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Four-dimensional Fano toric complete intersections

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We find at least 527 new four-dimensional Fano manifolds, each of which is a complete intersection in a smooth toric Fano manifold.

1. Introduction

Fano manifolds are the basic building blocks of algebraic geometry, both in the sense of the Minimal Model Program [1–4] and as the ultimate source of most explicit examples and constructions. There are finitely many deformation families of Fano manifolds in each dimension [5]. There is precisely 1 one-dimensional Fano manifold: the line; there are 10 deformation families of two-dimensional Fano manifolds: the del Pezzo surfaces and there are 105 deformation families of three-dimensional Fano manifolds [6–13]. Very little is known about the classification of Fano manifolds in higher dimensions.

In this paper, we begin to explore the geography of Fano manifolds in dimension 4. Four-dimensional Fano manifolds of higher Fano index have been classified [6,14–22]—there are 35 in total—but the most interesting case, where the Fano variety has index 1, is wide open. We use computer algebra to find many four-dimensional Fano manifolds that arise as complete intersections in toric Fano manifolds in codimension at most 4. We find at least 738 examples, 717 of which have Fano index 1 and 527 of which are new.

Suppose that Y is a toric Fano manifold and that L_1, \dots, L_c are nef line bundles on Y such that $-K_Y - \Lambda$ is ample, where $\Lambda = c_1(L_1) + \dots + c_1(L_c)$. Let $X \subset Y$ be a smooth complete intersection defined by a regular section of $\oplus_i L_i$. The Adjunction Formula gives that

$$-K_X = (-K_Y - \Lambda)|_X,$$

so X is Fano. We find all four-dimensional Fano manifolds X of this form such that the codimension c is at most 4.

Our interest in this problem is motivated by a program to classify Fano manifolds in higher dimensions using mirror symmetry [23]. For each four-dimensional Fano manifold X as above, therefore, we compute the essential ingredients for this program: the quantum period and regularized quantum differential equation associated to X , and a Laurent polynomial f that corresponds to X under mirror symmetry; we also calculate basic geometric data about X , the ambient space Y and f . The results of our computations in machine-readable form, together with full details of our implementation and all source code used, can be found in the electronic supplementary material.

2. Finding four-dimensional Fano toric complete intersections

Our method is as follows. Toric Fano manifolds Y are classified up to dimension 8 by Batyrev *et al.*¹ [24–29]. For each toric Fano manifold Y of dimension $d = 4 + c$, we:

- (i) compute the nef cone of Y ;
- (ii) find all $\Lambda \in H^2(Y; \mathbb{Z})$ such that Λ is nef and $-K_Y - \Lambda$ is ample and
- (iii) decompose Λ as the sum of c nef line bundles L_1, \dots, L_c in all possible ways.

Each such decomposition determines a four-dimensional Fano manifold $X \subset Y$, defined as the zero locus of a regular section of the vector bundle $\oplus_i L_i$. To compute the nef cone in step (i), we consider dual exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{L} & \longrightarrow & \mathbb{Z}^N & \xrightarrow{\rho} & \mathbb{Z}^d \longrightarrow 0 \\ & & & & & & \\ 0 & \longleftarrow & \mathbb{L}^* & \xleftarrow{D} & (\mathbb{Z}^N)^* & \xleftarrow{\rho^*} & (\mathbb{Z}^d)^* \longleftarrow 0, \end{array}$$

where the map ρ is defined by the N rays of a fan Σ for Y . There are canonical identifications $\mathbb{L}^* \cong H^2(Y; \mathbb{Z}) \cong \text{Pic}(Y)$, and the nef cone of Y is the intersection of cones

$$\text{NC}(Y) = \bigcap_{\sigma \in \Sigma} \langle D_i : i \notin \sigma \rangle,$$

where D_i is the image under D of the i th standard basis vector in $(\mathbb{Z}^N)^*$ [30, proposition 15.1.3]. The classes Λ in step (ii) are the lattice points in the polyhedron $P = \text{NC}(Y) \cap (-K_Y - \text{NC}(Y))$ such that $-K_Y - \Lambda$ lies in the interior of $\text{NC}(Y)$. Since $\text{NC}(Y)$ is a strictly convex cone, P is compact and the number of lattice points in P is finite. We implement step (iii) by first expressing Λ as a sum of Hilbert basis elements in $\text{NC}(Y)$ in all possible ways

$$\Lambda = b_1 + \dots + b_r \quad b_i \text{ an element of the Hilbert basis for } \text{NC}(Y), \quad (2.1)$$

where some of the b_i may be repeated; this is a knapsack-style problem. We then, for each decomposition (2.1), partition the b_i into c subsets S_1, \dots, S_c in all possible ways and define the line bundle L_i to be the sum of the classes in S_i .

