are almost identical to each other, with almost no pressure drop occurring in this region. Comparing $\hat{P}_0$ and $\hat{P}_1$, we see that the inclusion of the dampener results in a significant dampening of the pressure amplitudes coming from the upstream pump. The very small pressure drop between $\hat{P}_1$ and $\hat{P}_2$ results in almost no oscillatory flux travelling towards the scope, as seen in Figure 60(b), where we give the oscillatory part of the flux over time, defined as $\text{Re}(\hat{Q}_j e^{it})$, for $\hat{Q}_0$, $\hat{Q}_D$, and $\hat{Q}_1$ in blue, orange, and green respectively. At the junction, the significant majority of the oscillatory flow goes to the dampener, shown by the figure in which there is almost no difference between the imposed oscillatory flux $\hat{Q}_0$ and $\hat{Q}_D$. These results suggest the dampener is working as intended by significantly reducing oscillations in the system.

To test the effectiveness of the dampener further, we can again exploit the modular nature of the model and compare these results to the results of a model without a dampener. In the no dampener model, we remove the section of tubing connecting the junction to the dampener, and instead consider one continuous system without a junction. This changes the continuity of flux equation given by equation (172) to be $\hat{Q}_0 = \hat{Q}_1$. A schematic of the two models which are being compared is shown in Figure 61.

In Figure 62, we give the oscillatory component of the pressure before the scope $\text{Re}(\hat{P}_2 e^{it})$ over time, for the models with and without a dampener in blue and orange respectively. We see the model with a dampener displays a significant reduction in amplitude of pressure compared to one without.

4.10 Summary and discussion

By creating a mathematical model of the full ureteroscopy system, we have a tool that can be used to provide access to pressure and flow conditions throughout the network, in particular within the kidney, as a function of upstream conditions, such as the rotation rate of the peristaltic pump. The model enables the exploration of the consequences of

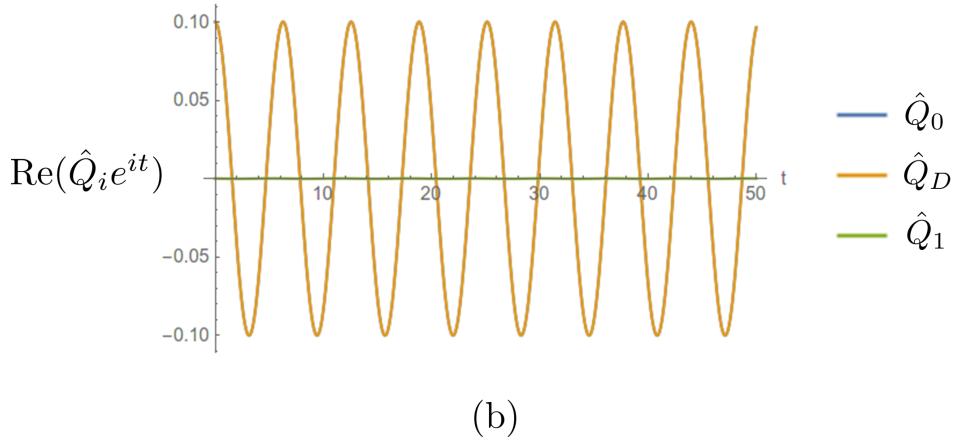
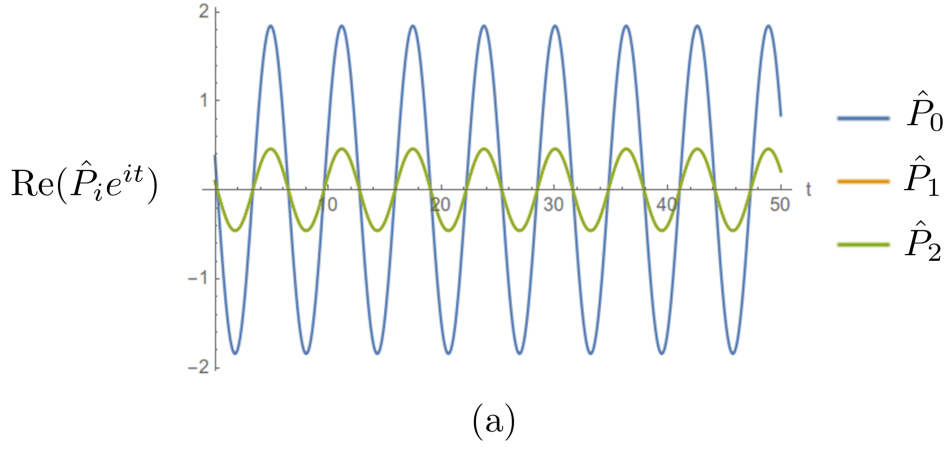


Figure 60: Oscillatory part of the (a) pressures $\hat{P}_i$ for $i = 0, 1, 2$ and (b) fluxes $\hat{Q}_i$ for $i = 0, 1, D$ over time.

varying the resistances in the network, for example by inserting a working tool into the working channel. The modelling of the air dampener allows us to directly test the impact it has on the system and determine its effectiveness in reducing oscillations.

We considered the impact inserting a working tool has on the system variables, and compared the model predictions with data to find that the model was over predicting the pressure increases when the tool was inserted. One possible explanation for this is that we are assuming the working tool is sitting concentrically and co-axially within the working channel. Previous studies (Piercy et al., 1933; Williams, 2019) have shown that this position offers the highest resistance to flow within the channel, which would explain the over-prediction in pressure, and that a fully offset working tool reduces the resistance. Work by Piercy et al. (1933) derives an equation for the flow rate through a circular pipe with an offset core, which Williams (2019) go on to apply to a ureteroscopic context. The details of these calculations are outlined in Appendix D.2. By introducing a parameter ϕ to represent the degree of offset, with $\phi = 0$ representing a tool in the centre and $\phi = 1$ representing a tool at the edge of the channel, Williams (2019) show that the resistance decreases monotonically with ϕ . It is found that at $\phi = 0.99$, the flow rate in the channel increases to around 2.5 times larger than what is seen for $\phi = 0.01$,

