SOME TOPICS IN THE THEORY OF FINITE GROUPS

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A regular 2-graph consists of a set $\Omega$ together with a (non-empty) set $t$ of three-element subsets of $\Omega$ such that any two-element subset of $\Omega$ is contained in the same number of elements of $t$, any four-element subset of $\Omega$ contains an even number of elements of $t$ and not every three-element subset of $\Omega$ is in $t$. These objects were introduced by G. Higman who used a regular 2-graph with 276 points to provide a combinatorial setting for the doubly transitive representation of Conway's sporadic simple group $G_3$.

In this thesis it is shown that regular 2-graphs are in one-one correspondence with equivalence classes of strong graphs (as defined by J.J. Seidel). Moreover, for each point of a regular 2-graph there is a natural way of defining a strongly regular graph on the remaining points. These graphical representations are used to obtain restrictions on the structure and on the parameters of a regular 2-graph. It is also possible, via the strong graphs, to represent a regular 2-graph as a configuration of equiangular lines in Euclidean space. Conversely, results about regular 2-graphs obtained in this thesis extend the results of J.J. Seidel on equiangular lines.

Regular 2-graphs are constructed which admit the following groups as doubly transitive groups of automorphisms: $\text{PSL}(2, q)$, $q \equiv 1 \pmod{4}$, $\text{Sp}(2m, 2)$, in both doubly transitive representations; $\text{PSU}(3, q^2)$, $q$ odd; all groups of Ree type.
together with $2G_2(3) = \text{Aut} (\text{PSL}(2,8))$; the sporadic simple
groups $G_2$ and $\text{HiS}$; the group $V.\text{Sp}(2m,2)$ which is the semi-
direct product of the group $V$ of translations of a vector
space of dimension $2m$ over the field $\text{GF}(2)$ by $\text{Sp}(2m,2)$. By
studying the centraliser ring of a monomial representation
associated with the doubly transitive representation it is
shown that (with the possible exception of some groups with
a regular normal subgroup) the above groups are the only
known groups which can act as doubly transitive groups of
automorphisms of a regular $2$-graph.

The bound obtained in (2,7,3) was also found (by another
method) by S. Todd who also observed the connection between
regular $2$-graphs and strongly regular graphs.

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It is a pleasure to acknowledge my gratitude to my supervisor, Professor G. Higman, F.R.S., both for his valuable advice and for his creation of the subject of this thesis. Professor Higman introduced regular 2-graphs to study the doubly transitive representation of degree 276 of the group $C_3$. The theorems of §§ 5.1, 5.2 were proved by Professor Higman (by different methods to those used here) and the construction of the regular 2-graphs in § 4.3 from the affine space of quadratic forms was also suggested by him.

The bound obtained in (2.7.3) was also found (by another method) by S. Todd who also observed the connection between regular 2-graphs and strong graphs.

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Finally, I wish to thank Mrs. E. Browne for her excellent typing of the manuscript and my wife Wendy for her patience and encouragement during the writing of this thesis.
The extension of $\text{PSL}(n,q)$ by the automorphisms of $\text{GF}(q)$ will be denoted by $\overline{\text{PSL}}(n,q)$ and a similar notation will be used for other classical simple groups extended by field automorphisms. The notation for sporadic simple groups is that of Tits [2]. Any other notation not explained in the text may be found in Dembowski [1] or Huppert [1].

The symbol (4.3.18) refers to the equation or theorem labelled (3.18) in Chapter 4; within Chapter 4 it will be referred to as (3.18).
CHAPTER 1

RESULTS FROM GRAPH THEORY

In this chapter we collect results on strong graphs and strongly regular graphs. Strong graphs were introduced by J.J. Seidel [2] generalising the concept of the strongly regular graph used by Bose [1]. Most of the results can be found in Seidel [2], [3] or Bose [1]. As a general reference for geometry and combinatorial theory we shall use Dembowski [1].

1. Strong graphs

A graph is an ordered pair \((\mathcal{V};\mathcal{E})\) where \(\mathcal{V}\) is a finite set and \(\mathcal{E}\) is a set of two element subsets of \(\mathcal{V}\). The elements of \(\mathcal{V}\) are called vertices (or points) and the elements of \(\mathcal{E}\) are called edges. A point \(a\) is joined to a point \(b\) if \(\{a, b\}\) is an edge. A graph \((\mathcal{V};\mathcal{E})\) is regular of valency \(v\) if every point is joined to exactly \(v\) other points.

Suppose that the points of \(\mathcal{V}\) are listed in some order and suppose that \(|\mathcal{V}| = n\). Relative to this order we define the adjacency matrix to be the \(n \times n\) matrix \(A = (a_{uv})\), where

\[
\begin{align*}
a_{uv} &= \begin{cases} 
-1 & \text{if } a \text{ is joined to } b \\
0 & \text{if } a = b \\
1 & \text{if } a \text{ is not joined to } b, \text{ for } a, b \in \mathcal{V}.
\end{cases}
\end{align*}
\]

The matrix \(A\) determines the set \(\mathcal{E}\) and conversely, so
we shall also use the symbol \((\mathcal{G}, A)\) to refer to the graph \((\mathcal{G}; E)\).

We now define some parameters associated with the graph \((\mathcal{G}, A)\). Suppose that \(a\) and \(\beta\) are points of \(\mathcal{G}\) such that 
\[ a_{\alpha \beta} = (-1)^k \] 
where \(k\) is 1 or 2. We define \(p_{ij}^k(a, \beta)\) to be the number of points \(\gamma\) such that 
\[ a_{\gamma \gamma} = (-1)^i \] 
and 
\[ a_{\gamma \beta} = (-1)^j \] 
where \(i, j \in \{1, 2\}\). We now define

\[(1.2)\]
\[ p_1(a, \beta) = p_{12}^1(a, \beta) + p_{21}^1(a, \beta) \quad \text{a joined to } \beta \]
\[ p_2(a, \beta) = p_{12}^2(a, \beta) + p_{21}^2(a, \beta) \quad \text{a not joined to } \beta. \]

(\text{cf. Dembowski [1], p. 281}).

A strong graph is a graph \((\mathcal{G}, A)\) such that the numbers \(p_1(a, \beta)\) and \(p_2(a, \beta)\) do not depend on the choice of \(a\) and \(\beta\). When this is the case we denote these numbers by \(p_1\) and \(p_2\) respectively. Note that \(p_1(a, \beta)\) is defined only when 
\[ a_{\alpha \beta} = (-1)^1. \] 
There are two types of strong graph for which \(p_1\) or \(p_2\) is not defined. These are the void graph defined by \(E = \emptyset\) and the complete graph in which \(E\) is the set of all two element subsets of \(\mathcal{G}\). These trivial cases are excluded by Seidel [2] in his definition of strong graph.

Let \(n_1(a)\) denote the number of points joined to \(a\) and let \(n_2(a)\) denote the number of points other than \(a\) not joined to \(a\). We have the following obvious relations:

\[(1.3)\]
\[ p_{ij}^k(a, \beta) = p_{ji}^k(\beta, a), \quad i, j, k \in \{1, 2\} \]

\[(1.4)\]
\[ n_1(a) = p_{11}^k(a, \beta) + p_{1j}^k(a, \beta) + s_{1k}, \quad \{i, j\} = \{1, 2\}, k \in \{1, 2\} \]
(1.5) \[ n = n_1(a) + n_2(a) + 1. \]

A **strongly regular graph** is a regular graph \((\Omega, A)\) such that the numbers \[ p_1^1(\alpha, \beta) \] and \[ p_2(\alpha, \beta) \] do not depend on the choice of \(\alpha\) and \(\beta\) in \(\Omega\). It is immediate from (1.4) that all the parameters \[ p_{1j}^k(\alpha, \beta) \] are independent of the choice of \(\alpha\) and \(\beta\) so we set \[ p_{1j}^k = p_{1j}(\alpha, \beta) \] and \[ n_1 = n_1(a) \], when \((\Omega, A)\) is strongly regular. For a point \(a \in \Omega\), let \(D_1(a)\) be the set of points joined to \(a\) and let \(D_2(a)\) be the set of points, other than \(a\), not joined to \(a\). Counting edges from \(D_1(a)\) to \(D_2(a)\) we obtain

\[ (1.6) \quad n_1 p_{jk}^1 = n_j p_{ik}^1 \quad 1, j, k \in \{1, 2\}. \]

**Strongly regular graphs** are a particular case of an **association scheme** (see Dombowski [1], § 7.1); they also arise in connection with rank 3 permutation groups (see D.G. Higman [1] and C. Sims [1]).

Let \(G\) be a transitive permutation group on a finite set \(\Omega\). Then we define the action of \(G\) on \(\Omega \times \Omega\) by \((\alpha, \beta)g = (\alpha g, \beta g)\) where \(\alpha, \beta \in \Omega, \ g \in G\). The **rank** of \(G\) is the number of orbits of \(G\) on \(\Omega \times \Omega\). Suppose that \(G\) has rank 3 and let the orbits of \(G\) on \(\Omega \times \Omega\) be \(I, D_1\) and \(D_2\) where \(I = \{(a, a) \mid a \in \Omega\}\).

For \(a \in \Omega\), define \(D_1(a) = \{ \beta \in \Omega \mid (a, \beta) \in D_1 \}\), \(i=1, 2\).

Then the orbits of \(G_a\) on \(\Omega\) are just \(\{a\}\), \(D_1(a)\) and \(D_2(a)\).

In general, if \(0\) is an orbit of \(G\) on \(\Omega \times \Omega\), then the **paired orbit** to \(0\) is \(0' = \{(\alpha, \beta) \mid (\beta, \alpha) \in 0\}\). It is clear that \(G\) has a self-paired orbit \(0'\) (i.e., \(0 = 0'\)) if and only if
G has even order.

Suppose now that G has rank 3 and even order. Then $D_1$ and $D_2$ are both self-paired. Let $E$ be the set of two element subsets $\alpha, \beta$ such that $(\alpha, \beta) \in D_1$. Then $(\Omega; E)$ is clearly a strongly regular graph. In the notation of D.G. Higman [1] we have $n = |\Omega|$, $k = |D_1(\alpha)|$, $\lambda = |D_2(\alpha)|$ and $|D_1(\alpha) \cap D_2(\beta)| = \begin{cases} \lambda & \beta \in D_1(\alpha) \\ \mu & \beta \in D_2(\alpha) \end{cases}$.

In our previous notation, $n_1 = k$, $n_2 = \lambda$, $p_{11}^1 = \lambda$, $p_{11}^2 = \mu$. Thus the examples of rank 3 groups given in D.G. Higman [1] give rise to a large collection of strongly regular graphs.

The following proposition justifies our terminology:

(1.7) **Proposition** A graph $(\Omega, A)$ is strongly regular if and only if it is both strong and regular.

**Proof.** If $(\Omega, A)$ is strongly regular it is obviously strong and regular. Conversely, suppose $(\Omega, A)$ is strong and regular with valency $n_1$. If $(\Omega, A)$ is complete or void we are done. Otherwise, choose $\alpha$ and $\beta$ in $\Omega$ such that $\alpha$ is joined to $\beta$.

It follows from (1.4) that $p_{12}^1(\alpha, \beta) = p_{21}^1(\alpha, \beta) = \frac{1}{2}p_1$ and then $p_{11}^1(\alpha, \beta)$ is independent of $\alpha$ and $\beta$. Similarly, $p_{11}^2(\alpha, \beta)$ is independent of $\alpha$ and $\beta$. Hence $(\Omega, A)$ is strongly regular. //

The following result is basic to a classification of strong graphs.
Proposition (Seidel [2]). A graph \((\mathcal{G}, A)\) of order \(n\) is strong if and only if these are real numbers \(\rho_1\) and \(\rho_2\) such that \(\rho_1 > \rho_2\) and

\[ (A-\rho_1 I)(A-\rho_2 I) = (n-1 + \rho_1 \rho_2)J \]

where \(J\) is the \(n \times n\) matrix all of whose entries are 1.

Proof. Let \(\rho_1\) and \(\rho_2\) be any real numbers and put

\[ B = (A-\rho_1 I)(A-\rho_2 I). \]

We write \(B = (b_{\alpha \beta})\). If \(a_{\alpha \beta} = (-1)^k\), then by definition of \(p_{1j}^k(a, \beta)\) we have

\[ b_{\alpha \beta} = (-1)^k (\rho_1 + \rho_2) + p_{11}^k(a, \beta) + p_{22}^k(a, \beta) - p_{12}^k(a, \beta) - p_{21}^k(a, \beta) \]

\[ = (-1)^k (\rho_1 + \rho_2) + (n-2) - 2p^k(a, \beta). \]

Suppose that \((\mathcal{G}, A)\) is a strong graph. If it has valency 0 or \(n\), then \(A = \varepsilon (J-I)\), where \(\varepsilon = \pm 1\) and then

\[ (A + \varepsilon I)(A-\rho I) = \varepsilon J(\varepsilon J - \varepsilon I - \rho I) = (n-1 + (-\varepsilon)\rho)J, \]

for any \(\rho\). Hence (1.9) holds in this case. We may now assume that \(p^1\) and \(p^2\) are both defined and choose \(\rho_1\) and \(\rho_2\) so that \(\rho_1 > \rho_2\) and

\[ \rho_1 + \rho_2 = p^1 - p^2 \]

\[ 1 - \rho_1 \rho_2 = p^1 + p^2 \]

Then \(b_{\alpha \beta} = n-1 + \rho_1 \rho_2\), that is \(B = (n-1 + \rho_1 \rho_2)J\).

Conversely, suppose that \(B = (n-1 + \rho_1 \rho_2)J\). Then we have

\[ \rho_1 + \rho_2 + (n-2) - 2p^1(a, \beta) = n-1 + \rho_1 \rho_2 \]

and

\[ - \rho_1 - \rho_2 + (n-2) - 2p^2(a, \beta) = n-1 + \rho_1 \rho_2. \]
If both $p^1$ and $p^2$ are defined, then

\[(1.11) \quad p^1 = -\frac{(p_1-1)(p_2-1)}{2} \]
\[\quad p^2 = -\frac{(p_1+1)(p_2+1)}{2} \]

and in any case $(\Omega, A)$ is a strong graph. //

2. A classification of strong graphs

Given a graph $(\Omega, A)$ the complementary graph is $(\Omega, -A)$. The complementary graph of a strong graph is obviously strong.

Two graphs $(\Omega, A_1)$ and $(\Omega, A_2)$ with the same point set $\Omega$ are equivalent (with respect to switching) if there exists a diagonal matrix $D$ whose diagonal entries are $\pm 1$ such that $D^{-1}A_1D = A_2$. If $D = (d_{\alpha\beta})$ and just $d_{aa}$ is $-1$, then $(\Omega, A_2)$ is obtained from $(\Omega, A_1)$ by changing all the adjacencies from $a$. We call this operation 'switching with respect to $a$'.

A graph equivalent to a complete graph is the disjoint union of two complete graphs; a graph equivalent to a void graph is called a complete bipartite graph; all of these will be called trivial strong graphs.

In classifying strong graphs we shall use equation (1.9). If the graph $(\Omega, A)$ is neither complete nor void, $p_1$ and $p_2$ of (1.9) are uniquely determined. For the complete graph we put $p_1 = 1$ and $p_2 = 1-n$; for the void graph we put $p_1 = n-1$ and $p_2 = -1$. It now follows that if $n-1 + p_1p_2 = 0$ in (1.9) then the adjacency matrix of any equivalent graph still satisfies the same equation. From now on we shall assume that $(\Omega, A)$ is a strong graph such that $(A-p_1I)(A-p_2I) = (n-1 + p_1p_2)J$. 
(2.1) **Lemma.** If \( n-1+p_1 p_2 \neq 0 \), then \((\omega, A)\) is regular and the eigenvalue \( \rho_0 = n_2-n_1 \) of \( A \) has multiplicity 1. The other eigenvalues of \( A \) are \( \rho_1 \) and \( \rho_2 \).

**Proof.** Since \( A \) is a real symmetric matrix we can diagonalise it. From (1.9) \( J \) will also be in diagonal form. Since the vector \((1,1,...,1)^T\) is an eigenvector of \( J \) it is an eigenvector of \( A \). Hence \((\omega, A)\) is regular and the eigenvalue corresponding to \((1,1,...,1)^T\) is obviously \( n_2-n_1 \). The remaining statements are now clear. //

We now define the following types of strong graph:

Type 1: \((\omega, A)\) is trivial.
Type 2: \((\omega, A)\) is non-trivial and \( n-1+p_1 p_2 = 0 \).
Type 3: \((\omega, A)\) is non-trivial and \( n-1+p_1 p_2 \neq 0 \).

(2.2) **Lemma.** If \((\omega, A)\) is non-trivial, then the numbers \( p_1 \) and \( p_2 \) are even.

**Proof.** (Seidel [2]). Choose \( a, \beta \in \omega \) such that \( a_{a\beta} = (-1)^k \). Then from (1.4) \( n_1(\alpha) + n_1(\gamma) = p^k \) (mod 2). If \( a_{a\gamma} = (-1)^k \) as well, then \( n_1(\alpha) + n_1(\gamma) = p^k \) (mod 2) so \( n_1(\beta) + n_1(\gamma) = 0 \) (mod 2). Thus one of \( p_1, p_2 \) is even. By taking the complement if necessary we may suppose \( p_1 = 0 \) (mod 2). Suppose \( p_1 = 1 \) (mod 2). Then \( a_{a\gamma} = 1 \) and \((\omega, A)\) must be a complete bipartite graph. //

For type 2 graphs it is clear that \( \rho_1 \) and \( \rho_2 \) are the only eigenvalues of \( A \). Let \( \mu_1 \) be the multiplicity of \( \rho_1 \) for \( i=0,1,2 \).
(2.3) **Lemma.** If \((\mathcal{L}, \Lambda)\) has type 2, then we have one of the following possibilities:

(a) \(p_1 = p_2, \mu_1 = \mu_2 = p_2, n = 2p_1 + 2, \rho_1 = -\rho_2 = \sqrt{(n-1)}\)

and \(\Lambda^2 = (n-1)I\), or

(b) \(p_1 \neq p_2, n = p_1^2 + p_2 + 2, \Lambda^2 = (p_1 - p_2)A - (n-1)I = 0\)

\[\rho_1 = \frac{1}{2}(p_1 - p_2) + \sqrt{(n^2 - 4p_1 p_2)}\]

\[\rho_2 = \frac{1}{2}(p_1 - p_2) - \sqrt{(n^2 - 4p_1 p_2)}\]

\[\mu_1 = \frac{1}{2}n[1 - (p_1 - p_2)/\sqrt{(n^2 - 4p_1 p_2)}]\]

\[\mu_2 = \frac{1}{2}n[1 + (p_1 - p_2)/\sqrt{(n^2 - 4p_1 p_2)}]\]

In particular, \(n\) is even and both \(\rho_1\) and \(\rho_2\) are odd integers.

**Proof.** Since the trace of \(A\) is 0 we have \(\rho_1^2 + \rho_2^2 = 0\) and \(\mu_1 \mu_2 = n\) as well as \(n-1 + \rho_1 \rho_2 = 0\). If \(p_1 = p_2\), then from (1.10) \(\rho_1 = -\rho_2\) and \(n = 2p_1 + 2\). Thus \(\mu_1 = \mu_2 = p_1 + 1\) and \(\Lambda^2 = (n-1)I\). It follows that \(\rho_1 = -\rho_2 = \sqrt{(n-1)}\) and so (a) holds.

Now suppose \(p_1 \neq p_2\). Again from (1.10) we have

\[n = p_1^2 + p_2 + 2\]

and from (1.9) and (1.10) \(\Lambda^2 = (p_1 - p_2)A - (n-1)I = 0\).

Thus \(\rho_1\) and \(\rho_2\) are the roots of \(x^2 - (p_1 - p_2)x - (n-1) = 0\).

The expressions for \(\rho_1, \rho_2, \mu_1\) and \(\mu_2\) are now clear. Since \(\mu_1\) and \(\mu_2\) are integers \(n^2 - 4p_1 p_2\) must be the square of an (even) integer so \(\rho_1\) and \(\rho_2\) are integers. From (1.11) and (2.2) it follows that \(\rho_1\) and \(\rho_2\) are odd. //

Because of this lemma we subdivide type 2 graphs into
types 2a and 2b according to whether $p_1^1=p_2^2$ or $p_1^1\neq p_2^2$.

(2.4) Lemma. Suppose that $(\mathcal{G},A)$ has type 3. Then $(\mathcal{G},A)$ is strongly regular with parameters $n_1^*,p_1^k$ and one of the following possibilities occurs:

(a) $\mu_1^*=\mu_2^*, \rho_0^*=0, n_1^*=n_2^*=n_1^*, \rho_1^*=-\rho_2^*=-\sqrt{n}$

$$p_1^1 = \frac{3}{2}n_1-1, \quad p_2^2 = \frac{3}{2}n_1 \quad \text{and} \quad A^2=nI-J, \text{ or}$$

(b) $\mu_1^*\neq\mu_2^*, A^2=(p_1^1-p_2^2)A=(p_1^1+p_2^2+1)I = (n-p_1^1-p_2^2-2)I$

$$\rho_1^* = \frac{3}{2} \left\{ (p_1^1-p_2^2) + \sqrt{[(p_1^1+p_2^2+2)^2 - 4p_1^1p_2^2]} \right\}$$

$$\rho_2^* = \frac{3}{2} \left\{ (p_1^1-p_2^2) - \sqrt{[(p_1^1+p_2^2+2)^2 - 4p_1^1p_2^2]} \right\}$$

$$\mu_1^* = \frac{3}{2} \left\{ (n_1+n_2^*) - \frac{(n_1+n_2^*)(p_1^1-p_2^2)-2(n_1-n_2^*)}{\sqrt{[(p_1^1+p_2^2+2)^2 - 4p_1^1p_2^2]}} \right\}$$

$$\mu_2^* = \frac{3}{2} \left\{ (n_1+n_2^*) + \frac{(n_1+n_2^*)(p_1^1-p_2^2)-2(n_1-n_2^*)}{\sqrt{[(p_1^1+p_2^2+2)^2 - 4p_1^1p_2^2]}} \right\}$$

In case (b), $\rho_1^*$ and $\rho_2^*$ are odd integers.

Proof. As in Lemma (2.3) we have $\rho_0^*+\rho_1^*\mu_1^*+\rho_2^*\mu_2^* = 0$ and $1+\mu_1^*+\mu_2^* = n$. From Lemma (2.1) $(\mathcal{G},A)$ is regular and $\rho_0^*=n_2^*-n_1^*$. Also, $p_1^1 = p_2^1 = \frac{3}{2}p_1^1$ and $p_2^2 = p_2^1 = \frac{3}{2}p_2^2$. Diagonalising (1.9) and using (1.10) we obtain

$$(\rho_0^*-\rho_1^*)(\rho_0^*+\rho_2^*) = n(n-1+p_1^1p_2^2) \text{ and } \rho_1^*, \rho_2^* \text{ are the solutions of } x^2=(p_1^1-p_2^2)x-(p_1^1+p_2^2+1) = 0.$$
Thus \( \mu_1 = n_1 \), \( \rho_1 = -\rho_2 \) and then \( \rho_1 = \sqrt{n} \). Finally, \( p^1 = n_1 \)
so from (1.4) \( p_{11}^1 = \frac{1}{2} n_1 - 1 \), \( p_{11}^2 = \frac{1}{2} n_1 \).

If \( \mu_1 \neq \mu_2 \) the expressions for \( \rho_1, \rho_2, \mu_1 \) and \( \mu_2 \) are obtained immediately from equations above. Since \( \mu_1 \) is an integer it follows that \( (p^1 + p^2 + 2)^2 - 4p^1 p^2 \) is the square of an even integer so \( \rho_1 \) and \( \rho_2 \) are integers. From (1.10) they are odd integers. //

We subdivide graphs of type 3 into types 3a and 3b according to whether \( \mu_1 = \mu_2 \) or \( \mu_1 \neq \mu_2 \).

Remarks 1. There exist type 3b graphs with \( n_1 = n_2 \). We give an example. Let \( \varphi \) be the set of 2 element subsets of a set of 7 elements, two subsets being joined if they have an element in common. This defines a strongly regular graph with \( n = 21 \), \( n_1 = 10 \), \( n_2 = 10 \), but \( p_{11} = 6 \) and \( p_{12} = 4 \).

2. There exist type 3b graphs with \( p^1 = p^2 \). An example is the Moore graph of valency 7 diameter 2. The parameters are \( n = 50 \), \( n_1 = 7 \), \( n_2 = 42 \), \( p_{11} = 0 \), \( p_{12} = 1 \), \( p^1 = p^2 = 12 \) (see D.G. Higman [2] or Berlekamp, van Lint and Seidel [1]).

3. There are strongly regular graphs of type 2. (See Chapter 2).

We now give a further restriction on the parameters of a graph of type 2a.

(2.5) Lemma. If \((G,A)\) has type 2a, then \( n = 2 \) (mod 4) and there are integers \( a, b \) such that \( n = 1 + a^2 + b^2 \).
Proof. (van Lint and Seidel [1]). From Lemma 2.3 $A^2 = (n-1)I$ and $n = 2p^1 + 2$ so $n \equiv 2 \pmod{4}$. We prove the following by induction on $n$: 'If $A$ is an $n \times n$ rational matrix such that $A^T A = mI$ where $m$ is an integer, and $n \equiv 2 \pmod{4}$, then $m$ is the sum of two squares'. This is true when $n = 2$. If $n \neq 2$, put $m = m_1^2 + m_2^2 + m_3^2 + m_4^2$ (Lagrange's theorem) and write

$$M = \begin{bmatrix}
m_1 & -m_2 & -m_3 & -m_4 \\
m_2 & m_1 & -m_4 & m_3 \\
m_3 & m_4 & m_1 & -m_2 \\
m_4 & -m_3 & m_2 & m_1
\end{bmatrix}$$

Then $M^T M = mI$ and multiplying a row and column of $M$ by $-1$ does not alter this equation. Write $A = \begin{bmatrix} A_1 & B \\ C & D \end{bmatrix}$ where $A_1$ is a $4 \times 4$ matrix and choose a suitable $M$ such that $M^T M = mI$ and $\det(A_1 - M) \neq 0$. Now put $A = D - C(A_1 - M)^{-1}B$.

Evaluating the product

$$((-B^T(A_1^{-1} - A^{-1}))_{n-4}) A^T A \begin{pmatrix} -(A_1 - M)^{-1}B \\ I_{n-4} \end{pmatrix}$$

in two ways we obtain $A^T A = mI_{n-4}$ and we have proved the induction step. ///

See also Hall [1] pp. 108-110 where the same argument is used to prove the Bruck-Reyser-Chowla theorem.

For future use we determine the strong graphs whose adjacency matrix has an eigenvalue 1 or -1. We define a graph $H(k, m) = (\emptyset, A)$ as follows: $\emptyset$ is the disjoint union of
k sets \( \Omega_1, \ldots, \Omega_k \) such that \( |\Omega_i| = m \) for \( i=1, \ldots, k \).

If \( a \) and \( \beta \) are points of \( \Omega_i \), \( a \) is joined to \( \beta \) whenever both \( a \) and \( \beta \) lie in \( \Omega_i \) for some \( i \). The graph \( H(k,m) \) is strongly regular with parameters \( n=km \), \( n_1=m-1 \), \( p_1^1=m-2 \), \( p_1^2=0 \). The matrix \( A \) has eigenvalues \( p_0=km-2m+1 \), \( p_1=1 \), \( p_2=-2m+1 \).

(2.6) Proposition. Let \((\Omega,A)\) be a strongly regular graph whose adjacency matrix \( A \) has an eigenvalue 1 or -1.

Then either \((\Omega,A)\) is trivial or it is the graph \( H(k,m) \) or its complement.

Proof. Suppose that \((\Omega,A)\) is neither complete nor void.

Passing to the complement if necessary we may assume \( p_1=1 \).

From (1.10) it follows that \( p_1^1=0 \). For \( a \in \Omega \) set \( D(a) = \{ a \} \cup \{ \beta | a_{a\beta} = -1 \} \). Since \( p_1=0 \) we have \( p_1^1(\beta,\gamma) = p_2^1(\beta,\gamma) = 0 \) for all \( \beta \) and \( \gamma \) in \( \Omega \) such that \( a_{a\beta} = -1 \). Thus any two distinct points of \( D(a) \) are joined. If \( a \) is not joined to \( \beta \), then no point of \( D(a) \) is joined to a point of \( D(\beta) \).

Thus \( \Omega \) is the disjoint union of complete graphs \( D(a) \).

If \((\Omega,A)\) is regular, then \( |D(a)| = |D(\beta)| \) for any \( a, \beta \) in \( \Omega \), so \((\Omega,A)\) is \( H(k,m) \) for \( m=|D(a)| \), \( k=n/m \). If \((\Omega,A)\) is not regular, then it is type 2 and \( n-1+p_2=0 \); thus \( p_2=n-2 \).

Choose \( a \) and \( \beta \) so that \( a_{a\beta}=1 \). Then \( p_1^2(a,\beta) = |D(a)| -1 \), \( p_2^1 = |D(\beta)| -1 \). Hence \( n-2 = p_2^2 = |D(a)| + |D(\beta)| -2 \). It follows that \((\Omega,A)\) is trivial. //

The automorphism group of \( H(k,m) \) is the wreath product \( S_m \wr S_k \).
The concept of regular 2-graph was introduced by G. Higman who proved that the sporadic simple group \( C_3 \) (3 of Conway [1]) is the automorphism group of a regular 2-graph with 276 points. In this chapter we define a regular 2-graph and then relate this concept to strong graphs and strongly regular graphs.

1. **Definitions**

A regular 2-graph is a pair \((\Omega, \mathfrak{t})\) where \(\Omega\) is a finite set of \(N\) points and \(\mathfrak{t}\) is a set of three-element subsets of \(\Omega\) such that the following three conditions hold:

(a) any two-element subset of \(\Omega\) is contained in exactly a elements of \(\mathfrak{t}\).

(b) any four-element subset of \(\Omega\) contains an even number of elements of \(\mathfrak{t}\).

(c) \(\mathfrak{t}\) is neither empty nor the set of all three-element subsets of \(\Omega\).

The elements of \(\mathfrak{t}\) are called coherent triangles. A subset \(C\) of \(\Omega\) is called coherent if all of its three-element subsets belong to \(\mathfrak{t}\).

If \(\Omega(3)\) denotes the set of all three-element subsets of \(\Omega\), then \((\Omega, \Omega(3) - \mathfrak{t})\) is a regular 2-graph; the complement of \((\Omega, \mathfrak{t})\). A subset of \(\Omega\) which is coherent in the complement
will be called incoherent (to be distinguished from non-coherent). We shall suppose that \(|\Omega| = N\), then any two element subset of \(\Omega\) is contained in just \(a'\) elements of \(\Omega(3) - \mathfrak{r}\) where \(a' = N - a - 2\).

The automorphism group of \(\Omega\) is the group of permutations of \(\Omega\) which take coherent triangles to coherent triangles. In contrast to graphs it is possible for regular 2-graphs to have doubly transitive automorphism groups.

\((1.1)\) **Proposition.** Let \((\Omega, \mathfrak{r})\) be a regular 2-graph. Then any coherent triangle is contained in the same number \(k\) of coherent four-element subsets of \(\Omega\) where \(N = 3a - 2k\).

**Proof.** Choose \(\{a, \beta, \gamma\} \in \mathfrak{r}\) and suppose that \(\{a, \beta, \gamma\}\) is contained in \(k\) coherent four-element subsets, \(\{a, \beta, \gamma, \delta_1\}\), \(i = 1, \ldots, k\). If \(x \notin \{a, \beta, \gamma, \delta_1, \ldots, \delta_k\}\), then just one of \(\{a, \beta, x\}\), \(\{a, \gamma, x\}\), \(\{\beta, \gamma, x\}\) belongs to \(\mathfrak{r}\). Hence \(N = 3 + k + 3(a - k - 1) = 3a - 2k\), so \(k\) is independent of the choice of element in \(\mathfrak{r}\). //

The numbers \(N, a\) and \(k\) will be called the parameters of \((\Omega, \mathfrak{r})\) and \(N, a'\) and \(k'\) will be the corresponding parameters for the complementary regular 2-graph. Whenever no ambiguity results we shall omit mentioning \(\mathfrak{r}\) and refer to the regular 2-graph \(\Omega\).

\((1.2)\) **Lemma.** The parameters are connected by the following equations:

\((1.3)\) \(N = a + a' + 2\).
(1.4) \[ N = 3a-2k = 3a'-2k' \]

(1.5) \[ N = 2k+2k'+6 \]

2. The associated strong graphs

In Chapter 1 we classified strong graphs into three types. We also introduced an equivalence relation on graphs (due to Seidel) and it follows from (1.1.9) that any graph equivalent to a strong graph of type 2 is again a strong graph of type 2. In this section we show that there is a one-to-one correspondence between regular 2-graphs and equivalence classes of strong graphs of type 2. We first establish a correspondence between regular 2-graphs and certain strongly regular graphs of type 3.

