

# Rational homotopy nilpotency of self-equivalences<sup>☆</sup>

Paolo Salvatore<sup>a,b,1</sup>

<sup>a</sup> *Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy*

<sup>b</sup> *Mathematical Institute, 24–29 St. Giles, OX1 3LB Oxford, UK*

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## Abstract

We study the homotopy nilpotency, after rationalization, of some spaces of self-homotopy equivalences of a finite, simply connected CW-complex. © 1997 Elsevier Science B.V.

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## 1. Introduction

A group  $G$  of homotopy self-equivalence classes of a finite CW-complex  $X$  is nilpotent, provided that it acts trivially on homology or homotopy groups of  $X$  [8]. After a question of Kahn [25], Arkowitz and Lupton [3] compute the nilpotency of such  $G$  in many cases, modulo torsion. Their approach relies strongly on rational homotopy theory.

If we pass from the *group*  $G$  to the *space*  $\Gamma$  of homotopy self-equivalences, whose path-components form  $G$ , the natural extension of the problem is to determine the homotopy nilpotency of  $\Gamma$ . This is the largest integer  $n$  such that  $n$ -commutators provide an essential map from  $\Gamma^n$  to  $\Gamma$ . This number is of course greater than or equal to the nilpotency of  $G$ . In general  $\Gamma$  fails to be homotopy nilpotent. As in the case of Arkowitz and Lupton, we restrict ourselves to studying this invariant after rationalizing  $\Gamma$ . As a first result we prove that modulo torsion the homotopy nilpotency of  $\Gamma$  is exactly the nilpotency of the algebra  $\pi_*(\Gamma)$ , endowed with the Samelson product. This equality (Theorem 3), due to Arkowitz and Curjel [1] for  $G$  trivial, follows from semidirect products considerations.

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<sup>1</sup> E-mail: salvatore@vmimat.mat.unimi.it; salvator@maths.ox.ac.uk.

After that we show (Theorem 14) that by an argument of Sullivan [30] the homotopy nilpotency of  $\Gamma$  is computable by commutator calculus of derivations of the minimal model of  $X$ . We have to be careful with derivations of degree 0, which are necessary when  $G$  is not trivial. Note that it is possible to rewrite the results of Arkowitz and Lupton using derivations of degree 0. As examples, we compute modulo torsion the homotopy nilpotency of  $\Gamma$ , when  $X$  has two infinite homotopy groups, or is a special unitary group, or is a wedge of odd spheres. In the last case the homotopy nilpotency is infinite. Our results are consistent with [11], where Felix and Thomas computed the additive structure of  $\pi_*(\Gamma)$  for some homogeneous spaces  $X$  when  $G$  is trivial.

The paper is organized as follows: Section 2 introduces the notion of homotopy nilpotency and the fundamental relation  $WL(X) \leq Hnil(\Omega X)$  between the homotopy nilpotency of a space of loops  $\Omega X$  and the maximum length of nontrivial Whitehead products in the homotopy groups of  $X$ . In Section 3 we prove (Theorem 3) that  $WL(X) = Hnil(\Omega X)$  when the CW-complex  $X$  has a rational universal covering. In Section 4 we define some spaces of self-equivalences, and give results about their homotopy nilpotency. In Section 5 we study the interaction of self-equivalences with localization theory, and in particular we describe the rational H-structure of our spaces of homotopy self-equivalences of  $X$  by means of derivations of the minimal model of  $X$ . In Section 6 we prove (Theorem 14) that the rational homotopy nilpotency of our spaces corresponds to the nilpotency of suitable Lie algebras, and give some examples.

## 2. Homotopy nilpotency

Let  $X$  be an associative H-group with multiplication  $\mu: X \times X \rightarrow X$ , strict unit  $e \in X$  and homotopy inverse  $\lambda: X \rightarrow X$ . Write  $\mu(x, y) = xy$ ,  $\lambda(x) = x^{-1}$ . The commutator map  $c_2: X \times X \rightarrow X$ , given by  $c_2(a, b) = aba^{-1}b^{-1}$ , is well defined up to homotopy. The higher commutator maps  $c_n: X^n \rightarrow X$  are defined, up to homotopy, by recursion:

$$c_n(a_1, \dots, a_n) = c_2(a_1, c_{n-1}(a_2, \dots, a_n)).$$

**Definition 1.** The homotopy nilpotency  $Hnil(X)$  of  $X$  is the maximum of those integers  $n$  such that the commutator map  $c_n$  is essential. If it is finite, we say that  $X$  is homotopy nilpotent.

