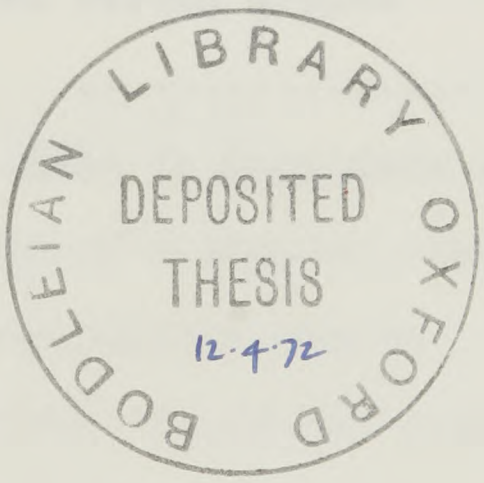
SOME TOPICS IN THE THEORY OF FINITE GROUPS

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ABSTRACT

A regular 2-graph consists of a set Ω together with a (non-empty) set $\underline{t}$ of three-element subsets of Ω such that any two-element subset of Ω is contained in the same number of elements of $\underline{t}$, any four-element subset of Ω contains an even number of elements of $\underline{t}$ and not every three-element subset of Ω is in $\underline{t}$. These objects were introduced by G. Higman who used a regular 2-graph with 276 points to provide a combinatorial setting for the doubly transitive representation of Conway's sporadic simple group C_3.

In this thesis it is shown that regular 2-graphs are in one-one correspondence with equivalence classes of strong graphs (as defined by J.J. Seidel). Moreover, for each point of a regular 2-graph there is a natural way of defining a strongly regular graph on the remaining points. These graphical representations are used to obtain restrictions on the structure and on the parameters of a regular 2-graph. It is also possible, via the strong graphs, to represent a regular 2-graph as a configuration of equiangular lines in Euclidean space. Conversely, results about regular 2-graphs obtained in this thesis extend the results of J.J. Seidel on equiangular lines.

Regular 2-graphs are constructed which admit the following groups as doubly transitive groups of automorphisms: $PSL(2,q)$, $q \equiv 1 \pmod 4$, $Sp(2m,2)$, in both doubly transitive representations; $PSU(3,q^2)$, q odd; all groups of Ree type

together with ${}^2G_2(3)$ = Aut(PSL(2,8)); the sporadic simple groups C_3 and HiS; the group V.Sp(2m,2) which is the semi-direct product of the group V of translations of a vector space of dimension 2m over the field GF(2) by Sp(2m,2). By studying the centraliser ring of a monomial representation associated with the doubly transitive representation it is shown that (with the possible exception of some groups with a regular normal subgroup) the above groups are the only known groups which can act as doubly transitive groups of automorphisms of a regular 2-graph.

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The bound obtained in (2.7.3) was also found (by another method) by S. Todd who also observed the connection between regular 2-graphs and strong graphs.

I wish to thank Dr. P. Cameron for many useful and interesting discussions.

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NOTATION

The extension of PSL(n,q) by the automorphisms of GF(q) will be denoted by PΣL(n,q) and a similar notation will be used for other classical simple groups extended by field automorphisms. The notation for sporadic simple groups is that of Tits [2]. Any other notation not explained in the text may be found in Dembowski [1] or Huppert [1].

The symbol (4.3.18) refers to the equation or theorem labelled (3.18) in Chapter 4; within Chapter 4 it will be referred to as (3.18).

C H A P T E R 1

RESULTS FROM GRAPH THEORY

In this chapter we collect results on strong graphs and strongly regular graphs. Strong graphs were introduced by J.J. Seidel [2] generalising the concept of the strongly regular graph used by Bose [1]. Most of the results can be found in Seidel [2],[3] or Bose [1]. As a general reference for geometry and combinatorial theory we shall use Dembowski [1].

1. Strong graphs

A graph is an ordered pair $(\Omega;E)$ where Ω is a finite set and E is a set of two element subsets of Ω. The elements of Ω are called vertices (or points) and the elements of E are called edges. A point α is joined to a point β if $\{\alpha,\beta\}$ is an edge. A graph $(\Omega;E)$ is regular of valency v if every point is joined to exactly v other points.

Suppose that the points of Ω are listed in some order and suppose that $|\Omega| = n$. Relative to this order we define the adjacency matrix to be the nxn matrix $A = (a_{\alpha\beta})$, where

(1.1)
$$a_{\alpha\beta} = \begin{cases} -1 & \alpha \text{ is joined to } \beta \\ 0 & \alpha = \beta \\ 1 & \alpha \text{ is not joined to } \beta, \end{cases} \quad \text{for } \alpha,\beta \in \Omega.$$

The matrix A determines the set E and conversely, so

we shall also use the symbol (Ω,A) to refer to the graph $(\Omega;E)$.

We now define some parameters associated with the graph (Ω,A). Suppose that α and β are points of Ω such that $a_{\alpha\beta} = (-1)^k$ where k is 1 or 2. We define $p^k_{ij}(\alpha,\beta)$ to be the number of points γ such that $a_{\alpha\gamma} = (-1)^i$ and $a_{\gamma\beta} = (-1)^j$ where $i,j \in \{1,2\}$. We now define

$$p^1(\alpha,\beta) = p^1_{12}(\alpha,\beta) + p^1_{21}(\alpha,\beta) \qquad \alpha \text{ joined to } \beta \tag{1.2}$$

$$p^2(\alpha,\beta) = p^2_{12}(\alpha,\beta) + p^2_{21}(\alpha,\beta) \qquad \alpha \text{ not joined to } \beta.$$

(cf. Dembowski [1], p. 281).

A <u>strong graph</u> is a graph (Ω,A) such that the numbers $p^1(\alpha,\beta)$ and $p^2(\alpha,\beta)$ do not depend on the choice of α and β. When this is the case we denote these numbers by p^1 and p^2 respectively. Note that $p^i(\alpha,\beta)$ is defined only when $a_{\alpha\beta} = (-1)^i$. There are two types of strong graph for which p^1 or p^2 is not defined. These are the <u>void graph</u> defined by $E = \emptyset$ and the <u>complete graph</u> in which E is the set of all two element subsets of Ω. These trivial cases are excluded by Seidel [2] in his definition of strong graph.

Let $n_1(\alpha)$ denote the number of points joined to α and let $n_2(\alpha)$ denote the number of points other than α not joined to α. We have the following obvious relations:

$$p^k_{ij}(\alpha,\beta) = p^k_{ji}(\beta,\alpha), \qquad i,j,k \in \{1,2\} \tag{1.3}$$

$$n_i(\alpha) = p^k_{i1}(\alpha,\beta) + p^k_{ij}(\alpha,\beta) + \delta_{ik}, \quad \{i,j\} = \{1,2\},\ k \in \{1,2\} \tag{1.4}$$

(1.5) $$n = n_1(\alpha) + n_2(\alpha) + 1.$$

A <u>strongly regular graph</u> is a regular graph (Ω, A) such that the numbers $p_{11}^1(\alpha,\beta)$ and $p_{11}^2(\alpha,\beta)$ do not depend on the choice of α and β in Ω. It is immediate from (1.4) that all the parameters $p_{ij}^k(\alpha,\beta)$ are independent of the choice of α and β so we set $p_{ij}^k = p_{ij}^k(\alpha,\beta)$ and $n_i = n_i(\alpha)$, when (Ω, A) is strongly regular. For a point $\alpha \in \Omega$, let $D_1(\alpha)$ be the set of points joined to α and let $D_2(\alpha)$ be the set of points, other than α, not joined to α. Counting edges from $D_1(\alpha)$ to $D_2(\alpha)$ we obtain

(1.6) $$n_i\, p_{jk}^i = n_j\, p_{ik}^j \qquad i,j,k \in \{1,2\}.$$

Strongly regular graphs are a particular case of an <u>association scheme</u> (see Dembowski [1], § 7.1); they also arise in connection with rank 3 permutation groups (see D.G. Higman [1] and C. Sims [1]).

Let G be a transitive permutation group on a finite set Ω. Then we define the action of G on $\Omega x \Omega$ by $(\alpha,\beta)g = (\alpha g, \beta g)$ where $\alpha,\beta \in \Omega$, $g \in G$. The <u>rank</u> of G is the number of orbits of G on $\Omega x \Omega$. Suppose that G has rank 3 and let the orbits of G on $\Omega x \Omega$ be I, D_1 and D_2 where $I = \{(\alpha,\alpha) \mid \alpha \in \Omega\}$. For $\alpha \in \Omega$, define $D_i(\alpha) = \{\beta \in \Omega \mid (\alpha,\beta) \in D_i\}$, $i=1,2$. Then the orbits of G_α on Ω are just $\{\alpha\}$, $D_1(\alpha)$ and $D_2(\alpha)$. In general, if O is an orbit of G on $\Omega x \Omega$, then the <u>paired orbit</u> to O is $O' = \{(\alpha,\beta) \mid (\beta,\alpha) \in O\}$. It is clear that G has a self-paired orbit $O \neq I$ (i.e., $O = O'$) if and only if

G has even order.

Suppose now that G has rank 3 and even order. Then D_1 and D_2 are both self-paired. Let E be the set of two element subsets α,β such that $(\alpha,\beta) \in D_1$. Then $(\Omega;E)$ is clearly a strongly regular graph. In the notation of D.G. Higman [1] we have $n = |\Omega|$, $k = |D_1(\alpha)|$, $\ell = |D_2(\alpha)|$ and $|D_1(\alpha) \cap D_2(\beta)| = \begin{cases} \lambda & \beta \in D_1(\alpha) \\ \mu & \beta \in D_2(\alpha) \end{cases}$.

In our previous notation, $n_1 = k$, $n_2 = \ell$, $p_{11}^1 = \lambda$, $p_{11}^2 = \mu$. Thus the examples of rank 3 groups given in D.G. Higman [1] give rise to a large collection of strongly regular graphs.

The following proposition justifies our terminology:

(1.7) Proposition A graph (Ω,A) is strongly regular if and only if it is both strong and regular.

Proof. If (Ω,A) is strongly regular it is obviously strong and regular. Conversely, suppose (Ω,A) is strong and regular with valency n_1. If (Ω,A) is complete or void we are done. Otherwise, choose α and β in Ω such that α is joined to β. It follows from (1.4) that $p_{12}^1(\alpha,\beta) = p_{21}^1(\alpha,\beta) = \frac{1}{2}p^1$ and then $p_{11}^1(\alpha,\beta)$ is independent of α and β. Similarly, $p_{11}^2(\alpha,\beta)$ is independent of α and β. Hence (Ω,A) is strongly regular. //

The following result is basic to a classification of strong graphs.

(1.8) Proposition (Seidel [2]). A graph (Ω,A) of order n is strong if and only if these are real numbers ρ_1 and ρ_2 such that $\rho_1 > \rho_2$ and

(1.9) $$(A-\rho_1 I)(A-\rho_2 I) = (n-1 + \rho_1\rho_2)J$$
where J is the $n \times n$ matrix all of whose entries are 1.

Proof. Let ρ_1 and ρ_2 be any real numbers and put $B = (A-\rho_1 I)(A-\rho_2 I)$. We write $B = (b_{\alpha\beta})$. If $a_{\alpha\beta} = (-1)^k$, then by definition of $p^k_{ij}(\alpha,\beta)$ we have

$$b_{\alpha\beta} = -(-1)^k(\rho_1+\rho_2) + p^k_{11}(\alpha,\beta) + p^k_{22}(\alpha,\beta) - p^k_{12}(\alpha,\beta) - p^k_{21}(\alpha,\beta)$$

$$= -(-1)^k(\rho_1+\rho_2) + (n-2) - 2p^k(\alpha,\beta).$$

Suppose that (Ω,A) is a strong graph. If it has valency 0 or n, then $A = \varepsilon(J-I)$, where $\varepsilon = \pm 1$ and then $(A+\varepsilon I)(A-\rho I) = \varepsilon J(\varepsilon J - \varepsilon I - \rho I) = (n-1 + (-\varepsilon)\rho)J$, for any ρ. Hence (1.9) holds in this case. We may now assume that p^1 and p^2 are both defined and choose ρ_1 and ρ_2 so that $\rho_1 \geq \rho_2$ and

(1.10) $$\rho_1 + \rho_2 = p^1 - p^2$$

$$-1 - \rho_1\rho_2 = p^1 + p^2$$

Then $b_{\alpha\beta} = n-1 + \rho_1\rho_2$, that is $B = (n-1 + \rho_1\rho_2)J$.

Conversely, suppose that $B = (n-1 + \rho_1\rho_2)J$. Then we have

$$\rho_1 + \rho_2 + (n-2) - 2p^1(\alpha,\beta) = n-1 + \rho_1\rho_2$$

and $$-\rho_1 - \rho_2 + (n-2) - 2p^2(\alpha,\beta) = n-1 + \rho_1\rho_2$$

If both p^1 and p^2 are defined, then

$$p^1 = -(\rho_1-1)(\rho_2-1)/2 \tag{1.11}$$

$$p^2 = -(\rho_1+1)(\rho_2+1)/2$$

and in any case (Ω,A) is a strong graph. //

2. A classification of strong graphs

Given a graph (Ω,A) the complementary graph is $(\Omega,-A)$. The complementary graph of a strong graph is obviously strong.

Two graphs (Ω,A_1) and (Ω,A_2) with the same point set Ω are equivalent (with respect to switching) if there exists a diagonal matrix D whose diagonal entries are ± 1 such that $D^{-1}A_1D = A_2$. If $D = (d_{\alpha\beta})$ and just $d_{\alpha\alpha}$ is -1, then (Ω,A_2) is obtained from (Ω,A_1) by changing all the adjacencies from α. We call this operation 'switching with respect to α'.

A graph equivalent to a complete graph is the disjoint union of two complete graphs; a graph equivalent to a void graph is called a complete bipartite graph; all of these will be called trivial strong graphs.

In classifying strong graphs we shall use equation (1.9). If the graph (Ω,A) is neither complete nor void, ρ_1 and ρ_2 of (1.9) are uniquely determined. For the complete graph we put $\rho_1 = 1$ and $\rho_2 = 1-n$; for the void graph we put $\rho_1 = n-1$ and $\rho_2 = -1$. It now follows that if $n-1 + \rho_1\rho_2 = 0$ in (1.9) then the adjacency matrix of any equivalent graph still satisfies the same equation. From now on we shall assume that (Ω,A) is a strong graph such that $(A-\rho_1 I)(A-\rho_2 I) = (n-1 + \rho_1\rho_2)J$.

(2.1) Lemma. If $n-1+\rho_1\rho_2 \neq 0$, then (Ω,A) is regular and the eigenvalue $\rho_0 = n_2-n_1$ of A has multiplicity 1. The other eigenvalues of A are ρ_1 and ρ_2.

Proof. Since A is a real symmetric matrix we can diagonalise it. From (1.9) J will also be in diagonal form. Since the vector $(1,1,\ldots,1)^T$ is an eigenvector of J_1 of multiplicity one it is an eigenvector of A. Hence (Ω,A) is regular and the eigenvalue corresponding to $(1,1,\ldots,1)^T$ is obviously n_2-n_1. The remaining statements are now clear. //

We now define the following types of strong graph:

Type 1: (Ω,A) is trivial.

Type 2: (Ω,A) is non-trivial and $n-1+\rho_1\rho_2 = 0$.

Type 3: (Ω,A) is non-trivial and $n-1+\rho_1\rho_2 \neq 0$.

(2.2) Lemma. If (Ω,A) is non-trivial, then the numbers p^1 and p^2 are even.

Proof. (Seidel [2]). Choose $\alpha,\beta \in \Omega$ such that $a_{\alpha\beta} = (-1)^k$. Then from (1.4) $n_1(\alpha) + n_1(\beta) \equiv p^k \pmod 2$. If $a_{\alpha\gamma} = (-1)^k$ as well, then $n_1(\alpha)+n_1(\gamma) \equiv p^k \pmod 2$ so $n_1(\beta)+n_1(\gamma) \equiv 0 \pmod 2$. Thus one of p^1,p^2 is even. By taking the complement if necessary we may suppose $p^2 \equiv 0 \pmod 2$. Suppose $p^1 \equiv 1 \pmod 2$. Then $a_{\beta\gamma} = 1$ and (Ω,A) must be a complete bipartite graph. //

For type 2 graphs it is clear that ρ_1 and ρ_2 are the only eigenvalues of A. Let μ_i be the multiplicity of ρ_i for $i=0,1,2$.

(2.3) Lemma. If (Σ,A) has type 2, then we have one of the following possibilities:

(a) $p^1 = p^2$, $\mu_1 = \mu_2 = p^2$, $n = 2p^1 + 2$, $\rho_1 = -\rho_2 = \sqrt{(n-1)}$ and $A^2 = (n-1)I$, or

(b) $p^1 \neq p^2$, $n = p^1+p^2+2$, $A^2 - (p^1-p^2)A - (n-1)I = 0$

$$\rho_1 = \tfrac{1}{2}[(p^1-p^2) + \sqrt{(n^2 - 4p^1p^2)}]$$

$$\rho_2 = \tfrac{1}{2}[(p^1-p^2) - \sqrt{(n^2 - 4p^1p^2)}]$$

$$\mu_1 = \tfrac{1}{2}n[1 - (p^1-p^2)/\sqrt{(n^2 - 4p^1p^2)}]$$

$$\mu_2 = \tfrac{1}{2}n[1 + (p^1-p^2)/\sqrt{(n^2 - 4p^1p^2)}]$$

In particular, n is even and both ρ_1 and ρ_2 are odd integers.

Proof. Since the trace of A is 0 we have $\rho_1\mu_1+\rho_2\mu_2 = 0$ and $\mu_1+\mu_2 = n$ as well as $n-1+\rho_1\rho_2 = 0$. If $p^1=p^2$, then from (1.10) $\rho_1 = -\rho_2$ and $n = 2p^1+2$. Thus $\mu_1=\mu_2=p^1+1$ and $A^2 = (n-1)I$. It follows that $\rho_1 = -\rho_2 = \sqrt{(n-1)}$ and so (a) holds.

Now suppose $p^1 \neq p^2$. Again from (1.10) we have $n = p^1+p^2+2$ and from (1.9) and (1.10) $A^2-(p^1-p^2)A - (n-1)I = 0$. Thus ρ_1 and ρ_2 are the roots of $x^2 - (p^1-p^2)x - (n-1) = 0$. The expressions for ρ_1, ρ_2, μ_1 and μ_2 are now clear. Since μ_1 and μ_2 are integers $n^2-4p^1p^2$ must be the square of an (even) integer so ρ_1 and ρ_2 are integers. From (1.11) and (2.2) it follows that ρ_1 and ρ_2 are odd. //

Because of this lemma we subdivide type 2 graphs into

types 2a and 2b according to whether $p^1=p^2$ or $p^1\neq p^2$.

(2.4) Lemma. Suppose that $(\mathcal{S},A)$ has type 3. Then $(\mathcal{S},A)$ is strongly regular with parameters n_i, p^k_{ij} and one of the following possibilities occurs:

(a) $\mu_1=\mu_2$, $\rho_o=0$, $n_1=n_2=\mu_1$, $\rho_1=-\rho_2=\sqrt{n}$

$p^1_{11} = \frac{1}{2}n_1-1$, $p^2_{11} = \frac{1}{2}n_1$ and $A^2=nI-J$, or

(b) $\mu_1\neq\mu_2$, $A^2-(p^1-p^2)A-(p^1+p^2+1)I = (n-p^1-p^2-2)J$

$$\rho_1 = \tfrac{1}{2}\left\{(p^1-p^2) + \sqrt{[(p^1+p^2+2)^2 - 4p^1p^2]}\right\}$$

$$\rho_2 = \tfrac{1}{2}\left\{(p^1-p^2) - \sqrt{[(p^1+p^2+2)^2 - 4p^1p^2]}\right\}$$

$$\mu_1 = \tfrac{1}{2}\left\{(n_1+n_2) - \frac{(n_1+n_2)(p^1-p^2)-2(n_1-n_2)}{\sqrt{[(p^1+p^2+2)^2 - 4p^1p^2]}}\right\}$$

$$\mu_2 = \tfrac{1}{2}\left\{(n_1+n_2) + \frac{(n_1+n_2)(p^1-p^2)-2(n_1-n_2)}{\sqrt{[(p^1+p^2+2)^2 - 4p^1p^2]}}\right\}$$

In case (b), ρ_1 and ρ_2 are odd integers.

Proof. As in Lemma (2.3) we have $\rho_o+\rho_1\mu_1+\rho_2\mu_2 = 0$ and $1+\mu_1+\mu_2 = n$. From Lemma (2.1) $(\mathcal{S},A)$ is regular and $\rho_o=n_2-n_1$. Also, $p^1_{12} = p^1_{21} = \frac{1}{2}p^1$ and $p^2_{12} = p^2_{21} = \frac{1}{2}p^2$. Diagonalising (1.9) and using (1.10) we obtain $(\rho_o-\rho_1)(\rho_o-\rho_2) = n(n-1+\rho_1\rho_2)$ and ρ_1,ρ_2 are the solutions of $x^2-(p^1-p^2)x - (p^1+p^2+1) = 0$.

If $\mu_1=\mu_2$, then $\rho_o= -\frac{1}{2}(n-1)(p^1-p^2)$. Since $n=n_1+n_2+1$ and $\rho_o=n_2-n_1$ we must have $\rho_o=0$ and $p^1=p^2$.

Thus $\mu_1=n_1$, $\rho_1 = -\rho_2$ and then $\rho_1 = \sqrt{n}$. Finally, $p^1=n_1$ so from (1.4) $p_{11}^1 = \frac{1}{2}n_1-1$, $p_{11}^2 = \frac{1}{2}n_1$.

If $\mu_1\neq\mu_2$ the expressions for ρ_1,ρ_2, μ_1 and μ_2 are obtained immediately from equations above. Since μ_1 is an integer it follows that $(p^1+p^2+2)^2 - 4p^1p^2$ is the square of an even integer so ρ_1 and ρ_2 are integers. From (1.10) they are odd integers. //

We subdivide graphs of type 3 into types 3a and 3b according to whether $\mu_1=\mu_2$ or $\mu_1\neq\mu_2$.

Remarks 1. There exist type 3b graphs with $n_1=n_2$. We give an example. Let Ω be the set of 2 element subsets of a set of 7 elements, two subsets being joined if they have an element in common. This defines a strongly regular graph with $n=21$, $n_1=10$, $n_2=10$, but $p_{11}^1=6$ and $p_{11}^2=4$.

2. There exist type 3b graphs with $p^1=p^2$. An example is the Moore graph of valency 7 diameter 2. The parameters are $n=50$, $n_1=7$, $n_2=42$, $p_{11}^1=0$, $p_{11}^2=1$, $p^1=p^2=12$ (see D.G. Higman [2] or Berlekamp, van Lint and Seidel [1]). (and Hoffman and Singleton IBM J. Res. and Develop. 4 (1960), 497-504)

3. There are strongly regular graphs of type 2. (See Chapter 2).

We now give a further restriction on the parameters of a graph of type 2a.

(2.5) Lemma. If (Ω,A) has type 2a, then $n \equiv 2 \pmod 4$ and there are integers a,b such that $n=1+a^2+b^2$.

Proof. (van Lint and Seidel [1]). From Lemma 2.3 $A^2=(n-1)I$ and $n=2p^1+2$ so $n \equiv 2 \pmod 4$. We prove the following by induction on n: 'If A is an nxn rational matrix such that $A^TA=mI$ where m is an integer >0 and $n \equiv 2 \pmod 4$, then m is the sum of two squares'. This is true when n=2. If $n \neq 2$, put $m = m_1^2+m_2^2+m_3^2+m_4^2$ (Lagrange's theorem) and write

$$M = \begin{bmatrix} m_1 & -m_2 & -m_3 & -m_4 \\ m_2 & m_1 & -m_4 & m_3 \\ m_3 & m_4 & m_1 & -m_2 \\ m_4 & -m_3 & m_2 & m_1 \end{bmatrix}$$

Then $M^TM=mI$ and multiplying a row and column of M by -1 does not alter this equation. Write $A = \begin{bmatrix} A_1 & B \\ C & D \end{bmatrix}$ where A_1 is a 4x4 matrix and choose a suitable M such that $M^TM=mI$ and $\det(A_1-M) \neq 0$. Now put $\overline{A} = D-C(A_1-M)^{-1}B$. Evaluating the product

$$(-B^T(A_1^T - M^T)^{-1}, I_{n-4})A^TA \begin{pmatrix} -(A_1-M)^{-1}B \\ I_{n-4} \end{pmatrix}$$

in two ways we obtain $\overline{A}^T\overline{A} = mI_{n-4}$ and we have proved the induction step. //

See also Hall [1] pp. 108-110 where the same argument is used to prove the Bruck-Ryser-Chowla theorem.

For future use we determine the strong graphs whose adjacency matrix has an eigenvalue 1 or -1. We define a graph $H(k,m) = (\Omega,A)$ as follows: Ω is the disjoint union of

k sets $\Omega_1,\ldots,\Omega_k$ such that $|\Omega_i| = m$ for $i=1,\ldots,k$. If α and β are points of Ω, α is joined to β whenever both α and β lie in Ω_i for some i. The graph $H(k,m)$ is strongly regular with parameters $n=km$, $n_1=m-1$, $p_{11}^1=m-2$, $p_{11}^2=0$. The matrix A has eigenvalues $\rho_0=km-2m+1$, $\rho_1=1$, $\rho_2=-2m+1$.

(2.6) Proposition. Let (Ω,A) be a strong graph whose adjacency matrix A has an eigenvalue 1 or -1. Then either (Ω,A) is trivial or it is the graph $H(k,m)$ or its complement.

Proof. Suppose that (Ω,A) is neither complete nor void. Passing to the complement if necessary we may assume $\rho_1=+1$. From (1.10) it follows that $p^1=0$. For $\alpha \in \Omega$ set $D(\alpha) = \{\alpha\} \cup \{\beta | a_{\alpha\beta} = -1\}$. Since $p^1=0$ we have $p_{12}^1(\beta,\gamma) = p_{21}^1(\beta,\gamma)=0$ for all β and γ in Ω such that $a_{\beta\gamma} = -1$. Thus any two distinct points of $D(\alpha)$ are joined. If α is not joined to β, then no point of $D(\alpha)$ is joined to a point of $D(\beta)$. Thus Ω is the disjoint union of complete graphs $D(\alpha)$. If (Ω,A) is regular, then $|D(\alpha)| = |D(\beta)|$ for any α,β in Ω, so (Ω,A) is $H(k,m)$ for $m= |D(\alpha)|$, $k=n/m$. If (Ω,A) is not regular, then it is type 2 and $n-1+\rho_2=0$; thus $p^2=n-2$. Choose α and β so that $a_{\alpha\beta}=1$. Then $p_{12}^2(\alpha,\beta) = |D(\alpha)| -1$, $p_{21}^2 = |D(\beta)| -1$. Hence $n-2 = p^2 = |D(\alpha)| + |D(\beta)| -2$. It follows that (Ω,A) is trivial. //

The automorphism group of $H(k,m)$ is the wreath product S_m wr S_k.

C H A P T E R 2

REGULAR 2-GRAPHS

The concept of regular 2-graph was introduced by G. Higman who proved that the sporadic simple group C_3 (.3 of Conway [1]) is the automorphism group of a regular 2-graph with 276 points. In this chapter we define a regular 2-graph and then relate this concept to strong graphs and strongly regular graphs.

1. Definitions

A regular 2-graph is a pair $(\Omega,\underline{t})$ where Ω is a finite set of N points and $\underline{t}$ is a set of three-element subsets of Ω such that the following three conditions hold:

(a) any two-element subset of Ω is contained in exactly a elements of $\underline{t}$.

(b) any four-element subset of Ω contains an even number of elements of $\underline{t}$.

(c) $\underline{t}$ is neither empty nor the set of all three-element subsets of Ω.

The elements of $\underline{t}$ are called coherent triangles. A subset C of Ω is called coherent if all of its three-element subsets belong to $\underline{t}$.

If $\Omega_{(3)}$ denotes the set of all three-element subsets of Ω, then $(\Omega,\Omega_{(3)}-\underline{t})$ is a regular 2-graph; the complement of $(\Omega,\underline{t})$. A subset of Ω which is coherent in the complement

will be called incoherent (to be distinguished from non-coherent). We shall suppose that $|\Omega| = N$, then any two element subset of Ω is contained in just a' elements of $\Omega(3) - \underline{t}$ where a' = N-a-2.

The automorphism group of Ω is the group of permutations of Ω which take coherent triangles to coherent triangles. In contrast to graphs it is possible for regular 2-graphs to have doubly transitive automorphism groups.

(1.1) Proposition. Let $(\Omega,\underline{t})$ be a regular 2-graph. Then any coherent triangle is contained in the same number k of coherent four-element subsets of Ω where N=3a-2k.

Proof. Choose $\{\alpha,\beta,\gamma\} \in \underline{t}$ and suppose that $\{\alpha,\beta,\gamma\}$ is contained in k coherent four-element subsets, $\{\alpha,\beta,\gamma,\delta_i\}$, i=1,...,k. If $x \notin \{\alpha,\beta,\gamma,\delta_1,\ldots,\delta_k\}$, then just one of $\{\alpha,\beta,x\}$, $\{\alpha,\gamma,x\}$, $\{\beta,\gamma,x\}$ belongs to $\underline{t}$.
Hence N = 3+k+3(a-k-1) = 3a-2k, so k is independent of the choice of element in $\underline{t}$. //

The numbers N,a and k will be called the parameters of $(\Omega,\underline{t})$ and N, a' and k' will be the corresponding parameters for the complementary regular 2-graph. Whenever no ambiguity results we shall omit mentioning $\underline{t}$ and refer to the regular 2-graph Ω.

(1.2) Lemma. The parameters are connected by the following equations:

(1.3) $N = a + a' + 2$,

(1.4) $N = 3a-2k = 3a'-2k'$

(1.5) $N = 2k+2k'+6$

2. The associated strong graphs

In Chapter 1 we classified strong graphs into three types. We also introduced an equivalence relation on graphs (due to Seidel) and it follows from (1.1.9) that any graph equivalent to a strong graph of type 2 is again a strong graph of type 2. In this section we show that there is a one-to-one correspondence between regular 2-graphs and equivalence classes of strong graphs of type 2. We first establish a correspondence between regular 2-graphs and certain strongly regular graphs of type 3.

Let $T=(\Omega,\underline{t})$ be a regular 2-graph with parameters N,a and k. Choose a point o in Ω and put $\Omega_o=\Omega-\{o\}$. Now define a graph $T_o=(\Omega_o,A_o)$ with adjacency matrix $A_o=(a^{(o)}_{\beta\gamma})$ where

(2.1)
$$a^{(o)}_{\beta\gamma} = \begin{cases} -1 & \{o,\beta,\gamma\} \in \underline{t} \\ 0 & \beta=\gamma \\ 1 & \{o,\beta,\gamma\} \notin \underline{t} \end{cases}$$

and β and γ belong to Ω_o.