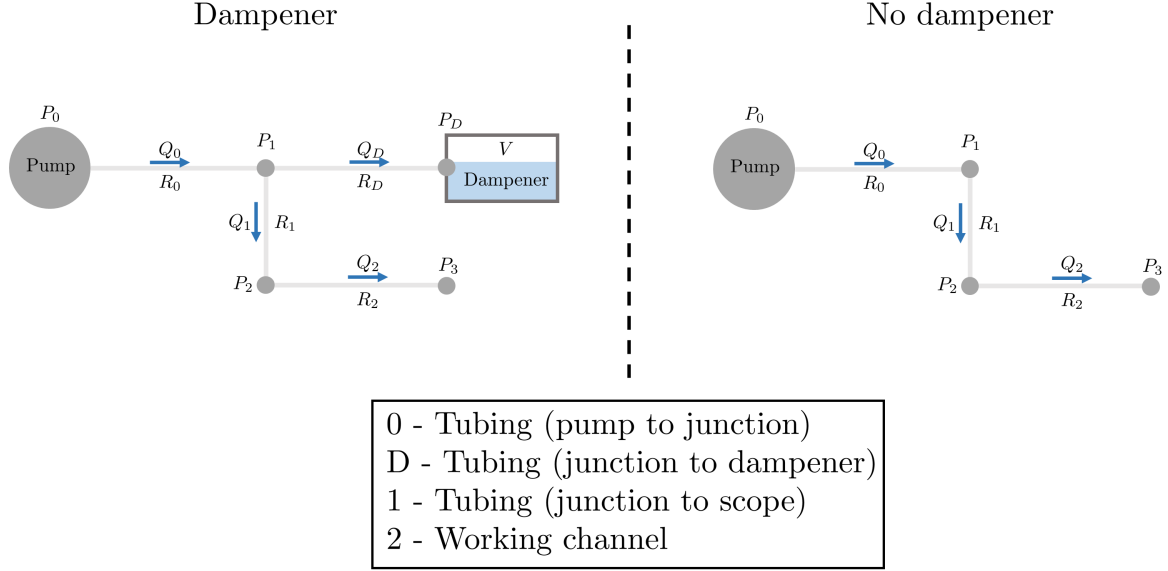


Figure 61: Schematic of the two models being compared, one with a dampener and one without.

translating to a reduction in flow resistance of 46.5%. To approximate this scenario, within our full system model we introduce a scaling factor for the resistance of working channel with working tool inserted, as given by equation (199). Using a scaling factor of 0.55, equivalent to a 45% reduction in resistance, in Figure 63, we give the updated comparison between model prediction and data, and see that this reduction in resistance has resulted in a more accurate prediction of the data, suggesting an almost fully offset working tool would account for the discrepancy seen.

Although we did not extract a pressure-flux relation from the cavity for the oscillatory system, we can consider how we might proceed. The introduction of oscillations within the cavity introduces two additional parameters of frequency and amplitude to consider, alongside the Reynolds number, significantly increasing the number of cavity simulations that are need to extract a pressure-flux relationship. Multiple pressure-flux relations would need to be determined, covering a range of both oscillations and frequencies. When extracting the pressure values from the cavity, a more in depth Fourier analysis would be needed in order to determine if the pressures oscillations decompose into multiple frequencies. The model is already designed to handle this, as we can account for ever smaller contributions in equation (158).

The inclusion of an air dampener to the system allows us to consider its viability in practice. We have shown that whilst a dampener does aid in reducing oscillations created by the upstream pump, we have seen it can lead to unwanted increased in flux throughout the system when removing a working tool. We have also shown that due to its significant reduction in oscillations, it may be difficult to impose any oscillations downstream of the scope within the kidney; something that Chapter 3 argues would be beneficial.

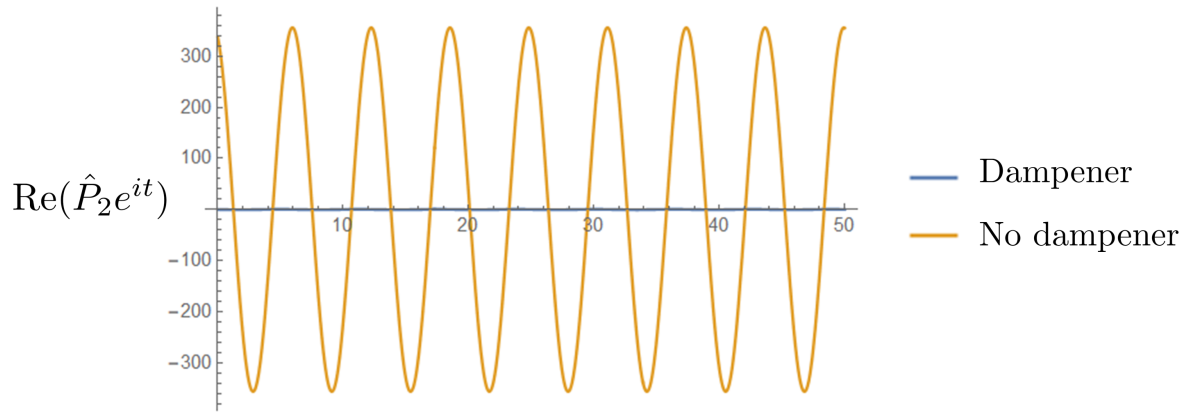


Figure 62: Oscillatory component of the pressure before the scope $\text{Re}(\hat{P}_2 e^{it})$ over time, for the models with and without a dampener in blue and orange respectively.