Clearly  $Hnil(X) \leq 1$  if and only if  $X$  is a H-commutative H-group. Given an arbitrary space  $A$ , the set of free homotopy classes  $[A, X]$  is a group. The nilpotency of these groups is related to the homotopy nilpotency of  $X$  by the formula  $Hnil(X) = \sup_A \{nil[A, X]\}$  [21]. Basic results concerning homotopy nilpotency are in [4]. Many computations have been done for Lie groups. For example a classical result gives  $Hnil(S^3) = 3$  [26], whilst recently Rao [27] has proved that  $Hnil(SO(3)) = \infty$ . Arkowitz and Curjel computed in [2]  $Hnil(S^3)$  for every H-multiplication on  $S^3$ . We are particularly interested in the space  $\Omega Y$  of based loops in a given space  $Y$ . How does the

lack of commutativity of loops depend on the geometry of  $Y$ ? The first result [4, 4.6] in this direction was:

**Proposition 2** (Bernstein, Ganea). *For a path-connected space  $X$ ,  $\text{Hnil}(\Omega X) \geq \text{WL}(X)$ , where  $\text{WL}(X)$ , the Whitehead length of  $X$ , is the maximum of those integers  $n$  such that  $n$ -iterated Whitehead products in the homotopy groups of  $X$  do not vanish.*

Namely  $n$ -iterated Whitehead products are obtained by the composition of a map from a certain  $n$ -product of spheres with the commutator map  $c_n$ . The proposition implies for example that  $\Omega(S^n \vee S^n)$  is not homotopy nilpotent since by the Hilton–Milnor Theorem [15]  $\text{WL}(S^n \vee S^n) = \infty$ .

### 3. Semidirect products

The purpose of this section is to show that if  $X$  has a rational space of finite type as universal covering, then the inequality of Proposition 2 becomes an equality.

**Theorem 3.** *Let  $X$  be a connected CW-complex such that  $\pi_i(X)$  is a finite-dimensional rational vector space for  $i > 1$ . Then  $\text{Hnil}(\Omega X) = \text{WL}(X)$ .*

Arkowitz and Curjel proved essentially this fact for  $X$  1-connected [1, Lemma 4.2]. The idea of our proof is to decompose  $\Omega X$  as a semidirect product of a rational space and a group. Hence the problem is transferred into the Hopf algebra setting. We need some preliminaries before the proof.

Given an H-group  $G$ , an H-homotopy action of a group  $\pi$  on  $G$  is a homomorphism  $\varphi: \pi \rightarrow \text{HAut}(G)$ , where  $\text{HAut}(G)$  is the group of the free homotopy classes of H-self-equivalences of  $G$ . The *semidirect product*  $G \rtimes \pi$  with respect to this action [24] is defined to be a space homeomorphic to  $G \times \pi$  where  $\pi$  has the discrete topology. This is again an H-group, with the product given as usual by  $(g, p) \cdot (g', p') = (g \cdot \varphi(p)(g'), p \cdot p')$ , and well defined up to homotopy. Observe that components of  $\Omega X$  correspond to elements of  $\pi_1(X)$ , and are all homotopically equivalent. We can identify the component containing the constant loop with  $\Omega \tilde{X}$ , where  $\tilde{X}$  is the universal covering of  $X$ . For  $\sigma \in \pi_1(X)$ , the conjugation map  $S_\sigma: \Omega \tilde{X} \rightarrow \Omega \tilde{X}$ , given by  $S_\sigma(\tau) = \sigma \tau \sigma^{-1}$ , is well defined up to homotopy and is an H-map because  $(\sigma \tau_1 \tau_2 \sigma^{-1}) \simeq (\sigma \tau_1 \sigma^{-1})(\sigma \tau_2 \sigma^{-1})$  in a continuous way. Hence  $S$  is an H-homotopy action of  $\pi_1(X)$  on  $\Omega \tilde{X}$ . Peschke in [24] proves that there is an H-equivalence of H-groups  $\Omega X \simeq (\Omega \tilde{X}) \rtimes \pi_1(X)$ .

We need the following:

**Lemma 4** [16, Chapter III, 1.15]. *If  $Y$  is a CW-complex,  $G$  a rational H-group of the homotopy type of a CW-complex, and both are connected and have finite type over  $\mathbb{Q}$ , then the homology functor induces a bijection between homotopy classes  $[Y, G]$  and coalgebra homomorphisms  $\text{Hom}(H_*(Y, \mathbb{Q}), H_*(G))$ .*

Here the coalgebras are rational, graded, cocommutative and coaugmented over  $\mathbb{Q}$ . The original version uses cohomology. The lemma follows from duality and the fact that, the  $k$ -invariants of  $G$  being trivial,  $G$  is a product of Eilenberg–MacLane spaces. Now the commutator map  $c_n$  of  $\Omega X$  is nullhomotopic if and only if its image lies in the component of the constant loop, and  $c_n$  restricted to each component of  $(\Omega X)^n$  is nullhomotopic. If we choose as  $Y$  in the lemma a generic component of  $(\Omega X)^n$  and  $G = \Omega \tilde{X}$ , we get:

**Proposition 5.** *If  $X$  is as in Theorem 3 then the following are equivalent:*

- (a)  $\text{Hnil}(\Omega X) \leq n$ ;
- (b)  $\text{nil}(\pi_1(X)) \leq n$  and  $H_*(c_{n+1}) = 0$  in positive degrees.

