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Requirement of Dissonance in Assisted Optimal State Discrimination

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A fundamental problem in quantum information is to explore what kind of quantum correlations is responsible for successful completion of a quantum information procedure. Here we study the roles of entanglement, discord, and dissonance needed for optimal quantum state discrimination when the latter is assisted with an auxiliary system. In such process, we present a more general joint unitary transformation than the existing results. The quantum entanglement between a principal qubit and an ancilla is found to be completely unnecessary, as it can be set to zero in the arbitrary case by adjusting the parameters in the general unitary without affecting the success probability. This result also shows that it is quantum dissonance that plays as a key role in assisted optimal state discrimination and not quantum entanglement. A necessary criterion for the necessity of quantum dissonance based on the linear entropy is also presented. PACS numbers: 03.65.Ta, 03.67.Mn, 42.50.Dv.

An important distinctive feature of quantum mechanics is that quantum coherent superposition can lead to quantum correlations in composite quantum systems like quantum entanglement[1], Bell nonlocality[2] and quantum discord[3,4]. Quantum entanglement has been extensively studied from various perspectives, and it has served as a useful resource for demonstrating the superiority of quantum information processing. For instance, entangled quantum states are regarded as key resources for some quantum information tasks, such as teleportation, superdense coding and quantum cryptography[5].

In contrast to quantum entanglement, quantum discord measures the amount of nonclassical correlations between two subsystems of a bipartite quantum system. A recent report regarding the deterministic quantum computation with one qubit (DQC1)[6,7] demonstrates that a quantum algorithm to determine the trace of a unitary matrix can surpass the performance of the corresponding classical algorithm in terms of computational speedup even in the absence of quantum entanglement between the the control qubit and a completely mixed state. However, the quantum discord is never zero. This result is somewhat surprising and it has engendered much interest in quantum discord in recent years. In particular, it has led to further studies on the relation of quantum discord with other measures of correlations. Moreover, it has been shown that it is possible to formulate an operational interpretation in the context of a quantum state merging protocol[8,9] where it can be regarded as the amount of entanglement generated in an activation protocol[10] or in a measurement process[11]. Also, a unified view of quantum correlations based on the relative entropy[12] introduces a new measure called quantum dissonance which can be regarded as the nonclassical correlations in which quantum entanglement has been totally excluded. For a separable state (with zero entanglement), its quantum dissonance is exactly equal to its discord.

It is always interesting to uncover non-trivial roles of nonclassical correlations in quantum information processing. The quantum algorithm in DQC1 has been widely regarded as the first example for which quantum discord, rather than quantum entanglement, plays a key role in the computational process. Moreover, a careful consideration of the natural bipartite split between the control qubit and the input state reveals that the quantum discord is nothing but the quantum dissonance of the system. This simple observation naturally leads to an interesting question: Can quantum dissonance serve as a similar key resource in some quantum information tasks? The affirmative answer was shown in an interesting piece of work by Roa, Retamal and Alid-Vaccarezza[13] where the roles of entanglement, discord, and dissonance needed for performing unambiguous quantum state discrimination assisted by an auxiliary qubit[14,15] was studied. This protocol for *assisted optimal state discrimination* (AOSD) in general requires both quantum entanglement and discord. However, for the case in which there exist equal *a priori* probabilities, the entanglement of the state of system-ancilla qubits is absent even though its discord is nonzero, and hence the unambiguous state discrimination protocol is implemented successfully only with quantum dissonance. This protocol therefore provides an example for which dissonance, and not entanglement, plays as a key role in a quantum information processing task.

In this work, we show more generally that quantum entanglement is not even necessary for AOSD. Moreover, we look at the roles of correlations in the AOSD under the most general settings by considering a generic AOSD protocol. We also show that only dissonance in general is required for AOSD and quantum entanglement is never needed.

Results

The general AOSD protocol. Suppose Alice and Bob share an entangled two-qubit state $|\zeta\rangle=\sqrt{p_+}|\psi_+\rangle|0\rangle_c+\sqrt{p_-}|\psi_-\rangle|1\rangle_c$ (see Fig. 1), where $p_\pm\in[0,1]$ and $p_++p_-=1$, $|\psi_\pm\rangle$ are two nonorthogonal states of the qubit of Alice (system qubit S), and $\{|0\rangle_c,|1\rangle_c\}$ are the orthonormal bases for the one of Bob (qubit C). The reduced state of system qubit $\rho=p_+|\psi_+\rangle\langle\psi_+|+p_-|\psi_-\rangle\langle\psi_-|$ is a realization of the model in[13] in which a qubit is prepared in the two nonorthogonal states $|\psi_\pm\rangle$ with *a priori* probabilities $p_\pm$. To discriminate the two states $|\psi_+\rangle$ or $|\psi_-\rangle$ unambiguously, the system is coupled to an auxiliary qubit A, prepared in a known initial pure state $|k\rangle_a$. Under a joint unitary transformation $\mathcal{U}$ between the system and the ancilla, one obtains