Now suppose that $R=(\Gamma,A)$ is a strongly regular graph with n points and parameters n_i, p^k_{ij} as in Chapter 1. Suppose further that $n_1=2p^2_{11}$. Choose a point o not in the set Γ and put $R^o=(\Gamma^o,\underline{t}^o)$ where $\Gamma^o=\Gamma\cup\{o\}$ and $\underline{t}^o$ is the set of the following three-element subsets of Γ^o:

(2.2)
$$\begin{array}{lll} \{o,\alpha,\beta\} & \alpha,\beta \in \Gamma & a_{\alpha\beta} = -1, \\ \{\alpha,\beta,\gamma\} & \alpha,\beta,\gamma \in \Gamma & a_{\alpha\beta}a_{\beta\gamma}a_{\gamma\alpha} = -1. \end{array}$$

(2.3) Theorem. If $T=(\Omega,\underline{t})$ is a regular 2-graph, then $T_0=(\Omega_0,A_0)$ is a strongly regular graph with $n_1=2p_{11}^2$. Conversely, if $R=(\Gamma,A)$ is a non-trivial strongly regular graph with $n_1=2p_{11}^2$, then $R^0=(\Gamma^0,\underline{t}^0)$ is a regular 2-graph. Moreover, $(T_0)^0 = T$ and $(R^0)_0 = R$, and we have the following correspondence between the parameters:

$n = N - 1$	$p_{11}^1 = k$	$p_{11}^2 = \frac{1}{2}a$
$n_1 = a$	$p_{12}^1 = \frac{1}{2}a'$	$p_{12}^2 = \frac{1}{2}a'$
$n_2 = a'$	$p_{22}^1 = \frac{1}{2}a'$	$p_{22}^2 = k'$

Proof. Begin with the regular 2-graph $T=(\Omega,\underline{t})$. Since any set $\{o,\alpha\}$ is contained in a elements of $\underline{t}$ we have $n_1 = a$ and $n_2 = a'$ for $T_0(\Omega_0,A_0)$. From (1.1) it follows that any edge of T_0 is contained in k triangles, i.e., $p_{11}^1 = k$. Next suppose that α and β belong to Ω_0 and $a_{\alpha\beta} = 1$. Since $\{o,\alpha,\beta\}$ is not coherent it is contained in $N-3-k'$ four-element subsets of Ω which contain just two coherent triangles. Since $\{\alpha,\beta\}$ is contained in a coherent triangles there are $N-3-k'-a$ four element subsets $\{o,\alpha,\beta,\gamma\}$ containing $\{o,\alpha,\beta\}$ such that just $\{o,\alpha,\gamma\}$ and $\{o,\beta,\gamma\}$ are coherent, i.e., $p_{11}^2 = N-3-k'-a$. Using (1.4) and (1.5) we have $p_{11}^2 = \frac{1}{2}a$. We have shown that T_0 is a strongly regular graph and $n_1 = 2p_{11}^2$. The remaining parameters can now be easily calculated.

Suppose that $R=(\Gamma,A)$ is a non-trivial strongly regular

graph with $n_1 = 2p_{11}^2$. It follows immediately from (2.2) that every four-element subset of Γ^o contains an even number of elements of $\underline{t}^o$. Now suppose $\{\alpha,\beta\} \subseteq \Gamma^o$. If $\alpha = o$, then from (2.2) $\{o,\beta\}$ is contained in exactly n_1 elements of $\underline{t}^o$. If neither α nor β equal o, then again from (2.2) $\{\alpha,\beta\}$ is contained in either $p_{22}^1 + p_{11}^1 + 1$ or $p_{12}^2 + p_{21}^2$ elements of $\underline{t}^o$. From (1.1.6) $n_2 = 2p_{12}^1$ and from (1.1.4) $n_1 = p_{11}^2 + p_{12}^2$ so $p_{12}^2 + p_{21}^2 = 2p_{12}^2 = n_1$ and $p_{22}^1 + p_{11}^1 + 1 = p_{12}^1 + p_{11}^1 + 1 = n_1$. Since R is non-trivial, $\underline{t}^o$ is neither empty nor the set of all the three-element subsets of Γ^o so we have proved that R^o is a regular 2-graph.

It is clear that $(T_o)^o = T$ and $(R^o)_o = R$. //

(2.5) Corollary. If $T = (\Omega,t)$ is a regular 2-graph and $T_o = (\Omega_o,A_o)$ then T_o is a strongly regular graph of type 3 whose adjacency matrix satisfies

$$(2.6) \qquad A_o^2 + (a-a')A_o - (N-1)I = -J.$$

If ρ_1 and ρ_2 are the eigenvalues of A_o other than $a'-a$, with multiplicities μ_1 and μ_2 respectively, then

$$(2.7) \qquad \rho_1 = \tfrac{1}{2}\{(a'-a) + \sqrt{[(a+a'+2)^2 - 4aa']}\}$$

$$\rho_2 = \tfrac{1}{2}\{(a'-a) - \sqrt{[(a+a'+2)^2 - 4aa']}\}$$

$$(2.8) \qquad \mu_1 = \tfrac{1}{2}\left\{(N-2) - \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}}\right\}$$

$$\mu_2 = \tfrac{1}{2}\left\{(N-2) + \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}}\right\}$$

Proof. From (1.2.4). //

Remark. The numbers N,a,a' are even.

Again suppose $T = (\Omega,\underline{t})$ to be a regular 2-graph with parameters N,a,k and form $T_o = (\Omega_o, A_o)$. Now define a graph $T_o' = (\Omega, A_o')$ by joining o to every point of Ω_o and keeping the other edges as in T_o. Let $\mathcal{S}(T)$ denote the class of graphs equivalent to T_o'.

Suppose that $\underline{X}$ is an equivalence class of strong graphs of type 2 with point set Ω and let $S = (\Omega, A)$ be an element of $\underline{X}$. Define $\mathcal{T}(\underline{X}) = (\Omega,\underline{t})$ where

$$(2.9) \qquad \underline{t} = \{ \{\alpha,\beta,\gamma\} \subseteq \Omega \mid a_{\alpha\beta}a_{\beta\gamma}a_{\gamma\alpha} = -1 \}.$$

(2.10) Theorem. If $T = (\Omega,\underline{t})$ is a regular 2-graph, then T_o' is a strong graph of type 2 and $\mathcal{S}(T)$ does not depend on the chosen point o. Conversely, if $\underline{X}$ is an equivalence class of strong graphs of type 2 and $S = (\Omega,A) \in \underline{X}$, then $\mathcal{T}(\underline{X})$ is a regular 2-graph and does not depend on the chosen element S of $\underline{X}$. Moreover, $\mathcal{T}(\mathcal{S}(T)) = T$ and $\mathcal{S}(\mathcal{T}(\underline{X})) = \underline{X}$.

Proof. Let $T = (\Omega,\underline{t})$ be a regular 2-graph. Let n, n_i, p_{ij}^k be the parameters of T_o. Consider the parameters $p^i(\alpha,\beta)$, of T_o'. If $\alpha,\beta \in \Omega_o$ and $a_{\alpha\beta} = -1$, then $p^1(\alpha,\beta) = p_{12}^1 + p_{21}^1 = a'$; if $a_{\alpha\beta} = 1$, then $p^2(\alpha,\beta) = p_{12}^2 + p_{21}^2 = a$. We also have $p^1(o,\alpha) = n_2 = a'$. Thus T_o' is a strong graph with parameters $N, p^1 = a'$ and $p^2 = a$. From (1.1.10) $\rho_1\rho_2 = -1-a-a' = -(N-1)$ so T_o' is a strong graph of type 2. Therefore, every graph in $\mathcal{S}(T)$ is a strong graph of type 2 with the same parameters as T_o' and whose adjacency matrix has the same minimal

polynomial as A'_o.

Now suppose $a_1, a_2 \in \Omega$. We must show that (Ω, A'_{a_1}) and (Ω, A'_{a_2}) are equivalent with respect to switching. Define D_1 to be $\{\gamma \in \Omega \mid \{a_1, a_2, \gamma\} \notin \underline{t}\}$ and D_2 to be $\{\gamma \in \Omega \mid \{a_1, a_2, \gamma\} \in \underline{t}\}$. Let (Ω, A) be the graph obtained from (Ω, A'_{a_1}) by switching with respect to each point of D_1. We claim that $(\Omega, A) = (\Omega, A'_{a_2})$. This depends on condition (b) in the definition of regular 2-graph. Suppose that $\alpha, \beta \in D_i$, i=1 or 2 and $\{\alpha, \beta\}$ is an edge in (Ω, A), then it is an edge in (Ω, A'_{a_1}) and therefore $\{a_1, \alpha, \beta\}$ is coherent whence $\{a_2, \alpha, \beta\}$ is coherent so that $\{\alpha, \beta\}$ is an edge in (Ω, A'_{a_2}) and conversely. If $\alpha \in D_1$, $\beta \in D_2$ and $\{\alpha, \beta\}$ is an edge in (Ω, A), then $\{\alpha, \beta\}$ is not an edge in (Ω, A'_{a_1}) so $\{a_1, \alpha, \beta\}$ is not coherent whence $\{a_2, \alpha, \beta\}$ is coherent so that $\{\alpha, \beta\}$ is an edge in (Ω, A'_{a_2}) and conversely. If $\{a_1, \alpha\}$ is an edge in (Ω, A), then $\alpha \in D_2$ or $\alpha = a_2$. In either case $\{a_1, \alpha\}$ is an edge in (Ω, A'_{a_2}) and conversely. Finally $\{a_2, \alpha\}$ is an edge in (Ω, A) for all $\alpha \neq a_2$.

Now suppose that (Ω, A) is a strong graph of type 2 with parameters N, p^1, p^2, and define $\underline{t}$ as in (2.9). It is immediate that any four-element subset of Ω contains an even number of elements of $\underline{t}$, and that $\underline{t}$ is neither empty nor the set of all three-element subsets of Ω. If $\alpha \neq \beta \in \Omega$, and $a_{\alpha\beta}=1$, then from (2.9), the number of elements of $\underline{t}$ containing $\{\alpha, \beta\}$ is $p_{12}^2(\alpha, \beta) + p_{21}^2(\alpha, \beta) = p^2$. If $a_{\alpha\beta}=-1$, then the number of elements of $\underline{t}$ containing $\{\alpha, \beta\}$ is $p_{11}^1(\alpha, \beta) + p_{22}^1(\alpha, \beta)$. From (1.1.4) $p_{11}^1(\alpha, \beta) + p_{22}^1(\alpha, \beta) = N - p^1 - 2$. Since (Ω, A) is type 2 it follows from (1.1.10) that $N - p^1 - 2 = p^2$.

Hence $(\Omega,\underline{t})$ is a regular 2-graph with parameters N, p^2, k where $k = \frac{1}{2}(3p^2 - N)$. It is easily seen that switching does not alter the definition of $\underline{t}$ in (2.9) so $\mathcal{S}(\underline{X})$ depends only on $\underline{X}$. By choosing an element $S \in \underline{X}$ which has a point a joined to every other point it is obvious that $\mathcal{T}(\mathcal{T}(\underline{X})) = \underline{X}$. Similarly, $\mathcal{T}(\mathcal{S}(T)) = T$. //

(2.11) Corollary. Let $T = (\Omega,\underline{t})$ be a regular 2-graph with parameters N, a and k. If $(\Omega,A) \in \mathcal{S}(T)$, then the parameters of (Ω,A) are

$$n = N, \qquad p^1 = a' \quad \text{and} \quad p^2 = a.$$

The matrix A has minimal polynomial

(2.12) $$x^2 + (a-a')x - (N-1) = 0$$

and the eigenvalues ρ_1, ρ_2 of A have multiplicities μ_1 and μ_2, where

(2.13) $$\rho_1 = \tfrac{1}{2}\{(a'-a) + \sqrt{[N^2 - 4aa']}\}$$

$$\rho_2 = \tfrac{1}{2}\{(a'-a) - \sqrt{[N^2 - 4aa']}\}$$

(2.14) $$\mu_1 = \tfrac{1}{2}\left\{N - \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}}\right\}$$

$$\mu_2 = \tfrac{1}{2}\left\{N + \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}}\right\}$$

Remarks. 1. Given a set Ω and a function $f : \Omega_{(n)} \to Z_2$ where $Z_2 = \{1,-1\}$, define $\delta f : \Omega_{(n+1)} \to Z_2$ by $\delta f(\{a_0,\ldots,a_n\}) = \prod_{i=0}^{n} f(\{a_0,\ldots,\hat{a}_i,\ldots,a_n\})$; δ is the coboundary operator. A graph (Ω,A) may be considered

as a function $f : \Omega_{(2)} \to Z_2$ where $f(\{\alpha,\beta\}) = a_{\alpha\beta}$ and we can define a 2-graph as a function $g : \Omega_{(3)} \to Z_2$, (the set of coherent triangles is $t = \{\{\alpha,\beta,\gamma\} \mid g(\{\alpha,\beta,\gamma\}) = -1\}$). Let $\underline{1}$ denote the function $\Omega_{(n)} \to Z_2$ sending every element to 1. Then $\delta^2 f = \underline{1}$. If $g : \Omega_{(3)} \to Z_2$ is a 2-graph, condition (b) of the definition of regular 2-graph corresponds to $\delta g = \underline{1}$; i.e., g is a co-cycle. In this situation we can write $g = \delta f$ where $f : \Omega(2) \to Z_2$ is a graph. Theorem (2.10) shows that regularity of g is equivalent to f being a strong graph (of type 2). If $\delta f_1 = \delta f_2$, then $\delta(f_1 f_2) = \underline{1}$ so $f_1 f_2 = \delta d$ where $d : \Omega \to Z_2$, i.e., f_1 is equivalent to f_2 via switching. Conversely, given a graph $f : \Omega_{(2)} \to Z_2$, δf is always a 2-graph satisfying (b) since $\delta(\delta f) = \delta^2 f = \underline{1}$.

2. If $R = (\Gamma, A)$ is a strongly regular graph with $n_1 = 2p_{11}^2$, then it is clear from the construction of R^o that any automorphism of R extends to an automorphism of R^o which fixes o. Conversely, any automorphism of R^o fixing o induces an automorphism of $R = (R^o)_o$. However, it can happen that for a regular 2-graph T, T_{a_1} and T_{a_2} are not isomorphic so the automorphism group of T is not transitive.

3. An example.

In this section we look at regular 2-graphs whose associated strong graphs are of type 2(a).

(3.1) Proposition. Let $T = (\Omega, t)$ be a regular 2-graph.

The following conditions are equivalent:

(i) The strong graphs of (T) have type 2(a).

(ii) $a = a'$

(iii) $k = k'$

(iv) $a = 2k+2$.

Proof. From (2.11), (1.2). //

It follows from (1.2.5) that if $T = (\Omega, \underline{t})$ is a regular 2-graph with $a = a'$, then $N-1$ is the sum of two squares. We now give an example with $N = 1 + p^e$ where p is a prime and $p^e \equiv 1 \pmod 4$ (see van Lint and Seidel [1], D.G. Higman [5]).

Let $q = p^e$ be a prime power with $q \equiv 1 \pmod 4$. Set $\Omega = GF(q) \cup \{\infty\}$. Define $X : \Omega \to \{-1, 0, 1\}$ by

$$X(a) = \begin{cases} +1 & a \text{ is a square in } GF(q)^X \\ 0 & a = o \\ -1 & \text{otherwise} \end{cases}$$

Since $q \equiv 1 \pmod 4$ we have $X(-1) = 1$. As in the previous section define a graph on Ω by $f : \Omega_{(2)} \to Z_2$ where $f(\{\alpha,\beta\}) = X(\alpha-\beta)$. Then $\delta f : \Omega_{(3)} \to Z_2$ defines a 2-graph on Ω which must be regular since $P\overline{\sum}L(2,q)$ acts doubly transitively on Ω and preserves squares. We have $\underline{t} = \{\{\alpha,\beta,\gamma\} \mid X(\alpha-\beta)X(\beta-\gamma)X(\gamma-\alpha) = -1\}$ and $(\Omega,\underline{t})$ is a regular 2-graph. The parameters of $(\Omega,\underline{t})$ are $N = q+1$, $a = a' = \frac{q-1}{2}$, $k = k' = \frac{q-5}{4}$. We shall write $T(q) = (\Omega,\underline{t})$.

(3.2) Theorem. The automorphism group of $T(q)$ is $P\overline{\sum}L(2,q)$ and this group acts doubly transitively on Ω. The stabilizer of the point ∞ is the group $\overline{\sum}(q)$ consisting of the permutations

$$x \mapsto ax^{\sigma} + \beta$$

of GF(q) where α is a non-zero square and

σ Aut GF(q).

Proof. Let G be the automorphism group of $(\Omega,\underline{t})$. Since $G \supseteq P\Sigma L(2,q)$ G acts doubly transitively on Ω. We show that $G_\infty = \Sigma(q)$. Let $g \in G_\infty$, then g is a permutation of GF(q) so we may represent g by a polynomial (of degree $\leq q-2$) with coefficients in GF(q). We want to show that $g \in \Sigma(q)$. We can suppose that $g(0) = 0$ and $g(1) = 1$ since $\Sigma(q)_0$ is transitive on non-zero squares. Moreover, $X(g(\alpha)-g(\beta)) = X(\alpha-\beta)$ so by the theorem of Carlitz [1] $g(x) = x^{p^j}$ for some $j, 0 \leq j < n$. Thus $G_\infty = \Sigma(q)$. //

Remarks. 1. If $q \equiv 3 \pmod 4$ there is a similar theorem of Kantor [1], p. 20 on the automorphism group of the Hadamard design constructed from GF(q).

2. $\Sigma(q)$ acts as a primitive rank 3 group of even order on GF(q) whose associated strongly regular graph has $n_1 = \frac{q-1}{2} = 2p_{11}^2$. The theorem illustrates how regular 2-graphs provide a setting for a transitive extension of such rank 3 groups.

3. A theorem of D.G. Higman [5] shows that if $T = (\Omega,\underline{t})$ is a regular 2-graph with automorphism group G such that G_α is a primitive rank 3 group and $a=a'=2p$, p a prime, then $(\Omega,\underline{t})$ is the regular 2-graph T(4p+1) described above.

4. There exist regular 2-graphs such that $a = a'$ but which are not isomorphic to any of the above examples T(q) (see Chapters 3 and 5).

4. Equiangular lines in Euclidean space

In this section we give a result of M. Gerzon which is needed later. In preparing for this result we show that regular 2-graphs may be represented by configurations of equiangular lines in Euclidean space (see van Lint and Seidel [1]).

Let V be a vector space of dimension r over the field R with an inner product $(\cdot\ ,\ \cdot)$. Suppose that $E = \{\ell_1,\ldots,\ell_n\}$ is a set of n one-dimensional subspaces of V such that the angle between any pair is $\emptyset$. We call such a set E a set of equiangular lines of V. Now choose unit vectors $v_1,\ldots,v_n$ such that v_i spans ℓ_i. Writing $a = \cos\emptyset$ we have

$$(4.1)\qquad (v_i,v_j) = \begin{cases} \pm a & i \neq j \\ 1 & i = j \end{cases}$$

The Gramian of the vectors $v_1,\ldots,v_n$ is the nxn matrix P whose (i,j)-th entry is (v_i,v_j).

(4.2) Proposition. The matrix P is symmetric and positive semi-definite of rank r_1 where r_1 is the dimension of the space spanned by $v_1,\ldots,v_n$. In particular, P has o as an eigenvalue with multiplicity $n-r_1$.

Proof. The rank of P is clearly r_1. Representing each v_i by the column vector of coefficients with respect to an orthonormal basis we have $P = N^TN$ where $N = [v_1,\ldots,v_n]$. Hence P is symmetric and positive semi-definite. //

From now on suppose that $v_1,\ldots,v_n$ span V. We define $A = a^{-1}(P-I)$ so that A has diagonal entries o and other entries ± 1. Thus we may interpret A as the adjacency matrix of a graph with n vertices; the vertices may be identified with the n vectors $v_1,\ldots,v_n$, joining v_i to v_j whenever $(v_i,v_j) = -a$.

The matrix A has smallest eigenvalue $\rho = -a^{-1}$ with multiplicity n-r. Let the remaining eigenvalues be $x_1,\ldots,x_r$.

(4.3) Proposition. We have

$$x_1 + \cdots + x_r = -(n-r)\rho \tag{4.4}$$

$$x_1^2 + \cdots + x_r^2 = n(n-1)-(n-r)\rho^2 \tag{4.5}$$

Moreover, if $a^2 < r^{-1}$, then

$$n \leq r\left(\frac{1-a^2}{1-a^2r}\right) \tag{4.6}$$

with equality if and only if $x_1 = \cdots = x_r$, i.e., if and only if A is the adjacency matrix of a strong graph.

Proof. (van Lint and Seidel [1]). Equations (4.4) and (4.5) follow from $\mathrm{Tr}\, A = 0$ and $\mathrm{Tr}\, A^2 = n(n-1)$ respectively. Equation (4.4) determines a plane in R^r and equation (4.5) determines a sphere. Since the plane meets the sphere we have

$$r^{-1}(n-r)^2\rho^2 \leq n(n-1) - (n-r)\rho^2 \tag{4.7}$$

If $a^2 < r^{-1}$, then (4.6) follows from (4.7) by putting $\rho = -a^{-1}$.

Equality holds in (4.6) if and only if the plane is tangent to the sphere and this is true if and only if A has just two distinct eigenvalues. //

The following result was noticed by P.M. Neumann:

(4.8) Proposition. If $n > 2r$, then the eigenvalue $\rho = -\alpha^{-1}$ of A is an odd rational integer

Proof. Since the entries of A are rational integers ρ is an algebraic integer and any algebraic conjugate of ρ is also an eigenvalue of A. Since the multiplicity of ρ is $n-r > \frac{1}{2}n$ ρ is equal to all of its conjugates. Hence ρ is a rational integer.

Let J be the nxn matrix every entry of which is 1. Then A+J-I has entries either 0 or 2. The eigenspace of J corresponding to the eigenvalue 0 has dimension n-1 so there is at least one eigenvector of A in the eigenspace of ρ which belongs to the eigenspace of 0 of J. Hence A+J-I has $\rho-1$ as an eigenvalue. But then $\frac{1}{2}(\rho-1)$ is both a rational number and an algebraic integer. Hence ρ is an odd integer. //

We now begin with a graph (Ω, A) and construct a set of equiangular lines. Suppose that the smallest eigenvalue of A is $\rho < -1$ and that ρ has multiplicity μ. Then the matrix $B = I-\rho^{-1}A$ has smallest eigenvalue 0 with multiplicity μ so we can find Q such that $Q^2 = B$, where Q is symmetric and positive semi-definite and has rank $n-\mu$. Let e_i, $i=1,\ldots,n$ be the column vector with n entries, 1 in the i-th

place and 0 elsewhere. The vectors $e_1,\ldots,e_n$ span the space $\mathbb{R}^n$. If x and y $\in \mathbb{R}^n$ define the inner product by $(x,y) = x^Ty$. Now put $v_i = Qe_i$, $i=1,\ldots,n$ and let P be the Gramian of $v_1,\ldots,v_n$. Since Q has rank $r = n-\mu$ the vectors $v_1,\ldots,v_n$ span a subspace of $\mathbb{R}^n$ of dimension r. We have

$$(4.9) \qquad (v_i,v_j) = v_i^Tv_j = e_i^TQQe_j$$
$$= e_i^TBe_j = e_i^T(I-\rho^{-1}A)e_j$$
$$= \delta_{ij} - \rho^{-1}a_{ij}$$

If we put $a = -\rho^{-1}$, then

$$(v_i,v_j) = \begin{cases} \pm a & i \neq j \\ 1 & i = j \end{cases}$$

so the lines $\ell_1,\ldots,\ell_n$ spanned by $v_1,\ldots,v_n$ respectively form a set of n equiangular lines in r dimensions with angle $\phi = \cos^{-1}a$. From (4.9) we have $P = I-\rho^{-1}A = B$ and so $A = a^{-1}(P-I)$.

If T is a regular 2-graph and $(\Omega,A) \in \mathcal{S}(T)$, then $A^2 - (\rho_1+\rho_2)A + \rho_1\rho_2 I = 0$ and $P^2 = (1-\rho_1\rho_2^{-1})P$ so that $\sqrt{P} = (1-\rho_1\rho_2^{-1})^{-1/2}P$.

We shall consider two sets of equiangular lines to be equivalent if one can be mapped onto the other by an orthogonal transformation. If $\{\ell_1,\ldots,\ell_n\}$ is a set of equiangular lines with vectors $v_1,\ldots,v_n$ as in (4.1) and if M is an orthogonal matrix, then the Gramian P of $v_1,\ldots,v_n$

is also the Gramian of $Mv_1,\ldots,Mv_n$. Hence the adjacency matrix $A = \alpha^{-1}(P-I)$ of the associated graph depends only on the equivalence class. Moreover, if we change the basis vector v_i of ℓ_i to $-v_i$ then the i-th row and column of A is multiplied by -1 and the new graph is equivalent to (Ω,A) by switching at the ith vertex.

Conversely, if we begin with the regular 2-graph T and $(\Omega,A),(\Omega,A') \in \mathcal{S}(T)$, then $A' = DAD$ where D is diagonal with diagonal entries ± 1. Let $P = I-\rho_2^{-1}A$ and $P' = I-\rho_2^{-1}A'$ $= DPD$, then $\sqrt{P'} = D\sqrt{P}D$. Now $v_i = \sqrt{P}\, e_i$ and $v_i' = \sqrt{P'}\, e_i = D\sqrt{P}De_i = \pm Dv_i$. Since $D^TD = I$ the equiangular lines determined by $v_1,\ldots,v_n$ and $v_1',\ldots,v_n'$ are equivalent. Thus we may associate an equivalence class of equiangular lines with a regular 2-graph.

(4.10) Theorem. (M. Gerzon). If $\{\ell_1,\ldots,\ell_n\}$ is a set of equiangular lines in the r-dimensional Euclidean space V, then $n \leq \frac{1}{2}r(r+1)$.

Proof. The inner product $(\cdot\,,\,\cdot)$ induces an isomorphism $V \otimes V \simeq \mathrm{Hom}(V,V)$ by $(v \otimes w)(u) = (v,u)w$. We identify $V \otimes V$ with $\mathrm{Hom}(V,V)$ so we can write $\mathrm{Tr}((v \otimes v)(w \otimes w)) = (v,w)^2$. Let $v_1,\ldots,v_n$ be unit vectors spanning the lines $\ell_1,\ldots,\ell_n$ and suppose that $|(v_i,v_j)| = \alpha$ when $i \neq j$.

We shall show that the vectors $v_1 \otimes v_1,\ldots,v_n \otimes v_n$ are linearly independent. Suppose that $\sum_{i=1}^{n} a_i(v_i \otimes v_i) = 0$, then multiplying by $v_j \otimes v_j$ and taking the trace we get $a_j + \alpha^2 \sum_{i\neq j} a_i = 0$ so that $(1-\alpha^2)a_j + \alpha^2 \sum_1^n a_i = 0$. Since $\alpha^2 \neq 1$ we have $a_i = a_j$ for all i,j. Let $a = a_i$, then

$[(1-a^2) + na^2]a = 0$ whence $a = 0$. Thus $v_1 \otimes v_1, \ldots, v_n \otimes v_n$ are linearly independent. Since the dimension of the space of symmetric tensors is $\frac{1}{2}r(r+1)$ we have $n \leq \frac{1}{2}r(r+1)$. //

(4.11) Corollary. If $T = (\Omega, \underline{t})$ is a regular 2-graph with parameters N, a, k and if (Ω, A) is a strong graph of $\mathcal{S}(T)$ such that A has eigenvalues ρ_1 and ρ_2 with multiplicities μ_1 and μ_2 respectively, then

$$N \leq \min \left\{ \tfrac{1}{2}\mu_1(\mu_1+1), \tfrac{1}{2}\mu_2(\mu_2+1) \right\}.$$

Proof. Passing to the complement if necessary we can assume $\rho_2 < \rho_1$ and $\mu_1 < \mu_2$. By (1.2.6) $\rho_2 \neq -1$ so the construction above yields N equiangular lines in $\mu_1 = N - \mu_2$ dimensions. //

5. The parameters of a regular 2-graph

Let $T = (\Omega, \underline{t})$ be a regular 2-graph with parameters $N, a,$ and k. The parameters of the complement are N, a' and k'. We shall assume that $a - a'$. Let A be the adjacency matrix of a graph of (T). Then A has eigenvalues ρ_1 and ρ_2 with multiplicities μ_1 and μ_2 respectively, given by (2.13) and (2.14). In this section we shall investigate the consequences of μ_1 and μ_2 being integers. In particular, we desire a bound for N in terms of k. We shall say that T is of type (a) if (Ω, A) is of type 2(a) and that T is of type (b) if (Ω, A) is of type 2(b). We first deal with the case $k = o$.

(5.1) Theorem. If T is a regular 2-graph with k=o, then

T <u>is isomorphic to</u> T(5) of § 3 <u>and</u> Aut(T) <u>is</u> PSL(2,5) $\simeq A_5$.

<u>Proof</u>. Suppose that $a \neq a'$. From (2.14) $\dfrac{N(a'-a)}{2\sqrt{[N^2-4a'a]}}$ is an integer. Put $N = 2M$, $a = 2s$; since $a' = 2a-2$ and $M = 3s$ we see that s^2+4s must be a square, say m^2. Then we have $(s+2)^2-m^2 = 4$ so $2s+4 = m_1+m_2$ where $m_1m_2 = 4$, but this contradicts $s > o$. Hence we have $a = a'$ whence $k = k' = o$ and $N = 6$ by (3.1). Thus T_o is a strongly regular graph with $n_1=n_2 = 2$ and $p_{11}^1 = o$, i.e., T_o is just a pentagon. This shows that T is uniquely determined. Since T(5) satisfies the conditions we have $T \simeq T(5)$, $Aut(T) \simeq PSL(2,5)$. //

(5.2) <u>Theorem</u>. <u>Let</u> T <u>be a regular</u> 2-<u>graph of type</u> (b) <u>such that</u> $k \neq 1$. <u>Then</u> $N \leq 10k - 24$.

<u>Proof</u>. Again put $N = 2M$, $a = 2s$, $a' = 2r$. Since $a \neq a'$ by assumption (2.14) implies that M^2-4rs is a square, say m^2. Now from (1.3) and (1.4) $M = 3s-k$ and $r = 2s-k-1$ so $9s^2 - 6sk + k^2 - 8s^2 + 4sk + 4s = m^2$.
Hence

$$(5.3) \qquad [s - (k-2)]^2 = -4(k-1)+ m^2.$$

It now follows that there exist integers m_1', m_2' such that $2[s - (k-2)] = m_1' + m_2'$, $2m = m'_1 - m_2'$ and $m_1'm_2' = -4(k-1)$. Since one of m_1', m_2' must be even both must be so we can put $m_1' = 2m_1$ and $m_2' = -2m_2$ and then

(5.4)
$$\begin{cases} s = (k-2) + m_1 - m_2 \\ m_1 m_2 = k-1 \\ m = m_1 + m_2 \end{cases}$$

Assuming $k \neq 1$ we have $m_1 - m_2 \leq (k-1)-1$ so that $s \leq 2(k-2)$. Thus $N \leq 10k - 24$. //

(5.5) Theorem. If T is a regular 2-graph with $k = 1$, then the parameters of T must be one of the following three possibilities:

N	a	a'	k	k'	ρ_1	ρ_2	μ_1	μ_2
10	4	4	1	1	3	-3	5	5
16	6	8	1	4	5	-3	6	10
28	10	16	1	10	9	-3	7	21

Proof. If T is of type (a), then we obtain the first line of the table. We now suppose that $a \neq a'$ and use the fact that μ_1 of (2.14) is an integer. Again set $N = 2M$, $a = 2s$, $a' = 2r$. Since $k = 1$ we have $r = 2s-2$ so $M^2 - 4rs = (s+1)^2$ and $M(r-s) = (3s-1)(s-2)$. Since μ_1 is an integer, $s+1$ must divide $(3s-1)(s-2)$ and so $s+1$ divides 12. We have only to eliminate the possibility $s = 11$, $N = 64$. In this case $\rho_1 = 21$, $\rho_2 = -3$, $\mu_1 = 8$ and $\mu_2 = 56$ and so $N \neq \frac{1}{2}\mu_1(\mu_1+1)$, contradicting (4.11). //

Remark. 1. All three possibilities of the theorem occur and the graphs are unique. The first line corresponds to the

graph T(9) of § 3. The existence and uniqueness of the other possibilities is discussed further in Chapters 3 and 5 (see Seidel [1]).

