

The effect of atomic response time in the theory of Doppler cooling of trapped ions

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ABSTRACT

We describe a simple approach to the problem of incorporating the response time of an atom or ion being Doppler-cooled into the theory of the cooling process. The system being cooled does not in general respond instantly to the changing laser frequencies it experiences in its rest frame, and this “dynamic effect” can affect significantly the temperatures attainable. It is particularly important for trapped ions when there is a slow decay out of the cooling cycle requiring the use of a repumping beam. We treat the cases of trapped ions with two and three internal states, then apply the theory to $^{40}\text{Ca}^+$. For this ion experimental data exist showing the ion to be cold under conditions for which heating is predicted if the dynamic effect is neglected. The present theory accounts for the observed behaviour.

KEYWORDS

Laser cooling, Optical Bloch equations, ion trap

Doppler cooling exploits the fact that an atom or ion counter-propagating with a laser beam and absorbing photons from it is subject to a retarding force [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The simplest practical system is that of an ion, trapped in a harmonic potential, illuminated by a single laser beam detuned to the red from a resonance transition [12, 13, 14, 15]. The ion is more likely to absorb photons when travelling towards the source than when moving in the opposite sense, so that averaged over a cycle of the motion the momentum of the ion is reduced. An equilibrium temperature is reached when the cooling is balanced by the heating effects associated with spontaneous emission and the stochastic nature of the absorption process.

The simplest assumption is that the internal states of the system being cooled remain in equilibrium with the radiation field, i.e., that the system can respond instantaneously to the changing Doppler shifts of the laser beams. We shall refer to this as the quasi-static approximation. The spontaneous transition rate on the resonance transition is often fast enough for this approximation to be justified. However, in some ions there is a significant time-lag in the atomic response which can significantly influence the outcome of the cooling process. We refer to this phenomenon as the “dynamic effect”. It can occur in any system, but particularly when there is a weak decay from the resonance level to a third level, so that a repumping laser is required to return the ion to the cooling cycle. While it is then generally a good approximation to neglect the momentum exchange with the repumping beam—there are many cooling cycles in

between the unwanted decays—the longer response time associated with the weaker decay makes it necessary to take the dynamic effect into account.

The effect has been demonstrated in experiments on the calcium ions $^{40}\text{Ca}^+$ and $^{43}\text{Ca}^+$ in a Paul trap [16]. The cooling transition, $4s^2S_{1/2} - 4p^2P_{1/2}$ at 397nm, has an Einstein A -coefficient A_{397} of $1.32 \times 10^8 \text{ s}^{-1}$ so that in a trap of frequency of order a few MHz one would not expect serious time-lag effects. However, the decay $4p^2P_{1/2} - 3d^2D_{3/2}$ at 866nm is much weaker, with $A_{866} = 8.4 \times 10^6 \text{ s}^{-1}$. Over a large frequency range of the repumping laser, in which the quasi-static theory suggests that the ion will heat indefinitely, we have observed experimentally that a $^{40}\text{Ca}^+$ ion remains cold. A simple theoretical treatment including the dynamic effect accounts for the observations. In section 1 below we outline such a treatment based on the optical Bloch equations, applying it first to two- and three-state systems (sections 2 and 3), then to the particular case of $^{40}\text{Ca}^+$ (section 4). In each case we give examples where a configuration of lasers which would cause heating of the motion according to quasi-static theory in fact produces cooling, or vice versa, and we explain why.

Our discussion adopts a semi-classical approach in which the ion's motional degree of freedom is treated classically. Consequently the model does not account for details of the quantised motion which are typically important near the ground state of motion or in the Lamb-Dicke limit. The phenomena we shall discuss are well handled to first approximation by our approach. A more fully quantised theory of laser cooling of a trapped ion has been studied extensively in the Lamb-Dicke limit, see for example [14, 17, 15, 18]. These treatments have been useful for studying the approach to steady state at the lowest temperatures, but have not remarked the behaviour discussed here.

In sections 1, 2 and 3 which are entirely theoretical we work in angular frequency units, but for the comparison with experiment in section 4 it is more convenient to use MHz.

1. Theory

To illustrate the principles involved we consider the simplest possible case, an ion which undergoes simple harmonic motion with secular frequency ω_s in an isotropic trap, subject to a laser beam red-detuned from a resonance transition of wavelength λ . Other repumping lasers may also be present. Let the cooling beam be propagating in the positive x -direction. We take the velocity component V_x of the ion to be positive in this direction also. We define the detuning Δ_c of the cooling laser from resonance to be positive if it is towards higher frequency. At first we neglect the dynamic effect. Let $R(V_x)$ be the rate at which resonance photons are absorbed. Then to a sufficient approximation for an ion at mK temperatures being cooled on a resonance transition we can write

$$R(V_x) = R_0 + R'V_x \quad (1)$$

where R_0 is the absorption rate for a stationary ion and R' is the derivative of R with respect to V_x at $V_x = 0$. Then it can be shown that the equilibrium temperature T is given by

$$\frac{k_B T}{h} = -\frac{\alpha}{\lambda} \left[\frac{R_0}{R'} \right] \quad (2)$$