Now observe that  $H_*(\Omega X)$  is a rational, graded, cocommutative, generally not connected Hopf algebra. By Peschke's result there is an isomorphism of Hopf algebras

$$H_*(\Omega X) \cong H_*(\Omega \tilde{X}) \otimes \mathbb{Z}[\pi_1(X)].$$

On the right-hand side  $\pi_1(X)$  is concentrated in degree 0, the tensor product of a  $\mathbb{Q}$ -vector space and an Abelian group gives a  $\mathbb{Q}$ -vector space, and the product in the algebra is defined by  $(h \otimes \sigma) \cdot (h' \otimes \sigma') = (h \cdot (S_\sigma)_*(h')) \otimes \sigma \cdot \sigma'$ .

The conjugation of  $\tilde{H} = H_*(\Omega \tilde{X})$  is the coalgebra automorphism  $\omega = H_*(\lambda) : \tilde{H} \rightarrow \tilde{H}$ , where  $\lambda$  is the homotopy inverse in  $\Omega \tilde{X}$ . It satisfies

- (i)  $\omega(ab) = (-1)^{|a||b|} \omega(b)\omega(a)$ ;
- (ii)  $\omega(x) = -x$  if  $x$  is a primitive in  $\tilde{H}$ .

The commutator homomorphism  $H_*(\tilde{c}_2)$  is obtained by the composition

$$\tilde{H} \otimes \tilde{H} \xrightarrow{(1 \otimes T \otimes 1)(\Delta \otimes \Delta)} \tilde{H} \otimes \tilde{H} \otimes \tilde{H} \otimes \tilde{H} \xrightarrow{\text{id} \otimes \text{id} \otimes \omega \otimes \omega} \tilde{H} \otimes \tilde{H} \otimes \tilde{H} \otimes \tilde{H} \xrightarrow{\mu(\mu \otimes \mu)} \tilde{H},$$

where  $T$  is the interchange homomorphism  $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$ . A direct computation shows that if  $x$  and  $y$  are primitive in  $\tilde{H}$  then  $H_*(\tilde{c}_2)(x \otimes y) = xy - (-1)^{|x||y|} yx$ . Now by the Milnor–Moore Theorem [23], since  $\tilde{X}$  is 1-connected and rational of finite type, the Lie algebra  $P\tilde{H}$  of the primitives of  $\tilde{H}$  is isomorphic to the algebra of the homotopy groups of  $\tilde{X}$ , endowed with the Whitehead product, with a shift in dimension, and  $P\tilde{H}$  generates  $\tilde{H}$  multiplicatively. One can carry out a similar construction for  $H_*(\Omega X) = \tilde{H} \otimes \mathbb{Z}[\pi_1(X)]$ , whose conjugation  $j$  is given by  $j(h \otimes \sigma) = (S_{\sigma^{-1}})_*(\omega(h)) \otimes \sigma^{-1}$ , and construct the commutator homomorphism  $H_*(c_2)$  as above. If  $e$  is the identity of  $\pi_1(X)$ , then we will identify  $H_*(\Omega \tilde{X}) \equiv H_*(\Omega \tilde{X}) \otimes e \subseteq H_*(\Omega X)$ , and  $\pi_1(X) \equiv 1 \otimes \pi_1(X) \subseteq 1 \otimes \mathbb{Z}[\pi_1(X)] \subseteq H_*(\Omega X)$ . Clearly  $H_*(\Omega X)$  is multiplicatively generated by  $\pi_1(X)$  and by the primitives of  $H_*(\Omega \tilde{X})$ , which we shall identify with  $\pi_*(\tilde{X})$ . By writing explicitly the commutator in the semidirect product one obtains that if  $p, q \in \pi_*(X) = \pi_1(X) \oplus \pi_*(\tilde{X})$  then  $H_*(c_2)(p \otimes q) = \pm[p, q]$ , where the brackets denote the Whitehead product in  $\pi_*(X)$ . The iterated application of this formula shows that if  $x_1, \dots, x_n \in \pi_*(X)$  then  $H_*(c_n)(x_1 \otimes \dots \otimes x_n) = \pm[x_1, [\dots, x_n] \dots]$ .