2. If a regular 2-graph has $\rho_2 = -3$, then it must be one of the graphs considered in the above theorem (this follows immediately from (2.13)). The corresponding strong graphs were considered by Seidel [1].

We next consider the graphs for which the bound for N obtained in (5.2) is attained.

(5.6) Theorem. If T is a regular 2-graph such that $N = 10k-24$ then the parameters of T must be one of the following eight possibilities:

N	a	a'	k	k'	ρ_1	ρ_2	μ_1	μ_2
16	8	6	4	1	3	-5	10	6
26	12	12	5	5	5	-5	13	13
36	16	18	6	9	7	-5	15	21
76	32	42	10	25	15	-5	19	56
96	40	54	12	33	19	-5	20	76
126	52	72	15	45	25	-5	21	105
176	72	102	20	65	35	-5	22	154
276	112	162	30	105	55	-5	23	253

Proof. We refer to the proof of Theorem (5.2). Since $N = 10k-24$ we must have $s = 2(k-2)$, $m_1 = k-1$, $m_2 = 1$ and $m = k$. The condition that μ_1 be an integer now implies

that $\frac{(5k-12)(k-5)}{k}$ is an integer and hence that k divides 60. It is clear that we cannot have k=1,2 or 3. This leaves k = 60 as the only case to rule out. In this case we have N = 576, a = 232, a' = 342, so $\rho_1 = 115$, $\rho_2 = -5$, $\mu_1 = 24$ and $\mu_2 = 552$ from (2.14). Thus $N \neq \frac{1}{2}\mu_1(\mu_1+1)$, contradicting (4.11). //

Remarks. 1. Notice that all the 2-graphs have $\rho_2 = -5$. Conversely, if T is a regular 2-graph with $\rho_2 = -5$, then (2.13) gives N = 10k-24.

2. Existence and uniqueness of the above graphs will be considered in Chapters 4 and 5. The first line corresponds to the complement of the second possibility of Theorem (5.5). The graph T(25) of § 3 gives a solution for the second line, however this is not the only solution. It is not known whether there are graphs with N = 76 or N = 96.

Suppose that $T = (\Omega, \underline{t})$ is a regular 2-graph of type (b) and $k \neq 1$. We put N = 2M, a = 2s, a' = 2r so that r = 2s-k-1, M = 3s-k. We explore the condition that μ_1 be an integer.

As in Theorem (5.2) we can write $m = \sqrt{[M^2-4rs]}$ where m is an integer. Since ρ_1 and ρ_2 are odd integers we can put $\rho_2 = -2\omega-1$ where ω is a positive integer.

(5.7) Lemma. $k = \omega+s - s/\omega$ so that ω divides s.

Proof. From (2.13) $\rho_2 = (r-s) - \sqrt{[M^2-4rs]}$ so we have

$9s^2-6sk+k^2-8s^2+4sk+4s = s^2+k^2+4\omega^2-2sk+4\omega s-4\omega k$

whence $k = \omega+s - s/\omega$. //

We may now put $s = \sigma\omega$ so that $m = \sigma+\omega$, $M = \sigma+2\sigma\omega-\omega$, $k = \omega+\sigma\omega-\sigma$ and $r-s = \sigma-\omega-1$. Hence μ_1 is an integer if and only if $\sigma+\omega$ divides $(\sigma-\omega-1)(\sigma+2\sigma\omega-\omega)$. This implies that $\sigma+\omega$ divides $2\omega(\omega+1)(2\omega+1)$. If $\sigma+\omega = 2\omega(\omega+1)(2\omega+1)$, then $M = \sigma+2\sigma\omega-\omega = 8\omega^4+16\omega^3+8\omega^2$ so that $N = [4\omega(\omega+1)]^2$. However, $\mu_1 = N - \frac{M(r-s)}{m}$ so $\mu_1 = 4\omega(\omega+1)$ and this contradicts (4.11). We now express the remaining parameters in terms of ω and σ and summarise our results in the following.

(5.8) Theorem. Let $T=(\Omega,t)$ be a regular 2-graph of type (b) with parameters N, a, k, etc. as above. Then there exist positive integers ω and σ such that:

$$N = 2(\sigma+2\sigma\omega-\omega)$$

$$a = 2\sigma\omega, \qquad a' = 2(\sigma-1)(\omega+1)$$

$$k = \omega+\sigma\omega-\sigma, \qquad k' = 2\sigma+\sigma\omega-2\omega-3$$

$$\rho_1 = 2\sigma-1, \qquad \rho_2 = -2\omega-1$$

$$\mu_1 = \frac{\sigma(2\omega+1)^2-\omega(2\omega+1)}{\sigma+\omega} \qquad \mu_2 = \frac{\omega(2\sigma-1)^2+\sigma(2\sigma-1)}{\sigma+\omega}$$

Moreover, $\sigma+\omega$ divides $2\omega(\omega+1)(2\omega+1)$ but $\sigma+\omega \neq 2\omega(\omega+1)(2\omega+1)$. //

Remarks. 1. Putting $\omega=1$ we have $a = 2\sigma$ and $\sigma+1 \mid 12$ so we obtain Theorem (5.5); putting $\omega=2$ we have $k=\sigma$ and $k \mid 60$ and we obtain Theorem (5.6).

2. Table I contains a list of parameters satisfying the above conditions.

We have shown in (5.2) that $N \leq 10k-24$ and in (5.6) we have determined the parameters for which $N = 10k-24$. We now enquire how large N may be if ρ_2 is given. It follows from (5.8) that the largest value of $\sigma+\omega$ is $\omega(\omega+1)(2\omega+1)$; i.e., $\sigma = \omega^2(2\omega+1)$ and then $\mu_1 = 4\omega^2+4\omega-1 = (2\omega+1)^2-2$. Since $N = 2\omega(\omega+1)(4\omega^2+4\omega-1)$ we have $N = \frac{1}{2}\mu_1(\mu_1+1)$ and there is no conflict with (4.11).

Since $2\omega(\omega+1)(2\omega+1)$ is always divisible by 3 we always have the solution $\sigma+\omega = \frac{2}{3}\omega(\omega+1)(2\omega+1)$.

6. Further strong regular graphs

Let $T = (\Omega, \underline{t})$ be a regular 2-graph and let (Ω, A) be a strong graph in $\mathcal{S}(T)$. In this section we shall suppose (Ω, A) to be regular and then determine the parameters n_i, p_{ij}^k. Since the only eigenvalues of A are ρ_1 and ρ_2 the eigenvalue $\rho_0 = n_2-n_1$ is either ρ_1 or ρ_2 and hence both ρ_1 and ρ_2 must be integers. We may therefore retain the notation used in (5.8). (Note that this holds for graphs of type (a) provided the eigenvalues are integers.)

(6.1) Proposition. Let $T = (\Omega, \underline{t})$ be a regular 2-graph and suppose that $\mathcal{S}(T)$ contains a strongly regular graph (Ω, A). Then there exist positive integers σ, ω such that the parameters of (Ω, A) are one of the following two sets:

(i) $n=N=2(\sigma+2\sigma\omega-\omega)$

$n_1 = \omega(2\sigma-1)$ $\qquad$ $n_2 = (\omega+1)(2\sigma-1)$

$p_{11}^1 = \sigma(\omega-1)$ $\qquad$ $p_{11}^2 = \omega(\sigma-1)$

$p_{12}^1 = (\sigma-1)(\omega+1)$ $\qquad$ $p_{12}^2 = \sigma\omega$

$p_{22}^1 = \sigma(\omega+1)$ $\qquad$ $p_{22}^2 = (w+2)(\sigma-1)$

$\rho_o = \rho_1 = 2\sigma-1,$ $\qquad$ $\rho_2 = -2\omega-1$

where $a = 2\sigma\omega$ and $k = \omega+\sigma\omega-\sigma$

(ii) $n=N=2(\sigma+2\sigma\omega-\omega)$

$n_1 = \sigma(2\omega+1)$ $\qquad$ $n_2 = (\sigma-1)(2\omega+1)$

$p_{11}^1 = \omega(\sigma+1)$ $\qquad$ $p_{11}^2 = \sigma(\omega+1)$

$p_{12}^1 = (\omega+1)(\sigma-1)$ $\qquad$ $p_{12}^2 = \sigma\omega$

$p_{22}^1 = \omega(\sigma-1)$ $\qquad$ $p_{22}^2 = (\sigma-2)(\omega+1)$

$\rho_o = \rho_2 = -2\omega-1$ $\qquad$ $\rho_1 = 2\sigma-1$

where $a = 2\sigma\omega$ and $k = \omega+\sigma\omega-\sigma$

Proof. Use (5.8) and (1.2.4). //

(6.2) Proposition. A strongly regular graph $S = (\Omega, A)$ is a graph of type 2 if and only if $n = 2(1+p_{12}^1+p_{12}^2)$; this is the case if and only if $\mathcal{J}(S)$ is a regular 2-graph.

Proof. From (1.2.3)(b) and (2.2.10). //

Remark. In the notation of § 1.1 the above condition reads:

(6.3) $$n = 2(2k - \lambda - \mu).$$

7. Coherent subsets of regular 2-graphs

In this section we consider the embedding of a coherent subset in a regular 2-graph. In an associated strong graph a coherent subset corresponds to a subgraph of type 1.

(7.1) Lemma. Let $T = (\Omega, \underline{t})$ be a regular 2-graph and suppose that M is a coherent subset of Ω. Then, for each point $x \in \Omega - M$ there is a unique partition of M into subsets $M_1(x)$ and $M_2(x)$ such that for $\alpha_i, \beta_i \in M_i(x)$, $i=1,2$ the set $\{\alpha_i, \beta_j, x\}$ belongs to $\underline{t}$ if and only if $i = j$.

Proof. We may assume that $M \cup \{x\}$ is not coherent so that we can choose $\alpha, \beta \in M$ such that $\{\alpha, \beta, x\}$ is not coherent. We put $M_1(x) = \{\gamma \in M \mid \{\alpha, \gamma, x\} \in \underline{t}\}$ and $M_2(x) = \{\gamma \in M \mid \{\beta, \gamma, x\} \in \underline{t}\}$. It is clear that $M_1(x)$ and $M_2(x)$ give the required partition and that this partition is unique. //

Now choose a fixed coherent subset M of Ω where $T = (\Omega, \underline{t})$ is a regular 2-graph. Suppose that $m = |M|$ and for $x \in \Omega - M$ choose the notation so that $|M_1(x)| \geq |M_2(x)|$ and put $m(x) = |M_1(x)|$. Suppose the parameters of T to be N, a, k, etc. as usual.

(7.2) Proposition. We have the following formula:

$$\sum_{x \in \Omega - M} m(x)(m - m(x)) = a'\binom{m}{2}.$$

Proof. We count the number of coherent triangles α,β,x where α and β $\in M$ and $x \in \Omega-M$ in two ways. Choosing α and β and then x gives $\binom{m}{2}(a-m+2)$ coherent triangles; choosing x then α and β gives $\sum_{x\in\Omega-M}\{\binom{m(x)}{2} + \binom{m-m(x)}{2}\}$ coherent triangles.

Now $|\Omega-M| = N-m$ so

$$\sum_{x\in\Omega-M}\{\binom{m(x)}{2} + \binom{m-m(x)}{2}\} = \tfrac{1}{2}\sum_{x\in\Omega-M}(m(x)^2-m(x)+m(m-1)+m(x^2)-2m.m(x)+m(x))$$

$$= \sum_{x\in\Omega-M} m(x)(m(x)-m) + (N-m)\binom{m}{2}.$$

Since $(N-m)-(a-m+2) = a'$ we have proved the proposition. //

(7.3) Corollary. We have $m \le 1-\rho_2$, and equality holds if and only if $m(x) = \frac{1}{2}m$ for all $x \in \Omega-M$ where ρ_2 is the smallest eigenvalue of a graph in $\mathcal{S}(T)$.

Proof. For all $x \in \Omega-M$ we have $m(x)(m-m(x)) \le (\frac{m}{2})^2$ and therefore $a'(\frac{m}{2}) \le (N-m)(\frac{m}{2})^2$. Thus $m^2 + (2a'-N)m-2a' \le 0$ whence

$$m \le \tfrac{1}{2}\{(N-2a') + \sqrt{[(a-a'+2)^2 + 4a']}\}$$

$$= \tfrac{1}{2}\{(a-a') + \sqrt{[(a+a'+2)^2 - 4aa']}\} + 1$$

$$= 1 - \rho_2, \text{ from (2.2.13).}$$

It is clear that $m = 1-\rho_2$ if and only if $m(x) = \frac{1}{2}m$ for all $x \in \Omega-M$. //

We shall see later that in many cases the bound for m as given in (7.3) is often attained. However, in these cases the bound ρ_1-1 for incoherent sets is not best possible so we now give bounds for m in terms of the multiplicities μ_1

and μ_2 (see (2.2.13)). In order to do this we make use of the correspondence between regular 2-graphs and equiangular lines described in § 2.4.

Let us suppose the points of Ω are $a_1,\ldots,a_N$ where $M = \{a_1,\ldots,a_m\}$. We can choose (Ω,A) in $\mathcal{S}(T)$ such that $A = (a_{ij})$, (where a_{ij} is $a_{a_i a_j}$ in our previous notation) and

$$(7.4) \qquad a_{ij} = \begin{cases} -1 & i \neq j \\ 0 & i = j \end{cases} \qquad 1 \leq i,j \leq m$$

As in § 2.4 we put $P = I - \rho_2^{-1}A$ so that the vectors $v_i = \sqrt{P}\, e_i$, $i=1,\ldots,N$ span equiangular lines $\ell_1,\ldots,\ell_N$ and the vector space spanned by $v_1,\ldots,v_N$ has dimension $\mu_1 = N-\mu_2$. From (2.4.9) and (7.4) we have

$$(7.5) \qquad (v_i,v_j) = \begin{cases} 1 & i = j \\ \rho_2^{-1} & i \neq j \end{cases} \qquad 1 \leq i,j \leq m$$

(7.6) Proposition. The vectors $v_1,\ldots,v_m$ are linearly independent unless $m = 1-\rho_2$ in which case we have $v_1+ \cdots +v_m = 0$.

Proof. Suppose that we have a relation $\sum_{i=1}^{m} b_i v_i = 0$. Then taking the inner product with v_j for $1 \leq j \leq m$ we obtain $\sum_{i=1}^{m} b_i(v_i,v_j) = 0$ and then (7.5) gives

$$\sum_{i=1}^{m} b_i\rho_2^{-1} - b_j\rho_2^{-1} + b_j = 0$$

Consequently, b_j does not depend on j so we may put $b_j = b$ for $1 \leq j \leq m$. Then $m\, b\rho_2^{-1} - b(\rho_2^{-1}-1) = 0$ and either $b = 0$

or $m = 1-\rho_2$ and $\sum_{i=1}^{m} v_i = o$. In any case if $m = 1-\rho_2$ and $x = \sum_{i=1}^{m} v_i$, then $(x,x) = o$ so $x = o$. //

Remark. This does not improve our bound for m since except for $N = 6$, $\mu_1 = 3$ and $\rho_2 = -\sqrt{5}$ we have $\mu_1 > 1-\rho_2$. To see this we may assume that T is type (b) and then use (2.5.8) which gives $-\mu_1\rho_2^{-1} = 1 + 2\omega(\sigma-1)(\sigma+\omega)^{-1} > 1$.

We now construct a set of equiangular lines from the complement of T or what amounts to the same thing, we put $Q = I-\rho_1^{-1}A$ with A as before so that the vectors $\omega_i = \sqrt{Q}\, e_i$, $i=1,\ldots,N$ span N equiangular lines in a space of dimension $\mu_2 = N-\mu_1$. As in (7.5) we have

$$(7.7) \qquad (\omega_i,\omega_j) = \begin{cases} 1 & i = j \\ \rho_1^{-1} & i \neq j, \quad 1 \leq i,j \leq m. \end{cases}$$

(7.8) Proposition. The vectors $\omega_1,\ldots,\omega_m$ are linearly independent.

Proof. Suppose that we have a relation $\sum_{i=1}^{m} b_i\omega_i = o$. As in (7.6) we have $\sum_{i=1}^{m} b_i\rho_1^{-1} - b_j\rho_1^{-1} + b_j = o$ for $1 \leq j \leq m$. Thus b_j does not depend on j so we put $b_j = b$. Then either $b = o$ or $m = 1-\rho_1$. But $\rho_1 > 1$ so we must have $b = o$. //

(7.9) Corollary. We have $m \leq \mu_2$. //

Choose a point $x \in \Omega-M$. We wish to determine the possible values for $m(x)$ when $m = \mu_2$. Therefore, we shall assume that $m = \mu_2$ for the remainder of this section. It

follows that the vectors $\omega_1,\ldots,\omega_m$ form a basis for the vector space of dimension μ_2 spanned by $\omega_1,\ldots,\omega_N$. Let v be the vector corresponding to x. We have $v = \sum_{i=1}^{m} b_i\omega_i$ and by switching at x if necessary we may suppose that

$$(7.10)\qquad (v,\omega_i) = \begin{cases} \rho_1^{-1} & i = 1,\ldots,r \\ -\rho_1^{-1} & i = r+1,\ldots,m \end{cases}$$

where $r = m(x)$.

Taking the inner product of v with $\omega_1,\ldots,\omega_m$ and v in turn we obtain

$$\rho_1^{-1} = \rho_1^{-1} \sum_{i=1}^{m} b_i - b_j\,\rho_1^{-1} + b_j \qquad j=1,\ldots,r$$

$$-\rho_1^{-1} = \rho_1^{-1} \sum_{i=1}^{m} b_i - b_j\,\rho_1^{-1} + b_j \qquad j=r+1,\ldots,m$$

$$1 = \rho_1^{-1} \sum_{i=1}^{r} b_i - \rho_1^{-1} \sum_{i=r+1}^{m} b_i$$

Thus for $1 \le j \le r$ we can put $b_j = c$ and for $r+1 \le j \le m$ we can put $b_j = d$. Then the above equations become

$$(7.11)\qquad 1 = rc + (m-r)d - c + \rho_1 c$$

$$(7.12)\qquad -1 = rc + (m-r)d - d + \rho_1 d$$

$$(7.13)\qquad \rho_1 = rc - (m-r)d.$$

From these equations we get

$$(7.14)\qquad c = (\rho_1 + 1)/(2r-1+\rho_1)$$

$$(7.15)\qquad d = (\rho_1 + 1)/(-2m+2r+1-\rho_1)$$

and then substituting these in (7.13) we get :

$$4r^2 - 4mr + (\rho_1^3 + (m-2)\rho_1^2 - 2m\rho_1 + \rho_1 + m) = 0.$$

The discriminant of this quadratic must be a square so we can write

$$m^2 - m(\rho_1^2 - 2\rho_1 + 1) - \rho_1(\rho_1^2 - 2\rho_1 + 1) = \ell^2,$$

i.e.,

$$\{m - \tfrac{1}{2}(\rho_1-1)^2\}^2 - \ell^2 = \{\tfrac{1}{2}(\rho_1^2-1)\}^2$$

so we can find integers ℓ_1 and ℓ_2 such that

$$2\{m-\tfrac{1}{2}(\rho_1-1)^2\} = \ell_1 + \ell_2$$

where $\ell_1\ell_2 = \frac{1}{4}(\rho_1^2-1)^2$. Since $\rho_1 \equiv 1 \pmod 2$, we have $\ell_1\ell_2 \equiv 0 \pmod 2$ and then both ℓ_1 and ℓ_2 must be even so we obtain the following

(7.16) Theorem. If $T = (\Omega,t)$ is a regular 2-graph and M is a coherent set of m points of Ω, then $m = \mu_2$ implies

$$m = \tfrac{1}{2}(\rho_1-1)^2 + \ell_1 + \ell_2$$

where $\ell_1\ell_2 = (\rho_1^2-1)^2/16$.

Moreover, for any $x \in \Omega-M$ the number $r = m(x)$ is uniquely determined as the larger root of

$$r^2-mr + \tfrac{1}{4}(\rho_1^3+(m-2)\rho_1^2-2m\rho_1 + \rho_1+ m) = 0,$$

that is $r = \frac{1}{2}(m+\ell_2-\ell_1)$, where we choose $\ell_2 - \ell_1$.

We apply (7.16) to several examples which will be used later.

(a) $\rho_1 = 3$. In this case $\ell_1\ell_2 = 4$ and we have then two solutions $m = 7$, $r = 5$ and $m = 6$, $r = 3$ (see (5.5)).

(b) $\rho_1 = 5$. In this case $\ell_1\ell_2 = 36$. By remark 1 to Theorem (5.6) the complementary graph to T has N = 10k-24 so $\mu_2 \leq 23$. Hence we have three solutions:

(i) m = 23, r = 16

(ii) m = 21, r = 13

(iii) m = 20, r = 10.

(c) $\rho_1 = 7$. We have $\ell_1\ell_2 = 144$. From (5.8) the largest possible value of μ_2 is 47 so we have only the following solutions:

(i) m = 44, r = 27

(ii) m = 43, r = 25

(iii) m = 42, r = 21. //

Note that there is always the solution $\ell_1 = \ell_2 = \frac{1}{4}(\rho_1^2-1)$, $\mu_2 = m = \rho_1^2 - \rho_1$ and $r = m/2$. In this case we see that $m = 1-\rho_2$ so that (7.3) and (7.6) apply. This situation can occur (see Chapter 4).

We complete the above investigation by considering two distinct points $x,y \in \Omega-M$. Suppose x corresponds to v as above and suppose that y corresponds to u. Let us write

(7.17) $\qquad (u,v) = \varepsilon \rho_1^{-1}$ where $\varepsilon = \pm 1$.

We have $v = c\sum_{i=1}^{r} \omega_i + d\sum_{i=r+1}^{m} \omega_i$ and by switching at y if necessary we can suppose that

(7.18) $\qquad (u,\omega_i) = \begin{cases} \rho_1^{-1} & 1 \leq i \leq s \text{ and } r+1 \leq i \leq t \\ -\rho_1^{-1} & s+1 \leq i \leq r \text{ and } t+1 \leq i \leq m \end{cases}$

Hence $u = c\sum_1^s \omega_i + d\sum_{s+1}^r \omega_i + c\sum_{r+1}^t \omega_i + d\sum_{t+1}^m \omega_i$

and $s+t = 2r$.

Thus $(u,v) = cs\rho_1^{-1} - c(r-s)\rho_1^{-1} + d(t-r)\rho_1^{-1} - d(m-t)\rho_1^{-1}$

and so it follows from (7.16) that

$$\varepsilon = 2(c-d)s - cr + 3dr - dm.$$

(7.19) Theorem. If the hypotheses of (7.16) hold and x and y are distinct points of $\Omega - M$, such that $r = m(x) = m(y)$ and $s = |M_1(x) \cap M_1(y)|$, then

$$s = \frac{\varepsilon + rc - 3rd + md}{2(c-d)}$$

where c and d are given in (7.14) and (7.15) and

$$\varepsilon = \begin{cases} 1 & (M_1(x) \cap M_1(y)) \cup \{x,y\} \text{ is coherent} \\ -1 & \text{otherwise.} \end{cases}$$

8. Higher regularity

We define a t-regular 2-graph as follows. Any regular 2-graph is 2-regular and a regular 2-graph $T = (\Omega, \underline{t})$ is t-regular where t is an integer, $t > 2$, if the following three conditions are satisfied.

(a) Ω contains a coherent set of t points.

(b) Any coherent set of t points is contained in k_t coherent sets of t+1 points.

(c) T is (t-1)-regular.

It was proved in (1.1) that any regular 2-graph is 3-regular with $k_3=k$. We put $k_2=a$. The following theorem generalises (1.1).

(8.1) Theorem. Let $T = (\Omega, t)$ be a t-regular 2-graph with t even. If Ω contains a coherent set of $t+1$ points then T is $(t+1)$-regular.

Proof. Let M be a coherent set of $t+1$ points and suppose that M is contained in ℓ coherent sets of $t+2$ points. We must show that ℓ is independent of the choice of M. We use the notation of Lemma (7.1).
Given $D \subseteq M$ we put $\sigma(D) = \{x \in \Omega - M \mid D = M_1(x)\}$ and if $|D| = p$ we let $\omega_p(D) = |\sigma(D)|$. The sets $\sigma(D)$ partition $\Omega - M$ $(\omega_p(D) = o$ for $p \leq \frac{1}{2}t)$.

Suppose that p is an integer such that $\frac{1}{2}t+1 \leq p \leq t$ and for all integers q such that $1 \leq q \leq t + 1 - p$ the number $\omega_{p+q}(D)$ does not depend on the choice of $p+q$ element set $D \subseteq M$. We can therefore write ω_{p+q} for $\omega_{p+q}(D)$.
Since $\omega_{t+1} = \ell$ the above assumption is true for $p = t$.
With p and q as above there are $\binom{t+1}{p}$ p element subsets of M and such a subset D is contained in $\binom{t+1-p}{q}$ $p+q$ element subsets of M and in k_p coherent sets of $p+1$ elements. Thus we have

$$(8.2) \qquad k_p = (t+1-p) + \sum_{q=1}^{t+1-p} \binom{t+1-p}{q} \omega_{p+q} + \omega_p(D).$$

It now follows that $\omega_p(D)$ does not depend on D and by induction this is true for all p such that $\frac{1}{2}t+1 \leq p \leq t$.

We can therefore write

$$(8.3) \qquad N = (t+1) + \sum_{p=\frac{1}{2}t+1}^{t+1} \binom{t+1}{p} \omega_p .$$

The next formula is established by induction.

$$(8.4) \quad \omega_p = \sum_{q=0}^{t+1-p} (-1)^q \binom{t+1-p}{q} k_{p+q} \qquad \tfrac{1}{2}t+1 \le p \le t-1.$$

First observe that from (8.2) we have

$$(8.5) \quad \omega_t = k_t - 1 - \ell \qquad \text{and} \quad \omega_{t+1} = \ell .$$

Next suppose that for all $\lambda' \le \lambda$ we have

$$\omega_{t-\lambda'} = \sum_{q=0}^{\lambda'+1} (-1)^q \binom{\lambda'+1}{q} k_{t-\lambda'+q}$$

where $1 \le \lambda \le \frac{1}{2}t-1$. This is certainly true for $\lambda = 1$ as this is just (8.2) with $p = t-1$.

From (8.2)

$$\omega_{t-(\lambda+1)} = k_{t-\lambda-1} - (\lambda+2) - \sum_{q=1}^{\lambda+2} \binom{\lambda+2}{q} \omega_{t-\lambda-1+q}$$

$$= k_{t-\lambda-1} - (\lambda+2) - \sum_{q=1}^{\lambda} \binom{\lambda+2}{q} \sum_{r=0}^{\lambda+2-q} (-1)^r \binom{\lambda+2-q}{r} k_{t-\lambda-1+q+r}$$

$$- (\lambda+2)(k_t - 1 - \ell) - \ell$$

$$= k_{t-\lambda-1} - \sum_{q=1}^{\lambda+2} \binom{\lambda+2}{q} \sum_{r=0}^{\lambda+2-q} (-1)^r \binom{\lambda+2-q}{r} k_{t-\lambda-1+q+r}$$

Let c_s be the coefficient of $k_{t-\lambda-1+s}$ where $q+r = s$.

Then

$$c_s = -\sum_{q=1}^{s} \binom{\lambda+2}{q} (-1)^{s-q} \binom{\lambda+2-q}{s-q}$$

$$= \binom{\lambda+2}{s} \sum_{q=1}^{s} (-1)^{s-q+1} \binom{s}{q}$$

$$= (-1)^s \binom{\lambda+2}{s}.$$

Thus (8.4) is established. Now put $\lambda = t-p$ in (8.3) to get

$$(8.6) \qquad N = (t+1)k_t - t\ell + \sum_{\lambda=1}^{\frac{1}{2}t-1} \binom{t+1}{\lambda+1} \omega_t - \lambda .$$

From (8.4) the coefficient of ℓ in (8.6) is $\sum_{\lambda=0}^{\frac{1}{2}t} \binom{t+1}{\lambda} (-1)^{\lambda}$ which is non-zero. Hence ℓ does not depend on the choice of D and the theorem is proved. //

(8.7) Corollary. Under the hypotheses of (8.1) we have

$$N = \sum_{s=\frac{1}{2}t+1}^{t+1} \left\{ (-1)^s \binom{t+1}{s} \sum_{p=\frac{1}{2}t+1}^{s} (-1)^p \binom{s}{p} \right\} k_s .$$

In particular, for $t = 4$ we have $5k_4 - 2k_5 = 4k-a$ and for $t = 6$ we have $7k_5 - 7k_6 + 2k_7 = 3k-a$.

Proof. From (8.6) and (8.4) we have

$$N = (t+1)k_t - tk_{t+1} + \sum_{\lambda=1}^{\frac{1}{2}t-1} \binom{t+1}{\lambda+1} \sum_{q=0}^{\lambda+1} (-1)^q \binom{\lambda+1}{q} k_{t+q-\lambda}$$

$$= \sum_{\lambda=0}^{\frac{1}{2}t} \binom{t+1}{\lambda} \sum_{q=0}^{\lambda} (-1)^q \binom{\lambda}{q} k_{t+1+q-\lambda}$$

Put $p = t+1-\lambda$, then

$$N = \sum_{p=\frac{1}{2}t+1}^{t+1} \binom{t+1}{p} \sum_{q=0}^{t+1-p} (-1)^q \binom{t+1-p}{q} k_{p+q}$$

The coefficient of k_s is c_s where

$$c_s = \sum_{p=\frac{1}{2}t+1}^{s} \binom{t+1}{p}(-1)^{s-p} \binom{t+1-p}{s-p} = \binom{t+1}{s} \sum_{p=\frac{1}{2}t+1}^{s} (-1)^{s-p} \binom{s}{p}$$

hence we have obtained the required expression for N.

If $t = 4$, then $N = 10k_3 - 15k_4 + 6k_5$ and from (1.4)

$N = 3a - 2k$, so $5k_4 - 2k_5 = 4k-a$. Similarly if $t = 6$ then

$N = 35k_4 - 84k_5 + 70k_6 - 20k_7$ so $7k_5 - 7k_6 + 2k_7 = 3k-a$. //

For the remainder of this section suppose that $T = (\Omega, \underline{t})$ is a 4-regular 2-graph and that M is a coherent set of m points. Counting the number of coherent sets $\{\alpha,\beta,\gamma,\delta,x\}$ with $\alpha,\beta,\gamma,\delta \in M$ and $x \notin M$ in two ways as in (7.2) we obtain:

$$(8.8) \qquad \sum_{x \notin M} \left\{ \binom{m(x)}{4} + \binom{m-m(x)}{4} \right\} = \binom{m}{4} (k_4 - m + 4)$$

Expanding this and using (7.2) we obtain

$$(8.9) \qquad \sum_{x \notin M} m(x)^2 (m-m(x))^2 = a' \binom{m}{2}(2m^2-9m+11) - 12\binom{m}{4}(N-k_4-4)$$

9. An equivalence relation

Let $T = (\Omega, \underline{t})$ be a regular graph and let ρ_1 and ρ_2 be the eigenvalues of the adjacency matrix of an associated

strong graph. Suppose that $m = 1-\rho_2$ is a positive integer. We shall consider the following hypotheses:

(A) There exists a coherent set of m points in Ω.

