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Grouping Agents with Persistent Types

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Abstract

Employees are divided into grades. Toyota places suppliers into only a small number of categories. This paper shows that grouping of privately informed and persistent agent types arises naturally in relational incentive contracts when agent type is continuous. Malcomson (2016) showed that full separation is not possible if, following full revelation of an agent's type, payoffs for principal and agent are on the Pareto frontier. This paper shows how much separation can be achieved. Specifically, it characterises the finest partitioning that can be achieved in each period with agent types for which first-best effort is unattainable. Separation may take time, with initial coarser partitions being subsequently refined, but does not continue indefinitely. When it stops, there remain a finite number of groups of agent types. Numerical illustrations for constant elasticity cost of effort show the maximum number is typically small despite agent type being continuous.

Keywords: Relational incentive contracts, persistent private information, renegotiation-proofness, type pooling

JEL classification: C73, D82, D86

1 Introduction

Relational incentive contracts with non-contractible performance (typically referred to as *effort*) have proved insightful for analysing a variety of economic relationships. For applications, see Malcomson (1999) on employment and Malcomson (2013) on supply relationships. In many such relationships, agents of different types are pooled in groups, with those in each group persistent over time and treated the same despite the differences between them. Employees are grouped in grades. Toyota, as described by Asanuma (1989), groups its suppliers into a small number of categories that receive differential treatment. This paper shows that persistent partial pooling into groups is fundamental to relational incentive contracts with privately-observed, continuous and persistent agent types.

Malcomson (2016) established that it is not possible in a relational incentive contract to separate all of a continuum of privately-informed agent types that are persistent over time, and for whom first-best effort is unachievable, if equilibria following full revelation of the agent's type are renegotiation-proof in the sense of having payoffs for principal and agent on the Pareto frontier. Then equilibria must have some pooling of types. That is, however, a negative result: it gives no indication of how much separation can be achieved. The present paper addresses that issue. It first shows that necessary conditions for a continuation equilibrium to be on the Pareto frontier (and, hence, for there to be no mutually beneficial renegotiation in the future) for agent types for which first-best effort is unattainable are that, at each future date, the agent type with the highest cost of effort in a pool is indifferent between delivering agreed effort and shirking. It then characterises the finest partitions of types that have that property at each future date. In these, separation may take time (even though it would happen straightaway if effort were fully contractible) with initial coarser partitions being subsequently refined. But it does not continue indefinitely. How long it continues depends on the discount factor: for discount factors above just over 0.6, it continues for at most two periods. Moreover, when further separation stops, there remain a finite number of groups of agent types. These finest partitions are independent of the distribution of agent types, a property that is convenient for practical applications where the distribution of privately-observed types is unknown to the researcher.

The finest partitions depend primarily on the discount factor and the convexity of the cost of effort function. This is illustrated with constant elasticity cost of effort functions, showing for what degree of convexity and discount factor combinations separation is possible and, when it is, into how many groups. The number of groups is quite small even though agent type is continuous: for a discount factor of 0.9, no separation is possible, and the costs of pooling substantial, unless the degree of convexity is relatively low and even then the maximum number of groups is no more

than six or so unless there is almost no convexity in the cost of effort. With quadratic cost of effort, the discount factor has to be as low as 2/3 for any separation to occur and partitioning into more than 2 groups is not possible.

Partial pooling of privately observed, continuous and persistent agent types can arise from the *ratchet effect* in dynamic models of procurement, see Laffont and Tirole (1988). There it occurs when a principal is legally constrained from committing to contract terms for future periods, even terms conditioned only on outcomes that can be contracted on when those future periods arrive, and makes "take it or leave it" offers that extract all future rent if the agent's type is revealed.[1] The relational contract approach in Malcomson (2016) and used here does not depend on the principal having all the bargaining power in negotiations. Partial pooling of privately-observed, continuous but non-persistent types arises from *dynamic enforcement* in the hidden information relational incentive contract model of Levin (2003). But, because types are *iid* draws each period, there is no systematic persistence of a particular agent in a particular pool. See Malcomson (2016) for a fuller discusion of these issues. The paper closest to the current paper is MacLeod and Malcomson (1988) but, as discussed along with other related literature in Section 7, that relies on contract restrictions that are not imposed here.

The structure of the paper is as follows. Section 2 sets out the model. Section 3 sets out the equilibrium conditions and summarises results in Malcomson (2016). Section 4 establishes necessary properties for a continuation equilibrium with pooled agent types to be on the Pareto frontier. Section 5 characterises finest partition equilibria with those properties. Section 6 applies the results to constant elasticity cost of effort functions. Section 7 discusses related literature. Section 8 contains concluding remarks. An appendix contains lemmas and proofs.

2 Model

The model is essentially that in Malcomson (2016). A principal uses an agent to perform a task in each of a potentially infinite number of periods. The principal's payoff in period t if matched with the agent is $e_t - w_t$, with $e_t \in [0, \bar{e}]$ the agent's effort and w_t the payment to the agent in period t. Effort cannot be verified by third parties, so a legally enforceable agreement for performance is not possible. The principal's payoff for a period not matched with the agent is $\underline{v} \geq 0$.

The agent's payoff in period t if matched with the principal is $w_t - ac(e_t)$, where $ac(e_t)$ is the cost of effort e_t to agent type $a \in [a^{\min}, a^{\max}]$, with a observed privately by

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[1]For discrete types, Gerardi and Maestri (2020) show how pooling depends on the distribution of types and Acharya and Ortner (2017) show how the extent of pooling can be reduced with time-varying productivity shocks.
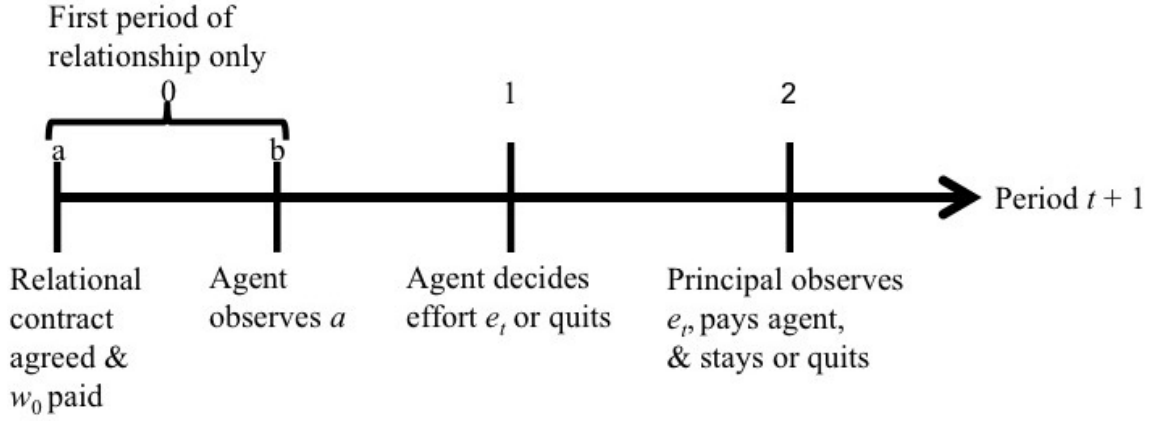
Figure 1: Timing of events in period t

the agent.[2] Agent type is distributed $F(\cdot)$ that admits an everywhere strictly positive density. The agent's payoff for a period not matched with the principal is $\underline{u} \geq 0$, with $\underline{u} + \underline{v} > 0$. Principal and agent have the same discount factor δ. The cost of effort function c is strictly convex with the following properties.

Assumption 1 (1) *$c(\cdot)$ is twice differentiable with $c'(\cdot) > 0$ and $c''(\cdot) > 0$; (2) $c(0) = 0$, $c'(0) < 1$ and $\bar{e}$ is sufficiently large that $c'(\bar{e}) > 1$; (3) there exists some $\tilde{e} \in [0, \bar{e}]$ for which $a^{\min} c(\tilde{e}) < \delta \left[\tilde{e} - \left(\underline{u} + \underline{v} \right) \right]$; and (4) for all $\tilde{e} \in [0, \bar{e}]$, $a^{\max} c(\tilde{e}) > \delta \left[\tilde{e} - \left(\underline{u} + \underline{v} \right) \right]$.*

Properties (1) and (2) in Assumption 1 are standard. Even though $c(0) = 0$, continuing a relationship with $e_t = 0$ for all t is not worthwhile because $\underline{u} + \underline{v} > 0$. Properties (3) and (4) ensure that, for reasons explained later, a relational contract can be sustained for some, but not all, agent types.

The timing of events in period t is shown in Figure 1. In the first period of the relationship ($t = 1$), the parties first decide (at stage 0a) whether to agree a relational contract and, if they do, make initial payment w_0. For the present purpose, the process by which an agreement is reached is immaterial. Then the agent (at stage 0b) observes a. The other stages are the same for all t. At stage 1, the agent either incurs effort e_t or ends the relationship. At stage 2, the principal observes e_t, pays the agent and decides whether to continue the relationship.[3]

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[2]In Malcomson (2016), the cost of effort e_t to agent type a takes the more general form $c(e_t, a)$ that is decreasing in a. The separable form used here makes the results more transparent and having it increasing in a is convenient for examples. The statements of results from Malcomson (2016) have been re-phrased accordingly. Effort can be multi-dimensional. It is treated as unidimensional only for simplicity.

[3]The timing used here has each party make decisions at only one stage in each period, which simplifies the analysis by avoiding having to keep track of the parties' payoffs at other stages within a period. A party's payoff from continuing the relationship is, however, at its lowest at its decision stage, so allowing a party to end the relationship at other stages would not affect its participation decision. See Malcomson (2016) for further discussion.

As in MacLeod and Malcomson (1989) and Levin (2003), payment has a fixed component $\underline{w}_t$ conditional on the relationship being continued by both parties for period t but not on the agent's effort at t. It also has a bonus component $w_t - \underline{w}_t$ that can be conditioned on effort at (and before) t but is not legally enforceable because effort is unverifiable. The magnitude and sign of $\underline{w}_t$ are unrestricted (negative $\underline{w}_t$ requires the agent to pay the principal) but, to avoid a decision by the agent at stage 2 of whether to accept the bonus, $w_t - \underline{w}_t$ is restricted to being non-negative. (This restriction does not restrict the set of payoffs attainable in equilibrium.)

Let $h_t = h_{t-1} \cup (e_{t-1}, \underline{w}_{t-1}, w_{t-1})$ for $t \geq 2$, with $h_1 = \{w_0\}$, denote the commonly observed history at stage 1 of period t conditional on the relationship not having ended before then. At that stage, the agent can condition actions on $(a, h_t, \underline{w}_t)$. At stage 2 of period t, the principal can condition actions on $(h_t, \underline{w}_t, e_t)$.[4]

The joint payoff gain to the principal and the agent from being matched in period t conditional on a is $s(e_t, a) = e_t - ac(e_t) - \left(\underline{u} + \underline{v}\right)$. *Efficient* or *first-best* effort for agent type a in the absence of private information $e^*(a)$ maximises this joint gain. Under Assumption 1, $e^*(a) \in (0, \bar{e})$ for all a and is uniquely given by

$$\frac{\partial s(e^*(a), a)}{\partial e} = 0, \quad \text{for all } a \in \left[a^{\min}, a^{\max}\right]. \tag{1}$$

For what follows, $a^{\max}$ is taken as defined by $s(e^*(a^{\max}), a^{\max}) = 0$. This is without significant loss of generality because no mutually beneficial relationship exists for an agent type with higher cost of effort even if first-best effort were achievable.

3   Equilibrium

With agent type private information, a natural minimum requirement for equilibrium is that it is perfect Bayesian. In describing equilibria, the history argument is omitted where that does not result in ambiguity: for pure strategy equilibria, h_t at each t is fully

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[4]Formally, a strategy σ^a for the agent consists of a decision rule for whether to accept/pay w_0 and, for stage 1 of each t, decision rules for each $\left(h_t, \underline{w}_t\right)$ for whether to continue the relationship and an effort choice conditional on continuation. A strategy σ^p for the principal consists of a decision rule for whether to pay/accept w_0 and, for stage 2 of each t, decision rules for each $(h_t, \underline{w}_t, e_t)$ for what bonus to pay, whether to continue the relationship for $t + 1$ and what $\underline{w}_{t+1}$ to commit to conditional on continuation by both parties. To avoid the measurability details that can arise with mixed strategies when action spaces are continuous (see Mailath and Samuelson (2006, Remark 2.1.1)), attention is restricted to pure strategies. A *relational contract* is a w_0 and a strategy pair (σ^p, σ^a).

In the model, the essential role for a court is to enforce payment $\underline{w}_t$ at stage 2 of each period t. For this, $\underline{w}_t$ need not be independent of efforts before t — it is sufficient that the principal states $\underline{w}_t$ publicly at the same time as paying the bonus for $t - 1$, with the same consequences to reneging on either. Legal enforcement of even that could be dropped without changing the results at the cost of complicating the exposition by having the agent decide whether to continue the relationship for period t, and having $\underline{w}_t$ paid, before the agent decides e_t. An alternative would be to specify a $\underline{w}_t$ that is independent of the effort elements of h_t, thus enabling it to be legally enforceable over the whole future from the start of the relationship, but that does not enlarge the set of payoffs attainable in the equilibria discussed here.

determined by the relational contract and the agent's type. For example, $e_t(a)$ denotes the effort for agent type a at t given the history that would be generated if both parties stuck to the equilibrium path up to t.

3.1 Equilibrium conditions

For both parties continuing an equilibrium path at t, let $U_t(a)$ denote the payoff gain over the whole future relationship to agent type a at stage 1 of period t, $P_t(a)$ the payoff gain over the whole future relationship to the principal at stage 2 of period t when matched with agent type a conditional on paying the bonus at t, $\underline{w}(h_t)$ the payment $\underline{w}_t$ for history h_t, $A^+(h_t)$ the set of agent types with history h_t, and $A(h_t)$ the subset of $A^+(h_t)$ of types for whom the relationship is to be continued at t. Then these payoff gains must satisfy

$$U_t(a) \geq \max[0, \underline{w}(h_t) - \underline{u}], \quad \text{for all } a \in A(h_t), h_t, t; \tag{2}$$

$$E_{a|h_t, \underline{w}(h_t), e_t} P_t(a) \geq 0, \quad \text{for all } h_t, e_t, t. \tag{3}$$

The first of these follows because agent type a can obtain payoff gain of zero by quitting the relationship at t and payoff gain $\underline{w}(h_t) - \underline{u}$ by continuing it, choosing zero effort and quitting at $t+1$. The agent's equilibrium path payoff gain at t must be at least as great as the higher of these. At t, the principal observing history $\left(h_t, \underline{w}(h_t), e_t\right)$ believes the agent is of some type for whom that history is on the equilibrium path. To be willing to continue the relationship, the expected payoff gain from continuing the relationship with all those types (the left-hand side in (3)) must be non-negative.

The payoff gains must also be consistent with the total output produced. Let $S_t(a)$ denote the joint gain to the principal and the agent (sometimes called the *surplus*) from continuing the relationship with agent types $a \in A(h_t)$ at stage 1 of period t, defined by

$$S_t(a) = e_t(a) - ac(e_t(a)) - \underline{u} - \underline{v} + \delta S_{t+1}(a), \quad \text{for all } a \in A(h_t), h_t, t. \tag{4}$$

The joint gain to starting a relational contract before the agent learns type is

$$S_0 = \int_{a^{\min}}^{a^{\max}} S_1(a) dF(a). \tag{5}$$

A necessary condition for a relational contract to start is that $S_0 \geq 0$. Moreover, provided $S_0 \geq 0$, there is always a w_0 such that the principal's and the agent's initial payoff gains before the agent learns type are both non-negative. Equilibrium requires that the agent receives that part of the joint gain not received by the principal. It follows that, for histories h_t for which the relationship is continued,

$$U_t(a) = -ac(e_t(a)) - \underline{u} + \underline{w}(h_t) + \delta S_{t+1}(a) - P_t(a), \quad \text{for all } a \in A(h_t), h_t, t. \tag{6}$$

This condition is the *budget balance constraint* from which the dynamic enforcement constraint in Levin (2003) is derived. (The bonus does not appear explicitly in this because $P_t(a)$. is measured *after* it has been paid.)

Equilibrium also requires that it is incentive compatible for agent type a to choose the equilibrium path effort $e_t(a)$ for type a. The following result follows directly from Proposition 1 in Malcomson (2016).

Proposition 1 *(Malcomson (2016)) Let $\tilde{U}(a', a, h_t)$ denote the maximal payoff gain available to agent type $a \in A(h_t)$ from choosing the effort for type $a' \in A(h_t)$ at t. Necessary conditions for choosing effort $e_t(a)$ to be best responses for $a \in A(h_t)$ at t are*

$$\tilde{U}(a, a, h_t) - \tilde{U}(a, a', h_t) \geq U_t(a) - U_t(a')$$
$$\geq \tilde{U}(a', a, h_t) - \tilde{U}(a', a', h_t), \text{ for all } a, a' \in A(h_t), h_t, t. \quad (7)$$

These conditions are also sufficient if, following deviation by the agent given history h_t from $e_t(a)$ for every $a \in A(h_t)$, the continuation equilibrium is that for a' the highest type in $A(h_t)$ except that (1) the principal pays no bonus at t and (2) $\underline{w}_{t+1}$ is such that a' would receive non-positive payoff gain from continuing the relationship at stage 1 of period $t + 1$.

The proof and a full discussion of this result are in Malcomson (2016). Together with the individual rationality conditions (2) and (3) and the budget balance condition (6), the incentive compatibility conditions in Proposition 1 constitute the requirements for a perfect Bayesian equilibrium as long as the principal's updating of beliefs about the agent's type corresponds to Bayes' Rule.

Proposition 1 is an extension to multiple periods of the incentive compatibility result from mechanism design for one-period models. If reference to h_t and t were deleted, (7) would be just the incentive compatibility conditions for a one-period model. The crucial difference from a one-period model is that a type a that chooses $e_t(a')$ at t is not committed to choosing $e_{t+1}(a')$ at $t + 1$ but can quit the relationship or deviate in some other way. For what follows, it is useful to rewrite (7) in a form that makes this explicit and to combine it with the budget balance condition (6). For this, it is convenient to define $C_t(a) = \sum_{\theta=t}^{\infty} \delta^{\theta-t} c(e_\theta(a))$ as the present discounted value at t of the cost of equilibrium effort for a for the whole future and to let $h_{t+1}(a)$ denote the history for type a generated by equilibrium path actions for the principal and agent type a up to $t + 1$.

