Instability of stratified two-fluid flow in a Venturi flow meter

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Abstract

In this dissertation the subject of stability of stratified two-fluid flows in an inclined channel is examined and modelled using classical fluid dynamics. The problem is motivated by the consideration of oil-water flows through a Venturi flow meter in the oil industry.

A background to both inviscid and viscous instability theories are given. An inviscid approach to the problem is undertaken and conditions for the onset of instability in the flow are derived. These conditions involve density contrast, Richardson number and critical slip velocity. Furthermore it is shown that a stable flow in an inclined channel may be made unstable by a small enough constriction.

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Chapter 1

Introduction

Stratified flows occur in many fields of science and a range of industry. Characteristic examples include the cooling of molten metal with water, oil and water flows in the oil industry, even dolphins swimming - the dolphin emits lubricating fluid which acts to reduce its hydrodynamic drag. We shall be concerned with the production of oil from subsurface reservoirs where oil and water are both extracted as part of the production process. The stratified mixture is pumped through pipes to the surface. For the purposes of maximising the oil yield from a particular oil reservoir it is important to know the relative flows of oil and water from different areas of the reservoir. These measurements are made in near-horizontal pipes at the bottom of the oil well.

To carry out this work Schlumberger Cambridge Research (SCR) have developed a Permanent Downhole Flow meter consisting of a Venturi flow meter similar to that shown in Figure 1.1.

The Venturi flow meter accelerates the flow at the constriction (backward-facing step) and there is a drop in pressure at this constriction. The fluid flow then slows through the diverging section and the flow downstream of the backward-step is unstable as can be seen from the experiments and numerical simulations. Schlumberger know that the total mass flux per unit area through this flow meter, Q_{Tot}, is given by

$$Q_{Tot} = k\sqrt{\rho_{mix}\Delta P},$$

where ΔP is the pressure difference across the backward-facing step, ρ_{mix} is the "average" density, and k is a constant of proportionality. As Schlumberger are interested in the value of this flux for the purposes of production optimisation, as mentioned above, SCR need to be able to measure ρ_{mix}, and in order to do so a gamma-ray source and detector are placed downstream of the Venturi flow meter. For an accurate measurement to be made the flow must be well-mixed at this point. The stratified flow in the pipe is stable but the presence of the flow meter causes an instability in the flow and hence the mixing of the oil and water.

Thus our task is to analyse, and understand, the conditions that make the stratified two-fluid flow in the Venturi flow meter unstable. The aim of this dissertation will be to set up stability analysis models of stratified flow in order to derive the conditions for instabilities to occur. These conditions will involve critical flow rates and Richardson number dependence.

Throughout this dissertation we shall make use of Squire's Theorem, which is stated below.

Squire's Theorem. 1.1 *To each unstable three-dimensional disturbance there corresponds a more unstable two-dimensional one.*

As Schlumberger wish the flow to be unstable we are interested in finding a lower bound on the conditions for instability and consequently, thanks to Squire's Theorem, we may restrict our attention to two-dimensional disturbances.

Figures 1.2 and 1.3 have been obtained from a direct numerical simulation of the flow through an axisymmetric Venturi flow meter. Discussions between ourselves and Dr John Ferguson, together with figures 1.2 and 1.3 in particular, suggested the following possible instability mechanisms:

1. Looking at figure 1.2 we see that the interface becomes wildly unstable downstream of the forward-facing step. It seems feasible that the flow could be stable with respect to the backward-facing step but unstable with respect to the forward-facing step.

2. The flow may develop Kelvin-Helmholtz instabilities in the constriction, since there is a greater velocity difference at the interface in the constriction than in the main pipe due to the faster flow in the constriction. These instabilities may then become magnified in the diverging channel and by the forward-facing step.

3. Figure 1.2 indicates that the first breakup of the interface appears in the expanding channel and it is possible that the interface becomes unstable in this section of the Venturi.

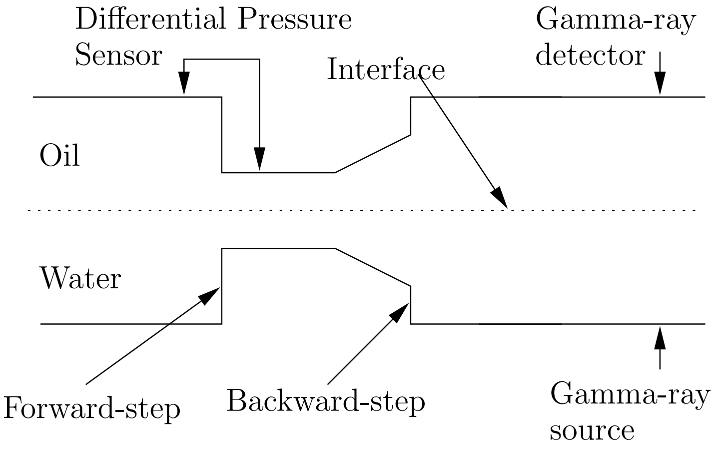

Figure 1.1: A Venturi flow meter.

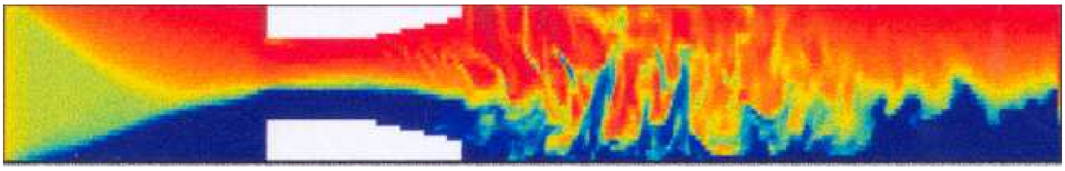

Figure 1.2: A snapshot in time of oil (red) - water (blue) volume fraction.

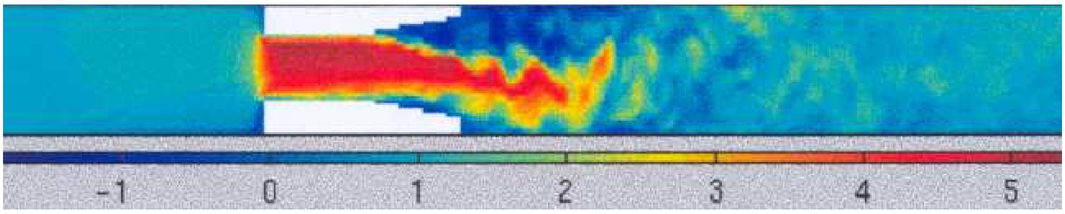

Figure 1.3: A snapshot in time of mixture axial velocity (m/s).

4. Figure 1.3 shows boundary layer separation at the walls near the start of the expanding channel. This leads to a shear layer in the flow that does not coincide with the interface. The Kelvin-Helmholtz instability of this shear layer could possibly drive instabilities and mixing at the interface.

In this dissertation we shall be mainly concerned with investigating mechanisms 2 and 3.

1.1 A Background to Viscous and Inviscid Stability Theory

Hydrodynamic stability has been at the centre of fluid mechanics for over a century. The work in this area was first addressed in the late nineteenth century by Kelvin, Helmholtz, Rayleigh and Reynolds. The early work of these key mathematicians and the progress made by others over the next one hundred years is well-presented in Drazin and Reid [4], Batchelor [3] and Acheson [2]. Much of the work on the stability of two-fluid flows was carried out in the 1960's by Yih [15], Thorpe [13], [14], Kao [9] and Drazin [5]. In [15] Yih shows that viscosity stratification can cause instability; Thorpe analyses the conditions for the onset of Kelvin-Helmholtz instabilities in flows with shear at the interface in [14], while in [13] he carries out experimental work producing instabilities in stratified flows; Kao investigates the stability of flow down an inclined plane for the case of a stratified fluid system consisting of two layers of viscous fluids with different densities in [9]; and in [5] Drazin considers the stabilising effect of density stratification on the horizontal shear layer between two parallel streams of uniform velocities.
In Appendix A we shall describe the background to both the viscous and inviscid instability theories, much of which comes from the authors mentioned above.

Since the Schlumberger problem involves a turbulent flow the boundary layers will not grow to fill the entire channel. Hence we are left with a base profile that is not fully developed and we therefore propose that, since any shear or boundary layers remain thin, an inviscid approach to the stability problem is more relevant.

Chapter 2

Instability of accelerating flow down an inclined plane

The presence of shear leads to Kelvin-Helmholtz instabilities which are characterised by the growth of waves at an interface between two fluids. In this chapter we shall develop a theory to predict the conditions for the onset of Kelvin-Helmholtz instability of an accelerating flow in a channel. The work here closely follows that of Thorpe [14].

2.1 Problem Formulation

We will consider the stratified flow in an inclined channel as shown in figure 2.1, with axes $\underline{\widehat{X}} = (\widehat{X}, \widehat{Y})$ as shown. The upper fluid has density ρ_+, depth h_+ and pressure p_+, and the lower fluid has density ρ_-, depth h_- and pressure p_-. The two immiscible fluids are separated by an interface at $\widehat{Y} = 0$. We shall analyse the stability of parallel base flows of the form

$$\underline{\widehat{u}}_{\pm}(\widehat{X}, \widehat{Y}, \widehat{t}) = (\widehat{u}_{\pm}(\widehat{Y}, \widehat{t}), 0).$$

The two fluids are initially pumped at velocities $(U_{0\pm}, 0)$ where $U_{0\pm}$ are constants.

The Euler equations that govern the flow of the inviscid fluids can be written in the form

$$\frac{\partial \underline{\widehat{u}}_{\pm}}{\partial \widehat{t}} + \left(\underline{\widehat{u}}_{\pm} . \underline{\widehat{\nabla}}\right) \underline{\widehat{u}}_{\pm} = -\frac{1}{\rho_{\pm}} \underline{\widehat{\nabla}} \widehat{p}_{\pm} + \underline{g}, \tag{2.1}$$

where $\underline{g} = -g(\sin\theta, \cos\theta)$.

To begin with we non-dimensionalise with typical flow speed U_{0+}, fluid depth h_+ and density ρ_+ by writing

$$
\begin{aligned}
\widehat{\underline{u}}_{\pm} &= U_{0+}\underline{u}_{\pm}, \\
\widehat{\underline{X}} &= h_+\underline{X}, \\
\widehat{p}_{\pm} &= \rho_+ U_{0+}^2 p_{\pm}^*, \\
\widehat{t} &= \frac{h_+}{U_{0+}} t.
\end{aligned}
$$

Next we incorporate the hydrostatic pressure directly into pressures $p_{\pm}$ by writing

$$
\begin{aligned}
p_+ &= p_+^* + \frac{h_+ g}{U_{0+}^2}(X\sin\theta + Y\cos\theta), \\
p_- &= p_-^* + \frac{h_+ g}{U_{0+}^2}(X\sin\theta + Y\cos\theta) + Ri(X\sin\theta + Y\cos\theta),
\end{aligned}
$$

where the non-dimensional parameter Ri, the Richardson number, is given by $Ri = \left(\frac{\rho_-}{\rho_+} - 1\right)\frac{gh_+}{U_+^2}$. In this way we obtain the following non-dimensional governing equations

$$
\begin{aligned}
\frac{\partial u_+}{\partial t} &= -\frac{\partial p_+}{\partial X}, && (2.2) \\
0 &= -\frac{\partial p_+}{\partial Y}, && (2.3) \\
\frac{\partial u_-}{\partial t} &= -\frac{1}{r}\frac{\partial p_-}{\partial X}, && (2.4) \\
0 &= -\frac{1}{r}\frac{\partial p_-}{\partial Y}, && (2.5)
\end{aligned}
$$

where the non-dimensional parameters, r, δ, β and Ri, appearing in the problem are defined by $r = \frac{\rho_-}{\rho_+}$, $\delta = \frac{h_-}{h_+}$, $\beta = \frac{U_{0-}}{U_{0+}}$, $Ri = \left(\frac{\rho_-}{\rho_+} - 1\right)\frac{gh_+}{U_{0+}^2}$.