$$\mathcal{U}|\psi_+\rangle|k\rangle_a=\sqrt{1-|\alpha_+|^2}|0\rangle|0\rangle_a+\alpha_+|\Phi\rangle|1\rangle_a, \tag{1a}$$

$$\mathcal{U}|\psi_-\rangle|k\rangle_a=\sqrt{1-|\alpha_-|^2}|1\rangle|0\rangle_a+\alpha_-|\Phi\rangle|1\rangle_a, \tag{1b}$$

where $|\Phi\rangle=\cos\beta|0\rangle+\sin\beta e^{i\delta}|1\rangle$, $\{|0\rangle,|1\rangle\}$ and $\{|0\rangle_a,|1\rangle_a\}$ are the bases for the system and the ancilla, respectively. The probability amplitudes α_+ and α_- satisfy $\alpha_+^*\alpha_-=\alpha$, where $\alpha=\langle\psi_+|\psi_-\rangle=|\alpha|e^{i\theta}$ is the *priori* overlap between the two nonorthogonal states. The unitary transformation can be constructed by performing an operation $\mathcal{W}=(|+\rangle\langle 0|+|-\rangle\langle 1|)\otimes|0\rangle_a\langle 0|+(|\Phi\rangle\langle 0|+|\bar{\Phi}\rangle\langle 1|)\otimes|1\rangle_a\langle 1|$ on the original one in Ref. 13, where $|\pm\rangle=(|0\rangle\pm|1\rangle)/\sqrt{2}$ and $|\bar{\Phi}\rangle=\sin\beta|0\rangle-\cos\beta e^{i\delta}|1\rangle$. It has the form as

$$\mathcal{U}=\frac{1}{1-|\alpha|^2}\Bigg[\left(\sqrt{1-|\alpha_+|^2}|0\rangle|0\rangle_a+\alpha_+|\Phi\rangle|1\rangle_a\right)\langle\tilde{\psi}_+|_a\langle k|+\left(\sqrt{1-|\alpha_-|^2}|1\rangle|0\rangle_a+\alpha_-|\Phi\rangle|1\rangle_a\right)\langle\tilde{\psi}_-|_a\langle k|\Bigg]+\mathcal{V},$$

where $|\tilde{\psi}_\pm\rangle=|\psi_\pm\rangle-|\psi_\mp\rangle\langle\psi_\mp|\psi_\pm\rangle$ are the components of $|\psi_\pm\rangle$ orthogonal to $|\psi_\mp\rangle$, and $\mathcal{V}=|\Upsilon_+\rangle\langle 0|_a\langle\bar{k}|+|\Upsilon_-\rangle\langle 1|_a\langle\bar{k}|$, with $|\Upsilon_\pm\rangle$ being two arbitrary states orthogonal to the right hands of Eq. (1) and $\langle\Upsilon_-|\Upsilon_+\rangle=0$, and ${}_a\langle\bar{k}|k\rangle_a=0$. Obviously, only the terms with $\langle\tilde{\psi}_\pm|_a\langle k|$ have effect on the initial state $|\psi_\pm\rangle|k\rangle_a$.

The state of the system-ancilla qubits is given by

$$\begin{aligned}\rho_{SA}=&\,p_+\mathcal{U}(|\psi_+\rangle\langle\psi_+|\otimes|k\rangle_a\langle k|)\mathcal{U}^\dagger\\&+p_-\mathcal{U}(|\psi_-\rangle\langle\psi_-|\otimes|k\rangle_a\langle k|)\mathcal{U}^\dagger,\end{aligned} \tag{2}$$

which depends on β and δ, and it is generally not equivalent to the corresponding one in[13] under local unitary transformations unless $|\Phi\rangle=|+\rangle$. The state discrimination is successful if the ancilla collapses to $|0\rangle_a$. This occurs with success probability given by

$$\begin{aligned}P_{\rm suc}&=\mathrm{Tr}\left[(\mathbb{1}_s\otimes|0\rangle_a\langle 0|)\rho_{SA}\right]\\&=p_+\left(1-|\alpha_+|^2\right)+p_-\left(1-|\alpha_-|^2\right),\end{aligned} \tag{3}$$