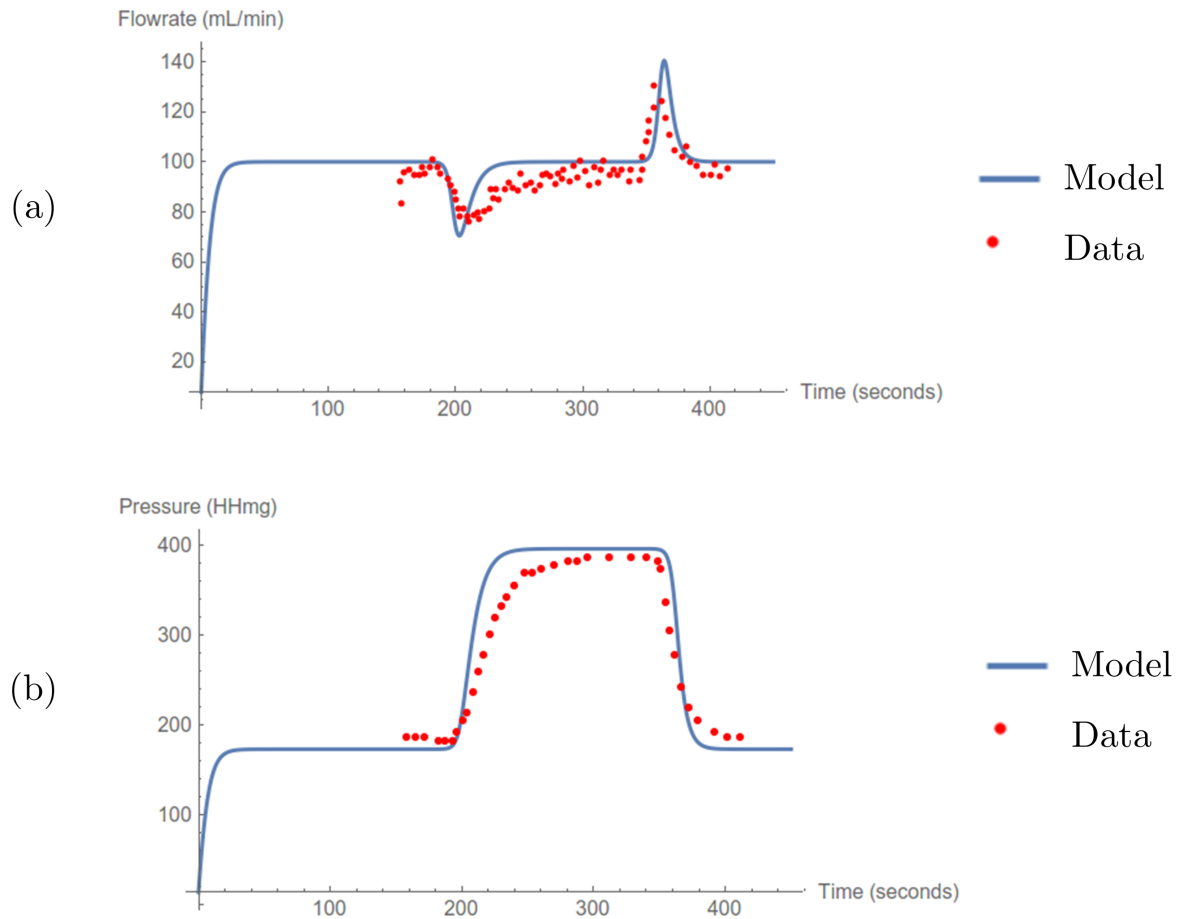


Figure 63: Comparison between the model predictions for (a) flow rate and (b) pressure in blue and the data obtained from experiments in red, where a scaling factor of 0.55 has been introduced in the modelling.

5 Summary and future work

5.1 Summary

In this thesis we have focused on the mathematical modelling of unsteady flows throughout the ureteroscopy fluid irrigation system. Motivated by the introduction of using a peristaltic pump to drive the irrigation fluid, we considered the impact that oscillatory flows have on the pressures and fluxes throughout the system. Through analytical and numerical techniques,

Beginning with Chapter 2, we created a mathematical model of the fluid flow throughout the cylindrical geometries of the ureteroscopy system, namely the tubing, ureteroscope working channel with working tool inserted, and access sheath with ureteroscope inserted. At the junctions between each of these three regions, we simplified the sometimes complex geometries, such as that of flow within the kidney cavity, and assumed continuity of pressure and flux. The flow was driven by an upstream pressure which oscillated in time, mimicking the behaviour of the peristaltic pump, with the downstream pressure set to atmospheric. Via an ansatz motivated from the form of the upstream flux, we decomposed the system into its time-independent and time-dependent parts. Due to the simplifications made in the modelling, we were able to find analytical solutions for the fluid velocities and pressures throughout the system. These consisted of classic pipe flow solutions describing the time-independent problem and solutions that followed from work by Womersley (1955) which utilised Bessel functions for time-dependent problem. Focusing on the key area of the kidney cavity downstream of the scope, we determined the magnitude, amplitude, and frequency of the pressure at this region and explored how these values are varied by changing the parameters of the system. We conclude that due to the small radii of the working channel offering high resistance to the flow, oscillations are significantly dampened by the time they reach the kidney downstream of the ureteroscope. It was found that if a working tool is not present within the working channel, the dampening effect is less severe due to the increased cross sectional area for the fluid to flow through. The oscillations of the velocities and pressures were more sensitive to the geometric parameters, however it was also found that they could be reduced by increasing the frequency of the imposed oscillations. We tested the continuity assumptions made at the junctions between each region by numerically solving for the flow within the more complicated geometry of a two-dimensional cavity, represented as a pipe expansion and subsequent contraction. The continuity assumptions made in the system modelling were found to be justified as we saw that the pressure drop across this cavity is small, and that the flow returns to being unidirectional shortly after exiting the cavity. This numerical simulation of cavity flow would be modified and examined in more detail in a later chapter. To close the chapter, we presented a model of a peristaltic pump and determined how the upstream parameters of magnitude, amplitude, and frequency are related to one another via the geometry and rotation rate of the pump driving the fluid throughout the system. We went on to combine this pump model with the cylindrical geometry system model, and compared the theory against experimental data for varying pump rotation rate. Good agreement between model and data was found, with the accuracy decreasing as pump rotation rate increases. We found that by adjusting the amount which a pump head compressed the tubing, good agreement was found between theory and data at the

higher rotation rates.

Due to the simplification of neglecting the kidney cavity in the previous chapter, in Chapter 3, we went on to model fluid flow in a two-dimensional, rectangular cavity with an oscillating, parabolic inlet flow. On the same side as the inlet the cavity had two outlets located above and below the inlet, where we prescribed zero-stress conditions. As well as modelling the fluid flow, we also considered the advection and diffusion of a passive tracer within said fluid, representing the transport of kidney stone dust. Via non-dimensionalisation of the problem, four key dimensionless parameters were identified, representing the magnitude, amplitude, and frequency of the inlet flow, as well as the diffusivity of the dust. The problem was constructed via a finite element formulation and solved numerically for fluid velocity and pressure and dust transport throughout the cavity. Since we anticipated that vortical fluid structures were of interest, we characterised a vortical structure metric which captures these features of the fluid structure. We also characterised the efficiency of the procedure by considering a dust wash-out time metric, with the faster the dust was washed out of the cavity the more efficient a procedure was. We found that a high frequency inlet oscillation resulted in flow patterns which are similar to the steady system, where a large central vortex was seen. Dust can become trapped in said vortex, leading to long wash-out times reliant on slow diffusive processes for transport. Increasing the magnitude of the inlet flow resulted in the central vortex increasing in size. By reducing the frequency of the inlet flow oscillations, we found that low frequencies caused disturbances to the central vortex. This created a mixing effect which resulted in freeing debris trapped within the vortex and a reduction in wash-out times. Larger amplitudes were shown to further amplify the mixing effect. We concluded that an inlet flow with a low frequency and high amplitude significantly improved wash-out times when compared to those of a high frequency or steady inlet flow. By exploring a wider range of parameters and calculating the wash-out times and vortical fluid structure metric for each, we created a linear fit which supported the conclusions. We compared the suggested oscillatory mixing technique to the current leading industry standard for excess debris removal, known as flushing. We found an oscillatory inlet to perform favourably over flushing, achieving the same wash-out times at a much lower flow rate. To test the robustness of results to changes in the idealised cavity geometry, we went on to compare wash-out times in six new two-dimensional domains. We found similar qualitative flow patterns in each, and that low frequency, high amplitude oscillations improve wash-out times in all scenarios, further supporting the previous results.