2.1.1 Determination of the base velocities

Equations (2.3) and (2.5) say that $p_{\pm}$ are functions only of X and t. Consequently $\frac{\partial p_{\pm}}{\partial X}$ are functions only of X and t. But, since $u_{\pm}$ are functions of Y and t only, then from (2.2) and (2.4) we deduce that $\frac{\partial p_{\pm}}{\partial X}$ are functions of t only and consequently $u_{\pm}$ are also functions of t only.

The pressure condition at the interface between the fluids gives $\widehat{p}_+ = \widehat{p}_-$ at $Y = 0$ in dimensional variables and so in non-dimensional variables we have

$$
p_+ - p_- + Ri(X\sin\theta + Y\cos\theta) = 0 \quad \text{on} \quad Y = 0. \qquad (2.6)
$$

From this pressure condition (2.6), and the fact the pressure gradients are functions of t only, we see that the difference in the pressure gradients is equal to $Ri\sin\theta$,

$$\frac{\partial p_+}{\partial X} - \frac{\partial p_-}{\partial X} + Ri\sin\theta = 0. \tag{2.7}$$

Integrating (2.2) and (2.4) with respect to t, and using the fact that $u_+ = 1$ and $u_- = \beta$ at $t = 0$, we find that the fluid velocities may be written

$$\begin{aligned} u_+(t) &= 1 - \int_0^t \frac{\partial p_+}{\partial X} dt \qquad (2.8)\\ u_-(t) &= \beta - \frac{1}{r}\int_0^t \frac{\partial p_-}{\partial X} dt. \qquad (2.9) \end{aligned}$$

Now, since fluid must be conserved in the flow, the total volume flux of fluid at time t must equal the volume flux at $t = 0$ and so

$$\int_0^1 u_+(t)dY + \int_{-\delta}^0 u_-(t)dY = 1 + \delta\beta. \tag{2.10}$$

Now (2.7), (2.8), (2.9) and (2.10) give

$$\begin{aligned} \int_0^t \frac{\partial p_+}{\partial X} dt &= -\frac{\delta t}{r+\delta} Ri\sin\theta \qquad (2.11)\\ \int_0^t \frac{\partial p_-}{\partial X} dt &= \frac{rt}{r+\delta} Ri\sin\theta. \qquad (2.12) \end{aligned}$$

And so from (2.8) and (2.9) we obtain the base velocities

$$\begin{aligned} u_+ &= 1 + \frac{\delta t}{r+\delta} Ri\sin\theta, \qquad (2.13)\\ u_- &= \beta - \frac{t}{r+\delta} Ri\sin\theta, \qquad (2.14) \end{aligned}$$

which we will now denote by U_+ and U_- respectively.
We assume the flow is irrotational and therefore define velocity potentials $\Phi_\pm(X, Y, t)$ such that $\underline{u}_\pm = \underline{\nabla}\Phi_\pm$, and as is usual we introduce small disturbances to the base flow by writing

$$\Phi_\pm(X, Y, t) = U_\pm(t)X + \epsilon\phi_\pm(X, Y, t), \tag{2.15}$$

where $\epsilon \ll 1$. Similarly we introduce a small perturbation to the interface by writing $Y = \epsilon s(X, t)$.

2.1.2 Governing Equations

The perturbed velocity potentials $\phi_\pm(X,Y,t)$ satisfy *Laplace's Equation*

$$\underline{\nabla}^2\phi_\pm = 0. \tag{2.16}$$

Equations (2.16) are subject to the boundary conditions of no normal flow at the walls

$$\begin{aligned} \frac{\partial\phi_+}{\partial Y} &= 0 \quad \text{on} \quad Y=1, &(2.17)\\ \frac{\partial\phi_-}{\partial Y} &= 0 \quad \text{on} \quad Y=-\delta, &(2.18)\end{aligned}$$

the kinematic condition at the interface, which says that particles on the interface stay on the interface

$$\left(\frac{\partial}{\partial t} + \underline{u}_\pm.\underline{\nabla}\right)(Y-\epsilon s(X,t)) = 0 \quad \text{at} \quad Y=\epsilon s(X,t), \tag{2.19}$$

and the dimensional pressure condition at the interface, which says that the difference in pressure across the interface must equal the stress induced by surface tension (see [7])

$$\widehat{p}_+ - \widehat{p}_- = -2\gamma\kappa \quad \text{at} \quad Y=\epsilon s(X,t), \tag{2.20}$$

where γ denotes the surface tension between the two fluids. We linearise (2.19), expanding about $Y=0$ and discarding terms of $O(\epsilon^2)$,

$$\frac{\partial\phi_\pm}{\partial Y} = \frac{\partial s}{\partial t} + U_\pm\frac{\partial s}{\partial X} \quad \text{at} \quad Y=0, \tag{2.21}$$

and (2.20) becomes (see Appendix A for details)

$$p_+ - p_- + Ri(X\sin\theta + Y\cos\theta) = \sigma\left(\epsilon\frac{\partial^2 s}{\partial X^2} + O\left(\epsilon^2\right)\right) \quad \text{at} \quad Y=0, \tag{2.22}$$

in non-dimensional variables, where the non-dimensional parameter σ is defined by $\sigma = \frac{\gamma}{\rho_+ h_+ U_{0+}^2}$. The non-dimensional pressures are given by Bernoulli's equation for an unsteady irrotational flow

$$\begin{aligned} p_+ + \frac{\partial\Phi_+}{\partial t} + \frac{1}{2}\left|\underline{\nabla}\Phi_+\right|^2 &= A_+(t), &(2.23)\\ \frac{1}{r}p_- + \frac{\partial\Phi_-}{\partial t} + \frac{1}{2}\left|\underline{\nabla}\Phi_-\right|^2 &= A_-(t), &(2.24)\end{aligned}$$

where $A_\pm$ are arbitrary functions of t and we shall take them to be zero without loss of generality. Now we linearise (2.22), with pressures given by (2.23) and (2.24), and we have to first order

$$r\left(X\frac{\partial U_-}{\partial t}+\frac{1}{2}U_-^2\right)-\left(X\frac{\partial U_+}{\partial t}+\frac{1}{2}U_+^2\right)+RiX\sin\theta=0, \tag{2.25}$$

which, because of the base flows (2.13) and (2.14), implies that

$$r=\frac{1}{\beta^2}. \tag{2.26}$$

And linearising to $O(\epsilon)$ we obtain

$$r\left(\frac{\partial\phi_-}{\partial t}+U_-\frac{\partial\phi_-}{\partial X}\right)-\left(\frac{\partial\phi_+}{\partial t}+U_+\frac{\partial\phi_+}{\partial X}\right)+Ris\cos\theta=\sigma\frac{\partial^2 s}{\partial X^2}. \tag{2.27}$$

Equation (2.26) tells us that given the density stratification and one fluid velocity the other fluid's velocity is fixed by the flow.

2.2 Solution

In this section we shall solve the equations (2.16), (2.17), (2.18), (2.21) and (2.27). We suppose that the disturbance to the interface, $s(X,t)$, takes the form

$$s(X,t)=\eta(t)e^{ikX},$$

where k is the wavenumber (so a general solution can be found by superposition) and that

$$\begin{aligned}\phi_+ &= \psi_+(t)\cosh k(Y-1)e^{ikX}, && (2.28)\\ \phi_- &= \psi_-(t)\cosh k(Y+\delta)e^{ikX}, && (2.29)\end{aligned}$$

to satisfy (2.17) and (2.18), the real parts of the right-hand sides being understood. Then, it may be shown (see Appendix B) that the solution for s, the disturbance to the interface, is given by

$$s(X,t)=\Re\left\{N(\tau)e^{i(kX+ct+dt^2)}\right\}, \tag{2.30}$$

where $\tau = \lambda t$,

$$\lambda = \left[\frac{2kRi\sin\theta(rT_+T_-)^{\frac{1}{2}}(1+\delta)}{(r+\delta)(rT_+ + T_-)}\right]^{\frac{1}{2}}, \tag{2.31}$$

$$c = -k\left[\frac{r\beta T_+ + T_-}{rT_+ + T_-}\right], \tag{2.32}$$

$$d = -\frac{kRi\sin\theta}{2(r+\delta)}\left[\frac{\delta T_- - rT_+}{rT_+ + T_-}\right], \tag{2.33}$$

$$T_+ = \tanh k, \tag{2.34}$$

$$T_- = \tanh k\delta, \tag{2.35}$$

and $\Re$ indicates that the real part is to be taken. $N(\tau)$ satisfies the equation

$$\frac{d^2N}{d\tau^2} - \left\{a + \frac{1}{4}(\tau + b)^2\right\}N = 0, \tag{2.36}$$

where

$$a = \frac{-\left[\sigma k^2 + Ri\cos\theta\right](r+\delta)(T_+T_-)^{\frac{1}{2}}}{2Ri\sin\theta(1+\delta)r^{\frac{1}{2}}} \tag{2.37}$$

$$b = \left[\frac{2k(r+\delta)(rT_+T_-)^{\frac{1}{2}}}{(rT_+ + T_-)Ri\sin\theta(1+\delta)}\right]^{\frac{1}{2}}(1-\beta) \tag{2.38}$$

We note that as $r > 1$ (since otherwise we would be in the trivially unstable case of heavy fluid above light fluid), $\beta < 1$ by (2.26), and so $b > 0$. Now if we put $\hat{\tau} = \tau + b$ then

$$\frac{d^2N}{d\hat{\tau}^2} - \left(a + \frac{1}{4}\hat{\tau}^2\right)N = 0, \tag{2.39}$$

and (2.39) is a standard form of the equation for the parabolic cylinder function - see Abramowitz and Stegun [1]. Two linearly independent solutions, $U(a,\hat{\tau})$, $V(a,\hat{\tau})$ are known. For $a < 0$ as is the case here, U is bounded and oscillatory for all $\hat{\tau} > 0$ and tends to zero as $\hat{\tau}$ tends to infinity, whilst V is oscillatory and bounded for $\hat{\tau} < 2\sqrt{(-a)}$ and increases monotonically for $\hat{\tau} > 2\sqrt{(-a)}$.