(B) Any coherent set of $1+m/2$ points is contained in a unique coherent set of m points.

(C) Any coherent set of $m/2$ points is contained in ℓ coherent sets of $1+m/2$ points.

Let $\mathcal{C}$ be the set of all coherent subsets of Ω which have $m/2$ points. For $D_1, D_2 \in \mathcal{C}$ we shall write $D_1 \sim D_2$ whenever $D_1 = D_2$ or $D_1 \cup D_2$ is a coherent set of m points.

(9.1) Theorem. Under hypotheses (A) and (B) $\sim$ is an equivalence relation on $\mathcal{C}$.

Proof. We use the results and notation of § 7, in particular, (7.3) applies. We have only to show that $\sim$ is transitive so suppose that $D_1 \sim D_2$ and $D_2 \sim D_3$ for $D_1, D_2, D_3 \in \mathcal{C}$. We can assume that no two of D_1, D_2, D_3 are equal. If $x \in D_1 \cap D_3$, then $D_2 \cup \{x\}$ is a coherent set of $1+m/2$ points so $D_1 = D_3$, contrary to assumption. Hence $|D_1 \cup D_3| = m$.

For $x \in D_3$ we can choose our notation so that $M = D_1 \cup D_2$, $D_1 = M_1(x)$ and $D_2 = M_2(x)$. Hence $D_1 \cup \{x\}$ is coherent and so $D_1 \sim D_3$. //

(9.2) Corollary. If hypotheses (A), (B) and (C) hold, then each equivalence class of $\sim$ contains $1+2\ell/m$ elements.

Proof. Let $\mathcal{E} = \{D_1, \ldots, D_t\}$ be an equivalence class of $\sim$ and

put $E = \bigcup_{i=1}^{t} D_i$. Since $D_i \cap D_j = \emptyset$ if $i \neq j$ we have $|E| = tm/2$. If $x \in E$, then $D_1 \cup \{x\}$ is coherent. Conversely, if $D_1 \cup \{x\}$ is coherent, then $D_1 \cup \{x\} \subseteq M$ where M is a coherent set of m points. Thus $D_1 \sim M - D_1$ and so $x \in E$. Hence $|E| = |D_1| + \ell$ and it follows that $t = 1 + 2\ell/m$. //

We now give two applications.

(9.3) Proposition. If T is a regular 2-graph such that $\rho_2 = -3$ (equivalently, $k=1$), then hypotheses (A), (B) and (C) hold with $m=4$ and $\ell = a$. Thus each equivalence class contains $1+a/2$ elements and there are $3(a-1)(3a-2)(a+1)^{-1}$ equivalence classes in all.

(9.4) Proposition. If T is a 4-regular 2-graph such that $\rho_2 = -5$ (equivalently, $N = 10k-24$) then hypotheses (A), (B) and (C) are satisfied with $m=6$ and $\ell = k$. Thus each equivalence class contains $1+k/3$ elements and there are $20(5k-12)(2k-5)(k-2)(k+3)^{-1}$ equivalence classes in all.

Proof. If $\rho_2 = -5$, then $N = 10k-24$ (see the remark following (5.6)) so $a = 4k-8$. From (8.7) $5k_4 - 2k_5 = 8$ and therefore $k_4 = 2$, $k_5 = 1$. Thus hypothesis (B) holds with $m=6$. It is trivial that (C) holds with $\ell = k$. //

C H A P T E R 3

RESULTS AND EXAMPLES FROM COMBINATORIAL THEORY

The purpose of this chapter is to provide a catalogue of examples of regular 2-graphs constructed from various combinatorial designs. In particular, we show that all three possibilities of Theorem (2.5.5) can occur.

We have shown in Chapter 2 that the existence of a regular 2-graph T is equivalent to the existence of a class of strong graphs $\mathcal{S}(T)$ of type 2 and also to the existence of a strongly regular graph T_0 of type 3 such that $n_1 = 2p_{11}^2$. Thus given a strongly regular graph R we consider the two possibilities:

(I) $n_1 = 2p_{11}^2$. In this case we construct a regular 2-graph as in (2.2.3) by adjoining a point.

(II) $n = 2(1 + p_{12}^1 + p_{12}^2) = 2(2n_1 - p_{11}^1 - p_{11}^2)$.

Since R is of type 2 we can construct a regular 2-graph on the same number of points as in (2.2.10). In this case there are two possibilities for the valency of R (see § 2.6).

The strong graphs and strongly regular graphs described in this chapter may be found in either Blackwelder [1], Goethals and Seidel [1] or Seidel [3].

Other examples of regular 2-graphs will be given in Chapter 4 in connection with doubly transitive groups and rank 3 groups.

The last section of this chapter is about certain designs obtained from regular 2-graphs.

The notation for the parameters of a regular 2-graph T is that of Chapter 2. Note that if $(\Omega, A) \in \mathcal{S}(T)$, then the eigenvalues of A are eigenvalues of the adjacency matrix of T_o but the multiplicities of these eigenvalues are decreased by 1 in passing to T_o.

1. Definitions

In this section we collect definitions of and basic facts about designs and related structures. For proofs we refer to Dembowski [1].

An <u>incidence structure</u> is a triple $\underline{s} = (\underline{p}, \underline{B}, I)$ where $\underline{p}$, $\underline{B}$ and I are finite sets such that $\underline{p} \cap \underline{B} = \emptyset$ and $I \subseteq \underline{p} \times \underline{B}$. The elements of $\underline{p}$ are called <u>points</u>, those of $\underline{B}$ are called <u>blocks</u>. We say that a point P is <u>incident</u> with a block b if $(P,b) \in I$. We set $v = |\underline{p}|$ and $b = |\underline{B}|$.

An incidence structure $\underline{s}$ is called a <u>tactical configuration</u> if there are natural numbers r and k such that every point is incident with r blocks and every block is incident with k points. In this case we have

$$vr = bk. \tag{1.1}$$

A tactical configuration $\underline{s}$ is called a <u>design</u> if $k \leq v-1$ and there is a natural number λ such that any two points are incident with λ blocks. In this case

$$r(k-1) = \lambda(v-1). \tag{1.2}$$

Let $\underline{s}$ be an incidence structure with points $P_1,\ldots,P_v$ and blocks $\underline{b}_1,\ldots,\underline{b}_b$. We define the <u>incidence matrix</u> $N = (n_{ij})$ of $\underline{s}$ by

$$(1.3) \qquad n_{ij} = \begin{cases} 1 & (P_i,b_j) \in I \\ 0 & \text{otherwise.} \end{cases}$$

If $\underline{s}$ is a design, then

$$(1.4) \qquad N\,N^T = (r-\lambda)I + \lambda J$$

where I is the vxv identity matrix and J is the vxv matrix every entry of which is 1.

The numbers v,b,r,k, λ are called the parameters of the design $\underline{s}$ and we shall sometimes refer to $\underline{s}$ as a (v,b,r,k,λ)-design. Apart from (1.1) and (1.3) we have Fisher's inequality

$$(1.5) \qquad b \geq v.$$

If $\underline{s}$ is a design with b=v then $\underline{s}$ is called <u>projective</u> (or <u>symmetric</u>). From (1.1) r=k.

(1.6) <u>If $\underline{s}$ is a projective design with parameters</u> v,k, <u>and</u> λ <u>then any two blocks are incident with exactly</u> λ <u>points</u>.

An <u>automorphism</u> α of an incidence structure $\underline{s} = (\underline{p},\underline{B},I)$ is a pair (σ,τ) where σ is a permutation of $\underline{p}$ and τ is a permutation of $\underline{B}$ such that if $(P,b) \in I$, then $(P\sigma,b\tau) \in I$.

(1.7) (Brauer) <u>Suppose that</u> G <u>is a group of automorphisms</u>

of the projective design s. Then the number of orbits of G as a permutation group on the points is equal to the number of orbits of G as a permutation group on blocks. If G is cyclic then the number of fixed points of G is equal to the number of fixed blocks of G. If G is transitive on points then the rank of G acting on points is equal to the rank of G on blocks.

Proof. Apply (12.1) of Feit [1] to the incidence matrix of s. //

An nxn matrix H with entries ± 1 is called a Hadamard matrix of order n if $HH^T = nI$. If $H = A \pm I$ where $A^T = A$ then H is called a symmetric Hadamard matrix with constant diagonal.

2. Triangular graphs

The three regular 2-graphs described below satisfy the conditions of Theorem (2.5.5) and show that all three possibilities of that theorem can occur. The uniqueness of these graphs is proved in Chapter 5. From (2.7.3) and (2.7.9) the size of a coherent set is bounded by the minimum of $1-\rho_2$ and μ_2 while the size of an incoherent set is bounded by $1+\rho_1$ and μ_1. In the examples of this section, these bounds are attained.

Let $\triangle$ be a set of m elements and let Ω be the set of 2 element subsets of $\triangle$. We join two elements of Ω whenever they have one element of $\triangle$ in common. The resulting graph

is strongly regular with the following parameters:

(2.1) $n = \frac{1}{2}m(m-1)$

$n_1 = 2(m-2)$ $\qquad p_{11}^1 = m-2$

$n_2 = \frac{1}{2}(m-2)(m-3)$ $\qquad p_{11}^2 = 4$

$\rho_1 = 3$ $\qquad \mu_1 = \frac{1}{2}m(m-3)$

$\rho_2 = -2m+7$ $\qquad \mu_2 = m-1$

It has been proved by D.G. Higman [3] that the automorphism group of this graph is the symmetric group S_m and the permutation representation has rank 3.

From condition (I) above we obtain a regular 2-graph T_{n+1} by adjoining a point o if and only if m = 6. The parameters are

(2.2) $N = 16$

$a = 8$ $\qquad a' = 6$

$k = 4$ $\qquad k' = 1$

$\rho_1 = 3$ $\qquad \mu_1' = 10$

$\rho_2 = -5$ $\qquad \mu_2' = 6$

where μ_1' and μ_2' refer to the multiplicities of ρ_1 and ρ_2 as eigenvalues of an adjacency matrix of a graph in $\mathcal{S}(T_{16})$.

The set $\{0, \{1,2\}, \{3,4\}, \{5,6\}\}$ is incoherent and this is as large as possible by (2.7.3). The set $\{0, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}\}$ is coherent and again by (2.7.3) this is as large as possible. But note that $\{0, \{1,2\}, \{2,3\}, \{1,3\}\}$ is a maximal coherent set

so T_{16}^{*} is not 4-regular.

We shall prove in Chapter 5 that the automorphism group of T_{16}^{*} is doubly transitive, has a regular normal subgroup and that the stabilizer of a point is S_6.

The complement of T_{16}^{*} will be denoted by T_{16}.

From condition (II) we can construct a regular 2-graph T_n^{*} on Ω if and only if $m^2-13m+40 = 0$; i.e., $m=5$ or $m=8$.

If $m=5$ the parameters of T_{10}^{*} are

(2.3)
$$N = 10$$
$$a = a = 4$$
$$k = k = 1.$$

These are the same parameters as those of T(9) of § 2.3 and in Chapter 5 we shall prove that any two regular 2-graphs with these parameters are isomorphic.

If $m=8$, the parameters of T_{28}^{*} are

(2.4)

$N = 28$	
$a = 16$	$a' = 10$
$k = 10$	$k' = 1$
$\rho_1 = 3$	$\mu_1' = 21$
$\rho_2 = -9$	$\mu_2' = 7$

The set $\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}\}$ is coherent and by (2.7.3) this is as large as possible. In fact T_{28}^{*} is 4-regular and hence 5-regular. But $\{\{1,2\}, \{2,3\}, \{1,3\}, \{4,5\}, \{5,6\}, \{4,6\}\}$ is a maximal coherent set so T_{28}^{*} is not 6-regular. We have $k_4=6$ and $k_5=3$.

In Chapter 5 we shall show that the automorphism group of this graph is Sp(6,2).

3. SLB graphs.

Let $\underline{s} = (\underline{p},\underline{B},I)$ be a $(v,b,r,h,1)$-design. Construct a graph on the set $\underline{B}$ of blocks by joining two blocks whenever they have a point in common. We obtain a strongly regular graph with the following parameters (see Bose [1]):
(see also S.S. Shrikhande: Biometrics 8 (1952), 66-72.)

(3.1)
$$n = r(hr-r+1)/h \qquad p_{11}^1 = r-2+(h-1)^2$$
$$n_1 = h(r-1) \qquad p_{11}^2 = h^2$$
$$n_2 = (r-h)(h-1)(r-1)/h \qquad \mu_1 = (r-h)(hr-r+1)/h$$
$$\rho_1 = 2h-1 \qquad \mu_2 = r(h-1)$$
$$\rho_2 = 2h-2r+1$$

We shall call this graph a singly linked block (or SLB) graph. If $h=2$ it is just the triangular graph of the previous section.

From condition (I) we can construct a regular 2-graph by adjoining a point if and only if $r = 2h+1$. The parameters are:

(3.2)
$$N = 4h^2$$
$$a = 2h^2 \qquad a' = 2h^2-2$$
$$k = h^2 \qquad k' = h^2-3$$
$$\rho_1 = 2h-1 \qquad \mu_1' = h(2h+1)$$
$$\rho_2 = -2h-1 \qquad \mu_2' = h(2h-1)$$

This arises from a design with parameters $(2h^2-h, 4h^2-1, 2h+1, h, 1)$. If $h=3$ the design is a Steiner triple system. There are 80 such (15, 35, 7, 3, 1)-designs and they give rise to 80 mutually non-isomorphic regular 2-graphs on 36 points (see Bussemaker and Seidel [1]).

The designs with $h=4,5$ or 7 exist and examples may be found in Hall [1]. They give rise to regular 2-graphs with $N = 64$, 100 or 196 respectively. It is clear from the construction that all these graphs contain coherent sets of $2h+2$ points and that this is as large as possible.

From condition (II) we can construct a regular 2-graph on the same number of points if and only if $r^2(h-1) - (4h^2-2h-1)r + 4h^3 - 2h = 0$. Thus either $h=2$ and $r=7$ or $h=2r$. The first case has occurred as (2.4) above. If $r=2h$ the regular 2-graph has parameters:

$$
\begin{aligned}
(3.3) \qquad N &= 2(2h^2-2h+1) = (2h-1)^2 + 1 \\
a &= a' = 2h(h-1) \\
k &= k' = h^2 - h - 1 \\
\rho_1 &= -\rho_2 = 2h-1 \\
\mu_1' &= \mu_2' = 2h^2 - 2h+1
\end{aligned}
$$

The regular 2-graph is of type (a) and from the construction there are always coherent sets of $2h$ points in this 2-graph. It arises from a design with parameters $(2h^2-2h+1, 2(2h^2-2h+1), 2h, h, 1)$. For example, a Steiner triple system with parameters (13,26,6,3,1) gives rise to a regular 2-graph with $N = 26$, $a = 12$, $k = 5$. We shall see

later that this is not isomorphic to T(25) of § 2.3.

4. Latin square graphs

Arrange m^2 elements in an mxm array and join two elements if they lie in the same row or column. This defines the graph $L_2(m)$. Alternatively, it is the line graph of the complete bipartite graph on m+m points. It is strongly regular and its automorphism group is the wreath product S_m wr S_2 (see D.G. Higman [3]).

Now suppose we have a set of r-2 mutually orthogonal mxm Latin squares, where $3 \leq r \leq m+1$ (see Hall [1], p. 189). We construct the graph $L_r(m)$ by first forming $L_2(m)$ as above and then joining two points if they correspond to the same symbol in some Latin square. The parameters of $L_r(m)$ for $2 \leq r \leq m+1$ are :

(4.1)
$$n = m^2$$
$$n_1 = r(m-1) \qquad p_{11}^1 = (r-1)(r-2)+m-2$$
$$n_2 = (m-r+1)(m-1) \qquad p_{11}^2 = r(r-1)$$
$$\rho_1 = 2r-1 \qquad \mu_1 = (m-r+1)(m-1)$$
$$\rho_2 = 2r-2m-1 \qquad \mu_2 = r(m-1)$$

In order to obtain a regular 2-graph on adjoining a point we must have m=2r-1. The parameters are then:

(4.2)
$$N = (2r-1)^2 + 1$$
$$a = a' = 2r(r-1)$$
$$k = k' = r^2-r-1$$
$$\rho_1 = -\rho_2 = 2r-1$$
$$\mu_1' = \mu_2' = 2r^2-2r+1$$

This is a regular 2-graph of type (a) containing coherent sets of $m+1 = 2r = 1 - \rho_2$ points. These always exist if m is a prime power since we can then construct $L_r(m)$ from an affine plane (Hall [1], p. 177).

From condition (II) we can construct a regular 2-graph on the same number of points provided $m^2-2(2r-1)m + 4r(r-1) = 0$, i.e., $m=2r$ or $m=2r-2$. If $m=2r$, the regular 2-graph has parameters

(4.3)
$$N = 4r^2$$
$$a = 2r^2 \qquad a' = 2r^2-2$$
$$k = r^2 \qquad k' = r^2-3$$
$$\rho_1 = 2r-1 \qquad \mu_1 = r(2r+1)$$
$$\rho_2 = -2r-1 \qquad \mu_2 = r(2r-1)$$

The parameters are the same as in (3.2); we shall consider these again in § 6. The graph contains coherent sets of $m = 1-\rho_2$ points. If $m = 2r-2$, the parameters are:

(4.4)
$$N = 4(r-1)^2$$
$$a = 2r(r-2) \qquad a' = 2(r-1)^2$$
$$k = r^2-2r-2 \qquad k' = (r-1)^2$$
$$\rho_1 = 2r-1 \qquad \mu_1' = (r-1)(2r-3)$$
$$\rho_2 = -2r+3 \qquad \mu_2' = (r-1)(2r-1).$$

Again, these graphs always contain coherent sets of $m = 1-\rho_2$ points. For example, there exist 5 mutually orthogonal Latin squares of order 12. Taking any 4 of them we get a regular 2-graph with N = 144, a = 72, k = 36 from (4.3) and taking all 5 together we get a regular 2-graph from (4.4)

with the complementary parameters N = 144, a = 70, k = 33. This example was pointed out by Professor J.J. Seidel.

5. Constructions from Hadamard matrices

In this section we summarise those constructions of Goethals and Seidel [1], that lead to regular 2-graphs.

(5.1) *If there exists a* (v,b,r,h,1)*- design and a Hadamard matrix of order* r+1, *then there exists a strong regular graph of type* 2 *with parameters*

$$n = v(r+1)$$

$$n_1 = \tfrac{1}{2}r(v+h) \qquad p_{11}^1 = \tfrac{1}{4}(r-1)(v+h+2)$$

$$n_2 = 2r(v+h-2) \qquad p_{11}^2 = \tfrac{1}{4}(r+1)(v+h)$$

$$\rho_1 = v+h-1 \qquad \mu_1 = b$$

$$\rho_2 = -r \qquad \mu_2 = v(r+1)-b.$$

The corresponding regular 2*-graph has parameters*

$$N = v(r+1)$$

$$a = \tfrac{1}{2}(v+h)(r-1) \qquad a' = \tfrac{1}{2}(h-1)(r+1)^2 + 1$$

$$k = \tfrac{1}{4}(h-1)(r-1)^2 + r-h-1 \qquad k' = \tfrac{1}{4}(r+1)(hr-r+3h-5)$$

and the 2*-graph contains coherent sets of* $r+1 = 1-\rho_2$ *points.*

Proof. (Goethals and Seidel, [1], p. 599). Let N be the incidence matrix of the design and let H be the Hadamard

matrix. We can suppose that the first column of H consists entirely of 1's and let L be the matrix formed by the remaining columns. Then $LL^T = rI + (I-J)$ and $JL = 0$. In each row of N we replace each 0 by a column of r+1 0's and we replace the r 1's by the successive columns of L. The resulting matrix is called P and we have $PP^T = rI + A$ where A is a symmetric matrix with diagonal elements 0 and ± 1 elsewhere. Now $JA = JPP^T - rJ = -rJ$ so we may interpret A as the adjacency matrix of a regular graph. Since $P^TP = h(r+1)I$, PP^T has rank $v(r+1)-b$ so A has smallest eigenvalue -r with multiplicity $v(r+1)-b$.

From (1.1) and (1.2) we have

$$\frac{b(r^2-1)}{r^2-b} = \frac{(r/h)(rh-r+1)(r^2-1)}{r^2-(r/h)(rh-r+1)}$$

$$= v(r+1).$$

Using (2.4.3) we see that A is the adjacency matrix of a strong graph with eigenvalues v+h-1 and -r. The parameters can now be calculated from (2.6.1) and (2.5.8). //

Since the pair design with parameters $(r+1, \frac{1}{2}r(r+1), r, 2, 1)$ always exists we can construct a regular 2-graph with parameters $N = (r+1)^2$, $a = \frac{1}{2}(r-1)(r+3)$ whenever there is a Hadamard matrix of order r+1. Notice that the parameters are the same as (3.2), (4.3) and (4.4). In particular, we have another construction for a regular 2-graph with N = 144.

Other examples can be constructed from some of the

designs listed in Hall [1], p. 291. For the remainder of this section we shall consider regular 2-graphs whose parameters are given by (3.2). The connection with Hadamard matrices is given in the following construction (Goethals and Seidel [1], p. 604).

Suppose that H is a symmetric Hadamard matrix with constant diagonal. Then $H = A + \varepsilon I$, where $\varepsilon = \pm 1$ and $H^2 = 4nI$ for some n, (we exclude the case where H is a 2x2 matrix).

Let $m = \sqrt{n}$, $\rho_1 = 2m - \varepsilon$ and $\rho_2 = -2m - \varepsilon$. Then $(A-\rho_1 I)(A-\rho_2 I) = 0$ so A may be interpreted as the adjacency matrix of a strong graph $S = (\Omega, A)$. It follows that m is an integer and the regular 2-graph $T = \mathcal{S}(S)$ has parameters

(5.2)
$$N = 4m^2$$
$$a = 2m^2 - 1 + \varepsilon \qquad a' = 2m^2 - 1 - \varepsilon$$
$$k = m^2 + 3(\varepsilon - 1)/2 \qquad k' = m^2 - 3(\varepsilon - 1)/2 - 3$$

If H is regular in the sense that $HJ = rJ$ for some r, then the graph S is strongly regular.

Conversely, suppose there exists a regular 2-graph T with parameters as in (5.2). Let $S = (\Omega, A)$ be a strong graph of $\mathcal{S}(T)$ and put $H = A + \varepsilon I$. Then $H^2 = 4m^2 I$ so H is a symmetric Hadamard matrix of order $4m^2$ with constant diagonal.

The following construction was suggested by P. Cameron who first considered the case m=12.

Let H be a Hadamard matrix of order m and let $G = (g_{ij;k\ell})$

be the $m^2 x m^2$ matrix whose rows and columns are indexed by the ordered pairs (i,j), $i,j = 1,\ldots,m$ where

$$(5.3) \qquad g_{ij;k\ell} = h_{ij}h_{i\ell}h_{kj}h_{k\ell} \qquad i,j,k,\ell = 1,\ldots,m.$$

Since $HH^T = mI$ we have

$$\sum_{i,j} g_{ij;k\ell} = \Big(\sum_{i,j} h_{ij}h_{i\ell}h_{kj}\Big)h_{k\ell}$$

$$= \sum_i h_{i\ell}\delta_{ik}m \cdot h_{k\ell}$$

$$= m\,h_{k\ell}^2 = m$$

and

$$\sum_{k,\ell} g_{ij;k}\,g_{k\ell;mn} = \sum_{k,\ell} h_{ij}h_{i\ell}h_{kj}h_{k\ell}h_{k\ell}h_{kn}h_{m\ell}h_{mn}$$

$$= h_{ij}h_{mn}\sum_{k,\ell} h_i\,h_{kj}h_{kn}h_m$$

$$= m^2\,\delta_{im}\,\delta_{jn}.$$

Thus $GJ = mJ$ and $G^2 = m^2I$. Now G is symmetric and $g_{ij;ij} = 1$ so G has constant diagonal. Thus G is a symmetric Hadamard matrix of order m^2 with constant diagonal. Moreover if $B = G-I$, then the corresponding strong graph (Ω,B) constructed as above is strongly regular since $BJ = (m-1)J$.

Finally in this section we give another construction due to Goethals and Seidel [1]. An nxn matrix S is called an <u>S-matrix</u> if $SS^T = nI-J$, $JS = SJ = 0$, $S^T = \pm S$, and S has diagonal elements 0 and ± 1 elsewhere. Suppose that there exists a symmetric S-matrix S_{n+1} of order n+1 and a skew

S-matrix S_{n-1} of order n-1. We may regard S_{n+1} as the adjacency matrix of a strongly regular graph which yields a regular 2-graph of type (a) on adjoining a point. Hence $n \equiv 0 \pmod 4$. Suppose moreover that S_{n-1} is symmetric with respect to its anti-diagonal then $(U S_{n-1})^T = U S_{n-1}$, where U is the permutation matrix with 1's on its antidiagonal. Note that such S-matrices always exist if n-1 and n+1 are odd prime powers (see Hall [1], p. 209).

If we put $K = U S_{n-1} \otimes S_{n+1} + U \otimes (J_{n+1} - I_{n+1}) - J_{n-1} \otimes I_{n+1}$ then $K = K^T$, $KJ = JK = J$ and $K^2 = n^2 I - J$. Thus K+I is the adjacency matrix of a strongly regular graph of order n^2-1 which yields a regular 2-graph by adjoining a point. If n=2m the parameters are given by (5.2). If j is the column vector with every entry 1 then

$$H = \begin{bmatrix} -1 & j^T \\ j & K \end{bmatrix}$$

is a symmetric Hadamard matrix with constant diagonal. Let F be the (n+1) x (n+1) diagonal matrix whose diagonal elements are alternately -1 and +1 and set $G = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \otimes F \end{bmatrix}$ and $\overline{H} = GHG$. Then $\overline{H}J = -nJ$ so the strong graph corresponding to $\overline{H}$ is regular with valency $n_1 = \frac{1}{2}n(n+1)$.

If n=12 we obtain a strongly regular graph of order 144 and valency 78. We may also obtain a strongly regular graph of order 144 from the construction (5.3). This graph has valency 66 so that both possibilities of (2.6.1) are realised.

6. Designs related to regular 2-graphs

Let $T = (\Omega, \underline{t})$ be a regular 2-graph and let (Ω_0, A_0) be the strongly regular graph obtained by deleting the point 0. From § 2.2 $A_0^2 = (a'-a)A_0 + (a+a'+1)I-J$ and $AJ = (a'-a)J$. We put

$$N = \tfrac{1}{2}(J + I - A_0). \tag{6.1}$$

The matrix N has entries 0 and 1 so we may enquire whether N is the incidence matrix of a projective design. Since $NJ = aJ$, $N = N^T$ and

$$N^2 = \tfrac{1}{4}\left((3a-a'+2)J + (a+a'+2)I + (a'-a-2)A_0 \right)$$

this will be the case if and only if

$$a' = a + 2. \tag{6.2}$$

If (6.2) holds the design described by N has parameters

$$v = b = 2a+3, \quad r = k = a+1, \quad \lambda = \tfrac{1}{2}a \tag{6.3}$$

From (2.5.8) we have $\sigma-1 = \omega+1 = m$ say and the parameters of the regular 2-graph are as in (5.2) with $a = 2m^2-2$. Thus we are dealing with the same class of 2-graphs as in § 5. The design is a Hadamard design (Dembowski [1], pp. 59, 111, 113) and therefore has an extension to an affine design with parameters

$$v = 2a+4, \quad b = 2(2a+3), \quad r = 2a+3, \quad k = a+2, \quad \lambda = a+1. \tag{6.4}$$

If $\alpha \in \Omega_0$, then $B(\alpha) = \{\alpha\} \cup \{\beta \in \Omega_0 \mid \{0, \alpha, \beta\}$ is coherent $\}$ is a block of the Hadamard design and the

correspondence $a \leftrightarrow B(a)$ is a null polarity which commutes with every automorphism of (Ω_o, A_o). This design was considered by D.G. Higman [1] in the context of rank 3 groups.

If in (6.1) we put $N = \frac{1}{2}(J-I-A_o)$ the design obtained is the complement of that considered above, provided $a = a'+2$.

Next suppose that $(\Omega, A) \in \mathcal{S}(T)$ is strongly regular. In this case we have $\rho_o = \rho_1$ or ρ_2 and $A^2 = (a'-a)A + (a+a'+1)I$. As in (6.1) put

$$N = \tfrac{1}{2}(J+I-A). \tag{6.5}$$

Then $NJ = -\frac{1}{2}(\rho_o + \rho_1\rho_2)J$ and

$$N^2 = \tfrac{1}{4}\left((a+a'+4-2\rho_o)J + (a+a'+2)I + (a'-a-2)A \right)$$

Thus N is the incidence matrix of a projective design if and only if (6.2) holds. If this is the case then $\sigma - 1 = \omega + 1 = m$, $\rho_1 = 2m+1$, $\rho_2 = -2m+1$, $\rho_o = 2mx+1$ where $x = \pm 1$ and the design has parameters:

$$v = b = 4m^2, \quad r = k = 2m^2 - mx, \quad \lambda = m^2 - mx. \tag{6.6}$$

If we put $N = \frac{1}{2}(J-I-A)$ in (6.5), then we obtain a design if $a = a'+2$. The parameters are still given by (6.6) since the two sets of parameters of (6.6) are complementary.

In some instances the designs obtained seem to be new. For example, from a Latin square of order 6 we may construct a strongly regular graph which leads to a design with

parameters $v = b = 36$, $r = k = 15$, $\lambda = 6$. This is listed as 'solution unknown' in Hall [1], p. 297.

The strongly regular graphs considered here are symmetric (v,k,λ)-graphs in the sense of Ahrens and Szekeres [1]. Other examples of these graphs will be considered in the next chapter.

C H A P T E R 4

DOUBLY TRANSITIVE GROUPS

In this chapter we construct five infinite families of regular 2-graphs which have doubly transitive automorphism groups. A sixth such family has been constructed in § 2.3. This is the family of regular 2-graphs T(q), q a prime power $\equiv 1 \pmod 4$; the automorphism group of T(q) being $P\Gamma L(2,q)$. We also construct 2-graphs from the doubly transitive representations of the sporadic simple groups HiS and C_3.

The main theorem of § 1 shows that in most cases the regular 2-graph can be constructed via the centraliser ring of a monomial representation associated with the doubly transitive permutation representation. This is used to construct regular 2-graphs for the groups of Ree type. For the other regular 2-graphs more information is obtained by giving a geometrical description.

1. A monomial representation and its centraliser ring

In this section we extend some of the results of Chapter V of Wielandt [1] to regular 2-graphs with doubly transitive automorphism groups by considering monomial representations as well as permutation representations (see Feit [1]).

Let G be a finite group. By a representation of G of degree n we shall mean a homomorphism of G into the group of nxn non-singular matrices over the field $\mathbb{C}$ of complex

numbers. If $\underline{R}$ is a representation of G of degree n we define the centraliser ring $C(\underline{R})$ of $\underline{R}$ to be the ring of all nxn matrices which commute with every matrix $\underline{R}(x), x \in G$. As usual, let J denote the (nxn) matrix, every entry of which is 1. A monomial matrix is a matrix M = D P where D is a diagonal matrix and P is a permutation matrix; a monomial representation of G is a representation $\underline{M}$ such that for every x in G $\underline{M}(x)$ is a monomial matrix.