We found 117 173 distinct triples $(X; Y; L_1, \dots, L_c)$, with a total of 17 934 distinct ambient toric varieties Y . Note that the representation of a given Fano manifold X as a toric complete intersection is far from unique: for example, if X is a complete intersection in Y given by a section

¹Øbro's method, which applies in all dimensions, starts with the observation that without loss of generality one cone in the fan for Y is the positive orthant and the sum of the primitive generators v_1, \dots, v_N for the rays of the fan lies in this cone; he then uses the facts that Y is smooth and Fano to give bounds on the coordinates of the v_i , and defines an order on such fans with the property that if $F_1 < F_2$ then the toric Fano varieties defined by F_1 and F_2 are non-isomorphic. By constructing the fans in increasing order, one never needs to check for fan isomorphism, and thus one never needs to refer back to fans already found.

of $L_1 \oplus \cdots \oplus L_c$ then it is also a complete intersection in $Y \times \mathbb{P}^1$ given by a section of $\pi_1^* L_1 \oplus \cdots \oplus \pi_1^* L_c \oplus \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$. Thus, we have found far fewer than 117 173 distinct four-dimensional Fano manifolds. We show below, by calculating quantum periods of the Fano manifolds X , that we find at least 738 non-isomorphic Fano manifolds. Since the quantum period is a very strong invariant—indeed no examples of distinct Fano manifolds $X \not\cong X'$ with the same quantum period $G_X = G_{X'}$ are known—we believe that we found precisely 738 non-isomorphic Fano manifolds. Eliminating the quantum periods found in [31], we see that at least 527 of our examples are new.

Remark 2.1. There exist Fano manifolds which do not occur as complete intersections in toric Fano manifolds. But in low dimensions, most Fano manifolds arise this way: 8 of the 10 del Pezzo surfaces, and at least 78 of the 105 smooth three-dimensional Fano manifolds, are complete intersections in toric Fano manifolds [32].

Remark 2.2. It may be the case that any d -dimensional Fano manifold which occurs as a toric complete intersection in fact occurs as a toric complete intersection in codimension d ; we know of no counterexamples. But even if this holds in dimension 4, our search will probably not find all four-dimensional Fano manifolds which occur as toric complete intersections. This is because, if one of the line bundles L_i involved is nef but not ample, then the Kähler cone for X can be strictly bigger than the Kähler cone for Y . In other words, it is possible for $-K_X$ to be ample on X even if $-K_Y - \Delta$ is not ample on Y . For an explicit example of this in dimension 3, see [32, §55].

3. Quantum periods and mirror Laurent polynomials

The quantum period G_X of a Fano manifold X is a generating function

$$G_X(t) = 1 + \sum_{d=1}^{\infty} c_d t^d \quad t \in \mathbb{C}, \quad (3.1)$$

for certain genus-zero Gromov–Witten invariants c_d of X which plays an important role in mirror symmetry. A precise definition can be found in [32, §B], but roughly speaking one can think of c_d as the ‘virtual number’ of rational curves C in X that pass through a given point, satisfy certain constraints on their complex structure, and satisfy $\langle -K_X, C \rangle = d$. The quantum period is discussed in detail in [23, 32]; for us what will be important is that the regularized quantum period

$$\hat{G}_X(t) = 1 + \sum_{d=1}^{\infty} d! c_d t^d \quad t \in \mathbb{C}, |t| \ll \infty \quad (3.2)$$

satisfies a differential equation called the *regularized quantum differential equation* of X :

$$L_X \hat{G}_X \equiv 0 \quad L_X = \sum_{m=0}^{m=N} p_m(t) D^m, \quad (3.3)$$

where the p_m are polynomials and $D = t(d/dt)$.

It has been proposed that Fano manifolds should correspond under mirror symmetry to Laurent polynomials which are *extremal* or of *low ramification* [23], in the sense discussed in §4. An n -dimensional Fano manifold X is said to be *mirror-dual* to a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ if the regularized quantum period of X coincides with the classical period of f :

$$\pi_f(t) = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{1}{1 - tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \quad t \in \mathbb{C}, |t| \ll \infty.$$

If a Fano manifold X is mirror-dual to the Laurent polynomial f then the regularized quantum differential equation (3.3) for X coincides with the Picard–Fuchs differential equation satisfied by π_f . The correspondence between Fano manifolds and Laurent polynomials is not one-to-one, but it is expected that any two Laurent polynomials f, g that are mirror-dual to the same Fano manifold are related by a birational transformation $\varphi: (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ called a *mutation*

or a *symplectomorphism of cluster type* [33–35]: $\varphi^*f = g$. We will write such a mutation as $f \xrightarrow{\varphi} g$. Mutations are known to preserve the classical period [33]: if $f \xrightarrow{\varphi} g$ then $\pi_f = \pi_g$.

Remark 3.1. In the paragraphs above we discuss *the* regularized quantum differential equation and *the* Picard–Fuchs differential equation. This involves choices of normalization. Our conventions are that the regularized quantum differential operator is the operator L_X as in (3.3) such that:

- (i) the order, N , of L_X is minimal;
- (ii) the degree of $p_N(t)$ is minimal;
- (iii) the leading coefficient of p_N is positive; and
- (iv) the coefficients of the polynomials p_0, \dots, p_N are integers with greatest common divisor equal to 1.

The Picard–Fuchs differential operator is the differential operator L_f such that:

$$L_f \pi_f \equiv 0 \quad L_f = \sum_{m=0}^{m=N} P_m(t) D^m,$$

where the P_m are polynomials and $D = t(d/dt)$, and that the analogues of conditions (i)–(iv) above hold.

