Braided Hopf Algebras, Double Constructions, and Applications

Robert Laugwitz
St Peter's College
University of Oxford

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This thesis is dedicated to my parents

Ina and *Uwe*

for their lifelong support

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Abstract

This thesis contains four related papers which study different aspects of double constructions for braided Hopf algebras. The main result is a categorical action of a braided version of the Drinfeld center on a Heisenberg analogue, called the Hopf center. Moreover, an application of this action to the representation theory of rational Cherednik algebras is considered.

Chapter 1: In this chapter, the Drinfeld center of a monoidal category is generalized to a class of mixed Drinfeld centers. This gives a unified picture for the Drinfeld center and a natural Heisenberg analogue. Further, there is an action of the former on the latter. This picture is translated to a description in terms of Yetter–Drinfeld and Hopf modules over quasi-bialgebras in a braided monoidal category. Via braided reconstruction theory, intrinsic definitions of braided Drinfeld and Heisenberg doubles are obtained, together with a generalization of the result of Lu (1994) that the Heisenberg double is a 2-cocycle twist of the Drinfeld double for general braided Hopf algebras.

Chapter 2: In this chapter, we present an approach to the definition of multiparameter quantum groups by studying Hopf algebras with triangular decomposition. Classifying all of these Hopf algebras which are of what we call weakly separable type, we obtain a class of pointed Hopf algebras which can be viewed as natural generalizations of multiparameter deformations of universal enveloping algebras of Lie algebras. These Hopf algebras are instances of a new version of braided Drinfeld doubles, which we call *asymmetric* braided Drinfeld doubles. This is a generalization of an earlier result by Benkart and Witherspoon (2004) who showed that two-parameter quantum groups are Drinfeld

doubles. It is possible to recover a Lie algebra from these doubles in the case where the group is free and the parameters are generic. The Lie algebras arising are generated by Lie subalgebras isomorphic to $\mathfrak{sl}_2$.

Chapter 3: The universal enveloping algebra $U(\mathfrak{tr}_n)$ of a Lie algebra associated to the classical Yang-Baxter equation was introduced in 2006 by Bartholdi–Enriquez–Etingof–Rains where it was shown to be Koszul. This algebra appears as the A_{n-1} case in a general class of braided Hopf algebras in work of Bazlov–Berenstein (2009) for any complex reflection group. In this chapter, we show that the algebras corresponding to the series B_n and D_n, which are again universal enveloping algebras, are Koszul. This is done by constructing a PBW-basis for the quadratic dual. We further show how results of Bazlov–Berenstein can be used to produce pairs of adjoint functors between categories of rational Cherednik algebra representations of different rank and type for the classical series of Coxeter groups.

Chapter 4: Quantum groups can be understood as braided Drinfeld doubles over the group algebra of a lattice. The main objects of this chapter are certain braided Drinfeld doubles over the Drinfeld double of an irreducible complex reflection group. We argue that these algebras are analogues of the Drinfeld–Jimbo quantum enveloping algebras in a setting relevant for rational Cherednik algebra. This analogy manifests itself in terms of categorical actions, related to the general Drinfeld–Heisenberg double picture developed in Chapter 2, using embeddings of Bazlov and Berenstein (2009). In particular, this work provides a class of quasitriangular Hopf algebras associated to any complex reflection group which are in some cases finite-dimensional.

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Chapter 1

Introduction

1.1 Motivational Background

This thesis falls into an area of algebra — specifically, representation theory — which is often referred to as *quantum algebra.* In particular, a central theme of this work is the construction of algebras (or Hopf algebras) which fit into the realm of this relatively young subfield of mathematics.[2]

The purpose of this chapter is to explain the main ideas which are central throughout the thesis,[3] and hence connect the different papers included into a general view on quantum algebra. This section is not intended to be a complete literature review on the rich areas of mathematical research this thesis draws from, or relates to.

The two main ideas appearing throughout this thesis are the study of braided Hopf algebras, and double constructions of such objects. The thesis develops a framework given by a categorical action of the monoidal Drinfeld center on the Heisenberg center, and gives a specific application to a setting relevant to rational Cherednik algebras. In this section, we motivate the main ingredients of this work.

1.1.1 Quantum Groups

One central idea in the study of algebraic geometry (in particular, algebraic groups) propagated by the Grothendieck school in the 1960s and 1970s is based on the

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[2] Quantum algebra had its origins in the late 1980s with the work of Drinfeld, Jimbo and others [36, 57].

[3] This thesis contains work made available in [67, 68, 69].

idea that when studying a geometric object X one should access it through the investigation of the ring $\mathcal{O}_X$ of functions on X. If the geometric space $X = G$ comes equipped with a group operation, one obtains the algebraic structure of a commutative Hopf algebra on $\mathcal{O}_G$ (see e.g. [24, Chapter I]). There is a converse to this construction: from any commutative Hopf algebra over a commutative ring R, we can obtain a group scheme over R. This is an anti-equivalence of categories (see e.g. [56, 2.3]).

$$\{\text{group schemes over } R\} \longleftrightarrow \{\text{commutative Hopf algebras over } R\}$$

Developments related to the study of integrable systems arising from quantum physics (in particular, conformal field theory, and many body problems) in the Leningrad school of theoretical physics in the 1980s (see e.g. [58] for a brief introduction) eventually led to the discovery of Hopf algebras which are non-commutative, and can be understood as deformations of classical objects such as the algebra $\mathcal{O}_G$ — these objects are often referred to as *quantum groups*. This development started with the discovery of quantized analogues $U_q(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra by Drinfeld and Jimbo around 1986 [36, 57]. Deformations $\mathcal{O}_q(G)$ of the quantum coordinate ring were introduced shortly thereafter (see [43] and others). It should be noted that, unlike in the commutative case, these non-commutative Hopf algebras do not have a corresponding object on the group side.

$$\begin{array}{ccc}
\{\ ?\ \} & \longleftrightarrow & \{\text{Hopf algebras over } R\} \\
\cup & & \cup \\
\{\text{group schemes over } R\} & \longleftrightarrow & \{\text{commutative Hopf algebras over } R\}
\end{array}$$

From this point of view — extending the Grothendieck philosophy — quantum groups do not exist as concrete mathematical objects. They can only be accessed via their quantized ring of functions (or the corresponding quantized universal enveloping algebra), and are hence thought of as *virtual* structures.

In this thesis, most of the work studies the universal enveloping algebras $U_q(\mathfrak{g})$ (with generalizations), rather than the quantum coordinate ring. We shall remark here that the study of $\mathcal{O}_q(G)$ has certain advantages. From the function algebra point of view it is natural to consider comodules over $\mathcal{O}_q(G)$. These can be viewed as integrable modules over $U_q(\mathfrak{g})$ via the fully faithful functor

$$\mathcal{O}_q(G)\text{-}\mathbf{CoMod} \hookrightarrow U_q(\mathfrak{g})\text{-}\mathbf{Mod},$$

induced by the natural pairing of $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$. This functor is far from being essentially surjective, which makes it necessary to study the representation theory of $U_q(\mathfrak{g})$ with additional assumptions on local finiteness, semisimplicity over the Cartan subalgebra, etc., in order to obtain a more manageable category of modules. Hence the category of comodules over $\mathcal{O}_q(G)$ is better behaved in certain aspects. We however study $U_q(\mathfrak{g})$ due to its description in terms of the braided Drinfeld double construction.

A successful approach to the theory of modules over $U(\mathfrak{g})$ is to introduce an intermediate category — the so-called *BGG category* $\mathcal{O}$ — which was introduced for the Lie algebra case in [21]:

$$\mathcal{O}_q(G)\text{-}\mathbf{CoMod} \hookrightarrow \mathcal{O} \hookrightarrow U_q(\mathfrak{g})\text{-}\mathbf{Mod}.$$

Studying category $\mathcal{O}$ allows the use of highest weight modules and has been adapted to both quantum groups [2] and rational Cherednik algebras [46]. Such techniques also play a reoccurring role in the work presented here.

1.1.2 Approaches to the Theory

In this thesis, we draw from different approaches to the theory of quantum groups: these are either more algebraic, or more categorical in nature. In general, we observe three main approaches to the theory of quantum groups:

- *The algebraic approach:* A common way to define quantum groups is using algebraic techniques. This uses Hopf algebra theory, in particular, classification of pointed Hopf algebras. The work in [69] which is included as Chapter 3 of this thesis uses similar techniques, studying Hopf algebras with a triangular decomposition.

  Another point of view on the algebraic construction of quantum groups is deformation theory of algebras, which does not play a role in this work.

Other techniques include the *Drinfeld twist* of a Hopf algebra by a 3-cycle (see 2.4.2), or the classification of Lie bialgebras (see e.g. [27, Chapter 6]) which also does not play a part in this thesis.

- *The categorical approach:* It was observed by Majid and other authors (for more detailed references see [88, Chapter 9]) in the early 1990s that constructions of quantum algebras can often be given using general categorical definitions (such as bosonization, cross products, and most prominently Drinfeld's quantum double construction [36, 90]). Central objects in this approach are *braided* Hopf algebras (or *braided groups*, [85, 86]). These are Hopf algebras defined in a general braided monoidal category, rather than the symmetric monoidal category of vector spaces. Perhaps the most prominent example of such a structure is the positive part $U_q(\mathfrak{n}_+)$ of the quantum enveloping algebra $U_q(\mathfrak{g})$.

  The connection of the theory of quantum groups to invariants of framed knots has possibly been the most well-studied and successful application of the abstract theory (see [62, 78, 103]). In fact, braided monoidal categories — which appear in quantum algebra via representation theory — can be studied using knot diagrams (see e.g. [61]). Connecting to the algebraic structure of braided Hopf algebras, one can use a diagrammatic calculus (see e.g. [88, 9.4]) which is closely connected with a generalized form of Tannakian reconstruction theory (generalizing [32, 107]) to a setting for working in braided monoidal categories. This is due to Majid [80, 83, 85].

  The categorical approach plays a major role in this thesis, especially in Chapter 2 which is a revised version of [68], but also in Section 3.3.4. We make use of reconstruction theory in order to define algebra and Hopf algebra objects. In Section 2.2.3 we include a version of braided reconstruction theory adapted to braided quasi-Hopf algebras, simultaneously generalizing [84] and [80].

- *Geometric and topological approaches:* Other approaches to the definition of quantum groups include geometric and topological techniques. A most notable construction is the geometric canonical basis of Lusztig [74, 75].

These constructions use the theory of perverse sheaves, and Hall algebras. Such geometric and topological approaches are not discussed in this thesis.

1.1.3 Yang–Baxter Equations

The Yang–Baxter equations are central in the theory of quantum groups. Their origin lies in their role as a consistency condition in the study of integrable systems, which arise from quantum physics (see [58] for a survey). The Yang–Baxter equations also appear in knot theory, C^*-algebras, and conformal field theory, thus providing a link between these areas.

In this thesis, the Yang–Baxter equations appear in two forms. The first form is the *quantum Yang–Baxter equation (QYBE)*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.1.1}$$

where $R_{ij} \in V^{\otimes 3}$ for finite-dimensional vector space V. This equation resembles the third Reidemeister move in knot theory [1, 101]. It plays an important part in Chapter 2 (see in particular Section 2.1.1). A second incarnation — the *classical Yang–Baxter equation (CYBE)* — is given by

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{1.1.2}$$

where again $r_{ij} \in V^{\otimes 3}$. It can be viewed as a classical limit of (1.1.1) (as explained in [58, 3.1]). This equation appears in this thesis in Chapter 4 (which contains the content of the preprint [67]), where we study universal enveloping algebras associated to Lie algebras which have generalizations of this equation to complex reflection groups as relations (due to [14, 16]).

1.1.4 Braided Hopf Algebras

A braided Hopf algebra is simply a Hopf algebra[4] object in a braided monoidal category (see e.g. [85, 86] or 2.1.6 where this structure is described concretely). The role of the symmetry in the category of vector spaces is replaced by a general braiding $\Psi\colon \otimes \to \otimes^{\mathrm{op}}$. As explained in Section 1.1.2, this generality is needed to

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[4] Hopf algebras first appeared in the work of Hopf in the 1940s studying cohomology of topological groups [55]. See also the early textbook by Sweedler [110].

capture essential examples in the theory of quantum groups, such as the positive part of $U_q(\mathfrak{g})$.

The idea of braided Hopf algebras is present in all parts of this thesis. In Chapter 2 we study constructions associated to general (quasi-)Hopf algebras in a braided monoidal category. The setting in Chapter 3 is in some sense more restricted: here we are working with dually paired Hopf algebras in a braided monoidal category which has the form of Yetter–Drinfeld modules over a Hopf algebra over k. In Chapters 4 and 5 we focus on very specific braided Hopf algebras, which are associated to complex reflection groups.

A vast class of examples of braided Hopf algebras is given by *Nichols algebras* (or *Nichols–Woronowicz algebras*). These are certain minimal quotients of braided tensor algebras of a braided vector space (see e.g. [9, Section 2] for a definition). This class was independently discovered by [96] and [115], although the braided Hopf algebra structure appeared later. Nichols algebras are central objects of study in different places of all chapters of this thesis.

1.1.5 Double and Center Constructions

At the core of this thesis are two different double constructions: the braided Drinfeld double, and the braided Heisenberg double. The non-braided versions of these algebras are fundamental constructions in quantum algebra. Braided generalizations of the two constructions have been introduced before. In the case of the Drinfeld double, this is due to Majid [88, 90] and referred to as the *double-bosonization* therein. Braided Heisenberg doubles are discussed for the special case of braided Hopf algebras which are primitively generated (and hence quotients of a braided tensor algebra) in the study of so-called *braided doubles* [16] by Bazlov and Berenstein. In fact, the braided Heisenberg double can alternatively be obtained as the bosonization of the *braided cross product* defined by Majid in [86, Proposition 2.6] (using the coregular action). From this point of view the construction goes back to work of Radford [99].

Drinfeld and Heisenberg doubles are rich constructions containing important classes of examples. For the Drinfeld double, the most notable examples include the Drinfeld double of a group (see 2.3.5.6). This appears in Dijkgraaf–Witten theory [33, 34], which is one of the most studied examples of a topological field theory. Even in more generality, the structure of the Drinfeld double can be used to construct topological field theories (cf. [19] for the geometric example) as their categories of modules are braided monoidal.[5] Another main example is given by the quantum groups $U_q(\mathfrak{g})$ which — in the form of Lusztig [75, 3.1] — have been shown to be braided Drinfeld doubles in [90] (cf. 2.3.5.11). The fundamental example of a Heisenberg double is the Weyl algebra of a vector space (see 2.3.5.10 for a detailed discussion of this example). From this point of view, braided Heisenberg doubles can be viewed as a vast generalization of Weyl algebras. In particular, they are *not* bialgebras apart from degenerate cases.

From the point of view of category theory (via reconstruction theory) double constructions correspond to center constructions. In the case of the Drinfeld double, the corresponding categorical construction is the *Drinfeld center*[6] (or *monoidal center*) which is due to [81] (see also [82]) where it is referred to as the *dual* monoidal category (see also [88, 9.1.5]).

We introduce a more general form of centers, called *mixed Drinfeld centers* (see Section 2.2.2), which are relative to a braided monoidal base category. A special case is a center construction corresponding to the Heisenberg double, called the *Hopf center* in Chapter 2, which is a new construction. The name comes from *Hopf modules* which can be used to describe the objects of the Hopf center in the case where the underlying monoidal category is given by modules over a bialgebra. Hopf modules, in turn, are a known description for modules over the Heisenberg double. A braided version of Hopf modules already appeared in [76] and [22]. For the Drinfeld center, such a description is given by so-called *Yetter–Drinfeld modules* which play a central role in different constructions used in this thesis. Yetter–Drinfeld modules are generalizations of *crossed G-modules* which correspond to the case over a group G and have been studied in algebraic topology since the work of J.H.C. Whitehead in the 1940s [111] (see also [112, 113]). From a Hopf algebra theoretic point of view, the Yetter–Drinfeld condition already appears in

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[5] Hence they are E_2-categories which give 2d topological field theories via the cobordism hypothesis, see [72, Theorem 4.1.24].

[6] Drinfeld independently studied a special case of Majid's construction and showed that it is braided monoidal (see also the categorical construction of [62]).

the work of Radford [99, 2.6(b)] in the 1980s before the introduction of braided monoidal categories in [60]. They also appear in the work [115] of Woronowicz for bialgebras over k, and in [22] in the braided case.

Our main result of Chapter 2 is a categorical action of the Drinfeld center on the Hopf center. Hence, one important theme of the thesis is the study of the interplay between the different double and center constructions. Chapter 5 discusses an application of this categorical action to the representation theory of rational Cherednik algebras.

A related point of view is to interpret the Drinfeld center as a categorical *Hochschild cohomology* of a monoidal category $\mathcal{M}$ (cf. [19, Definition 1.4]):

$$\mathbf{HH}^{\bullet}(\mathcal{M}) := \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}).$$

This is explained briefly in 2.1.1. In this way, the general theory of *mixed Drinfeld centers* developed in Section 2.2.2 can be seen as categorical Hochschild cohomology[7] with coefficients in bimodules other than the regular one.

1.1.6 Rational Cherednik Algebras

A class of algebras that has generated a vast amount of interest in different areas of mathematics (most notably algebraic and symplectic geometry) and mathematical physics after its introduction in [39], in 2002, is given by the *rational Cherednik algebras* $H_{t,c}(G)$. These algebras are deformations of the bosonization $\mathbb{C}[V \oplus V^*] \rtimes G$, where G is a finite group acting on V such that the quotient V/G is non-singular. This implies that G is a complex reflection group by the Shephard–Todd–Chevalley theorem [28, 108]. The quotient $V \oplus V^*/G$ is singular and the algebras $H_{t,c}(G)$ offer an approach to the resolution of its symplectic singularities via representation theory. More specifically, irreducible finite-dimensional representations over $H_{0,c}(G)$ form a smooth algebraic variety which is a Poisson deformation of $V \oplus V^*/G$.[8] For surveys on the representation theory of these algebras and their application

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[7] Hochschild Cohomology for algebras over k was introduced by Hochschild in the 1940s [54].

[8] The work of Bellamy in [18] gives a strong restriction on when a *smooth* Poisson deformation exists. This can only happen if the group G is of types $G(m, 1, n)$ or G_4 in the Shephard–Todd classification.

see e.g. [38, 41, 47, 106]. The algebras $H_{t,c}(G)$ have important geometric interpretations, for example in terms of the geometry of Hilbert schemes of points in the case $G = S_n$ (see [49, 50, 65]).

Due to their deformation-theoretic origin, rational Cherednik algebras can also be interpreted as *quantum objects* in a broader sense. Work in [16] by Bazlov and Berenstein produces an embedding of $H_{0,c}(G)$ for generic c into certain braided Heisenberg doubles. We use this work in Chapter 5 to apply the general theory of braided Heisenberg and Drinfeld doubles presented in Chapter 2 to a setting tailored for complex reflection groups. The main application is to produce endofunctors on $H_{0,c}(G)$-**Mod** which assemble into a categorical action. The study of the braided Hopf algebras of which we take the Drinfeld double in this construction and the resulting categorical action are the main topics of Chapters 4 and 5.

1.2 Thesis Overview

This thesis contains a set of related results in the areas of quantum algebra and representation theory. In this section, we provide a brief overview of its main results. We also explain the connection between the different parts of the thesis in more detail. Each of the individual chapters contains a more detailed introduction as in their versions in [67, 68, 69]. The material from Chapter 5 is unpublished to date.

One point of view on the thesis work is that we study categorical actions of particular monoidal categories (of Drinfeld center form) on certain categories (of Hopf center form). There are four main examples worth pointing out:

General picture (see 2.2.2.15): $\mathrm{Drin}_H(B)\text{-}\mathbf{Mod} \curvearrowright \mathrm{Heis}_H(B)\text{-}\mathbf{Mod}$,
Basic geometric example (2.3.5.3): $\mathcal{O}_{T*X}\text{-}\mathbf{Mod} \curvearrowright D_X\text{-}\mathbf{Mod}$,
Quantum picture (2.3.5.11): $U_q(\mathfrak{g})\text{-}\mathbf{Mod} \curvearrowright D_q(\mathfrak{g})\text{-}\mathbf{Mod}$,
Reflection algebras picture (5.2.4): $\mathcal{D}(G)\text{-}\mathbf{Mod} \curvearrowright H_{0,c}(G)\text{-}\mathbf{Mod}$.

Here, the algebra $D_q(\mathfrak{g})$ is the braided Heisenberg double of $U_q(\mathfrak{n}_+)$ for a given Cartan datum. This algebra contains the corresponding generalized quantum Weyl algebra of [59, 3.1.3] as a subalgebra. Hence, its category of modules can be thought of as *quantum D-modules*, and modules over $U_q(\mathfrak{g})$ act on this category.

1.2.1 Connection of the Different Parts

Chapter 2 forms the basis of this thesis. We work in the general setting of (quasi-) Hopf algebras in a braided monoidal category — or on the categorical level with monoidal categories fibered over braided monoidal categories. We study the main notions of braided Drinfeld and Heisenberg doubles (corresponding to relative Drinfeld and Hopf centers on the categorical level) which will be used in subsequent chapters. The main construction, a categorical action of the relative Drinfeld center on the relative Hopf center, will be applied in Chapter 5 to a specific setting for complex reflection groups.

The next chapter, Chapter 3, approaches quantum groups from a more algebraic point of view, via the partial classification of Hopf algebras with a triangular decomposition. The finding that, under suitable assumptions, these objects are a new form of braided Drinfeld doubles, which we call *asymmetric* braided Drinfeld doubles, provides a generalization of the Hopf algebras studied in Chapter 2.

Chapter 4 can be thought of as preliminary material for the final chapter (Chapter 5) since we investigate Koszulness of the quadratic braided Hopf algebras $U(\mathfrak{y}\mathfrak{b}_G)$. Their braided Drinfeld doubles $\mathcal{D}(G)$ will play a major role in the final chapter. This ultimate chapter draws from Chapter 2 and 4 to provide applications to the theory of rational Cherednik algebras. This work leaves a variety of open questions for further research and can hence be viewed as the conclusion of the thesis.

1.2.2 Braided Drinfeld and Heisenberg Doubles

Chapter 2 starts by providing a categorical picture giving the definition of the relative Hopf center $\mathcal{H}_{\mathcal{B}}(\mathcal{M}) = \mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, {}^{\mathrm{reg}}\mathcal{M}^{\mathrm{triv}})$ of a monoidal category $\mathcal{M} \to \mathcal{B}$, where $\mathcal{B}$ is braided monoidal, in 2.2.2. This definition provides a new Heisenberg analogue to the known description of the relative Drinfeld center $\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) = \mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ which slightly generalizes the construction of [81, 88, 90] in the sense that we make reference to an underlying base category $\mathcal{B}$. With this description, it is clear that there is a natural categorical action

$$\triangleright\colon \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \otimes \mathcal{H}_{\mathcal{B}}(\mathcal{M}) \longrightarrow \mathcal{H}_{\mathcal{B}}(\mathcal{M}),$$

given by precomposition of bimodule morphisms. The category $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is braided monoidal, and moreover is rigid provided that $\mathcal{M}$ itself is rigid. We next provide a version of Tannakian reconstruction theory for quasi-Hopf algebras in a braided monoidal category $\mathcal{B}$ in 2.2.3, generalizing work of [84] using techniques as in [80] or [88, 9.4.2].

It is well-known that modules over the Drinfeld double can be described in terms of Yetter–Drinfeld modules (see e.g. [88, 7.1.6], [64, XIII.4]), and modules over the Heisenberg double in terms of Hopf modules. In Section 2.2.4 we provide a description of the relative Drinfeld and Hopf centers in terms of Yetter–Drinfeld[9] and Hopf modules over a quasi-Hopf algebra in $\mathcal{B}$ and translate all results from the categorical level into this language.

In Section 2.3.2, we leave the generality of quasi-Hopf algebras and consider dually paired *strict* Hopf algebras B, C. We can then define the braided Drinfeld double $\mathrm{Drin}_H(C,B)$ and the braided Heisenberg double $\mathrm{Heis}_H(C,B)$ using reconstruction theory in Section 2.3.4, and provide explicit presentations in 2.3.5. The case of the braided Drinfeld double (together with its interpretation in terms of Yetter–Drinfeld modules) is due to [90], where it is referred to as the double-bosonization $U(\overline{B},H,C)$, and we compare this definition to a suitable Heisenberg version (which is equivalent to the bosonization $({}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C) \rtimes H$ of the braided cross product of the coregular action of [86], cf. Lemma 2.3.5.5). Our main result of this section is to use the categorical action to generalize the result of [71] that the Heisenberg double is a 2-cocycle twist of a Drinfeld double:

Corollary 2.3.8.5. *The algebra* $\mathrm{Heis}_H(C,B)$ *is a right 2-cocycle twist of the Hopf algebra* $\mathrm{Drin}_H(C,B)$.

We also include a version of the BGG category $\mathcal{O}$ [21] suitable for braided Drinfeld and Heisenberg doubles in 2.3.7, and study the representation theory of the braided Heisenberg double of $U_q(\mathfrak{n}_+)$ explicitly in the $\mathfrak{sl}_2$-case in 2.3.9. Finally, we define notions of twisted Drinfeld and Heisenberg doubles in 2.4. In case of the Drinfeld double, see the *quantum double* in [89], and for the Heisenberg double

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[9] Over $\mathcal{B} = \mathbf{Vect}_k$, such a description was given in [89] in the quasi-bialgebra case, while it is a new result for the Hopf center.

cf. to the bosonization for quasi-bialgebras in [25, Theorem 3.3, Section 5]. We include the new example of the twisted Heisenberg double $\mathrm{Heis}^{\omega}(G)$ of a group.

1.2.3 Pointed Hopf Algebras with Triangular Decomposition

A different part of this thesis is an approach to the classification of Hopf algebras that can be called "quantum groups" in a broader sense, presented in Chapter 3. A formal definition of the term "quantum group" is a long-standing open question of the field. The approach to this question presented here uses the work of [16] on classifying algebras with a quasitriangular decomposition of the form $U \otimes H \otimes U^*$. What is new is that we consider bialgebras and Hopf algebras with such structure. Multiparameter quantum groups are *pointed* Hopf algebras (see e.g. [9]) with a triangular decomposition. The work presented in Chapter 3 classifies indecomposable Hopf algebras with a triangular decomposition (see Definition 3.3.1.1) over a group algebra which satisfies the technical restriction of being of *weak diagonal type*:

Theorem 3.4.2.2. *Let A be any indecomposable Hopf algebra with a triangular decomposition of weak diagonal type over a group algebra kG generated by elements g_i, h_i for $i = 1, \ldots, n$. Then kG is abelian, and $A \cong T(V)/I \otimes kG \otimes T(V^*)/I^*$, for Yetter–Drinfeld modules $V = k\langle v_1, \ldots, v_n\rangle$, $V^* = k\langle f_1, \ldots, f_n\rangle$ over G, and $I \lhd T(V)$, $I^* \lhd T(V^*)$ ideals which form a triangular Hopf ideal in A. The algebra A is subject to the relations*

$$[f_i, v_j] = \gamma_{ij}(g_j - h_i), \qquad g v_i = (g \triangleright v_i) g, \qquad f_i g = g(f_i \triangleleft g),$$

where γ_{ij} are scalars s.t. $\gamma_{ij} = 0$ whenever g_i, g_j act by different characters on V. The coproducts are given by $\Delta(v_i) = v_i \otimes g_i + 1 \otimes v_i$, and $\Delta(f_i) = f_i \otimes 1 + h_i \otimes f_i$.

The multiparameter quantum groups $U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ of [43, 102] as given e.g. in [29] occur as examples in this classification. They satisfy that γ_{ij} is zero whenever $i \neq j$ and non-zero when $i = j$. Hopf algebras with this additional property can be constructed using a new, more general, form of the braided Drinfeld double called the *asymmetric* braided Drinfeld double which can defined even if the base Hopf algebra H is not quasitriangular. In this way, we give the following generalization of the result of [20], which shows that two-parameter quantum groups are Drinfeld doubles:

Theorem 3.4.3.2. *The multiparameter quantum groups $U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ are asymmetric braided Drinfeld doubles.*

In particularly well-behaved situations (namely, if the braiding parameters q_{ij} can be set equal to one to give a map $\mathbb{Z}[q_{ij}] \to \mathbb{Z}$), it is possible to recover a Lie algebra as a classical limit from a Hopf algebra occurring in the classification (see 3.4.4).

We also combine the results of Theorem 3.4.2.2 with work of Rosso and Andruskiewitsch–Schneider [10, 105] to obtain an abstract characterization of the quantum enveloping algebras $U_q(\mathfrak{g})$ in 3.5.2. If V is of what is called *positive generic type* in [9], then the only indecomposable bialgebras with triangular decomposition as $\mathcal{B}(V) \otimes k\mathbb{Z}^n \otimes \mathcal{B}(V^*)$ (under the non-degeneracy condition that the parameters γ_{ii} are non-zero) are isomorphic to $U_q(\mathfrak{g})$ for some semisimple, possibly infinite-dimensional, Lie algebra $\mathfrak{g}$.

1.2.4 Koszulness of Universal Enveloping Algebras Associated to Generalized Yang–Baxter Equations

Chapter 4 studies first applications of the theory of braided Heisenberg doubles to rational Cherednik algebras and is the content of [67]. Here, we work with Lie algebras $\mathfrak{yb}_G$ coming from generalized Yang–Baxter-equations associated to irreducible complex reflection groups. The A_n-case gives the Lie algebra $\mathfrak{tr}_n$ from [14] corresponding to the classical Yang–Baxter-equations, and its universal enveloping algebra $U(\mathfrak{tr}_n)$ is a quadratic cover of the Fomin–Kirillov algebra $\mathcal{E}_n$ from [44] which was introduced to study the combinatorics of the cohomology ring of the flag variety and Schubert calculus. The general definition of $\mathfrak{yb}_G$ is due to [16].

As a first main result, we obtain a generalization of the Koszulness of the algebra $U(\mathfrak{yb}_G)$ for type A in [14] to other types:

Theorem 4.3.2.6 (see also 4.3.3.4)**.** *The algebras $U(\mathfrak{yb}_G)$ for G a Coxeter group of type $B = C$, or D are Koszul.*

These Koszulness results – proved by constructing a PBW-basis for the Koszul duals – will simplify the study of the representation categories of their braided Drinfeld doubles. Moreover, the braided Heisenberg doubles $\mathrm{Heis}_{\mathbb{C}G}(U(\mathfrak{nb}_G))$ (as bimodules over rational Cherednik algebras) are used to construct adjoint pairs of functors between module categories $H_{0,c}(G)$-**Mod** for different Coxeter groups of the classical series. Such functors exist if the Dynkin diagram of one reflection group embeds into another:

Corollary 4.4.2.1. *There exist adjoint pairs of functors*

$$\mathrm{HInd}\colon H_{0,c}(S_n)\text{-}\mathbf{Mod} \rightleftarrows H_{0,c'}(G_m)\text{-}\mathbf{Mod} :\mathrm{HRes},$$

for G_m *a Coxeter group of type* B_n*,* D_n *or* S_{n+1}*, and* t*,* c*,* c' *scalars in* $\mathbb{C}$*.*

1.2.5 Quantum Groups Associated to Complex Reflection Groups

The final chapter, Chapter 5, can be seen as the conclusion of this thesis. It contains work which is still in progress, but essential to explain how the different parts of the thesis fit together. The main result is that the algebra morphisms of Bazlov–Berenstein in [16]

$$M_c\colon H_{0,t}(G) \longrightarrow \mathcal{H}(G),$$

which are embeddings if c is generic, can be used to restrict the categorical action of Chapter 2 to a categorical action on representations of rational Cherednik algebras at parameter $t = 0$:

Theorem 5.2.4.1. *There is a categorical action*

$$\begin{aligned} \rhd\colon \mathcal{D}(G)\text{-}\mathbf{Mod} \otimes H_{0,c}(G)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G)\text{-}\mathbf{Mod}, \\ (V, W) &\longmapsto V \otimes W, \end{aligned}$$

fibered over $\mathbf{Vect}_k$*.*

The algebra $\mathcal{H}(G)$ is the braided Heisenberg double $\mathrm{Heis}_{\mathbb{C}G}(U(\mathfrak{nb}_G^*), U(\mathfrak{nb}_G))$, and $\mathcal{D}(G)$ — the *reflection quantum group* — is the corresponding braided Drinfeld double $\mathrm{Drin}_{\mathrm{Drin}(G)}(U(\mathfrak{hb}_G^*), U(\mathfrak{hb}_G))$. We include variations of these algebras taking the quotient by different ideals of relations, including restricted versions $\overline{\mathcal{D}}(G)$, $\overline{\mathcal{H}}(G)$ which are sometimes finite-dimensional. This was computed to hold if G is equal to S_n for $n = 3, 4, 5$ in [44], while it is unknown for larger values of n. In fact, the corresponding quotient algebras $\mathcal{B}(Y_G)$ of $U(\mathfrak{hb}_G)$ are at the core of important conjectures in the theory of pointed Hopf algebras.

One questions is if the algebra $\mathcal{D}(C_n)$ are finite-dimensional over their center. In this case, there would be a rich supply of finite-dimensional simple modules. In the A_1-case, this holds and we explicitly classify all finite-dimensional simples, compute their tensor products and the categorical action in Section 5.3.3. This case already displays an interesting representation theory.

To conclude this thesis, we list some open problems centered around the algebras $\mathcal{D}(G)$ and their applications to rational Cherednik algebras in Section 5.4.1.

Chapter 2

Braided Drinfeld and Heisenberg Doubles

2.1 Introduction

2.1.1 Motivation

The Drinfeld double was originally introduced as the quantum double by Drinfeld in [36]. The construction was generalized to quasi-Hopf algebras in [89], and to braided Hopf algebras in [90].

There are different ways to motivate the introduction of the Drinfeld double. For example, it gives a way to construct morphism $\Psi\colon V\otimes V\to V\otimes V$ satisfying the *Yang–Baxter equation*

$$(\Psi\otimes\mathrm{Id}_V)(\mathrm{Id}_V\otimes\Psi)(\Psi\otimes\mathrm{Id}_V)=(\mathrm{Id}_V\otimes\Psi)(\Psi\otimes\mathrm{Id}_V)(\mathrm{Id}_V\otimes\Psi). \tag{2.1.1}$$

It also gives a natural way of associating to a Hopf algebra a quasitriangular Hopf algebra (i.e. one that is almost cocommutative).

The Heisenberg double can be given a similar interpretation, where the Yang–Baxter equation is replaced by the *Pentagon equation* (see [64])

$$(\Phi\otimes\mathrm{Id}_V)(\mathrm{Id}_V\otimes\Phi)(\Phi\otimes\mathrm{Id}_V)=(\Phi\otimes\mathrm{Id}_V)(\mathrm{Id}_V\otimes\Phi). \tag{2.1.2}$$

Every module V over the Heisenberg double comes with a map $\Phi\colon V\otimes V\to V\otimes V$ satisfying this equation.

In [64] it was also shown that solutions for (2.1.1) can be obtained from solutions for Eq. (2.1.2). One application of the results of this chapter is to show that given a solution (V,Ψ) to (2.1.1) and (W,Φ) to (2.1.2), $(V\otimes W,\Psi^{(1)}\otimes\Phi^{(1)}\otimes\Psi^{(2)}\otimes\Phi^{(2)})$ again has the structure of a solution to the pentagon equation. For finite-dimensional Hopf algebras, this follows from a twisting result of [71] which is generalized to the context of braided Hopf algebras (so-called *braided groups* in the braided cocommutative case in [85, 86] and other papers) in 2.3.8.3 and, more generally, monoidal categories in Corollary 2.2.2.15.

To argue why it is beneficial to define a braided version of the Drinfeld and Heisenberg double, it is helpful to consider an example. Let B is the coordinate ring $\mathcal{O}_X$ for $X=\mathbb{A}^n$. In this case, $\operatorname{Heis}(B)$ is the ring of differential operators on X, D_X. However, if we compute the Drinfeld double of B, then this simply gives $\mathcal{O}_X[\partial_1,\ldots,\partial_n]=\mathcal{O}_{T^*X}$. This is a commutative and cocommutative Hopf algebra. A more interesting object is obtained by considering B as a Hopf algebra in the category of YD-modules over the group C_2. This can be seen as a *super* algebra version of the coordinate ring. Computing the braided Drinfeld double $\operatorname{Drin}_{\operatorname{Drin}C_2}(B)$ gives a non-commutative Hopf algebra in which the commutator relation

$$[\partial_i,x_j]=(1-\delta_1-\delta_{-1})\delta_{i,j} \tag{2.1.3}$$

holds (see 2.3.5.10). From the point of view that the Drinfeld double gives examples of "Quantum groups" it is more natural to have a non-commutative and non-cocommutative Hopf algebra. Note that computing the Heisenberg double over $\operatorname{Drin}(C_2)$ gives $D_X\otimes kC_2$, so the answer for the Heisenberg double is essentially not changed as the commutator relation in the braided Heisenberg double remains

$$[\partial_i,x_j]=\delta_{i,j}. \tag{2.1.4}$$

Another application of the braided version is to give a clean description of the quantum groups $U_q(\mathfrak{g})$ as Drinfeld doubles (see [90]), while it was already observed in [36] that $U_q(\mathfrak{g})$ is a quotient of the Drinfeld double of $U_q(\mathfrak{n}_+)\rtimes U_q(\mathfrak{t})$. In our unified picture, we now have a natural Heisenberg analogue for the quantum groups, for which the commutator relation is

$$[E_i, F_j] = \frac{K^{-\frac{i\cdot i}{2}}}{q_i^{-1} - q_i}. \tag{2.1.5}$$

This algebra has no finite-dimensional representations (see 2.3.9).

Another generalization included in the constructions of this chapter is to allow quasi-Hopf algebras (introduced in [37]), which we consider in a braided monoidal category. Including this direction of generalization will enable us to consider examples such as the twisted Drinfeld double $\mathrm{Drin}^{\omega}(G)$ of a group G which is of relevance in mathematical physics as it occurs as data associated to particular orbifolds in Rational Conformal Field theory (see [33]). In [89], a construction of the Drinfeld double of a quasi-Hopf algebra is given via reconstruction theory. We add a Heisenberg analogue to this picture and generalize it to quasi-Hopf algebras in a braided monoidal category, thus combining the two directions of generalization (braiding and twisting).

In a derived setting (see e.g. [19]), the Drinfeld center gives a categorical version of *Hochschild cohomology* which is defined as

$$\mathbf{HH}^{\bullet}(\mathcal{M}) = \mathbf{HH}^{\bullet}(\mathcal{M},\mathcal{M}) := \mathcal{Z}(\mathcal{M}) = \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}},\mathcal{M}^{\mathrm{reg}}).$$

From this point of view, in Section 2.2, we consider a more general form of Drinfeld centers, which can be viewed as categorical Hochschild cohomology with values in other bimodule categories $\mathcal{V}$ over $\mathcal{M}$,

$$\mathbf{HH}^{\bullet}(\mathcal{M},\mathcal{V}) := \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M},\mathcal{V}).$$

Even though the constructions are given in a non-derived setting in the (2,1)-meta category **Cat** of categories, because of the intrinsic nature of the definitions in terms of bimodule categories, it is possible to obtain higher-categorical analogues working in the meta-category $(\infty,1)$-category of $(\infty,1)$-categories $\mathbf{Cat}_{\infty}$ instead using an appropriate formalism of bimodule categories (such as e.g. [73, 4.3.2]). Hochschild cohomology with coefficients in any $\mathcal{M}$-bimodule $\mathcal{V}$ will give module categories over $\mathbf{HH}^{\bullet}(\mathcal{M})$, and relative versions of these with respect to $\mathcal{M}$ living over a braided monoidal category $\mathcal{B}$. In particular, the Hopf center from Definition

2.2.2.7 — which is the Heisenberg analogue of the Drinfeld center — can be interpreted as $\mathbf{HH}^{\bullet}_{\mathcal{B}}(\mathcal{M}, {}^{\mathrm{reg}}\mathcal{M}^{\mathrm{triv}})$ in a derived setting. Here, coefficients are taken in the $\mathcal{M}$-bimodule ${}^{\mathrm{reg}}\mathcal{M}^{\mathrm{triv}}$ which has the regular action on the left and the trivial action (given by the underlying tensor product in $\mathcal{B}$) on the right.

The original motivation for the author to write this chapter lies in the applications of the categorical action of $\mathrm{Drin}_H(C,B)$-**Mod** on $\mathrm{Heis}_H(C,B)$-**Mod** to the *rational Cherednik algebras* $H_{t,c}(G)$ of [39]. For this, the morphisms from [16, 7.20]

$$M_c\colon H_{0,c}(G) \longrightarrow \mathrm{Heis}_{\mathbb{C}G}(U(\mathfrak{hb}^*_G), U(\mathfrak{hb}_G)),$$

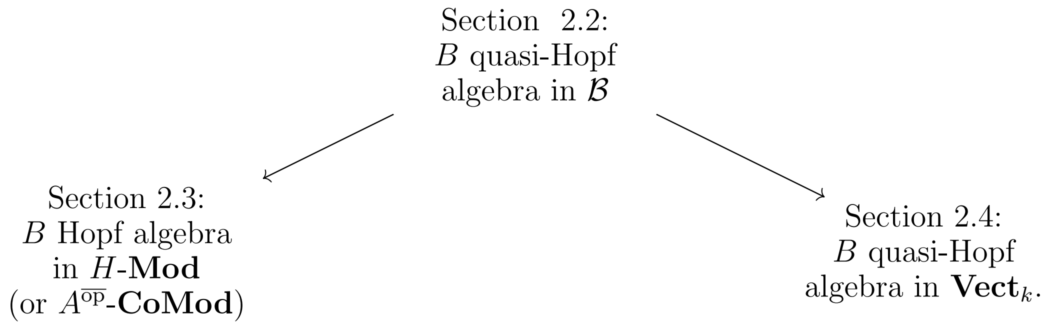
where $U(\mathfrak{hb}_G)$ is a generalization of the algebra $U(\mathfrak{tr}_n)$ from [14] to general complex reflection groups (cf. Chapter 4 for the notation) are used. Analogues also exist for parameters $t \neq 0$, or the restricted rational Cherednik algebras. The morphisms M_c can be used to restrict the categorical action studied in this chapter to gives actions (cf. Section 5.2.4)

$$\rhd_{t,c}\colon\ \mathrm{Drin}_{\mathrm{Drin}(G)}(U(\mathfrak{hb}^*_G), U(\mathfrak{hb}_G))\text{-}\mathbf{Mod} \otimes H_{t,c}(G)\text{-}\mathbf{Mod} \longrightarrow H_{t,c}(G)\text{-}\mathbf{Mod}.$$

It is work in progress to study these actions using category $\mathcal{O}$ techniques as in [46].

2.1.2 Summary

The discussion of Drinfeld and Heisenberg doubles in this chapter is done at different levels of generality which are structured by the sections:



In Section 2.2 we work on the level of a monoidal category $\mathcal{M}$ living over a braided monoidal category $\mathcal{B}$. That is, there exist quasi-monoidal functors $F\colon \mathcal{M} \to \mathcal{B}$ and $P\colon \mathcal{B} \to \mathcal{M}$ such that $FP \cong \mathrm{Id}_{\mathcal{B}}$. It is not necessary to make further assumptions on the braided monoidal category, such as a k-linear structure, although in typical examples, the categories will have fiber functors to the category of k-vector spaces for a field k.

We start by giving the most general definition of a *mixed relative Drinfeld center* in Section 2.2.2. This is done by translating the data of morphisms of $\mathcal{M}$-bimodules $\mathcal{M}^{\mathrm{reg}} \to {}^{G_1}\mathcal{V}^{G_2}$ to pairs (V, c), where V is an object of $\mathcal{V}$ and $c \in \mathrm{Isom}^{\otimes}(V \otimes G_2, G_1 \otimes V)$. Here, ${}^{G_1}\mathcal{V}^{G_2}$ is the $\mathcal{M}$-bimodule where the left module structure is induced by pulling the regular action back along G_1, and the right one along G_2. We will focus on two classes of examples: the relative Drinfeld center $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$, and its "Heisenberg analogue" (called the Hopf center) $\mathcal{H}_{\mathcal{B}}(\mathcal{M})$. The Drinfeld center corresponds to the pair of functors $(G_1, G_2) = (\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}})$, the Hopf center to $(G_1, G_2) = (\mathrm{triv}, \mathrm{Id}_{\mathcal{M}})$, where $\mathrm{triv} := PF$. The main observation is a natural action of the Drinfeld center on the Hopf center.

For the purposes of this chapter, we introduce a slightly more general version of Majid's braided reconstruction theory in Section 2.2.3, working with quasi-Hopf algebra objects in $\mathcal{B}$ (cf. also [52] in the weak case over $\mathbf{Vect}_k$). We further give a categorical interpretation of the concept of quasitriangularity in Section 2.2.5.

In Section 2.2.4, we consider monoidal categories of the form $\mathcal{M} = B\text{-}\mathbf{Mod}(\mathcal{B})$ for a quasi-bialgebra (or quasi-Hopf algebra) object B in $\mathcal{B}$. For such $\mathcal{M}$, the Drinfeld center can be reformulated as the category of Yetter–Drinfeld modules, while the Hopf center consists of Hopf modules. In the case of the Drinfeld center, this is well-known for Hopf algebras. A version for braided Hopf algebras is due to [90], and a version for quasi-Hopf algebras in $\mathbf{Vect}_k$ can be found in [89]. Working with strict Hopf algebras (trivial 3-cycles) many formulas simplify as summarized in Section 2.3.1.

As preparation for the definition of the braided Drinfeld and Heisenberg double requires working with two dually paired braided Hopf algebra C, B (see Section 2.1.7 for the conventions used). We embed the Drinfeld and Hopf center into larger categories of left C- and right B-modules which satisfy compatibility conditions resembling those of Yetter–Drinfeld[10] (respectively Hopf) modules (see Section

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[10] In the Yetter–Drinfeld module case, this larger category is the category of braided crossed B-C-bimodules used in [90, Appendix B] to interpret the double-bosonization categorically.

2.3.2). We also discuss a reformulation of Majid's concept of weak quasitriangularity in 2.3.3. This concept is needed to obtain the quantum groups as examples of braided Drinfeld doubles as in [90].

To summarize, we give five formulations of the action of the relative Drinfeld center on the relative Hopf center, for C, B dually paired Hopf algebras in $\mathcal{B}$:

$$\begin{tikzcd}
\mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}}) \arrow[r, phantom, "\cong"] & \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \arrow[r, phantom, "\cong"] & {}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B}) \arrow[r, hook] & {}^{C}\mathcal{YD}^{B}(\mathcal{B}) \arrow[r, phantom, "\cong"] & \mathrm{Drin}_{H}(C,B)\text{-}\mathbf{Mod} \\
\mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, {}^{\mathrm{reg}}\mathcal{M}^{\mathrm{triv}}) \arrow[u, hook] \arrow[r, phantom, "\cong"] & \mathcal{H}_{\mathcal{B}}(\mathcal{M}) \arrow[u, hook] \arrow[r, phantom, "\cong"] & {}_{B^{\overline{\mathrm{op}}}}\mathcal{H}^{B}(\mathcal{B}) \arrow[u, hook] \arrow[r, hook] & {}^{C}\mathcal{H}^{B}(\mathcal{B}) \arrow[u, hook] \arrow[r, phantom, "\cong"] & \mathrm{Heis}_{H}(C,B)\text{-}\mathbf{Mod} \arrow[u, hook]
\end{tikzcd}$$

In Section 2.3.9, we consider the example of the quantum groups $U_q(\mathfrak{g})$. In this example, our result will give a categorical action of the category of $U_q(\mathfrak{g})$-modules on the category of $D_q(\mathfrak{g})$-modules, which can be interpreted as a category of quantum differential operators.

We now have a machinery to also define twisted braided Drinfeld and Heisenberg doubles. There exist different concepts of twist in the literature which will appear at different places in this exposition. To provide an overview:

- A (right) 2-cocycle twist of a Hopf algebra is a way of obtaining new algebras H_σ from the datum of a (braided) Hopf algebra H together with a 2-cocycle σ. This is used in 2.3.8 to twist the braided Drinfeld double, giving the braided Heisenberg double.

- The *Drinfeld twist* of a (quasi-)Hopf algebra goes back to [35]. It provides a way to change the monoidal structure of a category of representations over a quasi-bialgebra in an equivalent way by means of conjugation by an element F of $B \otimes B$ (see e.g. [51, 88]). We include this concept in 2.4.2.

- A twisted version of the Drinfeld double of a group algebra can be found in the literature (see e.g. [33, 89, 114]). Underlying this notion of twist is the idea that a commutative bialgebra can be viewed as a quasi-bialgebra with respect to any 3-cycle. From this point of view, twisted versions of commutative Hopf algebras can be introduced in larger generality (see 2.4.2).

Applying the categorical action to the case of the twisted Hopf algebra $k^\omega[G]$ of functions on a group, there is a categorical action of modules over the twisted Drinfeld double $\mathrm{Drin}^\omega(G)$, which is the category of G^{ad}-equivariant ω-twisted vector bundles on the category of ω-twisted G^{reg}-equivariant twisted vector bundles on G. This is the topic of Section 2.4.3 which concludes this chapter.

2.1.3 Hints on Reading this Chapter

The basic structure of this chapter of the thesis is a transgression from category theory (Section 2.2) to representation theory of algebras (Sections 2.3 and 2.4). The link is given by braided reconstruction theory (Section 2.2.3).

The exposition is significantly easier if one works with strict monoidal categories $\mathcal{M}$ (i.e. Hopf algebras via reconstruction theory, using monoidal functors). For the purpose of considering twisted Drinfeld doubles, we include the formulas for the more general case of non-strict monoidal categories (and quasi-Hopf algebras, via quasi-monoidal functors). The readers only interested in the strict case can safely skip to Section 2.3 and when looking up the relevant proofs in Section 2.2 treat associativity and rigidity isomorphisms as identities.

Throughout Section 2.2, we find it most effective to do the proofs using graphical calculus. This however requires to work with a strict monoidal base categories $\mathcal{B}$ (while $\mathcal{M}$ may still be non-strict). We refer to Mac Lane's coherence theorem to justifying giving many proofs on this level. Often, in the proofs the computations are not given in detail. It is an essential standing exercise in reading this chapter to always draw diagrams for all statements and proofs that come up.

If the reader is only interested in the Drinfeld and Heisenberg doubles as (Hopf) algebras, Section 2.3.5 is a good point to start. In this section, concrete examples are provided as well.

2.1.4 Some Notational Conventions

In this chapter, k always denotes a field. The category of finite-dimensional k-vector spaces is denoted by $\mathbf{Vect}_k^{\mathrm{fd}}$, the category of possibly infinite-dimensional k-vector spaces by $\mathbf{Vect}_k$. We denote the symmetric monoidal category of (co)algebras in $\mathbf{Vect}_k$ by $\mathbf{Alg}$ (respectively $\mathbf{CoAlg}$).

More generally, the concepts of algebras $\mathbf{Alg}(\mathcal{M})$, coalgebras $\mathbf{CoAlg}(\mathcal{M})$, and their modules can be defined in any monoidal category $\mathcal{M}$. To illustrate the idea, the multiplication is a morphism $m\colon A\otimes A\to A$ in $\mathcal{M}$. It satisfies associativity and unitarity with respect to $1\colon I\to A$ which is a morphism in $\mathcal{M}$. These properties are commutative squares and can be expressed in $\mathcal{M}$. For more details on this approach see e.g. [88, 9.2.11ff.] or [86]. Given an algebra (or coalgebra) object in a monoidal category $\mathcal{M}$, we denote the category of left A-modules (respectively comodules) in $\mathcal{M}$ by A-$\mathbf{Mod}(\mathcal{M})$ (respectively A-$\mathbf{CoMod}(\mathcal{M})$) and right A-modules by $\mathbf{Mod}$-$A(\mathcal{M})$. If $\mathcal{M}=\mathbf{Vect}_k$, we omit mentioning the category $\mathcal{M}$ and simply write A-$\mathbf{Mod}$ (respectively A-$\mathbf{CoMod}$).

Functors of monoidal categories are always strong quasi-monoidal (sometimes even monoidal), cf. Section 2.2.1. For compositions of morphisms or functors, we write $f\circ g$ simply as fg. In the whole chapter, $\mathcal{B}$ will denote a strict monoidal braided category,[11] with braiding Ψ. The monoidal category $\mathcal{M}$ typically lives over $\mathcal{B}$. That is, there exists a quasi-monoidal functor $\mathcal{M}\to\mathcal{B}$ referred to as the *fiber functor*. We do not require $\mathcal{M}$ to be strict monoidal itself.

2.1.5 Bialgebra and Hopf Algebra Objects

In order to define bialgebras and Hopf algebras, one needs a braided monoidal category $\mathcal{B}$ with braiding Ψ. We will always treat the base category $\mathcal{B}$ as strict monoidal. The categories $\mathbf{Alg}(\mathcal{B})$ and $\mathbf{CoAlg}(\mathcal{B})$ of algebra and coalgebra objects then have a monoidal structure fibered over $\mathcal{B}$. We denote the product on $A\otimes B$ by $m_{A\otimes B}$ for two algebra objects A, B in $\mathcal{B}$. That is, $m_{A\otimes B}=(m_A\otimes m_B)(\mathrm{Id}_A\otimes\Psi_{A,B}\otimes\mathrm{Id}_B)$. Inductively, denote by $m_{B^{\otimes n}}$ the product on $B^{\otimes n}$. Dually, we denote the coalgebra structure on $C\otimes D$ for two coalgebras C, D in $\mathcal{B}$ by $\Delta_{C\otimes D}$. We will occasionally use the notation m^k, for the map $m(m\otimes\mathrm{Id})\ldots(m\otimes\mathrm{Id})\colon B^{\otimes k+1}\to B$ obtained by applying m k times.

In $\mathcal{B}$, we can define bialgebra objects as simultaneous algebras and coalgebras satisfying the bialgebra condition

$$\Delta m=(m\otimes m)(\mathrm{Id}\otimes\Psi\otimes\mathrm{Id})(\Delta\otimes\Delta). \tag{2.1.6}$$

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[11] For an introduction to braided monoidal categories see e.g. [63] or [88, 9.2].

We call a bialgebra (resp. Hopf algebra) object in $\mathcal{B}$ a *braided* bialgebra (or Hopf algebra). In order for $\mathbf{BiAlg}(\mathcal{B})$ to be monoidal, a braiding is not sufficient, but a symmetric monoidal structure is. However, the category B-$\mathbf{Mod}(\mathcal{B})$ is monoidal using the comultiplication. It is important to use this more general definition to study main examples such as the quantum groups (or more generally, Nichols algebras) later. It is also important that we do not restrict ourselves to finite-dimensional Hopf algebras over k.

2.1.6 Bialgebras vs. Quasi-Bialgebras

Let B be a bialgebra in $\mathcal{B}$. Then the category B-$\mathbf{Mod}(\mathcal{B})$ is strict monoidal with fiber functor over $\mathbf{Vect}_k$. That is, the underlying morphisms in $\mathcal{B}$ of the associativity transformation $\alpha\colon \otimes(\otimes \times \mathrm{Id}) \Longrightarrow \otimes(\mathrm{Id} \times \otimes)$ are identity morphisms. In some cases, one requires a higher level of generality (for example, when working with twists of Hopf algebras as in [37]) and wants to drop the assumption of $\mathcal{M}$ being strict over $\mathcal{B}$, and use the weaker notion of a quasi-monoidal functor. The natural notion arising via reconstruction theory (see 2.2.3) is that of a *quasi*-bialgebra. Following [89], we require that there exists an invertible element $\phi \in B \otimes B \otimes B$ (the *coassociator*) such that

$$m_{B\otimes B\otimes B}(\mathrm{Id}_B \otimes \Delta \otimes \phi)\Delta = m_{B\otimes B\otimes B}(\phi \otimes \Delta \otimes \mathrm{Id}_B)\Delta. \tag{2.1.7}$$

Such an element ϕ needs to satisfy the 3-cycle condition of a non-abelian homology theory (see e.g. [87, Section 6], [88, 2.3]):

$$\begin{aligned} &m_{B\otimes B\otimes B}(m_{B\otimes B\otimes B} \otimes \mathrm{Id}_{B\otimes B\otimes B})(1 \otimes \phi \otimes \mathrm{Id}_B \otimes \Delta \otimes \mathrm{Id}_B \otimes \phi \otimes 1)\phi \\ &= m_{B\otimes B\otimes B}(\mathrm{Id}_{B\otimes B} \otimes \Delta \otimes \Delta \otimes \mathrm{Id}_{B\otimes B})(\phi \otimes \phi) \end{aligned} \tag{2.1.8}$$

The counitary property still holds as in the bialgebra case, given that $(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id})\phi = 1 \otimes 1$. For a quasi-bialgebra B, the categories B-$\mathbf{Mod}(\mathcal{B})$ and $\mathbf{Mod}$-$B(\mathcal{B})$ are monoidal.

If B is a (quasi-)Hopf algebra object in $\mathcal{B}$, then B-$\mathbf{Mod}(\mathcal{B})$ is rigid given that $\mathcal{B}$ is rigid. That is, left dual objects exist[12] and are denoted by V^* for $V \in \mathcal{B}$. That is for example the case if $\mathcal{B} = \mathbf{Vect}_k^{\mathrm{fd}}$. For infinite-dimensional modules V,

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[12] See e.g. [88, Section 9.3].

we can still give the finite dual $V^\circ = \{\delta_v \mid v \in V\}$ a module structure, but there is no coevaluation map. We observe that left dual objects are unique up to canonical isomorphism.

When working with quasi-Hopf algebras, the antipode axioms are valid only up to elements $a, b \in H$ (cf. e.g. [66, XV.5], or Section 2.2.3 in the braided setting):

$$m^2(S \otimes a \otimes \mathrm{Id})\Delta = a\epsilon, \text{ and } m^2(\mathrm{Id} \otimes b \otimes S)\Delta = b\epsilon. \tag{2.1.9}$$

This requires the compatibility conditions

$$m^5(\mathrm{Id} \otimes b \otimes S \otimes a \otimes \mathrm{Id})\phi = 1, \text{ and } m^5(S \otimes a \otimes \mathrm{Id} \otimes b \otimes S)\phi^{-1} = 1, \tag{2.1.10}$$

with the coassociators. The formulas may be more clear when drawn as diagrams (using graphical calculus) or using generalized Sweedler's notation.[13] The notion of a quasitriangular quasi-Hopf algebra is also spelled out in [88, 2.4]. We also include a version for braided quasi-Hopf algebras in 2.2.3.

2.1.7 Dually Paired Hopf Algebras

Let $\mathcal{B}$ be a braided monoidal category with braiding Ψ (recall $\mathcal{B}$ is always treated as strict monoidal in this chapter). In this section, we want to discuss what notion of dually paired Hopf algebras is suitable for our purposes in the remainder of the chapter. Unlike working in the category of finite-dimensional vector spaces $\mathbf{Vect}_k$, a dual may not necessarily exist in this more general setting. We assume that C, B are braided Hopf algebras in $\mathcal{B}$ with a pairing, in the sense that there exists an evaluation map $\mathrm{ev}\colon C \otimes B \to I$ to the unit I in $\mathcal{B}$ compatible with the structure. This displays C as the left (categorical) dual of B. That is, using graphical calculus[14] the conditions from Figure 2.1.1 hold. If C, B are Hopf algebras, then we further assume that the antipodes are invertible and the duality $\mathrm{ev}(S \otimes \mathrm{Id}) = \mathrm{ev}(\mathrm{Id} \otimes S)$ holds.

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[13] We will follow [89, Preliminaries]) for conventions about Sweedler's notation (from [110]). We will use these conventions, including the Einstein sum convention from Section 2.3 onward.

[14] We follow similar conventions to [88, Chapter 9] about graphical calculus of Hopf algebra objects in $\mathcal{B}$. The drawings are created using inkscape.

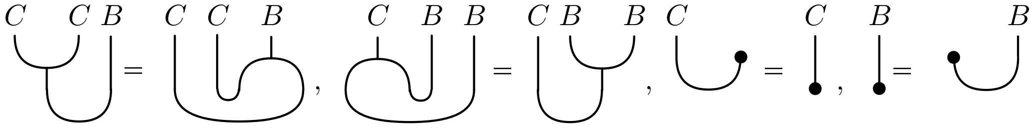


Figure 2.1.1: Dually paired Hopf algebras

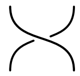
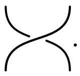
Remark 2.1.7.1. Note that we do not restrict ourselves to treating finite-dimensional Hopf algebras here. In particular, a coevaluation map $\mathrm{coev}\colon I \to B \otimes C$ may not exist as a morphism in $\mathcal{B}$. In later applications, we use infinite-dimensional Hopf algebras and their restricted duals, where the coevaluation map exists as a formal power series rather than a linear map given that the pairing is perfect and dual bases exist (2.3.6). The restricted dual of a Hopf algebra H is denoted by H° and consists of those functions that vanish on a (two-sided) ideal of H of finite codimension. This is not necessarily equal to the finite dual of an infinite-dimensional vector space and the pairing is not necessarily perfect.

Remark 2.1.7.2. The situation for quasi-bialgebras is asymmetric. The dual of a quasi-bialgebra has a multiplication which is not strictly associative.

Let us denote the braided category $\mathcal{B}$ with *inverse* braiding Ψ^{-1} by $\overline{\mathcal{B}}$. In $\mathcal{B}$, the categories B-$\mathbf{Mod}(\mathcal{B})$, B-$\mathbf{CoMod}(\mathcal{B})$ of left (co)modules (and their right versions) are monoidal. Given a dual pair B, C as above, we further observe that there exist monoidal functors

$$ {}_B\Phi\colon B\text{-}\mathbf{CoMod}(\mathcal{B}) \to {}^{\mathrm{cop}}C\text{-}\mathbf{Mod}(\overline{\mathcal{B}}), \qquad \Phi_C\colon \mathbf{CoMod}\text{-}C(\mathcal{B}) \to \mathbf{Mod}\text{-}{}^{\mathrm{cop}}B(\overline{\mathcal{B}}). $$

Here, ${}^{\mathrm{cop}}C$ denotes C with co-opposite coproduct $\Psi^{-1}\Delta$. This is a bialgebra (resp. Hopf algebra) in the braided monoidal category $\overline{\mathcal{B}}$ (with antipode S^{-1}). The functor ${}_B\Phi$ maps a comodule with coaction $\delta\colon V \to B \otimes V$ to V with action $(\mathrm{ev} \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)$, and Φ_C is defined analogously. We find it helpful to check such statements using graphical calculus, in which the braiding Ψ and its inverse Ψ^{-1} are denoted by

Ψ =  and Ψ^{-1} = .

Note that if B is finite-dimensional and the pairing ev is perfect, the functors ${}_B\Phi$ and Φ_C are part of equivalences of categories. In general, this is not the case.

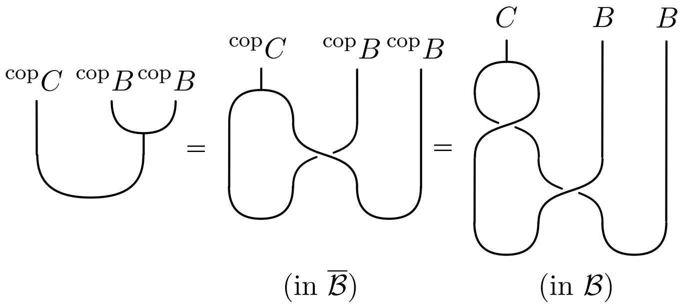


Figure 2.1.2: Pairing of co-opposite Hopf algebras

If we restrict to situations in which the braided monoidal category $\mathcal{B}$ admits a monoidal functor $\mathcal{B} \to \mathbf{Vect}_k$ to vector spaces over a field k, then we can talk about the pairing ev being perfect. If that is the case, the above functors will be fully faithful. In the general situation we *define* the pairing to be perfect if the functors ${}_B\Phi$ and Φ_C are fully faithful.

Remark 2.1.7.3. The bialgebras ${}^{\mathrm{cop}}C$ and ${}^{\mathrm{cop}}B$ are dually paired in a different way in $\mathcal{B}$ (see Figure 2.1.2). Here, it is important to distinguish whether we express a functional identity in $\mathcal{B}$ or in $\overline{\mathcal{B}}$. In this example, the second term is expressed using symbols in $\overline{\mathcal{B}}$, while the third term is the same expression written in $\mathcal{B}$.

2.2 The Categorical Picture

In this section, we introduce the two main categories of interest in this chapter in purely categorical terms. The first one, the *Drinfeld center* $\mathcal{Z}(\mathcal{M})$ of a monoidal category, is well known. The other one, the *Hopf center* $\mathcal{H}(\mathcal{M})$ is well-known in the case where $\mathcal{M} = H$-**Mod** is the category of modules over an ordinary Hopf algebra where it can be described as the category of Hopf modules over H. For the more general case $\mathcal{M} = H$-**Mod**$(\mathcal{B})$ where $\mathcal{B}$ is a braided monoidal category and H a bialgebra object in it, see e.g. [22]. We present a new description of $\mathcal{H}(\mathcal{M})$ as a special case of a *mixed* Drinfeld center construction.

Applying techniques from reconstruction theory, which is a generalization of Tannaka-duality, one can recover certain categories $\mathcal{C} \to \mathcal{V}$, where $\mathcal{V}$ is monoidal, as module (or comodule) categories H-**Mod**$(\mathcal{V}) \to \mathcal{V}$ in the category $\mathcal{V}$. This is used later to describe the categories $\mathcal{Z}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ in the case where $\mathcal{M} = \mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod})$ for a bialgebra (or Hopf algebra) B in the braided monoidal category H-**Mod** as module categories leading to the definition of the *braided* Drinfeld and Heisenberg doubles. For this, we need another description of the categories $\mathcal{Z}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ for $\mathcal{M} = B\text{-}\mathbf{Mod}(\mathcal{B})$ in terms of simultaneous modules and comodules over a quasi-bialgebra B satisfying certain compatibility conditions, leading to Yetter–Drinfeld and Hopf modules in 2.2.4. Finally, we give a categorical explanation of different concepts of quasitriangularity of a braided Hopf algebra in 2.2.5.

2.2.1 Bimodules over Monoidal Categories

In this section, we discuss the 2-category of $\mathcal{M}$-bimodules over a monoidal category. This will later serve us to define relative Drinfeld and Heisenberg centers in 2.2.2. One application is a natural strictification of a monoidal category (see Corollary 2.2.1.6).