Motivated by the modelling of the cylindrical geometries throughout the system in Chapter 2 and the kidney cavity in Chapter 3, in Chapter 4 we aim to combine all elements together into a full ureteroscopy fluid irrigation system model. We characterised the system via a series of pressure-flux relations drawn from the cylindrical geometries throughout the system. Cavity simulations were used to extract a pressure-flux relation from the cavity, which was fed into the full system model. The system was driven by an upstream flux that was calculated from the previous peristaltic pump modelling work, and the pressure at the end of the access sheath was set to atmospheric. We included the additional element of an air dampener into the system: a device which consists of a chamber of air connected to the fluid network, which serves to reduce natural oscillations created by the peristaltic pump. The problem was non-dimensionalised and the small parameter δ , the ratio between the time it takes for the dampener to adjust and the

frequency of oscillations, was exploited to expand the system and decompose it into quasi-steady and oscillatory parts. Analysing each of these parts separately, we first solve the quasi-steady system, and determine the time it takes for the system adjusts to steady state from its initial conditions, as well as how it adjusts as a working tool is inserted and removed from the working channel. Steady cavity simulations were used to extract a quadratic relationship between cavity flux and inlet and outlet pressures, with the complexities of relating a two-dimensional cavity back to a three-dimensional geometry considered. Parameters of the system were varied to explore how the adjustment times are impacted, and it was found that increase the upstream flux reduces these times. These quasi-steady results were support by experimental data of tool insertion and removal, where the resistance to the flow in the working channel was adjusted to show good agreement with the data. We went on to solve the oscillatory system, and found that the air dampener served to significantly reduce the oscillations produced by the upstream pump when compared against a system with no dampener.

The study has provided insight into the previously unexplored application of modelling oscillatory fluid flows during ureteroscopy irrigation. The analytical solution presented in Chapter 2 revealed a dramatic dampening in the oscillations coming from the peristaltic pump due to the small radius of the working channel, an effect which had not been considered by the industry, highlighted by the thought to include a dampener to achieve the same result. Determining the oscillation prevalence through-out the system revealed the counter-intuitive behaviour that the higher frequency flows displayed results more similar to the steady system, with the lower frequency range being the one that introduces different behaviours to the fluid irrigation. This result is compounded upon when modelling the cavity flow in Chapter 3, where the differing flow patterns coming from a low frequency inlet flow where shown to be a benefit for dust wash-out. The introduction of an air dampener model in the combined full system model allowed for the introduction of a new quasi-steady timescale, opening up the ability to determine the systems response times when varying resistances and driving flows, such as when inserting and removing a working tool. The quasi-steady timescale and the oscillatory timescale were found to be separable, allowing for the fluid irrigation system to be explored in both a steady and oscillatory driving flow scenario. Bringing the two dimensional pipe flow system model and the three dimensional cavity flow simulation together provided interesting challenges such as determining how to relate two outlet channels into a single outlet tube and handling the multiple solutions of the cavity flow.

We set out to answer two key questions: firstly, does the new method of irrigation fluid delivery differ significantly from the traditional hanging saline bag set-up? And secondly, can the new irrigation fluid delivery method be used to improve the effectiveness and efficiency of the ureteroscopy procedure? Addressing the first question, in Chapter 2 we have shown that while the peristaltic pump does induce natural oscillations in the flow, these are significantly dampened in the key area of the system downstream of the ureteroscope. This dampening effect is amplified further when a working tool is present in the working channel or when upstream frequencies are large. We have also explored the idea of using an air dampener to reduce these oscillations in Chapter 4. The theory shows that the dampener works as intended by significantly reducing the oscillations in the flow. However, the results drawn in Chapter 2 suggest that maybe the inclusion of a dampener is not wholly necessary, due to the natural dampening of the system. The exclusion of the

dampener is especially worth consider when we turn to the potential benefits of the new irrigation fluid delivery method. To determine said benefits, we explored the fluid flow and dust transport within a kidney cavity in Chapter 3. We suggested that by having finer control over irrigation fluid delivery, an oscillating fluid flow could be imposed into the network. We found that a cavity inlet flow with low frequency and high amplitude oscillations served to disturb accumulations of kidney stone dust which became trapped in fluid vortices, leading to faster wash-out times and hence a more efficient procedure. This technique was shown to outperform the current leading industry method for disturbing clouds of kidney stones dust.

5.2 Future work

5.2.1 Three-dimensional cavity flows

In Chapter 3, we have shown that the qualitative results are robust to changes in geometry in a two-dimensional setting. An obvious extension to the two-dimensional cavity flow is exploring the results in three dimensions. In reality, the structure of the kidney renal pelvis is complex (Figure 1), and considering the flow in three dimensions would be the first step towards capturing these complexities. We anticipate that further vortical structures will be present in the various cavities throughout the renal pelvis, which would likely lead to an increase in the amount trapped debris and increase wash-out times. Modelling such a scenario would give insight into whether the low frequency, high amplitude inlet oscillations also aid in disturbing the dust trapped in these additional vortices. A realistic, three-dimensional kidney domain would also give more accurate predictions for wash-out times.

5.2.2 Experimental support

Whilst some efforts were made to compare the theory presented throughout this thesis to experimental results, the work would benefit from a broader comparison to real world data. The cavity flow simulations displayed in Chapter 3 concluded that the vortical structures presented by the flow were disturbed by low frequency inlet oscillations, leading to improved wash-out times. We have seen from previous work that PIV experiments were performed to compared the steady cavity flow theory against (Williams et al., 2020a). Similar experiments run with an unsteady, oscillating inlet flow would help support the conclusions drawn to confirm whether the fluid structures found respond as expected to changes in the frequency and amplitude of the inlet oscillations. This work could be taken further by including a passive tracer in the fluid experiments to quantitatively compare wash-out times while qualitatively see whether clouds of dust form and subsequently disperse under low frequency oscillation conditions. In Chapter 4 we showed that the dampener had a significant impact of reducing oscillations from the upstream pump. Experiments could be performed through a similar fluid irrigation system as described in the theory for output both with and without a dampener. Supporting these results with bench-top experiments would give insight into whether including a dampener in the system performs as effectively as suggested.

