Geometry of smooth Gaussian fields

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To *Amma, Appa.*

Acknowledgments: First of all, I thank Dmitry for being an amazing advisor. I have enjoyed all the meetings with him and learnt a lot from the high-level explanations, especially in the initial years of my PhD. My officemates Julius and Rui were always available for discussions, especially when my ideas were too incoherent to pitch to Dmitry (which was most of the time). I also thank Yogesh for his support, beginning with my masters in Bangalore, and for his advice and maths discussions. Zhongmin and Bálint Tóth were also immensley helpful. I am grateful to Ben Hambly and Igor Wigman for going through the thesis and suggesting improvements.

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Abstract

Gaussian fields are ubiquitous in probability as they are scaling limits of many natural objects, and in applied science as they are instrumental in modelling natural phenomena. In this thesis, we study geometric features of smooth Gaussian fields, like the measure of level sets and the structure of critical points of the field. Large-scale geometry, i.e. studying geometric observables in a domain where the domain size goes to infinity, is of particular interest.

In the first chapter, we formally define smooth Gaussian fields and state their basic properties. Then we briefly explain connections to other topics in mathematics, including percolation theory, quantum chaos, and real algebraic geometry.

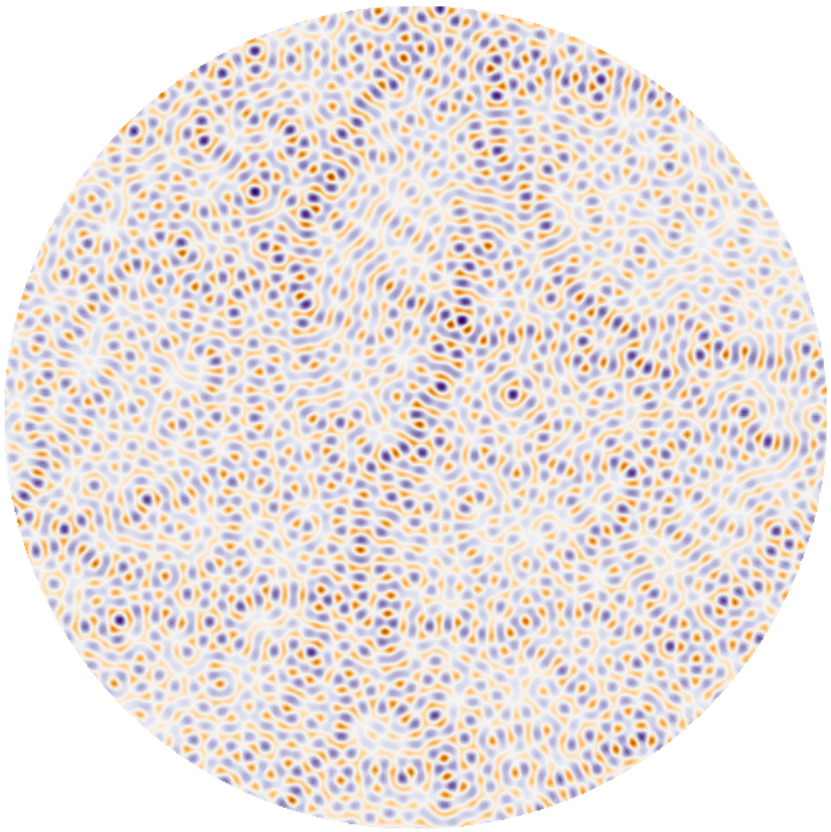
Next, we study the measure of level sets of stationary Gaussian fields. Given two Gaussian fields that are close, we ask how close the measures of their level sets are in a given domain. Here we bring novel ideas (in this context) from geometric analysis to answer this question. We prove the convergence of the Hausdorff measures of level sets of smooth Gaussian fields when the levels converge. Given two stationary fields f_1, f_2 on a probability space , we estimate the difference of Hausdorff measures of level sets in expectation, in terms of C^2-fluctuations of the field $F = f_1 - f_2$. We prove this using the mean curvature representation of the difference in measure of level sets. This approach is different from using a Kac-Rice type formula as the primary tool in the analysis. This chapter is based on joint work with Dmitry Belyaev [BH23]. We extend this result in several directions, thanks to a subsequent work of Peccati and Stecconi [PS25]. The extended result also applies to some non-stationary fields and improves the exponent of the bound to a (conjecturally) optimal one. Also, we require only C^0-fluctuations of the difference field F. The extended result is an ongoing work with Michele Stecconi and Francesca Pistolato.

In the third chapter, we study the critical point structure of smooth Gaussian fields. We consider the point process (in $\mathbb{R}^d$) of local maxima of Gaussian fields, with sufficient decay of correlation at infinity, above a level u. We show that this point process, rescaled appropriately, converges weakly to a Poisson point process in the limit $u \to \infty$. In the literature, high excursion sets of many smooth and non-smooth Gaussian processes of decaying correlation are well studied [LLR83]. Also, for Gaussian processes with some Markov property, like Brownian motion or the Gaussian free field, high points (after suitable rescaling) have been shown to converge to a Poisson process. Our proof relies on the classical observation that simple point

processes are characterised by avoidance probabilities (i.e. $\mathbb{P}(\eta(B) = 0)$ for Borel sets B). Then, we approximate avoidance probability with excursion probability, where the latter is well studied.

Another significant result in the thesis is a quantified version of the Poisson convergence of high critical points of smooth Gaussian fields. We show that, for Bargmann-Fock field in two dimensions, the total variation distance between a Poisson random variable and the number of local maxima of the field above a threshold u in an $R \times R$ box in $\mathbb{R}^2$ decays like $\exp(-\beta u^2)$, for some fixed $\beta > 0$. As an immediate consequence, when the level u is a function of R such that $u(R) \to \infty$ and $u(R)/\sqrt{\log R} \to 0$ as $R \to \infty$, we have a quantitative central limit theorem for the number of high local maxima. The proof is based on the Chen-Stein method for quantitative Poisson approximation [CX04]. The basic idea is that, for any point process η , if η and its Palm version are "close" in some (pseudo-)metric, then η is "close" to a Poisson point process. Here, we produce a close coupling of a (non-degenerate) stationary smooth field and its Palm version. Then, we study the difference in the number of critical points using smooth flows of critical points. We believe the same method/argument works for any non-degenerate stationary Gaussian field with sufficient correlation decay and in any dimension.

This thesis's critical point structure results are joint with Dmitry Belyaev.



A portrait of the random plane waves model.

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Chapter 1

Smooth Gaussian fields

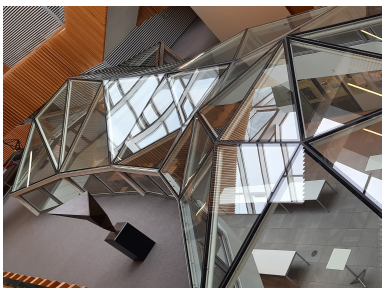
1.1 Motivation

At the end of the 18th century, the musician and physicist Chladni noticed that sounds of different pitch could be made by exciting a metal plate with the bow of a violin, depending on where the bow touched the plate (although Robert Hooke did similar experiments in the mid-17th century). The latter was fixed only in the centre, and when there was some sand on it, a curious pattern appeared for each pitch. The vibration patterns observed in Chladni figures correspond to standing wave modes of an elastic plate. Mathematically, these are described by eigenfunctions of the Laplacian (i.e. $\Delta u = -\lambda u$) satisfying under suitable boundary conditions. Hence, the study of Chladni figures reduces to analysing the nodal sets of Laplace eigenfunctions.

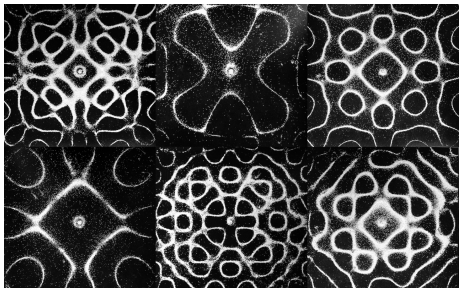
Studying the Laplace eigenfunctions and their geometry is a classical subject and of great interest to mathematicians and physicists, going back to at least the 19th century. The advent of quantum mechanics in the 20th century further fueled this area of mathematics. Much of quantum mechanics is concerned with the eigenvalue problem for the Schrodinger equation,

$$\left(-\frac{h^2}{2}\Delta + V\right)\psi = E\psi \tag{1.1}$$

where h is Planck's constant, V is the potential, ψ is the wave-function, E energy level of the wave-function. Many important questions about these involve studying the behaviour of "typical" eigenfunctions, which leads to Gaussian random fields. Examples of some of the models considered in this context include Random Plane Waves (RPW) (which are random eigenfunctions of the Laplacian in $\mathbb{R}^2$), Gaussian random linear combinations of deterministic eigenfunctions.



(a) South crystal in Oxford maths institute



(b) Chladni figures

Figure 1.1: (a) The crystal is a triangulation of the amplitude of the first overtone of that domain. (b) Sand accumulating on nodes corresponding to different resonating frequencies

Apart from this, Gaussian fields appear naturally in many practical applications. Especially when one encounters random surfaces in real life, like photographs, television pictures, topographic maps, and atmospheric pressure charts, studying the statistical properties of the contours on a surface is helpful [Swe62]. Random fields have found some applications in areas as diverse as oceanography [LH57], cosmology [BBKS86], and medical imaging [WMN+96] as well.

1.2 Smooth Gaussian fields

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $V \subset \mathbb{R}^n$ be an open set. A function $f : V \times \Omega \to \mathbb{R}$ is called a Gaussian function (more commonly, a Gaussian field) if

1. for each $x \in V$, the mapping $\omega \to f(x, \omega)$ is measurable as a mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$;

2. for each finite set of points $x_1, x_2, \ldots x_n \in V$ and for each $c_1, \ldots c_n \in \mathbb{R}$, the sum $\sum_j c_j f(x_j, \omega)$ is a Gaussian r.v. (being degenerate is allowed).

Let $k \in \mathbb{N}$. The Gaussian function f is called C^k-smooth if

3. for almost every $\omega \in \Omega$, the function $x \to f(x, \omega)$ belongs to the space $C^k(V)$.

Given a Gaussian field $f : V \to \mathbb{R}$ (which is a common abuse of notation), define its *covariance kernel* to be $K(x, y) := \mathbb{E}[f(x)f(y)]$ for $x, y \in V$. In later chapters, the covariance kernel will be denoted by $r(x, y) = \mathbb{E}[f(x)f(y)]$.

Theorem 1.2.1 (Kolmogorov's theorem, Appendix A of [NS16]). *Let $k \in \mathbb{N}$. Suppose that $K : V \times V \to \mathbb{R}$ is a positive definite symmetric function of class $C^{k,k}(V \times V)$ and, in addition, that*

$$B := \max_{|\alpha|,|\beta| \leq k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x,y)| < \infty.$$

Then there exists a (unique up to an equivalence of distribution) C^{k-} Gaussian function f on V with the covariance kernel K. Moreover, for a compact set $G \subset V$, and $p \geq 1$,

$$\mathbb{E}\left[\sup_G |f|^p\right] \leq C(G, V, k) B^{p/2}.$$

This implies if we have a 'nice/smooth' covariance kernel, we'll get a 'nice/smooth' enough field. We have plenty of examples of (interesting) positive definite functions, such as the Gaussian kernel (e^{-ax^2}), Sinc kernel ($\sin x/x$), etc. Without loss of generality, we assume that *all* Gaussian fields in this thesis are centred (i.e. $\mathbb{E}f(x) = 0, \forall x$) because we can subtract a deterministic function to get another Gaussian field.

We describe three ways to think about Gaussian fields. The first one is a more 'probabilistic' way. A Gaussian field is a stochastic process on an index set V, where the random variables $f_x = f(x), x \in V$ are jointly Gaussian. We know that, thanks to Kolmogorov's theorem, such processes are determined by pairwise covariances $K(x,y) = \mathbb{E}[f(x)f(y)]$. While it seems that the nature/topology of the index set is irrelevant from this viewpoint, it becomes important when we demand a nicer structure on the field.

A more 'analytic' approach to Gaussian fields is the following. We can think of $f(x)$ as a function drawn at random from some space of functions (say a Hilbert space). More or less equivalently, it can also be thought of as a random variable corresponding to a Gaussian measure on that function space. One canonical (and simple) way to 'write down' a Gaussian function is a random linear combination of deterministic functions, usually written as

$$f(x) = \sum a_i \psi_i(x)$$

where a_i's are i.i.d. standard normal variables and $\{\psi_i(x)\}$ are orthonormal basis of that Hilbert space.

We can switch from one viewpoint to the other in many cases. Consider $f = \sum a_i \psi_i$ as a formal series and compute $\mathbb{E}[f(x)f(y)]$ formally. We get,

$$K(x,y) = \sum \psi_i(x)\psi_i(y).$$

The other direction is a bit more involved. Call a continuous Gaussian field on $\mathbb{R}^n$ *stationary* or *translation invariant* if its covariance kernel $K(x, y)$ depends only on $x - y$, say $K(x, y) = \kappa(x - y)$. By Bochner's theorem, κ can be written as a Fourier transform of some finite symmetric (i.e. of the form $\mu(-A) = \mu(A)$) positive Borel measure ρ on $\mathbb{R}^n$, i.e.,

$$\kappa(x) = \int_{\mathbb{R}^n} e^{2\pi\sqrt{-1}(\lambda\cdot x)} d\rho(\lambda).$$

Call the measure ρ *the spectral measure of the field f.* Thinking of a Gaussian field in terms of its spectral measure is more geometric, while that of kernels is more analytic. The most important example of Gaussian fields we are interested in are stationary. The field is given by the Fourier transform of the white noise on the spectral measure. One description of white noise on the spectral measure is the following. Consider the Hilbert space $L^2_{sym}(\rho) = \{\psi \in L^2(\rho) : \psi(-t) = \overline{\psi(t)}\}$. Now, the Fourier transform of symmetric functions is real valued and forms a Hilbert space, call it $H = \mathcal{F}L^2_{sym}(\rho)$ with inner product inherited from the L^2 space. A white noise W on the measure ρ would be an isonormal process on H, i.e. an inner product preserving map from H taking values in a Gaussian Hilbert space. So our field has a representation,

$$f(x) = W(e^{2\pi\sqrt{-1}x\cdot t}).$$

By considering an orthonormal basis of H (which is the Fourier transform of an orthonormal basis of $L^2_{sym}(\rho)$), we can recover the 'analytic' picture of the field, at least on a formal level.

To think of the field in terms of spectral measure, let us start with a simple example. When the spectral measure is a two-point measure, the field is a sine wave in the direction of the line passing through these points with a Gaussian amplitude, whose variance is the measure of these points and wavelength is the inverse of the distance from the origin. That is, if $\mu = (\delta_t + \delta_{-t})/2$, where $t \in \mathbb{R}^2$, then the corresponding field is $f(x) = A\cos(2\pi x \cdot t)$. Suppose we have a spectral measure supported on $2n$ points. In that case, the field will be a random superimposition of sine waves in the direction of these points with amplitudes' variance proportional to the measure at those points. For example, [BM22, Theorem 4.1] gives a coupling of fields which are close when the spectral measures are close with high probability. Therefore, we can approximate fields by considering approximations of spectral measures, which gives us a more geometric picture.

Please refer to [AT09], [AW09], and the appendix of [NS16] for more on smooth Gaussian fields.

1.3 Connections to other topics in maths

Next, we briefly describe some topics in maths related to smooth Gaussian fields. We focus primarily on translation invariant or isotropic real-valued smooth Gaussian fields, which excludes interesting topics like random polynomials, Gaussian holomorphic functions, etc.

Spherical spin glass models: A spin glass is a disordered magnetic alloy which exhibits unusual behaviour. A simplistic mean-field model is as follows:
Let $N \in \mathbb{N}$ and a configuration $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in S^{N-1}(\sqrt{N})$ on the $(N-1)$-dim sphere of radius $\sqrt{N}$. The Hamiltonian of the model is a random function defined as

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j}\sigma_i\sigma_j$$

where $J_{i,j}$'s are i.i.d standard normal random variables. Here J's are thought to be the 'disorder' and σ_i is the spin value of i-th particle. The goal is to understand the landscape of H_N, such as the distribution of global maximum, as $N \to \infty$.

Notice that H_N is a smooth Gaussian field on the sphere whose distribution is invariant under unitary transformations (i.e. isotropic). Critical points play an essential role in the geometry of the random landscape. For example, [AAC13] shows that the logarithm of the number of critical points (called the complexity of the landscape) reveals information on the global minimum.

Understanding the landscape of high-dimensional random fields is challenging and deep. The first breakthrough was in the 1970s by Parisi [Par79] where he, non-rigorously, described the landscape of the Sherrington-Kirkpatrick model of spin glass using a variational approach. It was rigorously proven later in the 2000s by Talagrand [Tal06], building on the works of Guerra.

Another approach among many, including the Thouless-Anderson-Palmer (TAP) approach, to study the landscape is by examining the distribution of the number of critical points. Using the classical Kac-Rice formula, Fyodorov [Fyo04] observed that we can use the distribution of eigenvalues of GOE matrices for this purpose.

To be precise,

$$\mathbb{E}[\mathrm{Crt}_k] = \mathrm{vol}(S^{N-1}(\sqrt{N}))\mathbb{E}[|\det \mathrm{Hess}\ H_N(P)|\mathbb{1}[\mathrm{index}(\mathrm{Hess}\ (H_N(P))) = k]]$$

where Crt_k is the number of critical points of index k of H_N and P is any point on the sphere. Here index is the number of negative eigenvalues of the Hessian matrix at that point. Fyodorov's novel observation was that the Hessian is a GOE matrix (up to a scalar factor) plus $c\mathbf{Z}I_N$, where $\mathbf{Z}$ is an independent normal variable and c is a constant.

Please refer to [Fyo13] for an introduction to this exciting area, which links up smooth Gaussian fields, random matrix theory, and statistical mechanics.

Percolation theory: Let us first describe percolation theory briefly. One of the simplest and non-trivial percolation models is Bernoulli bond percolation. Consider the nearest neighbour graph of the planar lattice $\mathbb{Z}^2$. Remove every edge with a fixed probability $(1-p) \in [0,1]$, independently of each other. We are interested in the large-scale connectivity property of the random graph thus obtained. One of the first questions we ask about a percolation model is whether there is an infinite cluster. Let $\theta(p)$ be the probability that, say, the origin is in an infinite cluster and define $p_c := \inf_{p\in[0,1]}\{p : \theta(p) > 0\}$, the critical probability.

Many natural questions crop up immediately, some of them being: 1) Is p_c non-trivial? 2) Is there an infinite cluster at $p = p_c$? 3) If $0 < p_c < 1$, how does the phase transition occur? 4) When $\theta(p) = 0$, what is the typical size of clusters?
All four of the above questions are well understood in the case of planar Bernoulli bond percolation, thanks to the analysis available from dual graphs. But some of the more sophisticated questions are yet to be fully answered, like whether the scaling limit of this model (if it exists) is conformally invariant at $p = p_c$? For example, when you take the mesh size of the lattice to zero, are crossing events conformally invariant? It is conjectured that the critical percolation is conformally invariant in the scaling limit, independent of the planar lattice structure. In his celebrated paper [Smi01], Smirnov answered it positively for triangular lattices.

We can ask questions similar to those above in the context of smooth Gaussian fields. The tools/techniques used in discrete percolation models often translate directly or have an analogy in this setting, sometimes presenting many additional challenges. A notable conjecture in this regard is one made by Bogomolny and Schmidt in [BS02], which offers a bond percolation model for random plane waves (abbreviated as RPW,

defined later in (1.3)). They argue that the extrema of RPW can be thought of as a purterbed square lattice with maxima and minima alternating. The *nodal lines* (i.e. the zero set) form edges of the dual lattice, roughly speaking. Since nodal lines cannot intersect almost surely, the 'ties' between positive and negative regions of the field can be resolved both way equally probably. This paints a critical percolation picture of the random planar waves. [BK13] argued that the numericals suggested were off by around 5% .

The following form of the Bogomolny-Schmidt conjecture is believed to be true:

Conjecture 1.3.1. *All large-scale connectivity properties of the RPW nodal lines and domains are the same as for the critical percolation. In particular, all crossing events have the same scaling limits. The collection of all nodal lines has a scaling limit which is conformally invariant and the same as that of the critical percolation model.*

An extended version of this conjecture states that for a very general class of smooth Gaussian fields, obeying some regularity conditions, the nodal domains of the field should have the same large-scale properties as critical percolation clusters, and excursion sets for non-zero levels should behave like off-critical percolation clusters.

Refer to [Gri99, DC19] and references therein for more on percolation theory. See [Bel23] for a survey on the relationship between percolation theory and smooth Gaussian fields.

Machine learning: A deep neural network is one of the most important architectures in machine learning. It is at the core of many recent advancements in the field, such as computer vision (e.g. self-driving cars), natural language processing (e.g. ChatGPT), and structural biology (e.g. protein folding). The study of smooth Gaussian fields in relation to this architecture is essential, as explained below.

A feedforward neural network is a function

$$f : \mathbb{R}^d \to \mathbb{R}$$

defined by recursive compositions of affine transformations and pointwise nonlinearities. For an input $x \in \mathbb{R}^d$, the network with L hidden layers and widths $n_1, \ldots, n_L$

computes:

$$\begin{aligned} h^{(0)} &= x, \\ h^{(\ell)} &= \phi\left(W^{(\ell)} h^{(\ell-1)} + b^{(\ell)}\right), \quad \ell = 1, \ldots, L, \\ f(x) &= W^{(L+1)} h^{(L)} + b^{(L+1)}, \end{aligned}$$

where $W^{(\ell)} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$ and $b^{(\ell)} \in \mathbb{R}^{n_\ell}$ are weight matrices and bias vectors, and $\phi : \mathbb{R} \to \mathbb{R}$ is a fixed activation function (e.g., ReLU, tanh) applied elementwise.

Infinite width limit and Gaussian process convergence: A remarkable result is that a randomly initialised network converges to a Gaussian field in the limit of *infinite width* and *fixed depth*. Let each weight be independently drawn as

$$W_{ij}^{(\ell)} \sim \mathcal{N}\left(0, \frac{\sigma_w^2}{n_{\ell-1}}\right), \quad b_i^{(\ell)} \sim \mathcal{N}(0, \sigma_b^2),$$

and assume ϕ is Lipschitz continuous. As all layer widths $n_1, \ldots, n_L \to \infty$ jointly, the random function f converges in distribution to a Gaussian field f with a kernel $K^{(L)}$ defined recursively as:

$$\begin{aligned} K^{(0)}(x, x') &= \frac{1}{d} x^\top x', \\ K^{(\ell)}(x, x') &= \sigma_b^2 + \sigma_w^2 \, \mathbb{E}_{(u,v)\sim\mathcal{N}(0,\Sigma^{(\ell-1)})} \left[\phi(u)\phi(v)\right], \\ \Sigma^{(\ell-1)} &= \begin{bmatrix} K^{(\ell-1)}(x, x) & K^{(\ell-1)}(x, x') \\ K^{(\ell-1)}(x', x) & K^{(\ell-1)}(x', x') \end{bmatrix}. \end{aligned}$$

The limiting kernel $K^{(L)}$ depends on the depth L, the activation ϕ, and the initialization variances σ_w^2, σ_b^2. For smooth activation functions (e.g., erf, tanh), $K^{(L)}$ is a smooth, positive definite function of x and x'. The corresponding Gaussian field thus has almost surely smooth sample paths, which reflects the strong regularity bias of deep networks with random weights [LXN+18]. Recent quantified convergence results have also improved our understanding of these models [FHM+25].

In practice, the depth parameter L and dimension d are large, and the landscape f usually is highly non-convex. Ideas from spin glass theory have helped understand these high-dimensional landscapes, for example, see [Hop82]. Recently, progress has been made on understanding what the neural network does by studying the topological complexity of the landscape.

Real algebraic geometry: The first part of Hilbert's 16th problem asks for an analysis of the number of components and arrangement of an algebraic curve of degree n. Real algebraic hyperspaces in projective spaces are well studied and it

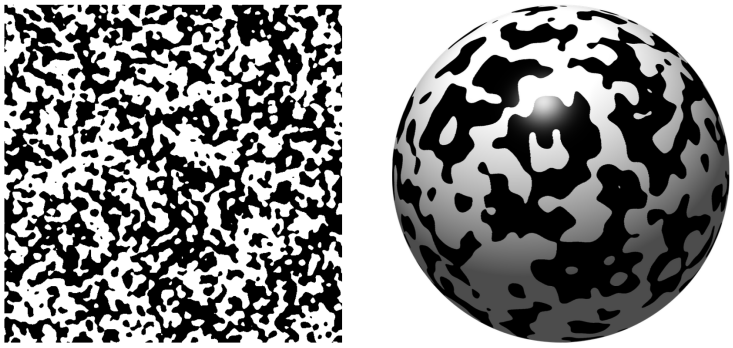


Figure 1.2: (Left) Bargman-Fock field sample. (Right) Gaussian ensemble of homogeneous polynomials of degree 300. The scale is $d^{-1/2}$ where d is the degree. Picture: Dmitry Beliaev

is interesting to investigate 'typical' behaviour of these objects. Kostlan ensembles, which is a Gaussian measure on the space of homogeneous polynomials of degree-d (say), is one of the natural objects to consider. An important case is the behaviour when the degree is large. Interesting percolation theoretic properties, like Russo-Seymour-Welsch estimates, have been observed in this model [BMW21]. Also, the Kostlan ensemble has a translation invariant local scaling limit as $d \to \infty$, called the Bargmann-Fock field.

Let us define the Bargmann-Fock field on $\mathbb{R}^2$ as follows

$$F(x) = F(x_1, x_2) = e^{-|x|^2/2} \sum_{m,n=0}^{\infty} \frac{a_{m,n}}{\sqrt{m!n!}} x_1^m x_2^n \tag{1.2}$$

where $a_{m,n}$ are i.i.d. standard Gaussian variables. This field is a scaling limit of many Gaussian fields, especially the ones with algebraic origins. Its spectral measure is the Gaussian measure on $\mathbb{R}^2$, hence the covariance kernel has super-exponential decay at infinity (or can be computed directly, $\mathbb{E}[F(x)F(y)] = e^{-|x-y|^2/2}$). Due to this rapid decay, it is sometimes amenable to percolation-theoretic methods.

In the last two decades, it was understood that methods and techniques to study smooth Gaussian fields, like the Kac-Rice formula, are useful to characterise generic topological properties of real algebraic varieties [GW16]. See also [Ler24], a recent short survey on this subject by A. Lerario.

Quantum Chaos: The dynamics of a classical particle in a bounded domain (either with some potential or no potential with reflecting boundaries) is a well-studied topic in maths and physics which goes back at least to the Newtonian era. Stability (or the

lack of) of these particles was of particular interest in dynamical systems. Consider a simple model system, that of a billiard particle. The description of the system in the language of *classical mechanics* is of a point particle moving without friction in a billiard table - a bounded planar enclosure where the particle reflects from the boundary so that the angle of incidence equals the angle of reflection. Now, a *quantum mechanical* description of this system at a given instant of time includes the wave function of the particle $\psi(x,t)$, which vanishes at the boundary of the billiard table.

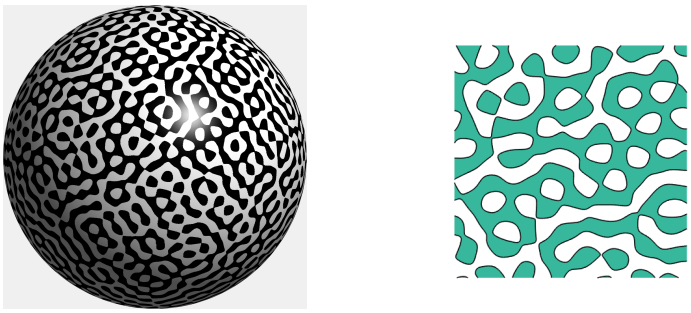
We can relate the two descriptions by taking $V = 0, E = 1, h = \lambda^{-1}$ and letting $h \to 0$ in equation (1.1), sometimes called the semiclassical limit. In 1977 [Ber77], Berry suggested that the classical behaviour of a particle on a generic billiard table is characterised by the local behaviour of wavefunctions in the semiclassical limit. Specifically, in the chaotic case, wavefunctions behave 'locally'(at a particular scale) as a uniform random superimposition of monochromatic waves in all directions, the random plane wave model (RPW) as we call it now.

The RPW has the following series representation,

$$F(x) = \sum_{\infty}^{\infty} C_n J_{|n|}(|x|) e^{in\theta} \tag{1.3}$$

where C_k are independent standard complex Gaussian random variables satisfying $C_{-k} = \overline{C_k}$ and J_k is the kth Bessel function. The covariance kernel is $J_0(|x|)$, an oscillating function that decays like $|x|^{-1/2}$. The corresponding spectral measure is the normalised arc-length on the unit circle in $\mathbb{R}^2$. Hence, sample functions are eigenfunctions of the Laplacian in the plane, almost surely. Also, as mentioned above, intuitively, the field is a random interference of monochromatic waves, uniform in all directions. RPW also appears as a scaling limit of various fields, for example 'band-limited' functions [BW18].

In the last two decades, numerous suggestions for observables have been made for the quantum chaos classification problem. Blum, Gnutzmann and Smilansky [BGS02] suggested that nodal domain count distribution as a criterion for this classification. Bogomolny and Schmidt [BS02] proposed a percolation model for the RPW model, which is the conjectured local behaviour of chaotic systems. It is also worth noting that some of the observables suggested, say eigenvalue gaps, are related to those of random matrix theory.