We determined the quantum period G_X , for each of the triples $(X; Y; L_1, \dots, L_c)$ from §2, as follows. For each such triple we found, using the Mirror Theorem for toric complete intersections [36] and a generalization of a technique due to V. Przyjalkowski, a Laurent polynomial f that is mirror-dual to X . This process is described in detail in §5. We then computed, for each triple, the first 20 terms of the power series expansion of $\hat{G}_X = \pi_f$ using the Taylor expansion

$$\pi_f(t) = \sum_{d=0}^{\infty} \alpha_d t^d,$$

where α_d is the coefficient of the unit monomial in f^d . We divided the 117 173 triples into 738 ‘buckets’, according to the value of the first 20 terms of the power series expansion of $\hat{G}_X = \pi_f$, and then proved that any two Fano manifolds X, X' in the same bucket have the same quantum period by exhibiting a chain of mutations $f \xrightarrow{\varphi_0} f_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{n-1}} f_n \xrightarrow{\varphi_n} g$ that connects the Laurent polynomials f and g mirror-dual to X and X' .

For each quantum period G_X , we computed the quantum differential operator L_X directly from the mirror Laurent polynomial f chosen above, using Lairez’s generalized Griffiths–Dwork algorithm [37]. The output from Lairez’s algorithm is a differential operator $L = \sum_{m=0}^N P_m(t) D^m$ with $P_0, \dots, P_N \in \mathbb{Q}[t]$ such that, with very high probability, $L\pi_f \equiv 0$. Such an operator L gives a recurrence relation for the Taylor coefficients $\alpha_0, \alpha_1, \alpha_2, \dots$ of π_f ; using this recurrence relation and the first 20 Taylor coefficients computed above, we solved for the first 2000 Taylor coefficients α_k . We then consider an operator

$$\bar{L} = \sum_{m=0}^{\bar{N}} \bar{P}_m(t) D^m,$$

where the \bar{P}_m are polynomials of degree at most \bar{R} , and impose the condition that $\bar{L}\pi_f \equiv 0$. The 2000 Taylor coefficients of π_f give 2000 linear equations for the coefficients of the polynomials \bar{P}_m and, provided that $(\bar{N} + 1)(\bar{R} + 1) \ll 2000$, this linear system is highly over-determined. Since we are looking for the Picard–Fuchs differential operator (see remark 3.1), we may assume that (\bar{N}, \bar{R}) is lexicographically less than $(N, \deg p_N)$. We searched systematically for such differential operators with $(\bar{N} + 1)(\bar{R} + 1) \ll 2000$, looking for the operator \bar{L} with lexicographically minimal (\bar{N}, \bar{R}) and clearing denominators so that the analogues of conditions (iii) and (iv) in remark 3.1 holds. We can say with high confidence that this operator \bar{L} is in fact the Picard–Fuchs operator L_f , although this

is not proven—partly because Lairez’s algorithm relies on a randomized interpolation scheme that is not guaranteed to produce an operator annihilating π_f , and partly because if L_f were to involve polynomials P_m of extremely large degree, 2000 terms of the Taylor expansion of π_f will not be enough to detect L_f . The operators \tilde{L} that we found satisfy a number of delicate conditions that act as consistency checks: for example they are of Fuchsian type (which is true for L_f , as L_f arises geometrically from a variation of Hodge structure). Thus, we are confident that $\tilde{L} = L_f$ in every case.² Since $\hat{G}_X = \pi_f$ and $L_X = L_f$ by construction, this determines, with high confidence, the quantum period G_X and the regularized quantum differential operator L_X .

Remark 3.2. The use of Laurent polynomials and Lairez’s algorithm is essential here. There is a closed formula [32, corollary D.5] for the quantum period of the Fano manifolds that we consider, and one could in principle use this together with the linear algebra calculation described above to compute (a good candidate for) the regularized quantum differential operator L_X . In practice, however, for many of the examples that we treat here, it is impossible to determine enough Taylor coefficients from the formula: the computations involved are well beyond the reach of current hardware, both in terms of memory consumption and runtime. By contrast, our approach using mirror symmetry and Lairez’s algorithm will run easily on a desktop PC.

Remark 3.3. The regularized quantum differential equation for X coincides with the (unregularized) quantum differential equation for an anticanonical Calabi–Yau manifold $Z \subset X$. The study of the regularized quantum period from this point of view was pioneered by Batyrev *et al.* [39,40] and an extensive study of fourth-order Calabi–Yau differential operators was made in [41]. We found 26 quantum differential operators with $N = 4$; these coincide with or are equivalent to the fourth-order Calabi–Yau differential operators with AESZ IDs 1, 3, 4, 5, 6, 15, 16, 17, 18, 19, 20, 21, 22, 23, 34, 369, 370 and 424 in the Calabi–Yau Operators Database [42], together with one new fourth-order Calabi–Yau differential operator (which corresponds to our period sequence with ID 469).