where $\mathbb{1}_s$ is the unit matrix for the system qubit. Without loss of generality, let us assume that $p_+\le p_-$ and denote $\bar{\alpha}=\sqrt{p_+/p_-}$. The analysis of the optimal success probability can be divided into two cases: (i) $|\alpha|<\bar{\alpha}$, $P_{\rm suc}$ is attained for $|\alpha_+|=\sqrt[4]{p_-/p_+}\sqrt{|\alpha|}$; (ii) $\bar{\alpha}\le|\alpha|\le 1$, $P_{\rm suc}$ is attained for $|\alpha_+|=1$ (or equivalently $|\alpha_-|=|\alpha|$). One has

$$P_{\rm suc,max}=1-2\sqrt{p_+p_-}|\alpha|,\quad\text{for case(i)}, \tag{4a}$$

$$P_{\rm suc,max}=\left(1-|\alpha|^2\right)p_-,\quad\text{for case(ii)}. \tag{4b}$$

Before proceeding further to explore the roles of correlations in the AOSD, we make the following remarks.

Remark 1. State discrimination of a subsystem in a reduced mixed state has practical interest in conclusive quantum teleportation where the resource is not prepared in a maximally entangled state (see Refs. 16–18). In the conclusive teleportation protocol, the sender Alice possesses an arbitrary one-qubit state $|\varphi\rangle_{\rm Alice}=a|0\rangle+b|1\rangle$, and she shares a non-maximally entangled state $|\Psi_+(\theta)\rangle=\cos\theta|00\rangle+\sin\theta|11\rangle$ with the receiver Bob. Under the protocol, one has

$$\begin{aligned}|\Psi_{\rm tel}\rangle&=|\varphi\rangle_{\rm Alice}\otimes|\Psi_+(\theta)\rangle\\&=\frac{1}{2}\Big\{|\Psi_+(\theta)\rangle\otimes|\varphi\rangle_{\rm Bob}+|\Psi_-(\theta)\rangle\otimes\sigma_z|\varphi\rangle_{\rm Bob}+\\&\qquad|\Phi_+(\theta)\rangle\otimes\sigma_x|\varphi\rangle_{\rm Bob}+|\Phi_-(\theta)\rangle\otimes(-i\sigma_y)|\varphi\rangle_{\rm Bob}\Big\},\end{aligned}$$

where $|\Psi_\pm(\theta)\rangle=\cos\theta|00\rangle\pm\sin\theta|11\rangle$, $|\Phi_\pm(\theta)\rangle=\sin\theta|01\rangle\pm\cos\theta|10\rangle$, and σ_x, σ_y, σ_z are Pauli matrices. The concurrences[19] of the states $|\Psi_\pm(\theta)\rangle$ and $|\Phi_\pm(\theta)\rangle$ are all equal to $C=|\sin 2\theta|$. The states $|\Psi_\pm(\theta)\rangle$ are orthogonal to the states $|\Phi_\pm(\theta)\rangle$, but $\{|\Psi_+(\theta)\rangle,|\Psi_-(\theta)\rangle\}$ (or $\{|\Phi_+(\theta)\rangle,|\Phi_-(\theta)\rangle\}$) are not mutually orthogonal. To teleport the unknown state $|\varphi\rangle_{\rm Alice}$ from Alice to Bob with perfect fidelity (equals to 1), state discrimination[16–18] is generally required. It should also be noted that only the maximally entangled states (with $\theta=\pi/4$) can realize the perfect teleportation with unit success probability.

Remark 2. Through quantum teleportation, we see that our model recover the scheme in[13], in which the principal qubit is randomly prepared in one of the two pure states $|\psi_+\rangle$ or $|\psi_-\rangle$. Let us conisder replacing the entangled resource $|\Psi_+(\theta)\rangle$ by maximally entangled states randomly prepared with a probabilities as $\{p_1:|\Psi_+(\frac{\pi}{4})\rangle, p_2:|\Psi_-(\frac{\pi}{4})\rangle, p_3:|\Phi_+(\frac{\pi}{4})\rangle, p_4:|\Phi_-(\frac{\pi}{4})\rangle\}$. Although they are all maximally entangled states and each of them is a resource for perfect teleportation, perfectly faithful teleportation cannot be realized in this case. It can be shown that the fidelity of teleportation is the one corresponding to the *average state*[16]