where α is a factor of order unity which depends on the particular system; for our purposes we shall take $\alpha = 1$. With a two-state system, the response curve is a simple Lorentzian, and (2) allows us to find T for any Δ_c . The minimum, T_{\min} , occurs when the transition is unsaturated, and occurs at $\Delta_c = -(A/2)$. If we now consider a more complex system, for example when there is a repumping laser, R' can be found by solving the optical Bloch equations (OBE) representing the ion in two equilibrium situations, one with laser frequencies corresponding to $V_x = 0$, the other with V_x slightly displaced from zero. The OBE (see, for example, [19, 20, 11, 21]) govern the time-dependence of the density matrix describing the ion under specified conditions. The elements can be expressed in terms of real quantities x_{ij} , y_{ij} such that the population of a state $|i\rangle$ is x_{ii} while the coherence between $|i\rangle$ and $|j\rangle$ is $x_{ij} + iy_{ij}$. The equations are then of the form

$$\dot{z}_m = \sum_n c_{mn} z_n \quad (3)$$

where z_n represents any of the x_{ij} or y_{ij} and the coefficients c_{mn} are all real. They contain the information specific to the particular problem, including the laser frequencies, linewidths, polarizations and intensities, the matrix elements for stimulated and spontaneous transitions, and the external magnetic field. For a system in equilibrium they therefore reduce to a set of simultaneous linear equations which can be solved relatively easily even when there are many states.

However, the validity of (1) depends on the neglect of the dynamic effect. In reality, the absorption rate at a given velocity depends on the history of the ion up to that point; for example, at $V_x = 0$ it will depend on whether the velocity is changing from positive to negative or vice-versa. R' as defined above ceases to be a useful parameter.

We therefore adopt a different approach. Let

$$V_x = V_0 \cos \omega_s t. \quad (4)$$

Then to a good approximation the value of R will also oscillate, according to

$$R = R_0 + V_0 (P \cos \omega_s t + Q \sin \omega_s t) \quad (5)$$

Cooling is then due to the in-phase component P . Equation (2) is replaced by

$$\frac{k_B T}{h} = -\frac{\alpha}{\lambda} \left[\frac{R_0}{P} \right] \quad (6)$$

Thus, as $\omega_s \rightarrow 0$, $Q \rightarrow 0$ also, and $P \rightarrow R'$, i.e., as calculated quasi-statically. When the dynamic effect is significant, we calculate P and Q using the time-dependent OBE. However, the assumption of the steady state solution (5) enables us to avoid explicitly integrating over time. The physical meaning of the parameter P is signalled by equations (4)–(6): it expresses the degree to which the influence of the velocity on the absorption rate is in phase with the oscillation of V_x .

We illustrate the method by considering an ion illuminated by two lasers, one on the cooling transition (subscript c), the other a repumper (subscript r). Otherwise the treatment is quite general; the levels involved may comprise any number of states, and the ion may be in a magnetic field. The extension to more (or less) complex systems is straightforward. The time-dependence all arises from the fact that there are oscillating

Doppler shifts $\delta\omega$ superposed on the detunings, given by

$$\delta\omega_c = V_0 k_c \cos \omega_s t \quad (7)$$

$$\delta\omega_r = \pm V_0 k_r \cos \omega_s t \quad (8)$$

where $k_c = 2\pi/\lambda_c$, $k_r = 2\pi/\lambda_r$. We take the positive sign in (8), corresponding to co-propagating lasers, since the experimental data presented in section 4 were obtained with this configuration. Thus, the c_{mn} in (3) are replaced by $C_{mn} + a_{mn} \cos \omega_s t$, where the C_{mn} are time-independent and the a_{mn} are zero except for the terms representing detunings, in which case they are either $V_0 k_c$ or $V_0 k_r$.

We first solve the time-independent equations with the C_{mn} , i.e., we use the mean values of the detunings. Let the solutions for this case be Z_n . Then provided the amplitudes of the variations in the matrix elements are very small compared with unity, we can write

$$z_n = Z_n + V_0 (u_n \cos \omega_s t + v_n \sin \omega_s t) \quad (9)$$

Thus, we assume that the harmonically oscillating detunings produce similarly oscillating density matrix elements, but with amplitudes and phases to be determined. Substituting (7), (8) and (9) into (3) and neglecting second order terms we obtain

$$u_m + \sum_n D_{mn} v_n = 0 \quad (10)$$

$$v_m - \sum_n [D_{mn} u_n + b_{mn} Z_n] = 0 \quad (11)$$

where

$$D_{mn} = C_{mn}/\omega_s \quad (12)$$

$$b_{mn} = a_{mn}/\omega_s. \quad (13)$$

We have in (10, 11) a set of linear equations with constant coefficients, independent of V_0 , which can be solved in the usual way to find the u_n and v_n , and hence P and Q . In the limit $\omega_s \rightarrow 0$, where dynamic effects are negligible, all the $v_n \rightarrow 0$ and the u_n are the same as those given by the quasi-static theory.

We note that both theories rely on the velocity of the ion remaining low enough for the basic equations, (1) for the quasi-static theory and (5) for the dynamic theory, to be good approximations. While this condition is fulfilled for the temperatures of interest here, refinements to the theory are needed to describe the behaviour of systems at elevated temperatures [22, 16].