Proposition 2 *Under the conditions of Proposition 1, (7) can be replaced by*

$$(a' - a) c(e_t(a)) + \delta \left\{ \tilde{U}(a, a, h_{t+1}(a)) - \max \left[\tilde{U}(a, a', h_{t+1}(a)), 0, \underline{w}(h_{t+1}(a)) - \underline{u} \right] \right\}$$
$$\geq U_t(a) - U_t(a') \geq (a' - a) C_t(a'), \text{ for all } a < a'; a, a' \in A(h_t), h_t, t. \quad (8)$$

To satisfy budget balance (6), it must also be that

$$ac(e_t(a)) + (a' - a)C_t(a') \leq \delta S_{t+1}(a) - P_t(a) - U_t(a') - \underline{u} + \underline{w}(h_t),$$

$$\text{for all } a < a'; a, a' \in A(h_t), h_t, t. \quad (9)$$

Essentially, (8) just rewrites (7) in a form that makes possible future choices explicit. If agent type $a < a'$ takes the action for type a' from t on, the difference in payoff gain is just $(a' - a)C_t(a')$ because monetary payments are valued the same by all agent types. Under the sufficiency conditions in Proposition 1, it is never a best response for $a < a'$ to deviate in the future to an action that is off the equilibrium path for all agent types with the same history so, as long as it is worthwhile for a' to continue the relationship, it is worthwhile for a (with lower cost of effort) to do so too. Thus requiring the right-hand inequality in (8) to hold for all $a < a'$ is sufficient to deter deviation by a to the effort for any $a' > a$. If, alternatively, agent type $a' > a$ takes the action for a at t, the difference in period t payoff gain is the difference in the cost of effort for that period, $(a' - a)c(e_t(a))$, again because monetary payments are valued the same by all agent types. But at $t + 1$, a' has the choice between continuing to take the action for a with payoff gain $\tilde{U}(a, a', h_{t+1}(a))$,[5] quitting the relationship immediately with payoff gain 0, or choosing an action different from that for any type pooled with a at t, collecting the fixed component of pay $\underline{w}(h_{t+1}(a))$ but forgoing the outside opportunity $\underline{u}$, and quitting at $t + 2$. In the last case, it is optimal to choose effort 0 because that incurs the lowest cost of effort, so the payoff gain is $\underline{w}(h_{t+1}(a)) - \underline{u}$. Agent type a' would make the choice with the highest of these payoff gains. Thus requiring the left-hand inequality in (8) to hold for all $a' > a$ is sufficient to deter deviation by a' to the effort for any $a < a'$. Condition (9) corresponds to (6) with the right-hand inequality in (7) used to replace $U_t(a)$ by $U_t(a')$.

3.2 Renegotiation with fully revealed agent type

In addition to equilibrium requirements, the contracting literature typically also imposes that contracts are renegotiation-proof, in the sense that it is not possible for the parties to renegotiate a contract at any stage to another that both prefer. If effort in the model here were verifiable with the principal committing to payment explicitly contracted on it, effort for each agent type could be set at first-best each period and payment set to ensure that (7) is satisfied. All agent types would then be fully separated and, with effort at first-best for all types, the issue of renegotiation would not arise. With effort unverifiable, however, the principal cannot commit to payments conditioned on it. Malcomson (2016) shows that there is then no perfect Bayesian equilibrium that separates all agent types for which first-best effort is unattainable in a

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[5]Type a' could, of course, choose the effort for some type other than, but with the same history h_{t+1} as, a but that is taken into account because (8) is required to hold for all $a < a'$.

relational contract if, following revelation of the agent's type, the parties renegotiate to a continuation equilibrium with payoffs for the principal and the revealed agent type on the Pareto frontier.

The essential reasoning is the following. If first-best effort is not attainable for an agent type, higher effort always gets closer to the Pareto frontier for that type. As shown in Levin (2003) and proved formally in Malcomson (2016) for the model used here, optimal effort for an agent whose type has been revealed is stationary, essentially because of the strict convexity of the agent's effort cost — to be incentive compatible, stationary effort requires less future gain from co-operation than time-varying effort with the same mean value. For stationary effort $e(a)$, the joint gain in (4) becomes

$$S_t(a) = \frac{1}{1-\delta}\left[e(a) - ac(e(a)) - \underline{u} - \underline{v}\right], \quad \text{for all } a \in A(h_t), h_t, t. \tag{10}$$

Used in the budget balance condition (6), that condition can be re-arranged as

$$U_t(a) - \underline{w}(h_t) + \underline{u} + P_t(a) = \frac{1}{1-\delta}\left[\delta e(a) - ac(e(a)) - \delta\left(\underline{u} + \underline{v}\right)\right],$$

$$\text{for all } a \in A(h_t), h_t, t. \tag{11}$$

In view of the constraints (2) and (3) applied to a revealed agent type a, the left-hand side of (11) has to be non-negative. By Assumption 1, there is no $e(a)$ for which the right-hand side is non-negative for $a = a^{\max}$, so a relational contract could not be sustained for this type. Also by Assumption 1, there exists some $e(a)$ for which the right-hand side of (11) is strictly positive for $a = a^{\min}$. Let $\bar{a}$ denote the highest agent type for which there is an $e(a)$ that makes the right-hand side of (11) non-negative and $\underline{a}$ denote the higher of $a^{\min}$ and the highest a for which first-best effort makes the right-hand side of (11) non-negative. Then, for $a \in (\underline{a}, \bar{a}]$, the highest feasible stationary effort $\hat{e}(a)$ is given by

$$\hat{e}(a) = \max e(a) \text{ such that } \delta e(a) - ac(e(a)) - \delta\left(\underline{u} + \underline{v}\right) = 0, \quad \text{for } a \in (\underline{a}, \bar{a}]. \tag{12}$$

Because the parties can redistribute gains from renegotiation at $t+1$ by setting $\underline{w}(h_{t+1})$ appropriately, effort on the Pareto frontier from $t+1$ on for any agent type $a \in (\underline{a}, \bar{a}]$ fully revealed at t is $\hat{e}(a)$, so renegotiation at $t+1$ will set effort at that level from $t+1$ on. From (11), that requires $U_{t+1}(a) = \underline{w}(h_{t+1}) - \underline{u}$.[6] Then, for period t for which a has the same history h_t as some type $a' > a$, (8) requires

$$c(e_t(a)) \geq c(e_t(a')) + \delta C_{t+1}(a'), \quad \text{for all } a < a'; a, a' \in A(h_t). \tag{13}$$

But $C_{t+1}(a')$ has to be bounded above zero for the relationship with a' to be continued

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[6]It follows from an argument in Goldlücke and Kranz (2013, Section 4.3) that continuation equilibria with these characteristics correspond to strong perfect equilibria and so also to strong renegotiation-proof equilibria in the sense of Farrell and Maskin (1989) — see Malcomson (2016).

in the future, so this condition implies that there must be a jump between the effort at t for a and that for a'. But with a continuous and $e(a)$ bounded both above and below, it is not possible to have a jump between the efforts for all agent types in an interval, so it is not possible to separate all types within that interval. The formal result in Malcomson (2016) shows that this remains true no matter how many periods separation continues, so it is not possible to separate all agent types $a \in \left(\underline{a}, \overline{a}\right]$.[7]

That is a negative result: it shows that full separation of agent types cannot be sustained, but not the extent of separation that can be sustained, in a renegotiation-proof equilibrium. There is, however, an issue of how to define renegotiation-proofness for infinite horizon games with private information. The standard concepts set out in Mailath and Samuelson (2006, Section 4.6.2) are defined only for games without private information about types and natural extensions to games with private information are not helpful in the context of the present model.[8] The approach taken here is to investigate the extent of separation that can be achieved with equilibria that satisfy necessary conditions for there to be no mutually beneficial renegotiation. While those necessary conditions do not guarantee that no mutually beneficial renegotiation is possible, the finest separation that could be achieved with any additional restrictions required for fully renegotiation-proof equilibria could not be finer than if only those necessary conditions are satisfied.

What the possibility of renegotiation does when agent type a for whom first-best effort is unattainable has been fully revealed at t is to impose the condition that $U_{t+1}(a) = \underline{w}(h_{t+1}) - \underline{u}$ and $C_{t+1}(a) = \frac{1}{1-\delta}c(\hat{e}(a))$, which prevents full revelation of type at t. The next section develops corresponding conditions when agent type is revealed to be in an interval of types.

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[7]Making $\underline{w}(h_t)$ legally enforceable over the whole future from the start of the relationship by having it independent of past efforts does not alter this conclusion. As long as $\underline{w}(h_t)$ is independent of effort at t, the highest $e(a)$ satisfying (11), $U_t(a) - \underline{w}(h_t) + \underline{u} \geq 0$ and $P_t(a) \geq 0$ requires those inequalities to hold with equality, as a result of which (13) follows. Requiring future effort to be only close to, not actually on, the Pareto frontier does not overcome this unless $U_{t+1}(a) - \underline{w}(h_{t+1}) + \underline{u} \geq \delta C_{t+1}(a')$ for all $a' > a$ for $a, a' \in A(h_t)$. It is similarly not overcome if first-best effort can be achieved for type a unless that enables the same condition to be satisfied.

[8]In what might seem a natural extension to the criterion of internal consistency in Bernheim and Ray (1989) and weak renegotiation-proofness in Farrell and Maskin (1989), all the perfect Bayesian continuation equilibria (PBCE) sustainable with the punishments in the sufficiency part of Proposition 1 satisfy that criterion if the beliefs of the principal following an action by the agent that is off an equilibrium path at t is that the agent is the highest cost type with the history h_t. The essential reason is that those punishments involve playing the same continuation equilibrium as on the equilibrium path but with a one-off payment from the deviating party, so no continuation equilibrium Pareto dominates any other. In what might seem a natural extension to the criterion of strong renegotiation-proofness in Farrell and Maskin (1989), none of these equilibria satisfy that criterion. The essential reason is that, because effort is continuous, the payoff sets are not closed and there is always a weakly renegotiation-proof PBCE that Pareto dominates any given PBCE. Strulovici (2020) has developed a concept of renegotiation-proofness applicable to infinite horizon games for an agent with private information but this applies when that information changes according to a diffusion process and the principal knows the agent's initial information. Issues also arise with the renegotiation-proofness concepts in Abreu et al. (1993) and in Bergin and MacLeod (1993).

4   Renegotiation with pooled agent types

This section investigates the implications of renegotiation with pooled agent types with the same history. The first result concerns agent types that remain pooled from some date on.

Proposition 3 *Conditional on all agent types $a \in A(h_\tau) \subseteq (\underline{a}, \overline{a}]$ with history h_τ choosing the same effort $e_t(a)$ at t for all $t \geq \tau$, any perfect Bayesian (continuation) equilibrium with payoffs for principal and agent at τ on the Pareto frontier has, for $\overline{a}(h_\tau)$ the highest cost type in $A(h_\tau)$, $e_t(a) = \hat{e}(\overline{a}(h_\tau))$ for all $a \in A(h_\tau)$ and $t \geq \tau$. Any such (continuation) equilibrium has, for all $t \geq \tau$, $P_t(a) = 0$ for all $a \in A(h_\tau)$, $U_t(\overline{a}(h_\tau)) = \underline{w}(h_t) - \underline{u} \geq 0$, $\delta S_t(\overline{a}(h_\tau)) = \overline{a}(h_\tau) c(\hat{e}(\overline{a}(h_\tau)))$, and*

$$U_t(a) = U_t(\overline{a}(h_\tau)) + \frac{1}{1-\delta}(\overline{a}(h_\tau) - a) c(\hat{e}(\overline{a}(h_\tau))), \text{ for } a \in A(h_\tau), \quad t \geq \tau. \tag{14}$$

Proposition 3 shows that, conditional on a set of types containing all those with the same history remaining pooled indefinitely, continuation equilibria on the Pareto frontier have the characteristics of a continuation equilibrium on the Pareto frontier for the type in the pool with the highest cost of effort. Underlying this is that first-best effort is decreasing in agent type. So if two agent types $a'' < a'$ with the same history h_τ are to choose the same effort from τ on and first-best effort for a' is not achievable, effort for a'' is necessarily below its first-best level, so the joint gain to principal and agent is always higher with higher effort. So, effort for $a \in A(h_t)$ is $\hat{e}(\overline{a}(h_\tau))$, where $\overline{a}(h_\tau)$ is the highest cost type in $A(h_\tau)$. This is a generalisation of the case in which the agent's type is fully revealed, which corresponds to an interval with just a single type. Types $a < \overline{a}(h_\tau)$ necessarily receive the payoff gain in (14) strictly greater than $\overline{a}(h_\tau)$ because of their lower cost of effort.

An important implication of Proposition 3 is that, to be on the Pareto frontier of payoffs for principal and agent if there is no further separation of agent types in $A(h_t)$ requires $U_t(\overline{a}(h_\tau)) = \underline{w}(h_t) - \underline{u}$ and $C_t(\overline{a}(h_\tau)) = \frac{1}{1-\delta}c(\hat{e}(\overline{a}(h_\tau)))$ for all $t \geq \tau$. Thus, if partitioning of an interval of types ever ceases in a continuation equilibrium without those properties, it is mutually beneficial for principal and agent to renegotiate to one with them. For these types of continuation equilibria, therefore, those conditions are necessary for renegotiation not to occur. The next result shows that the first of these equalities applies more generally.

Proposition 4 *A perfect Bayesian (continuation) equilibrium at t of the interval $(\underline{a}(h_t), \overline{a}(h_t)] = A(h_t) \subseteq (\underline{a}, \overline{a}]$, with $C_t(\overline{a}(h_t))$ the present discounted value of effort from t on for $\overline{a}(h_t)$, that has $\underline{w}(h_t) < U_t(\overline{a}(h_t)) + \underline{u}$ is Pareto-dominated by an alternative perfect Bayesian (continuation) equilibrium at t with fixed component of payment $\underline{\hat{w}}(h_t) = U_t(\overline{a}(h_t)) + \underline{u}$ if*

$$\overline{a}(h_t) \leq \int_{\underline{a}(h_t)}^{\overline{a}(h_t)} \frac{1}{c'(e(a))} \frac{dF(a)}{F(\overline{a}(h_t)) - F(\underline{a}(h_t))} \tag{15}$$

for e (a) given by

$$c\left(e\left(a\right)\right) = c\left(\hat{e}\left(a\right)\right) - \left(\frac{\bar{a}\left(h_t\right)}{a} - 1\right)\left[C_t\left(\bar{a}\left(h_t\right)\right) + \frac{U_t\left(\bar{a}\left(h_t\right)\right) - \underline{w}\left(h_t\right) + \underline{u}}{\bar{a}\left(h_t\right)}\right],$$

$$\text{for } a \in \left(\underline{a}\left(h_t\right), \bar{a}\left(h_t\right)\right]. \quad (16)$$

A sufficient condition that is independent of the distribution of agent types is

$$\bar{a}\left(h_t\right) \leq \frac{1}{c'\left(e\left(\underline{a}\left(h_t\right)\right)\right)}. \quad (17)$$

Proposition 4 establishes that perfect Bayesian (continuation) equilibria at t for which $U_t\left(\bar{a}\left(h_t\right)\right) > \underline{w}\left(h_t\right) - \underline{u}$ are Pareto-dominated by alternative (continuation) equilibria at t with $U_t\left(\bar{a}\left(h_t\right)\right) = \underline{w}\left(h_t\right) - \underline{u}$ if (15) is satisfied. The underlying reason is that increasing $\underline{w}\left(h_t\right)$ and $c\left(e_t\left(\bar{a}\left(h_t\right)\right)\right)$ with actions and payment from $t + 1$ on unchanged in such a way as to keep $U_t\left(\bar{a}\left(h_t\right)\right)$ unchanged increases the right-hand side of (9) by more than the left-hand side, thus permitting higher effort at t for all $a < \bar{a}\left(h_t\right)$ too. With effort below first-best for all agent types, this increases the joint payoff gain for all types. Condition (15), with $e\left(a\right)$ given by (16), is sufficient to ensure that the principal can receive some of the increased joint payoff gain. Condition (16) gives an upper bound on $c\left(e\left(a\right)\right)$, and hence on $c'\left(e\left(a\right)\right)$, derived from (9) holding with equality for $a' = \bar{a}\left(h_t\right)$ when the right-hand side is the maximum future joint gain from a relationship with type a. But the upper bound in (16) is not attainable for all types in equilibrium because it would require full revelation of all $a \in \left(\underline{a}\left(h_t\right), \bar{a}\left(h_t\right)\right]$, which is not possible so, while sufficient, (16) is not necessary. Condition (15) holds if the distribution of types $F\left(a\right)$ is sufficiently heavily weighted towards high cost of effort. But the distribution of agent types is often hard to pin down in practice. A more easily checked, but less tight, sufficient condition is given by (17), which replaces the mean value of $1/c'\left(e\left(a\right)\right)$ by $1/c'\left(e\left(\underline{a}\left(h_t\right)\right)\right)$, the lowest value it can take with $c\left(e\right)$ strictly convex and $e\left(a\right)$ non-increasing, as it must be in any perfect Bayesian (continuation) equilibrium.

5 Finest partition equilibria

This section explores equilibria that satisfy the necessary conditions for renegotiation-proofness in Propositions 3 and 4, in particular equilibria that partition an interval of agent types with the same history into subintervals, defined formally as follows.

Definition 1 *A partition (continuation) equilibrium at t is a perfect Bayesian (continuation) equilibrium in pure strategies in which agent types $a \in \left(\underline{a}\left(h_t\right), \bar{a}\left(h_t\right)\right] = A\left(h_t\right)$ with the same history h_t at t are partitioned by $\bar{a}\left(h_t\right) \geq a^1\left(h_t\right) > a^2\left(h_t\right) > \ldots > a^{n\left(h_t\right)}\left(h_t\right) > a^{n\left(h_t\right)+1}\left(h_t\right) = \underline{a}\left(h_t\right)$ such that all agent types $a \in \left(a^1\left(h_t\right), \bar{a}\left(h_t\right)\right]$ end the relationship at t and all agent types*

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$a \in \left(a^{i+1}(h_t), a^i(h_t) \right]$ *for* $i = 1, \ldots, n(h_t)$ *choose the same effort at t that is distinct from the effort at t chosen by agent types in every other sub-interval of the partition.*

Partition equilibria are convenient because pooled types form an interval.[9] The definition allows for $a^1(h_t) < \bar{a}(h_t)$, so a partition equilibrium permits the relationship to end for some high cost of effort types, something of particular relevance for $t = 1$ when some types have too high cost for it to be worthwhile continuing the relationship. To determine the finest partition equilibria that satisfy the necessary conditions in Propositions 3 and 4 note that, in a partition continuation equilibrium at t, $\bar{a}(h_t)$ corresponds to $a^i(h_{t-1})$, and $\underline{a}(h_t)$ to $a^{i+1}(h_{t-1})$, for some i for h_{t-1} the same as h_t up to $t-1$. An implication of Proposition 4 is, therefore, that any partition (continuation) equilibrium at $t-1$ that has continuation at t with $U_t\left(a^i(h_{t-1})\right) > \underline{w}(h_t) - \underline{u}$ for any i is Pareto-dominated at t by one with $U_t\left(a^i(h_{t-1})\right) = \underline{w}(h_t) - \underline{u}$ for that i if (15) or (17) is satisfied. Thus, to be renegotiation-proof, any partition (continuation) equilibria at $t-1$ must have $U_t\left(a^i(h_{t-1})\right) = \underline{w}(h_t) - \underline{u}$ for all i, which motivates the following definition. (Recall that $h_{\tau+1}(a)$ is the history at $\tau+1$ along the equilibrium path for agent type a.)