**Proof of Theorem 3.** We need only to show that  $\text{WL}(X) \geq \text{Hnil}(\Omega X)$ . Suppose that  $\text{WL}(X) = n - 1$ . Then obviously  $\text{nil}(\pi_1(X)) \leq n - 1$ , and  $H_*(c_n)$  vanishes on terms

of positive degree, of the form  $x_1 \otimes \cdots \otimes x_n$ , where  $x_i \in \pi_*(X)$ . If we are able to show that  $H_*(c_n)$  vanishes on every element of positive degree, i.e. for arbitrary  $x_i \in H_*(\Omega X)$ , then we are done. The conclusion is not immediate because the commutator map is not an algebra homomorphism. It is known in group theory that by the formula  $[x, yz] = [x, y][x, z][z, [x, y]]$  and by induction one obtains that if a group  $G$  is generated by a set  $T$  then every commutator of length  $n$  can be written as product of commutators of length  $\geq n$  of elements of  $T$ . Applying the same kind of induction to the H-group  $\Omega X$  and passing to homology, one can show that, given decompositions  $x_i = x_i^1 \cdots x_i^{m_i}$  in  $H_*(\Omega X)$  for  $i = 1, \dots, n$ ,

$$\begin{aligned} H_*(c_n)(x_1 \otimes \cdots \otimes x_n) \\ = \mu \circ (H_*(c_{a_1}) \otimes \cdots \otimes H_*(c_{a_k})) \circ \sigma \circ (\Delta(x_1^1) \otimes \cdots \otimes \Delta(x_n^{m_n})), \end{aligned} \quad (1)$$

where  $a_j \geq n$ ,  $\sigma$  interchanges factors and multiplies by  $\pm 1$ ,  $\mu$  is an iterated multiplication, each  $\Delta$  is an appropriate iterated diagonal. We can choose every  $x_i^j$  in  $\pi_*(X)$ , hence primitive, so that  $\Delta(x_i^j) = (x_i^j \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes x_i^j)$ . There is at least one of them, say  $x_a^b$ , of positive degree. After reordering, a factor of the form  $H_*(c_{a_i})(\cdots \otimes x_a^b \otimes \cdots) = [\dots, \dots, [x_a^b, \dots], \dots]$  appears in the right-hand side of (1). This factor is trivial by hypothesis, so  $H_*(c_n)(x_1 \otimes \cdots \otimes x_n) = 0$ .  $\square$

#### 4. Spaces of self-equivalences

Let  $\text{aut}(X)$  be the space of free homotopy self-equivalences of a CW-complex  $X$ . We want to study its homotopy nilpotency. This makes sense if  $\text{aut}(X)$  is an H-group. Now  $\text{aut}(X)$  is known to have the homotopy type of a CW-complex, and hence is an H-group by [32, Chapter X, Theorem 2.2], when either

- (i)  $X$  is a finite complex [22] or
- (ii)  $X$  has a finite number of nontrivial, finitely generated homotopy groups [17].

Therefore we will restrict ourselves to these cases.

The path-components of  $\text{aut}(X)$  form the group  $\text{Aut}(X) = \pi_0(\text{aut}(X))$  of classes of self-equivalences. Let  $B$  be the Dold functor [6], which takes topological monoids to spaces. For any topological monoid  $G$  there is a weak H-equivalence  $\Omega BG \simeq G$ . The space  $B(\text{aut}(X))$  classifies fibrations over CW-complexes with fiber  $X$ .

We define some subspaces of  $\text{aut}(X)$ .

(1)  $\text{aut}_1(X)$  is the path-component of  $\text{aut}(X)$  containing the identity of  $X$ . Its elements are the maps homotopic to the identity. The space  $B(\text{aut}_1(X))$  is the universal covering of  $B(\text{aut}(X))$  and classifies fibrations with fiber  $X$  over simply connected CW-complexes.

(2)  $\text{aut}_*(X)$  is the subspace of maps inducing the identity on the homology groups of  $X$ . The group  $\text{Aut}_*(X) = \pi_0(\text{aut}_*(X))$  is nilpotent in case (i) by [8, Theorem D]. The space  $B(\text{aut}_*(X))$  is the covering of  $B(\text{aut}(X))$  corresponding to the subgroup  $\text{Aut}_*(X) \subseteq \text{Aut}(X) \cong \pi_1(B(\text{aut}(X)))$ , and classifies fibrations with fiber  $X$  over CW-complexes where the fundamental group of the basis acts trivially on the homology groups of  $X$ .