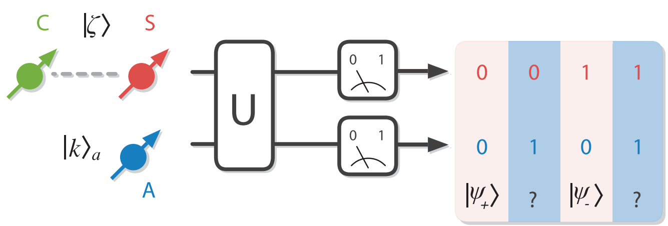

Figure 1 | The General AOSD Protocol Illustration. Alice and Bob share a pure entangled state $|\zeta\rangle$ of qubits S and C. To discriminate the two states $|\psi_+\rangle$ or $|\psi_-\rangle$ of S, Alice performs a joint unitary transformation $\mathcal{U}$ between qubits S and A, followed by two independent von Neumann measurements on the two qubits. Her state discrimination is successful if the outcome of A is 0, but unsuccessful if outcome 1.

$$\rho_{\rm res}=p_1|\Psi_+\left(\frac{\pi}{4}\right)\rangle\langle\Psi_+\left(\frac{\pi}{4}\right)|+p_2|\Psi_-\left(\frac{\pi}{4}\right)\rangle\langle\Psi_-\left(\frac{\pi}{4}\right)|$$
$$+p_3|\Phi_+\left(\frac{\pi}{4}\right)\rangle\langle\Phi_+\left(\frac{\pi}{4}\right)|+p_4|\Phi_-\left(\frac{\pi}{4}\right)\rangle\langle\Phi_-\left(\frac{\pi}{4}\right)|.$$

Consequently, the amount of entanglement contributing to teleportation is not just the average value of the entanglement which is $p_1\mathcal{C}\left(|\Psi_+\left(\frac{\pi}{4}\right)\rangle\right)+p_2\mathcal{C}\left(|\Psi_-\left(\frac{\pi}{4}\right)\rangle\right)+p_3\mathcal{C}\left(|\Phi_+\left(\frac{\pi}{4}\right)\rangle\right)+p_4\mathcal{C}\left(|\Phi_-\left(\frac{\pi}{4}\right)\rangle\right)$, but the entanglement of the average state as $\mathcal{C}(\rho_{\rm res})$. Therefore the amount of entanglement available depends crucially on the knowledge of the entangled state. The amount of quantum entanglement that is needed for the AOSD scheme considered here, as well as the one in Ref. 13, refers to the entanglement of the average state, $\mathcal{C}(\rho_{SA})$, and not to the average value of the entanglement as $p_+\mathcal{C}(\mathcal{U}\,|\psi_+\rangle|k\rangle_a)+p_-\mathcal{C}(\mathcal{U}\,|\psi_-\rangle|k\rangle_a)$.

We are now ready to investigate the roles of correlations in the AOSD. To this end, let us first calculate the concurrence of ρ_{SA}:

$$\mathcal{C}(\rho_{SA})=2\left[\mathcal{Y}_+^2\ \sin^2\beta+\mathcal{Y}_-^2\ \cos^2\beta-2\mathcal{Y}_+\mathcal{Y}_-\ \sin\beta\cos\beta\cos(\theta+\delta)\right]^{1/2},\tag{5}$$

with $\mathcal{Y}_\pm=\sqrt{1-|\alpha\pm|^2}\,|\alpha_\pm|p_\pm$. When $\beta=\pi/4$ and $\delta=0$, Eq. (5) reverts to the result in[13].

Let us impose the constraint $\mathcal{C}(\rho_{SA})=0$ for any α, α_+ and p_+. It is then easy to see that

$$\delta=-\theta,\quad \beta=\arctan(\mathcal{Y}_-/\mathcal{Y}_+)\,.\tag{6}$$

Based on Eq. (6), state (2) is a separable state as

$$\rho_{SA}=|\eta_1\rangle\langle\eta_1|\otimes|0\rangle_a\langle0|+|\Phi\rangle\langle\Phi|\otimes|\eta_2\rangle_a\langle\eta_2|,\tag{7}$$

where $|\eta_1\rangle$ and $|\eta_2\rangle_a$ are two unnormalized states as

$$|\eta_1\rangle=\frac{\sqrt{p_+p_-}}{\mathcal{Z}}\left(\sqrt{1-|\alpha_+|^2}\alpha_-|0\rangle-\sqrt{1-|\alpha_-|^2}\alpha_+|1\rangle\right),$$
$$|\eta_2\rangle_a=\frac{\sqrt{\mathcal{Y}_+^2+\mathcal{Y}_-^2}}{\mathcal{Z}}|0\rangle_a+\frac{\mathcal{Z}\alpha_+}{|\alpha_+|}|1\rangle_a.\tag{8}$$

where $\mathcal{Z}=\sqrt{p_+|\alpha_+|^2+p_-|\alpha_-|^2}$.