(a) Random spherical harmonic of high degree

(b) A closer look at random plane wave nodal lines

Figure 1.3: Both pictures by Dmitry Beliaev

Helmholtz equation: Consider $(\mathcal{M}, g)$ a closed Riemannian manifold of dimension 2 and a smooth function $f : \mathcal{M} \to \mathbb{R}$. Let Δ denote the Laplace-Beltrami operator on the manifold. The geometry of eigenfunctions of Δ is important in the analysis of PDEs on manifolds. Also, the geometry of these manifolds is related (conjecturally in some cases) to the eigenvalue problem, say of minimal embedded hypersurfaces [LY82]. One of Yau's notable conjectures states that the length of nodal lines of the eigenfunctions is comparable to the square root of the corresponding eigenvalues when they're large. Specifically, $\exists\ c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ such that

$$c_{\mathcal{M}}\sqrt{\lambda_j} \leq \text{length}(f_j^{-1}(0)) \leq C_{\mathcal{M}}\sqrt{\lambda_j} \tag{1.4}$$

where $(f_j, \lambda_j), j \geq 1$ are eigenpairs of Δ. See [LM19] for a survey on progress on Yau's conjecture. We can ask whether similar estimates as (1.4) hold on average for various Gaussian field models, for example the random band-limited function model.

Arithmetic random waves: The study of Laplace eigenvalues and eigenfunctions on the 2-torus is linked to that of lattice points on ellipses/circles in classical number theory. Let us define the random Gaussian Laplace toral eigenfunctions as follows. Consider the the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Now the eigenvalues are of the form $E_m = 4\pi^2 m$ where m is a sum of two integer squares. Let

$$\Lambda = \Lambda_m = \{\lambda \in \mathbb{Z}^2 : |\lambda|^2 = m\}$$

be the set of lattice points on the circle $\sqrt{m}\mathbb{S}^1$. For the eigenvalue E_m the collection of exponentials

$$\{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda_m}$$

forms a basis for the eigenspace. Consider the following (rescaled) Gaussian ensemble of eigenfunctions,

$$\psi_m(x) = \frac{1}{\sqrt{r_2(m)}} \sum_{\lambda \in \Lambda_m} a_\lambda e^{2\pi i \langle \lambda/m, x \rangle}, x \in \sqrt{m}\mathbb{T}^2$$

where $r_2(m) = |\Lambda_m|$ and a_λ is complex Gaussian with unit variance. Now the corresponding spectral measure is,

$$\nu_m = \frac{1}{r_2(m)} \sum_{\lambda \in \Lambda_m} \delta_{\lambda/\sqrt{m}}$$

where δ_x is the Dirac measure on the point x. This measure is supported on $(1/\sqrt{m})\Lambda_m \subset \mathbb{S}^1$. This family of spectral measures changes in a complicated way and does not converge as $m \to \infty$. However, along a generic (i.e. density one) subsequence, it weak*-converges to the uniform measure on $\mathbb{S}^1$. To the other extreme, there are subsequences (of zero density) along which the measure weak*-converges to $1/4(\delta_{\pm 1} + \delta_{\pm i})$, called the Cilleruelo measure. Cilleruelo-type fields serve as a motivation for us to study degenerate fields in more detail.

Applications in science and engineering:

1. Cosmology: Cosmology seeks to understand the large-scale structure and evolution of the universe, where isotropic random fields model statistically homogeneous and directionally uniform phenomena such as the cosmic microwave background (CMB) radiation. The CMB is commonly modelled as a Gaussian isotropic random field on the sphere, capturing temperature fluctuations across the sky.

   The Cosmic Microwave Background (CMB) temperature fluctuations observed on the celestial sphere are modelled as a realisation of a real-valued random field on the 2-sphere $\mathbb{S}^2$. Let $T : \mathbb{S}^2 \to \mathbb{R}$ denote the temperature fluctuation field. The field is typically expanded in spherical harmonics as follows:

$$T(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(\theta, \phi),$$

   where

   - (θ, ϕ) are spherical coordinates,
   - $Y_{\ell,m}$ are the spherical harmonics, forming an orthonormal basis of $L^2(\mathbb{S}^2)$,

- $a_{\ell,m}$ are complex-valued random coefficients.

The CMB field is assumed to be isotropic, with mean zero.

Under these assumptions, the coefficients $a_{\ell,m}$ are uncorrelated and satisfy:

$$\mathbb{E}[a_{\ell,m}\overline{a_{\ell',m'}}] = C_\ell \delta_{\ell,\ell'} \delta_{m,m'},$$

where C_ℓ is the angular power spectrum and δ denotes the Kronecker delta. The angular power spectrum C_ℓ encodes the variance of the field at each scale ℓ and contains critical information about early-universe physics (like inflation).

2. String theory: String theory posits that fundamental particles are one-dimensional vibrating strings, with their different vibrational modes corresponding to particle types. In the string landscape, an immense number of metastable vacua arise from different compactifications of extra dimensions. These vacua correspond to critical points of a high-dimensional random potential, often modelled as a Gaussian random field [Zel06]. Statistical properties of these critical points such as their density and stability are studied using tools like the Kac-Rice formula. This approach helps characterise the distribution of physically viable vacua and informs probabilistic predictions within the landscape.

Chapter 2

Measure of level sets

2.1 Introduction

2.1.1 Overview

Nodal volume of smooth Gaussian fields (i.e. Hausdorff measure of zero level sets, called 'nodal' sets) is a challenging topic in probability, and its research is very active. Initial advances (in the last 20 years) in understanding the distribution of nodal volume were motivated by Berry's prediction [Ber02] on the variance of the nodal volume of random plane waves. We refer to a survey on this by Wigman [Wig24]. A couple of recent works, related to the main result of this chapter, are regarding the 'differentiability' of nodal volume. Angst and Poly [AP20] studied the absolute continuity of nodal volumes of stationary fields in $\mathbb{R}^d$ with $d \geq 3$ using the Malliavin derivative of the functional. They used a novel representation of the Kac-Rice formula which is Lipschitz continuous in $(f, \nabla f, \text{Hess } f)$ where f is the Gaussian field. In 2024, Peccati and Stecconi [PS25] vastly generalised Angst and Poly's result to fields on arbitrary compact Riemannian manifolds. This was done using a geometric characterisation of the Malliavin-Sobolev derivative of the nodal volume functional. The main ingredient in the proof was a version of the 'first variation of area' formula from geometric analysis. Our main result in this chapter and its extension can be considered complementary to [PS25]. Very vaguely, our extended result can thought of as a Poincaré inequality for nodal volumes.

Another interesting recent paper regarding the study of nodal volume is by Vernotte [Ver24] on the fractal behaviour. This paper shows that the volume of macroscopic (i.e. greater than some diameter) nodal lines of some stationary smooth Gaussian fields scales like λ^s, for some $1 < s < 2$ in a large $\lambda \times \lambda$ box. This supports the percolation conjecture of the nodal sets of Gaussian fields. Interestingly, the proof

does not use any variant of the Kac-Rice formula (which is not surprising since we are dealing with non-local observables).

2.1.2 Plan and summary of the chapter

This chapter consists of three parts/sections. The **first part** is regarding the geometry of degenerate Gaussian random fields. Specifically, we are interested in the geometry of a sequence of non-degenerate Gaussian fields converging to a degenerate field. This is motivated by Cilleruelo-type fields as explained in Section 1.3.

Many results in the literature require the field to be non-degenerate, for example, the Kac-Rice formula as stated in [AW09], [AT09]. However, in some interesting cases like arithmetic random waves, where certain subsequences converge to a degenerate field, it is essential to prove variants of standard results in this particular case. We show continuity of the expected length of level lines, even when the limiting field is degenerate, for certain Gaussian fields.

We also investigate the geometry of the nodal lines in some special cases. First, we look at the case of 4-point measures converging to 2-point measures. Next, we consider a case arising from arithmetic ensembles and study the convergence of line restrictions of the field.

In the **second part**, we prove convergence of the Hausdor measure of level sets of smooth Gaussian fields when the levels converge. Given two coupled stationary fields f_1, f_2, we estimate the difference of Hausdor measures of level sets in expectation, in terms of C^2 -fluctuations of the field $F = f_1 - f_2$. The main idea in the proof is to represent the difference in volumes of level sets as an integral of the mean curvature of the hypersurface using the divergence theorem. This representation is classical in Riemannian geometry and has been used extensively to study minimal surfaces [Law80, Chapter 1]. The novelty is to get an average estimate of the difference in volumes in the context of Gaussian fields. As a byproduct, we give an explicit formula for the mean curvature of level sets at a given level.

In the **third part**, we extend the above main result in several aspects. First, the fields f_1, f_2 that we are comparing need not be stationary and are potentially generalisable to manifolds. Second, we need only fluctuations of $\sup |f_1 - f_2|$ as opposed to the C^2-fluctuations of $\|f_1 - f_2\|_{C^2}$. This is a bit surprising, given that the change in length depends on the curvature (i.e. requires second derivatives). Most importantly, we show that the exponent of the fluctuation is 1, which was 1/7

in our previous result, in the upper bound. This exponent in the upper bound seems optimal, which is what we expect from the geometric intuition.

2.2 Degenerate fields

We call a C^1-smooth Gaussian field f degenerate if the Gaussian vector $(f, \nabla f)$ is degenerate (i.e. does not admit a density function) at some point in the domain. Let ∂_j denote the partial derivative with respect to j-th variable in $\mathbb{R}^d$.

Degenerate Kac-Rice formula: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-smooth, stationary Gaussian field with $K(x, y) = K(x - y), x, y \in \mathbb{R}^2$ as the covariance kernel and $\mathrm{Var}(f(0)) = K(0, 0) = 1$. If $(f(x), \nabla f(x))$ is degenerate we claim that , by a linear change of coordinates in $\mathbb{R}^2$, we can assume that $\partial_2 f = 0$. First observe that $f(x)$ is independent of the Gaussian vector $\nabla f(x)$ for all $x \in \mathbb{R}^2$ because

$$-\mathbb{E}[f(0)\partial_1 f(0)] = -\mathbb{E}[f(0)\partial_2 f(0)] = \partial_1 K(0) = \partial_2 K(0) = 0.$$

So the Gaussian vector $(f(x), \nabla f(x))$ is degenerate means that $\nabla f(x)$ is degenerate. Now there's a constant $c \neq 0$ such that

$$\partial_1 f(x) = c\partial_2 f(x)$$

where the constant doesn't depend on x since stationarity of the field f implies stationarity of ∇f. Let

$$A = \begin{pmatrix} 1 & 0 \\ -\frac{1}{c} & 1 \end{pmatrix}$$

then the field $g(x) = f(Ax)$ has the property that $\partial_2 g = 0$.

Note that the zero set $\{f = 0\}$ is the union of straight lines in $\mathbb{R}^2$. Let $L(D; f)$ (which we write as $L(D)$, when the field is obvious) denote the length of the nodal lines of f in a Borel set $D \subset \mathbb{R}^2$.

Proposition 2.2.1. *Let f be a field as described above. For any Borel $D \subset \mathbb{R}^2$, we have*

$$\mathbb{E}[L(D)] = \frac{vol(D)}{\sqrt{2\pi}}\mathbb{E}|\nabla f|. \tag{2.1}$$

Proof. Let R be an $a \times b$ rectangle with sides parallel to the axes. Observe that, by construction, the nodal lines are vertical. Let h be the restriction of the field f to the x-axis. Then h is a non-degenerate, stationary one-dimensional Gaussian field, so applying Kac-Rice to h we get, $\mathbb{E}[N([0, a], h)] = \frac{a}{\sqrt{2\pi}}\mathbb{E}[|h'(0)|]$ where N denotes the number of zeros of h. But $\mathbb{E}|h'(0)| = \mathbb{E}|\nabla f(0)|$, so we have $\mathbb{E}L(R) = \frac{\mathrm{Area}(R)}{\sqrt{2\pi}}\mathbb{E}|\nabla f(0)|$ for a rectangle R.

Let $\mathcal{A}$ be the set of finite union of rectangles in $\mathbb{R}^2$. Now $\mathcal{A}$ is a $\pi-$system on $\mathbb{R}^2$. Also, $\mathbb{E}L(R) = \frac{\text{Area}(R)}{\sqrt{2\pi}}\mathbb{E}|\nabla f|$ holds for any $R \in \mathcal{A}$ since an element in $\mathcal{A}$ is a disjoint finite union of rectangles except for boundary.

Let $\mathcal{M} = \{R \in \mathcal{B}(\mathbb{R}^2) : \mathbb{E}L(R) = \frac{\text{Area}(R)}{\sqrt{2\pi}}\mathbb{E}|\nabla f(0)| \text{ holds}\}$. Since length and integrals are additive over disjoint regions, $\mathcal{M}$ is a $\lambda-$system. Since we already have $\mathcal{A} \subset \mathcal{M}$, by the $\pi - \lambda$ theorem we know $\sigma(\mathcal{A}) \subset \mathcal{M}$. But $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^2)$, so $\mathcal{M} = \mathcal{B}(\mathbb{R}^2)$.

Hence equation (2.1) hold for any Borel $D \subset \mathbb{R}^2$.

□

Note that the same trick of projecting the field to a lower-dimensional non-degenerate field works for any dimension.

Convergence of mean length: Consider a sequence $\{f_n\}_{n\geq 1}$ of C^1-smooth, stationary, non-degenerate Gaussian fields on $\mathbb{R}^2$ with respective kernels K_n. Let f be a stationary, degenerate field on $\mathbb{R}^2$ with $\partial_2 f = 0$ and with kernel K. Assume that $K_n \to K$ pointwise and $\partial^\alpha K_n(0) \to \partial^\alpha K(0)$ for $|\alpha| \leq 2$. This translates to saying that the spectral measures converge in law and their first two moments converge, because the kernel is the Fourier transform of the spectral measure.

Fix a rectangle R of unit area and denote by L_n, L the length of nodal lines of f_n, f in R, respectively.

Proposition 2.2.2. *With the convergence of fields $f_n \to f$ described above, we have $\mathbb{E}L_n \to \mathbb{E}L$.*

Proof: Since all fields involved are stationary, by the Kac-Rice formula, we have to show that $\mathbb{E}|\nabla f_n(0)| \to \mathbb{E}|\nabla f(0)|$. Let $\partial_{11}K_n(0) = p_n^2, \partial_{22}K_n(0) = q_n^2, \partial_{12}K_n(0) = r_n^2$. So we have $q_n, r_n \to 0$ as $n \to \infty$ and $p_n^2 \to \partial_{11}K(0)$. Now, after a change of variables $x \mapsto x/p_n, y \mapsto y/q_n$ in the standard formula for computing the mean, we have

$$\mathbb{E}|\nabla f_n(0)| = \frac{1}{2\pi\sqrt{1-r_n^2}} \int_{\mathbb{R}^2} \sqrt{p_n^2x^2 + q_n^2y^2} \exp\left(-\frac{1}{2(1-r_n^2)}(x^2 - 2r_nxy + y^2)\right)dxdy.$$

Note that $|x^2 - 2r_nxy + y^2| \leq 2(x^2+y^2)$. So the integrand above uniformly converges to $|\partial_{11}K(0)x| \exp(-1/2(x^2+y^2))$, because the exponential dominates the polynomials away from the origin. Hence, the integrals also converge, which completes the proof.

■

Note that additional symmetry of the covariance kernel of the field f also would have been sufficient for the convergence of length in mean. For example $K(x,-y) = K(x,y)$ for all $x, y \in \mathbb{R}$. This implies $\partial_{12}K(0) = 0$ because $\partial_{12}K(0) = \int_{\mathbb{R}^2} t_1 t_2 d\mu(t)$, where $t = (t_1, t_2)$ and μ is the spectral measure of f. Due to the imposed symmetry of the spectral measure, the integral is zero.

Geometry of nodal lines of f_n: Now that we have proved that the length of nodal lines converges, we look at the geometry of convergence. Specifically, we show that the proportion of the length of nodal lines of f_n that are almost vertical is nearly 1. To make computation simpler, as explained above, we assume the additional symmetry condition $K(x,-y) = K(x,y)$ for all $x, y \in \mathbb{R}$, so that $\partial_1 f_n(x)$ and $\partial_2 f_n(x)$ are independent.

For $\delta > 0$, let $A_n(\delta) = \{\arg(\nabla f_n(0)) \in (-\delta, \delta) + m\pi, m \in \mathbb{Z}\}$, i.e. $A_n(\delta)$ is the event where the tangent at 0 is vertical within an angle of δ. Consider a box of unit area. By the Kac-Rice formula, using stationarity of the fields, we have

$$\mathbb{E}L_n = \frac{1}{\sqrt{2\pi}} \left(\mathbb{E}[|\nabla f_n(0)| \mathbb{1}[A_n(\delta)]] + \mathbb{E}[|\nabla f_n(0)| \mathbb{1}[A_n(\delta)^c]] \right).$$

Note that $1/\sqrt{2\pi}\mathbb{E}[|\nabla f_n(0)| \mathbb{1}[A_n(\delta)^c]]$ is the contribution from the nodal lines which are not vertical within an angle δ. Chapter 3, section 2 of [AW09] explains that the Kac-Rice formula has many variants with similar proofs. One variant is the expected length of nodal lines with a gradient at that point in a certain direction. Recall that $\nabla f_n \sim N\left(0, \begin{pmatrix} p_n^2 & 0 \\ 0 & q_n^2 \end{pmatrix}\right)$. Now, $A_n(\delta)$ is contained in the event $\left\{ \left| \frac{\partial_2 f_n}{\partial_1 f_n} \right| \leq 2\delta \right\}$ for δ small enough. Now we have

$$\frac{p_n \partial_2 f_n}{q_n \partial_1 f_n} \sim \text{Cauchy}(0,1).$$

So $\mathbb{P}(A_n(\delta)) \geq \frac{2}{\pi} \arctan(\frac{2p_n\delta}{q_n})$, which implies $\mathbb{P}(A_n(\delta)^c) \leq C\frac{q_n}{\delta}$ where C is an absolute constant.

Now, by Cauchy-Schwarz inequality, we have

$$\mathbb{E}[|\nabla f_n(0)| \mathbb{1}[A_n(\delta)^c]]) \leq C_1 \sqrt{\frac{q_n}{\delta}}.$$

When $q_n \ll \delta$ we see that the segments of the nodal line contributing to length are nearly vertical. At this point, we note that the global geometry of the nodal lines, probability of a given component of the nodal line coonnecting top of the box to the bottom, of f_n's are usually non-trivial, see Figure 2.1.

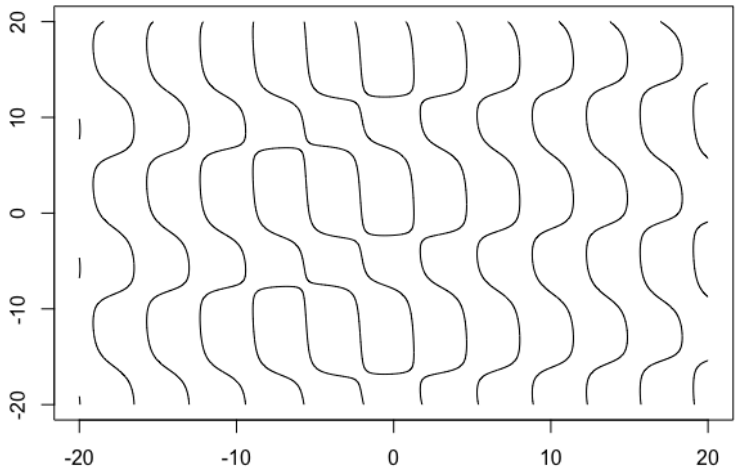


Figure 2.1: Nodal lines of a field with spectral measure supported on $\pm x, \pm(1,0), \pm y$ with equal mass, where $x = (\cos(\pi/7), \sin(\pi/7)), y = (\cos(\pi/7), -\sin(\pi/7))$.

Arithmetic ensemble: Here we want to show that the nodal lines of f_n and f are close, demonstrating using an example from an ensemble of random toral eigenfunctions. We look at the product $f \cdot f_n$, specifically its restriction g to a straight line. We want to show that the nodal intersections are typically pairs of very close points (to deduce that nodal lines of f_n and f are close in some nice box). By continuity, in some suitable box, g should have the same sign with high probability (as typically its sign should change twice in rapid succession).
To see this quantitatively, if X, Y are standard Gaussians of correlation τ, then the density of the product $Z = XY$ is given by

$$\psi_Z(z) = \frac{1}{\pi\sqrt{1-\tau^2}} \exp\left(\frac{\tau z}{1-\tau^2}\right) K_0\left(\frac{|z|}{1-\tau^2}\right) \tag{2.2}$$

where K_0 is the Bessel function of the second kind. Integrating ψ over $\mathbb{R}^+$,

$$\mathbb{P}(Z > 0) = \frac{\pi - \arccos(\tau)}{\pi} \tag{2.3}$$

i.e. this probability is close to 1 or 0 for τ close to ± 1.

Let us define the arithmetic ensemble that we want to consider. For $m \in \mathbb{N}$, define the ellipse

$$p(x,y) = x^2 + \frac{y^2}{a} = m, \qquad a \in \mathbb{N},\ a \geq 2. \tag{2.4}$$

Let $\nu_{p;m} = \frac{1}{\mathcal{N}_m}\sum_{\lambda\in\Lambda_m}\delta_\lambda$ be the discrete probability measure defined by assigning delta masses to the $\mathcal{N}$ projected lattice points of the level set $\{p = m\}$. For p as in equation (2.4), there exists a subsequence $(m_k)_{k\in\mathbb{N}}$ such that $\nu_{p;m_k} \Rightarrow (\delta_1 + \delta_{-1})/2$ (see [CC94, Theorem 2]).

Recall the notation for a in equation (2.4). We define a set of $\mathcal{N} \geq 6$ atoms (of equal mass) at the distinct points

$$w_j = \left(\frac{\sqrt{a}\cos(\varphi_j)}{\sqrt{a\cos^2(\varphi_j) + \sin^2(\varphi_j)}}, \frac{\sqrt{a}\sin(\varphi_j)}{\sqrt{a\cos^2(\varphi_j) + \sin^2(\varphi_j)}}\right), \qquad j = 1, \ldots, \mathcal{N}$$

where we assume that the angles φ_j satisfy

$$-\epsilon < \tan(\varphi_j) < \epsilon, \qquad j = 1, \ldots, \mathcal{N}$$

for some fixed small $\epsilon > 0$. We also assume these points to be invariant with respect to a rotation of π radians around the origin so that $\mathcal{N}$ is even. We order the points w_j so that the first $M := \mathcal{N}/2$ points are ϵ-close to the point $(1, 0)$. We then define G_ϵ as follows:

$$G_\epsilon(x, y) = \sum_{j=1}^{M} b_j \cos(\langle (x, y), w_j\rangle) + \sum_{j=1}^{M} c_j \sin(\langle (x, y), w_j\rangle), \tag{2.5}$$

where $b_j, c_j \sim N(0, 1/M)$ for $j = 1, \ldots, M$ and i.i.d.

With our coupling, in the degenerate field G_0 all w_j's are $(1, 0)$. The correlation coefficient $\tau(x, y)$ between G_0, G_ϵ at the point (x, y) is then given by

$$\mathbb{E}[G_\epsilon(x, y)G_0(x, y)] = \frac{1}{M}\sum_{j=1}^{M}(\cos(\langle (x, y), w_j\rangle)\cos(x) + \sin(\langle (x, y), w_j\rangle)\sin(x)) \tag{2.6}$$

$$= 1 + O(y^2\epsilon^2). \tag{2.7}$$

Replacing,

$$\mathbb{P}(Z > 0) = \frac{\pi - \arccos(\tau(x, y))}{\pi} = 1 + O(y\epsilon)$$

at a point (x, y) for the field $G_\epsilon(x, y)G_0(x, y)$.

Therefore, in a box of sides $o(1/\epsilon)$, at each point $G_\epsilon(x, y)G_0(x, y)$ is positive with high probability. If there are sign changes along a given line, there must be two of them in rapid succession, i.e., in any given direction, the nodal lines are close w.h.p. The only exception is when one of the fields degenerates in the direction of the given line. Hence, trivially, there would be no intersections of the nodal lines with the given straight line.

2.3 Main result

This section considers smooth Gaussian fields $f : \mathbb{R}^d \to \mathbb{R}$ with mild non-degeneracy conditions, of fixed dimension $d \geq 2$. Call a field *stationary* if the covariance kernel $K(x, y) = \mathbb{E}[f(x)f(y)]$ is translation invariant. For stationary fields, the kernel K is a Fourier transform of a finite symmetric Borel measure ρ, called the spectral measure.

Fix a domain $D = [-R, R]^d \subset \mathbb{R}^d$. Consider two C^2-smooth Gaussian fields $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ defined on the same probability space. Le the difference field $F = f_1 - f_2$ has the C^2-fluctuations

$$\sigma_D^2 := \sup_{x \in D} \sup_{|\alpha| \leq 2} \mathrm{Var}[\partial^\alpha F(x)].$$

Assumptions 2.3.1. *Assume that the fields f_1, f_2 are*

1. *stationary, C^2-smooth a.s.*
2. *non-degenerate, i.e. $(f_i, \nabla f_i)$ has density in $\mathbb{R}^{d+1}$ for $i = 1, 2$.*
3. *Morse functions a.s. That is, the Hessian of the function at critical points are non-degenerate a.s.*

Let $\mathcal{L}^n$ denote n-dimensional Lebesgue measure. Let $\mathcal{H}^n$ denote the n- dimensional Hausdorff measure, which is scaled so that $\mathcal{H}^n([0, 1]^n) = \mathcal{L}^n([0, 1]^n)$. Note that by the Bulinskaya lemma (see [NS16, section 5.3]), a.s. nodal sets are sub-manifolds of $\mathbb{R}^d$ of co-dimension one. So we interchangeably use the terms hypersurface measure and Hausdorff measure.

Theorem 2.3.2. *Let $\mathcal{H}^{d-1}(f_i^{-1}(a))$ denote the volume of level sets in the domain D. With the setup as above, we have*

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C(f_1, f_2)(R^d\sqrt{\log R})\sigma_D^{1/7}$$

assuming σ_D is small enough (say, $\sigma_D < 1$). Here, the constant $C(f_1, f_2)$ depends only on the laws of the fields and not the coupling.

The factor $\sqrt{\log R}$ appearing in the above theorem is from the quantitative version of Kolmogorov's existence theorem for smooth Gaussian fields as stated in [NS16, Appendix A]. We believe the optimal exponent of σ_D is 1 due to cancellations in the integral of mean curvature in the bulk.

We make some comments on the assumptions on the fields. We believe that the proof of Theorem 2.3.2 works for non-stationary fields with positive lower bounds on fluctuations of the field and its derivatives, with suitable modifications, but the computations become tedious. Only Corollary 2.3.4 uses the stationarity assumption crucially. Also, the assumption that the fields are a.s. Morse functions are not very restrictive, and many interesting non-degenerate fields we know are Morse functions a.s. It can be shown that stationary fields with spectral measures containing an open set are Morse a.s. If the field is isotropic, then also we can show that the field is Morse a.s. In particular, the random plane wave (RPW) model and Bargmann-Fock field (on $\mathbb{R}^d$) are Morse a.s.

One such field coupling is available using white noise coupling (see [BM22]). The coupling as in [BM22] gives the following estimate for the fluctuations of the field $F = f_1 - f_2$. We have,

$$\sigma_D^2 \leq C(R^d + 1) \inf_{\rho \in \mathcal{P}(\rho_1, \rho_2)} \int_{D \times D} (|s|^2 + |t|^2 + 1)^{2+1} |s - t|^2 d\rho(s, t) \tag{2.8}$$

where $\mathcal{P}(\rho_1, \rho_2)$ is the space of all symmetric couplings of ρ_1 and ρ_2 and C is an absolute constant. Here ρ_1 and ρ_2 are the spectral measures of f_1 and f_2 respectively. By the coupling techniques mentioned above, σ_D can be controlled by the transport distances between the measures in the domain (in the general case) or by the norm of differences in spectral densities (in special cases). Let us give an example where this is useful. Recall that the spectral measure of random planar waves is the uniform measure on the unit circle in $\mathbb{R}^2$. In the transport distance of (2.8), we can approximate this measure by a measure supported on finite points. This field corresponds to a finite superposition of pure sine waves. So we can obtain quantitative bounds on the difference in lengths.

To prove Theorem 2.3.2, first we study the convergence of the volume of level sets using the divergence theorem. Although expressing change in volume of a hypersurface in the normal direction in terms of mean curvature is classical, as previously mentioned, we need a version as in Proposition 2.3.3.

Proposition 2.3.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-degenerate, C^2-smooth Gaussian field which is a Morse function a.s. Let $\mathcal{H}^{d-1}(f^{-1}(a))$ denote the volume of the level set $f^{-1}(a)$ in D. Then, almost surely, we have*

$$\mathcal{H}^{d-1}(f^{-1}(b)) - \mathcal{H}^{d-1}(f^{-1}(a)) = \int_D \kappa \mathbb{1}_{f\in[a,b]} dvol - \oint_{\partial D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]} dS \quad (2.9)$$

where

$$\kappa = div\left(\frac{\nabla f}{|\nabla f|}\right)$$

is $(d-1)$ *times the mean curvature of the level set of* f *at* x *and* $\hat{\eta}$ *is the outward unit normal to the* $(d-1)$*-dimensional slabs of* ∂D*. We also have*

$$\mathcal{H}^{d-1}(f^{-1}(b)) \to \mathcal{H}^{d-1}(f^{-1}(a)), \text{ as } b \to a$$

almost surely and in L^1*.*

As a corollary, we get the following formula for the mean curvature of level sets at a given level. Usually, it is hard to get such an explicit formula for general fields.

Corollary 2.3.4. *With assumptions as in Theorem 2.3.2 on* f *rather than* f_1, f_2 *, for* $x \in \mathbb{R}^d$ *we have*

$$\mathbb{E}[\kappa(x)|f(x) = a] = -a\mathbb{E}[|\nabla f(x)|].$$

2.3.1 Proofs

Proof of Proposition 2.3.3. Note that f has only finitely many critical points in D a.s. We prove in subsection 2.3.2 that κ as a function on D is integrable almost surely. We can also assume that f has no critical points on ∂D. This is because of the Bulinskaya lemma (Lemma A.2.2), since ∂D is $(d-1)$-dimensional and for non-degenerate, smooth Gaussian f the gradient ∇f has (Gaussian) density on $\mathbb{R}^d$.

Case 1: a, b are regular values of f.