Let \( T=(G,\xi) \) be a regular 2-graph with parameters \( N,a \) and \( k \). Choose a point \( o \) in \( G \) and put \( G_0=G-\{o\} \). Now define a graph \( T_0=(G_0,A_0) \) with adjacency matrix \( A_0=(a_{\beta\gamma}^{(o)}) \) where

\[
(2.1) a_{\beta\gamma}^{(o)} = \begin{cases} 
-1 & \{o,\beta,\gamma\} \in \xi \\
0 & \beta = \gamma \\
1 & \{o,\beta,\gamma\} \notin \xi 
\end{cases}
\]

and \( \beta \) and \( \gamma \) belong to \( G_0 \).

Now suppose that \( R=(\Gamma,A) \) is a strongly regular graph with \( n \) points and parameters \( n_1, \beta^{(k)} \) as in Chapter 1. Suppose further that \( n_1=2p_1 \). Choose a point \( o \) not in the set \( \Gamma \) and put \( R^0=(\Gamma^0,\xi^0) \) where \( \Gamma^0=\Gamma-\{o\} \) and \( \xi^0 \) is the set of the following three-element subsets of \( \Gamma^0 \):

\[
(2.2) \quad \{o,\alpha,\beta\} \quad a_{\alpha\beta} \in \Gamma \quad a_{\alpha\beta} = -1, \\
\{\alpha,\beta,\gamma\} \quad a_{\alpha\gamma}, a_{\beta\gamma} \in \Gamma \quad a_{\alpha\beta}a_{\alpha\gamma}a_{\beta\gamma} = -1.
\]
(2.3) Theorem. If $T=(\Omega, \mathcal{T})$ is a regular 2-graph, then $T_0=(\Omega_0, \mathcal{A}_0)$ is a strongly regular graph with $n_1=2p_{11}^2$. Conversely, if $R=(\Gamma, \mathcal{A})$ is a non-trivial strongly regular graph with $n_1=2p_{11}^2$, then $R^0=(\Gamma^0, \mathcal{A}^0)$ is a regular 2-graph. Moreover, $(T_0)^0 = T$ and $(R^0)^0 = R$, and we have the following correspondence between the parameters:

$$
\begin{align*}
\text{n} &= N - 1 \\
\text{p}_{11} &= k \\
\text{p}_{12} &= \frac{1}{2}a' \\
\text{p}_{22} &= k' \\
\text{n}_1 &= a \\
\text{p}_{11}^2 &= \frac{1}{2}a \\
\text{p}_{12}^2 &= \frac{1}{2}a' \\
\text{p}_{22}^2 &= k' \\
\end{align*}
$$

Proof. Begin with the regular 2-graph $T=(\Omega, \mathcal{T})$. Since any set $\{o, a\}$ is contained in a elements of $\mathcal{T}$ we have $n_1 = a$ and $n_2 = a'$ for $T_0=(\Omega_0, \mathcal{A}_0)$. From (1.1) it follows that any edge of $T_0$ is contained in $k$ triangles, i.e., $p_{11}^1 = k$. Next suppose that $a$ and $\beta$ belong to $\Omega_0$ and $a_\alpha^\beta = 1$. Since $\{o, a, \beta\}$ is not coherent it is contained in $N$ four-element subsets of $\Omega$ which contain just two coherent triangles. Since $\{o, \alpha, \beta\}$ is contained in a coherent triangles there are $N - 3 - k'$ four-element subsets $\{o, a, \beta, \gamma\}$ containing $\{o, a, \beta\}$ such that just $\{o, a, \gamma\}$ and $\{o, \beta, \gamma\}$ are coherent, i.e., $p_{11} = N - 3 - k'$. Using (1.4) and (1.5) we have $p_{11}^2 = \frac{1}{2}a$. We have shown that $T_0$ is a strongly regular graph and $n_1 = 2p_{11}^2$. The remaining parameters can now be easily calculated.

Suppose that $R=(\Gamma, \mathcal{A})$ is a non-trivial strongly regular...
graph with $n_1 = 2p_{11}^2$. It follows immediately from (2.2) that every four-element subset of $\Gamma^0$ contains an even number of elements of $\kappa^0$. Now suppose $\{a, \beta\} \subseteq \Gamma^0$. If $a = 0$, then from (2.2) $\{a, \beta\}$ is contained in exactly $n_1$ elements of $\kappa^0$. If neither $a$ nor $\beta$ equal $0$, then again from (2.2) $\{a, \beta\}$ is contained in either $p_{12}^{1} + p_{11}^{1} + 1$ or $p_{12}^{2} + p_{21}^{2}$ elements of $\kappa^0$. From (1.1.6) $n_2 = 2p_{12}^{1}$ and from (1.1.4) $n_1 = p_{12}^{2} + p_{11}^{2}$ so $p_{12}^{2} + p_{21}^{2} = 2p_{12}^{1} = n_1$ and $p_{22}^{1} + p_{11}^{1} + 1 = p_{12}^{1} + p_{11}^{1} + 1 = n_1$. Since $\Gamma$ is non-trivial, $\kappa^0$ is neither empty nor all the three-element subsets of $\Gamma^0$ so we have proved that $R^0$ is a regular 2-graph.

It is clear that $(T_0)^0 = T$ and $(R^0)_0 = R$. //

(2.5) Corollary. If $T = (s, t)$ is a regular 2-graph and $T_0 = (s_0, A_0)$ then $T_0$ is a strongly regular graph of type 3 whose adjacency matrix satisfies

(2.6) $A_0^2 + (a - a')A_0 - (N-1)I = -J$.

If $\rho_1$ and $\rho_2$ are the eigenvalues of $A_0$ other than $a' - a$, with multiplicities $\mu_1$ and $\mu_2$ respectively, then

(2.7) $\rho_1 = \frac{1}{2} \left\{ (a' - a) + \sqrt{[(a+a'+2)^2 - 4aa']} \right\}$

(2.8) $\mu_1 = \frac{1}{2} \left\{ (N-2) - \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}} \right\}$

$\rho_2 = \frac{1}{2} \left\{ (a' - a) - \sqrt{[(a+a'+2)^2 - 4aa']} \right\}$

$\mu_2 = \frac{1}{2} \left\{ (N-2) + \frac{N(a'-a)}{\sqrt{[N^2 - 4aa']}} \right\}$

Proof. From (1.2.4). //
Remark. The numbers $N, a, a'$ are even.

Again suppose $T = (\Omega, \mathbf{t})$ to be a regular 2-graph with parameters $N, a, k$ and form $T_0 = (\Omega_0, A_0)$. Now define a graph $T'_0 = (\Omega_0, A'_0)$ by joining $o$ to every point of $\Omega_0$ and keeping the other edges as in $T_0$. Let $\mathcal{J}(T)$ denote the class of graphs equivalent to $T'_0$.

Suppose that $X$ is an equivalence class of strong graphs of type 2 with point set $\Omega$ and let $S = (\Omega, A)$ be an element of $X$. Define $\mathcal{J}(X) = (\Omega, \mathbf{t})$ where

$$ (2.9) \quad \mathbf{t} = \{ (\alpha, \beta, \gamma) \in \Omega \mid a_{\alpha \beta} a_{\gamma} = -1 \}. $$

$$ (2.10) \quad \text{Theorem. If } T = (\Omega, \mathbf{t}) \text{ is a regular 2-graph, then } T'_0 \text{ is a strong graph of type 2 and } \mathcal{J}(T) \text{ does not depend on the chosen point } o. \text{ Conversely, if } X \text{ is an equivalence class of strong graphs of type 2 and } S = (\Omega, A) \in X, \text{ then } \mathcal{J}(X) \text{ is a regular 2-graph and does not depend on the chosen element } S \text{ of } X. \text{ Moreover, } \mathcal{J}(\mathcal{J}(T)) = T \text{ and } \mathcal{J}(\mathcal{J}(X)) = X. $$

Proof. Let $T = (\Omega, \mathbf{t})$ be a regular 2-graph. Let $n, n_1, P_{1j}$ be the parameters of $T_0$. Consider the parameters $p_1(\alpha, \beta)$ of $T'_0$. If $\alpha, \beta \in \Omega_0$ and $a_{\alpha \beta} = -1$, then $p_1(\alpha, \beta) = p_{12}^1 + p_{21}^1 = a'$; if $a_{\alpha \beta} = 1$, then $p_2(\alpha, \beta) = p_{12}^2 + p_{21}^2 = a$. We also have $p_1(o, \alpha) = n_2 = a'$. Thus $T'_0$ is a strong graph with parameters $N, p_1 = a'$ and $p_2 = a$. From (1.1.10) $p_1 p_2 = -1 - a - a' = -(N-1)$ so $T'_0$ is a strong graph of type 2. Therefore, every graph in $\mathcal{J}(T)$ is a strong graph of type 2 with the same parameters as $T'_0$ and whose adjacency matrix has the same minimal
polynomial as $A_0$.

Now suppose $a_1, a_2 \in \Omega$. We must show that $(\Omega, A_{a_1})$ and $(\Omega, A_{a_2})$ are equivalent with respect to switching.

Define $D_1$ to be $\{ \gamma \in \Omega \mid \{a_1, a_2, \gamma\} \notin \mathcal{E} \}$ and $D_2$ to be $\{ \gamma \in \Omega \mid \{a_1, a_2, \gamma\} \in \mathcal{E} \}$. Let $(\Omega, A)$ be the graph obtained from $(\Omega, A_{a_1})$ by switching with respect to each point of $D_1$.

We claim that $(\Omega, A) = (\Omega, A_{a_2})$. This depends on condition (b) in the definition of regular 2-graph. Suppose that $a, \beta \in D_1$, $i=1$ or 2 and $\{a, \beta\}$ is an edge in $(\Omega, A)$, then it is an edge in $(\Omega, A_{a_i})$ and therefore $\{a_1, a_2, \beta\}$ is coherent whence $\{a_2, a, \beta\}$ is coherent so that $\{a, \beta\}$ is an edge in $(\Omega, A_{a_2})$ and conversely. If $a \in D_1$, $\beta \in D_2$ and $\{a, \beta\}$ is an edge in $(\Omega, A)$, then $\{a, \beta\}$ is not an edge in $(\Omega, A_{a_1})$ so $\{a_1, a, \beta\}$ is not coherent whence $\{a_2, a, \beta\}$ is coherent so that $\{a, \beta\}$ is an edge in $(\Omega, A_{a_2})$ and conversely. If $\{a_1, a\}$ is an edge in $(\Omega, A)$, then $a \in D_2$ or $a = a_2$. In either case $\{a_1, a\}$ is an edge in $(\Omega, A_{a_2})$ and conversely. Finally $\{a_2, a\}$ is an edge in $(\Omega, A)$ for all $a \neq a_2$.

Now suppose that $(\Omega, A)$ is a strong graph of type 2 with parameters $N, p_1, p_2$, and define $\mathcal{E}$ as in (2.9). It is immediate that any four-element subset of $\Omega$ contains an even number of elements of $\mathcal{E}$, and that $\mathcal{E}$ is neither empty nor the set of all three-element subsets of $\Omega$. If $a \neq \beta \in \Omega$, and $a_{a, \beta} = 1$, then from (2.9), the number of elements of $\mathcal{E}$ containing $\{a, \beta\}$ is $p_{12}^2(a, \beta) + p_{21}^2(a, \beta) = p^2$. If $a_{a, \beta} = -1$, then the number of elements of $\mathcal{E}$ containing $\{a, \beta\}$ is $p_{11}^1(a, \beta) + p_{22}^1(a, \beta)$. From (1.1, 4) $p_{11}(a, \beta) + p_{22}(a, \beta) = N - p - 2$. Since $(\Omega, A)$ is type 2 it follows from (1.1, 10) that $N - p - 2 = p^2$. 
Hence \((\Omega, \mathcal{A})\) is a regular 2-graph with parameters \(N, p^2, k\) where \(k = \frac{1}{2}(3p^2 - N)\). It is easily seen that switching does not alter the definition of \(\mathcal{A}\) in (2.9) so \(\mathcal{A}(\mathcal{X})\) depends only on \(\mathcal{X}\). By choosing an element \(S \in \mathcal{X}\) which has a point \(a\) joined to every other point it is obvious that \(\mathcal{F}(\mathcal{J}(\mathcal{X})) = \mathcal{X}\). Similarly, \(\mathcal{J}(\mathcal{A}(T)) = T\). //

**Corollary.** Let \(T = (\Omega, \mathcal{A})\) be a regular 2-graph with parameters \(N, a\) and \(k\). If \((\Omega, A) \in \mathcal{A}(T)\), then the parameters of \((\Omega, A)\) are

\[
n = N, \quad p^1 = a, \quad p^2 = a.\]

The matrix \(A\) has minimal polynomial

\[
x^2 + (a-a')x - (N-1) = 0\]

and the eigenvalues \(\lambda_1, \lambda_2\) of \(A\) have multiplicities \(\mu_1\) and \(\mu_2\), where

\[
\lambda_1 = \frac{1}{2} \left\{ (a'-a) + \sqrt{[N^2 - 4aa']} \right\} \\
\lambda_2 = \frac{1}{2} \left\{ (a'-a) - \sqrt{[N^2 - 4aa']} \right\}
\]

**Remarks.** 1. Given a set \(\Omega\) and a function \(f : \Omega(n) \to \mathbb{Z}_2\) where \(\mathbb{Z}_2 = \{1, -1\}\), define \(\delta f : \Omega(n+1) \to \mathbb{Z}_2\) by

\[
\delta f(\{a_0, \ldots, a_n\}) = \prod_{i=0}^{n} f(\{a_0, \ldots, \hat{a}_i, \ldots, a_n\}); \quad \delta\text{ is the coboundary operator. A graph } (\Omega, A) \text{ may be considered}\\
as a function $f : \mathcal{Q}(2) \to \mathbb{Z}_2$ where $f(\{a, \beta\}) = a_{\alpha}\beta$ and we can define a $2$-graph as a function $g : \mathcal{Q}(3) \to \mathbb{Z}_2$, (the set of coherent triangles is $\mathcal{T} = \{\{a, \beta, \gamma\} \mid g(\{a, \beta, \gamma\}) = -1\}$).

Let $\mathcal{L}$ denote the function $\mathcal{Q}(n) \to \mathbb{Z}_2$ sending every element to $1$. Then $g^2f = 1$. If $g : \mathcal{Q}(3) \to \mathbb{Z}_2$ is a $2$-graph, condition (b) of the definition of regular $2$-graph corresponds to $8g = 1$; i.e., $g$ is a co-cycle. In this situation we can write $g = 8f$ where $f : \mathcal{Q}(2) \to \mathbb{Z}_2$ is a graph. Theorem (2.10) shows that regularity of $g$ is equivalent to $f$ being a strong graph (of type 2). If $8f_1 = 8f_2$, then $8(f_1f_2) = 1$ so $f_1f_2 = 8d$ where $d : \mathcal{Q} \to \mathbb{Z}_2$, i.e., $f_1$ is equivalent to $f_2$ via switching. Conversely, given a graph $f : \mathcal{Q}(2) \to \mathbb{Z}_2$, $8f$ is always a $2$-graph satisfying (b) since $8(8f) = 8^2f = 1$.

2. If $R = (\Gamma, A)$ is a strongly regular graph with $n_1 = 2p_1^2$, then it is clear from the construction of $R^0$ that any automorphism of $R$ extends to an automorphism of $R^0$ which fixes $o$. Conversely, any automorphism of $R^0$ fixing $o$ induces an automorphism of $R = (R^0)_o$. However, it can happen that for a regular $2$-graph $T$, $T_{a_1}$ and $T_{a_2}$ are not isomorphic so the automorphism group of $T$ is not transitive.

3. An example.

In this section we look at regular $2$-graphs whose associated strong graphs are of type 2(a).

(3.1) Proposition. Let $T = (\mathcal{Q}, \mathcal{T})$ be a regular $2$-graph.

The following conditions are equivalent:
(1) The strong graphs of \((T)\) have type \(2(a)\).

(ii) \(a = a'\)

(iii) \(k = k'\)

(iv) \(a = 2k + 2\).

Proof. From (2.11), (1.2).

It follows from (1.2,5) that if \(T = (\Omega, \xi)\) is a regular 2-graph with \(a = a'\), then \(N-1\) is the sum of two squares.

We now give an example with \(N = 1 + p^0\) where \(p\) is a prime and \(p^0 = 1 \pmod{4}\) (see van Lint and Seidel [1], D.G. Higman [5]).

Let \(q = p^0\) be a prime power with \(q = 1 \pmod{4}\). Set \(\Omega = GP(q) \cup \{\infty\}\). Define \(X : \Omega \to \{-1, 0, 1\}\) by

\[
X(a) = \begin{cases} 
+1 & a \text{ is a square in } GP(q) \\
0 & a = 0 \\
-1 & \text{otherwise}
\end{cases}
\]

Since \(q = 1 \pmod{4}\) we have \(X(-1) = 1\). As in the previous section define a graph on \(\Omega\) by \(f : \Omega(2) \to \mathbb{Z}_2\) where

\[
f(\{a, \beta\}) = X(a - \beta).
\]

Then \(f : \Omega(3) \to \mathbb{Z}_2\) defines a 2-graph on \(\Omega\) which must be regular since \(F \Sigma L(2, q)\) acts doubly transitively on \(\Omega\) and preserves squares. We have

\[
\xi = \{ \{a, \beta, \gamma\} | X(a - \beta)X(\beta - \gamma)X(\gamma - a) = -1 \}\]

and \((\Omega, \xi)\) is a regular 2-graph. The parameters of \((\Omega, \xi)\) are \(N = q + 1\), \(a = a' = \frac{q-1}{2}\), \(k = k' = \frac{q-5}{4}\). We shall write \(T(q) = (\Omega, \xi)\).

(3,2) Theorem. The automorphism group of \(T(q)\) is \(F \Sigma L(2, q)\) and this group acts doubly transitively on \(\Omega\). The stabilizer of the point \(\infty\) is the group \(\Sigma(q)\) consisting of the permutations

\[
x \mapsto ax^\sigma + \beta
\]
of GF(q) where \( \alpha \) is a non-zero square and 
\( \sigma \) \( \text{Aut GF(q)} \).

**Proof.** Let \( G \) be the automorphism group of \((\Omega, \mathcal{E})\). Since 
\( G \supseteq \Sigma_2 \) \( \text{AGL}(2, q) \) \( G \) acts doubly transitively on \( \Omega \). We show that 
\( G_\infty = \Sigma(q) \). Let \( g \in G_\infty \), then \( g \) is a permutation of \( GF(q) \) 
so we may represent \( g \) by a polynomial (of degree \( \leq q-2 \)) 
with coefficients in \( GF(q) \). We want to show that \( g \in \Sigma(q) \).

We can suppose that \( g(0) = 0 \) and \( g(1) = 1 \) since \( \Sigma(q)_0 \) is 
transitive on non-zero squares. Moreover, \( X(g(a)-g(\beta)) \) 
\( = X(a-\beta) \) so by the theorem of Carlitz [1] \( g(x) = x^{j} \) for 
some \( j, 0 \leq j < n \). Thus \( G_\infty = \Sigma(q) \). 

**Remarks.** 1. If \( q = 3 \pmod{4} \) there is a similar theorem 
of Kantor [1], p. 20 on the automorphism group of the 
Hadamard design constructed from \( GF(q) \).

2. \( \Sigma(q) \) acts as a primitive rank 3 group of even order 
on \( GF(q) \) whose associated strongly regular graph has 
\( n_1 = \frac{q-1}{2} = 2p_1^2 \). The theorem illustrates how regular 
2-graphs provide a setting for a transitive extension of 
such rank 3 groups.

3. A theorem of D.G. Higman [5] shows that if \( T = (\Omega, \mathcal{E}) \) 
is a regular 2-graph with automorphism group \( G \) such that \( G_\alpha \) 
is a primitive rank 3 group and \( a=a'=2p \), \( p \) a prime, then 
\((\Omega, \mathcal{E})\) is the regular 2-graph \( T(4p+1) \) described above.

4. There exist regular 2-graphs such that \( a = a' \) but which 
are not isomorphic to any of the above examples \( T(q) \) (see 
Chapters 3 and 5).
4. **Equiangular lines in Euclidean space**

In this section we give a result of M. Gerzon which is needed later. In preparing for this result we show that regular 2-graphs may be represented by configurations of equiangular lines in Euclidean space (see van Lint and Seidel [1]).

Let \( V \) be a vector space of dimension \( r \) over the field \( \mathbb{R} \) with an inner product \( (\cdot, \cdot) \). Suppose that \( E = \{e_1, \ldots, e_n\} \) is a set of \( n \) one-dimensional subspaces of \( V \) such that the angle between any pair is \( \emptyset \). We call such a set \( E \) a set of equiangular lines of \( V \). Now choose unit vectors \( v_1, \ldots, v_n \) such that \( v_1 \) spans \( e_1 \). Writing \( c = \cos \emptyset \) we have

\[
(4.1) \quad (v_i, v_j) = \begin{cases} \frac{1}{c} & i \neq j \\ 1 & i = j \end{cases}
\]

The Gramian of the vectors \( v_1, \ldots, v_n \) is the \( nxn \) matrix \( P \) whose \((i,j)\)-th entry is \((v_i, v_j)\).

\( (4.2) \quad \text{Proposition. The matrix } P \text{ is symmetric and positive semi-definite of rank } r_1 \text{ where } r_1 \text{ is the dimension of the space spanned by } v_1, \ldots, v_n. \text{ In particular, } P \text{ has } c \text{ as an eigenvalue with multiplicity } n-r_1. \)

**Proof.** The rank of \( P \) is clearly \( r_1 \). Representing each \( v_i \) by the column vector of coefficients with respect to an orthonormal basis we have \( P = N^T N \) where \( N = [v_1, \ldots, v_n] \). Hence \( P \) is symmetric and positive semi-definite. //
From now on suppose that $v_1, \ldots, v_n$ span $V$. We define $A = \alpha^{-1}(P-I)$ so that $A$ has diagonal entries 0 and other entries $\pm 1$. Thus we may interpret $A$ as the adjacency matrix of a graph with $n$ vertices; the vertices may be identified with the $n$ vectors $v_1, \ldots, v_n$, joining $v_i$ to $v_j$ whenever $(v_i, v_j) = -\alpha$.

The matrix $A$ has smallest eigenvalue $\rho = -\alpha^{-1}$ with multiplicity $n-r$. Let the remaining eigenvalues be $x_1, \ldots, x_r$.

**Proposition.** We have

\( x_1 + \cdots + x_r = -(n-r)\rho \tag{4.4} \)

\( x_1^2 + \cdots + x_r^2 = n(n-1)-(n-r)\rho^2 \tag{4.5} \)

Moreover, if $\alpha^2 < r^{-1}$, then

\( n \leq r \left( \frac{1-\alpha^2}{1-\alpha^2 r} \right) \tag{4.6} \)

with equality if and only if $x_1 = \cdots = x_r$, i.e., if and only if $A$ is the adjacency matrix of a strong graph.

**Proof.** (van Lint and Seidel [1]). Equations (4.4) and (4.5) follow from $\text{Tr } A = 0$ and $\text{Tr } A^2 = n(n-1)$ respectively. Equation (4.4) determines a plane in $\mathbb{R}^r$ and equation (4.5) determines a sphere. Since the plane meets the sphere we have

\( r^{-1}(n-r)^2\rho^2 \leq n(n-1) - (n-r)\rho^2 \tag{4.7} \)

If $\alpha^2 < r^{-1}$, then (4.6) follows from (4.7) by putting $\rho = -\alpha^{-1}$. 

Equality holds in (4.6) if and only if the plane is tangent to the sphere and this is true if and only if $A$ has just two distinct eigenvalues. //

The following result was noticed by P.M. Neumann:

(4.8) Proposition. If $n > 2r$, then the eigenvalue $\rho = -\alpha^{-1}$ of $A$ is an odd rational integer.

Proof. Since the entries of $A$ are rational integers $\rho$ is an algebraic integer and any algebraic conjugate of $\rho$ is also an eigenvalue of $A$. Since the multiplicity of $\rho$ is $n-r > \frac{1}{2}n$, $\rho$ is equal to all of its conjugates. Hence $\rho$ is a rational integer.

Let $J$ be the $n \times n$ matrix every entry of which is 1. Then $A+J-I$ has entries either 0 or 2. The eigenspace of $J$ corresponding to the eigenvalue 0 has dimension $n-1$ so there is at least one eigenvector of $A$ in the eigenspace of $\rho$ which belongs to the eigenspace of 0 of $J$. Hence $A+J-I$ has $\rho+1$ as an eigenvalue. But then $\frac{1}{2}(\rho-1)$ is both a rational number and an algebraic integer. Hence $\rho$ is an odd integer. //

We now begin with a graph $(\mathbb{Z}, A)$ and construct a set of equiangular lines. Suppose that the smallest eigenvalue of $A$ is $\rho < -1$ and that $\rho$ has multiplicity $\mu$. Then the matrix $B = I-\rho^{-1}A$ has smallest eigenvalue 0 with multiplicity $\mu$ so we can find $Q$ such that $Q^2 = B$, where $Q$ is symmetric and positive semi-definite and has rank $n-\mu$. Let $e_i$, $i=1,\ldots,n$ be the column vector with $n$ entries, 1 in the $i$-th
place and $0$ elsewhere. The vectors $e_1, \ldots, e_n$ span the space $\mathbb{R}^n$. If $x$ and $y \in \mathbb{R}^n$ define the inner product by $(x, y) = x^T y$. Now put $v_i = Qe_i$, $i=1, \ldots, n$ and let $P$ be the Gramian of $v_1, \ldots, v_n$. Since $Q$ has rank $r = n-\mu$ the vectors $v_1, \ldots, v_n$ span a subspace of $\mathbb{R}^n$ of dimension $r$. We have

$$
(4.9) \quad (v_i, v_j) = v_i^T v_j = e_i^T Q e_j
$$

Conversely, if we have a set of vectors $v_1, \ldots, v_n$ with diagonal entries determined by $a_{ij}$, then $\sqrt{P} = Q^T D Q$ where $D$ has diagonal $\sqrt{a_{ii}}$ and off-diagonal $0$. Since $\sqrt{P}$ has rank $n$, the Gramians of the lines spanned by $v_1, \ldots, v_n$ are equivalent. Thus we can associate an equivalence class of equiangular lines with a regular 2-graph $(\mathcal{G}, \Lambda) \in \mathcal{J}(T)$.

We shall consider two sets of equiangular lines to be equivalent if one can be mapped onto the other by an orthogonal transformation. If $\{\zeta_1, \ldots, \zeta_n\}$ is a set of equiangular lines with vectors $v_1, \ldots, v_n$ as in (4.1) and $H$ is an orthogonal matrix, then the Gramian $P$ of $v_1, \ldots, v_n$
is also the Gramian of $Mv_1, \ldots, Mv_n$. Hence the adjacency matrix $A = c^{-1}(P-I)$ of the associated graph depends only on the equivalence class. Moreover, if we change the basis vector $v_i$ of $\ell_i$ to $-v_i$ then the $i$-th row and column of $A$ is multiplied by $-1$ and the new graph is equivalent to $(\Omega, A)$ by switching at the $i$th vertex.

Conversely, if we begin with the regular 2-graph $T$ and $(\Omega, A), (\Omega, A') \in \mathcal{S}(T)$, then $A' = DAD$ where $D$ is diagonal with diagonal entries $\pm 1$. Let $P = I - P_2^{-1}A$ and $P' = I - P_2^{-1}A' = DPD$, then $\sqrt{P'} = D\sqrt{PD}$. Now $v_1 = \sqrt{P} e_1$ and $v_1' = \sqrt{P'} e_1 = D\sqrt{PD}e_1 = \pm Dv_1$. Since $D^TD = I$ the equiangular lines determined by $v_1, \ldots, v_n$ and $v_1', \ldots, v_n'$ are equivalent. Thus we may associate an equivalence class of equiangular lines with a regular 2-graph.

(4.10) Theorem. (M. Gerzon). If $\{\ell_1, \ldots, \ell_n\}$ is a set of equiangular lines in the $r$-dimensional Euclidean space $V$, then $n \leq \frac{1}{2} r(r+1)$.

Proof. The inner product $(\cdot, \cdot)$ induces an isomorphism $V \otimes V \cong \text{Hom}(V, V)$ by $(v \otimes w)(u) = (v, u)w$. We identify $V \otimes V$ with $\text{Hom}(V, V)$ so we can write $\text{Tr}((v \otimes v)(w \otimes w)) = (v, w)^2$. Let $v_1, \ldots, v_n$ be unit vectors spanning the lines $\ell_1, \ldots, \ell_n$ and suppose that $|(v_i, v_j)| = \alpha$ when $i \neq j$.

We shall show that the vectors $v_1 \otimes v_1, \ldots, v_n \otimes v_n$ are linearly independent. Suppose that $\sum_{i=1}^n a_i(v_i \otimes v_i) = 0$, then multiplying by $v_j \otimes v_j$ and taking the trace we get $a_j + \alpha^2 \sum_{i \neq j} a_i = 0$ so that $(1-\alpha^2)a_j + \alpha^2 \sum_{i=1}^n a_i = 0$. Since $\alpha^2 \neq 1$ we have $a_i = a_j$ for all $i, j$. Let $a = a_1$, then
\[(1-a^2) + na^2]a = 0 \text{ whence } a = 0. \text{ Thus } v_1 \otimes v_1, \ldots, v_n \otimes v_n \text{ are linearly independent. Since the dimension of the space of symmetric tensors is } \frac{1}{2} r(r+1) \text{ we have } n \leq \frac{1}{2} r(r+1). \]

(4.11) Corollary. If \( T = (\omega, k) \) is a regular 2-graph with parameters \( N, a, k \) and if \( (\omega, A) \) is a strong graph of \( S(T) \) such that \( A \) has eigenvalues \( \rho_1 \) and \( \rho_2 \) with multiplicities \( \mu_1 \) and \( \mu_2 \) respectively, then
\[
N \leq \min \left\{ \frac{1}{2} \mu_1(\mu_1+1), \frac{1}{2} \mu_2(\mu_2+1) \right\}.
\]

Proof. Passing to the complement if necessary we can assume \( \mu_2 < \rho_1 \) and \( \mu_1 < \mu_2 \). By (1.2.6) \( \rho_2 \neq -1 \) so the construction above yields \( N \) equiangular lines in \( \mu_1=N-\mu_2 \) dimensions.

5. The parameters of a regular 2-graph

Let \( T = (\omega, k) \) be a regular 2-graph with parameters \( N, a, \) and \( k \). The parameters of the complement are \( N, a' \) and \( k' \). We shall assume that \( a = a' \). Let \( A \) be the adjacency matrix of a graph of \( (T) \). Then \( A \) has eigenvalues \( \rho_1 \) and \( \rho_2 \) with multiplicities \( \mu_1 \) and \( \mu_2 \) respectively, given by (2.13) and (2.14). In this section we shall investigate the consequences of \( \mu_1 \) and \( \mu_2 \) being integers. In particular, we desire a bound for \( N \) in terms of \( k \). We shall say that \( T \) is of type (a) if \( (\omega, A) \) is of type 2(a) and that \( T \) is of type (b) if \( (\omega, A) \) is of type 2(b). We first deal with the case \( k = 0 \).

(5.1) Theorem. If \( T \) is a regular 2-graph with \( k=0 \), then
T is isomorphic to $T(5)$ of § 3 and $\text{Aut}(T)$ is $\text{PSL}(2,5) \cong A_5$.

Proof. Suppose that $a \neq a'$. From (2.14) \[ \frac{N(a'-a)}{2\sqrt{N^2-4a'a}} \] is an integer. Put $N = 2M$, $a = 2s$; since $a' = 2a-2$ and $M = 3s$ we see that $s^2 + 4s$ must be a square, say $m^2$. Then we have $(s+2)^2 - m^2 = 4$ so $2s+4 = m_1 + m_2$ where $m_1 m_2 = 4$, but this contradicts $s > 0$. Hence we have $a = a'$ whence $k = k' = 0$ and $N = 6$ by (3.1). Thus $T_0$ is a strongly regular graph with $n_1 = n_2 = 2$ and $p_{11} = 0$, i.e., $T_0$ is just a pentagon. This shows that $T$ is uniquely determined.

Since $T(5)$ satisfies the conditions we have $T \cong T(5)$, $\text{Aut}(T) \cong \text{PSL}(2,5)$.

(5.2) Theorem. Let $T$ be a regular 2-graph of type (b) such that $k \neq 1$. Then $N \leq 10k - 24$.

Proof. Again put $N = 2M$, $a = 2s$, $a' = 2r$. Since $a \neq a'$ by assumption (2.14) implies that $M^2 - 4rs$ is a square, say $m^2$. Now from (1.3) and (1.4) $M = 3s-k$ and $r = 2s-k-1$ so $9s^2 - 6sk + k^2 - 8s^2 + 4sk + 4s = m^2$.