We now work in the meta-2-category of categories **Cat** under suitable locally smallness assumptions. This category is symmetric monoidal with respect to the Cartesian product of categories denoted by ×. Let $(\mathcal{M}, \otimes)$ be a monoidal category (not necessarily strict, but strictly unital with unit object I). Such categories can be thought of as unitary monoid objects in **Cat**. We denote the associativity isomorphism by $\alpha\colon \otimes(\otimes \times \mathrm{Id}) \Longrightarrow \otimes(\mathrm{Id} \times \otimes)$.

Recall that a *braided* monoidal category $\mathcal{V}$ is a monoidal category with a natural isomorphism $\Psi\colon \otimes \to \otimes^{\mathrm{op}}$ which satisfies the $\otimes$-compatibilities

$$\Psi_{V\otimes W,X} = \alpha_{X,V,W}(\Psi_{V,X} \otimes \mathrm{Id}_W)\alpha^{-1}_{V,X,W}(\mathrm{Id}_V \otimes \Psi_{W,X})\alpha_{V,W,X}, \tag{2.2.1}$$

$$\Psi_{V,W\otimes X} = \alpha^{-1}_{W,X,V}(\mathrm{Id}_W \otimes \Psi_{V,X})\alpha_{W,V,X}(\Psi_{V,W} \otimes \mathrm{Id}_X)\alpha^{-1}_{V,W,X}. \tag{2.2.2}$$

Let $\mathcal{M}$ and $\mathcal{V}$ be monoidal categories. We further recall that a *quasi-monoidal* functor[15] $G\colon \mathcal{M} \to \mathcal{V}$ is a functor such that there exists a natural isomorphism

$$\mu = \mu_G\colon G\otimes \overset{\sim}{\Longrightarrow} \otimes(G \times G).$$

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[15] Another term used in the literature is that of a *multiplicative* functor in [84, 88], or quasi-tensor functor, see e.g. [26, 42].

A *monoidal* functor[16] is, in addition, compatible with the associativity isomorphisms of $\mathcal{M}$ and $\mathcal{V}$, i.e. for any three objects X, Y and Z in $\mathcal{M}$ we have

$$\alpha_{G(X),G(Y),G(Z)}(\mu_{X,Y} \times \mathrm{Id}_{G(Z)})\mu_{X\otimes Y,Z} = (\mathrm{Id}_{G(X)} \times \mu_{Y,Z})\mu_{X,Y\otimes Z}G(\alpha_{X,Y,Z}). \tag{2.2.3}$$

Similarly to defining modules of unitary monoids (or rings), we can consider modules over monoidal categories. A *left module over* $\mathcal{M}$ is a category $\mathcal{V}$ with a functor $\rhd\colon \mathcal{M} \times \mathcal{V} \to \mathcal{V}$, as well as a natural isomorphism

$$\chi\colon \; \rhd(\mathrm{Id}_{\mathcal{M}} \times \rhd) \overset{\sim}{\Longrightarrow} \rhd(\otimes \times \mathrm{Id}_{\mathcal{V}}),$$

satisfying the following compatibility of χ with the associativity isomorphism α: for all objects X, Y, Z of $\mathcal{M}$, and V of $\mathcal{V}$,

$$\chi_{X,Y\otimes Z,V}(X \rhd \chi_{Y,Z,V}) = (\alpha_{X,Y,Z} \rhd V)\chi_{X\otimes Y,Z,V}\chi_{X,Y,Z\rhd V}. \tag{2.2.4}$$

For simplification of notation, we assume that $\rhd(I \times \mathrm{Id}_{\mathcal{V}}) = \mathrm{Id}_{\mathcal{V}}$. This notion compares to the definition used e.g. in [97, 2.3]. However, we consider modules to be strictly unital, i.e. suppress the unitarity isomorphism and do not require exactness properties (but will later work in a k-linear setting). A morphism of left modules over $\mathcal{M}$ is a functor $F\colon \mathcal{V} \to \mathcal{W}$, together with a natural isomorphism

$$\lambda_F\colon F\rhd \overset{\sim}{\Longrightarrow} \rhd(\mathrm{Id}_{\mathcal{M}} \times F).$$

We require that the following coherence between $\lambda = \lambda_F$ and $\chi^{\mathcal{V}}$, $\chi^{\mathcal{W}}$ is satisfied:

$$\lambda_{M\otimes N,V}F(\chi^{\mathcal{V}}_{M,N,V}) = \chi^{\mathcal{W}}_{M,N,F(V)}(M \rhd \lambda_{N,V})\lambda_{M,N\rhd V}, \quad M, N \in \mathcal{M}, V \in \mathcal{V}. \tag{2.2.5}$$

We denote the category of left modules over $\mathcal{M}$ by $\mathbf{Mod}_{\mathcal{M}}$. This category is in fact a 2-category, where 2-morphisms are given by natural transformations $F \Rightarrow G$ which commute λ_F, λ_G. Analogously, we define right $\mathcal{M}$-modules.

In the following, we will mainly consider bimodule categories over $\mathcal{M}$. A *bimodule* over $\mathcal{M}$ is a category $\mathcal{V}$ with a left and right $\mathcal{M}$-module structure which commute up to a coherent natural isomorphism. We usually denote the left action

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[16] Our natural notion of functors (or equivalences) of monoidal categories is that of monoidal functors. We will allow the fiber functor to be quasi-monoidal in view of working with quasi-bialgebras in Section 2.2.3, 2.2.4, and 2.4.

by $\triangleright\colon \mathcal{M}\times\mathcal{V}\to\mathcal{V}$ and the right action by $\triangleleft\colon \mathcal{V}\times\mathcal{M}\to\mathcal{V}$. The action coherences are natural isomorphisms

$$\begin{aligned}\chi\colon\ \triangleright(\mathrm{Id}_{\mathcal{M}}\times\triangleright) &\overset{\sim}{\Longrightarrow} \triangleright(\otimes\times\mathrm{Id}_{\mathcal{V}}), &\qquad \triangleright(I\times\mathrm{Id}_{\mathcal{V}}) &= \mathrm{Id}_{\mathcal{V}},\\ \xi\colon\ \triangleleft(\triangleleft\times\mathrm{Id}_{\mathcal{M}}) &\overset{\sim}{\Longrightarrow} \triangleleft(\mathrm{Id}_{\mathcal{V}}\times\otimes), &\qquad \triangleleft(\mathrm{Id}_{\mathcal{V}}\times I) &= \mathrm{Id}_{\mathcal{V}}\,.\end{aligned}$$

Again, the module structures are considered to be strictly unital to simplify the exposition. The coherence commuting the left and right action is a natural isomorphism

$$\zeta\colon\ \triangleleft(\triangleright\times\mathrm{Id}_{\mathcal{M}}) \overset{\sim}{\Longrightarrow} \triangleright(\mathrm{Id}_{\mathcal{M}}\times\triangleleft).$$

We require compatibilities between the coherences χ,ξ and ζ. The idea is that whenever two combinations of the functors $\triangleright$ and $\triangleleft$ can be transformed into one another using different combinations of the transformations χ,ξ and ζ, then these different combinations have to be equal. In view of Mac Lane's coherence theorem [77, VII.2] which shows that elementary coherence axioms of minimal tensor order are sufficient to explain all coherences, we require the compatibilities

$$\zeta_{M\otimes N,V,P}(\chi_{M,N,V}\triangleleft P) = \chi_{M,N,V\triangleleft P}(M\triangleright\zeta_{N,V,P})\zeta_{M,N\triangleright V,P}, \tag{2.2.6}$$

$$\zeta^{-1}_{M,V,N\otimes P}(M\triangleright\xi_{V,N,P}) = \xi_{M\triangleright V,N,P}(\zeta^{-1}_{M,V,N}\triangleleft P)\zeta^{-1}_{M,V\triangleleft N,P}. \tag{2.2.7}$$

In addition, the condition (2.2.4), and the analogous compatibility between α and ξ, and hence all coherences between α, χ, ξ, and ζ are required to be valid.

Morphisms of $\mathcal{M}$-bimodules are assumed to commute with the left and right actions up to coherent natural isomorphism. That is, a functor $F\colon\mathcal{V}\to\mathcal{W}$ is a *morphism of bimodules* if there exist natural isomorphisms

$$\rho_F\colon F\triangleleft \overset{\sim}{\Longrightarrow} \triangleleft(F\times\mathrm{Id}_{\mathcal{M}}), \qquad \lambda_F\colon F\triangleright \overset{\sim}{\Longrightarrow} \triangleright(\mathrm{Id}_{\mathcal{M}}\times F).$$

We often just write ρ, λ if only one morphism F is considered. These again have to be compatible with the natural isomorphisms χ,ξ and ζ. That is, the condition (2.2.5), the analogue for the compatibility of $\rho=\rho_F$ and $\xi^{\mathcal{V}}$, $\xi^{\mathcal{W}}$, as well as the following compatibility of ρ, λ, and $\zeta^{\mathcal{V}}$, $\zeta^{\mathcal{W}}$ are assumed:

$$\zeta^{\mathcal{W}}_{M,F(V)\triangleleft N}(\lambda_{M,V}\triangleleft N)\rho_{M\triangleright V,N} = (M\triangleright\rho_{V,N})\lambda_{M,V\triangleleft N}F(\zeta^{\mathcal{V}}_{M,V,N}). \tag{2.2.8}$$

We denote the category of bimodules over $\mathcal{M}$ by $\mathbf{BiMod}_{\mathcal{M}}$.

The category $\mathbf{BiMod}_{\mathcal{M}}$ in fact has the structure of a 2-category. 2-morphisms are natural transformations $\tau\colon F \Rightarrow G$ of module morphisms which commute with the bimodule coherences, i.e. $\rho_G\tau_{\triangleleft} = (\triangleleft(\tau \times \mathrm{Id}_{\mathrm{Id}_{\mathcal{M}}}))\rho_F\colon F\triangleleft \Rightarrow \triangleleft(G \times \mathrm{Id}_{\mathcal{M}})$ and $\lambda_G\tau_{\triangleright} = (\triangleright(\mathrm{Id}_{\mathrm{Id}_{\mathcal{M}}} \times \tau))\lambda_F\colon F\triangleright \Rightarrow \triangleright(\mathrm{Id}_{\mathcal{M}} \times G)$. It is helpful to write these conditions as commutative diagrams for any object $(X, M) \in \mathcal{V} \times \mathcal{M}$:

$$\begin{tikzcd}
F(X \triangleleft M) \arrow[r, "\tau_{X\triangleleft M}"] \arrow[d, "(\rho_F)_{X,M}"'] & G(X \triangleleft M) \arrow[d, "(\rho_G)_{X,M}"'] \\
F(X) \triangleleft M \arrow[r, "\triangleleft(\tau_X \times \mathrm{Id}_M)"] & G(X) \triangleleft M,
\end{tikzcd}
\qquad
\begin{tikzcd}
F(M \triangleright X) \arrow[r, "\tau_{M\triangleright X}"] \arrow[d, "(\lambda_F)_{M,X}"'] & G(M \triangleright X) \arrow[d, "(\lambda_G)_{M,X}"'] \\
M \triangleright F(X) \arrow[r, "\triangleright(\mathrm{Id}_M \times \tau_X)"] & M \triangleright G(X).
\end{tikzcd}$$

Example 2.2.1.1.

(i) The regular $\mathcal{M}$-bimodule $\mathcal{M}^{\mathrm{reg}}$ is defined using $\otimes\colon \mathcal{M}\otimes\mathcal{M} \to \mathcal{M}$ as module structure (both left and right), and $\xi = \alpha$, $\chi = \alpha^{-1}$, $\zeta = \alpha$ as structure maps.

(ii) The trivial $\mathcal{M}$-bimodule on $\mathcal{M}$ is given by $X \triangleright Y = Y$ and $X \triangleleft Y = Y$, and trivial action on morphisms too. We say that this bimodule is obtained by pulling the regular bimodule structure back along the functor $I\colon \mathcal{M} \to \mathcal{M}$ (factoring through the terminal and initial monoidal category $\mathcal{I}$ with one element and morphism). Here, $\xi = \chi = \zeta = \mathrm{Id}$.

(iii) More generally, for any pair of monoidal functors $G_1, G_2\colon \mathcal{M} \to \mathcal{V}$, we can give $\mathcal{V}$ an $\mathcal{M}$-bimodule structure ${}^{G_1}\mathcal{V}^{G_2}$ where the left action is induced by pulling the regular bimodule structure on $\mathcal{V}$ back along G_1, i.e. $X \triangleright V = G_1(X) \otimes V$, and the right action is induced by G_2 in the same way. For $\mu_i\colon G_i(\otimes) \Rightarrow \otimes(G_i \times G_i)$, we have

$$\chi := (\mu_{G_1}^{-1} \otimes \mathrm{Id})\alpha^{\mathcal{V}}, \qquad \xi : = (\mathrm{Id} \otimes \mu_{G_2}^{-1})(\alpha^{\mathcal{V}})^{-1}, \qquad \zeta := \alpha^{\mathcal{V}}.$$

Lemma 2.2.1.2. *Let $\mathcal{V}$ be an $\mathcal{M}$-bimodule. Then $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{V}, \mathcal{V})$ is a strict monoidal category via composition of functors and composition of structure maps, i.e.*

$$\rho^{\psi\phi}_{A,B} = \rho^{\psi}_{\phi(A),B}\psi(\rho^{\phi}_{A,B}), \qquad \lambda^{\psi\phi}_{A,B} = \lambda^{\psi}_{\phi(A),B}\psi(\lambda^{\phi}_{A,B}).$$

Proof. It is evident that the composition of bimodule morphisms and structure maps is strictly associative. It remains to check that the defined composite transformations $\lambda^{\psi\phi}$, $\rho^{\psi\phi}$ are compatible with χ, ξ, and ζ. This follows by combining the compatibilities of λ^{ϕ}, λ^{ψ}, ρ^{ϕ}, ρ^{ψ} with appropriate naturality conditions of these transformations. □

Lemma 2.2.1.3. *The category* $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ *is braided monoidal with braiding* $\Psi_{\psi,\phi}$ *given for an object* A *of* $\mathcal{M}$ *by*

$$\psi\phi(A) = \psi\phi(A \otimes I) \xrightarrow{\psi(\lambda^{\phi}_{A,I})} \psi(A \otimes \phi(I)) \xrightarrow{\rho^{\psi}_{A,\phi(I)}} \psi(A) \otimes \phi(I) \xrightarrow{\lambda^{\phi}_{\psi(A),I}} \phi(\psi(A) \otimes I) = \phi\psi(A).$$

Lemma 2.2.1.4. *Consider an* $\mathcal{M}$*-bimodule structure on* $\mathcal{M}$ *itself where either the left or the right action is given by the regular action and denote this bimodule by* $\mathcal{M}'$*. Then the functor*

$$F\colon \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}', \mathcal{M}') \to \mathcal{M}, \quad \phi \mapsto \phi(I)$$

is monoidal.

Proof. Assume that the left $\mathcal{M}$-action of $\mathcal{M}'$ is regular. Then for two morphisms of bimodules $\phi, \psi\colon \mathcal{M}' \to \mathcal{M}'$,

$$\mu_{\phi,\psi} : \psi\phi(I) = \psi(\phi(I) \otimes I) = \psi(\phi(I) \rhd I) \xrightarrow{\lambda^{\psi}_{\phi(I),I}} \phi(I) \rhd \psi(I) = \phi(I) \otimes \psi(I)$$

is an isomorphism. To check F is monoidal, we need to verify (2.2.3). This follows using the compatibility of the transformations λ of the functors with $\chi^{\mathrm{reg}} = \alpha^{-1}$. That is, with the above definition of μ^F, we can identify the equation

$$\alpha_{F(\tau),F(\phi),F(\psi)}(\mu_{\tau,\phi} \otimes F(\psi))\mu_{\phi\tau,\psi} = (F(\tau) \otimes \mu_{\phi,\psi})\mu_{\tau,\psi\phi}, \tag{2.2.9}$$

with

$$(\chi^{\mathrm{reg}}_{\tau(I),\phi(I),\psi(I)})^{-1}(\lambda^{\phi}_{\tau(I),I} \rhd \psi(I))\lambda^{\psi}_{\phi\tau(I),I} = (\tau(I) \rhd \lambda^{\psi}_{\phi(I),I})\lambda^{\psi\phi}_{\tau(I),I}. \tag{2.2.10}$$

The commutativity of this square follows from compatibility of χ, λ^{ψ} (applied to the objects $\tau(I)$, $\phi(I)$, I); combined with naturality of λ^{ψ} in the first component, applied to the morphism $\lambda^{\phi}_{\tau(I),I}$. □

Lemma 2.2.1.5. *For any monoidal category where I is a terminal object, the functor F from Lemma 2.2.1.4 is part of an equivalence of categories*

$$\mathbf{BiMod}_{\mathcal{M}}({}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}, {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}) \cong \mathcal{M}.$$

Proof. We define the inverse $\mathrm{Ind}\colon \mathcal{M} \to \mathbf{BiMod}_{\mathcal{M}}({}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}, {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I})$ by mapping an object X to the functor $\mathrm{Ind}(X)$ given by

$$Y \mapsto \mathrm{Ind}(X)(Y) := Y \otimes X = Y \rhd X.$$

For morphisms, we use the transformation $\mathrm{Ind}(F)_X := \mathrm{Id}_X \otimes f$. The structure transformations are

$$\begin{aligned} \lambda^{\mathrm{Ind}(X)}_{A,B} &:= \chi^{-1}_{A,B,X}\colon \mathrm{Ind}(X)(A \rhd B) \to A \rhd \mathrm{Ind}(X)(B), \\ \rho^{\mathrm{Ind}(X)}_{A,B} &:= \mathrm{Id}_{A \otimes X}\colon \mathrm{Ind}(X)(A \lhd B) \to \mathrm{Ind}(X)(A) \lhd B. \end{aligned}$$

In order to show that the two constructions are mutually inverse to each other, we first show that for any $\phi\colon {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I} \to {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}$, $\rho_{X,Y} = \mathrm{Id}_{\phi(X)}$. For this, consider the square

$$\begin{tikzcd}
\phi((X \lhd Y) \lhd X) \arrow[r, "\rho_{X \lhd Y,Z}"] \arrow[d, "\mathrm{Id}"'] & \phi(X \lhd Y) \lhd Z \arrow[r, "\rho_{X,Y \lhd Z}"] & (\phi(X) \lhd Y) \lhd Z \arrow[d, "\mathrm{Id}"'] \\
\phi(X \lhd (Y \otimes Z)) \arrow[rr, "\rho_{X,Y \otimes Z}"] & & \phi(X) \lhd (Y \otimes Z).
\end{tikzcd}$$

This square commutes by the coherence between ρ and ξ (which is Id in this case). Thus we obtain that

$$\rho_{X,Z}\rho_{X,Y} = \rho_{X \lhd Y,Z}\rho_{X,Y \lhd Z} = \rho_{X,Y \otimes Z}. \tag{2.2.11}$$

Hence, as these are isomorphisms, $\rho_{X,I} = \mathrm{Id}_{\phi(X)}$. Further, using the naturality square of ρ in the second component, we obtain that $\rho_{X,Y} = \rho_{X,Y'}$ whenever there exists a morphism $f\colon Y \to Y'$. Under the assumption that I is terminal, we always have an isomorphism $X \to I$ and hence $\lambda = \mathrm{Id}$.

Returning to the proof of the equivalence, it is clear that $F\,\mathrm{Ind} = \mathrm{Id}_{\mathcal{M}}$. We claim that $\lambda_{-,I}\colon \mathrm{Id} \Longrightarrow \mathrm{Ind}\,F$ is a natural isomorphism. As $\lambda = \mathrm{Id}$, it remains to check that the square

$$\begin{tikzcd}
\phi(X \rhd Y) \arrow[r, "\lambda_{X,Y}"] \arrow[d, "\lambda_{X\otimes Y,I}"'] & X \rhd \phi(Y) \arrow[d, "X\otimes\lambda_{Y,I}"'] \\
\begin{array}{l}\mathrm{Ind}(\phi(I))(X \rhd Y)\\ = (X\otimes Y) \rhd \phi(I)\end{array} \arrow[r, "\chi^{-1}_{X,Y,\phi(I)}"] & \begin{array}{l}X \rhd \mathrm{Ind}(\phi(I))(Y)\\ = X \rhd (Y \rhd \phi(I))\end{array}
\end{tikzcd}$$

commutes. But this follows from the coherence of χ and λ. □

It is not strictly necessary to assume that I is terminal. It is sufficient that the graph of the category (objects as vertices, morphisms as edges) is connected. If, for example, $\mathcal{M} = B\text{-}\mathbf{Mod}(\mathcal{B})$ for B a quasi-Hopf algebra object in $\mathcal{B}$, then $\mathcal{M}$ has I as terminal object provided that B does. If $\mathcal{M}$ is additive, then the graph of the category is connected (via the zero morphisms). Note that

$$\mathbf{BiMod}_{\mathcal{M}}({}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}, {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}) \cong \mathbf{Mod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}}) \cong \mathcal{M}$$

by the Lemma. The result can be interpreted as the following strictification:

Corollary 2.2.1.6. *Any monoidal category $\mathcal{M}$ with connected graph is equivalent to a strict monoidal category.*

Proof. We use the equivalence of Lemma 2.2.1.5. By Lemma 2.2.1.2, the monoidal category $\mathbf{BiMod}_{\mathcal{M}}({}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I}, {}^{\mathrm{Id}_{\mathcal{M}}}\mathcal{M}^{I})$ is strict. □

Example 2.2.1.7. If $\mathcal{M}$ is a monoidal category with connected graph, then

$$\mathbf{BiMod}_{\mathcal{M}}({}^{I}\mathcal{M}^{I}, {}^{I}\mathcal{M}^{I}) \cong \mathbf{Fun}(\mathcal{M}, \mathcal{M}).$$

This can be seen using the observation in the proof of Lemma 2.2.1.5 showing that under the given assumption on $\mathcal{M}$, $\rho = \lambda = \mathrm{Id}$. Hence any functor is a bimodule morphism for the trivial bimodule.

Let us fix an $\mathcal{M}$-bimodules $\mathcal{V}$ and consider the category of bimodule morphisms $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{V})$. To represent the data of a morphism $G\colon \mathcal{M}^{\mathrm{reg}} \to \mathcal{V}$ in a more compact way, denote the image $G(I)$ of the $\otimes$-unit I by V. Then it is sufficient

— by Proposition 2.2.1.8 below — to consider the composite coherence (called *centralizing isomorphism*)

$$c_X := V \lhd X \overset{\rho^{-1}_{I,X}}{\to} G(I \otimes X) = G(X) = G(X \otimes I) \overset{\lambda_{X,I}}{\to} X \rhd V,$$

for any object X of $\mathcal{M}$. The natural isomorphism obtained this way will be denoted by $c\colon V \lhd \mathrm{Id}_{\mathcal{M}} \Longrightarrow \mathrm{Id}_{\mathcal{M}} \rhd V$. Note that (V, c) is *monoidal* in the sense that

$$c_{X\otimes Y} = \chi_{X,Y}(X \rhd c_Y)\zeta_{X,V,Y}(c_X \lhd Y)\xi^{-1}_{X,Y}. \tag{2.2.12}$$

A morphism of $\mathcal{M}$-bimodules, $\vartheta\colon (V, c_V) \to (W, c_W)$ gives a morphism $\vartheta\colon V \to W$ satisfying that the square

$$\begin{tikzcd}
V \lhd X \arrow[r, "\vartheta\otimes\mathrm{Id}"] \arrow[d, "c_{V,X}"'] & W \lhd X \arrow[d, "c_{W,X}"'] \\
X \rhd V \arrow[r, "\mathrm{Id}\otimes\vartheta"] & X \rhd W
\end{tikzcd}$$

commutes for any object X of $\mathcal{M}$.

We denote the set of such natural isomorphisms $c\colon V \lhd \mathrm{Id}_{\mathcal{M}} \Longrightarrow \mathrm{Id}_{\mathcal{M}} \rhd V$ obeying the $\otimes$-compatibility (2.2.12) by $\mathrm{Isom}^{\otimes}(V \lhd \mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}} \rhd V)$. We use the notation

$$\mathrm{Isom}^{\otimes}(\mathcal{V} \lhd \mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}} \rhd \mathcal{V}) \tag{2.2.13}$$

to denote the category of pairs (V, c), where V varies over the objects $V \in \mathcal{V}$, introduced above.

Proposition 2.2.1.8. *There is an equivalence of categories*

$$\mathrm{Isom}^{\otimes}(\mathcal{V} \lhd \mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}} \rhd \mathcal{V}) \cong \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{reg}, \mathcal{V}).$$

Proof. First, we show that given a morphism of $\mathcal{M}$-bimodules $G\colon \mathcal{M}^{\mathrm{reg}} \to \mathcal{V}$, we can recover the data of G, ρ_G and λ_G from the pair (V, c). We can set $G' := X \rhd V$. Then $G' \cong G$ via $\rho^{-1}_{X,I}$. We can recover $\lambda_{X,Y}$ as

$$\chi^{-1}_{X,Y,V}\colon G'(X \otimes Y) = (X \otimes Y) \rhd V \to X \rhd (Y \rhd V) = X \rhd G'(X).$$

The natural isomorphism $\rho_{X,Y}$ can be recovered from (V, c) by considering the composite $\zeta^{-1}_{X,Y}(X \rhd c_Y)\chi^{-1}_{X,Y}$. The condition (2.2.4) implies that $\lambda^{G'}$ is compatible with the recovered χ.

One checks that, given any pair (V, c) as above, the procedure described in the first part gives an $\mathcal{M}$-bimodule morphism $G' = G_{(V,c)}$. Clearly, $G_{(V,c)}(I) = V$ and if we apply the above procedure to define the centralizing isomorphism, we recover c. The requirement of (V, c) being monoidal implies the required compatibilities of ρ, λ and ζ. $\square$

Remark 2.2.1.9. If $\mathcal{B}$ and $\mathcal{M}$ are additive, abelian or k-linear, then these properties are inherited by the categories $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{V}, \mathcal{W})$.

In the next section, bimodule categories will be used to define different kinds of centers of monoidal categories.

2.2.2 Mixed Relative Drinfeld Centers

The Drinfeld center is a canonical way of associating a braided monoidal category to a monoidal category $\mathcal{M}$. It can be defined, using the work of the previous subsection, via morphism categories of $\mathcal{M}$-bimodules.

Definition 2.2.2.1. Let $G\colon \mathcal{M} \to \mathcal{V}$ be a monoidal functor. Then $\mathcal{V}$ is an $\mathcal{M}$-bimodule using the functor G, i.e. for objects $M \in \mathcal{M}$, $V \in \mathcal{V}$, we have $M \rhd V := G(M) \otimes V$. The right action is defined analogously and the resulting $\mathcal{M}$-bimodule is denoted by $\mathcal{V}^G$. We say that the action on $\mathcal{V}^G$ is *induced* by pullback of the regular bimodule structure on $\mathcal{V}$ along G. This is ${}^G\mathcal{V}^G$ from Example 2.2.1.1.

The *Drinfeld center*[17] of $\mathcal{M}$ over $\mathcal{V}$ with respect to G is defined as

$$\mathcal{Z}^G(\mathcal{M}) := \mathrm{Isom}^{\otimes}(\mathcal{V} \otimes G, G \otimes \mathcal{V}).$$

The special case $G = \mathrm{Id}\colon \mathcal{M} \to \mathcal{M}$ is denoted by $\mathcal{Z}(\mathcal{M})$ and referred to as the *Drinfeld center of* $\mathcal{M}$. That is, $\mathcal{Z}(\mathcal{M}) = \mathrm{Isom}^{\otimes}(\mathcal{M} \otimes \mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}} \otimes \mathcal{M})$.

We want to emphasize two special cases that will be the main categories of interest in this chapter: One is the Drinfeld center $\mathcal{Z}(\mathcal{M})$ which is equivalent to $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ by 2.2.1.8. To define the other one, the Hopf center $\mathcal{H}(\mathcal{M})$, we have to generalize the definition of the Drinfeld center in two different ways. On one hand, we will provide an appropriate relative setting with respect to a

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[17] This definition and its monoidal structure are due to [81] (see also [88, 9.1.7]), where $\mathcal{Z}^G(\mathcal{M})$ is called the *dual* of $F\colon \mathcal{M} \to \mathcal{V}$.

(quasi-)monoidal fiber functor $F\colon \mathcal{M} \to \mathcal{B}$. On the other hand, we will allow to use two functors G_1 and G_2 instead of just one. This however leads to the loss of monoidacity.

Setting 2.2.2.2. First, we generalize to the setting over a braided monoidal category. Namely, we will always consider the monoidal category $\mathcal{M}$ together with a quasi-monoidal fiber functor $F\colon \mathcal{M} \to \mathcal{B}$ where $\mathcal{B}$ is a braided monoidal category, which has a left quasi-monoidal section $P\colon \mathcal{B} \to \mathcal{M}$ such that there exists $\tau\colon FP \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathcal{B}}$. We assume that τ is a monoidal natural transformation.

Further, we will consider *mixed* centers. For this we assume given two quasi-monoidal functors $G_1, G_2\colon \mathcal{M} \to \mathcal{V}$ which factor through the fiber functor F, i.e. for $i = 1, 2$ we have commutative diagrams of functors

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, "G_i"'] \arrow[dr, "F"] & \\
\mathcal{V} \arrow[rr, "F'"] & & \mathcal{B}.
\end{tikzcd}$$

Note that for any monoidal category, we can always consider it over the trivial braided monoidal category $\mathcal{I}$ with one element and one morphism. As this is both terminal and initial, we have unique functors $T\colon \mathcal{M} \to \mathcal{I}$ and $I\colon \mathcal{I} \to \mathcal{M}$, and $\tau = \mathrm{Id}_{\mathcal{I}}\colon PF \cong IT$.

Definition 2.2.2.3. In the same setting as above, the *mixed relative Drinfeld center* of $\mathcal{M}$ over $\mathcal{B}$ w.r.t. G_1, G_2 is defined to be

$$^{G_1}\mathcal{Z}_{\mathcal{B}}^{G_2}(\mathcal{M}) := \mathrm{Isom}_{\mathcal{B}}^{\otimes}(\mathcal{V} \otimes G_2, G_1 \otimes \mathcal{V}),$$

where $\mathrm{Isom}_{\mathcal{B}}^{\otimes}(\mathcal{V} \otimes G_2, G_1 \otimes \mathcal{V})$ is the full subcategory of $\mathrm{Isom}^{\otimes}(\mathcal{V} \otimes G_2, G_1 \otimes \mathcal{V})$ on objects (V, c) of $\mathrm{Isom}^{\otimes}(V \otimes G_2, G_1 \otimes V)$ which are (F, P)-*admissible.* That is,

$$\Psi_{F'(V),X} = (\tau_X \times \mathrm{Id}_{F'(V)})\mu^{F'}_{G_1P(X),V}F'(c_{P(X)})(\mu^{F'}_{V,G_2P(X)})^{-1}(\mathrm{Id}_{F'(V)} \times \tau_X^{-1}), \quad (2.2.14)$$

for each object X of $\mathcal{B}$, where Ψ is the braiding on $\mathcal{B}$. Note that we use $F'G_i = F$ for $i = 1, 2$ for these compositions to be well-defined.

If $G = G_1 = G_2$ is a monoidal functor, then we denote $\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M}) := {}^{G}\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M})$ and refer to it as the *relative Drinfeld center* of $\mathcal{M}$ over $\mathcal{B}$. If even $G_1 = G_2 = \mathrm{Id}_{\mathcal{M}}$, then we denote

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) := \mathcal{Z}_{\mathcal{B}}^{\mathrm{Id}_{\mathcal{M}}}(\mathcal{M})$$

and refer to it as the *Drinfeld center* of $\mathcal{M}$ over $\mathcal{B}$.

We are mainly interested in the case where $G_1 = G_2 = \mathrm{Id}_{\mathcal{M}}$ and $F = F'$ are monoidal. In this situation, (F, P)-admissible means that the underlying morphism of the braiding with an object in the image of the functor P is the same as the braiding in $\mathcal{B}$. As special cases, we recover $\mathcal{Z}^G(\mathcal{M})$ as defined in 2.2.2.1 as $\mathcal{Z}^G_{\mathcal{I}}(\mathcal{M})$, which is Majid's original construction of the dual from [81].

Warning 2.2.2.4. A quasi-monoidal functor $G\colon \mathcal{M} \to \mathcal{V}$ does *not* give a bimodule structure $\mathcal{V}^G$ unless it is monoidal. However, all compatibility conditions apart from the compatibility (2.2.4) of α and χ (and the analogue for the right action, i.e. the compatibility of α and ξ) are satisfied. For this reason, the category $\mathrm{Isom}_{\mathcal{B}}(\mathcal{V} \otimes G_2, G_1 \otimes \mathcal{V})$ is still well-defined for two such functors, and it caries an action of the Drinfeld center (as we will see in Proposition 2.2.2.11), provided that G_1 is a monoidal functor. We require this generality for working with quasi-Hopf algebras in Sections 2.2.3 and 2.2.4.

Example 2.2.2.5. Let $\mathcal{M} = \mathbf{Mod}\text{-}B(\mathcal{B})$, for a bialgebra B in $\mathcal{B}$, with $P = \mathrm{triv}$ the functor giving an object of $\mathcal{B}$ the trivial module structure, F the forgetful functor. Consider the category $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{B}}(\mathcal{M})$. This category consists of objects (V, δ), where V is a right B-module and δ is a left ${}^{\mathrm{op}}B$-comodule, such that the action and coaction commute. Here, δ is obtained as $c_B(\mathrm{Id}_B \otimes 1)$ from the commutativity isomorphism.

If, to specify further, B is a finite-dimensional bialgebra over k, then $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{B}}(\mathcal{M})$ is equivalent to the category of left modules over the bialgebra $B^* \otimes B^{\mathrm{op}}$.

Remark 2.2.2.6. If $\mathcal{B} = \mathbf{Vect}_k$, for a quasitriangular Hopf algebra, many authors do not impose the admissibility condition as under certain representability conditions (e.g. $\mathcal{M} = B\text{-}\mathbf{Mod}$), any object in the center will be admissible for F being the forgetful functor and $P = \mathrm{triv}$ the functor mapping a vector space to the trivial B-module on it. See e.g. [89, Lemma 2.1] for such a proof, which relies on the existence of elements in a vector space. In our general setting, elements of objects do not exist, hence the admissibility assumption. This condition is a generalization of the assumption used in [22, Section 3.6].

At the general level, admissibility will be crucial in the proof of Proposition 2.2.4.7 which is the main result of Section 2.2.4 where we describe the centers in terms of Yetter–Drinfeld and Hopf modules.

We can apply this more general definition to the pair of functors $G_1 = \mathrm{Id}_{\mathcal{M}}$, and $G_2 = PF$. The second center of interest in this chapter can now be defined.

Definition 2.2.2.7. We define the *relative Hopf center* for a monoidal functor $G\colon \mathcal{M} \to \mathcal{V}$ compatible with the fiber functors to be

$$\mathcal{H}^G_{\mathcal{B}}(\mathcal{M}) := {}^G\mathcal{Z}^{GPF}_{\mathcal{B}}(\mathcal{M}).$$

In particular, the *Hopf center* of $\mathcal{M}$ over $\mathcal{B}$ is

$$\mathcal{H}_{\mathcal{B}}(\mathcal{M}) := \mathcal{H}^{\mathrm{Id}_{\mathcal{M}}}_{\mathcal{B}}(\mathcal{M}).$$

In Section 2.3, we explore the relationship between the categories $\mathcal{Z}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ in the case where $\mathcal{M} = \mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod})$ is a category of right modules over a bialgebra (or Hopf algebra) $B \in H\text{-}\mathbf{Mod}$, where $H\text{-}\mathbf{Mod}$ is braided monoidal, using techniques from reconstruction theory. In this case, we have a fiber functor to $\mathbf{Vect}_k$, and $P = \mathrm{triv}$ is the functor mapping a vector space to the trivial B-module. We can proof the structural results about the Drinfeld and Hopf center at the level of generality of this section which is often easier. For instance, we show that the relative Drinfeld center has a natural monoidal structure[18] and that $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is braided.

Proposition 2.2.2.8. *For any monoidal functor $G\colon \mathcal{M} \to \mathcal{V}$, we can give $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M})$ a monoidal structure by setting $(V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V\otimes W})$, where*

$$c^{V\otimes W} := \alpha_{G(X),V,W}(c^V \otimes \mathrm{Id}_W)\alpha^{-1}_{V,G(X),W}(\mathrm{Id}_V \otimes c^W)\alpha_{V,W,G(X)}. \tag{2.2.15}$$

By construction, there is a monoidal functor $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M}) \to \mathcal{V}$.

Proof. This is shown by straightforward but lengthy checking of the axioms and compatibilities. The associativity isomorphism in $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M})$ is just the associativity isomorphism of $\mathcal{M}$, which can be checked to be compatible with the centralizing isomorphisms of threefold tensor products.

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[18] One can give the category $\mathcal{H}^G_{\mathcal{B}}(\mathcal{M})$ a monoidal structure which is not compatible with the forgetful functor, see e.g. [22] in the case $\mathcal{M} = \mathbf{Mod}\text{-}B(\mathcal{B})$ by taking relative tensor products $\otimes_B$.

Next, we have to check that $C^{V\otimes W}$ is monoidal in the sense of 2.2.12. The proof of this requires repeated application of naturality of c^V, c^W to Δ, the hexagonal axioms for α, and naturality of α with respect to c^V and c^V. It is clear that the tensor product of two (F,P)-admissible objects is again (F,P)-admissible. $\square$

The main advantage of the Drinfeld center is that it is braided.

Proposition 2.2.2.9. *The category $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ has a braiding*[19] *defined by*

$$\Psi = \Psi_{(V,c^V),(W,c^W)} := c^V_W \colon (V\otimes W, c^{V\otimes W}) \to (W\otimes V, c^{W\otimes V}),$$

for objects (V,c^V) and (W,c^W) in $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$.

Proof. This follows by applying naturality of c^V to the morphisms c^W_X for any object X of $\mathcal{M}$. Indeed, this gives that $c^V_{W\otimes X}(\mathrm{Id}_V \otimes c^W_X) = (c^W_X \otimes \mathrm{Id}_V)c^V_{W\otimes X}$. Now we use the monadicity of c as in (2.2.12) twice giving the required commutative square. $\square$

For monoidal functors G_1, G_2, we define the category $\mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, {}^{G_1}\mathcal{V}^{G_2})$ as the full subcategory on bimodule morphisms $\mathcal{M}^{\mathrm{reg}} \to {}^{G_1}\mathcal{V}^{G_2}$ that corresponds to $\mathrm{Isom}^{\otimes}_{\mathcal{B}}(\mathcal{V}\otimes G_2, G_1\otimes\mathcal{V})$ under the equivalence of Proposition 2.2.1.8.

Theorem 2.2.2.10. *There is an equivalence of braided monoidal categories*

$$\mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}}) \simeq \mathcal{Z}_{\mathcal{B}}(\mathcal{M}),$$

where $\mathbf{BiMod}^{\mathcal{B}}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ is a monoidal category via composition of functors.

Proof. Recall that by Proposition 2.2.1.8 there is an equivalence of categories for the larger categories without the (F,P)-admissibility requirement. The left hand side is by definition the subcategory corresponding to (F,P)-admissible objects under this equivalence. To show that the monoidal structure defined in 2.2.2.8 corresponds to the monoidal structure of composition of functors on the left hand side, observe that for $\phi,\psi\colon \mathcal{M}^{\mathrm{reg}} \to \mathcal{M}^{\mathrm{reg}}$, the commutativity isomorphism of the composition $\phi\psi$ is given by

$$c^{\phi\psi}_X = (\lambda_\phi)_{X,\psi(I)}\psi((\lambda_\psi)_{X,I})\psi((\rho_\psi^{-1})_{I,X})(\rho_\phi^{-1})_{\psi(I),X}. \tag{2.2.16}$$

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[19] A braiding on $\mathcal{Z}(\mathcal{M})$ was observed by Drinfeld in unpublished work (cf. note in [81, 3.4]).

In 2.2.1.8 we saw that for ϕ corresponding to (V,c) and ψ corresponding to (W,d), we have $\phi(\psi(I)) = \phi(W) = V \otimes W$, and ρ^{-1} is given by α as we use the regular action. Using these equalities, we find

$$c_X^{\phi\psi} = (\lambda_\phi)_{X,\psi(I)}(\phi(\mathrm{Id}_X \otimes c_X^{\psi}))(\rho_\phi^{-1})_{\psi(I),X} = c_X^{V\otimes W}, \tag{2.2.17}$$

comparing with the definition of the monoidal structure in $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ from 2.2.2.8. The fact that the braidings are related can be checked by a similar calculation. □

Note that the monoidal category $\mathbf{BiMod}_{\mathcal{M}}^{\mathcal{B}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ is strict as composition of functors is strictly associative, and the additional datum of compositions of the transformations ρ and λ is strictly associative too. However, the reinterpretation of the data of bimodule morphisms as pairs (V,c) yields a non-strict monoidal category if $\mathcal{M}$ is not strict.

Theorem 2.2.2.10 gives an easy way to find module categories over $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ from bimodule categories over $\mathcal{M}$. The resulting categorical actions is the main topic of this chapter and will be reinterpreted in various reformulations of the braided Drinfeld and Heisenberg double in the course of this chapter. The general statement is:

Proposition 2.2.2.11. *Let $\mathcal{V}$ be an M-bimodule. Then there exists a natural action by composition of functors*

$$\triangleright\colon \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \times \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{V}) \longrightarrow \mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{V}).$$

For $\mathcal{V}$ a monoidal category, G_1 a monoidal and G_2 a quasi-monoidal functor factoring as in 2.2.2.2, we obtain an action

$$\triangleright\colon \mathcal{Z}_{\mathcal{B}}^{G_1}(\mathcal{M}) \times {}^{G_1}\mathcal{Z}_{\mathcal{B}}^{G_2}(\mathcal{M}) \longrightarrow {}^{G_1}\mathcal{Z}_{\mathcal{B}}^{G_2}(\mathcal{M}).$$

Proof. The first statement follows by using the action of $\mathbf{BiMod}_{\mathcal{M}}^{\mathcal{B}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{M}^{\mathrm{reg}})$ on $\mathbf{BiMod}_{\mathcal{M}}(\mathcal{M}^{\mathrm{reg}}, \mathcal{V})$ by composition and relating it to the Drinfeld center by 2.2.2.10. Note that there is no dependence on $\mathcal{B}$ in this statement.

For the second part, the action $(V,c) \triangleright (W,d) = (V \otimes W, d^{V\otimes W})$ of a pair (V,c) in $\mathcal{Z}_{\mathcal{B}}^{G_1}(\mathcal{M})$ on a pair (W,d) in ${}^{G_1}\mathcal{Z}_{\mathcal{B}}^{G_2}(\mathcal{M})$ is defined as

$$d_X^{V\otimes W} := \alpha_{G_1(X),W,V}(c_X \otimes \mathrm{Id}_W)\alpha^{-1}_{V,G_1(X),W}(\mathrm{Id}_V \otimes d_X)\alpha_{V,W,G_2(X)}.$$

It follows by combining bimodule coherences and naturality that $d^{V\otimes W}$ again satisfies (2.2.12). Note that this does not require G_2 to be compatible with $\alpha^{\mathcal{M}}$, $\alpha^{\mathcal{V}}$. The isomorphism $\chi := (\alpha^{\mathcal{V}})^{-1}$ is the coherence for the action defined, which is clearly compatible with $\alpha^{\mathcal{V}}$. This action restricts to the (F,P)-admissible subcategories as, by definition, $d^{V\otimes W}$ is obtained by combining c and d. Hence if both of these centralizing isomorphisms are related to Ψ as in (2.2.14), then so is their tensor product $c^{V\otimes W}$, using that

$$\alpha^{\mathcal{B}}(\Psi_{F(V),F(G_1(X))}\otimes \mathrm{Id})(\alpha^{\mathcal{B}})^{-1}(\mathrm{Id}\otimes\Psi_{F(W),F(G_2(X))})\alpha^{\mathcal{B}} = \Psi_{F(V)\otimes F(W),F(G_2(X))}, \tag{2.2.18}$$

where we use that $F(G_1(X)) = F(G_2(X))$. □

Given that $\mathcal{V}$ has (left) duals (that is, the category $\mathcal{V}$ is *rigid*), we can show that the relative Drinfeld centers inherit the same structure.

Proposition 2.2.2.12. *Let $\mathcal{V}$ be a rigid category, $G\colon \mathcal{M}\to\mathcal{V}$ a monoidal functor. Then the center $\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M})$ is rigid with the dual of an object (V,c) given by V^* with commutativity isomorphism*

$$c^*_X := (\mathrm{ev}_V\otimes\mathrm{Id})(\alpha^{-1}\otimes\mathrm{Id}_{V^*})(\mathrm{Id}\otimes(c^V_X)^{-1}\otimes\mathrm{Id})(\alpha\otimes\mathrm{Id})\alpha^{-1}(\mathrm{Id}_{V^*\otimes G(X)}\otimes\mathrm{coev}_V). \tag{2.2.19}$$

The coevaluation and evaluation morphisms are the respective maps in $\mathcal{M}$.

Proof. Note first that a monoidal structure is necessary to talk about rigidity of a category. Hence the restriction $G = G_1 = G_2$. Now c^* is a well-defined natural transformation $V^*\otimes G \Rightarrow G\otimes V^*$. It is easy to see that c^* is $\otimes$-compatible and thus (V^*, c^*) gives an element of $\mathcal{Z}^G(\mathcal{M})$. Moreover ev_V and coev_V commute with $c\otimes c^*$ and $c^*\otimes c$. We will sketch in more detail how commutativity with ev_V can be proved. This will be straightforward after proving that

$$(\mathrm{ev}_V\otimes\mathrm{Id}_X)\alpha^{-1}_{V^*,V,X}(\mathrm{Id}_{V^*}\otimes c_X^{-1})\alpha_{V^*,X,V} = (\mathrm{Id}_X\otimes\mathrm{ev}_V)\alpha_{X,V^*,V}(c^*_X\otimes\mathrm{Id}_V). \tag{2.2.20}$$

Starting from the right hand side of the equation, using the definition of c^*, we extract the expression $(\mathrm{Id}_V\otimes\mathrm{ev}_V)\alpha^{-1}(\mathrm{coev}_V\otimes\mathrm{Id}_V)$. This is done using naturality of α^{-1} in coev, ev or c_X^{-1} and the hexagonal axioms. This expression equals Id_V by rigidity of $\mathcal{M}$.

It remains to check that the dual (according to Proposition 2.2.2.12) of an admissible object is admissible again. For this, note that for any X of $\mathcal{B}$ and (F, P)-admissible (V, c) in the center, we have that $F(c^{-1}_{P(X)})$ can be expressed in terms of the inverse braiding and structural isomorphisms. From this we can conclude admissibility of (V^*, c^*). ☐

The next lemma shows how to extract the inverse of the centralizing isomorphism from the definition of the dual.

Lemma 2.2.2.13. *If $\mathcal{M}$ has duals, then the inverse can be described using the definition of the dual as*

$$c_X^{-1} = (\mathrm{Id}\otimes(\mathrm{Id}\otimes \mathrm{ev}))(\mathrm{Id}_V\otimes\alpha)(\mathrm{Id}\otimes c_X^*\otimes \mathrm{Id})(\mathrm{Id}_V\otimes\alpha^{-1})\alpha_{V,V^*,X\otimes V}(\mathrm{coev}\otimes \mathrm{Id}_{X\otimes V}). \tag{2.2.21}$$

Proof. Key in the proof is to use (2.2.20). We first extract the right hand side of this equation in the right hand side of the claim. After applying (2.2.20), we use naturality of α^{-1} in c_X^{-1} and coev as well as the hexagonal axioms to transform the resulting composition of maps into

$$((\mathrm{Id}_V\otimes \mathrm{ev}_V)\otimes \mathrm{Id}_X)(\alpha\otimes \mathrm{Id}_X)((\mathrm{coev}\otimes \mathrm{Id}_V)\otimes \mathrm{Id}_X)c_X^{-1} = c_X^{-1}.$$ ☐

In fact, one can show that if $\mathcal{M}$ has right duals, then these can be used to show that every monoidal natural transformation $V\otimes \mathrm{Id}_{\mathcal{M}} \Longrightarrow \mathrm{Id}_{\mathcal{M}}\otimes V$ is automatically invertible. This fact will be used in Section 2.2.4. For this, recall that the right dual ${}^*V$ for an object V of $\mathcal{M}$ is an object together with morphisms $\mathrm{ev}'_V\colon V\otimes {}^*V \to I$ and $\mathrm{coev}'_V\colon I\to {}^*V\otimes V$, such that the axioms

$$(\mathrm{ev}'_V\otimes \mathrm{Id}_V)\alpha^{-1}_{V,{}^*V,V}(\mathrm{Id}_V\otimes \mathrm{coev}'_V) = \mathrm{Id}_V, \tag{2.2.22}$$

$$(\mathrm{Id}_{{}^*V}\otimes \mathrm{ev}'_V)\alpha_{{}^*V,V,{}^*V}(\mathrm{coev}'_V\otimes \mathrm{Id}_{{}^*V}) = \mathrm{Id}_{{}^*V}, \tag{2.2.23}$$

are satisfied.

Lemma 2.2.2.14. *Let $\mathcal{M}$ have right duals. Then for $(V, c)\in \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$,*

$$c_X^{-1} = (\mathrm{ev}'\otimes \mathrm{Id}_{V\otimes X})\alpha^{-1}(\mathrm{Id}\otimes\alpha)(\mathrm{Id}_X\otimes c_{{}^*X}\otimes \mathrm{Id}_{{}^*X})(\mathrm{Id}\otimes\alpha^{-1})(\mathrm{Id}_X\otimes(V\otimes \mathrm{coev}'_X)). \tag{2.2.24}$$

Proof. Applying the hexagonal axiom and naturality of α^{-1} in coev' and c_X we obtain

$$\begin{aligned}
&c_X(\mathrm{ev}' \otimes \mathrm{Id}_{V \otimes X})(\alpha^{-1})(\mathrm{Id} \otimes \alpha)(\mathrm{Id}_X \otimes c_{{}^*X} \otimes \mathrm{Id}_{{}^*X})(\mathrm{Id} \otimes \alpha^{-1})(\mathrm{Id}_X \otimes (V \otimes \mathrm{coev}'_X))\\
&\quad = (\mathrm{ev}' \otimes \mathrm{Id})(\alpha^{-1} \otimes \mathrm{Id})\alpha^{-1}(\mathrm{Id} \otimes \alpha^{-1})(\mathrm{Id}_{X \otimes {}^*X} \otimes c_X)\\
&\qquad (\mathrm{Id} \otimes \alpha)(\mathrm{Id}_X \otimes c_{{}^*X} \otimes \mathrm{Id}_{{}^*X})(\mathrm{Id} \otimes \alpha^{-1})(\mathrm{Id}_X \otimes (V \otimes \mathrm{coev}'_X)).
\end{aligned}$$

Applying first the definition of the monoidal rule (2.2.12) and naturality of $c_{{}^*X}$ in coev', followed by the right dual axioms, this expression becomes

$$\begin{aligned}
&(\mathrm{ev}' \otimes \mathrm{Id})(\alpha^{-1} \otimes \mathrm{Id})\alpha^{-1}(\mathrm{Id}_X \otimes c_{{}^*X \otimes X})(\mathrm{Id}_{X \otimes V} \otimes \mathrm{coev}')\\
&\quad = ((\mathrm{ev}'_X \otimes \mathrm{Id}_X) \otimes \mathrm{Id}_V)(\alpha^{-1} \otimes \mathrm{Id}_V)((\mathrm{Id}_X \otimes \mathrm{coev}'_X) \otimes \mathrm{Id}_V) = \mathrm{Id}_{X \otimes V}. \quad \square
\end{aligned}$$

At this general level, we can show that the relative Drinfeld center acts on the relative Hopf center. We will later use this result to obtain twisting results on the level of algebras (cf. 2.3.8).

Corollary 2.2.2.15. *For G monoidal, there is a left action of the monoidal category $\mathcal{Z}_{\mathcal{B}}^G(\mathcal{M})$ on $\mathcal{H}_{\mathcal{B}}^G(\mathcal{M})$ defined by $(V,c) \rhd (W,d) = (V \otimes W, d^{V \rhd W})$, where*

$$c^{V \rhd W} := \alpha_{G(X),V,W}(c_X \otimes \mathrm{Id}_W)\alpha^{-1}_{V,G(X),W}(\mathrm{Id}_V \otimes d_X), \tag{2.2.25}$$

for $(V,c) \in \mathcal{Z}_{\mathcal{B}}^G(\mathcal{M})$ and $(W,d) \in \mathcal{H}_{\mathcal{B}}^G(\mathcal{M})$.

Proof. This is a special case of the second part of Proposition 2.2.2.11, where $G = G_1 = G_2$.

The result can also easily be seen directly using the monoidal structure introduced in (2.2.2.8). The left action isomorphism χ will simply be the associativity isomorphism α in $\mathcal{V}$. It is clear that with this action, the resulting object will again be (F,P)-admissible. □

In particular, there is a natural action $\rhd\colon \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \times \mathcal{H}_{\mathcal{B}}(\mathcal{M}) \to \mathcal{H}_{\mathcal{B}}(\mathcal{M})$.

Remark 2.2.2.16. If $\mathcal{M}$, $\mathcal{V}$, $\mathcal{B}$ and all functors carry additional structure such as being additive, abelian (with left or right exact functors), or k-linear, then these structures are inherited by the mixed relative Drinfeld centers. The constructions discussing in this section however work in the generality of braided monoidal category without such structures. In Section 2.3 a k-linear setting is used.

2.2.3 Braided Reconstruction Theory

We now present a categorical reconstruction theorem which generalizes Tannaka–Krein duality — which is classically stated in the setting of a monoidal category $\mathcal{C}$ over $\mathbf{Vect}_k$ — to the setting of a quasi-monoidal fiber functor $F\colon \mathcal{C} \to \mathcal{B}$ where $\mathcal{B}$ is any braided monoidal category which we tread as strictly monoidal (identifying all ways of setting the brackets), but keep track of the associativity isomorphisms in $\mathcal{C}$. We assume that the functor F is strong quasi-monoidal.[20] More details about this can be found in [88, Section 9.4.2] (or [86, 3.2] for a comodule version) for the braided Hopf algebra case, and for the quasi-Hopf algebra case (but not braided) see [84, Section 2] for a comodule version.

In order to state the reconstruction theorem, we need to assume certain representability conditions on the fiber functor $F\colon \mathcal{C} \to \mathcal{B}$. The basic assumption is that the functor $\mathrm{Nat}_{\mathcal{C}}(-\otimes F, F)\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is representable. This means there exists an object $B \in \mathcal{B}$ such that

$$\mathrm{Nat}_{\mathcal{C}}(V \otimes F, F) \cong \mathrm{Hom}_{\mathcal{B}}(V, B), \quad \forall V \in \mathcal{B}. \tag{2.2.26}$$

For example, if $\mathcal{B} = \mathbf{Vect}_k$, $\mathcal{C} = B\text{-}\mathbf{Mod}$ and F is the forgetful functor, we have

$$\mathrm{Nat}_{B\text{-}\mathbf{Mod}}(F, F) \cong \mathrm{Nat}_{B\text{-}\mathbf{Mod}}(k \otimes F, F) \cong \mathrm{Hom}_{\mathbf{Vect}_k}(k, B) \cong B,$$

recovering B. To recover classical Tannaka–Krein duality, one considers the vector space $\mathrm{Nat}_{B\text{-}\mathbf{Mod}}(F, F)$ for $\mathcal{C} \to \mathbf{Vect}_k^{\mathrm{fd}}$ under suitable assumptions (see e.g. [32]).

Theorem 2.2.3.1. *Let $F\colon \mathcal{C} \to \mathcal{B}$ be a functor satisfying (2.2.26) with respect to some object B in $\mathcal{B}$. Then B is an algebra object in $\mathcal{B}$ and F factors as $\mathcal{C} \to B\text{-}\mathbf{Mod}(\mathcal{B}) \xrightarrow{F} \mathcal{B}$ where F is the forgetful functor.*

The object B is universal in the following sense: If B is another such algebra object in $\mathcal{B}$, then there exists a unique morphism $B' \to B$ such that the induced

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[20] Dropping the strongness assumption leads to weak quasi-Hopf algebras. A reconstruction theorem for such objects can be found in [52].

pullback functor B-$\mathbf{Mod}(\mathcal{B}) \to B'$-$\mathbf{Mod}(\mathcal{B})$ *makes the diagram*

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl] \arrow[dr] & \\
B\text{-}\mathbf{Mod}(\mathcal{B}) \arrow[rr] \arrow[dr] & & B'\text{-}\mathbf{Mod}(\mathcal{B}) \arrow[dl] \\
& \mathcal{B} &
\end{tikzcd} \tag{2.2.27}$$

commute.

Proof (Sketch). Indeed, the natural transformation $\sigma \colon B \otimes F \to F$ corresponding to Id_B under (2.2.26) gives an action of H on each object $F(X)$ for $X \in \mathcal{C}$. The product map $B \otimes B \to B$ corresponds to the natural transformation given by

$$\sigma_X(\mathrm{Id}_B \otimes \sigma_X) \colon B \otimes B \otimes F(X) \to F(X),$$

for $X \in \mathcal{C}$. The unit is given by the natural transformation $I \otimes F \Rightarrow F$ of the unit of the monoidal structure of $\mathcal{B}$. The morphism σ_X gives any object X of $\mathcal{C}$ a B-modules structure, which we will sometimes denote by $\rhd$. For the universal property, cf. the dual version in [84, 2.2]. ☐

In order to obtain more structure on B, we need to assume more structure on $\mathcal{C}$ and that this structure is preserved by the functor F. In addition, we will need *higher representabilities*. Given a morphism $\beta \colon V \to B^{\otimes n}$ we can define a natural transformation $\theta^n_V(\beta)$ as the composition

$$\begin{tikzcd}
V \otimes F(X_1) \otimes \ldots \otimes F(X_n) \arrow[r, "\beta \otimes \mathrm{Id}"] \arrow[d, "\theta^n_V(\beta)"'] & B^{\otimes n} \otimes F(X_1) \otimes \ldots \otimes F(X_n) \arrow[d] \\
F(X_1) \otimes \ldots \otimes F(X_n) & B \otimes F(X_1) \otimes \ldots \otimes B \otimes F(X_n), \arrow[l, "\sigma_{X_1} \otimes \ldots \otimes \sigma_{X_n}"']
\end{tikzcd} \tag{2.2.28}$$

where the right vertical arrow is obtained by the braiding in $\mathcal{B}$.

Definition 2.2.3.2 ([88, (9.40)])**.** We say that a functor $F \colon \mathcal{C} \to \mathcal{B}$ is *higher representable* if the functors $\mathrm{Nat}(V \otimes F^{\otimes n}, F^{\otimes n})$ are representable for $n \geqslant 0$ by the object $B^{\otimes n}$ such that a morphism β corresponds to $\theta^n_V(\beta)$.

This condition says that the representing object B for $\mathrm{Nat}(V \otimes F, F)$ induces representability of $\mathrm{Nat}(V \otimes F^{\otimes n}, F^{\otimes n})$ by $B^{\otimes n}$. This condition is not automatic in general. In a classical Tannaka–Krein duality setting, working with $\mathcal{B} = \mathbf{Vect}_k^{\mathrm{fd}}$, it will be automatically satisfied.

Theorem 2.2.3.3. *Let $F\colon \mathcal{M} \to \mathcal{B}$ be a quasi-monoidal fiber functor satisfying the higher representability conditions for some object $B \in \mathcal{B}$. Then B is a universal quasi-bialgebra object in $\mathcal{B}$ such that F factors as $\mathcal{M} \to B\text{-}\mathbf{Mod}(\mathcal{B}) \xrightarrow{F} \mathcal{B}$.*

Proof. The coproduct Δ is the morphism $B \to B \otimes B$ corresponding to the natural transformation $\delta\colon B \otimes F^2 \to F^2$ defined by $\delta_{X,Y} = \sigma_{X \otimes Y}$. For the counit, we define $F^{\otimes 0}$ to be the constant functor I with image the unit object I in $\mathcal{B}$. The data of a natural transformation $V \otimes F^{\otimes 0} \to F^{\otimes 0}$ consists of only one morphism $V \to I$. The counit is defined to be the morphism $a_I\colon B \to I$. The 3-cycle $\phi\colon I \to B^{\otimes 3}$ corresponds to the natural transformation $F(\alpha_{X,Y,Z})$, coming from the associativity isomorphism in $\mathcal{M}$. The quasi-coassociativity of Δ now follows (under translation with use of the higher representability condition) from the commutativity of the square

$$\begin{tikzcd}
B \otimes F((X \otimes Y) \otimes Z) \arrow[r, "\mathrm{Id}_B \otimes F(\alpha_{X,Y,Z})"] \arrow[d, "\sigma_{(X\otimes Y)\otimes Z}"'] & B \otimes F(X \otimes (Y \otimes Z)) \arrow[d, "\sigma_{X\otimes(Y\otimes Z)}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha_{X,Y,Z})"] & F(X \otimes (Y \otimes Z)),
\end{tikzcd} \tag{2.2.29}$$

for object X, Y, Z of $\mathcal{M}$, which uses naturality of σ. Moreover, the hexagonal axiom translates to the 3-cycle condition. The proof that Δ is an algebra homomorphism in $\mathcal{B}$ uses naturality of the braiding (see [88, Figure 9.16(b)]). Note that also for quasi-bialgebras, Δ is an algebra homomorphism, i.e. the bialgebra condition holds strictly (not up to isomorphism). □

If we assume even more structure on the category $\mathcal{M}$ and $\mathcal{B}$, we obtain more structure on the representing bialgebra B. We will need the following preliminary observation regarding the interplay of left and right duals in $\mathcal{B}$:

Lemma 2.2.3.4. *If $\mathcal{B}$ is a braided monoidal category (with associativity isomorphism α) which is rigid (i.e. left duals exist). Then the left dual of an object V is also a right dual.*

Proof. Recall that a right dual ${}^*V$ for an object V of $\mathcal{B}$ is an object together with morphisms $\mathrm{ev}'_V\colon V\otimes{}^*V \to I$ and $\mathrm{coev}'_V\colon I \to {}^*V\otimes V$, such that (2.2.22) is satisfied. Using the braiding on $\mathcal{B}$, we define $\mathrm{ev}'_V := \mathrm{ev}_V\,\Psi^{-1}$, and $\mathrm{coev}'_V := \Psi\,\mathrm{coev}_V$. This gives V^* the structure of a right dual. □

Let F be a quasi-monoidal functor. In order to recover the antipode of B, we need to assume the existence of a natural *duality isomorphism* $d_X\colon F(X)^* \to F(X^*)$. If F is monoidal, then $d_X = (\mathrm{ev}_{F(X)}\otimes \mathrm{Id}_{F(X^*)})(\mathrm{Id}_{F(X)^*}\otimes F(\mathrm{coev}_X))$. We require that the compatibility condition $d_{X\otimes Y} = d_Y\otimes d_X$ of the monoidal structure with the duality holds.

We say the quasi-monoidal functor F is *rigid* if d exists, and for the evaluation and coevaluation morphisms in $\mathcal{M}$, the conditions

$$\mathrm{ev}_{\mathcal{M}} = \mathrm{ev}_{F(X)}(d_X\otimes \mathrm{Id}), \qquad \mathrm{coev}_{\mathcal{M}} = (\mathrm{Id}\otimes d_X^{-1})\,\mathrm{coev}_{F(X)}, \tag{2.2.30}$$

are satisfied for any object X of $\mathcal{M}$.

Theorem 2.2.3.5.

(a) Let $\mathcal{B}$ be rigid. Then $\mathcal{M}$ and the functor F are rigid if and only if the representing object B is a quasi-Hopf algebra object in $\mathcal{B}$.

(b) Let $\mathcal{B}$ be a braided monoidal and rigid. Then $\mathcal{M}$ has left and right duals on the same object, and F is rigid, if and only if S has an invertible antipode.

(c) Let $\mathcal{B}$ be braided monoidal. Then $\mathcal{M}$ is braided monoidal if and only if we can define a second coproduct Δ^{cop} (see Definition 2.2.3.7) and a universal R-matrix turning B into a quasitriangular quasi-bialgebra (respectively quasi-Hopf algebra if we are in case (a)) in $\mathcal{B}$.

In the following, we will explore these structures more concretely and sketch the proofs. For part (a), we first define the map a as the map $I \to B$ corresponding to the natural isomorphism

$$(\mathrm{Id}\otimes F(\mathrm{ev}_X))(\mathrm{Id}\otimes d_X\otimes \mathrm{Id})(\mathrm{coev}_{F(X)}\otimes \mathrm{Id}),$$

while b is defined using

$$(\mathrm{Id}\otimes \mathrm{ev}_{F(X)})(\mathrm{Id}\otimes d_X^{-1}\otimes \mathrm{Id})(F(\mathrm{coev}_X)\otimes \mathrm{Id}).$$

This implies that

$$F(\mathrm{ev}_{F(X)}) = \mathrm{ev}_{F(X)}(\mathrm{Id}\otimes\sigma_X)(\mathrm{Id}\otimes a\otimes\mathrm{Id}), \tag{2.2.31}$$

$$F(\mathrm{coev}_{F(X)}) = \sigma_X\otimes\mathrm{Id}(b\otimes\mathrm{Id}_{F(X)\otimes F(X)^*})\,\mathrm{coev}_{F(X)}\,. \tag{2.2.32}$$

The dual action is defined as $\sigma_X^* := d_X^{-1}\sigma_{X^*}(\mathrm{Id}\otimes d_X)$. We then define the antipode as the morphism $S\colon B\to B$ corresponding to the natural transformation $B\otimes F\to F$ defined for an object X of $\mathcal{M}$ as

$$(\mathrm{Id}\otimes\mathrm{ev}_{F(X)})(\mathrm{Id}\otimes d_X^{-1})(\mathrm{Id}_{F(X)}\otimes\sigma_{X^*}\otimes\mathrm{Id})(\Psi\otimes d_X\otimes\mathrm{Id})(\mathrm{Id}_B\otimes\mathrm{coev}_{F(X)}\otimes\mathrm{Id}_{F(X)}).$$

That is, translating the action σ_{X^*} on the dual to an action on $F(X)$ using conjugation by d_X. We directly derive a formula for translating between the action on $F(X)$ and the dual action:

$$\mathrm{ev}_{F(X)}(\triangleright^*\otimes\mathrm{Id}_{F(X)}) = \mathrm{ev}_{F(X)}(\mathrm{Id}\otimes\triangleright)(\Psi\otimes\mathrm{Id}_{F(X)})(S\otimes\mathrm{Id}_{F(X^*)\otimes F(X)}). \tag{2.2.33}$$

We can now easily proof the antipode axioms (2.1.9). We also check directly that the duality conditions in H-**Mod**($\mathcal{B}$) are equivalent to the conditions (2.1.10). This completes the proof that having an antipode for B is equivalent to the existence of left duals in $\mathcal{M}$ via reconstruction.