Definition 2 *A finest constrained partition (continuation) equilibrium at t is a partition (continuation) equilibrium at all $\tau \geq t$ with*

1. $U_{\tau+1}\left(a^i(h_\tau)\right) = \underline{w}\left(h_{\tau+1}\left(a^i(h_\tau)\right)\right) - \underline{u}$ *for all $i \in \{1, \ldots, n(h_\tau)\}$ and h_τ for $\tau \geq t$;*

2. *the result in Proposition 3 respected for all $\tau \geq t$;*

3. $a^{i+1}(h_\tau)$ *for $i \geq 1$ as close as possible to $a^i(h_\tau)$ for all h_τ and $\tau \geq t$.*

To see the implications of this definition, consider incentive compatibility for separation of an interval of types $a \in \left(\underline{a}(h_t), \bar{a}(h_t) \right] = A(h_t)$ with the same history h_t at t. When $U_{t+1}(a) = \underline{w}(h_{t+1}(a)) - \underline{u}$, any higher cost of effort type a' taking the action for a at t can obtain the same payoff as a at $t+1$ by continuing the relationship, collecting the fixed payment $\underline{w}(h_{t+1})$ and choosing zero effort. With $\tilde{U}(a, a', h_{t+1}(a)) = U_{t+1}(a)$ along an equilibrium path, (8) in Proposition 2 becomes

$$(a' - a)\, c\left(e_t(a)\right) \geq U_t(a) - U_t(a')$$
$$\geq (a' - a)\left[c\left(e_t(a')\right) + \delta C_{t+1}(a') \right], \quad \text{for all } a < a';\, a, a' \in \left(\underline{a}(h_t), \bar{a}(h_t) \right]. \quad (18)$$

For given a, a' and continuation at $t+1$, (18) restricts how low $e_t(a)$ can be for given $e_t(a')$ for $a < a'$ to be separated. In contrast, condition (9) in Proposition 2 restricts how

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[9]These are not the only possible continuation equilibria exhibiting partial pooling. Laffont and Tirole (1993, p. 383) describe, in the context of a two-period procurement model, continuation equilibria that exhibit infinite reswitching in which actions that generate the same outcome are chosen by different types, but never by neighbouring types. That is, for any two types choosing the same action, there is always some intermediate type that chooses an action that generates a different outcome. Such equilibria do not, however, avoid the problem of separating agent types, see Malcomson (2016).

high $e_t(a)$ can be. The combination of these two determines how fine partition can be. Because $c(e_t(a'))$ has to be non-negative and $C_{t+1}(a')$ has to be sufficiently high to avoid mutually beneficial renegotiation in the future, these conditions require a gap between a and a' for the partition to be incentive compatible but, for a sufficiently wide interval of types, also permit partitioning into multiple sub-intervals. The next result establishes the properties of the finest partitions these conditions permit.

Proposition 5 *Consider the interval of types $\left(\underline{a}, a^{\max}\right]$ at $t = 1$. For given $a^1 \leq \bar{a}$ that is the highest cost of effort type for which the relationship is to be continued and $\delta \geq \left(\sqrt{5} - 1\right)/2 \approx 0.618$, a finest constrained partition equilibrium has a finite number of sub-intervals, with $e_1\left(a^1\right) = 0$ and with n and a^i for $i = 2, \ldots, n$ uniquely determined as follows:*

1. *If $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \leq \frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right)$, there is no partitioning of $\left(\underline{a}, a^1\right]$ in any period.*

2. *If $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \in \left(\frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right), \left(\frac{\delta}{1-\delta}\right)^2 a^1 c\left(\hat{e}\left(a^1\right)\right)\right]$, there is partitioning of $\left(\underline{a}, a^1\right]$ at $t = 1$ into two sub-intervals with a^2 given by $a^2 c\left(\hat{e}\left(a^2\right)\right) = \frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right)$ and no further partitioning after period 1.*

3. *If $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) > \frac{\delta}{1-\delta}\left(\underline{u} + \underline{v}\right) + \left(\frac{\delta}{1-\delta}\right)^2 a^1 c\left(\hat{e}\left(a^1\right)\right)$, there is partitioning of $\left(\underline{a}, a^1\right]$ at $t = 1$ at a^i with i odd satisfying*

$$a^3 c\left(\hat{e}\left(a^3\right)\right) = \underline{u} + \underline{v} + \frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right), \tag{19}$$

$$a^{i+2} c\left(\hat{e}\left(a^{i+2}\right)\right) = \frac{1}{1-\delta}a^i c\left(\hat{e}\left(a^i\right)\right), \text{ for } i = 3, 5, \ldots, n - 2, \tag{20}$$

$$a^{n+2} c\left(\hat{e}\left(a^{n+2}\right)\right) = \min\left\{\frac{1}{1-\delta}a^n c(\hat{e}\left(a^n\right)), \underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)\right\}, \tag{21}$$

with finite n determined endogenously by these conditions and with

$$c\left(e_1\left(a^{i+2}\right)\right) = c\left(e_1\left(a^i\right)\right) + \frac{\delta^2}{1-\delta}c\left(\hat{e}\left(a^i\right)\right), \text{ for } i = 1, 3, 5, \ldots, n - 2. \tag{22}$$

There is also partitioning of $\left(a^{i+2}, a^i\right]$ at $t = 2$ at a^{i+1} with $i + 1$ even satisfying

$$a^{i+1} c\left(\hat{e}\left(a^{i+1}\right)\right) = \frac{\delta}{1-\delta}a^i c\left(\hat{e}\left(a^i\right)\right), \text{ for } i = 1, 3, 5, \ldots, n - 2, \tag{23}$$

$$a^{n+1} c\left(\hat{e}\left(a^{i+2}\right)\right) = \frac{\delta}{1-\delta}a^n c\left(\hat{e}\left(a^n\right)\right), \text{ if } a^{n+2} c\left(\hat{e}\left(a^{n+2}\right)\right) > \frac{\delta}{1-\delta}a^n c(\hat{e}\left(a^n\right)), \tag{24}$$

with $e_2\left(a^i\right) = 0$ for i odd and $e_2\left(a^{i+1}\right)$ for $i + 1$ even given by

$$c\left(e_2\left(a^{i+1}\right)\right) = \frac{\delta}{1-\delta}c\left(\hat{e}\left(a^i\right)\right), \text{ for } i = 1, 3, 5, \ldots, n. \tag{25}$$

There is no further partitioning after period 2.

Proposition 5 characterises the partitioning in finest constrained partition equilibria of types for which first-best effort is unattainable for given a^1, the highest cost of effort type for which the relationship is to be continued. The partitioning there is independent of the distribution of agent types and of the division of the joint payoff gains from the relationship between principal and agent. Setting $a^1 < a^{\max}$ allows the relationship to be discontinued for some higher cost agent types. To induce $a > a^1$ to end the relationship requires $U_1\left(a^1\right) = 0$ which, from (2), requires choosing $\underline{w}\left(h_1\right) \leq \underline{u}$. When the interval of agent types $\left(\underline{a}, a^1\right]$ is sufficiently constrained, there is either no partitioning of types (as in Part 1 of Proposition 5) or partitioning into two sub-intervals in the first period of the relationship, with no subsequent partitioning (as in Part 2 of Proposition 5). When the interval of agent types $\left(\underline{a}, a^1\right]$ is less constrained, there is partitioning into sub-intervals in the first period of the relationship, with each of these sub-intervals partitioned into two further sub-intervals in the second period of the relationship but with no further partitioning after that second period (as in Part 3 of Proposition 5). In each case, the proposition uniquely specifies the partition points. The number of such partition points is always finite.

The kernel of the proof of Proposition 5 is that, as explained above, (18) puts a lower bound on $e_t\left(a\right)$ that requires a gap between $e_t\left(a\right)$ and $e_t\left(a'\right)$ for any $a < a'$ that are to be separated, whereas (9) in Proposition 2 puts an upper bound on $e_t\left(a\right)$. For the finest partition, the lower and upper bounds are equal which, replacing a by a^{i+1} and a' by a^i in (9) and (18), implies

$$\delta S_{t+1}\left(a^{i+1}\right) - P_t\left(a^{i+1}\right) - U_t\left(a^i\right) - \underline{u} + \underline{w}\left(h_t\right) = a^{i+1}\left[c\left(e_t\left(a^i\right)\right) + \delta C_{t+1}\left(a^i\right)\right]. \qquad (26)$$

It follows that a^{i+1} is closest to a^i for $P_t\left(a^{i+1}\right) = 0$ and $U_t\left(a^i\right)$ as low as possible for all i given the continuity of $U_t\left(a\right)$ at $a = a^i$ required by (18). For $i = 1$, finest partitioning also requires $U_t\left(a^1\right) = \underline{w}\left(h_t\right) - \underline{u}$ and $e_t\left(a^1\right) = 0$. If there is further partitioning of $\left(a^{i+1}, a^i\right]$ at $t + 1$, finest partitioning then requires $U_{t+1}\left(a^i\right) = \underline{w}\left(h_{t+1}\right) - \underline{u}$ and $e_{t+1}\left(a^i\right) = 0$. The rest of the proof is a matter of determining how many periods of partitioning there will be in order to specify the values of $S_{t+1}\left(a^{i+1}\right)$ and $C_{t+1}\left(a^i\right)$, from which the finest partition can then be constructed sequentially from (26). If $\left(\underline{a}, a^1\right]$ is wide enough, there can be partitioning at $t = 1$. There may also be further partitioning at $t = 2$ because setting $e_2\left(a^i\right) = 0$ for $i > 1$ allows for separation at $t = 2$ at some a closer to a^i than at $t = 1$ because $e_1\left(a^i\right)$ has to be strictly positive to satisfy (18) for $a' = a^i$ and $a = a^{i+1}$. This is always the case if partitioning at $t = 1$ is into three or more sub-intervals. But there is never further partitioning after $t = 2$ because the finest partition at $t = 1$ results in intervals of pooled types too small for there to be more than one period of further

partitioning.[10]

Proposition 5 is concerned with the case $\delta \geq \left(\sqrt{5} - 1 \right)/2 \approx 0.618$ because that seems the most relevant for practical purposes. If partitioning at $t = 1$ is into at least three sub-intervals, that condition is necessary as well as sufficient for partitioning to take place in at most two periods — see Lemma 5 in the Appendix. For $\delta \in (0.5, \left(\sqrt{5} - 1 \right)/2)$, finest constrained partition equilibria still exist but may have partitioning in more than two periods. A sufficient condition for partitioning to take place in no more than k periods is that $[\delta/\left(1 - \delta\right)]^k \geq 1/\left(1 - \delta\right)$. For $\delta \in (0, 0.5]$, no finest constrained partition equilibria exist. The intuitive reason can be seen from (23) in Proposition 5, which, for $i = 1$, specifies the requirement for the finest partition in the final period in which partitioning takes place when the equality is replaced by greater than or equal to. For partition to occur, it must be that $a^2 < a^1$ but, for $\delta \leq 1/2$, that requirement would allow a^2 to be arbitrarily close to a^1. So, for any given k, it would always be possible to partition further at $k + 1$ and thus further partitioning would never end. But if partitioning never ends, the joint payoff gain from starting a relationship with a^1 is negative and so could not be mutually beneficial.

It is important to be clear that finest constrained partition equilibria, where they exist, need not be optimal. Depending on the distribution of agent types, the parties may prefer an equilibrium with a less fine partition or with partitioning spread over more than two periods. Moreover, because Proposition 5 uses only necessary conditions for renegotiation-proofness, it is not guaranteed that a finest constrained partition equilibrium, even if agreed before the agent learns type, could not be mutually beneficially renegotiated once the agent has learnt type. But as long as the requirement to be on the Pareto frontier remains for each partition element satisfying any further restrictions, the essential characteristic driving the results in Proposition 5 that $U_{\tau+1}\left(a^i \left(h_\tau \right) \right) = \underline{w}\left(h_{\tau+1}\left(a^i \left(h_\tau \right) \right) \right) - \underline{u}$ for all τ will continue to apply. Furthermore, the partitioning in Proposition 5 is independent of the distribution of agent types (something often hard to pin down in practice) and the division between principal and agent of the joint payoff gains from the relationship. It can thus be used to indicate the

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[10]Proposition 5 does not cover the case in which

$$\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \in \left[\left(\frac{\delta}{1 - \delta}\right)^2 a^1 c\left(\hat{e}\left(a^1\right)\right), \frac{\delta}{1 - \delta}\left(\underline{u} + \underline{v}\right) + \left(\frac{\delta}{1 - \delta}\right)^2 a^1 c\left(\hat{e}\left(a^1\right)\right) \right],$$

which is intermediate between Parts 2 and 3. For that interval, no finest constrained partition equilibrium as defined exists for the following reason. If at $t = 1$ the parties anticipate no further partitioning at $t = 2$, then there is partitioning into two sub-intervals $\left(\underline{a}, a^2\right]$ and $\left(a^2, \bar{a}\right]$ at $t = 1$. But then, at $t = 2$, there can be further partitioning of each of $\left(\underline{a}, a^2\right]$ and $\left(a^2, \bar{a}\right]$ into two further sub-intervals, so play would actually result in further partitioning at $t = 2$. If, however, the parties at $t = 1$ had anticipated further partitioning at $t = 2$ with the lower effort at $t = 2$ that is required for further partitioning in that period, there would have been no partitioning at $t = 1$. Because the only restriction on $\underline{u} + \underline{v}$ is that it is strictly positive, this gap can be very small. There exist partition equilibria with a single period of partitioning into two sub-intervals but with the partition points not as close as possible at each date and which do not, therefore, satisfy the definition of a finest constrained partition equilibrium.

maximum number of groups of types that can be expected in empirical applications without specifying those.

6 Examples

This section considers examples of the finest constrained partition equilibria characterised in Proposition 5. Partitioning in these equilibria is independent of the division of the joint payoff gain between principal and agent and of the distribution of agent types, and the only functional form that affects it is that for the cost of effort $c(\tilde{e})$. It is illustrated here with the cost of effort function $c(\tilde{e}) = b\tilde{e}^{\beta}/\beta$, with $\beta > 1$ for it to be strictly convex and $b > 0$ for it to be strictly positive as required by Assumption 1.

For this functional form, efficient (first-best) effort specified by (1) is

$$e^{*}(a) = (ab)^{\frac{1}{1-\beta}}, \text{ for all } a \in \left[a^{\min}, a^{\max}\right]. \tag{27}$$

For a such that maximum feasible effort under a relational contract specified by (12) is less than first-best effort but great enough that a relational contract is potentially mutually beneficial (that is $a \in \left(\underline{a}, \overline{a}\right]$), that maximal feasible effort $\hat{e}(a)$ satisfies

$$\delta\hat{e}(a) - \frac{ab}{\beta}\hat{e}(a)^{\beta} - \delta(\underline{u} + \underline{v}) = 0, \text{ for } a \in \left(\underline{a}, \overline{a}\right]. \tag{28}$$

First-best effort may be attainable under a relational contract with some agent types, in which case the widest possible interval of agent types for which a relational contract can be sustained but first-best effort is not achievable has greatest lower bound $\underline{a}$ defined by $\hat{e}\left(\underline{a}\right) = e^{*}\left(\underline{a}\right)$. But it may also be that first-best effort is not achievable with any agent type. To allow for both possibilities, define $\underline{a}$ by $\hat{e}\left(\underline{a}\right) = (rab)^{\frac{1}{1-\beta}}$, where $r \geq 1$ captures the extent to which the effort attainable by $\underline{a}$ in a relational contract falls below first best. Then

$$\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) = \frac{\delta(\underline{u} + \underline{v})}{r\beta\delta - 1}. \tag{29}$$

The model requires this to be positive and thus $r > 1/\beta\delta$. For $\beta\delta > 1$, r can equal 1, in which case $\underline{a}$ can attain first-best effort. For $\beta\delta < 1$, r must be strictly greater than 1.[11]

The highest cost type $\overline{a}$ for which a relational contract is potentially sustainable is that for which the maximum value of the left-hand side of (28) is zero, which implies

$$\overline{a}c\left(\hat{e}\left(\overline{a}\right)\right) = \frac{1}{b}\left[\frac{\beta(\underline{u} + \underline{v})}{\beta - 1}\right]^{1-\beta}\frac{b}{\beta}\left[\frac{\beta(\underline{u} + \underline{v})}{\beta - 1}\right]^{\beta} = \frac{\delta\left(\underline{u} + \underline{v}\right)}{\beta - 1}. \tag{30}$$

In periods in which partitioning occurs, however, effort for the highest cost type in an interval to be separated is below the highest sustainable level. As a result, the joint

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[11]Derivations of equations in this section are in Appendix A.3.

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payoff gain, $S_1(\bar{a})$, at the start of a relational contract with $\bar{a}$ may be negative, so the highest cost type for which the relationship is continued may be $a^1 < \bar{a}$. If there are k periods of partitioning involving a^1, the joint payoff gain to a relationship with a^1 is

$$S_1\left(a^1\right) = -(\underline{u} + \underline{v})\sum_{j=0}^{k-1}\delta^j + \delta^k S_{1+k}\left(a^1\right) = -(\underline{u} + \underline{v})\sum_{j=0}^{k-1}\delta^j + \delta^{k-1}a^1 c\left(\hat{e}\left(a^1\right)\right),$$

the second equality following from Proposition 3. Because $ac(\hat{e}(a))$ is decreasing in a by Lemma 1 in the Appendix, the highest a^1 that satisfies both $S_1\left(a^1\right) \geq 0$ and $a^1 \leq \bar{a}$ defined in (30) is thus given by

$$a^1 c\left(\hat{e}\left(a^1\right)\right) = (\underline{u} + \underline{v})\max\left\{\frac{\delta}{\beta - 1}, \sum_{j=1}^{k}\delta^{j-k}\right\}$$

$$= (\underline{u} + \underline{v})\frac{\delta}{\beta - 1}\max\left\{1, (\beta - 1)\sum_{j=1}^{k}\delta^{j-k-1}\right\}. \tag{31}$$

Together with (29), that gives

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} = \frac{\beta - 1}{(r\delta\beta - 1)\max\left\{1, (\beta - 1)\sum_{j=1}^{k}\delta^{j-k-1}\right\}}. \tag{32}$$

This ratio is strictly positive for $r\delta\beta > 1$ and $\beta > 1$.