Hence

\[ [s - (k-2)]^2 = -4(k-1) + m^2. \]

It now follows that there exist integers $m_1', m_2'$ such that $2[s - (k-2)] = m_1' + m_2'$, $2m = m_1^2 - m_2^2$, and $m_1^2 m_2^2 = -4(k-1)$. Since one of $m_1', m_2'$ must be even both must be so we can put $m_1' = 2m_1$ and $m_2' = -2m_2$ and then
\[(5.4)\] \[
\begin{aligned}
\begin{cases}
  s = (k-2) + m_1 - m_2 \\
  m_1 m_2 = k-1 \\
  m = m_1 + m_2
\end{cases}
\end{aligned}
\]
Assuming \( k \neq 1 \) we have \( m_1 m_2 \leq (k-1)-1 \) so that \( s \leq 2(k-2) \).
Thus \( N \leq 10k - 24 \). //

\[(5.5)\] Theorem. If \( T \) is a regular 2-graph with \( k = 1 \), then the parameters of \( T \) must be one of the following three possibilities:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( a )</th>
<th>( a' )</th>
<th>( k )</th>
<th>( k' )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>-3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>-3</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>28</td>
<td>10</td>
<td>16</td>
<td>1</td>
<td>10</td>
<td>9</td>
<td>-3</td>
<td>7</td>
<td>21</td>
</tr>
</tbody>
</table>

Proof. If \( T \) is of type (a), then we obtain the first line of the table. We now suppose that \( a \neq a' \) and use the fact that \( \mu_1 \) of (2.14) is an integer. Again set \( N = 2M, a = 2s, a' = 2r \). Since \( k = 1 \) we have \( r = 2s-2 \) so \( M^2 - 4rs = (s+1)^2 \) and \( N(r-s) = (3s-1)(s-2) \). Since \( \mu_1 \) is an integer, \( s+1 \) must divide \( (3s-1)(s-2) \) and so \( s+1 \) divides 12. We have only to eliminate the possibility \( s = 11, N = 64 \). In this case \( \rho_1 = 21, \rho_2 = -3, \mu_1 = 8 \) and \( \mu_2 = 56 \) and so \( N \neq \frac{3}{2}\mu_1(\mu_1+1) \), contradicting (4.11). //

Remark. 1. All three possibilities of the theorem occur and the graphs are unique. The first line corresponds to the
graph T(9) of $\S$ 3. The existence and uniqueness of the other possibilities is discussed further in Chapters 3 and 5 (see Seidel [1]).

2. If a regular 2-graph has $\rho_2 = -3$, then it must be one of the graphs considered in the above theorem (this follows immediately from (2.13)). The corresponding strong graphs were considered by Seidel [1].

We next consider the graphs for which the bound for $N$ obtained in (5.2) is attained.

(5.6) **Theorem.** If $T$ is a regular 2-graph such that $N = 10k-24$ then the parameters of $T$ must be one of the following eight possibilities:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$a$</th>
<th>$a'$</th>
<th>$k$</th>
<th>$k'$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>-5</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>26</td>
<td>12</td>
<td>12</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>-5</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>36</td>
<td>16</td>
<td>18</td>
<td>6</td>
<td>9</td>
<td>7</td>
<td>-5</td>
<td>15</td>
<td>21</td>
</tr>
<tr>
<td>76</td>
<td>32</td>
<td>42</td>
<td>10</td>
<td>25</td>
<td>15</td>
<td>-5</td>
<td>19</td>
<td>56</td>
</tr>
<tr>
<td>96</td>
<td>40</td>
<td>54</td>
<td>12</td>
<td>33</td>
<td>19</td>
<td>-5</td>
<td>20</td>
<td>76</td>
</tr>
<tr>
<td>126</td>
<td>52</td>
<td>72</td>
<td>15</td>
<td>45</td>
<td>25</td>
<td>-5</td>
<td>21</td>
<td>105</td>
</tr>
<tr>
<td>176</td>
<td>72</td>
<td>102</td>
<td>20</td>
<td>65</td>
<td>35</td>
<td>-5</td>
<td>22</td>
<td>154</td>
</tr>
<tr>
<td>276</td>
<td>112</td>
<td>162</td>
<td>30</td>
<td>105</td>
<td>55</td>
<td>-5</td>
<td>23</td>
<td>253</td>
</tr>
</tbody>
</table>

**Proof.** We refer to the proof of Theorem (5.2). Since $N = 10k-24$ we must have $s = 2(k-2)$, $m_1 = k-1$, $m_2 = 1$ and $m = k$. The condition that $\mu_1$ be an integer now implies
that \( \frac{(5k-12)(k-5)}{k} \) is an integer and hence that \( k \) divides 60. It is clear that we cannot have \( k=1,2 \) or 3. This leaves \( k=60 \) as the only case to rule out. In this case we have \( N=576, a=232, a'=342 \), so \( \rho_1=115, \rho_2=-5, \mu_1=224 \) and \( \mu_2=552 \) from (2.14). Thus \( N \neq \frac{1}{2} \mu_1(\mu_1+1) \), contradicting (2.11). \\

Remarks. 1. Notice that all the 2-graphs have \( \rho_2=-5 \).

Conversely, if \( T \) is a regular 2-graph with \( \rho_2=-5 \), then (2.13) gives \( N=10k-2h \).

2. Existence and uniqueness of the above graphs will be considered in Chapters 4 and 5. The first line corresponds to the complement of the second possibility of Theorem (5.5). The graph \( T(25) \) of \( \S \,3 \) gives a solution for the second line, however this is not the only solution. It is not known whether there are graphs with \( N=76 \) or \( N=96 \).

Suppose that \( T=(\xi,\eta) \) is a regular 2-graph of type (b) and \( k \neq 1 \). We put \( N=2N, a=2s, a'=2r \) so that \( r=2s-k-1, M=3s-k \). We explore the condition that \( \mu_1 \) be an integer.

As in Theorem (5.2) we can write \( m=\sqrt{[N^2-4rs]} \) where \( m \) is an integer. Since \( \rho_1 \) and \( \rho_2 \) are odd integers we can put \( \rho_2=-2\omega-1 \) where \( \omega \) is a positive integer.

\[ (5.7) \quad \text{Lemma.} \quad k=\omega+s-s/\omega \text{ so that } \omega \text{ divides } s. \]

\[ \text{Proof.} \quad \text{From (2.13) } \rho_2=(r-s)-\sqrt{[N^2-4rs]} \text{ so we have} \]
\[ 9s^2 - 6sk + k^2 - 3s^2 + 4sk + 4s = s^2 + k^2 + 16s^2 - 2sk + 4os^2 + k^2 \]

whence \( k = \omega + s - s/\omega \).

We may now put \( s = \sigma_0 \) so that \( m = \sigma_0 \), \( M = \sigma_0 + 2\sigma_0 = \omega \), \( k = \omega + \sigma_0 - \sigma \) and \( r - s = \sigma - \omega - 1 \). Hence \( \mu_1 \) is an integer if and only if \( \sigma_0 \) divides \( (\sigma_0 - 1)(\sigma_0 + 2\sigma_0 - \omega) \). This implies that \( \sigma_0 \) divides \( 2\omega(\omega + 1)(2\omega + 1) \). If \( \sigma_0 = 2\omega(\omega + 1)(2\omega + 1) \), then \( M = \sigma_0 + 2\sigma_0 - \omega = 8\omega^4 + 16\omega^3 + 8\omega^2 \) so that \( N = [4\omega(\omega + 1)]^2 \). However, \( \mu_1 = \frac{M(r - s)}{m} \) so \( \mu_1 = 4\omega(\omega + 1) \) and this contradicts \((4,11)\).

We now express the remaining parameters in terms of \( \omega \) and \( \sigma \), and summarise our results in the following.

(5.8) Theorem. Let \( T = (\omega, t) \) be a regular 2-graph of type (b) with parameters \( N, a, k, \) etc. as above. Then there exist positive integers \( \omega \) and \( \sigma \) such that:

\[
N = 2(\sigma + 2\sigma_0 - \omega) \\
a = 2\sigma_0, \quad a' = 2(\sigma_0 - 1)(\omega + 1) \\
k = \omega + \sigma_0 - \sigma, \quad k' = 2\sigma + \sigma_0 - 2\omega - 3 \\
\rho_1 = 2\sigma - 1, \quad \rho_2 = -2\omega - 1 \\
\mu_1 = \frac{\sigma(2\omega + 1)^2 - \omega(2\omega + 1)}{\sigma + 2\omega} \quad \mu_2 = \frac{\omega(2\sigma - 1)^2 + \sigma(2\sigma - 1)}{\sigma + 2\omega}
\]

Moreover, \( \sigma_0 \) divides \( 2\omega(\omega + 1)(2\omega + 1) \) but \( \sigma_0 \neq 2\omega(\omega + 1)(2\omega + 1) \). //

Remarks. 1. Putting \( \omega = 1 \) we have \( a = 2\sigma \) and \( \sigma + 1 \mid 12 \) so we obtain Theorem (5.5); putting \( \omega = 2 \) we have \( k = \sigma \) and \( k \mid 60 \) and we obtain Theorem (5.6).

2. Table I contains a list of parameters satisfying the above conditions.
We have shown in (5.2) that $N \leq 10k - 24$ and in (5.6) we have determined the parameters for which $N = 10k - 24$. We now enquire how large $N$ may be if $\rho_2$ is given. It follows from (5.8) that the largest value of $\sigma + \omega$ is $\omega(\omega + 1)(2\omega + 1)$; i.e., $\sigma = \omega^2(2\omega + 1)$ and then $\mu_1 = 4\omega^2 + 4\omega - 1 = (2\omega + 1)^2 - 2$. Since $N = 2\omega(\omega + 1)(4\omega^2 + 4\omega - 1)$ we have $N = \frac{3}{2} \mu_1(\mu_1 + 1)$ and there is no conflict with (4.11).

Since $2\omega(\omega + 1)(2\omega + 1)$ is always divisible by 3 we always have the solution $\sigma + \omega = \frac{2}{3} \omega(\omega + 1)(2\omega + 1)$.

6. Further strong regular graphs

Let $T = (G, k)$ be a regular 2-graph and let $(G, A)$ be a strong graph in $\mathcal{S}(T)$. In this section we shall suppose $(G, A)$ to be regular and then determine the parameters $n_1, p_1, k$. Since the only eigenvalues of $A$ are $\rho_1$ and $\rho_2$ the eigenvalue $\rho_0 = n_2 - n_1$ is either $\rho_1$ or $\rho_2$ and hence both $\rho_1$ and $\rho_2$ must be integers. We may therefore retain the notation used in (5.8). (Note that this holds for graphs of type (a) provided the eigenvalues are integers.)

(6.1) Proposition. Let $T = (G, k)$ be a regular 2-graph and suppose that $\mathcal{S}(T)$ contains a strongly regular graph $(G, A)$. Then there exist positive integers $\sigma, \omega$ such that the parameters of $(G, A)$ are one of the following two sets:

(1) $n = n = 2(\sigma + 2\omega)$

Remark. In the notation of (6.1) the above condition reads:
\[
\begin{align*}
n_1 &= \omega(2\sigma-1) \\
n_2 &= (\omega+1)(2\sigma-1) \\
p_{11}^1 &= \sigma(\omega-1) \\
p_{11}^2 &= \omega(\sigma-1) \\
p_{12}^1 &= (\omega-1)(\omega+1) \\
p_{12}^2 &= \sigma \omega \\
p_{22}^1 &= \sigma(\omega+1) \\
p_{22}^2 &= (\omega+2)(\sigma-1) \\
p_0 &= p_1 = 2\sigma-1, \\
p_2 &= -2\omega-1
\end{align*}
\]

where \( a = 2\sigma \omega \) and \( k = \omega+\sigma\omega-\sigma \)

\[(ii) \quad n = \omega = 2(\sigma+2\sigma\omega-\omega) \]
\[
\begin{align*}
n_1 &= \sigma(2\omega+1) \\
n_2 &= (\sigma-1)(2\omega+1) \\
p_{11}^1 &= \omega(\sigma+1) \\
p_{11}^2 &= \sigma(\omega+1) \\
p_{12}^1 &= (\omega+1)(\sigma-1) \\
p_{12}^2 &= \sigma \omega \\
p_{22}^1 &= \omega(\sigma-1) \\
p_{22}^2 &= (\sigma-2)(\omega+1) \\
p_0 &= p_2 = -2\omega-1, \\
p_1 &= 2\sigma-1
\end{align*}
\]

where \( a = 2\sigma \omega \) and \( k = \omega+\sigma\omega-\sigma \)

\textbf{Proof.} Use (5.8) and (1.2.4).

\textbf{(6.2) Proposition.} A strongly regular graph \( S = (\Omega, \mathcal{A}) \) is a graph of type 2 if and only if \( n = 2(1+p_{12}^1+p_{12}^2) \);

this is the case if and only if \( \mathcal{J}(S) \) is a regular 2-graph.

\textbf{Proof.} From (1.2.3)\((b)\) and (2.2.10).

\textbf{Remark.} In the notation of §1.1 the above condition reads:
7. Coherent subsets of regular 2-graphs

In this section we consider the embedding of a coherent subset in a regular 2-graph. In an associated strong graph a coherent subset corresponds to a subgraph of type 1.

(7.1) Lemma. Let \( T = (\Omega, t) \) be a regular 2-graph and suppose that \( M \) is a coherent subset of \( \Omega \). Then, for each point \( x \in \Omega - M \) there is a unique partition of \( M \) into subsets \( M_1(x) \) and \( M_2(x) \) such that for \( \alpha, \beta \in M_1(x) \), \( i = 1, 2 \) the set \( \{\alpha, \beta, x\} \) belongs to \( t \) if and only if \( i = j \).

Proof. We may assume that \( M \cup \{x\} \) is not coherent so that we can choose \( \alpha, \beta \in M \) such that \( \{\alpha, \beta, x\} \) is not coherent.

We put \( M_1(x) = \{\gamma \in M \mid \{\alpha, \gamma, x\} \in t\} \) and \( M_2(x) = \{\gamma \in M \mid \{\beta, \gamma, x\} \in t\} \). It is clear that \( M_1(x) \) and \( M_2(x) \) give the required partition and that this partition is unique. \( \Box \)

Now choose a fixed coherent subset \( M \) of \( \Omega \) where \( T = (\Omega, t) \) is a regular 2-graph. Suppose that \( m = |M| \) and for \( x \in \Omega - M \) choose the notation so that \( |M_1(x)| \geq |M_2(x)| \) and put \( m(x) = |M_1(x)| \). Suppose the parameters of \( T \) to be \( N, a, k, \) etc. as usual.

We shall see later that in many cases this choice for a.

(7.2) Proposition. We have the following formula:

\[
\sum_{x \in \Omega - M} m(x)(m - m(x)) = a'(\frac{m}{2}).
\]
**Proof.** We count the number of coherent triangles $a, \beta, \chi$ where $a$ and $\beta \in \Omega - M$ in two ways. Choosing $\beta$ and then $\chi$ gives $\binom{m}{2}(a-m+2)$ coherent triangles; choosing $\chi$ then $\beta$ and $\chi$ gives $\sum_{x \in \Omega - M} \{ \binom{m(x)}{2} + \binom{m-m(x)}{2} \}$ coherent triangles.

Now $|\Omega - M| = N - m$ so

$$\sum_{x \in \Omega - M} \{ \binom{m(x)}{2} + \binom{m-m(x)}{2} \} = \frac{1}{2} \sum_{x \in \Omega - M} (m(x)^2 - m(x) + m(m-1) + m(x^2) - 2m \cdot m(x) + m(x))$$

$$= \sum_{x \in \Omega - M} m(x)(m(x)-m) + (N-m)\binom{m}{2}.$$  \hspace{1cm} (7.4)

Since $(N-m)(a-m+2) = a'$ we have proved the proposition. //

(7.3) **Corollary.** We have $m \leq 1-\rho_2$, and equality holds if and only if $m(x) = \frac{1}{2}m$ for all $x \in \Omega - M$ where $\rho_2$ is the smallest eigenvalue of a graph in $\Sigma(T)$.

**Proof.** For all $x \in \Omega - M$ we have $m(x)(m-m(x)) \leq \frac{m}{2}^2$ and therefore $a'(\frac{m}{2}) \leq (N-m)\frac{m}{2}^2$. Thus $m^2 + (2a'-N)m - 2a' \leq 0$ whence

$$m \leq \frac{1}{2} \left\{ (N-2a') + \sqrt{[(a-a')^2 + 4a']} \right\}$$

$$= \frac{1}{2} \left\{ (a-a') + \sqrt{[(a+a')^2 - 4aa']} \right\} + 1$$

$$= 1 - \rho_2,$$ from (2.2.13).

It is clear that $m = 1-\rho_2$ if and only if $m(x) = \frac{1}{2}m$ for all $x \in \Omega - M$. //

We shall see later that in many cases the bound for $m$ as given in (7.3) is often attained. However, in these cases the bound $\rho_1 - 1$ for incoherent sets is not best possible so we now give bounds for $m$ in terms of the multiplicities $\mu_1$.
and $\mu_2$ (see (2.2.13)). In order to do this we make use of the correspondence between regular 2-graphs and equiangular lines described in §2.4.

Let us suppose the points of $\mathbb{G}$ are $a_1, \ldots, a_N$ where $M = \{a_1, \ldots, a_m\}$. We can choose $(M, A)$ in $\mathcal{B}(T)$ such that $A = (a_{ij})$, (where $a_{ij}$ is $a_{ij}^*$ in our previous notation)

and

$$a_{ij} = \begin{cases} -1 & i \neq j, \quad 1 \leq i, j \leq m \\ 0 & i = j \end{cases}$$

(7.4)

As in §2.4 we put $P = I - \rho_2^{-1}A$ so that the vectors $v_1 = \sqrt{P} e_i, \quad i = 1, \ldots, N$ span equiangular lines $\ell_1, \ldots, \ell_N$

and the vector space spanned by $v_1, \ldots, v_N$ has dimension $\mu_1 = N - \mu_2$. From (2.4.9) and (7.4) we have

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ \rho_2^{-1} & i \neq j \quad 1 \leq i, j \leq m \end{cases}$$

(7.5)

(7.6) Proposition. The vectors $v_1, \ldots, v_m$ are linearly independent unless $m = 1 - \rho_2$ in which case we have $v_1 + \cdots + v_m = 0$.

Proof. Suppose that we have a relation $\sum_{i=1}^{M} b_i v_i = 0$.

Then taking the inner product with $v_j$ for $1 \leq j \leq m$ we obtain $\sum_{i=1}^{M} b_i \langle v_i, v_j \rangle = 0$ and then (7.5) gives

$$\sum_{i=1}^{M} b_i \rho_2^{-1} - b_j \rho_2^{-1} + b_j = 0$$

Consequently, $b_j$ does not depend on $j$ so we may put $b_j = b$

for $1 \leq j \leq m$. Then $m b \rho_2^{-1} - b (\rho_2^{-1} - 1) = 0$ and either $b = 0$
or \( m = 1-p_2 \) and \( \sum_{i=1}^{M} v_i = 0 \). In any case if \( m = 1-p_2 \)
and \( x = \sum_{i=1}^{M} v_i \), then \( (x, x) = 0 \) so \( x = 0 \). //

Remark. This does not improve our bound for \( m \) since except for \( N = 6, \mu_1 = 3 \) and \( p_2 = -\sqrt{5} \) we have \( \mu_1 > 1-p_2 \). To see this we may assume that \( T \) is type (b) and then use (2.5.8) which gives 
\[
-\mu_2^{-1} = 1 + 2\omega(\sigma -1)(\sigma + \omega)^{-1} > 1.
\]

We now construct a set of equiangular lines from the complement of \( T \) or what amounts to the same thing, we put \( Q = 1-p_1^{-1}A \) with \( A \) as before so that the vectors \( \omega_1 = \sqrt{Q} e_1, \) \( i=1,\ldots,N \) span \( N \) equiangular lines in a space of dimension \( \mu_2 = N-\mu_1 \). As in (7.5) we have
\[
(7.7) \quad (\omega_i, \omega_j) = \begin{cases} 
1 & i = j \\
\rho_1^{-1} & i \neq j, \quad 1 \leq i, j \leq m.
\end{cases}
\]

(7.8) Proposition. The vectors \( \omega_1, \ldots, \omega_m \) are linearly independent.

Proof. Suppose that we have a relation \( \sum_{i=1}^{M} b_i \omega_i = 0 \). As in (7.6) we have \( \sum_{i=1}^{M} b_i \rho_1^{-1} - b_j \rho_1^{-1} + b_j = 0 \) for \( 1 \leq j \leq m \). Thus \( b_j \) does not depend on \( j \) so we put \( b_j = b \). Then either \( b = 0 \) or \( m = 1-p_1 \). But \( \rho_1 > 1 \) so we must have \( b = 0 \). //

(7.9) Corollary. We have \( m \leq \mu_2 \). //

Choose a point \( x \in M \). We wish to determine the possible values for \( m(x) \) when \( m = \mu_2 \). Therefore, we shall assume that \( m = \mu_2 \) for the remainder of this section. It
follows that the vectors $\omega_1, \ldots, \omega_m$ form a basis for the vector space of dimension $\mu_2$ spanned by $\omega_1, \ldots, \omega_N$. Let $v$ be the vector corresponding to $x$. We have $v = \sum_{i=1}^{m} b_i \omega_i$ and by switching at $x$ if necessary we may suppose that

$$(7.10) \quad (v, \omega_i) = \begin{cases} \rho_1^{-1} & i = 1, \ldots, r \\ -\rho_1^{-1} & i = r+1, \ldots, m \end{cases}$$

where $r = m(x)$.

Taking the inner product of $v$ with $\omega_1, \ldots, \omega_m$ and $v$ in turn we obtain

$$(7.10) \quad (v, \omega_i) = \begin{cases} \rho_1^{-1} & i = 1, \ldots, r \\ -\rho_1^{-1} & i = r+1, \ldots, m \end{cases}$$

Thus for $1 \leq j \leq r$ we can put $b_j = c$ and for $r+1 \leq j \leq m$ we can put $b_j = d$. Then the above equations become

$$(7.11) \quad l = rc + (m-r)d - c + \rho_1 c$$

$$(7.12) \quad -l = rc + (m-r)d - d + \rho_1 d$$

$$(7.13) \quad \rho_1 = rc - (m-r)d.$$

From these equations we get

$$(7.14) \quad c = (\rho_1 + 1)/(2r-1+\rho_1)$$

$$(7.15) \quad d = (\rho_1 + 1)/(2m+2r+1-\rho_1)$$

and then substituting these in (7.13) we get:
The discriminant of this quadratic must be a square so we can write

$$m^2 - m(p_1^2 - 2p_1 + 1) - p_1(p_1^2 - 2p_1 + 1) = \ell^2,$$

i.e.,

$$\left\{ m - \frac{1}{2}(p_1-1)^2 \right\}^2 - \ell^2 = \left\{ \frac{1}{2}(p_1^2-1) \right\}^2$$

so we can find integers $\ell_1$ and $\ell_2$ such that

$$2 \left\{ m - \frac{1}{2}(p_1-1)^2 \right\} = \ell_1 + \ell_2$$

where $\ell_1 \ell_2 = \frac{1}{4}(p_1^2-1)^2/16$. Since $p_1 \equiv 1 \pmod{2}$, we have $\ell_1 \ell_2 \equiv 0 \pmod{2}$ and then both $\ell_1$ and $\ell_2$ must be even so we obtain the following

(7.16) **Theorem.** If $T = (\Omega, t)$ is a regular 2-graph and $M$ is a coherent set of $m$ points of $\Omega$, then $m = \mu_2$ implies

$$m = \frac{1}{2}(p_1-1)^2 + \ell_1 + \ell_2$$

where $\ell_1 \ell_2 = (p_1^2-1)^2/16$.

Moreover, for any $x \in \Omega - M$ the number $r = m(x)$ is uniquely determined as the larger root of

$$r^2 - mr + \frac{1}{4}(p_1^3 + (m-2)p_1^2 - 2mp_1 + p_1 + m) = 0,$$

that is $r = \frac{1}{2}(m + \ell_2 - \ell_1)$, where we choose $\ell_2 - \ell_1$.

We apply (7.16) to several examples which will be used later.

(a) $p_1 = 3$. In this case $\ell_1 \ell_2 = 4$ and we have then two solutions $m = 7$, $r = 5$ and $m = 6$, $r = 3$ (see (5.5)).
(b) \( \rho_1 = 5 \). In this case \( \ell_1 \ell_2 = 36 \). By remark 1 to Theorem (5.6) the complementary graph to \( T \) has \( N = 10k-24 \) so \( \mu_2 \leq 23 \). Hence we have three solutions:

(i) \( m = 23, \ r = 16 \)
(ii) \( m = 21, \ r = 13 \)
(iii) \( m = 20, \ r = 10 \).

(c) \( \rho_1 = 7 \). We have \( \ell_1 \ell_2 = 144 \). From (5.8) the largest possible value of \( \mu_2 \) is 47 so we have only the following solutions:

(i) \( m = 44, \ r = 27 \)
(ii) \( m = 43, \ r = 25 \)
(iii) \( m = 42, \ r = 21 \).

Note that there is always the solution \( \ell_1 \ell_2 = \frac{1}{4}(\rho_1^2 - 1) \), \( \mu_2 = m = \rho_1^2 - \rho_1 \) and \( r = m/2 \). In this case we see that \( m = 1 - \rho_2 \) so that (7.3) and (7.6) apply. This situation can occur (see Chapter 4).

We complete the above investigation by considering two distinct points \( x, y \in \mathbb{R}^n \). Suppose \( x \) corresponds to \( v \) as above and suppose that \( y \) corresponds to \( u \). Let us write

\[
(7.17) \quad (u,v) = \varepsilon \rho_1^{-1} \quad \text{where} \quad \varepsilon = \pm 1.
\]

We have \( v = c \sum_{i=1}^{s} \omega_i + d \sum_{i=r+1}^{m} \omega_i \) and by switching at \( y \) if necessary we can suppose that

\[
(7.18) \quad (u,\omega_i) = \begin{cases} 
\rho_1^{-1} & 1 \leq i \leq s \quad \text{and} \quad r+1 \leq i \leq t \\
-\rho_1^{-1} & s+1 \leq i \leq r \quad \text{and} \quad t+1 \leq i \leq m 
\end{cases}
\]
Hence \( u = c \sum_{1}^{s} \omega_{1} + d \sum_{s+1}^{r} \omega_{1} + c \sum_{r+1}^{t} \omega_{1} + d \sum_{t+1}^{m} \omega_{1} \)

and \( s+t = 2r \).

Thus \((u, v) = cs \rho_{1}^{-1} - c(r-s) \rho_{1}^{-1} + d(t-r) \rho_{1}^{-1} - d(m-t) \rho_{1}^{-1}\)

and so it follows from (7.16) that

\[ \varepsilon = 2(c-d)s - (c-r) + 3dr - dm. \]

(7.19) Theorem. If the hypotheses of (7.16) hold and \( x \) and \( y \) are distinct points of \( M \), such that \( r = m(x) = m(y) \) and \( s = |M_{1}(x) \cap M_{1}(y)| \), then

\[ s = \frac{\varepsilon + rc - 3rd + md}{2(c-d)} \]

where \( c \) and \( d \) are given in (7.14) and (7.15) and

\[ \varepsilon = \begin{cases} 1 & \text{if } (M_{1}(x) \cap M_{1}(y)) \cup \{x,y\} \text{ is coherent} \\ -1 & \text{otherwise.} \end{cases} \]

8. Higher regularity

We define a \( t \)-regular 2-graph as follows. Any regular 2-graph is 2-regular and a regular 2-graph \( T = (\Omega, \mathcal{T}) \) is \( t \)-regular where \( t \) is an integer, \( t > 2 \), if the following three conditions are satisfied.

(a) \( \Omega \) contains a coherent set of \( t \) points.
(b) Any coherent set of \( t \) points is contained in \( k_{t} \) coherent sets of \( t+1 \) points.
(c) \( T \) is \((t-1)\)-regular.
It was proved in (1.1) that any regular 2-graph is 3-regular with $k_3 = k$. We put $k_2 = a$. The following theorem generalises (1.1).

(8.1) **Theorem.** Let $\Omega = (\omega, t)$ be a $t$-regular 2-graph with $t$ even. If $\omega$ contains a coherent set of $t+1$ points then $T$ is $(t+1)$-regular.

**Proof.** Let $M$ be a coherent set of $t+1$ points and suppose that $M$ is contained in $\mathcal{L}$ coherent sets of $t+2$ points. We must show that $\mathcal{L}$ is independent of the choice of $M$. We use the notation of Lemma (7.1).

Given $D \subseteq M$ we put $\sigma(D) = \{ x \in \omega - M \mid D = M_1(x) \}$ and if $|D| = p$ we let $\omega_p(D) = |\sigma(D)|$. The sets $\sigma(D)$ partition $\omega - M$ ($\omega_p(D) = 0$ for $p \leq \frac{t}{2}$).

Suppose that $p$ is an integer such that $\frac{t}{2} + 1 \leq p \leq t$ and for all integers $q$ such that $1 \leq q \leq t + 1 - p$ the number $\omega_{p+q}(D)$ does not depend on the choice of $p+q$ element set $D \subseteq M$. We can therefore write $\omega_{p+q}$ for $\omega_{p+q}(D)$.

Since $\omega_{t+1} = \mathcal{L}$ the above assumption is true for $p = t$.

With $p$ and $q$ as above there are $\binom{t+1}{p}$ $p$ element subsets of $M$ and such a subset $D$ is contained in $\binom{t+1-p}{q}$ $p+q$ element subsets of $M$ and in $k_p$ coherent sets of $p+1$ elements. Thus we have

\[(8.2) \quad k_p = (t+1-p) + \sum_{q=1}^{t+1-p} \binom{t+1-p}{q} \omega_{p+q} + \omega_p(D).\]

It now follows that $\omega_p(D)$ does not depend on $D$ and by induction this is true for all $p$ such that $\frac{t}{2} + 1 \leq p \leq t$. 
We can therefore write
\[(8.3) \quad N = (t+1) + \sum_{p=\frac{1}{2}t+1}^{t+1} (t+1)_p \omega_p.\]

The next formula is established by induction.
\[(8.4) \quad \omega_p = \sum_{q=0}^{t+1-p} (-1)^q \frac{(t+1-p)_k}{q} \omega_{p+q}, \quad \frac{1}{2}t+1 \leq p \leq t-1.\]

First observe that from (8.2) we have
\[(8.5) \quad \omega_0 = k_{t-1-\ell} \quad \text{and} \quad \omega_{t+1} = \ell .\]

Next suppose that for all \(\lambda' \leq \lambda\) we have
\[
\omega_{t-\lambda'} = \sum_{q=0}^{\lambda'+1} (-1)^q \frac{(\lambda'+1)_q}{q} k_{t-\lambda'+q},
\]
where \(1 \leq \lambda \leq \frac{1}{2}t-1\). This is certainly true for \(\lambda = 1\) as this is just (8.2) with \(p = t-1\).

From (8.2)
\[
\omega_{t-(\lambda+1)} = k_{t-\lambda-1} - (\lambda+1) - \sum_{q=1}^{\lambda+2} \frac{(\lambda+2)_q}{q} \omega_{t-\lambda-1+q},
\]

\[
= k_{t-\lambda-1} - (\lambda+1) - \sum_{q=1}^{\lambda} \frac{(\lambda+2)_q}{q} \sum_{r=0}^{\lambda+2-q} (-1)^r \frac{(\lambda+2-q)_r}{r} k_{t-\lambda-1+q+r} - (\lambda+2)(k_{t-1-\ell}) - \ell
\]

\[
= k_{t-\lambda-1} - \sum_{q=1}^{\lambda+2} \frac{(\lambda+2)_q}{q} \sum_{r=0}^{\lambda+2-q} (-1)^r \frac{(\lambda+2-q)_r}{r} k_{t-\lambda-1+q+r}
\]

Let \(c_s\) be the coefficient of \(k_{t-\lambda-1+s}\) where \(q+r = s\).
Then
\[ c_s = \sum_{q=1}^{s} \binom{s}{q} (-1)^{s-q} \binom{s+2}{s-q} \]
\[ = \binom{s+2}{s} \sum_{q=1}^{s} (-1)^{s-q+1} \binom{s}{q} \]
\[ = (-1)^s \binom{s+2}{s}. \]

Thus (8.4) is established. Now put \( \lambda = t - p \) in (8.3) to get
\[ (8.6) \quad N = (t+1)k_t - t\ell + \sum_{\lambda=1}^{\frac{1}{2}t-1} \binom{t+1}{\lambda}w_t - \lambda. \]

From (8.4) the coefficient of \( \ell \) in (8.6) is \[ \sum_{\lambda=0}^{\frac{1}{2}t} \binom{t+1}{\lambda}(-1)^{\lambda} \]
which is non-zero. Hence \( \ell \) does not depend on the choice of \( D \) and the theorem is proved. //

(8.7) Corollary. Under the hypotheses of (8.4) we have

\[ N = \sum_{s=\frac{1}{2}t+1}^{t+1} \left\{ (-1)^s \binom{t+1}{s} \sum_{p=\frac{1}{2}t+1}^{s} (-1)^p \binom{s}{p} \right\} k_s. \]

In particular, for \( t = 4 \) we have \( 5k_4 - 2k_5 = 4k-a \)
and for \( t = 6 \) we have \( 7k_5 - 7k_6 + 2k_7 = 3k-a. \)

Proof. From (8.6) and (8.4) we have
\[ N = (t+1)k_t - tk_{t+1} + \sum_{\lambda=1}^{\frac{1}{2}t-1} \binom{t+1}{\lambda} \sum_{q=0}^{\lambda+1} (-1)^q \binom{\lambda+1}{q} k_{t+q-\lambda} \]
\[ = \sum_{\lambda=0}^{\frac{1}{2}t} \binom{t+1}{\lambda} \sum_{q=0}^{\lambda} (-1)^q \binom{\lambda}{q} k_{t+1+q-\lambda} \]

Put \( p = t+1-\lambda \), then
The coefficient of $k^s$ is $c_s$ where

$$c_s = \sum_{p=\frac{s}{3}t+1}^{s} (t+1) (-1)^{s-p} \binom{t+1-p}{s-p} = \binom{t+1}{s} \sum_{p=\frac{s}{3}t+1}^{s} (-1)^{s-p} \binom{s}{p}$$

hence we have obtained the required expression for $N$. 