For part (b), we need the following Lemma regarding the antialgebra and coalgebra morphism properties of the antipode.

Lemma 2.2.3.6. *The antipode S satisfies*

$$Sm = m\Psi(S\otimes S), \qquad \Delta S = (S\otimes S)\Psi\Delta. \tag{2.2.34}$$

If the inverse S^{-1} exists, then it satisfies

$$S^{-1}m = m\Psi^{-1}(S^{-1}\otimes S^{-1}), \qquad \Delta S^{-1} = (S^{-1}\otimes S^{-1})\Psi^{-1}\Delta. \tag{2.2.35}$$

Proof. It is an exercise to adapt the proof of [88, Figure 9.14] to quasi-Hopf algebras. This proof uses (2.1.10) for ϕ^{-1}. □

Using this Lemma, we can further observe conditions on S^{-1} which are equivalent to the antipode axioms (2.1.9):

$$m^2(\mathrm{Id}\otimes a\otimes S^{-1})\Psi^{-1}\Delta = a\varepsilon, \qquad m^2(S^{-1}\otimes b\otimes\mathrm{Id})\Psi^{-1}\Delta = b\varepsilon. \tag{2.2.36}$$

We now turn to the proof of (b): In Lemma 2.2.3.4, we saw that $\mathcal{B}$ has right duals. Given that the antipode S is invertible, we can use the left duals in $\mathcal{M}$ to define right duals. The right dual ${}^*F(X)$ is defined to be $F(X)^*$ with (co)evaluation maps

$$\operatorname{ev}' = \operatorname{ev}(\operatorname{Id}\otimes\sigma)(\operatorname{Id}\otimes S^{-1}a\otimes\operatorname{Id})\Psi^{-1}, \tag{2.2.37}$$

$$\operatorname{coev}' = \Psi(\sigma\otimes\operatorname{Id})(S^{-1}b\otimes\operatorname{Id})\operatorname{coev}. \tag{2.2.38}$$

It is not hard to check that the maps $\operatorname{ev}'_{F(X)}$, $\operatorname{coev}'_{F(X)}$ give the left dual $F(X)^*$ a right dual structure. The proofs use Lemma 2.2.3.6 and Lemma 2.2.3.4.

Conversely, given right duals ${}^*X$ in $\mathcal{M}$ on the same objects, we view the natural transformation d_X as $d_X\colon {}^*F(X)\to F({}^*X)$. We then apply reconstruction to the natural transformation

$$(\operatorname{Id}\otimes\operatorname{ev}_{F(X)})(\operatorname{Id}\otimes d_X^{-1}\otimes\operatorname{Id})(\operatorname{Id}\otimes\sigma_{{}^*X})(\Psi^{-1}\otimes d_X\otimes\operatorname{Id}_{F(X)})(\operatorname{Id}_B\otimes\operatorname{coev}_{F(X)}\otimes\operatorname{Id}_{F(X)}),$$

to give a map $S'\colon B\to B$. If we define the *right* dual action as

$${}^*\triangleright := d_X^{-1}\sigma_{{}^*X}(\operatorname{Id}_B\otimes d_X), \tag{2.2.39}$$

then we can derive the following property:

$$\operatorname{ev}_{F(X)}({}^*\triangleright\otimes\operatorname{Id}_{F(X)}) = \operatorname{ev}_{F(X)}(\operatorname{Id}_{F(X)^*}\otimes\triangleright)(\operatorname{Id}_{F(X)^*}\otimes S'\otimes\operatorname{Id}_{F(X)})(\Psi\otimes\operatorname{Id}_{F(X)}). \tag{2.2.40}$$

Further, the morphisms

$$\operatorname{coev}^r_{\mathcal{M}} := \Psi(\operatorname{Id}\otimes{}^*\triangleright)(\operatorname{Id}\otimes b\otimes\operatorname{Id})\operatorname{coev}_{F(X)}, \tag{2.2.41}$$

$$\operatorname{ev}^r_{\mathcal{M}} := \operatorname{ev}_{F(X)}({}^*\triangleright\otimes\operatorname{Id})(a\otimes\operatorname{Id}_{F(X)\otimes F(X)^*})\Psi^{-1} \tag{2.2.42}$$

are morphisms of left B-modules in $\mathcal{B}$. We can now show that the map S' is inverse to the antipode S recovered from the left module structure in $\mathcal{M}$. One checks that the right duality axioms correspond to the conditions (2.1.10) after application of the antipode S.

For the readers convenience, we include the definitions of the antipode (and its inverse) via reconstruction theory using graphical calculus in Figure 2.2.1. This

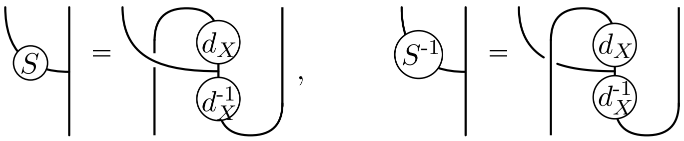

Figure 2.2.1: Antipode reconstruction

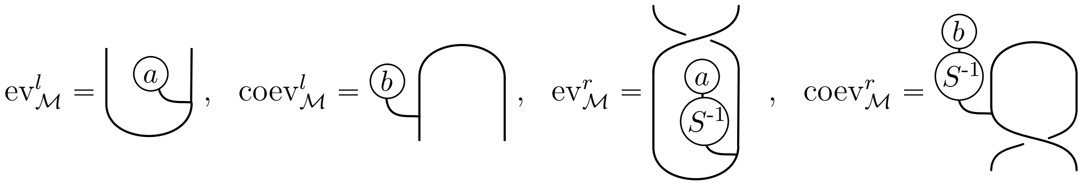

Figure 2.2.2: Duals of left modules

generalizes [88, Figure 9.15], where the definition of m, Δ, 1, ε are as given there. Further, the duality structure on $\mathcal{M} = B\text{-}\mathbf{Mod}(\mathcal{B})$ is given in Figure 2.2.2.

To describe the structure on B obtained from the braiding, and thus proof (c), we will need a general definition of an universal R-matrix in a braided monoidal category. For this, we have to consider the *opposite* coproduct Δ^{cop} of [86, 2.5].

Definition 2.2.3.7. The *opposite coproduct* Δ^{cop} is defined to be the morphism $B \to B \otimes B$ in $\mathcal{B}$ corresponding to the natural transformation τ given by

$$\tau_{X,Y} = \Psi^{-1}_{F(X),F(Y)} \sigma_{Y\otimes X} (\mathrm{Id} \otimes \Psi_{F(X),F(Y)}). \tag{2.2.43}$$

That is, the action of the opposite coproduct satisfies the identity

$$\begin{aligned}&(\rhd_V \otimes \rhd_W)(\mathrm{Id}_B \otimes \Psi \otimes \mathrm{Id}_{V\otimes W})(\Delta^{\mathrm{cop}} \otimes \mathrm{Id}_{V\otimes W})\\ &= (\rhd_V \otimes \rhd_W)(\mathrm{Id}_B \otimes \Psi^{-1} \otimes \mathrm{Id}_{V\otimes W})(\Psi^{-1}\Delta \otimes \mathrm{Id}_{V\otimes W}).\end{aligned} \tag{2.2.44}$$

Dually, we define the *opposite product* m^{op}.

We express this diagram using graphical calculus in Figure 2.2.3 (the opposite coproduct Δ^{cop} is labeled with cop in the diagram).

Remark 2.2.3.8. In the symmetric monoidal case, Δ^{cop} is the usual opposite coproduct $\Psi\Delta$, but this does not hold in a general braided monoidal category $\mathcal{B}$. The two following lemmas, which will be needed in Section 2.2.4, explain the relationship in the general case.

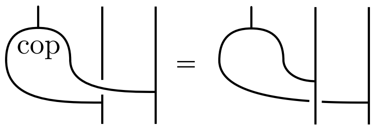

Figure 2.2.3: The opposite coproduct

Lemma 2.2.3.9. *Let B be a quasi-bialgebra in $\mathcal{B}$ with coassociator ϕ.*

(a) Denote by $B^{\overline{\mathrm{cop}}}$ the object B equipped with the same product but opposite coproduct Δ^{cop}. Then $B^{\overline{\mathrm{cop}}}$ is a quasi-bialgebra object in $\mathcal{B}$ (with coassociator ϕ^{-1}). By definition,

$$B^{\overline{\mathrm{cop}}}\text{-}\mathbf{Mod}(\mathcal{B}) \cong {}^{\mathrm{cop}}B\text{-}\mathbf{Mod}(\overline{\mathcal{B}}),$$

is an isomorphism of monoidal categories. If B is a quasi-Hopf algebra, then so is $B^{\overline{\mathrm{cop}}}$ with the same antipode S (and the same elements a, b as B).

(b) Denote by $B^{\overline{\mathrm{op}}}$ the object B with the same coproduct but opposite product m^{op}. Then $B^{\overline{\mathrm{op}}}$ is a quasi-bialgebra object in $\mathcal{B}$ (with coassociator ϕ). By definition,

$$B^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}(\mathcal{B}) \cong {}^{\mathrm{op}}B\text{-}\mathbf{CoMod}(\overline{\mathcal{B}}).$$

If B is a Hopf algebra, then so is $B^{\overline{\mathrm{op}}}$ with the same antipode S (and the same a, b as B).

Proof. This is an exercise in graphical calculus. □

We need to introduce further structure via reconstruction to state the axioms for a universal R-matrix in this general setting. Similar to the definition of the opposite coproduct, we introduce *twisted coassociators*. For any element $\sigma \in S_3$, we define $\phi^{\overline{\sigma}}$ by reconstruction corresponding to the transformation $\Psi_\sigma^{-1}F(\alpha)\Psi_\sigma$, where Ψ_σ is the composition of Ψs acting on two of the three tensor components according to a minimal expression of σ as a product of transpositions of adjacent indices. For example, if $\sigma = (13) = (12)(23)(12)$, then $\Psi_\sigma = (\Psi \otimes \mathrm{Id})(\mathrm{Id} \otimes \Psi)(\Psi \otimes \mathrm{Id})$. Such an expression of σ is not unique, but the resulting Ψ_σ is independent of choice.

After these preliminary observations and notations, we turn back to part (c) of the proof of 2.2.3.5. The universal R-matrix R corresponds to the natural transformation ϱ given by

$$\varrho_{X,Y} = \Psi^{-1}_{F(X),F(Y)} F(\Psi^{\mathcal{M}}_{X,Y})\colon F^{\otimes 2} \to F^{\otimes 2}.$$

It satisfies the following axioms, obtained from the braiding axioms (in $\mathcal{M}$):

$$m_{B\otimes B}(R \otimes \Delta) = m_{B\otimes B}(\Delta^{\mathrm{cop}} \otimes R), \tag{2.2.45}$$

$$(\mathrm{Id} \otimes \Delta)R = m^5_{B^{\otimes 3}}((\phi^{-1})^{\overline{(123)}} \otimes (\mathrm{Id} \otimes 1 \otimes \mathrm{Id})R \otimes \phi^{\overline{(12)}} \otimes (R \otimes 1) \otimes \phi^{-1}), \tag{2.2.46}$$

$$(\Delta^{\mathrm{cop}} \otimes \mathrm{Id})R = m^5_{B^{\otimes 3}}(\phi^{\overline{(13)}} \otimes (1 \otimes R) \otimes (\phi^{-1})^{\overline{(123)}} \otimes ((\mathrm{Id} \otimes 1 \otimes \mathrm{Id})R) \otimes \phi^{\overline{(12)}}). \tag{2.2.47}$$

Remark 2.2.3.10. For the purpose of Sections 2.3.4 and 2.4, we either have $B = \mathbf{Vect}_k$ which is symmetric monoidal, or $\phi = 1 \otimes 1 \otimes 1$, so these axioms simplify. We include this generality for completeness.

We are particularly interested in cases for which the functor $\mathcal{M} \to B\text{-}\mathbf{Mod}(\mathcal{B})$ is an equivalence. For this reason, we mention a version of the reconstruction theorem similar to Tannaka–Krein duality [81, Theorem 2.1] (where comodules are used).

Theorem 2.2.3.11. *Let $\mathcal{M}$ be a rigid monoidal category which is k-linear over $\mathbf{Vect}_k^{fd}$ and equivalent to as small one. Then there is a finite-dimensional quasi-Hopf algebra H over k with invertible antipode such that $\mathcal{M} \cong H\text{-}\mathbf{Mod}(\mathbf{Vect}_k^{fd})$.*

This theorem can be viewed as a special case of Theorems 2.2.3.3 and 2.2.3.5. As before, if $\mathcal{M}$ is braided, then H is quasitriangular. The following result may be useful when addressing the question whether $\mathcal{M} \cong B\text{-}\mathbf{Mod}(\mathcal{B})$:

Lemma 2.2.3.12 (cf. [52, Theorem 16])**.** *If the functor $F\colon \mathcal{M} \to \mathcal{B}$ is faithful, then the functor $\mathcal{M} \to B\text{-}\mathbf{Mod}(\mathcal{B})$ is fully faithful.*

2.2.4 Yetter–Drinfeld and Hopf Modules

For the reminder of this section, let $\mathcal{M} = \mathbf{Mod}\text{-}B(\mathcal{B})$ where B is a quasi-bialgebra object in $\mathcal{B}$. We want to reinterpret the categories $\mathcal{Z}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ in more familiar terms leading to the definition of Yetter–Drinfeld and Hopf modules.

Note that the quasi-monoidal forgetful functor $F\colon \mathcal{M} \to \overline{\mathcal{B}}$ always has a section triv given by the functor mapping an object X in $\mathcal{B}$ to the trivial module X^{triv} on it (with action given by the counit of $\mathcal{B}$). This functor is the identity on morphisms. In the following, we will often omit writing the functor F.

Remark 2.2.4.1. We are considering right B-modules in this section to make a description of the center in terms of right B-modules and left C-modules more convenient (for C dually paired with B) in Section 2.3.2.

A priori, it seems that we can choose whether to consider the forgetful functor F as mapping to $\mathcal{B}$ or $\overline{\mathcal{B}}$. It turns out that it is necessary to use $\overline{\mathcal{B}}$ for the description of the center in terms of YD -modules even though this choice may seem less natural.

Warning 2.2.4.2. The reconstruction theory in Section 2.2.3 for quasi-Hopf algebras is *not* symmetric with respect to switching to right modules. Right action by the coassociator ϕ gives the *inverse* associativity isomorphism α^{-1} rather than α. This happens because for right modules we read the action of a product of elements from left the right, while for left modules from right to left, and the 3-cycle condition (2.1.8) is *not* left-right-symmetric. Note that this problem does not occur in braided Hopf algebra reconstruction (the case $\phi = 1$) as then all conditions are symmetric.

This has the following consequences for reconstruction of duals in $\mathbf{Mod}\text{-}B(\mathcal{B})$:

$$\mathrm{ev}^l_{\mathcal{M}} = \mathrm{ev}_{F(X)}(\mathrm{Id}_{F(X)^*} \otimes \lhd)(\mathrm{Id}_{F(X)^* \otimes F(X)} \otimes S^{-1}b), \tag{2.2.48}$$

$$\mathrm{coev}^l_{\mathcal{M}} = (\lhd \otimes \mathrm{Id}_{F(X^*)})(\mathrm{Id}_{F(X)} \otimes S^{-1}a \otimes \mathrm{Id}_{F(X)^*})\,\mathrm{coev}_{F(X)}, \tag{2.2.49}$$

$$\mathrm{ev}^r_{\mathcal{M}} = \mathrm{ev}_{F(X)}(\mathrm{Id}_{F(X)^*} \otimes \lhd)(\mathrm{Id}_{F(X)^* \otimes F(X)} \otimes b)\Psi^{-1}, \tag{2.2.50}$$

$$\mathrm{coev}^r_{\mathcal{M}} = \Psi(\lhd \otimes \mathrm{Id}_{F(X)^*})(\mathrm{Id}_{F(X)} \otimes a \otimes \mathrm{Id}_{F(X)^*})\,\mathrm{coev}_{F(X)}\,. \tag{2.2.51}$$

These left and right (co)evaluations in $\mathbf{Mod}\text{-}B(\mathcal{B})$ are depictured in Figure 2.2.4.

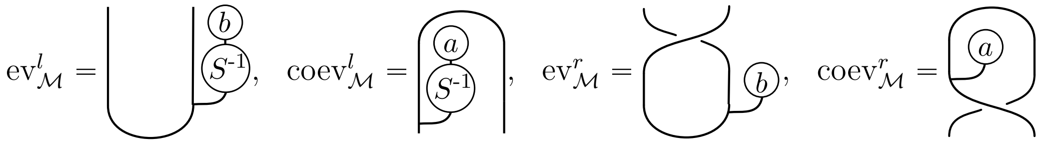

Figure 2.2.4: Right module duals

Definition 2.2.4.3. Let B, C be quasi-bialgebras in a braided monoidal category $\mathcal{B}$ with a morphism $G\colon B \to C$ of quasi-bialgebras.

(a) Define the category of *Yetter–Drinfeld modules* over (B, C) in $\mathcal{B}$, denoted ${}_{C^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$, as having objects V of $\mathcal{B}$ with a right action $\lhd$ of B, together with a map $\delta\colon V \to C \otimes V$ (called *quasi*-coaction) which satisfies the rules given using graphical calculus in Figure 2.2.5.

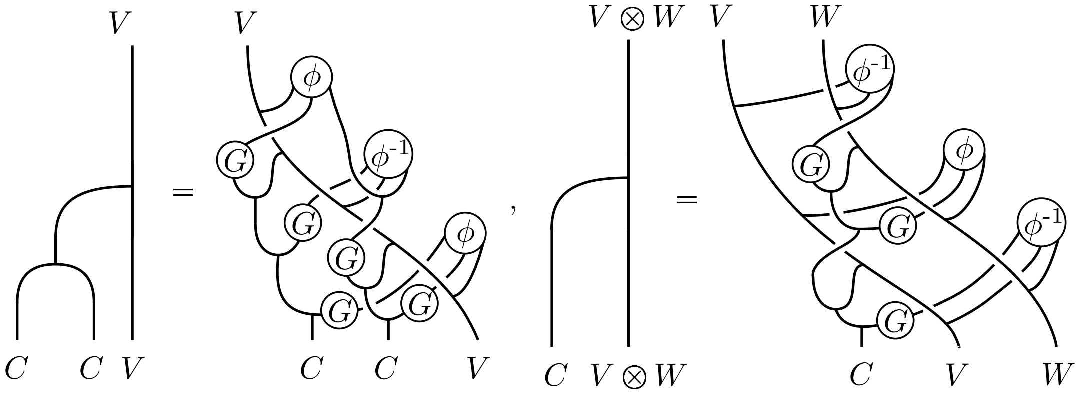

Figure 2.2.5: Quasi-comodule conditions

In the case where B and C are *strict* bialgebras (i.e. with trivial coassociator ϕ), these rules give that δ is a left $C^{\overline{\mathrm{op}}}$-coaction in $\mathcal{B}$. Further, the two structures δ and $\lhd$ satisfy the *Yetter–Drinfeld condition*

$$(m\otimes\mathrm{Id})(G\otimes\delta)\Psi^{-1}(\lhd\otimes\mathrm{Id})(\mathrm{Id}\otimes\Delta) = (m\otimes\mathrm{Id})(\mathrm{Id}\otimes G\otimes\lhd)(\mathrm{Id}\otimes\Psi\otimes\mathrm{Id})(\delta\otimes\Delta). \tag{2.2.52}$$

Morphisms in ${}_{C^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$ are those commuting with the B-module and $C^{\overline{\mathrm{op}}}$-quasi-comodule[21] structure.

---

[21] This definition is *not* the same as the quasi-comodules in [16, 2.1].

(b) Further, define the category of *Hopf modules* over (B, C) in $\mathcal{B}$, denoted by ${}_{C^{\overline{\mathrm{op}}}}\mathcal{H}^B(\mathcal{B})$, which consists of objects of $\mathcal{B}$ with a right action and a map δ as above (satisfying the first rule in Figure 2.2.5), such that the *Hopf condition*

$$\delta\triangleleft = (m \otimes \mathrm{Id})(\mathrm{Id} \otimes G \otimes \triangleleft)(\mathrm{Id} \otimes \Psi \otimes \mathrm{Id})(\delta \otimes \Delta) \tag{2.2.53}$$

is satisfied. Again, morphisms commute with the action and quasi-coaction.

Note that in the case where B and C are strict bialgebras, the Hopf module condition can be reformulated by saying that δ is a morphism of right B-modules, where C is a B-module via the action induced by $G\colon B \to C$. It is helpful to use graphical calculus to visualize the different conditions in Figure 2.2.6. The first picture displays the compatibility condition for YD-modules, the second one for Hopf modules. The reason why $C^{\overline{\mathrm{op}}}$ appears instead of C is to make the category of YD-modules into a monoidal category (see e.g [22, Lemma 3.3.2] for a direct proof in the non-quasi-case). This fails when using C. Note that the coassociator ϕ only appears in the comodule and monoidal rule for YD-modules but not in the compatibility condition.

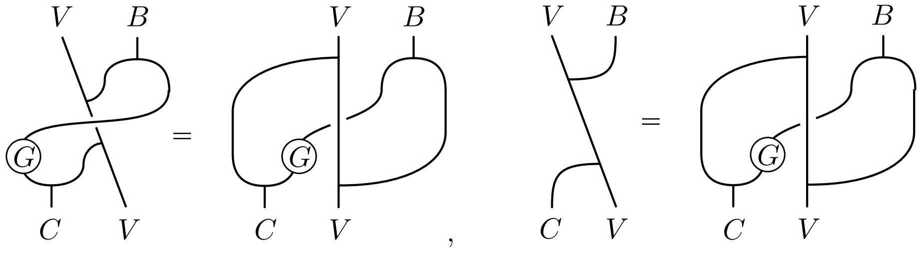

Figure 2.2.6: Right module YD- and Hopf-compatibility conditions

Considering left instead of right B-modules we obtain the category ${}^B_C\mathcal{YD}(\mathcal{B})$ with compatibility conditions from Figure 2.2.7.

Proposition 2.2.4.4. *Let B be a quasi-bialgebra in $\mathcal{B}$.*

(a) The category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$ is monoidal (combining the monoidal structures of **Mod**-$B(\mathcal{B})$ *and the second rule in Figure 2.2.5) and* pre-*braided*[22] *with*

---

[22] That is, the braiding is not necessarily a natural isomorphism.

pre-braiding given by

$$ {}_{B^{\overline{\mathrm{op}}}}\Psi^{B} := (\triangleleft \otimes \mathrm{Id}_V)(\mathrm{Id}_W \otimes \delta)\Psi^{-1}. \tag{2.2.54}$$

The forgetful functor ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B}) \to \overline{\mathcal{B}}$ *is quasi-monoidal*[23] *and preserves the pre-braiding.*

(b) The category ${}^{B}_{B}\mathcal{YD}(\mathcal{B})$ *is monoidal and pre-braided with pre-braiding*

$$ {}^{B}_{B}\Psi := (\triangleright \otimes \mathrm{Id})(\mathrm{Id} \otimes \Psi)(\delta \otimes \mathrm{Id}). \tag{2.2.55}$$

The forgetful functor ${}^{B}_{B}\mathcal{YD}(\mathcal{B}) \to \mathcal{B}$ *is quasi-monoidal and preserves the pre-braiding.*

Proof. These statements will all be proved in 2.2.4.7. Using graphical calculus (and 2.2.56), the braidings are given in Figure 2.2.8. □

Corollary 2.2.4.5.

(a) The forgetful functor ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B}) \longrightarrow$ **Mod**-$B(\mathcal{B})$ *is a monoidal functor of pre-braided monoidal categories.*

(b) The forgetful functor ${}^{B}_{B}\mathcal{YD}(\mathcal{B}) \longrightarrow B$-**Mod**$(\mathcal{B})$ *is a monoidal functor of pre-braided monoidal categories.*

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[23] The forgetful functor is monoidal if B is a bialgebra.

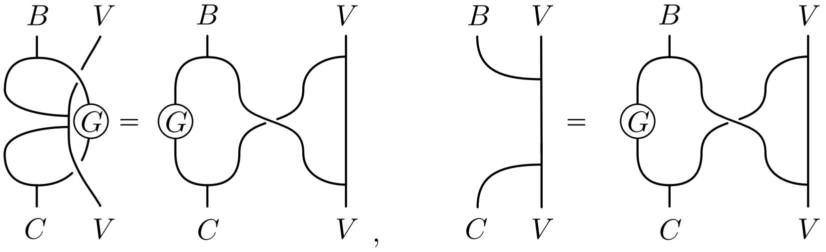

Figure 2.2.7: Left module YD- and Hopf-compatibility conditions

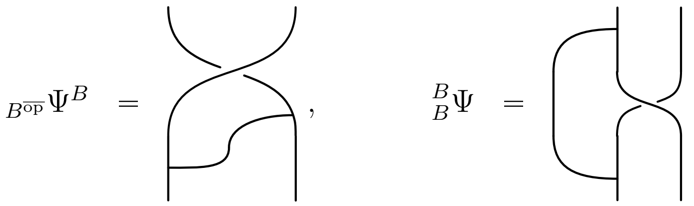

Figure 2.2.8: The braidings of YD-modules

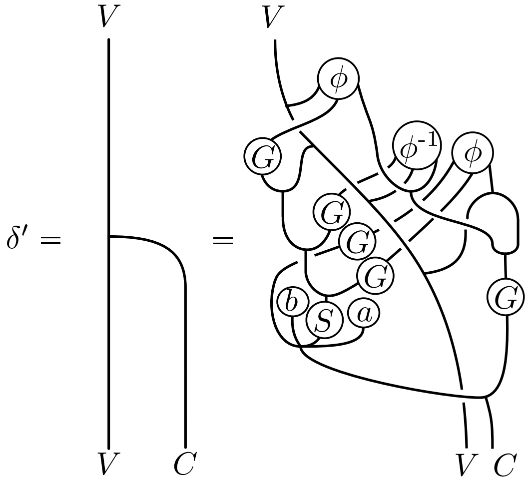

Figure 2.2.9: The right quasi-coaction δ'

Lemma 2.2.4.6. *Let B, C be quasi-Hopf algebras in $\mathcal{B}$ with a morphism $G\colon B \to C$ preserving the structure. For any YD-module V over (B, C) with quasi-coaction δ, and any C-module X, the map*

$$c(\delta)_X\colon V \otimes X \to X \otimes V, \quad c(\delta)_X := (\triangleleft \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)\Psi^{-1}_{X,V}$$

has an inverse $c(\delta)^{-1}_X\colon X \otimes V \to V \otimes X$ which equals $(\mathrm{Id}_V \otimes \triangleleft)(\Psi \otimes \mathrm{Id}_C)(\mathrm{Id}_X \otimes \delta')$, for a right *$C$-quasi-coaction δ'. We give the formula for δ' in Figure 2.2.9.*

Proof. We first show that $c(\delta)^{-1}$ is a right inverse to $c(\delta)$. For this, it suffices to show that $c(\delta)_C\delta' = 1 \otimes \mathrm{Id}_V$. Insert $1 \otimes 1 \otimes 1 = (m \otimes \mathrm{Id})(\mathrm{Id} \otimes S \otimes \mathrm{Id})((\Delta G) \otimes \mathrm{Id}_B \otimes G)\phi$ into the right hand side of this equation δ' so that the right hand side of (2.1.8) appears. Next, apply the 3-cycle condition and note that, using Figure 2.2.5, $m_{B^{\otimes 3}}(S \otimes a \otimes \mathrm{Id})(\Delta \otimes \mathrm{Id}_V)\delta = a$ appears. The expression then simplifies to $1 \otimes \mathrm{Id}_V$ using (2.1.10).

To show that $c(\delta)^{-1}$ is a left inverse, it suffices to show that $c(\delta)_C^{-1}\delta = \mathrm{Id}_C \otimes 1$. This is done in a similar way to the first part (using 3-cycle manipulations and the antipode axioms). □

Proposition 2.2.4.7. *Let $\mathcal{B}$ be a braided monoidal category, B a quasi-Hopf algebra in $\mathcal{B}$. Further assume that*

$$G\colon \mathbf{Mod}\text{-}C(\mathcal{B}) \to \mathbf{Mod}\text{-}B(\mathcal{B}) = \mathcal{M}$$

be a monoidal functor factoring through the forgetful functor F. Then the category $\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M})$ is isomorphic as a monoidal category to the category of YD-modules ${}_{C^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B})$. If $G = \mathrm{Id}_{\mathcal{M}}$, then the isomorphism is one of braided monoidal categories. Further, the category $\mathcal{H}_{\mathcal{B}}^{G}(\mathcal{M})$ is isomorphic to the category of Hopf modules ${}_{C^{\overline{\mathrm{op}}}}\mathcal{H}^{B}(\mathcal{B})$.

Proof. As G factors through F, G is monoidal. Further observe that by the universal property of reconstruction (2.2.27) the functor G is equivalent to a morphism of quasi-bialgebras $B \to C$ which we also denote by G.

First, consider the case of $\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M})$. We recall that all objects in $\mathcal{Z}_{\mathcal{B}}^{G}(\mathcal{M})$ are (F, triv)-admissible. To fix notation, we write $\mathrm{Nat}^{\otimes}(V \otimes G, G \otimes V)$ for monoidal transformations satisfying (2.2.12) which are not necessarily invertible. Using Nat instead of Isom in the notation of (2.2.13), we can define a functor

$$\delta(-)\colon\ \mathrm{Nat}_{\mathcal{B}}^{\otimes}(\mathcal{V} \otimes G, G \otimes \mathcal{V}) \to {}_{C^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B})$$

by mapping (V, c) to V with the quasi-coaction $\delta(V, c)\colon V \to C \otimes V$ defined as $c_B(\mathrm{Id}_V \otimes 1)$. On morphisms, δ is just the identity. Conversely, define $c(\delta)$ for a coaction δ on V to be the natural transformation $V \otimes G \to G \otimes V$ defined by $c(\delta)_V := (\triangleleft \otimes \mathrm{Id}_V)(\mathrm{Id}_{G(X)} \otimes G \otimes \mathrm{Id}_V)(\mathrm{Id}_{G(X)} \otimes \delta)\Psi^{-1}_{V,G(X)}\colon V \otimes G(X) \to G(X) \otimes V$ for any object X of $\mathcal{M}$, using the inverse braiding Ψ^{-1} in the base category $\mathcal{B}$.

We have to show that the functors δ are well-defined (i.e. map to the categories claimed). For this, we first verify that under the functor $\delta(-)$, the requirement that c is $\otimes$-compatible gives that $\delta(c)$ is a C-quasi-comodule and vice versa. Further we have to show that the Yetter–Drinfeld condition (2.2.52) for $\delta(V, c)$ and the

B-action on V corresponds precisely the condition that $c_B(\mathrm{Id}\otimes 1)$ is a morphism of B-comodules. The key observation to prove this is the identity

$$c_B = (m_B \otimes \mathrm{Id}_V)(G \otimes \mathrm{Id}_B \otimes \mathrm{Id}_V)(\mathrm{Id}\otimes \delta(c))\Psi^{-1}_{V,B}. \tag{2.2.56}$$

Note that this identity crucially depends on (F,P)-admissibility. It is computed by observing that $\mathrm{Id}_V = \lhd(\mathrm{Id}_V \otimes 1)$, where we view $\lhd$ as a morphism of B-modules $V^{\mathrm{triv}} \otimes B \to V$. The equality arises if we apply naturality of c to the regular action $\lhd\colon B^{\mathrm{triv}} \otimes B \to B$. Note that all the associativity isomorphisms (which enter the picture as right action by ϕ) vanish because they act on B^{triv}. In order to prove the observation above, we use 2.2.2.8 and rewrite it using (2.2.56).

The mappings δ and c are mutually inverse: It is clear that $\delta((V, c(\delta))) = \delta$ as $\Psi(\mathrm{Id}\otimes 1) = 1 \otimes \mathrm{Id}$. To verify that $c(\delta(V,c)) = c$ we use that the action $\lhd$ is a morphism of right B-modules $V^{\mathrm{triv}} \otimes B \to V$, where B has the regular action. Applying naturality of c, monadicity of c, and the assumption that (V,c) is admissible over $\overline{\mathcal{B}}$ implies that $c(\delta(V,c)) = c$. This establishes that $\delta(-)$, with inverse $c(-)$, form an isomorphism of categories.

For $c(\delta)$ to give an object of the center, it needs to be invertible. This is not true for general quasi-bialgebras, but can be assured using the assumption that the antipode S exists. For this, we use Lemma 2.2.4.6. Hence $\mathcal{Z}^G_{\mathcal{B}} \cong {}_{C^{\mathrm{op}}}\mathcal{YD}^B(\mathcal{B})$ is an isomorphism of categories via the mutually inverse functors $\delta(-)$ and $c(-)$.

Looking at the Hopf-center $\mathcal{H}^G_{\mathcal{B}}(\mathcal{M})$, we note that the one can run an analogous argument to establish an isomorphism with the category of Hopf modules ${}_{C^{\mathrm{op}}}\mathcal{H}^B(\mathcal{B})$. In this case, the compatibility condition obtained is (2.2.53).

Next, restricting to the case where $G = \mathrm{Id}_{\mathcal{M}}$, we note that the monoidal structure and braiding of ${}_{C^{\mathrm{op}}}\mathcal{YD}^B(\mathcal{B})$ are precisely the ones induced by the monoidal structure of $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M})$ thus making the isomorphism δ an isomorphism of braided monoidal categories. Monadicity of the equivalence can also be show for a general monoidal functor G.

If B is only a quasi-bialgebra, the equivalence does not hold as stated, as $\otimes$-compatible natural transformations are not necessarily invertible. We however still obtain a pre-braiding on the category of YD-modules. This establishes Proposition 2.2.4.4 of which we had postponed the proof. The key observation for translating properties and constructions in $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M})$ to the description in terms of Yetter–Drinfeld modules is (2.2.56). Using this, the monoidal structure in the Drinfeld center (cf. 2.2.2.8) translates precisely to the claimed monoidal structure on the category of Yetter–Drinfeld modules (which comes from the coaction on B and the second axiom in Figure 2.2.5). This completes the proof of the isomorphism of monoidal categories $\mathcal{Z}^G_{\mathcal{B}}(\mathcal{M}) \cong {}_{C^{\mathrm{op}}}\mathcal{YD}^B(\mathcal{B})$. □

The case we will be most interested in is $G = \mathrm{Id}_{\mathcal{M}}$. In this case the category $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is equivalent to the category of YD-modules over B in $\mathcal{B}$, and the category $\mathcal{H}_{\mathcal{B}}(\mathcal{M})$ is equivalent to the category of Hopf modules over B in $\mathcal{B}$.

Similarly, for left B-modules, the categories $\mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{Mod}(\mathcal{B}))$ and ${}^B_B\mathcal{YD}(\mathcal{B})$ (and the corresponding Hopf-versions), are isomorphic as (braided monoidal) categories. Consider the isomorphism of categories

$$\mathbf{Mod}\text{-}B(\mathcal{B}) \to B^{\overline{\mathrm{cop}}}\text{-}\mathbf{Mod}(\mathcal{B}), \quad (V, \triangleleft) \mapsto (V, \triangleright := \triangleleft \Psi^{-1}(S^{-1} \otimes \mathrm{Id}_V)).$$

As the center constructions are stable under isomorphism, we find that

$$\mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{Mod}(\mathcal{B})) \cong \mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B^{\overline{\mathrm{cop}}}(\mathcal{B})), \tag{2.2.57}$$

and the corresponding equivalence for the Hopf centers holds.

Remark 2.2.4.8. It is important to note for later applications that Proposition 2.2.4.7 does not rely on $\mathcal{B}$ being rigid. This will enable us to work with the category of countably infinite-dimensional vector spaces later.

A fundamental theorem, proved in this general form in [23, 3.5.2], states that provided that in the category $\mathcal{B}$ equalizers split, there is an equivalence of categories $\mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B})) \cong \mathcal{B}$, for B a Hopf algebra object in $\mathcal{B}$. We can recover part of this statement in the quasi-case:

Theorem 2.2.4.9. *Assume that $\mathcal{B}$ has split idempotents and let B be a quasi-Hopf algebra in $\mathcal{B}$. Then there exists a functor*

$$\mathrm{Res}\colon \mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B) \xrightarrow{\sim} \mathcal{B}, \qquad V \mapsto V^B,$$

with right inverse given by the fully faithful functor

$$\mathrm{Ind}\colon \mathcal{B} \to \mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B), \quad V \mapsto (B \otimes V, (\triangleleft \otimes \mathrm{Id}_V)(\mathrm{Id}_B \otimes \Psi), \delta \otimes \mathrm{Id}_V), \quad g \mapsto \mathrm{Id}_B \otimes g.$$

Proof. First check that the functor Ind gives Hopf modules as stated, which is an easy exercise in graphical calculus. Next, it is possible to see that the functor Ind is fully faithful. To prove fullness, use that each morphism $f\colon \mathrm{Ind}(V) \to \mathrm{Ind}(W)$ is of the form $\mathrm{Id}_B \otimes f'$, where $f' = (\varepsilon\otimes\mathrm{Id}_V)f(1\otimes\mathrm{Id}_V)$ (using an analogous computation as in [23]).

Next, given a Hopf module $(V, \triangleleft, \delta)$, we consider the morphism

$$e_V := \triangleleft(\triangleleft \otimes a^{-1})(\mathrm{Id}_V \otimes S^{-1})\Psi^{-1}(\mathrm{Id}_B \otimes \triangleleft)(\delta \otimes a). \tag{2.2.58}$$

Using the antipode axiom and the condition (2.2.53), we can show that e_V is an idempotent in $\mathcal{B}$. By assumption on $\mathcal{B}$, it splits as $e_V = \iota_V \pi_V$, where $\iota_V\colon V^B \to V$ and $\pi_V\colon V \to V^B$ for an object V^B of $\mathcal{B}$, s.t. $\pi_V \iota_V = \mathrm{Id}_{V^B}$. We can now define the functor Res using a choice of such a splitting. On morphisms, we map $f\colon V \to W$ to $\pi_W f \iota_V$. To show this gives a functor, we use the identity that $f e_V = e_W f$ for any morphism of Hopf modules. This follows directly from f commuting with δ and $\triangleright$.

It is easy to check that $\mathrm{Res}\,\mathrm{Ind} \cong \mathrm{Id}_{\mathcal{B}}$ directly. Observe that in this case, $\iota = (1 \otimes \mathrm{Id}_V)$, and $\pi = (\varepsilon \otimes \mathrm{Id}_V)$ for the object $\mathrm{Ind}(V) = B \otimes V$. ☐

Note that if both $\mathcal{M}$ and its strictification (from 2.2.1.6) are higher representable, then we can conclude that $\mathcal{H}_{\mathcal{B}}(\mathcal{M}) \cong \mathcal{B}$ is an equivalence of categories using the proof of [23, 5.3.2].

Remark 2.2.4.10. The construction of δ' in Lemma 2.2.4.6 gives a functor

$$\Theta\colon {}_{C^{\mathrm{op}}}\mathcal{YD}^B(\mathcal{B}) \to \mathcal{YD}^B_C(\mathcal{B}),$$

to the category of right YD-modules, which is the identity on morphisms. This functor is part of a monoidal equivalence of categories. This symmetry is not valid for Hopf modules. In fact, for $(V, c) \in \mathcal{H}^G_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$, the pair (V, c) is *not* a right Hopf module over (B, C).

Recall that, given $\mathcal{B}$ is rigid, then it has simultaneous left and right duals (cf. Proposition 2.2.3.4). Further, $\mathcal{M}$ has left left and right duals if and only if B is a quasi-Hopf algebra with invertible antipode (cf. 2.2.4.2). We reinterpret the results of Proposition 2.2.2.12 under the equivalence of Proposition 2.2.4.7 showing that in this case the categories of YD-module have duals.

Corollary 2.2.4.11. *Let B, C be quasi-Hopf algebras, $G\colon B \to C$. Then the category ${}_{C^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$ is rigid. The left dual action is given by*

$$\triangleleft^* = (\mathrm{ev}_V \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes \triangleleft \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*\otimes V} \otimes \Psi_{V^*,B})(\mathrm{Id}_{V^*} \otimes \mathrm{coev}_V \otimes S^{-1}), \tag{2.2.59}$$

and the dual quasi-coaction δ^* is defined, using graphical calculus where the right quasi-coaction δ' is denoted by $\vdash\!\!\frown$, as depictured in Figure 2.2.10.

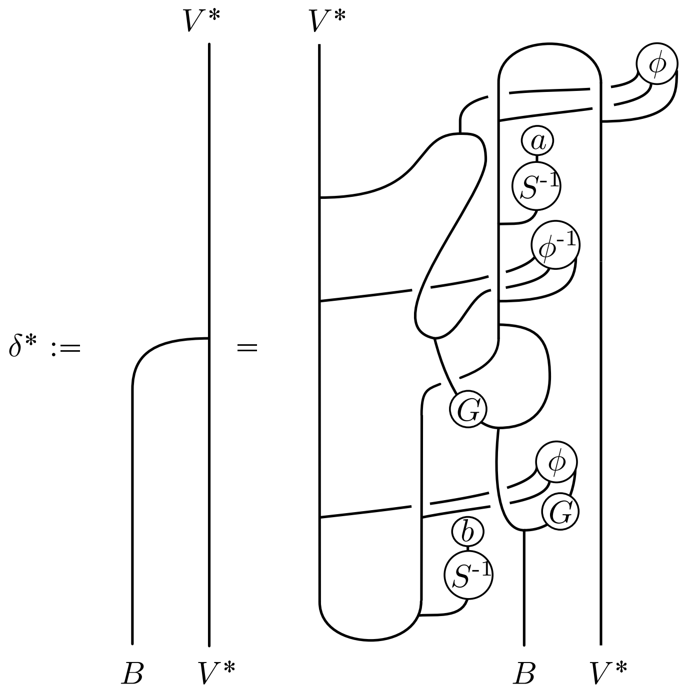

Figure 2.2.10: Dual quasi-coaction

Proof. The formula for δ^* follows by translating Proposition 2.2.2.12 under (2.2.56). We use δ' to express the inverse commutativity isomorphism in terms of YD-modules. One can then use the functor Θ to compute this from the data of the quasi-coaction δ. The dual right B-action on V^* is the usual dual action in $\mathcal{M} = \mathbf{Mod}\text{-}B(\mathcal{B})$. $\square$

Finally, it is now a direct corollary that the category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$ acts on the category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{H}^B(\mathcal{B})$ on the left, using Proposition 2.2.4.7 to translate Corollary 2.2.2.15 to the module-(quasi-)comodule description of this section.

2.2.5 Quasitriangularity

Note that for any monoidal category $\mathcal{M}$ over $\mathcal{B}$, there always exists a forgetful monoidal functor $F\colon \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \to \mathcal{M}$. In this section, we want to study the situation when this functor has a right inverse, i.e. a fully faithful monoidal functor $R\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ such that there exists an natural isomorphism $\gamma\colon FR \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathcal{M}}$. This will induce a braiding on the category $\mathcal{M}$ coming from the braiding Ψ of $\mathcal{Z} = \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$. Note that the natural isomorphism γ is required to be compatible with the monadicity transformations μ^F, μ^R. That means that the diagram

$$\begin{tikzcd}
FR(X\otimes Y) \arrow[r, "F(\mu^R_{X,Y})"] \arrow[dr, "\gamma_{X\otimes Y}"'] & F(R(X)\otimes R(Y)) \arrow[r, "\mu^F_{R(X),R(Y)}"] & FR(X)\otimes FR(Y) \arrow[dl, "\gamma_X\otimes\gamma_Y"] \\
& X\otimes Y &
\end{tikzcd} \tag{2.2.60}$$

commutes. Moreover, the functor $RF\colon \mathcal{Z} \to \mathcal{Z}$ is required to be a functor of braided monoidal categories. This means that the diagram

$$\begin{tikzcd}
RF(A\otimes B) \arrow[r, "R(\mu^F_{A,B})"] \arrow[d, "RF(\Psi_{A,B})"'] & R(F(A)\otimes F(B)) \arrow[r, "\mu^R_{F(A),F(B)}"] & RF(A)\otimes RF(B) \arrow[d, "\Psi_{RF(A),RF(B)}"'] \\
RF(B\otimes A) \arrow[r, "R(\mu^F_{B,A})"] & R(F(B)\otimes F(A)) \arrow[r, "\mu^R_{F(B),F(A)}"] & RF(B)\otimes RF(A)
\end{tikzcd} \tag{2.2.61}$$

is required to commute. We say that RF *preserves* the braiding Ψ in this case.

Definition 2.2.5.1. Let $\mathcal{M}$ be a monoidal category, $\mathcal{Z}$ braided monoidal, $F\colon \mathcal{Z} \to \mathcal{M}$ a monoidal functor. We call a monoidal functor $R\colon \mathcal{M} \to \mathcal{Z}$ which is a right inverse to F, with the data of γ satisfying the compatibilities (2.2.60) and (2.2.61) as above a *quasitriangular structure* on $\mathcal{M}$. This implies that R is fully faithful.

Lemma 2.2.5.2. *A quasitriangular structure $R\colon \mathcal{M} \to \mathcal{Z}$ gives a braiding on the category $\mathcal{M}$, $\mathcal{Z} \xrightarrow{F} \mathcal{M}$, such that the functors R and F preserve the braidings.*

Proof. For any two objects X and Y of $\mathcal{M}$, define the braiding by the composition

$$\Psi^{\mathcal{M}}_{X,Y} := \gamma_{Y\otimes X} F((\mu^R_{Y,X})^{-1}) F(\Psi^{\mathcal{Z}}_{R(X),R(Y)}) F(\mu^R_{X,Y}) \gamma^{-1}_{X\otimes Y}. \tag{2.2.62}$$

The functor F preserves this braiding. To see this, combine three commutative squares: The one defining $\Psi^{\mathcal{M}}$ applied to $F(A)$, $F(B)$, F applied to (2.2.61), and naturality of γ in $F(\Psi^{\mathcal{Z}}_{A,B})$. This gives

$$\begin{aligned}
&\gamma_{F(B\otimes A)}FR(\mu^F_{B,A})F(\mu^R_{F(B),F(A)})^{-1}F(\mu^R_{F(B),F(A)})\gamma^{-1}_{F(B)\otimes F(A)}\Psi^{\mathcal{M}}_{F(A),F(B)}\\
&= F(\Psi^{\mathcal{Z}}_{A,B})\gamma_{F(A\otimes B)}FR(\mu^F_{A,B})F(\mu^R_{F(A),F(B)})^{-1}F(\mu^R_{F(A),F(B)})\gamma^{-1}_{F(A)\otimes F(B)}.
\end{aligned}$$

We can simplify to

$$\begin{aligned}
&\gamma_{F(A\otimes B)}FR(\mu^F_{A,B})\gamma^{-1}_{F(A)\otimes F(B)}\\
&= \mu^F_{A,B}\gamma_{F(A)\otimes F(B)}F(\mu^R_{F(A),F(B)})^{-1}F(\mu^R_{F(A),F(B)})\gamma^{-1}_{F(A)\otimes F(B)}\\
&= \mu^F_{A,B}(\gamma_{F(A)}\otimes\gamma_{F(B)})(\gamma^{-1}_{F(A)}\otimes\gamma^{-1}_{F(B)}) = \mu^F_{A,B},
\end{aligned}$$

by first applying naturality of γ in $\mu^F_{A,B}$, and then applying (2.2.60) twice. Hence $F(\Psi^{\mathcal{Z}}_{A,B})\mu^F_{A,B} = \mu^F_{B,A}\Psi^{\mathcal{M}}_{F(A),F(B)}$ as required.

The proof that R preserves the braiding starts by composing the diagrams of R applied to the definition of $\Psi^{\mathcal{M}}_{X,Y}$ with (2.2.61), and naturality of $\Psi^{\mathcal{Z}}$ applied to $R(\gamma_X)$, $R(\gamma_Y)$. This gives the equality

$$\begin{aligned}
&\Psi^{\mathcal{Z}}_{R(X),R(Y)}(R(\gamma_X)\otimes R(\gamma_Y))\mu^R_{FR(X),FR(Y)}R(\mu^F_{R(X),R(Y)})RF(\mu^R_{X,Y})R(\gamma^{-1}_{X\otimes Y})\\
&= (R(\gamma_Y)\otimes R(\gamma_X))\mu^R_{FR(Y),FR(X)}R(\mu^F_{R(Y),R(X)})RF(\mu^R_{Y,X})R(\gamma^{-1}_{Y\otimes X})R(\Psi^{\mathcal{M}}_{X,Y}).
\end{aligned}$$

Applying first naturality of μ^R, and then (2.2.60) we can simplify to

$$\begin{aligned}
&(R(\gamma_X)\otimes R(\gamma_Y))\mu^R_{FR(X),FR(Y)}R(\mu^F_{R(X),R(Y)})RF(\mu^R_{X,Y})R(\gamma^{-1}_{X\otimes Y})\\
&= \mu^R_{X,Y}R(\gamma_X\otimes\gamma_Y)R(\mu^F_{R(X),R(Y)})RF(\mu^R_{X,Y})R(\gamma^{-1}_{X\otimes Y})\\
&= \mu^R_{X,Y}R(\gamma_{X\otimes Y})R(\gamma_{X\otimes Y})^{-1} = \mu^R_{X,Y}.
\end{aligned}$$

This proves $\mu^R_{Y,X}R(\Psi^{\mathcal{M}}_{X,Y}) = \Psi^{\mathcal{Z}}_{R(X),R(Y)}\mu_{X,Y}$ as claimed. $\square$

Theorem 2.2.5.3. *Let $\mathcal{M}$ be a monoidal category over a braided monoidal category $\mathcal{B}$ (with functors F, P as in 2.2.2.2). Then $\mathcal{M}$ is braided monoidal (and F, P preserve the braidings) if and only if $\mathcal{M}$ has a quasitriangular structure over $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$.*

Proof. Applying Lemma 2.2.5.2 to $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ shows that if such a quasitriangular structure exists, then $\mathcal{M}$ is braided. It is clear that the functors F, P preserve the braidings. For the converse, we use the functor $R\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ defined by mapping an object V of $\mathcal{M}$ to the pair $(V, \Psi^{\mathcal{M}}_{V,-})$, where $\Psi^{\mathcal{M}}$ is the given braiding in $\mathcal{M}$, and the identity on morphisms. $\square$

We can recover the usual definitions of quasitriangularity (including *co*-quasitriangularity, and *weak* quasitriangularity (due to Majid [90, Section 2]), see Section 2.3.3) from this more general definition using reconstruction theory. These results can be given in the general setting over a braided monoidal category $\mathcal{B}$. To do this, we assume that the functors F and R are *strictly* quasi-monoidal and inverse to each other (i.e. $\gamma = \mathrm{Id}_{\mathcal{M}}$, $\mu^F = \mathrm{Id}_{\mathcal{M}}$ and $\mu^R = \mathrm{Id}_{\mathcal{B}}$).

Proposition 2.2.5.4. *A quasi-bialgebra B in $\mathcal{B}$, is quasitriangular, i.e. an R-matrix satisfying (2.2.45)–(2.2.47) exists, if and only if a quasitriangular structure $R\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is given on the category $\mathcal{M} = B$-$\mathbf{Mod}(\mathcal{B})$.*

If we identify $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ with the category of YD-modules over B, the functor R can be identified with the functor[24] *mapping a B-module $(V, \triangleright)$ to the YD-module on V with action $\triangleright$ and quasi-coaction $\Psi(\triangleright \otimes \mathrm{Id}_B)(\mathrm{Id}_B \otimes \Psi)(R \otimes \mathrm{Id})$.*

Proof. Recall that by Lemma 2.2.5.2 the existence of the functor R implies that B-$\mathbf{Mod}(\mathcal{B})$ is braided, with braiding induced by R. By higher representability 2.2.3.2 this gives the existence of a universal R-matrix $R\colon I \to B \otimes B$ satisfying the axioms (2.2.45)–(2.2.47). Moreover, as the functor R preserves the braiding by construction, this gives the equality

$$\Psi(\triangleright_V \otimes \triangleright_W)(\mathrm{Id} \otimes \Psi \otimes \mathrm{Id})(R \otimes \mathrm{Id}_{V \otimes W}) = (\triangleright_W \otimes \mathrm{Id})(\mathrm{Id} \otimes \Psi)(\delta_V \otimes \mathrm{Id}), \quad (2.2.63)$$

as we assume $\mu^R = \mathrm{Id}$. From this, we conclude that the quasi-coaction needs to be as stated by apply naturality of the braiding to $1 \to B$, where B has the regular action. The YD-condition corresponds to the R-matrix axiom that conjugation by R transforms the coproduct into the opposite coproduct Δ^{cop} (see 2.2.45). The other axioms correspond to the quasi-coaction condition and the fact that the functor R is quasi-monoidal. Note that we need left module versions of these

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[24] Given an R-matrix for a Hopf algebra over k, such a functor was introduced in [79].

axioms here. These can however easily obtained from 2.2.2.8 and (2.2.12) using braided left module reconstruction as in 2.2.3.5. Conversely, given a universal R-matrix, one can run the argument backwards and check that the functors are obtained from the axioms (2.2.45)–(2.2.47). □

2.3 Braided Drinfeld and Heisenberg Doubles

In this section, we use reconstruction theory to obtain general definitions of the braided Drinfeld and Heisenberg doubles. In the case of the braided Drinfeld double, this is the double-bosonization in [90, Proposition 4.3]. For this, we will now leave the generality of quasi-Hopf algebras restricting to strict Hopf algebras. The main reason for this is that the picture becomes more symmetric, as the dual of a Hopf algebra is a Hopf algebra (which is *not* true for quasi-Hopf algebras). Recall the definition of dually paired Hopf algebras from 2.1.7. We note that the restriction to Hopf algebras leads to several simplifications. For instance, as mentioned before, the quasi-coaction $\delta\colon B \to B \otimes V$ of the definition of Yetter–Drinfeld modules now becomes a $B^{\overline{\mathrm{op}}}$-coaction. We will summarize the simpler formulae in 2.3.1. Next, we use the categorical definition of quasitriangularity from Section 2.2.5 to discuss the notion of weak quasitriangularity in the setting of dually paired bialgebras in the braided monoidal category $\mathcal{B}$.

Once this preliminary work has been done, we will take $\mathcal{B}$ to be H-**Mod** or H-**CoMod** where H is an ordinary Hopf algebra in $\mathbf{Vect}_k$ which is either quasitriangular or weak quasitriangular (but possibly infinite-dimensional). We then define braided Drinfeld and Heisenberg doubles via reconstruction theory and give explicit presentations using modified Sweedler's notation. We also interpret the analogues of the BGG-category $\mathcal{O}$ in this general context.

Finally, we explain how the categorical action from 2.2.2.15 gives that the braided Heisenberg double is a 2-cocycle twist of the braided Drinfeld double. This generalizes an earlier result of [71] and is the main result of this section.

2.3.1 Simplifications for Strict Hopf Algebras

As mentioned above, for B a Hopf algebra in $\mathcal{B}$, the category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B})$ consists of simultaneous B-modules and $B^{\overline{\mathrm{op}}}$-comodules in $\mathcal{B}$ which satisfy (2.2.52).

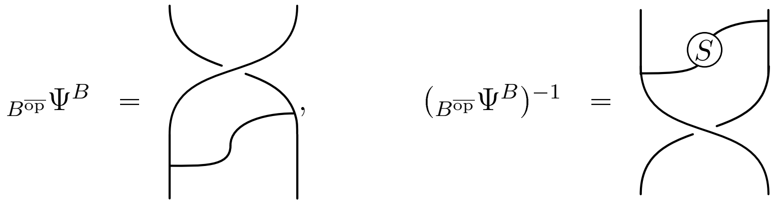

Figure 2.3.1: Braiding and inverse braiding for strict Hopf algebras

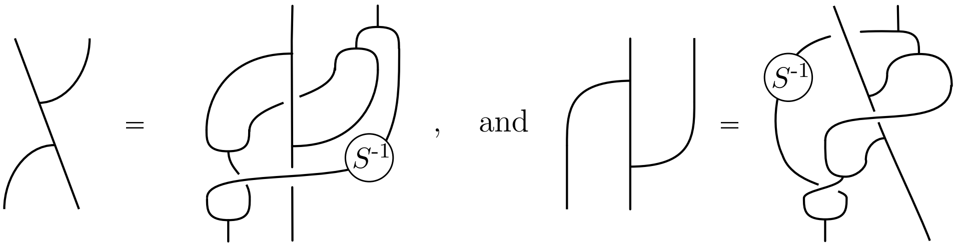

Figure 2.3.2: Equivalent YD-conditions for strict Hopf algebras

Further, the inverse braiding can now be given in Figure 2.3.1. Furthermore, we can introduce an equivalent form of the YD-conditions, given that the antipode is invertible:

Lemma 2.3.1.1. *The YD-condition for* ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^{B}(\mathcal{B})$ *is equivalent to either of the conditions of Figure 2.3.2.*

Proof. This is an exercise in graphical calculus using that $m(S^{-1}\otimes \mathrm{Id})\Psi^{-1}\Delta = 1\varepsilon$. $\square$

We interpret this relation as enabling us to permute the action past the coaction and vice versa (similar formulas can be given for quasi-Hopf algebras, but obstructions caused by the 3-cycle occur). For strict Hopf algebras, we can also give simpler formulae relating the (left) dual YD-modules V^* to the given structures on V.

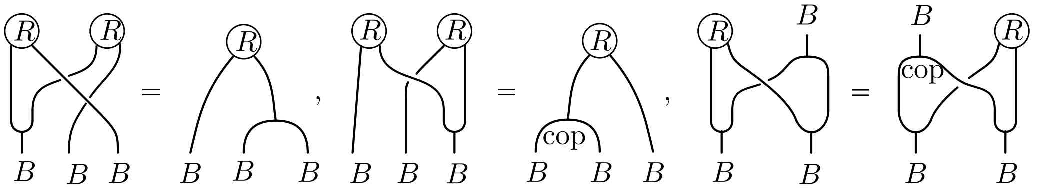

Figure 2.3.3: R-matrix axioms for braided Hopf algebras

Corollary 2.3.1.2. *Let B be a Hopf algebra with invertible antipode in $\mathcal{B}$, then the category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B})$ is rigid. Here, the dual action and coaction are given by*

$$\delta^* = (\mathrm{ev}_V \otimes S \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes \Psi_{B,V} \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes \delta \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes \mathrm{coev}_V) \tag{2.3.1}$$

$$\triangleleft^* = (\mathrm{ev}_V \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^*} \otimes \triangleleft \otimes \mathrm{Id}_{V^*})(\mathrm{Id}_{V^* \otimes V} \otimes \Psi_{V^*,B})(\mathrm{Id}_{V^*} \otimes \mathrm{coev}_V \otimes S^{-1}). \tag{2.3.2}$$

Proof. This is an immediate consequence of Corollary 2.2.4.11, setting $\phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$. ☐

In the setting of strict Hopf algebras in $\mathcal{B}$, the R-matrix axioms simplify to the ones in Figure 2.3.3. See [88, Figure 9.18] for a proof using graphical calculus. Conversely, given such a universal R-matrix for B, the category B-**Mod**$(\mathcal{B})$ (or **Mod**-$B(\mathcal{B})$) is braided monoidal (this is a special case of 2.2.3.5(c)).

Note that, unless we are in the symmetric monoidal case, the convolution inverse R^{-1} does not give an R-matrix up to application of the braiding. It is however possible to define $R^{\overline{\mathrm{op}}}$ by applying reconstruction theory to $\overline{\mathcal{M}}$ (that is, $\mathcal{M}$ with opposite braiding). The axioms for a dual R-matrix (giving a braiding on B-**CoMod**$(\mathcal{B})$) can be obtained by rotating this picture in the horizontal axis.

2.3.2 Yetter–Drinfeld Modules over Dually Paired Hopf Algebras

Given dually paired Hopf algebras C, B (see Section 2.1.7), we can embed the category ${}_{B^{\overline{\mathrm{op}}}}\mathcal{YD}^B(\mathcal{B}) = \mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ into a larger category of Yetter–Drinfeld modules over (C, B) by applying the functor

$${}_{B^{\overline{\mathrm{op}}}}\Phi \colon B^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}(\mathcal{B}) \to B\text{-}\mathbf{Mod}(\mathcal{B}), \quad (V, \delta) \mapsto (V, \triangleright_\delta),$$

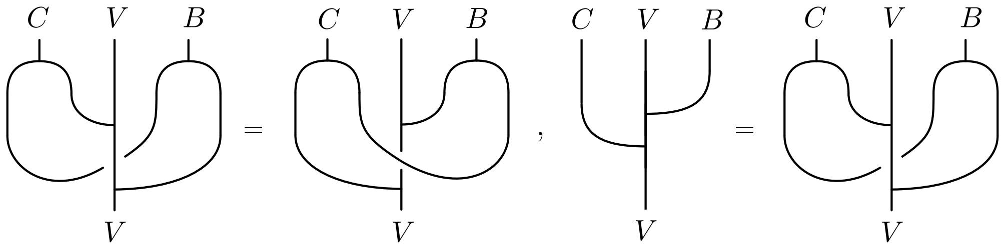

Figure 2.3.4: YD- and Hopf-compatibility for dually paired Hopf algebras

where $\triangleright_\delta = (\mathrm{ev} \otimes \mathrm{Id}_V)(\mathrm{Id}_B \otimes \delta)$, to the left comodule structure of the YD-module. The resulting category is denoted by ${}^C\mathcal{YD}^B(\mathcal{B})$ and consists of objects V of $\mathcal{B}$ with a left C-action $\triangleright$ and a right B-action $\triangleleft$ compatible via the YD-condition

$$\begin{aligned}&(\mathrm{ev} \otimes \triangleleft)(\mathrm{Id} \otimes \Psi_{V,B} \otimes \mathrm{Id})(\mathrm{Id} \otimes \triangleright \otimes \mathrm{Id})(\Delta_C \otimes \mathrm{Id} \otimes \Delta_B)\\ &= (\triangleright \otimes \mathrm{ev})(\mathrm{Id} \otimes \Psi_{C,V} \otimes \mathrm{Id})(\mathrm{Id} \otimes \triangleleft \otimes \mathrm{Id})(\Delta_C \otimes \mathrm{Id} \otimes \Delta_B)\end{aligned}. \tag{2.3.3}$$

Morphisms are required to commute with both the left C- and the right B-action. Note that the monoidal structure is given by the usual monoidal structures for left C- and right B-modules in $\mathcal{B}$. Note that the larger category is not braided in general any more (for this, we require that a coevaluation map $I \to B \otimes C$ exists in $\mathcal{B}$).

We can also embed the category of Hopf modules ${}_{B^{\overline{\mathrm{op}}}}\mathcal{H}^B(\mathcal{B}) = \mathcal{H}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ into a larger category ${}^C\mathcal{H}^B(\mathcal{B})$ by again applying the functor ${}_{B^{\overline{\mathrm{op}}}}\Phi$ to the left comodule structure. The resulting category consists of objects in $\mathcal{B}$ with a left C-module and a right B-module structure such that

$$\triangleright(\mathrm{Id}_B \otimes \triangleleft) = (\mathrm{ev} \otimes \triangleleft)(\mathrm{Id}_C \otimes \Psi_{V,B} \otimes \mathrm{Id}_B)(\mathrm{Id}_C \otimes \triangleright \otimes \mathrm{Id}_{B \otimes B})(\Delta_C \otimes \mathrm{Id}_V \otimes \Delta_B). \tag{2.3.4}$$

Again, morphisms are required to commute with both the left and right action. We add the compatibility conditions in the language of graphical calculus in Figure 2.3.4. Note that the YD-condition given by the first diagram is precisely the one used in [90, A.2].

Proposition 2.3.2.1. *The monoidal category ${}^C\mathcal{YD}^B(\mathcal{B})$ of Yetter–Drinfeld modules acts on the category ${}^C\mathcal{H}^B(\mathcal{B})$ of Hopf modules, where $V \triangleright W := V \otimes W$ with left C-action on $V \otimes W$ given by Δ_C and right B-action on $V \otimes W$ by Δ_B.*

Proof. It is easy to check using graphical calculus that $V \rhd W$ is a Hopf module over (C, B). It is also clear that a pair of morphisms $f\colon V \to V'$, $g\colon W \to W'$ induces a morphism $f \otimes g\colon V \rhd W \to V' \rhd W'$ as both f, g commute with the respective C, B actions, so their tensor product will commute with the tensor product actions. $\square$

Finally, we can also compute the center of monoidal categories of comodules in terms of Yetter–Drinfeld modules.