2. The two-state system

We first apply the analysis to the two-state system: lower state $|1\rangle$, upper state $|2\rangle$, transition wavelength λ_{12} (corresponding to a frequency ω_{12}), cooled by a single red-detuned laser beam of frequency ω_L . Although normally one cools on transitions in which the decay rate is fast enough to render the dynamic effect unimportant, in a tight trap this is not necessarily the case. We specify the system by the following

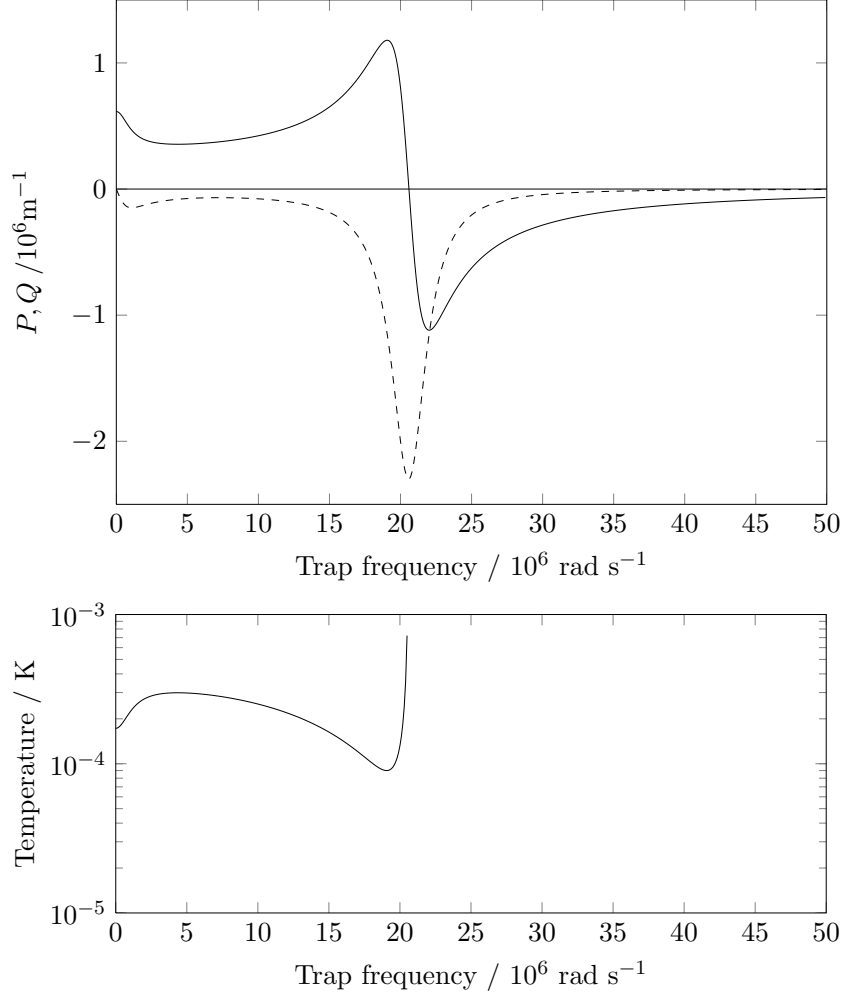


Figure 1. Results of the approximate method for solving the Bloch equations for a two-state system. In the upper plot, the full (dashed) line is the quantity P (Q), both defined in equation (5). P and Q give respectively the in-phase and out-of-phase amplitudes of the photon emission rate for an oscillating ion with velocity amplitude 1 ms^{-1} . The lower plot shows the equilibrium temperature T from equation (6); $T \sim 1/P$, so the peak value of P corresponds to the lowest temperature attainable. For $\omega_s > \Omega'_{12}$, where P is negative, the ion will heat. See text for discussion and the values of the parameters.

parameters: Rabi frequency on resonance Ω_{12} , Einstein A -coefficient for the transition A_{21} , and detuning $\Delta_{12} = \omega_L - \omega_{12}$. For simplicity we set the laser linewidth to zero since it plays no fundamental role in the process. Then the equilibrium temperature depends on u_2 , which characterises the in-phase amplitude of the upper state population. We take $P = A_{21}u_2$.

Figure 1 shows the results for P and Q as functions of ω_s for $\lambda_{12} = 400 \text{ nm}$, $\Omega_{12} = 20 \times 10^6 \text{ rad s}^{-1}$, $A_{21} = 2 \times 10^6 \text{ s}^{-1}$ and $\Delta_{12} = -5 \times 10^6 \text{ rad s}^{-1}$ (thus $\Omega_{12} = 10A_{21}$, $\Delta_{12} = -2.5A_{21}$). The red detuning ensures cooling according to the quasi-static theory. The equilibrium temperature found using (6) is also shown.