(1.1) Lemma. Let $\underline{M}$ be a monomial representation of G such that the non-zero entries of $\underline{M}(x)$ are ± 1 for each x in G. If $J \in C(\underline{M})$, then G has a subgroup of index 2 or $\underline{M}$ is a permutation representation.

Proof. For $x \in G$ we have $\underline{M}(x) = D(x)\underline{P}(x)$ and $J\underline{M}(x) = \underline{M}(x)J$. Thus $D(x) = \pm I$ and we can write $\underline{M}(x) = \mu(x)\underline{P}(x)$ where $\mu(x) = \pm 1$. Since $\underline{M}$ is a representation, μ is a linear character of G. If μ is not the principal character it defines a homomorphism of G onto a group of order 2. //

The next theorem shows how regular 2-graphs are obtained from doubly transitive groups.

(1.2) Theorem. Let G be a group which acts doubly transitively on a set Ω of n points such that

(i) n is even,

(ii) G has no subgroup of index 2,

(iii) If H is the stabiliser of a point then H has a subgroup K of index 2.

Let M_o be the non-principal linear representation of H which has K in its kernel and let M be the induced monomial representation of G. Then one of the following two possibilities occurs:

(a) M is irreducible and C(M) has dimension 1.

(b) C(M) has dimension 2 and contains a matrix A which is the adjacency matrix of a strong graph (Ω, A), such that the group G acts as a group of automorphisms of the associated regular 2-graph T. The multiplicities of the eigenvalues of A are equal to the degrees of the irreducible constituents of M and hence the parameters of T are uniquely determined by M. The matrices A and I form a basis for C(M).

Proof. Let $x \in H-K$, then $x^2 \in K$ and $H = K + Kx$. We shall identify the points of Ω with the cosets of H and we shall identify an orbit of a subgroup of G with the corresponding double coset. Since G acts doubly transitively on Ω we can find an involution t such that $G = H + HtH$ and then

$$G = K + Kx + KtK + KxtK + KtxK + KxtxK. \tag{1.3}$$

Hence G has rank at most 6 when acting on the cosets of K. The non-trivial orbits of K in (1.3) all have the same length $|K : K \cap K^t|$. Since n is even and $|G : K| = 2n$, G cannot have rank 6. Moreover

$$\begin{aligned} KtK = KxtK \quad &\text{if and only if} \quad KtxK = KxtxK \\ KtK = KtxK \quad &\text{if and only if} \quad KxtK = KxtxK \\ KtK = KxtxK \quad &\text{if and only if} \quad KtxK = KxtK. \end{aligned} \tag{1.4}$$

Thus G has rank 3 or 4 on the cosets of K. Let 1_K be the trivial representation of K. Then $(1_K)^H = 1_H + \underline{M}_o$ so $(1_K)^G = (1_H)^G + \underline{M}$. If $\underline{P}$ is the associated permutation representation of $\underline{M}$, then $\underline{P} = (1_H)^G = 1_G + \underline{X}$ where $\underline{X}$ is irreducible. If G has rank 3 on the cosets of K, then $\underline{M}$ must be irreducible and (a) holds. From now on we suppose that G has rank 4 on the cosets of K. Then $\underline{M}$ has two irreducible constituents $\underline{M}_1$ and $\underline{M}_2$ and $C(\underline{M})$ has dimension 2. Let $A = (a_{ij})$ be a matrix in $C(\underline{M})$ which is not a multiple of I and let j be an involution of G. We may suppose $\underline{M}(j)$ to have the form

(1.5)
$$\begin{bmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & & \\ & & * & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & & \\ & & * & \end{bmatrix}$$

Since $\underline{M}(j) A \underline{M}(j) = A$ we have $a_{11} = a_{22}$ and $a_{12} = a_{21}$. By double transitivity all the diagonal elements of A are equal and A is symmetric. By subtracting a suitable multiple of I we may assume that the diagonal elements of A are 0. Now using a matrix of the form

(1.6)
$$\begin{bmatrix} \pm 1 & 0 & 0 & \\ 0 & 0 & \pm 1 & \\ & & & \\ & & * & \end{bmatrix}$$

we obtain $\pm a_{12} = a_{13}$. Again by double transitivity all off diagonal elements are equal up to sign. Therefore, by

dividing by a_{12} we may assume that the off diagonal elements of A are ± 1. By Schur's lemma A has just two eigenvalues whose multiplicities are the degrees of $\underline{M}_1$ and $\underline{M}_2$. Thus A is the adjacency matrix of a strong graph (Ω, A) of type 1 or 2. Let T be the corresponding regular 2-graph. If T is trivial, then we can find a diagonal matrix D with diagonal entries ± 1 such that $J \in C(D^{-1}\underline{M}D)$. Since $D^{-1}\underline{M}D$ is monomial Lemma (1.1) contradicts assumption (ii). Hence T is non-trivial and (Ω, A) is of type 2.

For g in G we can write $\underline{M}(g) = \underline{P}(g)D(g)$ where D(g) is a diagonal matrix with diagonal entries ± 1. Then

$$\underline{P}(g)^{-1}A\underline{P}(g) = D(g)\underline{M}(g)^{-1}A\underline{M}(g)D(g)^{-1}$$

$$= D(g)AD(g)^{-1}$$

Since $(\Omega, D(g)AD(g)^{-1})$ is equivalent to (Ω, A) via switching, $\underline{P}(g)$ induces an automorphism of T. Since all the parameters of T can be calculated from the multiplicities of the eigenvalues of A all parts of the theorem are proved. //

Remarks. 1. If we omit condition (ii) from the hypothesis of the theorem then it may happen that (Ω, A) is a trivial graph. However, if this is the case, then from the proof of (1.1) $\underline{M}(g) = \mu(g)\underline{P}(g)$ so $\underline{M} = \mu + \underline{X}$. Hence we may replace (ii) by

(ii)' no constituent of $\underline{M}$ has degree 1.

and the conclusion of the theorem still holds.

2. If we omit condition (i) then either the conclusion of

the theorem still holds or G has rank 6 on the cosets of K and n is odd. Let π be the character of $\underline{P}$ and let π' be the character of $(1_K)^G$. Since H has three orbits on the cosets of K the inner product of π with π' is 3. Now $\pi = 1 + \phi$ where ϕ is irreducible so ϕ occurs in π' with multiplicity 2. Since the inner product of π' with itself is 6 π' has one further constituent μ which has degree $2n-2(n-1)-1 = 1$. Since μ is rational its square is the principal character. Hence G has a subgroup of index 2. We have $\underline{M} = \underline{X} + \mu$, regarding μ as a representation, so the centraliser ring $C(\underline{M})$ has dimension 2. However the strong graph (Ω, A) is trivial. This occurs, for example, in the Frobenius group of order 20 acting on 5 points.

We now consider the converse situation to (1.2). Let $T = (\Omega, \underline{t})$ be a regular 2-graph and suppose that $G = \mathrm{Aut}(T)$ acts doubly transitively on Ω. Let (Ω, A) be a strong graph of $\mathcal{S}(T)$ and let $\underline{P}$ denote the permutation representation of G on Ω. If x belongs to G, then $(\Omega, \underline{P}(x)^{-1} A \underline{P}(x))$ is in $\mathcal{S}(T)$ so there is a diagonal matrix D(x) (with entries ± 1) such that $\underline{P}(x)^{-1} A \underline{P}(x) = D(x) A D(x)$. Thus $\underline{M}(x) = \underline{P}(x) D(x)$ commutes with A. Let G_0 be the group of all monomial matrices with entries $0, \pm 1$ which commute with A. Then the subgroup $Z = \{ \pm I \}$ lies in the centre of G_0 and G is isomorphic to G_0/Z. Unfortunately G_0 does not always split over Z (see the next section). However if G_0 does split we shall write $G_0 = Z \times G$; in this case we may suppose that $\underline{M}$ defines a monomial representation of G. We now relate this to the equiangular lines defined

in § 2.4. Let μ_1 and μ_2 be the multiplicities of the eigenvalues ρ_1 and ρ_2 of A. Let $v_1,\ldots,v_N$ be the vectors spanning the equiangular lines defined by A in dimension μ as in (2.4. 9) where μ is either μ_1 or μ_2. The matrices M of G_o are orthogonal and induce orthogonal transformations in $\mathbb{R}^{\mu}$ which permute the equiangular lines among themselves. Moreover, the group G_o is just the group of symmetries of the configuration. If μ_1 (and hence μ_2) is odd then by taking the determinant of the representation of G_o in $\mathbb{R}^{\mu}$ we see that -I lies outside a subgroup of G_o of index 2. Hence Z splits in this case.

We have presented G_o as a group of monomial matrices so we may take the representation $\underline{M}$ to be the identity automorphism. Since the spaces of dimensions μ_1 and μ_2 spanned by the two sets of equiangular lines can be considered as invariant subspaces of the space on which G_o acts we see that $\underline{M}$ has two constituents $\underline{M}_1$ and $\underline{M}_2$ of degrees μ_1 and μ_2 respectively. By Theorem (1.2) and the remark $\underline{M}_1$ and $\underline{M}_2$ must be irreducible. Thus $\underline{M}$ is not a permutation representation and therefore, if H is the stabiliser of a point in the associated permutation representation of $\underline{M}$, then H has a subgroup K of index 2 and a linear representation $\underline{M}_o$ with K in its kernel such that $\underline{M}$ is similar to $\underline{M}_o{}^G$. If $G_o = Z \times G$, then we may restrict $\underline{M}$ to G so that the same conclusions hold for G.

We have proved the following partial converse to (1.2) :

(1.8) Theorem. Let $T = (\Omega, t)$ be a regular 2-graph with $G = \mathrm{Aut}(T)$ doubly transitive on Ω. Let G_0 be the group of symmetries of a set of equiangular lines associated with T and let Z be the subgroup generated by the central inversion. Then $Z \leq Z(G_0)$ and $G = G_0/Z$. Moreover, G_0 has a monomial representation M (with entries $0, \pm 1$) having just two irreducible constituents M_1 and M_2 which are the representations of G_0 on the vectors spanning the equiangular lines. If H is the stabiliser of a point in the associated permutation representation then H has a subgroup K of index 2 and M is similar to the representation induced from the non-principal linear representation of H with K in its kernel. The regular 2-graph derived from M as in Theorem (1.2) coincides with T. If $G_0 = Z \times G$ then the above conclusions hold with G_0 replaced by G. This will be the case if μ_1 and μ_2 are odd.

Remark. It is always the case that in G_0 the stabiliser H_0 of a line splits over Z. Thus $H_0 = Z \times H$ where H is the stabiliser of a vector.

Suppose that we are in case (b) of (1.2). By conjugating with a diagonal matrix we may suppose that the non-zero entries of the first row and column of A are all 1. Let A_0 be the matrix obtained by deleting the first row and column of A. From Chapter 2 we know that A_0 is the adjacency matrix of

strongly regular graph. We also have

(1.9) Theorem. The matrix A_0 belongs to the centraliser ring of the restriction of P to H.

Proof. If $g \in G$ and $M(g)$ has non-zero entry ω in position (1,1) then every other non-zero entry of $M(g)$ must be ω since $M(g)$ commutes with A. Thus $M(g) = \pm P(g)$ and therefore $P(g)$ commutes with A and the theorem is proved. //

2. Symplectic geometry over GF(2)

Let V be a vector space of dimension 2m over the field GF(2) and let B : VxV - GF(2) be a non-degenerate alternating form on V. To avoid a trivial case we shall suppose $m > 1$. Let $P = V^{\#}$ be the set of non-zero vectors of V. The group Sp(2m,2) acts as a rank 3 permutation group on P and the stabiliser of a vector x has the following non-trivial orbits:

$$(2.1) \qquad D_1(x) = \{ y \in P \mid B(x,y) = 0, \ x \neq y \}$$
$$D_2(x) = \{ y \in P \mid B(x,y) = 1 \}.$$

From Huppert [1], p. 221 we see that a strongly regular graph defined by this rank 3 group has parameters

$$(2.2) \qquad n = 2^{2m}-1$$
$$n_1 = 2^{2m-1}-2 \qquad p_{11}^1 = 2^{2m-2}-3$$
$$n_2 = 2^{2m-1} \qquad p_{11}^2 = 2^{2m-2}-1$$

$$\rho_1 = 2^m+1 \qquad \mu_1 = 2^{2m-1}-2^{m-1}-1$$

$$\rho_2 = -2^m+1 \qquad \mu_2 = 2^{2m-1}+2^{m-1}+1$$

Since $n_1 = 2p_{11}^2$ we obtain a regular 2-graph $S(2m) = (V,\underline{t})$ by adjoining a point which we may take to be $o \notin V$. In fact

$$(2.3) \qquad \underline{t} = \{ \{ x,y,z \} \mid B(x,y) + B(y,z) + B(z,x) = 0 \} .$$

The parameters for S(2m) are

$$(2.4) \qquad N = 2^{2m}$$

$$a = 2^{2m-1}-2 \qquad a' = 2^{2m-1}$$

$$k = 2^{2m-2}-3 \qquad k' = 2^{2m-2}.$$

Now let T(V) denote the group of translations of V (an elementary abelian 2-group isomorphic to V) and let H be the group Sp(2m,2) of linear transformations of V which fix B. From the description of $\underline{t}$ in (2.3) the group $G = T(V)\,H$ is a group of automorphisms of S(2m) and acts doubly transitively on V.

(2.5) Theorem. The automorphism group of S(2m) is the group G.

Proof. It is enough to show that $H = \mathrm{Aut}(S(2m)_o)$. We already have $H \leq \mathrm{Aut}(S(2m)_o)$. We shall show that the vector space structure of V can be obtained from the 2-graph structure and that any automorphism of $S(2m)_o$ is a linear

transformation preserving the bilinear form B.

Let M be a coherent subset of V containing o. From (2.3) M is totally isotropic. Thus a coherent set of maximal order is just a translate of a maximal totally isotropic subspace. We shall suppose M to be a maximal totally isotropic subspace. Since $|M| = 2^m = 1-\rho_2$ from § 2.7 we have $m(x) = 2^{m-1}$ for $x \notin M$. Choose the notation so that $o \in M_1(x)$. Given $a,b \in M$, $a,b \neq o$, define $L(a,b) = \cap \{M_1(x) \mid a,b \in M_1(x), \quad x \notin M\}$. Then L(a,b) is the(totally isotropic)plane containing a and b, a+b is uniquely determined by $L(a,b) = \{o,a,b,a+b\}$ and L(a,b) does not depend on the choice of M. If $\{o,a,b\}$ is incoherent, then the(hyperbolic)plane determined by a and b is the unique maximal incoherent set of 4 points containing o, a and b. Thus a+b is again uniquely determined by the 2-graph structure. For $g \in \mathrm{Aut}(S(2m)_o)$ and $a,b \in V$ we therefore have $(a+b)g = ag + bg$ and from (2.3) $B(a,b) = B(ag,bg)$. Hence $H = \mathrm{Aut}(S(2m)_o)$ and $G = \mathrm{Aut}(S(2m))$. //

In terms of rank 3 groups, this proof is given by D.G. Higman [1].

Since a hyperbolic plane is a maximal incoherent set of 4 points, the complement of S(2m) is not 4-regular.

However, there exist incoherent sets of size 2m+2. To see this, write $V = V_1 \perp \ldots \perp V_m$ where V_i is a hyperbolic plane with basis e_i,f_i. Then the vectors o, e_1, f_1, $e_1+f_1+e_2$, $e_1+f_1+f_2,\ldots,e_1+\ldots+e_m+f_1+\ldots+f_m$ form an incoherent set of size 2m+2. This is obviously as large

as possible.

The 2-graph S(2m) itself is not 4-regular unless m=2, in which case $k_4 = 0$. If $m > 2$ then o, e_1, e_2 and e_1+e_2 form a coherent set of 4 points contained in $2^{2m-2}-2$ coherent sets of 5 points while o, e_1, e_2 and e_3 form a coherent set of 4 points contained in $2^{2m-3}-4$ coherent sets of 5 points. However, (A) and (B) of § 2.9 are satisfied with 2^m for m since if L is a coherent set of $2^{m-1}+1$ points then acting with T(V) we may suppose that $o \in L$ and then the linear span of L is the unique coherent set of size 2^m containing L.

Since a' = a+2 we are in the situation discussed in § 3.6. We obtain a projective design with parameters

(2.6) $\quad v = 2^{2m}-1, \qquad k = 2^{2m-1}-1, \qquad = 2^{2m-2}-1.$

This is just the design of points and hyperplanes of the projective geometry of V (D.G. Higman [1]). The extension with parameters (6.4) is the design of points and hyperplanes of the affine geometry of V.

In order to exhibit further designs we shall construct two strongly regular graphs of $\mathcal{S}(S(2m))$. Let $Q : V \to GF(2)$ be a quadratic form such that

(2.7) $\qquad Q(x+y) + Q(x) + Q(y) = B(x,y).$

Let $\mathcal{Q}$ be the set of quadratic forms which satisfy (2.7). The group H = Sp(2m,2) has two orbits on the set $\mathcal{Q}$ (Dickson [1], p. 197): the orbit $\mathcal{Q}'$ of forms of index m

and the orbit 2^{-1} of forms of index m-1. (See also Dembowski [1], p. 46 and Dieudonne [1], p. 34). Choose $Q^a \in 2^a$ and define

$$(2.8) \qquad \underline{q}^a = \{ x \in P \mid Q^a(x) = 0 \} \quad \text{where } a \in \{1,-1\}.$$

Let $O^a(2m,2)$ denote the group of linear transformations of V which fix Q^a. From (2.7) $O^a(2m,2)$ is contained in Sp(2m,2). By Witt's theorem (Dieudonné [1], p.36) the sets $\underline{q}^a$ and $P-\underline{q}^a$ are orbits of $O^a(2m,2)$. Hence the groups $G^a = T(V)\, O^a(2m,2)$ have rank 3 on V.

We have (Dembowski [1], p. 46)

$$(2.9) \qquad |\underline{q}^a| = (2^{m-1}+a)(2^m-a) = 2^{2m-1}+a\cdot 2^{m-1}-1.$$

A subspace W of V is called singular (with respect to Q^a) if $Q^a(x) = 0$ for all $x \in W$. The number of singular subspaces of dimension 2 in V is (Dembowski [1], p.46)

$$(2.10) \qquad b = \tfrac{1}{3}(2^m-a)(2^{m-1}-a)(2^{m-2}+a)(2^{m-1}+a)$$

Hence the number of singular subspaces of dimension 2 of V which contain a given point of $\underline{q}^a$ is

$$(2.11) \qquad r = 3b/|\underline{q}^a| = (2^{m-1}-a)(2^{m-2}+a).$$

Now a plane which intersects $\underline{q}^a$ in more than one point is either singular or hyperbolic so by Witt's theorem $O^a(2m,2)$ has rank 3 on $\underline{q}^a$ and the length of one of the suborbits is 2r where r is given by (2.11). Hence the strongly regular graph determined by the rank 3 groups G^a has parameters:

(2.12) $n = 2^{2m}$

$$n_1 = 2^{2m-1}+a.2^{m-1}-1 \qquad p_{11}^1 = 2(2^{m-1}-a)(2^{m-2}+a)$$

$$n_2 = 2^{2m-1}-a.2^{m-1} \qquad p_{11}^2 = 2^{m-1}(2^{m-1}+a)$$

$$\rho_1 = 2^m+1 \qquad \rho_o = -a.2^m+1$$

$$\rho_2 = -2^m+1$$

From (2.3) and (2.7) we see that these strongly regular graphs belong to $\mathcal{S}(S(2m))$ and $G^a \leq G$. Thus both possibilities of (2.6.1) occur and from the construction of (3.6.5) we obtain two projective designs with parameters

(2.13) $v = 2^{2m}$, $k = 2^{2m-1}+a.2^{m-1}$, $\lambda = 2^{2m-2}+a.2^{m-1}$.

These designs will be considered again in the next section.

From Dickson [1], Ch. VIII and the above discussion we see that the stabiliser of a point in $O^a(2m,2)$ acting on $\underline{q}^a$ is an elementary abelian group of order 2^{2m-2} extended by $O^a(2m-2,2)$ and the stabiliser of a point in $O^a(2m,2)$ acting on $P-\underline{q}^a$ is $Z_2 \times Sp(2m-2,2)$.

Suppose that $m = m_1m_2$ and let X be the Galois group of $GF(2^{m_1})$ over GF(2). The group $Sp(2m_2,2^{m_1})$ acts on a vector space E of dimension $2m_2$ over $GF(2^{m_1})$. If A is the alternating form fixed by $Sp(2m_2,2^{m_1})$ then by restriction of scalars we may identify E with V as above and define B as $B(x,y) = Tr(A(x,y))$ where Tr is the trace from $GF(2^{m_1})$

to GF(2). It follows that the semi-direct product $\sum p(2m_2,2^{m_1}) = Sp(2m_2,2^{m_1})$. X is a subgroup of Sp(2m,2), (see Huppert [1], p. 228). In particular, we have $\sum L(2,2^m) \subseteq Sp(2m,2)$ since $SL(2,2^m)$ is isomorphic to $Sp(2,2^m)$. Since $\sum p(2m_2,2^{m_1})$ acts transitively on the non-zero vectors of E the group $T(V)\cdot\sum p(2m_2,2^{m_1})$ acts doubly transitively on the regular 2-graph S(2m).

Again write $V = V_1 \perp \cdots \perp V_m$ where each V_i is a hyperbolic plane. Suppose further that each V_i is an elliptic plane for the quadratic form Q. If m is even, then Q has index m while if m is odd, then Q has index m-1. Since the group of automorphisms of each V_i is $SL(2,2) \simeq S_3$ we find that the wreath product S_3 wr S_m is a subgroup of $O^1(2m,2)$ when m is even and of $O^{-1}(2m,2)$ when m is odd.

If X is a set of 2m+2 elements we may identify the vector space V with the set of partitions of X into pairs of even subsets, where addition is symmetric difference.

If $x = (X_1,X_2)$ and $y = (Y_1,Y_2)$ we may suppose that our alternating form B is defined by

$$(2.15) \qquad B(x,y) = \begin{cases} 0 & |X_1 \cap Y_1| \equiv 0 \pmod 2 \\ 1 & |X_1 \cap Y_1| \equiv 1 \pmod 2 \end{cases}$$

since this defines a non-degenerate alternating form (note that $|X_1 \cap Y_1| \equiv |X_1 \cap Y_2| \equiv |X_2 \cap Y_1| \equiv |X_2 \cap Y_2| \pmod 2$). From this construction for V we obtain the inclusion $S_{2m+2} \subseteq Sp(2m,2)$. Moreover, if $m \equiv 1 \pmod 2$ then we may define a quadratic form Q satisfying (2.7) by

$$(2.16) \qquad Q(x) = \begin{cases} 0 & |X_1| \equiv 0 \pmod 4 \\ 1 & |X_1| \equiv 2 \pmod 4 \end{cases}$$

so that for $m \equiv 1 \pmod 2$ we have $S_{2m+2} \subseteq O^a(2m,2)$ where $a+m \equiv 0 \pmod 4$.

These constructions will be used in Chapter 5.

Finally, let us apply the results of § 1 to the 2-graphs S(2m). Since apart from Sp(4,2), Sp(2m,2) does not have a subgroup of index 2 it follows from (1.8) that the covering group G_o of G does not split over its centre. We may describe the group G_o as follows. Let W be a vector space of dimension 2m+1 over GF(2), let Q be a non-degenerate quadratic form on W (see Dieudonné [1], p. 33) and let F be the associated alternating form. The subspace W_o of W orthogonal to W with respect to F has dimension 1. If O(2m+1,2) is the group of linear transformations leaving Q invariant then every element of O(2m+1,2) fixes W_o pointwise and O(2m+1,2) is isomorphic to Sp(2m,2), (Dickson [1], p. 200). Let us write $W_o = \langle e_o \rangle$ and define $\underline{q}$ as in (2.8). Let $\underline{q}'$ be the points of W not in $\underline{q}$ or W_o. Then the orbits of Sp(2m,2) on W are o, e_o, $\underline{q}$ and $\underline{q}'$ and we have $|\underline{q}| = |\underline{q}'| = 2^{2m}-1$. The translation t_{e_o} in direction e_o lies in the centre of $G_o = T(W)\cdot Sp(2m,2)$ and if we put $Z = \langle t_{e_o} \rangle$ we may identify $T(W)\cdot Sp(2m,2)/Z$ with $T(V)\cdot Sp(2m,2)$ where $V = W/W_o$. If $m \neq 2$, then G_o does not have a subgroup of index 2 we may use Theorem (1.2) to construct a regular 2-graph. By Theorem (1.9) this coincides with S(2m) since S(2m) was defined via the rank 3 representation of Sp(2m,2). This also holds for S(4) since t_{e_o} does not lie outside a subgroup of index 2.

3. Quadratic forms over GF(2)

We continue the notation of § 2: V is a vector space of dimension 2m on GF(2) with a non-degenerate alternating form B and $\mathcal{Q}$ is the set of quadratic forms satisfying (2.7). Again put G = T(V).H where H = Sp(2m,2). For $x \in V$ define the translation

$$t_x(y) = x + y \tag{3.1}$$

and the symplectic transvection

$$\emptyset_x(y) = y + B(x,y)x \tag{3.2}$$

It is well known (Dieudonné [1], p. 41) that H is generated by the involutions $\emptyset_x$, $x \in V$ and that these elements are a conjugacy class of H.

Throughout this section suppose that $a \in \{1,-1\}$. We shall construct a regular 2-graph on the set $\mathcal{Q}^a$ which has Sp(2m,2) as its (doubly transitive) automorphism group. In order to do this we construct several designs one of which is the projective design considered in the previous section. We shall show that its automorphism group is T(V).Sp(2m,2).

Choose Q_0 in $\mathcal{Q}^a$ and define

$$\underline{q}^a = \{ x \in V \mid x \neq 0,\ Q_0(x) = 0 \} \tag{3.3}$$

$$\underline{p}^a = \{ x \in V \mid Q_0(x) = 1 \}$$

For any Q in $\mathcal{Q}$ we can write $Q(x) + Q_0(x) = f(x)^2$ where

$f : V \to GF(2)$ is a linear functional. We may find an element a of V such that $f(x) = B(a,x)$ for all x in V. Hence any element of $\mathcal{Q}$ may be written uniquely as

$$(3.4) \qquad Q_a(x) = Q_o(x) + B(a,x)^2.$$

Thus $|\mathcal{Q}| = 2^{2m}$ and we may make $\mathcal{Q}$ into a vector space over GF(2) with zero Q_o by defining an addition $\dot{+}$ by

$$(3.5) \qquad (Q_1 \dot{+} Q_2)(x) = Q_1(x) + Q_2(x) + Q_o(x).$$

We shall consider $\mathcal{Q}$ as an affine space over GF(2) and show that G acts as a group of affine transformations of $\mathcal{Q}$. First define the action of $t_a \in T(V)$ on $\mathcal{Q}$ by

$$(3.6) \qquad (t_a Q)(x) = Q(x) + B(a,x)^2.$$

Then t_a acts as an affine transformation of $\mathcal{Q}$ and $t_a Q_b = Q_{a+b}$. Now define the action of $\sigma \in H$ on $\mathcal{Q}$ by

$$(3.7) \qquad (\sigma Q)(x) = Q(\sigma^{-1} x)$$

Again we have an affine transformation of $\mathcal{Q}$ and if $a \in V$ we have $(\sigma t_a \sigma^{-1} Q)(x) = Q(x) + B(\sigma a, x)$ so that

$$(3.8) \qquad \sigma t_a \sigma^{-1} = t_{\sigma a}.$$

Hence the group $G = T(V) \cdot H$ acts on the affine space . Let H_o be the subgroup of G fixing Q_o. Then $H_o \simeq Sp(2m,2)$ and H_o acts on $\mathcal{Q}$ the way H acts on V since T(V) acts regularly on $\mathcal{Q}$. Hence G acts doubly transitively on $\mathcal{Q}$. We may define an isomorphism

$$(3.9) \qquad \gamma : V \to \mathcal{Q} : a \mapsto Q_a$$

Changing Q_o amounts to altering γ by a translation. We may use γ to transfer the regular 2-graph structure of V to $\mathcal{2}$. The set $\underline{t}$ of coherent triangles so obtained does not depend on the choice of Q_o.

Now let $\mathcal{2}^{a}_{(3)}$ be the set of three element subsets of $\mathcal{2}^{\alpha}$ and define

$$\underline{t}^{a} = \underline{t} \cap \mathcal{2}^{a}_{(3)}. \tag{3.10}$$

We shall show that $(\mathcal{2}^{a}, \underline{t}^{a})$ is a regular 2-graph. To do this we construct a design $\underline{s}$ by taking $\mathcal{2}$ as the set of points and the images of $\mathcal{2}^{-a}$ under G as blocks. By the results of § 3.1 $\underline{s}$ is projective and G acts doubly transitively on blocks. Moreover, the permutation character of H acting on points is the same as that of H acting on blocks. Since T(V) acts regularly on the blocks and H fixes $\mathcal{2}^{-a}$ we see that H acts on blocks as it does on V. Hence the permutation character is $2\emptyset_o + \emptyset_1 + \emptyset_2$, where $\emptyset_o$ is the principal character and $\emptyset_1$ and $\emptyset_2$ are irreducible (from (2.1) H has rank 3 on V - { o }). We may now choose the notation so that the permutation character of H is $\emptyset_o + \emptyset_1$ on the points of $\mathcal{2}^{1}$ and $\emptyset_o + \emptyset_2$ on the points of $\mathcal{2}^{-1}$. We have proved

(3.11) H <u>acts doubly transitively on both</u> $\mathcal{2}^{1}$ and $\mathcal{2}^{-1}$.

Since G preserves coherent triangles in $\mathcal{2}$ and H acts doubly transitively on $\mathcal{2}^{a}$ it follows that $O^{a}(2m) = (\mathcal{2}^{a}, \underline{t}^{a})$ is a regular 2-graph.

We next consider the action of the transvection $\emptyset_a$ on

$\mathcal{Q}$ and relate this to t_a (cf. Jordan [1], p. 229).

$$(3.12) \qquad \phi_a Q = \begin{cases} t_a Q & Q(a) = 0 \\ Q & Q(a) = 1 \end{cases}$$

Proof. $(\phi_a Q)(x) = Q(\phi_a x)$

$$= Q(x + B(a,x)a)$$

$$= Q(x) + B(a,x)^2(Q(a) + 1). \quad //$$

Thus we have

(3.13) ϕ_a moves Q_b if and only if $Q_o(a) + B(a,b)^2 = 0$.

Since H is doubly transitive on $\mathcal{Q}^a$, given $Q_1, Q_2 \in \mathcal{Q}^a$ we can find a transvection ϕ_a such that $\phi_a Q_1 = Q_2$. From (3.12) we have $t_a Q_1 = Q_2$ and $Q_1(a) = 0$. It now follows that

(3.14) The quadratic forms Q_1 and Q_2 of $\mathcal{Q}$ have the same index if and only if there exists an element a in V such that $t_a Q_1 = Q_2$ and $Q_1(a) = Q_2(a) = 0$. The element a is uniquely determined by Q_1 and Q_2.