From Proposition 5, partitioning of $\left(\underline{a}, a^1\right]$ in a finest constrained partition equilibrium is possible if and only if

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} > \frac{\delta}{1 - \delta}. \tag{33}$$

For $\delta \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$, the interval of discount factors to which Proposition 5 applies, $r\beta\delta > 1$, and the cost of effort function in this example, this requires

$$\beta \in \begin{cases} \left(\frac{1}{r\delta}, \frac{2-\delta}{r\delta}\right), & \text{if } \delta \in \left(\frac{\sqrt{5}-1}{2}, \frac{-(1+r)+\sqrt{r^2+10r+1}}{2r}\right); \\ \left(\frac{1}{r\delta}, \frac{2\delta-1}{r\delta^2+\delta-1}\right), & \text{if } \delta \in \left[\frac{-(1+r)+\sqrt{r^2+10r+1}}{2r}, 1\right). \end{cases} \tag{34}$$

Partitioning into more than two sub-intervals is possible if and only if

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} > \frac{\delta}{1 - \delta}\left(\frac{\underline{u} + \underline{v}}{a^1 c\left(\hat{e}\left(a^1\right)\right)} + \frac{\delta}{1 - \delta}\right), \tag{35}$$
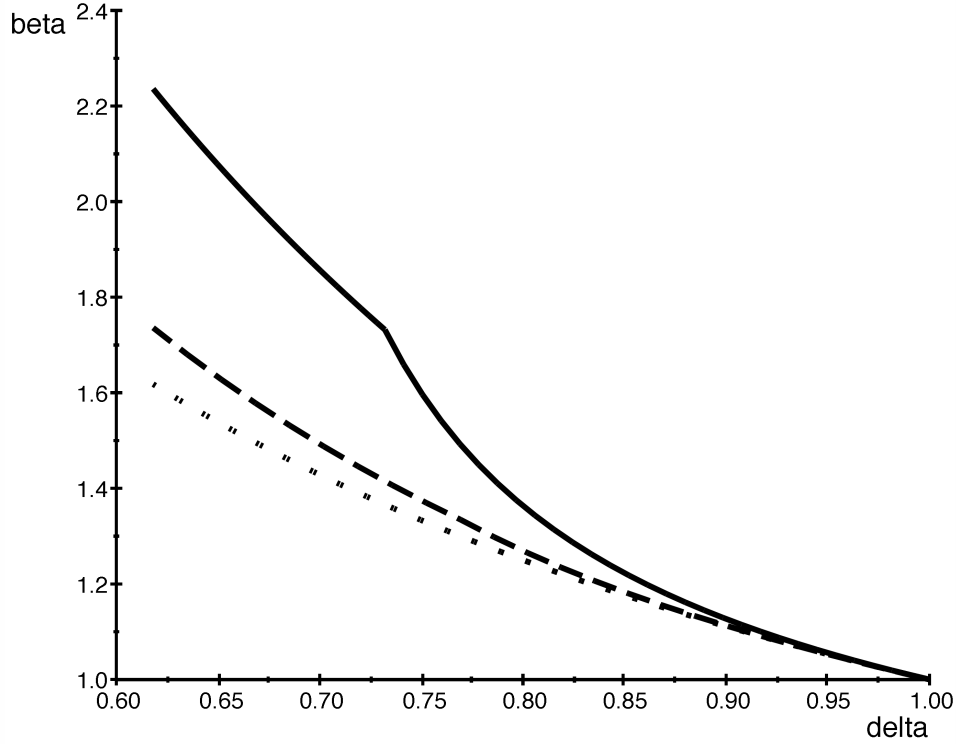
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Figure 2: Highest β for which single separation (solid curve) and multiple separations (dashed curve) feasible, and lowest permitted β for given δ (dotted curve) with $r = 1$

which requires

$$\beta \in \begin{cases} \left(\left(\frac{1}{r\delta}, \frac{1}{r\delta}\left[1 + \frac{(1-\delta)^2}{2}\right]\right), & \text{if } r < \frac{1+\delta}{\delta+\delta^2+\delta^3}\left[1 + \frac{(1-\delta)^2}{2}\right]; \\ \left(\frac{1}{r\delta}, \frac{1}{2r\delta}\left\{r\delta\left(1 - \frac{\delta^2}{1-\delta}\right) + 2 - \delta \right. \right. \\ \left. \left. + \sqrt{\left[r\delta\left(1 - \frac{\delta^2}{1-\delta}\right) + 2 - \delta\right]^2 + 4r\delta\frac{3\delta-2}{1-\delta}}\right\}\right), & \text{if } r \geq \frac{1+\delta}{\delta+\delta^2+\delta^3}\left[1 + \frac{(1-\delta)^2}{2}\right]. \end{cases}$$

(36)

For $r = 1$, these are illustrated for $\delta \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$ in Figure 2, in which the solid curve plots the highest β for which any partition of $\left(\underline{a}, a^1\right]$ is feasible, the dashed curve plots the highest β for which partition into more than two sub-intervals is feasible and the dotted curve plots the lowest β consistent with $r\delta\beta > 1$.[12] As implied by the model, the dashed curve is everywhere below the solid curve but only slightly so for values of δ close to one. (For all the curves, β converges to 1 as δ goes to 1.) So, for example, with quadratic cost of effort, δ has to be below 2/3 for any partitioning to occur and partitioning into more than 2 sub-intervals is not possible.

The appropriate discount factor for practical applications of the model will be lower than just the pure time discount factor if there is a positive probability that

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[12]All computations in this section were done with Scientific WorkPlace 6.
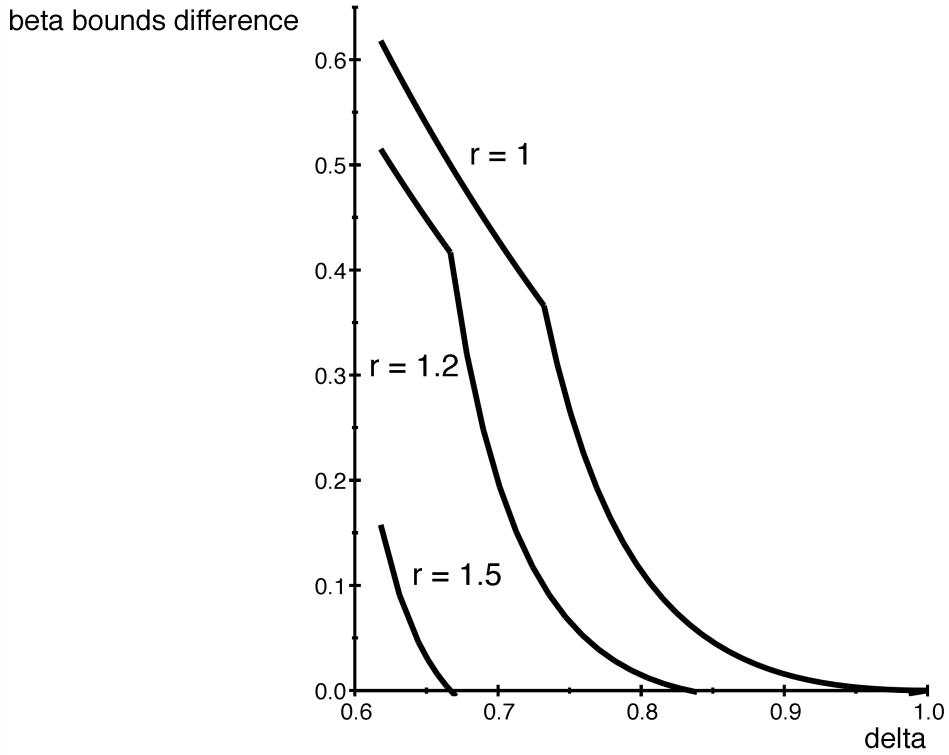
Figure 3: Difference between upper and lower bounds on β in (34) for different r

the relationship comes to an end for purely exogenous reasons or if there is positive probability that deviation by the agent is not detected by the principal. But it is clear from Figure 2 that, for discount factors towards the higher end of the relevant range, any separation of agent types is possible only for cost of effort functions with a degree of convexity that is both very low and in a very restricted range. To illustrate, for $\delta = 0.9$, the range of β for which separation occurs is $(1.1111, 1.1126)$. Even for discount factors towards the lower end of the relevant range, no separation of agent types is possible for highly convex cost of effort functions and partition into more than two sub-intervals is possible only for even less convex ones. As $\delta \to \left(\sqrt{5} - 1 \right)/2, \beta$ has to be less than 2.24 for any partitioning to be possible and less than 1.74 to have partitioning into more than two sub-intervals.

Figure 2 is drawn for the lowest value of $r, r = 1$. An increase in r shifts downward the ranges of β satisfying (34). But it also narrows the gap between the upper and lower bounds in (34). It thus both reduces the highest β and narrows the range of β for which any separation is possible. Figure 3 illustrates the difference between the upper and lower bounds in (34) for $r = 1, r = 1.2$ and $r = 1.5$. For the two higher values of r, no separation is possible for *any* strictly convex cost of effort function ($\beta > 1$) for those values of δ above which the corresponding curves in Figure 3 cut the x-axis. For $r = 2$ (not shown in Figure 3), there are no values of δ within the relevant range for which separation of agent types is possible.

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To have partition into more than two sub-intervals, Proposition 5 requires two periods of partitioning. From (19) and (20), to have partitioning into $m \geq 3$ sub-intervals at $t = 1$ requires

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} > \frac{a^m c\left(\hat{e}\left(a^m\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} = \left(\frac{\underline{u} + \underline{v}}{a^1 c\left(\hat{e}\left(a^1\right)\right)} + \frac{\delta}{1-\delta}\right)\left(\frac{1}{1-\delta}\right)^{m-2}, \text{ for } m \geq 3. \quad (37)$$

This, together with (31) and (32) with $k = 2$, can be used to derive the range of β for which a finest constrained partition equilibrium partitions into m sub-intervals at $t = 1$.[13] Table 1 gives the maximum (formally, the least upper bound) value of β consistent with m partition sub-intervals at $t = 1$ for specified values of δ and $r = 1$, together with the minimum (formally, the greatest lower bound) β for which $\underline{a}$ is positive and hence the cost of effort is positive for $\underline{a}$. (For given δ, $\min \beta$ is the same for all m.) The final row of Table 1 provides a measure of the output loss from the inability to separate agent types. Specifically, it gives the ratio $\hat{e}\left(\underline{a}\right) / \hat{e}\left(\bar{a}\right)$ of the output of type $\underline{a}$ to that of type $\bar{a}$ that would be achieved after separation of these types if separation were possible, evaluated at the lowest β for which no separation is possible. (For $\delta = 0.7$, that is $\beta = 1.4929$.) For $m \geq 3$, there is further partitioning at $t = 2$ of each of those sub-intervals into two sub-intervals, except possibly (depending on the value of β within the range) for the lowest cost of effort sub-interval, giving $2m$ or $2m - 1$ sub-intervals overall. So, for example, for $\delta = 0.7$, $m = 3$ for $\beta \in [1.4421, 1.4736)$. A key message from the table for values of δ typically used in empirical studies is that, for anything other than a small range of lowish degrees of convexity, the number of partition sub-intervals in a finest constrained partition equilibrium of those agent types for whom first-best effort cannot be achieved is small. For $\delta = 0.9$ for example, partitioning into more than two sub-intervals is not possible unless $\beta < 1.1125$ and, for this discount factor, β must be greater than 1.1111. Even for lower δ, the range of β in Table 1 has to be extremely restricted to get more than 6 or so sub-intervals overall ($m = 3$ at $t = 1$ and a further 3 at $t = 2$) — and the values in the table are for the widest possible range of types for which first-best effort cannot be achieved. Moreover, the output cost of not separating types can be substantial. Without separation, $\underline{a}$ and $\bar{a}$ both produce output $\hat{e}\left(\bar{a}\right)$. If separation had been possible, type $\bar{a}$ would have produced $\hat{e}\left(\bar{a}\right)$ once separation had been completed but, for the critical β used for the final row of the table, type $\underline{a}$ would have produced output many multiples of that.[14]

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[13]See Appendix A.3.5 for details.

[14]Straightforward but tedious calculation shows that the sufficient condition (17) in Proposition 4 is satisfied for all (δ, β) combinations in Table 1 for $\left(\underline{a}\left(h_t\right), \bar{a}\left(h_t\right)\right] = \left(\underline{a}, \bar{a}\right]$. It is thus certainly satisfied for the smaller intervals, with no higher $\bar{a}\left(h_t\right)$ and no higher $c'\left(e\left(\underline{a}\left(h_t\right)\right)\right)$, after partitioning at $t = 1$ for which Proposition 4 justifies the application of Proposition 5.

	$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$
$m = 2$	1.4929	1.2694	1.1126
$m = 3$	1.4736	1.2653	1.1125
$m = 4$	1.4421	1.2529	1.1112
$m = 5$	1.4326	1.2506	1.1111
$m = 6$	1.4298	1.2501	1.1111
$\min \beta$	1.4286	1.25	1.1111
$\hat{e}\left(\underline{a}\right)/\hat{e}\left(\overline{a}\right)$	7.6622	13.887	75.627

Table 1: Maximum and minimum β for m sub-intervals at $t = 1$ in finest partition equilibrium with $r = 1$ and output ratio $\hat{e}\left(\underline{a}\right)/\hat{e}\left(\overline{a}\right)$ evaluated at β in row for $m = 2$

7 Related literature

Proposition 5 established that, with a finest constrained partition equilibrium, even continuous agent types for whom first-best effort cannot be achieved are separated into only a finite number of sub-intervals and that this process may take time. So, even with continuous types, employees will be placed in a finite number of grades that include different abilities and suppliers grouped into a finite number of bands, as Asanuma (1989) explains for Toyota. The examples in the previous section suggest that the number of bands may well be quite small and the costs of pooling quite large.

There is a substantial literature on pooling of agent types when there is both moral hazard and adverse selection but none of it satisfactorily captures the patterns of concern here. Much of that literature is concerned with static, one-period models. In these, pooling can arise because the distribution of agent types has a non-monotone hazard rate, as discussed in Laffont and Tirole (1993, pp. 121-123), because of type-dependent outside options, because of violations of the Spence-Mirrlees condition, because of limited agent liability and for reasons of robustness in contracting. On these, see the discussion in Castro-Pires and Moreira (2020) and the survey of robustness in mechanism design and contracting by Carroll (2019). But one-period models are not appropriate for capturing the dynamics of relationships with persistent agent types — unless there is full pooling (which is certainly not the case in the applications considered here), information about agents' types is revealed by their choices, that information will impact on arrangements for the subsequent period and anticipation of this will affect decisions in the current period. The insights of these models apply to different contexts.

Relational contract models are inherently dynamic because it is the gain from continuation of the relationship in the future that enables non-verifiable aspects of agent performance to be sustained. Given the complexity of the tasks in many employment and supply relationships, it is not surprising that some aspects of performance are such as not to be enforceable by formal contracts or payment by results. (The essence of the

model used here does not depend on effort being unidimensional.) In the relational contract literature, Miller and Watson (2013) and Watson et al. (2020) study specifications for renegotiation different from just the requirement used here that it results in an outcome on the Pareto frontier but in their models there is no private information about type so there can be no pooling of such types. Kostadinov (2020) also studies renegotiation, but for a risk-averse agent with no private information. In MacLeod (2003), Fuchs (2007), Ishihara (2016) and Zhu (2018), there is private information but, as in Levin (2003), it is not persistent. In Kostadinov and Kuvalekar (2021), match quality is persistent but symmetrically unknown to both parties. Papers that study private information about persistent types include the following. In Halac (2012), the private information is about the principal's outside opportunities. In Li and Matouschek (2013), it is about the opportunity cost of paying the agent after observing output. In Martimort et al. (2017), it is about the agent's cost. In Nikolowa (2017), it is about the agent's ability. In Kartal (2018), it is about the discount factor. But in all these papers, there are just two possible types so there is no possibility of partial pooling of types.

In the relational contract literature, the paper closest to the current one is MacLeod and Malcomson (1988). That paper derives an equilibrium relational contract with a finite number of ranks that corresponds to a partition of continuous, privately-observed agent types that are persistent. But there are important differences. One is that MacLeod and Malcomson (1988) follow the efficiency wage model of Shapiro and Stiglitz (1984) in having no bonus component to payments. Moreover, in MacLeod and Malcomson (1988), agent types are not specific to the relationship, but equally valuable to competing principals, who can observe an agent's rank in a current relationship. Specifically, an agent dismissed for not complying with the relational contract in one rank is believed by competing principals to be appropriate for the rank below. This corresponds in the current paper to the payoff $\underline{u}$ being a function of information that is revealed. To induce effort with no bonus then requires a discrete difference in payoff between ranks that corresponds to the difference in payoff between the employed and the unemployed required to induce effort in Shapiro and Stiglitz (1984) and this difference in payoff requires discrete partitioning of types. With either bonuses or relationship-specific types, that reason for discrete partitioning would disappear. The model used in the present paper, which has both these, shows that partitioning of agent types into a finite number of intervals does not depend on the particular restrictions in MacLeod and Malcomson (1988). It is a more general phenomenon that is inherent to models with unverifiable performance and privately-informed and persistent agent types.

With the finest partition equilibria analysed here, once agent types have been separated in the initial periods, they remain in the same band over time, unlike in the relational contract model of Levin (2003), in which the agent's type is an *iid* random draw each period. Applied to employment, the model used here generalises the

models of Shapiro and Stiglitz (1984) and MacLeod and Malcomson (1989) to private information about workers' disutility of effort. Moreover, the pooling that arises does not depend on the parties being legally constrained from committing to future contract terms and the principal making "take it or leave it" contract offers, as in the ratchet effect model of Laffont and Tirole (1988).

8 Conclusion

This paper is motivated by the observation that, in many economic relationships, agents of different types are pooled in groups, with those in each group persistent over time and all treated the same despite differences between them. Employees are grouped in grades, with those in a grade all paid the same. Toyota, as described by Asanuma (1989), groups its suppliers into a small number of categories that receive differential treatment. Relational contracts with non-contractible performance elements have proved an insightful way to analyse many aspects of such long-term relationships but have not yet satisfactorily addressed the issue of persistent pooling. Malcomson (2016) showed that it is not possible to separate all of a continuum of privately-informed agent types that are persistent over time in a relational contract if continuation equilibria following full revelation of the agent's type are renegotiation-proof in the sense of being on the Pareto frontier of principal and agent payoffs. Although that result implies that any equilibrium must have at least some pooling of types, it gave no clue as to how extensive that pooling will be. This paper has investigated that by exploring the finest degree of separation that can be achieved in renegotiation-proof equilibria.

A central result is that, among equilibria that separate some agent types from others for whom first-best effort is not attainable, necessary conditions for a continuation equilibrium to be on the Pareto frontier (and, hence, for there to be no possibility of mutually beneficial renegotiation in the future) are that, at each future date, the agent type with the highest cost of effort in a pool is indifferent between delivering agreed effort and shirking. An implication is that there must be a jump in effort between separated pools with different costs of effort because, by imitating the lower cost type pool, higher cost types could, by shirking in the future, attain the same future payoff as the highest cost of those lower cost types. This has important implications for the finest partition of agent types that is achievable. First, it is not possible to separate types in sufficiently small intervals. Second, separation may take time (even though it would happen straightaway if performance were fully contractible) with initial coarser partitions being subsequently refined. But it does not continue indefinitely. How long it continues depends on the discount factor: for discount factors above just over 0.6, it continues for at most two periods. Third, when further separation stops, there remain a finite number of groups of agent types. From then on, agent types remain in the same group over time, unlike in relational contract models such as Levin (2003) with

agent type an *iid* random draw in each time period. Finally, these finest partitions are independent of the division between principal and agent of the joint payoff gains from the relationship and of the distribution of agent types, properties that are convenient for practical applications where either of these is unknown to the researcher.