(3) If  $X$  is simply connected and based, then its free self-equivalences act on its homotopy groups. We define  $\text{aut}_\#(X)$  as the subspace of maps inducing the identity on  $\pi_j(X)$  in case (i) for every  $j \leq \dim(X)$ , and in case (ii) for every  $j$ . The group  $\text{Aut}_\#(X) = \pi_0(\text{aut}_\#(X))$  is nilpotent in the cases (i) and (ii) by [8, Theorem A]. The space  $B(\text{aut}_\#(X))$  has similar properties to those of  $B(\text{aut}_*(X))$ .

We give some examples and results about the homotopy nilpotency of  $\text{aut}_1(X)$ . If  $X = S^1$  then  $\text{aut}_1(X)$  is H-equivalent to  $S^1$  and hence is H-commutative. If  $X$  is a Riemann surface of genus  $g \geq 2$  then  $\text{aut}_1(X)$  is homotopically trivial [13], and of course H-commutative.

**Example 6.**  $\text{Hnil}(\text{aut}_1(S^2)) \geq 3$ .

Consider the inclusion  $j: SO(3) \rightarrow \text{aut}_1(S^2)$ , and the evaluation  $\text{ev}: \text{aut}_1(S^2) \rightarrow S^2$ . Given the universal cover  $\phi: S^3 \rightarrow SO(3)$ , the composite  $\text{ev} \circ j \circ \phi$  is the Hopf fibration, so  $\text{ev} \circ j$  induces an isomorphism on homotopy groups  $\pi_i$  in dimension  $i \geq 3$ . The homotopy class  $r \in \pi_3(SO(3))$  representing the universal cover is such that  $\alpha = \langle r, \langle r, r \rangle \rangle \neq 0$  [26]. Therefore  $j_\#(\alpha)$  is a nontrivial Samelson product of length 3 in  $\pi_9(\text{aut}_1(S^2))$ , and we are done.

**Proposition 7.** *If  $G$  is an H-group, then  $\text{Hnil}(\text{aut}_1(G)) \geq \text{Hnil}(G)$ .*

The evaluation on the unit  $\text{ev}: \text{aut}_1(G) \rightarrow G$  admits an H-section  $s$  given by  $s(x)(t) = x \cdot t$ . The following diagram commutes up to homotopy.

$$\begin{array}{ccc} G^n & \xrightarrow{c_n} & G \\ s^n \downarrow & & \downarrow s \\ (\text{aut}_1(G))^n & \xrightarrow{\bar{c}_n} & \text{aut}_1(G) \end{array}$$

If  $c_n = \text{ev} \circ s \circ c_n$  is not trivial, then also  $\bar{c}_n$  is not trivial. Therefore for example  $\text{aut}_1(SO(3))$  is not homotopy nilpotent and  $\text{Hnil}(\text{aut}_1(S^3)) \geq 3$ .

**Proposition 8.** *If  $Y$  is a Postnikov piece then  $\text{aut}_1(Y)$  is H-nilpotent. Moreover*

$$\text{Hnil}(\text{aut}_1(Y)) \leq \max\{i \mid \pi_i(Y) \neq 0\}.$$

By [31, Theorem 1]  $\pi_i(\text{aut}_1(Y)) = 0$  for  $i > \max\{i \mid \pi_i(Y) \neq 0\}$ . Use [4, 4.12] with  $X = B\text{aut}_1(Y)$ .

## 5. Localization and self-equivalences

Given a set of primes  $P$ , we define the  $P$ -localization of a loop space  $G$  of the homotopy type of a CW-complex, whose classifying space  $BG$  is nilpotent, as  $G_P = \Omega((BG)_P)$ . The corresponding  $P$ -localization map is given by the composite  $G \rightarrow \Omega(BG) \rightarrow \Omega(BG_P)$ , where the first map inverts the canonical homotopy equivalence

$\Omega(BG) \simeq G$ . If  $G$  is connected we find again the ordinary notion of  $P$ -localization. We define the  $P$ -local homotopy nilpotency of  $G$  as  $\text{Hnil}_P(G) = \text{Hnil}(G_P)$ . If  $\pi$  is a nilpotent group, its  $P$ -local nilpotency is  $\text{nil}_P(\pi) = \text{nil}(\pi_P)$ . Note that if  $G$  is connected and finite dimensional, for example a connected Lie group, then  $\text{Hnil}_0(G) = 1$  because  $G_0$  is H-equivalent to  $\prod_i K(\pi_i(G) \otimes \mathbb{Q}, i)$  [9]. Fortunately the spaces of self-equivalences are far from being finite-dimensional, so that their rational homotopy nilpotency is a nontrivial invariant.

Localization commutes with self-equivalences in the following sense.