Let $R' = D \cap f^{-1}[a,b]$ with the unit outward normal $\hat{\eta} = -\nabla f/|\nabla f|$ on $f^{-1}(a)$, $\hat{\eta} = \nabla f/|\nabla f|$ on $f^{-1}(b)$ (assuming $a < b$), and outward normal on parts of $\partial D \cap f^{-1}(a,b)$. Assume that R' has no critical points of f, and we know that κ is continuous except at critical points of f. Applying the Greens formula for the function $\nabla f/|\nabla f|$ on R', we get

$$\int_{f^{-1}(b)\cap D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle dS + \int_{f^{-1}(a)\cap D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle dS + \oint_{\partial D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]} dS = \iint_{R'} \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) d\text{vol}. \quad (2.10)$$

But first two terms of the LHS of the above equation are $\mathcal{H}^{d-1}(f^{-1}(b)), -\mathcal{H}^{d-1}(f^{-1}(a))$ respectively. Hence, we get equation (2.9) in this case.

If R' contains critical points of f, then the number of critical points has to be finite. Let $\{x_1, x_2, \dots x_k\}$ be the critical points in R'. Now apply the divergence theorem to the field $\nabla f/|\nabla f|$ on $R' \setminus \cup_j B_\delta(x_j)$. Letting $\delta \to 0$, and using integrability of κ on D (see subsection 2.3.2), we again get equation (2.9).

Case 2: a or b (or both) are critical values of f.

First, let us show continuity of volume of level sets at all levels, including at critical values of f. Fix a critical value a of f. By the Morse lemma [GP74, page 42], f can be made a quadratic function at a critical point by re-parametrisation. Let p be a critical point, then there is a neighborhood U of p and a smooth chart $(y_1, y_2, \dots, y_d)$ such that $y_i(p) = 0$ and

$$f(y) = f(p) \pm y_1^2 \pm y_2^2 \cdots \pm y_d^2.$$

We know that the volume of level sets of quadratic functions is continuous. So, given a critical point p of f at level a, the volumes of the level sets of f in a neighbourhood U of p converge when the levels converge to a. When $x_0 \in f^{-1}(a)$ is a regular point, then there exists a neighbourhood U_{x_0} such that the volume of level sets is continuous. This follows from the implicit function theorem. Now, using compactness of $f^{-1}(a) \cap D$, we get that the volume of level sets is continuous at any arbitrary level.

Since the number of critical values of f is finite in D, any critical level in D can be approximated by regular levels of f in D. Let ϵ_n be a sequence converging to zero such that $(b - \epsilon_n), (a + \epsilon_n)$ are sequences of such regular values of f. By continuity of the volume of level sets, we have

$$\mathcal{H}^{d-1}(f^{-1}(b)) - \mathcal{H}^{d-1}(f^{-1}(a)) = \lim_{n\to\infty} [L(b - \epsilon_n) - L(a + \epsilon_n)].$$

Using case 1, we have the integral formula for the difference of the volumes of level sets. Note that

$$\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a+\epsilon_n, b-\epsilon_n]} \to \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]},$$
$$\kappa \mathbb{1}_{f\in[a+\epsilon_n, b-\epsilon_n]} \to \kappa \mathbb{1}_{f\in[a,b]}$$

pointwise. Hence, by the dominated convergence theorem, we also have equation (2.9) for case 2.

We have that $\mathcal{H}^{d-1}(f^{-1}(b)) \to \mathcal{H}^{d-1}(f^{-1}(a))$ as $b \to a$ a.s. by above discussion of continuity of length w.r.t levels. We also have $\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))] \to \mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))]$ when $b \to a$ by the Kac-Rice formula. Hence, by Scheffe's lemma, we have L^1 convergence.

□

Proof of Corollary 2.3.4. Take expectation on both sides of the equation (2.9). Switching integration and expectation by Fubini's theorem, we get

$$\begin{aligned}\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))]-&\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))] \\ &= \iint_D \mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}]d\text{vol} - \oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta}\right\rangle \mathbb{1}_{f\in[a,b]}\right] dS.\end{aligned}$$

Now, divide the above equation by $b-a$ and try taking the limit $b \to a$.

First, from the stationary Kac-Rice formula, we have

$$\lim_{b\to a} \frac{\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))] - \mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))]}{b-a} = -ap(a)\mathcal{L}^d(D)\mathbb{E}[|\nabla f|].$$

where p is the pdf of a standard Gaussian random variable. Next, from the continuity of the Gaussian regression formula, we get the following conditional expectations (see [AW09, Theorem 3.2] for an explanation). Consider the expression $\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}]$ and write it in the following form,

$$\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}] = \int_a^b \mathbb{E}[\kappa | f = u]p(u)du$$

Now note that $\mathbb{E}[\kappa|f=u]$ is continuous in u, hence $(b-a)^{-1}\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}] \to \mathbb{E}[\kappa|f=a]$ as $b \to a$. By the dominated convergence theorem, we have

$$\lim_{b\to a} \frac{1}{b-a} \iint_D \mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}]d\text{vol} = \iint_D \mathbb{E}[\kappa|f=a]p(a)d\text{vol}.$$

A similar argument works for the claim

$$\lim_{b\to a} \frac{1}{b-a} \oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta}\right\rangle \mathbb{1}_{f\in[a,b]}\right] d\text{S} = \oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta}\right\rangle |f=a\right] p(a)d\text{S}.$$

Combining these calculations, we have the following equation

$$-ap(a)\mathcal{L}^d(D)\mathbb{E}[|\nabla f|] + \oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta}\right\rangle |f=a\right] p(a)d\text{S} = \iint_D \mathbb{E}[\kappa|f=a]p(a)d\text{vol}. \tag{2.11}$$

Now, we claim that

$$\oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle | f = a\right] d\mathrm{S} = 0. \tag{2.12}$$

Since ∇f and f are pointwise independent r.v. (by stationary), integral on a $(d-1)$-dimensional slab in ∂D cancels that from the opposite slab (also by stationarity). So we have equation (2.12).

Again by stationarity of κ, the equation (2.11) reduces to

$$\mathbb{E}[\kappa | f = a]p(a) = -a\mathbb{E}[|\nabla f|]p(a).$$

This finishes the proof of Corollary 2.3.4.

□

Proof of Theorem 2.3.2. First, observe that $\mathcal{H}^{d-1}(f^{-1}(a)) \to 0$ almost surely as $a \to \infty$ or as $a \to -\infty$, since probability that f is unbounded on D is zero. Now, taking difference of equation (2.9) applied to f_1, f_2 and taking $b = 0$, $a \to -\infty$ we have,

$$\begin{aligned}\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0)) = &\iint_D [\kappa_1 \mathbb{1}_{f_1 \le 0} - \kappa_2 \mathbb{1}_{f_2 \le 0}]\, d\mathrm{vol} \\ &- \int_{\partial D} \left[\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \hat{\eta} \right\rangle \mathbb{1}_{f_1 \le 0} - \left\langle \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta} \right\rangle \mathbb{1}_{f_2 \le 0}\right] dS.\end{aligned} \tag{2.13}$$

We bound the bulk term and the boundary term of equation (2.13) separately.

Bulk term: First we have,

$$\begin{aligned}\left|\int_D [\kappa_1 \mathbb{1}_{f_1 \le 0} - \kappa_2 \mathbb{1}_{f_2 \le 0}]\, d\mathrm{vol}\right| \le &\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[f_1, f_2 < 0] d\mathrm{vol}\right| \\ &+ \left|\int_D \kappa_1 \mathbb{1}[f_1 f_2 < 0] d\mathrm{vol}\right| + \left|\int_D \kappa_2 \mathbb{1}[f_1 f_2 < 0] d\mathrm{vol}\right|.\end{aligned} \tag{2.14}$$

For the second term of equation (2.14), we show that, with probability close to one, $\mathcal{L}^d(f_1 f_2 < 0)$ is small and that the integral of curvature is bounded with high probability.

Note that $\mathbb{E}[|\kappa_1|^{1+\alpha}] < \infty$ for all $0 < \alpha < 1$ (see Section 2.3.2). Take $\alpha = 1/2$ when applying Hölder inequality in the following computation. Given a point $x \in D$,

recall that $\kappa_1(x)$ is the mean curvature of the level set $f^{-1}(c)$, where $x \in f^{-1}(c)$, at x.

$$\begin{aligned}\left|\mathbb{E}\int_D \kappa_1 \mathbb{1}[f_1 f_2 < 0] d\text{vol}\right| &\leq \mathbb{E}\left|\int_D \kappa_1 \mathbb{1}[f_1 f_2 < 0] d\text{vol}\right| \\ &\leq \int_D \mathbb{E}|\kappa_1 \mathbb{1}[f_1 f_2 < 0]| d\text{vol} \\ &\leq (\mathbb{E}|\kappa_1|^{3/2})^{2/3} \int_D \mathbb{P}[f_1(x) f_2(x) < 0]^{1/3} d\text{vol} \\ &\leq C_1 \cdot \mathcal{L}^d(D) \sup_D [(\arccos(\tau(x)))^{1/3}]\end{aligned} \tag{2.15}$$

where $\tau(x)$ is the correlation between $f_1(x)$ and $f_2(x)$,and the constant C_1 depends only on the law of the fields. Note that $\arccos(t) = \sqrt{2(1-t)} + O((1-t)^{3/2})$ near $t = 1$. We have that $|1 - \tau(x)| \leq \sigma_D^2/2$ for all $x \in D$. Hence, we have,

$$\mathbb{E}\left[\left|\int_D \kappa_1 \mathbb{1}[f_1 f_2 < 0] d\text{vol}\right|\right] \leq C_2 \mathcal{L}^d(D) \sigma_D^{1/3} \tag{2.16}$$

where the constant C_2 only depends on the spectral measure.

Next, we'll bound the term

$$\mathbb{E}\left[\left|\int_D (\kappa_1 - \kappa_2) \mathbb{1}[f_1, f_2 < 0] d\text{vol}\right|\right].$$

Notice that

$$\mathbb{E}\left[\left|\int_D (\kappa_1 - \kappa_2) \mathbb{1}[f_1, f_2 < 0] d\text{vol}\right|\right] \leq \mathbb{E}\left[\int_D |\kappa_1 - \kappa_2| d\text{vol}\right].$$

We split the computation into two cases: $||\nabla f_i|| < \delta$ for one of the $i = 1, 2$ and $||\nabla f_i|| > \delta$ for both i's (for some fixed $\delta > 0$).
Now,

$$\begin{aligned}\int_D \mathbb{E}\left[|\kappa_1 - \kappa_2| \mathbb{1}[||\nabla f_1|| < \delta]\right] d\text{vol} &\leq (\mathbb{E}|\kappa_1 - \kappa_2|^{4/3})^{3/4} \int_D \mathbb{P}(||\nabla f_1||^2 < \delta^2)^{1/4} d\text{vol} \\ &\leq C_3 \mathcal{L}^d(D) \sqrt{\delta}.\end{aligned} \tag{2.17}$$

In the first inequality, we used the fact that curvature has $1+\alpha$ moments for $\alpha \in [0, 1)$ and applied Hölder's inequality. Observe that $||\nabla f_1||^2$ has bounded pdf around zero, so $\mathbb{P}(||\nabla f_1||^2 < \delta^2) = O(\delta^2)$.
Define

$$\beta := ||f_1 - f_2||_{C^2(D)}.$$

We exploit the explicit representation of the curvature (2.23) in terms of derivatives of the field. Given that $||\nabla f_1||, ||\nabla f_2|| > \delta$ we have,

$$|\kappa_1 - \kappa_2| \leq \frac{1}{\delta^3}(\beta p_1 + \beta^2 p_2 + \beta^3 p_3)$$

where p_i's are polynomials in the first two derivatives of f_1 of degree at most 2. Hence,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\text{vol}\right| \leq \delta^{-3}\int_D \mathbb{E}[(\beta p_1 + \beta^2 p_2 + \beta^3 p_3)] d\text{vol}. \tag{2.18}$$

Using the Cauchy-Schwartz inequality and the fact that the laws of the polynomials p_is are translation invariant, we have the following estimate,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\text{vol}\right| \leq \frac{C_4 \mathcal{L}^d(D)}{\delta^3}(\sqrt{\mathbb{E}\beta^2} + \sqrt{\mathbb{E}\beta^4} + \sqrt{\mathbb{E}\beta^6}).$$

From Theorem 1.2.1, we have the moment estimates of β,

$$\mathbb{E}[\beta^p] \leq \tilde{C}\sigma_D^p.$$

The $\mathbb{E}\beta^2$ term dominates when the coupling of the fields f_1, f_2 close, so we have,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\text{vol}\right| \leq \frac{C_5 \mathcal{L}^d(D)}{\delta^3}(\sqrt{\mathbb{E}\beta^2}). \tag{2.19}$$

Boundary term: We attend to the boundary term of equation (2.13).

$$\int_{\partial D}\left[\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \hat{\eta}\right\rangle \mathbb{1}_{f_1 \leq 0} - \left\langle \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}_{f_2 \leq 0}\right] dS =$$
$$\int_{\partial D}\left[\left\langle \frac{\nabla f_1}{|\nabla f_1|} - \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}[f_1, f_2 < 0]\right] dS + \int_{\partial D}\left[\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \hat{\eta}\right\rangle \mathbb{1}[f_1 < 0, f_2 > 0]\right] dS$$
$$+ \int_{\partial D}\left[\left\langle \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}[f_2 < 0, f_1 > 0]\right] dS \tag{2.20}$$

The analysis of bounds of the first term of the RHS of equation (2.20) is similar to that of equation (2.17). We get that,

$$\left|\mathbb{E}\int_{\partial D}\left[\left\langle \frac{\nabla f_1}{|\nabla f_1|} - \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}[f_1, f_2 < 0]\right] dS\right| \leq C_6 \mathcal{L}^{d-1}(\partial D)(\delta_1^2 + \mathbb{E}\beta/\delta_1) \tag{2.21}$$

for $\delta_1 > 0$.

Now, the second term of RHS is bounded by $C \cdot \mathcal{L}^{d-1}(\partial D \cap \{f_1 f_2 < 0\})$ since $\nabla f_1/|\nabla f_1|$ is a unit vector. By a similar argument which led to equation (2.16), we have

$$\mathbb{E}\mathcal{L}^{d-1}(\partial D \cap \{f_1 f_2 < 0\}) \leq C_8 \mathcal{L}^{d-1}(\partial D)\sigma_D. \tag{2.22}$$

This is again dominated by the quantity on the RHS of equation (2.16).

Analysis of the final bound: We combine the bounds from (2.16), (2.17),(2.19), and (2.21). to get

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C\mathcal{L}^d(D)\left(\sigma_D^{1/3} + \sqrt{\delta} + \frac{\sqrt{\mathbb{E}\beta^2}}{\delta^3} + \frac{\delta_1^2}{R} + \frac{\mathbb{E}\beta}{\delta_1 R}\right).$$

Applying the quantitative bound from Theorem 1.2.1 give us $\mathbb{E}\beta \leq C_1(R)\sigma_D$ and $\sqrt{\mathbb{E}\beta^2} \leq C_2(R)\sigma_D$. From the Dudley's entropy bound for the field $f_1 - f_2$ on D, we have $C_1(R), C_2(R) \leq c\sqrt{\log R}$ for some constant $c > 0$.

Now, choosing $\delta = \sigma_D^{2/7}, \delta_1 = \sigma_D^{1/2}$, and assuming σ_D is small enough we have,

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C(R)\mathcal{L}^d(D)\sigma_D^{1/7}.$$

□

2.3.2 Integrability of mean curvature

Moments of the curvature r.v.: We show that the $\gamma = (d - \epsilon)$-moments are finite, where $0 < \epsilon < d$, for the random variable $\kappa(x)$ for a C^2-smooth Gaussian field f on $\mathbb{R}^d$ such that $(\nabla f(x), \text{Hess } (f)(x))$ is non-degenerate for all x. We do not assume stationarity of the field f. We have,

$$\kappa = \frac{|\nabla f|^2 \text{Tr}(\text{Hess } (f)) - \nabla f \text{Hess } (f) \nabla f^{\text{T}}}{|\nabla f|^3}. \tag{2.23}$$

Notice that,

$$|\kappa| \leq \frac{\text{Tr}(\text{Hess } f)}{\|\nabla f\|} + \frac{\|\text{Hess } (f)\|}{\|\nabla f\|} \leq c\frac{\|\text{Hess } (f)\|}{\|\nabla f\|}.$$

We use the fact that the norm of the Gaussian vector $\|\text{Hess } (f)\|$ has all positive moments finite and use the Holder's inequality as follows,

$$\begin{aligned}\mathbb{E}[|\kappa|^\gamma] \leq & C\mathbb{E}\left[\frac{\|\text{Hess } (f)\|^\gamma}{\|\nabla f\|^\gamma}\right] \\ \leq & C(\mathbb{E}\|\text{Hess } (f)\|^{\gamma p})^{1/p}(\mathbb{E}\|\nabla f\|^{-\gamma q})^{1/q}\end{aligned}$$

where $q = 1 + \delta$ and $1/p + 1/q = 1$. It is simple to check that if $\mathbf{X}$ is a Gaussian vector of n-dimensions, then $\mathbb{E}[\|\mathbf{X}\|^t] < \infty$ for any $t > -n$. So, for a given $\epsilon > 0$, choosing small enough δ such that $\gamma q < d$, we are done.

Integrability of the curvature function: Consider a deterministic C^2-Morse function f on a compact domain $D \subset \mathbb{R}^d$. As above, at every $x \in D$ which is a regular point of f, define κ to be the divergence of unit normal of f.
We prove that

$$\int_D |\kappa| d\text{vol} < \infty.$$

Note that except at critical points of f, κ is continuous. So just need to show that $\int_{B_r(x_0)} |\kappa| d\text{vol} < \infty$ for a critical point x_0 of f and a small enough ball $B_r(x_0)$ around x_0.

We have $\nabla f(x) = H(f)|_{x_0}(x - x_0) + O(||x - x_0||^2)$, by Taylor's series. Since f is Morse, we can invert $H(f)|_{x_0}$ to have

$$||\nabla f(x)|| \geq C \frac{||x - x_0||}{||H(f)_{x_0}^{-1}||}.$$

Since $\partial_{xx} f, \partial_{xy} f, \partial_{yy} f$ are all bounded on D and

$$|\partial_x f(x)| \leq c_1 ||x - x_0||, |\partial_y f(x)| \leq c_2 ||x - x_0||$$

near x_0 and again, exploiting the equation (2.23), we have

$$\int_{B_r(x_0)} |\kappa| d\text{vol} \leq \tilde{C} \int_{B_r(x_0)} \frac{1}{||x - x_0||} d\text{vol}.$$

But we have

$$\int_{B_r(x_0)} \frac{1}{||x - x_0||} d\text{vol} < \infty$$

for any $d \geq 2$. This concludes the proof of the integrability of the mean curvature function on D.

2.4 Extension of the result

Let (M, g) be a compact Riemannian manifold of dimension $d \geq 2$, possibly with boundary. Let $f\colon M \to \mathbb{R}$ be a C^2-smooth Gaussian random field on M, having unit variance ($f(x) \sim \mathcal{N}(0,1)$ for all $x \in M$) and such that g is the Adler-Taylor metric of f (see [AT09, Section 12.2]), that is

$$g_x(v, v) = \mathbb{E}\|d_x f(v)\|^2, \quad \forall v \in T_x M \tag{2.24}$$

where d_x denotes the differential at $x \in M$, i.e. the map $d_x f : T_x M \to \mathbb{R}$. If $M \subset \mathbb{R}^d$ with the Euclidean metric and f is a stationary Gaussian field, then both the metric tensor g_x and $\mathbb{E}\|d_x f\|^2$ are constants in x. So (2.24) implies that $\partial_{x_i} f$ has variance 1, for each coordinate x_i. This additional constraint on partial derivatives of f is not restrictive as, given a stationary field f, there exists a matrix A such that $f(Ax)$ has the normalised partial derivatives.

Let V denote the nodal volume of f, that is

$$V(f) := \mathrm{vol}^{d-1}(f^{-1}(0)). \tag{2.25}$$

Theorem 2.4.1. *Let $f_0, f_1 : M \to \mathbb{R}, d \geq 2$ be two C^2-smooth Gaussian fields defined on the same probability space, centered with unit variance. Define*

$$h := f_1 - f_0, \qquad f_t = f_0 + th, \text{ for all } t \in [0,1].$$

Assume that the vector $(f_t(x), \nabla f_t(x))$ is non-degenerate Gaussian for all $t \in [0,1]$. Then,

$$\mathbb{E}|V(f_0) - V(f_1)| \leq cvol(M)\sigma$$

where c is a constant depending on the laws of f_0, f_1 but not on the coupling and,

$$\sigma^2 := \max_{x \in M} Var(h(x)).$$

The main idea of the proof is similar to that of Theorem 2.3.2. That is, to represent the difference in volume of level sets as the integral of the mean curvature. But the difference here is that we now have a suitable representation for our needs, thanks to Theorem 2.4.2.

The vector $(f_t(x), \nabla f_t(x))$ has density for all $t \in [0,1]$ in the following scenarios. If $(f_0(x), \nabla f_0(x), f_1(x), \nabla f_1(x))$ has density then it is an exercise in linear algebra to show positive definiteness of the covariance matrix of $(f_t(x), \nabla f_t(x))$ for

all t. Another case is when f_0 and $h = f_1 - f_0$ are independent functions and $(f_i(x), \nabla f_i(x)), i = 0, 1$ are non-degenerate. Then by a convolution argument we can show that $(f_t(x), \nabla f_t(x))$ has density.

Proof. Let $U \subset C^2(M)$ be the subset of functions f for which 0 is a regular value, that is

$$U = \{f \in C^2(M) : f(x) = 0 \implies \nabla f(x) \neq 0\}. \tag{2.26}$$

Then we can use the following fact from [PS25]. Let us define

$$\tilde{\Delta} f := \Delta f - \text{Hess } f\,(\nu, \nu)\,; \qquad \nu = \frac{\nabla f}{\|\nabla f\|}; \tag{2.27}$$

Theorem 2.4.2 (Nodal volumes as differentiable mappings, Theorem 4.6 of [PS25])**.** *Assume that M is a C^2 Riemannian manifold with boundary ∂M and let $\eta \in \Gamma^\infty(TM|_{\partial M})$ denote the outward normal vector to the boundary. Define $U \subset C^2(M)$ as in* (2.26)*. Fix $f \in U$, let $Z := f^{-1}(0)$. Then, for all $h \in C^2(M)$, we have:*

$$\langle d_f V, h\rangle = -\int_Z h \cdot \frac{\tilde{\Delta} f}{\|df\|^2} dZ + \int_{\partial Z} h \cdot \frac{g(\eta, \nu)}{\|d(f|_{\partial M})\|} dS, \tag{2.28}$$

where the existence of the integrals on the right-hand side is part of the conclusion.

Let $h \in C^2(M)$ and define $f_t := f + th$, $Z_t := f_t^{-1}(0)$ and $u(t) := V(f_t) = \text{vol}^{d-1}(Z_t)$. A standard consequence of Bulinskaya's Lemma is that $\mathbb{P}(f + h \in U) = 1$ for any $h \in C^2(M)$ and that it is possible to apply the Kac-Rice formula to f_t.
We have

$$u'(t) = \langle d_{(f_t)} V, \dot{f}_t\rangle = \int_{Z_t} \left(-\frac{\tilde{\Delta} f + t\tilde{\Delta} h}{\|\nabla f_t\|^2}\right) \cdot h + \int_{\partial Z} h \cdot \frac{g(\eta, \nu)}{\|d(f_t|_{\partial M})\|} dS \tag{2.29}$$

The above and the following integrals are all with respect to the obvious Riemannian volume density, and dS denotes the volume element of the induced Riemannian manifold. The thing we want to estimate is the following:

$$\begin{aligned}
\mathbb{E}|V(f_1) - V(f_0)| =& \mathbb{E}|u(1) - u(0)| \leq \int_0^1 \mathbb{E}|u'(t)| dt \\
&\leq \int_0^1 \mathbb{E}\left[\int_{Z_t} \frac{|\tilde{\Delta} f + t\tilde{\Delta} h|}{\|\nabla f_t\|^2} \cdot |h|\right] dt + \int_0^1 \mathbb{E}\int_{\partial Z} \left|h \cdot \frac{g(\eta, \nu)}{\|d(f_t|_{\partial M})\|}\right| dS dt \\
&\overset{Kac-Rice}{=} \int_0^1 \int_M \mathbb{E}\left[\frac{|\tilde{\Delta} f + t\tilde{\Delta} h|}{\|\nabla f_t\|} \cdot |h| \;\middle|\; f_t(x) = 0\right] p_{f_t}(0) dx dt + \\
&\qquad \int_0^1 \int_{\partial M} \mathbb{E}\left[|h \cdot g(\eta, \nu)| \middle| f_t(x) = 0\right] p_{f_t}(0) dS(x) dt
\end{aligned}$$

Note that by our assumption on the non-degeneracy of f_t and compactness of M, we know that $p_{f_t}(0)$ is bounded in t. Call the bulk and boundary terms of the above equation H, G respectively. Let $p > 1, 1 < q < 2$ such that $1/p + 1/q = 1$, then applying Hölder's inequality we have

$$H \leq \int_0^1 \int_M \mathbb{E}\left[|h|^p \middle| f_t(x) = 0\right]^{1/p} \mathbb{E}\left[\left|\frac{|\tilde{\Delta} f + t\tilde{\Delta} h|}{\|\nabla f_t\|}\right|^q \;\middle|\; f_t(x) = 0\right]^{1/q} p_{f_t}(0) dx dt \tag{2.30}$$

Note that

$$\frac{|\tilde{\Delta} f + t\tilde{\Delta} h|}{\|\nabla f_t\|} = |\kappa(f_t)|$$

where κ denotes the mean curvature (up to a constant). Now we know from Section 2.3.2 that κ has $(d-\epsilon)$-moments.

Let $\sigma^2(x)$ be the variance of $h(x)$ and τ_t be the correlation between h and f_t. Then

$$h|f_t(x) = 0 \sim \mathcal{N}(0, (1 - \tau_t^2)\sigma^2(x)).$$

By the standard theory of absolute raw moments of the Gaussian distribution [Win14], we have

$$\mathbb{E}\left[|h|^p \middle| f_t(x) = 0\right] = c(\tau_t)\sigma^p.$$

The boundary term G has very similar bounds, since $\|d_x f(v)\|^2$ has all the moments. This finishes the proof.

□

A discussion: There are several interesting directions from here. We believe that, from a rough computation from the first variation of area formula, that there should be a matching lower bound. That is,

$$\mathbb{E}|V(f_0) - V(f_1)| \simeq \sigma.$$

One can also ask if there are similar estimates for higher moments. For example, can we show that, under some extra non-degeneracy conditions on f_0, f_1,

$$\mathbb{E}|V(f_0) - V(f_1)|^m \simeq \mathrm{vol}(M)^m \sigma^m?$$

One application of these estimates is that we can recover the known CLT result for nodal volume in large domains, for smooth Gaussian fields with enough correlation decay. If we have a third moment estimate for the difference of nodal volumes, we can follow the argument of [BMM24].

Chapter 3

Critical point structure of smooth fields

3.1 Overview

The distribution of critical points of random functions is an interesting topic, aside from its applications in science and engineering. Systematic study of critical points of smooth Gaussian fields dates back to the late 1960s [Nos69]. In the 1970s, the topic was explored further by Belyaev and Piterbarg, among others. See Adler's 1980s book [Adl10] for developments till the 80s. For an overview of results, see Piterbarg's monograph [Pit96], and Adler and Taylor's book [AT09] (for a more recent one). The following paragraph lists some interesting results in the last 15-20 years.

Characterising the distribution of the number of critical points in a bounded domain of a general smooth Gaussian field is a non-trivial task currently out of reach. Theoretically, the mean number and higher moments can be estimated from the Kac-Rice formulas. At a large scale, variance estimates for specific models like random spherical harmonics are available now [CW17]. Recently, Ancona, Gass, Letendre, and Stecconi [AGLS25] (building upon previous works of [GS23, Cuz75]) computed all cumulants of the number of critical points and, as an application, proved the law of large numbers and the central limit theorem (CLT). Previously, the Wiener chaos decomposition was employed to show the CLT for certain Gaussian fields [Nic17]. The spatial structure of the critical points, the high critical point/ excursion set, was studied in detail in the latter half of the 20th century. The study of the local structure of RPW critical points was initiated in [BCW19], which inspired the generalisation of the result to other models [BCW20, LLR23].

The first part concerns the weak convergence of high critical points of stationary smooth Gaussian fields to a homogeneous Poisson point process on $\mathbb{R}^d$ (see Theorem 3.2.2). The second part quantifies the Poisson convergence result of the first part (the main result being Theorem 3.4.1).

Poisson process convergence of high excursion sets/points of random functions is a classical topic [LLR83]. The following are some results about Poisson convergence of point processes of smooth Gaussian fields (including dimension one). In one dimension, the number of upcrossings above the level u of a smooth Gaussian process, when appropriately scaled, converges weakly to the homogeneous Poisson point process as $u \to \infty$ [LLR83, Chapter 9]. In dimension 2 or more, "exit points" above the level u of a smooth Gaussian field, scaled appropriately, converges weakly to the homogeneous Poisson point process as $u \to \infty$ [Pit96, Section 15]. More recently, it was shown that local maxima of certain smooth Gaussian fields on $\mathbb{R}^d$ with $d \geq 2$ satisfy the same conclusion [Qi22, Chapter 3].

In all of the examples above, the field's correlation decay required is faster than $1/\log$ of the distance between the points. A somewhat related set of results includes limit theorems for extremal processes for the class of processes with the Markov property, like the Gaussian free field, and branching Brownian motion. Arguin et al. [ABK13] showed that the extremal process of branching Brownian motion converges weakly to a clustered Poisson process. Oleskar-Taylor, Sousi [ST20] showed that the high points (level above $\alpha\mathbb{E}[\text{maxima}], 0 < \alpha_0 < \alpha$) of the discrete GFF in $d \geq 3$ converge in total variation distance to an independent Bernoulli process on the lattice. In essence, we can expect some Poisson limit for an extremal process if either the covariance decays fast enough at infinity or there is some Markov property.