Finest partitions depend primarily on the discount factor and the convexity of the cost of effort function. This is illustrated with constant elasticity of effort functions. With quadratic cost of effort, the discount factor has to be as low as 2/3 for any separation to occur and partitioning into more than 2 groups is not possible. For high discount factors (above 0.9), no separation of agent types is possible except with very moderate convexity in the cost of effort. The costs of pooling are also substantial. Moreover, for a discount factor of 0.9, the number of type groups is no more than 6 or so unless there is extremely little convexity in the cost of effort. This suggests that, in practical applications, the number of groups of types might be expected to be low despite agent types being continuous. Where the number of groups is in practice small, a relational contract approach looks a potentially attractive explanation.

Appendix

A.1   Lemmas

Lemma 1 *Effort function* $\hat{e}(a)$, *defined in (12), is strictly decreasing for all* $a \in \left(\underline{a}, \bar{a}\right)$ *and* $ac(\hat{e}(a))$ *is strictly decreasing in a for all* $a \in \left(\underline{a}, \bar{a}\right)$.

Proof. From (12), $\hat{e}(a)$ satisfies

$$\delta\hat{e}(a) - ac(\hat{e}(a)) - \delta\left(\underline{u} + \underline{v}\right) = 0, \text{ for } a \in \left(\underline{a}, \bar{a}\right]. \tag{A.1}$$

Differentiation with respect to a gives

$$[\delta - ac'(\hat{e}(a))]\frac{\partial}{\partial a}\hat{e}(a) - c(\hat{e}(a)) = 0, \text{ for } a \in \left(\underline{a}, \bar{a}\right). \tag{A.2}$$

It must be that $\delta - ac'(\hat{e}(a)) < 0$ for $a \in \left(\underline{a}, \bar{a}\right)$ because otherwise a higher effort would have satisfied the constraint in (12). Hence, with $c(\tilde{e}) > 0$ for $\tilde{e} > 0$, it follows from (A.2) that $\partial\hat{e}(a)/\partial a$ is strictly negative for $a \in \left(\underline{a}, \bar{a}\right)$. With $\hat{e}(a)$ strictly decreasing in a for all $a \in \left(\underline{a}, \bar{a}\right)$, it follows from (A.1) that $ac(\hat{e}(a))$ is too. ∎

Lemma 2 *Suppose a finest constrained partition (continuation) equilibrium at* τ *partitions* $\left(\underline{a}(h_t), \bar{a}(h_t)\right] = A(h_t) \subseteq \left(\underline{a}, \bar{a}\right]$ *into at least two sub-intervals at* $t \geq \tau$.

1. $a^i(h_t) \in \left(\underline{a}(h_t), \overline{a}(h_t)\right]$ satisfy

$$\delta S_{t+1}\left(a^{i+1}(h_t)\right) = a^i(h_t)C_t\left(a^i(h_t)\right) + U_t(a^i(h_t)) + \underline{u} - \underline{w}(h_t), \text{ for } i = 1,\ldots,n(h_t) - 1,$$
(A.3)

$$S_{t+1}\left(a^{i+1}(h_t)\right) - S_{t+1}\left(a^i(h_t)\right) = a^i(h_t)C_{t+1}\left(a^i(h_t)\right), \text{ for } i = 2,\ldots,n(h_t) - 1, \quad \text{(A.4)}$$

with $\underline{w}(h_t) = U_t(a^1(h_t)) + \underline{u}$ and $e_t\left(a^i(h_t)\right)$ satisfying

$$e_t\left(a^1(h_t)\right) = 0, \tag{A.5}$$
$$c\left(e_t\left(a^{i+1}(h_t)\right)\right) = C_t\left(a^i(h_t)\right), \text{ for } i = 1,\ldots,n(h_t) - 1. \tag{A.6}$$

2. If there is no further partitioning of $\left(a^{i+1}(h_t), a^i(h_t)\right]$ after t, then

$$\delta S_{t+1}\left(a^i(h_t)\right) = a^i(h_t) c\left(\hat{e}\left(a^i(h_t)\right)\right), \tag{A.7}$$
$$C_{t+1}(a) = \frac{1}{1-\delta} c\left(\hat{e}\left(a^i(h_t)\right)\right), \forall a \in \left(a^{i+1}(h_t), a^i(h_t)\right], \text{ for } i = 1,\ldots,n(h_t) - 1. \quad \text{(A.8)}$$

3. If $a^2(h_t)$ is such that
$$\delta S_{t+1}\left(a^2(h_t)\right) = a^2(h_t) c\left(\hat{e}\left(a^2(h_t)\right)\right), \tag{A.9}$$

there is no partitioning of $\left(a^2(h_t), a^1(h_t)\right]$ subsequent to t and

$$C_{t+1}\left(a^1(h_t)\right) = \frac{1}{1-\delta} c\left(\hat{e}\left(a^1(h_t)\right)\right). \tag{A.10}$$

Proof. Proposition 2 gives necessary conditions for a perfect Bayesian equilibrium that are also sufficient if (6) holds. By Definition 2, a finest constrained partition (continuation) equilibrium at τ has $U_{t+1}\left(a^i(h_t)\right) = \underline{w}(h_{t+1}) - \underline{u}$ for all $a^i(h_t)$ for all h_t and $t \geq \tau$ and $a \in \left(a^{i+1}(h_t), a^i(h_t)\right]$ choosing effort $e_t\left(a^i(h_t)\right)$, so conditions (8) and (9) for $a' = a^i(h_t)$ and $a = a^{i+1}(h_t)$ can be written

$$\left[a^i(h_t) - a^{i+1}(h_t)\right]c(e_t(a^{i+1}(h_t))) \geq U_t(a^{i+1}(h_t)) - U_t(a^i(h_t))$$
$$\geq \left[a^i(h_t) - a^{i+1}(h_t)\right]C_t(a^i(h_t)), \text{ for all } i, h_t, t \geq \tau, \quad \text{(A.11)}$$

and

$$a^{i+1}(h_t) c(e_t(a^{i+1}(h_t))) \leq \delta S_{t+1}(a^{i+1}(h_t)) - P_t\left(a^{i+1}(h_t)\right) - U_t\left(a^i(h_t)\right) - \underline{u} + \underline{w}(h_t)$$
$$- \left[a^i(h_t) - a^{i+1}(h_t)\right]C_t(a^i(h_t)), \text{ for all } i, h_t, t \geq \tau.$$

For there to exist $e_t(a^{i+1}(h_t))$ that satisfies both these, it must be that

$$\delta S_{t+1}(a^{i+1}(h_t)) - P_t\left(a^{i+1}(h_t)\right) - U_t\left(a^i(h_t)\right) - \underline{u} + \underline{w}(h_t) - \left[a^i(h_t) - a^{i+1}(h_t)\right]C_t(a^i(h_t))$$

$$\geq a^{i+1}(h_t)\, c(e_t(a^{i+1}(h_t))) \geq a^{i+1}(h_t)\, C_t(a^i(h_t)), \text{ for all } i, h_t, t \geq \tau. \quad \text{(A.12)}$$

and so

$$\delta S_{t+1}(a^{i+1}(h_t)) - P_t\left(a^{i+1}(h_t)\right) \geq a^i(h_t)\, C_t(a^i(h_t)) + U_t\left(a^i(h_t)\right) + \underline{u} - \underline{w}(h_t),$$

$$\text{for all } i, h_t, t \geq \tau.$$

In a perfect Bayesian equilibrium, the joint gain $S_{t+1}(a)$ must be non-increasing in a, so the closest $a^{i+1}(h_t) < a^i(h_t)$ can be to $a^i(h_t)$, as required for a finest constrained partition (continuation) equilibrium, is for this condition to hold with equality when $P_t\left(a^i(h_t)\right) = 0$ for all $a^i(h_t) \in \left(\underline{a}(h_t), \bar{a}(h_t)\right]$, giving (A.3). This implies that the inequalities in (8) hold with equality for all $a^i(h_t) \in \left(\underline{a}(h_t), \bar{a}(h_t)\right]$ which, with (9), implies that (6) also holds for all $a^i(h_t) \in \left(\underline{a}(h_t), \bar{a}(h_t)\right]$.

By definition, $C_t(a) = c(e_t(a)) + \delta C_{t+1}(a)$ so, from (A.3) for $i = 1$, $a^2(h_t)$ is closest to $a^1(h_t)$ with $\underline{w}(h_t) = U_t(a^1(h_t)) + \underline{u}$ and $e_t\left(a^1(h_t)\right) = 0$, which gives (A.5). Moreover, for (A.3) to hold, (A.12) must hold with both inequalities replaced by equalities, the second of which gives (A.6). Conditions (A.7) and (A.8) follow directly from Proposition 3.

To establish (A.4), subtract (A.3) for $i - 1$ on the right-hand side from (A.3) for i on the right-hand side to get

$$\delta\left[S_{t+1}\left(a^{i+1}(h_t)\right) - S_{t+1}\left(a^i(h_t)\right)\right] = a^i(h_t)C_t\left(a^i(h_t)\right) - a^{i-1}(h_t)C_t\left(a^{i-1}(h_t)\right)$$

$$+ U_t(a^i(h_t)) - U_t(a^{i-1}(h_t)), \text{ for all } i = 2, \dots, n(h_t) - 1. \quad \text{(A.13)}$$

With the right-hand inequality in (8) holding with equality for all $a^i(h_t) \in \left(\underline{a}(h_t), \bar{a}(h_t)\right]$,

$$U_t\left(a^i(h_t)\right) = U_t(a^{i-1}(h_t)) + \left[a^{i-1}(h_t) - a^i(h_t)\right]C_t\left(a^{i-1}(h_t)\right).$$

Use of this in (A.13) gives

$$\delta\left[S_{t+1}\left(a^{i+1}(h_t)\right) - S_{t+1}\left(a^i(h_t)\right)\right] = a^i(h_t)C_t\left(a^i(h_t)\right) - a^{i-1}(h_t)C_t\left(a^{i-1}(h_t)\right)$$

$$+ \left[a^{i-1}(h_t) - a^i(h_t)\right]C_t\left(a^{i-1}(h_t)\right), \text{ for all } i = 2, \dots, n(h_t) - 1.$$

By definition, $C_t(a) = c(e_t(a)) + \delta C_{t+1}(a)$ so, with the use of (A.6) to substitute for $C_t\left(a^{i-1}(h_t)\right)$, this simplifies to (A.4).

With $\left(\underline{a}(h_t), \bar{a}(h_t)\right]$ partitioned at t, $e_t\left(a^1(h_t)\right) = 0$ by (A.5). So $C_t\left(a^1(h_t)\right) = \delta C_{t+1}\left(a^1(h_t)\right)$ and by (A.3), with $\underline{w}(h_t) = U_t(a^1(h_t)) + \underline{u}$ and $e_t\left(a^1(h_t)\right) = 0$,

$$\delta S_{t+1}\left(a^2(h_t)\right) = a^1(h_t)\, C_t\left(a^1(h_t)\right) = \delta a^1(h_t)\, C_{t+1}\left(a^1(h_t)\right).$$

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Suppose there were further partitioning of $\left(a^2\left(h_t\right),a^1\left(h_t\right)\right]$ at $t+1$. Then also $\underline{w}\left(h_{t+1}\right) = U_{t+1}(a^1\left(h_t\right)) + \underline{u}$ and $e_{t+1}\left(a^1\left(h_t\right)\right) = 0$ by (A.5), so $C_{t+1}\left(a^1\left(h_t\right)\right) = \delta C_{t+2}\left(a^1\left(h_t\right)\right)$. So partitioning must be at $a \in \left(a^2\left(h_t\right),a^1\left(h_t\right)\right]$ for which, by (A.3) applied to $t+2$ and $t+1$ and $\delta < 1$,

$$\delta S_{t+2}\left(a\right) = a^1\left(h_t\right)C_{t+1}\left(a^1\left(h_t\right)\right) > \delta a^1\left(h_t\right)C_{t+1}\left(a^1\left(h_t\right)\right) = \delta S_{t+1}\left(a^2\left(h_t\right)\right).$$

Further partitioning would, therefore, require $S_{t+2}\left(a\right) > S_{t+1}\left(a^2\left(h_t\right)\right)$. But, when (A.9) holds, $S_{t+1}\left(a^2\left(h_t\right)\right)$ is at the highest feasible level for $a^2\left(h_t\right)$ and, since the highest feasible $S_t\left(a\right)$ is necessarily decreasing in a, this would require $a < a^2\left(h_t\right)$. But there is no such $a \in \left(a^2\left(h_t\right),a^1\left(h_t\right)\right]$, so no further partitioning of $\left(a^2\left(h_t\right),a^1\left(h_t\right)\right]$ at $t+1$ is possible. The same argument applies to all $\tau > t+1$, so there is no further partitioning of $\left(a^2\left(h_t\right),a^1\left(h_t\right)\right]$ subsequent to t and, by Proposition 3, (A.10) holds. ∎

Lemma 3 *Consider a finest constrained partition (continuation) equilibrium at t that partitions at t an interval of types $\left(\underline{a}\left(h_t\right),\bar{a}\left(h_t\right)\right] = A\left(h_t\right) \subseteq \left(\underline{a},a^{\max}\right]$ with the same history h_t for given $a^1\left(h_t\right) \le \min\{\bar{a}\left(h_t\right),\bar{a}\}$ that is the highest cost of effort type for which the relationship is to be continued. If, for $\delta > 1/2$ and $k \ge 1$, each partition interval of $\left(\underline{a}\left(h_t\right),a^2\left(h_t\right)\right]$ is further partitioned at each date up to and including $t+k-1$ but there is no further partitioning from $t+k$ on,*

$$a^2\left(h_t\right)c\left(\hat{e}\left(a^2\left(h_t\right)\right)\right) = \left(\underline{u}+\underline{v}\right)\sum_{j=2}^{k}\delta^{j-k} + \frac{\delta}{1-\delta}a^1\left(h_t\right)c\left(\hat{e}\left(a^1\left(h_t\right)\right)\right), \tag{A.14}$$

$$a^{i+1}\left(h_t\right)c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right) = \frac{1}{1-\delta}a^i\left(h_t\right)c(\hat{e}\left(a^i\left(h_t\right)\right)), \text{ for } i = 2,\ldots,n\left(h_t\right)-1, \tag{A.15}$$

$$a^{n(h_t)+1}\left(h_t\right)c\left(\hat{e}\left(a^{n(h_t)+1}\left(h_t\right)\right)\right)$$
$$= \min\left\{\frac{1}{1-\delta}a^{n(h_t)}\left(h_t\right)c(\hat{e}\left(a^{n(h_t)}\left(h_t\right)\right)),\underline{a}\left(h_t\right)c\left(\hat{e}\left(\underline{a}\left(h_t\right)\right)\right)\right\}, \tag{A.16}$$

with the convention that the summation term is zero for $k = 1$, and $e_t\left(a^i\left(h_t\right)\right)$ for $i = 2,\ldots,n\left(h_t\right)$ is given by

$$c\left(e_t\left(a^{i+1}\left(h_t\right)\right)\right) = c\left(e_t\left(a^i\left(h_t\right)\right)\right) + \frac{\delta^k}{1-\delta}c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right), \text{ for } i = 1,\ldots,n\left(h_t\right)-1. \tag{A.17}$$

   Proof. A partition point $a^i\left(h_t\right)$ at t becomes $a^1\left(h_{t+1}\right)$ at $t+1$ for some h_{t+1}. For each date $\tau > t$ with further partitioning of an interval with upper endpoint $a^1\left(h_{t+1}\right)$, $e_\tau\left(a^1\left(h_{t+1}\right)\right) = 0$ by (A.5) in Lemma 2 and, for each such date, the joint payoff gain is therefore $s_\tau\left(0,a\right) = -\left(\underline{u}+\underline{v}\right)$. Thus, if there are k periods of partitioning,

$$S_{t+1}\left(a^{i+1}\left(h_t\right)\right) = -\left(\underline{u}+\underline{v}\right)\sum_{j=0}^{k-2}\delta^j + \delta^{k-1}S_{t+k}\left(a^{i+1}\left(h_t\right)\right).$$

Once partitioning ceases, $\delta S_{t+k}\left(a^{i+1}\left(h_t\right)\right) = a^{i+1}\left(h_t\right)c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right)$ by Proposition 3. So,

for exactly k periods of partitioning,

$$S_{t+1}\left(a^{i+1}\left(h_t\right)\right) = -(\underline{u} + \underline{v}) \sum_{j=0}^{k-2} \delta^j + \delta^{k-2} a^{i+1}\left(h_t\right) c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right). \tag{A.18}$$

For $t+k-1$ the last period in which partitioning takes place, $C_{t+k}\left(a^i\left(h_t\right)\right) = \frac{1}{1-\delta} c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right)$ because, by Proposition 3, $e_\tau\left(a^i\left(h_t\right)\right) = \hat{e}\left(a^i\left(h_t\right)\right)$ for $\tau \geq t+k$. So, because $e_{t+j}\left(a^i\left(h_t\right)\right) = 0$, and thus also $c\left(e_{t+j}\left(a^i\left(h_t\right)\right)\right) = 0$, for $j = 1, \ldots, k-1$,

$$C_{t+1}\left(a^i\left(h_t\right)\right) = \frac{\delta^{k-1}}{1-\delta} c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right). \tag{A.19}$$

Application of (A.18) and (A.19) to (A.3) for $i = 1$ with $\underline{w}\left(h_t\right) = U_t(a^1\left(h_t\right)) + \underline{u}$ and $e_t\left(a^1\left(h_t\right)\right) = 0$, as required by Lemma 2, gives

$$\delta\left[-(\underline{u} + \underline{v}) \sum_{j=0}^{k-2} \delta^j + \delta^{k-2} a^2\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right)\right] = a^1\left(h_t\right)\left[0 + \delta\frac{\delta^{k-1}}{1-\delta} c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right)\right]$$

or, divided through by δ^{k-1},

$$a^2\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right) = \delta^{2-k}(\underline{u} + \underline{v}) \sum_{j=0}^{k-2} \delta^j + \frac{\delta}{1-\delta} a^1\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right),$$

which, for $\delta > 1/2$, implies $a^2\left(h_t\right) < a^1\left(h_t\right)$ as required for separation and corresponds to (A.14).