4. Ramification data

Consider now one of our regularized quantum differential operators

$$L_X = \sum_{m=0}^{m=N} p_m(t) D^m$$

as in (3.3), and its local system $\mathbb{V} \rightarrow \mathbb{P}^1 \setminus S$ of solutions. Here $S \subset \mathbb{P}^1$ is the set of singular points of the regularized quantum differential equation.

Definition 4.1 ([23]). Let $S \subset \mathbb{P}^1$ be a finite set and $\mathbb{V} \rightarrow \mathbb{P}^1 \setminus S$ a local system. Fix a basepoint $x \in \mathbb{P}^1 \setminus S$. For $s \in S$, choose a small loop that winds once anticlockwise around s and connect it to x via a path, thereby making a loop γ_s about s based at x . Let $T_s: \mathbb{V}_x \rightarrow \mathbb{V}_x$ denote the monodromy of \mathbb{V} along γ_s . The *ramification* of \mathbb{V} is

$$\text{rf}(\mathbb{V}) := \sum_{s \in S} \dim \left(\mathbb{V}_x / \mathbb{V}_x^{T_s} \right).$$

The ramification $\text{rf}(\mathbb{V})$ is independent of the choices of basepoint x and of small loops γ_s . A non-trivial, irreducible local system $\mathbb{V} \rightarrow \mathbb{P}^1 \setminus S$ has $\text{rf}(\mathbb{V}) \geq 2\text{rk}(\mathbb{V})$: see [23, §2].

Definition 4.2. Let $\mathbb{V} \rightarrow \mathbb{P}^1 \setminus S$ be a local system as above. The *ramification defect* of \mathbb{V} is the quantity $\text{rf}(\mathbb{V}) - 2\text{rk}(\mathbb{V})$. A local system of ramification defect zero is called *extremal*.

Definition 4.3. The *ramification* (respectively, *ramification defect*) of a differential operator L_X is the ramification (respectively, ramification defect) of the local system of solutions $L_X f \equiv 0$.

²This could be proved in any given case using methods of van Hoeij [38]; cf. [37, §8.2.2].

Table 1. Ramification defects for 575 of the 738 regularized quantum differential operators.

ramification defect	0	1	2	3
number of occurrences	92	290	167	26

To compute the ramification of L_X , we proceed as in [31]. One can compute Jordan normal forms of the local log-monodromies $\{\log T_s : s \in S\}$ using linear algebra over a splitting field k for $p_N(t)$. (Every singular point of L_X is defined over k .) This is classical, going back to Birkhoff [43], as corrected by Gantmacher [44, vol. 2, §10] and Turrittin [45]; a very convenient presentation can be found in the book of Kedlaya [46, §7.3]. In practice, we use the symbolic implementation of $\overline{\mathbb{Q}}$ provided by the computational algebra system Magma [47,48]. We computed ramification data for 575 of the 738 regularized quantum differential operators, finding ramification defects as shown in table 1; this lends some support to the conjecture, due to Golyshev [23], that a Laurent polynomial f which is mirror-dual to a Fano manifold should have a Picard–Fuchs operator L_f that is extremal or of low ramification. For the remaining 163 regularized quantum differential operators, the symbol $p_N(t)$ contains a factor of extremely high degree. This makes the computation of ramification data prohibitively expensive.

5. The Przyjalkowski method

We now explain, given complete intersection data $(X; Y; L_1, \dots, L_c)$ as in §2, how to find a Laurent polynomial f that is mirror-dual to X . This is a slight generalization of a technique that we learned from V. Przyjalkowski³ [50,51], and which is based on the mirror theorems for toric complete intersections due to Givental [36] and Hori *et al.* [52]. Recall the exact sequence

$$0 \longleftarrow \mathbb{L}^* \xleftarrow{D} (\mathbb{Z}^N)^* \xleftarrow{\rho^*} (\mathbb{Z}^d)^* \longleftarrow 0$$

from §2 and the elements $D_i \in \mathbb{L}^*$, $1 \leq i \leq N$, defined by the standard basis elements of $(\mathbb{Z}^N)^*$. Recall further that $\mathbb{L}^* \cong \text{Pic}(Y)$, so that each line bundle L_m defines a class in \mathbb{L}^* . Suppose that there exists a choice of disjoint subsets E, S_1, \dots, S_c of $\{1, 2, \dots, N\}$ such that:

- $\{D_j : j \in E\}$ is a basis for \mathbb{L}^* ;
- each L_m is a non-negative linear combination of $\{D_j : j \in E\}$; and
- $\sum_{k \in S_m} D_k = L_m$ for each $m \in \{1, 2, \dots, c\}$;

and choose distinguished elements $s_m \in S_m$, $1 \leq m \leq c$. Set $S_m^\circ = S_m \setminus \{s_m\}$. Writing the map D in terms of the standard basis for $(\mathbb{Z}^N)^*$ and the basis $\{D_j : j \in E\}$ for \mathbb{L}^* defines an $(N-d) \times N$ matrix (m_{ji}) of integers. Let (x_1, \dots, x_N) denote the standard coordinates on $(\mathbb{C}^\times)^N$, let $r = N - d$, and define q_1, \dots, q_r and F_1, \dots, F_c by

$$q_j = \prod_{i=1}^N x_i^{m_{ji}} \quad F_m = \sum_{k \in S_m} x_k.$$

Givental [36] and Hori *et al.* [52] have shown that

$$G_X = \int_{\Gamma} e^{tW} \frac{\bigwedge_{i=1}^N (dx_i/x_i)}{\bigwedge_{m=1}^c dF_m \wedge \bigwedge_{j=1}^r (dq_j/q_j)}, \quad (5.1)$$