The main features are as follows. At very low ω_s the value of P tends to $6.15 \times 10^5 \text{ m}^{-1}$, the value given by the quasi-static theory (equation (2)). As ω_s increases, there is an initial drop; the population of the upper state has a time constant of $1/A_{21}$ and cannot remain in equilibrium with the changing radiation field. In the frequency

domain, the separation of the Fourier components of the radiation field experienced by the ion increases until only the carrier lies under the response curve. There is a corresponding initial increase in the out-of-phase term Q , but this amplitude also then falls off as the damping effect of the slow spontaneous emission becomes more marked. However, this behaviour is superposed on a resonant feature, reminiscent of a simple harmonic oscillator. A two-state system can exhibit Rabi oscillations at the frequency $\Omega'_{12} = (\Omega_{12}^2 + \Delta_{12}^2)^{1/2}$, i.e., $20.62 \times 10^6 \text{ rad s}^{-1}$ in this case. The populations can then change rapidly on a time-scale much shorter than that of spontaneous decay because the effect is entirely due to stimulated processes. Normally, in an equilibrium situation, any such oscillations have been damped out by the randomizing effect of spontaneous emission. However, in the present case the changes in the radiation field give rise to forced oscillations which grow in amplitude as $\omega_s \rightarrow \Omega'_{12}$. Of course, spontaneous emission—on which the cooling process depends—still occurs; the analysis thus suggests that one can get enhanced cooling as one approaches the resonance, as shown in the figure. It should nevertheless be pointed out that this phenomenon is not universally exploitable. The parameters on which figure 1 is based were chosen to illustrate the two distinct physical effects which contribute to the curve, requiring A_{21} to be significantly less than Ω'_{12} , and hence also less than the trap frequency on resonance. This condition would not be fulfilled for the strong transitions normally used for cooling. If it is not satisfied, the initial drop and the broadened resonant feature overlap, and there is no appreciable improvement in cooling.

The phenomenon which we have discussed in this section is entirely unrelated to the dipole force and does not concern an ion near the ground state of motion. It is therefore not the subject of such works as [14, 17, 15, 18]. On the other hand, the phenomenon described in this section can be related to the one exploited for quantum logic gates in the proposal of Jonathan *et al.* [23, 24]. This relies on the resonance which occurs when the a.c. Stark shift matches the motional frequency.

3. The three-state system

We now turn to the three-state system, in which the cooling again takes place on the transition $|1\rangle\text{--}|2\rangle$ at λ_{12} but there is a weak decay from $|2\rangle$ to $|3\rangle$ at λ_{32} [20, 25]. The energy levels are arranged in a lambda-system with $|2\rangle$ at the top. It is straightforward to apply the analysis above to any given set of parameters; for the purposes of illustration we take $\lambda_{12} = 400 \text{ nm}$, $\lambda_{32} = 800 \text{ nm}$, $A_{21} = 125 \times 10^6 \text{ s}^{-1}$, $A_{23} = 10 \times 10^6 \text{ s}^{-1}$, $\Omega_{12} = 200 \times 10^6 \text{ rad s}^{-1}$, $\Omega_{32} = 50 \times 10^6 \text{ rad s}^{-1}$ and $\Delta_{12} = -35 \times 10^6 \text{ rad s}^{-1}$ (thus $A_{23} = 0.08A_{21}$, $\Omega_{12} = 1.6A_{21}$, $\Omega_{32} = 0.4A_{21}$, $\Delta_{12} = -0.28A_{21}$). With this value of Δ_{12} in a two-state system one would normally expect Doppler cooling to be effective. For simplicity we work in the regime $\omega_s \ll A_{21}$ to exclude the two-state effects outlined above. We are interested in how the behaviour of the system changes as ω_s increases; however, we first consider a particular trap frequency $\omega_s = 25 \times 10^6 \text{ rad s}^{-1}$, high enough for dynamic effects to be significant. Figure 2 shows the equilibrium temperature as a function of Δ_{32} according to the quasi-static theory and the dynamic theory. The upper state population for a stationary ion is also shown; the sharp minimum, a dark resonance, occurs at $\Delta_{12} = \Delta_{32}$. We note that for the three-state system the laser linewidths play an important role around this minimum [26], and we set them both to the value of $5.0 \times 10^6 \text{ rad s}^{-1}$. As the plot shows, the temperature according to the dynamic theory settles in both wings to a steady value, while the predictions of the quasi-static theory are quite different; in