5.2.3 Kidney stones modelling

The work in this thesis has focused on modelling kidney stone dust created during laser treatment via a passive tracer that diffuses and advects with the flow. In reality, the laser treatment is unlikely to completely reduce the stone to dust, with smaller fragments often remaining, which must be removed manually using a wire basket. The influence that these larger fragments have on the fluid flow is unknown. The complex fluid-structure interaction problem that arises from this modelling scenario was beyond the scope of this thesis. A possible starting point would be to consider a fixed fragment within the cavity domain, a problem akin to two-dimensional flow past a cylinder. The first steps of such a problem have been explored during the shape exploration in Section 3.10, where in the geometry *stone*, a circular section has been removed from the centre of the domain. The forces acting upon this surface due to the surrounding flow could then be calculated to infer how the fragment would move. An undesired consequence of the fluid irrigation is stone retropulsion, where the jet of fluid causes the stone fragment to be pushed away when the scope tip is pointed towards it, making it hard for the clinician to view the stone. The oscillatory nature of the flows we are considering may lead to interesting stone fragment behaviour.

5.3 Concluding remarks

This thesis has utilised mathematical modelling to provide critical insight into the relatively unexplored latest advancement in ureteroscopy fluid irrigation technology; flow driven via a peristaltic pump. Suggestions had been made that the natural oscillations caused by the introduction of a pump may lead to undesired behaviours in the flow, with a call to produce a system akin to the traditional, steady flow method. We have given a theoretical understanding of the oscillatory fluid flow during ureteroscope irrigation: modelling how these natural oscillations are produced and suggesting ways in which the amplitudes of the oscillations can be reduced. Further to this, we have gone on to show how this new technological advancement can be utilised to improve wash-out times, outperforming current dust clearance methods. We are confident that the evidence throughout this thesis supports the fact that we should not be aiming to replicate a system akin to the traditional, but to harness these advancements to our benefit, giving the first steps towards designing a system for a more effective and efficient ureteroscopy procedure.

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Appendix

B Chapter 2 appendix

B.1 Solution for the time-dependent part of the velocity

The solution for time-dependent part of the velocity in regions one, two, and three is given as

$$\hat{w}_1 = \frac{i(B_1 - B_0)}{\alpha^2} \left(1 - \frac{J_0(i^{3/2}r\alpha)}{J_0(i^{3/2}\alpha)} \right), \quad (205)$$

$$\begin{aligned} \hat{w}_2 = & \frac{i(B_2 - B_1)Y_0(-i^{3/2}r\alpha)J_0(i^{3/2}r_1\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))} \\ & - \frac{i(B_2 - B_1)Y_0(-i^{3/2}r\alpha)J_0(i^{3/2}r_2\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))} \\ & - \frac{i(B_2 - B_1)J_0(i^{3/2}r\alpha)Y_0(-i^{3/2}r_1\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))} \\ & + \frac{i(B_2 - B_1)J_0(i^{3/2}r\alpha)Y_0(-i^{3/2}r_2\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))} \\ & + \frac{i(B_2 - B_1)Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))} \\ & - \frac{i(B_2 - B_1)J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha)}{l_1\alpha^2(Y_0(-i^{3/2}r_1\alpha)J_0(i^{3/2}r_2\alpha) - J_0(i^{3/2}r_1\alpha)Y_0(-i^{3/2}r_2\alpha))}, \end{aligned} \quad (206)$$

$$\begin{aligned} \hat{w}_3 = & \frac{i(-B_2)Y_0(-i^{3/2}r\alpha)J_0(i^{3/2}r_3\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))} \\ & - \frac{i(-B_2)Y_0(-i^{3/2}r\alpha)J_0(i^{3/2}r_4\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))} \\ & - \frac{i(-B_2)J_0(i^{3/2}r\alpha)Y_0(-i^{3/2}r_3\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))} \\ & + \frac{i(-B_2)J_0(i^{3/2}r\alpha)Y_0(-i^{3/2}r_4\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))} \\ & + \frac{i(-B_2)Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))} \\ & - \frac{i(-B_2)J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha)}{l_2\alpha^2(Y_0(-i^{3/2}r_3\alpha)J_0(i^{3/2}r_4\alpha) - J_0(i^{3/2}r_3\alpha)Y_0(-i^{3/2}r_4\alpha))}, \end{aligned} \quad (207)$$

respectively, with the real part understood, where B_0 is given.

B.2 Pipe expansion flow results with a refined meshing structure

This appendix gives an example of how the results presented in section 2.6 can be improved upon by refining the mesh size. The refined mesh uses a target element size of

$dx = 0.05$, which is five times smaller than the original $dx = 0.25$ used in Chapter 2. In Figure 64, we compare the results for inlet, upper outlet, and lower outlet pressure for both the original mesh and refined mesh in blue and orange respectively, with $Re = 10$ and $L_{cav} = 0.6$. We see that the refined mesh provides a more smooth solution closer to the wall, however, discontinuities are still seen very close to the wall. This is expected due to the sharp corners located at these positions. The refined mesh comes at the cost of a significant increase in simulation run time, and so doing it for multiple parameter variations becomes very time consuming.