**Proposition 9** [7]. *Given a finite 1-connected CW-complex  $X$ , there exists a map*

$$h: \text{aut}(X) \rightarrow \text{aut}(X_P)$$

*that  $P$ -localizes on each component, up to weak homotopy equivalences, and such that the function  $\pi_0(h): \text{Aut}(X) \rightarrow \text{Aut}(X_P)$  is induced by the  $P$ -localization functor.*

Note that in general  $\pi_0(h)$  is not a  $P$ -localization of groups. The statement holds up to weak equivalence, since  $\text{aut}(X_P)$  does not necessarily have the homotopy type of a CW-complex. It is known [20] that  $h$  is a loop map such that  $Bh$  induces the fibrewise  $P$ -localization. If we restrict  $h$  to some subspaces of  $\text{aut}(X)$  we obtain:

**Proposition 10.** *If  $X$  is a finite 1-connected complex then  $h_*: \text{aut}_*(X) \rightarrow \text{aut}_*(X_P)$ ,  $h_\#: \text{aut}_\#(X) \rightarrow \text{aut}_\#(X_P)$  are  $P$ -localizations of loop spaces, up to weak homotopy equivalence.*

This makes sense because  $B\text{aut}_\#(X)$  and  $B\text{aut}_*(X)$  are nilpotent spaces by [8, Theorems B, D]. The homomorphisms  $\pi_i(h)$   $P$ -localize for  $i \geq 1$  by Proposition 9. Furthermore  $\pi_0(h_\#): \text{Aut}_\#(X) \rightarrow \text{Aut}_\#(X_P)$  and  $\pi_0(h_*): \text{Aut}_*(X) \rightarrow \text{Aut}_*(X_P)$   $P$ -localize respectively by [18] and [19].

In the remainder of this section we describe algebraic models, modulo torsion, of some spaces of self-equivalences. We assume basic knowledge of rational homotopy theory. For details see [9]. Let  $(A, d)$  be a rational differential commutative  $\mathbb{Z}$ -graded algebra. A graded linear map  $\theta: A \rightarrow A$  of degree  $n$  (write  $|\theta| = n$ ) is a *derivation* if, for any  $x, y$  in  $A$ ,  $\theta(xy) = \theta(x)y + (-1)^{|x||\theta|}x\theta(y)$ . Consider the  $\mathbb{Z}$ -graded differential Lie algebra  $\text{Der}(A)$  such that  $\text{Der}(A)_i = \{\text{derivations of } A \text{ of degree } -i\}$ . The product in  $\text{Der}(A)$  is given by  $[\theta, \theta'] = \theta \circ \theta' - (-1)^{|\theta||\theta'|}\theta' \circ \theta$ , and the differential  $\partial$  by  $\partial(\theta) = d \circ \theta - (-1)^{|\theta|}\theta \circ d$ . Let  $\text{Der}_+(A)$  be the subalgebra restricted to positive degree. A classical result of Sullivan states:

**Proposition 11** [11]. *If  $X$  is a 1-connected complex,  $AV$  its minimal model, then there is a graded Lie algebra isomorphism*

$$H_*(\text{Der}_+(AV), \partial) \cong \pi_*(\text{aut}_1(X))_0.$$

The right-hand term is endowed with the Samelson product. The idea is that the pointed homotopy classes of maps  $S^n \rightarrow \text{aut}_1(X) \subseteq X^X$  are in bijection with the homotopy

classes of those maps  $S^n \times X \rightarrow X$  that composed with the inclusion  $X \rightarrow S^n \times X$  yield the identity. If we pass to minimal models, these classes correspond to homomorphisms  $AV \rightarrow AV \otimes \Lambda y/y^2$  of the form  $x \mapsto x \otimes 1 + \theta(x) \otimes y$ , where  $\theta$  is a derivation, a cycle in  $\text{Der}_+(AV)$ , well defined up to a boundary.

We look for a generalization involving mapping spaces which are not necessarily connected. Let  $\text{DER}_\#(AV)$  be the rational vector space of those derivations of degree 0 commuting with  $d$  and taking the generators of degree less than or equal to  $\dim(X)$  into decomposables. Consider the differential Lie algebra  $\text{Der}_\#(AV) = \text{Der}_+(AV) \oplus \text{DER}_\#(AV)$ .