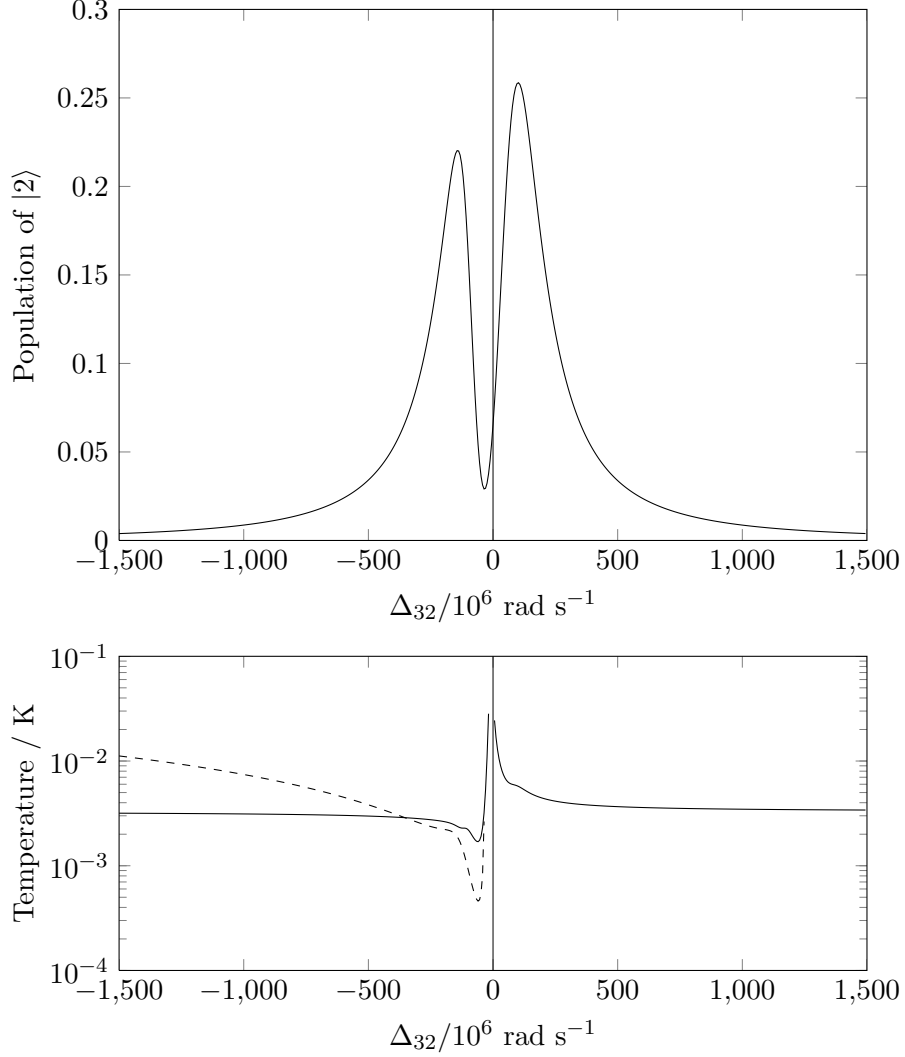


Figure 2. The three state system. The upper curve is the mean population of $|2\rangle$ as a function of the detuning Δ_{32} of the repumper. The lower plots give the corresponding equilibrium temperatures T according to the quasi-static theory (dashed curve) and the dynamic theory (full curve). The former is not plotted for Δ_{32} higher than the minimum of the dark resonance because the theory predicts indefinite heating at these frequencies. The dynamic theory predicts a high temperature on the high frequency side of the dark resonance but as the frequency increases cooling is quickly re-established. See text for discussion and the values of the parameters.

particular, in the blue wing the ion does not cool at all.

We explain this different behaviour by considering how the in-phase amplitudes of the components of the state populations u_1 , u_2 , u_3 change with ω_s . We distinguish three regions in figure 2: (i) large red detuning (ii) large blue detuning and (iii) $\Delta_{32} \sim \Delta_{12}$. Region (iii) is complicated by coherent effects, so we first consider the other cases. According to the quasi-static theory, the cooling process is less effective in (ii)(blue) than in (i)(red) because in (i) the repumping rate is higher when V_x is negative than when it is positive, tending to increase u_2 . The converse is true in (ii), so that in some circumstances u_2 can be negative. Negative u_2 is here owing to the fact that when the repumper is blue-detuned, the repumping process increases the scattering when the ion's velocity is positive, with the result that the ion is accelerated on average, even