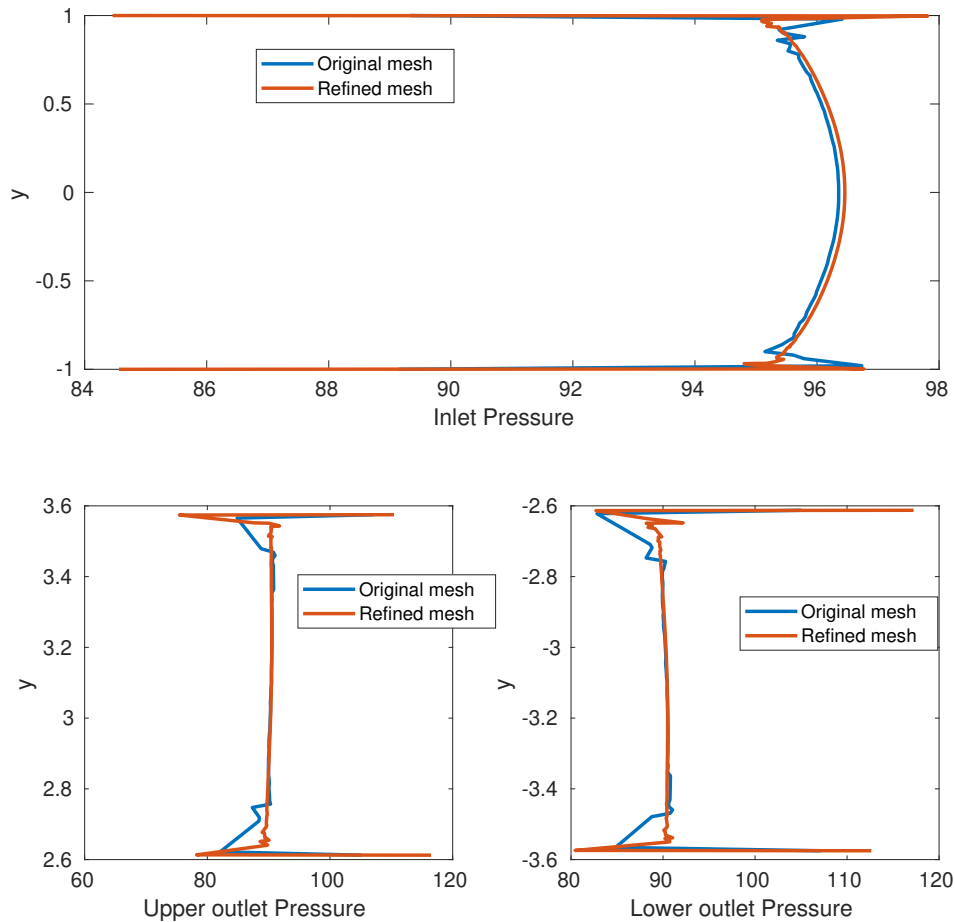


Figure 64: Solution of inlet, upper outlet, and lower outlet pressure for both the original mesh and refined mesh in blue and orange respectively, with $Re = 10$ and $L_{cav} = 0.6$.

C Chapter 3 appendix

C.1 Steady system

This appendix outlines the steady system which is solved to give an initial condition for the unsteady system. In order to be a valid initial condition for the unsteady problem,

the steady problem uses the same domain as laid out in the problem description. We non-dimensionalise the problem via the same scalings as given in equation (79).

The non-dimensional, steady Navier-Stokes equations which govern the fluid is then given by

$$Re ((\mathbf{u}_{\text{steady}} \cdot \nabla) \mathbf{u}_{\text{steady}}) = -\nabla p_{\text{steady}} + \nabla^2 \mathbf{u}_{\text{steady}} , \quad (208)$$

$$\nabla \cdot \mathbf{u} = 0 . \quad (209)$$

At the inlet to the cavity we prescribe fully-developed parabolic flow, given by

$$\mathbf{u}_{\text{steady}} = ((1 - y^2), 0) \quad \text{on} \quad \Gamma_{\text{in}} , \quad (210)$$

At the outlet boundaries we prescribe zero stress such that

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{out}} . \quad (211)$$

We impose no-slip on all impermeable walls, given by

$$\mathbf{u}_{\text{steady}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{walls}} . \quad (212)$$

We solve the system numerically via a finite element formulation, which is implemented using the open source finite element library Firedrake (Rathgeber et al., 2017). To solve, we use a stationary incompressible Navier-Stokes solver with Reynolds robust pre-conditioner (Farrell et al., 2019). This solution is used in equation (85) as the initial condition to the unsteady system, as well as for comparison purposes to the unsteady results.

C.2 Full list of parameter variations

This appendix gives the full list of parameter variations performed during the linear fit analysis, along with the resulting T_{90}/Re , $\Theta_{\text{avg}}Re$, and $\Theta_{\text{amp}}Re$, as given in Table 2.

C.3 Shape exploration dimensions

This appendix outlines the changes in domain geometry for the six domain shapes. All values are given as dimensionless quantities, where the original domain sizes remain unchanged unless stated. For *scopein*, the region from $y = (-a - b)/a$ to $y = (a + b)/a$ is protruded into the cavity by a length of 2.5. For *scopein*, the length of the cavity is reduced to $l_c/a = 6.425$, with the right-hand wall then generated as a semi-circle of radius 3.575. For *long*, the length of the cavity is increased to $l_c/a = 15$. For *large*, the length of the cavity is increased to $l_c/a = 16.667$ and the width is extended such that it goes from $y = (-a - b - d - 4.7585)/a$ to $y = (a + b + d + 4.7585)/a$. For *large*, the area which is removed from the domain is a circle centred at $(5, 0)$ of radius 1. Finally, *kidney* is generated to qualitatively represent a kidney calyx of height 41.667 and width 25, where the scope has width $2(a + b)/a$.

Re	Sc	ω	$\hat{U}$	T_{90}/Re	$\Theta_{avg} * Re$	$\Theta_{amp} * Re$
30	10	10	5	5.95	21.70	15.35
30	10	10	10	5.55	25.12	30.81
30	10	10	15	5.06	30.44	46.90
30	10	10	20	4.95	37.47	64.01
30	10	100	5	6.19	22.11	11.90
30	10	100	10	6.30	26.40	24.39
30	10	100	15	6.49	32.30	38.38
30	10	100	20	6.77	39.48	53.68
40	10	10	5	5.99	31.82	17.39
40	10	10	20	3.94	45.14	68.17
40	10	100	5	6.81	32.28	11.62
40	10	100	20	7.51	48.25	48.76
50	10	2.5	5	7.45	42.93	21.48
50	10	2.5	20	4.37	49.63	93.72
50	10	5	5	7.18	43.01	19.43
50	10	5	20	3.56	51.60	82.84
50	10	7.5	5	5.61	43.11	18.47
50	10	7.5	20	3.22	52.96	73.94
50	10	10	5	6.15	43.40	18.24
50	10	10	10	4.16	45.89	36.68
50	10	10	15	3.22	50.12	55.01
50	10	10	20	3.04	56.23	73.18
50	10	20	5	7.33	43.70	16.98
50	10	20	20	3.54	59.46	65.93
50	10	30	5	7.76	43.67	11.84
50	10	30	20	5.25	59.92	48.26
50	10	40	5	7.90	43.54	10.54
50	10	40	20	6.93	58.85	44.83
50	10	50	5	7.97	43.49	10.52
50	10	50	20	7.75	58.19	45.32
50	10	100	5	8.03	43.55	11.25
50	10	100	10	8.16	46.59	23.16
50	10	100	15	8.37	51.50	34.74
50	10	100	20	8.65	57.95	46.31
60	10	10	5	6.37	55.88	19.51
60	10	10	20	3.41	67.79	79.26
60	10	100	5	9.14	55.84	10.93
60	10	100	20	9.50	68.66	45.76
70	10	10	5	8.99	69.31	21.10
70	10	10	20	5.07	80.15	85.76
70	10	100	5	9.87	69.15	10.49
70	10	100	20	9.92	80.83	44.76

Table 2: The full list of parameter variations performed during linear fit analysis, along with the resulting T_{90} , Θ_{avg} , and Θ_{amp} for each.