If $t = 4$, then $N = 10k_3 - 15k_4 + 6k_5$ and from (1.4)

$N = 3a - 2k$, so $5k_4 - 2k_5 = 4k - a$. Similarly if $t = 6$ then

$N = 35k_4 - 85k_5 + 70k_6 - 20k_7$ so $7k_5 - 7k_6 + 2k_7 = 3k - a$. //

For the remainder of this section suppose that $T = (\Omega, t)$ is a 4-regular 2-graph and that $M$ is a coherent set of $m$ points. Counting the number of coherent sets $\{\alpha, \beta, \gamma, \delta, \chi\}$ with $\alpha, \beta, \gamma, \delta \in M$ and $\chi \not\in M$ in two ways as in (7.2) we obtain:

$$(8.8) \sum_{\chi \not\in M} \left\{ \binom{m(x)}{4} + \binom{m-m(x)}{4} \right\} = \binom{m}{4} (k_4 - m + 4)$$

Expanding this and using (7.2) we obtain

$$(8.9) \sum_{\chi \not\in M} m(x)^2 (m-m(x))^2 = a' \binom{m}{2} (2m^2 - 9m + 11) - 12 \binom{m}{4} (N - k_4 - 4)$$

9. **An equivalence relation**

Let $T = (\Omega, t)$ be a regular graph and let $\rho_1$ and $\rho_2$ be the eigenvalues of the adjacency matrix of an associated
strong graph. Suppose that \( m = 1 - p_2 \) is a positive integer.

We shall consider the following hypotheses:

(A) There exists a coherent set of \( m \) points in \( \Omega \).

(B) Any coherent set of \( 1 + m/2 \) points is contained in a unique coherent set of \( m \) points.

(C) Any coherent set of \( m/2 \) points is contained in \( \mathcal{E} \) coherent sets of \( 1 + m/2 \) points.

Let \( \mathcal{E} \) be the set of all coherent subsets of \( \Omega \) which have \( m/2 \) points. For \( D_1, D_2 \in \mathcal{E} \) we shall write \( D_1 \sim D_2 \) whenever \( D_1 = D_2 \) or \( D_1 \cup D_2 \) is a coherent set of \( m \) points.

(9.1) Theorem. Under hypotheses (A) and (B) \( \sim \) is an equivalence relation on \( \mathcal{E} \).

Proof. We use the results and notation of §7, in particular, (7.3) applies. We have only to show that \( \sim \) is transitive so suppose that \( D_1 \sim D_2 \) and \( D_2 \sim D_3 \) for \( D_1, D_2, D_3 \in \mathcal{E} \). We can assume that no two of \( D_1, D_2, D_3 \) are equal. If \( x \in D_1 \cap D_3 \), then \( D_2 \cup \{ x \} \) is a coherent set of \( 1 + m/2 \) points so \( D_1 = D_3 \), contrary to assumption. Hence \( |D_1 \cup D_3| = m \).

For \( x \in D_3 \) we can choose our notation so that \( M = D_1 \cup D_2 \), \( D_1 = M_1(x) \) and \( D_2 = M_2(x) \). Hence \( D_1 \cup \{ x \} \) is coherent and so \( D_1 \sim D_3 \). //

(9.2) Corollary. If hypotheses (A), (B) and (C) hold, then each equivalence class of \( \sim \) contains \( 1 + 2c/m \) elements.

Proof. Let \( \mathcal{E} = \{ D_1, \ldots, D_t \} \) be an equivalence class of \( \sim \) and
put \( E = \bigcup_{i=1}^{t} D_i \). Since \( D_i \cap D_j = \emptyset \) if \( i \neq j \) we have \( |E| = t m/2 \). If \( x \in E \), then \( D_1 \cup \{x\} \) is coherent. Conversely, if \( D_1 \cup \{x\} \) is coherent, then \( D_1 \cup \{x\} \leq M \) where \( M \) is a coherent set of \( m \) points. Thus \( D_1 \sim M-D_1 \) and so \( x \in E \).

Hence \( |E| = |D_1| + \ell \) and it follows that \( t = 1 + 2\ell/m \).

We now give two applications.

(9.3) Proposition. If \( T \) is a regular 2-graph such that \( \rho_2 = -3 \) (equivalently, \( k=1 \)), then hypotheses (A), (B) and (C) hold with \( m=4 \) and \( \ell=a \). Thus each equivalence class contains \( 1+a/2 \) elements and there are \( 3(a-1)(3a-2)(a+1)-1 \) equivalence classes in all.

(9.4) Proposition. If \( T \) is a 4-regular 2-graph such that \( \rho_2 = -5 \) (equivalently, \( N = 10k-24 \)) then hypotheses (A), (B) and (C) are satisfied with \( m=6 \) and \( \ell=k \). Thus each equivalence class contains \( 1+k/3 \) elements and there are \( 20(5k-12)(2k-5)(k-2)(k+3)-1 \) equivalence classes in all.

Proof. If \( \rho_2 = -5 \), then \( N = 10k-24 \) (see the remark following (5.6)) so \( a = 4k-8 \). From (8.7) \( 5k_4-2k_5 = 8 \) and therefore \( k_4=2, k_5=1 \). Thus hypothesis (B) holds with \( m=6 \). It is trivial that (C) holds with \( \ell=k \). //
The purpose of this chapter is to provide a catalogue of examples of regular 2-graphs constructed from various combinatorial designs. In particular, we show that all three possibilities of Theorem (2.5.5) can occur.

We have shown in Chapter 2 that the existence of a regular 2-graph \( T \) is equivalent to the existence of a class of strong graphs \( \mathcal{S}(T) \) of type 2 and also to the existence of a strongly regular graph \( T_0 \) of type 3 such that \( n_1 = 2p_{11}^2 \).

Thus given a strongly regular graph \( R \) we consider the two possibilities:

(I) \( n_1 = 2p_{11}^2 \). In this case we construct a regular 2-graph as in (2.2.3) by adjoining a point.

(II) \( n = 2(1 + p_{12} + p_{12}^2) = 2(2n_1 - p_{11}^1 - p_{11}^2) \).

Since \( R \) is of type 2 we can construct a regular 2-graph on the same number of points as in (2.2.10). In this case, there are two possibilities for the valency of \( R \) (see § 2.6).

The strong graphs and strongly regular graphs described in this chapter may be found in either Blackwelder [1], Goethals and Seidel [1] or Seidel [3].

Other examples of regular 2-graphs will be given in Chapter 4 in connection with doubly transitive groups and rank 3 groups.
The last section of this chapter is about certain designs obtained from regular 2-graphs.

The notation for the parameters of a regular 2-graph \( T \) is that of Chapter 2. Note that if \( (\Omega, A) \in \mathcal{F}(T) \), then the eigenvalues of \( A \) are eigenvalues of the adjacency matrix of \( T_0 \) but the multiplicities of these eigenvalues are decreased by 1 in passing to \( T_0 \).

\[ 1. \quad \text{Definitions} \]

In this section we collect definitions of and basic facts about designs and related structures. For proofs we refer to Dembowski [1].

An incidence structure is a triple \( \mathcal{g} = (P, B, I) \) where \( P \), \( B \) and \( I \) are finite sets such that \( P \cap B = \emptyset \) and \( I \subseteq P \times B \). The elements of \( P \) are called points, those of \( B \) are called blocks. We say that a point \( P \) is incident with a block \( b \) if \( (P, b) \in I \). We set \( v = |P| \) and \( b = |B| \).

An incidence structure \( \mathcal{g} \) is called a tactical configuration if there are natural numbers \( r \) and \( k \) such that every point is incident with \( r \) blocks and every block is incident with \( k \) points. In this case we have

\[
(1.1) \quad vr = bk.
\]

A tactical configuration \( \mathcal{g} \) is called a design if \( k \leq v-1 \) and there is a natural number \( \lambda \) such that any two points are incident with \( \lambda \) blocks. In this case

\[
(1.2) \quad r(k-1) = \lambda (v-1).
\]
Let $g$ be an incidence structure with points $P_1, \ldots, P_v$ and blocks $b_1, \ldots, b_b$. We define the incidence matrix $N = (n_{ij})$ of $g$ by

\[
(1.3) \quad n_{ij} = \begin{cases} 
1 & (P_i, b_j) \in I \\
0 & \text{otherwise.}
\end{cases}
\]

If $g$ is a design, then

\[
(1.4) \quad N N^T = (r-\lambda)I + \lambda J
\]

where $I$ is the $v \times v$ identity matrix and $J$ is the $v \times v$ matrix every entry of which is 1.

The numbers $v, b, r, k, \lambda$ are called the parameters of the design $g$ and we shall sometimes refer to $g$ as a $(v, b, r, k, \lambda)$-design. Apart from (1.1) and (1.3) we have Fisher's inequality

\[
(1.5) \quad b \geq v.
\]

If $g$ is a design with $b=v$ then $g$ is called projective (or symmetric). From (1.1) $r=k$.

(1.6) **If $g$ is a projective design with parameters $v, k, \lambda$ then any two blocks are incident with exactly $\lambda$ points.**

An automorphism $\alpha$ of an incidence structure $g = (P, B, I)$ is a pair $(\sigma, \tau)$ where $\sigma$ is a permutation of $P$ and $\tau$ is a permutation of $B$ such that if $(P, b) \in I$, then $(P\sigma, b\tau) \in I$.

(1.7) **(Brauer)** Suppose that $G$ is a group of automorphisms
of the projective design $g$. Then the number of orbits of $G$ as a permutation group on the points is equal to the number of orbits of $G$ as a permutation group on blocks. If $G$ is cyclic then the number of fixed points of $G$ is equal to the number of fixed blocks of $G$. If $G$ is transitive on points then the rank of $G$ acting on points is equal to the rank of $G$ on blocks.

Proof. Apply (12.1) of Feit [1] to the incidence matrix of $g$. //

An $nxn$ matrix $H$ with entries $\mathbb{I}$ is called a Hadamard matrix of order $n$ if $HH^T = nI$. If $H = A + I$ where $A^T = A$ then $H$ is called a symmetric Hadamard matrix with constant diagonal.

2. Triangular graphs

The three regular 2-graphs described below satisfy the conditions of Theorem (2.5.5) and show that all three possibilities of that theorem can occur. The uniqueness of these graphs is proved in Chapter 5. From (2.7.3) and (2.7.9) the size of a coherent set is bounded by the minimum of $1-\rho_2$ and $\mu_2$ while the size of an incoherent set is bounded by $1+\rho_1$ and $\mu_1$. In the examples of this section, these bounds are attained.

Let $\triangle$ be a set of $m$ elements and let $\mathcal{S}$ be the set of 2 element subsets of $\triangle$. We join two elements of $\mathcal{S}$ whenever they have one element of $\triangle$ in common. The resulting graph
is strongly regular with the following parameters:

(2.1) \[ n = \frac{1}{6}m(m-1) \]

\[ n_1 = 2(m-2) \quad p_{11}^1 = m-2 \]

\[ n_2 = \frac{1}{2}(m-2)(m-3) \quad p_{11}^2 = 4 \]

\[ \rho_1 = 3 \quad \mu_1 = \frac{1}{6}m(m-3) \]

\[ \rho_2 = -2m+7 \quad \mu_2 = m-1 \]

It has been proved by D.G. Higman [3] that the automorphism group of this graph is the symmetric group $S_m$ and the permutation representation has rank 3.

From condition (I) above we obtain a regular 2-graph $T_{n+1}$ by adjoining a point $o$ if and only if $m = 6$. The parameters are

(2.2) \[ N = 16 \]

\[ a = 8 \quad a' = 6 \]

\[ k = 4 \quad k' = 1 \]

\[ \rho_1 = 3 \quad \mu_1' = 10 \]

\[ \rho_2 = -5 \quad \mu_2' = 6 \]

where $\mu_1'$ and $\mu_2'$ refer to the multiplicities of $\rho_1$ and $\rho_2$ as eigenvalues of an adjacency matrix of a graph in $S(T_{16})$.

The set \{ 0, \{1,2\}, \{3,4\}, \{5,6\} \} is incoherent and this is as large as possible by (2.7.3). The set \{ 0, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\} \} is coherent and again by (2.7.3) this is as large as possible. But note that \{ 0, \{1,2\}, \{2,3\}, \{1,3\} \} is a maximal coherent set
so \( T^* \) is not 4-regular.

We shall prove in Chapter 5 that the automorphism group of \( T^*_16 \) is doubly transitive, has a regular normal subgroup and that the stabilizer of a point is \( S_6 \).

The complement of \( T^*_16 \) will be denoted by \( T^*_{16} \).

From condition (II) we can construct a regular 2-graph \( T^*_n \) on \( \mathcal{Q} \) if and only if \( m^2 - 13m + 40 = 0 \); i.e., \( m = 5 \) or \( m = 8 \).

If \( m = 5 \) the parameters of \( T^*_10 \) are

\[
(2,3) \quad N = 10 \\
a = 4 \\
k = 1.
\]

These are the same parameters as those of \( T(9) \) of \( \S \) 2.3 and in Chapter 5 we shall prove that any two regular 2-graphs with these parameters are isomorphic.

If \( m = 8 \), the parameters of \( T^*_28 \) are

\[
(2,4) \quad N = 28 \\
a = 16 \\
k = 10 \\
\rho_1 = 3 \\
\rho_2 = -9
\]

\[
a' = 10 \\
k' = 1 \\
\mu_1' = 21 \\
\mu_2' = 7
\]

The set \( \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\} \) is coherent and by (2.7.3) this is as large as possible. In fact \( T^*_28 \) is 4-regular and hence 5-regular. But

\[
\{1,2\}, \{2,3\}, \{1,3\}, \{4,5\}, \{5,6\}, \{4,6\}
\]

is a maximal coherent set so \( T^*_28 \) is not 6-regular. We have \( k_4 = 6 \) and \( k_5 = 3 \).
In Chapter 5 we shall show that the automorphism group of this graph is \( \text{Sp}(6,2) \).

3. **SLB graphs.**

Let \( \mathcal{g} = (p, B, I) \) be a \( (v, b, r, h, l) \)-design. Construct a graph on the set \( B \) of blocks by joining two blocks whenever they have a point in common. We obtain a strongly regular graph with the following parameters (see Bose \([1]\)):

\[
\begin{align*}
(3,1) & \\
\quad n = r(hr-r+1)/h & p_{11}^1 = r-2+(h-1)^2 \\
\quad n_1 = h(r-1) & p_{11}^2 = h^2 \\
\quad n_2 = (r-h)(h-1)(r-1)/h & \mu_1 = (r-h)(hr-r+1)/h \\
\quad \rho_1 = 2h-1 & \mu_2 = r(h-1) \\
\quad \rho_2 = 2h-2r+1
\end{align*}
\]

We shall call this graph a *singly linked block* (or SLB) graph. If \( h=2 \) it is just the triangular graph of the previous section.

From condition (I) we can construct a regular 2-graph by adjoining a point if and only if \( r = 2h+1 \). The parameters are:

\[
\begin{align*}
(3,2) & \\
\quad N = 4h^2 & a^\prime = 2h^2 - 2 \\
\quad a = 2h^2 & k^\prime = h^2 - 3 \\
\quad k = h^2 & \mu_1^\prime = h(2h+1) \\
\quad \rho_1 = 2h-1 & \mu_2^\prime = h(2h-1) \\
\quad \rho_2 = -2h-1
\end{align*}
\]
This arises from a design with parameters

\((2h^2-h, 4h^2-1, 2h+1, h, 1)\). If \(h=3\) the design is a Steiner triple system. There are 80 such \((15, 35, 7, 3, 1)\)-designs and they give rise to 80 mutually non-isomorphic regular 2-graphs on 36 points (see Bussemaker and Seidel [1]).

The designs with \(h=4, 5\) or 7 exist and examples may be found in Hall [1]. They give rise to regular 2-graphs with \(N = 64, 100\) or 196 respectively. It is clear from the construction that all these graphs contain coherent sets of \(2h+2\) points and that this is as large as possible.

From condition (II) we can construct a regular 2-graph on the same number of points if and only if \(r^2(h-1) - (4h^2-2h-1)r + 4h^3 - 2h = 0\). Thus either \(h=2\) and \(r=7\) or \(h=2r\). The first case has occurred as (2.4) above. If \(r=2h\) the regular 2-graph has parameters:

\[
\begin{align*}
(3.3) & \quad N = 2(2h^2-2h+1) = (2h-1)^2 + 1 \\
& \quad a = a' = 2h(h-1) \\
& \quad k = k' = h^2 - h - 1 \\
& \quad \rho_1 = -\rho_2 = 2h-1 \\
& \quad \mu_1 = \mu_2 = 2h^2 - 2h + 1
\end{align*}
\]

The regular 2-graph is of type (a) and from the construction there are always coherent sets of \(2h\) points in this 2-graph. It arises from a design with parameters \((2h^2-2h+1, 2(2h^2-2h+1), 2h, h, 1)\). For example, a Steiner triple system with parameters \((13, 26, 6, 3, 1)\) gives rise to a regular 2-graph with \(N = 26, a = 12, k = 5\). We shall see
later that this is not isomorphic to $T(25)$ of § 2.3.

4. **Latin square graphs**

Arrange $m^2$ elements in an $m \times m$ array and join two elements if they lie in the same row or column. This defines the graph $L_2(m)$. Alternatively, it is the line graph of the complete bipartite graph on $m+m$ points. It is strongly regular and its automorphism group is the wreath product $S_m \wr S_2$ (see D.G. Higman [3]).

Now suppose we have a set of $r-2$ mutually orthogonal $m \times m$ Latin squares, where $3 \leq r \leq m+1$ (see Hall [1], p. 189). We construct the graph $L_r(m)$ by first forming $L_2(m)$ as above and then joining two points if they correspond to the same symbol in some Latin square. The parameters of $L_r(m)$ for $2 \leq r \leq m+1$ are:

\begin{align*}
(4.1) \quad n &= m^2 \\
&= r(m-1) \\
&= (m-r+1)(m-1) \\
\rho_1 &= 2r-1 \\
\rho_2 &= 2r-2m-1 \\
\mu_1 &= (m-r+1)(m-1) \\
\mu_2 &= r(m-1)
\end{align*}

In order to obtain a regular 2-graph on adjoining a point we must have $m=2r-1$. The parameters are then:

\begin{align*}
(4.2) \quad N &= (2r-1)^2 + 1 \\
a &= a' = 2r(r-1) \\
k &= k' = r^2-r-1 \\
\rho_1 &= -\rho_2 = 2r-1 \\
\mu_1 &= \mu_2' = 2r^2-2r+1
\end{align*}
This is a regular 2-graph of type (a) containing coherent sets of \( m+1 = 2r = 1 - \rho_2 \) points. These always exist if \( m \) is a prime power since we can then construct \( L_r(m) \) from an affine plane (Hall [1], p. 177).

From condition (II) we can construct a regular 2-graph on the same number of points provided \( m^2-2(2r-1)m + 4r(r-1) = 0 \), i.e., \( m=2r \) or \( m=2r-2 \). If \( m=2r \), the regular 2-graph has parameters

\[
\begin{align*}
N &= 4r^2 \\
a &= 2r^2 \\
k &= r^2 \\
\rho_1 &= 2r-1 \\
\rho_2 &= -2r-1
\end{align*}
\]

The parameters are the same as in (3.2); we shall consider these again in § 6. The graph contains coherent sets of \( m = 1-\rho_2 \) points. If \( m = 2r-2 \), the parameters are:

\[
\begin{align*}
N &= 4(r-1)^2 \\
a &= 2r(r-2) \\
k &= r^2-2r-2 \\
\rho_1 &= 2r-1 \\
\rho_2 &= -2r+3
\end{align*}
\]

Again, these graphs always contain coherent sets of \( m = 1-\rho_2 \) points. For example, there exist 5 mutually orthogonal Latin squares of order 12. Taking any 4 of them we get a regular 2-graph with \( N = 144 \), \( a = 72 \), \( k = 36 \) from (4.3) and taking all 5 together we get a regular 2-graph from (4.4)
with the complementary parameters $N = 144$, $a = 70$, $k = 33$. This example was pointed out by Professor J.J. Seidel.

5. Constructions from Hadamard matrices

In this section we summarise those constructions of Goethals and Seidel [1], that lead to regular 2-graphs.

(5.1) **If there exists a $(v,b,r,h,1)$-design and a Hadamard matrix of order $r+1$, then there exists a strong regular graph of type 2 with parameters**

\[
\begin{align*}
n &= v(r+1) \\
\lambda_1 &= \frac{1}{4}r(v+h) \\
\lambda_2 &= 2r(v+h-2) \\
\mu_1 &= b \\
\mu_2 &= v(r+1)-b.
\end{align*}
\]

The corresponding regular 2-graph has parameters

\[
\begin{align*}
N &= v(r+1) \\
a &= \frac{1}{2}(v+h)(r-1) \\
k &= \frac{1}{4}(h-1)(r-1)^2 + r-h-1 \\
&+ \frac{1}{4}(r-1)^2 + 1.
\end{align*}
\]

and the 2-graph contains coherent sets of $r+1 = 1-\rho_2$ points.

**Proof.** (Goethals and Seidel, [1], p. 599). Let $N$ be the incidence matrix of the design and let $H$ be the Hadamard
matrix. We can suppose that the first column of $H$ consists entirely of 1's and let $L$ be the matrix formed by the remaining columns. Then $LL^T = rI + (I-J)$ and $JL = 0$. In each row of $N$ we replace each 0 by a column of $r+1$ 0's and we replace the $r$ 1's by the successive columns of $L$. The resulting matrix is called $P$ and we have $PP^T = rI + A$ where $A$ is a symmetric matrix with diagonal elements 0 and $\pm 1$ elsewhere. Now $JA = JP^T = rJ = -rJ$ so we may interpret $A$ as the adjacency matrix of a regular graph. Since $P^TP = h(r+1)I$, $PP^T$ has rank $v(r+1)-b$ so $A$ has smallest eigenvalue $-r$ with multiplicity $v(r+1)-b$.

From (1.1) and (1.2) we have

$$b(r^2-1) = \frac{(r/h)(rh-r+1)(r^2-1)}{r^2 - b} = \frac{r^2(r/h)(rh-r+1)}{r^2 - b} = v(r+1).$$

Using (2.4.3) we see that $A$ is the adjacency matrix of a strong graph with eigenvalues $v+h-1$ and $-r$. The parameters can now be calculated from (2.6.1) and (2.5.8). //

Since the pair design with parameters $(r+1, \frac{1}{2}r(r+1), r, 2, 1)$ always exists we can construct a regular 2-graph with parameters $N = (r+1)^2$, $a = \frac{1}{2}(r-1)(r+3)$ whenever there is a Hadamard matrix of order $r+1$. Notice that the parameters are the same as (3.2), (4.3) and (4.4). In particular, we have another construction for a regular 2-graph with $N = 144$. Other examples can be constructed from some of the
designs listed in Hall [1], p. 291. For the remainder of this section we shall consider regular 2-graphs whose parameters are given by (3.2). The connection with Hadamard matrices is given in the following construction (Goethals and Seidel [1], p. 604).

Suppose that $H$ is a symmetric Hadamard matrix with constant diagonal. Then $H = A + \varepsilon I$, where $\varepsilon = \pm 1$ and $H^2 = 4nI$ for some $n$, (we exclude the case where $H$ is a 2x2 matrix).

Let $m = \sqrt{n}$, $\rho_1 = 2m - \varepsilon$ and $\rho_2 = -2m - \varepsilon$. Then

$$(A - \rho_1 I)(A - \rho_2 I) = 0$$

so $A$ may be interpreted as the adjacency matrix of a strong graph $S = (\mathcal{G}, A)$. It follows that $m$ is an integer and the regular 2-graph $T = \mathcal{G}(S)$ has parameters

$$(\ref{5.2}) \quad \begin{align*}
N &= 4m^2 \\
\alpha &= 2m^2 - 1 + \varepsilon \\
\alpha' &= 2m^2 - 1 - \varepsilon \\
k &= m^2 + 3(\varepsilon - 1)/2 \\
k' &= m^2 - 3(\varepsilon - 1)/2 - 3
\end{align*}$$

If $H$ is regular in the sense that $HJ = rJ$ for some $r$, then the graph $S$ is strongly regular.

Conversely, suppose there exists a regular 2-graph $T$ with parameters as in (5.2). Let $S = (\mathcal{G}, A)$ be a strong graph of $\mathcal{G}(T)$ and put $H = A + \varepsilon I$. Then $H^2 = 4m^2I$ so $H$ is a symmetric Hadamard matrix of order $4m^2$ with constant diagonal.

The following construction was suggested by P. Cameron who first considered the case $m=12$.

Let $H$ be a Hadamard matrix of order $m$ and let $G = (g_{ij}; k\varepsilon)$
be the $m^2 \times m^2$ matrix whose rows and columns are indexed by the ordered pairs $(i,j)$, $i,j = 1, \ldots, m$ where

\begin{equation}
\mathbf{g}_{ij,k^\ell} = h_{ij} h_{i\ell} h_{k^j} h_{k^\ell}
\end{equation}

for $i,j,k^\ell = 1, \ldots, m$.

Since $\mathbf{H}^T \mathbf{H} = m \mathbf{I}$ we have

\[
\sum_{i,j} \mathbf{g}_{ij,k^\ell} = \left( \sum_{i,j} h_{ij} h_{i\ell} h_{k^j} h_{k^\ell} \right) h_{k^\ell} = \sum_{i} h_{i\ell} \delta_{ik} m \cdot h_{k^\ell} = m \ h_{k^\ell}^2 = m
\]

and

\[
\sum_{k^\ell} \mathbf{g}_{ij,k^\ell} \mathbf{g}_{k^\ell, mn} = \sum_{k^\ell} h_{ij} h_{i\ell} h_{k^j} h_{k^\ell} h_{k^\ell} k^n m \ h_{k^\ell} m
\]

\[
= h_{ij} \ h_{mn} \sum_{k^\ell} \ h_{i\ell} h_{k^j} h_{kn} m
\]

\[
= m^2 \delta_{im} \delta_{jn}.
\]

Thus $\mathbf{G} = m \mathbf{J}$ and $\mathbf{G}^2 = m^2 \mathbf{I}$. Now $\mathbf{G}$ is symmetric and $\mathbf{g}_{ij,ij} = 1$ so $\mathbf{G}$ has constant diagonal. Thus $\mathbf{G}$ is a symmetric Hadamard matrix of order $m^2$ with constant diagonal. Moreover if $\mathbf{B} = \mathbf{G} - \mathbf{I}$, then the corresponding strong graph $(\mathcal{G}, \mathcal{B})$ constructed as above is strongly regular since $\mathbf{B} \mathbf{j} = (m-1)\mathbf{j}$.

Finally in this section we give another construction due to Goethals and Seidel [1]. An $n \times n$ matrix $\mathbf{S}$ is called an $\mathbf{S}$-matrix if $\mathbf{S}^T \mathbf{S} = n \mathbf{I} - \mathbf{J}$, $\mathbf{J} \mathbf{S} = \mathbf{S} \mathbf{J} = 0$, $\mathbf{S}^T = \mathbf{S}$, and $\mathbf{S}$ has diagonal elements 0 and $\pm 1$ elsewhere. Suppose that there exists a symmetric $\mathbf{S}$-matrix $\mathbf{S}_{n+1}$ of order $n+1$ and a skew
S-matrix $S_{n-1}$ of order $n-1$. We may regard $S_{n+1}$ as the adjacency matrix of a strongly regular graph which yields a regular 2-graph of type (a) on adjoining a point. Hence $n = 0 \pmod{4}$. Suppose moreover that $S_{n-1}$ is symmetric with respect to its anti-diagonal then $(U S_{n-1})^T = U S_{n-1}$, where $U$ is the permutation matrix with 1's on its antidiagonal. Note that such S-matrices always exist if $n-1$ and $n+1$ are odd prime powers (see Hall [1], p. 209).

If we put $K = U S_{n-1} \otimes S_{n+1} + U \otimes (J_{n+1} - I_{n+1})$ then $K = K^T$, $KJ = JK = J$ and $K^2 = n^2 I - J$.

Thus $K+I$ is the adjacency matrix of a strongly regular graph of order $n^2 - 1$ which yields a regular 2-graph by adjoining a point. If $n=2m$ the parameters are given by (5.2). If $j$ is the column vector with every entry 1 then

$$H = \begin{bmatrix} -1 & j^T \\ j & k \end{bmatrix}$$

is a symmetric Hadamard matrix with constant diagonal. Let $F$ be the $(n+1) \times (n+1)$ diagonal matrix whose diagonal elements are alternately -1 and +1 and set $G = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \otimes F \end{bmatrix}$ and $H = GHG$. Then $HJ = -nj$ so the strong graph corresponding to $H$ is regular with valency $n_1 = \frac{1}{2}n(n+1)$.

If $n=12$ we obtain a strongly regular graph of order 144 and valency 78. We may also obtain a strongly regular graph of order 144 from the construction (5.3). This graph has valency 66 so that both possibilities of (2.6.1) are realised.
6. Designs related to regular 2-graphs

Let \( T = (\mathcal{V}, \mathcal{B}) \) be a regular 2-graph and let \((\mathcal{G}, A_0)\) be the strongly regular graph obtained by deleting the point 0. From § 2.2, \( A_0^2 = (a'-a)A_0 + (a+a'+1)I-J \) and

\[
A_J = (a'-a)J. \quad \text{We put}
\]

\[
(6.1) \quad N = \frac{1}{2}(J + I - A_0).
\]

The matrix \( N \) has entries 0 and 1 so we may enquire whether \( N \) is the incidence matrix of a projective design.

Since \( NJ = aJ \), \( N = N^2 \) and

\[
N^2 = \frac{1}{4} \left( (3a-a'+2)J + (a+a'+2)I + (a'-a-2)A_0 \right)
\]

this will be the case if and only if

\[
(6.2) \quad a' = a + 2.
\]

If (6.2) holds the design described by \( N \) has parameters

\[
(6.3) \quad v = b = 2a+3, \quad r = k = a+1, \quad \lambda = \frac{1}{2}a
\]

From (2.5.8) we have \( \delta-1 = \omega+1 = m \) say and the parameters of the regular 2-graph are as in (5.2) with \( a = 2m^2-2 \).

Thus we are dealing with the same class of 2-graphs as in § 5. The design is a Hadamard design (Dembowski [1], pp. 59, 111, 113) and therefore has an extension to an affine design with parameters

\[
(6.4) \quad v = 2a+4, \quad b = 2(2a+3), \quad r = 2a+3, \quad k = a+2, \quad \lambda = a+1.
\]

If \( a \neq 2 \), then \( B(a) = \{ a \} \cup \{ \beta \in \mathcal{G} \mid \{ o, a, \beta \} \}
\)

is coherent \( \Box \) is a block of the Hadamard design and the
correspondence \( a \leftrightarrow B(a) \) is a null polarity which commutes with every automorphism of \((\mathcal{G}, A_0)\). This design was considered by D.G. Higman [1] in the context of rank 3 groups.

If in (6.1) we put \( N = \frac{1}{3}(J-I-A_0) \) the design obtained is the complement of that considered above, provided \( a = a' + 2 \).

Next suppose that \((\mathcal{G}, A) \in \mathcal{A}(T)\) is strongly regular. In this case we have \( \rho_0 = \rho_1 \) or \( \rho_2 \) and \( A^2 = (a'-a)A + (a+a'+1)I \). As in (6.1) put

\[
N = \frac{1}{3}(J+I - A).
\]

Then \( NJ = -\frac{1}{3}(\rho_0 + \rho_1 \rho_2)J \) and

\[
N^2 = \frac{1}{4} \left( (a+a'+4-2\rho_0)J + (a+a'+2)I + (a'-a-2)A \right)
\]

Thus \( N \) is the incidence matrix of a projective design if and only if (6.2) holds. If this is the case then \( \sigma -1 = \omega + 1 = m, \rho_1 = 2m+1, \rho_2 = -2m+1, \rho_0 = 2m+1 \) where \( x = \pm 1 \) and the design has parameters:

\[
(6.6) \quad v = b = 4m^2, \quad r = k = 2m^2 - mx, \quad \lambda = m^2 - mx.
\]

If we put \( N = \frac{1}{3}(J-I-A) \) in (6.5), then we obtain a design if \( a = a' + 2 \). The parameters are still given by (6.6) since the two sets of parameters of (6.6) are complementary.