Proposition 2.3.2.2. *For a Hopf algebra object B in $\mathcal{B}$, there are isomorphisms of categories*

$$\begin{aligned}\mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{CoMod}(\mathcal{B})) &\cong {}_{B}\mathcal{YD}^{B^{\overline{\mathrm{cop}}}}(\mathcal{B}),\\ \mathcal{H}_{\mathcal{B}}(B\text{-}\mathbf{CoMod}(\mathcal{B})) &\cong {}_{B}\mathcal{H}^{B^{\overline{\mathrm{cop}}}}(\mathcal{B}).\end{aligned}$$

Proof. For $(V, c) \in \mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{CoMod}(\mathcal{B}))$, consider the map $\rhd := (\varepsilon \otimes \mathrm{Id}_V)c_B$. Using monadicity of c and the monoidal structure on the center, we find that $\rhd$ is a right $B^{\overline{\mathrm{cop}}}$-module. Dually to the proof of 2.2.4.7 (with the simplification that $\phi = 1 \otimes 1 \otimes 1$, we find that the datum of $\rhd$ allows to recover c, and that conversely any right $B^{\overline{\mathrm{cop}}}$-module that satisfies the YD-condition (2.2.52) gives an object of the center. The proof for the Hopf center is again analogous. $\square$

Corollary 2.3.2.3. *There exists an isomorphism of braided monoidal categories*

$$\mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{CoMod}(\mathcal{B})) \cong \overline{\mathcal{Z}_{\mathcal{B}}(B\text{-}\mathbf{Mod}(\mathcal{B}))}.$$

Proof. This can be proved by showing that the monoidal categories ${}_{B}\mathcal{YD}^{B^{\overline{\mathrm{cop}}}}(\mathcal{B})$ and ${}_{B}^{B}\mathcal{YD}(\mathcal{B})$ are isomorphic. To do this, we recall the equivalence of categories $\mathbf{Mod}\text{-}B^{\overline{\mathrm{cop}}}(\mathcal{B}) \cong \mathbf{Mod}\text{-}^{\mathrm{cop}}B(\mathcal{B})$ by definition of the opposite coproduct. The latter category is equivalent to $B\text{-}\mathbf{Mod}(\mathcal{B})$ by the functor

$$(V, \lhd) \mapsto (V, \rhd := \lhd\Psi(S \otimes \mathrm{Id}_V).$$

We use this functor to translate the datum of the right $B^{\overline{\mathrm{cop}}}$-module of an object of ${}_{B}\mathcal{YD}^{B^{\overline{\mathrm{cop}}}}(\mathcal{B})$ to a left B-module structure. It is an exercise to check, using the alternative YD-conditions of 2.3.1.1, that the resulting left module is YD-compatible with the comodule structure on V. Note that the braiding corresponds to the opposite braiding. As all translations used are isomorphisms (which are the identity on morphisms), this gives an equivalence of the centers as stated. $\square$

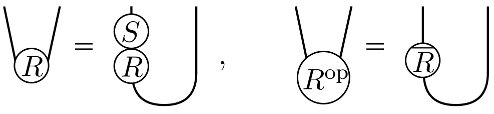

Figure 2.3.5: A weak quasitriangular structure for dually paired Hopf algebras

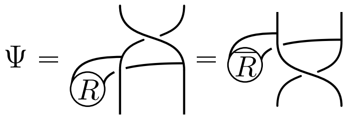

Figure 2.3.6: The opposite R-matrix

2.3.3 Weak Quasitriangularity

We now apply the general categorical viewpoint on quasitriangularity from Section 2.2.5 to a setting of dually paired Hopf algebras B, C in $\mathcal{B}$. The aim is to reinterpret Majid's concept of weak quasitriangularity in this context as it is a crucial feature used in defining braided Drinfeld doubles of braided Hopf algebras in comodule categories as algebras. In fact, it is needed in order to include quantum groups as an example of braided Drinfeld doubles (as done in [90]).

The aim is to rewrite the datum of a dual R-matrix via applying evaluation to the datum of morphisms $R\colon B^{\overline{\mathrm{op}}} \to C$. In Figure 2.3.5 we define two ways of doing this. Here, $R^{\overline{\mathrm{op}}}$ is the dual R-matrix obtain by reconstruction such that the condition of Figure 2.3.6 holds. We will in general need to remember the datum of two morphisms R and $\overline{R}$. It is furthermore required that ev is a pairing satisfying

$$\begin{aligned} \mathrm{ev}(m_C \otimes \mathrm{Id}_B) &= \mathrm{ev}(\mathrm{Id}_C \otimes \mathrm{ev} \otimes \mathrm{Id}_B)(\mathrm{Id}_{C\otimes C} \otimes \Delta_B), \\ \mathrm{ev}(\mathrm{Id}_C \otimes m_B^{\mathrm{op}}) &= (\mathrm{ev} \otimes \mathrm{ev})(\mathrm{Id}_C \otimes \Psi_{C,B} \otimes \mathrm{Id}_B)(\Delta_C \otimes \mathrm{Id}_{B\otimes B}) \end{aligned} \tag{2.3.5}$$

The reason for this is that we want to be able to write the braiding of $B^{\overline{\mathrm{op}}}$-comodules in terms of the induced C-module structure. In Figure 2.3.7, there are two ways of doing this, using the C-module structure on either tensorand of the braiding.

Being able to rewrite the dual R-matrix in this way is essential if we want to apply module reconstruction to braided categories of *co*modules, for the reason that we need to express all formulas in terms of the induced action (from a coaction)

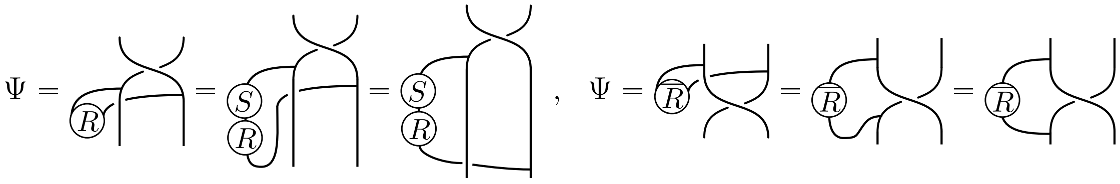

Figure 2.3.7: Braidings for weak quasitriangular structures

via ev. This is used in the construction of the quantum groups $U_q(\mathfrak{g})$ as braided Drinfeld doubles (cf. 2.3.5.11).

Our definition of weak quasitriangularity resembles the idea of the definition in [90, Section 2]. It also gives an intermediate notion between the stronger requirement of C having a universal R-matrix and the weaker assumption of $B^{\overline{\mathrm{op}}}$ having a dual R-matrix.

Lemma 2.3.3.1. *Given a pairing* ev *as in (2.3.5) and let* $R\colon B^{\overline{\mathrm{op}}} \to C$ *be a morphism of bialgebras in* $\mathcal{B}$*. Then for any given left* $B^{\overline{\mathrm{op}}}$*-comodule* (V,δ) *in* $\mathcal{B}$*,*

(i) $\rhd := (\mathrm{ev} \otimes \mathrm{Id}_V)(R \otimes \delta)$ *is a left* $B^{\overline{\mathrm{op}}}$*-module in* $\mathcal{B}$*, and*

(ii) $\lhd := (\mathrm{ev} \otimes \mathrm{Id}_V)(R \otimes \Psi)(\delta \otimes \mathrm{Id}_B)$ *is a right* $B^{\overline{\mathrm{opcop}}}$*-module in* $\mathcal{B}$*.*

Proof. This is not hard to check, using (2.3.5) and the definition of the (co)opposite product. Note that $B^{\overline{\mathrm{opcop}}} = (B^{\overline{\mathrm{op}}})^{\overline{\mathrm{cop}}}$ and hence is a bialgebra in $\mathcal{B}$. □

Proposition 2.3.3.2. *Let* $R\colon B^{\overline{\mathrm{op}}} \to C$ *be a morphism of Hopf algebras with an evaluation as in (2.3.5) such that the condition*

$$\begin{aligned}&(m_C^{op} \otimes \mathrm{ev}(R \otimes \mathrm{Id}_B))(\mathrm{Id}_C \otimes \Psi_{C,B} \otimes \mathrm{Id}_B)(\Delta_C \otimes \Delta_B)\\ &= (\mathrm{ev}(R \otimes \mathrm{Id}_B) \otimes m_B)(\mathrm{Id}_C \otimes \Psi_{C,B} \otimes \mathrm{Id}_B)(\Delta_C \otimes \Delta_B)\end{aligned} \tag{2.3.6}$$

holds. Then there exists two quasitriangular structures on the category of comodules $B^{\overline{\mathrm{op}}}$-**CoMod**$(\mathcal{B})$ *in* $\mathcal{Z}_{\mathcal{B}}(B^{\overline{\mathrm{op}}}$-**Mod**$(\mathcal{B}))$*, both with respect to the forgetful functor.*

(i) The functor mapping (V,δ) *to* $(V,\delta,\rhd := (\mathrm{ev} \otimes \mathrm{Id}_V)(R \otimes \delta))$ *as in 2.3.3.1(i).*

(ii) The functor mapping (V,δ) *to* $(V,\delta,\triangleright := (\mathrm{ev}\otimes\mathrm{Id}_V)(R\otimes\Psi)(\delta\otimes\mathrm{Id}_B)\Psi(S\otimes\mathrm{Id}_V))$*, which is the composition of the functor in 2.3.3.1(ii) with the equivalence 2.3.2.3.*

Proof. In Lemma 2.3.3.1, we checked that the stated compositions of maps indeed give left $B^{\overline{\mathrm{op}}}$-modules (respectively right $B^{\overline{\mathrm{opcop}}}$-modules). The condition (2.3.6) is precisely what is required for the resulting map Ψ as in Figure 2.3.7 to be morphisms of comodules. Hence, $B^{\overline{\mathrm{op}}}$-**CoMod**$(\mathcal{B})$ is braided. This gives the quasitriangular structure of (i) as described and a quasitriangular structure of $B^{\overline{\mathrm{op}}}$-**CoMod**$(\mathcal{B})$ in $\overline{\mathcal{Z}_{\mathcal{B}}(B^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}(\mathcal{B}))}$. By the equivalence 2.3.2.3, this corresponds to the quasitriangular structure (ii) as stated. □

Definition 2.3.3.3. A *weak quasitriangular pair* for dually paired Hopf algebras C, B is the datum of two morphisms of Hopf algebras $R, \overline{R}\colon B^{\overline{\mathrm{op}}} \to C$ in $\mathcal{B}$ which satisfy (2.3.6), s.t. with respect to a pairing ev of $B^{\overline{\mathrm{op}}}$ and C as in (2.3.5), the axioms from Figure 2.3.5 hold.

Note that such maps $R, \overline{R}$ are automatically convolution invertible with convolution inverses given by $R^{-1} := RS$ and $\overline{R}^{-1} := \overline{R}S$. The existence of quasitriangular structures as in 2.3.3.2 does *not* imply the existence of the maps R, $\overline{R}$. In fact, it only implies the existence of dual universal R-matrices. In the following, we will describe how one can obtain the maps R, $\overline{R}$ via reconstruction theory under certain representability conditions.

Lemma 2.3.3.4. *If the category* $B^{\overline{\mathrm{op}}}$-**CoMod**$(\mathcal{B})$ *can be (higher) represented in the sense of Section 2.2.3 as modules over a Hopf algebra* C *in* $\mathcal{B}$*, then there exists a weak quasitriangular pair* R, $\overline{R}$ *for the dually paired Hopf algebras* C, B.

Proof. Consider the image of $(B^{\overline{\mathrm{op}}})^{\mathrm{coreg}}$, $B^{\overline{\mathrm{op}}}$ with coregular coaction given by the coproduct, under $E\colon B^{\overline{\mathrm{op}}}$-**CoMod**$(\mathcal{B}) \to C$-**Mod**$(\mathcal{B})$. This gives a map $\gamma\colon C\otimes B \to B$. Note that the functor E factors through the forgetful functor to $\mathcal{B}$ by assumption. We define

$$\mathrm{ev} := \varepsilon\gamma\colon C\otimes B \to I.$$

Now let δ be any $B^{\overline{\mathrm{op}}}$-comodule structure on an object V of $\mathcal{B}$. Then δ is a morphism of $B^{\overline{\mathrm{op}}}$-comodules $V \to (B^{\overline{\mathrm{op}}})^{\mathrm{coreg}} \otimes V^{\mathrm{triv}}$. This follows simply from the comodule condition. Hence, δ is a morphism of C-modules w.r.t. the image under E and we derive that for the C-action (denoted by $\triangleright$) on V under E we have

$$\triangleright = (\varepsilon \otimes \mathrm{Id}_V)\delta\gamma = (\varepsilon \otimes \mathrm{Id}_V)(\triangleright \otimes \mathrm{Id}_V)(\mathrm{Id}_C \otimes \delta) = (\mathrm{ev} \otimes \mathrm{Id}_V)(\mathrm{Id}_C \otimes \delta). \tag{2.3.7}$$

Hence, the functor E is induced by the map ev. From this we derive directly that ev satisfies the axioms (2.3.5).

To find $\overline{R}$, we apply reconstruction (of C-modules) to the natural transformation

$$\overline{R}\colon B^{\overline{\mathrm{op}}} \otimes F \to F, \quad \overline{R}_{(V,\delta)} = (R^{\overline{\mathrm{op}}} \otimes \mathrm{Id}_V)(\mathrm{Id}_B \otimes \delta).$$

The morphism R can be obtained by applying reconstruction to

$$R\colon B^{\overline{\mathrm{op}}} \otimes F \to F, \quad R_{(V,\delta)} = (R \otimes \mathrm{Id}_V)(S^{-1} \otimes \delta).$$

hence $\mathrm{ev}(RS \otimes \mathrm{Id}) = R$ as required in Figure 2.3.5. $\square$

The following proposition relates the different concepts for quasitriangularity.

Proposition 2.3.3.5. *Let B, C be dually paired Hopf algebras in $\mathcal{B}$.*

(a) A universal R-matrix for C induces a weak quasitriangular structure for B, C, which induces a dual R-matrix on B.

(b) Let C be the left dual of B in the sense that a coevaluation map $\mathrm{coev}\colon I \to B \otimes C$ *exists, satisfying that the usual duality relations hold. Then the three concepts in (a) coincide.*

Proof. This is due to [90, Section 2]. To prove part (a), given a universal R-matrix for C, define

$$\overline{R} := (\mathrm{Id}_C \otimes \mathrm{ev})(R^{\overline{\mathrm{op}}} \otimes \mathrm{Id}_B). \tag{2.3.8}$$

To obtain R, we need to find a morphism $\widetilde{R}\colon I \to B \otimes B$ such that

$$\begin{aligned}&(\triangleright_V \otimes \triangleright_W)(\mathrm{Id} \otimes \Psi_{C,B} \otimes \mathrm{Id})(\widetilde{R} \otimes \mathrm{Id}_{V \otimes W})\\ &\qquad = \Psi_{F(W),F(V)} F(\Psi^{\mathcal{M}}_{V,W}) \Psi^{-1}_{F(W),F(V)} \Psi^{-1}_{F(V),F(W)}.\end{aligned} \tag{2.3.9}$$

This can be thought of as $R^{\overline{\mathrm{op}}\mathrm{op}}$ and exists by higher representability. Now define

$$R^{-1} := (\mathrm{Id}_C \otimes \mathrm{ev})(\widetilde{R} \otimes \mathrm{Id}_B). \tag{2.3.10}$$

It was already observed in Proposition 2.3.3.2 that a weak quasitriangular structure induced a dual R-matrix.

Part (b) follows from the observation that given a coevaluation map, the categories $B^{\overline{\mathrm{op}}}$-$\mathbf{CoMod}(\mathcal{B})$ and C-$\mathbf{Mod}(\mathcal{B})$ are canonically equivalent. □

Note that in a symmetric monoidal category this theory simplifies as $R^{\overline{\mathrm{op}}}$, $R = \widetilde{R}$ can be obtained from R using the symmetry.

We will later observe that in the case of the quantum group, a weak quasitriangular structure exists for the Hopf algebra $\mathbb{C}\mathbb{Z}^n$. This is essential in the interpretation of $U_q(\mathfrak{g})$ as a *double-bosonization* in [90, 4.3].

Remark 2.3.3.6. If $\mathcal{B} = \mathbf{Vect}_k$, then the functor $E\colon B^{\overline{\mathrm{op}}}$-$\mathbf{CoMod} \to C$-$\mathbf{Mod}$ is fully faithful if the pairing ev is perfect. Clearly, E is faithful as it is the identity on morphisms. Assume the pairing is perfect and consider a linear map $f\colon V \to W$. The expression[25]

$$\mathrm{ev}(g \otimes v^{(-1)}) \otimes f(v^{(0)}) - \mathrm{ev}(g \otimes (fv)^{(-1)}) \otimes (fv)^{(0)}, \qquad \forall v \in g \in C.$$

is zero for all if $E(f)$ is a morphism of C-modules. But if that means that the difference $v^{(-1)} \otimes f(v^{(0)}) - (fv)^{(-1)} \otimes (fv)^{(0)}$ lies in the right radical of ev. If ev is perfect these terms have to be zero and hence f is a morphism of $B^{\overline{\mathrm{op}}}$-comodules. This shows E is full.

2.3.4 Definition of Braided Drinfeld and Heisenberg Doubles

Assume that B and C are perfectly dually paired Hopf algebras in $\mathcal{B}$. Using the description of YD-modules over (B, C) from Section 2.3.2, we have obtained a fully faithful functor

$$\mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B})) \hookrightarrow {}^{C}\mathcal{YD}^{B}(\mathcal{B}).$$

---

[25] Using modified Sweedler's notation, cf. 2.3.5.

In the case where $\mathcal{B} = H$-**Mod** for a quasitriangular Hopf algebra H (or $\mathcal{B} = A$-**CoMod** for A and H perfectly dually paired Hopf algebras in $\mathbf{Vect}_k$ with a weak quasitriangular structure), we can now define braided Drinfeld and Heisenberg doubles via reconstruction theory (see Section 2.2.3). The advantage of this approach is that we can extract all formulae from braided diagrams and the structure maps of the (Hopf) algebras thus defined will satisfy all the axioms by general theory and no explicit checks have to be carried out. For the braided Drinfeld double this repeats Majid's construction of the *double-bosonization* in [90, Theorem 3.2] using left H-modules instead of right ones. In [90] the computations for the Hopf algebra structure are also carried out explicitly.

Remark 2.3.4.1. In the following, we will work over $\mathbf{Vect}_k$, which is the symmetric monoidal category of countably infinite-dimensional vector spaces. We want to work with an infinite-dimensional Hopf algebras B. Now B does not necessarily have a perfectly dually paired Hopf algebra C in the sense of 2.1.7. In fact, the maximal subalgebra of the vector space dual $B^* = \operatorname{Hom}_k(B, k)$ which is a Hopf algebra with the dual structure from B is

$$B^\circ = \{f \in B^* \mid \exists I \lhd B, \dim B/I < \infty\},$$

where I are Hopf ideals. However, $C = B^\circ$ is not necessarily perfectly paired with B. Using finite-dimensional representations, we can describe B° as the Hopf algebra of *matrix coefficients* (see e.g. [24, I.7]). From this description we obtain the condition that B° and B are perfectly paired (with respect to the natural pairing) if and only if no element of B acts by zero on all finite-dimensional modules of B. As we will often work with positively graded Hopf algebras (for example, studying Nichols algebras), we will include the following Lemma:

Lemma 2.3.4.2. *If B is a positively graded Hopf algebra with finite-dimensional graded pieces, then B° and B are perfectly paired.*

Proof. Write $B = \oplus_{n \geqslant 0} B_n$ and consider the finite-dimensional module $B_{<k} := B/\oplus_{n \geqslant k} B_n$. For each $b \in B$, we find that $b \in \oplus_{n \leqslant k} B_n$ for some k. Then $0 \neq b \cdot 1 \in B_{<k+1}$, so we have found a finite-dimensional module on which b acts non-trivially. ☐

Lemma 2.3.4.3. *For C and B dually (not necessarily perfectly) paired bialgebras in $\mathcal{B}$, the forgetful functor ${}^{C}\mathcal{YD}^{B}(\mathcal{B}) \to \mathbf{Vect}_k$ is higher representable on the vector space $C \otimes H \otimes B$. The same holds true for the functor ${}^{C}\mathcal{H}^{B}(\mathcal{B}) \to \mathbf{Vect}_k$.*

Proof. The structure of an object of the categories of YD- or Hopf modules over B, C can be encoded by the action maps of B, C and H. The compatibility conditions and all other axioms are expressed by conditions involving the action of these vector spaces on modules. This basic observation can be used to show (higher) representability. □

Definition 2.3.4.4.

(i) The *braided Drinfeld double* $\mathrm{Drin}_H(C, B)$ is the Hopf algebra obtained from ${}^{C}\mathcal{YD}^{B}(H\text{-}\mathbf{Mod})$ by reconstruction ([88, 9.4.1], or Theorem 2.2.3.3, 2.2.3.5) on $C \otimes H \otimes B$. If A, H have a weak quasitriangular structure, denote the algebra obtained by reconstruction on $C \otimes H \otimes B$ from ${}^{C}\mathcal{YD}^{B}(A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod})$ by $\mathrm{Drin}_A(C, B)$.

(ii) The *braided Heisenberg double* $\mathrm{Heis}_H(C, B)$ is the algebra obtained by reconstruction on $C \otimes H \otimes B$ from ${}^{C}\mathcal{H}^{B}(\mathcal{B})$. In the weak quasitriangular case, denote the resulting algebra by $\mathrm{Heis}_A(C, B)$.

Note that a PBW-decomposition is given by construction for these algebras. Moreover, we have that

$$
\begin{aligned}
\mathrm{Drin}_H(C, B)\text{-}\mathbf{Mod} &\cong {}^{C}\mathcal{YD}^{B}(H\text{-}\mathbf{Mod}),\\
\mathrm{Heis}_H(C, B)\text{-}\mathbf{Mod} &\cong {}^{C}\mathcal{H}^{B}(H\text{-}\mathbf{Mod}),
\end{aligned}
$$

where the first equivalence is one of monoidal categories. Note that in the case $\mathcal{B} = A\text{-}\mathbf{CoMod}$, such equivalences do *not* hold if A, H are infinite-dimensional. The reason is that in ${}^{C}\mathcal{YD}^{B}(A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod})$ the H-actions are induced by A-coactions. If the pairing of A and H is perfect, then they correspond to all locally finite (integrable) modules. But modules in $\mathrm{Drin}_A(C, B)$ can be more general (compare this to the requirement of studying weight modules of the quantum group).

Remark 2.3.4.5. We choose the notation $\mathrm{Drin}_H(C,B)$ indicating both B and C. This is because for given B we can consider different dually paired Hopf algebras (which may not be perfectly paired). This gives a more flexible definition allowing the treatment of algebras which have no perfectly paired dual Hopf algebra.

Remark 2.3.4.6. The braided Heisenberg double can in fact be obtained as the bosonization of the braided cross product (introduced in [86, Proposition 2.6]), using the coregular action of C on ${}^{\mathrm{op}}B$. This action is given by $(\mathrm{ev}\otimes \mathrm{Id}_B)(\mathrm{Id}_C \otimes \Delta_B)$ and can be checked to make ${}^{\mathrm{op}}B$ a left C-module coalgebra in $\mathcal{B}$, denoted by ${}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C$. There is an isomorphism of categories ${}^{C}\mathcal{H}^{B}(\mathcal{B}) \cong {}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C\text{-}\mathbf{Mod}(\mathcal{B})$ given by transforming the right B-action into a left ${}^{\mathrm{op}}B$-action via

$$(V, \triangleleft) \longmapsto (V, \triangleright := \triangleleft \Psi^{-1}). \tag{2.3.11}$$

Hence, the bosonization $({}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C) \rtimes H$ satisfies the universal property of Definition 2.3.4.4, and we conclude $\mathrm{Heis}_H(C,B) \cong ({}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C) \rtimes H$. From the next section onward, we will use a different presentation of this algebra in this chapter for comparison to the double-bosonization which we use to define $\mathrm{Drin}_H(C,B)$.

Before providing examples, we will write out explicit presentations for the abstractly defined doubles.

2.3.5 Explicit Presentations

In the following, we will use Sweedler's notation to write down presentations for the algebras just defined. For this, we denote the coproducts by $\Delta(x) = x_{(1)} \otimes x_{(2)}$ (for x an element of H, B or C). Note that in the case of B and C these are coproducts in the braided monoidal category H-**Mod** which are often referred to as *braided* coproducts. We write $x^{(-1)} \otimes x^{(0)}$ for left coactions. In this notation, summation over tensors is omitted. We maintain to use the notation $\triangleright$ and $\triangleleft$ for actions. When clarification is needed, we denote the products in H, B or C by $\cdot$ with a lower index indicating the algebra. We denote the R-matrix of H by $R = R^{(1)} \otimes R^{(2)}$ and its convolution inverse by $R^{-1} = R^{-(1)} \otimes R^{-(2)}$.

Proposition 2.3.5.1. *The algebra* $\mathrm{Drin}_H(C,B)$ *is generated by the subalgebras* H, B^{op} *(meaning that* $bb' = b' \cdot_B b$*) and* C *subject to the following relations:*

$$hc = (h_{(1)} \triangleright c)h_{(2)}, \qquad (\Leftrightarrow ch = h_{(2)}(S^{-1}h_{(1)} \triangleright c)) \tag{2.3.12}$$

$$hb = (h_{(2)} \triangleright b)h_{(1)}, \qquad (\Leftrightarrow bh = h_{(1)}(Sh_{(2)} \triangleright b)) \tag{2.3.13}$$

$$b_{(2)}R^{(1)}c_{(2)}\,\mathrm{ev}(c_{(1)} \otimes (R^{(2)} \triangleright b_{(1)})) = c_{(1)}R^{(2)}b_{(1)}\,\mathrm{ev}((R^{(1)} \triangleright c_{(2)}) \otimes b_{(2)}). \tag{2.3.14}$$

The coproducts are given by

$$\begin{aligned} &\Delta(h) = h_{(1)} \otimes h_{(2)}, \quad \Delta(b) = (R^{(2)} \triangleright b_{(1)}) \otimes b_{(2)}R^{(1)}, \\ &\Delta(c) = c_{(1)}R^{(2)} \otimes R^{(1)} \triangleright c_{(2)}. \end{aligned} \tag{2.3.15}$$

The counit is simply given by $\varepsilon(chb) = \varepsilon(c)\varepsilon(h)\varepsilon(b)$*. The antipode and inverse antipode are the anti-algebra morphisms given by:*

$$S(h) = S(h), \qquad S^{-1}(h) = S^{-1}(h), \tag{2.3.16}$$

$$S(b) = R^{-(2)}(R^{-(1)} \triangleright Sb), \qquad S^{-1}(b) = (R^{-(1)} \triangleright S^{-1}b)R^{-(2)}, \tag{2.3.17}$$

$$S(c) = R^{-(1)}(R^{-(2)} \triangleright Sc), \qquad S^{-1}(c) = (R^{-(2)} \triangleright S^{-1}c)R^{-(1)}. \tag{2.3.18}$$

The algebra $\mathrm{Heis}_H(C,B)$ *has the same algebra bosonization relations (2.3.12)–(2.3.13) as* $\mathrm{Drin}_H(C,B)$*, but relation (2.3.14) (referred to as the cross relation) is replaced by*

$$cb = b_{(2)}R^{(1)}c_{(2)}\,\mathrm{ev}(c_{(1)} \otimes (R^{(2)} \triangleright b_{(1)})). \tag{2.3.19}$$

Proof. These formulas are obtained using reconstruction in $\mathbf{Vect}_k$ (cf. [88, 9.4.1], [90, Appendix B]). For $h \in H$, $b \in B$ and $c \in C$, we define the action by

$$(c \otimes h \otimes b) \triangleright v := b \triangleright (h \triangleright (v \triangleleft b)). \tag{2.3.20}$$

The formulae for the antipode are acquired by defining e.g. Sb to be the element of $\mathrm{Drin}_H(C,B)$ which satisfies $\mathrm{ev}(b \triangleright f \otimes v) = (f \otimes S(b) \triangleright v)$ for all $f \in V^\circ$ and $v \in V$ where V° is the finite dual of the space V. □

Lemma 2.3.5.2. *The cross relation (2.3.14) in* $\mathrm{Drin}_H(C,B)$ *is equivalent to each of the following relations:*

$$\begin{aligned}cb =&R_1^{-(2)}b_{(2)}R^{(2)}(R_2^{-(2)} \triangleright c_{(2)})\,\mathrm{ev}(R_2^{-(1)}R_3^{-(1)} \triangleright c_{(3)} \otimes R_1^{-(1)} \triangleright S^{-1}(b_{(3)}))\\ &\mathrm{ev}(R_3^{-(2)} \triangleright c_{(1)} \otimes R^{(1)} \triangleright b_{(1)}),\end{aligned} \tag{2.3.21}$$

$$\begin{aligned}bc =&(R_1^{-(1)} \triangleright c_{(2)})R_1^{(2)}b_{(2)}R_2^{(1)}\,\mathrm{ev}(R_2^{-(1)}R_1^{(1)} \triangleright c_{(3)} \otimes b_{(3)})\\ &\mathrm{ev}(R_1^{-(2)}R_2^{-(2)} \triangleright S^{-1}(c_{(1)}) \otimes R_2^{(2)} \triangleright b_{(1)}).\end{aligned} \tag{2.3.22}$$

Proof. Apply reconstruction to Lemma 2.3.1.1 after reinterpreting the $B^{\overline{\mathrm{op}}}$-coaction as a C-action using the functor ${}_{B^{\overline{\mathrm{op}}}}\Phi$. ☐

Using this Lemma, we can write down general product formulas in PBW-form in $\mathrm{Drin}_H(C,B)$:

$$\begin{aligned}chb\cdot c'h'b' =&c(h_{(1)}R_1^{-(1)} \triangleright c'_{(2)})h_{(2)}R_1^{(2)}R_2^{(1)}h'_{(1)}b'(S(h'_{(2)})R_3^{-(1)} \triangleright b_{(2)})\\ &\mathrm{ev}(R_2^{-(1)}R_1^{(1)} \triangleright c'_{(3)} \otimes b_{(3)})\\ &\mathrm{ev}(R_1^{-(2)}R_2^{-(2)} \triangleright S^{-1}(c'_{(1)}) \otimes R_3^{-(2)}R_2^{(2)} \triangleright b_{(1)})\end{aligned} \tag{2.3.23}$$

The general product formula for $\mathrm{Heis}_H(C,B)$ in PBW-form is:

$$\begin{aligned}chb\cdot c'h'b' =&c(h_{(1)}R_1^{-(1)} \triangleright c'_{(2)})h_{(2)}R^{(1)}h'_{(1)}b'(S(h'_{(2)})R_2^{-(1)} \triangleright b_{(2)})\\ &\mathrm{ev}(R_1^{-(2)} \triangleright S^{-1}(c'_{(1)}) \otimes R^{(2)}R_2^{-(2)} \triangleright b_{(1)}).\end{aligned} \tag{2.3.24}$$

Example 2.3.5.3. Let $X = \mathbb{A}^n$ and $B = \mathbb{C}[x_1,\ldots,x_n]$ be its coordinate ring. We denote its restricted dual (as a Hopf algebra) by $C = \Theta_X = \mathbb{C}[\partial_1,\ldots,\partial_n]$, where $\mathrm{ev}(\partial_i, x_j) = \delta_{i,j}$. Both B and C are primitively generated Hopf algebras over k and perfectly paired via ev. One easily sees that $\mathrm{Heis}(\mathcal{O}_X) = \mathcal{D}_X = A_n$ is the ring of differential operators on X, the *nth Weyl algebra*. The Drinfeld double is simply $\mathrm{Drin}(\mathcal{O}_X) = \mathbb{C}[x_1,\ldots,x_n,\partial_1,\ldots,\partial_n]$ which can be identified with $\mathcal{O}_{T*X}$, the ring of functions on the tangent space. Hence there is an action of $\mathcal{O}_{T*X}$-**Mod** on $\mathcal{D}_X$-**Mod**.

We have an analogue of Proposition 2.3.5.1 in the case where $\mathcal{B} = A$-**CoMod**. Recall that in this case, we have convolution invertible morphisms of Hopf algebras $R, \overline{R}\colon A^{\overline{\mathrm{op}}} \to H$ such that

$$R(a \otimes a') = \mathrm{ev}(R^{-1}(a') \otimes a) = \mathrm{ev}(\overline{R}(a) \otimes a'), \tag{2.3.25}$$

$$R^{-1}(a \otimes a') = \mathrm{ev}(\overline{R}^{-1}(a) \otimes a') = \mathrm{ev}(R(a') \otimes a). \tag{2.3.26}$$

Proposition 2.3.5.4. *The Hopf algebra* $\mathrm{Drin}_A(C,B)$ *is generated by the subalgebras* H*,* B^{op} *(opposite product in* $\mathbf{Vect}_k$*) and* C *with the same bosonization relations (2.3.12)–(2.3.13) as* $\mathrm{Drin}_H(C,B)$ *and cross relation*

$$b_{(2)}\overline{R}({b_{(1)}}^{(-1)})c_{(2)}\,\mathrm{ev}(c_{(1)}\otimes {b_{(1)}}^{(0)}) = c_{(1)}R^{-1}({c_{(2)}}^{(-1)})b_{(1)}\,\mathrm{ev}({c_{(2)}}^{(0)}\otimes b_{(2)}). \quad (2.3.27)$$

The coproducts are given by

$$\begin{aligned}&\Delta(h) = h_{(1)}\otimes h_{(2)}, \quad \Delta(b) = {b_{(1)}}^{(0)}\otimes b_{(2)}\overline{R}({b_{(1)}}^{(-1)}),\\ &\Delta(c) = c_{(1)}R^{-1}({c_{(2)}}^{(-1)})\otimes {c_{(2)}}^{(0)}.\end{aligned} \quad (2.3.28)$$

The unit and counit are as before and the formulas for the antipode and inverse antipode are given by

$$S(h) = S(h), \qquad S^{-1}(h) = S^{-1}(h), \quad (2.3.29)$$

$$S(b) = R(b^{(-1)})S(b^{(0)}), \qquad S^{-1}(b) = S^{-1}(b^{(0)})R(b^{(-1)}), \quad (2.3.30)$$

$$S(c) = \overline{R}^{-1}(c^{(-1)})S(c^{(0)}), \qquad S^{-1}(c) = S^{-1}(c^{(0)})\overline{R}^{(-1)}(c^{(-1)}). \quad (2.3.31)$$

The cross-relation of $\mathrm{Heis}_A(C,B)$ *is given by*

$$cb = b_{(2)}\overline{R}({b_{(1)}}^{(-1)})c_{(2)}\,\mathrm{ev}(c_{(1)}\otimes {b_{(1)}}^{(0)}). \quad (2.3.32)$$

Proof. All expressions for $\mathrm{Drin}_H(C,B)$ (or $\mathrm{Heis}_H(C,B)$) can be translated into expressions for $\mathrm{Drin}_A(C,B)$ (or $\mathrm{Heis}_A(C,B)$) by using the following rules which are derived from the weak quasitriangular structure on A, H:

$$\begin{aligned}R^{(1)}\otimes R^{(2)}\triangleright v &= \overline{R}(v^{(-1)})\otimes v^{(0)},\\ (R^{(1)}\triangleright v)\otimes R^{(2)} &= v^{(0)}\otimes R^{-1}(v^{(-1)}),\\ R^{-(1)}\otimes R^{-(2)}\triangleright v &= \overline{R}^{-1}(v^{(-1)})\otimes v^{(0)},\\ (R^{-(1)}\triangleright v)\otimes R^{-(2)} &= v^{(0)}\otimes R(v^{(-1)}).\end{aligned}$$

□

Lemma 2.3.5.5. *There is a canonical isomorphism of* $\mathrm{Heis}_H(C,B)$ *with the presentation given in 2.3.5.1, with* $({}^{\mathrm{op}}B \rtimes_{\mathrm{coreg}} C)\rtimes H$*.*

Proof. This follows from the observations in Remark 2.3.4.6, together with the universal property of reconstruction in (2.2.27). □

Example 2.3.5.6. Let us consider the case where $H = k$ and B is a finite-dimensional Hopf algebra over k to discover the classical notions. In this case, $\mathrm{Drin}(H) := \mathrm{Drin}_k(H^*, H)$ is given on $H^* \otimes H^{\mathrm{op}}$ with product determined by the relation

$$cb = b_{(2)}c_{(2)} \,\mathrm{ev}(c_{(1)} \otimes b_{(1)}) \,\mathrm{ev}(c_{(3)} \otimes S^{-1}(b_{(3)})). \tag{2.3.33}$$

The coproduct, counit and antipode are simply the corresponding tensor product structures on $H^* \otimes H$. The R-matrix on $\mathrm{Drin}(H)$ is given by the coevaluation map of H. Note that there are two conventional differences of this definition to the definition of $\mathrm{Drin}(H)$ usually found in the literature (cf. e.g. [88, 7.1]). At first, H and H^* are *categorically* dually paired. That is $\mathrm{ev}(aa' \otimes bb') = \mathrm{ev}(a \otimes b') \,\mathrm{ev}(a' \otimes b)$. Second, the left modules are left-right Yetter–Drinfeld modules (that is, YD-compatible *right* H-action with a *left* H^*-coaction).

For this reason, we will often consider the Drinfeld double $\mathrm{Drin}(H^{\mathrm{op}})$ which is by definition $\mathrm{Drin}_k(H^{*\,\mathrm{cop}}, H^{\mathrm{op}})$. This Hopf algebra has the property that

$$\mathrm{Drin}(H^{\mathrm{op}})\text{-}\mathbf{Mod} = {}^{H}_{H}\mathcal{YD}.$$

It has H, H^* as subalgebras such that

$$cb = b_{(2)}c_{(2)} \,\mathrm{ev}(c_{(3)} \otimes b_{(1)}) \,\mathrm{ev}(c_{(3)} \otimes S(b_{(1)})). \tag{2.3.34}$$

The coproducts are given by $\Delta(b) = b_{(1)} \otimes b_{(2)}$ and $\Delta(c) = c_{(2)} \otimes c_{(1)}$. The universal R-matrix for $\mathrm{Drin}(H)$ is $\tau \,\mathrm{coev}_H$, where τ is the symmetric braiding in $\mathbf{Vect}_k$. We will write $\mathrm{coev}_H = e_\alpha \otimes f_\alpha \in H \otimes H^*$. The algebra $\mathrm{Drin}(H^{\mathrm{op}})$ recovers the classical Drinfeld double of H (as found in the literature, cf. e.g. [64, IX.4]) and has the same categorical interpretation.

For example, consider the Drinfeld double $\mathrm{Drin}(G) := \mathrm{Drin}(k[G]^{\mathrm{cop}}, kG^{\mathrm{op}})$, for G a finite group. It is generated by kG and the algebra of k-valued functions $k[G]$ (basis $\delta_h(g) = \delta_{h,g}$). Note that the coproduct is $\Delta(\delta_h) = \sum_{ab=h} \delta_a \otimes \delta_b$. The relations in this algebra are $g\delta_h = \delta_{ghg^{-1}}g$.

Example 2.3.5.7. A vast class of examples of braided Hopf algebras is given by *Nichols algebras*. These play an important part in the classification of *pointed Hopf algebras* (see e.g. [9] for a survey). An important feature is that they are primitively generated. That is, B is generated by elements $b \in B$ such that $\Delta(b) =$

$b\otimes 1+1\otimes b$, and the same is true for C. For such braided Hopf algebras, we obtain simpler formulae as the cross relations will turn out to be commutator relations. Denote the space of primitive elements of B by $P(B)$ and similarly the space of primitive elements in C by $P(C)$. Then $\mathrm{Drin}_H(C,B)$ is generated by H, $P(B)$ and $P(C)$ with respect to the bosonization relations (2.3.12) and (2.3.13) and the cross relation

$$[b,c] = R^{(2)}\operatorname{ev}(R^{(1)} \rhd c\otimes b) - R^{-(1)}\operatorname{ev}(R^{-(2)} \rhd c\otimes b), \tag{2.3.35}$$

for $b\in P(B)$, $c\in P(C)$. The coproducts are given on the generators by

$$\begin{aligned} \Delta(h) &= h_{(1)}\otimes h_{(2)}, && (2.3.36)\\ \Delta(b) &= 1\otimes b + (R^{(2)}\rhd b)\otimes R^{(1)}, && (2.3.37)\\ \Delta(c) &= c\otimes 1 + R^{(2)}\otimes R^{(1)}\rhd c. && (2.3.38)\end{aligned}$$

The condition (2.3.19) is equivalent to the commutator relation

$$[c,b] = R^{(1)}\operatorname{ev}(c\otimes R^{(2)}\rhd b), \tag{2.3.39}$$

for $c\in P(C)$ and $b\in P(B)$. Working over $\mathrm{Drin}(H)$ for an ordinary Hopf algebra H, we can view $B\in \mathbf{Hopf}(\mathrm{Drin}(H)\text{-}\mathbf{Mod})$ as a YD-module over H (an object of $\mathcal{Z}(\mathbf{Mod}\text{-}H)$. Then the relation (2.3.19) is equivalent to

$$[c,b] = b^{(-1)}\operatorname{ev}(c\otimes b^{(0)}). \tag{2.3.40}$$

To compare to the definition of the braided Heisenberg double from [16, Section 5], we have to apply algebra reconstruction on $C\otimes H\otimes B$ (rather than $C\otimes\mathrm{Drin}(H)\otimes B$). That is, we consider the subalgebra generated by H, B and C of $C\otimes\mathrm{Drin}(H)\otimes B$. We include this *restricted* version of the braided Heisenberg double.

Lemma 2.3.5.8. *Let* $B\in\mathbf{Hopf}(\mathcal{Z}(\mathbf{Mod}\text{-}H))$*, then the* restricted *braided Heisenberg double* $\overline{\mathrm{Heis}}_H(C,B)$ *of* H *over* B *is the algebra generated by* H, B^{op} *(in* $\mathbf{Vect}_k$*) and* C *subject to the relations*

$$\begin{aligned} hb &= (h_{(2)}\rhd b)h_{(1)}, && (2.3.41)\\ hc &= (h_{(1)}\rhd c)h_{(2)}, && (2.3.42)\\ cb &= b_{(2)}{b_{(1)}}^{(-1)}c_{(2)}\operatorname{ev}(c_{(1)}\otimes {b_{(1)}}^{(0)}). && (2.3.43)\end{aligned}$$

It is the subalgebra of $\mathrm{Heis}_{\mathrm{Drin}(H)}(C,B)$ *generated by* H, B^{op} *and* C.

Note that H is assumed to be finite-dimensional for $\mathrm{Drin}(H)$ to be quasitriangular. We observe that there is a restriction functor $\mathrm{Heis}_{\mathrm{Drin}(H)}(C,B)\text{-}\mathbf{Mod} \to \overline{\mathrm{Heis}}_H(C,B)\text{-}\mathbf{Mod}$.

Now the braided Heisenberg double defined in [16, Section 5] is isomorphic to $\overline{\mathrm{Heis}}_H(C^{\mathrm{cop}}, B^{\mathrm{op}})$ where the (co)opposites are taking in $\mathbf{Vect}_k$, and B, C are Nichols algebras. The cross relation for this algebra is (2.3.40).

Corollary 2.3.5.9. *The coproduct on* $\mathrm{Drin}_{\mathrm{Drin}(H)}(C,B)$ *induces a left action of the monoidal category* $\mathrm{Drin}_{\mathrm{Drin}(H)}(C,B)\text{-}\mathbf{Mod}$ *on* $\overline{\mathrm{Heis}}_H(C,B)\text{-}\mathbf{Mod}$.

Proof. For this to hold, we need to ensure that the restriction of the coproduct Δ viewed as an algebra map $\overline{\mathrm{Heis}}_H(C,B) \to \mathrm{Drin}_{\mathrm{Drin}(H)}(C,B) \otimes \mathrm{Heis}_{\mathrm{Drin}(H)}(C,B)$ maps to $\mathrm{Drin}_{\mathrm{Drin}(H)}(C,B) \otimes \overline{\mathrm{Heis}}_H(C,B)$. This can be checked on generators. Clearly $\Delta(h) \in H \otimes H$ satisfies this. Further, $\Delta(b) = {b_{(1)}}^{(0)} \otimes b_{(2)}{b_{(1)}}^{(-1)}$ and $\Delta(c) = c_{(1)} f_\alpha \otimes e_\alpha \triangleright c_{(2)}$ both lie in the subspace $(C \otimes H^* \otimes H \otimes B) \otimes (C \otimes H \otimes B)$. □

Example 2.3.5.10. In 2.3.5.3, the Drinfeld double of $\mathcal{O}_X$ collapses to simply be the tensor product Hopf algebra of $\mathcal{O}_X$ and its dual. We will now consider the symmetric algebra as a Hopf algebra object in ${}^{C_2}_{C_2}\mathcal{YD}$ for the group C_2 with two elements. We can also include the alternating algebra in the same setting as an example.

Let V be a finite-dimensional vector space. Consider the braidings $\tau\colon V \otimes V \to V \otimes V$, where $\tau(v \otimes w) = w \otimes v$, and $-\tau$. These can naturally be realized as braidings coming from C_2-YD-module structures on V, corresponding to $\delta(v) = s \otimes v$ where $C_2 = \{1, s\}$ with the trivial action for τ, respectively the sign representation $sv = -v$ for $-\tau$. Using this, we have that

$$S(V) = T(V) \Big/ (v \otimes w - w \otimes v) = T(V) \Big/ \ker(\mathrm{Id} + \tau),$$
$$\Lambda(V) = T(V) \Big/ (v \otimes w + w \otimes v) = T(V) \Big/ \ker(\mathrm{Id} - \tau),$$

are braided Hopf algebras in ${}^{C_2}_{C_2}\mathcal{YD} = \mathrm{Drin}(C_2)\text{-}\mathbf{Mod}$. As they are Nichols algebras, they are perfectly paired with $S(V^*)$ (respectively $\Lambda(V^*)$).

The restricted Heisenberg double $\overline{\mathrm{Heis}}_{kC_2}(S(V^*), S(V))$ is now $A_n \otimes kC_2$, but $\mathrm{Drin}_{\mathrm{Drin}(C_2)}(S(V^*), S(V))$ has the non-trivial commutator relation

$$[f, v] = (s - \delta_1 - \delta_s)\,\mathrm{ev}(f \otimes v). \tag{2.3.44}$$

Hence viewing $S(V)$ over $\mathrm{Drin}(C_2)$ causes the resulting Drinfeld double to be non-commutative.

For the exterior algebra, we have that $\mathrm{Heis}_{kC_2}(\Lambda(V^*), \Lambda(V))$ is generated by $v \in V$, $f \in V^*$ and $1, s$ with additional relations

$$sv = -vs, \quad sf = -fs, \quad [f, v] = \mathrm{ev}(f \otimes v). \tag{2.3.45}$$

In $\mathrm{Drin}_{\mathrm{Drin}(C_2)}(\Lambda(V^*), \Lambda(V))$ the commutator relation is

$$[f, v] = (s - \delta_1 + \delta_s)\,\mathrm{ev}(f \otimes v). \tag{2.3.46}$$

Example 2.3.5.11 (The quantum groups). Majid constructs Lusztig's version of $U_q(\mathfrak{g})$ (for generic q) as braided Drinfeld doubles in [90, 4.3]. We repeat the construction with the conventions of this chapter. For this, a (symmetric) Cartan datum $\cdot$ gives a paring of two lattice group algebras $A = k\mathbb{Z}[I] = k[g_i^{\pm 1}]$ and $H = k[K_i^{\pm 1}]$ via $\langle K_i, g_j \rangle = q^{2\frac{i \cdot j}{i \cdot i}}$ (where $k = \mathbb{C}(q)$). The dual R-matrix $R(g_i, g_j) = q^{i \cdot j}$ is given by a weak quasitriangular structure with $R(g_i) = K^{-\frac{i \cdot i}{2}}$ (and $\overline{R} = R^{-1}$). We can then define an A-comodule $F = k\langle F_1, \ldots, F_n \rangle$ by $F_i \mapsto g_i^{-1} \otimes F_i$ and denote its dual by $E = k\langle E_1, \ldots, E_n \rangle$. The tensor algebras (in A-**CoMod**) $B = T(F)$, $C = T(E)$ are dually paired via

$$\mathrm{ev}(E_i, F_i) = \frac{\delta_{i,j}}{(q_i^{-1} - q_i)}, \tag{2.3.47}$$

where $q_i := q^{\frac{i \cdot i}{2}}$. This pairing extends uniquely to a Hopf algebra pairing of the tensor algebras and it is a result by Lusztig (see [75, 1.2–1.4]) such that then left and right radical of this pairing are precisely the quantum Serre relations. Hence the quotients are $U_q(\mathfrak{n}_+)$ (of C) and $U_q(\mathfrak{n}_-)$ (of B), which are Nichols algebras. Now translating (2.3.35) under 2.3.5.4 we obtain the correct relation

$$[E_i, F_j] = \frac{K_i^{\frac{i \cdot i}{2}} - K_i^{-\frac{i \cdot i}{2}}}{(q_i - q_i^{-1})} \delta_{i,j}. \tag{2.3.48}$$

Further, the bosonization relations (2.3.12) and (2.3.13) give

$$K_i E_j = q^{2\frac{i \cdot j}{i \cdot i}} E_j K_i, \quad K_i F_j = q^{-2\frac{i \cdot j}{i \cdot i}} F_j K_i. \tag{2.3.49}$$

The coproducts are given by

$$\Delta(F_i) = F_i \otimes K^{-\frac{i\cdot i}{2}} + 1 \otimes F_i, \qquad \Delta(F_i) = E_i \otimes 1 + K^{\frac{i\cdot i}{2}} \otimes E_i. \tag{2.3.50}$$

Hence the resulting Hopf algebra is $U_q(\mathfrak{g})$.

The cross relation for the Heisenberg double $D_q(\mathfrak{g}) := \mathrm{Heis}_A(U_q(\mathfrak{n}_+), U_q(\mathfrak{n}_-))$ is given by

$$[E_i, F_j] = \frac{K_i^{-\frac{i\cdot i}{2}}}{q_i^{-1} - q_i}\delta_{i,j}. \tag{2.3.51}$$

We will look at the action of $U_q(\mathfrak{sl}_2)$-**Mod** on $D_q(\mathfrak{sl}_2)$-**Mod** in Section 2.3.9.

2.3.6 R-Matrices for the Drinfeld Double

While the Hopf algebra structure of $\mathrm{Drin}_H(C, B)$ can be defined for general infinite-dimensional Hopf algebras with a choice of a dually paired Hopf algebra, viewing the braided structure of $\mathrm{Drin}_H(C, B)$-**Mod**, and hence the R-matrix, at this level of generality is problematic as they require the existence of a coevaluation map $\mathrm{coev}\colon I \to B \otimes C$ in $\mathcal{B} = H$-**Mod**. In the infinite-dimensional case, given a perfect pairing, this map may still exist as a formal power series, given that there is an orthonormal bases of the dual Hopf algebra C to a basis of B.

Recall the braidings from Figure 2.3.1. Given the existence of a coevaluation map $\mathrm{coev}\colon I \to B \otimes C$, the category ${}^{C}\mathcal{YD}^{B}(\mathcal{B})$ is braided via the braiding

$$ {}^{C}\Psi^{B} := (\triangleleft \otimes \triangleright)(\mathrm{Id} \otimes \mathrm{coev} \otimes \mathrm{Id})\Psi^{-1}, \tag{2.3.52}$$

$$ ({}^{C}\Psi^{B})^{-1} := \Psi(\triangleleft \otimes \triangleright)(\mathrm{Id} \otimes \mathrm{Id}_B \otimes S \otimes \mathrm{Id})(\mathrm{Id} \otimes \mathrm{coev} \otimes \mathrm{Id}). \tag{2.3.53}$$

Given that B and C are (not necessarily finite-dimensional) Hopf algebras with a perfect duality pairing $\mathrm{ev}\colon C \otimes B \to k$ such that there is a basis $\{b_i\}_i$ of B with an orthogonal basis $\{c_i\}_i$ of C, we can consider the formal power series $\mathrm{coev}_B := \sum_i b_i \otimes c_i$. This is *not* a morphism $k \to B \otimes C$, but satisfies the duality axioms of the coevaluation map. Using this formal sum, we can write an R-matrix for $\mathrm{Drin}_H(C, B)$ as the power series (omitting summation)

$$R_{\mathrm{Drin}_H(C,B)} := c_i R^{-(2)} \otimes b_i R^{-(1)}, \quad R^{-1}_{\mathrm{Drin}_H(C,B)} := R^{(2)} c_i \otimes R^{(1)} S(b_i). \tag{2.3.54}$$

If $\mathcal{B} = A$-**CoMod**, the situation is even more problematic. Expressing the braiding

$$\Psi_{V,W}(v, w) = \mathrm{ev}(R^{-1}(v^{(-1)}) \otimes w^{(-1)}) w^{(0)} \otimes v^{(0)} \tag{2.3.55}$$

as an R-matrix, which is necessary in algebra reconstruction, involves a coevaluation map which will be an infinite sum unless H, A are finite-dimensional. We denote it by $\mathrm{coev}_H = h_j \otimes a_j$. Then

$$R = a_j \otimes R^{-1}(h_j) = \overline{R}(h_j) \otimes a_j, \qquad R^{-1} = \overline{R}^{-1}(h_j) \otimes a_j = a_j \otimes R(h_j). \tag{2.3.56}$$

Further, the R-matrix on $\mathrm{Drin}_A(C, B)$ is given by

$$R_{\mathrm{Drin}_A(C,B)} := c_i R(h_j) \otimes b_i a_j, \quad R^{-1}_{\mathrm{Drin}_A(C,B)} := a_j b_i \otimes R^{-1}(h_j) S(c_i). \tag{2.3.57}$$

To overcome the problematic of the braiding involving formal infinite sums, we will describe the properly braided subcategory of modules over the braided Drinfeld doubles corresponding to the center $\mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ in Section 2.3.7.

Example 2.3.6.1. Consider $U_q(\mathfrak{sl}_2)$. We can compute that $\mathrm{ev}(f^n \otimes e^n) = \frac{[n]_q!}{(q^{-1}-q)^n}$, where $[n]_q = 1 + q^{-2} + \ldots + q^{2(n-1)}$ and $[n]_q! = [1]_q \cdot \ldots \cdot [n]_q$. Hence

$$\mathrm{coev}_B = \sum_{n \geqslant 0} \frac{(q^{-1} - q)^n}{[n]_q!} F^n \otimes E^n. \tag{2.3.58}$$

However, writing the required coevaluation map for $I \to H \otimes A$ is problematic. A solution is to introduce the element $q^{h \otimes h} \in (A \otimes A)^*$. This element satisfies

$$\langle q^{h \otimes h} \otimes (g^n \otimes g^m) \rangle = q^{2nm}. \tag{2.3.59}$$

This can be generalized to any Cartan datum. See [90, p.171] for more details.

Example 2.3.6.2 ([90, 4.5]). The small quantum groups $u_q(\mathfrak{g})$ can also be realized as examples of braided Drinfeld doubles. For this, we assume that q is a primitive rth root of unity, where r is odd and $\mathrm{char}\, k$ does not divide r. As underlying Hopf algebras, we take the group algebras $A = H = k(C_r)^n$.

2.3.7 The Category $\mathcal{O}$

The R-matrices $R_{\mathrm{Drin}_H(C,B)}$ and $R_{\mathrm{Drin}_A(C,B)}$ from Section 2.3.6 induce the braiding on the subcategories $\mathcal{Z}_{H\text{-}\mathbf{Mod}}(\mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod}))$ respectively its subcategory $\mathcal{Z}_{A\text{-}\mathbf{CoMod}}(\mathbf{Mod}\text{-}B(A\text{-}\mathbf{CoMod}))$. For perfectly paired B and C, these Drinfeld centers are *almost* analogues of the BGG-category $\mathcal{O}$ for the quantum groups. To support this interpretation, recall that for infinite-dimensional dually paired bialgebras B and C in H-**Mod**, the essential image of the fully faithful functor

$${}_{B^{\overline{\mathrm{op}}}}\Psi\colon B^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}(H\text{-}\mathbf{Mod}) \to C\text{-}\mathbf{Mod}(H\text{-}\mathbf{Mod})$$

consists of those C-modules V such that for each $v \in V$, $C/\operatorname{Ann}(v)$ is finite-dimensional. Such modules are called *locally finite* or *admissible*. Hence, the full subcategory $\mathcal{Z}_{H\text{-}\mathbf{Mod}}(\mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod})) \subset \mathrm{Drin}_H(C,B)\text{-}\mathbf{Mod}$ consists of those YD-modules which are locally finite for the C-action, but not necessarily for the B-action. For example, one can define Verma modules in $\mathcal{Z}_{H\text{-}\mathbf{Mod}}(\mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod}))$.

When working in $\mathcal{B} = A\text{-}\mathbf{CoMod}$, one also has to restrict to $\mathrm{Drin}_A(C,B)$-modules for which the H-action is induced by an A-coaction, i.e. the H-action has to be locally finite too. Using these observations, we conclude the following lemma:

Lemma 2.3.7.1. *For the quantum group $U_q(\mathfrak{g})$ at q not a root of unity, the subcategory of*

$$\mathcal{Z}_{k\mathbb{Z}^n\text{-}\mathbf{CoMod}}(\mathbf{Mod}\text{-}U_q(\mathfrak{n}^+)(k\mathbb{Z}^n\text{-}\mathbf{CoMod}))$$

on finitely generated modules is the BGG-category $\mathcal{O}$ (denoted by $\mathcal{O}_q(\mathfrak{g})$) for quantum groups as defined in [2, 3.1].

Proof. An object V in $\mathcal{O}_q(\mathfrak{g})$ satisfy three properties:

(I) V is finitely generated over $U_q(\mathfrak{g})$

(II) V is a weight module, i.e. there exists a direct sum decomposition

$$V = \oplus_{\lambda \in \mathbb{Z}^n} V_\lambda, \qquad V_\lambda = \{v \in V \mid K_i \rhd v = q^{2\lambda_i}\}.$$

(III) V is locally finite as a $U_q(\mathfrak{n}_+)$-module.

If V comes from an object in $\mathcal{Z}_{k\mathbb{Z}^n\text{-}\mathbf{CoMod}}(\mathbf{Mod}\text{-}U_q(\mathfrak{n}^+)(k\mathbb{Z}^n\text{-}\mathbf{CoMod}))$, then (II) is automatically fulfilled because the H-action is induced by an $A = k\mathbb{Z}^n$-coaction. Further, (III) is fulfilled as the C-action is induced by a $B^{\overline{\text{op}}}$-coaction and hence locally finite. Further, any such module can be obtained from an object of the center as the pairing of B and C is perfect. Thus, if we restrict to finitely generated modules, we recover $\mathcal{O}_q(\mathfrak{g})$ as a subcategory of the center. □

Hence we can define the category $\mathcal{O}$ for a braided Drinfeld (or Heisenberg) double as the subcategory of finitely generated modules in $\mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ (respectively $\mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B})))$. We denote this category by $\mathcal{Z}^{\text{fg}}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ (respectively $\mathcal{H}^{\text{fg}}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B})))$.

Definition 2.3.7.2.

(a) For $\mathcal{B} = H\text{-}\mathbf{Mod}$, we define the category $\mathcal{O}^{\text{Drin}}_H(C, B)$ (or $\mathcal{O}^{\text{Heis}}_H(C, B)$) as the full subcategory of $\text{Drin}_H(C, B)\text{-}\mathbf{Mod}$ (respectively, $\text{Heis}_H(C, B)\text{-}\mathbf{Mod}$) of objects which are

  (I) finitely generated as $\text{Drin}_H(C, B)$-modules (resp. $\text{Heis}_H(C, B)$-modules),

  (II) semisimple as H-modules,

  (III) locally finite as C-modules.

(b) Working with $\mathcal{B} = A^{\overline{\text{op}}}\text{-}\mathbf{CoMod}$, we define the category $\mathcal{O}^{\text{Drin}}_A(C, B)$ (respectively, $\mathcal{O}^{\text{Heis}}_A(C, B)$) as the full subcategory of $\text{Drin}_A(C, B)\text{-}\mathbf{Mod}$ (respectively, $\text{Heis}_A(C, B)\text{-}\mathbf{Mod}$) on objects satisfying (I), (III) and

  (II)' The H-module structure is induced by an $A^{\overline{\text{op}}}$-comodule structure.

In the special setting of Section 2.3.6 where we have the existence of orthonormal bases $\{e_\alpha\}$ of B and $\{f_\alpha\}$ of C, we were able to define a formal power series $\text{coev} = e_\alpha \otimes f_\alpha$ serving as formal coevaluation map. Given this structure, we can link the categories $\mathcal{O}^{\text{Drin}}_H(C, B)$ (and the other versions) with $\mathcal{Z}^{\text{fg}}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$, generalizing Lemma 2.3.7.1.

Theorem 2.3.7.3. *Let C, B be perfectly paired Hopf algebras in H-**Mod** such that* coev *exists as a formal power series. Then there exist isomorphisms of categories*

$$\begin{aligned}\mathcal{O}_H^{\mathrm{Drin}}(C,B) &\cong \mathcal{Z}_{H\text{-}\mathbf{Mod}}^{\mathrm{fg},H-\mathrm{ss}}(\mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod})),\\ \mathcal{O}_H^{\mathrm{Heis}}(C,B) &\cong \mathcal{H}_{H\text{-}\mathbf{Mod}}^{\mathrm{fg},H-\mathrm{ss}}(\mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod})).\end{aligned}$$

*If C, B are such objects in $A^{\overline{\mathrm{op}}}$-**CoMod** where A, H have a weak quasitriangular structure, then*

$$\begin{aligned}\mathcal{O}_A^{\mathrm{Drin}}(C,B) &\cong \mathcal{Z}_{A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}}^{\mathrm{fg},A-\mathrm{ss}}(\mathbf{Mod}\text{-}B(A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod})),\\ \mathcal{O}_A^{\mathrm{Heis}}(C,B) &\cong \mathcal{H}_{A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}}^{\mathrm{fg},A-\mathrm{ss}}(\mathbf{Mod}\text{-}B(A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod})).\end{aligned}$$

Proof. The proof works as the proof of Lemma 2.3.7.1 in this more general setting. Semisimplicity is not automatic any more, so it needs to be imposed in general (hence the superscripts $H-\mathrm{ss}$ or $A-\mathrm{ss}$). Note that $\mathrm{coev} = e_\alpha \otimes f_\alpha$ can be used to define a $B^{\overline{\mathrm{op}}}$-comodule, given a locally finite C-module V via $\delta(v) := e_\alpha \otimes f_\alpha \triangleright v$, for all $v \in V$. This expression is always well-defined if V is locally finite (i.e. only finite sums of tensors occur). ☐

We would like to obtain an action of the category $\mathcal{O}$ for the braided Drinfeld double on the category $\mathcal{O}$ for the braided Heisenberg double. The problem with this is that the action does not necessarily preserve finite-generation of the module. In generality, we therefore need to restrict to finite-dimensional modules.

Corollary 2.3.7.4. *Under the assumptions on C, B as in 2.3.7.3, there are actions*

$$\begin{aligned}\triangleright &: \mathrm{Drin}_H(C,B)\text{-}\mathbf{Mod}^{\mathrm{fd}} \otimes \mathcal{O}_H^{\mathrm{Heis}}(C,B) \longrightarrow \mathcal{O}_H^{\mathrm{Heis}}(C,B),\\ \triangleright &: \mathrm{Drin}_A(C,B)\text{-}\mathbf{Mod}^{\mathrm{fd}} \otimes \mathcal{O}_A^{\mathrm{Heis}}(C,B) \longrightarrow \mathcal{O}_A^{\mathrm{Heis}}(C,B).\end{aligned}$$

Similar to the study of the category $\mathcal{O}$ in other contexts, we can introduce *standard* (or *Verma*) modules in the category $\mathcal{O}$ for braided Drinfeld and Heisenberg doubles. For this, a general theory as in [46] could be adapted. We will not introduce such a theory in general here, but provide a definition of a Verma module.

Definition 2.3.7.5. Let B, C be braided Hopf algebras in $\mathcal{B}$ and let S be a simple H-module (respectively $A^{\overline{\mathrm{op}}}$-comodule). Then we can define a $C^{\mathrm{op}} \rtimes H$-module as S with trivial C action via the antipode of C and the given H-action (if S is an A-comodule, we view S as an H-module via the pairing). The *Verma module* of $M(S)$ is defined as $M(S) := B \otimes_{C^{\mathrm{op}} \rtimes H} S$. This can either be done as a module over $\mathrm{Drin}_{\mathcal{B}}(C, B)$ or $\mathrm{Heis}_{\mathcal{B}}(C, B)$ (for $\mathcal{B} = H\text{-}\mathbf{Mod}$ or $A^{\overline{\mathrm{op}}}\text{-}\mathbf{CoMod}$).

It is not guaranteed that all $M(S)$ are in the respective category $\mathcal{O}$. However, under additional assumptions, such that the B and C are positively graded, by an inner grading as in [46, 2.1], the theory developed therein ensure that they are. In general, we can say that $M(S)$ are in the finitely-generated subcategory of $\mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ (respectively $\mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ in the Hopf case). One of the main applications is that all simple objects in $\mathcal{O}$ appear as quotients of such Verma modules, given an inner grading element exists.

2.3.8 Cocycle Twists

In this section, we observe that the left action of the category $\mathcal{Z}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ on $\mathcal{H}_{\mathcal{B}}(\mathbf{Mod}\text{-}B(\mathcal{B}))$ extends to an action of $\mathrm{Drin}_H(C, B)$-**Mod** on $\mathrm{Heis}_H(C, B)$-**Mod**. This action implies that $\mathrm{Heis}_H(C, B)$ is a right cocycle twist of the Hopf algebra $\mathrm{Drin}_H(C, B)$ generalizing an earlier result of [71] to the braided case. In particular, $\mathrm{Heis}_H(C, B)$ is a left $\mathrm{Drin}_H(C, B)$-comodule algebra.

Definition 2.3.8.1. Let $B \in \mathbf{BiAlg}(\mathcal{B})$, for $\mathcal{B}$ a braided monoidal category. A *right 2-cocycle of* B is a morphism $\sigma\colon B \otimes B \to I$ such that the morphism $m_\sigma := m^2_{B\otimes B}(\Delta \otimes \Delta \otimes \sigma)\colon B \otimes B \to B$ gives B an algebra structure which is denoted by B_σ and called the *right cocycle twist of* B by σ.

It is easy to check that a morphism $\sigma\colon B \otimes B \to I$ is a right 2-cocycle if and only if it satisfies the condition

$$\begin{aligned} &(\sigma \otimes \sigma)(m \otimes \Psi \otimes \mathrm{Id}_B)(\mathrm{Id}_B \otimes \Psi \otimes \Psi)(\Delta \otimes \Delta \otimes \mathrm{Id}_B) \\ &\quad = (\sigma \otimes \sigma)(\mathrm{Id}_B \otimes m \otimes \mathrm{Id}_{B\otimes B})(\mathrm{Id}_{B\otimes B} \otimes \Psi \otimes \mathrm{Id}_B)(\mathrm{Id}_B \otimes \Delta \otimes \Delta) \end{aligned} \tag{2.3.60}$$

on $B \otimes B \otimes B$, as well as the normalization conditions $\sigma(\mathrm{Id} \otimes 1) = \sigma(1 \otimes \mathrm{Id}) = \varepsilon$.

Lemma 2.3.8.2. *The coproduct of B can be viewed as a morphism $\Delta\colon B_\sigma \to B \otimes B_\sigma$ making B_σ a left B-comodule algebra in $\mathcal{B}$.*

Proof. This is an exercise in graphical calculus. $\square$

Corollary 2.3.8.3. *If σ is a right 2-cocycle, then Δ gives a left action of B-$\mathbf{Mod}(\mathcal{B})$ on B_σ-$\mathbf{Mod}(\mathcal{B})$ where the B_σ-action on $V \triangleright W$ (defined on the object $V \otimes W$) for a V-module V and a B_σ-module W is given by*

$$\triangleright_{V\otimes W} = (\triangleright_V \otimes \triangleright_W)(\mathrm{Id}_B \otimes \Psi_{B,V} \otimes \mathrm{Id}_W)(\Delta \otimes V \otimes W). \tag{2.3.61}$$

That is, given by the coproduct of B.