Note that the state (2) has rank two, and it is really the reduced state of the following tripartite pure state

$$|\Psi\rangle=\sqrt{p_+}\left(\mathcal{U}|\psi_+\rangle|k\rangle_a\right)|0\rangle_c+\sqrt{p_-}\left(\mathcal{U}|\psi_-\rangle|k\rangle_a\right)|1\rangle_c.\tag{9}$$

Its discord can be derived analytically as $D(\rho_{SA})=S(\rho_A)-S(\rho_{SA})+E(\rho_{SC})=S(\rho_A)-S(\rho_C)+E(\rho_{SC})$ using the Koashi-Winter identity[20], where $S(\rho)$ is the von Neumann entropy, $E(\rho_{SC})$ is the entanglement of formation[19] between the principal system and the qubit C. The explicit expression for the discord is

$$D(\rho_{SA})=\mathcal{H}(\tau_A)-\mathcal{H}(\tau_C)+\mathcal{H}(\tau_{SC}),\tag{10}$$

where

$\mathcal{H}(x)=-\frac{1+\sqrt{1-x}}{2}\ln\frac{1+\sqrt{1-x}}{2}-\frac{1-\sqrt{1-x}}{2}\ln\frac{1-\sqrt{1-x}}{2}$, τ_A is the tangle between A and SC, τ_C is the tangle between C and SA, and $\tau_{SC}=\mathcal{C}^2(\rho_{SC})$ is the concurrence between S and C in the state ρ_{SC}. One can obtains

$$\tau_A=\tau_{SA}+4p_+p_-\left(|\alpha_+|^2+|\alpha_-|^2-2|\alpha|^2\right),\tag{11a}$$

$$\tau_C=4p_+p_-\left(1-|\alpha|^2\right),\tag{11b}$$

$$\tau_{SC}=\tau_S-\tau_{SA}-\tau(|\Psi\rangle),\tag{11c}$$

with τ_S the tangle between S and AC, $\tau_{SA}=\mathcal{C}^2(\rho_{SA})$, and $\tau(|\Psi\rangle)$ the three-tangle[21]. The tangle between S and AC is given by

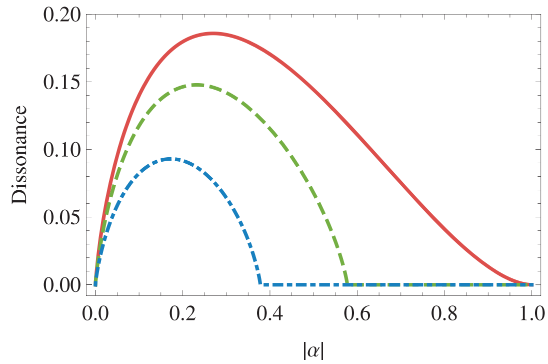

Figure 2 | Quantum dissonance in the AOSD. We plot the dissonance versus $|\alpha|$, for $p_+=1/2$ (solid line), 1/4 (dashed line), and 1/8 (dot-dashed line). Dissonance is greater than zero for case (i), and is zero for case (ii). The critical point for $D(\rho_{SA})=0$ occurs at $|\alpha|=\bar{\alpha}$.

$$\tau_S=4\left\{\left(p_-|\alpha_-|^2+p_+|\alpha_+|^2\right)\left[p_-\left(1-|\alpha_-|^2\right)\cos^2\beta+p_+\left(1-|\alpha_+|^2\right)\sin^2\beta\right]+p_+p_-\left(1-|\alpha_+|^2\right)\left(1-|\alpha_-|^2\right)\right\},\tag{12}$$

and the three-tangle is

$$\tau(|\Psi\rangle)=4p_+p_-\left|\sqrt{1-|\alpha_-|^2}\alpha_+\cos\beta+\sqrt{1-|\alpha_+|^2}\alpha_-\sin\beta e^{i\delta}\right|^2.\tag{13}$$