Our contribution is the study of local maxima of smooth Gaussian fields above level u. The first main result (Theorem 3.2.2) states that rescaled high local maxima converges in law to Poisson point process as the level $u \to \infty$. Our result extends [Qi22, Theorem 3.3.2], which includes important models such as random plane waves (RPW) and other monochromatic random waves in dimensions $d \geq 2$.

This weak convergence of high critical points raises natural questions: Considering local maxima above level u in a window $[-R, R]^d$, what is the Wasserstein distance between this point process and a suitable Poisson point process, given a (pseudo-)metric ? What is the role of the rate of decay of the covariance kernel $r(x, y)$ of the field? In Theorem 3.4.1, we show that the total variation distance (Wasserstein

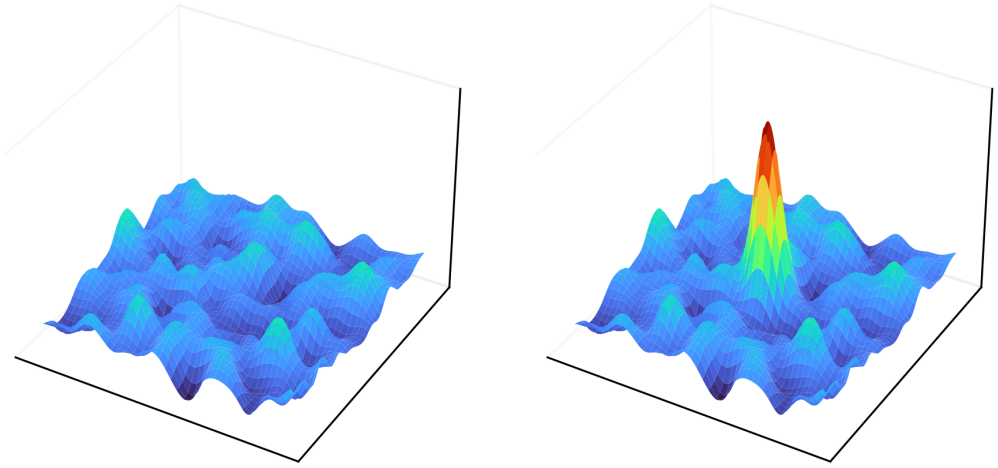


Figure 3.1: (Left) Bargmann-Fock field in a 20 by 20 box. (Right) Palm coupling of the field with maxima at origin with height at least 20.

distance with pseudo-metric identically zero) between the number of high critical points of the Bargmann-Fock field and a Poisson variable is exponentially small in the level u. The main idea of the proof is as follows.

An important characterisation Poisson point process is the following: A point process ζ is a Poisson process if and only if ζ conditioned to have a point at x has the same distribution as $(\zeta + \delta_x)$, for all x. Following Stein's idea to approximate a Gaussian distribution by looking at functionals, Chen-Xia [CX04] showed that if the functionals of the Palm measure of ζ at x and that of $(\zeta + \delta_x)$ are close, then ζ is close to being a Poisson point process. We will exploit this idea to quantify Poisson point process convergence of high critical points of stationary Gaussian fields.

In the Random Plane Wave (RPW) model, the spatial organization of the maxima exhibits a striking hierarchy. When all maxima are considered, their configuration resembles a quasi-lattice pattern, reflecting the strong short-range repulsion between critical points an effect rigorously analyzed by Beliaev, Cammarota, and Wigman [BCW19]. However, restricting attention to higher levels, say those exceeding two to three standard deviations, a filamentary network emerges, where maxima tend to cluster along elongated ridges of high field intensity. At even more extreme levels, around four and a half standard deviations and beyond, this rigidity breaks down: the high peaks become sparse and spatially uncorrelated, following statistics close to a Poisson point process. This transition from local order to apparent randomness- is clearly visible in Figure 3.2 .

In sharp contrast to the Random Plane Wave (RPW) model, the critical point structure of the BargmannFock field shows little sign of spatial rigidity. When all critical points are plotted, they exhibit an almost completely disordered, Poisson-like pattern, with no visible short-range repulsion or lattice organization (see Figure 3.3). This reflects the much faster decay of correlations in the BargmannFock field compared to the oscillatory, long-range structure of RPW. As a result, the BargmannFock field behaves more like a locally independent Gaussian surfaceits critical points scattered almost as if placed at random.

3.2 The Poisson point process convergence: A qualitative result

3.2.1 Setup and statement

Consider a C^4-smooth centered Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, with $d \geq 2$. Let $\mathbb{P}$ be the associated probability measure and $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. By $r(x, y) = \mathbb{E}[f(x)f(y)]$ we denote the covariance kernel of f.

For $u > 0$, we consider the point process of local maxima of f above the level u in $\mathbb{R}^d$, i.e. all local maximum $\zeta \in \mathbb{R}^d$ with $f(\zeta) > u$. By a theorem of Adler from 1970's (see Theorem 3.2.10), the density of this process is

$$cu^{d-1} \exp(-u^2/2)(1 + O(u^{-1})) \quad \text{as } u \to \infty$$

where the constant in $O(u^{-1})$ and c depend only on the law of f. The density goes to zero as $u \to \infty$, hence we rescale the point process to have asymptotic density one.

Define Φ_u, for $u > 0$, to be the point process such that for each Borel set $B \subseteq \mathbb{R}^d$,

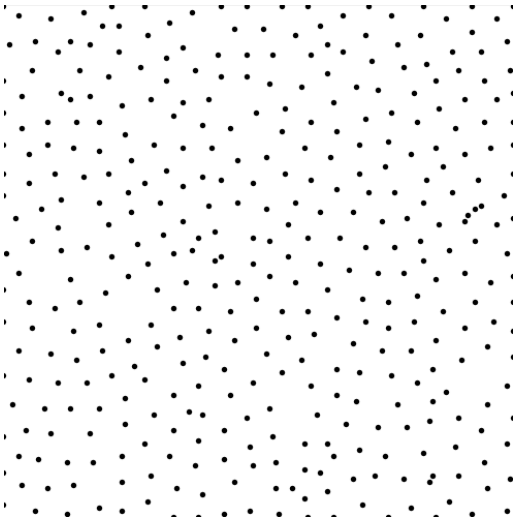
$$\Phi_u(B) = \text{ number of local maxima of } f \text{ above level } u \text{ in } \mu(u)B$$

where

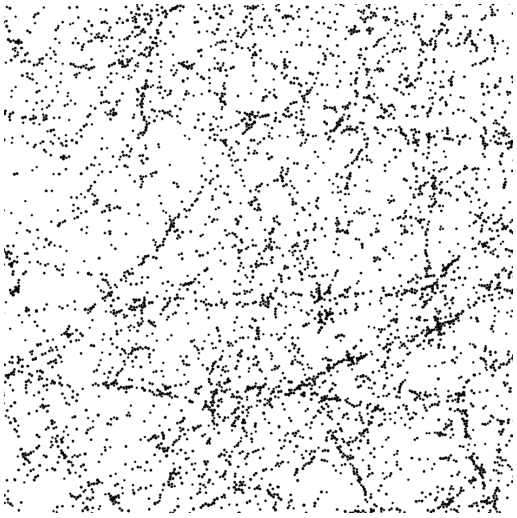
$$\mu(u) = (2\pi)^{\frac{d+1}{2d}} u^{\frac{1-d}{d}} \exp\left(\frac{u^2}{2d}\right).$$

μ chosen in such a way that the density of Φ_u is asymptotically equal to 1 (c.f. Theorem 3.2.10). We prefer to have an explicit scaling factor and asymptotic density rather than rescaling by the exact density, which does not have a simple closed form expression.

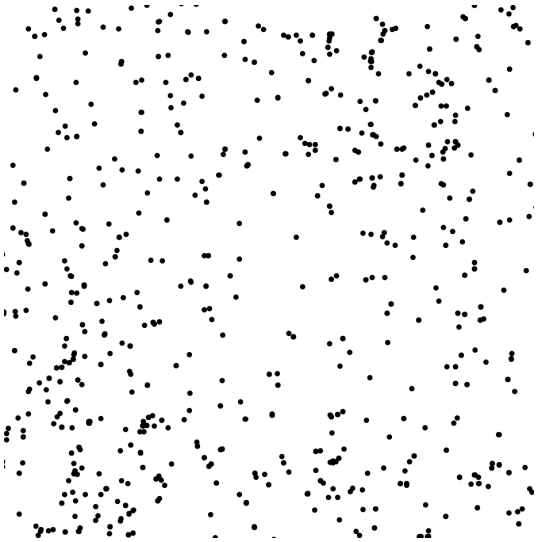
Assumptions 3.2.1. *We impose the following conditions on the Gaussian field f.*



(a) RPW: All maxima in a 20 by 20 box.



(b) RPW: Maxima above 2.5 in a 200 by 200 box.



(c) RPW: Field above 4.5 in 2000 by 2000 box, represented by dots.

Figure 3.2: Hierarchy of maxima in the Random Plane Wave model. All three pictures are from the same sample, but at different scale and level.

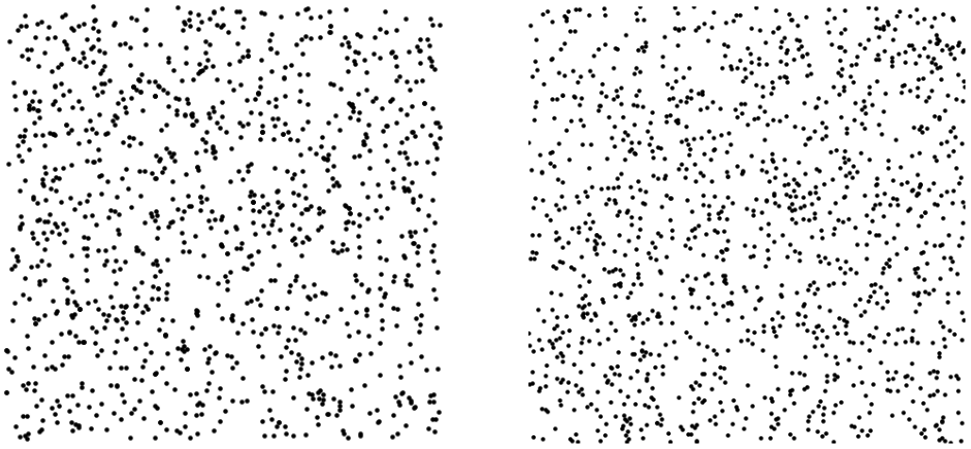


Figure 3.3: (Left) Bargmann-Fock critical points in dimension 2. (Right) Poisson point process with the same intensity.

1. *Centered* ($\mathbb{E}[f(x)] = 0$), *stationary* ($r(x,y) = r(x-y)$), *normalised* ($\mathbb{E}[f(x)^2] = 1$) *for all* $x, y \in \mathbb{R}^d$.

2. *Decay of correlation:* $r(x) = o((\log \|x\|)^{-1})$ *as* $x \to \infty$.

3. *The vector* $(f(0), \nabla f(0))$ *has density in* $\mathbb{R}^{d+1}$. *In addition, either the vector*

$$(f(0), \nabla f(0), \nabla^2 f(0))$$

   has density in $\mathbb{R}^{(d+1)+d(d+1)/2}$ *or* f *is an isotropic field, i.e.* $r(x) = r(\|x\|)$.

4. *Local structure:* $r(x) = 1 - \|x\|^2 + o(\|x\|^2)$ *as* $x \to 0$.

The stationarity assumption is very crucial to our argument. We utilise that the density of critical points is homogeneous in $\mathbb{R}^d$ and also that asymptotic densities (like μ^{-1}) of critical points are hard to come by for non-stationary fields. The decay of correlation assumption rate is optimal (i.e. necessary and sufficient) for Poisson process convergence for discretised versions (c.f [Pit96, Corollary 13.1]). Hence we believe that it is the same case for the smooth version as well. Note that the last condition is not very restrictive; it is just a convenient normalization since for any C^4-smooth field f satisfying other assumptions, there exists an invertible matrix M such that $r(Mx) = 1 - \|x\|^2 + o(\|x\|^2)$ as $x \to 0$. These normalisations (i.e. unit variance and local structure assumtpion) implies that the asymptotic density of Φ_u is one.

One observation regarding the covariance structure r is that

$$r(x, y) < 1 \quad \forall x \neq y.$$

This follows from stationarity of the field and the fact that $r(x) \to 0$ as $x \to \infty$. This is helpful when estimating the exceedance probability of the field over a large given threshold.

Theorem 3.2.2. *With the setup above and with the Assumptions 3.2.1 on the Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\Phi_u \to \Phi \quad \textit{in law as } u \to \infty$$

where Φ is the Poisson point process with intensity measure as Lebesgue measure on $\mathbb{R}^d$.

First, note that an invertible linear transform T of a Poisson point process (with intensity measure λ) is again a Poisson point process with new intensity measure $|\det(T)|\lambda$. So rescaling the field to satisfy the last condition in Assumption 3.2.1 does not change the result. This allows us to work with suitable normalized fields. Next, the Bargmann-Fock field and monochromatic random waves for dimension $d \geq 2$ satisfy the assumptions. Indeed, the covariance kernels have decay rates $\exp(-\|x\|^2/2)$ and $O(\|x\|^{-1/2})$ for Bargmann-Fock and monochromatic random waves respectively.

Another remark is that even though we have stated our result for the point process of local maxima of f, the same result holds even for *all* critical points above level u. This is because $\mu(u)^{-d}$ is still the asymptotic density of critical points above level u. For example, for random plane wave model, by Example 3.15 of [CS18] we have that intensity of local maxima above level u is $c_1 \cdot u \exp(-u^2/2)$ as $u \to \infty$ whereas the intensity of saddle points is asymptotically only $c_2 \cdot u^{-1} \exp(-3u^2/2)$.

For the Bargmann-Fock field in dimension 2, similar asymptotically explicit intensities for saddle points and local minima are calculated in Example 3.8 of [CS18]. Hence, we believe that intensities of critical points of lower index (not the top index i.e. local maxima) above a high level u is o-small of that of local maxima. This heuristic is further supported by the estimates in [AT09, Theorem 11.7.2] where the expected Euler characteristics of the excursion of f above u has the same asymptotics as that of local maxima intensity. By Morse theory, this is almost the same result we need.

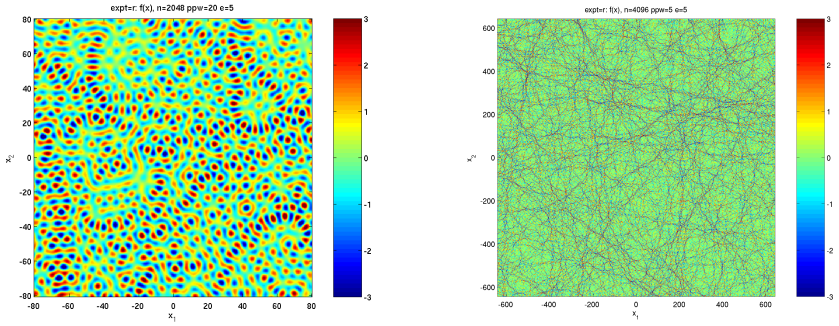


Figure 3.4: (Left) A sample of RPW in a box of sidelength around 200 wavelengths. (Right) The same with around 2000 wavelengths. Picture by Alex Barnett (Link: users.flatironinstitute.org/ ˜ahb/rpws/)

Our motivation to consider local maxima of Gaussian fields above a level was due to the apparent filament structure of local maxima above 2 to 3 standard deviation of RPW field (see Figure 3.4).

Finally, we present a simple heuristic behind the statement of the theorem 3.2.2. As the level u tends to infinity, the density of the critical points above this level tends to zero and the typical distance between them tends to infinity. Since the covariance kernel goes to zero and is independent of u, for sufficiently large u these points are so far away from each other that they are essentially independent of each other. This means that when we rescale them to have density 1 they become close to a Poisson point process. Unfortunately, it is not that easy to turn this heuristic into a rigorous argument. So our proof follows a slightly different path.

Plan of the proof

It is well known at least since the 1970's that avoidance probabilities (i.e. $\mathbb{P}(\eta(B) = 0)$ for Borel sets B) characterise simple point process (i.e. point processes with mass concentrated only on atoms where all delta measures have equal weights). Now, weak convergence of these point processes can be studied by scrutinising avoidance probabilities and intensity measures.

Definition 3.2.3 (DC-ring)**.** *Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^d$. A ring $\mathcal{L} \subset \mathcal{B}$ is called a DC-ring ('dissecting covering' ring) if for any compact set K from $\mathcal{B}$, and arbitrary $\epsilon > 0$, there exists a finite covering of K by some sets $l \in \mathcal{L}$ such that diam $l \leq \epsilon$.*

Let $\mathcal{L}$ be a ring generated by rectangles

$$\prod_{i=1}^{d}[t_i, t_i + s_i), \quad s_i \geq 0, i = 1, 2, \ldots, d$$

which will be a DC-ring with the property that $\Phi(\partial l) = 0$ a.s. for any $l \in \mathcal{L}$.

Theorem 3.2.4 (c.f. Theorem 4.18 of [Kal17])**.** *If*

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{P}(\Phi_u(l) = 0) = \mathbb{P}(\Phi(l) = 0), \\ &\limsup_{R\to\infty} \mathbb{E}\Phi_u(l) \leq \mathbb{E}\Phi(l) \end{aligned} \tag{3.1}$$

for all $l \in \mathcal{L}$, then we have

$$\Phi_u \to \Phi \quad \text{as } u \to \infty$$

in law.

We'll show (3.1) for $L = [0, 1]^d$ but the argument works for any $l \in \mathcal{L}$.

First, we approximate avoidance probabilities of the sequence Φ_u by the non-exceedance probabilities of the field f (Lemma 3.2.5). We then approximate the non-exceedance probabilities on rectangles $\mathbb{P}(\sup_{\mu(u)\cdot L} f > u)$ by that on a grid which is fine enough (Lemma 3.2.6). Then we show that for a regular enough field f with unit variance, the excursion set $\{f > u\}$ is captured by a grid with spacing of order u^{-1} for large u. Now we compare the non-exceedance probabilities of the field f to that of the field f_0 which is an i.i.d copy of f on each fixed box. This is done by the comparison method for Gaussian vectors [Pit96, Theorem 1.1] and is the same as the proof of [Pit96, Theorem 15.2]. Lastly, from Lemma 3.2.8 we show that non-exceedance probabilities of the field f_0 converge to the avoidance probabilities of the Poisson point process, which proves the first part of (3.1).

The second part of (3.1) deals with the expected number of critical points of a given index of smooth Gaussian fields. It is a classical problem in Gaussian analysis (see [Adl10]). Thanks to Kac-Rice formulas, we know precise estimates of these quantities, even explicit results in some cases. Using these estimates, we will show that

$$\lim_{u\to\infty} \mathbb{E}[\Phi_u(L)] = \mathbb{E}[\Phi(L)].$$

3.2.2 Proof of Theorem 3.2.2

Recall that L is a unit box in $\mathbb{R}^d$ and let $L_u := \mu(u) \cdot L$. Define

$$P_f(u, S) = \mathbb{P}\left(\sup_{t\in S} f(t) \le u\right) \quad \text{and} \quad \overline{P}_f(u, S) = \mathbb{P}\left(\sup_{t\in S} f(t) \ge u\right).$$

Define $\widetilde{L_u} := \{x : \operatorname{dist}(\partial L_u, x) < 1\}$, i.e. $1-$neighbourhood of the boundary of L_u. Now we approximate the avoidance probability of the point process with non-exceedance probabilities.

Lemma 3.2.5. *With the above setup, we have*

$$\mathbb{P}(\Phi_u(L) = 0) = P_f(u, L_u) + o(1) \quad \text{as } u \to \infty.$$

Proof. First, observe that $\mathbb{P}(\Phi_u(L) = 0) \ge P_f(u, L_u)$. From the fact that each connected component of $\{f(x) \ge u\}$ must have a local maximum, we have

$$\{\Phi_u(L) > 0\} \supseteq \left\{\sup_{L_u} f \ge u, \sup_{\widetilde{L_u}\setminus L_u} f < u\right\}.$$

Note that the RHS ensures that L_u has at least one component of $\{f(x) \ge u\}$ lying completely inside it. Although the boundary ∂L_u siffices instead of $\widetilde{L_u} \setminus L_u$, we need a "thickened" version of the boundary (i.e. a positive measure in the ambient dimension) to directly apply Theorem A.5.1. Hence,

$$\mathbb{P}(\Phi_u(L) = 0) \le P_f(u, L_u) + \mathbb{P}\left(\sup_{L_u} f \ge u, \sup_{\widetilde{L_u}\setminus L_u} f \ge u\right).$$

Now,

$$\mathbb{P}\left(\sup_{L_u} f \ge u, \sup_{\widetilde{L_u}\setminus L_u} f \ge u\right) \le \overline{P}_f(u, \widetilde{L_u} \setminus L_u).$$

Note that $\operatorname{vol}(\widetilde{L_u} \setminus L_u) = O(\mu(u)^{d-1})$ for large u. Applying Theorem A.5.1, using stationarity of the field, we have

$$\begin{aligned}\overline{P}_f(u, \widetilde{L_u} \setminus L_u) \le& C \cdot \operatorname{vol}(\widetilde{L_u} \setminus L_u) u^{d-1} \exp(-u^2/2)\\ =& O(\mu(u)^{-1}) \text{ as } u \to \infty.\end{aligned}$$

□

As explained, we discretise the domain and approximate the non-exceedance probabilities on this grid. Fixing $b > 0$, define $\mathcal{R}_{b,u} = bu^{-1}\mathbb{Z}^d$.

Lemma 3.2.6. *For any $\epsilon > 0$, there exists $b, u_0 > 0$ such that for all $u > u_0$,*

$$P_f(u, L_u \cap \mathcal{R}_{b,u}) - P_f(u, L_u) \le \epsilon.$$

Proof. We have

$$P_f(u, L_u \cap \mathcal{R}_{b,u}) - P_f(u, L_u) = \mathbb{P}\left(\sup_{L_u \cap \mathcal{R}_{b,u}} f \leq u, \sup_{L_u} f > u\right).$$

Let $T := \lceil \mu(u) \rceil$, divide the $[0,T]^d$ into unit boxes. By the union bound and the stationarity of the field f,

$$\begin{aligned}\mathbb{P}\left(\sup_{L_u \cap \mathcal{R}_{b,u}} f \leq u, \sup_{L_u} f > u\right) &\leq \mathbb{P}\left(\sup_{T \cap \mathcal{R}_{b,u}} f \leq u, \sup_{T} f > u\right) \\ &\leq c\mu(u)^d \mathbb{P}\left(\sup_{[0,1]^d \cap \mathcal{R}_{b,u}} f \leq u, \sup_{[0,1]^d} f > u\right).\end{aligned}$$

Divide the cube $[0,1]^d$ into smaller cubes congruent to $[0, u^{-1}]^d$. By the union bound and the stationarity of the field f,

$$\mathbb{P}\left(\sup_{L_u \cap \mathcal{R}_{b,u}} f \leq u, \sup_{L_u} f > u\right) \leq cu^d \mu(u)^d \mathbb{P}\left(\sup_{[0,u^{-1}]^d \cap \mathcal{R}_{b,u}} f \leq u, \sup_{[0,u^{-1}]^d} f > u\right). \tag{3.2}$$

From Appendix A.7, specifically (A.2) with $M = [0,1]^d$ so that $u^{-1}M = [0, u^{-1}]^d$, we have

$$\begin{aligned}\lim_{u \to \infty} \frac{1}{\sqrt{2\pi}} u e^{u^2/2} \mathbb{P}\left(\sup_{[0,u^{-1}]^d \cap \mathcal{R}_{b,u}} f \leq u, \sup_{[0,u^{-1}]^d} f > u\right) = \\ \int_0^\infty e^v \mathbb{P}\left(\sup_{[0,1]^d \cap (b\mathbb{Z}^d)} (\chi(t) - \|t\|^2) \leq v, \sup_{[0,1]^d} (\chi(t) - \|t\|^2) > v\right) dv \end{aligned} \tag{3.3}$$

where χ is a continuous Gaussian field defined in intro of Appendix A.7. As $b \to 0$, the integrand of the RHS of (3.3) tends to zero sample paths of χ are continuous a.s. Hence, by the dominated convergence theorem, the integral itself tends to zero as $b \to 0$. Note that, since

$$u^d \mu(u)^d = u^{-1} e^{-u^2/2},$$

the RHS of (3.2) bounded by RHS of (3.3) (up to a constant factor) when u is large enough. So choosing $b > 0$ small enough proves the lemma.

□

Divide the rectangle L_u into smaller ones by following construction. Divide each side of L_u into segments of length 'a' alternated by that of δ. Define $\lambda_{a,u}$ to be the union of cubes of side length a. Note that the distance between the cubes is greater than δ. The following lemma says that if the gap between the cubes of $\lambda_{a,u}$ are small enough, then the non-exceedance probabilities of the discretisation of L_u and of $\lambda_{a,u}$ are close.

Lemma 3.2.7. *For any $a, \epsilon > 0$ given, there exists $\delta > 0$, such that, for all u large enough we have,*

$$P_f(u, \lambda_{a,u} \cap \mathcal{R}_{b,u}) - P_f(u, L_u \cap \mathcal{R}_{b,u}) \leq \epsilon.$$

Proof. We have that

$$P_f(u, \lambda_{a,u} \cap \mathcal{R}_{b,u}) - P_f(u, L_u \cap \mathcal{R}_{b,u}) \leq \mathbb{P}\left(\sup_{\lambda_{a,u} \cap \mathcal{R}_{b,u}} f \leq u, \sup_{L_u \cap \mathcal{R}_{b,u}} f > u \right).$$

Now using the stationarity of the field,

$$\begin{aligned}
\mathbb{P}\left(\sup_{\lambda_{a,u} \cap \mathcal{R}_{b,u}} f \leq u, \sup_{L_u \cap \mathcal{R}_{b,u}} f > u \right) &\leq \overline{P}_f(u, L_u \setminus \lambda_{a,u}) \\
&\leq \mathrm{vol}(L_u \setminus \lambda_{a,u}) \overline{P}_f(u, L) \\
&\leq C_1 \delta \frac{(\mu(u))^d}{(a+\delta)} \overline{P}_f(u, L) \\
&\leq C_2 \delta ((\mu(u)R)^d) u^{d-1} \exp(-u^2/2) \\
&\leq C_3 \delta
\end{aligned}$$

where C_3 is a constant which does not depend on u. □

Let f_0 be a field defined on $\lambda_{a,u}$ such that on the cubes of side length a, the field is made up of independent copies of f. We now show that the non-exceedance probability of f_0 converges to the avoidance probability of the Poisson point process in L.

Lemma 3.2.8. *We have*

$$P_{f_0}(u, \lambda_{a,u}) \to \exp(-vol(L)) \quad \text{as } u \to \infty.$$

Proof. Let N be the number of cubes of side length a in $\lambda_{a,R}$. Then,

$$P_{f_0}(u, \lambda_{a,u}) = (1 - \overline{P}_f(u, [0,a]^d))^N$$

by the independence of the field on these cubes. Taking the logarithm, it is enough to estimate

$$N \log(1 - \overline{P}_f(u, [0,a]^d)) = -N\overline{P}_f(u, [0,a]^d) + O(N\overline{P}_f(u, [0,a]^d)^2).$$

Now by Theorem 3.2.10,

$$\overline{P}_f(u, [0,a]^d) = a^d \mu(u)^{-d}(1 + O(u^{-1}))$$

and

$$N = \left(\frac{\mu(u)}{a+\delta}\right)^d + O((\mu(u))^{d-1}).$$

Hence,

$$N\overline{P}_f(u,[0,a]^d) = \left(\frac{a}{a+\delta}\right)^d + o(1) \text{ and } N\overline{P}_f(u,[0,a]^d)^2 = o(1).$$

We have the result since L is a unit box and we can take δ arbitrarily small. □

The following lemma would imply the second condition in (3.1).

Lemma 3.2.9. *We have,*

$$\lim_{R\to\infty} \mathbb{E}[\Phi_u(L)] = \mathbb{E}[\Phi(L)].$$

Proof. Let us first consider the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ has a density in $\mathbb{R}^{(d+1)+d(d+1)/2}$. We follow the convenetion that we consider the upper triangle part of the matrix $\nabla^2 f(0)$ in the vector $(f(0), \nabla f(0), \nabla^2 f(0))$. As mentioned, we will use the following theorem by Adler

Theorem 3.2.10 ([Adl10] Theorem 6.3.1)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, C^2-smooth Gaussian field such that $(f(x), \nabla f(x), \nabla^2 f(x))$ is non-degenerate for all $x \in \mathbb{R}^d$. Further, assume that $f(x)$ has zero mean and unit variance. Let $M_u(f, S)$ denote the number of local maxima of f in $S \subset \mathbb{R}^d$ with $f > u$. Then,*

$$\mathbb{E}[M_u(f,S)] = \frac{vol(S)\det(\Lambda_f)^{1/2}u^{d-1}}{(2\pi)^{(d+1)/2}} \exp(-u^2/2)(1+O(u^{-1}))$$

where Λ_f is the covariance matrix of ∇f and $O(u^{-1})$ is independent of choice of S.

For the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ has a density in $\mathbb{R}^{(d+1)+d(d+1)/2}$, 3.2.10 suffices. If the field is isotropic and if $(f(0), \nabla f(0), \nabla^2 f(0))$ is degenerate (where the upper triangle of the Hession of f is vectorised), then the field has to be a monochromatic random wave (MRW) (see [CS18, Prop 3.10]). From Example 3.15 of [CS18], we can calculate the limit of $\mathbb{E}[\Phi_u(L)]$ for the case $d = 2$. But explicit expressions for height densities are hard to get for $d \geq 3$ directly. So we shift the MRW field by an independent normal random variable, so that the joint vector of the field, its gradient, and Hessian has a density. Then we use the explicit asymptotic as in 3.2.10.