So the flow is stable as long as $\hat{\tau} < 2\sqrt{(-a)} \quad \forall k$, i.e. as long as $(\tau + b)^2 < -4a \quad \forall k$. The condition is equivalent to

$$\left[U_+(t) - U_-(t)\right]^2 < \left[\sigma k + \frac{Ri\cos\theta}{k}\right]\left[\frac{rT_+ + T_-}{r}\right] \quad \forall k. \tag{2.40}$$

And so the flow becomes unstable if

$$[U_+(t) - U_-(t)]^2 > \min_k \left\{ \left[\sigma k + \frac{Ri\cos\theta}{k} \right] \left[\frac{rT_+ + T_-}{r} \right] \right\}. \tag{2.41}$$

The first wavenumber to become unstable, k_c, is that value of k that minimises the right-hand side of (2.41). The time, t_c, that the flow becomes unstable to perturbations with this wave-number is the earliest time that the two terms in equation (2.41) become equal.
Therefore we have shown that, for a given fluid stratification and inclination angle, a shear of a certain magnitude can be supported. When the velocity difference becomes large enough the interface becomes unstable.

2.2.1 Deep Fluids

In this section we shall look at the limit $h_- = h_+ \to \infty$. We are forced to re-dimensionalise as when we take the limit $h_+ \to \infty$ there is no representative length scale in the problem. If we now write

$$\begin{aligned} \underline{\widehat{u}}_\pm &= U_{0+}\underline{u}_\pm, \\ \widehat{k} &= \frac{1}{h_+}k, \end{aligned}$$

where the hat variables represent the variables in the dimensional co-ordinate system, and take the limit $h_- = h_+ \to \infty$ then the condition for the flow to be unstable, (2.41), becomes

$$\left[\widehat{U}_+(t) - \widehat{U}_-(t)\right]^2 > \min_{\widehat{k}} \left\{ \left[\gamma\widehat{k} + \frac{(\rho_- - \rho_+)g\cos\theta}{\widehat{k}} \right] \left[\frac{\rho_- + \rho_+}{\rho_+\rho_-} \right] \right\}. \tag{2.42}$$

So for deep fluids (2.42) says that the flow becomes unstable at the earliest time, t_c, that the magnitude of the shear is such that

$$\left[\widehat{U}_+(t) - \widehat{U}_-(t)\right]^2 > 2\left[\gamma(\rho_- - \rho_+)g\cos\theta\right]^{\frac{1}{2}} \left[\frac{\rho_- + \rho_+}{\rho_+\rho_-} \right], \tag{2.43}$$

and the first wavenumber to go unstable is given by

$$k_c = \left[\frac{(\rho_- - \rho_+)g\cos\theta}{\gamma} \right]^{\frac{1}{2}}. \tag{2.44}$$

(2.43) and (2.44) are both in agreement with Kelvin [10], who showed this for the case $\theta = 0$.

2.2.2 Instability in the constriction

Looking at (2.41) it seems possible, since a constriction would accelerate the flow and therefore increase the square of the velocity difference (slip velocity) across the interface, to have a geometry, such as in figure 2.2, where the flow is unstable in the constriction but stable elsewhere.

Now since fluid must be conserved, both above and below the interface, the volume flux must be constant in both fluids and so

$$\begin{aligned} U_{1+}h_{1+} &= U_{2+}h_{2+}, & (2.45)\\ U_{1-}h_{1-} &= U_{2-}h_{2-}. & (2.46)\end{aligned}$$

We shall discuss the special case where the interface is in the centre of the pipe. Then,

$$1 = \delta = \frac{h_{1-}}{h_{1+}} = \frac{h_{2-}}{h_{2+}}. \tag{2.47}$$

We also note that

$$\frac{U_{1+}}{U_{2+}} = \frac{h_{2+}}{h_{1+}} < 1. \tag{2.48}$$

Since we require the flow to be stable before (and after) the constriction, we have

$$\left[U_{1+}(t) - U_{1-}(t)\right]^2 < \min_k \left\{ \left[\sigma_1 k + \frac{Ri_1 \cos\theta}{k}\right] \left[\frac{rT_+ + T_-}{r}\right] \right\}. \tag{2.49}$$

The non-dimensional parameters, σ_1, σ_2, Ri_1, Ri_2, used in this section are defined by

$$\begin{aligned} \sigma_1 &= \frac{\gamma}{\rho_+ h_{1+} U_{1+}^2}, & (2.50)\\ Ri_1 &= \frac{g h_{1+}}{U_{1+}^2}(r-1), & (2.51)\\ \sigma_2 &= \frac{\gamma}{\rho_+ h_{2+} U_{2+}^2}, & (2.52)\\ Ri_2 &= \frac{g h_{2+}}{U_{2+}^2}(r-1). & (2.53)\end{aligned}$$

Now from (2.45), (2.46) and (2.47) we find that the square of the slip velocity in the constriction can be expressed in terms of the square of the slip velocity in the main pipe in the following way

$$\left[U_{2+}(t) - U_{2-}(t)\right]^2 = \left(\frac{h_{1+}}{h_{2+}}\right)^2 \left[U_{1+}(t) - U_{1-}(t)\right]^2, \tag{2.54}$$

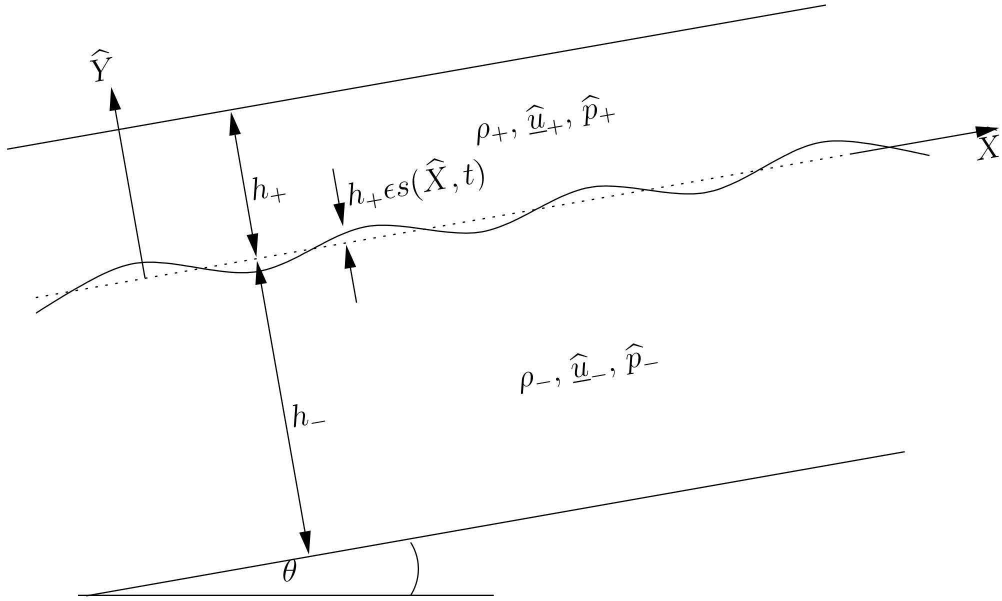

Figure 2.1: Inviscid flow in an inclined channel.

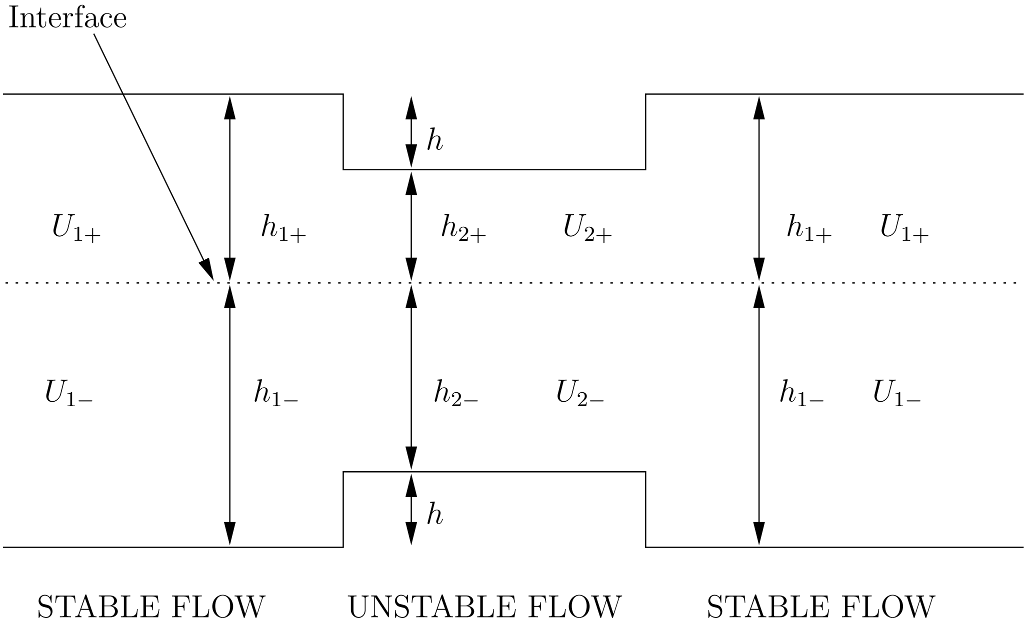

Figure 2.2: Pipe with constriction.

whilst

$$\begin{aligned}
&\min_k \left\{ \left[\sigma_2 k + \frac{Ri_2 \cos\theta}{k} \right] \left[\frac{rT_+ + T_-}{r} \right] \right\} \\
&= \left(\frac{h_{2+}}{h_{1+}} \right) \min_k \left\{ \left[\sigma_1 k + \left(\frac{h_{2+}}{h_{1+}} \right)^2 \frac{Ri_1 \cos\theta}{k} \right] \left[\frac{rT_+ + T_-}{r} \right] \right\} \\
&< \left(\frac{h_{2+}}{h_{1+}} \right) \min_k \left\{ \left[\sigma_1 k + \frac{Ri_1 \cos\theta}{k} \right] \left[\frac{rT_+ + T_-}{r} \right] \right\} \quad \text{by (2.48).}
\end{aligned} \tag{2.55}$$

For a constriction with small enough $\frac{h_{2+}}{h_{1+}}$, by consideration of (2.48), (2.49), (2.54) and (2.55), we can now obtain

$$[U_{2+}(t) - U_{2-}(t)]^2 > \min_k \left\{ \left[\sigma_2 k + \frac{Ri_2 \cos\theta}{k} \right] \left[\frac{rT_+ + T_-}{r} \right] \right\}, \tag{2.56}$$

and so comparing (2.48) and (2.56) we see that it is possible to achieve the situation where the flow is stable in the main pipe but unstable in the constriction. Therefore we have shown that, for the special case $\delta = 1$, the flow can be made unstable by a small enough constriction. Indeed it seems reasonable that a small enough constriction could cause instability even when the interface is not in the middle of the channel.