We are looking for a converse of the statement in Corollary 2.3.8.3. Indeed, the following holds:

Proposition 2.3.8.4.

(i) Let $D \in \mathbf{BiAlg}(\mathcal{B})$ and $H \in \mathbf{Alg}(\mathcal{B})$ such that $\mathcal{D} = D$-$\mathbf{Mod}(\mathcal{B})$ acts on $\mathcal{H} = H$-$\mathbf{Mod}(\mathcal{B})$ on the left. Assume further that the square

$$\begin{tikzcd} D\text{-}\mathbf{Mod}(\mathcal{B}) \otimes H\text{-}\mathbf{Mod}(\mathcal{B}) \arrow[r, "\triangleright"] \arrow[d, "F_{\mathcal{D}}\otimes F_{\mathcal{H}}"'] & H\text{-}\mathbf{Mod}(\mathcal{B}) \arrow[d, "F_{\mathcal{H}}"'] \\ \mathbf{Vect}_k \otimes \mathbf{Vect}_k \arrow[r, "\otimes"] & \mathbf{Vect}_k \end{tikzcd} \tag{2.3.62}$$

commutes and that $\mathrm{Nat}(V, F_{\mathcal{B}} \otimes F_{\mathcal{H}})$ is reconstructible by $D \otimes H$ (cf. 2.2.3.2). Then the action is given by a morphism $\delta\colon H \to D \otimes H$ which makes H an algebra object in D-$\mathbf{CoMod}(\mathcal{B})$.

(ii) If $F(H) = F(D)$ in $\mathcal{B}$ and the coaction δ is given by the coproduct Δ of D, then the product on H is a right cocycle twist by the 2-cocycle $\varepsilon_D m_H$.

Proof. Part (i) is an argument in reconstruction theory. By representability assumptions, the action $\triangleright$ is given by a map $\delta\colon H \to D \otimes H$. The requirement that $\triangleright$ is an action on the category $\mathcal{H}$ translates to the property that H is an algebra object in D-$\mathbf{CoMod}(\mathcal{B})$.

To prove part (ii), observe that

$$\begin{aligned} m_H &= (\mathrm{Id}_D \otimes \varepsilon)\Delta_D m_H = (\mathrm{Id}_D \otimes \varepsilon)\delta m_H \\ &= (\mathrm{Id}_D \otimes \varepsilon)(m_D \otimes m_H)(\mathrm{Id}_D \otimes \Psi_{D,D} \otimes \mathrm{Id}_H)(\Delta_D \otimes \delta), \\ &= (\mathrm{Id}_D \otimes \varepsilon)(m_D \otimes m_H)(\mathrm{Id}_D \otimes \Psi_{D,D} \otimes \mathrm{Id}_D)(\Delta_D \otimes \Delta) \end{aligned}$$

where we use that that $\delta\colon H \to D \otimes H$ is a morphism of algebras and that $\delta = \Delta$ as maps in $\mathcal{B}$. Hence $H = D_\sigma$ for $\sigma = \varepsilon m_H$. $\square$

Returning to the situation of Corollary 2.2.2.15, we observed that the categorical action can be extended to an action of ${}^C\mathcal{YD}^B(\mathcal{B})$ on ${}^C\mathcal{H}^B(\mathcal{B})$ in Corollary 2.3.2.1. The action is given by the coproducts of B and C. For $\mathcal{B} = H$-**Mod** this implies that the action of $\mathrm{Drin}_H(C,B)$-**Mod** on $\mathrm{Heis}_H(C,B)$-**Mod** is given by the coproduct of $\mathrm{Drin}_H(C,B)$ viewed as a morphism of algebras

$$\Delta\colon\ \mathrm{Heis}_H(C,B) \to \mathrm{Drin}_H(C,B) \otimes \mathrm{Heis}_H(C,B),$$

making $\mathrm{Heis}_H(C,B)$ a comodule over $\mathrm{Drin}_H(C,B)$. The same observation applies when working with $A^{\overline{\mathrm{op}}}$-**CoMod** instead.

Corollary 2.3.8.5. *The algebra* $\mathrm{Heis}_H(C,B)$ *is a right cocycle twist of the Hopf algebra* $\mathrm{Drin}_H(C,B)$ *via the 2-cocycle given by* $\sigma(chb \otimes c'h'b') = \varepsilon(c)\varepsilon(h)\,\mathrm{ev}(S^{-1}c' \otimes b)\varepsilon(h')\varepsilon(b')$.

Remark 2.3.8.6. Note that we can also define left 2-cocycles and a left twist (or cycles and cycle twists). A dual R-matrix for B gives a left 2-cocycle on B. This cocycle was used in [71] to twist the Drinfeld double. Note that we use a different cocycle here but they coincide in the case $H = k$.

2.3.9 The Braided Heisenberg Double of $U_q(\mathfrak{n}_+)$ for $\mathfrak{sl}_2$

In Example 2.3.5.11, we introduced the braided Heisenberg double $D_q(\mathfrak{g})$ of $U_q(\mathfrak{n}_+)$. We now want to study the categorical action of $D_q(\mathfrak{sl}_2)$-$\mathbf{Mod}^{\mathrm{fd}}$ on the category $\mathcal{O}_q^{\mathrm{Heis}}(\mathfrak{sl}_2) := \mathcal{O}^{\mathrm{Heis}}_{k[g^{\pm1}]}(U(\mathfrak{sl}_2)^*, U(\mathfrak{sl}_2))$ as an example. We first observe that $D_q(\mathfrak{sl}_2)$ has no finite-dimensional modules using a standard argument from the theory of highest weight representations.

Lemma 2.3.9.1. *In $D_q(\mathfrak{sl}_2)$, the commutator relation*

$$[E, F^m] = \frac{[m]_q K^{-1}}{q^{-1} - q} F^{m-1}. \tag{2.3.63}$$

holds, where $[n]_q = 1 + q^{-2} + q^{-4} + \ldots + q^{-2(n-1)}$.

Proof. By induction on m, using that $[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E, F]$. $\square$

First, we note that in the $\mathfrak{sl}_2$ case, the pairing $\langle g, K\rangle = q$ is perfect for generic q. Hence, the functor

$$\Phi\colon k[g^{\pm 1}]\text{-}\mathbf{CoMod} \longrightarrow k[K^{\pm 1}]\text{-}\mathbf{Mod}$$

induced by the pairing is fully faithful. The essential image of Φ is the semisimple category generated by one-dimensional simples $k_{q_n} = kv_n$, where $K \rhd v_n = q^n v_n$. Consider the Verma module $M(n) := M(k_{q^n})$ of weight q^n. More generally, write $M(\lambda)$ for the Verma associated to the simple module k_λ, on which K acts by the scalar $\lambda \in k\backslash 0$.

Lemma 2.3.9.2. *The Verma module $M(\lambda)$ has a k basis given by $F^m v_n = F^m \rhd v_n$ for $k \in \mathbb{N}$. The action is given by*

$$K \rhd F^m v_n = \lambda q^{-2m} v_n, \qquad E \rhd F^m v_n = \frac{\lambda^{-1}[m]_q}{q^{-1} - q} F^{m-1} v_n. \tag{2.3.64}$$

Corollary 2.3.9.3. *The Verma modules $M(\lambda)$ are simple for all $\lambda \in k\backslash 0$.*

Proof. Lemma 2.3.9.2 shows that $M(\lambda)$ decomposes as the sum of simple $k[K^{\pm 1}]$-modules as

$$V = \bigoplus_{m \geqslant 0} k_{\lambda q^{-2m}}.$$

Let $W \leqslant V$ be any submodule. Let l be minimal such that there exists $w \in W$ with $w \in k_{\lambda q^{-l}}$. Such an element exists, as we can take any inhomogeneous element and produce a homogeneous element $w' \in W$ by using the biggest k such that $E^k w' \neq 0$ which then must be homogeneous. This follows as the scalar $\frac{\lambda^{-1}[m]_q}{q^{-1}-q} \neq 0$ for all m, λ. Now observe that $w = \mu F^l v_n$ and hence $v_n \in W$ as it is proportional to $E^l F^l v_n$. $\square$

Corollary 2.3.9.4. *The algebra $D_q(\mathfrak{sl}_2)$ has no finite-dimensional representations.*

Proof. Let V be a finite-dimensional simple representation. In particular, $V = \bigoplus_{i=1}^n k_{\lambda_i}$ as a $k[K^{\pm 1}]$-module. Let v be any weight vector (i.e. a vector in k_{λ_i} for some i), then there exists a sufficiently large j such that $E^j v \in k\langle v, E^1 v, \ldots, E^{j-1} v\rangle$. But these are all vectors of different weight, so $E^j v = 0$. We may assume $w := E^{j-1} v \neq 0$ and hence w is a highest weight vector (i.e. one annihilated by E) which generates V. Hence there exists a surjective morphism of $D_q(\mathfrak{sl}_2)$-modules $\pi\colon M(\lambda) \twoheadrightarrow V$. But then $\ker \pi = 0$ as $M(\lambda)$ has no non-trivial submodules. $\square$

We now consider the action of $U_q(\mathfrak{sl}_2)$-$\mathbf{Mod}^{\mathrm{fd}}$ on $\mathcal{O}_q^{\mathrm{Heis}}(\mathfrak{sl}_2)$ from Corollary 2.3.7.4. This gives an analogue of the Quantum Clebsch–Gordan rule.

Proposition 2.3.9.5. *The categorical action of the irreducible $U_q(\mathfrak{sl}_2)$-module $L(n)$ of weight q^n on the Verma $M(m)$ of the category $\mathcal{O}_q^{\mathrm{Heis}}(\mathfrak{sl}_2)$ decomposes as a sum of simples as*

$$L(n) \rhd M(m) = \bigoplus_{k=0}^{n} M(m+n-2k). \tag{2.3.65}$$

Proof. Denote the highest weight vector of $L(n)$ by v and the one of $M(m)$ by w. Using the formulas for the coproduct on $U_q(\mathfrak{sl}_2)$, which serves as the coaction δ giving the categorical action, one checks that $v \otimes w$ is a highest weight vector of weight $n+m$. Note that the weight space $(L(n) \rhd M(m))_{n+m-2k}$ has dimension $k+1$, for $k = 0, \ldots, n$. Further note that each highest weight vector in $L(n) \rhd M(m)$ gives the corresponding Verma module as a direct summand. Hence, we can see inductively that weight space of weight $n+m-2k$ contains precisely one linearly independent highest weight vector for any $k = 0, \ldots, n$. $\square$

We observe that the category $\mathcal{O}_q^{\mathrm{Heis}}(\mathfrak{sl}_2)$ has the same integral weight lattice parametrizing Verma modules as for $\mathcal{O}_q(\mathfrak{sl})$ and that the action of $U_q(\mathfrak{sl}_2)$-$\mathbf{Mod}^{\mathrm{fd}}$ resembles the one on $\mathcal{O}_q(\mathfrak{sl}_2)$ with the crucial difference being that the Heisenberg version of category $\mathcal{O}$ has no finite-dimensional simples and hence all Vermas are simple. We expect the results of this section to generalize to the quantum groups of other semisimple Lie algebras $\mathfrak{g}$.

2.4 Twisted Drinfeld Doubles

Section 2.3 does not use the full generality of the definition of YD-modules in 2.2.4.3 as we are restricting to the case that $\phi = 1 \otimes 1 \otimes 1$ is the trivial 3-cycle. The preparations done in Section 2.2.4 however allow us, more generally, to define the (braided) Drinfeld and Heisenberg double of a quasi-Hopf algebra. This provides a Heisenberg analogue to the Drinfeld double of a quasi-Hopf algebra introduced in [89].

In this Section, we will first give a description in terms of generators and relations of the Drinfeld double of a quasi-Hopf algebra in $\mathbf{Vect}_k$. Next, we provide definitions of *twisted* Drinfeld and Heisenberg doubles. There are two different points of view on twisting used here. One is the *Drinfeld twist* of an ordinary Hopf algebra (see 2.4.2), the other one considers a commutative Hopf algebra as a quasi-Hopf algebra with respect to any 3-cycle (see 2.4.2.2). We close this section by discussing the example of the twisted Heisenberg double of a group in 2.4.3, including an adaptation of the groupoid interpretation of [114] for the twisted Heisenberg double.

2.4.1 The Drinfeld and Heisenberg Double of a Quasi-Hopf Algebra

Even though every monoidal category is equivalent to a strict one (cf. 2.2.1.6) it is still important to keep track of the associativity isomorphisms $\alpha \colon \otimes(\otimes \times \mathrm{Id}) \Rightarrow \otimes(\mathrm{Id} \times \otimes)$ in certain situations. For instance, the class of quasi-Hopf algebras is closed under *Drinfeld twist* $\Delta_F : m^3_{H \otimes H}(F \otimes \Delta \otimes F^{-1})$ of the coproduct for an arbitrary invertible element $F \in H \otimes H$ (see e.g. [88, 2.4]). If F is a 2-cocycle and H a Hopf algebra, then the twist H_F is again a Hopf algebra (see e.g. [51]).

In this section, we provide formulas for the Drinfeld and Heisenberg double of a quasi-Hopf algebra H, with 3-cycle ϕ. For this, we leave the generality of working with quasi-Hopf algebras in braided monoidal categories $\mathcal{B}$. However, via reconstruction theory, one can also obtain formulas in this most general case, if $\mathcal{B} = H$-**Mod** for a quasitriangular finite-dimensional Hopf algebra H (or even for a weakly quasitriangular pair A, H and $\mathcal{B} = A$-**CoMod**). For this, we need the following definition:

Definition 2.4.1.1. A *dual paired object* for a quasi-bialgebra $B \in \mathcal{B}$ is an object C of $\mathcal{B}$ together with a morphism $\mathrm{ev}\colon C \otimes B \to I$ such that there exist maps $\Delta_C\colon C \to C \otimes C$, $m_C\colon C \otimes C \to C$, $1_C\colon I \to C$, $\varepsilon_C\colon C \to I$, such that

$$\begin{aligned} \mathrm{ev}(m_C \otimes B) &= \mathrm{ev}(\mathrm{Id} \otimes \mathrm{ev} \otimes \mathrm{Id})(\mathrm{Id}_{C\otimes C} \otimes \Delta_B), \\ \mathrm{ev}(C \otimes m_B) &= \mathrm{ev}(\mathrm{Id} \otimes \mathrm{ev} \otimes \mathrm{Id})(\Delta_C \otimes \mathrm{Id}_{B\otimes B}), \\ \mathrm{ev}(1_C \otimes \mathrm{Id}_B) &= \varepsilon_B, \qquad \mathrm{ev}(\mathrm{Id}_C \otimes 1_B) = \varepsilon_C. \end{aligned} \tag{2.4.1}$$

Using graphical calculus, these are the same conditions as in Figure 2.1.1. Note however that C is *not* a quasi-bialgebra as the product is only associative up to a 3-*cocycle* (obtained by duality from ϕ). We refer an object with the structure of C as a *dual quasi-bialgebra*. If moreover B has the structure of a quasi-Hopf algebra, and there exists a morphism $S_C\colon C \to C$ such that $\mathrm{ev}(S_C \otimes \mathrm{Id}_B) = \mathrm{ev}(\mathrm{Id}_C \otimes S_B)$. In this case, we say that C is a *dual quasi-Hopf algebra*.

Definition 2.4.1.2. Let B be a quasi-bialgebra in H-**Mod**, where H is a finite-dimensional Hopf algebra, with a dually paired object C. The *braided Drinfeld double* of B, C over H, denoted by $\mathrm{Drin}_H(C, B)$, is defined as the quasi-Hopf algebra obtained by reconstruction of the category ${}_{B^{\mathrm{op}}}\mathcal{YD}^B(\mathcal{B})$ on $C \otimes H \otimes B$. The *braided Heisenberg double* $\mathrm{Heis}_H(C, B)$ is defined by reconstruction of ${}_{B^{\mathrm{op}}}\mathcal{H}^B(\mathcal{B})$ on $C \otimes H \otimes B$.

Similarly, we can define $\mathrm{Drin}_A(C, B)$ and $\mathrm{Heis}_A(C, B)$ if we are given a weak quasitriangular structure on dually paired Hopf algebras A and H.

We only give explicit presentations of $\mathrm{Drin}_H(C, B)$ and $\mathrm{Heis}_H(C, B)$ in the case where $\mathcal{B} = \mathbf{Vect}_k$ to simplify the exposition. Formulas for the general case can be obtained in a similar way, but involve several occurrences of the R-matrix.

Proposition 2.4.1.3. *Let B be a quasi-bialgebra in $\mathbf{Vect}_k$ with 3-cycle ϕ, and dually paired object C. Then $\mathrm{Drin}_H(C, B)$ is generated as an algebra by elements of C, H and B^{op} (opposite in $\mathbf{Vect}_k$) subject to the relations (2.3.12)–(2.3.14) and for $c, d \in C$ the relation*

$$\begin{aligned} cd =& \phi_2^{(3)} c_{(2)} \phi^{-(2)} d_{(3)} \phi_1^{(1)} \,\mathrm{ev}(c_{(1)} \otimes \phi_2^{(2)})\, \mathrm{ev}(c_{(3)} \otimes \phi^{-(3)})\, \mathrm{ev}(c_{(4)} \otimes \phi_1^{(3)}) \\ & \mathrm{ev}(d_{(1)} \otimes \phi_2^{(1)})\, \mathrm{ev}(d_{(2)} \otimes \phi^{-(1)})\, \mathrm{ev}(d_{(4)} \otimes \phi_1^{(2)}). \end{aligned} \tag{2.4.2}$$

The algebra $\mathrm{Drin}_H(C, B)$ *is a quasi-bialgebra with coproducts given by*

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \qquad \Delta(b) = b_{(1)} \otimes b_{(2)}, \tag{2.4.3}$$

$$\begin{aligned} \Delta(c) = {} & \phi_2^{-(2)} c_{(2)} \phi^{(1)} \phi_1^{-(1)} \otimes \phi_2^{-(3)} \phi^{(3)} c_{(4)} \phi_1^{-(2)} \operatorname{ev}(c_{(1)} \otimes \phi_2^{-(1)}) \\ & \operatorname{ev}(c_{(3)} \otimes \phi^{(2)}) \operatorname{ev}(c_{(5)} \otimes \phi_1^{-(3)}). \end{aligned} \tag{2.4.4}$$

With these formulas, Δ *gives a quasi-coproduct with respect to the 3-cycle* ϕ *of* B*. The counit is given by* $\varepsilon(chb) = \varepsilon(c)\varepsilon(h)\varepsilon(c)$*. If* B *is a quasi-Hopf algebra, a formula for the antipode can be obtained by combining* δ' *from Figure 2.2.9 and* δ^* *in Figure 2.2.10.*

The braided Heisenberg double $\mathrm{Heis}_H(C, B)$ *of the quasi-bialgebra* B *is the algebra generated by* C*,* H *and* B^{op} *subject to the relations (2.3.12), (2.3.13) and the cross relation (2.3.19), as well as the relation (2.4.2) from above.*

The Drinfeld double of a quasi-Hopf algebra was already introduced — using a left module version — in [89].

2.4.2 Twisted Hopf Algebras

In this section, we study the Drinfeld and Heisenberg double for two notions of twist of an ordinary Hopf algebra. The first one is often referred to as a *Drinfeld twist*, see e.g. [88, 4.2] for details. The second one generalizes, in the sense of [89], the construction of the twisted Drinfeld double of a group algebra $\mathrm{Drin}^{\omega}(G)$.

Definition 2.4.2.1. Let B be an ordinary bialgebra and $F = F^{(1)} \otimes F^{(2)} \in B \otimes B$ an invertible element. Then the *Drinfeld twist* B_F of B is the quasi-bialgebra defined on the algebra B with the quasi-coalgebra structure

$$\Delta_F(b) = F\Delta(b)F^{-1}, \qquad \phi = F_{23}((\mathrm{Id} \otimes \Delta)F)((\Delta \otimes \mathrm{Id})F^{-1})F_{12}^{-1}, \tag{2.4.5}$$

where the counit is not changed. If moreover B is a Hopf algebra, then the same antipode gives an antipode for B_F with respect to $a = S(F^{-(1)})F^{-(2)}$, and $b = F^{(1)}S(F^{(2)})$. If B is quasitriangular, then so is B_F with universal R-matrix $F_{21}RF^{-1}$.

For a proof that B_F is indeed a quasi-bialgebra (or Hopf algebra) see [88, Theorem 2.4.2]. In fact, it is shown more generally that any Drinfeld twist of a quasi-Hopf algebra is again a quasi-Hopf algebra. If F is a 2-cycle (satisfying a dual condition to (2.3.60)), then B_F is again a Hopf algebra. The Drinfeld twist is also referred to as a *gauge transformation* in the literature. It is a basic result that the categories B-**Mod** and B_F-**Mod** are equivalent as monoidal categories.

Another point of view on twisting is to view a bialgebra B as a quasi-bialgebra B^ϕ with respect to some 3-cycle ϕ which commutes with the two-fold coproduct $(\Delta \otimes \mathrm{Id})\Delta$. For example, if B is commutative, B^ϕ is a quasi-bialgebra for any 3-cycle. This way, the usual notion of a *twisted* Drinfeld double of a group algebra can be obtained (see [33, 89]). We will consider this example together with the corresponding Heisenberg double in the following section. Using either versions of twist, we can apply Proposition 2.4.1.3 to compute the Drinfeld and Heisenberg doubles of the corresponding twisted Hopf algebras. For a general definition, let C, B be dually paired Hopf algebra.

Definition 2.4.2.2. The *twisted Drinfeld double* $\mathrm{Drin}^\omega(C, B)$, with respect to a 3-cycle on B which commutes with the two-fold coproducts, is defined as the bialgebra (respectively, Hopf algebra) $\mathrm{Drin}_k(C, B^\omega)$ using the notation of Definition 2.4.1.2.

The *twisted Heisenberg double* $\mathrm{Heis}^\omega(C, B)$ is defined to be $\mathrm{Heis}_k(C, B^\omega)$.

Hence, we can give a presentation for the twisted double using Proposition 2.4.1.3. More generally, the same construction works if we start with dually paired braided Hopf algebras in $\mathcal{B} = H$-**Mod** (or A-**CoMod** in the weakly quasitriangular case) and ω a 3-cycle on B which commutes with all two-fold coproducts in $B \otimes B \otimes B \in \mathbf{Alg}(\mathcal{B})$ giving a definition of $\mathrm{Drin}^\omega_H(C, B) := \mathrm{Drin}_H(C, B^\omega)$ and its Heisenberg analogue. We will not use this general case in the present section.

As a direct corollary, we have that the monoidal category $\mathrm{Drin}^\omega_H(C, B)$-**Mod** acts on the category $\mathrm{Heis}^\omega_H(C, B)$-**Mod**.

2.4.3 The Twisted Heisenberg Double of a Group

We finish this exposition by showing, as an example, how the categorical action of the monoidal category of the Drinfeld center on the Hopf center can be used to obtain an action of the category of G^{ad}-equivariant ω-twisted vector bundles on the category of ω-twisted G^{reg}-equivariant vector bundles on G.

Let G be a finite group. Then 3-cycles in the sense of (2.1.8) on the Hopf algebra $k[G]$ of function on G correspond to 3-cocycles in the group cohomology of G. That is, for such ω,

$$\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k) = \omega(g,h,kl)\omega(gh,k,l), \qquad \forall g,h,k,l \in G, \tag{2.4.6}$$

where ω is normalized such that $\omega(g,h,k) = 1$ as soon as one of the group elements equals 1.

As $k[G]$ is commutative, $k^{\omega}[G]$, as introduced in the previous section, is a quasi-Hopf algebra for any 3-cocycle ω. In [89], the Drinfeld center of $k^{\omega}[G]$-**CoMod** is shown to be equivalent to the category of modules over the quasi-Hopf algebra $\mathrm{Drin}^{\omega}(G)$, which we refer to as the *twisted* Drinfeld double of the group G. There are two different points of view on its representation category. The first one is:

Proposition 2.4.3.1 ([89]). *The category* $\mathrm{Drin}^{\omega}(G)$-**Mod** *is equivalent to the category of G-graded vector spaces which are G^{ad}-equivariant with respect to twisted representations of G. The representations are twisted by the cocycle $\tau(\omega)$ in $Z^2(G, {}^{\mathrm{ad}}k[G])$ valued in $k[G]$ with the left adjoint action of G, which is defined by*

$$\tau(\omega)(g,h)(k) = \frac{\omega(g,h,h^{-1}g^{-1}kgh)\omega(k,g,h)}{\omega(g,g^{-1}kg,h)}. \tag{2.4.7}$$

That is, the action of a homogeneous element v of degree d is given by

$$g \rhd (h \rhd v) = \tau(\omega)(g,h)(d)gh \rhd v. \tag{2.4.8}$$

The equivariance condition corresponds to the YD-condition that the degree of $g \rhd v$ is gdg^{-1}. We refer to vector spaces V with this structure as ω-twisted G^{ad}-equivariant vector bundles over G.

Modules over the twisted Heisenberg double are twisted representations of G (with respect to the same 2-cocycle $\tau(\omega)$. The only difference is that the equivariance condition is that the degree of $g \rhd v$ is gd if v has degree d. Hence we speak of G^{reg}-equivariant vector bundles over G. The categorical action of G^{ad}-equivariant

ω-twisted vector bundles on G^{reg}-equivariant ones is now a direct consequence of the categorical action discussed before.

Another point of view on the category $\mathrm{Drin}^{\omega}(G)$-**Mod** is given in [114] where representations of the twisted Drinfeld double $\mathrm{Drin}^{\omega}(G)$ are identified with $\tau(\omega)$-twisted representations of the action groupoid $\underline{G}^{\mathrm{ad}}$ of G acting on itself by conjugation. The 2-cocycle $\tau(\omega)$ is obtained by the transgression map

$$\tau\colon Z^3(G, U(1)) \longrightarrow Z^2(\underline{G}^{\mathrm{ad}}).$$

However, the same map also produces 2-cocycles for the action groupoid $\underline{G}^{\mathrm{reg}}$ of G acting on itself by the regular action. Hence, we can identify the category $\mathrm{Heis}^{\omega}(G)$-**Mod** with the category of $\tau(\omega)$-twisted representations over $\underline{G}^{\mathrm{reg}}$.

Chapter 3

Pointed Hopf Algebras with Triangular Decomposition

3.1 Introduction

3.1.1 What Are Quantum Groups?

An important problem in the theory of quantum groups is to give some definition of a class of these objects that captures known series of quantum groups, such as the quantum enveloping algebras $U_q(\mathfrak{g})$ of [36], and their finite-dimensional analogues, as examples. This was for example formulated in [24, Problem II.10.2]:

> *"Given a finite-dimensional Lie algebra $\mathfrak{g}$, find axioms for Hopf algebras to qualify as quantized enveloping algebras of this particular $\mathfrak{g}$."*

A possible hint to the structure of quantum groups is that the quantum enveloping algebras $U_q(\mathfrak{g})$ (as well as the small quantum groups $u_q(\mathfrak{g})$ and multiparameter versions) are *pointed Hopf algebras*. Such Hopf algebras were studied by several authors (see e.g. [9]). Classification results as in [11] suggest a strong resemblance of all finite-dimensional pointed Hopf algebras over abelian groups with small quantum groups. Another paper [10] gives a characterization of quantum groups at generic parameters using pointed Hopf algebras of finite Gelfand–Kirillov dimension with infinitesimal braiding of positive generic type.

A further hint to the structure of quantum groups is that they can be decomposed in a triangular way (via the PBW theorem) as

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_+) \otimes k\mathbb{Z}^n \otimes U_q(\mathfrak{n}_-).$$

Here, the positive and negative part are perfectly paired braided Hopf algebras, and the relation with the group algebra $k\mathbb{Z}^n$ is governed by semidirect product relations. The positive (and negative) part are so-called *Nichols algebras.*

A third aspect — observed already in the original paper [36] — is that quantum groups are (quotients of) *quantum* or *Drinfeld doubles.* It was shown in [90] that $U_q(\mathfrak{g})$ in fact is a *braided* Drinfeld double (which are referred to as a *double bosonization* there). It was proved in [20] that also two-parameter quantum groups are Drinfeld doubles.

In this chapter, we aim to provide an axiomatic approach to the definition of (multiparameter) quantum groups by combining the pointed Hopf algebra and the triangular decomposition approach. Under the additional assumption of what we call a triangular decomposition of *weakly separable type*, the only indecomposable examples are close generalizations of multiparameter quantum groups. In particular, assuming further non-degeneracy, they are examples of a more general version of braided Drinfeld doubles, which we refer to as *asymmetric* braided Drinfeld doubles. Further, under certain assumptions on the group and the parameters, we can recover Lie algebras from these Hopf algebras, after introducing a suitable integral form.

3.1.2 This Chapter's Results

This chapter starts by recalling the necessary technical background, including a brief overview on classification results of finite-dimensional pointed Hopf algebras, as well as structural results by [16] on algebras with triangular decomposition, in Section 3.2. Next, we give the definition of a bialgebra with a triangular decomposition over a Hopf algebra H in Section 3.3. This adapts the two-step approach used for algebras in [16] to the study of bialgebras. Namely, we first consider the *free* case of a bialgebra $T(V) \otimes H \otimes T(V^*)$ where the positive and negative parts ($T(V)$, respectively $T(V^*)$) are tensor algebras, and then specify by what ideals (called *triangular* Hopf ideals) we can take the quotient.

The core of this chapter is formed by a partial classification of bialgebras with triangular decomposition over a group algebra kG. We again proceed in two steps. First, we determine all pointed bialgebras with free positive and negative part over kG in Section 3.4.2, and then look at pairs of ideals I, I^* such that the quotient $A/\langle I, I^*\rangle$ is still a bialgebra in Section 3.4.3. We find that indecomposable examples are automatically pointed Hopf algebras, and can only arise over finitely-generated abelian groups. Multiparameter quantum groups share these features. Indeed, the only possible commutator relations (3.2.10) closely resemble those of multiparameter quantum groups:

$$[f_i, v_j] = \gamma_{ij}(k_j - l_i) \in kG, \qquad \forall i = 1, \ldots, n. \tag{3.1.1}$$

We further observe that there exists a natural generalization of the definition of a braided Drinfeld double to the setting of primitively generated braided Hopf algebras in the category of Yetter–Drinfeld modules (YD-modules) over H. For this, the base Hopf algebra H does not need to be quasitriangular. We need two braided Hopf algebras which are only required to be dually paired considered as braided Hopf algebra in the category of modules (rather than YD-modules). That is, the requirement that is weakened compared to the definition of a braided Drinfeld double (as in [90] or Chapter 2) is that the comodule structures do not need to be dually paired. We refer to this generalization as the *asymmetric braided Drinfeld double*. It gives a natural way of producing Hopf algebras with triangular decomposition — which are not necessarily quasitriangular. We show in Theorem 3.4.3.2 that the Hopf algebras arising in the classification 3.4.2.2 are of this form (provided that the parameters γ_{ii} are non-zero).

In Section 3.4.4 we show that from these asymmetric braided Drinfeld doubles of separable type we can recover Lie algebras provided that there exists a well-defined morphism of rings to $\mathbb{Z}$ when setting the parameters equal to 1. Hence, in the spirit of the question asked in Section 3.1.1, we can relate the outcome of our classification back to Lie algebras, which are always generated by Lie subalgebras isomorphic to $\mathfrak{sl}_2$.

Here is an overview of the increasingly stronger assumptions on the Hopf algebras A and H used in the classification:

- Section 3.3: H any Hopf algebra over a field k, A a bialgebra with triangular decomposition
- Section 3.4: $H = kG$, A a bialgebra with triangular decomposition
  - Section 3.4.1-3.4.2: A is of weakly separable type and indecomposable after 3.4.1.3
  - Section 3.4.3: A is indecomposable of separable type, the scalars γ_{ii} are non-zero.
  - Section 3.4.4: In addition to the assumptions of 3.4.3, we require that $\operatorname{char} k = 0$, and that setting the parameters equal to 1 gives a well-defined homomorphism of rings to $\mathbb{Z}$.

The final Section 3.5 contains different classes of indecomposable pointed Hopf algebras with triangular decomposition over a group kG that arise as examples in the main classification. The first class we discuss are the multiparameter quantum groups $U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ introduced by [43] (adapting the presentation in [29]). They are asymmetric braided Drinfeld doubles, which is a generalization of the result of [20] showing that two-parameter quantum groups are Drinfeld doubles. In section 3.5.2 we bring results of [105] on growth condition (finite Gelfand–Kirillov dimension) and classification of Nichols algebras from [10] into the picture. We use these results to characterize the Drinfeld–Jimbo type quantum groups at generic parameters q within the classification of this chapter under the additional assumption that the triangular decomposition is what we call *symmetric*. Further, classes of finite-dimensional pointed Hopf algebras by Radford can naturally be included as examples in this framework (Section 3.5.3).

To conclude this chapter, we suggest in Section 3.5.4 that future research could focus on the search for Hopf algebras with triangular decomposition over other Hopf algebras H (replacing the group algebra kG). This might give interesting monoidal categories, or even knot invariants in other contexts. As the first — most classical — example, if we take H to be a polynomial ring $k[x_1, \ldots, x_n]$. In this case, the only examples are universal enveloping algebras of Lie algebras.

3.1.3 Notations and Conventions

In this chapter, adapted Sweedler's notation (see e.g. [110, 1.2]) is used to denote coproducts and coactions omitting sums. Unless otherwise stated, we work with Hopf algebras over an arbitrary field k. A Hopf algebra always has an invertible antipode S. The category of left YD-modules over a Hopf algebra H is denoted by ${}^{H}_{H}\mathcal{YD}$, while left modules are denoted by H-**Mod**, and right modules by **Mod**-H.

We denote the module spanned by generators S over a commutative ring R as $R\langle S\rangle$, while $R[S]$ denotes the R-algebra generated by elements S (subject to some specified relations). Groups generated by elements of a set S are denoted by $\langle S\rangle$.

3.2 Background

3.2.1 Pointed Hopf Algebras

Let the coproduct $\Delta\colon H \to H \otimes H$ make H a coalgebra over a field k. We can consider *simple* subcoalgebras $A \leqslant H$. That is, $\Delta(A) \leqslant A \otimes A$ and there are no proper subobjects of this type in A. A basic observation is that if $\dim A = 1$, then A can be written as kg, for a generator $g \in H$ such that $\Delta(g) = g \otimes g$. Such elements are called *grouplike*. Indeed, if H is a Hopf algebra, then the set of all grouplike elements $G(H)$ has a group structure. A Hopf algebra is *pointed* if all simple subcoalgebras are one-dimensional. This notion can be traced back to [110, 8.0] and classifying all finite-dimensional pointed Hopf algebras can be taken as a first step in the classification of all finite-dimensional Hopf algebras (see e.g. [3] for a recent survey).

In the late 1980s and early 1990s, large classes of pointed Hopf algebras have been discovered with the introduction of the quantum groups (and their small analogues). Due to the vast applications of and attention to these Hopf algebras in the literature, the study of pointed Hopf algebras has become an important algebraic question.

3.2.2 Link-Indecomposability

In the early 1990s, Montgomery asked the question, which groups may occur as $G(H)$ where H is an *indecomposable* pointed Hopf algebras. In [95], an appropriate notion of indecomposability is discussed in different ways. We will briefly recall the description in terms of *link-indecomposability* which is equivalent to indecomposability as a coalgebra and indecomposability of the Ext-quiver of simple comodules.

Given a pointed Hopf algebra H, we define a graph Γ_H with vertices being the simple subcoalgebras of H (that is, the grouplike elements). There is an edge $h \to g$ if there exists a (g, h)-skew-primitive element $v \in H$, i.e. $\Delta(v) = v \otimes g + h \otimes v$, which is not contained in $kG(H)$. We say that H is *indecomposable* if Γ_H is connected. As an example, group algebras kG are only indecomposable if $G = 1$. The quantum group $U_q(\mathfrak{sl}_2)$ is indecomposable if the coproducts are e.g. defined as $\Delta(E) = E \otimes 1 + K \otimes E$ and $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$. There are other versions of the coproduct which are not indecomposable (see [95]).

3.2.3 Classification Results for Pointed Hopf Algebras

It was understood early that some pointed Hopf algebras can be obtained as bosonizations $A = \mathcal{B}(V) \rtimes kG$ of so-called *Nichols* (or *Nichols-Woronowicz*) algebras $\mathcal{B}(V)$ associated to YD-modules over a group G (see e.g. [9, Section 2] for definitions). In this case, the coproducts are given by $\Delta(v) = v^{(0)} \otimes v^{(-1)} + 1 \otimes v$ using Sweedler's notation. That is, if v is a homogeneous element, then $\Delta(v) = v \otimes g + 1 \otimes v$ for the degree $g \in G(A)$ of v and A is indecomposable over the group generated by $g \in G$ with $V_g \neq 0$. Thus, the question of finding finite-dimensional pointed Hopf algebras is linked to finding finite-dimensional Nichols algebras.[26] Although both questions remain open in general, vast progress has been made in a series of papers by Andruskiewitsch and Schneider (see [9, 11]) for abelian groups G, and more recently for symmetric and alternating groups [6], or groups of Lie type [4, 5]. See [3] for more detailed references.

Let us briefly recall the classification results of [11] here in order to provide the basis for comparison to our own classification in Section 3.4 later. To fix notation, let $\mathcal{D}$ denote a *finite Cartan datum*. That is, a finite abelian group Γ, a Cartan

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[26] However, a pointed Hopf algebra is not necessary bosonizations of this form. Important tools available are the coradical filtration (see e.g. [94, 5.2]) and the *lifting method* of Andruskiewitsch and Schneider [9, Section 5].

matrix $A = (a_{ij})$ of dimension $n \times n$ with a choice of group elements g_i, characters χ_i for $i = 1, \ldots, n$. Then define $q_{ij} := \chi_j(g_i)$ and impose the conditions that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad q_{ii} \neq 1. \tag{3.2.1}$$

We can associate to the Cartan matrix A a root system Φ (with positive roots Φ^+). The simple roots α_i of Φ can be indexed by $i = 1, \ldots, n$. Denote by χ the set of connected components of the corresponding diagram, and by Φ_J the root system restricted to the component $J \in \chi$, and write $i \sim j$ if i and j are in the same connected component. Denote further

$$g_\alpha := \prod_{i=1}^{n} g_i^{n_i}, \qquad \chi_\alpha := \prod_{i=1}^{n} \chi_i^{n_i}, \qquad \text{for a root } \alpha = \sum_{i=1}^{n} n_i\alpha_i.$$

To state the classification of finite-dimensional pointed Hopf algebras, some technical assumptions need to be made.

(a) Assume that the parameters q_{ii} are roots of *odd* order N_i.

(b) $N_i = N_J$ is constant on each connected component, $i \in J$.

(c) If $J \in \chi$ is of type G_2, then 3 does not divide N_J.

To construct pointed Hopf algebra from a Cartan datum $\mathcal{D}$, we need two families of parameter.

(d) Let $\lambda = (\lambda_{ij})$ be an $n \times n$-matrix of elements in k such that for all $i \nsim j$, $g_ig_j = 1$ or $\chi_i\chi_j \neq \varepsilon$ implies $\lambda_{ij} = 0$.

(e) Further let $\mu = (\mu_\alpha)_{\Phi^+}$ be elements in k such that for any $\alpha \in \Phi_J^+$, for $J \in \chi$,

Definition 3.2.3.1 ([11, 5.4])**.** Given the a Cartan datum $\mathcal{D}$ with families of parameters λ, μ as above, there is a Hopf algebra $u = u(\mathcal{D}, \lambda, \mu)$. The algebra u is generated by elements $g \in \Gamma$ (define $u_\alpha(\mu) \in k\Gamma$, see [11, 2.14] for $\alpha \in \Phi^+$), and x_i for $i = 1, \ldots, n$, subject to the relations

$$gx_i = \chi_i(g)x_ig, \qquad \text{for all } i,\ g \in \Gamma, \tag{3.2.2}$$

$$\underline{\mathrm{ad}}(x_i)^{1-a_{ij}} = 0, \qquad \text{for } i \neq j,\ i \sim j, \tag{3.2.3}$$

$$\underline{\mathrm{ad}}(x_i)(x_j) = \lambda_{ij}(1 - g_ig_j), \qquad \text{for all } i < j,\ i \nsim j, \tag{3.2.4}$$

$$x_\alpha^{N_J} = u_\alpha(\mu), \qquad \text{for all } \alpha \in \Phi_J^+,\ J \in \chi. \tag{3.2.5}$$

Here, $\underline{\mathrm{ad}}(x)(y)$ is the *braided* commutator $xy - m \circ \Psi(x \otimes y)$ where m denotes multiplication and Ψ is the YD-braiding. The comultiplication is given by $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$.

Theorem 3.2.3.2 ([9, 0.1])**.** *Under the above assumptions (a)–(e) on a Cartan datum $\mathcal{D}$ with parameters λ, μ, the Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is pointed with $G(u) = \Gamma$ and of finite dimension. Moreover, if $|G|$ is not divisible by $2, 3, 5$ or 7, then any finite-dimensional pointed Hopf algebra is of this form.*

3.2.4 Algebras with Triangular Decomposition (Free Case)

A triangular decomposition of algebras means that an intrinsic PBW decomposition exists, similar to universal enveloping algebras of Lie algebras. This is a common feature of quantum groups and rational Cherednik algebras, but more generally shared by all braided Drinfeld or Heisenberg doubles (cf. Section 2.3.4). Here, we are using the definitions introduced in [16] to study such algebras with triangular decomposition (so-called *braided doubles*).

From a deformation theoretic point of view, triangular decomposition can be viewed as follows. Let V, V^* be dually paired finite-dimensional vector spaces and H a Hopf algebra over a field k, such that V is a left H-module, and V^* carries the dual right H-action. That is, for the evaluation map $\langle\ ,\ \rangle \colon V^* \otimes V \to k$, we have

$$\langle f \triangleleft h, v\rangle = \langle f, h \triangleright v\rangle, \qquad \forall f \in V^*, v \in V, h \in H. \tag{3.2.6}$$

Now define $A_0(V, V^*)$ to be the algebra on $T(V) \otimes H \otimes T(V^*)$ with relations

$$fh = h_{(1)}(f \triangleleft h_{(2)}), \qquad hv = (h_{(1)} \triangleright v)h_{(2)}, \tag{3.2.7}$$

(i.e. the bosonizations $T(V) \rtimes H$ and $H \ltimes T(V^*)$ are subalgebras), and $[f, v] = 0$.

In [16, 3.1], a family of deformations of $A_0(V, V^*)$ over $\mathrm{Hom}_k(V \otimes V^*, H)$ is defined. The algebra $A_\beta(V, V^*)$ over a parameter $\beta \colon V \otimes V^* \to H$ is defined using the same generators in V, V^* and H with the same bosonization relations, but the commutator relation

$$[f, v] = \beta(f, v). \tag{3.2.8}$$

In order to obtain flat deformations we restrict to maps β such that the multiplication

$$m\colon T(V)\otimes H\otimes T(V^*)\xrightarrow{\sim} A_\beta(V,V^*),\qquad v\otimes h\otimes f\mapsto vhf,$$

is an isomorphism of k-vector spaces.

Definition 3.2.4.1. In the case where m is an isomorphism of k-vector spaces, we say that $A_\beta(V,V^*)$ is a *free braided double*.

Theorem 3.2.4.2 ([16, Theorem 3.3])**.** *The algebra $A_\beta(V,V^*)$ is a free braided double if and only if there exists a k-linear map $\delta\colon V\to H\otimes V$, $\delta(v)=v^{[-1]}\otimes v^{[0]}$ which is YD-compatible with the H-action on V, i.e. for any $h\in H$*

$$h_{(1)}v^{[-1]}\otimes(h_{(2)}\rhd v^{[0]})=(h_{(1)}\rhd v)^{[-1]}h_{(2)}\otimes(h_{(1)}\rhd v)^{[0]}. \tag{3.2.9}$$

In this case, we call (V,δ) a quasi-YD-module *and we have*

$$[f,v]=\beta(f\otimes v)=v^{[-1]}\langle f,v^{[0]}\rangle. \tag{3.2.10}$$

3.2.5 Triangular Ideals

So far, the braided Hopf algebras $T(V)$ and $T(V^*)$ were assumed to be free. We can bring additional relations into the picture, defining *braided double* that are not necessarily free. Let $I\lhd T(V)$ and $I^*\lhd T(V^*)$ be ideals. We want to determine when the quotient map

$$m\colon T(V)/I\otimes H\otimes T(V^*)/I^*\xrightarrow{\sim} A_\beta(V,V^*)/\langle I,I^*\rangle$$

is still an isomorphism of k-vector spaces. In [16, Appendix A] it is show that this is the case if and only if $J:=\langle I,I^*\rangle$ is a so-called *triangular ideal*. That is, $J=I\otimes H\otimes T(V^*)+T(V)\otimes H\otimes I^*$, where $I\lhd T^{>0}(V)$, $I^*\lhd T^{>0}(V^*)$ such that I and I^* are H-invariant and

$$T(V^*)I\leqslant J,\qquad I^*T(V)\leqslant J. \tag{3.2.11}$$

This is equivalent to the commutator $[f,I]$ and $[I^*,v]$ being contained in J for all degree one elements $v\in V$, $f\in V^*$. For each quasi-YD-module, there exists a unique largest triangular ideal $I_{\max}$, and thus a unique maximal quotient referred to as a *minimal braided double*.

If δ is a YD-module, then the maximal quotient $T(V)/I_{\max}$ is the Nichols algebra $\mathcal{B}(V)$ of V, and the braided double on $\mathcal{B}(V)\otimes H\otimes\mathcal{B}(V^*)$ is a generalization of the Heisenberg double, a so-called *braided Heisenberg double*.

For the purpose of this chapter, we need ideals I such that $T(V)/I$ is a braided bialgebra, where V is a YD-module. That is, not a bialgebra object in the category of k-vector spaces but in the category of YD-modules over kG (see e.g. [9, 1.2–1.3]). However, if I is a homogeneous ideal in $T^{>1}(V)$ which is a coideal and a YD-submodule, then $T(V)/I$ is a braided Hopf algebra. We denote the collection of such ideals by $\mathcal{I}_V$. In fact $I_{\max} \in \mathcal{I}_V$ as the Nichols algebra $\mathcal{B}(V)$ is a braided Hopf algebra.

3.3 Hopf Algebras with Triangular Decomposition

In this section, we let k be a field of arbitrary characteristic and H a Hopf algebra over k. We introduce a notion of a Hopf algebra with triangular decomposition.

3.3.1 Definitions

We refer to the grading of a braided double $T(V)/I \otimes H \otimes T(V^*)/I^*$ given by

$$\deg v = 1, \quad \deg f = -1, \quad \deg h = 0, \qquad \forall v \in V,\ f \in V^*,\ h \in H.$$

as the *natural grading*. We want to study Hopf algebras with triangular decomposition preserving this grading.

Definition 3.3.1.1. A bialgebra (or Hopf algebra) A with *triangular decomposition* over a Hopf algebra H is a braided double $H = T(V)/I \otimes H \otimes T(V^*)/I^*$ which is a bialgebra (respectively Hopf algebra) such that

- H is a subcoalgebra of A with respect to the original coproduct of H, (3.3.1)
- the subspaces $T(V) \otimes H$ and $H \otimes T(V^*)$ are closed under the coproduct of A, (3.3.2)
- the coproduct and counit are morphisms of graded algebras for the natural grading. (3.3.3)

(In the Hopf case, the antipode S is required to preserve the natural grading and the subspaces $T(V) \otimes H$ and $H \otimes T(V^*)$.)

Note that (3.3.3) implies that $\varepsilon(v) = \varepsilon(f) = 0$ for all $v \in V$, $f \in V^*$. We further observe that assumption (3.3.2) and (3.3.3) combined with the counit property, give that $\Delta(V) \leqslant H \otimes V + V \otimes H$ as well as $\Delta(V^*) \leqslant H \otimes V^* + V^* \otimes H$. Consider the compositions δ_r, δ_l with projections in

$$\begin{tikzcd}
& V \arrow[dl, "\delta_l"'] \arrow[d, "\Delta"] \arrow[dr, "\delta_r"] & \\
H \otimes V & H \otimes V \oplus V \otimes H \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & V \otimes H.
\end{tikzcd}$$

The coalgebra axioms imply that δ_l and δ_r are left (respectively right) H-coactions. In particular, as the semidirect product relations in A are preserved by Δ, δ_l (and δ_r) are left (respectively right) YD-compatible with the given action of H on V. Similarly, we can obtain a left and right YD-module structure over H on the dual V^* from the coproduct. These are denoted by δ_l^* and δ_r^*.

Definition 3.3.1.2. Given a bialgebra A with triangular decomposition over H, we define the *right (respectively, left) YD-structure* of A to be δ_r (respectively, δ_l). We refer to δ_r^* and δ_l^* as the right and left *dual* YD-structures.

To fix Sweedler's notation for the different coactions, denote $\delta_r(v) = v^{(0)} \otimes v^{(-1)}$ and $\delta_l(v) = v^{\overline{(-1)}} \otimes v^{\overline{(0)}}$ and use similar notations for $f \in V^*$. We will reformulate the definition of a bialgebra with triangular decomposition in terms of conditions on the YD-structures of A in (3.3.6)–(3.3.10) in the free case first.

Lemma 3.3.1.3. *A bialgebra with triangular decomposition A is a Hopf algebra with triangular decomposition if and only if*

$$\begin{aligned} S(v^{(0)})v^{(-1)} + (S(v^{\overline{(-1)}})_{(1)} \rhd v^{\overline{(0)}})S(v^{\overline{(-1)}})_{(2)} &= 0, \\ v^{(0)}S(v^{(-1)}) + (v^{\overline{(-1)}}{}_{(1)} \rhd S(v^{\overline{(0)}}))v^{\overline{(-1)}}{}_{(2)} &= 0, \end{aligned} \qquad \forall v \in V, \tag{3.3.4}$$

$$\begin{aligned} f^{\overline{(-1)}}S(f^{\overline{(0)}}) + S(f^{(-1)})_{(1)}(f^{(0)} \lhd S(f^{(-1)})_{(2)}) &= 0, \\ S(f^{\overline{(-1)}})f^{\overline{(0)}} + f^{(-1)}{}_{(1)}(S(f^{(0)}) \lhd f^{(-1)}{}_{(2)}) &= 0, \end{aligned} \qquad \forall f \in V^*. \tag{3.3.5}$$

In this case, the antipode extends uniquely to all of A.

Proof. This follows (under use of the semidirect product relations) by restating the antipode axioms for the coproduct of a Hopf algebra with triangular decomposition, which has the form $\Delta(v) = v^{(0)} \otimes v^{(-1)} + v^{\overline{(-1)}} \otimes v^{\overline{(0)}}$. Note that $\varepsilon(v) = 0$ as we require the counit to be a morphism of graded algebras. ☐

3.3.2 The Free Case

Let A be a *free* braided double, i.e. $A = T(V) \otimes H \otimes T(V^*)$. We can now state necessary and sufficient conditions on the YD-structures of A to make the algebra A a bialgebra with triangular decomposition. In the following, we stick to the notation of [16, Definition 2.1] denoting the quasi-coaction determining the commutator relation between elements of V and V^* by $\delta(v) = v^{[-1]} \otimes v^{[0]}$, for $v \in V$.

Lemma 3.3.2.1. *A free braided double A on $T(V) \otimes H \otimes T(V^*)$ is a bialgebra with triangular decomposition if and only if there exist YD-structures δ_l, δ_r, δ_l^*, and δ_r^* such that the following conditions hold for $v \in V$, $f \in V^*$:*

$$(f^{(0)} \lhd v^{\overline{(-1)}}) \otimes (f^{(-1)} \rhd v^{\overline{(0)}}) = f \otimes v, \tag{3.3.6}$$

$$(f^{\overline{(-1)}} \rhd v^{(0)}) \otimes (f^{\overline{(0)}} \lhd v^{(-1)}) = v \otimes f, \tag{3.3.7}$$

$$v^{(0)}f^{(0)} \otimes (f^{(-1)}v^{(-1)} - v^{(-1)}f^{(-1)}) = 0, \tag{3.3.8}$$

$$(f^{\overline{(-1)}}v^{\overline{(-1)}} - v^{\overline{(-1)}}f^{\overline{(-1)}}) \otimes v^{\overline{(0)}}f^{\overline{(0)}} = 0, \tag{3.3.9}$$

$$\begin{aligned} &v^{(0)[-1]}\langle f^{(0)}, v^{(0)[0]}\rangle \otimes f^{(-1)}v^{(-1)} \\ &+ f^{\overline{(-1)}}v^{\overline{(-1)}} \otimes v^{\overline{(0)}[-1]}\langle f^{\overline{(0)}}, v^{\overline{(0)}[0]}\rangle \end{aligned} = v^{[-1]} \otimes v^{[-1]}\langle f, v^{[0]}\rangle. \tag{3.3.10}$$

Proof. The conditions are easily checked to be equivalent — under use of the relations in A and the PBW theorem — to the requirement that (3.2.10) is preserved by Δ. This gives the relations (3.3.8)–(3.3.10), as well as

$$\begin{aligned} v^{\overline{(-1)}}{}_{(1)}(f^{(0)} \triangleleft v^{\overline{(-1)}}{}_{(2)}) \otimes (f^{(-1)}{}_{(1)} \triangleright v^{\overline{(0)}}) f^{(-1)}{}_{(2)} &= v^{\overline{(-1)}} f^{(0)} \otimes v^{\overline{(0)}} f^{(-1)}, \\ (f^{\overline{(-1)}}{}_{(1)} \triangleright v^{(0)}) f^{\overline{(-1)}}{}_{(2)} \otimes v^{(-1)}{}_{(1)}(f^{\overline{(0)}} \triangleleft v^{(-1)}{}_{(2)}) &= v^{(0)} f^{\overline{(-1)}} \otimes v^{(-1)} f^{\overline{(0)}}. \end{aligned}$$

These relations are equivalent to (3.3.6) and (3.3.7) under use of the counit of H, applying the coaction axioms.

Conversely, given δ_r, δ_l as well as their dual counterparts δ_r^*, δ_l^*, the bosonization relations are preserved by the coproduct defined as

$$\Delta(v) = v^{(0)} \otimes v^{(-1)} + v^{\overline{(-1)}} \otimes v^{\overline{(0)}}, \quad \Delta(f) = f^{(0)} \otimes f^{(-1)} + f^{\overline{(-1)}} \otimes f^{\overline{(0)}}, \quad (3.3.11)$$

for $v \in V$, $f \in V^*$ by YD-compatibility. □

It will become apparent in Section 3.4 what constraints on the structure of A conditions (3.3.6)–(3.3.10) give working over a group, and over a polynomial ring in Section 3.5.4.

3.3.3 Triangular Hopf Ideals

We are looking for triangular ideals $J = I \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^*$ (cf. [16, Appendix A] or Section 3.2.5) which are also coideals, and hence A/J is a triangular bialgebra or Hopf algebra.

Using the description of the coproduct Δ in terms of the left and right YD-structures on A, the triangular ideas J that are also coideals are simply those triangular ideals for which I (and I^*) are YD-submodules for both δ_l and δ_r (respectively, δ_l^* and δ_r^*).

If A is a triangular Hopf algebra with antipode given as in Lemma 3.3.1.3, then every triangular ideal which is also a coideals is automatically a Hopf ideal.

Definition 3.3.3.1. We denote the collection of ideals of the form

$$J = I \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^*$$

for $I \triangleleft T(V)$ and $I^* \triangleleft T(V^*)$ which are also YD-submodules for δ_r, δ_l (respectively for δ_r^*, δ_l^*) by $\mathcal{I}_\Delta(A)$. Such ideals J are called *triangular Hopf ideals*.

3.3.4 Asymmetric Braided Drinfeld Doubles

A special class of Hopf algebras with triangular decomposition can be provided by braided Drinfeld doubles of primitively generated Hopf algebras over a quasitriangular base Hopf algebra H. This form of the Drinfeld double was introduced as the *double bosonization* in [88, 90], see also Section 2.3.5 for the presentation used here. We now give a more general definition of an *asymmetric* braided Drinfeld double which is suitable to capture the more general class of Hopf algebras that we find in Section 3.4, including multiparameter quantum groups, as examples. In this construction, the base Hopf algebra H need not be quasitriangular, and the asymmetric braided Drinfeld double is also not quasitriangular in general.

To define the braided Drinfeld double of dually paired braided Hopf algebras C and B in the category $\mathrm{Drin}(H)$-**Mod** $= {}^H_H\mathcal{YD}$ we require that $\langle\ ,\ \rangle\colon C\otimes B\to k$ is a morphism of YD-modules. This implies that the actions and coactions on C and B are dual to one-another (by means of the antipode of H). A weaker requirement is that we consider the images of C and B under the forgetful functor

$$F\colon {}^H_H\mathcal{YD} \longrightarrow H\text{-}\mathbf{Mod},$$

and require that $F(C)$ and $F(B)$ are dually paired Hopf algebras in H-**Mod** (with the induced braiding under F), while C and B may not be dually paired in ${}^H_H\mathcal{YD}$. Hence the coactions on C and B do not necessarily have to be related via the antipode, but the actions and resulting braidings need to be related by duality. This is captured by the following definition, where we denote the left coactions by $c\mapsto c^{(-1)}\otimes c^{(0)}$ and $b\mapsto b^{(-1)}\otimes b^{(0)}$ respectively.

Definition 3.3.4.1. We say that two braided Hopf algebras C, B in ${}^H_H\mathcal{YD}$ are *weakly dually paired* if there exists a morphism of H-modules $\langle\ ,\ \rangle\colon C\otimes B\to k$ such that

$$\langle cc', b\rangle = \langle c', b_{(1)}\rangle\langle c, b_{(2)}\rangle, \qquad \langle c, bb'\rangle = \langle c_{(1)}, b'\rangle\langle c_{(2)}, b\rangle, \tag{3.3.12}$$

for all $c, c'\in C$, and $b, b'\in B$; as well as

$$(c^{(-1)}\triangleright b)c^{(0)} = b^{(0)}(b^{(-1)}\triangleright c). \tag{3.3.13}$$

This weaker duality is equivalent to an analogue of condition (3.3.7). To see this, we can regard the left H-coaction on B as a right H^{cop}-coaction. Given a left H-action $\triangleright$, we define a right H^{cop}-action $\triangleleft := \triangleright(S^{-1} \otimes \mathrm{Id})\tau$ (where τ denotes the $\otimes$-symmetry in $\mathbf{Vect}_k$). The resulting structures make B a right YD-module over H^{cop}. The condition (3.3.7) can be rephrased as requiring for all $b \in B, c \in C$ that

$$\begin{aligned} & b^{(0)}c^{(-1)} \otimes b^{(-1)}c^{(0)} = c^{(-1)}b^{(0)} \otimes c^{(0)}b^{(-1)}, && (3.3.14)\\ \Longleftrightarrow \quad & bc = (c^{(-1)} \triangleright b^{(0)})(c^{(0)} \triangleleft b^{(-1)}), &&\\ \Longleftrightarrow \quad & (c^{(-1)} \triangleright b)c^{(0)} = b^{(0)}(c \triangleleft S(b^{(-1)})) = b^{(0)}(b^{(-1)} \triangleright c), && \end{aligned}$$

which gives condition (3.3.13). We can visualize condition (3.3.14) using graphical calculus (with the conventions from Chapter 2):

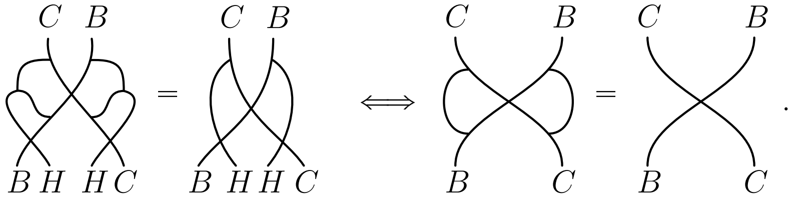

Figure 3.3.1: Left and right braiding compatibility condition

Assuming the axiom (3.3.13), we can define an analogue of the braided Drinfeld double on the k-vector space $B \otimes H \otimes C$ (rather than using $B \otimes \mathrm{Drin}(H) \otimes C$) with this weaker requirement of duality on C and B. The definition of the *asymmetric* braided Drinfeld double can be given using Tannakian reconstruction theory by describing their category of modules. This is similar to the approach used for the braided Drinfeld double in [90, Appendix B] (cf. also Section 2.3.2).

Definition 3.3.4.2. Let C, B be weakly dually paired braided Hopf algebras in ${}^{H}_{H}\mathcal{YD}$. We define the category ${}^{C}\mathcal{YD}^{B}_{\mathrm{asy}}(H)$ of *asymmetric YD-modules* over C, B as having objects V which are left H-modules (also viewed as right modules by means of the inverse antipode), equipped with a left C-action and a right B-action (by morphisms of H-modules) which satisfy the compatibility condition

$$((c_{(2)} \triangleright v) \triangleleft {b_{(1)}}^{(-1)}) \triangleleft b_{(2)} \langle c_{(1)}, {b_{(1)}}^{(0)} \rangle = c_{(1)} \triangleright ({c_{(2)}}^{(-1)} \triangleright (v \triangleleft b_{(1)})) \langle {c_{(2)}}^{(0)}, b_{(2)} \rangle, \tag{3.3.15}$$

for all $v \in V, b \in B, c \in C$. Morphisms in ${}^C\mathcal{YD}^B_{\mathrm{asy}}(H)$ are required to commute with the actions of H, B and C.

It may help to visualize the condition (3.3.15) using graphical notation:

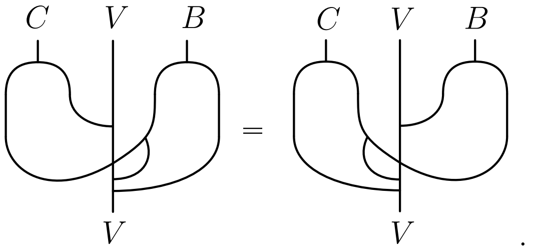

Figure 3.3.2: Asymmetric Yetter–Drinfeld modules

Proposition 3.3.4.3. *The category* ${}^C\mathcal{YD}^B_{\mathrm{asy}}(H)$ *is monoidal, with a commutative diagram of monoidal fiber functors*

$$\begin{tikzcd}
 & \mathbf{Mod}\text{-}B(H\text{-}\mathbf{Mod}) \arrow[dr] & & \\
{}^C\mathcal{YD}^B_{\mathrm{asy}}(H) \arrow[ur] \arrow[dr] & & H\text{-}\mathbf{Mod} \arrow[r] & \mathbf{Vect}_k. \\
 & C\text{-}\mathbf{Mod}(H\text{-}\mathbf{Mod}) \arrow[ur] & &
\end{tikzcd}$$

Proof. This monadicity statement can for example be checked directly using graphical calculus. Note that condition (3.3.14) is crucial. The fiber functors simply forget the additional structure at each step. ☐

Definition 3.3.4.4. The *asymmetric braided Drinfeld double* $\mathrm{Drin}^{\mathrm{asy}}_H(C, B)$ is defined as the algebra obtained by Tannakian reconstruction[27] on $B \otimes H \otimes C$ applied to the functor ${}^C\mathcal{YD}^B_{\mathrm{asy}}(H) \longrightarrow \mathbf{Vect}_k$. Hence $\mathrm{Drin}^{\mathrm{asy}}_H(C, B)$-**Mod** and ${}^C\mathcal{YD}^B_{\mathrm{asy}}(H)$ are canonically equivalent as categories.

---

[27] See e.g. [88, 9.4.1] or Section 2.2.3.

Proposition 3.3.4.5. *An explicit presentation for the asymmetric braided Drinfeld double* $\mathrm{Drin}^{\mathrm{asy}}_{H}(C,B)$ *on the* k*-vector space* $B\otimes H\otimes C$ *can be given as follows: the multiplication on* B *is opposite, and for* $c\in C$*,* $b\in B$ *and* $h\in H$ *we have*

$$hb = (h_{(2)} \rhd b)h_{(1)}, \tag{3.3.16}$$

$$hc = (h_{(1)} \rhd c)h_{(2)}, \tag{3.3.17}$$

$$b_{(2)}S^{-1}({b_{(1)}}^{(-1)})c_{(2)}\langle c_{(1)}\otimes {b_{(1)}}^{(0)}\rangle = c_{(1)}{c_{(2)}}^{(-1)}b_{(1)}\langle {c_{(2)}}^{(0)}\otimes b_{(2)}\rangle. \tag{3.3.18}$$

The coproducts are given by

$$\Delta(h) = h_{(1)}\otimes h_{(2)}, \tag{3.3.19}$$

$$\Delta(b) = {b_{(1)}}^{(0)}\otimes b_{(2)}S^{-1}({b_{(1)}}^{(-1)}), \tag{3.3.20}$$

$$\Delta(c) = c_{(1)}{c_{(2)}}^{(-1)}\otimes {c_{(2)}}^{(0)}, \tag{3.3.21}$$

and the antipode is

$$S(h) = S(h), \qquad S(b) = S^{-1}(b^{(0)})b^{(-1)}, \qquad S(c) = S(c^{(-1)})S(c^{(0)}), \tag{3.3.22}$$

using the respective given structures on H*,* B*, and* C*.*

Proof. This follows under application of reconstruction (in $\mathbf{Vect}_k$) applied to ${}^{C}\mathcal{YD}^{B}_{\mathrm{asy}}(H)$. See e.g Section 2.2.3 for formulas on how to obtain the structures, including the antipode (Figure 2.1). □

An important feature of the braided Drinfeld double is that it has braided categories of representations. For the *asymmetric* braided Drinfeld double to be quasitriangular, we need H to be quasitriangular. If H is not quasitriangular, this can be achieved by working with over $\mathrm{Drin}(H)$ instead of H as a base Hopf algebra.

From now on, we restrict to the important special case where B and C are primitively generated by finite-dimensional YD-modules. This way, we obtain examples of Hopf algebras with a triangular decomposition over H.