By Theorem 3.2.10, for any Borel set $B \subset \mathbb{R}^d$

$$\begin{aligned}\mathbb{E}[\Phi_u(B)] &= \mathbb{E}[M_u(f, \mu(u)B)] \\ &= \frac{\mathrm{vol}(\mu(u)B)}{(2\pi)^{(d+1)/2}} u^{d-1} \exp(-u^2/2)(1 + O(u^{-1})) \\ &= \mathrm{vol}(B)(1 + O(u^{-1})) \\ &\to \mathbb{E}[\Phi(B)] \quad \text{as } u \to \infty.\end{aligned} \tag{3.4}$$

Here, we have used the fact that the determinant of the covariance matrix of ∇f is 1, which follows from point 4 of Assumption 3.2.1.

Now we consider the monochromatic random waves (MRW) case. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an MRW field. Let $\epsilon > 0$ and consider a random variable N, independent of the field f, which is a standard Gaussian r.v. Define,

$$f_\epsilon(x) := f(x) + \epsilon N, \quad x \in \mathbb{R}^d.$$

Observe that f_ϵ is still a centred, stationary field and that $f_\epsilon(0), \nabla f_\epsilon(0), \nabla^2 f_\epsilon(0)$ is a Gaussian vector with density. Define $M(u, g)$ to be the number of local maxima of a Gaussian field g in $[0, 1]^d$.
Now we have, by an application of the Kac-Rice formula,

$$\mathbb{E}[M(u, f_\epsilon)] = \int_{\mathbb{R}} \mathbb{E}[M(u - \epsilon b, f)|N = b]\phi(b)d\mathrm{b}.$$

where ϕ is the pdf of a standard normal variate. Also,

$$M(u - \epsilon b, f) \longrightarrow M(u, f) \quad \text{a.s.} \quad \text{as} \quad \epsilon \to 0.$$

Note that $M(u, f)$ is integrable and monotonic with respect to u, given that u is large, so using the dominated convergence theorem,

$$\mathbb{E}[M(u - \epsilon b, f)] \to \mathbb{E}[M(u, f)], \quad \epsilon \to 0.$$

Since $\mathbb{E}[M(u, f)]$ is uniformly bounded in u, apply the dominated convergence theorem for $\mathbb{E}[M(u - \epsilon b, f)]\phi(b)$ to get,

$$\lim_{\epsilon \to 0} \mathbb{E}[M(u, f_\epsilon)] = \mathbb{E}[M(u, f)].$$

Computing $\mathbb{E}[M(u, f_\epsilon)]$ is the same as (3.4) by applying Theorem 3.2.10.

□

Proof of Theorem 3.2.2. First, observe that all the proofs of Lemmas 3.2.5 to 3.2.8 goes through even when L is a finite union of finite rectangles. For any given $\epsilon > 0$, there exists a, b, δ, u_0 such that for all $u > u_0$,

$$|\mathbb{P}(\Phi_u(L) = 0) - P_f(u, \lambda_{a,u} \cap \mathcal{R}_{b,u})| \leq \epsilon.$$

If we show that $|P_f(u, \lambda_{a,u} \cap \mathcal{R}_{b,u}) - P_{f_0}(u, \lambda_{a,u} \cap \mathcal{R}_{b,u})| \to 0$ as $u \to \infty$ then, together with Lemma 3.2.9 and Theorem 3.2.4 implies the conclusion of Theorem 3.2.2.

Let K_i be a renumbering of cubes with edges of length a which comprise $\lambda_{a,u}$, $i = 1, 2, \ldots, N$. Let covariance of the field f_0 on $\lambda_{a,u}$ be denoted by $r_0(t, s)$. Define $\lambda'_{a,u,b} = \lambda_{a,u} \cap \mathcal{R}_{b,u}$. Then by Theorem A.4.1, we have

$$\begin{aligned} |P_f(u, \lambda'_{a,u,b}) - P_{f_0}(u, \lambda'_{a,u,b})| \leq \frac{1}{\pi} \sum_{t,s \in \lambda'_{a,u,b}} & |r(t-s) - r_0(t-s)| \\ \times \int_0^1 (1 - (hr(t,s))^2)^{-1/2} \exp\left(-\frac{u^2}{1 + hr(t,s)}\right) dh. & \end{aligned} \tag{3.5}$$

Denote the summand on the RHS of the above equation by $\beta(t, s)$. If $t, s \in K_i$ for some i, then $r(t, s) = r_0(t, s)$, hence $\beta(t, s) = 0$.
Now consider the case that t, s belong to different K_i and K_j such that $|t - s| \leq \mu(u)^{\gamma_1}$, where $\gamma_1 > 0$ is a constant chosen later. Since t, s belong different cubes, we have $|t - s| > \delta$, hence $|1 - r(t, s)| > \gamma_2 > 0$. So,

$$\frac{1}{1 + r(t, s)} > \frac{1}{2} + \frac{\gamma_2}{4}.$$

Now,

$$\begin{aligned} \sum_{\substack{t \in K_i, s \in K_j, i \neq j, \\ |t-s| < \mu(u)^{\gamma_1}}} \beta(t, s) &\leq C_1 \sum |r(t, s)| \exp\left(-\frac{u^2}{1 + r(t, s)}\right) \\ &\leq C_2 (\mu(u))^d \mu(u)^{\gamma_1 d} \exp(-(1 + \gamma_2/2)u^2/2) \\ &\leq C_3 (u^{1-d} \exp(u^2/2))^{1+\gamma_1} \exp(-(1 + \gamma_2/2)u^2/2) \\ &\to 0 \quad \text{as} \quad u \to \infty \quad \text{if} \quad 0 < \gamma_1 < \gamma_2/2. \end{aligned} \tag{3.6}$$

C_i's are different constants not depending on u.

Lastly, we consider the case where $|t-s| \geq \mu(u)^{\gamma_1}$. We have,

$$\begin{aligned}\sum_{\substack{t\in K_i, s\in K_j, i\neq j,\\ |t-s|\geq\mu(u)^{\gamma_1}}} \beta(t,s) &\leq C_1 \sum |r(t,s)| \exp\left(-\frac{u^2}{1+r(t,s)}\right) \\ &\leq C_2(\mu(u))^{2d} r'(\mu(u)^{\gamma_1}) \exp\left(-\frac{u^2}{1+r'(\mu(u)^{\gamma_1})}\right) \\ &\leq C_3 u^{2-2d} r'(\mu(u)^{\gamma_1}) \exp\left(\frac{r'(\mu(u)^{\gamma_1})u^2}{1+r'(\mu(u)^{\gamma_1})}\right)\end{aligned} \tag{3.7}$$

where

$$r'(h) := \max_{|t|\geq h} |r(t,0)|, \quad h \in (0,\infty).$$

Observing that the assumption on the decay of correlation (point 2 of Assumption 3.2.1) implies that $u^2 r'(\mu(u)^{\gamma_1}) \to 0$ as $u \to \infty$. This also implies $u^{2-2d} r'(\mu(u)^{\gamma_1}) \to 0$.

Hence,

$$\sum_{\substack{t\in K_i, s\in K_j, i\neq j,\\ |t-s|\geq\mu(u)^{\gamma_1}}} \beta(t,s) \to 0 \quad \text{as} \quad u \to \infty.$$

□

3.3 Level above the expected maxima

As explained above, the excursion of the field above level $u(n) = \sqrt{2\alpha d \log n}$ for some $\alpha > 1$ in an $n \times n$ box is empty with high probability. In this case, we can strengthen the result to show that the discrete excursion set is close the identically zero process, in total variation distance.

Assumptions 3.3.1. *Consider a C^2-smooth Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, with covariance kernel r which satisfies the following conditions.*

1. *f is stationary, unit variance, zero mean.*
2. *$r(t) = o(1)$ as $t \to \infty$ (Correlation decay).*

Note that f being stationary and $r(t) \to 0$ as $t \to \infty$ implies that $r(s-t) \neq 1$ for all $s \neq t$.

Theorem 3.3.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth Gaussian field satisfying Assumptions 3.3.1. Let $I_n = [0, n]^d \cap \mathbb{Z}^d, D = [-1/2, 1/2]^d, D_t = t + D$ for $t \in I_n$. For $u(n) = u > 0$, let*

$$X_{t,n} = X_t = \mathbb{1}\left[\max_{x \in D_t} f(x) > u\right].$$

Let $X' \equiv 0$ be the zero process. For fixed $\alpha > 1$, let

$$u(n) = \sqrt{2\alpha d \log n}.$$

Then, for some constant $c_0 > 0$

$$\frac{\|\mathcal{L}(X) - \mathcal{L}(X')\|_{TV}}{n^{(1-\alpha)d}(\log n)^{(d-1)/2}} \to c_0 \quad as \quad n \to \infty.$$

Proof. Since X' is the zero process, we have

$$\|\mathcal{L}(X) - \mathcal{L}(X')\|_{TV} = \mathbb{P}\left(\bigcup_{t \in I_n} \{X_t = 1\}\right).$$

Denote $p_t = \mathbb{P}(X_t = 1)$, $p_{st} = \mathbb{P}(X_s = 1, X_t = 1)$. By inclusion-exclusion principle (also called the Bonferroni inequality),

$$\sum_{I_n} p_t - \frac{1}{2}\sum_{s \neq t} p_{st} \leq \|\mathcal{L}(X) - \mathcal{L}(X')\|_{TV} \leq \sum_{I_n} p_t \tag{3.8}$$

By Theorem A.5.1, we have

$$p_t = c_0 \cdot u^{d-1} \exp(-u^2/2)(1 + o(1))$$

for all $t \in I_n$ using the stationarity of the field f. By plugging $u = \sqrt{2\alpha d \log n}$, we get

$$p_t = c_0 n^{-\alpha d} (\log n)^{(d-1)/2} (1 + o(1)).$$

Notice that

$$\mu_n := \mathbb{E}[\text{ number of } t \in I_n \text{ with } X_t = 1] = \sum_{I_n} p_t = n^d p_0.$$

Now we show that $\sum_{s \neq t} p_{st} = o(\mu_n)$. Let $\zeta > 0$ be an arbitrary number. We give an upper bound for p_{st} by considering the cases $|s-t| \leq \sqrt{d}$, $\sqrt{d} < |s-t| \leq n^\zeta$ and $|s-t| \geq n^\zeta$. We have

$$\begin{aligned} p_{st} =& \mathbb{P}(X_s = 1, X_t = 1) \\ =& \mathbb{P}\left(\max_{D_s} f > u, \max_{D_t} f > u\right) \\ \leq& \mathbb{P}\left(\max_{x \in D_s, y \in D_t} f(x) + f(y) > 2u\right). \end{aligned} \tag{3.9}$$

Now we use Theorem A.5.1 for the field $F(x,y) = f(x) + f(y)$. Note that if $h = \operatorname{dist}(D_s, D_t) > 0$ then

$$\operatorname{Var}(f(x) + f(y)) \leq 4 - 2\beta,$$

for all $x \in D_s, y \in D_t$, where

$$\beta = \min_{|x-y| \geq h} (1 - r(x,y)) > 0.$$

Hence, we have,

$$p_{st} \leq c u^{d-1} \exp\left(-\frac{4u^2}{2(4-2\beta)}\right) = c u^{d-1} \exp\left(-\frac{u^2}{2} \times \frac{1}{1-\beta/2}\right).$$

For the case $\sqrt{d} < |s-t| \leq n^\zeta$, there exists a constant $\delta > 0$ depending only on the covariance r such that $\beta > \delta$ for all n.
Hence,

$$\sum_{\sqrt{d} < |s-t| \leq n^\zeta} p_{st} \leq c n^d n^{\zeta d} u^{d-1} \exp\left(-\frac{u^2}{2} \times \frac{1}{1-\delta/2}\right) \tag{3.10}$$

$$\leq c n^d n^{\zeta d} n^{-\frac{\alpha d}{1-\delta/2}} (\log n)^{(d-1)/2} \tag{3.11}$$

$$\leq c n^{(1-\alpha-\delta_0)d} = o(\mu_n) \tag{3.12}$$

for $\zeta < \delta/2$ and a fixed $0 < \delta_0 < \delta/2$.

Let

$$\tilde{r}(h) := \max_{|x-y|\geq h} r(x-y)$$

and notice that $\tilde{r}(h) \to 0$ as $h \to \infty$ by our assumption. We have

$$\sum_{|s-t|\geq n^\zeta} p_{st} \leq n^d(n^d - n^{\zeta d})n^{-\frac{2\alpha d}{1+\tilde{r}(n^\zeta)}} \tag{3.13}$$

$$\leq n^{2(1-\alpha)d+o(1)} = o(\mu_n). \tag{3.14}$$

Finally we bound the sum of p_{st} for the case $|s-t| \leq \sqrt{d}$. This is the case where $dist(D_s, D_t) = 0$, i.e. the squares D_s and D_t are touching. Fix $\epsilon > 0$ and let $D^\epsilon_{s,t}$ denote the union of ϵ-neighborhoods of boundaries of D_s and D_t. Let Q^ϵ_t denote D_t with ϵ-neighborhood of boundary of D_t removed. The event $X_s = 1, X_t = 1$ can happen if for some $x \in D^\epsilon_{s,t}$, $f(x) > u$ or that f exceeds u in both Q^ϵ_s and Q^ϵ_t. Hence, by the union bound on those two events,

$$p_{st} \leq c_1\epsilon n^{-\alpha d}(\log n)^{(d-1)/2} + c_2 n^{-\frac{\alpha d}{1-\delta_1}}(\log n)^{(d-1)/2} \tag{3.15}$$

for some $\delta_1 > 0$ which depends only on ϵ. Since $1/(1-\delta_1) > 1+\delta_1$ for small δ_1, for large enough n,

$$\sum_{0<|s-t|\leq\sqrt{d}} p_{st} \leq c_3\epsilon n^{(1-\alpha)d-\delta_2}(\log n)^{(d-1)/2} = o(\mu_n) \tag{3.16}$$

where $\delta_2 = \alpha d\delta_1$. By (3.8), we have

$$\mu_n - |o(\mu_n)| \leq \|\mathcal{L}(X) - \mathcal{L}(X')\|_{TV} \leq \mu_n.$$

□

Remarks: Using more quantitative version of Theorem A.5.1 (i.e. results from [AT09, Chapter 14]) we can show that, for some fields like Bargmann-Fock field,

$$\left| c_0 - \frac{\|\mathcal{L}(X) - \mathcal{L}(X')\|_{TV}}{n^{(1-\alpha)d}(\log n)^{(d-1)/2}} \right| = O\left(\frac{1}{\sqrt{\log n}}\right) = O(1/u) \quad \text{as} \quad n \to \infty.$$

Also, we get similar upper bounds as in Theorem 3.3.2 if we compare the process X_t with a Bernoulli X' with same marginals (i.e. $X_t \overset{d}{=} X'_t$) and independent entries X'_s using the Chen-Stein's method as in [ST20, Theorem 2.1].

3.4 Quantitative convergence

In this section, we want to quantify the convergence of high local maxima of a field converging to Poisson point process inside a box of size R. We will restrict ourselves to dimension $d = 2$ in this section but we believe the result holds for any $d \geq 2$. Given a smooth Gaussian field f on $\mathbb{R}^2$, for $R, u > 0$, define

$$\Psi_{R,u} := \Phi_u|_{D_{R,u}} \text{ where } D_{R,u} = [-R\mu(u)^{-1}/2, R\mu(u)^{-1}/2]^2$$

i.e., we restrict the point process Φ_u of rescaled local maxima of f above level u to the domain $D_{R,u}$. Note that $\Psi_{R,u}$ still has density approximately one.

For a point process η on a domain $D \subset \mathbb{R}^2$, denote by $|\eta|$ the number of points in D. Also, let $\mathcal{L}(X)$ denote the law of a random variable X. We know that for Theorem 3.2.10, for large $u > 0$,

$$\mathbb{E}|\Psi_{R,u}| \simeq cR^2 u \exp(-u^2/2).$$

We want to compare $\mathcal{L}(|\Psi_{R,u}|)$ to a Poisson random variable with suitable parameter. Let $U_{R,u}$ denote a Poisson random variable with mean $\mathbb{E}|\Psi_{R,u}|$.

Theorem 3.4.1. *Let $R > 0$ and $u > 0$ such that $u \leq 2\sqrt{\log R}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the Bargmann-Fock field and $\Psi_{R,u}$ be the point process associated to f as defined above. Then, for $R > 0$ large enough,*

$$d_{TV}(\mathcal{L}(|\Psi_{R,u}|), \mathcal{L}(U_{R,u})) \leq c \exp(-\beta u^2)$$

for some constants $c, \beta > 0$ that does not depend on u, R.

The above theorem can be interpreted as a quantitative central limit theorem in certain cases. Observe that, if U is a Poisson random variable with mean λ, then

$$\frac{U - \lambda}{\sqrt{\lambda}} \to Z \sim \mathcal{N}(0, 1) \text{ in law as } \lambda \to \infty.$$

Also, note that the total variation distance is shift and scale invariant. That is, if $P_{s,t}, Q_{s,t}$ denote that laws of random variables

$$\frac{X - s}{t}, \frac{Y - s}{t}$$

respectively, then

$$d_{TV}(P_{0,1}, Q_{0,1}) = d_{TV}(P_{s,t}, Q_{s,t}) \quad \text{for all } s, t \text{ with } t \neq 0.$$

Hence we need to ensure that, in Theorem 3.4.1, the level u depends on R such that $\mathbb{E}[|\Psi_{R,u}|] \to \infty$ as $R \to \infty$.

Theorem 3.4.2 (Quantitative central limit theorem)**.** *Let $u = u(R)$ be a function of R such that*

$$u(R) \to \infty \text{ and } \frac{u(R)}{\sqrt{\log R}} \to 0 \text{ as } R \to \infty.$$

With assumptions and notations as in Theorem 3.4.1, we have

$$d_{TV}\left(\mathcal{L}\left(\frac{|\Psi_{R,u}| - \mathbb{E}[|\Psi_{R,u}|]}{\sqrt{\mathbb{E}|\Psi_{R,u}|}}\right), \mathcal{L}\left(\frac{U_{R,u} - \mathbb{E}[|\Psi_{R,u}|]}{\sqrt{\mathbb{E}|\Psi_{R,u}|}}\right)\right) \leq c\exp(-\beta u(R)^2)$$

for large enough R. Here the constants $c, \beta > 0$ does not depend on R or the function $u(R)$.

Remarks:

- Notice that $|\Psi_{R,u}|$ is asymptotically (as $u \to \infty$) the number of local maxima of f above the level u in $[-R, R]^2$. This is because we scale the point process by $\mu(u)$ where $\mu(u)^{-2}$ is the *asymptotic* intensity of the local maxima of f above level u, and not scale by the exact intensity. We can rewrite Theorems 3.4.1 and 3.4.2 in terms of the number of local maxima of f above level u in $[-R, R]^2$, with same proof. Also, for the proof we need to spatially rescale the point process of local maxima.

- As pointed out in the remarks after Theorem 3.2.2, the quantitative estimates in Theorem 3.4.1 and 3.4.2 also hold for the point process of all critical points of f above level u instead of just local maxima.

- We conjecture that, for Gaussian fields with covariance kernel $r \in L^1(\mathbb{R}^d)$, the upper bound of Theorem 3.4.1 holds with a suitable constant β ,where the constant $\beta > 0$ would depend on the law of the field f, such that the rate of decay of the covariance kernel and derivatives of the kernel at the origin. Also, notice in the proof that the exponential upper bound in Theorem 3.4.1 comes from the decay of the tail of the Gaussian random variable rather than the decay of the covariance of the Bargmann-Fock field.

- Regarding the random plane wave (RPW) model, we believe we can get an upper bound for the total variation distance, which tends to zero as $R \to \infty$, although with a different rate than in the integrable kernel case. A potential technical obstacle to extending the result to the RPW model is the applicability of the Kac-Rice formula while bounding the expected difference in critical points of the field and its Palm version.

- In the proof, we will mostly use the following properties of the field rather than the explicit covariance structure for most of the computation.

  1. f is a unit variance, zero mean, stationary field, isotropic.
  2. (Non-degeneracy) The Gaussian vector

  $$(f(x), \nabla f(x), \text{Hess } f(x))$$

  non-degenerate. See [BMM20, Appendix A].
  3. Almost surely, f is a C^4-smooth function (it is in fact analytic).

Poisson approximation in Wasserstein distance: Denote by $\mathcal{H}$ the space of non-negative integer-valued locally finite measures on a domain $\Gamma \subset \mathbb{R}^2$. For a point process ζ denote by ζ^α a Palm version of ζ at $\alpha \in \Gamma$ (see appendix A for definition of Palm measures). Palm conditioning is the "horizontal" condition (i.e. $\nabla f(x) = 0$ with $|x| < \epsilon$ where $\epsilon \to 0$), as opposed to the Gaussian regression conditioning (i.e. $|\nabla f(0)| < \epsilon$ where $\epsilon \to 0$). If $0 \in \Gamma$, denote $\zeta^0 = \tilde{\zeta}$ for the Palm version of ζ at the origin. Later, we will adapt the same notation for Palm version of a Gaussian field f conditioned to have a local maxima at 0.

Let us define a pseudo-metric on $\mathcal{H}$ and then study the 1-Wasserstein distance on probability measures on $\mathcal{H}$ with respect to this pseudo-metric. Let ρ_0 be any bounded pseudo-metric on $\mathcal{H}$.
For two simple point configurations $\xi_1, \xi_2 \in \mathcal{H}$ such that

$$\xi_1 = \sum_{i=1}^{n} \delta_{x_i}, \quad \xi_2 = \sum_{i=1}^{m} \delta_{z_i}$$

with $m \geq n$, define

$$\rho_1(\xi_1, \xi_2) = \min_{\pi} \sum_{i=1}^{n} \rho_0(x_i, z_{\pi(i)}) + (m - n).$$

where π runs over S_n, symmetric group over n elements.
Let $\mathcal{F}$ denote the set of 1-Lipschitz functions with respect to ρ_1 on $\mathcal{H}$. Then the 1-Wasserstein metric on probability measures on $\mathcal{H}$ is given by,

$$\rho_2(Q_1, Q_2) = \sup_{f \in \mathcal{F}} \left| \int f dQ_1 - \int f dQ_2 \right|$$

Let $\mathcal{L}\Phi_{R,u}$ denote the law of the point process $\Psi_{R,u}$ and $\mathrm{Pois}(U)$ be the Poisson point process with intensity measure U, take U to be the Lebesgue measure on $D_{R,u}$. Note that when the pseudo-metric $\rho_0 \equiv 0$, then

$$\rho_2(\mathcal{L}\Psi_{R,u}, \mathrm{Pois}(U)) = d_{TV}(|\Psi_{R,u}|, |\mathrm{Pois}(U)|).$$

Another interesting choice for the pseudo-metric ρ_0 is

$$\rho_0(x, y) = \min(\|x - y\|, 1).$$

In this case, $\rho_2(\mathcal{L}\Psi_u, \mathrm{Pois}(U))$ measures how 'far' we have to move the points of Φ_u, on average, to make it look like Poisson point process. In this case, we suspect that the upper bound we will have extra factor $\mathrm{vol}(D_{R,u})$.

Proof of Theorem 3.4.1. For each $x \in D_{R,u}$ let A_x denote the open ball of radius $\tau(u)$ in $D_{R,u}$ around x. Let $V_{R,u} := \mathrm{vol}(D_{R,u})$, which is the expected total mass of $\Psi_{R,u}$. Recall that Ψ^x denote the Palm version of a point process Φ at x. Applying Theorem B.2.1 with $\lambda = V_{R,u}$, we have

$$\begin{aligned} d_{TV}(\mathcal{L}|\Psi_{R,u}|, U_{R,u}) \leq \frac{1 - e^{-V_{R,u}}}{V_{R,u}} \Bigg(\mathbb{E} \int_{D_{R,u}} (\Psi_{R,u}(A_x) - 1)\Psi_{R,u}(\mathrm{d}x) + \\ \int_{D_{R,u}} \mathbb{E}||\Psi^x_{R,u}(A^c_x)| - |\Psi_{R,u}(A^c_x)||\mathrm{d}x + \int_{D_{R,u}} \mathrm{vol}(A_x)\mathrm{d}x \Bigg) \end{aligned} \tag{3.17}$$

Notice that only the second term of the RHS above depends on the coupling of the field f and its Palm version $\widetilde{f}$. Using the stationarity of the field f, we have,

$$\begin{aligned} d_{TV}(\mathcal{L}|\Psi_{R,u}|, U_{R,u}) \leq \Bigg(\frac{1}{V_{R,u}} \mathbb{E} \int_{D_{R,u}} (\Psi_{R,u}(A_x) - 1)\Psi_{R,u}(\mathrm{d}x) + \\ \mathbb{E}||\Psi^0_{R,u}(A^c_0)| - |\Psi_{R,u}(A^c_0)|| + \mathrm{vol}(A_0)\big) \end{aligned} \tag{3.18}$$

We will choose the radius paramater $\tau(u)$ in the course of the proof.

There are three main components of the proof. The first term in the RHS above is the expected number of 'cluster' points of the process $\Psi_{R,u}$ (i.e. points of $\Psi_{R,u}$ which are within distance $\tau(u)$ of each other). The second component is to come up with a coupling of the field f and its Palm version $\widetilde{f}$ such that $\|f(x) - \widetilde{f}(x)\|_{C^4}$ is small for $x \gg 1$. The third component is the bound on the difference of the number of high critical points of these two fields in L^1.

3.4.1 Bounding the number of cluster points

We have

$$\int_{D_{R,u}} (\Psi_{R,u}(A_x) - 1)\Psi_{R,u}(\mathrm{d}x) = \sum_{x \in \Psi_{R,u}} \Psi_{R,u}(A_x) - 1.$$

For a box M, if $\Psi_{R,u}$ has k points in the $\tau(u)$-neighborhood of the box M, then

$$\int_{M} (\Psi_{R,u}(A_x) - 1)\Psi_{R,u}(\mathrm{d}x) \leq k(k-1).$$

The reasoning is that each point in M can have at most $(k-1)$ points within distance $\tau(u)$, and there are at most k points in M. Hence, by linearity of expectation and stationarity of the field f,

$$\mathbb{E} \int_{D_{R,u}} (\Psi_{R,u}(A_x) - 1)\Psi_{R,u}(\mathrm{d}x) \leq \frac{V_{R,u}}{\tau(u)^2} \mathbb{E}[\eta(\eta - 1)] \tag{3.19}$$

where

$$\eta = \text{ number of local maxima of } f \text{ above level } u \text{ in a square of side } \tau(u)\mu(u).$$

Here, $V_{R,u}/\tau(u)^2$ is the number of squares of side $\tau(u)\mu(u)$ that fit in $R \times R$ box.

We now compute $\mathbb{E}[\eta(\eta-1)]$ using the Kac-Rice formula. We have, from the second factorial moment of the Kac-Rice formula [AW09, Theorem 6.3], denoting $g_u = \tau(u)\mu(u)$,

$$\begin{aligned}\mathbb{E}[\eta(\eta-1)] = \int_{[-g_u/2, g_u/2]^2} \int_{[-g_u/2, g_u/2]^2} \mathbb{E}\left[|\det \text{Hess } f(x)||\det \text{Hess } f(y)|\right. \\ \left.\mathbb{1}[f(x) \geq u, f(y) \geq u] \middle| \nabla f(x) = 0 = \nabla f(y)\right] \psi_{x,y}(0,0)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{3.20}$$

where $\psi_{x,y}$ is the pdf of $(\nabla f(x), \nabla f(y))$. By stationarity of the field f,

$$\begin{aligned}\mathbb{E}[\eta(\eta-1)] = g_u^2 \int_{[-g_u/2, g_u/2]^2} \mathbb{E}\left[|\det \text{Hess } f(0)||\det \text{Hess } f(x)|\right. \\ \left.\mathbb{1}[f(0) \geq u, f(x) \geq u] \middle| \nabla f(0) = 0 = \nabla f(x)\right] \psi_x(0,0)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{3.21}$$

where p_x is the pdf of $(\nabla f(0), \nabla f(x))$. Now, the integrand of the above equation might blow up when x is near 0 since the vector $(\nabla f(0), \nabla f(x))$ becomes a degenerate Gaussian variable. To handle this, there is a trick called the divided difference method, introduced in [Cuz75].

Let $(\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis of $\mathbb{R}^2$ such that $x = |x|\mathbf{e}_1$. Then, as $x \to 0$,

$$\begin{aligned}\det \text{Cov}(\nabla f(x), \nabla f(0)) =& \|x\|^4 \det \text{Cov}(\nabla f(0), \frac{\nabla f(x) - \nabla f(0)}{\|x\|}) \\ \simeq& \|x\|^4 \det \text{Cov}(\nabla f(0), \partial_{\mathbf{e}_1} \nabla f(0)).\end{aligned}$$

Now, conditionally on $\nabla f(x) = \nabla f(0) = 0$, by Taylor expansion,

$$\begin{aligned}\det \text{Hess } f(x) =& \det(\partial_{\mathbf{e}_1}\nabla f(x) \quad \partial_{\mathbf{e}_2}\nabla f(x)) \\ =& \|x\| \det\left(\frac{\partial_{\mathbf{e}_1}\nabla f(x) - \partial_{\mathbf{e}_1}\nabla f(0)}{\|x\|} \quad \partial_{\mathbf{e}_2}\nabla f(x)\right) \\ \simeq& \frac{\|x\|}{2}\det(\partial^2_{\mathbf{e}_1}\nabla f(x) \quad \partial_{\mathbf{e}_2}\nabla f(x)).\end{aligned} \tag{3.22}$$

Similarly,

$$\det \text{Hess } f(0) \simeq \frac{\|x\|}{2}\det(\partial^2_{\mathbf{e}_1}\nabla f(0) \quad \partial_{\mathbf{e}_2}\nabla f(0)) \tag{3.23}$$

Since

$$\psi_x(0,0) = 1/\sqrt{\det(2\pi \text{Cov}(\nabla f(x), \nabla f(0)))}$$

we have that, when x is close to 0,

$$\begin{aligned}&\mathbb{E}\left[|\det \text{Hess } f(0)||\det \text{Hess } f(x)|\mathbb{1}[f(0)\geq u, f(x)\geq u]\big|\nabla f(0) = 0 = \nabla f(x)\right]\psi_x(0,0) \\ &\qquad \simeq \frac{1}{4}\frac{\mathbb{E}\left[\det(\partial^2_{\mathbf{e}_1}\nabla f(0), \partial_{\mathbf{e}_2}\nabla f(0))^2\mathbb{1}[f(0)\geq u]\big|\nabla f(0) = 0 = \partial_{\mathbf{e}_1}\nabla f(0)\right]}{\sqrt{\det 2\pi \text{Cov}(\nabla f(0), \partial_{\mathbf{e}_1}\nabla f(0))}}\end{aligned} \tag{3.24}$$

hence it is bounded.