Dissonance for cases (i) and (ii). For case (i), upon the substitution $|\alpha_+|=\sqrt[4]{p_-/p_+}\sqrt{|\alpha|}$, $p_-=1-p_+$, and Eqs. (6)(11)(12)(13) into Eq. (10), one has the analytical expression for the dissonance, which depends only on $|\alpha|$ and p_+. In Fig. 2, we plot the curves of the dissonance versus $|\alpha|$ for $p_+=1/2$, 1/4, 1/8, respectively (see the curves with $D(\rho_{SA})>0$). For case (ii), because $|\alpha_+|=1$, one has $\beta=\pi/2$ and the state ρ_{SA} is

$$\rho_{SA}=|1\rangle\langle1|\otimes\rho_a,\tag{14}$$

with $\rho_a=p_+|1\rangle_a\langle1|+p_-|\mu\rangle_a\langle\mu|$, $|\mu\rangle_a=\sqrt{1-|\alpha|^2}\,|0\rangle_a+\alpha_-e^{i\delta}|1\rangle_a$. The state (14) is clearly a direct-product state hence its dissonance is zero. In Fig. 2, for case (ii), we also plot the curves of dissonance versus $|\alpha|$ for the same p_+'s (see the curves with $D(\rho_{SA})=0$). Fig. 2 shows that dissonance is a key ingredient for AOSD other than

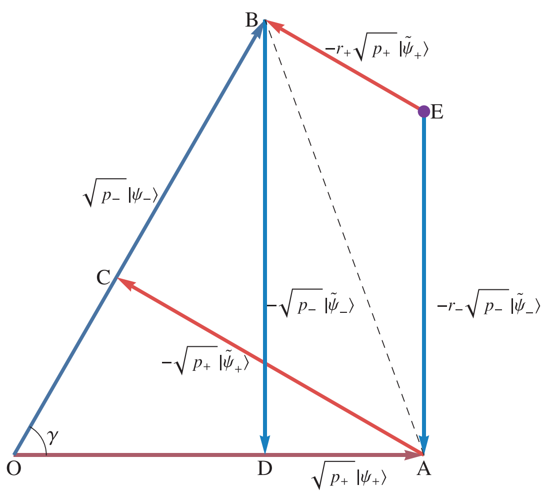

Figure 3 | Geometric picture for optimal success probability based on POVM strategy. The sides $|OA|=\sqrt{p_+}$, $|OB|=\sqrt{p_-}$, the angle $\gamma=\arccos|\alpha|$, and $AC\perp OB$, $BD\perp OA$, $EB\perp OB$, $EA\perp OA$. For $|\alpha|<\bar{\alpha}$, the point E locates inside of the angle $\angle AOB$; for $|\alpha|\geq\bar{\alpha}$, the point E coincides with the point B for $p_+<p_-$ (or A for $p_+>p_-$).

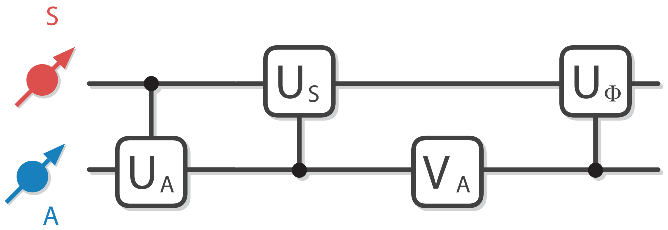

Figure 4 | Realization of the General Unitary Transformation. For the initial states $|\psi_+\rangle = |0\rangle$, $|\psi_-\rangle = \sqrt{1-|\alpha|^2}|1\rangle + \alpha|0\rangle$ and $|k\rangle_a = |0\rangle_a$, the unitary transformation $\mathcal{U}$ in Eq. (1) can be realized in four steps: (i) controlled-U_A with the system S being the control qubit; (ii) controlled-U_S where the system S is controlled by the ancilla A; (iii) local unitary V_A on the auxiliary qubit; (iv) controlled-U_Φ with the same control qubit and target as the second step. The single qubit operations $U_A = |\phi_A\rangle_a\langle 0| + |\bar{\phi}_A\rangle_a\langle 1|$, $U_S = |\phi_S\rangle\langle 0| + |\bar{\phi}_S\rangle\langle 1|$, $V_A = |\phi_V\rangle_a\langle 0| + |\bar{\phi}_V\rangle_a\langle 1|$, and $U_\Phi = |\Phi\rangle\langle 0| + |\bar{\Phi}\rangle\langle 1|$, with $|\phi_A\rangle_a = \left(\sqrt{\left(1-|\alpha_+|^2\right)\left(1-|\alpha_-|^2\right)}|0\rangle_a + \sqrt{|\alpha_+|^2+|\alpha_-|^2-2|\alpha|^2}|1\rangle_a\right)\Big/\sqrt{1-|\alpha|^2}$, $|\phi_S\rangle = \left(\sqrt{1-|\alpha_-|^2}\alpha_+^*|0\rangle + \sqrt{1-|\alpha_+|^2}\alpha_-^*|1\rangle\right)\Big/\sqrt{|\alpha_+|^2+|\alpha_-|^2-2|\alpha|^2}$, and $|\phi_V\rangle_a = \sqrt{1-|\alpha_+|^2}|0\rangle_a + \alpha_+|1\rangle_a$. Here, the states with a bar, $|\bar{\phi}_A\rangle_a$, $|\bar{\phi}_S\rangle_a$, and $|\bar{\phi}_V\rangle_a$, denote $(i\sigma_{y,a}|\phi_A\rangle_a)^*$, $(-i\sigma_y|\phi_S\rangle)^*$, and $(i\sigma_{y,a}|\phi_V\rangle_a)^*$.