³Przyjalkowski informs us that he learned this, for the case of the cubic threefold, from Katzarkov [49] and Orlov (V. Przyjalkowski 2014, personal communication).

where $W = x_1 + \cdots + x_N$ and Γ is a certain cycle in the submanifold of $(\mathbb{C}^\times)^N$ defined by

$$q_1 = \cdots = q_r = 1 \quad F_1 = \cdots = F_c = 1.$$

Introducing new variables y_i for $i \in \bigcup_{m=1}^c S_m^\circ$, setting

$$x_i = \begin{cases} \frac{1}{1 + \sum_{k \in S_m^\circ} y_k} & \text{if } i = s_m \\ \frac{y_i}{1 + \sum_{k \in S_m^\circ} y_k} & \text{if } i \in S_m^\circ. \end{cases}$$

and using the relations $q_1 = \cdots = q_r = 1$ to eliminate the variables x_j , $j \in E$, allows us to write $W - c$ as a Laurent polynomial f in the variables

$$\{y_i : i \in \bigcup_{m=1}^c S_m^\circ\} \quad \text{and} \quad \{x_i : i \notin E \text{ and } i \notin \bigcup_{m=1}^c S_m^\circ\}.$$

The mirror theorem (5.1) then implies that $\hat{G}_X = \pi_f$, or in other words that f is mirror-dual to X .

The Laurent polynomial f produced by Przyjalkowski's method depends on our choices of E, S_1, \dots, S_c , and s_1, \dots, s_c , but up to mutation this is not the case:

Theorem 5.1 ([53]). *Let Y be a toric Fano manifold and let L_1, \dots, L_c be nef line bundles on Y such that $-K_Y - \Lambda$ is ample, where $\Lambda = c_1(L_1) + \cdots + c_1(L_c)$. Let $X \subset Y$ be a smooth complete intersection defined by a regular section of $\oplus_i L_i$. Let f and g be Laurent polynomial mirrors to X obtained by applying Przyjalkowski's method to $(X; Y; L_1, \dots, L_c)$ as above, but with possibly different choices for the subsets E, S_1, \dots, S_c and the elements s_1, \dots, s_c . Then there exists a mutation φ such that $f \xrightarrow{\varphi} g$.*

Example 5.2. Let Y be the projectivization of the vector bundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$ over \mathbb{P}^2 . Choose a basis for the two-dimensional lattice \mathbb{L}^* such that the matrix (m_{ji}) of the map D is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Consider the line bundle $L_1 \rightarrow Y$ defined by the element $(2, 1) \in \mathbb{L}^*$, and the Fano hypersurface $X \subset Y$ defined by a regular section of L_1 . Applying Przyjalkowski's method to the triple $(X; Y; L_1)$ with $E = \{3, 4\}$, $S_1 = \{1, 2, 5\}$ and $s_1 = 1$ yields the Laurent polynomial

$$f = \frac{(1 + y_2 + y_5)^2}{y_2 x_6 x_7} + \frac{1 + y_2 + y_5}{y_5 x_6 x_7} + x_6 + x_7$$

mirror-dual to X . Applying the method with $E = \{3, 4\}$, $S_1 = \{1, 6\}$ and $s_1 = 1$ yields

$$g = x_2 + \frac{(1 + y_6)^2}{x_2 y_6 x_7} + \frac{1 + y_6}{x_5 y_6 x_7} + x_5 + x_7.$$

We have that $f \xrightarrow{\varphi} g$ where the mutation $\varphi: (\mathbb{C}^\times)^4 \rightarrow (\mathbb{C}^\times)^4$ is given by

$$(x_2, x_5, y_6, x_7) \mapsto \left(\frac{x_2}{x_5 y_6}, \frac{1}{y_6}, x_7, x_2 + x_5 \right) = (y_2, y_5, x_6, x_7).$$

Remark 5.3. Observe that, for a complete intersection of dimension n and codimension c , Przyjalkowski's method requires partitioning $n + c$ variables into c disjoint subsets. If $(n + c)/c < 2$ then at least one of the subsets must have size one and so the corresponding variable, x_j say, is eliminated from the Laurent polynomial via the equation $x_j = 1$. One could therefore have obtained the resulting Laurent polynomial from a complete intersection with smaller codimension: new Laurent polynomials are found only when $(n + c)/c \geq 2$, that is, when the codimension is at most the dimension. In particular, all possible mirrors to four-dimensional Fano toric complete intersections given by the Przyjalkowski method occur for complete intersections in toric manifolds of dimension at most 8.