This controls the integrand in (3.21) close to the origin. Next, we estimate it away from the origin. For $\delta > 0$, the pdf $\psi_x(0,0)$ is bounded above uniformly for $x \in B(0,\delta)^C$.

We have, for $p, q > 1$ with $1/p + 1/q = 1$, by Hölder's inequality,

$$\begin{aligned}&\mathbb{E}[|\det \text{Hess } f(0)||\det \text{Hess } f(x)|\mathbb{1}[f(0)\geq u, f(x)\geq u]\big|\nabla f(0) = 0 = \nabla f(x)] \leq \\ &\qquad (\mathbb{E}[(|\det \text{Hess } f(0)||\det \text{Hess } f(x)|)^q\big|\nabla f(0) = 0 = \nabla f(x)])^{1/q} \\ &\qquad (\mathbb{E}[\mathbb{1}[f(0)\geq u, f(x)\geq u]\big|\nabla f(0) = 0 = \nabla f(x)])^{1/p}.\end{aligned}$$

From the divided difference method, observe that

$$(\mathbb{E}[(|\det \text{Hess } f(0)||\det \text{Hess } f(x)|)^q\big|\nabla f(0) = 0 = \nabla f(x)])^{1/q}\psi_x(0,0) \tag{3.25}$$

is *uniformly* bounded in $x \in \mathbb{R}^2$ and for any fixed $q > 1$. Hence we will bound this quantity in the integrand by a constant and focus on the quantity

$$(\mathbb{E}[\mathbb{1}[f(0)\geq u, f(x)\geq u]\big|\nabla f(0) = 0 = \nabla f(x)])^{1/p}.$$

For the moment, let us compute $\mathbb{P}(f(0) > u, f(x) > u)$ and modify the argument to suit conditional expectation calculation. We have,

$$\mathbb{P}(f(0)>u, f(x)>u) \leq \frac{c}{\sqrt{1-r(x)^2}} \int_{y_1>u} \int_{y_2>u} \exp\left(-\frac{1}{2(1-r(x)^2)}(y_1^2 - 2r(x)y_1y_2 + y_2^2)\right).$$

By polar coordinate substitution,

$$y_1 = t\cos\theta$$
$$y_2 = t\sin\theta$$

we have,

$$\begin{aligned}\mathbb{P}(f(0)>u, f(x)>u) \leq \frac{c}{\sqrt{1-r(x)^2}} \Bigg(&\int_0^{\pi/4} \int_{t>u/\sin\theta} \\ &\exp\left(-\frac{1}{2(1-r(x)^2)} t^2(1-2r(x)\sin(2\theta))\right) t\mathrm{d}t\mathrm{d}\theta + \\ &\int_{\pi/4}^{\pi/2} \int_{t>u/\cos\theta} \exp\left(-\frac{1}{2(1-r(x)^2)} t^2(1-2r(x)\sin(2\theta))\right) t\mathrm{d}t\mathrm{d}\theta\Bigg).\end{aligned}$$

Now,

$$\begin{aligned}&\frac{c}{\sqrt{1-r(x)^2}} \left(\int_0^{\pi/4} \int_{t>u/\sin\theta} \exp\left(-\frac{1}{2(1-r(x)^2)} t^2(1-2r(x)\sin(2\theta))\right) t\mathrm{d}t\mathrm{d}\theta\right) \leq \\ &\qquad c\sqrt{1-r(x)^2} \left(\int_0^{\pi/4} \frac{\exp\left(-\frac{u^2(1-r(x)\sin(2\theta))}{2(1-r(x)^2)\sin^2\theta}\right)}{(1-r(x)\sin(2\theta))} \mathrm{d}\theta\right).\end{aligned}$$

For $r(x) \neq 1$, we have

$$\begin{aligned}&\frac{c}{\sqrt{1-r(x)^2}} \left(\int_0^{\pi/4} \int_{t>u/\sin\theta} \exp\left(-\frac{1}{2(1-r(x)^2)} t^2(1-2r(x)\sin(2\theta))\right) t\mathrm{d}t\mathrm{d}\theta\right) \leq \\ &\qquad c\sqrt{(1+r(x))/(1-r(x))} \int_0^{\pi/4} \exp\left(-\frac{u^2(1-r(x)\sin(2\theta))}{2(1-r(x)^2)\sin^2\theta}\right) \mathrm{d}\theta.\end{aligned}$$

The function (in θ)

$$(1-r(x)\sin 2\theta)/\sin^2\theta$$

in $[0, \pi/4]$ is decreasing, given $r(x) < 1$. Hence,

$$\begin{aligned}&\frac{c}{\sqrt{1-r(x)^2}} \left(\int_0^{\pi/4} \int_{t>u/\sin\theta} \exp\left(-\frac{1}{2(1-r(x)^2)} t^2(1-2r(x)\sin(2\theta))\right) t\mathrm{d}t\mathrm{d}\theta\right) \leq \\ &\qquad c\sqrt{(1+r(x))/(1-r(x))} \exp\left(-\frac{u^2}{1+r(x)}\right).\end{aligned}$$

We have a similar estimate for the other integral, so we have

$$\mathbb{P}(f(0) > u, f(x) > u) \leq c\sqrt{(1+r(x))/(1-r(x))}\exp\left(-\frac{u^2}{1+r(x)}\right).$$

Let us compute the following conditional Gaussian

$$(f(0), f(x))|\nabla f(0) = 0, \nabla f(x) = 0.$$

Since we explicitly know the covariance structure of Bargmann-Fock field, the covariance matrix is,

$$\begin{pmatrix} 1 & r(x) \\ r(x) & 1 \end{pmatrix} - (1 - e^{-t^2}(1 + (t^2-1)^2) + e^{-2t^2(t^2-1)^2})^{-1} \times \\ \begin{pmatrix} t^2e^{-t^2}(1-e^{-t^2}) & t^2(t^2-1)(1-e^{t^2})e^{-5t^2/2} \\ t^2(t^2-1)(1-e^{t^2})e^{-5t^2/2} & t^2e^{-t^2}(1-e^{-t^2}) \end{pmatrix} \tag{3.26}$$

where $\|x\|^2 = t^2$. We used Sympy python for the symbolic computation.
Observe that, as $t \to 0$,

$$\begin{aligned} 1 - e^{-t^2}(1 + (t^2-1)^2) + e^{-2t^2(t^2-1)^2} &= 3t^4 + O(t^6) \\ t^2e^{-t^2}(1-e^{-t^2}) &= t^4 + O(t^6) \end{aligned} \tag{3.27}$$

Hence, the variance of $f(0)$ conditional on $(\nabla f(0), \nabla f(x))$ near $x = 0$ is away from 1. Since there are no degeneracies except when $x = 0$, the covariance matrix of the conditional distribution is continuous with respect to x.
So there's a constant $K > 0$ such that for $x \in B(0, K)$ the conditional variance of $f(0)$ stays less than, say, 3/4. So,

$$\begin{aligned} \mathbb{E}[\mathbb{1}[f(0) > u, f(x) > u|\nabla f(0) = 0 = \nabla f(x)]] &\leq \mathbb{E}[\mathbb{1}[f(0) > u|\nabla f(0) = 0 = \nabla f(x)]] \\ &\leq cu^{-1}\exp(-2u^2/3) \end{aligned} \tag{3.28}$$

for $x \in B(0, K)$ and large enough u.

Let $w(x)$ denote the correlation of $f(0)$ and $f(x)$ conditioned on $(\nabla f(0), \nabla f(x))$. We claim that there exists a constant $0 < K_1 < 1$ such that $|w(x)| \leq K_1$ for all $x \in B(0, K)^C$. Since f is a Gaussian field with spectral measure has a a non-trivial open set in the support, we have that, for any distinct points $(x_1, x_2, \ldots, x_m)$

$$(f(x_1), f(x_2), \ldots, f(x_m), \nabla f(x_1), \nabla f(x_2), \ldots, \nabla f(x_m))$$

is non-degenerate.

Hence, $|w(x)| \neq 1$ for $x \neq 0$. Also notice that variance of $(f(0), f(x))|\nabla f(0), \nabla f(x)$ tends to 1 as $x \to \infty$ (because $t^n e^{-t^2/2} \to 0$ as $t \to \infty$ for any n integer). Similarly, the covariance tends to zero as $x \to \infty$. Hence $w(x) \to 0$ as $x \to \infty$. If a continuous function $b(x)$ on a compact set has $b < 1$, then there's a constant $C < 1$ such that $b(x) < C$ on that set. Hence, we proved the claim that there exists a constant $0 < K_1 < 1$ such that $|w(x)| \leq K_1$ for all $x \in B(0, K)^C$.

Using the bound (3.28) in (3.21) we have

$$\mathbb{E}[\eta(\eta - 1)] \leq C g_u^2 \left(u^{-1} \exp\left(-\frac{2u^2}{3p} \right) + g_u^2 \exp\left(-\frac{u^2}{p(1 + K_1)} \right) \right). \tag{3.29}$$

The first term on RHS of the bound is from integrating the integrand of the RHS of (3.21) over $B(0, K)$ where the variance of $f(0)$ conditionally is around $2/3$. The second term is naively bounding the integral of (3.21) by the maximum times the volume.

Take $p > 1$ close enough so that, for some $0 < \delta < 2$

$$\frac{1}{p(1 + K_1)} \geq \frac{1}{2 - \delta}.$$

Choose $\tau(u)$ such that

$$g_u = u \quad \text{i.e. } \tau(u) = u^{3/2} e^{-u^2/4}.$$

Finally, substituting all these for a large enough R, we get

$$\frac{1}{V_{R,u}} \mathbb{E} \int_{D_{R,u}} (\Psi_{R,u}(A_x) - 1) \Psi_{R,u}(\mathrm{d}x) \leq C u \exp\left(-u^2 \frac{\delta}{2(2 - \delta)} \right) \tag{3.30}$$

The next task is to come up with a close coupling of $\Psi_{R,u}$ and $\Psi^0_{R,u}$ for large R.

3.4.2 Coupling the field and its Palm version

Let $\widetilde{f}$ be the Palm condition of the field f to have a local maxima at $x = 0$ with value at least $u(R)$. One way to couple the point processes $\Psi_{R,u}$ and $\Psi^0_{R,u}$ is to couple the fields f and $\widetilde{f}$, then consider the critical point processes corresponding to these fields. This section might be of general interest, so we consider stationary smooth Gaussian fields on $\mathbb{R}^d$ rather than the particular Bargmann-Fock field.

Let us study the distribution of the Gaussian field f conditioned on $x = 0$ being a local maximum. Here, the field is conditioned in a way that it corresponds to the Palm conditioning (see the appendix). We follow the arguments in [Lin72, Section

1.2]. The difference is that we require $f(0) \geq u$ rather than conditioning on $f(0) = u$. We also give a representation for the conditioned field for the case of the RPW model where $(f(x), \nabla f(x), \text{Hess } (f(x)))$ is degenerate.

Consider the following event:

$A(u,h) = \{f \text{ has a local maximum with height at least } u \text{ at some point } \mathbf{s} \text{ with } |\mathbf{s}| \leq h\}$.

Let $\chi = (x^1, x^2, \ldots, x^m) \in \mathbb{R}^{d \times m}$ be m distinct points in $\mathbb{R}^d$ and $\tau = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$. The goal is to compute the m-point marginal distribution

$$f(x^i), i = 1, 2 \ldots m | A(u,h) \quad \text{as } h \to 0.$$

From [Adl10, Section 6.7, page 150] it follows that

$$\mathbb{P}(f(x^i) \leq t_i, i = 1, 2, \ldots m | A(u,h)) \to \frac{\mathbb{E}[N_1(\tau, u)]}{\mathbb{E}[N_1(u)]} \quad \text{as } h \to 0 \tag{3.31}$$

where

$$N_T(u) = \text{ the number of local maxima } \mathbf{s} \text{ with } |\mathbf{s}| \leq T \text{ and } f(\mathbf{s}) \geq u, \text{ and}$$

$$N_T(\tau, u) = \text{ those maxima } \mathbf{s} \text{ for which } f(\mathbf{s} + x^i) \leq t_i, i = 1, 2, \ldots m.$$

Here's a brief reason why. Define

$$A(\tau, u, h) = \big\{f \text{ has a local maximum in } B(0,h) \text{ of height at least } u, \text{ and } f(x^i) \leq t_i,\ i = 1, \ldots, k\big\}.$$

By stationarity and localization, for small h,

$$\frac{\mathbb{P}(A(\tau, u, h))}{h^d} \approx \mathbb{E}\, N_1(\tau, u), \qquad \frac{\mathbb{P}(A(u,h))}{h^d} \approx \mathbb{E}\, N_1(u).$$

Hence

$$\frac{\mathbb{P}(A(\tau, u, h))}{\mathbb{P}(A(u,h))} \longrightarrow \frac{\mathbb{E}[N_1(\tau, u)]}{\mathbb{E}[N_1(u)]}.$$

Under ergodicity, the same ratio also represents the almost-sure limit of these configurations around high peaks.

To compute explicitly the RHS of (3.31), we need some regularity properties of the field f.

3.4.2.1 The non-degenerate case

Assume that $(f(x), \nabla f(x), \text{Hess } (f(x))$ is a non-degenerate Gaussian vector. Here, the Hessian is vectorised as follows,

$$\text{Hess } (f(x)) = (\partial_{ii} f(x), i = 1, 2, \dots d, \partial_{ij} f(x), 1 \leq i < j \leq d).$$

Denote the covariance matrix of the vector $(f(0), \nabla f(0), \text{Hess } (f(0)))$, partitioned in a natural way, by

$$\Sigma = \begin{pmatrix} 1 & 0 & S_{02} \\ 0 & S_{11} & 0 \\ S_{20} & 0 & S_{22} \end{pmatrix} \tag{3.32}$$

where S_{11}, S_{22} are the internal covariance matrices of $\nabla f(0)$ and Hess $(f(0))$ respectively. Define

$$S_1(x) = (-r_i(x), 1 \leq i \leq d),$$

$$S_2(x) = (r_{ii}(x), 1 \leq i \leq d, r_{ij}(x), 1 \leq i < j \leq d)$$

where $r_i = \partial_i r$ and $r_{ij} = \partial_{ij} r$. The plan is to use the Kac-Rice formula to get an expression for $\mathbb{E}[N_1(\tau, u)], \mathbb{E}[N_1(u)]$, for which we need more notation.

Define the following probability densities and (Gaussian) conditional densities

$p_\chi(t, v, Z, x)$	for	$f(0), \nabla f(0), \text{Hess } (f(0)), f(x^1), f(x^2), \dots, f(x^m)$
$p_\chi(x \vert t, v, Z)$	for	$f(x^1), f(x^2), \dots, f(x^m) \vert f(0) = t, \nabla f(0) = v, \text{Hess } (f(0)) = Z$
$p(t, v, Z)$	for	$f(0), \nabla f(0), \text{Hess } (f(0))$
$p(Z, t \vert v)$	for	$\text{Hess } (f(0)), f(0) \vert \nabla f(0) = 0.$

Then by the Kac-Rice formula, we have

$$\frac{\mathbb{E}[N_1(\tau, u)]}{\mathbb{E}[N_1(u)]} = \frac{\int_{y_1 \leq t_1} \cdots \int_{y_m \leq t_m} \int_{t \geq u} \int_{Z \prec 0} |\det Z| p_\chi(t, 0, Z, y) dZ dt dy}{\int_{t \geq u} \int_{Z \prec 0} |\det Z| p(t, 0, Z) dZ dt} \tag{3.33}$$

where $Z \prec 0$ means Z is a negative definite matrix. Hence the conditional distribution of $f(x^1), \dots, f(x^m)$ given that f has a local maxima with height at least u at 0 has the density

$$\frac{\int_{t \geq u} \int_{Z \prec 0} |\det Z| p_\chi(t, 0, Z, y) dZ dt}{\int_{t \geq u} \int_{Z \prec 0} |\det Z| p(t, 0, Z) dZ dt}. \tag{3.34}$$

Define

$$q_u(t, Z) = \frac{\det Z p(Z, t|0)}{\int_{t \geq u} \int_{Z \prec 0} \det Z p(Z, t|0) dZ dt} \mathbb{1}[Z \prec 0, t \geq u]. \tag{3.35}$$

Hence, the density (3.34) is

$$\int_{t \geq u} \int_{Z \prec 0} q_u(t, Z) p_\chi(\tau | t, 0, Z) dZ dt.$$

So, 'freezing' the values of $f(0), \nabla f(0), \text{Hess } (f(0))$ the joint distribution of $f(x^1), \ldots, f(x^m)$ is Gaussian. Let us compute the mean and covariance matrix of the Gaussian with density $p_\chi(\cdot|t, 0, Z)$. Define the following quantities $A(x) \in \mathbb{R}, b(x) \in \mathbb{R}^{d(d+1)/2}$:

$$(A(x), b(x)) = (r(x), S_2(x)) \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}^{-1}.$$

By the Gaussian regression formula, we have that the mean of a Gaussian with density $p_\chi(\cdot|t, 0, Z)$ is

$$tA(x^i)Z^T b(x^i) \text{ for } i = 1, 2, \ldots m.$$

To compute the covariance matrix, we use the fact that $(f(0), \text{Hess } (f(0)))$ is independent of $\nabla f(0)$ by stationarity of the field f. Again, by the Gaussian regression formula, the (i, j)th entry of the covariance matrix is

$$C(x^i, x^j) = r(x^i - x^j) - (r(x^i), S_2(x^i)) \begin{pmatrix} 1 & S_{02} \\ S_{20} & S_{22} \end{pmatrix}^{-1} \begin{pmatrix} r(x^j) \\ S_2(x^j) \end{pmatrix} - S_1(x^i) S_{11}^{-1} S_1(x^j). \tag{3.36}$$

This proves the following proposition.

Proposition 3.4.3. *(c.f. [Lin72, Theorem 1.2]) Given a local maximum at $x = 0$ with height at least u, the conditional process $f(x)$ has the same law as the following process:*

$$\widetilde{f}(x) = \xi A(x) + Zb(x) + \overline{f}(x)$$

where $\overline{f}$ is a non-stationary zero mean Gaussian field with covariance function $C(\cdot, \cdot)$ and (ξ, Z) is random vector with density q_u which is independent of $\overline{f}$.

Let us couple the fields $\widetilde{f}$ and f such that $\|\widetilde{f}(x) - f(x)\|$ is small for $\|x\| \gg 1$. Recall the construction of the reproducing kernel Hilbert space (RKHS) of the kernel r. Let ρ be the spectral measure associated with the kernel r, then the RKHS H is given by

$$H = \mathcal{F} L^2_{sym}(\rho)$$

where $\mathcal{F}$ denotes the Fourier transform, and $L^2_{sym}(\rho)$ are the symmetric $L^2(\rho)$ functions. The inner product of the Hilbert space H is given by,

$$\langle \mathcal{F}\phi, \mathcal{F}\psi \rangle_H = \langle \phi, \psi \rangle_{L^2(\rho)}.$$

Now the field f is a white noise in the RKHS H, i.e. for an orthonormal basis $\{\psi_i\}$ of H, formally we have

$$f = \sum_i a_i \psi_i$$

where a_i's are i.i.d standard Gaussian variables.

Since we assume that the field f is C^3-smooth, we have up to 6 finite moments of ρ. This immediately implies that all of $r, r_i(1 \le i \le d), r_{ij}(1 \le i, j \le d)$ belong to H since

$$r_i = \int x_i e^{\sqrt{-1}x\cdot} d\rho(x), \quad r_{ij} = \int x_i x_j e^{\sqrt{-1}x\cdot} d\rho(x).$$

We have the following observation using the Fourier representation for r and its derivatives,

$$\begin{aligned}
\langle r, r_i\rangle_H &= -r_i(0) = 0 \quad && \text{for } 1 \le i \le d \\
\langle r_{kl}, r_i\rangle_H &= -r_{ikl}(0) = 0 \quad && \text{for } 1 \le i, k, l \le d \\
\langle r, r_{ij}\rangle_H &= r_{ij}(0) \quad && \text{for } 1 \le i, j \le d \\
\langle r_{ij}, r_{kl}\rangle_H &= r_{ijkl}(0) \quad && \text{for } 1 \le i, j, k, l \le d.
\end{aligned}$$

Observe that the covariance matrix for $(f(0), \nabla f(0), \text{Hess } (f(0)))$ which is Σ is given by the same entries, i.e.

$$\mathbb{E}[f(0)\partial_{x_i} f(0)] = -r_i(0) = 0, \quad \mathbb{E}[f(0)\partial_{x_i x_j} f(0)] = r_{ij}(0)$$

and so on. Hence $\{r, r_i, r_{ij} : 1 \le i, j \le d\}$ is a linearly independent set in H. Otherwise, it would imply that the covariance matrix Σ is degenerate.

Now by Gram-Schmidt orthonormalisation of the set $\{r, r_i, r_{ij} : 1 \le i, j \le d\}$ with respect to the inner product $\langle , \rangle_H$, say you get the orthonormal set

$$\{\phi_0, \phi_1, \cdots, \phi_d, \phi_{11}, \cdots, \phi_{ij}, \cdots, \phi_{dd}\}$$

which is presented by

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_d \\ \phi_{11} \\ \vdots \\ \phi_{ij} \\ \vdots \\ \phi_{dd} \end{pmatrix} = \mathbf{B} \begin{pmatrix} r \\ r_1 \\ \vdots \\ r_d \\ r_{11} \\ \vdots \\ r_{ij} \\ \vdots \\ r_{dd} \end{pmatrix}$$

Let $b_1, b_2, \ldots, b_n$ be the rows of the matrix $\mathbf{B}^{-1}$ (so $n = 1 + d + d(d+1)/2$). Then by the construction of the ϕ_i's, we have that

$$b_i \cdot b_j = \langle r_\#, r_* \rangle_H$$

where the inner product on LHS is the Euclidean one in $\mathbb{R}^n$ and indices $\#, *$ are the corresponding ones i, j (say $i = 1$ corresponds to $r_\# = r$, $i = 1 + d + d(d+1)/2$ corresponds to $r_\# = r_{dd}$). As observed before, the entries of Σ are given by the inner products $\langle r_\#, r_* \rangle_H$, hence we have

$$\mathbf{B}^{-1}(\mathbf{B}^{-1})^T = \Sigma.$$

Extend the orthonormal set $\{\phi_0, \cdots, \phi_{dd}\}$ to an orthonormal basis $\{\psi_i\}_{i=0}^\infty$ of H, where the first $n = 1 + d + d(d+1)/2$ elements match. Define the field f as

$$f = \sum_{i=0}^{\infty} a_i \psi_i \tag{3.37}$$

where a_i's are i.i.d standard Gaussian variables. Define

$$\overline{f} = \sum_{i \geq n} a_i \psi_i \tag{3.38}$$

so that

$$f = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} + \overline{f}.$$

Now we have,

$$\begin{aligned} \mathrm{Cov}(f(x) - \overline{f}(x), f(y) - \overline{f}(y)) &= \begin{pmatrix} \psi_0(x) & \psi_1(x) & \cdots & \psi_{n-1}(x) \end{pmatrix} \begin{pmatrix} \psi_0(y) \\ \psi_1(y) \\ \vdots \\ \psi_{n-1}(y) \end{pmatrix} \\ &= \begin{pmatrix} r(x) & \cdots & r_{dd}(x) \end{pmatrix} \mathbf{B}^T \mathbf{B} \begin{pmatrix} r(y) \\ \vdots \\ r_{dd}(y) \end{pmatrix} \\ &= \begin{pmatrix} r(x) & \cdots & r_{dd}(x) \end{pmatrix} \Sigma^{-1} \begin{pmatrix} r(y) \\ \vdots \\ r_{dd}(y) \end{pmatrix}. \end{aligned}$$

Hence, $\overline{f}$ defined here has the covariance $C(,)$ as in Proposition 3.4.3.

Now, let f be the field defined as in (3.37) and $\overline{f}$ be as in (3.38), i.e. using the orthonormal basis $\{\psi_i\}_{i=0}^\infty$ and the noise $(a_0, a_1, a_2, \ldots)$. Let (ξ, Z) be a random vector as in Proposition 3.4.3 which is independent of the sequence $a_0, a_1, a_2, \ldots$. Then,

$$\widetilde{f} = \xi A(x) - Z b(x) + \overline{f}(x)$$

is a random function which is C^2-smooth, non-Gaussian, has a local maxima at $x = 0$ with height at least u.

3.4.2.2 Isotropic case

Consider a Gaussian function $f : \mathbb{R}^d \to \mathbb{R}$ which is C^2-smooth, stationary, unit variance. Additionally assume that $(f(x), \nabla f(x))$ is non-degenerate for $x \in \mathbb{R}^d$. By stationarity, we can prove that $(f(x), \text{Hess } (f(x)))$ is independent of $(\nabla f(x))$. Hence, if the vector $(f(x), \nabla f(x), \text{Hess } (f(x)))$ is degenerate then it implies that $(f(x), \text{Hess } (f(x)))$ is degenerate.

Assume that $(f(0), \text{Hess } (f(0)))$ is degenerate. Under mild symmetry conditions on the covariance kernel, such as for $d = 2, r(x_1, x_2) = r(x_1, -x_2)$, we can show that f is a monochromatic random wave up to linear change of coordinates of the domain (i.e. the spectral measure of f is supported on $(S)^{d-1} \subset \mathbb{R}^d$). If we insist that f is isotropic, then f has spectral measure which is uniform on $\mathbb{S}^{d-1}$ ($d = 2$ case corresponds to RPW model) [see [CS18, Proposition 3.3]].

In this section assume that f is a Gaussian field with spectral measure given by the uniform measure on $\mathbb{S}^{d-1}$. The construction of the coupling of the conditioned field and the original field is similar to that of the non-degenerate case above, so we will not give all the details. Also, we retain the same notation as in the previous section.

We will work with the vector $(\nabla f(x), \text{Hess } (f(x)))$ which is non-degenerate, instead of

$$(f(x), \nabla f(x), \text{Hess } (f(x))).$$

Recall that $\chi = (x^1, x^2, \ldots, x^m) \in \mathbb{R}^{d \times m}$ are m distinct points in $\mathbb{R}^d$ and $\tau = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$. Rename/redefine the following quantities while keeping the other notation the same. Let

$$\Sigma = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$$

where S_{11} and S_{22} are covariance matrices of $\nabla f(x)$ and Hess $(f(x))$ respectively. Define the following probability densities and (Gaussian) conditional densities

$p_\chi(v, Z, x)$	for	$\nabla f(0), \text{Hess } (f(0)), f(x^1), f(x^2), \ldots, f(x^m)$
$p_\chi(x \vert v, Z)$	for	$f(x^1), f(x^2), \ldots, f(x^m) \vert \nabla f(0) = v, \text{Hess } (f(0)) = Z$
$p(v, Z)$	for	$\nabla f(0), \text{Hess } (f(0))$
$p(Z \vert v)$	for	$\text{Hess } (f(0)) \vert \nabla f(0) = v.$

Noting that $f(x) = -\mathrm{Tr}(\mathrm{Hess}\ (f(x)))$, we have, again by the Kac-Rice formula,

$$\begin{aligned}\frac{\mathbb{E}[N_1(\tau,u)]}{\mathbb{E}[N_1(u)]} =& \frac{\mathbb{E}\left[|\det \mathrm{Hess}\ (f(0))|\mathbb{1}[Q]\middle|\nabla f(0)=0\right]}{\mathbb{E}[|\det \mathrm{Hess}\ (f(0))|\mathbb{1}[\mathrm{ind}(\mathrm{Hess}\ (f(0)))=d]\mathbb{1}[f(0)\geq u]]}\\ =& \frac{\int_{y_1\leq t_1}\cdots\int_{y_m\leq t_m}\int_{Z\prec 0,\mathrm{Tr}(Z)\leq -u}|\det Z|p_\chi(0,Z,y)dZdy}{\int_{Z\prec 0,\mathrm{Tr}(Z)\leq -u}|\det Z|p(0,Z)dZ}\end{aligned}$$

where

$$\mathbb{1}[Q] = \mathbb{1}[\mathrm{ind}(\mathrm{Hess}\ (f(0)))=d]\mathbb{1}[f(0)\geq u]\prod_{i=1}^{m}\mathbb{1}[f(x^i)\leq t_i].$$

Following a similar argument as in the non-degenerate case, we get the following result regarding the conditional distribution of f.

Proposition 3.4.4. *Let f be an isotropic Gaussian field with spectral measure being uniform measure on $\mathbb{S}^{d-1}$, and other assumptions on f as in the beginning of this subsection. Then, given a local maximum with height at least u at the origin, the conditional process $f(x)$ has the same law as the following process:*

$$\widetilde{f}(x) = ZS_{22}^{-1}S_2(x)' + \overline{f}(x)$$

where $\overline{f}$ is a non-stationary zero mean Gaussian field with covariance function $C(\cdot,\cdot)$ and Z is random vector with density q_u which is independent of $\overline{f}$. Here

$$q_u(Z) := \frac{\det Zp(Z|0)}{\int_{Z\prec 0,\,Tr(Z)\leq -u}\det Zp(Z|0)dZ}\mathbb{1}[Z\prec 0,\,Tr(Z)\leq -u] \quad \textit{and}$$

$$C(x,y) = r(x-y) - S_2(x)S_{22}^{-1}S_2(y)' - S_1(x)S_{11}^{-1}S_1(y)'.$$

Let us now couple the conditioned field $\widetilde{f}$ and the original field f such that $\|f(x) - \widetilde{f}(x)\|$ is small for $\|x\| \gg 1$. Again, we follow the argument of the non-degenerate case. The only difference is that instead of extending the orthonormal basis generated by

$$\{r, r_i, r_{ij} : 1\leq i,j\leq d\}$$

we look at the linearly independent set $\{r_i, r_{ij} : 1\leq i,j\leq d\}$ because $r = -(r_{11} + r_{22} + \cdots + r_{dd})$. We take $n = d + d(d+1)/2$ (one less than previously defined), and orthonormal basis $\{\psi_i\}_{i=1}^{\infty}$ of the RKHS H, the noise $(a_1, a_2, \ldots)$ to define

$$f = \sum_{i=1}^{\infty} a_i\psi_i \quad \text{and} \quad \overline{f} = \sum_{i>n} a_i\psi_i.$$

Define the conditioned field $\widetilde{f}$ to be

$$\widetilde{f} = ZS_{22}^{-1}S_2(x)' + \overline{f}(x)$$

where Z has the density q_u and is independent of the noise $(a_1, a_2, \ldots)$. Verification that $\overline{f}$ has the desired covariance structure is the same as in the non-degenerate case.