entanglement for case (i), and that the classical state can accomplish the task of AOSD for case (ii).

Geometric picture. It can be observed that the optimal success probability $P_{\mathrm{suc,max}}$ in Eq. (4) can be analyzed in two different regions: $|\alpha| < \bar{\alpha}$ and $|\alpha| \geq \bar{\alpha}$. Here based on the positive-operator-valued measure (POVM) strategy[15], we provide a geometric picture of $P_{\mathrm{suc,max}}$. Since the success probability, the concurrence and the discord of state ρ_{SA} under the constraints in Eq. (6) are all independent of the phase θ of α, one can simply set $\theta = 0$, and regard the states $|\psi_\pm\rangle$ as two unit vectors in $\mathbb{R}^2$ with the angle $\gamma = \arccos|\alpha|$ between them. The square roots of the *a priori* probabilities, i.e., $\sqrt{p_+}$ and $\sqrt{p_-}$, behave like wave amplitudes, and the effects of the coherence can be seen from the states $|\zeta\rangle$ and $|\Psi\rangle$. In Fig. 3, we plot two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with $\angle BOA = \gamma$ to denote $\sqrt{p_+}|\psi_+\rangle$ and $\sqrt{p_-}|\psi_-\rangle$, respectively. The two POVM elements that identify the states $\sqrt{p_\pm}|\psi_\pm\rangle$ can be implemented as $\Pi_\pm = r_\pm|\tilde{\psi}_\pm\rangle\langle\tilde{\psi}_\pm|/(\langle\tilde{\psi}_\pm|\tilde{\psi}_\pm\rangle)$, with $r_\pm \geq 0$. The vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ correspond to the unnormalized states $|\tilde{\psi}_\pm\rangle$ with the coefficients $-\sqrt{p_\pm}$. The third POVM element giving the inconclusive result is $\Pi_0 = \mathbb{1}_s - \Pi_+ - \Pi_-$. The elements $\Pi_{\pm,0}$ are required to be positive - this is a constraint on the POVM strategy. Finally, the probability of successful discrimination is $P_{POVM} = (r_+p_+ + r_-p_-)(1-|\alpha|^2)$, which is

$$P_{POVM} = \left(\overrightarrow{OA} - \overrightarrow{OB}\right)\cdot\left(r_-\overrightarrow{BD} - r_+\overrightarrow{AC}\right). \tag{15}$$

When $|\alpha| < \bar{\alpha}$, the optimal P_{POVM} is attained at $r_\pm = \left(1 - \cos\gamma\sqrt{p_\mp/p_\pm}\right)/\sin^2\gamma$. The vectors $r_-\overrightarrow{BD} = \overrightarrow{EA}$ and $r_+\overrightarrow{AC} = \overrightarrow{EB}$, where E is the intersection point of AE and BE (see Fig. 3). The maximum value of P_{POVM} is the square of $|AB|$, nanmely $P_{POVM} = |AB|^2 = 1 - 2\sqrt{p_+p_-}|\alpha|$, which recovers Eq. (4a). When $|\alpha| = \bar{\alpha}$, the point E coincides with B for $p_+ < p_-$ (or A for $p_+ > p_-$), for the optimal P_{POVM} one has $r_- = 1$ (or $r_+ = 1$) and $P_{POVM} = p_-(1-|\alpha|^2)$. For $|\alpha| = \bar{\alpha}$ and $p_+ < p_-$, E lies outside of the angle $\angle AOB$ and $\overrightarrow{EB}$ is opposite to $\overrightarrow{AC}$. Consequently, we do not get a physically realizable value of r_+. The optimal P_{POVM} strategy then occurs at $r_- = 1$ and $r_+ = 0$ (i.e., E coincides with B), one has $P_{POVM} = -\overrightarrow{OB}\cdot\overrightarrow{BD} = p_-\left(1-|\alpha|^2\right)$, which is Eq. (4b).