Since $Q_a = t_a Q_o$ we have

$$(3.15) \qquad \mathcal{Q}^a = \{ Q_a \mid Q_o(a) = 0 \}$$

$$\mathcal{Q}^{-a} = \{ Q_a \mid Q_o(a) = 1 \}.$$

We are now able to relate the design $\underline{s}$ to the projective design constructed in § 2. The complement of that design has V for its set of points and the translates of $\underline{p}^a$ for

its blocks. But $\gamma \underline{p}^a = 2^{-a}$ so that this is just the design $\underline{s}$. It follows from (2.9) that

$$(3.16) \qquad |2^a| = 2^{2m-1} + a.2^{m-1}.$$

Hence the parameters of $\underline{s}$ are

$$(3.17) \quad v=b=2^{2m}, \qquad r=k=2^{2m-1}-a.2^{m-1}, \qquad \lambda = 2^{2m-2}-a.2^{m-1}.$$

Given $Q_a \in 2^a$ we want to calculate the number of points $Q_b \in 2^a$ such that $\{Q_o, Q_a, Q_b\} \in \underline{t}^a$. This is the number of points b such that $B(a,b) = 0$ and $Q_o(b) = 0$. Since $Q_o(a) = 0$ this is just twice the number of singular planes (with respect to Q_o) which contain a. By (2.11) this is $2(2^{m-1}-a)(2^{m-2}+a)$ and hence

(3.18) Theorem. The parameters of the regular 2-graph $O^a(2m)$ are

$$N = 2^{2m-1}+a.2^{m-1}$$

$$a = 2^{2m-2}+a.2^{m-1}-2 \qquad a' = 2^{2m-2}$$

$$k = 2^{2m-3}+a.2^{m-1}-3 \qquad k' = 2^{2m-3}-a.2^{m-2}$$

$$\rho_1 = (3-a).2^{m-2}+1 \qquad \mu_1 = \tfrac{1}{3}(2^m-1)(2^{m-2}(3+a)+a)$$

$$\rho_2 = -(3+a).2^{m-2}+1 \qquad \mu_2 = \tfrac{1}{3}(2^m+1)(2^{m-2}(3-a)+a)$$

The subgroup of H fixing Q_o is just $H_o \cap H = O^a(2m,2)$ and $H_o \cap H$ acts on the points of 2 the way $O^a(2m,2)$ acts on V. Thus the strongly regular graph obtained from $O^a(2m)$

by deleting a point is just the graph obtained from the rank 3 representation of $O^a(2m,2)$ on $\underline{q}^a$. Its parameters are:

(3.19) $n = 2^{2m-1}+a.2^{m-1}-1$

$n_1 = 2^{2m-2}+a.2^{m-1}-2$ $\qquad p_{11}^1 = 2^{2m-3}+a.2^{m-1}-3$

$n_2 = 2^{2m-2}$ $\qquad p_{11}^2 = 2^{2m-3}+a.2^{m-2}-1.$

There are just four possibilities for the restriction of Q_o to a plane of V. We can choose a basis e,f for the plane so that it has one of the following forms:

(3.20) singular $\qquad Q_o(e) = Q_o(f) = B(e,f) = 0$

(3.21) elliptic $\qquad Q_o(e) = Q_o(f) = B(e,f) = 1$

(3.22) hyperbolic $\qquad Q_o(e) = Q_o(f) = 0,\ B(e,f) = 1$

(3.23) totally isotropic $\qquad Q_o(e) = Q_o(f) = 1,\ B(e,f) = 0.$

By Witt's theorem $O^a(2m,2)$ is transitive on each type of plane and hence $O^a(2m,2)$ acts as a rank 3 group on both $\underline{q}^a$ and $\underline{p}^a$. Using (2.10), (2.11), (2.12) and the fact that V contains $\frac{1}{3}\,2^{2m-2}(2^m-1)$ hyperbolic planes (with respect to B) we obtain the following table:

(3.24)

	#planes containing a given singular point	#planes containing a given non-singular point
singular	$(2^{m-1}-a)(2^{m-2}+a)$	-
elliptic	-	$2^{m-2}(2^{m-1}-a)$
hyperbolic	2^{2m-2}	$2^{m-2}(2^{m-1}+a)$
totally isotropic	$2^{m-2}(2^{m-1}-a)$	$2^{2m-2}-1$

This determines the parameters n_1 and n_2 of the strongly regular graph obtained from the action of $O^a(2m,2)$ on $\underline{p}^a$. Since $O^a(2m,2) = H_0 \cap H$ acts on $\underline{p}^\alpha$ as it does on $\mathcal{Q}^{-a}$ we see that the regular 2-graph obtained from the strongly regular graph by the construction of (2.2.9) coincides with $O^{-a}(2m)$. Hence the remaining parameters may be calculated from (2.6.1) and we have:

(3.25)
$$n = 2^{2m-1}-a.2^{m-1}$$
$$n_1 = 2^{2m-2}-1 \qquad p_{11}^1 = 2^{2m-3}-2$$
$$n_2 = 2^{2m-2}-a.2^{m-1} \qquad p_{11}^2 = 2^{2m-3}+a.2^{m-2}.$$

This realises one of the possibilities of (2.6.1). The other possibility for the parameters of a strongly regular graph of $\mathcal{S}(O^{-a}(2m))$ is

(3.26)
$$n = 2^{2m-1}-a.2^{m-1}$$
$$n_1 = 2^{2m-2}-3.a.2^{m-2}-1 \qquad p_{11}^1 = 2^{2m-3}-3a\ 2^{m-2}-2$$
$$n_2 = 2^{2m-2}+a.2^{m-2} \qquad p_{11}^2 = 2^{2m-3}-a.2^{m-1}$$

If m is even we may always construct such a graph. We leave the construction to the end of this section. However, if m is odd such a construction is not always possible. For example, we have the following well-known result.

(3.27) Theorem. There does not exist a strongly regular graph whose parameters are given by (3.26) with m=3 and a=1.

Proof. If such a graph exists we may construct a regular 2-graph from it with the parameters of $O^{-1}(6)$, namely N=28, a=10, k=1, $\rho_1=9$, $\rho_2=-3$, $\mu_1=7$ and $\mu_2=21$. But $p_{11}^1=0$ so the regular 2-graph would contain a coherent set of 10 points contradicting (2.7.5). //

If m=3 and a=-1 a strongly regular graph with parameters (3.2.6) can be constructed from the rank 3 representation of $P\Gamma U(3,3^2)$ on 36 points (see Bussemaker and Seidel [1], p. 15).

(3.28) Theorem. Any coherent set of $O^a(2m)$ is contained in a maximal coherent set. Maximal coherent sets have size $1-\rho_2 = (3+a).2^{m-2}$ and H acts transitively on maximal coherent sets.

Proof. By transitivity of H we may suppose the coherent set C to contain Q_o. Now using γ we may suppose that C is contained in $\underline{q} \cup \{o\}$. Hence the subspace generated by C is still coherent. By definition, $\underline{q}^a \cup \{o\}$ contains a subspace of dimension m if a=1 and dimension m-1 if a = -1. By (2.7.3) these are coherent sets of maximal size. The rest of the theorem now follows from Witt's theorem. //

In order to determine incoherent subsets of $O^a(2m)$ write $V = V_1 \perp \cdots \perp V_m$ where each V_i is a hyperbolic plane (with respect to B) with basis e_i, f_i. As in § 2 we have an incoherent set $\{o, e_1, f_1, e_1 + f_1 + e_2, e_1 + f_2 + f_2, \ldots\}$ in S(2m). We may suppose that the V_i are alternately

hyperbolic and elliptic except possibly that V_{m-1} and V_m have the same type. We thus obtain the following table for the maximum size of an incoherent set in $O^a(2m)$. The columns are headed with m (mod 4).

(3.29)

	0	1	2	3
$a = 1$	2m+2	2m+1	2m	2m
$a = -1$	2m	2m	2m+2	2m+1

For example, if m=3 we have sets of size 6 or 7 according to a=1 or a=-1. Thus the bounds obtained in § 2.7 are reached in these cases. Similarly, if m=4 and a=1 we again achieve the bound given in § 2.7.

We wish to show that the automorphism group of $O^a(2m)$ is H = Sp(2m,2). This will be done by relating the transvections $\emptyset_a$ to the structure of $O^a(2m)$. First consider the design $\underline{s}'$ obtained from $\underline{s}$ by deleting the block $\mathcal{Q}^{-a}$ and all of its points. Since H fixes $\mathcal{Q}^{-a}$ it acts as a group of automorphisms of $\underline{s}'$. The parameters of $\underline{s}'$ are

(3.30)
$$v = 2^{2m-1} + a.2^{m-1}$$
$$b = 2^{2m}-1$$
$$r = 2^{2m-1}-a.2^{m-1}$$
$$k = 2^{2m-2}$$
$$\lambda = 2^{2m-2}-a.2^{m-1}$$

Let us denote the block $t_a 2^{-a} \cap 2^a$ by C_a.

(3.31) $$C_a = \{ Q \in 2^a \mid Q(a) = 1 \}$$

$$= \{ Q \in 2^a \mid \emptyset_a Q = Q \}$$

Proof. If $Q_b \in 2^a$ then $Q_o(b) = 0$ so $Q_b(a) = Q_o(a) + B(a,b) = Q_o(a+b)$. Hence $Q_b \in t_a 2^{-a}$ if and only if $Q_b(a) = 1$. The second equality now follows from (3.12). //

(3.32) Theorem. Given Q_1 and Q_2 in 2^a there exists a in V such that $\emptyset_a Q_1 = Q_2$. If $Q \in 2^a$ then $\{Q_1, Q_2, Q\}$ is coherent if and only if $\emptyset_a Q \neq Q$.

Proof. The existence of a comes from (3.14). Let us write $Q_1 = Q_b$ and $Q_2 = Q_{a+b}$ for some b. From (3.12) we have $Q_b(a) = 0$ and $Q_{a+b}(a) = 0$ and from (3.15) we have $Q_o(a) = Q_o(a+b) = 0$. If $Q = Q_c$, then $Q_o(c) = 0$. Thus $B(a,b) = 0$ and $Q_c(a) = B(a,c)^2$. Since $B(b,a+b)+B(a+b,c)+B(c,b) = B(a,c)$ the result follows from (3.12). //

We may define an equivalence relation on unordered pairs (Q_1,Q_2), $Q_1, Q_2 \in 2^a$ by making (Q_1,Q_2) equivalent to (Q_3,Q_4) if both appear in the cycle structure of some transvection. When k=1 this reduces to the equivalence relation defined in § 2.9.

Let Γ be the set of subsets of 2^a defined by :

(3.33) $$C(Q_1,Q_2) = \{ Q \in 2^a \mid \{Q_1, Q_2, Q\} \text{ is not coherent}, \text{ where } Q_1 \neq Q_2 \in 2^a.$$

Theorem (3.32) has the following important consequence:

(3.34) $$\Gamma = \{ C_a \mid a \in V \}.$$

It now follows that any automorphism of $O^a(2m)$ induces an automorphism of the design $\underline{s}'$. Moreover it is clear that only the identity of $\mathrm{Aut}(O^a(2m))$ fixes every block of $\underline{s}'$. Note that $\underline{s}'$ can also be defined as an extension of the tactical configuration formed by the suborbits of $O^a(2m,2)$ acting on $\underline{g}^a$ (see D.G. Higman [1]).

(3.35) *For* $\sigma \in H$ *we have* $\sigma C_a = C_{\sigma a}$ *and* $\sigma \emptyset_a \sigma^{-1} = \emptyset_{\sigma a}$.

Proof. From (3.31). //

(3.36) *Theorem.* *If* $a,b \in V$, $a \neq b$, $a,b \neq c$, *then*

$$|C_a \cap C_b| = \begin{cases} 2^{2m-3} & B(a,b) = 0 \\ 2^{2m-3} - a.2^{m-2} & B(a,b) = 1 \end{cases}$$

Proof. First suppose that $B(a,b) = 0$. Let C_a' and C_b' denote the complements of C_a and C_b in $\mathcal{Q}$, respectively. Operating with a transvection if necessary we may suppose that $Q_o \in C_a' \cap C_b'$ so that $Q_o(a) = Q_o(b) = 0$. Now $Q_c \in C_a' \cap C_b'$ if and only if $Q_c(a) = Q_c(b) = 0$. This will be the case if and only if $B(a,c) = B(b,c) = 0$. Since $Q_b(a) = Q_a(b) = 0$ we have $Q_a, Q_b \in C_a' \cap C_b'$ and $Q_c \in C_a' \cap C_b'$ if and only if $\{o,a,b,c\}$ is coherent. Therefore $|C_a' \cap C_b'| = 3+k = 2^{2m-3} + a.2^{m-1}$ and then $|C_a \cap C_b| = 2^{2m-3}$.

Now suppose that $B(a,b) = 1$. Again suppose that

$Q_o \in C'_a \cap C'_b$. Then $Q_{a,b} \notin C'_a \cap C'_b$ so arguing as above we get $|C'_a \cap C'_b| = p_{11}^2$ where p_{11}^2 refers to the graph obtained by deleting a point. From (3.19) we obtain $|C'_a \cap C'_b| = 2^{2m-3} + a.2^{m-2}$ whence $|C_a \cap C_b| = 2^{2m-3} - a.2^{m-2}$. //

We now define a graph $R = (\Gamma : E)$ by joining C_a to C_b whenever $|C_a \cap C_b| = 2^{2m-3}$. By construction R depends only on the 2-graph structure of $O^a(2m)$ and any automorphism of the design $\underline{s}'$ induces an automorphism of R. From (3.36) and (3.31) it follows that R is isomorphic to $S(2m)_o$ of § 2. Hence the next theorem follows from Theorem (2.5).

(3.37) Theorem.

(a) The automorphism group of $O^a(2m)$ is $H = Sp(2m,2)$.

(b) The automorphism group of the design $\underline{s}$ is $T(V) . Sp(2m,2)$.

We end this section by constructing strongly regular graphs with parameters given by (3.26) when m is even. Now let V denote a vector space of dimension m over GF(4), m even. Suppose that B is a non-degenerate alternating form on V and let $\mathcal{Q}$ be the set of quadratic forms Q such that $Q(x+y) + Q(x) + Q(y) = B(x,y)$. By restriction of scalars V becomes a vector space of dimension 2m over GF(2), Tr B is a non-degenerate alternating form on V (with respect to GF(2)) and if $Q \in \mathcal{Q}$, then Tr Q is a quadratic form on V. Hence we are back in our original situation.

However we may now define transvections $\phi_a^\mu \in Sp(m,4)$ by

(3.38) $\phi_a^\mu(x) = x + \mu B(a,x)a \qquad \mu \in GF(4),\ a \in V$

From Dickson [1], p. 197, Sp(m,4) has two orbits 2^1 and 2^{-1} in 2 consisting of forms of index $\frac{1}{2}m$ and $\frac{1}{2}m-1$ respectively. As above, T(V).Sp(m,4) acts on 2 and 2 is an affine space over GF(2). Again choose $Q_0 \in 2^a$ and define $\underline{g}^a$ and $\underline{p}^a$ as in (3.3). Then from Dembowski [1], p. 46

(3.39) $|\underline{g}^a| = (2^m - a)(2^{m-2} + a).$

Thus for $\mu \neq 0$

(3.40) $|\{x \in V \mid Q_0(x) = \mu\}| = \frac{1}{2}(|V| - |\underline{g}^a|) = 2^{2m-2} - a.2^{m-2}.$

As in (3.12) we obtain

(3.41) $\phi_a^\mu Q = t_{\beta a} Q$ <u>where</u> $\beta^2 = \mu^2 Q(a) + \mu.$

Hence $(\phi_a^\mu Q)(\beta a) = Q(\beta a) = Tr(\mu\, Q(a))$.

Conversely, if $Q_0(a) = Q_a(a) \in GF(2)$, then we can choose μ so that $\phi_a^\mu Q_0 = Q_a$. We have therefore proved:

(3.42) $$2^a = \{Q_a \mid Q_0(a) \in GF(2)\}$$

$$2^{-a} = \{Q_a \mid Q_0(a) \notin GF(2)\}.$$

Now suppose that $\phi_a^\mu \in K_0 \cap K$ where $K = Sp(m,4)$ and K_0 is the subgroup of $T(V)\cdot K$ fixing Q_0. From (3.41) we have $\mu^2 Q_0(a) + \mu = 0$. Therefore, if $Q_b \in 2$, then $\phi_a^\mu Q_b = Q_c$ where $c = \phi_a^\mu(b)$ and then $Q_0(c) = Q_0(b)$. It

follows that $K_0 \cap K = O^a(m,4)$ has at least two orbits on each of $\mathcal{Q}^{-a}$ and $\mathcal{Q}^a - \{Q_0\}$.

Again consider the design $\mathcal{E}$ whose point set is $\mathcal{Q}$ and whose blocks are translates of $\mathcal{Q}^{-a}$. The permutation character π of K acting on blocks is the same as that of K acting on V. Since K has rank 3 on the projective space of V (over GF(4)) we see that K has rank 7 on the non-zero points of V. By the result of Brauer in § 3.1 the permutation character of K acting on $\mathcal{Q}$ is again π. If the non-principal irreducible characters of π all appear with multiplicity 1 then $K_0 \cap K$ would act transitively on $\mathcal{Q}^{-a}$ which is not the case. Hence $\pi = 2\emptyset_0 + 2\emptyset_1 + \emptyset_2 + \emptyset_3$ where $\emptyset_0$ is the principal character and $\emptyset_1$, $\emptyset_2$ and $\emptyset_3$ are irreducible. We may choose the notation so that the characters of K acting on $\mathcal{Q}^1$ and $\mathcal{Q}^{-1}$ are $\emptyset_0 + \emptyset_1 + \emptyset_2$ and $\emptyset_0 + \emptyset_1 + \emptyset_3$ respectively. Hence

(3.43) K <u>acts as a rank</u> 3 <u>group on both</u> $\mathcal{Q}^1$ <u>and</u> $\mathcal{Q}^{-1}$.

From (3.39), (3.40) and (3.41) we have

$$|\mathcal{Q}^a| = 2^{2m-1} + a \cdot 2^{m-1} \tag{3.44}$$

The orbits of $K_0 \cap K$ on $\mathcal{Q}^a$ are

$$\{Q_0\}, \{Q_a \mid Q_0(a) = 0\}, \{Q_a \mid Q_0(a) = 1\}. \tag{3.45}$$

We use the rank 3 action of K to define a strongly regular graph on $\mathcal{Q}^a$ as in § 1.1. In terms of the quadratic forms we join Q_a to Q_b whenever $Q_a(a+b) = 0$. Now define a three element subset of $\mathcal{Q}^a$ to be coherent

whenever it contains an odd number of edges of the graph (as in (2.2.5)).

If Q_a and Q_b are elements of 2^a, then $Q_o(a), Q_o(b) \in GF(2)$ and $Q_a(a+b) = Q_o(a) + Q_o(b) + TrB(a,b)$. Hence $\{Q_o, Q_a, Q_b\}$ is coherent if and only if $TrB(a,b) = 0$. It follows that we have constructed the regular 2-graph $O^a(2m)$. From (2.6.1) and (3.39) the parameters of the strongly regular graph are

$$(3.46) \qquad n = 2^{2m-1}+a.2^{m-1}$$

$$n_1 = 2^{2m-2}+3a.2^{m-2}-1 \qquad p_{11}^1 = 2^{2m-3}+3a.2^{m-2}-2$$

$$n_2 = 2^{2m-2}-a.2^{m-2} \qquad p_{11}^2 = 2^{2m-3}+a.2^{m-1}.$$

We have completed the construction of the graph promised after (3.26).

Remark. We may carry out the above construction with the field $GF(2^e)$ in place of $GF(4)$. We again get the projective design $\underline{s}$ from the set of points 2^a and blocks $t_a 2^{-a}$. In the case $2^e = 8$ the group $\bar{}p(2m,8)$ has rank 3 representations on 2^1 and 2^{-1} but the strongly regular graphs do not belong to $\mathcal{S}(O^a(2m))$.

Another approach to this question is to begin with a vector space W of dimension 2m+1 over $GF(2^e)$ and a non-degenerate quadratic form Q on W whose associated bilinear form is F. The group $O(2m+1,2^e)$ is isomorphic to $Sp(2m,2^e)$. If $W_o = \langle e_o \rangle$ is the radical of W there are two possibilities for the restriction of Q to a hyperplane not containing W_o.

Thus in the projective space of W $Sp(2m,2^e)$ has two orbits on hyperplanes not containing the point W_o and in the dual situation $Sp(2m,2^e)$ has two orbits on an affine space of dimension 2m over $GF(q^e)$. We may identify this affine space with 2 above and the orbits are then 2^1 and 2^{-1}. In particular, we see that the design constructed in Dembowski [1], p. 95 is the design $\underline{s}$ and the full automorphism group is $T(V).Sp(2e,2)$.

4. Unitary geometry

In this section we shall construct, for each odd prime power q, a regular 2-graph U(q) which admits the group $P\Gamma U(3,q^2)$ as a doubly transitive group of automorphisms.

Let K be the field $GF(q^2)$ and let V be a three-dimensional vector space over K. Then K has an automorphism $x \to \bar{x} = x^q$ whose fixed field is $K_1 = GF(q)$. We shall suppose that $q = p^a$ where p is an odd prime. Choose a basis e_1, e_2, e_3 for V. If $x = x_1e_1 + x_2e_2 + x_3e_3$ and $y = y_1e_1 + y_2e_2 + y_3e_3$, then define

$$(4.1) \qquad s(x,y) = x_1\bar{y}_3 + x_2\bar{y}_2 + x_3\bar{y}_1.$$

Then s is a non-degenerate sesquilinear form. Given a subspace $U \subseteq V$ we define the unitary complement of U to be

$$(4.2) \qquad U^{\pi} = \{ v \in V \mid s(u,v) = 0 \quad \text{for all } u \in U \}.$$

Since s is non-degenerate $\dim U + \dim U^{\pi} = 3$ and the map $\pi : U \to U^{\pi}$ defines a unitary polarity of the projective

plane $\underline{P}(V)$. A subspace U of V is <u>absolute</u> if $U \subseteq U^{\pi}$ or $U^{\pi} \subseteq U$. We denote the set of absolute points of $\underline{P}(V)$ by $\mathcal{Q}$.

The vector x represents an absolute point of $\underline{P}(V)$ if and only if $s(x,x) = 0$. Taking x_1, x_2, x_3 as homogeneous coordinates in $\underline{P}(V)$, let L_∞ be the line with equation $x_1 = 0$ and let ∞ be the point $(0,0,1)$. Then L_∞ determines an affine plane E with coordinates (x,y) where $x = x_2 x_1^{-1}$, $y = x_3 x_1^{-1}$. In this notation we have

$$(4.3) \qquad \mathcal{Q} = \{\infty\} \cup \{(x,y) \mid N(x) + Tr(y) = 0\}$$

where $N, Tr : K \to K_1$ are the norm and trace respectively. Since $|\ker Tr| = q$ we have

$$(4.4) \qquad |\mathcal{Q}| = q^3+1.$$

(4.5) <u>Proposition.</u> <u>Any line through</u> ∞ <u>distinct from</u> L_∞ <u>intersects</u> $\mathcal{Q}$ <u>in</u> q <u>points other than</u> ∞.

<u>Proof.</u> In affine coordinates such a line has an equation $a + bx = 0$ with $b \neq 0$. We then have q choices for y such that $Tr(y) + N(x) = 0$. //

The group of linear transformations of V leaving s invariant is $GU(3,q^2)$. The factor group $PGU(3,q^2)$ of $GU(3,q^2)$ by its centre acts as a group of permutations of $\mathcal{Q}$. In describing $PGU(3,q^2)$ we follow Huppert [1], p. 242.*

Since $\infty^{\pi} = L_\infty$ the subgroup N of $PGU(3,q^2)$ which fixes ∞ acts as a group of transformations of E and we may describe

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the elements of N by their action on E.

Put $Q = \{ Q(a,b) \mid N(a) + Tr(b) = 0 \}$ where

$$(4.5) \qquad (x,y)^{Q(a,b)} = (x+a,\ y+b - \bar{a}x)$$

and $H = \{ H(k) \mid k \neq o \}$ where

$$(4.6) \qquad (x,y)^{H(k)} = (\bar{k}x, \bar{k}ky).$$

Then Q is an Sp-subgroup of G, H is a cyclic group of order q^2-1 and $N = Q.H$ is the normaliser of Q in G. The structure of N is determined by the following equations.

$$(4.7) \quad Q(a,b)Q(c,d) = Q(a+c, b+d-\bar{c}a)$$

$$(4.8) \quad H(k)\ H(m) = H(km)$$

$$(4.9) \quad H(k)^{-1}\ Q(a,b)\ H(k) = Q(\bar{k}a, \bar{k}kb).$$

We define a further element of $PGU(3,q^2)$ by the (projective) matrix

$$(4.10) \qquad \omega = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$$

We have $0^\omega = \infty$ and $\infty^\omega = 0$, where $0 = (o,o) \in E$. If $y \neq o$, then $(x,y)^\omega = (-xy^{-1}, y^{-1})$. Since Q acts regularly on $\Omega - \{\infty\}$ we see that $PGU(3,q^2)$ acts doubly transitively on Ω.

Suppose that p_1, p_2 and p_3 are distinct points of Ω

and x,y and z are elements of V representing p_1,p_2 and p_3 respectively. If $q \equiv -1 \pmod 4$ we define $\{p_1,p_2,p_3\}$ to be coherent whenever $s(x,y)s(y,z)s(z,x)$ is a square in K, while if $q \equiv 1 \pmod 4$ we define $\{p_1,p_2,p_3\}$ to be coherent whenever $s(x,y)s(y,z)s(z,x)$ is not a square in K. Let $\underline{t}$ be the set of all coherent three-element subsets of Ω. Since $k\bar{k}$ is always a square and $\bar{k}$ is a square whenever k is a square the definition of coherence does not depend on the representatives or the order chosen for the points of Ω.

(4.11) Theorem. $U(q) = (\Omega,\underline{t})$ is a regular 2-graph with the following parameters:

$$N = q^3+1$$

$$a = \tfrac{1}{2}(q-1)(q^2+1) \qquad a' = \tfrac{1}{2}(q+1)(q^2-1)$$

$$k = \frac{1}{4}(q^3-3q^2+3q-5) \qquad k' = \frac{1}{4}(q^3+3q^2-3q-5)$$

$$\rho_1 = q^2 \qquad \mu_1 = q^2-q+1$$

$$\rho_2 = -q \qquad \mu_2 = q(q^2-q+1)$$

Proof. Since $PGU(3,q^2)$ acts doubly transitively on Ω and preserves coherence it is immediate that any two-element subset of Ω is contained in the same number of elements of $\underline{t}$. Moreover in order to check condition (b) of the definition of regular 2-graph we may take the four-element set to consist of ∞, o, p_1, p_2 with representatives e_3, e_1, x, y. Since the product of two non-squares is a square it is clear that $\{\infty, o, p_1, p_2\}$ can contain only

an even number of elements of $\underline{t}$. Finally, $\underline{t}$ is neither empty nor the set of all three-element subsets of Ω. Hence U(q) is a regular 2-graph. Since $s(e_3,x) = \bar{x}_1$ the strongly regular graph $U(q)_\infty$ may be described as the graph on $\Omega - \{\infty\}$ obtained by joining p_1 to p_2 whenever s(x,y) is a square in K if $q \equiv -1 \pmod 4$ or a non-square in K if $q \equiv 1 \pmod 4$. (Cf. Remark 1, § 2.2.)

Suppose that $q \equiv -1 \pmod 4$. The number of coherent triangles containing $\{\infty, o\}$ is the number of points (x,y) in E such that y is a square and N(x) + Tr(y) = o. Since $q \equiv -1 \pmod 4$ every element $y \in K$ such that Tr(y) = o is a square. Also, $Tr:K^+ \to K_1^+$ and $N:K^\times \to K_1^\times$ are homomorphisms so the number of coherent triangles is $(\frac{1}{2}(q^2-1)-(q-1))\times (q+1)+(q-1) = \frac{1}{2}(q-1)(q^2+1)$.

Similarly, if $q \equiv 1 \pmod 4$ the number of incoherent triangles containing $\{o,\infty\}$ is $\frac{1}{2}(q^2-1)(q+1)$ since in this case if y is a square then $Tr(y) \neq o$. Since $N = q^3+1$ the remaining parameters of U(q) can be calculated from the equation of § 2.2. //

We have defined U(q) so that $\{\infty, o, (x,y)\}$ is coherent if Tr(y) = o. Thus $\{\infty, o\} \cup \{(o,y) \mid Tr(y) = o\}$ is a coherent set of q+1 points. From (2.7.9) it is a maximal coherent set.

Let X be the Galois group of K (over GF(p)). If $\sigma \in X$ we define the action of σ on V by $(x_1,x_2,x_3)\sigma = (x_1\sigma, x_2\sigma, x_3\sigma)$. Thus $s(x,y)\sigma = s(x\sigma, y\sigma)$ and σ therefore induces an automorphism of U(q). It follows that the

semi-direct product $P\Gamma U(3,q^2) = PGU(3,q^2).X$ is contained in $Aut(U(q))$. However, $P\Gamma U(3,q^2)$ need not be the full automorphism group of $U(q)$. For example, when $q=3$ the parameters of $U(3)$ are $N=28$, $a=10$, $k=1$ and in Chapter 5 we shall show that there is just one regular 2-graph with these parameters. Hence $U(3) \simeq O^-(6)$ and $Aut(U(3)) \simeq Sp(6,2)$. An interesting consequence of this is the embedding $P\Gamma U(3,3^2) \subseteq Sp(6,2)$. This is also a consequence of the facts: $P\Gamma U(3,3^2) \simeq G_2(2)$, $Sp(6,2) \simeq O(7,2)$, $G_2(2)$ is the group of automorphisms of the Cayley algebra over $GF(2)$ and hence $G_2(2) \subseteq O(7,2)$.

Using the double transitivity of $P\Gamma U(3,q^2)$ on Ω we can easily describe the geometry of Ω and $\underline{P}(V)$. Through each point p of Ω there is just one absolute line (viz. p^π). The other q^2 lines through p intersect Ω in $q+1$ points (by (4.5)). Since any two points of Ω determine a unique line we obtain $q^2(q^2-q+1)$ (non-absolute) lines that meet Ω in $q+1$ points and since $(q^3+1)+q^2(q^2-q+1) = q^4+q^2+1$ we have accounted for all the lines of $\underline{P}(V)$. It follows that the points Ω and the non-absolute lines $\mathcal{L}$ form a design $\underline{X}$ with parameters

(4.12) $\quad v = q^3+1,\ b = q^2(q^2-q+1).\ r=q^2,\ \ k = q+1,\ \ \lambda = 1.$

Since Q acts regularly on $\Omega - \{\infty\}$ we identify the point (a,b) of $\Omega - \{\infty\}$ with $Q(a,b)$. From (4.7) the map $\sigma : Q(a,b) \mapsto a$ is a homomorphism of Q onto K and $\ker \sigma = \{ Q(o,b) \mid Tr(b) = o \} = Z(Q)$. We may identify H with the multiplicative group of K and the action of $H(k)$

on $Q/Z(Q) \simeq K$ is $a \mapsto \bar{k}a$ (from (4.9)). The action of $H(k)$ on $Z(Q)$ is $Q(o,b) \mapsto Q(o,k\bar{k}b)$. Since both these actions of H are irreducible we have $Z(Q) = \Phi(Q) = Q'$ and it is easy to check that Q has exponent p. Thus Q is a special group of exponent p. Choose a fixed element $\neq o$ such that $\varepsilon + \bar{\varepsilon} = o$. If $b+\bar{b} = o$, then $b\varepsilon \in K_1$. Thus the map $\tau : Z(Q) \to K_1 : Q(o,b) \mapsto b$ identifies $Z(Q)$ with K_1. From (4.7) we have

$$(4.13) \qquad [Q(a,b),Q(c,d)] = Q(o,\bar{a}c-a\bar{c}).$$

Using our maps σ and τ and regarding K as a two-dimensional space of K_1 we obtain a non-degenerate alternating form $f:KxK \to K_1$ defined by

$$(4.14) \qquad f(a,b) = (\bar{a}b-a\bar{b})\varepsilon .$$

The structure of Q may be recovered from the map f.