In some instances the designs obtained seem to be new. For example, from a Latin square of order 6 we may construct a strongly regular graph which leads to a design with
parameters $v = b = 36$, $r = k = 15$, $\lambda = 6$. This is listed as 'solution unknown' in Hall [1], p. 297.

The strongly regular graphs considered here are symmetric $(v,k,\lambda)$-graphs in the sense of Ahrens and Szekeres [1]. Other examples of these graphs will be considered in the next chapter.

This is the family of regular 2-graphs $\Gamma$ of order power $= 1$ (and $2$), the automorphism group of $\Gamma$ being $PG(2,q)$. We also construct 2-graphs from the doubly transitive representations of the sporadic simple groups $HS$ and $L_3$.

The main theorem of §1 shows that in most cases the regular 2-graph can be constructed via the stabilizer ring of a cosetial representation associated with the doubly transitive permutation representation. This is used to construct regular 2-graphs for the groups of $BC$ type. For the other regular 2-graphs more information is obtained by giving a geometrical description.

1. A cosetial representation and its stabilizer ring

In this section we collect some of the results of Chapter V of Higman [1] on regular $k$-graphs with doubly transitive automorphism groups by considering cosetial representations as well as permutation representations (see Hall [2,3]).

Let $G$ be a finite group. A representation of $G$ of degree $n$ we shall mean a homomorphism of $G$ into the group of non-singular matrices over the field $F$ of complex
CHAPTER 4

DOUBLY TRANSITIVE GROUPS

In this chapter we construct five infinite families of regular 2-graphs which have doubly transitive automorphism groups. A sixth such family has been constructed in § 2.3. This is the family of regular 2-graphs $T(q)$, $q$ a prime power $\equiv 1 \pmod{4}$; the automorphism group of $T(q)$ being $P_{-\infty}(2,q)$. We also construct 2-graphs from the doubly transitive representations of the sporadic simple groups $H_6$ and $G_3$.

The main theorem of § 1 shows that in most cases the regular 2-graph can be constructed via the centraliser ring of a monomial representation associated with the doubly transitive permutation representation. This is used to construct regular 2-graphs for the groups of Ree type. For the other regular 2-graphs more information is obtained by giving a geometrical description.

1. A monomial representation and its centraliser ring

In this section we extend some of the results of Chapter V of Wielandt [1] to regular 2-graphs with doubly transitive automorphism groups by considering monomial representations as well as permutation representations (see Feit [1]).

Let $G$ be a finite group. By a representation of $G$ of degree $n$ we shall mean a homomorphism of $G$ into the group of $n \times n$ non-singular matrices over the field $\mathbb{C}$ of complex
numbers. If \( R \) is a representation of \( G \) of degree \( n \) we define the centraliser ring \( C(R) \) of \( R \) to be the ring of all \( nxn \) matrices which commute with every matrix \( R(x), x \in G \). As usual, let \( J \) denote the \( (nxn) \) matrix, every entry of which is 1. A monomial matrix is a matrix \( M = D \cdot P \) where \( D \) is a diagonal matrix and \( P \) is a permutation matrix; a monomial representation of \( G \) is a representation \( \rho \) such that for every \( x \) in \( G \) \( \rho(x) \) is a monomial matrix.

(1.1) Lemma. Let \( \rho \) be a monomial representation of \( G \) such that the non-zero entries of \( \rho(x) \) are \( \pm 1 \) for each \( x \) in \( G \). If \( J \in C(\rho) \), then \( G \) has a subgroup of index 2 or \( \rho \) is a permutation representation.

Proof. For \( x \in G \) we have \( \rho(x) = D(x)P(x) \) and \( J\rho(x) = \rho(x)J \).
Thus \( D(x) = \pm I \) and we can write \( \rho(x) = \mu(x)P(x) \) where \( \mu(x) = \pm 1 \). Since \( \rho \) is a representation, \( \mu \) is a linear character of \( G \). If \( \mu \) is not the principal character it defines a homomorphism of \( G \) onto a group of order 2. //

The next theorem shows how regular 2-graphs are obtained from doubly transitive groups.

(1.2) Theorem. Let \( G \) be a group which acts doubly transitively on a set \( \Omega \) of \( n \) points such that

(i) \( n \) is even,
(ii) \( G \) has no subgroup of index 2,
(iii) If \( H \) is the stabiliser of a point then \( H \) has a subgroup \( K \) of index 2.
Let $M_0$ be the non-principal linear representation of $H$ which has $K$ in its kernel and let $M$ be the induced monomial representation of $G$. Then one of the following two possibilities occurs:

(a) $M$ is irreducible and $C(M)$ has dimension 1.

(b) $C(M)$ has dimension 2 and contains a matrix $A$ which is the adjacency matrix of a strong graph $(G, A)$, such that the group $G$ acts as a group of automorphisms of the associated regular 2-graph $T$. The multiplicities of the eigenvalues of $A$ are equal to the degrees of the irreducible constituents of $H$ and hence the parameters of $T$ are uniquely determined by $K$. The matrices $A$ and $I$ form a basis for $C(M)$.

**Proof.** Let $x \in H - K$, then $x^2 \in K$ and $H = K + Kx$. We shall identify the points of $G$ with the cosets of $H$ and we shall identify an orbit of a subgroup of $G$ with the corresponding double coset. Since $G$ acts doubly transitively on $G$ we can find an involution $t$ such that $G = H + HtH$ and then

$$G = K + Kx + KtK + KxtK + KtxK + KtxtK.$$ 

Hence $G$ has rank at most 6 when acting on the cosets of $K$.

The non-trivial orbits of $K$ in (1.3) all have the same length $|K : K \cap K^t|$. Since $n$ is even and $|G : K| = 2n$, $G$ cannot have rank 6. Moreover

$$KtK = KxtK \quad \text{if and only if} \quad KtxK = KxtxK$$

$$KtK = KtxK \quad \text{if and only if} \quad KxtK = KtxtK$$

$$KtK = KxtxK \quad \text{if and only if} \quad KtxK = KtxK.$$
Thus G has rank 3 or 4 on the cosets of K. Let \( l^*_K \) be the trivial representation of K. Then \( (l^*_K)^H = l^*_H + M \) so 
\[
(l^*_K)^G = (l^*_H)^G + M.
\]
If \( \pi \) is the associated permutation representation of \( M \), let \( \overline{\pi} (l^*_H)^G = l^*_G + X \) where \( X \) is irreducible.

If G has rank 3 on the cosets of K, then \( \overline{M} \) must be irreducible and (a) holds. From now on we suppose that G has rank 4 on the cosets of K. Then \( \overline{M} \) has two irreducible constituents \( M_1 \) and \( M_2 \) and \( \operatorname{dim}(\overline{M}) = 2 \). Let \( A = (a_{ij}) \) be a matrix in \( \operatorname{M}(\overline{M}) \) which is not a multiple of I and let \( J \) be an involution of G. We may suppose \( \overline{M}(J) \) to have the form

\[
(1.5)
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\quad \text{or} \quad 
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Since \( \overline{M}(J)A \overline{M}(J) = A \) we have \( a_{11} = a_{22} \) and \( a_{12} = a_{21} \).

By double transitivity all the diagonal elements of \( A \) are equal and \( A \) is symmetric. By subtracting a suitable multiple of I we may assume that the diagonal elements of \( A \) are 0. Now using a matrix of the form

\[
(1.6)
\begin{bmatrix}
\pm 1 & 0 & 0 \\
0 & 0 & \mp 1
\end{bmatrix}
\]

we obtain \( \pm a_{12} = a_{13} \). Again by double transitivity all off diagonal elements are equal up to sign. Therefore, by
dividing by \( a_{12} \) we may assume that the off diagonal elements of \( A \) are \( 11 \). By Schur's lemma \( A \) has just two eigenvalues whose multiplicities are the degrees of \( \Phi_1 \) and \( \Phi_2 \). Thus \( A \) is the adjacency matrix of a strong graph \((\Omega, A)\) of type 1 or 2. Let \( T \) be the corresponding regular 2-graph. If \( T \) is trivial, then we can find a diagonal matrix \( D \) with diagonal entries \( 11 \) such that \( J \in C(D^{-1}MD) \). Since \( D^{-1}MD \) is monomial Lemma (1.1) contradicts assumption (ii). Hence \( T \) is non-trivial and \((\Omega, A)\) is of type 2.

For \( g \) in \( G \) we can write \( \Phi(g) = \Phi(g)D(g) \) where \( D(g) \) is a diagonal matrix with diagonal entries \( 11 \). Then

\[
\Phi(g)^{-1}A\Phi(g) = D(g)\Phi(g)^{-1}A\Phi(g)D(g)^{-1}
= D(g)AD(g)^{-1}
\]

Since \((\Omega, D(g)AD(g)^{-1})\) is equivalent to \((\Omega, A)\) via switching, \( \Phi(g) \) induces an automorphism of \( T \). Since all the parameters of \( T \) can be calculated from the multiplicities of the eigenvalues of \( A \) all parts of the theorem are proved. //

Remarks. 1. If we omit condition (ii) from the hypothesis of the theorem then it may happen that \((\Omega, A)\) is a trivial graph. However, if this is the case, then from the proof of (1.1) \( \Phi(g) = \mu(g)\Phi(g) \) so \( \Phi = \mu X \). Hence we may replace (ii) by

(ii)' no constituent of \( \Phi \) has degree 1.

and the conclusion of the theorem still holds.

2. If we omit condition (i) then either the conclusion of
the theorem still holds or \( G \) has rank 6 on the cosets of \( K \) and \( n \) is odd. Let \( \pi \) be the character of \( \mathbb{P} \) and let \( \pi' \) be the character of \( (1, K)^G \). Since \( H \) has three orbits on the cosets of \( K \) the inner product of \( \pi \) with \( \pi' \) is 3. Now \( \pi = 1 + \phi \) where \( \phi \) is irreducible so \( \phi \) occurs in \( \pi' \) with multiplicity 2. Since the inner product of \( \pi' \) with itself is 6 \( \pi' \) has one further constituent \( \mu \) which has degree \( 2n-2(n-1)-1 = 1 \). Since \( \mu \) is rational its square is the principal character. Hence \( G \) has a subgroup of index 2.

We have \( M = X + \mu \), regarding \( \mu \) as a representation, so the centraliser ring \( C(M) \) has dimension 2. However the strong graph \( (s, A) \) is trivial. This occurs, for example, in the Frobenius group of order 20 acting on 5 points.

We now consider the converse situation to (1.2).

Let \( T = (\mathbb{Q}, h) \) be a regular 2-graph and suppose that \( G = \text{Aut}(T) \) acts doubly transitively on \( \mathbb{Q} \). Let \((\mathbb{Q}, A)\) be a strong graph of \( S(T) \) and let \( \mathbb{P} \) denote the permutation representation of \( G \) on \( \mathbb{Q} \). If \( x \) belongs to \( G \), then \((\mathbb{Q}, \mathbb{P}(x)^{-1}A \mathbb{P}(x))\) is in \( S(T) \) so there is a diagonal matrix \( D(x) \) (with entries \( \pm 1 \)) such that \( \mathbb{P}(x)^{-1}A \mathbb{P}(x) = D(x)AD(x) \). Thus \( \mathbb{P}(x) = D(x) \mathbb{P}(x)D(x) \) commutes with \( A \). Let \( G_0 \) be the group of all monomial matrices which commute with \( A \). Then the subgroup \( Z = \{ \pm I \} \) lies in the centre of \( G_0 \) and \( G \) is isomorphic to \( G_0/Z \). Unfortunately \( G_0 \) does not always split over \( Z \) (see the next section).

However if \( G_0 \) does split we shall write \( G_0 = Z \times G \); in this case we may suppose that \( M \) defines a monomial representation of \( G \). We now relate this to the equiangular lines defined
Let $\mu_1$ and $\mu_2$ be the multiplicities of the eigenvalues $\rho_1$ and $\rho_2$ of $A$. Let $v_1, \ldots, v_N$ be the vectors spanning the equiangular lines defined by $A$ in dimension $\mu$, as in (2.4. 9) where $\mu$ is either $\mu_1$ or $\mu_2$. The matrices $M$ of $G_0$ are orthogonal and induce orthogonal transformations in $\mathbb{R}^\mu$ which permute the equiangular lines among themselves. Moreover, the group $G_0$ is just the group of symmetries of the configuration. If $\mu_1$ (and hence $\mu_2$) is odd then by taking the determinant of the representation of $G_0$ in $\mathbb{R}^\mu$ we see that $-I$ lies outside a subgroup of $G_0$ of index 2. Hence $Z$ splits in this case.

We have presented $G_0$ as a group of monomial matrices so we may take the representation $M$ to be the identity automorphism. Since the spaces of dimensions $\mu_1$ and $\mu_2$ spanned by the two sets of equiangular lines can be considered as invariant subspaces of the space on which $G_0$ acts we see that $M$ has two constituents $M_1$ and $M_2$ of degrees $\mu_1$ and $\mu_2$ respectively. By Theorem (1.2) and the remark $M_1$ and $M_2$ must be irreducible. Thus $M$ is not a permutation representation and therefore, if $H$ is the stabiliser of a point in the associated permutation representation of $M$, then $H$ has a subgroup $K$ of index 2 and a linear representation $M_0$ with $K$ in its kernel such that $M$ is similar to $M_0^G$. If $G_0 = Z \times G$, then we may restrict $M$ to $G$ so that the same conclusions hold for $G$.

We have proved the following partial converse to (1.2):

From Chapter 2 we know that $A_0$ is the adjacency matrix of the first row and columns of $A$ are all 1. Let $A_0$ be the matrix obtained by deleting the first row and columns of $A$. Then $A_0$ is a diagonal matrix.
Theorem. Let $T = (\mathcal{S}, t)$ be a regular 2-graph with $G = \text{Aut}(T)$ doubly transitive on $\mathcal{S}$. Let $G_0$ be the group of symmetries of a set of equiangular lines associated with $T$ and let $Z$ be the subgroup generated by the central inversion. Then $Z \leq Z(G_0)$ and $G = G_0/Z$. Moreover, $G_0$ has a monomial representation $M$ (with entries $0, \pm 1$) having just two irreducible constituents $M_1$ and $M_2$ which are the representations of $G_0$ on the vectors spanning the equiangular lines. If $H$ is the stabiliser of a point in the associated permutation representation then $H$ has a subgroup $K$ of index 2 and $M$ is similar to the representation induced from the non-principal linear representation of $H$ with $K$ in its kernel. The regular 2-graph derived from $M$ as in Theorem (1.2) coincides with $T$. If $G_0 = Z \times G$ then the above conclusions hold with $G_0$ replaced by $G$. This will be the case if $\mu_1$ and $\mu_2$ are odd.

Remark. It is always the case that in $G_0$ the stabiliser $H_0$ of a line splits over $Z$. Thus $H_0 = Z \times H$ where $H$ is the stabiliser of a vector.

Suppose that we are in case $(b)$ of (1.2). By conjugating with a diagonal matrix we may suppose that the non-zero entries of the first row and column of $A$ are all 1. Let $A_0$ be the matrix obtained by deleting the first row and column of $A$. From Chapter 2 we know that $A_0$ is the adjacency matrix of
strongly regular graph. We also have

(1.9) **Theorem.** The matrix $A_0$ belongs to the centraliser ring of the restriction of $P$ to $H$.

**Proof.** If $g \in G$ and $\overline{m}(g)$ has non-zero entry $\omega$ in position (1,1) then every other non-zero entry of $\overline{m}(g)$ must be $\omega$ since $\overline{m}(g)$ commutes with $A$. Thus $\overline{m}(g) = \overline{i}(g)$ and therefore $\overline{P}(g)$ commutes with $A$ and the theorem is proved. //

2. **Symplectic geometry over $GF(2)$**

Let $V$ be a vector space of dimension $2m$ over the field $GF(2)$ and let $B : V \times V \to GF(2)$ be a non-degenerate alternating form on $V$. To avoid a trivial case we shall suppose $m \geq 1$. Let $P = V^*$ be the set of non-zero vectors of $V$. The group $Sp(2m,2)$ acts as a rank 3 permutation group on $P$ and the stabiliser of a vector $x$ has the following non-trivial orbits:

(2.1) $D_1(x) = \{ y \in P \mid B(x,y) = 0, \ x \neq y \}$

$D_2(x) = \{ y \in P \mid B(x,y) = 1 \}$.

From Huppert [1], p. 221 we see that a strongly regular graph defined by this rank 3 group has parameters

(2.2) $n = 2^{2m-1}$

$\begin{align*}
\delta_1 &= 2^{2m-2} - 2 \\
\delta_2 &= 2^{2m-2} - 1 \\
\mu_{11} &= 2^{2m-3} \\
\mu_{12} &= 2^{2m-1} \\
\mu_{22} &= 2^{2m-3} - 1
\end{align*}$

**Proof.** It is easy to see that $\delta_1$ already have a $\mu$-autodifiniteness. We shall show that the vector space structure and that any subalgebra of $Gil(\gamma)$ is a linear
Since \( n_1 = 2p_1^2 \) we obtain a regular 2-graph \( S(2m) = (V, t) \) by adjoining a point which we may take to be \( o \in V \). In fact

\[
(2.3) \quad t = \{ \{ x, y, z \} \mid B(x, y) + B(y, z) + B(z, x) = 0 \}.
\]

The parameters for \( S(2m) \) are

\[
(2.4) \quad N = 2^{2m}, \quad a = 2^{2m-1} - 2, \quad a' = 2^{2m-1}, \\
k = 2^{2m-2} - 3, \quad k' = 2^{2m-2}.
\]

Now let \( T(V) \) denote the group of translations of \( V \) (an elementary abelian 2-group isomorphic to \( V \)) and let \( H \) be the group \( Sp(2m, 2) \) of linear transformations of \( V \) which fix \( B \). From the description of \( t \) in (2.3) the group \( G = T(V) H \) is a group of automorphisms of \( S(2m) \) and acts doubly transitively on \( V \).

(2.5) **Theorem.** The automorphism group of \( S(2m) \) is the group \( G \).

**Proof.** It is enough to show that \( H = \text{Aut}(S(2m)_o) \). We already have \( H \subseteq \text{Aut}(S(2m)_o) \). We shall show that the vector space structure of \( V \) can be obtained from the 2-graph structure and that any automorphism of \( S(2m)_o \) is a linear
transformation preserving the bilinear form $B$. Let $M$ be a coherent subset of $V$ containing $o$. From (2.3) $M$ is totally isotropic. Thus a coherent set of maximal order is just a translate of a maximal totally isotropic subspace. We shall suppose $M$ to be a maximal totally isotropic subspace. Since $|M| = 2^m = 1 - p_2$ from § 2.7 we have $m(x) = 2^{m-1}$ for $x \not\in M$. Choose the notation so that $o \in M_1(x)$. Given $a, b \in M$, $a, b \neq o$, define $L(a, b) = \cap \{M_1(x) \mid a, b \in M_1(x), x \not\in M\}$. Then $L(a, b)$ is the (totally isotropic) plane containing $a$ and $b$, $a+b$ is uniquely determined by $L(a, b) = \{o, a, b, a+b\}$ and $L(a, b)$ does not depend on the choice of $M$. If $\{o, a, b\}$ is incoherent, then the (hyperbolic) plane determined by $a$ and $b$ is the unique maximal incoherent set of 4 points containing $o$, $a$ and $b$. Thus $a+b$ is again uniquely determined by the 2-graph structure. For $g \in \text{Aut}(S(2m)_o)$ and $a,b \in V$ we therefore have $(a+b)g = ag + bg$ and from (2.3) $B(a, b) = B(ag, bg)$. Hence $H = \text{Aut}(S(2m)_o)$ and $G = \text{Aut}(S(2m))$. //

In terms of rank 3 groups, this proof is given by D.G. Higman [1].

Since a hyperbolic plane is a maximal incoherent set of 4 points, the complement of $S(2m)$ is not 4-regular.

However, there exist incoherent sets of size $2m+2$. To see this, write $V = V_1 \perp \ldots \perp V_m$ where $V_1$ is a hyperbolic plane with basis $e_1, f_1, e_1 + f_1, e_2, e_1 + f_1 + f_2, \ldots, e_1 + \ldots + e_m + f_1 + \ldots + f_m$ form an incoherent set of size $2m+2$. This is obviously as large
as possible.

The 2-graph $S(2m)$ itself is not 4-regular unless $m=2$, in which case $k_4 = 0$. If $m > 2$ then $o$, $e_1$, $e_2$ and $e_1 + e_2$ form a coherent set of 4 points contained in $2^{2m-2} - 2$ coherent sets of 5 points while $o$, $e_1$, $e_2$ and $e_3$ form a coherent set of 4 points contained in $2^{2m-3} - 4$ coherent sets of 5 points. However, (A) and (B) of § 2.9 are satisfied since if $L$ is a coherent set of $2^{m-1} + 1$ points then acting with $T(V)$ we may suppose that $o \in L$ and then the linear span of $L$ is the unique coherent set of size $2^m$ containing $L$.

Since $a' = a + 2$ we are in the situation discussed in § 3.6. We obtain a projective design with parameters

\[(2.6) \quad v = 2^{2m-1}, \quad k = 2^{2m-1} - 1, \quad \lambda = 2^{2m-2} - 1.\]

This is just the design of points and hyperplanes of the projective geometry of $V$ (D.G. Higman [1]). The extension with parameters $(6.4)$ is the design of points and hyperplanes of the affine geometry of $V$.

In order to exhibit further designs we shall construct two strongly regular graphs of $\mathcal{S}(S(2m))$. Let $Q : V \to GF(2)$ be a quadratic form such that

\[(2.7) \quad Q(x+y) + Q(x) + Q(y) = B(x,y).\]

Let $2$ be the set of quadratic forms which satisfy (2.7). The group $H = Sp(2m, 2)$ has two orbits on the set $2$ (Dickson [1], p. 197): the orbit $2'$ of forms of index $m$
and the orbit 2^{-1} of forms of index m-1. (See also Dembowski [1], p. 46 and Dieudonne [1], p. 34). Choose \( Q^a \in 2^a \) and define

\[
(2.8) \quad q^a = \{ x \in P \mid Q^a(x) = 0 \} \quad \text{where} \quad a \in \{1,-1\}.
\]

Let \( O^a(2m,2) \) denote the group of linear transformations of \( V \) which fix \( Q^a \). From (2.7) \( O^a(2m,2) \) is contained in \( \text{Sp}(2m,2) \). By Witt's theorem (Dieudonne [1], p.36) the sets \( q^a \) and \( P \setminus q^a \) are orbits of \( O^a(2m,2) \). Hence the groups \( G^a = T(V) O^a(2m,2) \) have rank 3 on \( V \).

We have (Dembowski [1], p. 46)

\[
(2.9) \quad |q^a| = (2^{m-1+a})(2^{m-1-a}) = 2^{2m-1+a}2^{m-1-a}.
\]

A subspace \( W \) of \( V \) is called singular (with respect to \( Q^a \)) if \( Q^a(x) = 0 \) for all \( x \in W \). The number of singular subspaces of dimension 2 in \( V \) is (Dembowski [1], p.46)

\[
(2.10) \quad b = \frac{1}{3}(2^m-a)(2^{m-1-a})(2^{m-2+a})(2^{m-1+a})
\]

Hence the number of singular subspaces of dimension 2 of \( V \) which contain a given point of \( q^a \) is

\[
(2.11) \quad r = \frac{3b}{|q^a|} = (2^{m-1-a})(2^{m-2+a}).
\]

Now a plane which intersects \( q^a \) in more than one point is either singular or hyperbolic so by Witt's theorem \( O^a(2m,2) \) has rank 3 on \( q^a \) and the length of one of the suborbits is 2r where r is given by (2.11). Hence the strongly regular graph determined by the rank 3 group \( G^a \) has parameters:
From (2.3) and (2.7) we see that these strongly regular graphs belong to $S(2m)$ and $G^\alpha \leq G$. Thus both possibilities of (2.6.1) occur and from the construction of (3.6.5) we obtain two proactive designs with parameters

$$v = 2^{2m}, k = 2^{2m-1} + a \cdot 2^{m-1}, \lambda = 2^{2m-2} + a \cdot 2^{m-1}.$$

These designs will be considered again in the next section.

From Dickson [1], Ch. VIII and the above discussion we see that the stabiliser of a point in $O^\alpha(2m,2)$ acting on $G^\alpha$ is an elementary abelian group of order $2^{2m-2}$ extended by $O^\alpha(2m-2,2)$ and the stabiliser of a point in $O^\alpha(2m,2)$ acting on $P - g^\alpha$ is $Z_2 \times Sp(2m-2,2)$.

Suppose that $m = m_1 m_2$ and let $X$ be the Galois group of $GF(2^{m_1})$ over $GF(2)$. The group $Sp(2m_2,2^{m_1})$ acts on a vector space $E$ of dimension $2m_2$ over $GF(2^{m_1})$. If $A$ is the alternating form fixed by $Sp(2m_2,2^{m_1})$ then by restriction of scalars we may identify $E$ with $V$ as above and define $B$ as $B(x,y) = Tr(A(x,y))$ where $Tr$ is the trace from $GF(2^{m_1})$.
to $\text{GF}(2)$. It follows that the semi-direct product $\Sigma p(2m_2, 2^{m_1}) = \text{Sp}(2m_2, 2^{m_1})$. $X$ is a subgroup of $\text{Sp}(2m, 2)$, (see Huppert [1], p. 228). In particular, we have $\Sigma L(2, 2^m) \leq \text{Sp}(2m, 2)$ since $\text{SL}(2, 2^m)$ is isomorphic to $\text{Sp}(2, 2^m)$. Since $\Sigma p(2m_2, 2^{m_1})$ acts transitively on the non-zero vectors of $E$ the group $T(V) \cdot \Sigma p(2m_2, 2^{m_1})$ acts doubly transitively on the regular 2-graph $S(2m)$.

Again write $V = V_1 \perp \cdots \perp V_m$ where each $V_i$ is a hyperbolic plane. Suppose further that each $V_i$ is an elliptic plane for the quadratic form $Q$. If $m$ is even, then $Q$ has index $m$ while if $m$ is odd, then $Q$ has index $m-1$. Since the group of automorphisms of each $V_i$ is $\text{SL}(2, 2^m) \times S_3$ we find that the wreath product $S_3 \wr S_m$ is a subgroup of $O(2m_2, 2^{m_1})$ when $m$ is even and of $O^{-1}(2m_2, 2)$ when $m$ is odd.

If $X$ is a set of $2^{m+2}$ elements we may identify the vector space $V$ with the set of partitions of $X$ into pairs of even subsets, where addition is symmetric difference.

If $x = (X_1, X_2)$ and $y = (Y_1, Y_2)$ we may suppose that our alternating form $B$ is defined by

$$B(x, y) = \begin{cases} 0 & |X_1 \cap Y_1| \equiv 0 \pmod{2} \\ 1 & |X_1 \cap Y_1| \equiv 1 \pmod{2} \end{cases}$$

since this defines a non-degenerate alternating form (note that $|X_1 \cap Y_1| \equiv |X_1 \cap Y_2| \equiv |X_2 \cap Y_1| \equiv |X_2 \cap Y_2| \equiv 0 \pmod{2}$). From this construction for $V$ we obtain the inclusion $S_2m+2 \subseteq \text{Sp}(2m, 2)$. Moreover, if $m \equiv 1 \pmod{2}$ then we may define a quadratic form $Q$ satisfying (2.7) by

$$Q(x) = \begin{cases} 0 & |X_1| \equiv 0 \pmod{4} \\ 1 & |X_1| \equiv 2 \pmod{4} \end{cases}$$
so that for \( m \equiv 1 \pmod{2} \) we have \( S_{2m+2} \leq O^\alpha(2m,2) \) where \( \alpha + m \equiv 0 \pmod{4} \).

These constructions will be used in Chapter 5.

Finally, let us apply the results of § 1 to the 2-graphs \( S(2m) \). Since apart from \( \text{Sp}(4,2) \), \( \text{Sp}(2m,2) \) does not have a subgroup of index 2 it follows from (1.3) that the covering group \( G_\circ \) of \( G \) does not split over its centre. We may describe the group \( G_\circ \) as follows. Let \( W \) be a vector space of dimension \( 2m+1 \) over \( GF(2) \), let \( Q \) be a non-degenerate quadratic form on \( W \) (see Dieudonné [1], p. 33) and let \( F \) be the associated alternating form. The subspace \( W_0 \) of \( W \) orthogonal to \( W \) with respect to \( F \) has dimension 1. If \( O(2m+1,2) \) is the group of linear transformations leaving \( Q \) invariant then every element of \( O(2m+1,2) \) fixes \( W_0 \) pointwise and \( O(2m+1,2) \) is isomorphic to \( \text{Sp}(2m,2) \), (Dickson [1], p. 200). Let us write \( W_0 = \langle e_0 \rangle \) and define \( q \) as in (2.8). Let \( q' \) be the points of \( W \) not in \( q \) or \( W_0 \). Then the orbits of \( \text{Sp}(2m,2) \) on \( W \) are \( o, e_0, q \) and \( q' \) and we have \( |q| = |q'| = 2^{2m} - 1 \). The translation \( t_{e_0} \) in direction \( e_0 \) lies in the centre of \( G_\circ = T(W) \cdot \text{Sp}(2m,2) \) and if we put \( Z = \langle t_{e_0} \rangle \) we may identify \( T(W) \cdot \text{Sp}(2m,2)/Z \) with \( T(V) \cdot \text{Sp}(2m,2) \) where \( V = W/W_0 \). If \( m \neq 2 \), then \( G_\circ \) does not have a subgroup of index 2 we may use Theorem (1.2) to construct a regular 2-graph. By Theorem (1.9) this coincides with \( S(2m) \) since \( S(2m) \) was defined via the rank 3 representation of \( \text{Sp}(2m,2) \). This also holds for \( S(4) \) since \( t_{e_0} \) does not lie outside a subgroup of index 2.
3. Quadratic forms over GF(2)

We continue the notation of § 2: $V$ is a vector space of dimension $2m$ on GF(2) with a non-degenerate alternating form $B$ and $\mathcal{Q}$ is the set of quadratic forms satisfying (2.7). Again put $G = T(V).H$ where $H = \text{Sp}(2m,2)$.

For $x \in V$ define the translation

$$t_x(y) = x + y$$

and the symplectic transvection

$$\sigma_x(y) = y + B(x,y)x$$

It is well known (Dieudonne [1], p. 41) that $H$ is generated by the involutions $\sigma_x$, $x \in V$ and that these elements are a conjugacy class of $H$.

Throughout this section suppose that $\alpha \in \{1, -1\}$. We shall construct a regular 2-graph on the set $\mathcal{Q}^\alpha$ which has $\text{Sp}(2m,2)$ as its (doubly transitive) automorphism group. In order to do this we construct several designs one of which is the projective design considered in the previous section. We shall show that its automorphism group is $T(V).\text{Sp}(2m,2)$.

Choose $Q_0$ in $\mathcal{Q}^\alpha$ and define

$$\mathcal{Q}^\alpha = \{ x \in V \mid x \neq 0, \ Q_0(x) = 0 \}$$

and

$$\mathcal{Q}^\alpha = \{ x \in V \mid Q_0(x) = 1 \}$$

For any $Q$ in $\mathcal{Q}$ we can write $Q(x) + Q_0(x) = f(x)^2$ where
$f : V \to \text{GF}(2)$ is a linear functional. We may find an element $a$ of $V$ such that $f(x) = B(a, x)$ for all $x$ in $V$.

Hence any element of $I$ may be written uniquely as

$$(3.4) \quad Q_a(x) = Q_0(x) + B(a, x)^2.$$ 

Thus $|I| = 2^{2m}$ and we may make $I$ into a vector space over $\text{GF}(2)$ with zero $Q_0$ by defining an addition $+$ by

$$(3.5) \quad (Q_1 + Q_2)(x) = Q_1(x) + Q_2(x) + Q_0(x).$$

We shall consider $I$ as an affine space over $\text{GF}(2)$ and show that $G$ acts as a group of affine transformations of $I$. First define the action of $t_a \in T(V)$ on $I$ by

$$(3.6) \quad (t_a Q)(x) = Q(x) + B(a, x)^2.$$ 

Then $t_a$ acts as an affine transformation of $I$ and $t_a Q_b = Q_{a+b}$. Now define the action of $\sigma \in H$ on $I$ by

$$(3.7) \quad (\sigma Q)(x) = Q(\sigma^{-1} x)$$

Again we have an affine transformation of $I$ and if $a \in V$ we have $(\sigma t_a \sigma^{-1} Q)(x) = Q(x) + B(\sigma a, x)$ so that

$$(3.8) \quad \sigma t_a \sigma^{-1} = t_{\sigma a}.$$ 

Hence the group $G = T(V) \cdot H$ acts on the affine space $I$.