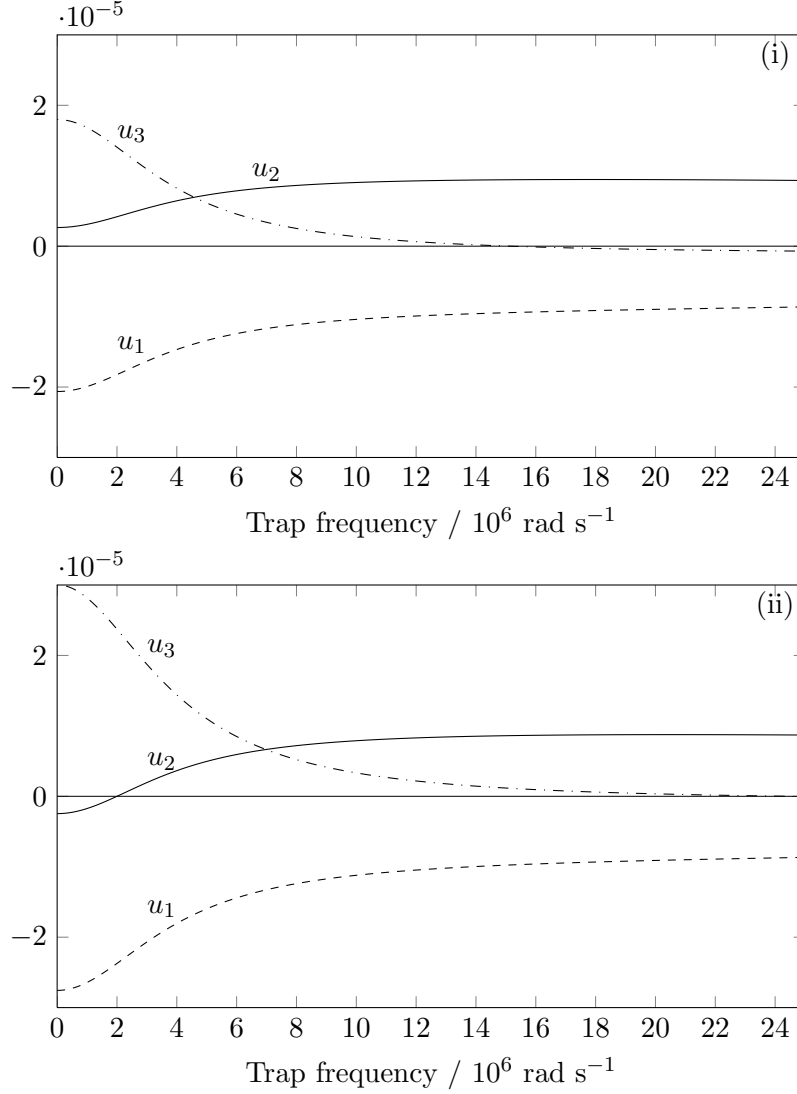


Figure 3. Amplitudes of the in-phase fluctuations u_1, u_2, u_3 for repumper detunings of (i) $-1500 \times 10^6 \text{ rad s}^{-1}$; (ii) $+1500 \times 10^6 \text{ rad s}^{-1}$. At very low frequency, the population in both cases mainly swings backwards and forwards between the states $|1\rangle$ and $|3\rangle$, but the small amplitude u_2 is negative in (ii) leading to heating, as predicted by the quasi-static theory. However, for $\omega_s \gtrsim A_{23}$ the population of $|3\rangle$ is unable to follow the rapidly changing radiation field and becomes constant, so $u_3 \rightarrow 0$. The remaining population then oscillates between $|1\rangle$ and $|2\rangle$ in phase with the motion in the trap as it would for a two-state system (provided that dynamic effects in the $|1\rangle$ - $|2\rangle$ system can be neglected). Parameters as for figure 2.

though the main cooling laser remains red-detuned. Indeed, for the conditions given, the quasi-static theory predicts that u_2 is always negative in region (ii), so no cooling can occur.

The amplitudes u_1, u_2 and u_3 are shown as functions of ω_s in figure 3 for $\Delta_{32} = \pm 1500 \times 10^6 \text{ rad s}^{-1}$. In both cases the fact that the repumper is far from resonance causes there to be a significant fraction of the population in $|3\rangle$. At very low ω_s , this population oscillates (in antiphase with the population of $|1\rangle$) at a frequency ω_s . However, according to the dynamic theory, for trap frequencies too high for the spontaneous transition rate $|2\rangle \rightarrow |3\rangle$ to follow, $u_3 \rightarrow 0$, i.e., the population of $|3\rangle$ becomes constant and we have effectively a two-state system. The equilibrium tem-

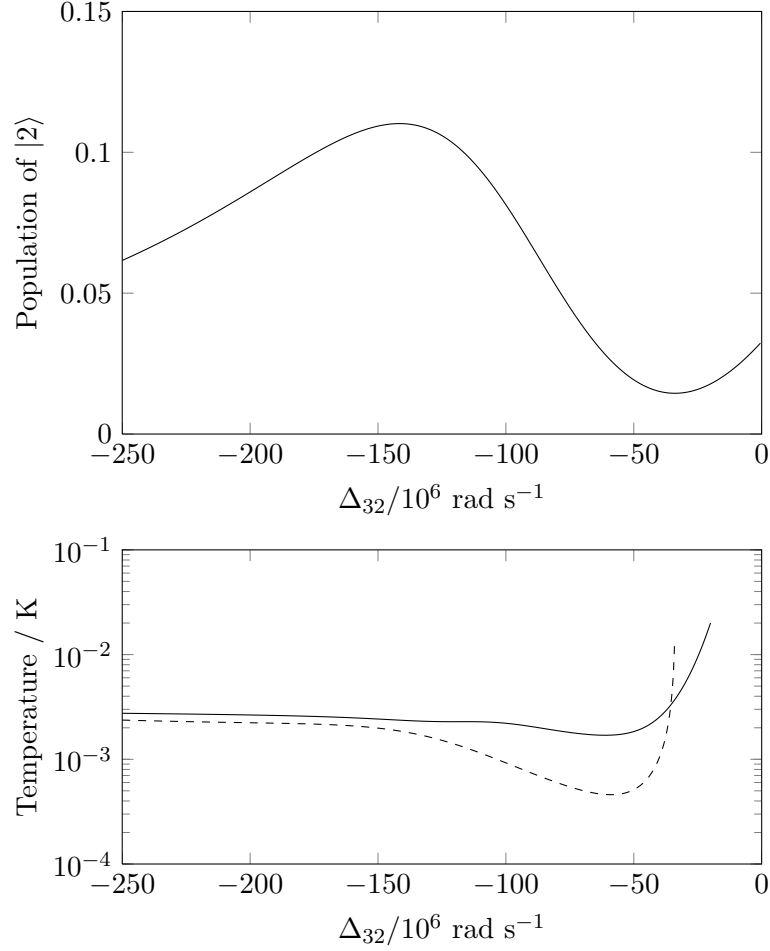


Figure 4. Close-up of central region of figure 2, showing the predictions of the quasi-static theory and those of the dynamic theory with $\omega_s = 25 \times 10^6 \text{ rad s}^{-1}$ in region (iii) (see text). The differences are less marked than in regions (i) and (ii) because u_3 does not tend to zero at high ω_s as coherent transfer does not involve spontaneous emission.

perature becomes almost independent of ω_s in (i) and (ii) tending to the same value in both regions.