6. Examples

(a) The cubic fourfold

Let X be the cubic fourfold. This arises in our classification from the complete intersection data $(X; Y; L)$ with $Y = \mathbb{P}^5$ and $L = \mathcal{O}_{\mathbb{P}^5}(3)$. The Przyjalkowski method yields [54, §2.1] a Laurent polynomial

$$f = \frac{(1+x+y)^3}{xyzw} + z + w$$

mirror-dual to X , and elementary calculation gives

$$\pi_f(t) = \sum_{d=0}^{\infty} \frac{(3d)!(3d)!}{(d!)^6} t^{3d}.$$

Thus $\hat{G}_X = \pi_f$, and the corresponding regularized quantum differential operator is

$$L_X = D^4 - 729t^3(D+1)^2(D+2)^2.$$

The local log-monodromies for the local system of solutions $L_X g \equiv 0$ are

$$\begin{aligned} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{at } t = 0 \\ & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{at } t = \frac{1}{9} \\ & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{at the roots of } 81t^2 + 9t + 1 = 0 \\ & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{at } t = \infty \end{aligned}$$

and the operator L_X is extremal.

(b) A (3,3) complete intersection in \mathbb{P}^6

Let X be a complete intersection in $Y = \mathbb{P}^6$ of type $(3, 3)$. This arises in our classification from the complete intersection data $(X; Y; L_1, L_2)$ with $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^6}(3)$. The Przyjalkowski method yields a Laurent polynomial

$$f = \frac{(1+x+y)^3(1+z+w)^3}{xyzw} - 36$$

mirror-dual to X , and [32, corollary D.5] gives

$$\hat{G}_X = \pi_f(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(3l)!(3l)!(k+l)!}{k!(l!)^7} (-36)^k t^{k+l}.$$

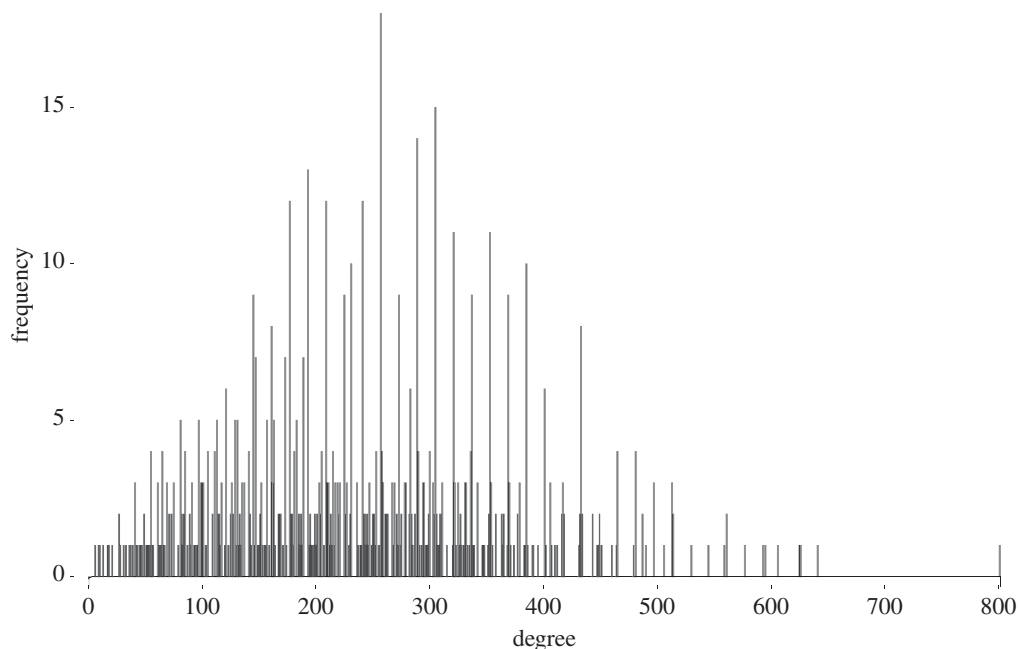


Figure 1. The distribution of degrees (frequency plot).

The corresponding regularized quantum differential operator L_X is

$$\begin{aligned}
 & (36t + 1)^4(693t - 1)D^4 \\
 & + 18t(36t + 1)^3(13\,860t + 61)D^3 \\
 & + 9t(36t + 1)^2(3\,492\,720t^2 + 57\,672t + 77)D^2 \\
 & + 144t(36t + 1)(11\,226\,600t^3 + 377\,622t^2 + 2754t + 1)D \\
 & + 15\,552t^2(1\,796\,256t^3 + 98\,496t^2 + 1605t + 7).
 \end{aligned}$$

The local log-monodromies for the local system of solutions $L_{Xg} \equiv 0$ are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{693}$$

$$\begin{pmatrix} \frac{2}{3} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{at } t = -\frac{1}{36}$$

and so the operator L_X is extremal.