Application of (A.18) and (A.19) to (A.4) for $i > 1$ gives

$$- (\underline{u} + \underline{v}) \sum_{j=0}^{k-2} \delta^j + \delta^{k-2} a^{i+1}\left(h_t\right) c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right)$$

$$+ (\underline{u} + \underline{v}) \sum_{j=0}^{k-2} \delta^j - \delta^{k-2} a^i\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right) = \frac{\delta^{k-1}}{1-\delta} a^i\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right)$$

or

$$\delta^{k-2} a^{i+1}\left(h_t\right) c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right) = \left(\delta^{k-2} + \frac{\delta^{k-1}}{1-\delta}\right) a^i\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right)$$

or

$$a^{i+1}\left(h_t\right) c\left(\hat{e}\left(a^{i+1}\left(h_t\right)\right)\right) = \left(1 + \frac{\delta}{1-\delta}\right) a^i\left(h_t\right) c\left(\hat{e}\left(a^i\left(h_t\right)\right)\right),$$

which corresponds to (A.15). Condition (A.16) also follows from this because, by definition of $n\left(h_t\right)$, $a^{n(h_t)+1}$ cannot be below $\underline{a}\left(h_t\right)$.

Condition (A.17) follows from (A.6), (A.19) and $C_t\left(a^i\left(h_t\right)\right) = c\left(e_t\left(a^i\left(h_t\right)\right)\right) + \delta C_{t+1}\left(a^i\left(h_t\right)\right)$. ∎

Lemma 4 *If partitioning of $\left(\underline{a}\,(h_{\tau-1}),\bar{a}\,(h_{\tau-1})\right]$ at $\tau-1$ in a perfect Bayesian equilibrium satisfies (A.14), (A.15) and (A.16) in Lemma 3 for $t = \tau-1$ with $k = 2$, partitioning of $\left(\underline{a}\,(h_\tau),\bar{a}\,(h_\tau)\right] = \left(a^{i+1}\,(h_{\tau-1}),a^i\,(h_{\tau-1})\right]$ for $i \in \{1,\ldots,n\,(h_{\tau-1})\}$ that satisfies Lemma 3 for $t = \tau$ with $k = 1$ is into at most two sub-intervals for $\delta > 1/2$.*

Proof. Partitioning $\left(\underline{a}\,(h_\tau),\bar{a}\,(h_\tau)\right] = \left(a^{i+1}\,(h_{\tau-1}),a^i\,(h_{\tau-1})\right]$ into three sub-intervals at τ requires $a^3\,(h_\tau) > \underline{a}\,(h_\tau) = a^{i+1}\,(h_{\tau-1})$ and so, because $ac\,(\hat{e}\,(a))$ is decreasing in a by Lemma 1,

$$a^3\,(h_\tau)\,c\left(\hat{e}\left(a^3\,(h_\tau)\right)\right) < a^{i+1}\,(h_{\tau-1})\,c(\hat{e}\left(a^{i+1}\,(h_{\tau-1})\right)). \tag{A.20}$$

For $t = \tau$ with $k = 1$, (A.15) and (A.14) require

$$a^3\,(h_\tau)\,c\left(\hat{e}\left(a^3\,(h_\tau)\right)\right) = \frac{1}{1-\delta}a^2\,(h_\tau)\,c\left(\hat{e}\left(a^2\,(h_\tau)\right)\right)$$

$$= \frac{\delta}{(1-\delta)^2}a^1\,(h_\tau)\,c\left(\hat{e}\left(a^1\,(h_\tau)\right)\right)$$

$$\geq \frac{\delta}{(1-\delta)^2}a^i\,(h_{\tau-1})\,c\left(\hat{e}\left(a^i\,(h_{\tau-1})\right)\right),$$

$$\text{for some } i \in \{1,\ldots,n\,(h_{\tau-1})\}. \tag{A.21}$$

For $t = \tau-1$ and $i \geq 2$, (A.15) and (A.16) require

$$a^{i+1}\,(h_{\tau-1})\,c(\hat{e}\left(a^{i+1}\,(h_{\tau-1})\right)) \leq \frac{1}{1-\delta}a^i\,(h_{\tau-1})\,c(\hat{e}\left(a^i\,(h_{\tau-1})\right)), \text{ for all } i \in \{2,\ldots,n\,(h_{\tau-1})\}.$$

With $\delta > 1/2$, this and (A.21) are inconsistent with (A.20), so partitioning at τ can be into at most two sub-intervals. This establishes the result for $i \geq 2$.

For $i = 1$, partitioning at $\tau-1$ with $k = 2$ requires, by (A.14) for $t = \tau-1$,

$$a^2\,(h_{\tau-1})\,c\left(\hat{e}\left(a^2\,(h_{\tau-1})\right)\right) = \underline{u} + \underline{v} + \frac{\delta}{1-\delta}a^1\,(h_{\tau-1})\,c\left(\hat{e}\left(a^1\,(h_{\tau-1})\right)\right).$$

For partitioning at τ, $k = 1$ so (A.21) implies that (A.20) requires

$$\frac{\delta}{(1-\delta)^2}a^1\,(h_{\tau-1})\,c\left(\hat{e}\left(a^1\,(h_{\tau-1})\right)\right) < \underline{u} + \underline{v} + \frac{\delta}{1-\delta}a^1\,(h_{\tau-1})\,c\left(\hat{e}\left(a^1\,(h_{\tau-1})\right)\right)$$

or

$$\frac{\delta}{1-\delta}\left(\frac{1}{1-\delta} - 1\right)a^1\,(h_{\tau-1})\,c\left(\hat{e}\left(a^1\,(h_{\tau-1})\right)\right) < \underline{u} + \underline{v}$$

or

$$\left(\frac{\delta}{1-\delta}\right)^2 a^1\,(h_{\tau-1})\,c\left(\hat{e}\left(a^1\,(h_{\tau-1})\right)\right) < \underline{u} + \underline{v}. \tag{A.22}$$

A perfect Bayesian equilibrium requires the relationship with $a^1\,(h_{\tau-1})$ to be on-going and thus such that $S_\tau\left(a^1\,(h_{\tau-1})\right) \geq 0$. For a finest partition continuation equilibrium at

τ, separation at τ implies $e_\tau\left(a^1\left(h_{\tau-1}\right)\right) = c(e_\tau\left(a^1\left(h_{\tau-1}\right)\right)) = 0$ and, from (4),

$$S_\tau\left(a^1\left(h_\tau\right)\right) = S_\tau\left(a^1\left(h_{\tau-1}\right)\right) = -\left(\underline{u}+\underline{v}\right) + \delta S_{\tau+1}\left(a^1\left(h_{\tau-1}\right)\right)$$
$$\leq -\left(\underline{u}+\underline{v}\right) + a^1\left(h_{\tau-1}\right) c\left(\hat{e}(a^1\left(h_{\tau-1}\right))\right),$$

the inequality following because $a^1\left(h_{\tau-1}\right) c\left(\hat{e}\left(a^1\left(h_{\tau-1}\right)\right)\right)$ is, by Proposition 3, an upper bound on $\delta S_{\tau+1}\left(a^1\left(h_{\tau-1}\right)\right)$. So to have $S_\tau\left(a^1\left(h_{\tau-1}\right)\right) \geq 0$ requires

$$a^1\left(h_{\tau-1}\right) c\left(\hat{e}\left(a^1\left(h_{\tau-1}\right)\right)\right) \geq \underline{u}+\underline{v}.$$

When this is satisfied, (A.22) cannot be satisfied if

$$\left(\frac{\delta}{1-\delta}\right)^2 \geq 1,$$

which always holds for $\delta > 1/2$ as specified. This establishes the result for $i = 1$. ∎

Lemma 5 *If partitioning at* $t = \{\tau - 2, \tau - 1\}$ *satisfies Lemma 3 with* $k = 2$ *for* $\tau - 2$ *and* $k = 1$ *for* $\tau - 1$, *a sufficient condition for no further partitioning at* τ *of* $\left(\underline{a}\left(h_\tau\right), \bar{a}\left(h_\tau\right)\right] = \left(a^{i+1}\left(h_{\tau-1}\right), a^i\left(h_{\tau-1}\right)\right]$ *for all* $i = 1, \ldots, n\left(h_{\tau-1}\right)$ *to satisfy (A.14) in Lemma 3 for* $t = \tau$ *is* $\delta \geq \left(\sqrt{5}-1\right)/2$. *If partitioning at* $\tau - 2$ *is into more than two sub-intervals, this condition is also necessary.*

Proof. Further partitioning at τ of $\left(\underline{a}\left(h_\tau\right), \bar{a}\left(h_\tau\right)\right] = \left(a^{i+1}\left(h_{\tau-1}\right), a^i\left(h_{\tau-1}\right)\right]$ is possible if and only if $a^2\left(h_\tau\right) > \underline{a}\left(h_\tau\right) = a^{i+1}(h_{\tau-1}) > \underline{a}\left(h_{\tau-1}\right)$ and so, because $ac\left(\hat{e}\left(a\right)\right)$ is decreasing in a by Lemma 1, if and only if

$$a^2\left(h_\tau\right) c\left(\hat{e}\left(a^2\left(h_\tau\right)\right)\right) < \underline{a}\left(h_{\tau-1}\right) c\left(\hat{e}\left(\underline{a}\left(h_{\tau-1}\right)\right)\right). \tag{A.23}$$

Sufficiency. To satisfy (A.14) for $t = \tau$ given $a^1\left(h_\tau\right) \leq a^i\left(h_{\tau-1}\right)$, further partitioning of $\left(\underline{a}\left(h_\tau\right), \bar{a}\left(h_\tau\right)\right] = \left(a^{i+1}\left(h_{\tau-1}\right), a^i\left(h_{\tau-1}\right)\right]$ at τ would have to satisfy

$$a^2\left(h_\tau\right) c\left(\hat{e}\left(a^2\left(h_\tau\right)\right)\right) = \frac{\delta}{1-\delta} a^1\left(h_\tau\right) c(\hat{e}\left(a^1\left(h_\tau\right)\right)) \geq \frac{\delta}{1-\delta} a^i\left(h_{\tau-1}\right) c\left(\hat{e}\left(a^i\left(h_{\tau-1}\right)\right)\right). \tag{A.24}$$

With $k = 1$ for $t = \tau - 1$ in (A.14) and $a^1\left(h_{\tau-1}\right) \leq \bar{a}\left(h_{\tau-1}\right) = a^j\left(h_{\tau-2}\right)$, for some $j \in \{1, \ldots, n\left(h_{\tau-2}\right)\}$,

$$a^i\left(h_{\tau-1}\right) c\left(\hat{e}\left(a^i\left(h_{\tau-1}\right)\right)\right) \geq \frac{\delta}{1-\delta} a^1\left(h_{\tau-1}\right) c\left(\hat{e}\left(a^1\left(h_{\tau-1}\right)\right)\right)$$
$$\geq \frac{\delta}{1-\delta} a^j\left(h_{\tau-2}\right) c\left(\hat{e}\left(a^j\left(h_{\tau-2}\right)\right)\right)$$
$$\text{for some } j \in \{1, \ldots, n\left(h_{\tau-2}\right)\}, i = 2, \ldots, n\left(h_{\tau-1}\right).$$

Together with (A.24), this implies that, for $i = 2, \ldots, n\,(h_{\tau-1})$,

$$a^2\,(h_\tau)\,c\left(\hat{e}\left(a^2\,(h_\tau)\right)\right) \geq \left(\frac{\delta}{1-\delta}\right)^2 a^j\,(h_{\tau-2})\,c\left(\hat{e}\left(a^j\,(h_{\tau-2})\right)\right)$$

$$\text{for some } j \in \{1, \ldots, n\,(h_{\tau-2})\}, i = 2, \ldots, n\,(h_{\tau-1}). \quad \text{(A.25)}$$

In equilibrium, it must be that $S_{\tau-2}\left(a^1\,(h_{\tau-2})\right) \geq 0$. With $k = 2$ at $\tau - 2$, $e_{\tau-2}\left(a^1\,(h_{\tau-2})\right) = c\left(e_{\tau-2}\left(a^1\,(h_{\tau-2})\right)\right) = e_{\tau-1}\left(a^1\,(h_{\tau-2})\right) = c\left(e_{\tau-1}\left(a^1\,(h_{\tau-2})\right)\right) = 0$ so, from (4) for $t = \tau - 2$,

$$S_{\tau-2}\left(a^1\,(h_{\tau-2})\right) = -(1+\delta)\,(\underline{u} + \underline{v}) + \delta^2 S_\tau\left(a^1\,(h_{\tau-2})\right)$$

$$= -(1+\delta)\,(\underline{u} + \underline{v}) + \delta a^1\,(h_{\tau-2})\,c\left(\hat{e}\left(a^1\,(h_{\tau-2})\right)\right),$$

the second equality following from Proposition 3. So to have $S_{\tau-2}\left(a^1\,(h_{\tau-2})\right) \geq 0$ requires

$$\frac{\delta}{1+\delta} a^1\,(h_{\tau-2})\,c\left(\hat{e}\left(a^1\,(h_{\tau-2})\right)\right) \geq (\underline{u} + \underline{v}).$$

Thus, (A.14) applied to $t = \tau - 2$ with $k = 2$ requires

$$a^2\,(h_{\tau-2})\,c\left(\hat{e}\left(a^2\,(h_{\tau-2})\right)\right) \leq \left(\frac{\delta}{1+\delta} + \frac{\delta}{1-\delta}\right) a^1\,(h_{\tau-2})\,c\left(\hat{e}\left(a^1\,(h_{\tau-2})\right)\right)$$

$$= \frac{2\delta}{(1+\delta)\,(1-\delta)} a^1\,(h_{\tau-2})\,c\left(\hat{e}\left(a^1\,(h_{\tau-2})\right)\right).$$

Since $\frac{2\delta}{(1+\delta)(1-\delta)} < \frac{1}{1-\delta}$ for $\delta < 1$, (A.14), (A.15) and (A.16) applied to $t = \tau - 2$ all require

$$a^{j+1}\,(h_{\tau-2})\,c\left(\hat{e}\left(a^{j+1}\,(h_{\tau-2})\right)\right) \leq \frac{1}{1-\delta} a^j\,(h_{\tau-2})\,c\left(\hat{e}\left(a^j\,(h_{\tau-2})\right)\right), \text{ for } j = 1, \ldots, n\,(h_{\tau-2}). \quad \text{(A.26)}$$

So for $\underline{a}\,(h_{\tau-1}) = a^{j+1}\,(h_{\tau-2})$, this gives an upper bound on $\underline{a}\,(h_{\tau-1})\,c\left(\hat{e}\left(\underline{a}\,(h_{\tau-1})\right)\right)$ of

$$\underline{a}\,(h_{\tau-1})\,c\left(\hat{e}\left(\underline{a}\,(h_{\tau-1})\right)\right) \leq \frac{1}{1-\delta} a^j\,(h_{\tau-2})\,c\left(\hat{e}\left(a^j\,(h_{\tau-2})\right)\right), \text{ for } j = 1, \ldots, n\,(h_{\tau-2}). \quad \text{(A.27)}$$

From (A.25) and (A.27), (A.23) can be satisfied for $i = 2, \ldots, n\,(h_{\tau-1})$ only if

$$\left(\frac{\delta}{1-\delta}\right)^2 < \frac{1}{1-\delta}.$$

For $\delta \geq \left(\sqrt{5} - 1\right)/2$, this never holds so further partitioning at τ of $\left(a^{i+1}\,(h_{\tau-1}), a^i\,(h_{\tau-1})\right]$ for any $i \in \{2, \ldots, n\,(h_{\tau-1})\}$ is not possible. Then, by (A.7) in Lemma 2, $\delta S_\tau\,(a^2\,(h_{\tau-1})) = a^2\,(h_{\tau-1})\,c\,(\hat{e}\,(a^2\,(h_{\tau-1})))$ and, by Part 3 of Lemma 2, there is no further partitioning at τ of $\left(a^2\,(h_{\tau-1}), a^1\,(h_{\tau-1})\right]$ either.

   Necessity. For $a^i\,(h_{t-1})$ to continue the relationship at $t - 1$ in equilibrium for any t, it must be that $a^1\,(h_t) = a^i\,(h_{t-1})$, so $a^1\,(h_\tau) = a^i\,(h_{\tau-1})$ and $a^1\,(h_{\tau-1}) = a^j\,(h_{\tau-2})$. In that case, (A.24) and (A.25) hold with equality. If partitioning at $\tau - 2$ is into more than

two sub-intervals, there exists a j such that (A.26) holds with equality. In that case, the condition in the lemma is necessary as well as sufficient. ■

A.2 Proofs of propositions

Proof of Proposition 2 Consider a such that $a \in A(h_t(a'))$ but $a \notin A(h_{t+1}(a'))$ that has taken the action for a' at t. At $t + 1$, a has the following possible actions: (1) choose an effort for some $a'' \in A(h_{t+1}(a'))$, in which case the deviation at t is not revealed to the principal at $t + 1$ and the agent's payoff gain at $t + 1$ is $\tilde{U}(a'', a, h_{t+1}(a'))$ — in this case, the agent's best choice at $t + 1$ gives payoff gain $\max_{a'' \in A(h_{t+1}(a'))} \tilde{U}(a'', a, h_{t+1}(a'))$; (2) quit the relationship at $t + 1$ giving payoff gain 0 at $t + 1$; and (3) continue the relationship at $t + 1$ but choose an effort different from that for any $a'' \in A(h_{t+1}(a'))$. With (3), that a deviation has occurred will be revealed to the principal at $t + 1$ so, at $t + 2$, the agent can either (3a) quit the relationship giving payoff gain at $t + 1$ of $U_{t+1}(a) = \underline{w}(h_{t+1}(a')) - \underline{u}$, or (3b) accept the punishment specified in Proposition 1 with a continuation equilibrium such that $a''' = \max_{a'' \in A(h_{t+1}(a'))} a''$ would receive non-positive payoff gain from continuing the relationship at $t + 2$. Because in case (3) continuation payoffs are independent of the deviation, the best response conditional on deviation is to set effort to its lowest feasible level of zero.

For $a < a'$ taking the action for a' at t, it is a best response at $t + 1$ to take the action yielding $\max_{a'' \in A(h_{t+1}(a'))} \tilde{U}(a'', a, h_{t+1}(a'))$, that is choose (1) over either (2) or (3), as long as not deviating is a best response for a' because the cost of effort is lower for a than for a' and so the difference in payoff gain between a and a' from (1) is strictly positive and at least as high as from (2) and (3). Because (8) is required to hold over all $a' > a$, and thus for $a' = a''$, the maximisation over a'' in this is superfluous. Thus, because monetary payments have the same value to all agent types, requiring the right-hand inequality in (8) to hold for all $a < a'$ with payoff difference $\tilde{U}(a', a, h_t) - \tilde{U}(a', a', h_t) = (a' - a) C_t(a')$ corresponds to the right-hand inequality in (7).