2.3 Results and Discussion

In this chapter we have derived conditions for the onset of instability in the flow and showed that a small enough constriction can be enough to cause instability in the flow. From figure 2.3 we can see that larger Richardson numbers are able to stabilise larger slip velocities in the horizontal case and in the inclined case larger Richardson numbers are able to postpone the time at which the inevitable instability first occurs. Equation (2.41) also tells us that for small θ the magnitude of the shear that can be supported does not change greatly and this concurs with the experimental evidence obtained by SCR. Figures 2.3 and 2.4 have been produced with typical parameters for oil-water flows in the Venturi flow meter obtained from Schlumberger. These parameters are $\sigma = 2.9 \times 10^{-3}$, $r = 1.25$ and $\delta = 1$, we have also picked the case $\theta = 0$.

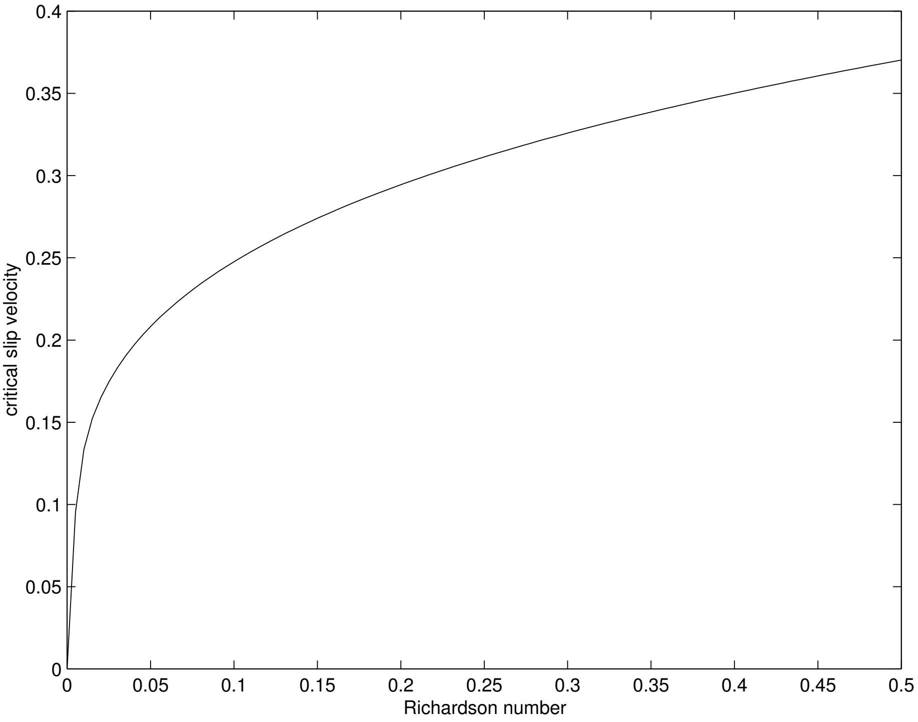

Figure 2.3: Plot showing how the critical slip velocity varies with Richardson number.

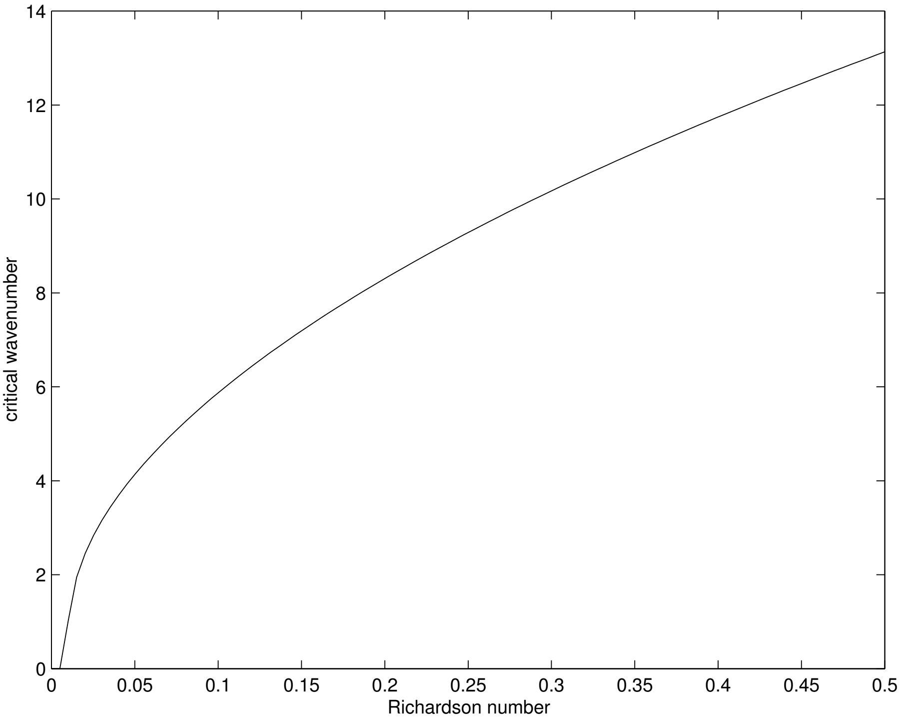

Figure 2.4: Plot showing how the critical wavenumber varies with Richardson number.

Chapter 3

Instability in the Expanding Channel

3.1 Problem Formulation

In this chapter we shall be concerned with the stability of the flow in the diverging section of the Venturi. As such we shall consider the flow in a wedge as in figure 3.1, with axes $\underline{\widehat{X}} = (\widehat{X}, \widehat{Y})$ as shown. Motivated by Jeffrey-Hamel type flow we consider base flows in the physical plane of the form $\underline{\widehat{u}} = (\widehat{u}_r, \widehat{u}_\theta)$, where $\widehat{u}_\theta = 0$ and

$$\widehat{u}_r = \left\{ \begin{array}{ll} \frac{U_+}{r} & Y > 0 \\ & \\ \frac{U_-}{r} & Y < 0. \end{array} \right\}, \tag{3.1}$$

where $U_\pm$ are both constants. The physical plane has co-ordinates $(\widehat{X}, \widehat{Y})$. We let $\widehat{Z} = \widehat{X} + i\widehat{Y}$ and then transform the wedge to a channel by means of the following conformal map

$$\widehat{Z} = e^{\widehat{\zeta}}, \tag{3.2}$$

where $\widehat{\zeta} = \widehat{\xi} + i\widehat{\eta}$. The new geometry is the familiar channel as shown in figure 3.2, with axes $(\widehat{\xi}, \widehat{\eta})$. The Euler equations that govern the flow of the inviscid fluids can be written in the form

$$\frac{\partial \underline{\widehat{u}}}{\partial \widehat{t}} + \left(\underline{\widehat{u}}_\pm . \underline{\widehat{\nabla}} \right) \underline{\widehat{u}}_\pm = -\frac{1}{\rho_\pm} \underline{\widehat{\nabla}} \widehat{p}_\pm + \underline{g}, \tag{3.3}$$

where $\underline{g} = -\underline{\widehat{\nabla}} \left(g e^{\widehat{\xi}} \sin \widehat{\eta} \right)$, and the vector operators are in the new co-ordinate system and in which $\underline{\widehat{\nabla}} = e^{-\widehat{\xi}} \left(\frac{\partial}{\partial \widehat{\xi}}, \frac{\partial}{\partial \widehat{\eta}} \right)$. And so we are now in a very similar position to that

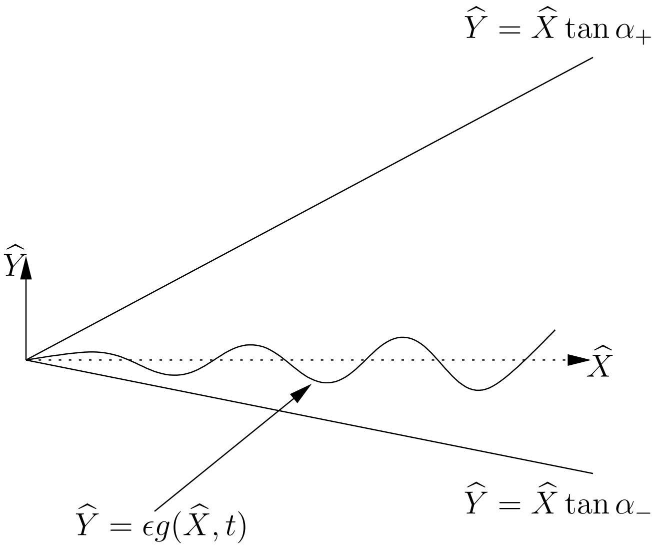

Figure 3.1: The diverging channel.

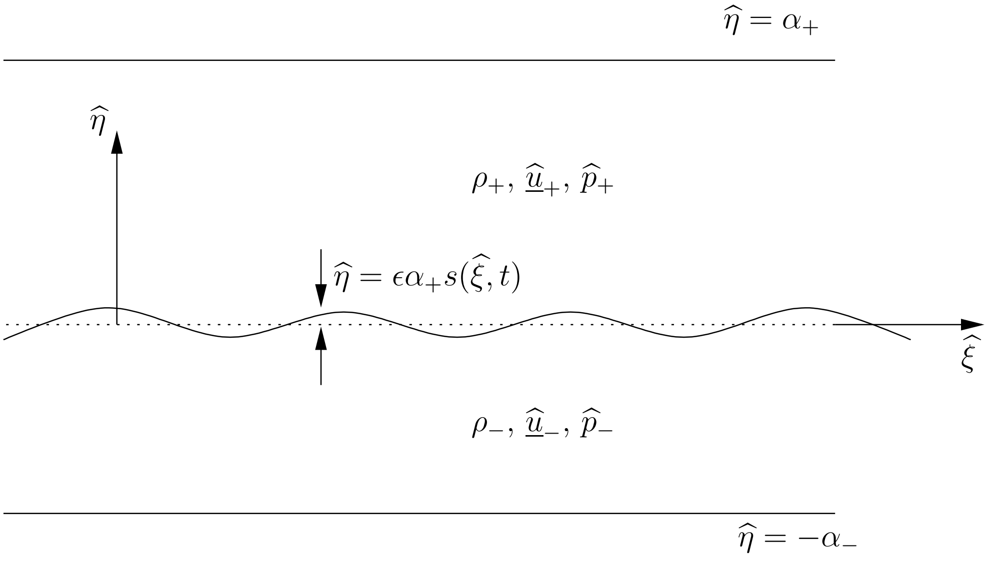

Figure 3.2: The new geometry.

in Chapter 3, with a problem of flow in a channel. Firstly we shall non-dimensionalise by writing

$$\begin{aligned} \widehat{\underline{u}}_{\pm} &= U_+ \underline{u}_{\pm}, \\ \widehat{\zeta} &= \alpha_+ \zeta, \\ \widehat{t} &= \frac{\alpha_+}{U_+} t, \\ \widehat{p}_{\pm} &= \rho_+ U_+^2 p_{\pm}^*, \end{aligned}$$

where the typical fluid flow speed is U_+, density ρ_+ and angle α_+. Now we proceed as in chapter 2 and incorporate the hydrostatic pressures directly into the pressures $p_{\pm}$ by writing

$$\begin{aligned} p_+ &= p_+^* + \frac{\alpha_+ g e^{\alpha_+ \xi}}{U_+^2} \sin(\alpha_+ \eta), \\ p_- &= p_-^* + \frac{\alpha_+ g e^{\alpha_+ \xi}}{U_+^2} \sin(\alpha_+ \eta) + Ri e^{\alpha_+ \xi} \sin(\alpha_+ \eta), \end{aligned}$$

where the non-dimensional parameter Ri, the Richardson number, is given by $Ri = \left(\frac{\rho_-}{\rho_+} - 1\right) \frac{g\alpha_+}{U_+^2}$.