Lemma 3.3.4.6. *Let* V*,* V^* *be left YD-modules over* H*, such that the action on* V^* *is dual to the action on* V*. Then braided tensor (co)algebras* $T(V)^{\mathrm{op}}$ *and*

$T(V^*)^{\mathrm{cop}}$ are dually paired[28] *in the monoidal category of right modules over H. Further assume that the compatibility condition (3.3.14) holds.*

Then the asymmetric braided Drinfeld double $\mathrm{Drin}_H^{\mathrm{asy}}(T(V^*)^{\mathrm{cop}}, T(V)^{\mathrm{op}})$ is given on $A = T(V) \otimes H \otimes T(V^*)$ subject to the usual bosonization relations (3.2.7) and the cross relation

$$[f, c] = S^{-1}(v^{(-1)})\langle f, v^{(0)}\rangle - f^{(-1)}\langle f^{(0)}, v\rangle. \tag{3.3.23}$$

The coalgebra structure is given by

$$\Delta(v) = v^{(0)} \otimes S^{-1}(v^{(-1)}) + 1 \otimes v, \qquad \Delta(f) = f \otimes 1 + f^{(-1)} \otimes f^{(0)}. \tag{3.3.24}$$

The counit is given by $\varepsilon(v) = \varepsilon(f) = 0$ and the antipode can be computed using the conditions from equations (3.3.4) and (3.3.5) as

$$S(v) = -v^{(0)}v^{(-1)}, \qquad S(f) = -S(f^{(-1)})f^{(0)}. \tag{3.3.25}$$

We can also consider quotients of the form A/J for any triangular Hopf ideal $J \in \mathcal{I}_\Delta(A)$. The quotient of A by the maximal triangular Hopf ideal in $\mathcal{I}_\Delta(A)$ is denoted by $U_H(V, V^*)$.

Lemma 3.3.4.7. *Let $A = \mathrm{Drin}_H^{\mathrm{asy}}(T(V)^{\mathrm{op}}, T(V^*)^{\mathrm{cop}})$ for V, V^* as in Lemma 3.3.4.6. Then the maximal ideal $I_{\max}(A)$ in $\mathcal{I}_\Delta(A)$ is given by*

$$I_{\max}(A) = I_{\max}(V) \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I_{\max}(V^*),$$

where $I_{\max}(V)$ is the maximal ideal for the left coaction on V, and $I_{\max}(V^*)$ is the maximal ideal for the left coaction on V^*. Hence

$$m \colon \mathcal{B}(V) \otimes H \otimes \mathcal{B}(V^*) \xrightarrow{\sim} U_H(V, V^*)$$

is an isomorphism of k-vector spaces (PBW theorem).

Proof. This is clear as we know that $T(V)^{\mathrm{op}}/I_{\max}(V)$ and $T(V^*)^{\mathrm{cop}}/I_{\max}(V^*)$ are weakly dually paired braided Hopf algebras and their asymmetric braided Drinfeld double is given by the quotient $\mathrm{Drin}_H^{\mathrm{asy}}(T(V)^{\mathrm{op}}, T(V^*)^{\mathrm{cop}})/I_{\max}(A)$, which must be the minimal double $U_H(V, V^*)$. □

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[28] We choose the opposite $T(V)^{\mathrm{op}}$ and coopposite $T(V^*)^{\mathrm{cop}}$ in order to avoid having to take the opposite multiplication in the resulting double (cf. 3.3.4.5). As tensor algebras are cocommutative, this choice does not affect the formulas for the coproduct.

A perfect pairing between the positive and negative part of $U_H(V,V^*)$ implies the existence of a formal power series coev satisfying the axioms of coevaluation. This can be used to give a braiding on a suitable category of modules over $U_H(V,V^*)$ (where $\mathcal{B}(V)$ acts integrally), and all modules have the structure of being YD-modules over H.

3.3.5 Symmetric Triangular Decompositions

The rest of this section will be devoted to the question of recovering the braided Drinfeld double over a quasitriangular base Hopf algebra H as a special case of the asymmetric braided Drinfeld double. For this, we introduce the idea of a Hopf algebra with a *symmetric* triangular decomposition:

Definition 3.3.5.1. Given a bialgebra with triangular decomposition. If the associated coactions satisfy that the right coaction δ_r^* of V^* is the dual coaction to δ_l, i.e.

$$\langle f^{(0)} \otimes v \rangle f^{(-1)} = \langle f \otimes \overline{v^{(0)}} \rangle \overline{v^{(-1)}}, \tag{3.3.26}$$

and the coactions δ_r and δ_l^* are compatible in the same way, then we call the triangular decomposition *symmetric*.

In the case where H is a quasitriangular Hopf algebra, we can recover a special case of the definition of the braided Drinfeld double given in Example 2.3.5.7 from the more general form given in Definition 3.3.4.4, and the resulting triangular decomposition will be symmetric. For this, note that the universal R-matrix and its inverse give functors (due to [79])

$$\begin{aligned} R^{-1}\colon H\text{-}\mathbf{Mod} &\longrightarrow {}^H_H\mathcal{YD}, \qquad (V,\triangleright) \longmapsto (V,\triangleright,(\mathrm{Id}_H \otimes \triangleright)(R^{-1}\otimes \mathrm{Id}_V)),\\ R\colon \mathbf{Mod}\text{-}H &\longrightarrow {}^H_H\mathcal{YD}, \qquad (V,\triangleleft) \longmapsto (V,\triangleleft,(\triangleleft \otimes \mathrm{Id}_H)(\mathrm{Id}_V \otimes R)). \end{aligned}$$

Given a right H-module V we can hence give V the left YD-module structure using R^{-1}, and V^* the dual YD-module structure. Note that (3.3.14) is satisfied in this case. With these structures, the relation (3.3.23) becomes

$$\begin{aligned} [f,c] &= S^{-1}(R^{-(2)})\langle f, v \triangleleft R^{-(1)}\rangle - R^{-(1)}\langle R^{-(2)} \triangleright f, v\rangle\\ &= R^{(2)}\langle R^{(1)} \triangleright f, v\rangle - R^{-(1)}\langle R^{-(2)} \triangleright f, v\rangle. \end{aligned}$$

This is precisely the condition of Example 2.3.5.7. Note that we use $R = (S^{-1} \otimes \mathrm{Id}_H)R^{-1}$. This proves the following:

Proposition 3.3.5.2. *Braided Drinfeld doubles of braided Hopf algebras over a quasitriangular Hopf algebras are a special case of Definition 3.3.4.4 with symmetric triangular decomposition.*

Note that a partial converse statement also holds: Given an asymmetric braided Drinfeld double that is symmetric, then it can be displayed as a braided Drinfeld double in the sense of [68, 90], but unless H is quasitriangular (and the coaction induced by the R-matrix), we need to view if over the base Hopf algebra $\mathrm{Drin}(H)$. If the positive and negative part are perfectly paired, then we can give a formal power series describing the R-matrix and an appropriate subcategory (corresponding to the Drinfeld center) is braided.

Particularly interesting examples of such braided Drinfeld doubles include the quantum groups $U_q(\mathfrak{g})$ for generic q, and the small quantum groups $u_q(\mathfrak{g})$ (see [90, Section 4]). Their construction uses the concept of a weak quasitriangular structure for which a similar statement to 3.3.5.2 can be made. We will see in Section 3.5 that multiparameter quantum groups can be viewed as examples of asymmetric braided Drinfeld doubles that are not symmetric. Further, all the pointed Hopf algebras classified in the main result of this chapter (Theorem 3.4.2.2), under the additional assumption that the braiding is of separable type and some commutators do not vanish, are asymmetric braided Drinfeld doubles.

3.4 Classification over a Group

In this section, we denote by $A = T(V) \otimes kG \otimes T(V^*)$ a bialgebra with triangular decomposition over a group algebra kG. Note that we do not assume G to be finite.

3.4.1 Preliminary Observations

Hopf algebras that are generated by grouplike and skew primitive elements are always pointed. We show that assuming a Hopf algebra has triangular decomposition over a group and is of what we call *weakly separable type*, it is generated by skew-primitive elements and hence pointed.

Lemma 3.4.1.1. *For a bialgebra A with triangular decomposition over kG as above, there exists a basis $v_1, \ldots, v_n$ of V and $f_1, \ldots, f_n$ of V^*, as well as invertible matrices M and N such that*

$$\Delta(v_i) = v_i \otimes g_i + \sum_j M_{ij} h_j \otimes v'_j, \qquad \Delta(f_i) = f_i \otimes a_i + \sum_j N_{ij} b_j \otimes f'_j, \tag{3.4.1}$$

where $v'_1, \ldots, v'_n$ is another basis of V, and $f'_1, \ldots, f'_n$ of V^*.

Proof. Let $v_1, \ldots, v_n$ be a homogeneous basis for the YD-compatible grading δ_r and $v'_1, \ldots, v'_n$ a homogeneous basis for δ_l. The form (3.4.1) of the coproducts is obtained by letting M be the base change matrix from $\{v_i\}$ to $\{v'_i\}$. The same argument works for the dual V^*, denoting the base change matrix from $\{f_i\}$ to $\{f'_i\}$ by N. □

Lemma 3.4.1.2. *A bialgebra A with a triangular decomposition over kG as above is a Hopf algebra, with antipode S given on generators of the form v_i, f_i as in (3.4.1) by*

$$S(v_i) = -\sum_j M_{ij}(h_j^{-1} \triangleright v'_j) h_j^{-1} g_i^{-1}, \quad S(f_i) = -\sum_j N_{ij}(f'_j \triangleleft b_j) b_j^{-1} a_i^{-1}. \tag{3.4.2}$$

Proof. The antipode axioms require that S is of the form stated, using that kG is a Hopf subalgebra, cf. (3.3.4)–(3.3.5). As $T(V)$ and $T(V^*)$ are free, defining S on the generators extends uniquely to an antialgebra and coalgebra map on all of A. □

Definition 3.4.1.3. A Hopf algebra A with triangular decomposition over a group is called of *weakly separable type* if the degrees right $g_i, \ldots, g_n$ of V are pairwise distinct group elements, and the same holds for the left degrees $h_1, \ldots, h_n$ of V as well as the dual degrees.

We observe that being of weakly separable type over a group implies that V and V^* have 1-dimensional homogeneous components. This gives that for a homogeneous basis element v_i of degree a_i, $g \triangleright v_i \neq 0$ is homogeneous of degree

ga_ig^{-1} which hence has to be a scalar multiple of a basis element $v_{g(i)}$ where $g(i)$ is an index $1, \ldots, n$. Hence we obtain an action of G on $\{1, \ldots, n\}$. To fix notation, we write

$$g \rhd v_i = \lambda_i(g) v_{g(i)}, \qquad\qquad f_i \lhd g = \mu_i(g) f_{g(i)}. \tag{3.4.3}$$

We will see that for A of weakly separable type, the bases change matrices M, N are diagonal matrices and can be chosen to be the identity matrix by rescaling of the diagonal bases. This implies that A is generated by primitive and group-like elements and hence pointed. It is a conjecture in [9, Introduction] that all finite-dimensional pointed Hopf algebras over a field of characteristic zero are in fact generated by skew-primitive and group-like elements.

Proposition 3.4.1.4. *If A is of weakly separable type, then there exists a basis $\{v_i\}$ of V and $\{f_i\}$ of V^* consisting of (g_i, h_i)-skew primitive elements, i.e.*

$$\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i, \qquad\qquad \Delta(f_i) = f_i \otimes a_i + b_i \otimes f_i, \tag{3.4.4}$$

and the antipode on a these skew-primitive elements is given by $S(v_i) = (h_i^{-1} \rhd v_i)h_i^{-1}g_i^{-1}$, $S(f_i) = (f_i \lhd b_i)b_i^{-1}a_i^{-1}$.

Proof. Consider the right and left coactions δ_r and δ_l from Section 3.3.1. Choosing a basis $v_1, \ldots, v_n$ homogeneous for δ_l and $v'_1, \ldots, v'_n$ homogeneous for δ_r, (3.4.1) gives

$$\Delta(v_i) = v_i \otimes g_i + \sum_j M_{ij} h_j \otimes v'_j, \tag{3.4.5}$$

where $M = (M_{ij})$ is the base change matrix. By coassociativity, we find that

$$\sum_{j,k} M_{ij}(M^{-1})_{jk} h_j \otimes v_k \otimes g_k = \sum_j M_{ij} h_j \otimes v'_j \otimes g_i. \tag{3.4.6}$$

By weak separability of δ_r and δ_l we now have for each $j = 1, \ldots, n$:

$$\sum_k M_{ij}(M^{-1})_{jk} v_k \otimes g_k = M_{ij} v'_j \otimes g_i. \tag{3.4.7}$$

Note that $M_{ij} \neq 0$ for at least some i. This implies that $(M^{-1})_{jk} = 0$ unless $k = i$ as the g_i are all distinct. Further, if $M_{ij} \neq 0$, then v_i and v'_j are proportional.

This can only be true for at most one i for given index j by weak separability. Hence by reordering the basis $v'_1, \ldots, v'_n$ we find that M is a diagonal matrix and can rescale the basis $\{v'_i\}$ such that M is the identity matrix. Hence we have $\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i$. The antipode conditions for A give (using Lemma 3.3.1.3) that S is of the form claimed. □

Remark 3.4.1.5. The bases $\{v_i\}$ and $\{f_i\}$ do not necessarily need to be orthogonal with respect to the pairing $\langle\ ,\ \rangle$. We will see in Theorem 3.4.2.2 that if the characters λ_i are all distinct, then the bases can be chosen to be dual bases.

Notation 3.4.1.6. In the following, we fix a basis $v_1, \ldots, v_n$ for V and $f_1, \ldots, f_n$ for V^* such that

$$\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i, \qquad \Delta(f_i) = f_i \otimes a_i + b_i \otimes f_i, \qquad i = 1, \ldots, n. \tag{3.4.8}$$

A direct observation from Proposition 3.4.1.4 is that the algebra A is generated by primitive and grouplike elements (which are precisely the group G) and hence pointed. Even in the general case (not assuming that A is of weakly separable type), we have the following restrictions on the group structure.

Proposition 3.4.1.7. *In the group G, the relations $[g_i, a_j] = [h_i, a_j] = 1$ and $[h_i, b_j] = [g_i, b_j] = 1$ hold for all $i, j = 1, \ldots, n$. In particular, if A has a symmetric triangular decomposition, then the subgroup of G generated by all degrees is abelian.*

Further, the following identities for the characters of the group action hold:

$$\mu_j(h_i) = \lambda_i(a_j)^{-1}, \qquad\qquad \mu_j(g_i) = \lambda_i(b_j)^{-1}. \tag{3.4.9}$$

Proof. The commutator relations follow by applying (3.3.8) and (3.3.9) to a pair of homogeneous basis elements of V and V^* with respect to δ_l, δ_r^* (or δ_r, δ_l^*). Then, even without weak separability, it follows from (3.3.6) and (3.3.7) that $h_i(j) = j$, $a_j(i) = i$, $g_i(j) = j$ and $b_j(i) = i$ by the PBW theorem. This implies the relations (3.4.9). In the symmetric case, $a_i = g_i^{-1}$ and $b_i = h_i^{-1}$ which forces the subgroup generated by all degrees to be abelian. □

3.4.2 Classification in the Free Case of Weakly Separable Type

We are now in the position, that we can classify all Hopf algebras A with triangular decomposition of weakly separable type (cf. Definition 3.4.1.3). This will enable us to view the Hopf algebras arising from this classification as analogues of multiparameter quantum groups in Section 3.5. We start by considering the case $A = T(V) \otimes kG \otimes T(V^*)$ which is referred to as the *free* case.

Proposition 3.4.2.1. *For the Hopf algebra A with triangular decomposition of weakly separable type to be indecomposable as a coalgebra it is necessary that G is generated by elements $k_1, \ldots, k_n$, $l_1, \ldots, l_n$ such that there exist generators v_i of V and f_i of V^* which are skew-primitive of the form*

$$\Delta(v_i) = v_i \otimes k_i + 1 \otimes v_i, \qquad \Delta(f_i) = f_i \otimes 1 + l_i \otimes f_i, \tag{3.4.10}$$

with $[k_i, l_j] = 1$ for all i, j. For the characters of the actions on the homogeneous components of V and V^* we require that

$$\mu_j(k_i) = \lambda_i(l_j)^{-1}. \tag{3.4.11}$$

Proof. To determine when pointed Hopf algebras are indecomposable as coalgebras, consider the graph Γ_A described in 3.2.2. Assume that A has generators given as in 3.4.1.6. We claim that the connected components of Γ_A are in bijection with the double cosets of the subgroup

$$Z := \langle g_1^{-1}h_1, \ldots, g_n^{-1}h_n, a_1^{-1}b_1, \ldots, a_n^{-1}b_n \rangle$$

in G which partition G. Indeed, using that the elements gv_i and gf_i are skew-primitive of type (gg_i, gh_i) and (ga_i, gb_i), we find that the connected component of g contains, for $i = 1, \ldots, n$, the strands

$$\ldots \longrightarrow g(g_i^{-1}h_i)^{-2} \longrightarrow g(g_i^{-1}h_i)^{-1} \longrightarrow g \longrightarrow g(g_i^{-1}h_i)^{1} \longrightarrow g(g_i^{-1}h_i)^{2} \longrightarrow \ldots$$

for $i = 1, \ldots, n$ and the same strand with $a_i^{-1}b_i$ instead of $g_i^{-1}h_i$ (and with g multiplied on the right). Moreover, as the elements gv_i, gf_i, v_ig, f_ig (and possibly linear combinations of products of them, which would again be of type given by elements in Z) are the only skew-primitive elements in A, and thus give the only arrows in Γ_A, two elements g and h are in the same connected component if and only if $z_1 g z_2 = z_3 h z_4$, for some $z_i \in Z$. Thus, A is indecomposable if and only if G equals the connected component of 1 in the graph Γ_A, hence if $G = Z$ which is the finitely-generated group generated by the elements $k_i := h_i^{-1} g_i$, $l_i := a_i^{-1} b_i$ for $i = 1, \ldots, n$. Hence, in order to obtain indecomposability, the coproducts are of the form as stated in (3.4.10). This is achieved by replacing the generators v_i by $v_i h_i^{-1}$ and f_i by $a_i^{-1} f_i$. The rest of the statements follow directly from Proposition 3.4.1.7. □

Theorem 3.4.2.2. *For an indecomposable pointed Hopf algebra A as in Theorem 3.4.2.1 of weakly separable type, the commutator relation (3.2.10) is of the form*

$$[f_i, v_j] = \gamma_{ij}(k_j - l_i) \qquad \forall 1 \leqslant i, j \leqslant n, \tag{3.4.12}$$

where γ_{ij} are scalars in k such that $\gamma_{ij} = 0$ whenever $\lambda_i \neq \lambda_j$ in which case also $\langle f_i, v_j \rangle = 0$. Conversely, any choice of such scalars gives a pointed Hopf algebra of this form.

Proof. With the work done in Proposition 3.4.1.3, it remains to verify that the form of the commutator relation (3.2.10) is as stated. Recall that in [16, 3.1], the commutator relation is given by means of a quasi-coaction. That is a morphism $\delta\colon V \to kG \otimes V$ satisfying (3.2.9) and (3.2.10). Such a morphism has the general form

$$\delta(v_j) = v_j^{[-1]} \otimes v_j^{[0]} = \sum_{k,g} \alpha^j_{k,g}\, g \otimes v_k, \qquad \alpha^i_{k,g} \in k, \tag{3.4.13}$$

on the basis elements. Then (3.3.10), which is required for A to be a bialgebra, rewrites as

$$\sum_{k,g} \alpha^j_{k,g}(g \otimes k_j + l_i \otimes g)\langle f_i, v_k \rangle = \sum_{k,g} \alpha^j_{k,g}\, g \otimes g \langle f_i, v_k \rangle.$$

For each i, there exists k such that $\langle f_i, v_k \rangle \neq 0$. For given i, we denote the set of indices such that $\langle f_i, v_k \rangle \neq 0$ by I_i. For such $k \in I_i$, we find that $\alpha^j_{k,g} = 0$ for $g \neq k_j, l_i$, and $\alpha^j_{k,k_j} = -\alpha^j_{k,l_i}$. Thus, we obtain that δ is of the form

$$\delta(v_j) = v_j^{[-1]} \otimes v_j^{[0]} = \sum_{i=1}^{n} \gamma_{ij}(k_j - l_i) \otimes v'_i, \tag{3.4.14}$$

where $\gamma_{ij} = \sum_{k\in I_i} \alpha^j_{k,k_j} 1/\langle f_i, v_k\rangle |I_i|$ and $\{v'_i\}$ is the dual basis of V to $\{f_i\}$. Conversely, given arbitrary scalars γ_{ij} for $i, j = 1, \ldots, n$, we can define a quasi-coaction by the same formula (3.4.14). Then δ is YD-compatible with the given action of G on V if and only if (cf. to condition (A) in [16, Theorem A])

$$\gamma_{ij}\mu_i(g)(gk_j - gl_i) = g[f_i \lhd g, v_j] \overset{\text{(A)}}{=} [f_i, g \rhd v_j]g = \gamma_{ij}\lambda_j(g)(k_jg - l_ig).$$

As A is indecomposable of weakly separable type, G is abelian and hence this condition is equivalent to $\lambda_j = \mu_i$ whenever $\gamma_{ij} \neq 0$. But by duality of the action, if $\langle f_i, v_j\rangle \neq 0$ then $\lambda_i = \mu_j$.

As for given $i = 1, \ldots, n$, $\langle f_i, v_j\rangle \neq 0$ for some j we have that $\lambda_i = \mu_j$ for at least some j, and vice versa. Hence, the set of characters and dual characters are in bijection. We can change the numbering and assume without loss of generality (recall that we are in the weakly separable case) to obtain

$$\lambda_i = \mu_i. \tag{3.4.15}$$

From now on, we will hence only use the notation λ_i. □

The situation, where $\{v_i\}$ and $\{f_i\}$ are orthogonal bases deserves particular attention. In this case, the scalars $\gamma_{ij} = 0$ for $i \neq j$. The following concept of separability ensure this.

Definition 3.4.2.3. Let A have a triangular decomposition of weakly separable type over a group G. If the characters $\lambda_1, \ldots, \lambda_n$ are distinct for different indices, we will speak of a triangular decomposition of *separable type*.

Remark 3.4.2.4. At this point, a comparison to [11, 2.4] and [10, 4.3] seems appropriate. The condition (3.4.12) is equivalent to the so-called *linking relation* (3.2.4) after a change of generators $f_i \leftrightarrow l_i^{-1} f_i$, since in the form of 3.2.3.1 all generators have coproducts $\delta(v_i) = v_i \otimes 1 + g_i \otimes v_i$. Such a change of generators causes the commutators ad $= [\ ,\ ]$ to become braided commutators $\underline{\mathrm{ad}} = \mathrm{Id}_{V^{\otimes 2}} - \Psi$. The scalars λ_{ij} satisfy the condition (d) in 3.2.3, where for the characters $\chi_i\chi_j \neq \varepsilon$ implies $\lambda_{ij} = 0$. This is the analogue of our condition $\lambda_i \neq \lambda_j$ implying $\gamma_{ij} = 0$.

This linking relation also appears in the quantum group characterization of [10, Theorem 4.3]. Hence we can conclude that the classification in this section gives

Hopf algebras with similar relations as appearing in the work of Andruskiewitsch and Schneider.

Example 3.4.2.5. The most degenerate case, where $\gamma_{ij} = 0$, gives the Hopf algebra $(T(V) \oplus T(V^*)) \rtimes kG$ where the tensor algebras are again computed in the category of YD-modules over kG.

Assuming the non-degeneracy that $\gamma_{ii} \neq 0$, we can adapt the terminology of [16, 5.5] that the braided doubles in this case come from *mixed* YD-structures. A mixed YD-structure is a quasi-coaction δ that is a weighted sum $\sum t_i \delta_i$, where δ_i are YD-modules compatible with the same action, and t_i are generic scalars. The quasi-YD-module in the theorem is the sum $\delta = \delta_r - (\delta_l^*)^*$, where $(\delta_l^*)^*$ is the YD-module given by $v_j \mapsto l_j \otimes v_j$, which is dual to δ_l^*. We will see that in this case all the Hopf algebras arising are certain *asymmetric* braided Drinfeld doubles (as defined in 3.3.4).

In the symmetric case, these algebras are in fact braided Drinfeld doubles. In particular, their adequately defined module categories (resembling the category $\mathcal{O}$, see Section 2.3.7) are braided.

3.4.3 Interpretation as Asymmetric Braided Drinfeld Doubles

So far, we have only classified *free* braided doubles over kG. That is, as a k-vector space $A \cong T(V) \otimes kG \otimes T(V^*)$ via the multiplication map. To capture examples such as quantum groups, it is necessary to consider quotients of A by ideals $J = \langle I, I^* \rangle$ such that $A/J \cong T(V)/I \otimes kG \otimes T(V^*)/I^*$ is still a Hopf algebra (and thus pointed). Here $I \lhd T(V)$ and $I^* \lhd T(V^*)$ are ideals and also coideals, and $J \in \mathcal{I}_\Delta(A)$. We will now refine our considerations from Section 3.3.3 to find for what ideals I and I^* this is the case. We will use the notation

$$q_{ij} := \lambda_j(k_i). \tag{3.4.16}$$

Then, by (3.4.11), we have that $\lambda_j(l_i) = q_{ji}^{-1}$, and the matrix $q = (q_{ij})$ describes the braiding on V fully, i.e. it is of *diagonal type*.

The collection of triangular Hopf ideals $\mathcal{I}_\Delta(A)$ introduced in Section 3.2.5 can be described more concretely for A satisfying the following restrictions: we assume that the parameters $\gamma_{ii} \neq 0$ for all i and that V (and hence V^*) are of separable type, and that $k_i \neq l_i$. Recall that in this situation, the algebras of the classification 3.4.2.2 are displayed as what is referred to in [16, 5.5] as arising from *mixed* YD-structures. More specifically, the quasi-coaction $\delta = \delta_r - (\delta_l^*)^*$, where δ_l^* denotes the coaction on V that is obtained by dualizing the left coaction δ_l^* on V^* (this is possible as G is abelian).

By Lemma 3.3.4.7, the ideals in $\mathcal{I}_\Delta(A)$ are of the form $J = I \otimes kG \otimes T(V^*) + T(V) \otimes kG \otimes I^*$ where I is an ideal in the collection $\mathcal{I}_{(V,\delta_r)}$ for V with the right coaction given by δ_r, and I^* is in $\mathcal{I}_{(V^*,\delta_l^*)}$ for the left dual coaction δ_l^* on V^*.

Note that by (3.4.11) the braiding Ψ_r coming from δ_r and Ψ_l from $(\delta_l^*)^*$ on V are given by

$$\Psi_r(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \qquad \Psi_l(v_i \otimes v_j) = q_{ji}^{-1} v_j \otimes v_i. \tag{3.4.17}$$

That is, $\Psi_l = \Psi_r^{-1}$, the inverse braiding. We hence drop the subscripts l, r.

Example 3.4.3.1. In the quantum groups $A = U_q(\mathfrak{g})$, the braiding satisfies the symmetry $q_{ij} = q^{i \cdot j} = q^{j \cdot i} = q_{ji}$ as the Cartan datum is symmetric. This implies that the relations in I are symmetric under reversing the order of tensors $v_1 \otimes \ldots \otimes v_n \leftrightarrow v_n \otimes \ldots \otimes v_1$. This can be verified explicitly by observing that in $U_q(\mathfrak{g})$ the ideal I is generated by q-Serre relations, which carry such a symmetry.

Theorem 3.4.3.2. *All quotients by triangular Hopf ideals $J \in \mathcal{I}_\Delta(A)$ of algebras A occurring in the classification 3.4.2.2, where A is of separable type with $\gamma_{ii} \neq 0$ for all i, are asymmetric braided Drinfeld doubles. If J is maximal of this form, then $A/J \cong U_{kG}(V, V^*)$.*

Proof. We have seen that the commutator relations are of the form $[f_i, v_j] = \delta_{ij}\gamma_{ii}(k_i - l_j)$. This is precisely the form of the asymmetric braided Drinfeld double of V with right YD-module structure given by the right grading, and V^* with left YD-module structure given by the left dual grading. The pairing is given by $\langle f_i, v_j \rangle = \delta_{ij}\gamma_{ii}$ here. We have to check that the braided Hopf algebras $T(V)$ and $T(V^*)$ of YD-modules over G are dually paired when viewed in the category of left kG-modules. This however follows from condition (3.4.11). Taking the maximal quotient by a triangular ideal (or the left and right radical of the pairing) gives the asymmetric braided Drinfeld double $U_{kG}(V, V^*)$. □

If some of the parameters γ_{ii} are zero, then the pointed Hopf algebras obtained are not an asymmetric braided Drinfeld double any more (in the sense of Definition 3.3.4.4).

3.4.4 Recovering a Lie Algebra

We assume that char $k = 0$ in this section and study Hopf algebras with triangular decomposition of separable type which are of the form $U_{kG}(V, V^*)$ (see Theorem 3.4.3.2). The aim is to set the characters λ_i and the group elements k_i, l_i equal to 1. This way, we want to recover a Lie algebra $\mathfrak{g}$ for any of the indecomposable pointed Hopf algebras of the form $U_{kG}(V, V^*)$, relating back to the question asked in the introduction of finding quantum groups for a given Lie algebra. The tool available for this is the Milnor–Moore theorem from [93] (see also [94, Theorem 5.6.5]) which shows that any cocommutative connected Hopf algebras is of the form $U(\mathfrak{g})$ for a (possibly infinite-dimensional) Lie algebra $\mathfrak{g}$.

There are technical problems with this naive approach. To set the elements q_{ij} — which will be replaced by formal parameters — equal to one, we need to give an appropriate integral form to avoid that the modules collapse to zero. This rules out examples like e.g. $k[x]/(x^n)$ (and, more generally, the small quantum groups) which are braided Hopf algebras in the category of YD-modules over $k\mathbb{Z}$, as here a generator of the group acts by a primitive nth root of unity q on x, and $\mathbb{Z}[q] \subset k$ is a cyclotomic ring.

As a first step, we introduce appropriate integral forms of $U_{kG}(V, V^*)$, for which we need the square roots of q_{ij}. We consider the subring $Z := \mathbb{Z}[q_{ij}^{\pm 1/2}]_{i,j} \subset k$ adjoining all square roots of the numbers q_{ij} and their inverses. This will now be treated as formal parameters with certain relations between them, coming from the relations we have among them in k.

Assumption 3.4.4.1. In this section, we assume that the ideal $\langle q_{ij}^{\pm 1/2} - 1 \mid i, j = 1, \ldots, n \rangle$ in Z is a proper ideal, and hence $p\colon Z \to \mathbb{Z}$, $q_{ij}^{\pm 1/2} \mapsto 1$ is a well-defined morphism of rings.

This assumption is crucial in the formal limiting process. It, for example, prevents examples in which $q^n + q^{n-1} + \ldots + q + 1 = 0$ as in cyclotomic rings.

To produce an integral form, we replace a given YD-module V over kG of separable type as in the previous sections, by a YD-module over ZG. For this, we can choose a G-homogeneous basis $v_1, \ldots, v_n$ and a homogeneous dual basis $f_i, \ldots, f_n$ such that (possibly after rescaling)

$$\langle f_i, v_j \rangle = \frac{1}{q_{ii}^{1/2} - q_{ii}^{-1/2}} \delta_{ij}, \qquad \forall i, j. \tag{3.4.18}$$

An important observation is that the Woronowicz symmetrizers, which are used to compute the Nichols ideal $I_{\max}(V)$, have coefficients in Z. Hence their kernels will be Z-modules. That is, for V^{int} defined as $Z\langle v_1, \ldots, v_n \rangle$, which is a YD-module over the group ring ZG, the Woronowicz symmetrizer $\operatorname{Wor}^n_{\text{int}} \Psi$ is a Z-linear map $V^{\text{int} \otimes n} \to V^{\text{int} \otimes n}$. Hence $I_{\max}(V^{\text{int}}) := \ker \operatorname{Wor}_{\text{int}} \Psi$ is an ideal in $T(V^{\text{int}})$, the tensor algebra over Z.

In order to provide an integral form of $U_{kG}(V, V^*)$, we will change the presentation by introducing new commuting generators, namely $[f_i, v_i] =: t_i$. One verifies that the following commutator relations hold over k, as we are given the relation $t_i = \frac{1}{q_{ii}^{1/2} - q_{ii}^{-1/2}}(k_i - l_i)$ when working over the field:

$$[f_i, t_j] = \delta_{i,j}(q_{ii}^{1/2} k_i f_i + q_{ii}^{-1/2} l_i f_i), \tag{3.4.19}$$

$$[v_i, t_j] = -\delta_{i,j}(q_{ii}^{-1/2} k_i v_i + q_{ii}^{1/2} l_i v_i). \tag{3.4.20}$$

Definition 3.4.4.2. The *integral form* $U_{ZG}(V^{\text{int}}, V^{\text{int}*})$ of $U_{kG}(V, V^*)$ is defined as the graded Hopf algebra over the ring Z generated by $v_1, \ldots, v_n$, of degree 1, $f_1, \ldots, f_n$ of degree -1, and the group elements $k_1, \ldots, k_n, l_1, \ldots, l_n \in G$, and additional elements $t_1, \ldots, t_n$ of degree 0, subject to the relations of $I_{\max}(V^{\text{int}})$ and $I^*_{\max}(V^{\text{int}})$, bosonization relations

$$gv_i = (g \triangleright v_i)g, \qquad f_i g = g(f_i \triangleleft g), \tag{3.4.21}$$

as well as the relations (3.4.19), (3.4.20) and

$$gv_i = (g \triangleright v_i)g, \qquad f_i g = g(f_i \triangleleft g), \tag{3.4.22}$$

$$q_{ii}^{1/2}(k_i - l_i) = (q_{ii} - 1)t_i, \tag{3.4.23}$$

$$[f_i, v_j] = \delta_{i,j} t_i, \tag{3.4.24}$$

$$[t_i, t_j] = 0. \tag{3.4.25}$$

The coproducts are given as before on the generators f_i, v_i, k_i, l_i and $\Delta(t_i) = t_i \otimes k_i + l_i \otimes t_i$.

Note that as $A = U_{ZG}(V^{\mathrm{int}}, V^{\mathrm{int}\,*})$ is a Hopf algebra over the commutative ring Z, the coproduct is a map $A \to A \otimes_Z A$. For the quantum groups $U_q(\mathfrak{g})$ at generic parameter, the integral form in this case is so-called *non-restricted* integral form (see e.g. [27, 9.2]) which goes back to De Concini–Kac [31]. To set the parameters equal to one, and to consider extensions of Hopf algebras to fields, we use the following Lemma:

Lemma 3.4.4.3. *Let* $\phi\colon R \to S$ *be a morphism of commutative algebras. We denote the category of Hopf algebras over* R *by* $\mathbf{Hopf}_R$*. Then base change along* ϕ *induces a functor*

$$\mathbf{Hopf}_\phi\colon \mathbf{Hopf}_R \longrightarrow \mathbf{Hopf}_S, \qquad\qquad A \longmapsto A \otimes_R S.$$

Proof. Given a Hopf algebra A which is an R-algebra, i.e. there is a morphism $R \to A$, we induce the multiplication and comultiplication on $S \mapsto A \otimes_R S$ using the isomorphism

$$(A \otimes_R S) \otimes_S (A \otimes_R S) \cong (A \otimes_R A) \otimes_R S.$$

It is easy to check that the Hopf algebra axioms are preserved under base change. $\square$

Proposition 3.4.4.4. *There is an isomorphism of graded Hopf algebras*

$$U_{ZG}(V^{\mathrm{int}}, V^{\mathrm{int}\,*}) \otimes_Z k \xrightarrow{\sim} U_{kG}(V, V^*).$$

Proof. Recall that $Z \leqslant k$ by construction. Extending to k, we are able to divide by $q_{ii} - 1$ in (3.4.23), and recover the original commutator and bosonization relations in $U_{kG}(V, V^*)$. It remains to verify that

$$I_{\max}(V^{\mathrm{int}}) \otimes_Z k = \ker \mathrm{Wor}_{\mathrm{int}} \Psi \otimes_Z k = \ker \mathrm{Wor}\, \Psi = I_{\max}(V).$$

This follows by noting that k is flat as a Z-module (since the function field $K(Z)$ is flat over Z as a localization, and k is free over $K(Z)$), and $V^{\mathrm{int}} \otimes_Z k \cong V$ as k-vector spaces. $\square$

Definition 3.4.4.5. We define the *classical limit* of $U_{kG}(V,V^*)$ as the algebra

$$U_k^{\mathrm{cl}}(V,V^*) := (U_{ZG}(V^{\mathrm{int}},V^{\mathrm{int}\,*})\otimes_Z \mathbb{Z})\otimes_{\mathbb{Z}} k \Big/ \langle \ker \varepsilon_G\rangle,$$

using the morphism $p\colon Z \to \mathbb{Z}$ mapping all $q_{ij}^{\pm 1/2}$ to 1, and the two sided ideal $\langle \ker \varepsilon_G\rangle$ generated by the kernel of the augmentation map $\varepsilon_G\colon kG \to k$ mapping all group elements to 1. Note that this ideal is a Hopf ideal.

That is, to obtain the classical form we first set the parameters $q_{ij}^{\pm 1/2}$ equal to 1 in the integral form and then extend the resulting $\mathbb{Z}$-module to a k-vector space, and finally set the group elements equal to 1 along the counit $\varepsilon_G\colon kG \to k$. We obtain a primitively generated Hopf algebra, and hence a Lie algebra, this way:

Proposition 3.4.4.6. *The classical limit $U_k^{\mathrm{cl}}(V,V^*)$ is a connected Hopf algebra, generated by primitive elements. Hence, for the Lie algebra $\mathfrak{p}_V$ of primitive elements, $U(\mathfrak{p}_V) = U_k^{\mathrm{cl}}(V,V^*)$. This algebra is generated by triples f_i, v_i, t_i which form a subalgebra isomorphic to $U(\mathfrak{sl}_2)$.*

Proof. Lemma 3.4.4.3 ensures that $U_l^{\mathrm{cl}}(V,V^*)$ is a Hopf algebra over k, and freeness of V^{int} over Z ensures that the positive and negative part do not collapse to the zero space. In particular, the k-vector space $V^{\mathrm{int}} \oplus V^{\mathrm{int}\,*}$ embeds into the Lie algebra $\mathfrak{p}_V$ of primitive elements. In the classical limit, we obtain the relations

$$[f_i, v_j] = \delta_{i,j} t_i, \qquad [f_i, t_j] = 2\delta_{i,j} f_i, \qquad [v_i, t_i] = -2\delta_{i,j} v_i. \tag{3.4.26}$$

Hence every triple f_i, v_i, t_i generates a Lie subalgebra of $\mathfrak{p}_V$ isomorphic to $\mathfrak{sl}_2$. Note that $U_k^{\mathrm{cl}}(V,V^*)$ is generated by primitive elements:

$$\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i, \qquad \Delta(v_i) = v_i \otimes 1 + 1 \otimes v_i.$$

We also compute

$$\Delta(t_i) = \Delta([f_i, v_i]) = [f_i, v_i] \otimes k_i + l_i \otimes [f_i, v_i] = t_i \otimes k_i + l_i \otimes t_i.$$

Hence, t_i is skew primitive in $U_{ZG}^{\mathrm{int}}(V,V^*)$ and primitive in the classical limit. Thus, $U_k^{\mathrm{cl}}(V,V^*)$ is a pointed Hopf algebra over the trivial group. That is, a *connected* pointed Hopf algebra. It is further cocommutative and Theorem 5.6.5 in [94] implies that such a Hopf algebra is of the form $U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of primitive elements as $\operatorname{char} k = 0$. □

Note that $U_k^{\mathrm{cl}}(V, V^*)$ is a braided double over the polynomial ring $S(T)$, where $T = k\langle t_1, \ldots, t_n \rangle$ (which is not necessarily n-dimensional). The action is given by $t_j \triangleright v_i = 2\delta_{i,j} v_i$ The quasi-coaction is given by $\delta(v_i) = t_i \otimes v_i$ which is *not* a coaction, hence $U_{ZG}^{\mathrm{int}}(V, V^*)$ is *not* a braided Heisenberg double. It is also not an asymmetric braided Drinfeld double.

Example 3.4.4.7. For $U_q(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, viewed as a braided Drinfeld double, the classical limit is $U(\mathfrak{g})$.

We can also compute examples that do not give finite-dimensional semisimple Lie algebras. As a general rule, the relations between the parameters q_{ij} determine the relations in the Lie algebra. It is easy to construct free examples, for which there are no relations between the $v_1, \ldots, v_n$ by choosing algebraically independent parameters q_{ij}. The work of [105] and [10] give restrictions on examples satisfying the growth condition of finite Gelfand–Kirillov dimension. We will view their results in the setting of this chapter in Section 3.5.2.

3.5 Classes of Quantum Groups

In this section, we relate the classification from Section 3.4 to various classes of examples which are often regarded as quantum groups. This includes the multiparameter quantum groups studied by [12, 43, 102, 109] and others, in Section 3.5.1, a characterization of Drinfeld–Jimbo quantum groups in Section 3.5.2, and classes of examples of pointed Hopf algebras from the work of Radford in Section 3.5.3. The classification in Theorem 3.4.2.2 points out natural generalizations of these classes of examples. We finally sketch how one can define analogues of quantum groups using triangular decompositions over other Hopf algebras than kG.

3.5.1 Multiparameter Quantum Groups

Let k be a field of characteristic zero. For the purpose of this section, let $\lambda \in k$ be generic, and $p_{ij} \in k$ for $1 \leqslant i < j \leqslant n$. Assume that $p_{ii} = 1$ and $p_{ji} = p_{ij}^{-1}$. Following [12, 29] and to fix notation, we set

$$\kappa_j^{(i)} = \begin{cases} p_{ij}, & \text{if } i<j, \\ \lambda, & \text{if } i=j, \\ \frac{\lambda}{p_{ji}}, & \text{if } i>j. \end{cases} \qquad \lambda_j^{(i)} = \begin{cases} \frac{\lambda}{p_{ij}}, & \text{if } i<j, \\ \lambda, & \text{if } i=j, \\ p_{ji}, & \text{if } i>j. \end{cases}$$

We will provide a variation of the presentation of [12, 29] in order to display multiparameter quantum groups as a Hopf algebra with triangular decomposition.

Example 3.5.1.1 (Multiparameter quantum groups). Let $F = k\langle f_1, \ldots, f_{n-1}\rangle$ be the YD-module over a group algebra G with commuting generators $k_1, \ldots, k_{n-1}$, $l_1, \ldots, l_{n-1}$. Denote the dual by $E = k\langle e_1, \ldots, e_{n-1}\rangle$, where the pairing is given by $\langle e_i, f_j\rangle = (1-\lambda)\delta_{ij}$. The YD-structure is of separable type, given by assigning the right degree k_i to f_i, and the left degree l_i to e_i, and actions

$$k_i \triangleright f_j = \lambda_j(k_i) f_j = \frac{\lambda_{j+1}^{(i)}\lambda_j^{(i+1)}}{\lambda_j^{(i)}\lambda_{j+1}^{(i+1)}} f_j, \tag{3.5.1}$$

$$l_i \triangleright f_j = \lambda_j(l_i) f_j = \frac{\kappa_j^{(i)}\kappa_{j+1}^{(i+1)}}{\kappa_{j+1}^{(i)}\kappa_j^{(i+1)}} f_j, \tag{3.5.2}$$

for $i, j = 1, \ldots n-1$. We define the *multiparameter quantum group* $U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ to be the asymmetric braided Drinfeld double $U_{kG}(F, E)$.

Note that the definition of $U_{kG}(F, E)$ is possible as (3.4.11) holds, i.e.

$$q_{ij} := \lambda_j(k_i) = \frac{\lambda_{j+1}^{(i)}\lambda_j^{(i+1)}}{\lambda_j^{(i)}\lambda_{j+1}^{(i+1)}} = \frac{\kappa_{i+1}^{(j)}\kappa_i^{(j+1)}}{\kappa_i^{(j)}\kappa_{i+1}^{(j+1)}} = \lambda_i(l_j)^{-1}.$$

The commutator relation in $U_{kG}(F, E)$ is given by

$$[E_i, F_j] = (1-\lambda)\delta_{ij}(k_i - l_i). \tag{3.5.3}$$

Our definition of the multiparameter quantum group is justified by the following isomorphism to an indecomposable subalgebra of the multiparameter quantum group considered in the literature:

Proposition 3.5.1.2. *There is an isomorphism of Hopf algebras $U_{kG}(F, E) \cong U'$ where U' is a subalgebra of the multiparameter quantum group $U = U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$.*

Proof. We prove the theorem by first considering the morphism

$$\phi\colon T(E)\otimes kG\otimes T(F)\longrightarrow U.$$

Such a morphism will descent to an injective morphism $\overline{\phi}\colon U_{kG}(F,E)\to U$ by the following Lemma 3.5.1.3. We further note that the image $\operatorname{Im}\overline{\phi}=:U'$ is a Hopf subalgebra isomorphic to $U_{kG}(F,E)$. Denote the generators of U by E_i, F_i for $i=1,\ldots,n-1$ and group elements K_i, L_i for $i=1,\ldots,n$ (see [29, 4.8]). The map ϕ is defined by $\phi(e_i)=\lambda E_i K_{i+1}^{-1}K_i$, $\phi(f_i):=F_i$, $\phi(k_i)=L_{i+1}L_i^{-1}$, and $\phi(l_i):=K_{i+1}^{-1}K_i$. One checks directly that the relations in the free braided double $T(E)\otimes kG\otimes T(F)$ are preserved under this map, using the presentation in [29, 4.8] for U. □

Lemma 3.5.1.3. *The quantum Serre relations in the positive part of* $A=U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ *are given by the largest ideal in* $\mathcal{I}_\Delta(A)$*, making the positive part a Nichols algebra. This ideal is generated by the braided commutators*

$$\underline{\operatorname{ad}}(E_i)^{1-a_{ij}}(E_j)=\underline{\operatorname{ad}}(F_i)^{1-a_{ij}}(F_j)=0, \tag{3.5.4}$$

where $\underline{\operatorname{ad}}(E_i)(E_j)=E_iE_j-q_{ij}E_jE_i$.

Proof. It follows from Lemma 3.3.4.7 that the maximal ideal J in $\mathcal{I}_\Delta(A)$ is given by $J=\langle I,I^*\rangle$ where I is the Nichols ideal of the YD-module F.

In U, the explicit description of the ideal the quotient of the positive (respectively negative) part is generated by quantum Serre relations. This follows from Lemma 4.5 in [29]. For this, it is crucial that λ is not a root of unity. The proof uses the observation in [102], or [12] for the deformed function algebra, that multiparameter quantum groups, using quantum coordinate rings, can be obtained via a 2-cocycle from a one-parameter quantum groups. The fact that the quantum Serre relations generate the ideal J follows from Theorem 4.4 in [29] where it is shown that these relations generate the radical of the pairing of $T(F)$ with $T(E)$ extending the pairing of E and F. □

The result that the multiparameter quantum group $U_{\lambda,\underline{p}}(\mathfrak{gl}_n)$ is the asymmetric braided Drinfeld double $U_{kG}(F,E)$ can be seen as a generalization of the result in [20] where the two-parameter quantum groups were shown to be Drinfeld doubles.

3.5.2 Characterizations of Quantum Groups

Let $\operatorname{char} k = 0$ in this section. In Section 3.4 we observed that for an algebra A with triangular decomposition to be an indecomposable pointed Hopf algebra, $G(A)$ needs to be abelian acting on V by scalars. That means, in the terminology of [9] that the YD-braiding $\Psi(v \otimes w) = v^{(-1)} \triangleright w \otimes v^{(0)}$ is of *diagonal type*, i.e. there exist non-zero scalars q_{ij} such that $\Psi(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$ for a basis $\{v_1, \ldots, v_n\}$.

We assume that the braidings arise from YD-module structures over an abelian group G in this section. That is, $q_{ij} = \lambda_j(k_i)$ for the characters λ_i by which G acts on kv_i and group elements k_i such that $\delta(v_i) = v_i \otimes k_i$. It is a basic observation that the braided Hopf algebras $T(V)/I$ for $I \in \mathcal{I}_V$, including the Nichols algebras for V, only depend on the braiding on V (rather than the concrete choice of λ_i, k_i). However, different diagonal braidings (V, Ψ) and (V, Ψ') give isomorphic braided Hopf algebras $T(V)/I$. Such isomorphisms can be obtained using the notion of *twist equivalence* for diagonal braidings (which is a special case of the more general concept of twisting an algebra by a 2-cocycle).

Definition 3.5.2.1. Two braided k-vector spaces of diagonal type (V, Ψ), (V', Ψ') (given by scalars q_{ij}, q'_{ij}) *twist equivalent* if $V \cong V'$, $q_{ii} = q'_{ii}$, and $q_{ij}q_{ji} = q'_{ij}q'_{ji}$.

Lemma 3.5.2.2. *If (V, Ψ), (V', Ψ') are twist equivalent of diagonal type, then $T(V) \cong T(V')$ as braided Hopf algebras in the category of braided k-vector spaces, preserving the natural grading.*

Proof. For a proof see e.g. [9, 3.9–3.10]. We can find generators v_i of V and v'_i of V' such that the isomorphism ϕ is determined by $v_i \mapsto v'_i$. Defining a 2-cocycle σ by $\sigma(v_i \otimes v_j) = q'_{ij} q_{ij}^{-1}$ for $i < j$ and 1 otherwise, we find that the product $v_i v_j$ maps to the product twisted by σ. Note that the isomorphism is *not* an isomorphism in the category of YD-modules over kG unless $(V', \Psi') = (V, \Psi)$. $\square$

For an ideal $I \in \mathcal{I}_V$, denote the corresponding ideal under the isomorphism $T(V) \cong T(V')$ from Lemma 3.5.2.2 by I'. Then we conclude that $T(V)/I \cong T(V')/I'$ is also an isomorphism of braided Hopf algebras. In particular, $\mathcal{B}(V) \cong \mathcal{B}(V')$ for the corresponding Nichols algebras.

Lemma 3.5.2.3. *If* (V,Ψ) *and* (V',Ψ') *are twist equivalent, such that*

$$G=\langle k_1,\dots,k_n\rangle\cong\langle k'_1,\dots,k'_n\rangle=G'$$

via $k_i\mapsto k'_i$*, then* $U_{kG}(V,V^*)\cong U_{kG'}(V',V'^*)$ *as Hopf algebras.*

Proof. By Lemma 3.5.2.2, $T(V)/I\cong T(V')/I'$ and $T(V^*)/I^*\cong T(V'^*)/I'^*$. By the assumptions on the group generators, $k_i\mapsto k'_i$ extends to an isomorphism $kG\cong kG'$. Thus we can define a morphism $U_{kG}(V,V^*)\to U_{kG}(V',V'^*)$ which is an isomorphism of k-vector spaces. Further, preservation of the bosonization condition can be checked on generators using the isomorphism ϕ from Lemma 3.5.2.2. Finally, the commutator relation (3.4.12) is preserved using the isomorphism on kG. $\square$

Diagonal braidings are a very general class of braidings. Quantized enveloping algebras at generic parameters however are based on braidings of specific type, called Drinfeld–Jimbo type. Following [10], there are different classes of braidings which we distinguish:

Definition 3.5.2.4 ([10, Definition 1.1])**.** Let (q_{ij}) be the $n\times n$-matrix of a braiding of diagonal type.

(a) The braiding given by (q_{ij}) is *generic* if q_{ii} is not a root of unity for any $i=1,\dots,n$.

(b) In the case $k=\mathbb{C}$ we say the braiding (q_{ij}) is *positive* if it is generic and all diagonal elements q_{ii} are positive real numbers.

(c) The braiding (q_{ij}) is of *Cartan type* if $q_{ii}\neq 1$ for all i and there exists a $\mathbb{Z}$-valued $n\times n$-matrix (a_{ij}) with values $q_{ii}=2$ on the diagonal and $0\leqslant -a_{ij}<\operatorname{ord}q_{ii}$ for $i\neq j$, such that

$$q_{ij}q_{ji}=q_{ii}^{a_{ij}}\qquad\text{for all } i,j. \tag{3.5.5}$$

That implies that (a_{ij}) is a generalized Cartan matrix which may have several connected components. We denote the collection of these by χ.

(d) The braiding (q_{ij}) is of *Drinfeld–Jimbo type (DJ-type)* if q_{ij} are generic (no roots of unity) and there exist positive integers $d_1, \dots, d_n$ such that for all i, j, $d_i a_{ij} = d_j a_{ji}$ (hence the matrix (a_{ij}) is symmetrizable, and for any $J \in \chi$, there exists a scalar $q_J \neq 0$ in k such that $q_{ij} = q_J^{d_i a_{ij}}$ for any $i \in I$, and $j = 1, \dots, n$.

Some observations can be made about the Nichols algebras associated to braided vector spaces of DJ-type. First, observe that for a braiding of Cartan type with connected components $I_1, \dots, I_n \in \chi$, we have that $\mathcal{B}(V)$ is the braided tensor product $\mathcal{B}(V_{I_1}) \otimes \dots \otimes \mathcal{B}(V_{I_n})$ ([8, Lemma 4.2]). Further, for V with braiding (q_{ij}) of DJ-type, the Nichols algebra can be computed explicitly as the quantum Serre relations ([105, Theorem 15]):

$$\mathcal{B}(V) = k\langle x_1, \dots, x_n \mid \underline{\mathrm{ad}}(x_i)^{1-a_{ij}}(x_j) = 0, \forall i \neq j \rangle.$$

We now bring the growth condition of finite *Gelfand–Kirillov dimension* (GK-dimension) into the picture, using characterization results of [105] of Nichols algebras with this property.

Lemma 3.5.2.5 ([105]). *Let $k = \mathbb{C}$. Let (q_{ij}) be the matrix of a braiding of diagonal type which is* generic *such that the Nichols algebra $\mathcal{B}(V)$ has finite Gelfand–Kirillov dimension. Then (q_{ij}) is of Cartan type.*

Moreover, if the braiding is positive then the braiding is twist equivalent to a braiding of DJ-type, and this condition is equivalent to finite GK-dimension.

Proof. See [10], Corollary 2.12 and Theorem 2.13. □

Corollary 3.5.2.6. *Let $A = U_{\mathbb{C}G}(V, V^*)$, for V or separable type, with generic positive braiding (q_{ij}). Then the following are equivalent*

(i) $A \cong U_q(\mathfrak{g})$ for $\mathfrak{g}$ a semisimple Lie algebra.

(ii) The braided $\mathbb{C}$-vector space V with braiding (q_{ij}) is twist equivalent to a braiding of DJ-type with finite type Cartan matrix.

(iii) $\mathcal{B}(V)$ has finite Gelfand–Kirillov dimension.

Proof. The equivalence of (ii) and (iii) is the statement of Lemma 3.5.2.5 due to [105]. Using Lemma 3.5.2.3 we find that (ii) implies (i), while it is clear that (i) implies (ii). In fact, the GK-dimension of $\mathcal{B}(V)$ for V of DJ-type equals the number of positive roots [10, 2.10(ii)]. □

Corollary 3.5.2.7. *The only indecomposable bialgebras with a symmetric triangular decomposition on* $\mathcal{B}(V) \otimes k\mathbb{Z}^n \otimes \mathcal{B}(V^*)$ *of separable type, such that* $V = \mathbb{C}\langle v_1, \dots, v_n\rangle$ *is of positive diagonal type, and that no* v_i *commutes with all of* V^* *are isomorphic to* $U_q(\mathfrak{g})$ *for some semisimple Lie algebra* $\mathfrak{g}$.

Proof. This follows from the classification 3.4.2.2, combined with the result of Rosso. The Lie algebra $\mathfrak{g}$ is determined by the Cartan matrix one obtains under twist equivalence in Lemma 3.5.2.5. The technical condition that no v_i commutes with all of V^* ensures that $[f_i, v_i] \neq 0$ for a dual basis $f_1, \dots, f_n$ of V^*, resembling the so-called non-degeneracy condition that the scalars $\gamma_{ii} \neq 0$ in Theorem 3.4.3.2. □

This is a characterization for quantum groups at generic parameters. The work surveyed in [9, 11] on pointed Hopf algebras over finite-dimensional Hopf algebras can be viewed as a characterization of small quantum groups. The triangular decomposition can be view as the case where the graph Γ described in 3.2.3 has two connected components, such that the corresponding generators for the two components give dually paired braided Hopf algebras.

The characterization suggests that if we are looking for examples outside of DJ-type, we can consider braidings of generic Cartan type which are not positive. In fact, [10, 2.6] gives an example that is generic of Cartan type, but not of DJ-type. We compute the associated quantum group here:

Example 3.5.2.8. Let $G = \langle k_1, k_2\rangle \cong C_\infty \times C_\infty$ be a free abelian group with two generators. We define a two-dimensional YD-module V over G on generators v_1 of degree k_1, v_2 of degree k_2 via

$$k_1 \triangleright v_1 = qv_1, \qquad k_1 \triangleright v_2 = q^{-1}v_2, \qquad k_2 \triangleright v_1 = q^{-1}v_1, \qquad k_2 \triangleright v_2 = -qv_2.$$

Lemma 2.1 in [10] shows that

$$\mathcal{B}(V) = \langle v_1, v_2 \mid \underline{\mathrm{ad}}(v_1)^3(v_2) = \underline{\mathrm{ad}}(v_2)^3(v_1) = 0\rangle.$$

The asymmetric braided Drinfeld double $U_{\mathbb{C}G}(V,V^*)$ is in fact a braided Drinfeld double if we define V^* to be the dual YD-module. It is the Hopf algebra given on $\mathcal{B}(V)\otimes\mathbb{C}G\otimes\mathcal{B}(V^*)$, subject to the relations

$$\begin{aligned}
[f_1,v_i] &= \delta_{1,i}\frac{k_1-k_1^{-1}}{q^{1/2}-q^{-1/2}}, & [f_2,v_i] &= \delta_{2,i}\frac{k_2-k_2^{-1}}{iq^{1/2}+iq^{-1/2}},\\
k_1v_2 &= q^{-1}v_2k_1, & k_2v_1 &= q^{-1}v_1k_2,\\
k_1v_1 &= qv_1k_1, & k_2v_2 &= -qv_2k_2,\\
k_1f_2 &= qf_2k_1, & k_2f_1 &= qf_1k_2,\\
k_1f_1 &= q^{-1}f_1k_1, & k_2f_2 &= -q^{-1}f_2k_2,
\end{aligned}$$

and with coproducts

$$\Delta(v_i)=v_i\otimes k_i+1\otimes v_i, \qquad \Delta(f_i)=f_i\otimes 1+k_i^{-1}\otimes f_i.$$

Apart from such examples, we can also include examples where free and nilpotent generators are combined, hence capturing features of both small and generic quantum groups. Here is such an example of minimal size:

Example 3.5.2.9. Let $G=C_\infty\times C_p=\langle g_\infty\rangle\times\langle g_p\rangle$ the product of an infinite cyclic group and one of order p. We define 2-dimensional YD-module over G as $\mathbb{C}v_\infty\oplus\mathbb{C}v_p$, where v_∞ has degree g_∞, and v_p has degree g_p. The group action is given by

$$\begin{aligned}
g_p\rhd v_p &= \xi_p v_p, & g_p\rhd v_\infty &= \eta_p v_\infty,\\
g_\infty\rhd v_p &= \xi_\infty v_p, & g_\infty\rhd v_\infty &= \eta_\infty v_\infty,
\end{aligned}$$

where scalars with a subscript p are primitive p-th roots of unity, and scalars with subscript ∞ are no roots of unity. We can now compute the Nichols algebra with generators v_p and v_∞. It is given by $\mathcal{B}(V)=\mathbb{C}\langle v_p,v_\infty\rangle/\langle v_p^p\rangle$. We denote the YD-dual by V^* with generators f_p, f_∞. The braided Drinfeld double of the braided Hopf algebra $\mathcal{B}(V)$ on $\mathcal{B}(V)\otimes k(C_p\times C_\infty)\otimes\mathcal{B}(V^*)$ is a quantum group that combines both $u_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned}
[f_p, v_i] &= \delta_{i,p}\frac{g_p - g_p^{-1}}{\xi_p^{1/2} - \xi_p^{-1/2}}, & [f_\infty, v_i] &= \delta_{\infty,i}\frac{g_\infty - g_\infty^{-1}}{\eta_\infty^{1/2} - \eta_\infty^{-1/2}},\\
g_pv_p &= \xi_pv_pg_p, & g_pv_\infty &= \eta_pv_\infty g_p,\\
g_\infty v_p &= \xi_\infty v_pg_\infty, & g_\infty v_\infty &= \eta_\infty v_\infty g_\infty,\\
g_pf_p &= \xi_p^{-1}f_pg_p, & g_pf_\infty &= \eta_p^{-1}f_\infty g_p,\\
g_\infty f_p &= \xi_\infty^{-1}f_pg_\infty, & g_\infty f_\infty &= \eta_\infty^{-1}f_\infty g_\infty.
\end{aligned}$$

and with coproducts

$$\Delta(v_i) = v_i \otimes g_i + 1 \otimes v_i, \qquad \Delta(f_i) = f_i \otimes 1 + g_i^{-1} \otimes f_i, \qquad \text{for } i = p, \infty.$$

3.5.3 Classes of Pointed Hopf Algebras by Radford

In [100], a class of pointed Hopf algebras $U_{(N,\nu,\omega)}$ was introduced (see also [45] for generalizations). These Hopf algebras are associated to the datum of a positive integer N and $1 \leqslant \nu < N$ such that N does not divide ν^2, and $\omega \in k$ is a primitive Nth root of unity in a field k. Denote $q := \omega^\nu$ and $r = |q^\nu| = \left|\omega^{\nu^2}\right|$. We let C_N denote a cyclic group of order N generated by an element a.

The algebra $U_{(N,\nu,\omega)}$ is the braided Drinfeld double of the YD-module Hopf algebra $U_+ := k[x]/(x^r)$ over C_p, with grading given by $x \mapsto a^\nu \otimes x$ and action $a \triangleright x = q^{-1}x$. Note that U_+ is the Nichols algebra of the one-dimensional YD-module kx. The coalgebra structure is given by $\Delta(x) = x \otimes a^\nu + 1 \otimes x$, and $\Delta(y) = y \otimes 1 + a^{-\nu} \otimes y$ for the dual generator y. Note further that the other Hopf algebra $H_{(N,\nu,\omega)}$ introduced by Radford is simply the bosonization $U_+ \rtimes kC_N$ in this set-up. The algebras $U_{(N,\nu,\omega)}$ and $H_{(N,\nu,\omega)}$ are not indecomposable unless $\nu = 1$. To obtain indecomposable pointed Hopf algebras, we can consider the subalgebras generated by x, y and a^ν (respectively, x and a^ν). Since these only depend on the choices of r and q we denote these Hopf algebras by $U_{(r,q)}$ (respectively, $H_{(r,q)}$). Note that $U_{(r,1,q)} = U_{(r,q)}$.

3.5.4 Quantum Group Analogues in Other Contexts

To conclude this chapter, we would like to adapt the point of view that quantum groups can also be studied over other Hopf algebras H than the group algebra.

For this, one can, motivated by the results of this chapter, look for Hopf algebras A with triangular decomposition over H. The property over a group that A is of separable type can be generalized by requiring that the YD-modules V with respect to the left and right coactions δ_r and δ_l are a direct sum of distinct one-dimensional simples.

As a first example, we can consider the case where H itself is primitively generated, i.e. $H = k[x_1, \dots, x_n]$ over a field of characteristic zero. If A is a bialgebra with triangular decomposition over H, then for $v \in V$, $\Delta(v) \in V \otimes H + H \otimes V$ implies that $\Delta(v)$ in fact equals $v \otimes 1 + 1 \otimes v$ using the counitary condition. This gives that A is generated by primitive elements and hence is a pointed Hopf algebra that is connected (i.e. the group like elements are the trivial group). Now A is in particular cocommutative, so Theorem 5.6.5 in [94] implies (for $\operatorname{char} k = 0$) that $A = U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of primitive elements in A. From this point of view, all quantum groups over $H = k[x_1, \dots, x_n]$ are simply the classical universal enveloping algebras. Investigating Hopf algebras with triangular decomposition over other Hopf algebras H can be the subject of future research.

Chapter 4

Koszulness of Enveloping Algebras Associated to Generalized Yang-Baxter Equations

4.1 Introduction

4.1.1 Motivation

The Yang-Baxter equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.1.1}$$

play a major role in the study of integrable system which are of importance in quantum field theory and statistical mechanics (see e.g. [58]). The *classical* Yang–Baxter equations

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{4.1.2}$$

can be obtained as a classical limit (see *loc.cit.*, Section 3) of these equations. There exist natural generalizations to n indices

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0, \tag{4.1.3}$$

for $1 \leqslant i < j < k \leqslant n$. In [14], the Lie algebra $\mathfrak{tr}_n$ with generators r_{ij} subject to the relations (4.1.3) and $[r_{ij}, r_{kl}] = 0$ for four distinct indices is considered. The corresponding discrete group such that the associated Malcev Lie algebras are $\mathfrak{tr}_n$ are also studied in [14]. These groups are the pure flat braid groups of [70].

It was shown in [14] that the universal enveloping algebras $U(\mathfrak{tr}_n)$ are Koszul. This result is in analogy with the Koszulness of the Drinfeld–Kohno Lie algebras, which have a similar presentation, with generators r_{ij}, and relations $[r_{ij}, r_{kl}] = 0$, $[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] = 0$ (see e.g. [40, 3.10]).