If $u,v \in \Omega$ let $\ell_{u,v}$ denote the other points of Ω which are incident with the block determined by u and v. Since the block determined by o and ∞ is the line of E with equation x=o we have $\ell_{o,\infty} = \{(o,y) \mid y + \bar{y} = o\}$, and we may identify $\ell_{o,\infty}$ with the non-identity elements of $Z(Q)$. Also, $\ell_{o,\infty}$ is the orbit of H of length q-1, the remaining non-trivial orbits of H having lengths q^2-1. Since $PGU(3,q^2)$ is doubly transitive on points it is transitive on blocks. The block $\{o,\infty\} \cup \ell_{o,\infty}$ was previously noted to be a maximal coherent set of U(q). Hence all blocks of $\underline{x}$ are maximal coherent sets. The converse is not true.

However in the remainder of this section we shall show that in the case of U(5) it is possible to use the 2-graph structure to determine which coherent sets are blocks.

(4.15) Theorem. Any automorphism of U(5) is an automorphism of the design $\underline{X}$.

Proof. We consider coherent sets containing o and ∞. The subgroup of $P\Gamma U(3,5^2)$ fixing o and ∞ is HX of order 48. Its orbits on the remaining points have lengths 4, 24, 48, 48. Let D_1 be the orbit of length 4 and D_2 the orbit of length 48 consisting of points (x,y) such that y is not a square in K. Then $p \in D_1 \cup D_2$ if and only if $\{o, \infty, p\}$ is coherent.

Let $B_o = \{o, \infty\} \cup D_1$. Then B_o is the block containing o and ∞. Choose $p_1 = (o,y_1) \in D_1$ and let B be the set of points $p \in \Omega$ such that $\{o, \infty, p_1, p\}$ is a coherent set of 4 points. From (4.11) $|B| = k = 15$. We have $p_2 = (x_2,y_2) \in B$ if and only if y_2 and $\bar{y}_2+y_1$ are both non-squares. It follows that $p_2 \in B$ if and only if the block with equation $y = y_2$ is contained in B.

Let us write $B_{01} = \{o, \infty, p_1\}$ and $B_{02} = B_o - B_{01}$. Then $B_{02} \subseteq B$ and the remaining points of B must lie in two disjoint blocks B_1 and B_2. In the notation of (2.7.1) we have, for $p \in B_1 \cup B_2$, $B_{01} = B_{01}(p)$ and $B_{02} = B_{02}(p)$. Now define $B_{ij} = B_{ij}(\infty)$ where $1 \le i,j \le 2$. It is easy to see that for any $p \in B_o$ we have $B_{ij} = B_{ij}(p)$, $1 \le i,j \le 2$ and as in the proof of (2.9.1), the union of any two of the sets B_{ij}, $o \le i \le 2$, $1 \le j \le 2$, is coherent. Moreover,

any three-element set whose elements come from distinct sets B_{ij} is incoherent.

Now choose a point $p_2 \in D_2$. We shall show that the configuration formed by the set of points $C = \{ p \in \Omega \mid \{ o, \infty, p_2, p_1 \}$ is coherent $\}$ does not have the same structure as that above. Choose $a \in GF(25)$ such that $a^2 = a+3$. Then a generates the multiplicative group of GF(25) and $\bar{\alpha} = 4a+1$. We may suppose that $p_2 = (4a+1, 3a)$ since $3a = a^{19}$ is not a square and $p_2 \quad \Omega$. Consider the coherent set $M = \{o, \infty, p_2, p_3\}$ where $p_3 = (2a, 2a)$. The fifteen elements of Ω which form a coherent set with $\{o, \infty, p_2\}$ are :

(4.16) $(o, a+2)$	$(4a+2, a)$	$(4a+1, 2a+3)$
$(4a+3, 2a)$	$(3a+4, 4a)$	$(3a+1, 2a+3)$
$(2a, 2a)$	$(2a+4, 3a)$	$(4a+4, 4a+1)$
$(4a+4, a)$	$(1, 4a)$	$(3a+3, a+4)$
$(4a, 3a)$	$(2a+2, a+4)$	$(a+1, 4a+1)$

It is now easily checked that M is a maximal coherent set. However the configuration of the B_{ij}'s above does not contain any maximal coherent set of 4 elements. It follows that the block containing o and ∞ can be determined from the 2-graph structure. Hence any automorphism of U(5) is an automorphism of $\underline{X}$. //

(4.17) Corollary. $\mathrm{Aut}(U(5)) = P\Gamma U(3, 5^2)$.

Proof. We shall say that the lines L and M of $\underline{P}(V)$ are orthogonal if L^w lies on M. If L and M are orthogonal we

shall write $L \perp M$. The block B_o above corresponds to the non-absolute line L_o whose equation in E is $x = o$. Hence $L_o^{\perp} = \langle e_2 \rangle$ and the lines orthogonal to L_o have equations $y=a$ for some $a \in K$. Such a line is non-absolute if and only if $Tr(a) \neq o$. Let us call the corresponding maximal coherent sets the blocks orthogonal to B_o. It follows that B_o together with the 20 blocks orthogonal to it form a partition of Ω. We have seen above that to each partition of B_o into a pair of three-element subsets there is a pair of disjoint blocks. If o and ∞ lie on the same side of the partition, the blocks have equations $y=a$ where a is a non-square and $Tr(a) \neq o$. If o and ∞ lie on different sides of the partition the blocks have equations $y=a$ where a is a square.

Hence the 10 partitions of B_o determine the 20 blocks orthogonal to B_o. Since we may determine the blocks from the 2-graph structure it follows that we may also determine the orthogonality relation from the 2-graph structure.

We are now able to reconstruct the projective plane $\underline{P}(V)$ from the 2-graph U(5). Let Π be the set consisting of the points of Ω and the symbols [L] where L is a non-absolute line and let Λ be the set of non-absolute lines and the symbols [p] where $p \in \Omega$. We define an incidence relation $I \subseteq \Pi \times \Lambda$ by

(4.18)	p I [p']	if p = p',	$p,p' \in \Omega$
	p I L	if $p \in L$	$p \in \Omega$, $L \in \mathcal{S}$
	[L] I[p]	if $p \in L$	$p \in \Omega$, $L \in \mathcal{S}$
	[L] I L'	if $L \perp L'$	$L,L' \in \mathcal{S}$.

It is clear that we may identify $\overline{\Pi}$ with the points of $\underline{P}(V)$, Λ with the lines of $\underline{P}(V)$ and I with the usual incidence relation. Also, the polarity π is given by $\pi : p \to [p]$ and $[L] \to L$. It follows that any automorphism of the regular 2-graph U(5) induces an automorphism of $(\overline{\Pi}, \Lambda, I)$ which commutes with π. Hence $\mathrm{Aut}(U(5)) = P\Gamma U(3,5^2)$. //

Remarks. 1. It is possible to show in general that the orthogonality relation - is uniquely determined by the incidence structure of the design $\underline{x}$. Hence the key point in extending (4.17) to all U(q), $q \neq 3$ is to show that the blocks of $\underline{x}$ can be obtained from the 2-graph structure.

2. The 2-graphs U(q) provide a new infinite family of equiangular lines in q^2-q+1 dimensions. The strongly regular graphs $U(q)_\infty$ are also new.

5. Groups of Ree type

Let G be a group of Ree type (satisfying the conditions of Ward [1]) or ${}^2G_2(3)$, and let Q be an S_3-subgroup of G. Then $|Q| = q^3$ where q is an odd power of 3. The group $N = N_G(Q)$ is the semi-direct product of Q by a cyclic group H of order q-1. The representation of G on the set Ω of cosets of N is doubly transitive of degree q^3+1. Since G has no subgroup of index 2 it satisfies the conditions of Theorem (1.2) and therefore G has a monomial representation $\underline{M}$ of degree q^3+1 with either one or two irreducible constituents. It follows easily from the character table of G in Ward [1]

that $\underline{M}$ has two irreducible constituents with degrees q^2-q+1 and $q(q^2-q+1)$. Hence part (b) of (1.2) applies and there exists a regular 2-graph $R(q) = (\Omega, \underline{t})$ which contains G in its automorphism group. Since the multiplicities of the eigenvalues of an adjacency matrix of a strong graph in $\mathcal{S}(R(q))$ are q^2-q+1 and $q(q^2-q+1)$ it follows that the parameters of R(q) are the same as those of U(q), (see (4.11)).

Let j be the unique involution of the group H. It can be seen from the character table of Ward [1] that j has exactly q+1 fixed points in Ω. We may therefore construct a block design $\underline{Y}$ as in Lüneburg [1]: the points of $\underline{Y}$ are the elements of Ω, the blocks are the sets of fixed elements of involutions fixing at least one element. The parameters of $\underline{Y}$ are the same as those of the design $\underline{X}$ of §4. It follows as in Lüneburg [1] that the stabiliser of the block B fixed by j is $C(j) = \langle j \rangle \times K$ where $K \simeq PSL(2,q)$ and all involutions of G are conjugate. Since K acts doubly transitively on B either B is coherent or the restriction of the 2-graph R(q) to B is a regular 2-graph of type (a). But $q \equiv 3 \pmod 4$ so this last possibility cannot occur. Hence B is a coherent set. Since G acts transitively on blocks it follows that every block of $\underline{Y}$ is a (maximal) coherent set.

The existence of the Ree groups ${}^2G_2(q)$ implies the existence of a family R(q) of regular 2-graphs having the same parameters as the regular 2-graphs U(q). The group ${}^2G_2(3)$ is isomorphic to $\sum L(2,8)$ and R(3) is a regular 2-graph with 28 points. By a result in Chapter 5 $R(3) \simeq U(3) \simeq O^{-1}(6)$

so $\mathrm{Aut}(R(3)) \simeq Sp(6,2)$ and $\Sigma L(2,8) \subseteq Sp(6,2)$ as was proved in § 2. In fact, ${}^2G_2(3)$ is the only group of Ree type with $q = 3$ (see Janko [1]). It is interesting to note that the designs $\underline{x}$ and $\underline{y}$ associated with $U(3)$ and $R(3)$ are not isomorphic since it is easy to see that $\mathrm{Aut}\underline{x} = P\Gamma U(3,3^2)$ (using the methods of the previous section). Both these designs satisfy the conditions of § 3.3 and consequently give rise to regular 2-graphs with the parameters of $S(6)$ of § 2.

6. The groups C_3 and HiS

Suppose that $v_1,\ldots,v_n$ are vectors in $\mathbb{R}^r$ which span equiangular lines with angle $\cos^{-1}\alpha$. If equality holds in (2.4.6) then as in § 2.4 we may make the set Ω of equiangular lines into a regular 2-graph by defining $\{\langle v_i\rangle, \langle v_j\rangle, \langle v_k\rangle\}$ to be coherent whenever

$$(6.1) \qquad (v_i,v_j)(v_j,v_k)(v_k,v_i) < 0.$$

We shall use this construction to obtain several regular 2-graphs with $\rho_2 = -\alpha = -5$. The parameters of these 2-graphs must therefore occur in (2.5.6).

As general references for this section we use Conway [1] and Todd [1].

Let $\Delta = \{0,1,\ldots,23\}$ and let $\mathcal{B}$ be the set of blocks of the Steiner system $S(5,8,24)$ defined on Δ. Let ^{24} be spanned by the orthonormal basis e_i, $i \in \Delta$ and define e_s as $\sum_{i \in s} e_i$ whenever $s \subseteq \Delta$. The Leech lattice Λ

is the lattice spanned by the vectors $2e_B$, $B \in \mathcal{B}$ and $e_\Delta - 4e_o$. The group C_3 (in Conway's notation, .3) can be taken to be the subgroup of the orthogonal group of $\mathbb{R}^{24}$ which preserves Λ as a whole and fixes $4e_o + e_\Delta = x$. There are 276 unordered pairs $\{y,z\}$ such that $x = y+z$, $y,z \in \Lambda$ and $(y,y) = (z,z) = 32$: namely, 23 with y of the form $4e_o + 4e_i$, $1 \le i \le 23$, and 253 with y of the form $2e_B$ where $B \in \mathcal{B}$ and $o \in B$. The vectors $2y-x$ lie in the hyperplane orthogonal to x and the lines spanned by these vectors are clearly permuted amongst themselves by C_3. This set Ω of 276 vectors in $\mathbb{R}^{23}$ consists of :

$$(6.2) \qquad v_i = 8e_i + 4e_o - e_\Delta \qquad 1 \le i \le 23,$$

$$v_B = 4e_B - 4e_o - e_\Delta \qquad B \in \mathcal{B}, \ o \in B.$$

The blocks of $\mathcal{B}$ which contain o constitute the Steiner system S(4,7,23) and any two distinct blocks which contain o intersect in either 2 or 4 points. Hence the vectors of Ω have the following inner products:

$$(6.3) \qquad (v_i, v_j) = 16 \qquad i \neq j$$

$$(v_i, v_B) = \begin{cases} 16 & i \in B \\ -16 & i \notin B \end{cases}$$

$$(v_{B_1}, v_{B_2}) = \begin{cases} 16 & |B_1 \cap B_2| = 4 \\ -16 & |B_1 \cap B_2| = 2 \end{cases}$$

$$(v,v) = 80 \qquad v \in \Omega$$

Thus we have obtained 276 equiangular lines in $\mathbb{R}^{23}$

with angle $\emptyset$ where $\cos\emptyset = \frac{1}{5}$. By (6.1) we have a regular 2-graph C(276) defined on Ω with parameters

(6.4)
$$\begin{aligned} N &= 276 \\ a &= 112 \qquad a' = 162 \\ k &= 30 \qquad k' = 105 \\ \rho_1 &= 55 \qquad \mu_1 = 23 \\ \rho_2 &= -5 \qquad \mu_2 = 253 \end{aligned}$$

It is clear that the structure of C(276) depends only on S(4,7,23) and that the group M_{23} which fixes o can be considered as a group of automorphisms of C(276) with orbits $D = \{ v_i \mid 1 \leq i \leq 23 \}$ and $E = \{ v_B \mid o \in B \}$. The set D is a maximal incoherent set. In the next chapter we shall prove that C(276) is the only regular 2-graph on 276 points with an incoherent set of size 23. We anticipate this result here in order to obtain the detailed structure of C(276).

By our construction the group C_3 is contained in Aut(C(276)). The group M^cL of M^cLaughlin [1] was defined as the subgroup of index 2 in the automorphism group $\overline{M}$ of a strongly regular graph R with parameters $n_1 = 112$, $n_2 = 162$, $p_{11}^1 = 30$ and $p_{11}^2 = 56$. By (2.2.3) we may construct a regular 2-graph T, with parameters (6.4), by adjoining a point to R. We next show that T contains an incoherent set of 23 points. The stabiliser of a point in R is the group $PSU(4,3^2)$ and the subgroup stabilising a point in the orbit of length 162 is PSL(3,4) with orbits of length 1,56 and 105. The group PSL(3,4) acts on the orbit of length 105

as it does on the flags of the 21 point plane Π, a flag (p,L) being joined in R to a flag (q,M) if and only if one of the following holds:

(6.5) (i) $p \in M$, $q \notin L$

(ii) $p \notin M$, $q \in L$.

Select a point $o \in \Pi$, let $L_1,\ldots,L_5$ be the lines through o and let $p_{i1},\ldots,p_{i4}$ be the remaining points of the line L_i, $1 \le i \le 5$. Then no two of the flags (p_{ij},L_i) are joined. These 20 flags together with the 2 points fixed by PSL(3,4) and the point adjoined to R form an incoherent set in T. It follows that we may identify T with C(276). In particular, $\overline{M}$ is the stabiliser of a point in Aut(C(276)) and by comparing orders we obtain $C_3 = \mathrm{Aut}(C(276))$.

Let $y = 4e_o - 4e_1 \in \Lambda$ and let $\underline{P}$ be the set of elements of Ω orthogonal to y; then $\underline{P}$ consists of

(6.6) $v_B = 4e_B - e_{\Delta} - 4e_o$, $B \in \mathcal{B}$, $o \in B$, $1 \notin B$.

Since there are 176 blocks containing o and omitting 1 we have found 176 equiangular lines in $\mathbb{R}^{22}$ with angle $\emptyset$. Using (6.1) we obtain a regular 2-graph H(176) defined on $\underline{P}$ with the following parameters:

(6.7)

N	= 176		
a	= 72	a'	= 102
k	= 20	k'	= 65
ρ_1	= 35	μ_1	= 22
ρ_2	= -5	μ_2	= 154.

The subgroup of M_{24} fixing 0 and 1 is M_{22} which is 3-fold transitive on the remaining points of $\triangle$ and has an orbit of length 77 on $\mathcal{B}$ consisting of the blocks containing o and 1 and an orbit of length 176 consisting of the blocks containing o but not 1. If we join two vectors $u,v \in \mathcal{R} - \underline{P} - \{v_1\}$ whenever $(u,v) = -16$ and then join v_1 to the remaining vectors of D we obtain the Higman-Sims' graph (see D.G. Higman and C. Sims [1]) with M_{22} acting on the graph and stabilising v_1. From Theorem 6 of Conway [1] the subgroup of M_{22} fixing a point in $\underline{P}$ is A_7 and the representation of M_{22} on $\underline{P}$ has rank 3. The corresponding strongly regular graph has parameters $n_1 = 70$, $n_2 = 105$, $p_{11}^1 = 18$ and $p_{11}^2 = 34$ and therefore belongs to $\mathcal{S}(H(176))$, (see M. Smith [1]). Now let $\underline{Q}$ be the set of vectors

$$w_B = 4e_B - e_{\Delta} - 4e_1, \quad B \in \mathcal{B}, \quad 0 \notin B, \ 1 \in B. \tag{6.8}$$

It is clear that M_{22} acts transitively on this set and that the stabiliser of a point is A_7. Moreover, there is an involution in M_{24} interchanging 0 with 1, $\underline{P}$ with $\underline{Q}$, inducing an outer automorphism of M_{22} and fusing the two classes of A_7's. Let H be the A_7 in M_{22} fixing $w_B \in \underline{Q}$. As in Conway [1], p. 228 we may identify the action of H on $\Delta - B$ with its action on the projective space PG(3,2). If $\mathcal{C}$ is the set of blocks $B' \in \mathcal{B}$ such that $0 \in B'$, $1 \notin B'$, then the 35 blocks $B' \in \mathcal{C}$ such that $|B \cap B'| = 4$ correspond to the 35 lines of PG(3,2) and the 15 blocks $B' \in \mathcal{C}$ such that $|B \cap B'| = 0$ correspond to the planes. The remaining 126 blocks intersect B in two points and correspond to non-ruled

quadrics in PG(3,2). In order to see that H is transitive on these 126 blocks we consider the set stabiliser of $\{a,b\} \subseteq B - \{\infty\}$. Firstly, this is a subgroup K of H isomorphic to S_5 and with orbits of length 2 and 5. Its action on the points of $\Delta - B$ corresponds to that of $P\Gamma L(2,4)$ on its vector space and it permutes the 16 blocks intesecting B in $\{a,b\}$ in orbits of length 6 and 10. (Furthermore, the subgroup of M_{24} fixing B and $\{a,b\}$ is $2^4 \cdot S_6 \simeq 2^4 \cdot Sp(4,2)$ and the subgroup fixing a point outside B is an S_6 with orbits of length 6 and 10 on the blocks intersecting B in $\{a,b\}$. However, the subgroup fixing one of these blocks is another S_6 permuting the remaining blocks transitively but having orbits of length 6 and 10 on $\Delta - B$. The stabiliser of a point in $\Delta - B$ is therefore an S_5 considered as $O^-(4,2)$.)

Corresponding to the orbits of length 15, 35 and 126 of H we have the following three possibilities for the inner product:

$$(6.9) \qquad (v_{B'}, w_B) = \begin{cases} -32 & |B' \cap B| = 0 \\ 32 & |B' \cap B| = 4 \\ 0 & |B' \cap B| = 2. \end{cases}$$

We may identify $\underline{P}$ with the points and $\underline{Q}$ with the quadrics of the version of the geometry of G. Higman [1] which was introduced by C. Sims [1] and used by M. Smith [1]; a point $v_{B'}$ being incident with a quadric w_B whenever $(v_{B'}, w_B) \neq 0$. In order to obtain G. Higman's original geometry we proceed as follows.

Consider the vectors $z \in \Lambda$ such that $(x,z) = 16$ and $(z,z) = 32$. Since C_3 fixes x it permutes these vectors z amongst themselves. Using Table 4 of Conway [1] we obtain five possible shapes for z :

(6.10)	I	$(4;-4, 0^{22})$	23
	II	$(0; 2^8, 0^{15})$	23.11.2
	III	$(1;3,-1^7,1^{15})$	23.11.16
	IV	$(2; -2^2, 2^5, 0^{16})$	23.11.7.3
	V	$(3; -1^{11}, 1^{12})$	23.7.8

Now y is of type I and the subgroup of C_3 fixing y must act on the regular 2-graph H(176) and on its complement in Ω, the Higman-Sims' graph. It follows that the only possibility is for C_3 to act transitively on the vectors in (6.10). Since the number of these vectors is $|C_3:H_iS|$ the subgroup fixing any one of them is isomorphic to H_iS. In particular $H_iS \subseteq \mathrm{Aut}(H(176))$ and the vectors of Ω not orthogonal to any vector of (6.10) form a Higman-Sims' graph. We shall label the set of vectors of Ω orthogonal to z in (6.10) by the type of z. Consider first a geometry of type II with $z = e_D$, $o \notin D$. The 176 vectors of the geometry are the following:

(6.11)	v_i	$i \in D$	(8)
	v_B	$\lvert B \cap D \rvert = 2$, $o \in B$,	(168)

The subgroup of our previous M_{23} which acts on this geometry is A_8. If $a,b \in D$ there are 6 blocks B such that

$B \cap D = \{a,b\}$ and $o \in B$. We may identify these blocks with the set $[ab]^*$ of G. Higman [1]. The subgroup of A_8 fixing $\{a,b\}$ is S_6 and this S_6 acts on $B - \{a,b\}$ and $[ab]^*$ in different ways. The identification of this geometry with G. Higman's original is now complete.

We observe that the conics of the geometry are maximal coherent sets and that any element u outside a conic induces a partitition with m(u) = 2. (Unfortunately, these are not the only maximal coherent sets of 8 points.)

Returning to our original geometry (6.6) of type I we see that the 126 vectors $v_{B'}$ such that $|B' \cap B| = 2$ for a fixed quadric w_B lie in $\mathbb{R}^{22}$ (by (6.9)) and hence can be made into a regular 2-graph by (6.1), whose parameters are the same as U(5) of § 4. Moreover, from G. Higman [1], the stabiliser in HiS of a quadric is $P\Sigma U(3,5^2)$ so this group must act on the regular 2-graph just obtained. Since there is only one possible representation of $P\Sigma U(3,5^2)$ on 126 points we see that the monomial representation considered in § 1 is uniquely determined and hence the 2-graph must be U(5). We have proved that $\mathrm{Aut}(U(5)) = P\Gamma U(3,5^2)$ but $P\Gamma U(3,5^2)$ does not have a representation of degree 50 so the subgroup of Aut(H(176)) fixing U(5) must be $P\Sigma U(3,5^2)$.

The stabiliser of a point is again $P\Sigma U(3,5^2)$ and the two classes of $P\Sigma U(3,5^2)$'s are fused by the outer automorphism of HiS extending that of M_{22}. The action of $P\Sigma U(3,5^2)$ can be studied via the geometry (6.11) of type II, choosing D so that $1 \in D$. The group $P\Sigma U(3,5^2)$ fixing v_1 act transitively on the 50 conics containing v_1. As shown by M. Smith [1],

these conics may be identified with the vertices of the Moore graph of width 2 and valency 7, and the 175 points other than v_1 may be identified with the edges of the Moore graph.

Maximal incoherent sets of various sizes may be obtained from the remaining types of geometry. Again let D be a block such that $o \notin D$ and $1 \in D$. The vectors of a geometry of type III may be taken as:

(6.12)

v_i	$i \notin D$	(15)
v_B	$o \in B,\ 1 \notin B,\ \lvert B \cap D \rvert = 2$	(126)
v_B	$o \in B,\ 1 \in B,\ \lvert B \cap D \rvert = 4$	(35)

The 15 elements v_i form a maximal incoherent set and the three types of vectors in (6.12) form the three orbits for $A_7 - M_{23}$ as described above. By adjoining the point v_1 to the orbit of length 35 we obtain $O^1(6)$ as a subgraph of C(276).

Suppose that C is a block such that 0,1,2 C. The vectors of a geometry of type IV may be taken as:

(6.13)

v_i	$i \notin C$	(16)
v_B	$o \in B,\ 1,2 \notin B,\ \lvert B \cap C \rvert = 2$	(80)
v_B	$o \in B,\ \lvert \{1,2\} \cap B \rvert = 1, \lvert B \cap C \rvert = 4$	(80)

The 16 vectors v_i form a maximal coherent set; the subgroup of M_{23} occurring is $2^4.S_4$.

Now let F be the symmetric difference of two blocks with two points in common such that $o \in F$. The vectors of

a geometry of type V may be taken to be

(6.14) v_i $\quad i \in F$ (11)

v_B $\quad o \in B, |B \cap F| = 4$ (165)

In this case we have obtained a maximal incoherent set of 11 points and it is easy to see that the group M_{11} acts on H(176) with orbits of length 11 and 165.

7. Known doubly transitive groups

In the previous sections we have shown that many doubly transitive groups act as groups of automorphisms of regular 2-graphs. We now present a list of doubly transitive groups G of degree N and show that with the possible exception of groups with a regular normal subgroup, any group for which we have not constructed a regular 2-graph cannot act on a regular 2-graph with N points. This list was obtained at a seminar given by M. O'Nan at Oxford, January 1971. I know of no other doubly transitive group.

(7.1)	Group	Degree
1.	G has a regular normal subgroup	
2.	$PSL(n,q) \leq G \leq P\Gamma L(n,q)$	$1+q+\cdots+q^{n-1}$
3.	$PSU(n,q) \leq G \leq P\Gamma U(n,q)$	$1+q^3$
4.	G is of Ree type, e.g., ${}^2G_2(q)$	$1+q^3$ $\quad q=3^{2m+1}$
5.	$Sz(q) \leq G \leq \Gamma Sz(q)$	$1+q^2$, $\quad q=2^{2m+1}$
6.	$G = A_n$ or S_n	n
7.(a)	$G = Sp(2m,2)$	$2^{2m-1}+2^{m-1}$
7.(b)	$G = Sp(2m,2)$	$2^{2m-1}-2^{m-1}$

8.(a)	M_{11}, M_{12}, M_{22}, Aut M_{22}, M_{23}, M_{24}	11,...,24
8.(b)	M_{11}	12
9.	PSL(2,11)	11
10.	A_7	15
11.	HiS	176
12.	C_3	276

If G has odd degree or if G is more than doubly transitive, then G cannot act on a regular 2-graph. This eliminates cases 5, 6, 8, 9 and 10. Cases 3, 4, 7(a), 7(b), 11 and 12 have been dealt with in this chapter. It remains to consider cases 1 and 2. In § 2.3 we constructed regular 2-graphs for PSL(2,q), $q \equiv 1 \pmod 4$. If PSL(2,q) acts on a regular 2-graph and $q \equiv 3 \pmod 4$, then the regular 2-graph must be of type (a). But then we obtain a contradiction from (1.2.5). Suppose that $n > 2$ and that PSL(n,q) acts on a regular 2-graph with $N = 1+q+\cdots+q^{n-1}$. The stabiliser of two points permutes the remaining points in orbits of length q-1 and N-q-1, the q-1 points lying on the line determined by the fixed points. But now a set of four points just three of which lie on a line cannot contain an even number of coherent triangles.

Thus we have dealt with all except case 1. The automorphism groups of the regular 2-graphs S(2m) of § 2 belong to case 1 and in § 2 we constructed a number of subgroups of Aut(S(2m)) which again belong to case 1. If $(\Omega, \underline{t})$ is a regular 2-graph and G is a doubly transitive group of automorphisms of $(\Omega, \underline{t})$ with a regular normal subgroup K, then

$|K| = 2^n$ for some n and $G = K.G_\alpha$ for $\alpha \in \Omega$. The group K is elementary abelian and $G_\alpha \subseteq GL(n,2)$ but it is a difficult problem to determine which subgroups of GL(n,2) are transitive on the non-identity elements of K.

8. Applications to equiangular lines

The problem of determining the maximum number of equiangular lines in $\mathbb{R}^r$ was first tackled by Haantjes [1] and was then taken up by Seidel and others (see Seidel [3], van Lint and Seidel [1]). The results of this chapter provide several infinite families of equiangular lines in Euclidean space which in many cases improve existing results. For example, the existence of U(5) together with (2.4.6) shows that the maximum number of equiangular lines (at angle $\cos^{-1}(\frac{1}{5})$) in $\mathbb{R}^{21}$ is 126.

In this section we use the results of § 6 to construct equiangular lines in less than 21 dimensions. Unfortunately we do not obtain equality in (2.4.6) so these configurations do not yield regular 2-graphs.

(8.1) Theorem. There exist 90 equiangular lines in $\mathbb{R}^{20}$.

Proof. We use the notation of § 6. Let w_{B_1} be a quadric as in (6.8). By (6.9) there are 126 vectors v_B, orthogonal to w_{B_1}. Now choose a second quadric w_{B_2} and consider the vectors orthogonal to both. Since the stabiliser in HiS of w_{B1} and w_{B_2} has orbits of length 36 and 90 on the vectors orthogonal to w_{B_1}, we see that there are 90 equiangular lines $\langle v_B, \rangle$ in $\mathbb{R}^{20}$.

The next result was discovered by D.S. Asche.

(8.2) Theorem. There exist 72 equiangular lines in $\mathbb{R}^{19}$.

Proof. Choose blocks B_1 and B_2 such that $o \notin B_1, B_2$ and $B_1 \cap B_2 = \{1,2\}$. A point v_B of H(176) is orthogonal to w_{B_1} if $o \in B$, $1 \notin B$ and $|B \cap B_1| = 2$. Of these 126 vectors, 90 are orthogonal to $e_o - e_2$ and 90 are orthogonal to w_{B_2}. Since there are 18 blocks of S(5,6,12) which contain a given point and contain neither of a given pair of points we see that there are 72 vectors of the above 126 which are orthogonal to both $e_o - e_2$ and w_{B_2}. //

Remark. Since each of the 72 blocks found above contain 7 points of $\Delta - \{0, 1, 2\}$ we see that there is some point of $\Delta - \{0, 1, 2\}$ contained in no more than 24 blocks (in fact each point of $\Delta - \{0, 1, 2\}$ must be contained in 24 blocks). Hence there are at least 48 equiangular lines in $\mathbb{R}^{18}$.

C H A P T E R 5

UNIQUENESS THEOREMS

The results of § 2.7 show that the existence of a large coherent set in a regular 2-graph has a considerable influence on the structure. In this chapter we use these results to show that certain regular 2-graphs are uniquely determined by their order and the existence of a suitable coherent set.

The uniqueness of the regular 2-graphs with k=1 was first proved by Seidel [1] in terms of strong graphs and by G. Higman in terms of regular 2-graphs. The uniqueness of C(276) was proved by G. Higman under the additional assumption of 4-regularity.

Throughout this chapter we shall often use (2,7.1) and (2.7.3) without comment. We shall also use the fact that all coherent triangles of a regular 2-graph are determined once we have specified the coherent triangles containing a given point.