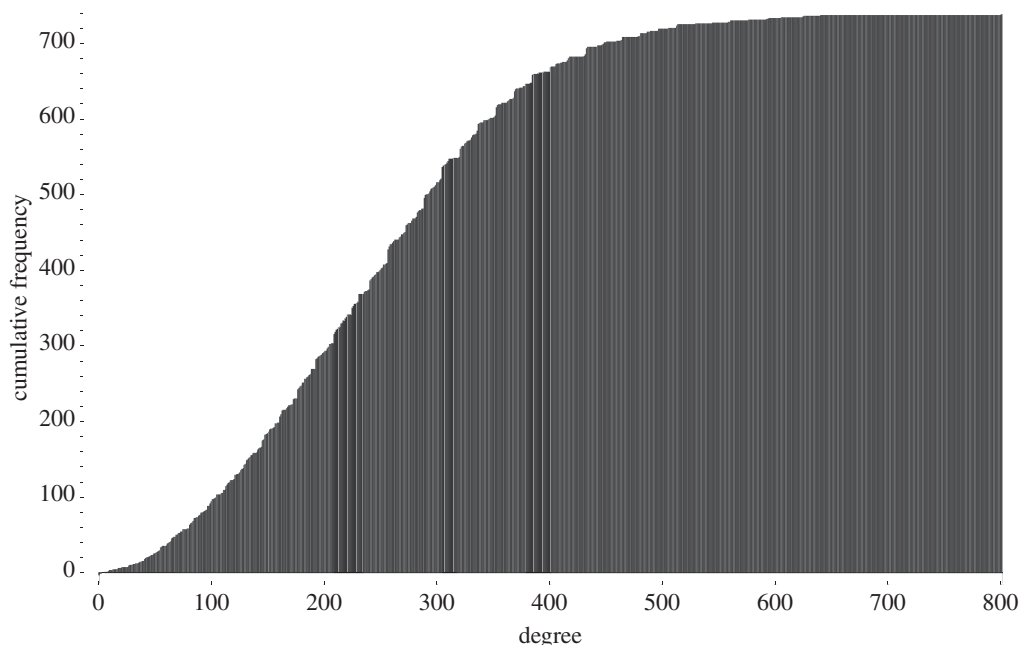


Figure 2. The distribution of degrees (cumulative frequency plot).

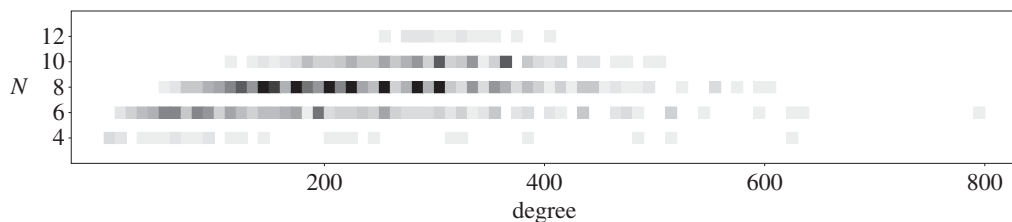


Figure 3. The distribution of degrees with N .

7. Results and analysis

We close by indicating how basic numerical invariants—degree and size of cohomology—vary across the 738 families of Fano manifolds that we have found. The degree $(-K_X)^4$ varies from 5 to 800, as shown in figures 1 and 2.

We do not have direct access to the size of the cohomology algebra of our Fano manifolds X , as many of the line bundles occurring in the complete intersection data $(X; Y; L_1, \dots, L_c)$ are not ample and so the Lefschetz theorem need not apply. But the order N of the regularized quantum differential operator is a good proxy for the size of the cohomology. N is the rank of a certain local system—an irreducible piece of the Fourier–Laplace transform of the restriction of the Dubrovin connection (in the Frobenius manifold given by the quantum cohomology of X) to the line in $H^\bullet(X)$ spanned by $-K_X$ —and in the case where this local system is irreducible, which is typical, N will coincide with the dimension of $H^\bullet(X)$. For our examples, N lies in the set $\{4, 6, 8, 10, 12\}$. Figure 3 shows how N varies with the degree $(-K_X)^4$, with darker grays indicating a larger number of examples with that N and degree.

The isolated example on the right of figure 3, with $N=6$ and degree 800, is the blow up of $\mathbb{P}(1, 1, 1, 1, 3)$ at a point. Figure 4 again shows how N varies with the degree $(-K_X)^4$, but this time with toric Fano manifolds highlighted in red. Figure 5 shows how the Euler number $\chi(T_X)$

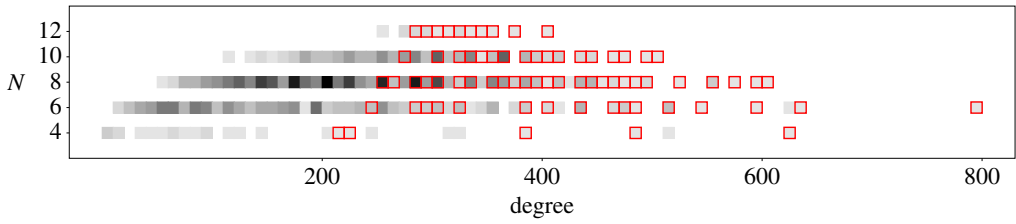


Figure 4. The distribution of degrees with N , with toric Fano manifolds highlighted.