**Proposition 12.** *If  $X$  is a finite 1-connected complex,  $AV$  its minimal model, then there is a graded bijection  $\alpha: H_*(\text{Der}_\#(AV), \partial) \cong \pi_*(\text{aut}_\#(X))_0$  that relates Samelson products and commutators as follows:*

For  $u, v \in H_*(\text{Der}_\#(AV))$ ,

(i) if  $|u|, |v| > 0$  then  $\langle \alpha(u), \alpha(v) \rangle = \alpha([u, v])$ ;

(ii) if  $|u| = |v| = 0$  then  $\alpha^{-1}(\langle \alpha(u), \alpha(v) \rangle) = \Phi(\Phi(u, v), \Phi(-u, -v))$ , where

$$\Phi(u, v) = \log(e^u \circ e^v) = u + v + \frac{1}{2}[u, v] + \frac{1}{12}[u, [u, v]] + \dots$$

is the Baker–Campbell–Hausdorff formula [30];

(iii) if  $0 = |u| < |v|$  then

$$\alpha^{-1}(\langle \alpha(u), \alpha(v) \rangle) = e^u v e^{-u} - v = \sum_{n \geq 1} \frac{1}{n!} [u, [u, \dots (n \text{ times}) \dots, [u, v] \dots]].$$

**Proof (Sketch).** Let  $\text{Aut}_\#(AV)$  be the group of the automorphisms of  $AV$  inducing the identity modulo decomposables in degree less than or equal to  $\dim(X)$ . There is a bijection between  $\text{DER}_\#(AV)$  and  $\text{Aut}_\#(AV)$ , given by the exponential  $u \mapsto e^u = I + u + u^2/2! + \dots$ . Under this bijection automorphisms homotopic to the identity correspond to boundaries [30], and the composition of automorphisms corresponds to the B–C–H formula. We claim that two derivations  $x, y$ , cycles of degree 0, correspond to homotopic automorphisms if and only if they are homologous. That amounts to checking that  $x - y$  is a boundary if and only if  $\Phi(x, -y)$  is a boundary. Use induction and the finite number of summands modulo boundaries in the B–C–H formula in a fixed degree. Therefore a bijection

$$H_0(\text{Der}_\#(AV)) \cong (\text{Aut}_\#(AV)/\simeq) \cong \text{Aut}_\#(X)_0$$

is induced. Eventually, the action of  $\pi_0(\text{aut}_\#(X))_0$  on  $\pi_n(\text{aut}_\#(X))_0$  is compatible with the action of  $\text{DER}_\#(AV)$  on  $\text{Der}_\#(AV)_n$ , given by  $u(v) = e^u v e^{-u}$ , so that the topological Samelson product corresponds to that induced by (iii).

Similarly, let  $\text{DER}_*(AV)$  be the set of those derivations of  $AV$  of degree 0 commuting with  $d$  and inducing the trivial homomorphism in cohomology, and define  $\text{Der}_*(AV) = \text{Der}_+(AV) \oplus \text{DER}_*(AV)$ . Then Proposition 12 holds if we replace  $\#$  by  $*$ .  $\square$



It is not difficult to prove that in Propositions 11 and 12 it suffices to take a model of  $X$  that is free as algebra, but not necessarily minimal.

**Proposition 13.** *If  $(AT, d)$  is a free model of  $X$  and  $(AV, d')$  is a minimal model of  $X$  then there is a Lie algebra isomorphism  $H_*(\text{Der}(AV)) \cong H_*(\text{Der}(AT))$ .*

The theory of models [10] says that  $(AT, d) \cong (AV, d') \otimes (\Lambda(U \oplus \bar{d}U), \bar{d})$ , where  $\Lambda(U \oplus \bar{d}U)$  is free contractible since  $\bar{d}: U \cong \bar{d}U$ . One can check inductively that relative homology groups corresponding to the inclusion  $\text{Der}(AV) \hookrightarrow \text{Der}(AT)$  vanish. Let  $(T, d_1)$  be the complex of indecomposables of  $(AT, d)$ . There is an isomorphism  $H^*(T) \cong \text{Hom}(\pi_*(X), \mathbb{Q})$ . We define consequently  $\text{DER}_\#(AT)$  as the set of derivations of degree 0 commuting with  $d$  and inducing the trivial homomorphism on  $H^i(T)$  for  $i \leq \dim(X)$ .

## 6. Applications and examples

**Theorem 14.** *Let  $X$  be a finite 1-connected CW-complex. Let  $(AV, d)$  be a free model of  $X$ . Then*

- (a)  $\text{Hnil}_0(\text{aut}_1(X)) = \text{nil}(H_*(\text{Der}_+(AV, d)))$ ;
- (b)  $\text{Hnil}_0(\text{aut}_\#(X)) = \text{nil}(H_*(\text{Der}_\#(AV, d)))$ ;
- (c)  $\text{nil}_0(\text{Aut}_\#(X)) = \text{nil}(H_0(\text{Der}_\#(AV, d)))$ .