Let $H_0$ be the subgroup of $G$ fixing $Q_0$. Then $H_0 \cong \text{Sp}(2m, 2)$ and $H_0$ acts on $I$ the way $H$ acts on $V$ since $T(V)$ acts regularly on $I$. Hence $G$ acts doubly transitively on $I$.

We may define an isomorphism

$$(3.9) \quad \gamma : V \to I : a \mapsto Q_a.$$
Changing $Q_0$ amounts to altering $\gamma$ by a translation.

We may use $\gamma$ to transfer the regular 2-graph structure of $V$ to $2$. The set $T$ of coherent triangles so obtained does not depend on the choice of $Q_0$.

Now let $2^a$ be the set of three element subsets of $2^\alpha$ and define

\[(3.10) \quad T^a = T \cap 2^a.\]

We shall show that $(2^a, T^a)$ is a regular 2-graph. To do this we construct a design $\delta$ by taking $2$ as the set of points and the images of $2^{-a}$ under $G$ as blocks. By the results of \S 3.1 $\delta$ is projective and $G$ acts doubly transitively on blocks. Moreover, the permutation character of $H$ acting on points is the same as that of $H$ acting on blocks. Since $T(V)$ acts regularly on the blocks and $H$ fixes $2^{-a}$, we see that $H$ acts on blocks as it does on $V$. Hence the permutation character is $\mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2$, where $\mathcal{O}_0$ is the principal character and $\mathcal{O}_1$ and $\mathcal{O}_2$ are irreducible (from (2.1) $H$ has rank 3 on $V - \{0\}$). We may now choose the notation so that the permutation character of $H$ is $\mathcal{O}_0 + \mathcal{O}_1$ on the points of $2^1$ and $\mathcal{O}_0 + \mathcal{O}_2$ on the points of $2^{-1}$. We have proved

\[(3.11) \quad H \text{ acts doubly transitively on both } 2^1 \text{ and } 2^{-1}.\]

Since $G$ preserves coherent triangles in $2$ and $H$ acts doubly transitively on $2^a$ it follows that $0^a(2m) = (2^a, T^a)$ is a regular 2-graph.

We next consider the action of the transvection $\psi_a$ on
and relate this to \( \phi \) (cf. Jordan [1], p. 229).

\[ (3.12) \quad \phi_a Q = \begin{cases} \phi_a Q & Q(a) = 0 \\ Q & Q(a) = 1 \end{cases} \]

**Proof.** \((\phi_a Q)(x) = Q(\phi_a x)\)

\[ = Q(x + B(a,x)a) \]
\[ = Q(x) + B(a,x)^2(Q(a) + 1). \]

Thus we have

\[ (3.13) \quad \phi_a \text{ moves } Q \text{ if and only if } Q_0(a) + B(a,b)^2 = 0. \]

Since \( H \) is doubly transitive on \( 2^a \), given \( Q_1, Q_2 \in 2^a \), we can find a transvection \( \phi_a \) such that \( \phi_a Q_1 = Q_2 \). From (3.12) we have \( \phi_a Q_1 = Q_2 \) and \( Q_1(a) = 0 \). It now follows that

\[ (3.14) \quad \text{The quadratic forms } Q_1 \text{ and } Q_2 \text{ of } 2 \text{ have the same index if and only if there exists an element } a \text{ in } V \]
\[ \text{such that } \phi_a Q_1 = Q_2 \text{ and } Q_1(a) = Q_2(a) = 0. \]

The element \( a \) is uniquely determined by \( Q_1 \) and \( Q_2 \).

Since \( Q_a = \phi_a Q_0 \) we have

\[ (3.15) \quad 2^{a_c} = \{ Q_a \mid Q_0(a) = 0 \} \]
\[ 2^{a_c} = \{ Q_a \mid Q_0(a) = 1 \}. \]

We are now able to relate the design \( S \) to the projective design constructed in \( \S \ 2 \). The complement of that design has \( V \) for its set of points and the translates of \( 2^c \) for
its blocks. But \( \gamma P^a = 2^{-a} \) so that this is just the design \( S \). It follows from (2.9) that

\[(3.16) \quad |2^a| = 2^{2m-1} + 2^{m-1}.
\]

Hence the parameters of \( S \) are

\[(3.17) \quad v=b=2^{2m}, \quad r=k=2^{2m-1} - 2^{m-1}, \quad \lambda = 2^{2m-2} - 2^{m-1}.
\]

Given \( Q_a \in 2^a \) we want to calculate the number of points \( Q_b \in 2^a \) such that \( \{Q_0, Q_a, Q_b\} \in 2^a \). This is the number of points \( b \) such that \( B(a, b) = 0 \) and \( Q_0(b) = 0 \). Since \( Q_0(a) = 0 \) this is just twice the number of singular planes (with respect to \( Q_0 \)) which contain \( a \). By (2.11) this is \( 2(2^{m-1} - a)(2^{m-2} + a) \) and hence

\[(3.18) \quad \text{Theorem. The parameters of the regular 2-graph } 0^a(2m) \text{ are}
\]

\[
N = 2^{2m-1} + 2^{m-1}
\]

\[
a = 2^{2m-2} + 2^{m-1} - 2
\]

\[
a' = 2^{m-2}
\]

\[
k = 2^{2m-3} + 2^{m-1} - 3
\]

\[
k' = 2^{2m-3} - 2^{m-2}
\]

\[
\rho_1 = (3-a).2^{m-2} + 1
\]

\[
\rho_2 = -(3+a).2^{m-2} + 1
\]

\[
\mu_1 = \frac{1}{3}(2^{m-1})(2^{m-2} - (3+a) + a)
\]

\[
\mu_2 = \frac{1}{3}(2^{m+1})(2^{m-2} - (3-a) + a)
\]

The subgroup of \( H \) fixing \( Q_0 \) is just \( H_0 \cap H = 0^a(2m, 2) \) and \( H_0 \cap H \) acts on the points of \( 2 \) the way \( 0^a(2m, 2) \) acts on \( V \). Thus the strongly regular graph obtained from \( 0^a(2m) \)
by deleting a point is just the graph obtained from the rank 3 representation of \( 0^a(2m,2) \) on \( q^a \). Its parameters are:

\[
\begin{align*}
(3.19) \quad n &= 2^{2m-1+a} \cdot 2^{m-1} - 1 \\
n_1 &= 2^{2m-2+a} \cdot 2^{m-1} - 2 \\
p_{11} &= 2^{2m-3+a} \cdot 2^{m-1} - 3 \\
n_2 &= 2^{2m-2} \\
p_{11} &= 2^{2m-3+a} \cdot 2^{m-2} - 1.
\end{align*}
\]

There are just four possibilities for the restriction of \( Q_0 \) to a plane of \( V \). We can choose a basis \( e, f \) for the plane so that it has one of the following forms:

\[
\begin{align*}
(3.20) \quad \text{singular} & \quad Q_0(e) = Q_0(f) = B(e,f) = 0 \\
(3.21) \quad \text{elliptic} & \quad Q_0(e) = Q_0(f) = B(e,f) = 1 \\
(3.22) \quad \text{hyperbolic} & \quad Q_0(e) = Q_0(f) = 0, B(e,f) = 1 \\
(3.23) \quad \text{totally isotropic} & \quad Q_0(e) = Q_0(f) = 1, B(e,f) = 0.
\end{align*}
\]

By Witt's theorem \( 0^a(2m,2) \) is transitive on each type of plane and hence \( 0^a(2m,2) \) acts as a rank 3 group on both \( q^a \) and \( p^a \). Using (2.10), (2.11), (2.12) and the fact that \( V \) contains \( \frac{1}{3} 2^{2m-2}(2^m-1) \) hyperbolic planes (with respect to \( B \)) we obtain the following table:

\[
\begin{array}{ccc}
\text{singular} & (2^{m-1-a})(2^{m-2+a}) & - \\
\text{elliptic} & - & 2^{m-2}(2^{m-1-a}) \\
\text{hyperbolic} & 2^{2m-2} & 2^{m-2}(2^{m-1+a}) \\
\text{totally isotropic} & 2^{m-2}(2^{m-1-a}) & 2^{2m-2}-1
\end{array}
\]
This determines the parameters \( n_1 \) and \( n_2 \) of the strongly regular graph obtained from the action of \( O_0(2m,2) \) on \( \mathbb{P}^n \). Since \( O_0(2m,2) = H_0 \cap H \) acts on \( \mathbb{P}^n \) as it does on \( \mathbb{P}^m \), we see that the regular 2-graph obtained from the strongly regular graph by the construction of (2.2.9) coincides with \( O_0(2m) \). Hence the remaining parameters may be calculated from (2.6.1) and we have:

\[
\begin{align*}
(3.25) \quad n &= 2^{2m-1-a}2^{m-1} \\
 n_1 &= 2^{2m-2-a}2^{m-1} \\
 n_2 &= 2^{2m-2+a}2^{m-1} \\
p_{11} &= 2^{2m-3-2} \\
p_{11} &= 2^{2m-3+a}2^{m-2}.
\end{align*}
\]

This realises one of the possibilities of (2.6.1). The other possibility for the parameters of a strongly regular graph of \( O_0(2m,2) \) is

\[
\begin{align*}
(3.26) \quad n &= 2^{2m-1-a}2^{m-1} \\
n_1 &= 2^{2m-2-3+a}2^{m-2} \\
n_2 &= 2^{2m-2+a}2^{m-2} \\
p_{11} &= 2^{2m-3-3a}2^{m-2} \\
p_{11} &= 2^{2m-3-a}2^{m-1}.
\end{align*}
\]

If \( m \) is even we may always construct such a graph. We leave the construction to the end of this section. However, if \( m \) is odd such a construction is not always possible. For example, we have the following well-known result.

\[
(3.27) \quad \text{Theorem. There does not exist a strongly regular graph whose parameters are given by (3.26) with } m=3 \text{ and } a=1.
\]
If such a graph exists we may construct a regular 2-graph from it with the parameters of $O_1^{-1}(6)$, namely $N=28$, $a=10$, $k=1$, $\rho_1=9$, $\rho_2=-3$, $\mu_1=7$ and $\mu_2=21$. But $p_{11}=0$ so the regular 2-graph would contain a coherent set of 10 points contradicting (2.7.5). //

If $m=3$ and $a=-1$ a strongly regular graph with parameters (3.2.6) can be constructed from the rank 3 representation of $\mathbf{H}U(3,2^2)$ on 36 points (see Bussemaker and Seidel [1], p. 15).

For example, if we have sets of size $a$ and $\hat{a}$

(3.28) Theorem. Any coherent set of $O^a(2m)$ is contained in a maximal coherent set. Maximal coherent sets have size $1-\rho_2 = (3+a)2^{m-2}$ and $H$ acts transitively on maximal coherent sets.

Proof. By transitivity of $H$ we may suppose the coherent set $C$ to contain $Q_0$. Now using $\gamma$ we may suppose that $C$ is contained in $q \cup \{ e \}$. Hence the subspace generated by $C$ is still coherent. By definition, $q^a \cup \{ e \}$ contains a subspace of dimension $m$ if $a=1$ and dimension $m-1$ if $a=-1$. By (2.7.3) these are coherent sets of maximal size. The rest of the theorem now follows from Witt's theorem. //

In order to determine incoherent subsets of $O^a(2m)$ write $V = V_1 \perp \cdots \perp V_m$ where each $V_i$ is a hyperbolic plane (with respect to $B$) with basis $e_i, f_i$. As in § 2 we have an incoherent set $\{ c, e_1, f_1, e_1 + f_1 + e_2, e_1 + f_2 + f_2, \cdots \}$ in $S(2m)$. We may suppose that the $V_i$ are alternately
hyperbolic and elliptic except possibly that $V_{m-1}$ and $V_m$ have the same type. We thus obtain the following table for the maximum size of an incoherent set in $O^a(2m)$. The columns are headed with $m \mod 4$.

(3.29)  

<table>
<thead>
<tr>
<th>$a = 1$</th>
<th>2m+2</th>
<th>2m+1</th>
<th>2m</th>
<th>2m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = -1$</td>
<td>2m</td>
<td>2m</td>
<td>2m+2</td>
<td>2m+1</td>
</tr>
</tbody>
</table>

For example, if $m=3$ we have sets of size 6 or 7 according to $a=1$ or $a=-1$. Thus the bounds obtained in §2.7 are reached in these cases. Similarly, if $m=4$ and $a=1$ we again achieve the bound given in §2.7.

We wish to show that the automorphism group of $O^a(2m)$ is $H = \text{Sp}(2m,2)$. This will be done by relating the transvections $\mathcal{O}_a$ to the structure of $O^a(2m)$. First consider the design $\mathcal{G}'$ obtained from $\mathcal{G}$ by deleting the block $2^{-a}$ and all of its points. Since $H$ fixes $2^{-a}$ it acts as a group of automorphisms of $\mathcal{G}'$. The parameters of $\mathcal{G}'$ are

(3.30)  

\[
\begin{align*}
\nu &= 2^{2m-1} + a \cdot 2^{m-1} \\
b &= 2^{2m-1} \\
r &= 2^{2m-1} - a \cdot 2^{m-1} \\
k &= 2^{2m-2} \\
\lambda &= 2^{2m-2} - a \cdot 2^{m-1}
\end{align*}
\]
Let us denote the block \( t_a 2^a \cap 2^a \) by \( c_a \).

\[(3.31) \quad c_a = \{ Q \in 2^a \mid Q(a) = 1 \} = \{ Q \in 2^a \mid \phi_a Q = Q \} \]

**Proof.** If \( Q_b \in 2^a \) then \( Q_0(b) = 0 \) so
\[Q_b(a) = Q_0(a) + B(a,b) = Q_0(a+b). \]
Hence \( Q_b \in t_a 2^a \) if and only if \( Q_b(a) = 1 \). The second equality now follows from (3.12).

\[(3.32) \quad \text{Theorem. Given } Q_1 \text{ and } Q_2 \text{ in } 2^a \text{ there exists a} \]
\[V \text{ such that } \phi_a Q_1 = Q_2. \]
If \( Q \in 2^a \) then
\[\{ Q_1, Q_2, Q \} \text{ is coherent if and only if } \phi_a Q \neq Q. \]

**Proof.** The existence of a comes from (3.14). Let us write
\[Q_1 = Q_b \text{ and } Q_2 = Q_{a+b} \text{ for some } b. \]
From (3.12) we have
\[Q_b(a) = Q_0(a) \quad \text{and} \quad Q_{a+b}(a) = Q_0(a+b) = Q_0(a+b) = 0. \]
If \( Q = Q_c \), then \( Q_0(c) = 0. \text{ Thus } B(a,b) = 0 \text{ and } Q_c(a) = B(a,c)^2. \]
Since \( B(b,a+b) + B(a+b,c) + B(c,b) = B(a,c) \) the result follows from (3.12).

We may define an equivalence relation on unordered pairs \((Q_1, Q_2)\), \(Q_1, Q_2 \in 2^a\) by making \((Q_1, Q_2)\) equivalent to \((Q_3, Q_4)\) if both appear in the cycle structure of some transvection. When \( k = 1 \) this reduces to the equivalence relation defined in \( \S 2.9. \)

Let \( \cap \) be the set of subsets of \( 2^a \) defined by:

\[(3.33) \quad \cap(Q_1, Q_2) = \{ Q \in 2^a \mid \{ Q_1, Q_2, Q \} \text{ is not coherent} \}, \]
where \( Q_1 \neq Q_2 \in 2^a. \)
Theorem (3.32) has the following important consequence:

\[(3.34) \quad \Gamma = \{ C_a \mid a \in V \}. \]

It now follows that any automorphism of \(O^a(2m)\) induces an automorphism of the design \(\mathcal{g}'\). Moreover it is clear that only the identity of \(\text{Aut}(O^a(2m))\) fixes every block of \(\mathcal{g}'\). Note that \(\mathcal{g}'\) can also be defined as an extension of the tactical configuration formed by the suborbits of \(O^a(2m, 2)\) acting on \(\mathcal{g}^a\) (see D. G. Higman [1]).

\[(3.35) \quad \text{For } \sigma \in H \text{ we have } \sigma C_a = C_{\sigma a} \text{ and } \sigma \not\in a \sigma^{-1} = \not C_a. \]

**Proof.** From (3.31). 

\[(3.36) \quad \text{Theorem. If } a, b \in V, a \neq b, a, b \neq 0, \text{ then} \]

\[
\begin{align*}
|C_a \cap C_b| &= \left\{ \begin{array}{ll}
2^{2m-3} & \text{if } B(a, b) = 0 \\
2^{2m-3} - a, 2^{m-2} & \text{if } B(a, b) = 1
\end{array} \right. \\
B(a, b) &= \begin{cases}
0 & \text{if } B(a, b) = 0 \\
1 & \text{if } B(a, b) = 1
\end{cases}
\]

**Proof.** First suppose that \(B(a, b) = 0\). Let \(C'_a\) and \(C'_b\) denote the complements of \(C_a\) and \(C_b\) in \(\mathcal{D}\), respectively. Operating with a transvection if necessary we may suppose that \(Q_o \in C'_a \cap C'_b\) so that \(Q_o(a) = Q_o(b) = 0\). Now \(Q_c \in C'_a \cap C'_b\) if and only if \(Q_c(a) = Q_c(b) = 0\). This will be the case if and only if \(B(a, c) = B(b, c) = 0\).

Since \(Q_b(a) = Q_b(b) = 0\) we have \(Q_a, Q_b \in C'_a \cap C'_b\) and \(Q_c \in C'_a \cap C'_b\) if and only if \(\{a, b, c\}\) is coherent. Therefore \(|C'_a \cap C'_b| = 3 + k = 2^{2m-3} + a, 2^{m-1}\) and then \(|C_a \cap C_b| = 2^{2m-3}\).

Now suppose that \(B(a, b) = 1\). Again suppose that
$Q_0 \in C'_a \cap C'_b$. Then $Q_a \cap Q_b \not\subseteq C'_a \cap C'_b$ so arguing as above we get $|C'_a \cap C'_b| = P_{11}^2$ where $P_{11}^2$ refers to the graph obtained by deleting a point. From (3.19) we obtain $|C'_a \cap C'_b| = 2^{2m-3} + a \cdot 2^{m-2}$ whence $|C_a \cap C_b| = 2^{2m-3} + a \cdot 2^{m-2}$.

We now define a graph $R = (V : E)$ by joining $C_a$ to $C_b$ whenever $|C_a \cap C_b| = 2^{2m-3}$. By construction $R$ depends only on the 2-graph structure of $G_n^g(2m)$ and any automorphism of the design $g'$ induces an automorphism of $R$. From (3.36) and (3.31) it follows that $R$ is isomorphic to $S(2m)_2$ of § 2. Hence the next theorem follows from Theorem (2.5).

(3.37) **Theorem.**

(a) The automorphism group of $G_n^g(2m)$ is $H = Sp(2m,2)$.

(b) The automorphism group of the design $g$ is $T(V) \cdot Sp(2m,2)$.

We end this section by constructing strongly regular graphs with parameters given by (3.26) when $m$ is even. Now let $V$ denote a vector space of dimension $m$ over $GF(q)$, $m$ even. Suppose that $B$ is a non-degenerate alternating form on $V$ and let $Q$ be the set of quadratic forms $Q$ such that $Q(x+y) + Q(x) + Q(y) = B(x,y)$. By restriction of scalars $V$ becomes a vector space of dimension $2m$ over $GF(2)$, $Tr B$ is a non-degenerate alternating form on $V$ (with respect to $GF(2)$) and if $Q \in Q$, then $Tr Q$ is a quadratic form on $V$. Hence we are back in our original situation.

However we may now define transvections $y_a^b \in Sp(m,4)$ by
\[(3.38) \quad \mathcal{V}_a(x) = x + \mu B(a,x)a \quad \mu \in GF(4), \ a \in V\]

From Dickson [1], p. 197, Sp(m,4) has two orbits $2^1$ and $2^{-1}$ in $2$ consisting of forms of index $\frac{3}{2}m$ and $\frac{1}{2}m^{-1}$ respectively. As above, $T(V).Sp(m,4)$ acts on $2$ and $2$ is an affine space over $GF(2)$. Again choose $Q_0 \in 2^2$ and define $\mathcal{V}_a$ and $\mathcal{V}_b$ as in (3.3). Then from Dembowski [1], p. 46

\[(3.39) \quad |\mathcal{V}_a^n| = (2^{m-a})(2^{m-2} + a).\]

Thus for $\mu \neq 0$

\[(3.40) \quad \left| \left\{ x \in Q_0(x) = \mu \right\} \right| = \frac{1}{3}(2^m - 1) = 2^{2m-2} \cdot 2^{m-2}.\]

As in (3.12) we obtain

\[(3.41) \quad \mathcal{V}_a^n = t_{\beta_a} Q \quad \text{where} \quad \beta^2 = \mu^2 Q(a) + \mu.\]

Hence $(\mathcal{V}_a^n Q)(\beta_a) = Q(\beta_a) = Tr(\mu Q(a))$.

Conversely, if $Q_0(a) = Q_a(a) \in GF(2)$, then we can choose $\mu$ so that $\mathcal{V}_a^n Q_0 = Q_a$. We have therefore proved:

\[(3.42) \quad 2^c = \left\{ Q_a \mid Q_0(a) \in GF(2) \right\}\]

\[(3.43) \quad 2^{-c} = \left\{ Q_a \mid Q_0(a) \notin GF(2) \right\}.\]

Now suppose that $\mathcal{V}_a^n \in K_0 \cap K$ where $K = Sp(m,4)$ and $K_0$ is the subgroup of $T(V).K$ fixing $Q_0$. From (3.41) we have $\mu^2 Q_0(a) + \mu = 0$. Therefore, if $Q_b \in 2^c$, then $\mathcal{V}_a^n Q_b = Q_c$ where $c = \mathcal{V}_a^n(b)$ and then $Q_0(c) = Q_0(b)$. It
follows that $K_0 \cap K = 0^a(m,4)$ has at least two orbits on each of $2^{-a}$ and $2^a - \{Q_0\}$.

Again consider the design $g$ whose point set is $2$ and whose blocks are translates of $2^{-a}$. The permutation character $\pi$ of $K$ acting on blocks is the same as that of $K$ acting on $V$. Since $K$ has rank $3$ on the projective space of $V$ (over $GF(4))$ we see that $K$ has rank $7$ on the non-zero points of $V$. By the result of Brauer in § 3.1 the permutation character of $K$ acting on $V$ is again $\pi$. If the non-principal irreducible characters of $\pi$ all appear with multiplicity $1$ then $K_0 \cap K$ would act transitively on $2^{-a}$ which is not the case. Hence $\pi = 2Q_0 + 2Q_1 + Q_2 + Q_3$ where $Q_0$ is the principal character and $Q_1$, $Q_2$ and $Q_3$ are irreducible. We may choose the notation so that the characters of $K$ acting on $2^1$ and $2^{-1}$ are $Q_0 + Q_1 + Q_2$ and $Q_0 + Q_1 + Q_3$ respectively. Hence

(3.43) $K$ acts as a rank $3$ group on both $2^1$ and $2^{-1}$.

From (3.39), (3.40) and (3.41) we have

(3.44) $|2^a| = 2^{2m-1} + a \cdot 2^{m-1}$

The orbits of $K_0 \cap K$ on $2^a$ are

(3.45) $\{Q_0\}, \{Q_a \mid Q_0(a) = 0\}, \{Q_a \mid Q_0(a) = 1\}$.

We use the rank $3$ action of $K$ to define a strongly regular graph on $2^a$ as in § 1.1. In terms of the quadratic forms we join $Q_a$ to $Q_b$ whenever $Q_a(a+b) = 0$.

Now define a three element subset of $2^a$ to be coherent
whenever it contains an odd number of edges of the graph (as in (2.2.5)).

If \( Q_a \) and \( Q_b \) are elements of \( 2^a \), then \( Q_o(a), Q_o(b) \in GF(2) \) and \( Q_a(a+b) = Q_o(a) + Q_o(b) + \text{Tr}B(a,b) \). Hence \( \{Q_o, Q_a, Q_b\} \) is coherent if and only if \( \text{Tr}B(a,b) = 0 \). It follows that we have constructed the regular 2-graph \( O^a(2m) \). From (2.6.1) and (3.39) the parameters of the strongly regular graph are

\[
(3.46) \quad n = 2^{2m-1+α,2^m-1}
\]

\[
n_1 = 2^{2m-2+α,2^m-2} \quad p_1^1 = 2^{2m-3+α,2^m-2-2}
\]

\[
n_2 = 2^{2m-2+α,2^m-2} \quad p_1^2 = 2^{2m-3+α,2^m-1}.
\]

We have completed the construction of the graph promised after (3.26).

**Remark.** We may carry out the above construction with the field \( GF(2^e) \) in place of \( GF(4) \). We again get the projective design \( \mathcal{A} \) from the set of points \( 2 \) and blocks \( t_a2^a \). In the case \( 2^e = 8 \) the group \( \text{Sp}(2m,8) \) has rank 3 representations on \( 2^1 \) and \( 2^{-1} \) but the strongly regular graphs do not belong to \( \mathcal{S}(O^a(2m)) \).

Another approach to this question is to begin with a vector space \( W \) of dimension \( 2m+1 \) over \( GF(2^e) \) and a non-degenerate quadratic form \( Q \) on \( W \) whose associated bilinear form is \( F \). The group \( O(2m+1,2^e) \) is isomorphic to \( \text{Sp}(2m,2^e) \). If \( W_0 = \langle e_0 \rangle \) is the radical of \( W \) there are two possibilities for the restriction of \( Q \) to a hyperplane not containing \( W_0 \).
Thus in the projective space of \( W \), \( \text{Sp}(2m,2^e) \) has two orbits on hyperplanes not containing the point \( W_0 \) and in the dual situation \( \text{Sp}(2m,2^e) \) has two orbits on an affine space of dimension \( 2m \) over \( GF(q^e) \). We may identify this affine space with \( \mathbb{Z} \) above and the orbits are then \( 2^1 \) and \( 2^{-1} \). In particular, we see that the design constructed in Dembowski [1], p. 95 is the design \( g \) and the full automorphism group is \( T(V), \text{Sp}(2e,2) \).

4. Unitary geometry

In this section we shall construct, for each odd prime power \( q \), a regular 2-graph \( U(q) \) which admits the group \( \text{PGL}(3,q^2) \) as a doubly transitive group of automorphisms.

Let \( K \) be the field \( GF(q^2) \) and let \( V \) be a three-dimensional vector space over \( K \). Then \( K \) has an automorphism \( x \to x^q \) whose fixed field is \( K_1 = GF(q) \). We shall suppose that \( q = p^a \) where \( p \) is an odd prime. Choose a basis \( e_1, e_2, e_3 \) for \( V \). If \( x = x_1e_1 + x_2e_2 + x_3e_3 \) and \( y = y_1e_1 + y_2e_2 + y_3e_3 \), then define

\[
(4.1)\quad s(x, y) = x_1\bar{y}_3 + x_2\bar{y}_2 + x_3\bar{y}_1.
\]

Then \( s \) is a non-degenerate sesquilinear form. Given a subspace \( U \subseteq V \) we define the unitary complement of \( U \) to be

\[
(4.2)\quad U^\perp = \{ v \in V \mid s(u, v) = 0 \quad \text{for all} \quad u \in U \}.
\]

Since \( s \) is non-degenerate \( \dim U + \dim U^\perp = 3 \) and the map \( u : U \to U^\perp \) defines a unitary polarity of the projective
plane $\mathbb{P}(V)$. A subspace $U$ of $V$ is **absolute** if $U \subseteq U^W$ or $U^W \subseteq U$. We denote the set of absolute points of $\mathbb{P}(V)$ by $\mathfrak{A}$.

The vector $x$ represents an absolute point of $\mathbb{P}(V)$ if and only if $s(x,x) = 0$. Taking $x_1, x_2, x_3$ as homogeneous coordinates in $\mathbb{P}(V)$, let $L_\infty$ be the line with equation $x_1 = 0$ and let $\infty$ be the point $(0,0,1)$. Then $L_\infty$ determines an affine plane $E$ with coordinates $(x,y)$ where $x = x_2 x_1^{-1}$, $y = x_3 x_1^{-1}$. In this notation we have

$$(4.3) \quad \mathfrak{A} = \{ \infty \} \cup \{ (x,y) \mid N(x) + \text{Tr}(y) = 0 \}$$

where $N$, $\text{Tr} : K \to K$ are the norm and trace respectively. Since $|\ker \text{Tr}| = q$ we have

$$(4.4) \quad |\mathfrak{A}| = q^{3+1}. $$

$$(4.5) \quad \text{Proposition.} \quad \text{Any line through } \infty \text{ distinct from } L_\infty \text{ intersects } \mathfrak{A} \text{ in } q \text{ points other than } \infty.$$

**Proof.** In affine coordinates such a line has an equation $a + bx = 0$ with $b \neq 0$. We then have $q$ choices for $y$ such that $\text{Tr}(y) + N(x) = 0$. //

The group of linear transformations of $V$ leaving $s$ invariant is $\text{GU}(3,q^2)$. The factor group $\text{PGU}(3,q^2)$ of $\text{GU}(3,q^2)$ by its centre acts as a group of permutations of $\mathfrak{A}$. In describing $\text{PGU}(3,q^2)$ we follow Huppert [1], p. 242.

Since $\infty^W = L$ the subgroup $N$ of $\text{PGU}(3,q^2)$ which fixes $\infty$ acts as a transformation of $E$ and we may describe

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the elements of \( N \) by their action on \( E \).

Put \( Q = \{ Q(a,b) \mid N(a) + \text{Tr}(b) = 0 \} \) where

\[
(4.5) \quad (x,y)^{Q(a,b)} = (x+a, y+b - \bar{a}x)
\]

and \( H = \{ H(k) \mid k \neq 0 \} \) where

\[
(4.6) \quad (x,y)^{H(k)} = (\bar{kx}, \bar{ky}).
\]

Then \( Q \) is an \( Sp \)-subgroup of \( G \), \( H \) is a cyclic group of order \( q^2-1 \) and \( N = Q \cdot H \) is the normaliser of \( Q \) in \( G \). The structure of \( N \) is determined by the following equations.

\[
(4.7) \quad Q(a,b)Q(c,d) = Q(a+c, b+d-\bar{c}a)
\]

\[
(4.8) \quad H(k) \cdot H(m) = H(km)
\]

\[
(4.9) \quad H(k)^{-1} \cdot Q(a,b) \cdot H(k) = Q(\bar{ka}, \bar{kb}).
\]

We define a further element of \( \text{PGU}(3, q^2) \) by the (projective) matrix

\[
(4.10) \quad \omega = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\]

We have \( 0^\omega = \infty \) and \( \infty^\omega = 0 \), where \( 0 = (0,0) \in E \).

If \( y \neq 0 \), then \( (x,y)^\omega = (-xy^{-1}, y^{-1}) \). Since \( Q \) acts regularly on \( \mathbb{F}_q \setminus \{ \infty \} \) we see that \( \text{PGU}(3, q^2) \) acts doubly transitively on \( \mathbb{F}_q \).

Suppose that \( p_1, p_2 \) and \( p_3 \) are distinct points of \( \mathbb{F}_q \).
and $x, y$ and $z$ are elements of $V$ representing $p_1, p_2$ and $p_3$ respectively. If $q \equiv -1 \pmod{4}$ we define $\{p_1, p_2, p_3\}$ to be coherent whenever $s(x, y)s(y, z)s(z, x)$ is a square in $K$, while if $q \equiv 1 \pmod{4}$ we define $\{p_1, p_2, p_3\}$ to be coherent whenever $s(x, y)s(y, z)s(z, x)$ is not a square in $K$. Let $\mathcal{T}$ be the set of all coherent three-element subsets of $\Omega$.

Since $kk$ is always a square and $\overline{k}$ is a square whenever $k$ is a square the definition of coherence does not depend on the representatives or the order chosen for the points of $\Omega$.

(4.11) Theorem. $U(q) = (\Omega, \mathcal{T})$ is a regular 2-graph with the following parameters:

$N = q^3 + 1$

$a = \frac{1}{2}(q-1)(q^2+1)$

$a' = \frac{1}{2}(q+1)(q^2-1)$

$k = \frac{1}{4}(q^3 - 3q^2 + 3q - 5)$

$k' = \frac{1}{4}(q^3 + 3q^2 - 3q - 5)$

$\rho_1 = q^2$

$\mu_1 = q^2 - q + 1$

$\rho_2 = -q$

$\mu_2 = q(q^2 - q + 1)$

Proof. Since $\text{PGU}(3, q^2)$ acts doubly transitively on $\Omega$ and preserves coherence it is immediate that any two-element subset of $\Omega$ is contained in the same number of elements of $\mathcal{T}$. Moreover in order to check condition (b) of the definition of regular 2-graph we may take the four-element set to consist of $\infty, o, p_1, p_2$ with representatives $e_3, e_4, x, y$. Since the product of two non-squares is a square it is clear that $\{\infty, o, p_1, p_2\}$ can contain only
an even number of elements of $\mathfrak{t}$. Finally, $\mathfrak{t}$ is neither empty nor the set of all three-element subsets of $\mathfrak{a}$.