C.4 Numerical convergence of the fluid solver

To test the accuracy and convergence of the fluid solver we vary the time-step dt and mesh size dx , where a value of $dx = 1$ refers to a target mesh size at each node in the domain of 1 dimensionless length. Using the fluid velocity $\mathbf{u}(u, v)$ as the value of interest, we calculate the 2-norm in each case, such that

$$\|\mathbf{u}\|_2 = \sqrt{\sum_{\mathbf{x}} |\mathbf{u}(u(\mathbf{x}, t; dx, dt), v(\mathbf{x}, t; dx, dt))|^2 d\mathbf{x}} . \quad (213)$$

We then compare the difference between the current time-step or mesh size with half of its value. Using this value, we integrating over time and divide by the norm of the solution at the current time-step/mesh size to give a measure for the relative error between the current time-step/mesh size and the next, defined as

$$\varepsilon_{\mathbf{u}_{dt}} = \frac{\int_0^T \|\mathbf{u}_{dt} - \mathbf{u}_{dt/2}\|_2}{\int_0^T \|\mathbf{u}_{dt}\|_2} , \quad (214)$$

$$\varepsilon_{\mathbf{u}_{dx}} = \frac{\int_0^T \|\mathbf{u}_{dx} - \mathbf{u}_{dx/2}\|_2}{\int_0^T \|\mathbf{u}_{dx}\|_2} , \quad (215)$$

for time-step and mesh size respectively.

In Figure 65(a), we present $\varepsilon_{\mathbf{u}_{dt}}$ for $dt = 2, 1, 0.5, 0.25, 0.125, 0.0625$ with $dx = 0.25$ and in Figure 65(b), we present $\varepsilon_{\mathbf{u}_{dx}}$ for $dx = 2, 1, 0.5, 0.25, 0.125$ with $dt = 0.1$. In each case the parameters are set as $Re = 50$, $\omega = 10$, and $\hat{U} = 20$, with an end time of $T = 100$. Alongside this, in Figures 65(c) and 65(d), we give the total run-time of each calculation for varying time-step and mesh size respectively. Both cases display convergence as the time-step and mesh size is decreased, whilst total run-time increases. With the aim of minimising both error and run-time, we choose to set a time-step of $dt = 0.1$ which has a relative error of $\varepsilon_{\mathbf{u}_{dt}} = 0.02$ and a mesh size of $dx = 0.25$ which has a relative error of $\varepsilon_{\mathbf{u}_{dx}} = 0.013$.

C.5 Numerical convergence of the dust transport solver

Similar to Appendix C.4, we test the accuracy and convergence of the dust transport solver by varying the time-step dt and mesh size dx , and calculating the 2-norm for the concentration c , such that

$$\|c\|_2 = \sqrt{\sum_{\mathbf{x}} |c(\mathbf{x}, t; dx, dt)|^2 d\mathbf{x}} . \quad (216)$$

Again, comparing the difference between the current time-step or mesh size with a smaller value, we integrate this value over time and divide by the norm of the solution at the current time-step/mesh size to give a measure for the relative error between the current

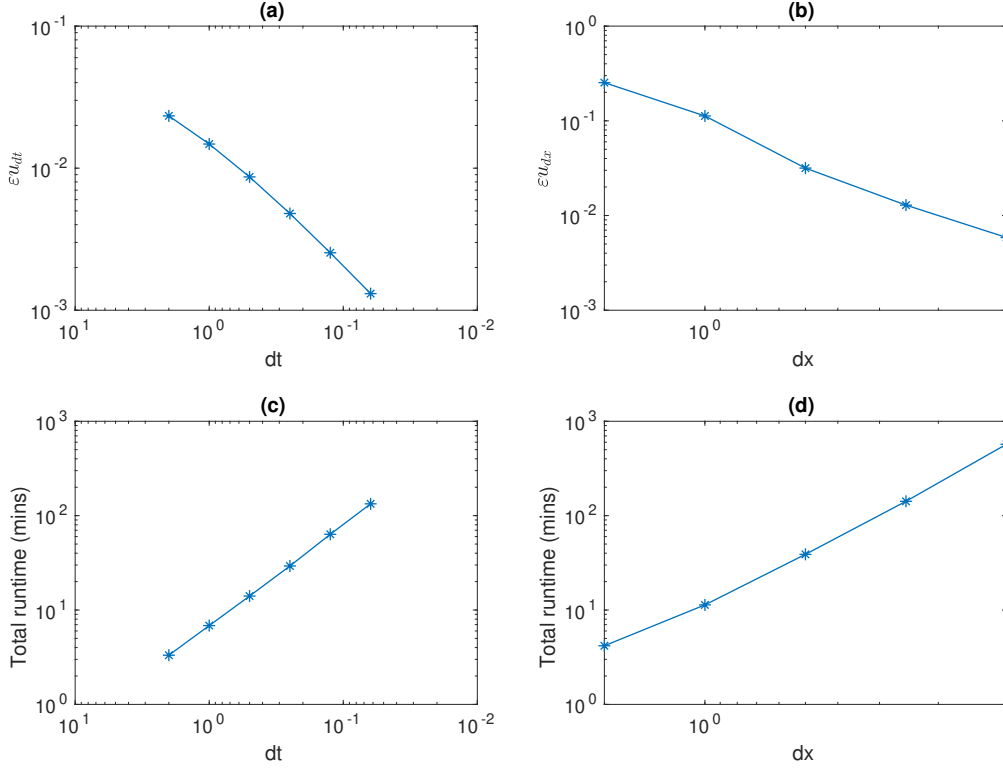


Figure 65: (a) Log-log plot of εu_{dt} for decreasing time-step with total run-time given in (c), (b) Log-log plot of εu_{dx} for decreasing mesh size with total run-time given in (d). All cases have parameter values of $Re = 50$, $\omega = 10$, and $\hat{U} = 20$, with an end time of $T = 100$.

time-step/mesh size and the next, defined as

$$\varepsilon c_{dt} = \frac{\int_0^T \|c_{dt} - c_{dt/10}\|_2}{\int_0^T \|c_{dt}\|_2}, \quad (217)$$

$$\varepsilon c_{dx} = \frac{\int_0^T \|c_{dx} - c_{dx/2}\|_2}{\int_0^T \|c_{dx}\|_2}, \quad (218)$$

for time-step and mesh size respectively. Note that since the fluid time-step of $dt_{textfluid} = 0.1$ is an upper bound for the dust time-step, we compare the dust time-step against a time-step one tenth smaller than it in order to examine smaller time-step values.