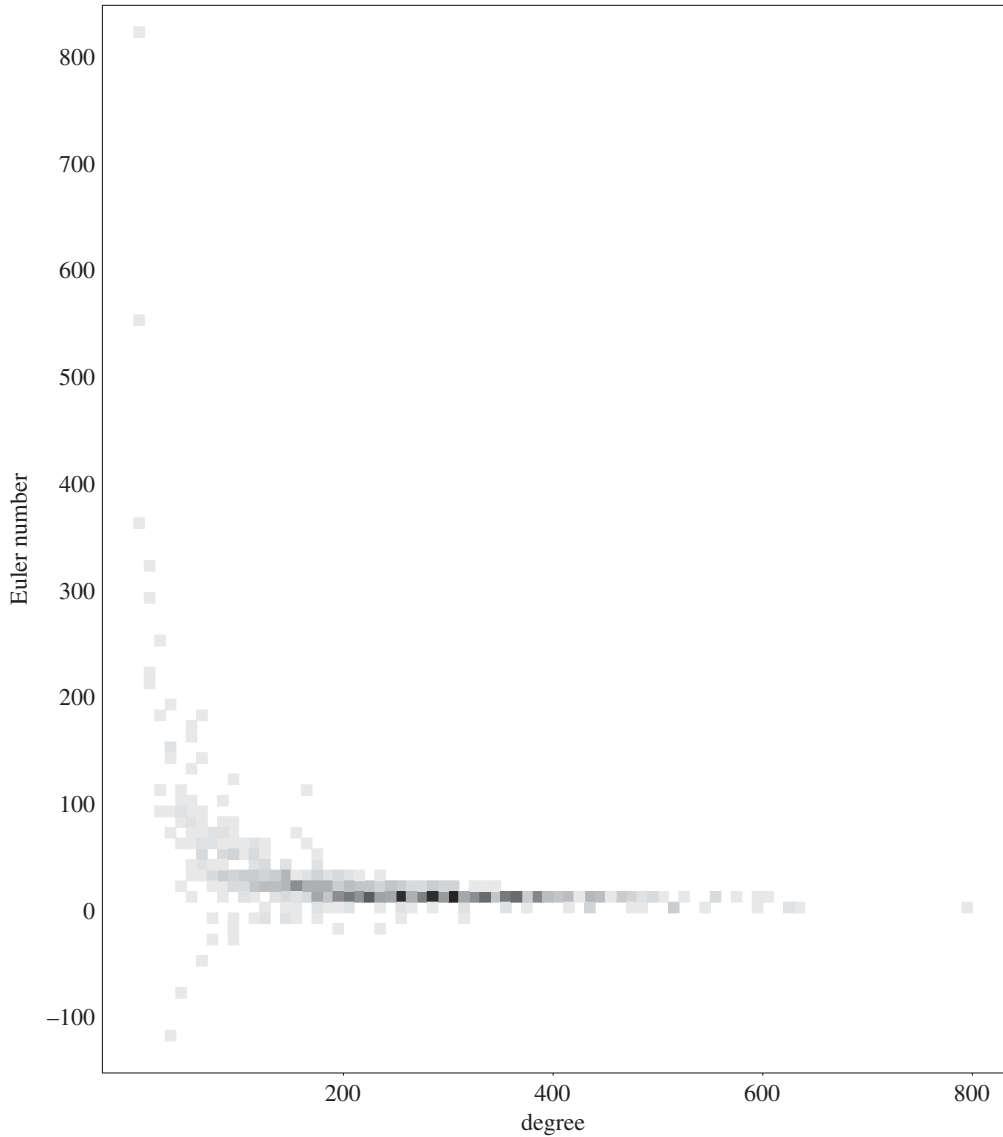


Figure 5. The distribution of degrees with Euler number.

varies with the degree $(-K_X)^4$, with darker grays indicating a larger number of examples with that Euler number and degree. The three examples with the largest Euler number χ are a quintic hypersurface in \mathbb{P}^5 , with $\chi = 825$; a complete intersection of type $(2, 4)$ in \mathbb{P}^6 , with $\chi = 552$; and a

complete intersection of type $(3, 3)$ in \mathbb{P}^6 , with $\chi = 369$. The three examples with the most negative Euler number are $\mathbb{P}^1 \times V_4^3$ where V_4^3 is a quartic hypersurface in \mathbb{P}^4 , with $\chi = -112$; $\mathbb{P}^1 \times V_6^3$ where V_6^3 is a complete intersection of type $(2, 3)$ in \mathbb{P}^5 , with $\chi = -72$; and $\mathbb{P}^1 \times V_8^3$ where V_8^3 is a complete intersection of type $(2, 2, 2)$ in \mathbb{P}^6 , with $\chi = -48$.

Ethics statement. This research did not involve human or animal subjects.

Data accessibility. The Electronic Supplementary Material contains the results of our computations, in machine readable form, together with full source code written in Magma [47]. See the files called `README.txt` for details. The source code and data, but not the text of this paper, are released under a Creative Commons CC0 license (*Creative Commons CC0 license*, <https://creativecommons.org/publicdomain/zero/1.0/> and <https://creativecommons.org/publicdomain/zero/1.0/legalcode>.): see the files called `COPYING.txt` for details. If you make use of the source code or data in an academic or commercial context, you should acknowledge this by including a reference or citation to this paper.

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