1. $O^1(4)$, S(4) and $O^{-1}(6)$

Let T = $(\Omega, \underline{t})$ be a regular 2-graph with k=1. We shall show that T is isomorphic to one of the 2-graphs of the title of this section. There are three possibilities for the parameters of T and these are given in (2.5.5). We also have available the equivalence relation ~ of § 2.9 defined on

unordered pairs of elements of Ω.

(1.1) Theorem. Let $T = (\Omega, t)$ be a regular 2-graph with $N=10$. Then T is unique (up to isomorphism) and $\mathrm{Aut}\ T \simeq Sp(4,2)$.

Proof. Since $N=10$ we must have $k=1$ and $a=4$. Choose ω and a_{oo} in Ω and let $L = \{\omega, a_{oo}, a_{o1}, a_{o2}\}$ and $M = \{\omega, a_{oo}, a_{1o}, a_{2o}\}$ be the two coherent sets of four points containing $\{\omega, a_{oo}\}$. If $\beta, \gamma \notin L$ and β and γ induce the same partition of L, then $\{\beta, \gamma, \delta\}$ is coherent for all $\delta \in L$. Since $a=4$ the four points not in L or M may be uniquely labelled a_{ij}, $i, j \in \{1,2\}$, such that $\{\omega, a_{io}, a_{ij}\}$ and $\{\omega, a_{oj}, a_{ij}\}$ are coherent. It follows that a_{ij} is joined to a_{mn} in the graph T_ω if and only if $i=m$ or $j=n$. Hence T_ω is the graph $L_2(3)$ of § 3.4 and T is isomorphic to the regular 2-graph obtained by adjoining a point to $L_2(3)$. Since $O^1(4)$ has 10 points we have $T \simeq O^1(4)$ and $\mathrm{Aut}\ T \simeq Sp(4,2)$. //

The regular 2-graphs $T(9)$ of § 2.3 and T_{10} of § 3.2 both have $N=10$ so $T(9) \simeq T_{10} \simeq O^1(4)$. Thus if $G = \mathrm{Aut}\ T$, then $G \simeq P\Sigma L(2,9)$ and $S_5 \subseteq G$, whence $G \simeq S_6$. Since $\mathrm{Aut}\ L_2(3) = S_3$ wr S_2 we have $G_\omega \simeq S_3$ wr S_2. Also, it follows that T is isomorphic with its complement and hence possesses a 'polarity' - i.e., a permutation of Ω which interchanges coherent and incoherent triangles. The polarity π induces an outer automorphism of G and we have $G\langle\pi\rangle \simeq P\Gamma L(2,9)$.

(1.2) Theorem. Any regular 2-graph $T = (\Omega, t)$ with $N=16$ is isomorphic to either $S(4)$ or its complement. In

particular Aut T = E.H where E is an elementary abelian group of order 16, $H \simeq Sp(4,2)$ and $E \trianglelefteq E.H$, $E \cap H = 1$.

Proof. By taking the complement if necessary we may suppose the parameters of T to be N=16, a=6 and k=1. Our aim is to construct an incoherent set of six points.

Let $\{\omega,\alpha,\beta,\gamma\}$ be coherent and suppose $\{\omega,\alpha\} \sim \{\alpha_1,\alpha_1'\}$. Choose $\{\beta_1,\beta_1'\} \sim \{\beta_2,\beta_2'\} \sim \{\omega,\beta\}$ so that $\{\omega,\alpha,\beta_1\}$ and $\{\omega,\alpha,\beta_2\}$ are coherent. (Note that α_1 and α_1' induce the same partition of $\{\beta_1,\beta_1',\beta_2,\beta_2'\}$.) Now choose $\{\alpha_2,\alpha_2'\} \sim \{\omega,\alpha\}$ so that $\{\omega,\alpha_2,\beta_1\}$ is coherent and then choose $\{\gamma_1,\gamma_1'\} \sim \{\gamma_2,\gamma_2'\} \sim \{\omega,\gamma\}$ so that $\{\omega,\alpha_1,\gamma_1\}$ and $\{\omega,\alpha_1,\gamma_2\}$ are coherent. Since just one of $\{\omega,\beta_1,\gamma_1\}$ and $\{\omega,\beta_1,\gamma_2\}$ is coherent we may choose the notation so that $\{\omega,\beta_1,\gamma_1\}$ is coherent. (The fourth point of the coherent 4-set containing $\{\omega,\alpha_1,\beta_1\}$ must be either γ_1 or γ_2.) By construction we have $\{\omega,\alpha_1\}$ $\{\alpha,\alpha_1'\}$ $\{\beta_1,\gamma_1\}$ and it follows that the remaining pair in the equivalence class is $\{\beta_2,\gamma_2\}$. Similarly, the pairs $\{\omega,\alpha_1'\}$, $\{\alpha,\alpha_1\}$, $\{\beta_1',\gamma_1'\}$ and $\{\beta_2',\gamma_2'\}$ are equivalent.

We shall show that $M = \{\omega,\gamma,\alpha_1',\alpha_2',\beta_1,\beta_2\}$ is incoherent. By our construction all that remains to be done is to show that $\{\omega,\beta_2,\alpha_2'\}$ is incoherent. Since either $\{\omega,\beta_1,\gamma_2\}$ or $\{\omega,\beta_1,\gamma_2'\}$ is coherent, one of the pairs $\{\alpha_2,\gamma_2\}$ or $\{\alpha_2,\gamma_2'\}$ is equivalent to $\{\omega,\beta_1\}$. However we cannot have $\{\omega,\beta_1\} \sim \{\alpha_2,\gamma_2\}$ since this would force $\{\omega,\alpha_1,\beta_1,\gamma_2\}$ to be coherent and we would have $\{\omega,\alpha_1\} \sim \{\beta_1,\gamma_2\}$, a

contradiction. Thus the pairs equivalent to $\{\omega,\beta_1\}$ are $\{\beta,\beta_1'\}$, $\{\alpha_1,\gamma_1\}$ and $\{\alpha_2,\gamma_2'\}$. If $\{\omega,\beta_2,\alpha_2'\}$ were coherent, then $\{\omega,\beta_2,\gamma_2\}$ would be contained in the coherent sets $\{\omega,\beta_2,\gamma_2,\alpha_2'\}$ and $\{\omega,\beta_2,\gamma_2,\alpha_1\}$ which contradicts k=1. This proves that M is an incoherent set of six points.

If M is any incoherent set of six points in T, then it follows from (2.7.16) and (2.7.19) that we may identify the points of Ω outside M with the 10 partitions of M into pairs of subsets of three elements such that the coherent triangles are defined as follows. If $\alpha,\beta,\gamma \in M$ and x,y,z are partitions then $\{\alpha,\beta,\gamma\}$ is incoherent, $\{\alpha,\beta,x\}$ is coherent if α and β lie on opposite sides of the partition, $\{\alpha,x,y\}$ is coherent if we can write $x = (X_1,X_2)$ and $y = (Y_1,Y_2)$ such that $\alpha \in X_1 \cap Y_1$ and $|X_1 \cap Y_1| = 1$, $\{x,y,z\}$ is coherent if we can write $x = (X_1,X_2)$ $y = (Y_1,Y_2)$ and $z = (Z_1,Z_2)$ such that $|X_1 \cap Y_1| = |Y_1 \cap Z_1| = |X_1 \cap Z_1| = 1$.

Thus T is unique up to isomorphism. In particular, $T \simeq S(4)$ and the rest of the theorem follows from (4.2.5). //

The last part of the proof shows that an incoherent set of six points can be transformed into any other by an element of Aut T. Moreover, the stabiliser of such a set is obviously S_6 and the points outside form a regular 2-graph on 10 points.

(1.3) Theorem. Any regular 2-graph $T = (\Omega,t)$ with N=28 is isomorphic to either $O^{-1}(6)$ or its complement. In particular, $\mathrm{Aut}\ T \simeq Sp(6,2)$.

Proof. We may suppose the parameters of T to be N=28, a=10

and k=1. Let $\{a_1,a_1'\},\ldots,\{a_6,a_6'\}$ be the elements of an equivalence class of $\sim$. If β is a further point, then we may choose the notation so that $\{\beta,a_1,a_i'\}$ is coherent for $i=2,\ldots,6$. It follows that $\{a_1,\ldots,a_6,\beta\}$ is an incoherent set of seven points. If M is an incoherent set of seven points in Ω then by (2.7.16) and (2.7.19) we may identify the remaining points of Ω with the set $M_{(2)}$ of unordered pairs of M, such that for $\alpha,\beta \in M$ and $x,y,z \in M_{(2)}$, $\{\alpha,\beta,x\}$ is coherent whenever $|\{\alpha,\beta\} \cap x| = 1$; $\{\alpha,x,y\}$ is coherent whenever $\alpha \notin x \cup y$, and $\{x,y,z\}$ is coherent whenever $|x \cap y| + |y \cap z| + |z \cap x| \equiv 0 \pmod 2$.

Since T is uniquely determined the theorem follows from (4.3.37). //

The regular 2-graphs T_{28}, $O^{-1}(6)$, U(3) and R(3) all have N=28 and k=1 so we have $T_{28} \simeq O^{-1}(6) \simeq U(3) \simeq R(3)$ and S_8, $P\Gamma U(3,3^2)$, ${}^2G_2(3) \leq Sp(6,2)$. The above proof shows that G = Aut T acts transitively on the incoherent sets of seven points and that the stabiliser of such a set is S_7.

The configuration of the 28 bitangents to a plane quartic curve is just the regular 2-graph $O^{-1}(6)$ and for this reason a regular 2-graph may be considered as a generalisation of this well known figure. Moreover the strongly regular graph obtained by deleting a point is the graph obtained from the 27 lines on a cubic surface by joining two lines if they meet. These configurations have been studied for more than one hundred years (see Dickson [2], Jordan [1] and the references given there) and it is

interesting to note that the description of the 28 bitangents given by Dickson [2] can be easily translated into the language of regular 2-graphs. A syzygetic triple of bitangents is just a coherent triangle, an Aronhold set is an incoherent set of seven points, and a Steiner set is an equivalence class of $\sim$. The representation as T_{28} is given on p. 357 of Dickson [2] and the representation as $O^{-1}(6)$ is given on p. 373. In fact, the representation as $O^{-1}(6)$ derives from the theory of Riemann theta functions (see Baker [1]) and in this context the regular 2-graphs S(2m) of § 4.2 were also studied, a coherent triangle being called a syzygetic triple. The regular 2-graphs $O^{-1}(2p)$ where $p = \frac{1}{2}(n-1)(n-2)$ also arise from the problem of the contact of curves (see Dickson [2], p. 376).

The regular 2-graph S(4) which we studied above can be identified with the configuration of 16 singular points and 16 singular tangent planes to the Kummer quartic surface (Jordan [1], p. 331) and also with the 16 lines on the Clebsch quartic surface (Jordan [1], p. 305).

We may note that in the strongly regular graph obtained by deleting a point of $O^{-1}(6)$, an equivalence class (= Steiner set) which does not involve the deleted point corresponds to a double-six of lines on the cubic surface. Also, the 16 points not in an equivalence class form a regular 2-graph S(4) and through each point ω of $O^{-1}(6)$ there are 40 triples of regular 2-graphs isomorphic to $O^{1}(4)$, any pair of a triple intersecting in $\{\omega\}$ (Jordan [1], p. 317, Dickson [1], § 283, 2^{o}).

2. C(276)

In § 4.6 we defined C(276) in terms of the Steiner system S(5,8,24) and showed that it contained an incoherent set of 23 points.

(2.1) Theorem. Let $T = (\Omega, t)$ be a regular 2-graph with $N = 276$ and $a = 162$ which contains a coherent set M of 23 points. Then T is isomorphic to the complement of C(276) and if M' is any other coherent set of 23 points, then there is an element $\sigma \in \mathrm{Aut}\ T$ such that $\sigma M = M'$.

Proof. From (2.7.16) any point $x \notin M$ induces a partition of M into sets $M_1(x)$ and $M_2(x)$ of size 16 and 7 respectively. We shall call the sets $M_2(x)$, $x \notin M$, the blocks of M. From (2.7.19) we see that any two distinct blocks intersect in either 1 or 3 points. Thus any 4 points of M are contained in at most one block. Since there are 253 blocks and any block contains 35 four-element subsets, whereas M contains 253.35 four-element subsets it follows that any four element subset of M is contained in a unique block. Hence the points and blocks of M form the Steiner system S(4,7,23). Since it is well known that S(4,7,23) is unique up to isomorphism (Witt [1]), we may identify the blocks of M with the blocks of S(5,8,24) which contain a given point. Comparing (2.7.19) and (4.5.4) we see that our regular 2-graph T is isomorphic to the complement of C(276). Moreover, the isomorphism takes M onto the set $D = \{ v_i \mid 1 \leq i \leq 23 \}$. //

From now on let T denote the complement of C(276) and let G = Aut T. We shall use the notation of § 4.6. By construction, we have M_{23} - G and M_{23} has orbits D and E of length 23 and 253. It follows that σ of (2.1) may be chosen to take any given ordered set of four points of M to any given ordered set of four points of M'. We use this to prove that G acts doubly transitively on Ω. This proof was first given by G. Higman.

Choose a block $B \in \mathcal{B}$ such that $o \in B$ and choose $p \in B$, $p \neq o$. The group $2^4.S_6$ acts doubly transitively on the 16 blocks $B_1,\ldots,B_{16}$ which intersect B in $\{o,p\}$, hence any pair of these blocks have four points in common. Therefore, the set

(2.2) $D' = \{v_B\} \cup \{v_i \mid i \in B, i \neq o,p\} \cup \{v_{B_j} \mid 1 \leq j \leq 16\}$

is coherent.

Thus, by (2.1)

(2.3) G <u>is transitive on</u> Ω.

Now suppose that $C \in \mathcal{B}$ and $o \in C$. If $1 \in C$, put $B = C$, choose $p \in B$, $p \neq 0,1$ and define D' as in (2.2). If $1 \notin C$, choose p C, $p \neq 1$ and $B \in \mathcal{B}$ such that $B \cap C = \{o,p\}$ and $1 \in B$ then define D' as in (2.2) with $C = B_1$. It follows that in either case we can find an element $\sigma \in G$ which fixes v_1 and takes v_C into D. Thus

(2.4) G <u>is doubly transitive on</u> Ω.

Suppose that $\{v_1,v_2,v_C\}$ is coherent. If $1,2 \in C$,

then define D' as in (2.2) with B = C and $p \neq 0,1,2$. If $1,2 \notin C$ we can choose $B \in \mathcal{B}$ with $1,2 \in B$ and $B \cap C = \{o,p\}$ for some p. In this case we define D' as in (2.2) with $B_1 = C$. By (2.1) there is an element $\sigma \in G$ which fixes v_1 and v_2 and takes v_C into D. Thus the stabiliser of two points u and v is transitive on the 162 points w such that $\{u,v,w\}$ is coherent. Let L be the set of points w such that $\{u,v,w\}$ is incoherent. Taking $u = v_1$ and $v = v_2$, L consists of the 112 points v_B such that $o \in B$ and $|\{1,2\} \cap B| = 1$. We may identify the points of $\Delta - \{0,1,2\}$ with the points of the 21-point plane so that the above blocks correspond to hyperovals. Thus the group M_{21} which fixes 0,1 and 2 has two orbits of length 56 on L. It follows that $G_{u,v}$ is either transitive on L or has two orbits of length 56. Now take $u = v_1$ and $v = v_C$ such that $0,1 \quad C$. Then L consists of the 16 points v_i, $i \notin C$, the 16 points v_B such that $B \cap C = \{0,1\}$ and the 80 points v_B such that $0 \in B$, $1 \notin B$ and $|B \cap C| = 4$. The group $2^4.A_6$ which fixes 0,1 and C has orbits of length 16, 16 and 80 on L. It follows from (2.4) that $G_{u,v}$ acts transitively on L. Hence

(2.5) G_u *is a rank* 3 *group on* $\Omega - \{u\}$ *with subdegrees* 1, 112 *and* 162.

(2.6) *Theorem.* *The group* G *is transitive on ordered coherent sets of four points and* T *is both* 4- *and* 5-*regular with* $k_4 = 72$ *and* $k_5 = 51$.

Proof. We may suppose the coherent set to be $\{v_1,v_2,v_3,v_C\}$.

If $1,2,3 \in C$, then define D' by (2.2) with $B = C$ and $p \neq 0,1,2,3$. If $1,2,3 \notin C$, then choose $B \in \mathcal{B}$ with $1,2,3 \in B$ and $B \cap C = \{o,p\}$ for some p. Now define D' as in (2.2) with $B' = C$. Then from (2.1) there is an element $\sigma \in G$ fixing v_1,v_2,v_3 and taking v_C into D. This proves the first part of the theorem and T is therefore 4-regular. The points which make a coherent set with $\{v_1,v_2,v_3,v_4\}$ are the remaining 19 points of D, the point v_B where B is the unique block containing 0,1,2,3,4 and the 52 points v_B where $o \in B$, $1,2,3,4 \notin B$. Hence $k_4 = 72$ and from (2.8.1) and (2.8.7) T is 5-regular with $k_5 = 51$. //

We now calculate the number of coherent sets of 23 points which contain the set $X = \{v_1,v_2,v_3,v_C\}$ where $0,1,2,3 \in C$. Let $I = C - \{0,1,2,3\}$, $P = \{v_i \mid i \in I\}$, $Q = \{v_B \mid B \in \mathcal{B}, B \cap C = \{0,1,2,3\}\}$, and $\pi(i) = \{v_B \mid B \in \mathcal{B}, B \cap C = \{0,i\}\}$ where $i \in I$. The elements of the sets P, Q, $\pi(i)$ are the 72 points which form a coherent set with X. As usual we may interpret the 16 points outside C as the points of an affine space over GF(2). The group $2^4.S_6$ fixing C and $\{0,i\}$ acts as $2^4.Sp(4,2)$ on this space and the blocks of the elements of $\pi(i)$ correspond to the translates of a quadric. In particular, the 16 points and 16 blocks form a projective design with parameters v=16, k=6 and $\lambda = 2$ (cf. §§4.2,4.3). It is now easy to see that a block of $\pi(j)$, $j \neq i$, intersects exactly 10 blocks of $\pi(i)$ in 4 points. Let $\Pi = \cup\{\pi(i) \mid i \in I\}$.

Suppose that D' is a coherent set of 23 points containing X and that D' contains no point of Q. Since D'

cannot contain all the points of P we may choose $v_i \in P-D'$ and consider the partition it induces on D'. It follows that $|D' \cap \pi(i)| = 7$ or 16. If $|D' \cap \pi(i)| = 7$, then we could repeat the process with another point of P-D' and this obviously leads to a contradiction. Hence we must have $|D' \cap \pi(i)| = 16$ and $D' = X \cup (P - \{v_i\}) \cup \pi(i)$. We may now assume that D' contains an element v_B of Q. Since B corresponds to a line of the affine space described above there are for each $i \in I$ just four blocks B' such that $v_{B'} \in \pi(i)$ and $|B \cap B'| = 4$. We cannot have $Q \subseteq D'$ so choose $v_{B_1} \in Q - D'$ and consider the partition it induces on D'. If we put $Q = \{v_{B_1}, v_{B_2}, v_{B_3}, v_{B_4}\}$ and $\mu(j) = \{v_B \in \Pi \mid |B \cap B_j| = 4\}$, then $|D' \cap \mu(1)| = 7$ or 16. Arguing as before we must have $|D' \cap \mu(1)| = 16$ and $D' = X \cup (Q - \{v_{B_1}\}) \cup \mu(1)$. We have proved that X is contained in 8 coherent sets of 23 points. Since it is clear that M_{23} is the stabiliser of D in G we have $|G| = \frac{276.275.162.105}{23.22.21.20} .8. |M_{23}|$, so that

$$(2.7) \qquad |G| = 2^{10}.3^{7}.5^{3}.7.11.23 = |C_3|.$$

(2.8) G <u>is a simple group</u>.

<u>Proof</u>. Let K be a minimal normal subgroup of G. Since G is doubly transitive, K is transitive on Ω and cannot be soluble. Since $23 \mid |K|$, K is simple and we have $M_{23} \leq K$. Let P be an S_{23}-subgroup of K. Since G has a faithful rational representation of degree 23 we have $C_G(P) = P$. By Sylow's theorem and the Frattini argument we have K = G. //

Remark. Using the fact that the stabiliser of a point contains M_{22} and the stabiliser of two points contains PSL(3,4) it can be shown that the stabiliser of a point has a simple subgroup of index 2.

3. $O^1(8)$.

(3.1) Theorem. If $T = (\Omega, t)$ is a 4-regular 2-graph with $N=136$, $a=64$ and a coherent set of 10 points, then T is isomorphic to $O^1(8)$ and Aut $T \simeq Sp(8,2)$.

Proof. Let M be a coherent set of 10 points. From (2.7.3) any point not in M partitions M into a pair of five-element subsets. Since 4-regularity implies 5-regularity we must have $k_5 = 6$ and hence $k_4 = 12$ from (2.8.7). Since there are 126 partitions of a set of 10 points into a pair of five-element subsets, each point x of Ω-M corresponds to a unique partition of M. We shall write $x = (M_1, M_2)$, where $M = M_1 \cup M_2$ is the partition of M induced by x. If $x_1 = (M_1, M_1')$ and $\alpha, \beta, \gamma, \delta \in M_1$, then there are 5 elements $x_2 = (M_2, M_2')$ such that $\{\alpha, \beta, \gamma, \delta, x_1, x_2\}$ is coherent and we may therefore choose the notation for x_2 so that $|M_1 \cap M_2| = 4$. Thus the set $\{x_1, x_2, a\}$, $a \in M$, $x_1, x_2 \notin M$ is coherent if and only if we may choose the notation so that $x_1 = (M_1, M_1')$, $x_2 = (M_2, M_2')$, $|M_1 \cap M_2| = 4$ and $a \in M_1 \cap M_2$. We have shown that T is uniquely determined. From (4.3.29) the 2-graph $O^1(8)$ contains a coherent set of 10 points. Moreover, the automorphism group of $O^1(8)$ is transitive on ordered coherent sets of four elements. Thus $O^1(8)$ satisfies the

initial conditions and the theorem is proved. //

From the description of $O^1(8)$ obtained in the above proof we see that S_{10} acts on $O^1(8)$ with orbits of length 10 and 126. The inclusion $S_{10} \leq Sp(8,2)$ was also obtained in § 4.3. It follows that S_{10} acts on the 2-graph $O^{-1}(8)$. The 120 points of $O^{-1}(8)$ can be identified with the 120 three-element subsets of a set of ten elements. If we join two such subsets whenever they have just one point in common we obtain a strongly regular graph with $n=120$, $n_1=63$, $n_2=56$, $p_{11}^1=30$ and $p_{11}^2=27$. By (2.6.2) this gives rise to a regular 2-graph which must be $O^{-1}(8)$. Another strongly regular graph with the same parameters is obtained from the rank 3 representation of $Sp(6,2)$ on $P\Gamma U(3,3^2)$. It is possible to prove that the corresponding regular 2-graph is not $O^{-1}(8)$.

Table 1

The parameters of regular 2-graphs with less than 400 points

The following table gives the possible parameters for a regular 2-graph on N points with $N \leq 400$ and $a \leq a'$ as obtained from (2.5.8), omitting those with non-integral eigenvalues. Where a regular 2-graph exists we give an example or a reference to the appropriate theorem.

N	a	k	a'	k'	ρ_1	ρ_2	μ_1	μ_2	
10	4	1	4	1	3	-3	5	5	$O^1(4)$
16	6	1	8	4	5	-3	6	10	S(4)
26	12	5	12	5	5	-5	13	13	$T(5^2)$
28	10	1	16	10	9	-3	7	21	$O^{-1}(6)$
36	16	6	18	9	7	-5	15	21	$O^{-1}(6)$
50	24	11	24	11	7	-7	25	25	$T(7^2)$
64	30	13	32	16	9	-7	28	36	S(6)
76	32	10	42	25	15	-5	19	57	
82	40	19	40	19	9	-9	41	41	$T(9^2)$
96	40	12	54	33	19	-5	20	76	
100	48	22	50	25	11	-9	45	55	(3.3.2)
120	54	21	64	36	17	-7	35	85	$O^{-1}(8)$
122	60	29	60	29	11	-11	61	61	$T(11^2)$
126	52	15	72	45	25	-5	21	105	U(5)
136	64	28	70	37	15	-9	51	85	$O^1(8)$
144	70	33	72	36	13	-11	66	78	(3.4.3)

N	a	k	a'	k'	ρ_1	ρ_2	μ_1	μ_2	
148	66	25	80	46	21	-7	37	111	
170	84	41	84	41	13	-13	85	85	$T(13^2)$
176	72	20	102	65	35	-5	22	154	H(176)
190	88	37	100	55	21	-9	57	133	
196	96	46	98	49	15	-13	91	105	(3.3.2)
210	100	45	108	57	19	-11	77	133	§ 5.3
226	112	55	112	55	15	-15	113	113	Goethals and Seidel [1]
244	112	46	130	73	27	-9	61	183	
246	108	39	136	81	35	-7	41	205	
256	126	61	128	64	17	-15	120	136	S(8)
276	112	30	162	105	55	-5	23	253	C(276)
280	128	52	150	85	31	-9	63	217	
288	126	45	160	96	41	-7	42	246	
290	144	71	144	71	17	-17	145	145	$T(17^2)$
320	150	65	168	92	29	-11	88	232	
324	160	78	162	81	19	-17	153	171	
344	150	53	192	116	49	-7	43	301	U(7)
352	160	64	190	109	39	-9	66	286	
362	180	89	180	89	19	-19	181	181	$T(19^2)$
364	170	73	192	106	33	-11	91	273	
376	182	85	192	100	25	-15	141	235	
378	180	81	196	105	29	-13	117	261	

REFERENCES

R.W. Ahrens and G. Szekeres : [1] On a combinatorial generalization of 27 lines associated with a cubic surface. J. Austral. Math. Soc. 7 (1967), 64-80.

H.F. Baker : [1] 'Abelian functions'. Cambridge University Press, 1897.

E.R. Berlekamp, J.H. van Lint and J.J. Seidel : [1] A strongly regular graph derived from the perfect ternary Golay code (to be published).

W.C. Blackwelder : [1] On constructing balanced incomplete block designs from association matrices with special reference to association schemes of two and three classes. J. Comb. Theory 7 (1969), 15-36.

R.C. Bose : [1] Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math. 13 (1963), 389-419.

F.C. Bussemaker and J.J. Seidel : [1] Symmetric Hadamard matrices of order 36. Technological University Eindhoven Report 70-WSK-02.

L. Carlitz : [1] A theorem on permutations in a finite field. Proc. Amer. Math. Soc. 11 (1960), 456-459.

J.H. Conway : [1] Three lectures on exceptional groups. Chapter VII of 'Finite simple groups', edited by M.B. Powell and G. Higman, Academic Press, 1971.

P. Dembowski : [1] 'Finite Geometries'. Springer, 1968.

L.E. Dickson : [1] 'Linear groups with an exposition of the Galois field theory', Dover, 1958.

L.E. Dickson : [2] Applications of finite groups. Part III of 'Theory and applications of finite groups', by G.A. Miller, H.F. Blichfeldt and L.E. Dickson. Dover, 1961.

J. Dieudonné : [1] 'La géométrie des groupes classiques', 2e édition, Springer, 1963.

W. Feit : [1] 'Characters of finite groups'. Benjamin, 1967.

J.M. Goethals and J.J. Seidel : [1] Strongly regular graphs derived from combinatorial designs. Canad. J. Math. 22 (1970), 597-614.

J.M. Goethals and J.J. Seidel : [2] Orthogonal matrices with zero diagonal. Canad. J. Math. 19 (1967), 1001-1010.

J. Haantjes : [1] Equilateral point-sets in elliptic two- and three-dimensional spaces. Nieuw Arch. Wisk. 22 (1948), 355-362.

M. Hall : [1] 'Combinatorial theory'. Blaisdell, 1967.

D.G. Higman : [1] Finite permutation groups of rank 3, Math. Z. 86 (1964), 145-156.

D.G. Higman : [2] Primitive rank 3 groups with a prime subdegree. Math. Z. 91 (1966), 70-86.

D.G. Higman : [3] Characterizations of families of rank 3 permutation groups by the subdegrees I. Archiv der Math. 21 (1970), 151-157.

D.G. Higman : [4] Characterizations of families of rank 3 permutation groups by the subdegrees II. Archiv der Math. 21 (1970), 353-361.

D.G. Higman : [5] Solvability of a class of rank 3 permutation groups. Nagoya Math. J. 41 (1971), 89-96.

D.G. Higman and C.C. Sims : [1] A simple group of order 44, 352, 000. Math. Z. 105 (1968), 110-113.

G. Higman : [1] On the simple group of D.G. Higman and C.C. Sims. Ill. J. Math. 13 (1969), 74-84.

B. Huppert : [1] 'Endliche Gruppen I'. Springer, 1967.

Z. Janko : [1] A characterization of the smallest group of Ree associated with the Lie-algebra of type (G_2). J. Algebra 4 (1966), 293-299.

C. Jordan : [1] 'Traité des substitutions et des équations algébriques'. Gauthiers-Villars, 1870.

W.M. Kantor : [1] 2-transitive symmetric designs. Trans. Amer. Math. Soc. 142 (1969), 1-28

J.H. van Lint and J.J. Seidel : [1] Equilateral point sets in elliptic geometry. Nederl. Akad. Wetensch. Proc. Ser. A 69 (1966), 335-348.

H. Lüneburg : [1] Some remarks concerning the Ree groups of type (G_2). J. Algebra 3 (1966), 256-259.

H. Lüneburg : [2] 'Transitive Erweiterungen endlicher Permutationsgruppen'. Lecture Notes in Mathematics, Vol. 84, Springer, 1969.

Jack McLaughlin : [1] A simple group of order 898,128,000. pp. 109-111 of 'Theory of finite groups', edited by R. Brauer and C.-H. Sah. Benjamin, 1969.

J.J. Seidel : [1] Strongly regular graphs of L_2-type and of triangular type. Nederl. Akad. Wetensch. Proc. Ser. A 70 (1967), 188-196.

J.J. Seidel : [2] Strongly regular graphs with (-1,1,0) adjacency matrix having eigenvalue 3. Linear Algebra and Appl. 1 (1968), 281-298.

J.J. Seidel : [3] Strongly regular graphs. 'Proc. 3rd Waterloo Conference in Combinatorics', edited by W.T. Tutte, Academic Press, 1969.

C.C. Sims : [1] Graphs and finite permutation groups. Math. Z. 95 (1967), 76-86.

C.C. Sims : [2] On the isomorphism of two groups of order 44,352,000. pp. 101-108 of 'Theory of finite groups', edited by R. Brauer and C.-H. Sah. Benjamin, 1969.

M. Smith : [1] On some finite simple groups. D.Phil. Thesis, University of Oxford, 1969.

J. Tits : [1] Les groupes simples de Suzuki et Ree. Séminaire Bourbaki, N°. 210, 1960.

J. Tits : [2] Groupes finis simples sporadiques. Séminaire Bourbaki, N°. 375, 1970.

J.A. Todd : [1] A representation of the Mathieu group M_{24} as a collineation group. Annali di Mat. Serie IV, 71 (1966), 219-238.

H.N. Ward : [1] On Ree's series of simple groups. Trans. Amer. Math. Soc. 121 (1966), 62-89.

H. Wielandt : [1] 'Finite permutation groups'. Academic Press, 1964.

E. Witt : [1] Über Steinersche Systeme. Abh. Hamburg 12 (1938), 265-275.