We introduce velocity potentials $\Phi_{\pm}(\xi, \eta, t)$ such that $\underline{u}_{\pm} = \underline{\nabla}\Phi_{\pm}$. As is usual in instability analysis we let the velocity potentials include both the base flow and a perturbation. So,

$$\Phi_+(\xi, \eta, t) = \xi + \epsilon\phi_+(\xi, \eta, t), \tag{3.4}$$

$$\Phi_-(\xi, \eta, t) = \xi\beta + \epsilon\phi_-(\xi, \eta, t), \tag{3.5}$$

where the non-dimensional parameter, β, that appears above is defined by $\beta = \frac{U_-}{U_+}$.

We shall investigate the existence of normal mode solutions of the form

$$\phi_{\pm}(\xi, \eta, t) = f_{\pm}(\eta) e^{i(k\xi - \omega t)}. \tag{3.6}$$

Finally we shall let the perturbed interface be given by $\eta = \epsilon s(\xi, t)$, where $\epsilon \ll 1$.

3.1.1 Governing Equations

As in chapter 2, the perturbed velocity potentials $\phi(\xi, \eta, t)$ satisfy *Laplace's Equation*

$$\underline{\nabla}^2 \phi_{\pm} = 0, \tag{3.7}$$

subject to the boundary conditions of no normal flow at the walls, the kinematic condition at the interface $\eta = \epsilon s(\xi, t)$ and the condition that the difference in pressure across the interface must equal the stress induced by surface tension. These conditions are directly analogous to equations (2.17), (2.18), (2.19) and (2.20). The non-dimensional pressures are given by Bernoulli

$$
\begin{aligned}
p_+ + \frac{\partial \Phi_+}{\partial t} + \frac{1}{2}\left|\underline{\nabla}\Phi_+\right|^2 = A_+(t), \qquad (3.8)\\
\frac{1}{r}p_- + \frac{\partial \Phi_-}{\partial t} + \frac{1}{2}\left|\underline{\nabla}\Phi_-\right|^2 = A_-(t), \qquad (3.9)
\end{aligned}
$$

where $A_\pm$ are arbitrary functions of t and we shall take them to be zero without loss of generality. And so as before we linearise these boundary conditions, ignoring terms of $O(\epsilon^2)$ and find

$$
\begin{aligned}
\frac{\partial \phi_+}{\partial \eta} &= 0 \quad \text{at} \quad \eta = 0, && (3.10)\\
\frac{\partial \phi_-}{\partial \eta} &= 0 \quad \text{at} \quad \eta = 0, && (3.11)\\
e^{-2\alpha_+\xi}\frac{\partial \phi_+}{\partial \eta} &= \frac{\partial s}{\partial t} + e^{-2\alpha_+\xi}\frac{\partial s}{\partial \xi} \quad \text{at} \quad \eta = 0, && (3.12)\\
e^{-2\alpha_+\xi}\frac{\partial \phi_-}{\partial \eta} &= \frac{\partial s}{\partial t} + \beta e^{-2\alpha_+\xi}\frac{\partial s}{\partial \xi} \quad \text{at} \quad \eta = 0, && (3.13)
\end{aligned}
$$

$$
\begin{aligned}
r\left(\frac{\partial \phi_-}{\partial t} + \beta e^{-2\alpha_+\xi}\frac{\partial \phi_-}{\partial \xi}\right) - \left(\frac{\partial \phi_+}{\partial t} + e^{-2\alpha_+\xi}\frac{\partial \phi_+}{\partial \xi}\right) + Rie^{\alpha_+\xi}\alpha_+ s \\
= \sigma e^{-\alpha_+\xi}\left(\alpha_+\frac{\partial s}{\partial \xi} + \frac{\partial^2 s}{\partial \xi^2})\right) \quad \text{at} \quad \eta = 0. \qquad (3.14)
\end{aligned}
$$

Also as in section 2.1 we find that the pressure condition yields, to first order,

$$
r = \frac{1}{\beta^2}, \qquad (3.15)
$$

which tells us that given a density stratification and one fluid velocity the other fluid velocity is fixed by the flow. We shall investigate small wave disturbances to the interface of the form

$$
s(\xi, t) = Ae^{i(k\xi - \omega t)}, \qquad (3.16)
$$

and in order for $\phi_\pm(\xi, \eta, t)$ to satisfy Laplace's equation (3.7) and the no flow boundary conditions (3.10) -(3.11) we take perturbed velocity potentials of the form

$$
\begin{aligned}
\phi_+(\xi, \eta, t) &= B_+ \cosh k(\eta - 1)e^{i(k\xi - \omega t)}, && (3.17)\\
\phi_-(\xi, \eta, t) &= B_- \cosh k(\eta + \delta)e^{i(k\xi - \omega t)}. && (3.18)
\end{aligned}
$$

Then from the linearised form of the kinematic condition, (3.12) and (3.13), we see that

$$
\begin{aligned}
-ke^{-2\alpha_+\xi}B_+\sinh k &= -Ai\left(\omega - ke^{-2\alpha_+\xi}\right), &(3.19)\\
ke^{-2\alpha_+\xi}B_-\sinh k\delta &= -Ai\left(\omega - \beta ke^{-2\alpha_+\xi}\right). &(3.20)
\end{aligned}
$$

Whilst the linearised pressure condition, (3.14), becomes

$$
\begin{aligned}
r\left(-i\omega + \beta ike^{-2\alpha_+\xi}\right)B_-\cosh k\delta - \left(-i\omega + ike^{-2\alpha_+\xi}\right)B_+\cosh k \\
+ Rie^{\alpha_+\xi}\alpha_+A = \sigma e^{-\alpha_+\xi}\left(\alpha_+ik - k^2\right)A, \qquad (3.21)
\end{aligned}
$$

where the non-dimensional parameter σ is defined by $\sigma = \frac{\gamma}{\rho_+U_+^2\alpha_+}$. Now eliminating A, $B_\pm$ from (3.19) - (3.21) we find that we obtain the dispersion relation

$$
\begin{aligned}
r\left(\omega - \beta ke^{-2\alpha_+\xi}\right)^2\coth k\delta + \left(\omega - ke^{-2\alpha_+\xi}\right)^2\coth k - \alpha_+Rike^{-\alpha_+\xi} \\
-\sigma e^{-3\alpha_+\xi}\left(k^3 - \alpha_+ik^2\right) = 0, \qquad (3.22)
\end{aligned}
$$

and we see that ω, k must be functions of ξ which at the outset we assumed they were not. This leads us to examine whether separable solutions exist.

3.1.2 Separable Solutions

We shall seek separable solutions for the perturbed velocity potentials $\phi_\pm(\xi,\eta,t)$, and the disturbance to the interface $s(\xi,t)$, of the form

$$
\begin{aligned}
\phi_\pm(\xi,\eta,t) &= f_\pm(\eta)m_\pm(\xi)n_\pm(t), &(3.23)\\
s(\xi,t) &= M(\xi)N(t). &(3.24)
\end{aligned}
$$

Then *Laplace's equation* (3.7), along with the boundary conditions of no normal flow at the walls (3.8), (3.9) and the fact that the velocity potentials must be bounded functions of ξ as the flow domain is unbounded in ξ, tells us that

$$
\begin{aligned}
m_\pm(\xi) &= A_\pm\cos k(\xi - C_\pm), &(3.25)\\
f_+(\eta) &= B_+\cosh k(\eta - 1), &(3.26)\\
f_-(\eta) &= B_-\cosh k(\eta + \delta), &(3.27)
\end{aligned}
$$

where $A_\pm$, $B_\pm$, $C_\pm$ are all constants and $k \in \mathbb{R}$.
Now substituting (3.23) and (3.24) into the linearised kinematic conditions (3.12), (3.13), we find that

$$
\begin{aligned}
e^{-2\alpha_+\xi}f_+'m_+n_+ &= M\dot{N} + e^{-2\alpha_+\xi}M'N \quad \text{at} \quad \eta = 0, &(3.28)\\
e^{-2\alpha_+\xi}f_-'m_-n_- &= M\dot{N} + \beta e^{-2\alpha_+\xi}M'N \quad \text{at} \quad \eta = 0, &(3.29)
\end{aligned}
$$

And so, balancing functions of ξ in the terms on the right hand sides of (3.28), (3.29) we find that $M \propto \exp\left(e^{2\alpha+\xi}\right)$ and then it is clear that we are unable to balance the right and left-hand sides in terms of ξ as $m_{\pm} = A_{\pm} \cos k(\xi - C_{\pm})$.

Therefore we can conclude that no unsteady separable solutions exist to the problem of flow in a wedge.

Chapter 4

Conclusions and Future Work

In this final chapter we briefly summarise the main results and discuss those extensions on which we believe further work would be particularly fruitful.

4.1 Summary of Work

This dissertation was motivated by the need to model the stability of oil-water flows through a Venturi flow meter. In chapter 1 we introduced the problem and its relevance to science and industry. Possible instability mechanisms were suggested with reference to direct numerical simulations of the flow. In chapter 2 we formulated the model for the accelerating flow in an inclined channel and derived conditions for the onset of instability in the flow. From these conditions we showed that the critical slip velocity is affected by the Richardson number but does not depend strongly on small angles of inclination. We also showed that a small enough constriction could cause a stable stratified flow to become unstable in the constriction. In chapter 3 we looked at the stability problem of stratified radial flow in a wedge and showed that no unsteady separable solution to the perturbed problem exists.

4.2 Future Work

Whilst this dissertation has examined some aspects of the original problem of stratified flow through a Venturi flow meter, there are others which are still to be investigated.

The work we have carried out in chapter 2, deriving conditions for the onset of instability in accelerating stratified flow in an inclined channel, predicts that whatever the value of the parameters in the problem and therefore the critical velocity slip given by (2.41), the flow will always become unstable at some time t. This is because we have not attempted to model the turbulent diffusion processes in the real flow and as a result our model exhibits no steady state in the inclined cases. These statistically steady states, in which turbulent diffusion balances gravitational acceleration, are assumed to exist in practice and therefore it seems at least plausible that it is possible to set up indefinitely stable inclined flows, if the slip velocity is held below the critical level as a result of the turbulent diffusion processes. Work should be undertaken to model this turbulent diffusion process and the stability of turbulent and non-parallel flows in general.