Discussion

In summary, based on a sufficiently general AOSD protocol, we found that the entanglement between the principal qubit and the ancilla is completely unnecessary. Moreover, this quantum entanglement can be arbitrarily zero by adjusting the parameters in the joint unitary transformation without affecting the success probability. Theoretically, this fact clearly indicates that dissonance plays a key role in assisted optimal state discrimination other than entanglement. Experimentally, the absence of entanglement can be more easily observed because there is no restriction on the *a priori* probabilities. In Fig. 4, we present a realization of the unitary transformation $\mathcal{U}$ in Eq. (1) for the initial states $|\psi_+\rangle = |0\rangle$, $|\psi_-\rangle = \sqrt{1-|\alpha|^2}|1\rangle + \alpha|0\rangle$ and $|k\rangle_a = |0\rangle_a$ by using single-qubit gates and two-qubit controlled-unitary gates. These gates can be demonstrated experimentally in many systems[23,24] in recent years. The success probability of state discrimination is determined by steps (i) to (iii), which transform the system-ancilla state into

$$|\psi_+\rangle|k\rangle_a \to \sqrt{1-|\alpha_+|^2}|0\rangle|0\rangle_a + \alpha_+|0\rangle|1\rangle_a, \tag{16a}$$

$$|\psi_-\rangle|k\rangle_a \to \sqrt{1-|\alpha_-|^2}|1\rangle|0\rangle_a + \alpha_-|0\rangle|1\rangle_a. \tag{16b}$$

It is not affected by the controlled-U_Φ in step (iv), which can adjust the correlations in state (2).

Let us also reiterate a necessary criterion for the requirement of dissonance in AOSD based on linear entropy. Under the general protocol, Alice and Bob share the entangled state $|\zeta\rangle$, encoded in the basis of the polarization of the qubit, Bob can acquire knowledge of the linear entropy $\mathcal{S}_L(\rho)$ of Alice's qubit. If $\mathcal{S}_L(\rho) > 1/2$, he can be sure that Alice needs dissonance for her AOSD (see Fig. 5). Finally, we would like to mention that local distinguishability of multipartite orthogonal quantum states was studied in Ref. 22 where again the local discrimination of entangled states does not require any entanglement.

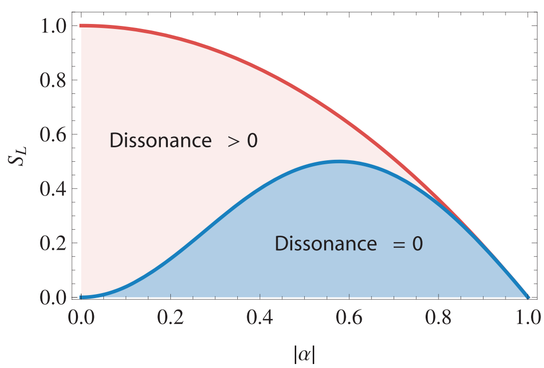

Figure 5 | Necessary criterion for requiring dissonance in AOSD based on linear entropy or purity. The linear entropy reads $\mathcal{S}_L(\rho) = 2 - 2\,\mathrm{Tr}\rho^2 = 4p_+p_-\left(1-|\alpha|^2\right)$, and the purity $\mathcal{P}(\rho) = 1 - \mathcal{S}_L(\rho)$. For a given amount of $|\alpha|$, when $\mathcal{S}_L(\rho) \leq 4\left(1-|\alpha|^2\right)|\alpha|^2\Big/\left(1+|\alpha|^2\right)^2$, the dissonance for the AOSD is zero (see the region below the dashed line). we note that $\mathcal{S}_L^{\max}(\rho) = 1/2$ when $|\alpha| = 1/\sqrt{3}$. This means that if $\mathcal{S}_L(\rho) > 1/2$, then the dissonance is necessarily needed for the AOSD.

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F.L.Z and J.L.C. initiated the idea. F.L.Z. derived the formulas and prepared the figures. J.L.C., F.L.Z., L.C.K. and V.V. wrote the main manuscript text. All authors contributed to the derivation and the manuscript.

Additional information

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