3.4.3 Difference in number of points in perturbed fields

Now that we have a coupling of the field f and its Palm version $\widetilde{f}$ at $x = 0$, define $\Psi^0_{R,u}$ as the rescaled point process of local maxima of $\widetilde{f}$ above $u(R)$ in the domain $D_{R,u}$, just as f. This is a coupling of $(\Psi^0_{R,u}, \Psi_{R,u})$.

Define the following functions:

$$F_t(x) := f(x) + th(x), \text{ and } h(x) = \widetilde{f}(x) - f(x), \quad t \in [0,1].$$

We will analyse the quantity $\rho_1(\Psi^0_{R,u}, \Psi_{R,u})$ by comparing the critical point structure of F_0 and F_1 in the box $[-R/2, R/2]^2$. One way to compare the critical point structures of F_0 and F_1 is to interpolate (as defined above) and see how the positions of the critical points change.

Define *a continuous flow x_t of a critical point under F_t* to be a continuous function $x. : [0, \delta] \to \mathbb{R}^2$ with $\nabla F_t(x_t) = 0$, for some $0 < \delta \leq 1$.

Almost sure properties of F_t:

1. $F.$ is a C^2-smooth function in (t, x) almost surely since both $f, \widetilde{f}$ are at least C^2-smooth in x.

2. Let us define the flow, for $x_t \in \mathbb{R}^2$,

$$\frac{dx_t}{dt} = -\text{Hess } (F_t(x_t))^{-1} \cdot \nabla h(x_t), \quad t \in [0,1]. \tag{3.39}$$

   This defines a smooth flow away from non-invertible points of Hess (F_t). By applying chain rule, you can check that $\nabla F_t(x_t)$ is constant in t. Hence, if you start the flow with a critical point of F_0, then x_t is a critical point of $F_t, \forall t$ as long as the flow is defined.

3. The index of a critical point x_0 changes during the F_t-flow only if

$$\det(\text{Hess } (F_t(x_t))) = 0 \text{ for some } t \in (0,1).$$

   This is because to change the index, one of the eigenvalues of Hess (F_t) has to change the sign, hence the determinant has to be zero at some $t \in (0,1)$.

4. Note that a critical point of F_t is an intersection of the hypersurfaces $\{\partial_{x_i} F_t = 0\}, i = 1, 2$. This intersection is stable if it is a transversal intersection (in this case, transversal intersection corresponds to non-degeneracy of the Hessian of F_t at that point). Hence, there's creation/destruction of a critical point under perturbation only if
$$\det(\text{Hess } (F_t)) = 0.$$

5. Also, two critical points of F_t cannot merge into one during the flow unless Hess (F_t) is degenerate at some point in between. This is because if the intersection of the hypersurfaces $\{\partial_{x_i} F_t = 0\}, i = 1, 2$ is transversal, then the number of intersection points (which corresponds to critical points in this case) is stable under perturbation (as it is a topological quantity). See Chapter 2 of [GP74] for details.

Now, the quantity $\mathbb{E}||\Psi^0_{R,u}(A_0^c)| - |\Psi_{R,u}(A_0^c)||$ is bounded by the sum of expectations of number of points of the following types (let $D'_R = [-R/2, R/2]^2 \setminus (\mu(u)A_0)$):

1. $\{x \in D'_R : \text{Hess } (F_t(x)) \text{ is degenerate for some } t \in [0, 1], f(x) > u, \widetilde{f}(x) > u\}$

   This set corresponds to the degeneracies that arise during the critical point flow, including the creation/destruction of points as described above.

2. $\{x \in D'_R : \nabla F_t(x) = 0 \text{ for some } t \in [0, 1], f(x) = u \text{ or } \widetilde{f}(x) = u\}$

   These are the points which flow in/out of the domain $\{f(x) > u\} \cap \{\widetilde{f}(x) > u\}$

3. $\{x \in \partial(D'_R) : \nabla F_t(x) = 0 \text{ for some } t \in [0, 1], f(x) \geq u \text{ or } \widetilde{f}(x) \geq u\}$

   Points which flow in/out of the domain $[-R/2, R/2]^2 \setminus (\mu(u)A_0)$.

4. $\{x \in D'_R : \nabla f(x) = 0 \text{ or } \nabla \widetilde{f}(x) = 0, x \in (\{f > u\} \cap \{\widetilde{f} < u\}) \cup (\{f < u\} \cap \{\widetilde{f} > u\})\}$.

We will use the Kac-Rice formulas to bound the number of the above-listed points, named P_i for $i = 1, 2, 3, 4, 5$ as below. Recall the notation from the coupling section above and define
$$f_1(x) := f(x) - \overline{f}(x), \quad f_2(x) = \widetilde{f}(x) - \overline{f}(x), \quad x \in \mathbb{R}^d.$$

Note that the random functions $f_1(x), f_2(x), \overline{f}(x)$ are jointly independent. Consequently, all derivatives of these functions are jointly independent. We need the following lemma to ensure the existence of some of the densities of the random vectors (i.e. non-degeneracy conditions) to be used in Kac-Rice formula.

Lemma 3.4.5. *For large enough R, u, for all $x \in D'_R$ the vector*

$$(\overline{f}(x), \nabla \overline{f}(x), \mathit{Hess}\ \overline{f}(x))$$

is a non-degenrate Gaussian vector in $\mathbb{R}^6$.

Proof. Notice that

$$(f(x), \nabla f(x), \text{Hess } f(x)) = (f_1(x), \nabla f_1(x), \text{Hess } f_1(x)) + (\overline{f}(x), \nabla \overline{f}(x), \text{Hess } \overline{f}(x)).$$

Let

$$\begin{aligned} X &= (f_1(x), \nabla f_1(x), \text{Hess } f_1(x)) \\ Y &= (\overline{f}(x), \nabla \overline{f}(x), \text{Hess } \overline{f}(x)) \\ Z &= (f(x), \nabla f(x), \text{Hess } f(x)) \end{aligned}$$

and $\Sigma_X, \Sigma_Y, \Sigma_Z$ be their respective covariance matrices. We want to show that Σ_Y is invertible matrix for large $\|x\|$. By construction, $f_1(x)$ is a linear combination of $r(x)$ and its derivatives up to second order with Gaussian coefficients. As a result, each entry of the covariance matrix Σ_X of $(f_1(x), \nabla f_1(x), \text{Hess } f_1(x))$ is bounded above by $c\|r(x)\|_4^2$, where

$$\|r(x)\|_4 := \max_{0 \le |\alpha| \le 4} |\partial^\alpha r(x)|.$$

Hence, the maximum value of absolute eigenvalue of Σ_X is at most $c\|r(x)\|_4^2$. Since $r(x) = e^{-\|x\|^2/2}$, we have

$$\|r(x)\|_4 \to 0 \quad \text{ as } \|x\| \to \infty.$$

Since $f(x)$ is stationary field the covariance matrix Σ_Z is constant in x. One of the property of Bargmann-Fock field is that Z is invertible. Let c_0 be the lowest absolute eigenvalue of Σ_Z. Now for any unit vector $\theta \in \mathbb{R}^6$, since f_1 and $\overline{f}$ are independent functions

$$\theta^T \Sigma_Z \theta = \theta^T \Sigma_X \theta + \theta^T \Sigma_Y \theta.$$

For any $\epsilon > 0$, for large enough $\|x\|$, we have $\theta^T \Sigma_X \theta < \epsilon$ for all θ. Also, $\theta^T \Sigma_Z \theta > c_0$ for all unit vector θ. Hence, $\theta^T \Sigma_X \theta \ge c_0 - \epsilon$ which proves our claim. □

Notice that for any independent random vectors X_1 and Y_1, if X_1 has density in $\mathbb{R}^d$ then $X_1 + Y_1$ also has density in $\mathbb{R}^d$. Indeed, if ρ_{X_1} is the density of X_1 and μ_{Y_1} is the law of Y_1 then the convolution $\rho_{X_1} * \mu_{Y_1}$ is the density of $X_1 +$

Y_1. Combining this observation and Lemma 3.4.5 we can show that, for instance, $(f(x), \nabla F_t(x), \text{Hess } F_t(x))$ has density for all t and $x \in D'_R$ because

$$(f(x), \nabla F_t(x), \text{Hess } F_t(x)) = (\text{Hess } \overline{f}(x), \nabla \overline{f}(x), \overline{f}(x)) + (f_1(x), (1-t)\nabla f_1(x) + t\nabla f_2(x), (1-t)\text{Hess } f_1(x) + t\text{Hess } f_2(x)).$$

Bound on P_1: Define

$$P_1 := \{(t, x) \in [0,1] \times D'_R : \nabla F_t(x) = 0, f(x) = u\}$$

and we want to compute $\mathbb{E}[P_1]$. By the Kac-Rice formula, we have

$$\mathbb{E}[P_1] = \int_0^1 \int_{D'_R} \mathbb{E}[|\det(\mathrm{d}(\nabla F_t(x), f(x)))| \big| \nabla F_t(x) = 0, f(x) = u] p_{t,x}(0, u) \mathrm{d}x \mathrm{d}t \tag{3.40}$$

where $p_{t,x}$ is the pdf of $(\nabla F_t(x), f(x))$. Since $(f_1, f_2, \overline{f})$ are independent, we first condition on f_1, f_2 and compute the integrand above. Let

$$(f_1(x), \nabla f_1(x), \text{Hess } f_1(x)) = (a_1, v_1, M_1) \quad (f_2, \nabla f_2(x), \text{Hess } f_2(x)) = (a_2, v_2, M_2).$$

Then the integrand in (3.40) is,

$$Z(t, a_1, v_1, M_1, v_2, M_2) = \mathbb{E}\left[|t| | \det \begin{pmatrix} (v_2 - v_1) & 0 \\ \text{Hess } \overline{f}(x) + (1-t)M_1 + tM_2 & (v_2 - v_1)^T \end{pmatrix} | \, | \right. \\ \left. \nabla \overline{f}(x) = (t-1)v_1 - tv_2, \overline{f}(x) = u - a_1 \right] \tag{3.41}$$

integrated over the vectors $(f_1(x), \nabla f_1(x), \text{Hess } f_1(x))$ and $(f_2, \nabla f_2(x), \text{Hess } f_2(x))$. Now, we look at the distribution of

$$\text{Hess } \overline{f}(x) | \overline{f}(x) = \bar{a}, \nabla \overline{f}(x) = \bar{v}.$$

We have that, since f is Bargmann-Fock field in 2-dimensions, for $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$

$$C(p_1, p_2) := \mathbb{E}[\overline{f}(p_1)\overline{f}(p_2)] = \exp\left(-\frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2)\right) - \\ \exp\left(-\frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2)\right)\left[1 + \frac{1}{2}(x_1 x_2)^2 + \frac{1}{2}(y_1 y_2)^2 + 2x_1 y_1 x_2 y_2\right] \tag{3.42}$$

which follows from Section 3.4.2. Denote the covariance of partial derivatives by subscripts to C, for example,

$$C_{11,2}(p_1, p_2) := \mathbb{E}[\partial_{11}\overline{f}(p_1)\partial_2 \overline{f}(p_2)].$$

When $p_1 = p_2$, by abuse of notation, we write $C(p_1, p_1) = C(p_1)$.
Let

$$\|r(x)\|_4 := \max_{0\le|\alpha|\le 4} |\partial^\alpha r(x)|.$$

Write Hess $\overline{f} = (\partial_{11}\overline{f}, \partial_{22}\overline{f}, \partial_{12}\overline{f})$. We choose A_0 such that, on $(R/\sqrt{N}A_0)^C$, $\|r(x)\|_{C^4}$ is small. Hence we compare covariances Cov(Hess $\overline{f}(x), (\overline{f}(x), \nabla\overline{f}(x)))$ with that of Cov(Hess $f(x), (f(x), \nabla f(x)))$.

Observe that $(f(x), \text{Hess } f(x))$ and $\nabla f(x)$ are independent. Also, $\text{Cov}(\partial_{11}f(x), f(x)) = \text{Cov}(\partial_{22}f(x), f(x)) = -1$ and $\text{Cov}(\partial_{12}f(x), f(x)) = 0$.

We have,

$$\Sigma_{22} = \text{Var}(\overline{f}(x), \nabla\overline{f}(x))) = I + O(\|r(x)\|_4)\mathbf{1}$$

and

$$\Sigma_{12} = \text{Cov}(\text{Hess } \overline{f}(x), (\overline{f}(x), \nabla\overline{f}(x))) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\|r(x)\|_4)\mathbf{1}$$

where $\mathbf{1}$ are matrices with all entries 1, with suitable dimensions. So,

$$\text{Hess } \overline{f}(x)|\overline{f}(x) = \bar{a}, \nabla\overline{f}(x) = \bar{v}$$

is a Gaussian vector with mean

$$\Sigma_{12}\Sigma_{22}^{-1}\begin{pmatrix}\bar{a}\\ \bar{v}\end{pmatrix} = \begin{pmatrix}-\bar{a}\\ -\bar{a}\\ 0\end{pmatrix} + O(\|r(x)\|_4)\mathbf{1}\begin{pmatrix}\bar{a}\\ \bar{v}\end{pmatrix}.$$

By standard Gaussian regression,

$$(\text{Hess } \overline{f}(x)|\overline{f}(x) = \bar{a}, \nabla\overline{f}(x) = \bar{v}) - \Sigma_{12}\Sigma_{22}^{-1}\begin{pmatrix}\bar{a}\\ \bar{v}\end{pmatrix}$$

is a Gaussian vector which does not depend on $\bar{a}$ or $\bar{v}$. And, the variance of this vector stays away from zero for $x \in D'_R$.

Observe that the determinant of the form

$$\det\begin{pmatrix} X & 0 \\ M & Y\end{pmatrix}$$

is a bilinear form in (X, Y) and since M is 2×2 matrix, co-efficients of this bilinear form will be linear in entries of M. Hence, the expression (3.41) is a quadratic form in $(v_2 - v_1)$ with coefficients being degree one polynomials in $(u - a_1, (t-1)v_1 - tv_2, M_1, M_2)$. So, we have integrated out $(\overline{f}(x), \nabla\overline{f}(x), \text{Hess } \overline{f}(x))$.

Our next goal is to compute the following quantity,

$$\mathbb{E}[Z(t, f_1(x), \nabla f_1(x), \text{Hess } f_1(x), \nabla f_2(x), \text{Hess } f_2(x)) p((t-1)\nabla f_1(x) - t\nabla f_2(x), u - f_1(x))] \tag{3.43}$$

where p is now the pdf of the Gaussian vector $(\nabla \overline{f}(x), \overline{f}(x))$. Again, since f_1, f_2 are independent random functions, first we will integrate over f_1 and its derivatives. Observe that, since $\nabla \overline{f}(x)$ is a zero mean Gaussian vector

$$p((t-1)\nabla f_1(x) - t\nabla f_2(x), u - f_1(x)) \leq p(0, u - f_1(x)).$$

We know that conditioning on a Gaussian random variable only decreases its variance. Hence,

$$p(0, u - f_1(x)) \leq \overline{P}(u - f_1(x))$$

for u large enough, where $\overline{P}$ is the pdf of $\overline{f}(x)$.

Now the Gaussian vector $((f_1(x), \nabla f_1(x), \text{Hess } f_1(x))$ is a linear transform of i.i.d standard Gaussians with coefficients $\partial^\alpha r(x), 0 \leq |\alpha| \leq 4$. Conditioning $(\nabla f_1(x), \text{Hess } f_1(x))$ on $f_1(x) = a_1$ only shifts the variance matrix by order $O(\|r(x)\|_4)$ and the mean a_1. After integrating out $(\nabla f_1(x), \text{Hess } f_1(x))$ in expression (3.43) while conditioning $f_1(x) = a_1$, we get (in terms of a_1)

$$w(a_1) \exp\left(-\frac{1}{2\sigma^2(x)}(u - a_1)^2\right) \tag{3.44}$$

where w is a polynomial of degree at most 2 and $\sigma^2(x) = \text{Var}(\overline{f}(x))$.

Since $f_1(x) + \overline{f}(x) = f(x)$, we have

$$\text{Var}\overline{f}(x) = 1 - \text{Var} f_1(x).$$

Let X be a centered Gaussian r.v. with variance $(1 - \sigma^2)$. Then, for σ close to 1,

$$\begin{aligned} \mathbb{E}\left[X^2 \exp\left(-\frac{1}{2\sigma^2}(u - X)^2\right)\right] &\leq c\sigma \exp(-1/2u^2)(\sigma^2(1-\sigma^2) + (1-\sigma^2)^2 u^2) \\ &\leq c(1-\sigma^2)(\sigma^2 + (1-\sigma^2)u^2) \exp(-1/2u^2). \end{aligned} \tag{3.45}$$

Replacing X by $f_1(x)$ and noting that

$$\text{Var}(f_1(x)) \leq \|r(x)\|_4^2$$

we have by expression (3.44)

$$\mathbb{E}\left[w(f_1(x)) \exp\left(-\frac{1}{2\sigma^2(x)}(u - f_1(x))^2\right)\right] \leq c\|r(x)\|_4^2(1 + \|r(x)\|_4^2 u^2) \exp(-1/2u^2).$$

At last, we integrate out the variables $(\nabla f_2(x), \text{Hess } f_2(x))$. Note that this random vector is not Gaussian, and the second moment is of order $u^2\|r(x)\|_4^2$. By definition, each component of the vector $(\nabla f_2(x), \text{Hess } f_2(x))$ is a linear transform of (ξ, Z), where the pdf is defined in (3.35). Now, $\mathbb{E}[w(Z)|\xi = a] = \mathbb{E}[\det(Z_1)w(Z_1)]$ where Z_1 is a Gaussian matrix distributed as

$$\text{Hess } f(0)|f(0) = a.$$

Hence, if w is a polynomial of degree m, then

$$\mathbb{E}[w(Z, \xi)] \leq cu^m$$

for u large enough. Now, as pointed out earlier, the expression (3.41) is a quadratic form in $(v_2 - v_1)$ with coefficients being degree one polynomials in $(u - a_1, (t-1)v_1 - tv_2, M_1, M_2)$, now we have integrated out (a_1, v_1, M_1). So, after integrating other terms, the expression (3.41) is a polynomial in v_2, M_2 of degree at most three.

Combining all the bounds above, we have that the integrand of the RHS of (3.40) is bounded above by,

$$u^3(\|r(x)\|_4^2)\exp(-1/2u^2). \tag{3.46}$$

Hence,

$$\mathbb{E}[P_1] \leq u^3 \exp(-1/2u^2) \int_{D'_R} \|r(x)\|_4^2 \mathrm{d}x.$$

Bound on P_2: Define

$$P_2 := \{(t, x) \in [0, 1] \times D'_R : \nabla F_t(x) = 0, \widetilde{f}(x) = u\}$$

and we want to give a similar upper bound for $\mathbb{E}[P_2]$. The computation is also similar to the upper bound for $\mathbb{E}[P_1]$, so we note the changes needed to the above computation.
We have,

$$\mathbb{E}[P_2] = \int_0^1 \int_{D'_R} \mathbb{E}[|\det(\mathrm{d}(\nabla F_t(x), \widetilde{f}(x)))| \big| \nabla F_t(x) = 0, \widetilde{f}(x) = u] p_{t,x}(0, u) \mathrm{d}x \mathrm{d}t \tag{3.47}$$

where $p_{t,x}$ is the pdf of $(\nabla F_t(x), \widetilde{f}(x))$. Since $(f_1, f_2, \overline{f})$ are independent, we first condition on f_1, f_2 and compute the integrand above. Again, let

$$(f_1(x), \nabla f_1(x), \text{Hess } f_1(x)) = (a_1, v_1, M_1) \quad (f_2, \nabla f_2(x), \text{Hess } f_2(x)) = (a_2, v_2, M_2).$$

Then the integrand above is,

$$Z(t, v_1, M_1, a_2, v_2, M_2) = \mathbb{E}\left[|1-t|| \det \begin{pmatrix} (v_2 - v_1) & 0 \\ \text{Hess } \overline{f}(x) + (1-t)M_1 + tM_2 & (v_2 - v_1)^T \end{pmatrix} || \right.$$
$$\left. \nabla \overline{f}(x) = (t-1)v_1 - tv_2, \overline{f}(x) = u - a_2 \right] \tag{3.48}$$

integrated over the vectors $(f_1(x), \nabla f_1(x), \text{Hess } f_1(x))$ and $(f_2, \nabla f_2(x), \text{Hess } f_2(x))$. First integrating out the Gaussian vector $(\overline{f}(x), \nabla \overline{f}(x), \text{Hess } \overline{f}(x))$, we have that the above expression is a quadratic form in $(v_2 - v_1)$ with coefficients being degree one polynomials in $(u - a_2, (t-1)v_1 - tv_2, M_1, M_2)$.
Our next goal is to compute the following quantity,

$$\mathbb{E}[Z(t, \nabla f_1(x), \text{Hess } f_1(x), f_2(x), \nabla f_2(x), \text{Hess } f_2(x)) p((t-1)\nabla f_1(x) - t\nabla f_2(x), u - f_2(x))] \tag{3.49}$$

where p is now the pdf of the Gaussian vector $(\nabla \overline{f}(x), \overline{f}(x))$. Observe that, since $\nabla \overline{f}(x)$ is a zero mean Gaussian vector

$$p((t-1)\nabla f_1(x) - t\nabla f_2(x), u - f_2(x)) \leq p(0, u - f_2(x)).$$

We know that conditioning on a Gaussian random variable only decreases its variance. Hence,

$$p(0, u - f_2(x)) \leq \overline{P}(u - f_2(x))$$

for u large enough, where $\overline{P}$ is the pdf of $\overline{f}(x)$. Since $\text{Var}(\overline{f}(x))$ is bounded away from zero uniformly for large R when $x \in D'_R$, we have that $\overline{P}$ is uniformly bounded. Hence, it is enough to bound

$$\mathbb{E}[Z(t, \nabla f_1(x), \text{Hess } f_1(x), f_2(x), \nabla f_2(x), \text{Hess } f_2(x))].$$

Since the expression (3.48) a quadratic form in $(v_2 - v_1)$ with coefficients being degree one polynomials in $(u - a_2, (t-1)v_1 - tv_2, M_1, M_2)$, we can see that

$$\mathbb{E}[Z(t, \nabla f_1(x), \text{Hess } f_1(x), f_2(x), \nabla f_2(x), \text{Hess } f_2(x))] \leq u^3 \|r(x)\|_4^2 \tag{3.50}$$

just like in the previous case. A noticeable difference here is that the upper bound

$$\mathbb{E}[P_2] \leq u^3 \int_{D'_R} \|r(x)\|_4^2 \mathrm{d}x$$

is missing the factor $\exp(-1/2u^2)$.

Bound on P_3: Define

$$P_3 = \{x \in D'_R : \nabla f(x) = 0, x \in (\{f > u\} \cap \{\widetilde{f} < u\})\}.$$

we have,

$$\int_{D'_R} \mathbb{E}\left[\det(\text{Hess } f(x))\mathbb{1}[\{f > u\} \cap \{\widetilde{f} < u\}]\middle|\nabla f(x) = 0\right] p_x(0)\mathrm{d}x \tag{3.51}$$

where p_x is the pdf of $\nabla f(x)$. Again, let

$$(f_1(x), \nabla f_1(x), \text{Hess } f_1(x)) = (a_1, v_1, M_1) \quad (f_2, \nabla f_2(x), \text{Hess } f_2(x)) = (a_2, v_2, M_2).$$

Let,

$$Z(a_1, v_1, M_1, a_2) = \mathbb{E}[\det(\text{Hess } (\overline{f}(x)) + M_1)\mathbb{1}[u - a_1 < \overline{f}(x) < u - a_2]|\nabla\overline{f}(x) = v_1].$$

Now,

$$\mathbb{E}[[\det(\text{Hess } (\overline{f}(x)) + M_1)|\overline{f}(x) = \bar{a}, \nabla\overline{f}(x) = \bar{v}]$$

is a degree one polynomial in $\bar{a}$. Also, we have

$$\mathbb{P}(u - a_1 < \overline{f}(x) < u - a_2) \leq c|a_1 - a_2|$$

for large enough u, since $\overline{f}(x)$ has pdf and it is uniformly bounded above in $x \in D'_R$. Hence,

$$z(a_1, v_1, M_1, a_2) = w(u, a_1, v_1, M_1)|a_1 - a_2|$$

where w is a multivariate polynomial (of degree one in u). So, finally integrating remaining variables such as $f_1(x), f_2(x)$ we have

$$\mathbb{E}[P_3] \leq cu^2 \int_{D'_R} \|r(x)\|_4 \mathrm{d}x.$$

The expectation of the following quantities also has similar upper bounds,

$$\{x \in D'_R : \nabla f(x) = 0, f(x) > u, \widetilde{f}(x) < u\}$$
$$\{x \in D'_R : \nabla \widetilde{f} = 0, f(x) > u, \widetilde{f}(x) < u\}$$
$$\{x \in D'_R : \nabla \widetilde{f}(x) = 0, f(x) < u, \widetilde{f}(x) > u\}.$$

Bound on P_4: Let,

$$P_4 = \{(t, x) \in [0, 1] \times \partial(D'_R) : \nabla F_t(x) = 0, f(x) \geq u\}.$$

Note that $\partial D'_R$ consists of two connected components, namely, a circle of radius $R/\sqrt{N} * \tau(u)$ and a square of side length R both centered at the origin.
First, let us compute the expected number of points hitting the circle in the flow. There's a Kac-Rice formula for the manifold version, see [AT09, Chapter 12]. To use this version, we need to calculate gradient of $\nabla F_t(x)$ when x belongs to the circle (a manifold). In our case, this corresponds to taking the derivative of $\nabla F_t(x)$ with respect to the angle θ in polar coordinates. By the Kac-Rice formula, and by the fact that Bargmann-Fock field is isotropic, we need to bound

$$|R\tau(u)/\sqrt{N}|\mathbb{E}\left[|\det\begin{pmatrix}\partial_1 h(x) & \partial_2 h(x)\\ \cdots & \cdots\end{pmatrix}|\mathbb{1}[f(x)>u]|\nabla F_t(x)=0\right]$$

where the second row of the matrix is a linear combination of the second derivatives of $f, \widetilde{f}$. Ignoring the factor $\mathbb{1}[f(x) > u]$ and conditioning first on f_1, f_2 and its derivatives, we integrate out $\overline{f}(x)$ and its derivatives as previously done. Since

$$\mathbb{E}[|\partial_j h(x)|] \le cu\|r(x)\|_4$$

we have,

$$\int_{\partial B(0,R\tau(u)/\sqrt{N})} |R\tau(u)/\sqrt{N}|\mathbb{E}\left[|\det\begin{pmatrix}\partial_1 h(x) & \partial_2 h(x)\\ \cdots & \cdots\end{pmatrix}|\mathbb{1}[f(x)>u]|\nabla F_t(x)=0\right]\mathrm{d}x \le cR^2\tau(u)^2N^{-1}u\|r(R\tau(u)/\sqrt{N})\|_4. \tag{3.52}$$

We now estimate an upper bound for the expected number of points hitting the square in the flow. As pointed out, it is a square of side length R centred at the origin. Hence, we need to bound

$$\int_0^1\int_{-R/2}^{R/2}\mathbb{E}\left[\left|\det\begin{pmatrix}\partial_1 h(x_1,R/2) & \partial_2 h(x_1,R/2)\\ \cdots & \cdots\end{pmatrix}\right||\nabla F_t(x_1,R/2)=0\right]\mathrm{d}x_1\mathrm{d}t.$$

Again using

$$\mathbb{E}[|\partial_j h(x)|] \le cu\|r(x)\|_4$$

we have

$$\mathbb{E}[P_4] \le cR^2\tau(u)^2N^{-1}u\|r(R\tau(u)/\sqrt{N})\|_4 + cRu\|r(R/2)\|_4$$

The expectation of the following quantity also has the same upper bound,

$$\{(t,x)\in[0,1]\times\partial(D'_R) : \nabla F_t(x)=0, \widetilde{f}(x)\ge u\}.$$

Bound on P_5: Let

$$P_5 = \{(t,x)\in[0,1]\times D'_R : \nabla F_t(x)=0, \det(\text{Hess } F_t(x))=0, f(x)>u, \widetilde{f}(x)>u\}.$$

We know that,

$$\text{Hess } F_t(x) = \text{Hess } f(x) + t\text{Hess } h(x).$$

Writing ij as subscripts for ij-th second partial derivatives, we have

$$\frac{d}{dt}\det(\text{Hess } F_t(x)) = h_{11}(x)(f_{22} + th_{22}) + h_{22}(x)(f_{11} + th_{11}) - 2h_{12}(x)(f_{12} + th_{12}).$$

By the Kac-Rice formula,

$$\mathbb{E}[P_5] = \int_0^1 \int_{D'_R} \mathbb{E}\left[|\det \begin{pmatrix} \partial_1 h(x) & \partial_2 h(x) & \frac{d}{dt}\det \text{Hess } F_t(x) \\ \dots & \dots & \dots \end{pmatrix}| \mathbb{1}[f(x) > u, \widetilde{f}(x) > u] \right.$$
$$\left. \Big| \nabla F_t(x) = 0, \det(\text{Hess } F_t(x)) = 0 \right] p_x(0,0)\mathrm{d}x\mathrm{d}t \tag{3.53}$$

where p_x is the pdf of $(F_t(x), \det(\text{Hess } F_t(x)))$. Observe that the determinant function is linear in the first row. Ignoring the factor $\mathbb{1}[f(x) > u, \widetilde{f}(x) > u]$, the integrand of the RHS in the above equation is linear in $h(x)$ and its derivatives (up to second derivatives). Hence, integrating out $\overline{f}(x)$ and its derivatives as previously done, we have

$$\mathbb{E}[P_5] \leq cu \int_{D'_R} \|r(x)\|_4 \mathrm{d}x.$$

3.4.4 Gathering all the bounds

From the expectation bounds on the number of points $P_1, \dots, P_5$, we have

$$\begin{aligned} \mathbb{E}||\Psi^0_{R,u}| - |\Psi_{R,u}|| \leq c \Bigg(& u^3 \exp\left(-\frac{1}{2}u^2\right) \int_{D'_{R,u}} \|r(x)\|_4^2 \mathrm{d}x \\ & u^3 \int_{D'_{R,u}} \|r(x)\|_4^2 \mathrm{d}x + u^2 \int_{D'_{R,u}} \|r(x)\|_4 \mathrm{d}x + \\ & u \int_{D'_{R,u}} \|r(x)\|_4 \mathrm{d}x + u^2 \|r(u)\|_4 + Ru\|r(R/2)\|_4 \Bigg). \end{aligned} \tag{3.54}$$

Recall that

$$\begin{aligned} \|r(x)\|_4 &= \max_{|\alpha| \leq 4} \|\partial^\alpha r(x)\| \\ &\leq c\|x\|^4 \exp\left(-\frac{1}{2}\|x\|^2\right). \end{aligned} \tag{3.55}$$

By a polar coordinate transformation in 2-dim, we have

$$\begin{aligned} \int_{D'_{R,u}} \|x\|^4 \exp(-1/2\|x\|^2)\mathrm{d}x &\leq c \int_u^\infty t^5 \exp(-1/2t^2)\mathrm{d}t \\ &\leq c' u^4 \exp(-1/2u^2) \end{aligned} \tag{3.56}$$

for large enough u.