In chapter 3 we analysed the stability characteristics of radial stratified flow in a wedge and showed that there were no unsteady separable solutions to the perturbed problem. More analysis could be carried out in this problem looking for non-separable solutions.

In chapter 1 we suggested four possible instability mechanisms and so far we have only looked at two of these. As suggested in the introduction the flow may be stable with respect to the backward-facing step but unstable with respect to the forward-facing step. Numerical simulations also indicate that the remaining possible mechanism, that of shear layers in the flow caused by boundary layer separation in the expanding channel, is a strong candidate for causing the instability seen in the Venturi flow meter. These shear layers do not coincide with the interface and the Kelvin-Helmholtz instability of this shear layer may lead to perturbations in the flow that drive the instability and mixing at the interface. It would seem that much useful work could be undertaken in analysing these mechanisms.

4.3 Final Comments

The possible extensions to this work, outlined above, would aid understanding of the problem of instability of stratified two-fluid flow in a Venturi flow meter. In this dissertation, however, we have come some way to understanding the problem, particularly with the possible instability caused by the constriction and the conditions derived for the onset on instability in an inclined channel involving Richardson number and critical slip velocity.

Appendix A

A Background to Viscous and Inviscid Stability Theory

In this appendix we shall describe the background to both the viscous and inviscid instability theories much of which comes from the authors mentioned in section 1.1.

A.1 Inviscid Stability Analysis

In this section we consider the flow as set out in figure A.1, with axes $\underline{\widehat{X}} = (\widehat{X}, \widehat{Y})$ as shown. The two immiscible fluids are separated by an interface at $\widehat{Y} = 0$. The stability of parallel base flows of the form

$$\underline{\widehat{u}}_{\pm} = (\widehat{U}_{\pm}(\widehat{Y}), 0)$$

will be analysed. The upper fluid has density ρ_+ and depth h_+, and the lower fluid has density ρ_- and depth h_-.
The Euler equations that govern the flow of an inviscid fluid can be written in the form

$$\begin{aligned}
\frac{\partial \underline{\widehat{u}}_+}{\partial \widehat{t}} + \left(\underline{\widehat{u}}_+ . \underline{\widehat{\nabla}}\right) \underline{\widehat{u}}_+ &= -\frac{1}{\rho_+} \underline{\widehat{\nabla}} p_+ - g \underline{\widehat{\nabla}} \widehat{Y}, && \text{(A.1)}\\
\underline{\widehat{\nabla}} . \underline{\widehat{u}}_+ &= 0, && \text{(A.2)}\\
\frac{\partial \underline{\widehat{u}}_-}{\partial \widehat{t}} + \left(\underline{\widehat{u}}_- . \underline{\widehat{\nabla}}\right) \underline{\widehat{u}}_- &= -\frac{1}{\rho_-} \underline{\widehat{\nabla}} \widehat{p}_- - g \underline{\widehat{\nabla}} \widehat{Y}, && \text{(A.3)}\\
\underline{\widehat{\nabla}} . \underline{\widehat{u}}_- &= 0. && \text{(A.4)}
\end{aligned}$$

where $\underline{\widehat{u}}_{\pm}$ denote the velocity in the upper and lower layers respectively.

To begin with we non-dimensionalise equations (A.1) - (A.4) by writing

$$\begin{aligned}
\widehat{\underline{u}}_\pm &= U_0 \underline{u}_\pm, \\
\widehat{\underline{X}} &= h_+ \underline{X}, \\
\widehat{p}_\pm &= \rho_+ U_0^2 p^*_\pm, \\
\widehat{t} &= \frac{h_+}{U_0} t,
\end{aligned}$$

where U_0 is the typical flow speed, h_+ the typical fluid depth and ρ_+ the typical fluid density. Next we incorporate the hydrostatic pressures directly into the pressures $p_\pm$ by writing

$$\begin{aligned}
p_+ &= p^*_+ + \frac{h_+ g}{U_0^2} Y, \\
p_- &= p^*_- + \frac{\rho_-}{\rho_+} \frac{h_+ g}{U_0^2} Y.
\end{aligned}$$

In this way we obtain the following non-dimensional governing equations

$$\frac{\partial \underline{u}_+}{\partial t} + \left(\underline{u}_+ . \underline{\nabla}\right) \underline{u}_+ = -\underline{\nabla} p_+, \tag{A.5}$$

$$\frac{\partial \underline{u}_-}{\partial t} + \left(\underline{u}_- . \underline{\nabla}\right) \underline{u}_- = -\frac{1}{r} \underline{\nabla} p_-, \tag{A.6}$$

where the non-dimensional parameter r, the density ratio, is given by $r = \frac{\rho_-}{\rho_+}$.
We now make small perturbations to the base flow and so we write

$$u_\pm = U_\pm + \epsilon u'_\pm, \tag{A.7}$$

$$v_\pm = \epsilon v'_\pm, \tag{A.8}$$

$$p_\pm = P_\pm + \epsilon p'_\pm, \tag{A.9}$$

where P is the dimensionless pressure for the base flow, $\underline{u}_\pm = (u_\pm, v_\pm)$ and $\epsilon \ll 1$. The perturbed interface is given by $Y = \epsilon s(X, t)$.
The equation of continuity allows us to introduce stream functions $\psi_\pm$ for the perturbed flow. So we define $u'_\pm = \psi_{\pm Y}$, $v'_\pm = -\psi_{\pm X}$.

Substitution of (A.7), (A.8), (A.9) into (A.5) and (A.6) gives the linearised equations governing the disturbance motion,

$$\begin{aligned}
\frac{\partial u'_+}{\partial t} + U_+ \frac{\partial u'_+}{\partial X} + v'_+ \frac{\partial U_+}{\partial Y} &= -\frac{\partial p'_+}{\partial X}, && \text{(A.10)}\\
\frac{\partial v'_+}{\partial t} + U_+ \frac{\partial v'_+}{\partial X} &= -\frac{\partial p'_+}{\partial Y}. && \text{(A.11)}\\
\frac{\partial u'_-}{\partial t} + U_- \frac{\partial u'_-}{\partial X} + v'_- \frac{\partial U_-}{\partial Y} &= -\frac{1}{r}\frac{\partial p'_-}{\partial X}, && \text{(A.12)}\\
\frac{\partial v'_-}{\partial t} + U_- \frac{\partial v'_-}{\partial X} &= -\frac{1}{r}\frac{\partial p'_-}{\partial Y}. && \text{(A.13)}
\end{aligned}$$

We explore sinusoidal perturbations of the form

$$\psi_\pm = F_\pm(Y)\exp[i(kx - \omega t)], \qquad \text{(A.14)}$$

$$p'_\pm = G_\pm(Y)\exp[i(kx - \omega t)], \qquad \text{(A.15)}$$

where $\omega = \omega_r + i\omega_i$, with $\omega_i < 0$ representing damped disturbances and $\omega_i > 0$ representing growing disturbances. Substituting (A.14) and (A.15) into (A.10), (A.11), (A.12) and (A.13), and using the stream functions, we find

$$\begin{aligned}
-i\omega F'_+ + ikU_+F'_+ - ikU'_+F_+ &= ikG_+, && \text{(A.16)}\\
-k\omega F_+ + k^2U_+F_+ &= -G'_+, && \text{(A.17)}\\
-i\omega F'_- + ikU_-F'_- - ikU'_-F_- &= \frac{ik}{r}G_-, && \text{(A.18)}\\
-k\omega F_- + k^2U_-F_- &= -\frac{1}{r}G'_-, && \text{(A.19)}
\end{aligned}$$

where $'$ denotes differentiation with respect to Y. Eliminating F and G produces the *Rayleigh Stability equations*

$$F''_\pm - \left[\frac{U''_\pm}{(U_\pm - c)} + k^2\right]F_\pm = 0, \qquad \text{(A.20)}$$

where the wave-speed c is given by $c = \frac{\omega}{k}$.

The above equations must be solved subject to the boundary conditions of no normal flow at the walls and so

$$F_+(1) = 0, \qquad \text{(A.21)}$$

$$F_-(-\delta) = 0, \qquad \text{(A.22)}$$

where the non-dimensional parameter δ, the ratio of fluid depths, is defined to be $\delta = \frac{h_-}{h_+}$.

At the interface, $Y = \epsilon s(X,t)$, we have the kinematic condition that says that particles on the interface stay on the interface and so

$$\left(\frac{\partial}{\partial t} + \underline{u}_{\pm}.\underline{\nabla}\right)(Y - \epsilon s) = 0 \quad \text{at} \quad Y = \epsilon s(X,t). \tag{A.23}$$

Linearising (A.23), by expanding about $Y = 0$ and discarding terms of $O(\epsilon^2)$ we obtain

$$\frac{\partial s}{\partial t} + U_{\pm}\frac{\partial s}{\partial X} = -\frac{\partial \psi_{\pm}}{\partial X} \quad \text{at} \quad Y = 0. \tag{A.24}$$

The two Rayleigh equations are coupled via the standard pressure condition, that says that the difference in pressures across the interface must equal the stress induced by surface tension (see [7]), and so in dimensional variables

$$\widehat{p}_{+} - \widehat{p}_{-} = -2\gamma\kappa, \tag{A.25}$$

applied at the interface, $Y = 0$, where γ denotes the surface tension between the two fluids and the curvature κ is given by the equation

$$2\kappa = -\underline{\nabla}.\left(\frac{\underline{\nabla}(Y - \epsilon s)}{|\underline{\nabla}(Y - \epsilon s)|}\right) = -\frac{\epsilon}{h_{+}}\frac{\partial^2 s}{\partial X^2} + O(\epsilon^2) \tag{A.26}$$

which is valid provided surface gradients are not too large. And so (A.25) becomes

$$p_{+} - p_{-} = \epsilon\sigma\frac{\partial^2 s}{\partial X^2} + O(\epsilon^2), \tag{A.27}$$

in non-dimensional variables where the non-dimensional parameter σ is defined by $\sigma = \frac{\gamma}{\rho_{+}h_{+}U_0^2}$.

Now using the identity $\left(\underline{u}_{\pm}.\underline{\nabla}\right)\underline{u}_{\pm} = \left(\underline{\nabla} \wedge \underline{u}_{\pm}\right) \wedge \underline{u}_{\pm} + \underline{\nabla}\left(\frac{1}{2}\underline{u}_{\pm}^2\right)$, we can rewrite (A.5) and (A.6) to get the non-dimensional Bernoulli equations

$$\frac{\partial \underline{u}_{+}}{\partial t} + \left(\underline{\nabla} \wedge \underline{u}_{+}\right) \wedge \underline{u}_{+} = -\nabla\left(p_{+} + \frac{1}{2}\underline{u}_{+}^2\right), \tag{A.28}$$

$$\frac{\partial \underline{u}_{-}}{\partial t} + \left(\underline{\nabla} \wedge \underline{u}_{-}\right) \wedge \underline{u}_{-} = -\nabla\left(\frac{1}{r}p_{-} + \frac{1}{2}\underline{u}_{-}^2\right). \tag{A.29}$$

If the assumption is made that the flow is irrotational, then $\underline{\nabla} \wedge \underline{u}_{\pm} = 0$ and there exist velocity potentials $\Phi_{\pm}$ such that $\underline{u}_{\pm} = \underline{\nabla}\Phi_{\pm}$. In this case (A.28) and (A.29) simplify to give

$$\frac{\partial \Phi_{+}}{\partial t} + p_{+} + \frac{1}{2}\underline{u}_{+}^2 = A_{+}(t), \tag{A.30}$$

$$\frac{\partial \Phi_{-}}{\partial t} + \frac{1}{r}p_{-} + \frac{1}{2}\underline{u}_{-}^2 = A_{-}(t), \tag{A.31}$$

where $A_{\pm}$ are arbitrary functions of t and may be taken to be zero without loss of generality.