In [16, 7.12], the algebra $U(\mathfrak{tr}_n)$ is reinterpreted as a certain quadratic cover[29] of the Nichols algebra $\mathcal{B}(Y_G)$ of the Yetter–Drinfeld module Y_G which has a basis parametrized by reflections $(ij) \in S_n$. The module structure is (up to a cocycle) given by conjugation of reflections and the grading is given by the corresponding group element (ij) of the generator. This description gives natural generalizations of $U(\mathfrak{tr}_n)$ for any complex reflection group G (see Section 4.2 for a summary of their construction). These algebras are in particular braided Hopf algebras in the category of YD-modules over G and can be viewed as the universal enveloping algebras of certain Lie algebras, which are described in terms of generators and relations (see Section 4.2.2). For explicit presentations in the cases of the classical series D_n and B_n, see Section 4.2.3, 4.2.4. We will refer to these generalizations as *BEER-algebras.*

The main reason for our interest in the algebras $U(\mathfrak{yb}_G)$ lies in their applications to *rational Cherednik algebras* $H_{t,c}(G)$. These algebras were introduced in [39] as deformations of the algebra $\mathbb{C}[T^*(V)] \rtimes kG$ for $t = 0$, and $D(V) \rtimes kG$ for $t \neq 0$. The algebras $H_{t,c}(G)$ and their symplectic generalizations have since been at the core of an active area of current research (see e.g. [39, 48, 106] and many more). One important tool in the study of these algebras is the *Dunkl embedding*

$$\Theta_c\colon H_{0,c}(G) \hookrightarrow \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}^{\mathrm{reg}}] \rtimes G, \qquad \Theta_c\colon H_{1,c}(G) \hookrightarrow D(\mathfrak{h}^{\mathrm{reg}}) \rtimes G.$$

In [16, Section 7] different embeddings were introduced, namely

$$M_c\colon H_{0,c}(G) \hookrightarrow \mathrm{Heis}_{\mathbb{C}G}(U(\mathfrak{yb}_G)), \qquad M_c\colon H_{t,c}(G) \hookrightarrow \mathrm{Heis}_{\mathbb{C}G}(U(\mathfrak{yb}_G))_{t'}.$$

The analogy is that the ring of differential operators is the Heisenberg double[30] of the ring of regular functors (for $\mathbb{A}^n$). In some sense, the embeddings M_c are

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[29] The minimal quadratic cover of $\mathcal{B}(Y_G)$ is the algebra $\mathcal{E}_n$ of [44]. It is a quotient of $U(\mathfrak{tr}_n)$.

[30] The subscript t' denotes a certain deformation parameter (cf. [16, Theorem 7.25]).

more symmetric than the Dunkl embedding, which is the identity on $\mathfrak{h}$ while it maps $y \in \mathfrak{h}^*$ to the corresponding Dunkl operator D_y. In the embeddings of [16], elements of both $\mathfrak{h}$ and its dual are mapped to similar expressions which are summations over all reflections $s \in \mathcal{S}$. More concretely, in the case $t = 0$,

$$D_y = \sum_{s \in \mathcal{S}} c_s(y, \alpha_s) \frac{1-s}{\alpha_s}, \qquad M_c(x) = \sum_{s \in \mathcal{S}} c_s(\alpha_s^*, x)\underline{s}, \qquad M_c(y) = \sum_{s \in \mathcal{S}} (y, \alpha_s)\overline{s}.$$

In this chapter, we want to demonstrate one possible application of the embedding M_c. Namely, the construction of induction (right exact) and restriction (left exact) functors of modules over rational Cherednik algebras between different types (for the classical series A_n, B_n and C_n). We focus on the case $t = 0$, although similar constructions can be done in the case $t \neq 0$ as well.

4.1.2 Summary

The chapter starts with a brief review of the construction of the generalizations of the algebras $U(\mathfrak{tr}_n)$ due to [16] in Section 4.2. We give an interpretation as universal enveloping algebras of a certain Lie algebra $\mathfrak{yb}_G$ and explicit descriptions of the examples corresponding to the D_n- and B_n-series. We will denote the corresponding Weyl groups by the same letters $A_{n-1} = S_n$, B_n, D_n.

In Section 4.3, we prove that the algebras $U(\mathfrak{yb}_{G_n})$ are Koszul[31] for $G_n = D_n$ and B_n. This is done by explicitly constructing a PBW-basis of the corresponding quadratic dual in both cases separately.

In the final Section 4.4, we consider maps between the braided Heisenberg doubles of $U(\mathfrak{yb}_G)$. There are injective algebra morphisms

$$\begin{aligned} \phi_A^D \colon \mathrm{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) &\hookrightarrow \mathrm{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n})), \\ \phi_A^B \colon \mathrm{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) &\hookrightarrow \mathrm{Heis}_{\mathbb{C}B_n}(U(\mathfrak{yb}_{B_n})), \end{aligned}$$

but also injective algebra morphisms

$$\tau_n^n \colon \mathrm{Heis}_{\mathbb{C}G_n}(U(\mathfrak{yb}_{G_n})) \hookrightarrow \mathrm{Heis}_{\mathbb{C}G_n}(U(\mathfrak{yb}_{G_n})),$$

---

[31] It was pointed out by Y. Bazlov that Koszulness of these algebras has been conjectured by A.N. Kirillov in 2006 based on computational evidence.

where G can be replaced by either A, D or B. These morphisms can be combined with the result of [16] that the rational Cherednik algebras $H_{0,c}(G)$ map to the braided Heisenberg doubles of $U(\mathfrak{nb}_G)$. Hence we can use the diagrams

$$\begin{tikzcd}
& \mathrm{Heis}_{\mathbb{C}G_n}(U(\mathfrak{nb}_{G_n})) & \\
H_{0,c}(S_n) \arrow[ur] & & H_{0,c}(G_n), \arrow[ul, "M_c"']
\end{tikzcd}$$

where $G_n = D_n$, B_n or G_{n-1}, to give the Heisenberg doubles a bimodule structure over different rational Cherednik algebras and use these to define functors

$$\begin{aligned}
\mathrm{HInd}_A^D \colon H_{0,c}(S_n)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G_n)\text{-}\mathbf{Mod},\\
\mathrm{HInd}_n^{n+1} \colon H_{0,c}(G_n)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G_{n+1})\text{-}\mathbf{Mod},\\
\mathrm{HInd}_D^A \colon H_{0,c}(G_n)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(S_n)\text{-}\mathbf{Mod},\\
\mathrm{HInd}_{n+1}^n \colon H_{0,c}(G_{n+1})\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G_n)\text{-}\mathbf{Mod},
\end{aligned}$$

as well as the corresponding restriction functors HRes in the other directions. By the general tensor-Hom-adjunction, the induction and restriction functors form adjoint pairs. Hence all versions of the functors HInd are right exact.

4.1.3 Further Applications

Another reason for this chapter is the role $U(\mathfrak{nb}_G)$ plays in the construction of categorical actions of the braided Drinfeld doubles of $U(\mathfrak{nb}_G)$ over $\mathrm{Drin}(G)$ on the category of representations of the rational Cherednik algebras $H_{t,c}(G)$. The construction of the action uses a special case of a more general phenomenon that the monoidal category of modules over the braided Drinfeld double of a braided Hopf algebra acts on modules over the corresponding braided Heisenberg double (this generalization of an earlier result of [71] is the content of Chapter 2). In Chapter 5, we combine this general categorical action with the embeddings M_c of [16] and obtain a categorical action of $\mathrm{Drin}_{\mathrm{Drin}(G)}(U(\mathfrak{nb}_G))$-$\mathbf{Mod}$ on $H_{t,c}(G)$-$\mathbf{Mod}$.

4.2 BEER-Algebras for Complex Reflection Groups

4.2.1 Notation

Let V be a finite-dimensional $\mathbb{C}$-vector space, $G \leqslant \mathrm{GL}(V)$ a finitely-generated *irreducible* complex reflection group with finite set of reflections

$$\mathcal{S} = \{s \in G \mid \dim_{\mathbb{C}}(1-s) = 1\}.$$

For each $s \in \mathcal{S}$, $V = \ker(1-s) \oplus \operatorname{im}(1+s) = V_1 \oplus V_\chi$ as a decomposition of eigenspaces. We can choose vectors $\alpha_s \in V$, $\alpha_s^* \in V^*$ such that

$$s(v) = v - \langle \alpha_s^*, v\rangle \alpha_s, \qquad \forall v \in V. \tag{4.2.1}$$

Here, $\alpha_s^* \otimes \alpha_s$ corresponds to $1-s \in \mathrm{End}(V)$ and is hence independent of choice. As $g \triangleright (1-s) = 1 - gsg^{-1}$, we have that $g \triangleright \alpha_s^* \otimes \alpha_s = \alpha^*_{gsg^{-1}} \otimes \alpha_{gsg^{-1}}$ and we can, following [16], define λ to be the function such that

$$g \triangleright \alpha_s^* = \lambda(g,s)\alpha^*_{gsg^{-1}}, \qquad \forall g \in G, s \in \mathcal{S}. \tag{4.2.2}$$

For a finite Coxeter group, $\lambda(g,s)$ can be chose to have values in $\{\pm 1\}$.

4.2.2 Generalized BEER-Algebras

We briefly recall the construction of the quadratic algebras $U(\mathfrak{y}\mathfrak{d}_G)$ associated to a complex reflection group G from [16, Chapter 7].

Definition 4.2.2.1. Let Y_G denote the Yetter–Drinfeld module over kG generated by $\underline{s}$, for $s \in \mathcal{S}$, where

$$g \triangleright \underline{s} = \lambda(g,s)\underline{gsg^{-1}}, \tag{4.2.3}$$

$$\delta(\underline{s}) = s \otimes \underline{s} \in kG \otimes Y_G. \tag{4.2.4}$$

For the braiding $\Psi\colon Y_G \otimes Y_G \to Y_G \otimes Y_G$, we consider the ideals

$$I_G^{\mathrm{quad}} := \langle \ker(\mathrm{Id}_{Y_G \otimes Y_G} + \Psi)\rangle \lhd T(Y_G),$$
$$I_G := I_G^{\Lambda,\mathrm{quad}} := \langle I^{\mathrm{quad}} \cap \Lambda Y_G\rangle \lhd T(Y_G).$$

We consider the following Hopf algebras in the category of YD-modules over kG,

$$\mathcal{B}^{\mathrm{quad}}(Y_G) := T(V)/I_G^{\mathrm{quad}}, \qquad \mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G) := T(V)/I_G.$$

The algebra $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)$ will be referred to as the *BEER-algebra for G*.

Note that the definition does not depend on the choices of α_s, α_s^* as different choices will give isomorphic YD-modules (cf. [16, Remark 7.16]). Moreover, λ can be extended to a $\mathrm{Drin}(G)$-character by setting

$$g \otimes \delta_t \triangleright \underline{s} = \delta_{t,s}\lambda(g,s)\underline{gsg^{-1}}.$$

This is equivalent to $\lambda(gh,s) = \lambda(g,hsh^{-1})\lambda(h,s)$.

Remark 4.2.2.2. The algebra $\mathcal{B}^{\mathrm{quad}}(Y_G)$ is a generalization of the algebra $\mathcal{E}_n$ of [44] (which is the A_{n-1}-case) to arbitrary finite complex reflection groups.

As for type A in [14], we can consider a description as a universal enveloping algebra of a Lie algebra for the generalized BEER-algebras. Since $I_G \subseteq \Lambda V$, we can consider Lie algebra $\mathfrak{yb}_G$ defined as the quotient of the free Lie algebra $\mathcal{L}(\mathcal{S})$ on generators $\underline{s}$, for $s \in \mathcal{S}$, by the ideal generated by I_G.

Example 4.2.2.3.

(i) The cyclic group with two elements C_2 is a complex reflection group. For this group, $U(\mathfrak{yb}_{C_2}) = \mathbb{C}[\underline{s}]$, which is a Koszul algebra.

(ii) More generally, for $G = S_n$, the Lie algebra $\mathfrak{yb}_G$ is $\mathfrak{tr}_n$ from [14], for which $U(\mathfrak{tr}_n)$ was shown to be Koszul.

Lemma 4.2.2.4. *There is an isomorphism of algebras $U(\mathfrak{yb}_G) \cong \mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)$ for any complex reflection group G.*

Proof. This can be seen by use of the universal properties of quotients of free Lie (respectively associative) algebras. □

Note that the algebra $U(\mathfrak{yb}_G)$ carries a natural grading, where the generators $\underline{s}$ have degree 1 and the relations are in degree 2, so the BEER-algebras are quadratic. Remark that the isomorphisms of $U(\mathfrak{yb}_G)$ and $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)$ is one of algebras, *not* of Hopf algebras. The coproduct of the latter is defined in the category of YD-modules over G. Hence $U(\mathfrak{yb}_G)$ has two different Hopf algebra structures. When considering the braided Heisenberg double, we will always use the braided coproduct.

4.2.3 The D_n-Case

We will give explicit relations for the algebras $U(\mathfrak{y}\mathfrak{b}_G)$ in the cases $G = D_n$, B_n for which we will investigate Koszulness in Section 4.4, starting with the D_n-case. The Weyl group of type D_n is $D_n := C_2^{n-1} \rtimes S_n$, acting on $\mathbb{C}^n$, with reflections parametrized by the positive roots $e_i \pm e_j$ for $1 \leqslant i < j \leqslant n$. The corresponding reflections are

$$s_{e_i - e_j} = (ij) \in S_n \leqslant D_n, \quad i < j, \qquad\qquad s_{e_i + e_j} = s_i s_j (ij), \quad i < j,$$

where s_i changes the sign of e_i. We denote the corresponding generators of $U(\mathfrak{y}\mathfrak{b}_{D_n})$ by $\underline{(ij)}$ for $s_{e_i - e_j}$ and $\underline{\underline{(ij)}}$ for $s_{e_i + e_j}$, where $1 \leqslant i < j \leqslant n$. The character λ can be described by first remembering the order of the indices in $\sigma \rhd (ij) = (\sigma(i), \sigma(j))$ and multiplying by (-1) if the order of the indices of the transposition $(\sigma(i), \sigma(j))$ is reversed. That gives the module structure (for a transposition $\sigma = (k, l)$)

$$\sigma \rhd \underline{(ij)} = \begin{cases} \underline{(\sigma(i), \sigma(j))}, & \text{if } \sigma(i) < \sigma(j), \\ -\underline{(\sigma(j), \sigma(i))}, & \text{if } \sigma(i) > \sigma(j), \end{cases} \tag{4.2.5}$$

$$\sigma \rhd \underline{\underline{(ij)}} = \begin{cases} \underline{\underline{(\sigma(i), \sigma(j))}}, & \text{if } \sigma(i) < \sigma(j), \\ -\underline{\underline{(\sigma(j), \sigma(i))}}, & \text{if } \sigma(i) > \sigma(j), \end{cases} \tag{4.2.6}$$

$$s_k s_l \sigma \rhd \underline{\underline{(ij)}} = \begin{cases} -\underline{\underline{(\sigma(i), \sigma(j))}}, & \text{if } \sigma = (ij), \\ \underline{\underline{(\sigma(i), \sigma(j))}}, & \text{if } \{k, l\} \cap \{i, j\} = \varnothing, \\ \underline{(\sigma(i), \sigma(j))}, & \text{if } \sigma(i) < \sigma(j), |\{k, l\} \cap \{i, j\}| = 1, \\ -\underline{(\sigma(j), \sigma(i))}, & \text{if } \sigma(i) > \sigma(j), |\{k, l\} \cap \{i, j\}| = 1, \end{cases} \tag{4.2.7}$$

$$s_k s_l \sigma \rhd \underline{(ij)} = \begin{cases} -\underline{(\sigma(i), \sigma(j))}, & \text{if } \sigma = (ij), \\ \underline{(\sigma(i), \sigma(j))}, & \text{if } \{k, l\} \cap \{i, j\} = \varnothing, \\ \underline{\underline{(\sigma(i), \sigma(j))}}, & \text{if } \sigma(i) < \sigma(j), |\{k, l\} \cap \{i, j\}| = 1, \\ -\underline{\underline{(\sigma(j), \sigma(i))}}, & \text{if } \sigma(i) > \sigma(j), |\{k, l\} \cap \{i, j\}| = 1. \end{cases} \tag{4.2.8}$$

Lemma 4.2.3.1. *The ideal $I_{D_n}^{\mathrm{quad}}$ is generated by the relations for $\{i, j\} \cap \{k, l\} = \varnothing$:*

$$[\underline{(ij)}, \underline{(kl)}] = 0, \tag{4.2.9}$$

$$[\underline{\underline{(ij)}}, \underline{\underline{(kl)}}] = 0, \tag{4.2.10}$$

$$[\underline{(ij)}, \underline{\underline{(kl)}}] = 0, \tag{4.2.11}$$

$$\underline{(ij)}^2 = 0, \tag{4.2.12}$$

$$\underline{\underline{(ij)}}^2 = 0, \tag{4.2.13}$$

$$\underline{(ij)}\,\underline{\underline{(ij)}} + \underline{\underline{(ij)}}\,\underline{(ij)} = 0. \tag{4.2.14}$$

And for $1 \leqslant k < j < l \leqslant n$*:*

$$\underline{(jl)}\underline{(kj)} = \underline{(kj)}\underline{(kl)} + \underline{(kl)}\underline{(jl)}, \tag{4.2.15}$$

$$\underline{(kj)}\underline{(jl)} = \underline{(jl)}\underline{(kl)} + \underline{(kl)}\underline{(kj)}, \tag{4.2.16}$$

$$\underline{\underline{(jl)}}\underline{\underline{(kj)}} = \underline{\underline{(kj)}}\underline{(kl)} + \underline{(kl)}\underline{\underline{(jl)}}, \tag{4.2.17}$$

$$\underline{\underline{(kj)}}\underline{\underline{(jl)}} = \underline{(jl)}\underline{\underline{(kl)}} + \underline{\underline{(kl)}}\underline{(kj)}, \tag{4.2.18}$$

$$\underline{\underline{(jl)}}\underline{(kj)} = \underline{(kj)}\underline{\underline{(kl)}} + \underline{\underline{(kl)}}\underline{\underline{(jl)}}, \tag{4.2.19}$$

$$\underline{\underline{(kj)}}\underline{(jl)} = \underline{\underline{(jl)}}\underline{\underline{(kl)}} + \underline{\underline{(kl)}}\underline{(kj)}, \tag{4.2.20}$$

$$\underline{(jl)}\underline{\underline{(kj)}} = \underline{\underline{(kj)}}\underline{\underline{(kl)}} + \underline{\underline{(kl)}}\underline{(jl)}, \tag{4.2.21}$$

$$\underline{\underline{(kj)}}\underline{(jl)} = \underline{(jl)}\underline{\underline{(kl)}} + \underline{\underline{(kl)}}\underline{\underline{(kj)}}. \tag{4.2.22}$$

Proof. This follows from computing the braiding Ψ. The first relations (4.2.9)–(4.2.14) are easy to see using the formulas for the module structure. Further, consider the circle

$$\underline{(kj)(kl)} \xmapsto{\Psi} \underline{(jl)(kj)} \xmapsto{\Psi} \underline{(kl)(jl)} \xmapsto{\Psi} -\underline{(kj)(kl)}$$

from which we derive the relation (4.2.15). The relation (4.2.16) is derived from

$$\underline{(kj)(jl)} \xmapsto{\Psi} \underline{(kl)(kj)} \xmapsto{\Psi} -\underline{(jl)(kl)} \xmapsto{\Psi} -\underline{(kj)(jl)}.$$

The other relations (4.2.17)–(4.2.22) are obtained by the same loops with all possible combinations of $\underline{(\quad)}$ and $\underline{\underline{(\quad)}}$.

The observation that Ψ always maps elements of the basis of $Y_G \otimes Y_G$ consisting of tensor products of generators to basis elements (up to scalar ±) can be used to verify that that these relations span $\ker(\mathrm{Id} + \Psi)$ (in fact, give a basis). ☐

Hence, Lemma 4.2.3.1 gives an explicit description in terms of generators and relations of the D_n-analogue of the algebra $\mathcal{E}_n$ in [44].

Lemma 4.2.3.2. *The ideal* I_{D_n} *is generated by the following commutator relations for* $\{i, j\} \cap \{k, l\} = \varnothing$*, and* $k < j < l$*:*

$$[\underline{(ij)}, \underline{(kl)}] = 0, \tag{4.2.23}$$

$$[\underline{\underline{(ij)}}, \underline{\underline{(kl)}}] = 0, \tag{4.2.24}$$

$$[\underline{(ij)}, \underline{\underline{(kl)}}] = 0, \tag{4.2.25}$$

$$[\underline{(jl)}, \underline{(kj)}] = [\underline{(kj)}, \underline{(kl)}] + [\underline{(kl)}, \underline{(jl)}], \tag{4.2.26}$$

$$[\underline{\underline{(jl)}}, \underline{\underline{(kj)}}] = [\underline{\underline{(kj)}}, \underline{(kl)}] + [\underline{(kl)}, \underline{\underline{(jl)}}], \tag{4.2.27}$$

$$[\underline{\underline{(jl)}}, \underline{(kj)}] = [\underline{(kj)}, \underline{\underline{(kl)}}] + [\underline{\underline{(kl)}}, \underline{\underline{(jl)}}], \tag{4.2.28}$$

$$[\underline{(jl)}, \underline{\underline{(kj)}}] = [\underline{\underline{(kj)}}, \underline{\underline{(kl)}}] + [\underline{\underline{(kl)}}, \underline{(jl)}]. \tag{4.2.29}$$

Proof. This in proved by intersecting I_G^{quad} with ΛY_G using the basis from Lemma 4.2.3.1. The relations (4.2.9)–(4.2.11) are anti-symmetric hence give (4.2.23)–(4.2.25). No linear combinations of the relations (4.2.12)–(4.2.14) give antisymmetric relations as these relations are symmetric. The relations (4.2.26)–(4.2.29) are obtained as differences of the corresponding pairs of relations among (4.2.15)–(4.2.22). □

4.2.4 The B_n-Case

We will now give an explicit presentation of $U(\mathfrak{hd}_G)$ in the B_n-case. The reflections in $B_n = C_2^n \rtimes S_n$ are in one-to-one correspondence with the positive roots for B_n which are e_i in addition to $e_i \pm e_j$ $(1 \leqslant i < j \leqslant n)$ which also appear in the D_n-case. We have $s_i = s_{e_i}$ for $i = 1, \ldots, n$, which is

$$s_i(e_k) = \begin{cases} e_k, & k \neq i, \\ -e_k, & k = i. \end{cases} \tag{4.2.30}$$

We denote the generator of Y_G corresponding to s_k by r_k. To describe the action of B_n on Y_{B_n} note the structure is the same on generators $\underline{(ij)}$ and $\underline{\underline{(ij)}}$ as in (4.2.5). In addition, we have

$$s_k \triangleright \underline{(ij)} = \begin{cases} \underline{(ij)}, & \text{if } k \neq i, j, \\ \underline{\underline{(ij)}}, & \text{else,} \end{cases} \tag{4.2.31}$$

$$s_k \triangleright \underline{\underline{(ij)}} = \begin{cases} \underline{\underline{(ij)}}, & \text{if } k \neq i, j, \\ \underline{(ij)}, & \text{else,} \end{cases} \tag{4.2.32}$$

$$(ij) \triangleright r_k = \begin{cases} r_k, & \text{if } k \neq i, j, \\ r_j, & \text{if } k = i, \\ r_i, & \text{if } k = j, \end{cases} \tag{4.2.33}$$

$$s_i s_j (ij) \triangleright r_k = \begin{cases} r_k, & \text{if } k \neq i, j, \\ r_j, & \text{if } k = i, \\ r_i, & \text{if } k = j. \end{cases} \tag{4.2.34}$$

Lemma 4.2.4.1. *The ideal $I_{B_n}^{\mathrm{quad}}$ is generated by the relations (4.2.9)–(4.2.22) plus the additional relations for $k \neq i, j$:*

$$[r_k, \underline{(ij)}] = 0, \tag{4.2.35}$$

$$[r_k, \underline{\underline{(ij)}}] = 0, \tag{4.2.36}$$

$$[r_j, r_k] = 0, \tag{4.2.37}$$

$$r_i\underline{(ij)} - \underline{\underline{(ij)}}r_i + r_j\underline{\underline{(ij)}} - \underline{(ij)}r_j = 0, \tag{4.2.38}$$

$$r_j\underline{(ij)} - \underline{\underline{(ij)}}r_j + r_i\underline{\underline{(ij)}} - \underline{(ij)}r_i = 0. \tag{4.2.39}$$

Proof. First note that the subspace of $Y_{B_n} \otimes Y_{B_n}$ spanned by the $\underline{(ij)}$, $\underline{\underline{(ij)}}$ is closed under Ψ and isomorphic to $Y_{D_n} \otimes Y_{D_n}$. All further relations arise by considering Ψ applied to $r_i \otimes \underline{(ij)}$, $r_i \otimes \underline{\underline{(ij)}}$ or $r_i \otimes r_j$. ☐

Lemma 4.2.4.2. *The ideal I_{B_n} is generated by the commutator relations (4.2.23)–(4.2.29), together with the relations (4.2.35)–(4.2.36), and*

$$[r_i + r_j, \underline{(ij)} + \underline{\underline{(ij)}}] = 0. \tag{4.2.40}$$

Proof. This is easy to see using Lemma 4.2.3.2 and 4.2.4.1. ☐

The explicit presentation of $U(\mathfrak{yb}_{D_n})$ and $U(\mathfrak{yb}_{B_n})$ will enable us to show that these algebras are Koszul in Section 4.3 by finding a PBW-basis for the quadratic dual.

4.3 Koszulness of BEER-algebras

4.3.1 The Quadratic Dual

The quadratic dual $A^!$ of a quadratic algebra $A = T(V)/\langle R \rangle$ is defined as

$$T(V^*)/\langle R^\perp \rangle$$

using the orthogonal complement $R^\perp$ of the relations R in $V^* \otimes V^*$. Returning to the algebra $U(\mathfrak{yb}_G)$, we have that $S^2(Y_G) \leqslant I_G^\perp$. Hence, the quadratic dual $U(\mathfrak{yb}_G)^!$ is antisymmetric and therefore a finite-dimensional PI algebra.

In the following, we will construct a PBW-basis for the quadratic dual $U(\mathfrak{yb}_G)^!$ which implies that $U(\mathfrak{yb}_G)^!$ is Koszul [98, Section 5], and hence $U(\mathfrak{yb}_G)$ is Koszul as well by standard theory of Koszul algebras [17, 2.8–2.9].

We denote the generator dual to $\underline{(ij)}$ of $U(\mathfrak{yb}_G)$ by $\overline{(ij)} \in U(\mathfrak{yb}_G)^!$ for $G = A_{n-1}, D_n, B_n$, and the dual elements to $\underline{\underline{(ij)}}$ by $\overline{\overline{(ij)}}$. Using the result of [14], we see that $U(\mathfrak{tr}_n)^!$ is the anti-symmetric algebra generated by the relations

$$\overline{(ij)(jk)} = \overline{(ik)(jk)} = \overline{(jk)(ij)} = \overline{(ij)(ik)}, \qquad \text{for all } i < j < k. \tag{4.3.1}$$

4.3.2 Koszulness for the D_n-Case

Lemma 4.3.2.1. *The quadratic dual $U(\mathfrak{nb}_{D_n})^!$ is the antisymmetric algebra generated by $\overline{(ij)}$, $\overline{\overline{(ij)}}$ for $1 \leqslant i < j \leqslant n$, which are, for $k < j < l$, subject to the relations (4.3.1) and*

$$\overline{\overline{(jl)(kl)}} = \overline{\overline{(jl)}(kj)} = \overline{\overline{(kl)}(kj)}, \quad (4.3.2) \qquad \overline{\overline{(kj)(jl)}} = \overline{(kl)\overline{(jl)}} = \overline{\overline{(kj)}(kl)}, \quad (4.3.4)$$

$$\overline{\overline{(kl)(kj)}} = \overline{(jl)\overline{(kl)}} = \overline{\overline{(kj)}(jl)}, \quad (4.3.3) \qquad \overline{(ij)\overline{(ij)}} = 0. \quad (4.3.5)$$

Proof. This follows by computing the orthogonal complement of $\ker(\Psi + \mathrm{Id}_{Y_{D_n} \otimes Y_{D_n}})$ using Lemma 4.2.3.1. □

Proposition 4.3.2.2. *A vector space basis $U(\mathfrak{nb}_{D_n})^!$ is given by products of monomials of the form*

$$\overline{(j,i_1)(j,i_2)} \dots \overline{(j,i_k)}, \tag{4.3.6}$$

$$\overline{(j,i_1)(j,i_2)} \dots \overline{(j,i_{p-1})(j,i_{p+1})} \dots \overline{(j,i_k)\overline{(j,i_p)}}, \qquad \forall p = 2, \dots, k, \tag{4.3.7}$$

$$\overline{(i_1,i_2)(i_1,i_3)} \dots \overline{(i_1,i_k)\overline{(j,i_1)}}. \tag{4.3.8}$$

for disjoint *index sets $\{j < i_1 < i_2 < \dots < i_k\}$. The monomials of this form are arranged according to the value of the respective smallest elements j to give basis elements. We refer to monomials of the form (4.3.6)–(4.3.8) as* reduced *or* basic *monomials.*

Example 4.3.2.3. If $n = 4$, the monomials

$$\begin{array}{llllll}
\overline{(12)(13)(14)}, & \overline{(12)(34)}, & \overline{\overline{(12)}(34)}, & \overline{(12)(13)}, & \overline{(12)}, & \overline{\overline{(12)}}, \\
\overline{(12)(14)\overline{(13)}}, & \overline{(13)(24)}, & \overline{\overline{(13)}(24)}, & \overline{(12)(14)}, & \overline{(13)}, & \overline{\overline{(13)}}, \\
\overline{(12)(13)\overline{(14)}}, & \overline{(14)(23)}, & \overline{\overline{(14)}(23)}, & \overline{(12)\overline{(13)}}, & \overline{(14)}, & \overline{\overline{(14)}}, \\
\overline{(23)(24)\overline{(12)}}, & \overline{(12)\overline{(34)}}, & \overline{\overline{(12)(34)}}, & \overline{(12)\overline{(14)}}, & \overline{(23)}, & \overline{\overline{(23)}}, \\
\overline{(23)\overline{(12)}}, & \overline{(13)\overline{(24)}}, & \overline{\overline{(13)(24)}}, & \overline{(23)(24)}, & \overline{(24)}, & \overline{\overline{(24)}}, \\
\overline{(24)\overline{(12)}}, & \overline{(14)\overline{(23)}}, & \overline{\overline{(14)(23)}}, & \overline{(23)\overline{(24)}}, & \overline{(34)}, & \overline{\overline{(34)}}, \\
\overline{(34)\overline{(23)}}, & & & & &
\end{array}$$

give a full list of all basic monomials in $U(\mathfrak{nb}_{D_4})^!$. Hence the Hilbert–Poincaré polynomial of this 38-dimensional algebra is

$$P^!_{D_4}(t) = 1 + 12t + 21t^2 + 4t^3.$$

From this we can compute the Hilbert–Poincaré polynomial of $U(\mathfrak{ydb}_{D_4})$ as $P_{D_4}(t) = (P^!_{D_4}(-t))^{-1}$, giving

$$P_{D_4}(t) = 1 + 12t + 123t^2 + 1228t^3 + 12201t^4 + 121116t^5 + O(t^6).$$

To proof Proposition 4.3.2.2, we provide an algorithm to transform an arbitrary monomial in the generators into a reduced one. Given a monomial

$$M = \prod_{i=1}^{k} \overline{(a_i, b_i)} \prod_{j=1}^{l} \overline{\overline{(c_j, d_j)}},$$

we consider the graph Γ with vertices $\{a_1, b_1 \dots, a_k, b_k, c_1, d_1, \dots, c_l, d_l\}$ and edges $a_i \leftrightarrow b_i$, $c_j \leftrightarrow d_j$. The monomial M can be decomposed into a product of monomials for each connected component of the graph Γ, using the antisymmetric relations. For the following algorithm we assume without loss of generality that Γ is connected and the set of vertices is $\{1, \dots, n\}$.

Algorithm 4.3.2.4.

(1) If M is a product of generators $\overline{(a_i, b_i)}$ only (i.e. $l = 0$), then $M = 0$ or $M = \pm\overline{(1,2)}\,\overline{(1,3)}\dots\overline{(1,n)}$ using relation (4.3.1), cf. [14]. Hence M is a scalar multiple of a monomial of the form (4.3.6).

(2) If $l \neq 0$, i.e. some $\overline{\overline{(c_i, d_i)}}$ occurs, we can use antisymmetry, the relations (4.3.2)–(4.3.4), and step (1) to bring M into the form

$$M = \pm\overline{(a, b_1)}\,\overline{(a, b_2)}\dots\overline{(a, b_m)}\,\overline{\overline{(c, d)}},$$

where $a < b_1 < \dots < b_m$ or $M = 0$. If $\{c, d\} \subset \{a, b_1, \dots, b_m\}$, then $M = 0$ (from (4.3.5)). Otherwise, we distinguish the following cases:

(2a) If $c = a = 1$, then

$$M = \pm\overline{(1,2)}\,\overline{(1,3)}\dots\overline{(1,d-1)}\,\overline{(1,d+1)}\dots\overline{(1,n)}\,\overline{\overline{(1,d)}},$$

and hence has the form $\pm$(4.3.7).

(2b) If $c = 1$, $d = a$, then

$$M = \pm\overline{(2,3)(2,4)}\dots\overline{(2,n)\overline{(1,2)}},$$

so it has the form $\pm$(4.3.8).

(2c) If $c = b_i$ for some i, then $a = 1$ and

$$M = \pm\overline{(1,2)(1,2)}\dots\overline{(1,d-1)(1,d+1)}\dots\overline{(1,n)\overline{(1,d)}},$$

using the relation $\overline{(1,b_i)\overline{(b_i,d)}} = \overline{(1,b_j)\overline{(1,d)}}$ from (4.3.4). Hence M has the form $\pm$(4.3.7).

(2d) If $c = 1$ and $d = b_i$ for some i, then $a = 2$, and

$$M = \pm\overline{(2,3)(2,4)}\dots\overline{(2,n)\overline{(1,2)}},$$

using $\overline{(2,b_i)\overline{(1,b_i)}} = -\overline{(2,b_i)\overline{(1,2)}}$ which follows from (4.3.3). This implies that M has the form $\pm$(4.3.8).

(2e) If $c \neq 1$ and $d = b_i$, then

$$M = \pm\overline{(1,2)(1,2)}\dots\overline{(1,d-1)(1,d+1)}\dots\overline{(1,n)\overline{(1,d)}},$$

using $\overline{(1,b_i)\overline{(c,b_i)}} = -\overline{(1,b_i)\overline{(1,c)}}$ from (4.3.4). Hence M is of the form $\pm$(4.3.7).

The algorithm proves Proposition 4.3.2.2. Next, we will show that the basis from Proposition 4.3.2.2 is a PBW-basis. That is, there exists a choice of a total order on the set of generators $g_1, \dots, g_k$ such that the basis is of the following form: We denote by $\mathcal{T}$ the set of pairs of indices (i, j) such that $g_i g_j$ cannot be expressed as a linear combination of elements in (lexicographically) smaller order. In this case, the PBW-basis is given by

$$\mathcal{B} = \{g_{i_1} \dots g_{i_l} \mid (i_k, i_{k+1}) \in \mathcal{T}, \forall k = 1, \dots, l-1\}.$$

Note that for a general total order on the generators, the set $\mathcal{B}$ is spanning, but not necessarily linearly independent.

We introduce the total ordering on the generators $\overline{(ij)}$, $\overline{\overline{(ij)}}$ by requiring $\overline{(ij)} < \overline{\overline{(kl)}}$ of any i, j, k, l and further ordering the $\overline{(ij)}$ (and $\overline{\overline{(ij)}}$) lexicographically.

Lemma 4.3.2.5. *For the above order on the generators of $U(\mathfrak{nb}_{D_n})^!$, the corresponding set $\mathcal{T}$ consists of the pairs*

$$\overline{(ab)}\,\overline{(cd)},\quad \overline{(ab)}\,\overline{\overline{(cd)}},\quad \overline{\overline{(ab)}}\,\overline{\overline{(cd)}},\quad \overline{(kj)}\,\overline{(kl)},\quad \overline{(kj)}\,\overline{\overline{(kl)}},\quad \overline{(jl)}\,\overline{\overline{(kj)}}.$$

for a, b, c, d pairwise distinct (such that $a < c$) and $k < j < l$.

Proof. For pairs with four distinct indices, the only relations are the anti-commutativity relations. Hence precisely the given pairs in the first column are in $\mathcal{T}$ as they have the lexicographically smaller order.

For pairs which share one index, consider the relations (4.3.1)–(4.3.4). All possible pairs appear in these relations. The ones with the smallest lexicographic order are precisely the ones in $\mathcal{T}$. All pairs with the same index set are zero. □

Theorem 4.3.2.6. *The basis of Proposition 4.3.2.2 is the PBW-basis with respect to the above order on the generators. In particular, $U(\mathfrak{nb}_{D_n})$ is a Koszul algebra.*

Proof. This can be verified by direct checking using the description of the set $\mathcal{T}$ from Lemma 4.3.2.5. □

4.3.3 Koszulness for the B_n-Case

We will now extend the Koszulness result to the B_n-case. For this, recall the presentation of $U(\mathfrak{nb}_{B_n})^!$ from Section 4.2.4.

Lemma 4.3.3.1. *The quadratic dual $U(\mathfrak{nb}_{B_n})^!$ is the antisymmetric algebra generated by $\overline{(ij)}$, $\overline{\overline{(ij)}}$ for $1 \leqslant i < j \leqslant n$, and r^i, for $i = 1, \ldots, n$, which are subject to the relations (4.3.1)–(4.3.5) and*

$$r^i\overline{(ij)} = r^i\overline{\overline{(ij)}} = r^j\overline{(ij)} = r^j\overline{\overline{(ij)}}, \qquad \forall i < j. \tag{4.3.9}$$

Proof. This follows computing the orthogonal complement of I_{B_n} using the basis from Lemma 4.2.4.1, cf. Lemma 4.3.2.1. □

Proposition 4.3.3.2. *A basis for $U(\mathfrak{nb}_{B_n})^!$ is given by lexicographically arranged products of monomials of the form (4.3.6)–(4.3.8) as well as monomials of the form*

$$\overline{(j,i_1)}\,\overline{(j,i_2)}\ldots\overline{(j,i_k)}r_j. \tag{4.3.10}$$

We can adjust Algorithm 4.3.2.4 to the B_n-case to prove Proposition 4.3.3.2.

Algorithm 4.3.3.3.

(1) Given a monomial M in the generators for $U(\mathfrak{y}\mathfrak{b}_{B_n})^!$ such that Γ is connected on the set $\{1, \ldots, n\}$, use antisymmetry relations to bring it into the form

$$M = \pm \prod_i^p \overline{(a_i, b_i)} \prod_j^q \overline{\overline{(c_j, d_j)}} \prod_k^s r^k.$$

(2) Use Algorithm 4.3.2.4 to bring $M' = \prod_i^p \overline{(a_i, b_i)} \prod_j^q \overline{\overline{(c_j, d_j)}}$ into the form of a basis element of $U(\mathfrak{y}\mathfrak{b}_{D_n})^!$ up to sign.

(3) Using the relations (4.3.9), $M = 0$ unless $s = 0, 1$. If $s = 0$, we are done as $M = M'$ is a basis element of the form (4.3.6)–(4.3.8) up to sign.

(4) If $s = 1$, use the relations (4.3.9) to interchange all generators $\overline{\overline{(ij)}}$ with the corresponding generators $\overline{(ij)}$. Using step (2), we can then bring M into the form

$$M = \pm\overline{(1,2)(1,3)} \ldots \overline{(1,n)} r^k.$$

Using (4.3.9) again, we can replace r^k by r^1. Hence, M has the form $\pm$(4.3.10) in this case.

We can give a total order on the generators of $U(\mathfrak{y}\mathfrak{b}_{B_n})^!$ by ordering the $\overline{(ij)}$ (and $\overline{\overline{(ij)}}$) lexicographically, setting $\overline{(a,b)} < \overline{\overline{(c,d)}}$ for all indices, and $r^k > \overline{\overline{(c,d)}}$ for all k, c, d. Where the r^k are ordered $r^1 < r^2 < \ldots < r^n$.

Using this total order the corresponding set $\mathcal{T}$ consists of the pairs from Lemma 4.3.2.5 and the additional pairs

$$\overline{(ij)} r^k, \qquad \overline{\overline{(ij)}} r^k, \qquad \overline{(ij)} r^i, \qquad \text{for pairwise distinct indices } i, j, k$$

Theorem 4.3.3.4. *The basis of Proposition 4.3.3.2 is the PBW-basis with respect to the above order on the generators. In particular, $U(\mathfrak{y}\mathfrak{b}_{B_n})$ is a Koszul algebra.*

Example 4.3.3.5. We can now compute the Hilbert polynomial for $U(\mathfrak{y}\mathfrak{b}_{B_n})$ and its quadratic dual using the PBW-basis. For $n = 4$, it is

$$P^!_{B_4}(t) = 1 + 72t + 51t^2 + 5t^3,$$
$$P_{B_4}(t) = 1 + 72t + 5133t^2 + 365909t^3 + 26084025t^4 + 1859414106t^5 = O(t^6).$$

4.3.4 Consequences

Koszulness of the algebras $U(\mathfrak{hb}_G)$ enables us to compute the extension algebras explicitly. If is well known that for a Koszul algebra U,

$$\operatorname{Ext}^*_U(k,k) = \bigoplus_{l,m\geqslant o} \operatorname{Ext}^{l,m}_U(k,k) = \bigoplus_{l\geqslant 0} \operatorname{Ext}^l_U(k,k) \cong U^!,$$

which is thus generated by $\operatorname{Ext}^1_U(k,k)$. A free resolution for U is given by the *Koszul complex* [17, 98]. The module categories can be related by *Koszul duality* [17, Theorem 1.2.6] giving an equivalence of triangulated categories between the bounded derived categories of finite-dimensional graded modules over U and $U^!$. The latter is a finite-dimensional PI algebra.

4.4 Embeddings of Heisenberg Doubles of BEER-Algebras and Functors of Rational Cherednik Algebra Representations

In this section, we use the notion of perfect subquotient from [16] to obtain maps between Heisenberg doubles of the BEER-algebras $U(\mathfrak{hb}_G)$ for different G of the classical series A_n, B_n, D_n. These can be used to construct functors between categories of representations of rational Cherednik algebras. To ensure irreducibility, we restrict to $n \geqslant 3$ in the D_n-case for the applications to rational Cherednik algebras.

4.4.1 Maps of Heisenberg Doubles of Different Series

Note that $A_{n-1} = S_n$ is both a subgroup (and quotient) of D_n (and B_n) using the group homomorphisms

$$S_n \xhookrightarrow{j^D_A} D_n \xrightarrow{p^D_A} S_n, \qquad\qquad S_n \xhookrightarrow{j^B_A} B_n \xrightarrow{p^B_A} S_n.$$

These morphisms supply induction functors of Yetter–Drinfeld modules:

$$\operatorname{Ind}^D_A\colon {}^{S_n}_{S_n}\mathcal{YD} \longrightarrow {}^{D_n}_{D_n}\mathcal{YD}, \qquad (V,\triangleright,\delta) \longmapsto (V, \triangleright(p^D_A\otimes \operatorname{Id}_V), (j^D_A\otimes\operatorname{Id}_V)\delta), \quad (4.4.1)$$

$$\operatorname{Ind}^B_A\colon {}^{S_n}_{S_n}\mathcal{YD} \longrightarrow {}^{B_n}_{B_n}\mathcal{YD}, \qquad (V,\triangleright,\delta) \longmapsto (V, \triangleright(p^B_A\otimes \operatorname{Id}_V), (j^B_A\otimes\operatorname{Id}_V)\delta). \quad (4.4.2)$$

Via reconstruction theory, these gives rise to maps

$$d_A^D\colon \operatorname{Drin}(S_n) \longrightarrow \operatorname{Drin}(D_n), \qquad d_A^B\colon \operatorname{Drin}(S_n) \longrightarrow \operatorname{Drin}(B_n).$$

Comparing generators gives rise to maps

$$\operatorname{Ind}_A^D Y_{S_n} \xhookrightarrow{\iota_A^D} Y_{D_n} \overset{\pi_A^D}{\twoheadrightarrow} \operatorname{Ind}_A^D Y_{S_n}, \qquad \operatorname{Ind}_A^B Y_{S_n} \xhookrightarrow{\iota_A^B} Y_{B_n} \overset{\pi_A^B}{\twoheadrightarrow} \operatorname{Ind}_A^B Y_{S_n},$$

of YD-modules over D_n (respectively, B_n). These pairs of maps are *perfect subquotients* in the terminology of [16]. Such perfect subquotients are used to induce maps between the corresponding braided Heisenberg doubles of the braided Hopf algebras $U(\mathfrak{yb}_G)$ (as YD-modules over D_n, respectively B_n).

Lemma 4.4.1.1. *The pair (ι_A^D, π_A^D) is a perfect subquotient of YD-modules over D_n, and (ι_A^B, π_A^B) gives a perfect subquotients of YD-modules over B_n.*

Proof. Consider the pair (ι_A^D, π_A^D). To show it is a subquotient, we need to show that $(\mathrm{Id} \otimes \pi_A^D)\delta_{D_n}\iota_A^D = (j_A^D \otimes \mathrm{Id})\delta_{S_n}$, where δ_G denotes the G-coaction on Y_G. But this is clear as $\delta_{D_n}(\underline{(ij)}) = (ij) \otimes \underline{(ij)} = \delta_{S_n}(\underline{(ij)})$.

Next, to check the subquotients are *perfect*, we need for the maximal triangular ideals[32] that

$$I(\operatorname{Ind}_A^D Y_{S_n}, (j_A^D \otimes \mathrm{Id})\delta_{S_n}) = T(\iota_A^D)^{-1} I(Y_{D_n}, \delta_{D_n}).$$

The right hand side can be identified with $T(Y_{S_n}) \cap I(Y_{D_n}, \delta_{D_n})$. Then it is easy to see using Lemma 4.2.3.1 that the relations among the generators $\underline{(ij)}$ are the same in Y_{D_n} as in Y_{S_n} which implies the equality. The proof for Y_{B_n} is analogous. □

We remark that the braided Heisenberg double from [16] is defined on $U(\mathfrak{yb}_G^*) \otimes kG \otimes U(\mathfrak{yb}_G)$, and the pairing between $U(\mathfrak{yb}_G^*)$ and $U(\mathfrak{yb}_G)$ is *not* perfect. Note that as the Drinfeld doubles are quasitriangular, we have a map $\operatorname{Drin}(G) \to {}^{\operatorname{Drin}(G)}_{\operatorname{Drin}(G)}\mathcal{YD}$ using the universal R-matrix $\sum_{g \in G} \delta_g \otimes g$. This is used this to compute the Heisenberg double of $U(\mathfrak{yb}_G)$ over $\operatorname{Drin}(G)$, but we can also compute the Heisenberg double over $\mathbb{C}G$. The version over $\mathbb{C}G$ is for the purpose of Section 4.4.1, while the version over $\operatorname{Drin}(G)$ is relevant for Chapter 5.

We also use the notations $\overline{(ij)}$, $\overline{\overline{(ij)}}$ for generators of $U(\mathfrak{yb}_{G_n})$ even though this notation was already used for the quadratic dual.

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[32] For a YD-module (V, δ) over H, the maximal triangular ideal $I(V, \delta)$ is the maximal homogeneous ideals in degree $\geqslant 2$ which is also a coideal, cf. [16].

Corollary 4.4.1.2. *There exist injective algebra morphisms*

$$\begin{aligned}
\phi_A^D\colon \operatorname{Heis}_{\operatorname{Drin}(S_n)}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Heis}_{\operatorname{Drin}(D_n)}(U(\mathfrak{yb}_{D_n})),\\
\phi_A^B\colon \operatorname{Heis}_{\operatorname{Drin}(S_n)}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Heis}_{\operatorname{Drin}(B_n)}(U(\mathfrak{yb}_{B_n})),\\
\phi_A^D\colon \operatorname{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n})),\\
\phi_A^B\colon \operatorname{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Heis}_{\mathbb{C}B_n}(U(\mathfrak{yb}_{B_n})).
\end{aligned}$$

Proof. A morphism $\operatorname{Heis}_{\operatorname{Drin}(D_n)}(U(\mathfrak{tr}_n)) \to \operatorname{Heis}_{\operatorname{Drin}(D_n)}(U(\mathfrak{yb}_{D_n}))$ of algebras exists by applying the general theory of perfect subquotients in [16] to the ones constructed in Lemma 4.4.1.1. The algebra morphism stated is obtained by pre-composing with the algebra morphism $\operatorname{Id}_{U(\mathfrak{tr}_n)} \otimes d_A^D \otimes \operatorname{Id}_{U(\mathfrak{tr}_n^*)}$. The construction for B_n is the same, using d_A^B instead. More concretely, the morphism is given by mapping the generator $\underline{(ij)} \in U(\mathfrak{tr}_n)$ to $\underline{(ij)} \in U(\mathfrak{yb}_{D_n})$ (and mapping the dual generators $\overline{(ij)} \in U(\mathfrak{tr}_n^*)$ to $\overline{(ij)} \in U(\mathfrak{yb}_{D_n}^*)$), the group element $(ij) \in S_n$ maps to $(ij) \in D_n$, and the function $\delta_{(ij)} \in \mathbb{C}[S_n]$ maps to $\delta_{(ij)} = p_A^{D*}\delta_{(ij)} \in \mathbb{C}[D_n]$. Hence we see that ϕ_A^D is injective. The proof for B_n in place of D_n is again analogous.

Both morphisms ϕ_A^D and ϕ_A^B admit versions over $\mathbb{C}G$ if we regard Y_G as YD-modules over G rather than its Drinfeld double. This first gives a morphism $\operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{tr}_n)) \longrightarrow \operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n}))$ which will give the desired morphism by pre-composition with $\operatorname{Id}_{U(\mathfrak{tr}_n)} \otimes j_A^D \otimes \operatorname{Id}_{U(\mathfrak{tr}_n^*)}$. □

We can also use the subquotient result 4.4.1.1 to obtain morphisms of Hopf algebras between the corresponding braided Drinfeld doubles (or *double-bosonizations* in [90]). These will be relevant in Chapter 5 in order to provide categorical actions of the monoidal category of modules over the braided Drinfeld double of type A_{n-1} on modules over rational Cherednik algebras of different types.

$$\begin{aligned}
\phi_A^D\colon \operatorname{Drin}_{\operatorname{Drin}(S_n)}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Drin}_{\operatorname{Drin}(D_n)}(U(\mathfrak{yb}_{D_n})),\\
\phi_A^B\colon \operatorname{Drin}_{\operatorname{Drin}(S_n)}(U(\mathfrak{tr}_n)) &\hookrightarrow \operatorname{Drin}_{\operatorname{Drin}(B_n)}(U(\mathfrak{yb}_{B_n})).
\end{aligned}$$

Note that unlike the braided Heisenberg double, the braided Drinfeld double of $U(\mathfrak{yb}_G)$ *cannot* be computed over $\mathbb{C}G$.

4.4.2 Application to Rational Cherednik Algebras

Note first that in the cases A_n, D_n and B_n, all reflections are conjugate. Therefore, the parameter $c = (c_s)_{s\in\mathcal{S}}$ is just a constant. In the following, we can also consider different constants for the functors obtained in the $t = 0$ case as all rational Cherednik algebras embed into the same Heisenberg double.

For this, the maps

$$\begin{aligned}\phi_A^D &\colon \operatorname{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) \hookrightarrow \operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n})),\\ \phi_A^B &\colon \operatorname{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) \hookrightarrow \operatorname{Heis}_{\mathbb{C}B_n}(U(\mathfrak{yb}_{B_n})),\end{aligned}$$

from Lemma 4.4.1.1 can be combined with the maps

$$M_c(G)\colon H_{0,c}(G) \longrightarrow \operatorname{Heis}_{\mathbb{C}G}(U(\mathfrak{yb}_G)).$$

from [16] to induce functors

$$\begin{aligned}\operatorname{HInd}_A^D &\colon H_{0,c}(S_n)\text{-}\mathbf{Mod} \longrightarrow H_{0,c}(D_n)\text{-}\mathbf{Mod},\\ \operatorname{HInd}_A^B &\colon H_{0,c}(S_n)\text{-}\mathbf{Mod} \longrightarrow H_{0,c}(B_n)\text{-}\mathbf{Mod}.\end{aligned}$$

The functors are given by

$$\operatorname{HInd}_A^D = M_c(D_n)^* \phi^D_{A*} M_c(S_n)_*, \qquad \operatorname{HInd}_A^B = M_c(B_n)^* \phi^B_{A*} M_c(S_n)_*,$$

where $(-)_*$ is the extension functor $\operatorname{Heis}_{\mathbb{C}G}(U(\mathfrak{yb}_G)) \otimes_{H_{t,c}(S_n)} (-)$ (pushforward), and $(-)^*$ is the restriction functor (pullback). Note that the functor HInd_A^D is given by the $H_{0,c}(D_n)$-$H_{0,c}(S_n)$-bimodule $\operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n}))$ (and HInd_A^B is given by the bimodule using B_n).

We can also consider the functors

$$\begin{aligned}\operatorname{HInd}_D^A &\colon H_{0,c}(D_n)\text{-}\mathbf{Mod} \longrightarrow H_{0,c}(S_n)\text{-}\mathbf{Mod},\\ \operatorname{HInd}_B^A &\colon H_{0,c}(B_n)\text{-}\mathbf{Mod} \longrightarrow H_{0,c}(S_n)\text{-}\mathbf{Mod}.\end{aligned}$$

These functors are given by tensoring with $\operatorname{Heis}_{\mathbb{C}D_n}(U(\mathfrak{yb}_{D_n}))$ viewed as a $H_{0,c}(S_n)$-$H_{0,c}(D_n)$-bimodule for type D_n (and using instead B_n of D_n for type B). By construction, the induction functors have natural right adjoints (using the Tensor-Hom adjunction) which we denote by $\operatorname{HRes}_A^D, \operatorname{HRes}_D^A$, (or using B instead of D).

For example, the functor $\mathrm{HRes}^D_A\colon H_{0,c}(D_n)\text{-}\mathbf{Mod} \longrightarrow H_{0,c}(S_n)\text{-}\mathbf{Mod}$ is given by mapping an object V to the $H_{0,c}(D_n)$-module $\mathrm{Hom}_{H_{0,c}(S_n)}(\mathrm{Heis}_{\mathbb{C}D_n}(U(\mathfrak{y}\mathfrak{b}_{D_n}), V)$, where the module structure is given by $(a \rhd f)(b) = f(a \rhd b)$.

Corollary 4.4.2.1. *All versions of the functors* HInd *have right adjoints. Hence they preserve colimits and are, in particular, right exact.*

It is possible to obtain versions of the above functors for the case $t \neq 0$ and for the restricted rational Cherednik algebras, again using the embeddings of [16] and the observations about subquotients. In the case $t \neq 0$, the bimodules are given by twisted Heisenberg doubles of $U(\mathfrak{y}\mathfrak{b}_G)$. Caution about the parameters is however in order in these cases. For the restricted versions, the actual Nichols algebras $\mathcal{B}(Y_G)$ are used instead of $U(\mathfrak{y}\mathfrak{b}_G)$.

4.4.3 Maps of Heisenberg Doubles of the Same Series

In addition to maps between Heisenberg doubles of different series, we can also consider the maps

$$\tau_n^{n+1}\colon \mathrm{Heis}_{\mathbb{C}S_n}(U(\mathfrak{tr}_n)) \longrightarrow \mathrm{Heis}_{\mathbb{C}S_{n+1}}(U(\mathfrak{tr}_{n+1})),$$

where we map $\mathbb{C}S_n \hookrightarrow \mathbb{C}S_{n+1}$ by acting on $\{1,\ldots,n\}$, and $\underline{(ij)} \mapsto \underline{(ij)}$, $\overline{(ij)} \mapsto \overline{(ij)}$. Note that this map is an injective morphism of algebras. Similarly, such maps exist for types B_n, D_n. Again, it is also possible to give morphisms of Drinfeld doubles

$$\tau_n^{n+1}\colon \mathrm{Drin}_{\mathrm{Drin}(S_n)}(U(\mathfrak{tr}_n)) \longrightarrow \mathrm{Drin}_{\mathrm{Drin}(S_{n+1})}(U(\mathfrak{tr}_{n+1})).$$

As in Section 4.4.2, we can combine the maps τ_n^{n+1} with the maps $M_c(G)$ to obtain functors

$$\begin{aligned}\mathrm{HInd}_n^{n+1}\colon H_{0,c}(G_n)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G_{n+1})\text{-}\mathbf{Mod},\\ \mathrm{HInd}_{n+1}^{n}\colon H_{0,c}(G_{n+1})\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G_n)\text{-}\mathbf{Mod},\end{aligned}$$

where G_n is S_n, B_n or D_n. The induction functor HInd_n^{n+1} is defined by considering the algebra $\mathrm{Heis}_{\mathbb{C}G_{n+1}}(U(\mathfrak{y}\mathfrak{b}_{G_{n+1}}))$ as a $H_{0,c}(G_{n+1})$-$H_{0,c}(G_n)$-bimodule. The left $H_{0,c}(G_{n+1})$-action is given by restriction along the morphism $M_c(G_{n+1})$, where the right $H_{0,c}(G_n)$-structure is given by restriction along the composite $\tau_n^{n+1}M_c(G_n)$. Similarly, the functor HInd_{n+1}^{n} is given by tensoring with the $H_{0,c}(G_n)$-$H_{0,c}(G_{n+1})$-bimodule $\mathrm{Heis}_{\mathbb{C}G_{n+1}}(U(\mathfrak{hb}_{G_{n+1}}))$.

As before, we can define right adjoints, denoted by $\mathrm{HRes}_{\mathrm{n}}^{\mathrm{n+1}}$, HRes_{n+1}^{n} which imply that the induction functors are right exact.

4.4.4 A More Complete Picture

A more general method to obtain morphisms between the braided Heisenberg doubles of algebras $U(\mathfrak{hb}_G)$ for finite Coxeter groups can be given by comparing Dynkin diagrams. If one Dynkin diagram embeds into another one, then the BEER-algebra of the smaller one will be perfect subquotient of the BEER-algebra of the group corresponding to the larger Dynkin diagram. In such a situation, we obtain a map between the braided Heisenberg doubles, so that restriction and induction functors can be defined as above. For example, the diagram of type A_{n-1} embeds into both the diagram of type D_n and B_n giving rise to the morphisms in Section 4.4.1. There are also embeddings of the Dynkin diagrams of smaller rank into the Dynkin diagrams of larger rank for the classical series, giving the morphisms of braided Heisenberg doubles in Section 4.4.3. Analogues of such morphisms can also be defined for types B_n and D_n.

Chapter 5

Quantum Groups Associated to Complex Reflection Groups

5.1 Introduction

This chapter constitutes the conclusion of the thesis. It contains first results on work in progress that connects the categorical action of Chapter 2 with the study of the braided Hopf algebras associated to generalized Yang–Baxter equations of Chapter 4 into a categorical action on module categories over rational Cherednik algebras $H_{0,c}(G)$ at parameter $t = 0$.

5.1.1 Motivation

Let G be a complex reflection group. The rational Cherednik algebras $H_{t,c}(G)$ have been introduced as deformations of $\mathbb{C}[V \oplus V^*] \rtimes G$ in [39] in 2002 and have since attracted a great amount of attention in the literature due to their applications in geometry and other fields (such as algebraic combinatorics, or mathematical physics).[33] As a special case, the deformation of $\mathbb{C}[V \oplus V^*] \rtimes G$ given by the semidirect product $D(V) \rtimes G$ of the group algebra $\mathbb{C}G$ with the *Weyl algebra* (or ring of differential operators $D(V)$ on V) arises by setting $t = 1$ and $c = 0$. Hence, $H_{t,c}(G)$ can be seen as a class of generalizations of Weyl algebras associated to G. A main technical tool in the study of the representation theory of $H_{t,c}(G)$ at

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[33] See also references in 1.1.6.

parameter $t = 1$ is to use the so-called *Dunkl embeddings* into $D(V^{\mathrm{reg}}) \rtimes G$ and techniques of highest weight categories (see [46]).

In this chapter, we focus on the case where $t = 0$, which has a very different representation theory than the case $t = 1$. We combine two different results: the first one are algebra morphisms of [16] from $H_{0,c}(G)$ into certain braided Heisenberg doubles $\mathcal{H}(G)$ of braided Hopf algebra (associated to G via generalized Yang–Baxter equations); the second one is the categorical action (developed in Chapter 2) of modules over the corresponding braided Drinfeld doubles $\mathcal{D}(G)$ on modules over $\mathcal{H}(G)$. The main result is Theorem 5.2.4.1 — a categorical action of $\mathcal{D}(G)$-modules on modules over $H_{0,c}(G)$.

There are two main analogies here: we view the embeddings of Bazlov–Berenstein as versions of the Dunkl embeddings, and the braided Drinfeld doubles appearing in the categorical action as analogues of *quantum groups*.[34] To explain this point of view, note that the quantum groups $U_q(\mathfrak{g})$ are braided Drinfeld doubles of certain Nichols algebras [90, 4.3]. Hence, they have a triangular decomposition of the form

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_-) \otimes \mathbb{C}\mathbb{Z}^n \otimes U_q(\mathfrak{n}_+).$$

The braided Drinfeld doubles that appear as main actors in this chapter are denoted by $\mathcal{D}(G)$. They have an analogous decomposition as

$$\mathcal{D}(G) = U(\mathfrak{hb}_G) \otimes \mathrm{Drin}(\mathbb{C}[G]) \otimes U(\mathfrak{hb}^*_G).$$

Note that instead of the group algebra $\mathbb{C}\mathbb{Z}^n$ of the lattice, the Drinfeld double of $\mathbb{C}[G]$ appears as the “Cartan” part. The double is needed here since, unlike in the quantum group case, the action of G on the positive and negative part is *not* induced by the coaction via a dual R-matrix. Note that we always compute the braided Drinfeld double with respect to the braided coproduct on $U(\mathfrak{hb}_G)$ (cf. remark after Lemma 4.2.2.4).

The main application of this construction is that every finite-dimensional module over the algebra $\mathcal{D}(G)$ gives an endofunctor of $H_{0,c}(G)$-**Mod**. Using this structure, one obtains addition symmetries in the representation theory of the rational Cherednik algebras.

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[34] See also the comparison presented in 1.2.

5.1.2 Summary

We start this chapter with the general construction of the categorical action of the monoidal category $\mathcal{D}(G)$-**Mod** on the category $H_{0,c}(G)$-**Mod** in Section 5.2. For this, we introduce different versions of the braided Hopf algebras of which we take the braided Drinfeld and Heisenberg doubles in order to produce the algebras $\mathcal{D}(G)$ and $\mathcal{H}(G)$ in Section 5.2.1. We recall the results of [16] used to map $H_{0,c}(G)$ into $\mathcal{H}(G)$ in Section 5.2.3. The main result is the categorical action of Theorem 5.2.4.1.

Section 5.3 contains first results on the representation theory of $\mathcal{D}(G)$. In particular, we study the easiest case $G = C_2$, the cyclic group with two elements, in Section 5.3.3. Even in this case, there is a rich supply of simple finite-dimensional modules and a complex, non-semisimple, representation theory.

To conclude this chapter, and hence the thesis, we include a discussion of ideas and open problems for further research in Section 5.4. In particular, we consider the case of a general cyclic group C_n viewed as a complex reflection group. The investigation of the representation theory of $\mathcal{D}(C_n)$ is work in progress.

5.2 Definitions and Main Construction

We will now present the main construction of this chapter, namely all versions of the Hopf algebras $\mathcal{D}(G)$ which we interpret as a type of quantum group associated to an irreducible complex reflection group G. We will further construct a categorical action of modules over these quantum groups on module categories over rational Cherednik algebras $H_{0,c}(G)$ at parameter $t = 0$.

To fix the setting for this section, let $G \leqslant \mathrm{GL}_n(V)$ be an irreducible complex reflection group. We denote the set of complex reflections by $\mathcal{S} = \{s \in G \mid \dim_{\mathbb{C}} \mathrm{Im}(1-s) = 1\}$. Note that $s^2 = 1$ is only guaranteed if G is a Coxeter group. We can choose roots $\alpha_s^* \in V^*$ and coroots $\alpha_s \in V$, for each $s \in \mathcal{S}$, such that $1 - s = \alpha_s^* \otimes \alpha_s \in \mathrm{End}_{\mathbb{C}}(V)$. This implies that

$$s \triangleright v = v - \langle \alpha_s^*, v \rangle \alpha_s. \tag{5.2.1}$$

As $1-s$ is stable under conjugation by elements $g \in G$, we find that $g \triangleright (\alpha_s^* \otimes \alpha_s) = \alpha_{gsg^{-1}}^* \otimes \alpha_{gsg^{-1}}$ and hence there exist non-zero scalars $\lambda(g,s)$ such that

$$g \triangleright \alpha_s = \lambda(g,s)\alpha_{gsg^{-1}}, \qquad g \triangleright \alpha_s^* = \lambda(g,s)^{-1}\alpha_{gsg^{-1}}^*. \tag{5.2.2}$$

Note that λ can be extended to give a $\mathrm{Drin}(G)$-character by setting $\lambda(g,h) = 0$ if $h \notin \mathcal{S}$. The character satisfies the rule

$$\lambda(gh,s) = \lambda(g,hsh^{-1})\lambda(h,s). \tag{5.2.3}$$

Hence $\lambda(g,s)^{-1} = \lambda(g^{-1},gsg^{-1})$. If G is a Coxeter group, then λ can be chosen to have values $\{\pm 1\}$.

Definition 5.2.0.1 ([39])**.** The *rational Cherednik algebra* $H_{t,c}(G)$ is defined, for a parameter $t \in \mathbb{C}$, and $c = (c_s)_{s\in\mathcal{S}} \in \mathbb{C}[\mathcal{S}]^G$ a conjugation invariant function $\mathcal{S} \to G$, as the algebra generated by elements $v, v' \in V$, $f, f' \in V^*$ and $g \in G$ such that

$$[v,v'] = 0, \qquad [f,f'] = 0, \tag{5.2.4}$$

$$gv = (g \triangleright v)g, \qquad gf = (g \triangleright f)g, \tag{5.2.5}$$

$$\begin{aligned}[f,v] &= t\langle f,v\rangle 1 + \sum_{s\in\mathcal{S}} c_s \langle f,(1-s)v\rangle s \\ &= t\langle f,v\rangle 1 + \sum_{s\in\mathcal{S}} c_s \langle f,\alpha_s\rangle\langle \alpha_s^*, v\rangle s.\end{aligned} \tag{5.2.6}$$

If $t = 0$, we also consider the *restricted* rational Cherednik algebra $\overline{H}_{0,c}(G)$ which is the quotient $H_{0,c}(G)/\langle S(V)^G \oplus S(V^*)^G\rangle$ by the G-equivariant polynomials in V and V^* (which are central elements only if $t = 0$).

Note that $H_{t,c}(G) \cong H_{1,t^{-1}c}$ if $t \neq 0$. Hence it suffices to consider the two cases $t = 0$ and $t = 1$. The representation theory is very different in these two cases (see e.g. [39] or surveys on the topic such as [38, 47, 106]). If $t = 0$, $H_{0,c}(G)$ is finite over its center and hence has a rich supply of finite-dimensional simple modules. If however $t = 1$, then finite-dimensional modules are rare and only exist at special parameters c. The representation theory is studied using highest weight module techniques in this case.