Hence $U(q)$ is a regular 2-graph. Since $s(e_3, x) = \overline{x_1}$ the strongly regular graph $U(q)\infty$ may be described as the graph on $\mathfrak{a} = \{\infty\}$ obtained by joining $p_1$ to $p_2$ whenever $s(x, y)$ is a square in $K$ if $q = -1 \pmod{4}$ or a non-square in $K$ if $q \equiv 1 \pmod{4}$. (Cf. Remark 1, § 2.2.)

Suppose that $q = -1 \pmod{4}$. The number of coherent triangles containing $\{\infty, 0\}$ is the number of points $(x, y)$ in $E$ such that $y$ is a square and $N(x) + \text{Tr}(y) = 0$. Since $q = -1 \pmod{4}$ every element $y \in K$ such that $\text{Tr}(y) = 0$ is a square. Also, $\text{Tr}: K^+ \to K_1^+$ and $N: K \to K_1$ are homomorphisms so the number of coherent triangles is $(\frac{1}{2}(q^2-1)-(q-1))(q+1)+(q-1) = \frac{1}{2}(q-1)(q^2+1)$.

Similarly, if $q = 1 \pmod{4}$ the number of incoherent triangles containing $\{0, \infty\}$ is $\frac{1}{2}(q^2-1)(q+1)$ since in this case if $y$ is a square then $\text{Tr}(y) \neq 0$. Since $N = q^3+1$ the remaining parameters of $U(q)$ can be calculated from the equation of § 2.2. //

We have defined $U(q)$ so that $\{\infty, 0, (x, y)\}$ is coherent if $\text{Tr}(y) = 0$. Thus $\{\infty, 0\} \cup \{(o, y) \mid \text{Tr}(y) = 0\}$ is a coherent set of $q+1$ points. From (2.7.9) it is a maximal coherent set.

Let $X$ be the Galois group of $K$ (over $\text{GF}(p)$). If $\sigma \in X$ we define the action of $\sigma$ on $V$ by $(x_1, x_2, x_3)\sigma = (x_1\sigma, x_2\sigma, x_3\sigma)$. Thus $s(x, y)\sigma = s(x\sigma, y\sigma)$ and $\sigma$ therefore induces an automorphism of $U(q)$. It follows that the
semi-direct product $P \cong U(3, q^2) = PGU(3, q^2).X$ is contained in Aut(U(q)). However, $P \cong U(3, q^2)$ need not be the full automorphism group of U(q). For example, when $q = 3$ the parameters of U(3) are $N = 28$, $a = 10$, $k = 1$ and in Chapter 5 we shall show that there is just one regular 2-graph with these parameters. Hence $U(3) \simeq O^- (6)$ and Aut(U(3)) $\simeq Sp(6, 2)$.

An interesting consequence of this is the embedding $P \cong U(3, 3^2) \subseteq Sp(6, 2)$. This is also a consequence of the facts: $P \cong U(3, 3^2) \simeq G_2(2)$, $Sp(6, 2) \simeq O(7, 2)$, $G_2(2)$ is the group of automorphisms of the Cayley algebra over $GF(2)$ and hence $G_2(2) \subseteq O(7, 2)$.

Using the double transitivity of $P \cong U(3, q^2)$ on $\Omega$ we can easily describe the geometry of $\Omega$ and $P(V)$. Through each point $p$ of $\Omega$ there is just one absolute line (viz. $p^\pi$). The other $q^2$ lines through $p$ intersect $\Omega$ in $q+1$ points (by (4.5)). Since any two points of $\Omega$ determine a unique line we obtain $q^2(q^2 - q + 1)$ (non-absolute) lines that meet $\Omega$ in $q+1$ points and since $(q^3 + 1) + q^2(q^2 - q + 1) = q^4 + q^2 + 1$ we have accounted for all the lines of $P(V)$. It follows that the points $\Omega$ and the non-absolute lines $\Sigma$ form a design $\mathcal{D}$ with parameters

\begin{equation}
(4.12) \quad v = q^3 + 1, \quad b = q^2(q^2 - q + 1), \quad r = q, \quad k = q + 1, \quad \lambda = 1.
\end{equation}

Since $G$ acts regularly on $\Omega - \{\infty\}$ we identify the point $(a, b)$ of $\Omega - \{\infty\}$ with $Q(a, b)$. From (4.7) the map $\sigma : Q(a, b) \mapsto a$ is a homomorphism of $Q$ onto $K$ and $\ker \sigma = \{ Q(o, b) \mid Tr(b) = 0 \} = Z(Q)$. We may identify $H$ with the multiplicative group of $K$ and the action of $H(k)$
on \( \mathbb{Q}/\mathbb{Z}(Q) \sim K \) is a \( \sim \mathbb{K}a \) (from (4.9)). The action of 
\( H(k) \) on \( \mathbb{Z}(Q) \) is \( Q(o,b) \rightarrow Q(o,kkb) \). Since both these 
actions of \( H \) are irreducible we have \( \mathbb{Z}(Q) = 1 \) \( \mathbb{Q} \) = \( \mathbb{Q} \) \nand it is easy to check that \( Q \) has exponent \( p \). Thus \( Q \) is 
a special group of exponent \( p \). Choose a fixed element \( \neq o \) 
such that \( e + e = o \). If \( b+b = o \), then \( b \in K_1 \). Thus the 
map \( \tau : \mathbb{Z}(Q) \rightarrow K_1: Q(o,b) \rightarrow b \) identifies 
\( \mathbb{Z}(Q) \) with \( K_1 \). 
From (4.7) we have 

\[
(4.13) \quad [Q(a,b),Q(c,d)] = Q(o,\overline{ac}-a\overline{c}).
\]

Using our maps \( \sigma \) and \( \tau \) and regarding \( K \) as a two-
dimensional space of \( K_1 \) we obtain a non-degenerate alternating 
form \( f : K \times K \rightarrow K_1 \) defined by 

\[
(4.14) \quad f(a,b) = (\overline{ab}-a\overline{b})e.
\]

The structure of \( \Omega \) may be recovered from the map \( f \).

If \( u,v \in \Omega \) let \( \ell_{u,v} \) denote the other points of \( \Omega \) which 
are incident with the block determined by \( u \) and \( v \). Since 
the block determined by \( o \) and \( \infty \) is the line of \( E \) with 
equation \( x = o \) we have \( \ell_0,\infty = \{(o,y) \mid y + \overline{y} = o\} \), and 
we may identify \( \ell_0,\infty \) with the non-identity elements of 
\( \mathbb{Z}(Q) \). Also, \( \ell_0,\infty \) is the orbit of \( H \) of length \( q-1 \), the 
remaining non-trivial orbits of \( H \) having lengths \( q^2-1 \). Since 
\( P\Gamma U(3,q^2) \) is doubly transitive on points it is transitive 
on blocks. The block \( \{o,\infty\} \cup \ell_0,\infty \) was previously noted 
to be a maximal coherent set of \( U(q) \). Hence all blocks of 
\( \chi \) are maximal coherent sets. The converse is not true.
However in the remainder of this section we shall show that in the case of $U(5)$ it is possible to use the 2-graph structure to determine which coherent sets are blocks.

(4.15) Theorem. Any automorphism of $U(5)$ is an automorphism of the design $X$.

Proof. We consider coherent sets containing $0$ and $\infty$. The subgroup of $PGL(3,5^2)$ fixing $0$ and $\infty$ is $H_X$ of order 48. Its orbits on the remaining points have lengths 4, 24, 48, 48. Let $D_1$ be the orbit of length 4 and $D_2$ the orbit of length 48 consisting of points $(x,y)$ such that $y$ is not a square in $K$. Then $p \in D_1 \cup D_2$ if and only if $\{0, \infty, p\}$ is coherent.

Let $B_0 = \{0, \infty\} \cup D_1$. Then $B_0$ is the block containing $0$ and $\infty$. Choose $p_1 = (0, y_1) \in D_1$ and let $B$ be the set of points $p \in \Omega$ such that $\{0, \infty, p_1, p\}$ is a coherent set of 4 points. From (4.11) $|B| = k = 15$. We have $p_2 = (x_2, y_2) \in B$ if and only if $y_2$ and $y_2 + y_1$ are both non-squares. It follows that $p_2 \in B$ if and only if the block with equation $y = y_2$ is contained in $B$.

Let us write $B_{01} = \{0, \infty, p_1\}$ and $B_{02} = B_0 - B_{01}$. Then $B_{02} \leq B$ and the remaining points of $B$ must lie in two disjoint blocks $B_1$ and $B_2$. In the notation of (2.7.1) we have, for $p \in B_1 \cup B_2$, $B_{01} = B_{01}(p)$ and $B_{02} = B_{02}(p)$. Now define $B_{1j} = B_{1j}(\infty)$ where $1 \leq i, j \leq 2$. It is easy to see that for any $p \in B_0$ we have $B_{1j} = B_{1j}(p)$, $1 \leq i, j \leq 2$ and as in the proof of (2.9.1), the union of any two of the sets $B_{1j}$, $0 \leq i \leq 2$, $1 \leq j \leq 2$, is coherent. Moreover,
any three-element set whose elements come from distinct sets $B_i$ is incoherent.

Now choose a point $p_2 \in D_2$. We shall show that the configuration formed by the set of points

$$C = \{ p \in \mathbb{P} | \{ o, \infty, p_2, p_1 \} \text{ is coherent} \}$$

does not have the same structure as that above. Choose $a \in \text{GF}(25)$ such that $a^2 = a + 3$. Then $a$ generates the multiplicative group of $\text{GF}(25)$ and $\overline{a} = 4a + 1$. We may suppose that $p_2 = (4a + 1, 3a)$ since $3a = a^{19}$ is not a square and $p_2 \neq o$.

Consider the coherent set $M = \{ o, \infty, p_2, p_3 \}$ where $p_3 = (2a, 2a)$. The fifteen elements of $M$ which form a coherent set with $\{ o, \infty, p_2 \}$ are:

$$(4, 16), (0, a + 2), (4a + 2, a), (4a + 1, 2a + 3), (4a + 3, 2a), (3a + 4, 4a), (3a + 1, 2a + 3), (2a, 2a), (2a + 4, 3a), (4a + 4, 4a + 1), (4a + 4, a), (1, 4a), (3a + 3, a + 4), (4a + 3, a), (2a + 2, a + 4), (a + 1, 4a + 1).$$

It is now easily checked that $M$ is a maximal coherent set. However the configuration of the $B_i$'s above does not contain any maximal coherent set of $4$ elements. It follows that the block containing $o$ and $\infty$ can be determined from the 2-graph structure. Hence any automorphism of $U(5)$ is an automorphism of $\mathbb{X}$. //

(4.17) **Corollary.** $\text{Aut}(U(5)) = \text{PG}U(3, 5^2)$.

**Proof.** We shall say that the lines $L$ and $M$ of $\mathbb{P}(V)$ are **orthogonal** if $L^M$ lies on $M$. If $L$ and $M$ are orthogonal we
shall write \( L \perp M \). The block \( B_0 \) above corresponds to the non-absolute line \( L_0 \) whose equation in \( E \) is \( x = 0 \). Hence \( L_0 = \langle e_2 \rangle \) and the lines orthogonal to \( L_0 \) have equations \( y = a \) for some \( a \in \mathbb{K} \). Such a line is non-absolute if and only if \( \text{Tr}(a) \neq 0 \). Let us call the corresponding maximal coherent sets the blocks orthogonal to \( B_0 \). It follows that \( B_0 \) together with the 20 blocks orthogonal to it form a partition of \( \mathcal{S} \). We have seen above that to each partition of \( B_0 \) into a pair of three-element subsets there is a pair of disjoint blocks. If \( o \) and \( \infty \) lie on the same side of the partition, the blocks have equations \( y = a \) where \( a \) is a non-square and \( \text{Tr}(a) \neq 0 \). If \( o \) and \( \infty \) lie on different sides of the partition the blocks have equations \( y = a \) where \( a \) is a square.

Hence the 10 partitions of \( B_0 \) determine the 20 blocks orthogonal to \( B_0 \). Since we may determine the blocks from the 2-graph structure it follows that we may also determine the orthogonality relation from the 2-graph structure.

We are now able to reconstruct the projective plane \( \mathbb{P}(V) \) from the 2-graph \( \Gamma(5) \). Let \( \mathcal{T} \) be the set consisting of the points of \( \mathcal{S} \) and the symbols \([L]\) where \( L \) is a non-absolute line and let \( \wedge \) be the set of non-absolute lines and the symbols \([p]\) where \( p \in \mathcal{S} \). We define an incidence relation \( I \subseteq \mathcal{T} \times \wedge \) by:

\[
\begin{align*}
(h, 18) & \quad p I [p'] & \text{if} \ p = p', \quad & p, p' \in \mathcal{S} \\
p I L & \quad & \text{if} \ p \in L \quad & p \in \mathcal{S}, \ L \in \mathcal{S} \\
[L] I [p] & \quad & \text{if} \ p \in L \quad & p \in \mathcal{S}, \ L \in \mathcal{S} \\
[L] I L' & \quad & \text{if} \ L \perp L' \quad & L, L' \in \mathcal{S}.
\end{align*}
\]
It is clear that we may identify $\Pi$ with the points of $\mathbb{P}(V)$, $\Lambda$ with the lines of $\mathbb{P}(V)$ and $I$ with the usual incidence relation. Also, the polarity $n$ is given by $n: p \rightarrow [p]$ and $[L] \rightarrow L$. It follows that any automorphism of the regular 2-graph $U(5)$ induces an automorphism of $(\Pi, \Lambda, I)$ which commutes with $\pi$. Hence $\text{Aut}(U(5)) = \text{PGL}(3, 5^2)$.

Remarks. 1. It is possible to show in general that the orthogonality relation is uniquely determined by the incidence structure of the design $\mathcal{X}$. Hence the key point in extending (4.17) to all $U(q)$, $q \neq 3$ is to show that the blocks of $\mathcal{X}$ can be obtained from the 2-graph structure.

2. The 2-graphs $U(q)$ provide a new infinite family of equiangular lines in $q^2 - q + 1$ dimensions. The strongly regular graphs $U(q)_\infty$ are also new.

5. Groups of Ree type

Let $G$ be a group of Ree type (satisfying the conditions of Ward [1]) and let $Q$ be an $S_3$-subgroup of $G$. Then $|Q| = q^3$ where $q$ is an odd power of 3. The group $N = N_G(Q)$ is the semi-direct product of $Q$ by a cyclic group $H$ of order $q - 1$. The representation of $G$ on the set $\mathcal{S}$ of cosets of $N$ is doubly transitive of degree $q^3 + 1$. Since $G$ has no subgroup of index 2 it satisfies the conditions of Theorem (1.2) and therefore $G$ has a monomial representation $M$ of degree $q^3 + 1$ with either one or two irreducible constituents. It follows easily from the character table of $G$ in Ward [1].
that $M$ has two irreducible constituents with degrees $q^2-q+1$ and $q(q^2-q+1)$. Hence part (b) of (1.2) applies and there exists a regular 2-graph $R(q) = (\emptyset, \emptyset)$ which contains $G$ in its automorphism group. Since the multiplicities of the eigenvalues of an adjacency matrix of a strong graph in $\mathcal{L}(R(q))$ are $q^2-q+1$ and $q(q^2-q+1)$ it follows that the parameters of $R(q)$ are the same as those of $U(q)$, (see (4.11)).

Let $j$ be the unique involution of the group $H$. It can be seen from the character table of Ward [1] that $j$ has exactly $q+1$ fixed points in $\Omega$. We may therefore construct a block design $\mathcal{X}$ as in Lüneburg [1]: the points of $\mathcal{X}$ are the elements of $\Omega$, the blocks are the sets of fixed elements of involutions fixing at least one element. The parameters of $\mathcal{X}$ are the same as those of the design $\mathcal{X}$ of §4. It follows as in Lüneburg [1] that the stabiliser of the block $B$ fixed by $j$ is $G(j) = \langle j \rangle \times K$ where $K \simeq \text{PSL}(2,q)$ and all involutions of $G$ are conjugate. Since $K$ acts doubly transitively on $B$ either $B$ is coherent or the restriction of the 2-graph $R(q)$ to $B$ is a regular 2-graph of type (a). But $q \equiv 3 \pmod{4}$ so this last possibility cannot occur. Hence $B$ is a coherent set. Since $G$ acts transitively on blocks it follows that every block of $\mathcal{X}$ is a (maximal) coherent set.

The existence of the Hee groups $^2G_2(q)$ implies the existence of a family $R(q)$ of regular 2-graphs having the same parameters as the regular 2-graphs $U(q)$. The group $^2G_2(3)$ is isomorphic to $\Sigma L(2,8)$ and $R(3)$ is a regular 2-graph with 28 points. By a result in Chapter 5 $R(3) \simeq U(3) \simeq O^{-1}(6)$
so $\text{Aut}(\text{H}(3)) \cong \text{Sp}(6,2)$ and $\Sigma L(2,8) \subseteq \text{Sp}(6,2)$ as was proved in §2. In fact, $^2G_2(3)$ is the only group of Ree type with $q = 3$ (see Janko [1]). It is interesting to note that the designs $x$ and $y$ associated with $U(3)$ and $R(3)$ are not isomorphic since it is easy to see that $\text{Aut}x = P^f U(3,3^2)$ (using the methods of the previous section). Both these designs satisfy the conditions of §3.3 and consequently give rise to regular 2-graphs with the parameters of $S(6)$ of §2.

6. The groups $G_2$ and $R(3)$

Suppose that $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^F$ which span equiangular lines with angle $\cos^{-1} a$. If equality holds in (2.4.6) then as in §2.4 we may make the set $\Omega$ of equiangular lines into a regular 2-graph by defining $\{<v_i>, <v_j>, <v_k>\}$ to be coherent whenever

$$\langle v_i, v_j \rangle \langle v_j, v_k \rangle \langle v_k, v_i \rangle < 0.$$  

(6.1)

We shall use this construction to obtain several regular 2-graphs with $\rho_2 = -a = -5$. The parameters of these 2-graphs must therefore occur in (2.5.6).

As general references for this section we use Conway [1] and Todd [1].

Let $\triangle = \{0, 1, \ldots, 23\}$ and let $\mathcal{B}$ be the set of blocks of the Steiner system $S(5,8,24)$ defined on $\triangle$. Let $\mathcal{B}_4$ be spanned by the orthonormal basis $e_i, i \in \triangle$ and define $e_s$ as $\sum_{i \in s} e_i$ whenever $s \subseteq \triangle$. The Leech lattice $\Lambda$
is the lattice spanned by the vectors $2e_B$, $B \in \mathcal{B}$, and $e_\Delta - 4e_o$. The group $G_3$ (in Conway's notation, $\cdot 3$) can be taken to be the subgroup of the orthogonal group of $\mathbb{R}^{24}$ which preserves $\bigwedge$ as a whole and fixes $4e_o + e_\Delta = x$.

There are 276 unordered pairs $\{y,z\}$ such that $x = y + z$, $y, z \in \bigwedge$ and $(y,y) = (z,z) = 32$: namely, 23 with $y$ of the form $4e_o + 4e_1$, $1 \leq i \leq 23$, and 253 with $y$ of the form $2e_B$ where $B \in \mathcal{B}$ and $o \in B$. The vectors $2y - x$ lie in the hyperplane orthogonal to $x$ and the lines spanned by these vectors are clearly permuted amongst themselves by $G_3$.

This set $\Omega$ of 276 vectors in $\mathbb{R}^{23}$ consists of:

$$(6,2) \quad v_i = 8e_i + 4e_o - e_\Delta \quad 1 \leq i \leq 23,$$

so small projective spaces.

The blocks of $\mathcal{B}$ which contain $o$ constitute the Steiner system $S(4,7,23)$ and any two distinct blocks which contain $o$ intersect in either 2 or 4 points. Hence the vectors of $\Omega$ have the following inner products:

$$(6,3) \quad (v_i, v_j) = 16 \quad i \neq j,$$

$$(v_i, v_B) = \begin{cases} 16 & i \in B \\ -16 & i \notin B \end{cases}$$

$$(v_B, v_{B_2}) = \begin{cases} 16 & |B_1 \cap B_2| = 4 \\ -16 & |B_1 \cap B_2| = 2 \end{cases}$$

$$(v, v) = 80 \quad v \in \Omega$$

Thus we have obtained 276 equiangular lines in $\mathbb{R}^{23}$. The group $G_3$ is isomorphic to the group $\text{PSL}(3,4)$ and the subgroup stabilizing $o$ acts on the orbit of length 162 in $\text{PSL}(3,4)$ with any other length at least 103...
with angle $\phi$ where $\cos \phi = \frac{1}{2}$. By (6.1) we have a regular 2-graph $C(276)$ defined on $\Omega$ with parameters

$$\begin{align*}
(6.4) & \quad N = 276 \\
& \quad a = 112 \quad a' = 162 \\
& \quad k = 30 \quad k' = 105 \\
& \quad \rho_1 = 55 \quad \mu_1 = 23 \\
& \quad \rho_2 = -5 \quad \mu_2 = 253
\end{align*}$$

It is clear that the structure of $C(276)$ depends only on $S(4,7,23)$ and that the group $M_{23}$ which fixes $0$ can be considered as a group of automorphisms of $C(276)$ with orbits $D = \{ v_1 \mid 1 \leq i \leq 23 \}$ and $E = \{ v_B \mid 0 \in B \}$. The set $D$ is a maximal incoherent set. In the next chapter we shall prove that $C(276)$ is the only regular 2-graph on 276 points with an incoherent set of size 23. We anticipate this result here in order to obtain the detailed structure of $C(276)$.

By our construction the group $C_3$ is contained in $\text{Aut}(C(276))$. The group $M^0$ of McLaughlin [1] was defined as the subgroup of index 2 in the automorphism group $\overline{M}$ of a strongly regular graph $R$ with parameters $n_1 = 112$, $n_2 = 162$, $p_{11} = 30$ and $p_{11} = 56$. By (2.2.3) we may construct a regular 2-graph $T$, with parameters (6.4), by adjoining a point to $R$. We next show that $T$ contains an incoherent set of 23 points. The stabiliser of a point in $R$ is the group $\text{PSU}(4,2)$ and the subgroup stabilising a point in the orbit of length 162 is $\text{PSL}(3,4)$ with orbits of length 1, 56 and 105. The group $\text{PSL}(3,4)$ acts on the orbit of length 105.
as it does on the flags of the 21 point plane $\Pi$, a flag $(p, L)$ being joined in $R$ to a flag $(q, M)$ if and only if one of the following holds:

\begin{align*}
(6.5) \quad (i) & \quad p \in M, \, q \notin L \\
(\text{ii}) & \quad p \notin M, \, q \in L.
\end{align*}

Select a point $o \in \Pi$, let $L_1, \ldots, L_5$ be the lines through $o$ and let $p_{11}, \ldots, p_{14}$ be the remaining points of the line $L_1, \ 1 \leq i \leq 5$. Then no two of the flags $(p_{1j}, L_1)$ are joined. These 20 flags together with the 2 points fixed by $\text{PSL}(3,4)$ and the point adjoined to $R$ form an incoherent set in $T$. It follows that we may identify $T$ with $C(276)$. In particular, $\overline{\text{H}}$ is the stabiliser of a point in $\text{Aut}(C(276))$ and by comparing orders we obtain $G_3 = \text{Aut}(C(276))$.

Let $y = 4e_0 - 4e_1 \in \Lambda$ and let $P$ be the set of elements of $\Lambda$ orthogonal to $y$; then $P$ consists of

\begin{equation}
(6.6) \quad v_B = 4e_B - e_o - 4e_1, \quad B \in \mathcal{B}, \ o \in B, \ 1 \notin B.
\end{equation}

Since there are 176 blocks containing $o$ and omitting $1$ we have found 176 equiangular lines in $\mathbb{R}^{22}$ with angle $\phi$. Using (6.1) we obtain a regular 2-graph $H(176)$ defined on $P$ with the following parameters:

\begin{align*}
(6.7) \quad N &= 176 \\
a &= 72 & a' &= 102 \\
k &= 20 & k' &= 65 \\
\rho_1 &= 35 & \mu_1 &= 22 \\
\rho_2 &= -5 & \mu_2 &= 154.
\end{align*}
The subgroup of $\text{M}_{24}$ fixing 0 and 1 is $\text{M}_{22}$ which is 3-fold transitive on the remaining points of $\triangle$ and has an orbit of length 77 on $B$ consisting of the blocks containing 0 and 1 and an orbit of length 176 consisting of the blocks containing 0 but not 1. If we join two vectors $u,v \in \mathbb{R} - \{v_1\}$ whenever $(u,v) = -16$ and then join $v_1$ to the remaining vectors of $D$ we obtain the Higman-Sims' graph (see D.G. Higman and C. Sims [1]) with $\text{M}_{22}$ acting on the graph and stabilising $v_1$.

From Theorem 6 of Conway [1] the subgroup of $\text{M}_{22}$ fixing a point in $\mathbb{P}$ is $A_7$ and the representation of $\text{M}_{22}$ on $\mathbb{P}$ has rank 3. The corresponding strongly regular graph has parameters $n_1 = 70$, $n_2 = 105$, $p_{11} = 18$ and $p_{12} = 34$ and therefore belongs to $\mathcal{B}(\text{H}(176))$, (see M. Smith [1]). Now let $\mathcal{Q}$ be the set of vectors

$$(6,8) \quad w_B = 4\mathbb{e}_B - 4\mathbb{e}_B - 4\mathbb{e}_A - 4\mathbb{e}_A, \quad B \in B, \quad 0 \notin B, \quad 1 \in B.$$ 

It is clear that $\text{M}_{22}$ acts transitively on this set and that the stabiliser of a point is $A_7$. Moreover, there is an involution in $\text{M}_{24}$ interchanging 0 with 1, $\mathbb{P}$ with $\mathcal{Q}$, inducing an outer automorphism of $\text{M}_{22}$ and fusing the two classes of $A_7$'s. Let $K$ be the $A_7$ in $\text{M}_{22}$ fixing $w_B \in \mathcal{Q}$. As in Conway [1], p. 228 we may identify the action of $K$ on $\mathbb{P} \cup B$ with its action on the projective space $\text{PG}(3,2)$. If $\mathcal{C}$ is the set of blocks $B' \in \mathcal{C}$ such that $0 \in B'$, $1 \notin B'$, then the 35 blocks $B' \in \mathcal{C}$ such that $|B \cap B'| = 1$ correspond to the 35 lines of $\text{PG}(3,2)$ and the 15 blocks $B' \in \mathcal{C}$ such that $|B \cap B'| = 0$ correspond to the planes. The remaining 126 blocks intersect $B$ in two points and correspond to non-ruled...
quadrics in PG(3,2). In order to see that $H$ is transitive on these 126 blocks we consider the set stabiliser of \[ \{a, b\} \leq B - \{j\} . \] Firstly, this is a subgroup $K$ of $H$ isomorphic to $S_5$ and with orbits of length 2 and 5. Its action on the points of $\Delta - B$ corresponds to that of $\text{PGL}(2,4)$ on its vector space and it permutes the 16 blocks intersecting $B$ in $\{a, b\}$ in orbits of length 6 and 10. (Furthermore, the subgroup of $\text{M}_{24}$ fixing $B$ and $\{a, b\}$ is $2^4: S_6 \cong 2^3: S_3(4,2)$ and the subgroup fixing a point outside $B$ is an $S_6$ with orbits of length 6 and 10 on the blocks intersecting $B$ in $\{a, b\}$. However, the subgroup fixing one of these blocks is another $S_6$ permuting the remaining blocks transitively but having orbits of length 6 and 10 on $\Delta - B$. The stabiliser of a point in $\Delta - B$ is therefore an $S_5$ considered as $G(4,2)$.)

Corresponding to the orbits of length 15, 35 and 126 of $H$ we have the following three possibilities for the inner product:

\[
(6.9) \quad (v_B, w_B) = \begin{cases} 
32 & |B' \cap B| = 0 \\
32 & |B' \cap B| = 4 \\
0 & |B' \cap B| = 2.
\end{cases}
\]

We may identify $P$ with the points and $Q$ with the quadrics of the version of the geometry of G. Higman [1] which was introduced by C. Sims [1] and used by M. Smith [1]; a point $v_B$ being incident with a quadric $w_B$ whenever $(v_B, w_B) \neq 0$. In order to obtain G. Higman's original geometry we proceed as follows.
Consider the vectors $z \in \Lambda$ such that $(x,z) = 16$ and $(z,z) = 32$. Since $C_3$ fixes $x$ it permutes these vectors $z$ amongst themselves. Using Table 4 of Conway [1] we obtain five possible shapes for $z$:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coordinates</th>
<th>Shape Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$(4^2; -4^2, 0^{22})$</td>
<td>23</td>
</tr>
<tr>
<td>II</td>
<td>$(0; 2^8, 0^{15})$</td>
<td>23.11.2</td>
</tr>
<tr>
<td>III</td>
<td>$(1^3; 3, -1^7, 1^{15})$</td>
<td>23.11.16</td>
</tr>
<tr>
<td>IV</td>
<td>$(2; -2^2, 2^5, 0^{16})$</td>
<td>23.11.7.3</td>
</tr>
<tr>
<td>V</td>
<td>$(3; -1^{11}, 1^{12})$</td>
<td>23.7.8</td>
</tr>
</tbody>
</table>

Now $y$ is of type I and the subgroup of $C_3$ fixing $y$ must act on the regular 2-graph $H(176)$ and on its complement in $\Sigma$, the Higman-Sims' graph. It follows that the only possibility is for $C_3$ to act transitively on the vectors in (6.10). Since the number of these vectors is $|C_3:H_1S|$ the subgroup fixing any one of them is isomorphic to $H_1S$.

In particular $H_1S \leq \text{Aut}(H(176))$ and the vectors of $\Sigma$ not orthogonal to any vector of (6.10) form a Higman-Sims' graph. We shall label the set of vectors of $\Sigma$ orthogonal to $z$ in (6.10) by the type of $z$. Consider first a geometry of type II with $z = e_D, o \not\in D$. The 176 vectors of the geometry are the following:

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Vectors</th>
<th>Shape Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$v_i$</td>
<td>$i \in D$</td>
</tr>
<tr>
<td>II</td>
<td>$v_B$</td>
<td>$</td>
</tr>
</tbody>
</table>

The subgroup of our previous $M_{23}$ which acts on this geometry is $A_9$. If $a, b \in D$ there are 6 blocks $B$ such that...
\[ B \cap D = \{a, b\} \] and \( o \in B \). We may identify these blocks with the set \([ab]^*\) of G. Higman [1]. The subgroup of \( A_8 \) fixing \( \{a, b\} \) is \( S_6 \) and this \( S_6 \) acts on \( B - \{a, b\} \) and \([ab]^*\) in different ways. The identification of this geometry with G. Higman's original is now complete.

We observe that the conics of the geometry are maximal coherent sets and that any element \( u \) outside a conic induces a partition with \( m(u) = 2 \). (Unfortunately, these are not the only maximal coherent sets of 8 points.)

Returning to our original geometry (6.6) of type I we see that the 126 vectors \( v_B \) such that \( |B' \cap B| = 2 \) for a fixed quadric \( w_B \) lie in \( R^{22} \) (by (6.9)) and hence can be made into a regular 2-graph by (6.1), whose parameters are the same as \( U(5) \) of \( \S 4 \). Moreover, from G. Higman [1], the stabiliser in \( \text{His} \) of a quadric is \( F\Sigma U(3, 5^2) \) so this group must act on the regular 2-graph just obtained. Since there is only one possible representation of \( F\Sigma U(3, 5^2) \) on 126 points we see that the monomial representation considered in \( \S 1 \) is uniquely determined and hence the 2-graph must be \( U(5) \). We have proved that \( \text{Aut}(U(5)) = P\Gamma U(3, 5^2) \) but \( P\Gamma U(3, 5^2) \) does not have a representation of degree 50 so the subgroup of \( \text{Aut}(H(176)) \) fixing \( U(5) \) must be \( F\Sigma U(3, 5^2) \).

The stabiliser of a point is again \( F\Sigma U(3, 5^2) \) and the two classes of \( P\Sigma U(3, 5^2) \)'s are fused by the outer automorphism of \( \text{His} \) extending that of \( M_{22} \). The action of \( F\Sigma U(3, 5^2) \) can be studied via the geometry (6.11) of type II, choosing \( D \) so that \( l \in D \). The group \( F\Sigma U(3, 5^2) \) fixing \( v_1 \) act transitively on the 50 conics containing \( v_1 \). As shown by M. Smith [1],
these conics may be identified with the vertices of the Moore graph of width 2 and valency 7, and the 175 points other than \( v_1 \) may be identified with the edges of the Moore graph.

Maximal incoherent sets of various sizes may be obtained from the remaining types of geometry. Again let \( D \) be a block such that \( 0 \notin D \) and \( 1 \in D \). The vectors of a geometry of type III may be taken as:

\[
\begin{align*}
(6.12) \quad v_1 & \quad \text{i} \notin D \\
B \quad o \in B, \quad l \notin B, \quad |B \cap D| = 2 \\
B \quad o \in B, \quad l \in B, \quad |B \cap D| = 4
\end{align*}
\]

The 15 elements \( v_1 \) form a maximal incoherent set and the three types of vectors in (6.12) form the three orbits for \( A_7 = N_{23} \) as described above. By adjoining the point \( v_1 \) to the orbit of length 35 we obtain \( G(6) \) as a subgraph of \( C(276) \).