In region (iii), the differences in the predictions of the quasi-static and the dynamic theories are less marked; u_3 can be significant at any trap frequency because in the dark resonance region population can transfer coherently between $|1\rangle$ and $|3\rangle$ without spontaneous emission being involved. At the minimum of the dark resonance, where $\Delta_{12} = \Delta_{32}$, the population of $|2\rangle$ for a stationary atom with lasers of zero linewidth would be zero and even for our conditions it is very small irrespective of ω_s . We show a close-up of the temperature plot for this region in figure 4. The marked drop in temperature as the bottom of the dark resonance is approached from lower detuning can be understood from equation (6); a low value of R_0 occurs together with a large value of P . This phenomenon has been exploited in cooling $^{43}\text{Ca}^+$ to below the Doppler limit [16]; see also [27, 28, 29].

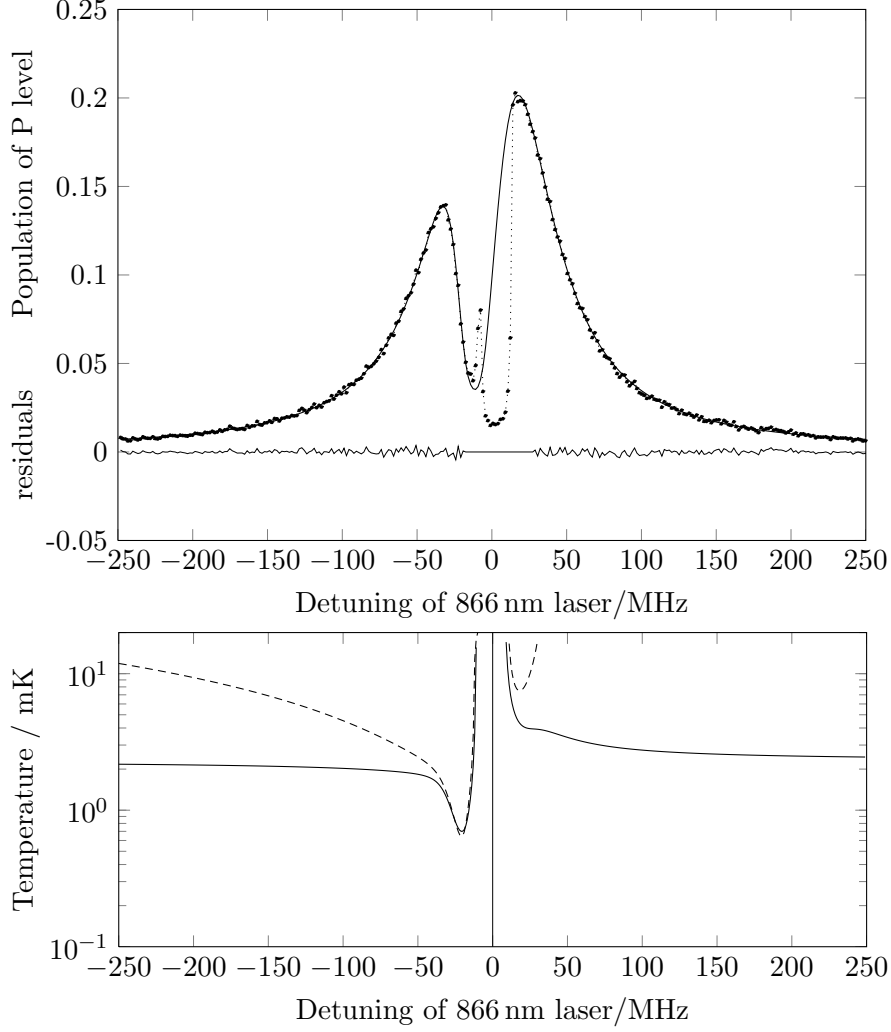


Figure 5. Upper plot: population of the $4^2P_{1/2}$ level of $^{40}\text{Ca}^+$ as a function of the frequency of the repumper laser at 866 nm, as deduced from the observed fluorescence at 397 nm. Points and dotted line: experimental data. The experimental parameters are given in the text. Full curve: fit to the data, with a theory profile appropriate to a stationary ion. Lower curve: residuals from best fit. On the high frequency side of the dark resonance there is a region where the ion is hot, leading to a marked departure from the theoretical curve. As the residual curve indicates, this region was omitted from the fit. Lower plot: The equilibrium temperature as given by the quasi-static theory (dashed curve) and the dynamic theory (full curve). For discussion see text.