Now, by assumption, $u \leq \log R$, hence

$$R \exp(-R^2/4) \leq c' \exp(-u^2/2).$$

Hence, we have

$$\mathbb{E}||\Psi^0_{R,u}| - |\Psi_{R,u}|| \leq cu^6 \exp(-1/2u^2). \tag{3.57}$$

Also,

$$\mathrm{vol}(A_0) = \tau(u)^2 \leq cu^3 \exp(-1/2u^2). \tag{3.58}$$

Finally substituting the bounds (3.30),(3.57), (3.58) to the RHS of (3.18) we have

$$d_{TV}(|\mathcal{L}\Psi_{R,u}|, U_{R,u}) \leq C \left(u \exp\left(-u^2 \frac{\delta}{2(2-\delta)} \right) + u^3 \exp(-1/2u^2) + u^6 \exp(-1/2u^2) \right).$$

Hence for $\beta < \min\{\frac{\delta}{2(2-\delta)}, 1/2\}$ we have,

$$d_{TV}(|\mathcal{L}\Psi_{R,u}|, U_{R,u} \leq C \exp(-\beta u^2).$$

This finishes the proof of Theorem 3.4.1.

Chapter 4

Future work

Filament structure of RPW

See Figure 4.1. What explains the apparent filament structure of the random plane wave (RPW) landscape? Is it just a numerical artefact? Our result in the previous chapter establishes that local maxima for RPW at high levels is Poissonian. On the other hand, the critical points of RPW have a somewhat 'rigid' structure, but not repulsive at small scale, and quite different from the Poisson process [BCW19]. Looking at another criterion, Tacey [Tac23] showed that the L^2 norm restricted to *any* long line is very close to that of the entire domain, which suggests that the L^2 norm is not concentrated on any line. But this was expected since these filaments spread in all directions. Another interesting observation in [OGH87] is that the 'stringiness' disappears for a non-isotropic random model of eigenfunctions of the Laplacian on $\mathbb{R}^2$. So, the filaments can't be solely attributed to 'rigidity' properties of those eigenfunctions, or the correlation structure of the RPW, since non-isotropic models can have similar properties (say, oscillation, rate of decay, etc.).

Critical points of high-dimensional fields

In the previous chapter, we explored the Poisson convergence of high critical points of stationary smooth Gaussian fields in fixed low dimension. Can the same techniques (Stein's method for Poisson approximation) work when dimension $d \to \infty$? From Auffinger et al. [AAC13], we know that the correct scaling is as follows.

Let $f_d : \mathbb{R}^d \to \mathbb{R}$ be a sequence of isotropic Gaussian random fields of unit variance. Then we consider, inside a ball of radius R in $\mathbb{R}^d$ (which is a constant, i.e. doesn't depend on d), consider the critical points of f_d (of fixed index k or all critical points),

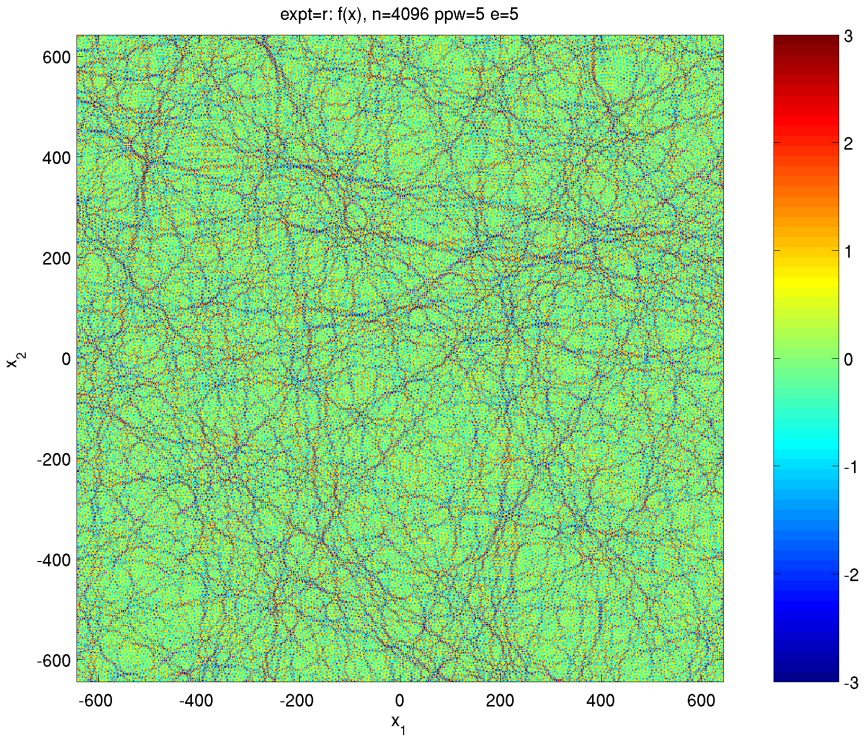


Figure 4.1: Filament structure of random plane wave. Picture by Alex Barnett

above the level

$$u(d) \simeq \sqrt{d} \quad \text{as } d \to \infty.$$

Depending on whether R is above or below a threshold R_c, the number is either exponentially large or exponentially small as $d \to \infty$. The topological complexity of the field (i.e. asymptotics of the logarithm of the number of critical points) is important in determining the landscape (e.g. in spin glass models). How about the spatial structure of these critical points? At what threshold R is there a Poisson-like behaviour?

Appendix A

Basic tools

A.1 Kac-Rice formula

Kac-Rice formulas are one of the central tools in studying random fields, which help us compute local observables. For simplicity, consider a one-dimensional smooth field f, and we count the number of zeros of the field in a given bounded interval $I = [a, b]$. Let $x_0 \in (a, b)$ be a zero and a regular point of f, i.e. $f'(x_0) \neq 0$. Given $\epsilon > 0$, for a sufficiently small neighborhood U of x_0, we have

$$\frac{1}{2\epsilon} \int_U |f'(x)| \mathbb{1}_{[|f(x)| \leq \epsilon]} dx = 1$$

which follows from the fundamental theorem of calculus. Now, assuming all zeroes of f are regular and $f(a)f(b) \neq 0$, the number of zeros $N_I(f)$ is given by

$$N_I(f) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{[a,b]} |f'(x)| \mathbb{1}_{[|f(x)| \leq \epsilon]} dx$$

Now, applying the same technique to a random field f and taking expectation, we have the following.

$$\begin{aligned} \mathbb{E}\left[N_I(f)\right] &= \mathbb{E}\left[\lim_{\epsilon \to 0} \int_{[a,b]} |f'(x)| \frac{1}{2\epsilon} \mathbb{1}_{[|f(x)| \leq \epsilon]}\right] dx \\ &= \lim_{\epsilon \to 0} \left[\int_{[a,b]} \mathbb{E}\left[|f'(x)| \frac{1}{2\epsilon} \mathbb{1}_{[|f(x)| \leq \epsilon]}\right] dx\right] \\ &= \int_{[a,b]} \mathbb{E}[|f'(x)||f(x) = 0] p_{f(x)}(0) dx \end{aligned} \tag{A.1}$$

where p_X is the pdf of a Gaussian variable X. The integral, limit and expectation swaps can be justified when the random field follows some regularity. Even the higher moments can be computed by similar techniques.

Theorem A.1.1 (Kac-Rice formula, ([AW09], Theorem 3.2))**.** *Let f be a C^1-smooth Gaussian field on an interval I. Let k be a positive integer. Assume that for every k pairwise distinct points $t_1, t_2, \ldots, t_k$ the joint distribution of $f(t_1), f(t_2), \ldots, f(t_k)$ does not degenerate. Then,*

$$\mathbb{E}[N_I^{[k]}] = \int_{I^k} \mathbb{E}\left[\prod_{i=1}^{k} |f'(t_i)| | f(t_1), f(t_2), \ldots f(t_k) = 0\right] p_{f(t_1),f(t_2)\ldots f(t_k)}(0) \prod_{i=1}^{k} dt_i$$

where $m^{[k]} = m(m-1)\cdots(m-k+1)$ given $m \geq k$, and 0 otherwise.

Observe that $f'(x)$ is also a Gaussian field and its regularity depends on that of the field f. Also we know that $(f'(t_1), f'(t_2), \ldots, f'(t_k), f(t_1), f(t_2)\ldots, f(t_k))$ is a Gaussian vector. Hence, the conditional expectation in the above theorem can be explicitly computed using the covariance kernel and its derivatives.
Note that the formula is particularly simpler when the field is stationary. We have,

$$\mathbb{E}[N_I] = \mathrm{Vol}(I)\mathbb{E}|f'(0)|/\sqrt{2\pi}$$

where the field is normalised to be $\mathrm{Var} f(0) = 1$.

We can generalise Theorem A.1.1 in many directions. We can consider fields in higher dimensions, say $f : \mathbb{R}^d \to \mathbb{R}^k$ and ask for the expected geometric measure of the level sets of f when $k \leq d$. There are versions of the Kac-Rice formula for non-Gaussian fields. We can generalise it to fields on a Riemannian manifold as well. In this article, we used the Kac-Rice formula for the mean lengths of level lines.

The Kac-Rice formulas are also crucial in the analysis of critical points. Geometry (and topology) of level/excrusion sets crucially depend on the distribution (in a deterministic sense) of critical points of the field (see Morse theorems in classical topology). Also, in many cases, even for non-local functionals like the number of connected components of level sets, a pretty good estimate can be obtained from looking at the critical points.

One interesting connection to random matrix theory for computing the expected number of critical points was made by Fyodorov [Fyo04] in the context of the theory of spin glasses. Note that by Kac-Rice formulas, the expected number of critical points can be computed by an integral of the conditional expectation of the Hessian of the field. The novel idea of Fyodorov was to express the law of Hessian in terms of Gaussian Orthonormal Ensemble (GOE) matrices, where explicit computations are available.

To give an example, we quote a proposition from Cheng and Schwartzmann [CS18]. Let X be an isotropic Gaussian field on $\mathbb{R}^d$ and $\mu(X)$ be the number of local maxima of X inside a unit volume ball. Then, under certain conditions on X, we have

$$\mathbb{E}[\mu(X)] = \Gamma((d+1)/2)\frac{2^{(d+1)/2}}{\pi^{(d+1)/2}\eta^N}\mathbb{E}_{GOE}^{N+1}\left[\exp(-\lambda_{d+1}^2/2)\right]$$

where expectation on RHS is w.r.t to GOE $(d+1)\times(d+1)$ ensembele, λ_{d+1} is the maximum eigenvalue of GOE, η is an explicit constant of the field X.

Refer to chapters 3 and 6 of [AW09] or [AT09] for more on Kac-Rice formulas.

A.2 Bulinskaya lemma

While studying nodal geometry of smooth fields, it is desirable that nodal sets are stable under small perturbations. The bad event we want to eliminate is that the random Gaussian function and its gradient are simultaneously small at some point.

Lemma A.2.1. *Let U be an open set in $\mathbb{R}^n$ and let $g : U \to \mathbb{R}^{n+1}$ be a random function. Assume that $g \in C^1(U)$ a.s. and that the vector $g(x)$ has a density on $\mathbb{R}^{n+1}$ that is bounded uniformly over x in compact subsets of U. Then $g^{-1}(0)$ is almost surely empty.*

Now, applying the above lemma to C^1-Gaussian field $(f, \nabla f)$ and assuming it has a density (i.e. non-degeneracy of f), we get that, almost surely, 0 is a regular value of f. One immediate consequence is that we know that nodal sets are submanifolds, almost surely.

In [NS16], Nazarov and Sodin stated a quantitative version of Bulinskaya's lemma. This helps bound the probability of "bad events".

Lemma A.2.2. *Let $f : U \to \mathbb{R}$ be a C^2-smooth Gaussian field which is non-degenerate. Fix a compact subset $C \subset U$. Given $\delta > 0$, there exists $\tau > 0$ (possibly depending on C) such that*

$$\mathbb{P}\left(\min_{x\in C}\max\{|f(x)|, |\nabla f(x)|\} < \tau\right) < \delta.$$

A.3 Borell-TIS inequality

Theorem A.3.1 ([AT09],Theorem 2.1.1)**.** *Let $f(x)$ be a Gaussian process on D. Assume that the process is almost surely bounded on D. Define $||f|| = ||f||_D = \sup_D f(x)$. Then $\mathbb{E}[||f||] < \infty$, and*

$$\forall u > 0,\ \mathbb{P}(||f|| - \mathbb{E}[||f||] > u) \leq e^{-u^2/2\sigma_D^2}$$

where $\sigma_D^2 = \sup_D \mathbb{E}[f(x)^2]$

This theorem holds for a continuous Gaussian process, but in the smooth setting, we are guaranteed that on any bounded D, the supremum is a.s. finite, due to Kolmogorov's theorem.

Using Borell-TIS inequality, we can bound the tail probability of C^k-norm of a smooth Gaussian field, see [BM22, Lemma 2.1] for example. Fixing the law of the field f and the domain D, $\mathbb{P}(||f||_{C^k} > l)$ is essentially bounded by $\exp(-c \cdot l^2)$ where c depends on variances of f and its derivatives on D. This is a bit surprising, since the tail of a single Gaussian random variable also has a similar upper bound.

A.4 Method of comparison

One of the fundamental questions which pops up regularly when studying the excursion of Gaussian fields is the following: given two Gaussian vectors in $\mathbb{R}^d$ with a covariance matrix that is close enough, how close are the excursion probabilities? Here we recall a generalisation of the classic Berman's inequality, taken from [Pit96].

Say we're given n sequence of real numbers, that we call *discritising levels*,

$$\mathbf{u}(k) = (\cdots < u_{-1}(k) < u_0(k) < u_1(k) < \cdots) \quad k = 1, 2, 3, \ldots n.$$

Consider the $\sigma-$algebra $\mathcal{U}$ generated by the n-dim rectangles

$$\Pi_{\mathbf{i}} = \{(x_1, x_2, \ldots, x_n) : x_k \in [u_{i_k}, u_{i_k+1}(k)\}$$

where $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ is a multi-index. Let $\mathbf{X}_0 = (X_0(1), X_0(2), \ldots, X_0(n))$ and $\mathbf{X}_1 = (X_1(1), X_1(2), \ldots, X_1(n))$ be two independent Gaussian vectors in $\mathbb{R}^n$ with zero mean. Consider an interpolation of these vectors,

$$\mathbf{X}_h = \sqrt{h}\mathbf{X}_1 + \sqrt{1-h}\mathbf{X}_0 \quad 0 \leq h \leq 1.$$

Denote by $R_h = \{r_h(i,j) : 1 \leq i, j \leq n\}$ the covariance matrix of $\mathbf{X}_h$.

Theorem A.4.1 (Theorem 1.2, [Pit96]). *With notations as above, if $r_0(k,k) = r_1(k,k)$ for all k and $|r_0(k,l)| < 1$ for $k \neq l$, then for any $B \in \mathcal{U}$, we have*

$$|\mathbb{P}(\mathbf{X}_0 \in B) - \mathbb{P}(\mathbf{X}_1 \in B)| \leq 2 \sum_{k>l}^{n} |r_0(k,l) - r_1(k,l)| \sum_{i,j} \int_0^1 \phi(u_i(k), u_j(k); r_h(k,l)) dh$$

where $\phi(x, y; r)$ is the density of 2-dim Gaussian with covariance r.

A.5 Asymptotic excursion probability

In this section, we state a result on the asymptotic excursion probability of stationary smooth Gaussian fields. Let $X : \mathbb{R}^d \to \mathbb{R}$ be a zero mean, unit variance, stationary C^2-smooth Gaussian field. Further assume that $(X(s), \nabla X(s))$ is a non-degenerate Gaussian vector.

Theorem A.5.1 (Theorem 7.1,[Pit96]). *Let $r(t,s)$ be the covariance function of the field X such that $r(t,s) < 1$ for $t \neq s$. Let $A \subset \mathbb{R}^d$ be a Jordan set of positive measure. Then,*

$$\mathbb{P}\left(\max_{t \in A} X(t) > u\right) = C vol(A) u^{d-1} \Psi(u)(1 + o(1)) \qquad \text{as} \quad u \to \infty.$$

Here, the constant C depends only on the field and not on level u, $1 - \Psi$ is the cdf of a standard Gaussian.

A.6 Maximum of Gaussian fields

It is a classical fact in probability that the expected maximum of n i.i.d. standard Gaussian random variables behaves asymptotically like $\sqrt{2 \log n}$ as $n \to \infty$. Also, it can be shown that, even if the random variables are dependent, it cannot exceed $\sqrt{2 \log n}$. What is a bit surprising is that a large number of Gaussian fields models with correlation decay 'fast enough' also have exact asymptotic $\sqrt{2 \log n}$. Examples include 2-dim discrete Gaussian free field, Sherrington-Kirkpatrick model energy landscape, etc [Cha16]. They are expected to converge to the Gumbel distribution for the asymptotic distribution of the (centred, normalised) maximum.

We have the same asymptotic for stationary smooth Gaussian fields.

Theorem A.6.1 (Theorem 14.1, [Pit96]). *Let $X : \mathbb{R}^d \to \mathbb{R}$ be centered, unit variance, C^2-smooth Gaussian field with covariance $r(t) = \mathbb{E}[X(0)X(t)]$. Assume that, for some $\alpha > 0$,*

$$\int_{\mathbb{R}^d} |r(t)|^\alpha dt < \infty.$$

Then,

$$\mathbb{P}\left(\max_{t\in[0,R]^d}(X(t)-l_R)l_R<x\right)=\exp(-\exp(-x))$$

where l_R is the largest solution of the equation

$$\frac{R^d\det(\Lambda_X)^{1/2}}{(2\pi)^{d-1}}l^{d-1}\exp(-l^2/2)=1$$

and Λ_X is the covariance matrix of $\nabla X(0)$.

From this theorem, we can get the exact asymptotic of the expected mean,

$$\frac{\mathbb{E}[\max_{t\in[0,R]^d}X(t)]}{\sqrt{2d\log R}}\to 1 \qquad \text{as} \quad R\to\infty.$$

A.7 Discrete approximation of excursion probability

Here we show that for a C^1-smooth stationary Gaussian field $f:\mathbb{R}^d\to\mathbb{R}$ the excursion probability of $f>u$ on a box B approximated by that on the grid $B\cap u^{-1}\mathbb{Z}^d$ with high probability. These approximations were first proved by Pickands [Pic69] and later generalised to what is now called the double sum method (see Section 6 of [Pit96] for a discussion).

Let $\chi:\mathbb{R}^d\to\mathbb{R}$ be a continuous Gaussian field with mean $\mathbb{E}[\chi(s)]=-\|s\|^2$ and covariance

$$\mathrm{Cov}(\chi(t),\chi(s))=2\langle t,s\rangle$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^d$. The existance of the continuous trajectories (in fact, smooth trajectories) of χ follows from Theorem 1.2.1. We now use the following lemma from [Pit96] for the special case of C^1-smooth Gaussian fields. Lemma 6.1 of [Pit96] holds for much more general continuous stationary Gaussian fields such as those with α-Holder continuous paths with $\alpha\in(0,1)$ (e.g. Ornstein-Uhlenbeck processes on $\mathbb{R}^d$)

Lemma A.7.1 (Lemma 6.1 of [Pit96])**.** *Let $f:\mathbb{R}^d\to\mathbb{R}$ be a zero-mean, stationary, C^1-smooth Gaussian field. Assume that the covariance satisfies*

$$\mathbb{E}[f(x)f(0)]=1-\|x\|^2+o(\|x\|^2) \qquad \text{as } x\to 0.$$

Then for any compact set $M\subset\mathbb{R}^d$,

$$\mathbb{P}\left(\max_{x\in u^{-1}M}\frac{f(x)}{1+\|x\|^2}>u\right)=\frac{1}{\sqrt{2\pi}}C(M)u^{-1}e^{-u^2/2}(1+o(1)) \qquad \text{as } u\to\infty$$

where

$$C(M)=\mathbb{E}\left[\exp\left(\max_{t\in M}(\chi(t)-\|t\|^2)\right)\right]<\infty.$$

The 'local structure of the covariance' can be achived by rescaling the domain. That is, for any C^1-smooth stationary centered field f with $f(0)$ having unit variance, there exists a fixed invertible matrix A such that

$$\mathbb{E}[f(Ax)f(0)] = 1 - \|x\|^2 + o(\|x\|^2).$$

Observing that

$$\begin{aligned} C(M) =& \mathbb{E}\left[\exp\left(\max_{t\in M}(\chi(t) - \|t\|^2)\right)\right] \\ &= \int_0^\infty e^v \mathbb{P}\left(\sup_M(\chi(t) - \|t\|^2) > v\right) dv \end{aligned}$$

the proof of Lemma 6.1 in [Pit96] yields the following, for any fixed $b > 0$,

$$\lim_{u\to\infty} \frac{1}{\sqrt{2\pi}} u e^{u^2/2} \mathbb{P}\left(\sup_{u^{-1}M\cap(bu^{-1}\mathbb{Z}^d)} f(x) \le u,\ \sup_{u^{-1}M} f(x) > u\right) = \int_0^\infty e^v \mathbb{P}\left(\sup_{M\cap(b\mathbb{Z}^d)} \chi(t) - \|t\|^2 \le v, \sup_M \chi(t) - \|t\|^2 > v\right) dv. \quad \text{(A.2)}$$

Appendix B

Point processes and Palm measures

B.1 Palm measures of point processes

Palm measures provide a rigorous framework for analysing point processes from the perspective of a typical point. While the distribution of a point process Φ on a measurable space S describes the statistical behaviour of configurations of points, the *Palm measure* conditions on the occurrence of a point at a specific location and allows us to understand the structure surrounding such points. References for this section of the appendix are Kallenberg's classical book [Kal17] and Last and Penrose's book [LP17].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let Φ be a simple point process on a Borel space S (typically $S = \mathbb{R}^d$), taking values in the space $\mathcal{N}(S)$ of locally finite counting measures on S. For a measurable function $f : S \times \mathcal{N}(S) \to [0, \infty)$, the *Campbell formula* states:

$$\mathbb{E}\left[\sum_{x \in \Phi} f(x, \Phi)\right] = \int_S \mathbb{E}[f(x, \Phi)]\, \lambda(dx),$$

where λ is the intensity measure of Φ.

If Φ is a *stationary* point process on $\mathbb{R}^d$ with constant intensity $\rho > 0$, then the *Palm measure* $\mathbb{P}^x$ is defined as the probability law of the process conditioned to have a point at x. The Palm-Campbell formula becomes

$$\mathbb{E}\left[\sum_{x \in \Phi} f(x, \Phi)\right] = \int_{\mathbb{R}^d} \mathbb{E}^x[f(x, \Phi)]\, \rho\, dx,$$

where $\mathbb{E}^x$ denotes expectation under $\mathbb{P}^x$.

Under stationarity, studying the Palm measure at the origin, denoted $\mathbb{P}^0$, suffices. For measurable functionals $f : \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}$, the Palm measure can be formally

defined via disintegration:

$$\mathbb{E}\left[\sum_{x\in\Phi} f(\theta_x\Phi)\right] = \rho \int_{\mathbb{R}^d} \mathbb{E}^x[f(\Phi)]\, dx,$$

where $\theta_x\Phi$ is the translation of Φ by $-x$.

B.2 Poisson Process Characterisation via Stein's Method

The Poisson point process plays a central role in the theory of stochastic geometry and spatial statistics, serving as the canonical model for complete spatial randomness. Various techniques exist for approximating more complex point processes using a Poisson process. One of the most powerful and flexible is *Stein's method*, introduced initially for approximating the Poisson distribution, and later extended to spatial point processes. This section provides a mathematical overview of the Poisson process characterisation via Steins method, and its connections to Palm measures.

B.2.1 The Classical Chen-Stein Method for Poisson Approximation

Let W be an integer-valued random variable with $\mathbb{E}[W] = \lambda$. The goal of Poisson approximation is to compare the distribution of W to that of a Poisson random variable $Z \sim \text{Poisson}(\lambda)$. The classical SteinChen method characterises the Poisson distribution as the unique solution to the identity

$$\mathbb{E}[\lambda f(W+1) - W f(W)] = 0 \quad \text{for all suitable test functions } f : \mathbb{N} \to \mathbb{R}.$$

This identity provides a means of bounding the total variation distance between the distribution of W and Poisson(λ):

$$d_{\text{TV}}(\mathcal{L}(W), \text{Poisson}(\lambda)) \leq \sup_{f\in\mathcal{F}} |\mathbb{E}[\lambda f(W+1) - W f(W)]|,$$

where $\mathcal{F}$ is a class of bounded test functions. Chen [Che75] developed a general method for applying this identity, especially when $W = \sum_{i=1}^n X_i$ is a sum of (possibly dependent) Bernoulli random variables.

Barbour and Brown [BHJ92] and later Schuhmacher and coauthors extended Stein's method to the setting of point processes. Let Φ be a point process on a Polish space S with finite intensity measure $\lambda(dx)$. We wish to compare the law of Φ to that of a Poisson point process Π with the same intensity measure.

A fundamental observation is that the Poisson process is characterised as the unique point process Φ satisfying:

$$\mathbb{E}\left[\int_S f(x, \Phi + \delta_x)\, \lambda(dx)\right] = \mathbb{E}\left[\sum_{x \in \Phi} f(x, \Phi)\right],$$

for all suitable test functions $f : S \times \mathcal{N}(S) \to \mathbb{R}$. This identity is equivalent to the statement that the *reduced Palm measure* $\mathbb{P}^{!x}$ coincides with the original measure $\mathbb{P}$ for all x, a defining property of the Poisson process:

$$\mathbb{P}^{!x} = \mathbb{P}, \quad \text{for all } x \in S.$$

B.2.2 Chen-Stein Approximation Using Palm Measures

Let Ξ be a point process on a locally compact second countable Hausdorff space Γ with finite mean measure λ, and let $A_\alpha \subseteq \Gamma$ be measurable neighbourhoods indexed by $\alpha \in \Gamma$. Denote by $\Xi(\Gamma)$ the total number of points of Ξ in Γ. Chen and Xia [CX04] obtain a total variation bound between the law of $\Xi(\Gamma)$ and a Poisson random variable with mean $\lambda = \lambda(\Gamma)$. Their result is as follows.

Theorem B.2.1 ([CX04], Theorem 3.1)**.** *Let* $\mathrm{Pois}(\lambda)$ *denote a Poisson random variable with mean* λ*, and write* d_{TV} *for the total variation metric. Then*

$$d_{\mathrm{TV}}\big(\mathcal{L}(\Xi(\Gamma)), \mathrm{Po}(\lambda)\big) \le \frac{1 - e^{-\lambda}}{\lambda} \mathbb{E} \int_{\alpha \in \Gamma} \big(\Xi(A_\alpha) - 1\big)\, \Xi(d\alpha) + \min\{\varepsilon_1, \varepsilon_2\} + \frac{1 - e^{-\lambda}}{\lambda} \int_{\alpha \in \Gamma} \lambda(A_\alpha)\, \lambda(d\alpha), \quad \text{(B.1)}$$

where

$$\varepsilon_1 = \big(1 \wedge \sqrt{2e/\lambda}\big) \int_{\alpha \in \Gamma} \mathbb{E}\big|G(\alpha, \Xi(\alpha)) - \varphi(\alpha)\big|\, \nu(d\alpha),$$
$$\varepsilon_2 = \frac{1 - e^{-\lambda}}{\lambda} \int_{\alpha \in \Gamma} \mathbb{E}\Big|\, \big|\Xi(\alpha)\big| - \big|\big(\Xi^\alpha\big)(\alpha)\big|\Big|\, \lambda(d\alpha).$$

Here G *and* φ *are functions appearing in the Stein equation for the Poisson process,* ν *is a reference measure, and* Ξ^α *denotes the Palm version of* Ξ *at* α*. The bound for* ε_1 *holds when* Ξ *is simple.*

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