5.2.1 Braided Hopf Algebras Associated to G

We will define the algebra $\mathcal{D}(G)$ as the braided Drinfeld double of the braided Hopf algebra $U(\mathfrak{ynb}_G)$ using the presentation in Chapter 2. The braided Hopf algebra $U(\mathfrak{ynb}_G)$ can be viewed as an object in the category $\mathrm{Drin}(\mathbb{C}[G])$-**Mod**. The quasitriangular Hopf algebra $H := \mathrm{Drin}(\mathbb{C}[G])$ has a presentation given as follows (cf. e.g. [13, 3.2]): it is generated as an algebra over $\mathbb{C}$ by $g \in G$ and functions δ_g such that $\delta_g(h) = \delta_{g,h}$. The defining relations is

$$g\delta_h = \delta_{ghg^{-1}}g. \tag{5.2.7}$$

The coproduct structure is given by

$$\Delta(g) = g \otimes g, \qquad \Delta(\delta_h) = \sum_{ab=h} \delta_b \otimes \delta_a, \tag{5.2.8}$$

and the counit is $\varepsilon g = 1$, $\varepsilon(\delta_h) = \delta_{h,1}$. The antipode is hence $S(g) = g^{-1}$ and $S(\delta_h) = \delta_{h^{-1}}$. The universal R-matrix is given by

$$R = \sum_{g \in G} g \otimes \delta_g. \tag{5.2.9}$$

We note that modules over H are left Yetter–Drinfeld modules over G.

We can now construct $U(\mathfrak{ynb}_G)$ and other versions of it as braided Hopf algebras in H-**Mod**. These are primitively generated braided Hopf algebras and can be constructed using the theory of Nichols algebras. For the minimal quadratic versions, this is due to [91, Appendix A] in the S_n case, and realized by [15], [16] in the general case.

Definition 5.2.1.1. Define the H-module $Y_G = \mathbb{C}\langle \underline{s} \mid s \in \mathcal{S}\rangle$ as a Yetter–Drinfeld module with G-action and G-coaction

$$g \triangleright \underline{s} = \lambda(g,s)\underline{gsg^{-1}}, \tag{5.2.10}$$

$$\delta(\underline{s}) = s \otimes \underline{s}. \tag{5.2.11}$$

The dual $Y_G^* = \mathbb{C}\langle \overline{s} \mid s \in \mathcal{S}\rangle$ is defined using the pairing $\langle \overline{s}, \underline{u}\rangle = \delta_{s,u}$, i.e.

$$g \triangleright \overline{s} = \lambda(g,s)^{-1}\overline{gsg^{-1}}, \tag{5.2.12}$$

$$\delta(\overline{s}) = s^{-1} \otimes \overline{s}. \tag{5.2.13}$$

In the following, we consider the dual pairs of braided Hopf algebras in the category B-**Mod** given by

$\mathcal{B}(Y_G)$,	$\mathcal{B}(Y_G^*)$,	Nichols algebras,
$\mathcal{B}^{\mathrm{quad}}(Y_G)$,	$\mathcal{B}^{\mathrm{quad}}(Y_G^*)$,	quadratic algebras,
$\mathcal{B}_\Lambda(Y_G)$,	$\mathcal{B}_\Lambda(Y_G^*)$,	only antisymmetric relations,
$\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G) := U(\mathfrak{yb}_G)$,	$\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G^*) := U(\mathfrak{yb}_G^*)$,	universal enveloping algebras.

Note that the Nichols algebras are perfectly paired[35]. The notation $U(\mathfrak{yb}_G)$ is used for the quadratic algebra with antisymmetric relations as this is — as an algebra — the universal enveloping algebra of a graded Lie algebra, generated by the degree one elements (see Chapter 4 for more details).

Some of these braided Hopf algebras can be recognized in the existing literature. The algebra $U(\mathfrak{yb}_G)$ for $G = S_n$ (type A) is the algebra $U(\mathfrak{tr}_n)$ of [14]. It is shown to be the Malchev Lie algebra of the pure virtual braid group. The algebra $\mathcal{B}^{\mathrm{quad}}(Y_G)$ is the Fomin–Kirillov algebra $\mathcal{E}_n$ from [44]. These algebras are subject to open conjectures in the theory of Nichols algebras. For example, it is not known if $\mathcal{E}_n$ is finite or infinite-dimensional for $n \geqslant 6$. It has been conjectured by Majid and others that $\mathcal{E}_n$ is in fact infinite-dimensional for $n \geqslant 6$ (see remarks after Proposition 3.1 in [91]). For smaller values of n, $\mathcal{E}_n$ is finite-dimensional (see [44, (2.8)]). In is remarked in [92, Section 6] that $\mathcal{E}_n$ is a Nichols algebra for $n \leqslant 4$, but this question is not answered for $n \geqslant 5$.

Theorem 5.2.1.2 ([14], and Chapter 4)**.** *The algebras $U(\mathfrak{yb}_G)$ are Koszul for the classical series A, $B = C$ and D of Weyl groups.*

It is proved in [104] that the quadratic quotient algebra $\mathcal{E}_n$ of $U(\mathfrak{yb}_G)$ is *not* Koszul for $n \geqslant 3$. This is shown be proving that $\mathcal{E}_3$ is not Koszul, and for larger n, $\mathcal{E}_3$ is an algebra retract of $\mathcal{E}_n$.

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[35] This is proved in [7, Theorem 3.2.29] and [90, 2.5] in general.

5.2.2 Quantum Groups Associated to G

We can now define the main actors of this chapter: the braided Drinfeld doubles $\mathcal{D}(G)$. We interpret these Hopf algebras as quantum group analogues associated to complex reflection groups.

Definition 5.2.2.1. Let $H = \mathrm{Drin}(\mathbb{C}[G])$ for a complex reflection group G. We defined the following Hopf algebras

$$\begin{aligned}
\overline{\mathcal{D}}(G) &:= \mathrm{Drin}_H(\mathcal{B}(Y_G^*), \mathcal{B}(Y_G)),\\
\mathcal{D}^{\mathrm{quad}}(G) &:= \mathrm{Drin}_H(\mathcal{B}^{\mathrm{quad}}(Y_G^*), \mathcal{B}^{\mathrm{quad}}(Y_G)),\\
\mathcal{D}_\Lambda(G) &:= \mathrm{Drin}_H(\mathcal{B}_\Lambda(Y_G^*), \mathcal{B}_\Lambda(Y_G)),\\
\mathcal{D}(G) &:= \mathrm{Drin}_H(U(\mathfrak{nb}_G^*), U(\mathfrak{nb}_G)).
\end{aligned}$$

The Hopf algebras $\mathcal{D}(G)$ and $\mathcal{D}_\Lambda(G)$ are referred to as *reflection quantum groups.*

Lemma 5.2.2.2. *The algebras $\mathcal{D}(G)$, $\mathcal{D}^{\mathrm{quad}}(G)$, $\mathcal{D}_\Lambda(G)$, and $\overline{\mathcal{D}}(G)$ are all generated by elements $\underline{s}$, $\overline{u}$ for $s, u \in \mathcal{S}$, $g \in G$, and $\delta_h \in \mathbb{C}[G]$ subject to the common relations*

$$g\delta_h = \delta_{ghg^{-1}}g, \tag{5.2.14}$$

$$\begin{aligned} g\overline{u} &= \lambda(g,u)^{-1}\overline{gug^{-1}}g,\\ g\underline{s} &= \lambda(g,s)\underline{gsg^{-1}}g, \end{aligned} \tag{5.2.15}$$

$$\begin{aligned} \delta_h\overline{u} &= \overline{u}\delta_{hu^{-1}},\\ \delta_h\underline{s} &= \underline{s}\delta_{s^{-1}h}, \end{aligned} \tag{5.2.16}$$

$$[\overline{u}, \underline{s}] = s\delta_{s,u} - \sum_{g\in G}\lambda(g,u)^{-1}\delta_{gug^{-1},s}\delta_g. \tag{5.2.17}$$

In addition, different versions of $\mathcal{D}(G)$ satisfy the corresponding Nichols type relations in the positive and negative part. We have the following diagram of quotient morphisms of Hopf algebras:

$$\begin{tikzcd}
 & \mathcal{D}^{\mathrm{quad}}(G) \arrow[dr, two heads] & \\
\mathcal{D}(G) \arrow[ur, two heads] \arrow[dr, two heads] & & \overline{\mathcal{D}}(G). \\
 & \mathcal{D}_\Lambda(G) \arrow[ur, two heads] &
\end{tikzcd} \tag{5.2.18}$$

In all of these Hopf algebras, the coproducts are given by

$$\Delta(g) = g \otimes g, \qquad \Delta(\delta_h) = \sum_{ab=h} \delta_b \otimes \delta_a, \tag{5.2.19}$$

$$\Delta(\underline{s}) = \underline{s} \otimes s + 1 \otimes \underline{s}, \qquad \Delta(\overline{u}) = \overline{u} \otimes 1 + \sum_{g \in G} \lambda(g,u)^{-1} \delta_g \otimes \overline{gug^{-1}}. \tag{5.2.20}$$

The antipode is given by

$$S(g) = g^{-1}, \qquad S(\delta_g) = \delta_g^{-1}, \tag{5.2.21}$$

$$S(\underline{s}) = -s\underline{s}, \qquad S(\overline{u}) = -\sum_{g \in G} \lambda(g,u)^{-1} \delta_g \overline{gug^{-1}}. \tag{5.2.22}$$

One analogy with quantized universal enveloping algebras of Lie algebras is that both can be viewed as braided Drinfeld doubles (for the quantum groups, this is due to [90]). Hence there is a strong resemblance in the relations of these algebra. The quantum group $U_q(\mathfrak{g})$ compares to $\mathcal{D}(G)$ or $\mathcal{D}_\Lambda(G)$, while the small version $u_q(\mathfrak{g})$ plays a similar role to $\overline{\mathcal{D}}(G)$ (we will see that they are also quasitriangular). A significant difference is that the quantum groups live over an abelian group $\mathbb{Z}^n$ (or $\mathbb{Z}/e\mathbb{Z}^n$), and the action is induced from the coaction via a dual R-matrix, while $\mathcal{D}(G)$ lives over a non-abelian group and the action is not induced in such a way. This explains why we have to compute the Drinfeld doubles over $\mathrm{Drin}(\mathbb{C}[G])$ rather than just $\mathbb{C}G$ or $\mathbb{C}[G]$. The choice to use $\mathrm{Drin}(\mathbb{C}[G])$ (rather than $\mathrm{Drin}(G)$) will be important in the categorical action in Sections 5.2.4.

Proposition 5.2.2.3. *The algebra $\overline{\mathcal{D}}(G)$ is quasitriangular with universal R-matrix given by the (formal) power series*

$$R_{\overline{\mathcal{D}}(G)} = \sum_{\alpha} f_\alpha R_H^{-(2)} \otimes e_\alpha R_H^{-(1)} = \sum_{\alpha, g \in G} f_\alpha \delta_g \otimes e_\alpha g^{-1}, \tag{5.2.23}$$

where $\sum_\alpha e_\alpha \otimes f_\alpha$ is the coevaluation given by orthonormal bases $\{e_\alpha\}$ of $\mathcal{B}(Y_G)$ and $\{f_\alpha\}$ of $\mathcal{B}(Y_G^*)$. Explicit orthonormal bases are not known for general G.

Proof. As $\overline{\mathcal{D}}(G)$ is a braided Drinfeld double of Nichols algebras (which have a perfect self-dual paring), we can find a coevaluation element $\mathrm{coev}_{\overline{\mathcal{D}}(G)} = \sum_\alpha e_\alpha \otimes f_\alpha$, which is a formal power series unless $\mathcal{B}(Y_G)$ is finite-dimensional. Now we use (2.3.54) to give a the formula (5.2.23) for the universal R-matrix. $\square$

However, a basis is hard to compute for $\mathcal{B}(Y_G)$ in most examples (like the symmetric groups) because it requires knowing the Nichols relations explicitly. It is a conjecture in the theory of Nichols algebras that the Fomin-Kirillov algebras are in fact quadratic. This has been verified for the cases $n \leqslant 4$. We extend this conjecture to all complex reflection groups:

Conjecture 5.2.2.4. *The Nichols ideal $I_{\max}(Y_G)$ is generated by homogeneous relations in degree less or equal $m_G := \max\{\mathrm{ord}(s) \mid s \in \mathcal{S}\}$.*

For any Coxeter group, $m_G = 2$. Hence the conjecture says that $\mathcal{D}(G) = \mathcal{D}^{\mathrm{quad}}(G)$.

5.2.3 Symmetric Dunkl Embeddings

In the last section, we introduced the braided Drinfeld doubles $\mathcal{D}(G)$. We now describe some results of [16] giving embeddings of $H_{0,c}(G)$ into the corresponding braided Heisenberg doubles $\mathcal{H}(G)$. We first introduce the algebras $\mathcal{H}(G)$ and give presentations. We can do this over $\mathbb{C}G$ (rather than $\mathrm{Drin}(\mathbb{C}[G])$) using what is called the *restricted* version of the braided Heisenberg double in Proposition 2.3.5.8.

Definition 5.2.3.1. Let $H = \mathbb{C}G$. We defined the following algebras

$$\begin{aligned}
\overline{\mathcal{H}}(G) &:= \overline{\mathrm{Heis}}_H(\mathcal{B}(Y_G^*), \mathcal{B}(Y_G)),\\
\mathcal{H}^{\mathrm{quad}}(G) &:= \overline{\mathrm{Heis}}_H(\mathcal{B}^{\mathrm{quad}}(Y_G^*), \mathcal{B}^{\mathrm{quad}}(Y_G)),\\
\mathcal{H}_\Lambda(G) &:= \overline{\mathrm{Heis}}_H(\mathcal{B}_\Lambda(Y_G^*), \mathcal{B}_\Lambda(Y_G)),\\
\mathcal{H}(G) &:= \overline{\mathrm{Heis}}_H(U(\mathfrak{yb}_G^*), U(\mathfrak{yb}_G)).
\end{aligned}$$

These algebras share the bosonization relations with the corresponding Drinfeld doubles. The only difference is the cross relation (5.2.24) below and that the functions on G are not part of the presentation:

Lemma 5.2.3.2. *The algebras $\mathcal{H}(G)$, $\mathcal{H}^{\mathrm{quad}}(G)$, $\mathcal{H}_\Lambda(G)$, and $\overline{\mathcal{H}}(G)$ are all generated by elements $\underline{s}$, $\overline{u}$ for $s, u \in \mathcal{S}$, and $g \in G$, subject to the common relations (5.2.15) and the commutator relation*

$$[\overline{u}, \underline{s}] = s\delta_{s,u}. \tag{5.2.24}$$

Again, the different versions of $\mathcal{H}(G)$ satisfy the corresponding Nichols type relations in the positive and negative part. We have the following commutative diagram of quotient morphisms of algebras:

$$\begin{tikzcd}
& \mathcal{H}^{\mathrm{quad}}(G) \arrow[dr, two heads] & \\
\mathcal{H}(G) \arrow[ur, two heads] \arrow[dr, two heads] & & \overline{\mathcal{H}}(G). \\
& \mathcal{H}_{\Lambda}(G) \arrow[ur, two heads] &
\end{tikzcd} \tag{5.2.25}$$

We recall that Heisenberg doubles can be viewed as a generalized form of the Weyl algebras (which are Heisenberg doubles of symmetric algebras, cf. Example 2.3.5.3).

Theorem 5.2.3.3 ([16, Theorem 7.20])**.** *For any parameter $c = (c_s)$, there is a morphism of algebras $M_c\colon H_{0,c}(G) \to \mathcal{H}(G)$ such that*

$$M_c(g) = g, \qquad \forall g \in G, \tag{5.2.26}$$

$$M_c(v) = \sum_{s \in \mathcal{S}} c_s \langle \alpha_s^*, v \rangle \underline{s}, \qquad \forall v \in V, \tag{5.2.27}$$

$$M_c(f) = \sum_{s \in \mathcal{S}} \langle f, \alpha_s \rangle \overline{s}, \qquad \forall f \in V^*. \tag{5.2.28}$$

If c is generic (for example, if $c_s \neq 0$ for all $s \in \mathcal{S}$), then M_c is injective even after composition with the quotient map $\mathcal{D}(G) \to \mathcal{D}_{\Lambda}(G)$.

We think of the embeddings M_c as more symmetric analogues of the important *Dunkl embeddings* used in the theory of rational Cherednik algebras to embed $H_{t,c}(G)$ into either $\mathcal{D}(V^{\mathrm{reg}}) \rtimes G$ or $\mathbb{C}[V \oplus V^*] \rtimes G$ depending on whether $t \neq 0$ or $t = 0$. The reason for this analogy is the resemblance in the formulas of M_c to those of the Dunkl embeddings for elements of V:

$$D_y = \sum_{s \in \mathcal{S}} c_s \langle y, \alpha_s \rangle \frac{1 - s}{\alpha_s}. \tag{5.2.29}$$

Note that on functions (elements of V^*) the Dunkl embeddings are simply the identity. In [16], an embedding is also defined in the $t \neq 0$ case using a twisted version of $\mathcal{H}(G)$ which is not used in this chapter.

Remark 5.2.3.4. Although harder to compute, it is in fact better to consider the algebra $\mathcal{D}_\Lambda(G)$ instead of $\mathcal{D}(G)$, as for example if $G = C_n$ is a cyclic group, the algebra $U(\mathfrak{nb}_G)$ is the tensor algebra (which is free), but when considering $\mathcal{B}_\Lambda(Y_G)$ interesting relations arise. However, for Coxeter groups, $\mathcal{D}(G)$ at least gives a good approximation of $\mathcal{D}_\Lambda(G)$ for which only quadratic relations need to be computed (cf. Conjecture 5.2.2.4).

Corollary 5.2.3.5. *The morphism M_c factors through the restricted rational Cherednik algebra to give a morphism $\overline{M}_c \colon \overline{H}_{0,c}(G) \to \overline{\mathcal{H}}(G)$. It is injective if c is generic.*

5.2.4 The Categorical Action

Theorem 5.2.4.1. *There is a coaction $\delta_c \colon H_{0,c}(G) \to \mathcal{D}(G) \otimes H_{0,c}(G)$ which is a morphism of algebras, such that*

$$\delta_c(g) = g \otimes g, \qquad \forall g \in G, \tag{5.2.30}$$

$$\delta_c(v) = \sum_{s \in \mathcal{S}} c_s \langle \alpha_s^*, v \rangle \underline{s} \otimes s + 1 \otimes v, \qquad \forall v \in V, \tag{5.2.31}$$

$$\delta_c(f) = \sum_{s \in \mathcal{S}} \langle f, \alpha_s \rangle \overline{s} \otimes 1 + \sum_{g \in G} \delta_g \otimes g \rhd f, \qquad \forall f \in V^*. \tag{5.2.32}$$

Proof. This result is based on the coaction $\delta \colon \mathcal{H}(G) \to \mathcal{D}(G) \otimes \mathcal{H}(G)$ from Corollary 2.3.5.9 which is a morphism of algebras. The map δ_c is defined as the composition δM_c, where we identify $H_{0,c}(G)$ with its image under M_c. This implies that it is a morphism of algebras. Even if this identification depends on c being generic, δ_c is a morphism of algebras even if this is not the case.

We can check explicitly on the generators that the identification of $H_{0,c}(G)$ with its image in $\mathcal{H}(G)$ preserves the property that δ is a coaction, implying that δ_c also is a coaction. ☐

This theorem gives a categorical action

$$\begin{aligned} \rhd \colon \mathcal{D}(G)\text{-}\mathbf{Mod} \otimes H_{0,c}(G)\text{-}\mathbf{Mod} &\longrightarrow H_{0,c}(G)\text{-}\mathbf{Mod}, \\ (V, W) &\longmapsto V \otimes W, \end{aligned}$$

induced using the map δ_c. We also obtain categorical actions for modules over other other versions of $\mathcal{D}(G)$, using the projections in (5.2.18). Further, this also gives a categorical action of $\overline{\mathcal{D}}(G)$-**Mod** on $\overline{H}_{0,c}(G)$-**Mod**.

We note that the existence of the categorical actions give us monoidal functors

$$F_{-}\colon \mathcal{D}(G)\text{-}\mathbf{Mod} \longrightarrow \operatorname{Fun}(H_{0,c}(G)\text{-}\mathbf{Mod}, H_{0,c}(G)\text{-}\mathbf{Mod}).$$

Hence, understanding the representation theory of the reflection quantum groups $\mathcal{D}(G)$ gives endofunctors on $H_{0,c}(G)$-**Mod**. In the following section, we will provide first results on the study of the representation theory of $\mathcal{D}(G)$.

5.3 Representation Theory

5.3.1 General Observations

Neither the category $\mathcal{D}(G)$-**Mod** nor $\overline{\mathcal{D}}(G)$-**Mod** are semisimple. The tensor products are not a sum of simples either. This can already be seen in the case $G = S_2$, which we study very explicitly in Section 5.3.3.

We declare an element $v \in V$ to have degree (g, χ) if $\delta(v) = g \otimes v$ and $h \rhd v = \chi(h)v$ for $h \in C_G(g)$ and χ a character of this centralizer. In other words, we use the forgetful functor $\mathcal{D}(G)$-**Mod** → $\operatorname{Drin}\mathbb{C}[G]$-**Mod** and decompose the image of a module V as a direct sum of simples in $\operatorname{Drin}\mathbb{C}[G]$-**Mod**,

$$V = \bigoplus_i V_{g_i^G, \chi_i},$$

where $V_{g_i^G,\chi_i}$ is the simple $\operatorname{Drin}\mathbb{C}[G]$-modules described by the conjugacy class of g_i and a $C_G(g_i)$-character χ_i. We say that g_i'-homogeneous vectors in this simple module have *degree* (g_i', χ_i), for $g_i' \in g_i^G$.

In the case where $\mathcal{D}(G)$ (or $\mathcal{D}_\Lambda(G)$) is finitely generated over its center, we can study representations using the variety of central characters of simple modules. This observation enables us to identify the Azumaya locus and make the following observations, using techniques similar to the ones used in [24, III] to study the representation theory of quantum groups at roots of unity.

Conjecture 5.3.1.1. *At least in the case where* $G = C_n$ *(for some* n*), the Azumaya locus of* $\mathcal{D}_\Lambda(G)$ *can be described as those maximal ideals* $\mathfrak{m}$ *such that* $\mathcal{D}_\Lambda(G)/\mathfrak{m}$ *is a simple module of maximal dimension* n*.*

To prove this claim, we suggest to use techniques such as in [24, III.1.6]. It remains to show that $\mathcal{D}_\Lambda(G)$ is a prime ring that is finite over a central subalgebra (its center). This would be implied if $\overline{\mathcal{D}}(G)$ can be shown to be finite-dimensional, that elements of the Nichols ideal are central in $\mathcal{D}_\Lambda(G)$ (cf. discussion in Section 5.4.2), and that $\mathcal{B}_\Lambda(Y_G)$ is an integral domain.

We test Conjecture 5.3.1.1 in the C_2-case in Section 5.3.3 by describing the algebra $\mathcal{D}(C_2) = \mathcal{D}_\Lambda(C_2)$ and its center explicitly.

5.3.2 Category $\mathcal{O}$

In order to make the non-semisimple category or representations over $\mathcal{D}(G)$ (or $\mathcal{D}_\Lambda(G)$) more manageable, we introduce versions of the BGG category $\mathcal{O}$ for this setting, using the triangular decompositions of these Hopf algebras. All constructions can equally be done with $\mathcal{D}_\Lambda(G)$ instead of $\mathcal{D}(G)$. We adapt the convention that we use $\mathcal{D}(G)$ when working with a Coxeter group, and $\mathcal{D}_\Lambda(G)$ otherwise.

Definition 5.3.2.1. Let $A \leqslant H$ be a graded subalgebra of a graded algebra. We say V is *locally nilpotent* over A if for each $v \in V$ there exist $n \in \mathbb{N}$ such that $A_{>n} \triangleright v = 0$ (cf. [46, 2.2]).

For modules over $\mathcal{D}(G)$, we can consider the subalgebra $\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G^*) \rtimes \mathrm{Drin}(G)$ and its dual counterpart $\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G) \rtimes \mathrm{Drin}(G)$, and require either of the actions be locally nilpotent. This gives two different versions of category $\mathcal{O}$ denoted by $\overline{\mathcal{O}}_G$ and $\underline{\mathcal{O}}_G$, respectively.

Definition 5.3.2.2. The category $\overline{\mathcal{O}}_G$ is defined to be the category of modules over $\mathcal{D}(G)$ which are finitely generated and locally nilpotent with respect to the graded subalgebra $\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G^*) \rtimes \mathrm{Drin}(G)$. Similarly, $\underline{\mathcal{O}}_G$ is defined on finitely-generated $\mathcal{D}(G)$-modules on which $\mathcal{B}_\Lambda^{\mathrm{quad}}(Y_G) \rtimes \mathrm{Drin}(G)$ acts locally nilpotently.

Lemma 5.3.2.3. *A finitely-generated module over* $\mathcal{D}(G)$ *is in the category* $\overline{\mathcal{O}}_G$ *if and only if for all* $v \in V$*, there exists* $n > 0$ *such that* $\overline{s_1} \cdot \ldots \cdot \overline{s_n} \triangleright v = 0$ *for all* $s_1, \ldots, s_n \in \mathcal{S}$*.*

A definition using local finiteness (similarly to 2.3.7) would be more general, and in fact every finite-dimensional simple would clearly be contained in both of $\overline{\mathcal{O}}_G$ and $\underline{\mathcal{O}}_G$, which is not the case using local nilpotency. We will justify this choice later. As in highest weight theory, we can define (co)standard modules which are always in $\overline{\mathcal{O}}_G$ (respectively $\underline{\mathcal{O}}_G$):

Definition 5.3.2.4. Let S be a simple finite-dimensional left YD-module over G. Note that there is an augmentation map $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G) \to \mathbb{C}$, the counit, and the dual also has an augmentation map. Hence S is a $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G^*) \rtimes \mathrm{Drin}(G)$ module S^+ (and similarly a $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G) \rtimes \mathrm{Drin}(G)$-module S^-) using this augmentation map — i.e. $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G^*)$ (respectively $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)$) acts trivially. We define

$$\begin{aligned}\Delta_S &:= \mathrm{Ind}^{\mathcal{D}(G)}_{\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G^*)\rtimes \mathrm{Drin}(G)} S^+ = \mathcal{D}(G) \otimes_{\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G^*)\rtimes \mathrm{Drin}(G)} S^+,\\ \nabla_S &:= \mathrm{Ind}^{\mathcal{D}(G)}_{\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)\rtimes \mathrm{Drin}(G)} S^- = \mathcal{D}(G) \otimes_{\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G)\rtimes \mathrm{Drin}(G)} S^-.\end{aligned}$$

We call Δ_S the *standard* (or *Verma*) module associated to S, and ∇_S the *costandard* module of S.

By construction, all Δ_S are in $\overline{\mathcal{O}}_G$, and all ∇_S are in $\underline{\mathcal{O}}_G$. They are special modules in these categories, so-called *highest* (respectively *lowest*) *weight modules*, i.e. there exist a highest weight vector v. That is, $\mathcal{B}^{\mathrm{quad}}_{\Lambda}(Y_G^*) = U(\mathfrak{h}\mathfrak{b}_G^*)$ acts trivially on $\mathbb{C}v$, and v generates the module. The *weight* of a highest weight module is the $\mathrm{Drin}(G)$-character $\mathbb{C}v$.

It should be remarked that in our setting, we will not be able to apply many of the general results of [46, Chapter 2] due to lack of an inner grading element. In particular, the Vermas do not have a unique simple quotient. We will see in the A_1-case that all finite-dimensional simples in $\overline{\mathcal{O}}_G$ are quotients of Vermas, but there are infinitely many different simple quotients of one fixed Verma in $\overline{\mathcal{O}}_G$, even in the A_1-case. However, there is a unique graded quotient module in the A_1-case and we expect this observation to also hold for other groups. As a general guideline for the study of $\overline{\mathcal{O}}_G$, we can recover the part of the theory of the category $\mathcal{O}$ for rational Cherednik algebras that holds for $t = 0$.

In the case where $A \leqslant H$ is also a graded sub-coalgebra, we observe that the tensor product of two locally nilpotent H-modules for A is again locally nilpotent. The tensor product is however not finitely-generated in general. Hence the category $\mathcal{O}$ is not closed under tensor products. However, if one module is finite-dimensional, then the tensor product with an arbitrary module of category $\mathcal{O}$ is again in category $\mathcal{O}$.

Lemma 5.3.2.5. *Denote by* $\overline{\mathcal{O}}_G^{\mathrm{fd}}$ *the full subcategory of* $\overline{\mathcal{O}}_G$ *on finite-dimensional modules. Then* $\overline{\mathcal{O}}_G^{\mathrm{fd}}$ *is a monoidal category. Moreover,* $\overline{\mathcal{O}}_G$ *is a module over* $\overline{\mathcal{O}}_G^{\mathrm{fd}}$ *by restricting the regular action.*

We also consider a *graded* version of category $\mathcal{O}$ which, as we will see, is easier to study than the general category $\mathcal{O}$:

Definition 5.3.2.6. We define $\overline{\mathcal{O}}_G^{\mathrm{gr}}$ to be the category consisting of $\mathbb{Z}$-graded $\mathcal{D}(G)$-modules in $\overline{\mathcal{O}}_G$, and $\underline{\mathcal{O}}_G^{\mathrm{gr}}$ to be the category of $\mathbb{Z}$-graded modules in $\underline{\mathcal{O}}_G$.

We can also define category $\mathcal{O}$ for the rational Cherednik algebras (at $t = 0$), as in [46]. That is, $\overline{\mathcal{O}}_c$ consists of finitely-generated modules over $H_{0,c}(G)$ which are locally nilpotent for $S(V^*)$, and $\underline{\mathcal{O}}_c$ is defined analogously.

Corollary 5.3.2.7. *There is a categorical action of the category of finite-dimensional modules in* $\overline{\mathcal{O}}_G$ *(over* $\mathcal{D}(G)$*) on the category* $\overline{\mathcal{O}}_c$ *for* $H_{0,c}(G)$*. This action is given by restricting the action from Theorem 5.2.4.1, and further restricts to an action of the corresponding graded category* $\mathcal{O}$ *analogues (as well as the lowest weight versions).*

Proof. This follows from the fact that the coaction map δ_c given in the categorical action preserves the triangular decomposition of the algebras. The version for the graded category $\mathcal{O}$ analogues follows from the simple observation that the coaction map δ_c is a graded morphism with respect to the natural $\deg \underline{s} = 1$, $\deg \overline{s} = -1$, and $\deg g = \deg \delta_g = 0$, for $s \in \mathcal{S}$, $g \in G$. $\square$

A central question for future investigation is how the functors F_V, for a $\mathcal{D}(G)$-module V behave:

Question 5.3.2.8. What properties do the functors F_V have with respect to exactness and preserving (co)limits? What are the images of Verma modules?

5.3.3 The Toy Case A_1

In the A_1-case, which corresponds to the cyclic group with two elements $C_2 = S_2 = \{1, s\}$, we can compute the representation theory of $\mathcal{D}(G)$ and the categorical action explicitly. It hence serves as a test or toy case for more general series of examples.

Example 5.3.3.1. The algebra $\mathcal{D}(C_2)$ is generated by $\underline{s}$, $\overline{s}$, s, δ_1, and δ_s subject to the relations

$$\begin{aligned} s\overline{s} &= -\overline{s}s, & \delta_s\overline{s} &= \overline{s}\delta_1, & s\delta_s &= \delta_s s, \\ s\underline{s} &= -\underline{s}s, & \delta_s\underline{s} &= \underline{s}\delta_1, & s\delta_1 &= \delta_1 s, \\ [\overline{s}, \underline{s}] &= s - \delta_1 + \delta_s. \end{aligned}$$

The coproduct is given on the generators by

$$\begin{aligned} \Delta(s) &= s \otimes s, & \Delta(1) &= 1 \otimes 1, \\ \Delta(\delta_s) &= \delta_s \otimes \delta_1 + \delta_1 \otimes \delta_s, & \Delta(\delta_1) &= \delta_1 \otimes \delta_1 + \delta_s \otimes \delta_s, \\ \Delta(\overline{s}) &= \overline{s} \otimes 1 + (\delta_1 - \delta_s) \otimes \overline{s}, \\ \Delta(\underline{s}) &= \underline{s} \otimes s + 1 \otimes \underline{s}. \end{aligned}$$

It is easy to see that $\mathcal{B}(Y_{A_1}) = \mathbb{C}[\underline{s}]/(\underline{s}^2)$ and $U(\mathfrak{hb}_{A_1}) = \mathbb{C}[\underline{s}]$. Therefore we find that $\dim \overline{\mathcal{D}}(A_1) = 8$.

The center of $\mathcal{D}(C_2)$ is generated over $\mathbb{C}$ by the elements 1, $D := s(\delta_1 - \delta_s)$, $A := \overline{s}^2$, $B := \underline{s}^2$, and $C = \mathrm{eu} := \underline{s}\overline{s} + \overline{s}\underline{s}$. Hence the variety of central characters is

$$\mathcal{Z}_{C_2} := \operatorname{Spec} Z(\mathcal{D}(C_2)) := \operatorname{Spec}\left(\mathbb{C}[A, B, C, D]/C^2 - 4AB - 2D - 2\right).$$

If V is a simple module, all central elements act by a scalar using Schur's Lemma. We see that $V = V_1 \oplus V_s$, where V_g is the homogeneous component of degree $g \in C_2$. We compute the action of D (acting by $d \in \mathbb{C}$) on such a simple module. For this, let $v_g \in V_g$.

$$D \triangleright v_1 = s(\delta_1 - \delta_s)v_1 = sv_1 = dv_1, \tag{5.3.1}$$

$$D \triangleright v_s = s(\delta_1 - \delta_s)v_s = -sv_s = dv_s. \tag{5.3.2}$$

This implies $d = \pm 1$ as $s^2 = 1$ and $sv_1 = \pm v_1$ and $sv_s = \mp v_s$. Hence we find that the image of the morphism

$$\gamma_{C_2} \colon \operatorname{Irrep} \mathcal{D}(C_2) \to \mathcal{Z}_{C_2}$$

is contained in a disjoint union of two subvarieties

$$\mathcal{Z}^1_{C_2} \sqcup \mathcal{Z}^{-1}_{C_2} = \operatorname{Spec}\left(\mathbb{C}[A,B,C]/C^2 - 4AB\right) \sqcup \operatorname{Spec}\left(\mathbb{C}[A,B,C]/C^2 - 4AB - 4\right),$$

corresponding to $d = 1$ and $d = -1$. It should be noted that the first component $\mathcal{Z}^1_{C_2}$ is the same as the character variety for the rational Cherednik algebra $H_{0,0}(C_2)$. It is singular at the origin. The the second component $\mathcal{Z}^{-1}_{C_2}$ corresponds to the central character variety of $H_{0,1}(C_2)$, which is non-singular. Thus $\mathcal{D}(C_2)$ captures both the singular Poisson variety and its non-singular deformation. We will show next, that the Azumaya locus of $\mathcal{D}(C_2)$ in fact is the non-singular locus of this disjoint union. Simple representations at regular points have dimension two, while there are two one-dimensional simples at the only singular point:

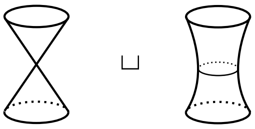

Figure 5.3.1: The character variety in the A_1-case

Proposition 5.3.3.2. *A full list of simple $\mathcal{D}(C_2)$-modules is given as follows:*

dim 1 *There are two 1-dimensional simples: $L^{\text{triv}} := L_{(1,+)}$ and $L^{\text{sgn}} := L_{(s,-)}$, generated by a vector of degree 1 (where s acts trivially) in L^{triv}, and degree s (where s acts by -1) in L^{sgn}, respectively. These have central character $(a,b,c,d) = (0,0,0,1)$, the singular point in $\mathcal{Z}^1_{C_2}$.*

dim 2 *Let $(a,b,c,\pm 1)$ be any point in $\mathcal{Z}^1_{C_2} \backslash (0,0,0,1) \sqcup \mathcal{Z}^{-1}_{C_2}$, i.e. any regular point. Then for each solution $\alpha_1, \alpha_2, \beta_1, \beta_2$ of the equations*

$$\alpha_1\alpha_2 = a, \qquad \beta_1\beta_2 = b, \qquad \alpha_1\beta_2 + \alpha_2\beta_1 = c, \tag{5.3.3}$$

the representation given by

$$\begin{tikzcd}
v_1 \arrow[r, bend left, "\overline{s}=\alpha_1", "\underline{s}=\beta_1"'] & v_s \arrow[l, bend left, "\underline{s}=\beta_2", "\overline{s}=\alpha_2"'] ,
\end{tikzcd}$$

that is,

$$\overline{s}v_1 = \alpha_1 v_s, \qquad \overline{s}v_s = \alpha_2 v_1, \qquad \underline{s}v_1 = \beta_1 v_s, \qquad \underline{s}v_s = \beta_2 v_1, \tag{5.3.4}$$

where v_1 has degree $(1,+)$ if $d = 1$, and $(1,-)$ if $d = -1$, is simple. Two such modules are isomorphic if and only if there exists a non-zero scalar λ such that

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\lambda\alpha'_1, \lambda^{-1}\alpha'_2, \lambda\beta'_1, \lambda^{-1}\beta'_2).$$

For any such point $(a, b, c, \pm 1)$, there exists a unique simple representation up to isomorphism, which we denote by $L^{\pm 1}_{(a,b,c)}$.

Proof. Given a simple $\mathcal{D}(C_2)$-module V, we can decompose it as $V = V_1 \oplus V_s$ into a direct sum of homogeneous C_2-graded pieces. Both the commutator $[\overline{s}, \underline{s}]$ and $C = \overline{s}\underline{s} + \underline{s}\overline{s}$ act by scalars on V_1 and V_s. Hence, the elements $\underline{s}\overline{s}$ and $\overline{s}\underline{s}$ act by scalars on V_1, V_s. Hence there are two submodules in V: The one generated by some element v_1 in V_1 which has dimension at least three, and the one generated by some $v_s \in V_s$. As V is simple, we conclude that it has to be at most two dimensional, $V = \mathbb{C}v_1 \oplus \mathbb{C}v_s$. Only in the case $a = b = c = 0$ we can have one-dimensional simples. It can be checked directly that for any other central character, there is a unique two-dimensional simple module up to isomorphism as described above. □

Next, we consider which of the modules are highest or lowest weight modules, and hence contained in category $\overline{\mathcal{O}}_{C_2}$ (respectively $\underline{\mathcal{O}}_{C_2}$) for C_2. By inspecting the simple modules from Proposition 5.3.3.2 on which $\overline{s}$ acts by zero, we find that the following finite-dimensional simples are in $\overline{\mathcal{O}}_{C_2}$:

$$L_{(1,+)}, \qquad L_{(s,-)}, \qquad L^{1}_{(a,0,0)}\ (a \neq 0), \qquad L^{-1}_{(a,0,\pm 2)}\ (a \in \mathbb{C}). \tag{5.3.5}$$

These correspond to the lines in the character varieties highlighted in Figure 5.3.2.

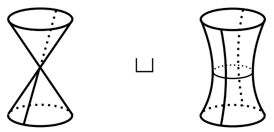

Figure 5.3.2: Category $\overline{\mathcal{O}}_G$ simples in the A_1-case

We recognize that these simples are heads of the four Verma modules in $\overline{\mathcal{O}}_{C_2}$:

$$0 \longrightarrow \Delta_{(s,-)} \xrightarrow{\overline{s}\rhd} \Delta_{(1,+)} \longrightarrow L_{(1,+)} \longrightarrow 0, \tag{5.3.6}$$

$$0 \longrightarrow \Delta_{(1,+)} \xrightarrow{\overline{s}\rhd} \Delta_{(s,-)} \longrightarrow L_{(s,-)} \longrightarrow 0, \tag{5.3.7}$$

$$0 \longrightarrow \Delta_{(1,+)} \xrightarrow{(\overline{s}^2-a)\rhd} \Delta_{(1,+)} \longrightarrow L^1_{(a,0,0)} \longrightarrow 0, \qquad a \neq 0, \tag{5.3.8}$$

$$0 \longrightarrow \Delta_{(s,-)} \xrightarrow{(\overline{s}^2-a)\rhd} \Delta_{(s,-)} \longrightarrow L^1_{(a,0,0)} \longrightarrow 0, \qquad a \neq 0 \tag{5.3.9}$$

$$0 \longrightarrow \Delta_{(1,-)} \xrightarrow{(\overline{s}^2-a)\rhd} \Delta_{(1,-)} \longrightarrow L^{-1}_{(a,0,-2)} \longrightarrow 0, \tag{5.3.10}$$

$$0 \longrightarrow \Delta_{(s,+)} \xrightarrow{(\overline{s}^2-a)\rhd} \Delta_{(s,+)} \longrightarrow L^{-1}_{(a,0,2)} \longrightarrow 0. \tag{5.3.11}$$

Note that there are indecomposable quotients of $\Delta_{(s,-)}$ and $\Delta_{(1,+)}$ of arbitrary length (via quotienting out by the submodules generated by the actions of $\overline{s}^{2k} - a$, or $\overline{s}^{2k+1}$. For the Vermas $\Delta_{(1,-)}$ and $\Delta_{(s,+)}$ there are only indecomposable quotients of even dimensions. We point out that there are graded two-dimensional indecomposables $V^2_{(1,-)}$, $V^2_{(s,-)}$ which are not simple:

$$0 \longrightarrow \Delta_{(1,-)} \xrightarrow{(\overline{s}^2)\rhd} \Delta_{(1,-)} \longrightarrow V^2_{(1,-)} \longrightarrow 0, \tag{5.3.12}$$

$$0 \longrightarrow \Delta_{(s,+)} \xrightarrow{(\overline{s}^2)\rhd} \Delta_{(s,+)} \longrightarrow V^2_{(s,+)} \longrightarrow 0. \tag{5.3.13}$$

These indecomposable are extensions of $L_{(1,+)}$ by $L_{(s,-)}$, or vice versa:

$$0 \longrightarrow L_{(1,+)} \xrightarrow{(\overline{s})\rhd} V^2_{(s,+)} \longrightarrow L_{(s,+)} \longrightarrow 0, \tag{5.3.14}$$

$$0 \longrightarrow L_{(s,-)} \xrightarrow{(\overline{s})\rhd} V^2_{(1,+)} \longrightarrow L_{(1,+)} \longrightarrow 0. \tag{5.3.15}$$

We can now compute tensor products of representations and write them as short exact sequences of simples, and hence determine the K-groups. For this, we restrict to finite-dimensional simples in $\overline{\mathcal{O}}_{C_2}$ (or $\overline{\mathcal{O}}^{\mathrm{gr}}_{C_2}$) only.

Proposition 5.3.3.3. *We can decompose the tensor products of simple $\mathcal{D}(C_2)$-modules, for $a + a' \neq 0$, as follows:*

$$0 \longrightarrow L^1_{(a+a',0,0)} \longrightarrow L^{-1}_{(a,0,\pm 2)} \otimes L^{-1}_{(a',0,\pm 2)} \longrightarrow L^1_{(a+a',0,0)} \longrightarrow 0, \tag{5.3.16}$$

where any combination of ± 2 is permitted. These short exact sequences do not split. However,

$$L^{\mathrm{sgn}} \otimes L^{\pm 1}_{(a,b,c)} \cong L^{\pm 1}_{(a,b,-c)}, \tag{5.3.17}$$

$$L^1_{(a,0,0)} \otimes L^1_{(a',0,0)} \cong L^1_{(a+a',0,0)} \oplus L^1_{(a+a',0,0)}, \tag{5.3.18}$$

$$L^1_{(a,0,0)} \otimes L^{-1}_{(a',0,\pm 2)} \cong L^{-1}_{(a+a',0,2)} \oplus L^{-1}_{(a+a',0,-2)}. \tag{5.3.19}$$

Note that L^{triv} is the unit of the monoidal structure on $\mathcal{D}(C_2)$-modules. In the case $a + a' = 0$ we have that

$$\begin{aligned} L^1_{(a,0,0)} \otimes L^1_{(-a,0,0)} &\cong L^{-1}_{(a,0,\pm 2)} \otimes L^{-1}_{(-a,0,\pm 2)} \\ &\cong L^1_{(a,0,0)} \otimes L^{-1}_{(-a,0,\pm 2)} \cong V^2_{(1,-)} \oplus V^2_{(s,+)}. \end{aligned} \tag{5.3.20}$$

Proof. The key idea is to find vectors on which $Z(\mathcal{D}(C_2))$ acts by a scalar. This is always true for the elements A, B, D. Hence, we need to find vectors in the tensor product on which

$$\Delta(C) = C \otimes s + (\delta_1 - \delta_s) \otimes C + s\bar{s} \otimes \underline{s} - 2\underline{s}(\delta_1 - \delta_2) \otimes \bar{s}s, \tag{5.3.21}$$

acts by a scalar (0 or ± 2). If we find a basis for the tensor product on which C acts by scalars, or two vectors on which C acts by distinct scalars, then it splits as a direct sum of simples (unless $a + a' = 0$). □

These computations of decompositions of tensor products enable us to compute the K-groups of the category $\overline{\mathcal{O}}_{C_2}$.

Corollary 5.3.3.4. *The K-group $K(\overline{\mathcal{O}}_{C_2})$ has generators $1 = [L^{\mathrm{triv}}]$ as well as*

$$\bar{1} := [L^{\mathrm{sgn}}], \qquad L^0_a := [L^1_{(a,0,0)}] \ (a \neq 0), \qquad L^{\pm 2}_a := [L^{-1}_{(a,0,\pm 2)}], \tag{5.3.22}$$

where [ ] *denotes the class of an object. These generators are subject to the relations for* $a + a' \neq 0$*:*

$$\overline{1} \cdot L_a^c = L_a^{-c}, \text{ for } c = 0, \pm 2, \qquad \overline{1}^2 = 1, \tag{5.3.23}$$

$$L_a^0 \cdot L_{-a}^0 = L_a^{\pm 2} \cdot L_{-a}^{\pm 2} = L_a^0 \cdot L_{-a}^{\pm 2} = 2(1 + \overline{1}), \tag{5.3.24}$$

$$L_a^0 \cdot L_{a'}^0 = L_a^{\pm 2} \cdot L_{a'}^{\pm 2} = 2L_{a+a'}^0, \tag{5.3.25}$$

$$L_a^0 L_{a'}^{\pm 2} = L_{a+a'}^2 + L_{a+a'}^{-2}. \tag{5.3.26}$$

The graded K-group $K(\overline{\mathcal{O}}_{C_2}^{\mathrm{gr}})$ *is the polynomial ring* $k[\overline{1}]$*.*

The category $\overline{\mathcal{O}}_{C_2}^{\mathrm{gr}}$ can be described more explicitly: For each integer $k \geqslant 0$, there exist two indecomposable modules of dimension k (one of highest weight $(1, +)$ and $(s, -)$) which are the quotients of the Verma modules $\Delta_{(1,+)}$ and $\Delta_{(s,-)}$ by the ideal generated by the action of $\overline{s}^k$.

Example 5.3.3.5. The rational Cherednik algebra $H_{0,c}(C_2)$ in the A_1-case is generated by x, y, s subject to the relations

$$[y, x] = 2cs, \qquad sx = -xs, \qquad sy = -ys, \qquad s^2 = 1. \tag{5.3.27}$$

Without loss of generality, we can distinguish the cases where $c = 1$ and $c = 0$. We start by considering the case $c = 1$: The center is given by

$$Z(H_{0,c(C_2)}) = \left\{ A = x^2, B = y^2, C = \frac{xy + yx}{2} \middle| AB = C^2 - 1 \right\}.$$

Denote $\mathcal{Z}_c := \operatorname{Spec} Z(H_{0,c}(C_2))$. The finite-dimensional simples over $H_{0,1}(C_2)$ are two-dimensional (see [39, 11.35]) and parametrized by central characters (a, b, c). We write $L_{(a,b,c)} = \mathbb{C}\langle v_+, v_- \rangle$ for the unique two-dimensional module such that

$$sv_\pm = \pm v_\pm, \qquad xv_+ = \alpha v_-, \qquad xv_- = \beta v_+, \tag{5.3.28}$$

$$yv_+ = \gamma v_-, \qquad yv_- = \delta v_+. \tag{5.3.29}$$

Here, $(\alpha, \beta, \gamma, \delta)$ are scalars such that

$$\alpha\beta = a, \qquad \gamma\delta = b, \qquad \beta\gamma = c + 1, \qquad \alpha\delta = c - 1. \tag{5.3.30}$$

Recall that the category $\overline{\mathcal{O}}_c$ for $H_{0,c}(C_2)$ is defined in the same way as for $\mathcal{D}(C_2)$ by requiring x to act locally finitely. We observe that the only simples contained in $\overline{\mathcal{O}}_1$ are $L_{(0,b,\pm 1)}$. In fact, $\delta = 0$ if and only if $c = -1$, and $\gamma = 0$ if and only if $c = 1$. Either δ or γ are always non-zero. Also, if $\delta = 0$, then $\beta \neq 0$ and if $\gamma = 0$, $\alpha \neq 0$. We write these simples $L_{(0,b,\pm 1)}$ for any scalar b as diagrams:

$$\begin{tikzcd} L_{(0,b,1)} : v_+ \arrow[r, bend left, "x=-2", "y=b"'] & v_- \arrow[l, bend left, "x=0", "y=1"'] \end{tikzcd}, \qquad \begin{tikzcd} L_{(0,b,-1)} : v_+ \arrow[r, bend left, "x=0", "y=1"'] & v_- \arrow[l, bend left, "x=2", "y=b"'] \end{tikzcd}.$$

The coaction δ_c is given by

$$\delta_c(s) = s \otimes s, \quad \delta_c(x) = 2\underline{s} \otimes s + 1 \otimes x, \quad \delta_c(y) = \overline{s} \otimes 1 + (\delta_1 - \delta_s) \otimes y. \tag{5.3.31}$$

In a first step, we will compute the categorical action of the simples in category $\mathcal{O}$ for $\mathcal{D}(C_2)$ on the ones for $H_{0,1}(C_2)$:

Lemma 5.3.3.6. *The categorical action of $L_{(s,-)}$ on $H_{0,1}(C_2)$-**Mod** is given by*

$$L^{\mathrm{sgn}} \rhd L_{(a,b,c)} \cong L_{(a,b,-c)}.$$

Moreover, the action on category $\overline{\mathcal{O}}_1$ of $\mathcal{D}(G)$-simples gives

$$L^1_{(a,0,0)} \rhd L_{(0,b,\pm 1)} \cong L_{(0,a+b,1)} \oplus L_{(0,a+b,-1)}.$$

The action of the other $\mathcal{D}(C_2)$-simples gives non-split extensions of the form:

$$\begin{aligned}
0 \longrightarrow L_{(0,a+b,1)} \longrightarrow L^{-1}_{(a,0,2)} \rhd L_{(0,b,1)} \longrightarrow L_{(0,a+b,-1)} \longrightarrow 0,\\
0 \longrightarrow L_{(0,a+b,1)} \longrightarrow L^{-1}_{(a,0,-2)} \rhd L_{(0,b,-1)} \longrightarrow L_{(0,a+b,-1)} \longrightarrow 0,\\
0 \longrightarrow L_{(0,a+b,-1)} \longrightarrow L^{-1}_{(a,0,-2)} \rhd L_{(0,b,1)} \longrightarrow L_{(0,a+b,1)} \longrightarrow 0,\\
0 \longrightarrow L_{(0,a+b,-1)} \longrightarrow L^{-1}_{(a,0,2)} \rhd L_{(0,b,-1)} \longrightarrow L_{(0,a+b,1)} \longrightarrow 0.
\end{aligned}$$

Proof. The strategy to find the decompositions of the tensor product is as follows: We note that

$$\begin{aligned}
\delta(x^2) &= 4\underline{s}^2 \otimes 1 + 1 \otimes x^2, && (5.3.32)\\
\delta(y^2) &= \overline{s}^2 \otimes 1 + 1 \otimes y^2, && (5.3.33)\\
\delta\left(\tfrac{xy+yx}{2}\right) &= (\underline{s}\overline{s} + \overline{s}\underline{s}) \otimes s + (\delta_1 - \delta_2) \otimes \tfrac{xy+yx}{2} && (5.3.34)\\
&\quad + \overline{s} \otimes x - 2\underline{s}(\delta_1 - \delta_s) \otimes ys. && (5.3.35)
\end{aligned}$$

This implies that x^2 and y^2 act on each vector in the tensor product by scalars. There always exist a vector in the tensor product on which $\frac{xy+yx}{2}$ also acts by a scalar. Such a vector generates a two-dimensional simple module. If $\frac{xy+yx}{2}$ acts by different scalars two vectors, then the tensor product splits as a direct sum of simples, otherwise it is a non-split extension. $\square$

Definition 5.3.3.7. We say that an object V *weakly generates* a category $\mathcal{C}$ over a monoidal category $\mathcal{M}$ with respect to a given categorical action $\triangleright\colon \mathcal{M} \otimes \mathcal{C} \to \mathcal{C}$ if for every simple object X of $\mathcal{C}$ there exists an object N of $\mathcal{M}$ such that X appears as a subobject of $N \triangleright V$.

Corollary 5.3.3.8. *The category $\overline{\mathcal{O}}_1$ for the rational Cherednik algebra $H_{0,1}(C_2)$ can, for example, be weakly generated by one simple object $L_{(0,0,1)}$ under the action of $\mathcal{D}(G)$-*$\mathbf{Mod}$.

Finally, we investigate the case $c = 0$: In this case, we have the central character variety

$$\mathcal{Z}_0 = \operatorname{Spec} Z(H_{0,0}(C_2)) = \operatorname{Spec}\left(\mathbb{C}[A,B,C]/AB - C^2\right).$$

We denote points satisfying this equation by (a, b, c). To find the finite-dimensional simple modules, we distinguish the following cases:

(1) If $(a, b, c) = (0, 0, 0)$, then there are two one-dimensional simples: $L_+ = \mathbb{C}v_+$, and $L_- = \mathbb{C}v_-$. Note that this is the only singular point of $\mathcal{Z}(H_{0,0}(C_2))$.

(2) If $a, b, c \neq 0$, then we can write the unique two-dimensional module $L_{(a,b,c)}$ corresponding to (a, b, c) as

$$\begin{tikzcd} v_+ \arrow[r, bend left, "x=1", "y=c/a"'] & v_- \arrow[l, bend left, "x=a", "y=c"'] \end{tikzcd}.$$

(3a) If $a = c = 0, b \neq 0$, we find one two-dimensional simple $L_{(0,b,0)}$:

$$\begin{tikzcd} v_+ \arrow[r, bend left, "x=0", "y=1"'] & v_- \arrow[l, bend left, "x=0", "y=b"'] \end{tikzcd}.$$

(3b) If $b = c = 0, a \neq 0$, we find one two-dimensional simple $L_{(a,0,0)}$:

$$\begin{tikzcd}
v_+ \arrow[r, bend left, "x=1", "y=0"'] & v_- \arrow[l, bend left, "x=a", "y=0"'] .
\end{tikzcd}$$

The only simples in category $\overline{\mathcal{O}}_0$ are $L_\pm$, and $L_{(0,b,0)}$. The coaction $\delta_c \colon H_{0,0}(C_2) \to \mathcal{D}(C_2) \otimes H_{0,0}(C_2)$ is given by

$$s \mapsto s \otimes s, \qquad x \mapsto 1 \otimes x, \qquad y \mapsto \overline{s} \otimes 1 + (\delta_1 - \delta_2) \otimes y.$$

Lemma 5.3.3.9. *The categorical action of simples over $\mathcal{D}(C_2)$ on simples over $H_{0,0}(C_2)$ is given by:*

$$\begin{align}
L^{\mathrm{sgn}} \rhd L_\pm &= L_\mp, \tag{5.3.36}\\
L^{\mathrm{sgn}} \rhd L_{(a,b,c)} &= L_{(a,b,-c)}, \tag{5.3.37}\\
L^1_{(a,0,0)} \rhd L_\pm &= L_{(0,a,0)}, \tag{5.3.38}\\
L^1_{(a,0,0)} \rhd L_{(0,b,0)} &= L_{(0,a+b,0)} \oplus L_{(0,a+b,0)}, \tag{5.3.39}\\
L^{-1}_{(a,0,\pm 2)} \rhd L_\pm &= L_{(0,a,0)}, \tag{5.3.40}\\
L^{-1}_{(a,0,\pm 2)} \rhd L_{(0,b,0)} &= L_{(0,a+b,0)} \oplus L_{(0,a+b,0)}. \tag{5.3.41}
\end{align}$$

Hence, we observe that $H_{0,0}(C_2)$ is generated by L_+ (or L_-) under the categorical action. The simples in the Azumaya locus can also be generated by $L_{(0,1,0)}$. It should be noted that the simples $L^{-1}_{(a,0,\pm 2)}$ act in the same way as $L^1_{(a,0,0)}$.

5.4 Conclusion

To conclude the thesis, we include a few remarks and open questions about the representation theory of the Hopf algebras $\mathcal{D}(G)$. These include ideas for work in progress, and future studies.

5.4.1 Open Questions

Here is a collection of open problems to address in order to gain a better understanding of the categorical action of $\mathcal{D}(G)$-**Mod** on $H_{0,c}(G)$-**Mod**. The structure of the category of $\mathcal{D}(G)$-modules is very complex. Hence it may be better to first restrict to its (graded) category $\mathcal{O}$.

Problem 5.4.1.1. *Collect all structural properties of category* $\overline{\mathcal{O}}_G$ *without the existence of an inner grading element as in [46]. What additional properties does* $\overline{\mathcal{O}}_G^{\mathrm{gr}}$ *have?*

Conjecture 5.4.1.2. *The category* $\overline{\mathcal{O}}_G^{\mathrm{gr}}$ *is a highest weight category in the sense of [30, Section 3].*

Given such an analysis, we can study the endofunctors induced by the finite-dimensional simples of $\overline{\mathcal{O}}_G^{\mathrm{gr}}$ on category $\overline{\mathcal{O}}_c$ (or its graded analogue) for the rational Cherednik algebras. For this, there are some interesting questions to ask:

Problem 5.4.1.3. *Identify the Euler element of* $\mathcal{D}(G)$ *in general. Recall that this is the central element* $\mathrm{eu} = \underline{s}\overline{s} + \overline{s}\underline{s}$ *in the* C_2*-case. This element is key to the study of the representation theory of* $\mathcal{D}(C_2)$*.*

Problem 5.4.1.4. *How does the action of finite-dimensional simples on category* $\overline{\mathcal{O}}_G$ *for* $H_{0,c}(G)$ *behave with respect to the central characters? Decompose the images of simples.*

Moving on from there, we may answer some questions about all simple modules of $\mathcal{D}(G)$. For this, it is important to understand when $\mathcal{D}(G)$ is finite-dimensional over its center. We expect this to be true for cyclic groups.

Question 5.4.1.5. For which groups G is the algebra $\mathcal{D}_\Lambda(G)$ is finite-dimensional over its center.

Given that an algebra is finite-dimensional over its center, we can describe finite-dimensional simples in terms of the central characters as done in the A_1-case in 5.3.3.

5.4.2 The Cyclic Group Case

As a next step of investigation, we intend to study the case of the general cyclic groups C_n. We collect a number of observations and problems for future work in this section.

Example 5.4.2.1. The cyclic group $C_n = \langle g \rangle$ of order n can be viewed as a complex reflection group. For this, let g act on $V = \mathbb{C}x$ by a primitive n-th root of unity ξ. Then $\mathcal{S} = \{g, \ldots, g^{n-1}\}$, and so $\mathbb{C}[\mathcal{S}]^G = \mathbb{C}^{n-1}$. Denote the dual of V by $V^* = \mathbb{C}y$. Then we can choose

$$\alpha_{g^i} = x, \qquad \alpha^*_{g^i} = (1 - \xi^i)y, \qquad \forall i = 1, \ldots, n-1. \tag{5.4.1}$$

We denote $\underline{\imath} := \underline{g^i}$, $\bar{\imath} := \overline{g^i}$, observe that $g^i \rhd \underline{\jmath} = \xi^i \underline{\jmath}$, and hence $\lambda(g^i, g^j) = \xi^i$. Hence the Drinfeld double of Y_{C_n} is generated by $\underline{\imath}$, $\bar{\imath}$, g, and δ_{g^i} subject to the relations:

$$g\underline{\imath} = \xi \underline{\imath} g, \qquad \delta_{g^i}\underline{\jmath} = \underline{\jmath}\delta_{g^{i+j}}, \qquad g\delta_h = \delta_h g, \tag{5.4.2}$$

$$g\bar{\imath} = \xi^{-1}\bar{\imath} g, \qquad \delta_{g^i}\bar{\jmath} = \bar{\jmath}\delta_{g^{i-j}}, \qquad [\bar{\imath}, \underline{\jmath}] = 0, \qquad i \neq j, \tag{5.4.3}$$

$$[\bar{\imath}, \underline{\imath}] = g^i - \sum_{l=0}^{n-1} \xi^l \delta_{g^l}, \tag{5.4.4}$$

as well as the Nichols relations (see Lemma 5.4.2.2). The coproducts are given by

$$\Delta(\underline{\imath}) = \underline{\imath} \otimes g^i + 1 \otimes \underline{\imath}, \qquad \Delta(\bar{\imath}) = \bar{\imath} \otimes 1 + \sum_{l=1}^{n-1} \xi^{-l}\delta_{g^l} \otimes \bar{\imath}. \tag{5.4.5}$$

Next, we observe that the braiding on Y_{C_n} is of diagonal type:

$$\Psi(\underline{\imath} \otimes \underline{\jmath}) = \xi^i \underline{\jmath} \otimes \underline{\imath}.$$

Lemma 5.4.2.2. *The following relations hold in* $I_{\max}(Y_{C_n})$ *for* $1 \leqslant i \neq j \leqslant n-1$ *:*

$$\underline{\mathrm{ad}}(\underline{\imath})^k(\underline{\jmath}) = 0, \qquad \textit{for } ik \equiv -j \mod n, \tag{5.4.6}$$

$$\underline{\imath}^{n/\gcd(i,n)} = 0. \tag{5.4.7}$$

Proof. The proof uses Lemma 3.7 of [9]. For a diagonal braidings, it states that $\underline{\mathrm{ad}}(\underline{\imath})^r(\underline{\jmath}) = 0$ if and only if

$$(r)!_{\xi^i} \prod_{0 \leqslant k \leqslant r-1} (1 - \xi^{i(k+1)+j}) = 0. \qquad \square$$

Question 5.4.2.3.

(i) For which n is $\mathcal{B}(Y_{C_n})$ finite-dimensional?

(ii) Do the relations (5.4.6) and (5.4.7) generate the Nichols ideal $I_{\max}(Y_{C_n})$?

In the smallest cases $n = 2, 3$, we can provide positive answers to questions 5.4.2.3(i) and (ii). In fact, $\mathcal{B}_\Lambda(Y_{C_2}) = \mathbb{C}[\underline{1}]$ which is an integral domain, and the relation $\underline{1}^2 = 0$ generates the Nichols ideal. In the case $n = 3$ we find:

Example 5.4.2.4. The algebra $\mathcal{B}(Y_{C_3})$ is generated by $\underline{1}$, $\underline{2}$ subject to the relations

$$\underline{1}^3 = 0, \qquad \underline{2}^3 = 0, \qquad \underline{12} = \xi\underline{21}.$$

This algebra is 9-dimensional, with Hilbert–Poincaré series

$$1 + 2t + 3t^2 + 2t^3 + t^4.$$

The algebra $\mathcal{B}_\Lambda(Y_{C_3})$ satisfies the relations

$$\underline{1}[\underline{2}, \underline{1}] + \xi[\underline{1}, \underline{2}]\underline{1} = 0, \qquad [\underline{2}, \underline{1}]\underline{2} + \xi\underline{2}[\underline{1}, \underline{2}] = 0.$$

Question 5.4.2.5. Describe the antisymmetric relations $I_\Lambda(Y_{C_n})$. Is $\mathcal{B}_\Lambda(Y_{C_n})$ an integral domain?

The coaction δ_c is given

$$\delta_c(g^i) = g^i \otimes g^i, \tag{5.4.8}$$

$$\delta_c(x) = \sum_{i=1}^{n-1} c_i(1 - \xi^i)\underline{i} \otimes g^i + 1 \otimes x, \tag{5.4.9}$$

$$\delta_c(y) = \sum_{i=1}^{n-1} \bar{i} \otimes 1 + \sum_{i=1}^{n} \xi^i \delta_{g^i} \otimes y. \tag{5.4.10}$$

5.4.3 The Symmetric Group Case

We expect the representation theory of $\mathcal{D}(S_n)$ to be hard to investigate for general $n \geqslant 3$ since the algebras $\mathcal{B}(Y_{S_n})$, and $\mathcal{E}_n = \mathcal{B}^{\text{quad}}(Y_{S_n})$ are subject to several conjectures about pointed Hopf algebras (see e.g. [9, 44, 91]). Here is one basic observation, namely that there are canonical embeddings of $\mathcal{D}(C_2)$ into $\mathcal{D}(S_n)$:

Proposition 5.4.3.1. *For any choice of $1 \leqslant i \neq j \leqslant n-1$, there exists a subalgebra of $\mathcal{D}(S_n)$ isomorphic to $\mathcal{D}(C_2)$, given on the elements*

$$\overline{(ij)}, \qquad \underline{(ij)}, \qquad (ij), \qquad \Sigma_{(ij)} := \sum_{\sigma \in C_G(ij)\setminus 1} \lambda(\sigma, (ij))\delta_\sigma.$$

Proof. Note first that ${\Sigma_{(ij)}}^2 = 1$, and it commutes with with (ij) in $\mathrm{Drin}(\mathbb{C}[S_n])$. Thus, $\mathbb{C}\langle (ij), \Sigma_{(ij)} \rangle \cong \mathrm{Drin}(\mathbb{C}[C_2])$. Moreover,

$$[\overline{(ij)}, \underline{(ij)}] = (ij) - \delta_1 + \Sigma_{(ij)}.$$

Observe that for the coproduct of $\overline{s}$,

$$\Delta(\overline{s}) = \overline{s} \otimes 1 + (\delta_1 - \Sigma_{(ij)}) \otimes \overline{s}.$$

Hence, $\mathbb{C}\langle \overline{(ij)}, \underline{(ij)}, (ij), \Sigma_{(ij)} \rangle$ is a Hopf subalgebra isomorphic to $\mathcal{D}(C_2)$. □

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