4. The ions $^{40}\text{Ca}^+$ and $^{43}\text{Ca}^+$

We have described 2- and 3-state systems because of their simplicity, but one can use the same approach with multistate ions. Indeed, it was the failure of the quasi-static theory to account for our observations on the alkali-like ions $^{40}\text{Ca}^+$ and $^{43}\text{Ca}^+$ that led to the development of the theory presented here. As noted in the introduction, these ions have a cooling transition at 397 nm ($4s^2S_{1/2} - 4p^2P_{1/2}$) with $A_{397} = 1.32 \times 10^8 \text{ s}^{-1}$, and there is a weak decay at 866 nm ($3d^2D_{3/2} - 4p^2P_{1/2}$) with $A_{866} = 8.4 \times 10^6 \text{ s}^{-1}$. We illustrate with a typical fluorescence spectrum observed from $^{40}\text{Ca}^+$ when the 866 nm laser is scanned (figure 5). The theoretical profile used in the fit is that appropriate to a stationary ion; a region around the dark resonance has been omitted from the fit

because of the evident distortion of the profile due to heating. The trap frequency was 816 kHz, and the B -field was 1.5 gauss. The fitted parameters are: 397 nm detuning -12.9 MHz, 397 nm intensity $6.9I_S$, 866 nm intensity $22.6I_S$, where the saturation intensity $I_S = (8\pi^3\hbar\Gamma)/\lambda^3$, with $\Gamma = A_{397} + A_{866}$, the FWHM of the $4^2P_{1/2}$ level. The lasers are co-propagating and both are linearly polarized perpendicular to the B -field. The model uses linewidths of order 1 MHz per beam. The system behaves very similarly to the 3-state case; indeed, the differences between figures 2 and 5 are mainly due to the different experimental parameters. On the low frequency side of the dark resonance, both theories predict cooling, though over most of this range the dynamic theory suggests a lower temperature. On the steep rise out of the dark resonance on the high frequency side, both theories predict indefinite heating. According to the dynamic theory, there is then a rapid recovery at higher frequencies. The quasi-static theory predicts indefinite heating after a short region of moderate cooling, clearly at odds with the experimental observations. In particular, if the quasi-static theory were a correct description, there would be virtually no fluorescence at frequencies above the dark resonance. The fit to the experimental profile is much more consistent with the dynamic theory.

We obtained an added confirmation of this interpretation by implementing [temperature diagnostics of two kinds in two different experiments](#). The first consisted in observing the image of the trapped ion fluorescence on a CCD camera. The image became blurred in the region of the bottom of the dark resonance, indicating a high amplitude oscillation of the ion, and therefore a high temperature (~ 800 mK). The image became sharp again in both wings of the 866nm laser scan, indicating a low temperature in both cases (< 20 mK, [limited by the spatial resolution of the imaging system](#)).

A more [precise](#) experimental study using the $^{43}\text{Ca}^+$ ion has been reported [16], in which the temperature of the ion was measured as a function of the various experimental parameters. The aim was to take advantage of a dark resonance to Doppler-cool to a temperature low enough for further side-band cooling to take the ion to the ground state of the trap. The results are discussed in [22]. We do not pursue this case further here, since there are complicating features which require detailed discussion: $^{43}\text{Ca}^+$ has a far more complex system of internal states than $^{40}\text{Ca}^+$, the ion was in a B -field of 146 gauss (required to realise a clock transition for quantum information processing) and three laser beams were needed to prevent optical pumping to inaccessible states. However, the dynamic theory is an essential feature of the theoretical treatment and the study gives further evidence of its validity.

5. Wider application

Since Doppler cooling is widely used in both Paul traps and other types of trap, both for charged and for neutral species, one would like to know to what extent the methods of this paper will apply more widely. The fact that the species being laser-cooled will have internal dynamics, such that a quasi-static theory is not always appropriate, is obviously widely applicable. The more specific question is, whether equations (4)–(13) provide an accurate way to calculate the temperature to first approximation in a wide range of cases. A trapped particle being laser-cooled undergoes momentum diffusion, so its velocity does not undergo precisely simple harmonic motion (SHM). The simple harmonic assumption is a good approximation when the timescale of the momentum diffusion is slow compared to the vibrational frequency. And even when the motion

is far from simple harmonic, eqs (4)–(13) may still offer a useful way to predict the steady-state temperature to reasonable approximation.

In the case of the magneto-optical trap, confinement and cooling arise together from the interaction between atom and light. Usually the confinement is not tight enough for the dynamic effect studied in this paper to be important, but in principle the conditions can arise such that it is.

A further phenomenon relevant to the Paul trap is the component of ion motion at the r.f. drive frequency, called micromotion. Although the position excursions in micromotion are typically small, the velocity excursions are similar to those of the secular motion. It can happen that the time-scale of the internal dynamics of an ion is intermediate between the secular and the micromotion frequencies, with parameters such that the dynamic theory predicts heating for SHM at the secular frequency and cooling for SHM at the micromotion frequency. As this situation is of some interest, we carried out an example calculation to explore it. The result indicated that it can arise that in such a situation the overall effect is cooling. This calculation employed numerical integration of the OBE and motional equations modelling a three-level ion subject to laser cooling and electrostatic forces in a Paul trap.

6. Conclusion

In this paper we have presented a simple technique which allows the response time of an atom or ion being Doppler-cooled to be incorporated into the theory of the cooling process. We have discussed the results for two- and three-state systems, and showed that experimental data for $^{40}\text{Ca}^+$ which is not explicable on the basis of quasi-static theory is readily understood using the new approach.

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