For $a > a'$ taking the action for a' at t, (3a) is a better response than (3b) under the sufficiency conditions of Proposition 1. Thus, the best choice over (1) to (3) at $t + 1$ with the roles of a and a' reversed to have $a < a'$ as in (8) corresponds to

$$\max \left\{ \max_{a'' \in A(h_{t+1}(a))} \tilde{U}(a'', a', h_{t+1}(a)), 0, \underline{w}(h_{t+1}(a)) - \underline{u} \right\}, \quad \text{for } a < a'.$$

Because (8) is required to hold over all $a < a'$, the maximisation over a'' in this is again superfluous. So, again because monetary payments have the same value to all agent types, requiring the payoff gain differences to satisfy the left-hand inequality in (7) corresponds to requiring them to satisfy the left-hand inequality in (8) for all $a < a'$.

That (9) is necessary follows from use of the right-hand inequality in (8) to substitute for $U_t(a)$ in (6) and re-arrangement. ■

Proof of Proposition 3   Conditional on all $a \in A(h_\tau) \subseteq (\underline{a}, \overline{a}]$ choosing the same effort $e_t(a)$ at t for all $t \geq \tau$, $E_{a \in A(h_\tau)} P_t(a) = P_t(\overline{a}(h_\tau))$ for all $t \geq \tau$. By definition of $(\underline{a}, \overline{a}]$, first-best effort $e^*(a)$ is unattainable for any $a \in A(h_\tau)$ and, because $a < \overline{a}(h_\tau)$ has lower cost of effort than $\overline{a}(h_\tau)$, any effort sustainable for $\overline{a}(h_\tau)$ is sustainable for $a < \overline{a}(h_\tau)$. So, because the joint gain to principal and agent can always be redistributed by choice of $\underline{w}(h_t)$ for $t \geq \tau$, achieving the Pareto frontier conditional on all $a \in A(h_\tau)$ choosing the same effort at t requires $e_t(a)$ at the highest sustainable level for $\overline{a}(h_\tau)$. By Proposition 5 in Malcomson (2016), this highest sustainable is stationary and given by $\hat{e}(\overline{a}(h_\tau))$ defined in (12). That condition (14) holds for any such continuation equilibrium follows from the difference in the cost of effort between $\overline{a}(h_\tau)$ and $a < \overline{a}(h_\tau)$. The remaining properties specified in the proposition follow from Lemma 1 in Malcomson (2016). ∎

Proof of Proposition 4   A perfect Bayesian (continuation) equilibrium at t must satisfy individual rationality (2), incentive compatibility (7) and budget balance (6) for all agent types $a \in (\underline{a}(h_t), \overline{a}(h_t)]$ and individual rationality (3) for the principal. Suppose such an equilibrium has $U_t(\overline{a}(h_t)) > \underline{w}(h_t) - \underline{u}$. Consider an alternative with effort and payments for $\tau \geq t+1$ and bonus $w(h_t \cup \underline{w}(h_t) \cup e_t(\overline{a}(h_t))) - \underline{w}(h_t)$ unchanged for all $a \in (\underline{a}(h_t), \overline{a}(h_t)]$, which implies that $S_{t+1}(a)$, $P_t(a)$ and $C_{t+1}(a)$ are also unchanged for all $a \in (\underline{a}(h_t), \overline{a}(h_t)]$, but increases effort at t to $\check{e}(a)$ that increases the cost of effort by the same amount $\Delta c = \left[U_t(\overline{a}(h_t)) - \underline{w}(h_t) + \underline{u}\right] / \overline{a}(h_t)$ for all $a \in (\underline{a}(h_t), \overline{a}(h_t)]$ and increases $\underline{w}(h_t)$ by $\overline{a}(h_t) \Delta c$. Then, from (6),

$$\Delta U_t(a) = [\overline{a}(h_t) - a]\Delta c \geq 0, \forall a \in (\underline{a}(h_t), \overline{a}(h_t)],$$

ensuring that individual rationality (2) remains satisfied. Moreover, the left-hand side of (9) changes by $[a + (a' - a)]\Delta c = a'\Delta c$ and the right-hand side by $[\overline{a}(h_t) + a' - \overline{a}(h_t)]\Delta c = a'\Delta c$, so (9) remains satisfied for all $a \in (\underline{a}(h_t), \overline{a}(h_t)]$. Furthermore, the changes in the cost of effort in the expressions on either side of the inequalities in (7) are all $[\overline{a}(h_t) - a]\Delta c$, so incentive compatibility for all agent types $a \in (\underline{a}(h_t), \overline{a}(h_t)]$ remains satisfied. Furthermore, from (4) and that $e = c^{-1}(c(e))$, the change in the joint payoff gain is given by

$$\Delta S_t(a) = \left[c^{-1}(c(e_t(a)) + \Delta c) - c^{-1}(c(e_t(a))) - a\right]\Delta c$$

$$> \left[\frac{1}{c'(\check{e}(a))} - a\right]\Delta c, \forall a \in (\underline{a}(h_t), \overline{a}(h_t)],$$

where $\check{e}(a) = c^{-1}(c(e_t(a)) + \Delta c)$ and the inequality follows from $c(\cdot)$ strictly increasing and strictly convex. This is positive for $\check{e}(a)$ below first-best. The change in the

principal's payoff gain at stage 1 of period t for given a is thus

$$\Delta S_t(a) - \Delta U_t(a) > \left[\frac{1}{c'(\check{e}(a))} - \bar{a}(h_t)\right]\Delta c, \forall a \in \left(\underline{a}(h_t), \bar{a}(h_t)\right].$$

So the expected payoff gain to the principal over all $a \in \left(\underline{a}(h_t), \bar{a}(h_t)\right]$ is positive if (15) is satisfied for $e(a) = \check{e}(a)$. For each a, $c(\check{e}(a))$, and thus $c'(\check{e}(a))$, is bounded above by equality in (9) for $a' = \bar{a}(h_t)$ with $\delta S_{t+1}(a) = ac(\hat{e}(a))$ (by Proposition 3, the highest value it can take in equilibrium), $P_t(a) = 0$, $U_t(a') = \underline{w}(h_t) - \underline{u}$ and $C_t(a') = C_t(\bar{a}(h_t)) + \Delta c$. This gives the upper bound on $\check{e}(a)$ of $e(a)$ given by (16) which, because $c(e)$ is strictly convex, gives an upper bound on $c'(\check{e}(a))$ and hence a lower bound on $\Delta S_t(a) - \Delta U_t(a)$. Finally, because $c(e)$ is strictly convex and, in any perfect Bayesian (continuation) equilibrium, $e(a)$ has to be non-increasing, (15) is certainly satisfied if (17) is. ∎

Proof of Proposition 5 For the proof, recall that, from Lemma 1, $ac(\hat{e}(a))$ is strictly decreasing in a for all $a \in \left(\underline{a}, \bar{a}\right)$.

Part 1. From Lemma 3, to have any partitioning at $t = 1$ requires there to exist $a^2 \in (\underline{a}, \bar{a}]$ that satisfies the condition for $a^2(h_t)$ in (A.14). For $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \leq \frac{\delta}{1-\delta}a^1c\left(\hat{e}\left(a^1\right)\right)$, there exists no such a^2 for any $k \geq 1$, thus establishing Part 1 of the proposition.

Part 2. For $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \in \left(\frac{\delta}{1-\delta}a^1c\left(\hat{e}\left(a^1\right)\right), \left(\frac{\delta}{1-\delta}\right)^2 a^1c\left(\hat{e}\left(a^1\right)\right)\right]$, there exists $a^2 \in (\underline{a}, \bar{a}]$ that satisfies the condition for $a^2(h_t)$ in (A.14) in Lemma 3 with $k = 1$ but no $a^3 \in (\underline{a}, \bar{a}]$ that then satisfies the condition for $a^3(h_t)$ in (A.15) for any $k \geq 1$. So, conditional on there being no further partitioning after $t = 1$, partitioning $(\underline{a}, \bar{a}]$ into two sub-intervals at a^2 satisfying $a^2c(\hat{e}(a^2)) = \frac{\delta}{1-\delta}a^1c\left(\hat{e}\left(a^1\right)\right)$ is a finest constrained partition equilibrium. With $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \leq \left(\frac{\delta}{1-\delta}\right)^2 a^1c\left(\hat{e}\left(a^1\right)\right)$, it is also the case that $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) \leq \frac{\delta}{1-\delta}a^2c(\hat{e}(a^2))$ so, with $a^1(h_2) = a^2$, no further partition of $\left(\underline{a}, a^2\right]$ at $t = 2$ can satisfy the condition for $a^2(h_2)$ in (A.14) in Lemma 3. Then, by Part 2 of Lemma 2, $\delta S_2(a^2) = a^2c(e(a^2))$ and thus, by Part 3 of Lemma 2, no further partitioning of $\left(a^2, a^1\right]$ is possible either. So there is no further partitioning of either $\left(a^2, a^1\right]$ or $\left(\underline{a}, a^2\right]$ after $t = 2$. Note also that $k > 1$ cannot be a finest partition equilibrium because, by (A.14) in Lemma 3, this would imply a higher $a^2c(\hat{e}(a^2))$ at $t = 1$ and there would be no possibility of further partition at $t = 2$.

Part 3. For $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right) > \frac{\delta}{1-\delta}\left(\underline{u} + \underline{v}\right) + \left(\frac{\delta}{1-\delta}\right)^2 a^1c\left(\hat{e}\left(a^1\right)\right)$, suppose first partitioning only at $t = 1$ is a finest constrained partition equilibrium. Then k in Lemma 3 equals 1 and, by (A.14), (A.15) and (A.16),

$$a^2c(\hat{e}(a)) = \frac{\delta}{1-\delta}a^1c\left(\hat{e}\left(a^1\right)\right),$$

$$a^3c\left(\hat{e}\left(a^3\right)\right) = \min\left\{\frac{1}{1-\delta}a^2c\left(\hat{e}\left(a^2\right)\right), \underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)\right\}.$$

Now consider a further partition at $t = 2$ of $\left(\underline{a}(h_2), a^1(h_2)\right] = (a^3, a^2]$ conditional on no

partition after $t = 2$, that is, with $k = 1$ at $t = 2$. By (A.14) with $a^1(h_2) = a^2$,

$$a^2(h_2)c\left(\hat{e}\left(a^2(h_2)\right)\right) = \frac{\delta}{1-\delta}a^2c\left(\hat{e}\left(a^2\right)\right) = \left(\frac{\delta}{1-\delta}\right)^2 a^1 c\left(\hat{e}\left(a^1\right)\right) < a^3 c\left(\hat{e}\left(a^3\right)\right),$$

so partition of $(a^3, a^2]$ at $t = 2$ is possible. Moreover, if $\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)$ is sufficiently large that partitioning at $t = 1$ is into more than two sub-intervals, each sub-interval $\left(a^{i+1}, a^i\right]$ for $i = 2, \ldots, n(h_1) - 1$ can be further partitioned in this way at $t = 2$. These contradict $k = 1$ at $t = 1$. Thus $k = 1$ cannot be a finest constrained partition equilibrium.

Suppose now $k = 2$. Then, from Lemma 3,

$$a^2(h_1)c\left(\hat{e}\left(a^2(h_1)\right)\right) = \underline{u} + \underline{v} + \frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right)$$

and $a^i(h_1)c\left(\hat{e}\left(a^i(h_1)\right)\right)$ for $i > 2$ is given by (A.15) and (A.16). Consider a further partition at $t = 2$ of $\left(\underline{a}(h_2), a^1(h_2)\right] = \left(a^{i+1}(h_1), a^i(h_1)\right]$ for $i \in \{1, \ldots, n(h_1) - 1\}$ conditional on no partitioning after $t = 2$, that is, with $k = 1$ at $t = 2$. By (A.14) with $a^1(h_2) = a^i(h_1)$,

$$\begin{aligned} a^2(h_2)c\left(\hat{e}\left(a^2(h_2)\right)\right) &= \frac{\delta}{1-\delta}a^i(h_1)c\left(\hat{e}\left(a^i(h_1)\right)\right) \\ &< \frac{1}{1-\delta}a^i(h_1)c\left(\hat{e}\left(a^i(h_1)\right)\right) = a^{i+1}(h_1)c\left(\hat{e}\left(a^{i+1}(h_1)\right)\right), \forall i \in \{1, \ldots, n(h_1) - 1\}, \end{aligned}$$

so partition of $\left(a^{i+1}(h_1), a^i(h_1)\right]$ at $t = 2$ is possible. But, as claimed in Part 3 of the proposition, this partition at $t = 2$ cannot be into more than two sub-intervals by Lemma 4. Condition (22) follows from (A.17) in Lemma 3 with $k = 2$ and (25) follows from (A.17) with $k = 1$ because at $t = 2$ there is only one period of further partition and, when there is partitioning in period t, $e_t\left(a^1(h_t)\right) = c\left(e_t\left(a^1(h_t)\right)\right) = 0$. Moreover, by Lemma 5, no further partitioning can take place after $t = 2$. This establishes that, conditional on $k = 2$, the outcome of partitioning specified in Part 3 of the proposition satisfies the conditions for a finest constrained partition equilibrium provided no finer partitioning at $t = 1$ can be achieved by choosing $k > 2$.

Consider next the possibility of an equilibrium with more than two periods of partitioning at $t = 1$, that is $k > 2$. By Parts 2 and 3 of Lemma 2, there can be no further partitioning of $\left(a^2(h_t), a^1(h_1)\right]$ unless there is also further partitioning of $(a^3(h_t), a^2(h_2)]$, so (A.14) in Lemma 3 applies with k corresponding to the number of periods of partitioning. From (A.14),

$$a^2(h_1)c\left(\hat{e}\left(a^2(h_1)\right)\right) = \left(\underline{u} + \underline{v}\right)\sum_{j=2}^{k}\delta^{j-k} + \frac{\delta}{1-\delta}a^1 c\left(\hat{e}\left(a^1\right)\right).$$

This gives $a^2(h_1)c\left(\hat{e}\left(a^2(h_1)\right)\right)$ strictly greater than $a^3 c\left(\hat{e}\left(a^3\right)\right)$ as specified in (19) in the proposition, and thus $a^2(h_1)$ strictly less than a^3, and so further from a^1. Moreover, in view of (20) and (A.15), the same would apply to all other partition points at $t = 1$. So

$k = 2$ at $t = 1$ gives a finer partition than $k > 2$.

Finally, consider the number of partition sub-intervals for $k = 2$. Condition (20) implies

$$a^{i+2}c\left(\hat{e}\left(a^{i+2}\right)\right) - a^i c\left(\hat{e}\left(a^i\right)\right) = \frac{1}{1-\delta}a^i c\left(\hat{e}\left(a^i\right)\right), \text{ for } i = 3, 5, \ldots, n-2. \qquad \text{(A.28)}$$

By Lemma 1, $ac\left(\hat{e}\left(a\right)\right)$ is increasing in a, so $a^{i+2}c\left(\hat{e}\left(a^{i+2}\right)\right) > a^i c\left(\hat{e}\left(a^i\right)\right)$ by an amount bounded away from zero and increasing in i. With $ac\left(\hat{e}\left(a\right)\right)$ bounded above by $\underline{a}c\left(e^*\left(\underline{a}\right)\right)$ which is finite, that is sufficient to ensure a finite number of sub-intervals to the partition. ∎

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A.3 Workings for examples

A.3.1 Derivation of Equation (29)

For the effort function $c(\tilde{e}) = b\tilde{e}^{\beta}/\beta$ and $\hat{e}(\underline{a}) = (r\underline{a}b)^{\frac{1}{1-\beta}}$, (28) becomes

$$\delta(r\underline{a}b)^{\frac{1}{1-\beta}} - \frac{ab}{\beta}\left[(r\underline{a}b)^{\frac{1}{1-\beta}}\right]^{\beta} - \delta(\underline{u}+\underline{v}) = 0$$

or

$$\delta r^{\frac{1}{1-\beta}}(\underline{a}b)^{\frac{1}{1-\beta}} - \frac{1}{\beta}r^{\frac{\beta}{1-\beta}}(\underline{a}b)^{\frac{1}{1-\beta}} - \delta(\underline{u}+\underline{v}) = 0$$

or

$$r^{\frac{1}{1-\beta}}(\delta - \frac{1}{\beta}r^{\frac{\beta-1}{1-\beta}})(\underline{a}b)^{\frac{1}{1-\beta}} = \delta(\underline{u}+\underline{v})$$

or

$$(\underline{a}b)^{\frac{1}{1-\beta}} = \frac{\delta(\underline{u}+\underline{v})}{\delta - \frac{1}{\beta r}}r^{-\frac{1}{1-\beta}} \tag{A.29}$$

or

$$\underline{a}b = \frac{1}{r}\left(\frac{\underline{u}+\underline{v}}{1 - \frac{1}{\delta\beta r}}\right)^{1-\beta}.$$

For $\hat{e}(\underline{a}) = (r\underline{a}b)^{\frac{1}{1-\beta}}$,

$$\underline{a}c(\hat{e}(\underline{a})) = \underline{a}b(r\underline{a}b)^{\frac{\beta}{1-\beta}}/\beta = \frac{1}{\beta}r^{\frac{\beta}{1-\beta}}(\underline{a}b)^{\frac{1}{1-\beta}}.$$

Use of (A.29) in these gives

$$\hat{e}(\underline{a}) = \frac{\underline{u}+\underline{v}}{1 - \frac{1}{\delta\beta r}}$$

and

$$\underline{a}c(\hat{e}(\underline{a})) = \frac{1}{\beta}r^{\frac{\beta-1}{1-\beta}}\left(\frac{\underline{u}+\underline{v}}{1 - \frac{1}{\delta\beta r}}\right),$$

which simplifies to (29).

A.3.2 Derivation of Equation (30)

The $\hat{e}(a)$ that maximises the left-hand side of (28) for given a is given by

$$\delta - ab\hat{e}(a)^{\beta-1} = 0$$

or

$$\hat{e}(a) = \left(\frac{\delta}{ab}\right)^{\frac{1}{\beta-1}} = \left(\frac{ab}{\delta}\right)^{\frac{1}{1-\beta}}. \tag{A.30}$$

Used in (28), this defines $\bar{a}$ by

$$\delta\left(\frac{\bar{a}b}{\delta}\right)^{\frac{1}{1-\beta}} - \frac{\bar{a}b}{\beta}\left[\left(\frac{\bar{a}b}{\delta}\right)^{\frac{1}{1-\beta}}\right]^{\beta} - \delta(\underline{u} + \underline{v}) = 0$$

or

$$\delta\left(\frac{\bar{a}b}{\delta}\right)^{\frac{1}{1-\beta}} - \frac{1}{\beta}\delta\left(\frac{\bar{a}b}{\delta}\right)^{1+\frac{\beta}{1-\beta}} - \delta(\underline{u} + \underline{v}) = 0$$

or, dividing by δ,

$$\left(1 - \frac{1}{\beta}\right)\left(\frac{\bar{a}b}{\delta}\right)^{\frac{1}{1-\beta}} = (\underline{u} + \underline{v})$$

or

$$\left(\frac{\bar{a}b}{\delta}\right)^{\frac{1}{1-\beta}} = \frac{\beta(\underline{u} + \underline{v})}{\beta - 1}$$

or

$$\bar{a}b = \delta\left[\frac{\beta(\underline{u} + \underline{v})}{\beta - 1}\right]^{1-\beta}.$$

Used in (A.30), this gives

$$\hat{e}(\bar{a}) = \frac{\beta(\underline{u} + \underline{v})}{\beta - 1}$$

and so

$$\bar{a}c(\hat{e}(\bar{a})) = \frac{\delta}{b}\left[\frac{\beta(\underline{u} + \underline{v})}{\beta - 1}\right]^{1-\beta}\frac{b}{\beta}\left[\frac{\beta(\underline{u} + \underline{v})}{\beta - 1}\right]^{\beta},$$

which simplifies to (30).