In Figure 66(a), we present εc_{dt} for $dt = 0.1, 0.01, 0.001$ with $dx = 0.25$ and in Figure 66(b), we present εc_{dx} for $dx = 2, 1, 0.5, 0.25, 0.125$ with $dt = 0.01$. In each case the parameters are set as $Re = 50$, $\omega = 10$, $\hat{U} = 20$, and $Sc = 10$, with a fluid time-step of $dt_{fluid} = 0.1$ and an end time of $T = 100$. Alongside this, in Figures 66(c) and 66(d), we give the total run-time of each calculation for varying time-step and mesh size respectively. Again we see the solution converges as the time-step and mesh size are decreased, whilst total run-time increases. We choose to set a time-step of $dt = 0.01$ which has a relative error of $\varepsilon c_{dt} = 0.026$ and a mesh size of $dx = 0.25$ which has a

relative error of $\varepsilon_{c_{dx}} = 0.019$.

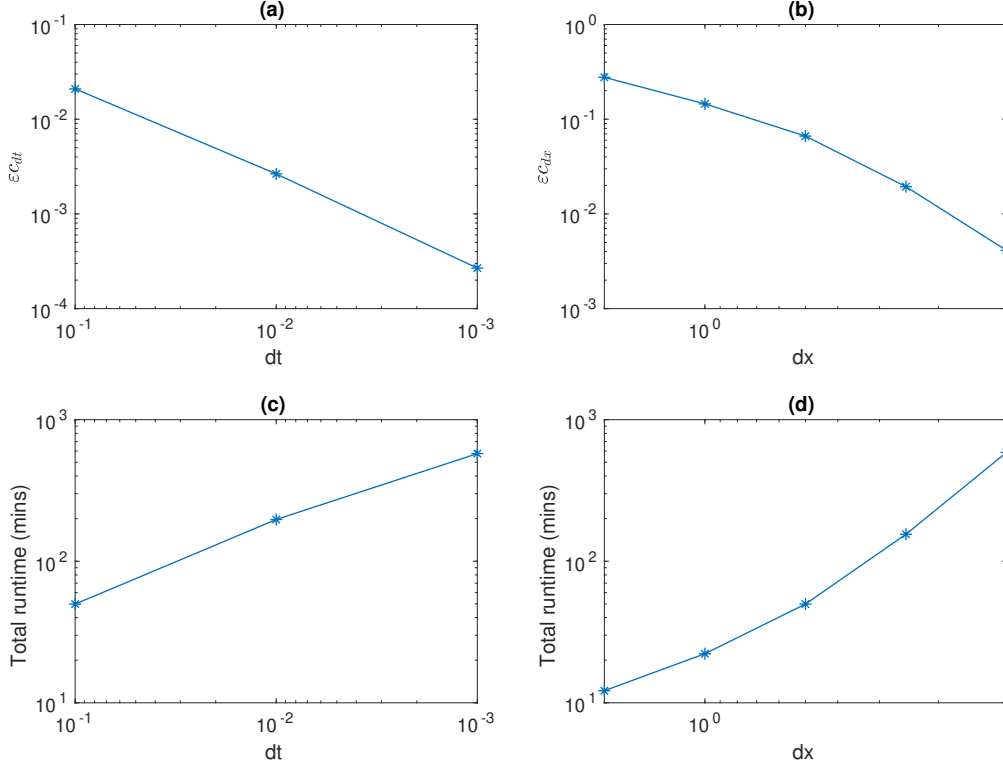


Figure 66: (a) Log-log plot of $\varepsilon_{c_{dt}}$ for decreasing time-step with total run-time given in (c), (b) Log-log plot of $\varepsilon_{c_{dx}}$ for decreasing mesh size with total run-time given in (d). All cases have parameter values of $Re = 50$, $\omega = 10$, $\hat{U} = 20$, and $Sc = 10$, with a fluid time-step of $dt_{\text{fluid}} = 0.1$ and an end time of $T = 100$.

D Chapter 4 appendix

D.1 Table of parameters

Unless stated otherwise, the parameter values used to generate plots of results are as given in Table 3. We note that the radius of the tubing is consistent throughout sections 0, 1 and D, giving $r_0^* = r_1^* = r_D^*$. The value of c^* is chosen such that the pressure in the dampener is atmospheric.

D.2 Calculation of flow rate for an offset working tool

This appendix outlines work by Piercy et al. (1933) in deriving the flux through a circular pipe with an offset core, and the application of this work to the ureteroscopic field in Appendix 3.B.1 of Williams (2019) and throughout.

Defining a flow rate of $Q = 1/R dp/dz$ where R is the resistance to the flow and dp/dz is the pressure gradient, the flow rate through a circular pipe with an offset circular core

Table 3: Table of parameters used in the full systems model.

Parameter	Description	Value	Units
r_0^*	Radius of tubing	0.3175	(cm)
r_{21}^*	Radius of working tool	0.0217	(cm)
r_2^*	Radius of working channel	0.06	(cm)
r_{41}^*	Radius of ureterscope	0.15675	(cm)
r_{42}^*	Radius of access sheath	0.2145	(cm)
L_0^*	Length of tubing	30	(cm)
L_1^*	Length of tubing	30	(cm)
L_D^*	Length of tubing	10	(cm)
L_2^*	Length of ureterscope	79	(cm)
L_4^*	Length of access sheath	36	(cm)
$\bar{q}_{in}^*$	Steady upstream flux	1.667	(cm/s)
$\hat{q}_{in}^*$	Amplitude of oscillatory component of upstream flux	0.1667	(cm/s)
ω^*	Frequency of oscillatory component of upstream flux	5	(1/s)
ρ^*	Density of irrigation fluid	0.997	(g/cm ³)
μ^*	Dynamic viscosity of irrigation fluid	0.0089	(g/cm·s)
V_0^*	Initial volume of air	60	(ml)
c^*	Ideal gas law constant	60 p_{atm}	(cm ² g/s ²)

is given by (Piercy et al., 1933)

$$Q = \frac{\pi}{8} \left(a^4 - b^4 + \frac{4c^2d^2}{\eta_a - \eta_b} - 8c^2d^2 \sum_{n=1}^{\infty} \frac{ne^{-n(\eta_a + \eta_b)}}{\sinh(n(\eta_b - \eta_a))} \right), \quad (219)$$

for $dp/dz = -1$, where a and b are the radii of the outer and inner pipe respectively, $d = \phi(a - b)$ is the distance between the pipe and the core, ϕ is the degree of offset with 0 being a centred core and 1 being a core at the edge, and

$$\eta_a = \cosh^{-1} \left(\frac{\gamma(1 + \phi^2) + (1 - \phi^2)}{2\phi\gamma} \right), \quad (220)$$

$$\eta_b = \cosh^{-1} \left(\frac{\gamma(1 - \phi^2) + (1 + \phi^2)}{2\phi} \right), \quad (221)$$

$$\gamma = b/a, \quad (222)$$

$$c = b \sinh \eta_b. \quad (223)$$