We can now work with the velocity potential and so the pressure condition at the interface becomes more manageable.

A.2 Stability of two-fluid viscous stratified flow down an inclined plane

In this section we follow the work of Yih [15] and Kao [9] on the linear stability of parallel channel flow of two superposed fluids of different density and viscosity. The two immiscible fluids are separated by an interface at $\widehat{Y} = 0$. With co-ordinate axes, $\underline{\widehat{X}} = (\widehat{X}, \widehat{Y})$, as in figure A.2, we shall consider the stability of perturbations to base flows of the form

$$\underline{\widehat{u}}_{\pm} = (\widehat{U}_{\pm}(\widehat{Y}), 0).$$

So our base flows are flows parallel to the X axis. The upper fluid has density ρ_+, viscosity μ_+ and depth h_+; and the lower fluid has density ρ_-, viscosity μ_- and depth h_-. The channel is also inclined at an angle θ to the horizontal.

Firstly we non-dimensionalise, as in section A.1, with typical fluid flow speed U_0, depth h_+ and density ρ_+. Again we incorporate the hydrostatic pressures directly into the non-dimensional pressures $p_{\pm}$ by writing

$$\begin{aligned} p_+ &= p_+^* + \frac{h_+ g}{U_0^2}(X\sin\theta + Y\cos\theta), \\ p_- &= p_-^* + \frac{h_+ g}{U_0^2}(X\sin\theta + Y\cos\theta) + Ri(X\sin\theta + Y\cos\theta), \end{aligned}$$

where the non-dimensional parameter Ri, the Richardson number, is defined by $Ri = \left(\frac{\rho_-}{\rho_+} - 1\right)\frac{g h_+}{U_0^2}$. In this way we obtain the non-dimensional Navier-Stokes equations that govern the flow of the viscous fluids which can be written in the form

$$\frac{\partial \underline{u}_+}{\partial t} + \left(\underline{u}_+ . \underline{\nabla}\right)\underline{u}_+ = -\underline{\nabla} p_+ + \frac{1}{Re_+}\nabla^2 \underline{u}_+, \tag{A.32}$$

$$\frac{\partial \underline{u}_-}{\partial t} + \left(\underline{u}_- . \underline{\nabla}\right)\underline{u}_- = \frac{1}{r}\left[-\underline{\nabla} p_- + \frac{m}{Re_+}\nabla^2 \underline{u}_-\right], \tag{A.33}$$

where the non-dimensional parameters, r, m and Re_+, appearing above are defined by $r = \frac{\rho_-}{\rho_+}$, $m = \frac{\mu_-}{\mu_+}$ and $Re_+ = \frac{\rho_+ U_0 h_+}{\mu_+}$.

As is usual we now resolve the motion into the primary motion and a perturbation.

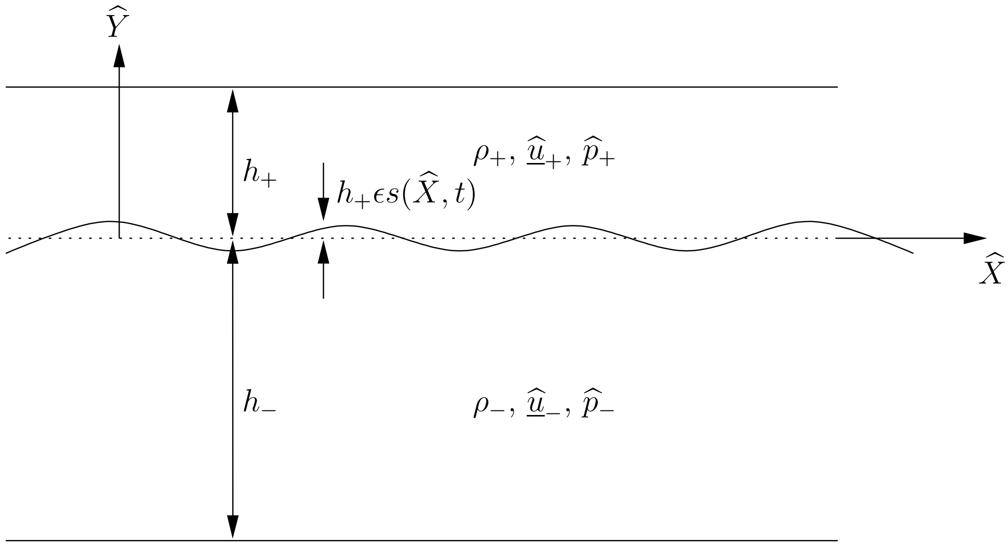

Figure A.1: Inviscid Channel flow.

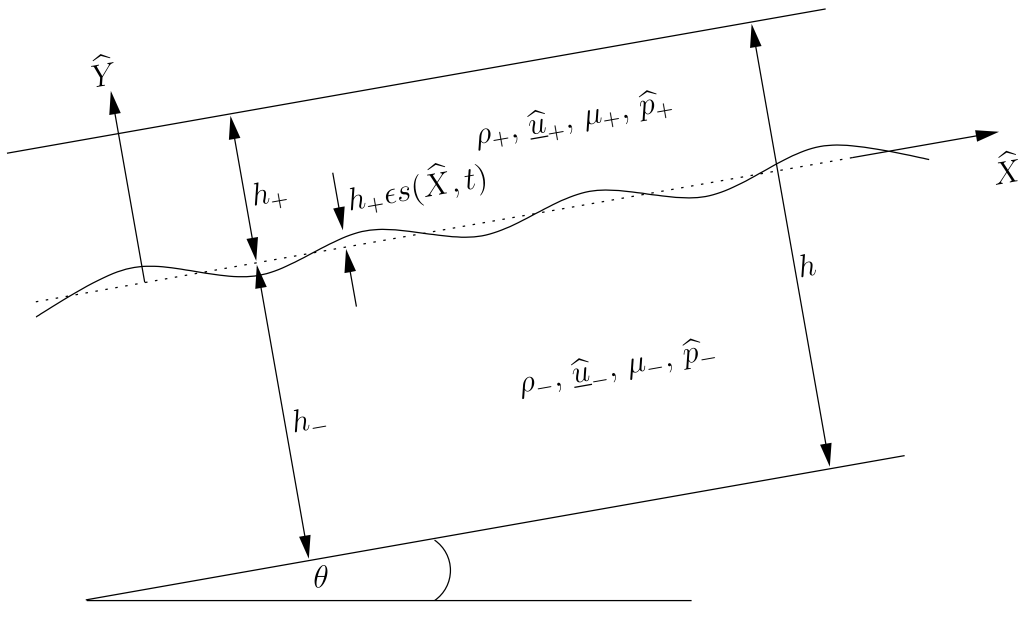

Figure A.2: Viscous flow in an inclined channel.

So we write

$$\begin{aligned} u_\pm &= U_\pm + \epsilon u'_\pm, && \text{(A.34)}\\ v_\pm &= \epsilon v'_\pm, && \text{(A.35)}\\ p_\pm &= P_\pm + \epsilon p'_\pm, && \text{(A.36)} \end{aligned}$$

where P is the dimensionless pressure for the base flow, $\underline{u}_\pm = (u_\pm, v_\pm)$, and $\epsilon \ll 1$. The perturbed interface is given by $Y = \epsilon s(X,t)$. The equation of continuity now allows us to introduce stream functions $\psi_\pm$ for the perturbed flow and so we write $u'_\pm = \psi_{\pm Y}$, $v'_\pm = -\psi_{\pm X}$.

We now assume that the perturbations take a sinusoidal form and write

$$\begin{aligned} \left(\psi_+, p'_+\right) &= (\phi(Y), f(Y))\exp[i(kx-\omega t)], && \text{(A.37)}\\ \left(\psi_-, p'_-\right) &= (\chi(Y), g(Y))\exp[i(kx-\omega t)], && \text{(A.38)} \end{aligned}$$

where $\omega = \omega_r + i\omega_i$ with $\omega_i < 0$ representing damped disturbances and $\omega_i > 0$ representing growing disturbances.

Substituting (A.34), (A.35), (A.36) into (A.32) and (A.33), gives the linearised equations governing the disturbance motion (analogous to (A.10) - (A.13)) and then using (A.37) and (A.38) we have

$$\begin{aligned} -i\omega\phi' + ikU_+\phi' - ik\phi U'_+ &= -ikf + \frac{1}{Re_+}\left[-k^2\phi' + \phi'''\right], && \text{(A.39)}\\ -k\omega\phi + U_+k^2\phi &= -f' + \frac{1}{Re_+}\left[ik^3\phi - ik\phi''\right], && \text{(A.40)}\\ -i\omega\chi' + ikU_-\chi' - ik\chi U'_- &= -ikg + \frac{m}{rRe_+}\left[-k^2\chi' + \chi'''\right], && \text{(A.41)}\\ -k\omega\chi + U_-k^2\chi &= -g' + \frac{m}{rRe_+}\left[ik^3\chi - ik\chi''\right], && \text{(A.42)} \end{aligned}$$

where $'$ denotes differentiation with respect to Y. Eliminating f and g produces the *Orr-Sommerfeld equations*

$$\begin{aligned} \phi'''' - 2k^2\phi'' + k^4\phi &= iRe_+\left[(U_+k-\omega)(\phi''-k^2\phi) - U''_+k\phi\right], && \text{(A.43)}\\ \chi'''' - 2k^2\chi'' + k^4\chi &= i\frac{Re_+r}{m}\left[(U_-k-\omega)(\chi''-k^2\chi) - U''_-k\chi\right]. && \text{(A.44)} \end{aligned}$$

The above two equations must be solved subject to eight boundary conditions, two at the top of the channel, two at the bottom and four at the interface. The boundary conditions at the interface provide the coupling between the Orr-Sommerfeld equations.