Suppose that \( C \) is a block such that \( 0,1,2 \in C \). The vectors of a geometry of type IV may be taken as:

\[
\begin{align*}
(6.13) \quad v_1 & \quad \text{i} \notin C \\
B \quad o \in B, \quad 1,2 \notin B, \quad |B \cap C| = 2 \\
B \quad o \in B, \quad \{|1,2\} \cap B| = 1, \quad |B \cap C| = 4
\end{align*}
\]

The 16 vectors \( v_1 \) form a maximal coherent set; the subgroup of \( N_{23} \) occurring is \( 2^4 \cdot S_4 \).

Now let \( F \) be the symmetric difference of two blocks with two points in common such that \( o \in F \). The vectors of
a geometry of type \( V \) may be taken to be

\[
V_i \quad i \in F
\]

\[
V_B \quad \sigma \in B, \quad |B \cap F| = 4
\]

In this case we have obtained a maximal incoherent set of 11 points and it is easy to see that the group \( M_{11} \) acts on \( R(176) \) with orbits of length 11 and 165.

7. Known doubly transitive groups

In the previous sections we have shown that many doubly transitive groups act as groups of automorphisms of regular 2-graphs. We now present a list of doubly transitive groups \( G \) of degree \( N \) and show that with the possible exception of groups with a regular normal subgroup, any group for which we have not constructed a regular 2-graph cannot act on a regular 2-graph with \( N \) points. This list was obtained at a seminar given by M. O'Nan at Oxford, January 1971. I know of no other doubly transitive group.

\[
\begin{array}{ll}
\text{(7.1)} & \text{Group} \\
1. & G \text{ has a regular normal subgroup} \\
2. & \text{PSL}(n,q) \leq G \leq \text{PGL}(n,q) \\
3. & \text{PSU}(n,q) \leq G \leq \text{P\Gamma U}(n,q) \\
4. & G \text{ is of Ree type, e.g., } 2G_2(q) \\
5. & \text{Sz}(q) \leq G \leq \text{Sz}(q) \\
6. & G = A_n \text{ or } S_n \\
7.(a) & G = \text{Sp}(2m,2) \\
7.(b) & G = \text{Sp}(2m,2)
\end{array}
\]

\[
\begin{array}{ll}
\text{Degree} & \\
1+q^+ \ldots + q^{n-1} & \\
1+q^3 & \\
1+q^3 & \\
1+q^2, \quad q=2^{m+1} & \\
& \\
& \\
2^{2m-1}+2^{m-1} & \\
2^{2m-1}+2^{m-1} & \\
\end{array}
\]
If $G$ has odd degree or if $G$ is more than doubly transitive, then $G$ cannot act on a regular 2-graph. This eliminates cases 5, 6, 8, 9 and 10. Cases 3, 4, 7(a), 7(b), 11 and 12 have been dealt with in this chapter. It remains to consider cases 1 and 2. In § 2.3 we constructed regular 2-graphs for $\text{PSL}(2,q)$, $q \equiv 1 \pmod{4}$. If $\text{PSL}(2,q)$ acts on a regular 2-graph and $q \equiv 3 \pmod{4}$, then the regular 2-graph must be of type (a). But then we obtain a contradiction from (1.2.5).

Suppose that $n > 2$ and that $\text{PSL}(n,q)$ acts on a regular 2-graph with $N = 1+q+\cdots+q^{n-1}$. The stabiliser of two points permutes the remaining points in orbits of length $q-1$ and $N-q-1$, the $q-1$ points lying on the line determined by the fixed points. But now a set of four points just three of which lie on a line cannot contain an even number of coherent triangles.

Thus we have dealt with all except case 1. The automorphism groups of the regular 2-graphs $S(2m)$ of § 2 belong to case 1 and in § 2 we constructed a number of subgroups of $\text{Aut}(S(2m))$ which again belong to case 1. If $(\mathcal{G}, \mathbf{t})$ is a regular 2-graph and $G$ is a doubly transitive group of automorphisms of $(\mathcal{G}, \mathbf{t})$ with a regular normal subgroup $K$, then
$|K| = 2^n$ for some $n$ and $G = K.G_a$ for $a \in \mathbb{Z}$. The group $K$ is elementary abelian and $G_a \subseteq \text{GL}(n, 2)$ but it is a difficult problem to determine which subgroups of $\text{GL}(n, 2)$ are transitive on the non-identity elements of $K$.

8. Applications to equiangular lines

The problem of determining the maximum number of equiangular lines in $\mathbb{R}^r$ was first tackled by Haantjes [1] and was then taken up by Seidel and others (see Seidel [3], van Lint and Seidel [1]). The results of this chapter provide several infinite families of equiangular lines in Euclidean space which in many cases improve existing results. For example, the existence of $U(5)$ together with (2.4.6) shows that the maximum number of equiangular lines in $\mathbb{R}^{21}$ is 126.

In this section we use the results of § 6 to construct equiangular lines in less than 21 dimensions. Unfortunately we do not obtain equality in (2.4.6) so these configurations do not yield regular 2-graphs.

(8.1) **Theorem.** There exist 90 equiangular lines in $\mathbb{R}^{20}$.

**Proof.** We use the notation of § 6. Let $w_{B_1}$ be a quadric as in (6.8). By (6.9) there are 126 vectors $v_B$, orthogonal to $w_{B_1}$. Now choose a second quadric $w_{B_2}$ and consider the vectors orthogonal to both. Since the stabiliser in $H_{13}$ of $w_{B_1}$ and $w_{B_2}$ has orbits of length 36 and 90 on the vectors orthogonal to $w_{B_1}$, we see that there are 90 equiangular lines $\langle v_B \rangle$ in $\mathbb{R}^{20}$. 
The next result was discovered by D.S. Asche.

(8.2) **Theorem.** There exist 72 equiangular lines in $\mathbb{R}^{19}$.

**Proof.** Choose blocks $B_1$ and $B_2$ such that $o \notin B_1, B_2$ and $B_1 \cap B_2 = \{1, 2\}$. A point $v_B$ of $H(176)$ is orthogonal to $w_{B_1}$ if $o \in B, 1 \notin B$ and $|B \cap B_1| = 2$. Of these 126 vectors, 90 are orthogonal to $e_0 - e_2$ and 90 are orthogonal to $w_{B_2}$. Since there are 18 blocks of $S(5,6,12)$ which contain a given point and contain neither of a given pair of points we see that there are 72 vectors of the above 126 which are orthogonal to both $e_0 - e_2$ and $w_{B_2}$. //

**Remark.** Since each of the 72 blocks found above contain 7 points of $\Delta - \{0, 1, 2\}$ we see that there is some point of $\Delta - \{0, 1, 2\}$ contained in no more than 24 blocks (in fact each point of $\Delta - \{0, 1, 2\}$ must be contained in 24 blocks). Hence there are at least 48 equiangular lines in $\mathbb{R}^{18}$.

1. $Q^1(k), S(k)$ and $Q^{-1}(d)$

Let $T = (S, S)$ be a regular 3-graph with $so$. We shall show that $T$ is isomorphic to one of the 3-graphs of the title of this section. There are three possibilities for the parameter of $T$ and these are given in (2.5, 6, 7). We shall have available the equivalence relation $\sim$ of (2.2) which en
CHAPTER 5

UNIQUENESS THEOREMS

The results of § 2.7 show that the existence of a large coherent set in a regular 2-graph has a considerable influence on the structure. In this chapter we use these results to show that certain regular 2-graphs are uniquely determined by their order and the existence of a suitable coherent set.

The uniqueness of the regular 2-graphs with $k=1$ was first proved by Seidel [1] in terms of strong graphs and by G. Higman in terms of regular 2-graphs. The uniqueness of $C(276)$ was proved by G. Higman under the additional assumption of 4-regularity.

Throughout this chapter we shall often use (2.7.1) and (2.7.3) without comment. We shall also use the fact that all coherent triangles of a regular 2-graph are determined once we have specified the coherent triangles containing a given point.

1. $0^{-1}(4)$, $S(4)$ and $0^{-1}(6)$

Let $T = (\Omega, \mathcal{T})$ be a regular 2-graph with $k=1$. We shall show that $T$ is isomorphic to one of the 2-graphs of the title of this section. There are three possibilities for the parameter of $T$ and these are given in (2.5.5). We also have available the equivalence relation $\sim$ of § 2.9 defined on
unordered pairs of elements of $\Omega$.

(1.1) Theorem. Let $T = (\Omega, t)$ be a regular 2-graph with $N=10$. Then $T$ is unique (up to isomorphism) and $\text{Aut } T \cong \text{Sp}(4,2)$.

Proof. Since $N=10$ we must have $k=4$ and $a=4$. Choose $\omega$ and $\alpha_{oo}$ in $\Omega$ and let $L = \{\omega, \alpha_{oo}, \alpha_{01}, \alpha_{02}\}$ and $M = \{\omega, \alpha_{oo}, \alpha_{01}, \alpha_{02}\}$ be the two coherent sets of four points containing $\{\omega, \alpha_{oo}\}$.

If $\beta, \gamma \in L$ and $\beta$ and $\gamma$ induce the same partition of $L$, then $\{\beta, \gamma, \delta\}$ is coherent for all $\delta \in L$. Since $a=4$ the four points not in $L$ or $M$ may be uniquely labelled $a_{ij}, i, j \in \{1, 2\}$, such that $\{\omega, a_{10}, a_{11}\}$ and $\{\omega, a_{00}, a_{11}\}$ are coherent.

It follows that $a_{ij}$ is joined to $a_{mn}$ in the graph $T_\omega$ if and only if $i=m$ or $j=n$. Hence $T_\omega$ is the graph $L_2(3)$ of $S.4$ and $T$ is isomorphic to the regular 2-graph obtained by adjoining a point to $L_2(3)$. Since $0^1(4)$ has 10 points we have $T \cong 0^1(4)$ and $\text{Aut } T \cong \text{Sp}(4,2)$.

The regular 2-graphs $T(9)$ of $S.2$ and $T_{10}$ of $S.3$ both have $N=10$ so $T(9) \cong T_{10} \cong 0^1(4)$. Thus if $G = \text{Aut } T$, then $G \cong \text{PEL}(2,9)$ and $S_5 \leq G$, whence $G \cong S_6$. Since $\text{Aut } L_2(3) = S_3 \wr S_2$, we have $G_\omega \cong S_3 \wr S_2$. Also, it follows that $T$ is isomorphic with its complement and hence possesses a 'polarity' - i.e., a permutation of $\Omega$ which interchanges coherent and incoherent triangles. The polarity $\pi$ induces an outer automorphism of $G$ and we have $G < \pi > \cong \text{PEL}(2,9)$.

(1.2) Theorem. Any regular 2-graph $T = (\Omega, t)$ with $N=16$ is isomorphic to either $S(4)$ or its complement. In
particular $\text{Aut } T = E \cdot H$ where $E$ is an elementary abelian group of order 16, $H \cong \text{Sp}(4,2)$ and $E \leq E \cdot H$, $E \cap H = 1$.

Proof. By taking the complement if necessary we may suppose the parameters of $T$ to be $N=16$, $a=6$ and $k=1$. Our aim is to construct an incoherent set of six points.

Let $\{ \omega, \alpha, \beta, \gamma \}$ be coherent and suppose $\{ \omega, \alpha \} \sim \{ \alpha_1, \alpha'_1 \}$. Choose $\{ \beta_1, \beta'_1 \} \sim \{ \beta_2, \beta'_2 \} \sim \{ \omega, \beta \}$ so that $\{ \omega, \alpha, \beta_1 \}$ and $\{ \omega, \alpha, \beta_2 \}$ are coherent. (Note that $\alpha_1$ and $\alpha'_1$ induce the same partition of $\{ \beta_1, \beta'_1, \beta_2, \beta'_2 \}$.) Now choose $\{ \alpha_2, \alpha'_2 \} \sim \{ \omega, \alpha \}$ so that $\{ \omega, \alpha_2, \beta_3 \}$ is coherent and then choose $\{ \gamma_1, \gamma'_1 \} \sim \{ \gamma_2, \gamma'_2 \} \sim \{ \omega, \gamma \}$ so that $\{ \omega, \alpha_1, \gamma_1 \}$ and $\{ \omega, \alpha_1, \gamma_2 \}$ are coherent. Since just one of $\{ \omega, \beta_1, \gamma_1 \}$ and $\{ \omega, \beta_1, \gamma_2 \}$ is coherent we may choose the notation so that $\{ \omega, \beta_1, \gamma_1 \}$ is coherent. (The fourth point of the coherent 4-set containing $\{ \omega, \alpha_1, \beta_1 \}$ must be either $\gamma_1$ or $\gamma_2$.) By construction we have $\{ \omega, \alpha_1 \} \{ \alpha, \alpha'_1 \} \{ \beta_1, \gamma_1 \}$ and it follows that the remaining pair in the equivalence class is $\{ \beta_2, \gamma_2 \}$. Similarly, the pairs $\{ \omega, \alpha'_1 \}, \{ \alpha, \alpha'_1 \}, \{ \beta_1, \gamma'_1 \}$ and $\{ \beta_2, \gamma'_2 \}$ are equivalent.

We shall show that $M = \{ \omega, \gamma, \alpha_1, \alpha'_1, \beta_1, \beta'_2 \}$ is incoherent. By our construction all that remains to be done is to show that $\{ \omega, \beta_2, \alpha'_2 \}$ is incoherent. Since either $\{ \omega, \beta_1, \gamma_2 \}$ or $\{ \omega, \beta_1, \gamma'_2 \}$ is coherent, one of the pairs $\{ \alpha_2, \gamma_2 \}$ or $\{ \alpha_2, \gamma'_2 \}$ is equivalent to $\{ \omega, \beta_1 \}$.

However we cannot have $\{ \omega, \beta_1 \} \sim \{ \alpha_2, \gamma_2 \}$ since this would force $\{ \omega, \alpha_1, \beta_1, \gamma_2 \}$ to be coherent and we would have $\{ \omega, \alpha_1 \} \sim \{ \beta_1, \gamma_2 \}$, a
contradiction. Thus the pairs equivalent to \( \{ \omega, \beta_1 \} \) are \( \{ \beta, \beta_1 \}, \{ \alpha_1, \gamma_1 \} \) and \( \{ \alpha_2, \gamma_2 \} \). If \( \{ \omega, \beta_2, \alpha_1 \} \) were coherent, then \( \{ \omega, \beta_2, \gamma_2 \} \) would be contained in the coherent sets \( \{ \omega, \beta_2, \gamma_2, \alpha_1 \} \) and \( \{ \omega, \beta_2, \gamma_2, \alpha_2 \} \) which contradicts \( k=1 \). This proves that \( M \) is an incoherent set of six points.

If \( M \) is any incoherent set of six points in \( T \), then it follows from (2.7.16) and (2.7.19) that we may identify the points of \( \Omega \) outside \( M \) with the 10 partitions of \( M \) into pairs of subsets of three elements such that the coherent triangles are defined as follows. If \( \alpha, \beta, \gamma \in M \) and \( x, y, z \) are partitions then \( \{ \alpha, \beta, \gamma \} \) is incoherent, \( \{ \alpha, \beta, x \} \) is coherent if \( \alpha \) and \( \beta \) lie on opposite sides of the partition, \( \{ \alpha, x, y \} \) is coherent if we can write \( x = (X_1, X_2) \) and \( y = (Y_1, Y_2) \) such that \( \alpha \in X_1 \cup Y_1 \) and \( |X_1 \cup Y_1| = 1 \), \( \{ x, y, z \} \) is coherent if we can write \( x = (X_1, X_2) \) \( y = (Y_1, Y_2) \) and \( z = (Z_1, Z_2) \) such that \( |X_1 \cup Y_1| = |X_1 \cup Z_1| = |X_1 \cup Z_1| = 1 \).

Thus \( T \) is unique up to isomorphism. In particular, \( T \cong S(4) \) and the rest of the theorem follows from (4.2.5).

The last part of the proof shows that an incoherent set of six points can be transformed into any other by an element of \( \text{Aut } T \). Moreover, the stabiliser of such a set is obviously \( S_6 \) and the points outside form a regular 2-graph on 10 points.

(1.3) **Theorem.** Any regular 2-graph \( T = (\Omega, \mathcal{K}) \) with \( N=28 \) is isomorphic to either \( \Omega^{-1}(6) \) or its complement.

**Proof.** We may suppose the parameters of \( T \) to be \( N=28, a=10 \)}.
and $k=1$. Let \( \{a_1, a_1', \ldots, a_6, a_6'\} \) be the elements of an equivalence class of \( \sim \). If $\beta$ is a further point, then we may choose the notation so that \( \{\beta, a_1, a_1'\} \) is coherent for $i=2, \ldots, 6$. It follows that \( \{a_1, \ldots, a_6, \beta\} \) is an incoherent set of seven points. If \( M \) is an incoherent set of seven points in \( \Omega \) then by (2.7.16) and (2.7.19) we may identify the remaining points of \( \Omega \) with the set \( M(2) \) of unordered pairs of \( M \), such that for $a, \beta \in M$ and $x, y, z \in M(2)$, \( \{a, \beta, x\} \) is coherent whenever \( |\{a, \beta\} \cap x| = 1 \); \( \{a, x, y\} \) is coherent whenever $a \notin x \cup y$, and \( \{x, y, z\} \) is coherent whenever \( |x \cap y| + |y \cap z| + |z \cap x| \equiv 0 \pmod{2} \).

Since $T$ is uniquely determined the theorem follows from (4.3.37).

The regular 2-graphs $T_{28}$, $G^{-1}(6)$, $U(3)$ and $R(3)$ all have $n=28$ and $k=1$ so we have $T_{28} \equiv G^{-1}(6) \equiv U(3) \equiv R(3)$ and $S_8 \cap U(3, 3^2) \leq G_2(3) \equiv Sp(6, 2)$. The above proof shows that $G = \text{Aut} T$ acts transitively on the incoherent sets of seven points and that the stabiliser of such a set is $S_7$.

The configuration of the 28 bitangents to a plane quartic curve is just the regular 2-graph $G^{-1}(6)$ and for this reason a regular 2-graph may be considered as a generalisation of this well known figure. Moreover the strongly regular graph obtained by deleting a point is the graph obtained from the 27 lines on a cubic surface by joining two lines if they meet. These configurations have been studied for more than one hundred years (see Dickson [2], Jordan [1] and the references given there) and it is
interesting to note that the description of the 28 bi-
tangents given by Dickson [2] can be easily translated into
the language of regular 2-graphs. A syzygetic triple of
bitangents is just a coherent triangle, an Aronhold set is
an incoherent set of seven points, and a Steiner set is an
equivalence class of \( \sim \). The representation as \( T_{28} \) is
given on p. 357 of Dickson [2] and the representation as
\( 0^{-1}(6) \) is given on p. 373. In fact, the representation as
\( 0^{-1}(6) \) derives from the theory of Riemann theta functions
(see Baker [1]) and in this context the regular 2-graphs
\( S(2m) \) of § 4.2 were also studied, a coherent triangle being
called a syzygetic triple. The regular 2-graphs \( 0^{-1}(2p) \)
where \( p = \frac{1}{2}(n-1)(n-2) \) also arise from the problem of the
contact of curves (see Dickson [2], p. 376).

The regular 2-graph \( S(4) \) which we studied above can be
identified with the configuration of 16 singular points and
16 singular tangent planes to the Kummer quartic surface
(Jordan [1], p. 331) and also with the 16 lines on the
Clebsch quartic surface (Jordan [1], p. 305).

We may note that in the strongly regular graph obtained
by deleting a point of \( 0^{-1}(6) \), an equivalence class
( = Steiner set) which does not involve the deleted point
corresponds to a double-six of lines on the cubic surface.
Also, the 16 points not in an equivalence class form a
regular 2-graph \( S(4) \) and through each point \( \omega \) of \( 0^{-1}(6) \)
there are 40 triples of regular 2-graphs isomorphic to \( 0^{1}(4) \),
any pair of a triple intersecting in \( \{ \omega \} \) (Jordan [1], p. 317,
Dickson [1], § 283, 2°).
2. \( \mathbb{C}(276) \)

In § 4.6 we defined \( \mathbb{C}(276) \) in terms of the Steiner system \( S(5,8,24) \) and showed that it contained an incoherent set of 23 points.

\[(2.1) \text{Theorem. Let } T = (\mathbb{G}, \mathbb{B}) \text{ be a regular 2-graph with } N = 276 \text{ and } a = 162 \text{ which contains a coherent set } M \text{ of 23 points. Then } T \text{ is isomorphic to the complement of } \mathbb{C}(276) \text{ and if } M' \text{ is any other coherent set of 23 points, then there is an element } \sigma \in \text{Aut } T \text{ such that } \sigma M = M'. \]

Proof. From (2.7.16) any point \( x \notin M \) induces a partition of \( M \) into sets \( M_1(x) \) and \( M_2(x) \) of size 16 and 7 respectively. We shall call the sets \( M_2(x), x \notin M \), the blocks of \( M \). From (2.7.19) we see that any two distinct blocks intersect in either 1 or 3 points. Thus any 4 points of \( M \) are contained in at most one block. Since there are 253 blocks and any block contains 35 four-element subsets, whereas \( M \) contains 253 \times 35 four-element subsets it follows that any four element subset of \( M \) is contained in a unique block. Hence the points and blocks of \( M \) form the Steiner system \( S(4,7,23) \). Since it is well known that \( S(4,7,23) \) is unique up to isomorphism (Witt [1]), we may identify the blocks of \( M \) with the blocks of \( S(5,8,24) \) which contain a given point. Comparing (2.7.19) and (1.5.4) we see that our regular 2-graph \( T \) is isomorphic to the complement of \( \mathbb{C}(276) \). Moreover, the isomorphism takes \( M \) onto the set \( D = \{ v_i \mid 1 \leq i \leq 23 \} \).
From now on let $T$ denote the complement of $G(276)$ and let $G = \text{Aut } T$. We shall use the notation of § 4.6. By construction, we have $M_{23} - G$ and $M_{23}$ has orbits $D$ and $E$ of length 23 and 253. It follows that $\sigma$ of (2.1) may be chosen to take any given ordered set of four points of $M$ to any given ordered set of four points of $M'$. We use this to prove that $G$ acts doubly transitively on $\omega$. This proof was first given by G. Higman.

Choose a block $B \in \mathcal{B}$ such that $c \in B$ and choose $p \in B$, $p \neq o$. The group $2^4.S_6$ acts doubly transitively on the 16 blocks $B_1, \ldots , B_{16}$ which intersect $B$ in $\{o, p\}$, hence any pair of these blocks have four points in common.

Therefore, the set

$$(2.2) \quad D' = \{ v_B \} \cup \{ v_{1i} \mid i \in B, i \neq o, p \} \cup \{ v_{1j} \mid 1 \leq j \leq 16 \}$$

is coherent.

Thus, by (2.1)

$$(2.3) \quad G \text{ is transitive on } \omega.$$
then define $D'$ as in (2.2) with $B = C$ and $p \neq 0, 1, 2$. If $1, 2 \notin C$ we can choose $B \in B$ with $1, 2 \in B$ and $B \cap C = \{0, p\}$ for some $p$. In this case we define $D'$ as in (2.2) with $B_1 = C$. By (2.1) there is an element $\sigma \in G$ which fixes $v_1$ and $v_2$ and takes $v_C$ into $D$. Thus the stabiliser of two points $u$ and $v$ is transitive on the 162 points $w$ such that $\{u, v, w\}$ is coherent. Let $L$ be the set of points $w$ such that $\{u, v, w\}$ is incoherent. Taking $u = v_1$ and $v = v_2$, $L$ consists of the 112 points $v_B$ such that $0 \in B$ and $|\{1, 2\} \cap B| = 1$. We may identify the points of $\Delta - \{0, 1, 2\}$ with the points of the 21-point plane so that the above blocks correspond to hyperovals. Thus the group $M_{21}$ which fixes 0, 1 and 2 has two orbits of length 56 on $L$. It follows that $G_{u,v}$ is either transitive on $L$ or has two orbits of length 56. Now take $u = v_1$ and $v = v_C$ such that $0, 1 \notin C$. Then $L$ consists of the 16 points $v_i$, $i \neq C$, the 16 points $v_B$ such that $B \cap C = \{0, 1\}$ and the 80 points $v_B$ such that $0 \in B$, $1 \neq B$ and $|B \cap C| = 4$. The group $2^4 : A_6$ which fixes 0, 1 and C has orbits of length 16, 16 and 80 on $L$. It follows from (2.4) that $G_{u,v}$ acts transitively on $L$.

Hence

(2.5) $G_u$ is a rank 3 group on $\Omega - \{u\}$ with subdegrees 1, 112 and 162.

(2.6) Theorem. The group $G$ is transitive on ordered coherent sets of four points and $T$ is both 4- and 5-

regular with $k_4 = 72$ and $k_5 = 51$.

Proof. We may suppose the coherent set to be $\{v_1, v_2, v_3, v_C\}$. 


If $1,2,3 \in C$, then define $D'$ by (2.2) with $B = C$ and $p \neq 0,1,2,3$. If $1,2,3 \not\in C$, then choose $B \in \mathcal{B}$ with $1,2,3 \in B$ and $B \cap C = \{0,p\}$ for some $p$. Now define $D'$ as in (2.2) with $B' = C$. Then from (2.1) there is an element $\sigma \in G$ fixing $v_1,v_2,v_3$ and taking $v_C$ into $D$. This proves the first part of the theorem and $T$ is therefore $4$-regular.

The points which make a coherent set with $\{v_1,v_2,v_3,v_4\}$ are the remaining 19 points of $D$, the point $v_B$ where $B$ is the unique block containing $0,1,2,3,4$ and the 52 points $v_B$ where $0 \in B$, $1,2,3,4 \not\in B$. Hence $k_4 = 72$ and from (2.8.1) and (2.8.7) $T$ is $5$-regular with $k_5 = 51$. 

We now calculate the number of coherent sets of 23 points which contain the set $X = \{v_1,v_2,v_3,v_C\}$ where $0,1,2,3 \in C$. Let $I = C - \{0,1,2,3\}$, $P = \{v_1 \mid i \in I\}$, $Q = \{v_B \mid B \in \mathcal{B}\}$, $B \cap C = \{0,1,2,3\}$, and $\pi(i) = \{v_B \mid B \in \mathcal{B}, B \cap C = \{0,1,3\}\}$ where $i \in I$. The elements of the sets $P$, $Q$, $\pi(i)$ are the 72 points which form a coherent set with $X$. As usual we may interpret the 16 points outside $C$ as the points of an affine space over $GF(2)$. The group $2^4.5_6$ fixing $C$ and \{0,1\} acts as $2^4.\text{Sp}(4,2)$ on this space and the blocks of the elements of $\pi(i)$ correspond to the translates of a quadric.

Proof. Let $A$ be a graph with vertex set $\{0,1,2,3\}$ and $\lambda = 2$. The group $2^4.5_6$ fixing $C$ and \{0,1\} acts as $2^4.\text{Sp}(4,2)$ on this space and the blocks of $\pi(i)$ correspond to the translates of a quadric.

In particular, the 16 points and 16 blocks form a projective design with parameters $v=16$, $k=6$ and $\lambda = 2$ (cf. §4.2,4.3).

It is now easy to see that a block of $\pi(j)$, $j \neq i$, intersects exactly 10 blocks of $\pi(i)$ in $4$ points. Let $\Pi = \cup \{\pi(i) \mid i \in I\}$.

Suppose that $D'$ is a coherent set of 23 points containing $X$ and that $D'$ contains no point of $Q$. Since $D'$
cannot contain all the points of $P$ we may choose $v_i \in P-D'$ and consider the partition it induces on $D'$. It follows that $|D' \cap \pi(i)| = 7$ or $16$. If $|D' \cap \pi(i)| = 7$, then we could repeat the process with another point of $P-D'$ and this obviously leads to a contradiction. Hence we must have $|D' \cap \pi(i)| = 16$ and $D' = X \cup (P - \{v_1\}) \cup \pi(i)$. We may now assume that $D'$ contains an element $v_B$ of $Q$. Since $B$ corresponds to a line of the affine space described above there are for each $i \in I$ just four blocks $B'$ such that $v_B \in \pi(i)$ and $|B \cap B'| = 4$. We cannot have $Q \subseteq D'$ so choose $v_{B_1} \in Q - D'$ and consider the partition it induces on $D'$. If we put $Q = \{v_{B_1}, v_{B_2}, v_{B_3}, v_{B_4}\}$ and $\mu(\cdot) = \{v_B \in \Pi \mid |B \cap B'_j| = 4\}$, then $|D' \cap \mu(\cdot)| = 7$ or $16$.

Arguing as before we must have $|D' \cap \mu(\cdot)| = 16$ and $D' = X \cup (Q - \{v_{B_1}\}) \cup \mu(\cdot)$. We have proved that $X$ is contained in 8 coherent sets of 23 points. Since it is clear that $M_{23}$ is the stabiliser of $D$ in $G$ we have $|G| = \frac{276.275.162.105}{23.22.21.20} . 8. (M_{23})$, so that

$$\text{(2.7)} \quad |G| = 2^{10} . 7 . 5 . 7 . 11 . 23 = |G_3| .$$

$$\text{(2.8)} \quad G \text{ is a simple group.}$$

\textbf{Proof.} Let $K$ be a minimal normal subgroup of $G$. Since $G$ is doubly transitive, $K$ is transitive on $\Omega$ and cannot be soluble. Since $23 \mid |K|$, $K$ is simple and we have $M_{23} \leq K$.

Let $P$ be an $S_{23}$-subgroup of $K$. Since $G$ has a faithful rational representation of degree 23 we have $G_0(P) = P$.

By Sylow's theorem and the Frattini argument we have $K = G$. \textit{\ 11}
Remark. Using the fact that the stabiliser of a point contains $H_{22}$ and the stabiliser of two points contains $PSL(3,4)$ it can be shown that the stabiliser of a point has a simple subgroup of index 2.

3. $O^1(8)$.

(3.1) **Theorem.** If $T = (\mathbb{F}, t)$ is a 4-regular 2-graph with $N=136$, $a=64$ and a coherent set of 10 points, then $T$ is isomorphic to $O^1(8)$ and $\text{Aut } T \cong \text{Sp}(8,2)$.

**Proof.** Let $M$ be a coherent set of 10 points. From (2.7.3) any point not in $M$ partitions $M$ into a pair of five-element subsets. Since 4-regularity implies 5-regularity we must have $k_5 = 6$ and hence $k_4 = 12$ from (2.8.7). Since there are 126 partitions of a set of 10 points into a pair of five-element subsets, each point $x$ of $\mathbb{F} - M$ corresponds to a unique partition of $M$. We shall write $x = (M_1, M_2)$, where $M = M_1 \cup M_2$ is the partition of $M$ induced by $x$. If $x_1 = (M_1', M_1)$ and $\alpha, \beta, \gamma, \delta \in M_1$, then there are 5 elements $x_2 = (M_2, M_2')$ such that $\{\alpha, \beta, \gamma, \delta, x_1, x_2\}$ is coherent and we may therefore choose the notation for $x_2$ so that $|M_1 \cap M_2| = 4$. Thus the set $\{x_1, x_2, a\}$, $a \in M$, $x_1, x_2 \notin M$ is coherent if and only if we may choose the notation so that $x_1 = (M_1, M_1')$, $x_2 = (M_2, M_2')$, $|M_1 \cap M_2| = 4$ and $\alpha \in M_1 \cap M_2$. We have shown that $T$ is uniquely determined. From (4.3.29) the 2-graph $O^1(8)$ contains a coherent set of 10 points. Moreover, the automorphism group of $O^1(8)$ is transitive on ordered coherent sets of four elements. Thus $O^1(8)$ satisfies the
initial conditions and the theorem is proved.

From the description of $O^1(8)$ obtained in the above proof we see that $S_{10}$ acts on $O^1(8)$ with orbits of length 10 and 126. The inclusion $S_{10} \leq Sp(8,2)$ was also obtained in § 4.3. It follows that $S_{10}$ acts on the 2-graph $O^{-1}(8)$. The 120 points of $O^{-1}(8)$ can be identified with the 120 three-element subsets of a set of ten elements. If we join two such subsets whenever they have just one point in common we obtain a strongly regular graph with $n=120$, $n_1=63$, $n_2=56$, $p_1=30$ and $p_2=27$. By (2.6.2) this gives rise to a regular 2-graph which must be $O^{-1}(8)$. Another strongly regular graph with the same parameters is obtained from the rank 3 representation of $Sp(6,2)$ on $PGU(3,3^2)$. It is possible to prove that the corresponding regular 2-graph is not $O^{-1}(8)$. 
Table 1

The parameters of regular 2-graphs with less than 400 points

The following table gives the possible parameters for a regular 2-graph on N points with $N \leq 400$ and $a \leq a'$ as obtained from (2.5.8), omitting those with non-integral eigenvalues. Where a regular 2-graph exists we give an example or a reference to the appropriate theorem.

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