A.3.3 Derivation of Equation (34)

For $k = 1$, (32) becomes

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} = \frac{\beta - 1}{(r\delta\beta - 1)\max\{1, (\beta - 1)/\delta\}}.$$

The first term in the max in the denominator is larger for $\beta \leq 1+\delta$, the second for higher β. Thus, for $\beta > 1 + \delta$, the condition (33) for partitioning to be possible is satisfied for

$$\beta < \frac{2 - \delta}{r\delta}, \text{ for } \beta > 1 + \delta,$$

which applies for

$$\delta < \frac{-(1 + r) + \sqrt{r^2 + 10r + 1}}{2r}.$$

For $\beta \leq 1 + \delta$, (33) is satisfied for

$$\beta < \frac{2\delta - 1}{r\delta^2 + \delta - 1}, \text{ for } \beta \leq 1 + \delta,$$

which applies for

$$\delta \geq \frac{-(1+r) + \sqrt{r^2 + 10r + 1}}{2r}.$$

Equation (34) combines these conditions.

A.3.4 Derivation of Equation (36)

For $k = 2$, (32) becomes

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} = \frac{\beta - 1}{(r\delta\beta - 1)\max\left\{1, (\beta - 1)\left(\delta^{-1} + \delta^{-2}\right)\right\}}$$

and, from (31),

$$\frac{\delta}{1 - \delta}\left(\frac{\underline{u} + \underline{v}}{a^1 c\left(\hat{e}\left(a^1\right)\right)} + \frac{\delta}{1 - \delta}\right) = \frac{\delta}{1 - \delta}\left(\frac{\beta - 1}{\delta\max\left\{1, (\beta - 1)\left(\delta^{-1} + \delta^{-2}\right)\right\}} + \frac{\delta}{1 - \delta}\right).$$

The first term in the max in the denominators of these applies for $\beta \leq 1 + \frac{\delta^2}{1+\delta} = (1 + \delta + \delta^2)/(1 + \delta)$, the second for higher β. It must also be that $\beta > \frac{1}{r\delta}$.

For $\beta > (1 + \delta + \delta^2)/(1 + \delta)$, the condition (35) for more than two sub-intervals at $t = 1$ is thus

$$\frac{\underline{a}c\left(\hat{e}\left(\underline{a}\right)\right)}{a^1 c\left(\hat{e}\left(a^1\right)\right)} = \frac{1}{(r\delta\beta - 1)\left(\delta^{-1} + \delta^{-2}\right)} > \frac{\delta}{1 - \delta}\left(\frac{1}{\delta\left(\delta^{-1} + \delta^{-2}\right)} + \frac{\delta}{1 - \delta}\right)$$

or

$$\frac{1}{(r\delta\beta - 1)\left(\delta^{-1} + \delta^{-2}\right)} > \frac{\delta}{1 - \delta}\left(\frac{1 - \delta + \delta^2\left(\delta^{-1} + \delta^{-2}\right)}{\delta(1 - \delta)\left(\delta^{-1} + \delta^{-2}\right)}\right)$$

or

$$\frac{1}{r\delta\beta - 1} > \frac{\delta}{1 - \delta}\left(\frac{1 - \delta + \delta + 1}{\delta(1 - \delta)}\right)$$

or

$$(1 - \delta)^2 > 2(r\delta\beta - 1)$$

or

$$\beta < \frac{1}{r\delta}\left[1 + \frac{(1 - \delta)^2}{2}\right], \text{ for } \beta > \frac{1 + \delta + \delta^2}{1 + \delta}.$$

Equality in this condition always satisfies $\beta > 1/(r\delta)$. So the right-hand side is an upper bound on β if and only if

$$\frac{1}{r\delta}\left[1 + \frac{(1 - \delta)^2}{2}\right] > \frac{1 + \delta + \delta^2}{1 + \delta}$$

or

$$r < \frac{1 + \delta}{\delta + \delta^2 + \delta^3}\left[1 + \frac{(1 - \delta)^2}{2}\right].$$

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This gives the condition on β for multiple sub-intervals in a finest partition equilibrium that

$$\beta < \frac{1}{r\delta}\left[1 + \frac{(1-\delta)^2}{2}\right], \quad \text{for } r < \frac{1+\delta}{\delta + \delta^2 + \delta^3}\left[1 + \frac{(1-\delta)^2}{2}\right]. \tag{A.31}$$

For $\beta \le (1 + \delta + \delta^2)/(1 + \delta)$, the condition (35) for more than two sub-intervals at $t = 1$ is

$$\frac{\beta - 1}{r\delta\beta - 1} > \frac{\beta - 1}{1 - \delta} + \left(\frac{\delta}{1-\delta}\right)^2$$

or, multiplying through by $1 - \delta$ and $r\delta\beta - 1$ (both strictly positive),

$$(1 - \delta)(\beta - 1) > (r\delta\beta - 1)\left(\beta - 1 + \frac{\delta^2}{1-\delta}\right)$$

or

$$\beta(1 - \delta) - (1 - \delta) > r\delta\beta^2 + r\delta\beta\left(-1 + \frac{\delta^2}{1-\delta}\right) - \beta + 1 - \frac{\delta^2}{1-\delta}$$

or

$$r\delta\beta^2 + \beta\left(-r\delta + r\delta\frac{\delta^2}{1-\delta} - 1 - 1 + \delta\right) + 1 - \frac{\delta^2}{1-\delta} + 1 - \delta < 0$$

or

$$r\delta\beta^2 - \beta\left[r\delta\left(1 - \frac{\delta^2}{1-\delta}\right) + 2 - \delta\right] + 2 - \delta - \frac{\delta^2}{1-\delta} < 0$$

or, because $\beta\delta r > 1$ is also required,

$$r\delta\beta^2 - \beta\left[r\delta\left(1 - \frac{\delta^2}{1-\delta}\right) + 2 - \delta\right] - \frac{3\delta - 2}{1-\delta} < 0, \quad \text{for } \beta \in \left(\frac{1}{r\delta}, \frac{1 + \delta + \delta^2}{1 + \delta}\right). \tag{A.32}$$

The derivative of the left-hand side with respect to β is

$$2r\delta\beta - \left[r\delta\left(1 - \frac{\delta^2}{1-\delta}\right) + 2 - \delta\right],$$

which is positive for

$$\beta > \frac{1}{2}\left(1 - \frac{\delta^2}{1-\delta}\right) + \frac{2-\delta}{2r\delta}.$$

Because it is required that $\beta > 1/(r\delta)$, this condition will certainly be satisfied if

$$\frac{1}{r\delta} > \frac{1}{2}\left(1 - \frac{\delta^2}{1-\delta}\right) + \frac{2-\delta}{2r\delta}$$

or

$$\frac{1}{r\delta} > \frac{1}{2}\left(1 - \frac{\delta^2}{1-\delta}\right) + \frac{1}{r\delta} - \frac{1}{2r}$$

or, cancelling terms and multiplying through by 2,

$$0 > \left(1 - \frac{\delta^2}{1-\delta}\right) - \frac{1}{r}$$

or

$$\frac{1}{r} > 1 - \frac{\delta^2}{1 - \delta} = \frac{1 - \delta - \delta^2}{1 - \delta}.$$

But the right-hand side of this is negative for $\delta \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$, so the derivative of the left-hand side of (A.32) with respect to β is certainly positive for δ within this range. Thus the upper bound β^* on β implied by (A.32) is the smaller of the highest β that makes the left-hand side equal to zero and $(1 + \delta + \delta^2) / (1 + \delta)$, so

$$\beta^* = \min\left\{\frac{1}{2r\delta}\left[r\delta\left(1 - \frac{\delta^2}{1 - \delta}\right) + 2 - \delta + \sqrt{\left[r\delta\left(1 - \frac{\delta^2}{1 - \delta}\right) + 2 - \delta\right]^2 + 4r\delta\frac{3\delta - 2}{1 - \delta}},\right.$$

$$\left.\frac{1 + \delta + \delta^2}{1 + \delta}\right\}, \quad \text{if } \beta^* > \frac{1}{r\delta}. \quad \text{(A.33)}$$

The upper bound for β for given δ is obviously lower when $\beta \leq 1 + \frac{\delta^2}{1+\delta} = (1 + \delta + \delta^2) / (1 + \delta)$ than when the inequality is reversed, so (A.31) gives the relevant upper bound whenever the condition on r in it is satisfied and (A.33) gives the relevant upper bound whenever that condition on r is not satisfied. Hence the values of β that permit partition at $t = 1$ into more than two sub-intervals are as specified in (36).

A.3.5 Workings for Table 1

From (31) and (32), (37) requires

$$\frac{\beta - 1}{(r\delta\beta - 1)\max\left\{1, (\beta - 1)\sum_{j=1}^{k}\delta^{j-k-1}\right\}} > \left(\frac{\beta - 1}{\delta\max\left\{1, \sum_{j=1}^{k}\delta^{j-k-1}\right\}} + \frac{\delta}{1 - \delta}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}.$$

For $k = 2$ and $r = 1$, this can be written

$$\frac{\beta - 1}{(\delta\beta - 1)\max\left\{1, (\beta - 1)(\delta^{-1} + \delta^{-2})\right\}}$$

$$> \left(\frac{\beta - 1}{\delta\max\left\{1, (\beta - 1)(\delta^{-1} + \delta^{-2})\right\}} + \frac{\delta}{1 - \delta}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}. \quad \text{(A.34)}$$

The first term in the max in the denominators of these applies for $\beta \leq 1 + \frac{\delta^2}{1+\delta} = (1 + \delta + \delta^2) / (1 + \delta)$, the second for higher β. It must also be that $\beta > \frac{1}{\delta}$. Note that

$$1 + \frac{\delta^2}{1 + \delta} \gtreqless \frac{1}{\delta} \iff \delta(1 + \delta) + \delta^3 \gtreqless 1 + \delta \iff \delta^3 + \delta^2 - 1 \gtreqless 0 \iff \delta \gtreqless \delta^* \approx 0.75488.$$

Thus $\beta \leq 1 + \frac{\delta^2}{1+\delta} = (1 + \delta + \delta^2) / (1 + \delta)$ is possible only for $\delta \geq \delta^* \approx 0.75488$.

For $\beta \leq (1 + \delta + \delta^2)/(1 + \delta)$, (A.34) therefore reduces to

$$\frac{\beta - 1}{\delta\beta - 1} > \left(\frac{\beta - 1}{\delta} + \frac{\delta}{1 - \delta}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}.$$

Multiplication by δ and $\delta\beta - 1$ (both positive) gives

$$\delta\beta - 1 + 1 - \delta > (\delta\beta - 1)\left(\beta - 1 + \frac{\delta^2}{1 - \delta}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}.$$

Multiplication by $(1 - \delta)^{m-2}$ then gives

$$(\delta\beta - 1)(1 - \delta)^{m-2} + (1 - \delta)^{m-1} > (\delta\beta - 1)(\beta - 1) + (\delta\beta - 1)\frac{\delta^2}{1 - \delta}$$

or, re-arranging,

$$(\delta\beta - 1)(\beta - 1) + (\delta\beta - 1)\left[\frac{\delta^2}{1 - \delta} - (1 - \delta)^{m-2}\right] - (1 - \delta)^{m-1} < 0$$

or

$$\delta\beta^2 - \beta - \delta\beta + 1 + \delta\beta\left[\frac{\delta^2}{1 - \delta} - (1 - \delta)^{m-2}\right] - \left[\frac{\delta^2}{1 - \delta} - (1 - \delta)^{m-2}\right] - (1 - \delta)^{m-1} < 0$$

or

$$\delta\beta^2 - \beta\left\{1 + \delta - \delta\left[\frac{\delta^2}{1 - \delta} - (1 - \delta)^{m-2}\right]\right\} + \left\{1 - \frac{\delta^2}{1 - \delta} + (1 - \delta)^{m-2} - (1 - \delta)^{m-1}\right\} < 0$$

or

$$\delta\beta^2 - \beta\left\{1 + \delta\left[1 - \frac{\delta^2}{1 - \delta} + (1 - \delta)^{m-2}\right]\right\} + \left\{1 - \frac{\delta^2}{1 - \delta} + (1 - \delta)^{m-2}[1 - (1 - \delta)]\right\} < 0$$

or

$$\delta\beta^2 - \beta\left\{1 + \delta\left[1 - \frac{\delta^2}{1 - \delta} + (1 - \delta)^{m-2}\right]\right\} + \left\{1 - \frac{\delta^2}{1 - \delta} + \delta(1 - \delta)^{m-2}\right\} < 0. \qquad (A.35)$$

For notational convenience, let

$$x = 1 + \delta\left[1 - \frac{\delta^2}{1 - \delta} + (1 - \delta)^{m-2}\right].$$

Then (A.35) can be written

$$\delta\beta^2 - x\beta + x - 1 + (1 - \delta)\left(1 - \frac{\delta^2}{1 - \delta}\right) < 0$$

or

$$\delta\beta^2 - x\beta + (x - \delta - \delta^2) < 0. \qquad (A.36)$$

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The left-hand side of this equals zero for $\delta > 0$ and

$$\beta^* = \frac{x \pm \sqrt{x^2 - 4\delta(x - \delta - \delta^2)}}{2\delta} = \frac{x \pm \sqrt{x^2 - 4\delta x + 4\delta^2 + 4\delta^3}}{2\delta}$$

$$= \frac{x \pm \sqrt{(2\delta - x)^2 + 4\delta^3}}{2\delta},$$

from which it follows that, with $\delta > 0$, the square root term is necessarily real. The derivative of the left-hand side of (A.36) with respect to β is $2\delta\beta - x$, which is positive for $\beta = \beta^*$ with the positive square root term and negative for β^* with the negative square root term. So, with the positive square root term, $\beta < \beta^*$ is required to satisfy (A.36) and, with the negative square root term, $\beta > \beta^*$ is required to satisfy (A.36). For this case to apply, therefore, β must satisfy

$$\max\left\{\frac{1}{\delta}, \frac{x - \sqrt{(2\delta - x)^2 + 4\delta^3}}{2\delta}\right\} < \beta < \min\left\{\frac{x + \sqrt{(2\delta - x)^2 + 4\delta^3}}{2\delta}, \frac{1 + \delta + \delta^2}{1 + \delta}\right\}.$$

For $\beta > (1 + \delta + \delta^2)/(1 + \delta)$, (A.34) corresponds to

$$\frac{1}{(\delta\beta - 1)(\delta^{-1} + \delta^{-2})} > \left(\frac{1}{\delta(\delta^{-1} + \delta^{-2})} + \frac{\delta}{1 - \delta}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}$$

or, putting the terms on the right-hand side over a common denominator,

$$\frac{1}{(\delta\beta - 1)(\delta^{-1} + \delta^{-2})} > \left(\frac{1 - \delta + \delta^2(\delta^{-1} + \delta^{-2})}{\delta(1 - \delta)(\delta^{-1} + \delta^{-2})}\right)\left(\frac{1}{1 - \delta}\right)^{m-2}$$

or

$$\delta(1 - \delta) > (\delta\beta - 1)(1 - \delta + \delta + 1)\left(\frac{1}{1 - \delta}\right)^{m-2}$$

or

$$\delta(1 - \delta)^{m-1} > 2(\delta\beta - 1)$$

or

$$\beta < \frac{1}{\delta} + \frac{1}{2}(1 - \delta)^{m-1}.$$

For this case to apply, therefore, β must satisfy

$$\max\left\{\frac{1}{\delta}, \frac{1 + \delta + \delta^2}{1 + \delta}\right\} < \beta < \frac{1}{\delta} + \frac{1}{2}(1 - \delta)^{m-1}.$$

Such a β certainly exists for all $\delta < \delta^* \approx 0.75488$ because then the relevant term in the max is $1/\delta$.

For $\delta < \delta^* \approx 0.75488$ then, the upper bound on β is

$$\frac{1}{\delta} + \frac{1}{2}(1 - \delta)^{m-1}.$$

For $\delta = 0.7$, this gives:

for $m = 3$, $\beta < 1.4736$; for $m = 4$, $\beta < 1.4421$; for $m = 5$, $\beta < 1.4326$; for $m = 6$, $\beta < 1.4298$, as given in Table 1.

For $\delta \geq \delta^* \approx 0.75488$, the upper bound on β is

$$\frac{x + \sqrt{(2\delta - x)^2 + 4\delta^3}}{2\delta},$$

where $x = 1 + \delta\left[1 - \frac{\delta^2}{1-\delta} + (1 - \delta)^{m-2}\right]$.

For $\delta = 0.8$, this gives:

for $m = 3$, $\beta < 1.2653$; for $m = 4$, $\beta < 1.2529$; for $m = 5$, $\beta < 1.2506$; for $m = 6$, $\beta < 1.2501$;

For $\delta = 0.9$, it gives:

for $m = 3$, $\beta < 1.1125$; for $m = 4$, $\beta < 1.1112$; for $m \approx 5$, $\beta < 1.1111$; for $m = 6$, $\beta < 1.1111$, as given in Table 1.

The final line of Table 1 gives the ratio of $\hat{e}\left(\underline{a}\right)/\hat{e}\left(\overline{a}\right)$ for the value of δ in the column and the greatest lower bound for β for any partition to take place. From above,

$$\frac{\hat{e}\left(\underline{a}\right)}{\hat{e}\left(\overline{a}\right)} = \frac{\underline{u} + \underline{v}}{1 - \frac{1}{\delta\beta r}} \frac{\beta - 1}{\beta(\underline{u} + \underline{v})} = \frac{\beta - 1}{\beta - \frac{1}{\delta r}}.$$

For Table 1, $r = 1$. Then:

for $\delta = 0.7$, $\frac{\hat{e}\left(\underline{a}\right)}{\hat{e}\left(\overline{a}\right)} = \frac{1.4929 - 1}{1.4929 - \frac{1}{0.7}} = 7.6622$;

for $\delta = 0.8$, $\frac{\hat{e}\left(\underline{a}\right)}{\hat{e}\left(\overline{a}\right)} = \frac{1.2694 - 1}{1.2694 - \frac{1}{0.8}} = 13.887$;

for $\delta = 0.9$, $\frac{\hat{e}\left(\underline{a}\right)}{\hat{e}\left(\overline{a}\right)} = \frac{1.1126 - 1}{1.1126 - \frac{1}{0.9}} = 75.627$.