There must be no slip at the walls and so the boundary conditions at the top and the bottom of the channel are

$$\phi(1) = 0, \tag{A.45}$$

$$\phi'(1) = 0, \tag{A.46}$$

$$\chi(-\delta) = 0, \tag{A.47}$$

$$\chi'(-\delta) = 0, \tag{A.48}$$

where $\delta = \frac{h_-}{h_+}$. The boundary conditions at the interface ensure continuity of velocity and of stress. Continuity of v' implies

$$\phi(0) = \chi(0). \tag{A.49}$$

Continuity of u must be applied at the interface $Y = \epsilon s(X,t)$. But the kinematic condition that says that particles on the interface stay on the interface and linearising as in section A.1 we find

$$\left(\frac{\partial}{\partial t} + U_+ \frac{\partial}{\partial X}\right) s(X,t) = -\frac{\partial \psi_+}{\partial X} = -ik\phi \exp[i(kX - \omega t)] \quad \text{at} \quad Y = 0. \tag{A.50}$$

Therefore letting $s(X,t) = A\exp[i(kX - \omega t)]$, we find from (A.50) that the interface takes the form

$$s(X,t) = \frac{k\phi(0)}{c'} \exp[i(kX - \omega t)],$$

where $c' = \frac{\omega}{k} - U_+(0)$.
Continuity in u then demands $u_+ = u_-$ at $Y = \epsilon s(X,t)$ and so expanding in terms of ϵ we find that $U_+(0) = U_-(0)$ to first order and to order ϵ

$$U_+'(0)\frac{\phi(0)}{c'} + \phi'(0) = U_-'(0)\frac{\chi(0)}{c'} + \chi'(0). \tag{A.51}$$

The dimensional stress tensors, $\widehat{\sigma}_{ij}^{\pm}$, is (see [12])

$$\widehat{\sigma}_{ij}^{\pm} \;=\; -\widehat{p}_{\pm}\delta_{ij} + \mu\left[\frac{\partial \widehat{u}_{\pm i}}{\partial \widehat{X}_j} + \frac{\partial \widehat{u}_{\pm j}}{\partial \widehat{X}_i}\right]$$

and so writing the stress tensors, $\widehat{\sigma}_{ij}^{\pm}$, in terms of non-dimensional variables we find

$$\begin{aligned}\hat{\sigma}_{ij}^{+} &= \left[\rho_+ g h_+ (X\sin\theta + Y\cos\theta) - \rho_+ U_0^2 p_+\right]\delta_{ij} + \frac{\mu_+ U_0}{h_+}\left[\frac{\partial u_{+i}}{\partial X_j} + \frac{\partial u_{+j}}{\partial X_i}\right],\\ \hat{\sigma}_{ij}^{-} &= \left[(\rho_+ g h_+ + \rho_+ U_0^2 Ri)(X\sin\theta + Y\cos\theta) - \rho_+ U_0^2 p_-\right]\delta_{ij}\\ &\quad + \frac{\mu_- U_0}{h_+}\left[\frac{\partial u_{-i}}{\partial X_j} + \frac{\partial u_{-j}}{\partial X_i}\right].\end{aligned} \tag{A.52}$$

Now continuity of shear stress demands $\widehat{\sigma}_{12}^{+} = \widehat{\sigma}_{12}^{-}$ at $Y = \epsilon s(X,t)$ and so to leading order

$$U_+'(0) = mU_-'(0),$$

and to order ϵ

$$\frac{\phi(0)}{c'}U_+''(0) + \phi''(0) + k^2\phi(0) = m\left[\frac{\chi(0)}{c'}U_-''(0) + \chi''(0) + k^2\chi(0)\right]. \tag{A.53}$$

Now the normal stress condition is slightly more complicated. The difference of the normal stresses must be balanced by the normal stress induced by surface tension (see [7]). That is

$$\left(\widehat{\sigma}_{22}^{+} - \widehat{\sigma}_{22}^{-}\right) = -\frac{\gamma\epsilon}{h_+}\frac{\partial^2 s}{\partial X^2} + O(\epsilon^2) \quad \text{at} \quad Y = \epsilon s(X,t) \quad \text{(c.f. (A.27))}. \tag{A.54}$$

To leading order we find that

$$P_+(0) - P_-(0) + RiX\sin\theta = 0$$

and to order ϵ

$$\begin{aligned}m(\chi''' - k^2\chi') &+ ikrRe_+(c'\chi + U_-'\chi) - (\phi''' - k^2\phi') - ikRe_+(c'\phi' + U_+'\phi)\\ &+ 2ik(-ik\phi' + mik\chi') = ikRe_+(Ri\cos\theta + k^2T)\frac{\phi}{c'}\end{aligned} \tag{A.55}$$

where the non-dimensional parameters, T and Ri, appearing above are defined to be $T = \frac{\gamma}{\rho_+ h_+ U_0^2}$, $Ri = \left(\frac{\rho_-}{\rho_+} - 1\right)\frac{gh_+}{U_0^2}$, and Ri is called the Richardson number.

The differential system governing the stability problem consists of (A.43)-(A.55). It defines an eigenvalue problem in that for each k, given m, n, r, Ri, Re_+ and T, certain values must be taken by c' and therefore ω for the solution not to be identically zero. The flow is then unstable, neutrally stable, or stable to disturbances of wavelength k according to whether ω_i is positive, zero or negative respectively.

A.2.1 Discussion

In [15] Yih goes on to analyse the stability characteristics for plane Poiseuille and Couette flow; and in [9] Kao looks at the stability of steady viscous flow down an inclined plane. However the Schlumberger problem involves a turbulent flow and so the boundary layers will not grow to fill the entire channel. Hence we are left with a base profile that is not fully developed and we therefore propose that, since any shear or boundary layers remain thin, an inviscid approach to the stability problem is more relevant.

Appendix B

Derivation of the effect of acceleration

We suppose that the interface takes the form

$$s = \eta(t) e^{ikX}$$

and that

$$\begin{aligned} \phi_+ &= \psi_+(t) \cosh k(Y-1) e^{ikX}, & \text{(B.1)} \\ \phi_- &= \psi_-(t) \cosh k(Y+\delta) e^{ikX}, & \text{(B.2)} \end{aligned}$$

to satisfy (2.17) and (2.18) where the real parts of the right-hand sides are understood. Then (2.21) becomes

$$\begin{aligned} \frac{d\eta}{dt} + ikU_+\eta &= -k\psi_+ \sinh k, & \text{(B.3)} \\ \frac{d\eta}{dt} + ikU_-\eta &= k\psi_- \sinh k\delta, & \text{(B.4)} \end{aligned}$$

and so

$$\begin{aligned} \frac{d^2\eta}{dt^2} + ikU_+\frac{d\eta}{dt} + ik\frac{\partial U_+}{\partial t} &= -k\frac{\partial \psi_+}{\partial t} \sinh k, & \text{(B.5)} \\ \frac{d^2\eta}{dt^2} + ikU_-\frac{d\eta}{dt} + ik\frac{\partial U_-}{\partial t} &= k\frac{\partial \psi_-}{\partial t} \sinh k\delta. & \text{(B.6)} \end{aligned}$$

Hence substituting for ϕ_+, ϕ_- from (B.1) and (B.2), and then for $\psi_\pm$, $\frac{\partial \psi}{\partial t}$ from (B.3) - (B.6) into (2.27) we find

$$\left(\frac{r}{T_-} + \frac{1}{T_+}\right)\frac{d^2\eta}{dt^2} + 2ik\left(\frac{rU_-}{T_-} + \frac{U_+}{T_+}\right)\frac{d\eta}{dt}$$
$$+ \left[Rik\cos\theta + ik\left(\frac{r}{T_-}\frac{\partial U_-}{\partial t} + \frac{1}{T_+}\frac{\partial U_+}{\partial t}\right) + \sigma k^3 - k^2\left(\frac{rU_-^2}{T_-} + \frac{U_+^2}{T_+}\right)\right]\eta = 0, \quad \text{(B.7)}$$

where $T_+ = \tanh k$, $T_- = \tanh k\delta$. If we now put

$$\eta(t) = \Re\left\{N(\tau)e^{i(ct+dt^2)}\right\}, \tag{B.8}$$

where $\tau = \lambda t$,

$$\begin{aligned}
\lambda &= \left[\frac{2kRi\sin\theta(rT_+T_-)^{\frac{1}{2}}(1+\delta)}{(r+\delta)(rT_++T_-)}\right]^{\frac{1}{2}}, && \text{(B.9)}\\
c &= -k\left[\frac{r\beta T_+ + T_-}{rT_++T_-}\right], && \text{(B.10)}\\
d &= -\frac{kRi\sin\theta}{2(r+\delta)}\left[\frac{\delta T_- - rT_+}{rT_++T_-}\right], && \text{(B.11)}
\end{aligned}$$

then (B.7) reduces to

$$\frac{d^2N}{d\hat{\tau}^2} - \left(a + \frac{1}{4}\hat{\tau}^2\right)N = 0, \tag{B.12}$$

where $\hat{\tau} = \tau + b$, and

$$\begin{aligned}
a &= \frac{-\left[\sigma k^2 + Ri\cos\theta\right](r+\delta)(T_+T_-)^{\frac{1}{2}}}{2Ri\sin\theta(1+\delta)r^{\frac{1}{2}}} && \text{(B.13)}\\
b &= \left[\frac{2k(r+\delta)(rT_+T_-)^{\frac{1}{2}}}{(rT_++T_-)Ri\sin\theta(1+\delta)}\right]^{\frac{1}{2}}(1-\beta). && \text{(B.14)}
\end{aligned}$$

Bibliography

[1] M. Abramowitz & I. A. Stegun. Handbook of mathematical functions, *Dover Publications*, 1965.

[2] D. J. Acheson. Elementary Fluid Dynamics, *Oxford University Press*, 1990.

[3] G. K. Batchelor. An introduction to fluid dynamics, *Cambridge University Press*, 1967.

[4] P. G. Drazin & W. H. Reid. Hydrodynamic Stability, *Cambridge University Press*, 1981.

[5] P. G. Drazin. The stability of a Shear layer in an unbounded heterogeneous inviscid fluid, *J. Fluid Mech*, **4**, 214-224, 1958.

[6] P. M . Eagles. The Stability of a family of Jeffrey-Hamel solutions for divergent channel flow, *J. Fluid Mech*, **24**, 191-207, 1966.

[7] A. C. Fowler. Mathematical Models in the Applied Sciences, *Cambridge University Press*, 1997.

[8] L. E. Fraenkel. On the method of matched asymptotic expansions, Part I, *Quart. Appl. Math.*, **26**, 57-69, 1968.

[9] T. W. Kao. Stability of two-layer viscous flow down an inclined plane, *The Physics of Fluids*, **8**, 812-20, 1965.

[10] Lord Kelvin. Hydrokinetic solutions and observations, *Phil. Mag.*, **42**, 362-77, 1871.

[11] J. W. Miles. On the stability of heterogeneous shear flows, Part 2, *J. Fluid Mech.*, **16**, 209-27, 1963.

[12] H. Ockendon & J. R. Ockendon. Viscous Flow, *Cambridge Texts in Applied Maths*, 1995.

[13] S. A. Thorpe. A method of producing a shear flow in a stratified fluid, *J. Fluid Mech.*, **32**, 693-704, 1968.

[14] S. A. Thorpe. Experiments on the instability of stratified shear flows: immiscible fluids, *J. Fluid Mech.*, **39**, 25-48, 1969.

[15] C.-S. Yih. Instability due to viscosity stratification, *J. Fluid Mech.*, **27**, 337-352, 1967.