**Proof.** On the right-hand side  $\text{nil}$  indicates the nilpotency of a Lie algebra.

(a) From Theorem 3 the left-hand side is equal to the Whitehead length of  $\text{Baut}_1(X)_0$ , since  $\text{aut}_1(X)_0$  is H-equivalent to  $\Omega \text{Baut}_1(X)_0$ . Hence the left-hand side is equal to the maximum length of nontrivial iterated Samelson products in  $\pi_*(\text{aut}_1(X))_0$ , which by Proposition 11 coincides with the right-hand side.

(b) The first step is as in (a). We have to check that the maximum length of nontrivial Samelson products coincides with that of the commutators. Suppose that the commutator length is finite, say  $k$ . Then by formulae (i) (ii) (iii) in Proposition 12  $\alpha^{-1}(\langle \alpha(x), \alpha(y) \rangle) \equiv \pm[x, y]$  modulo commutators of length at least 3. Therefore by induction one gets that for any combination of brackets  $\alpha^{-1}(\langle \alpha(x_1), \dots, \alpha(x_l) \rangle) \equiv \pm[x_1, \dots, x_l]$  modulo commutators of length at least  $l + 1$ . One concludes by taking  $l = k$  and  $l = k + 1$ . If such  $k$  does not exist, consider Postnikov approximations  $X^n$ , and the corresponding models  $AV_{\leq n}$ . Given  $m$ , for  $n$  big enough and  $0 \leq i \leq m$  one has that  $H_i(\text{Der}_\#(AV_{\leq n})) \cong H_i(\text{Der}_\#(AV))$  and  $\pi_i(\text{aut}_\#(X^n)) \cong \pi_i(\text{aut}_\#(X))$ . So one concludes that the Samelson length is infinite using the previous case and comparing the limits as  $n$  tends to infinity.

(c) Restrict the proof of (b) to the 0 level. Use Proposition 10 and the skew-isomorphism  $\text{Aut}_\#(X_0) \cong \text{Aut}_\#(AV)/\simeq$ .  $\square$

**Remarks.** (1) A similar result holds if we replace  $\#$  by  $\star$ .

(2) This theorem allows us to compute in a finite number of steps the invariants on the left, provided we have a free model  $\Lambda V$  with a finite number of generators. For example this is possible when  $X$  is a homogeneous space. In particular  $\text{Hnil}_0(\text{aut}_1(X)) \leq 1$  when  $X = G/H$  is a 1-connected homogeneous space with  $\text{rank}(H) = \text{rank}(G)$  [11].

(3) By (c) one can reinterpret the results of [3] in terms of derivations of degree 0.

We apply the theorem to some computations:

**Example 15.**  $\text{Hnil}_0(\text{aut}_1(SU(n))) = \text{Hnil}_0(\text{aut}_\#(SU(n))) = n - 1$ .

The minimal model of  $SU(n)$  is given by  $(\Lambda(x_3, \dots, x_{2n-1}), 0) = (\Lambda V_n, 0)$ . Hence  $\partial = 0$  on derivations and  $H_*(\text{Der}_+(\Lambda V_n)) = \text{Der}_+(\Lambda V_n)$ . We need the following:

**Definition 16.** An *elementary derivation*  $(\sigma, a)$  takes the generator  $\sigma$  to the monomial  $a$ , and the other generators to 0. If the derivation has degree 0, then  $a$  must be decomposable.

Note that the elementary derivations form a basis of  $\text{Der}_\#(\Lambda V_n)$ , since  $\Lambda V_n$  is free on  $V_n$ , so it suffices to consider commutators of elementary derivations. It is easy to verify that

- (a)  $[(\sigma, \alpha), (\tau, \beta)] = (\sigma, \alpha) \circ (\tau, \beta)$  if  $|\sigma| < |\tau|$ ;
- (b)  $[(\sigma, \alpha), (\tau, \beta)] = 0$  if  $|\sigma| = |\tau|$ .

Define  $H_n = \text{Hnil}_0(\text{aut}_1(SU(n)))$ . The iterated commutator satisfies

$$[\dots[(x_1, 1), (x_3, x_1)], \dots(x_{2n-1}, x_{2n-3})] = (x_{2n-1}, 1) \neq 0,$$

so  $H_n \geq n - 1$ . We see by induction that  $H_n \leq n - 1$ . For  $n = 1$ ,  $SU(1)$  is a point and  $H_1 = 0$ . Suppose that  $H_{n-1} = n - 2$ . Hence a possible nontrivial commutator  $\chi$  of more than  $n - 2$  elementary derivations in  $\text{Der}_+(\Lambda(x_3, \dots, x_{2n-1}))$  must involve the generator  $x_{2n-1}$ . By expressions (a) and (b) and Jacobi identities we may suppose that  $\chi$  is the sum of elements of the form  $h_1 \circ \dots \circ h_{t-1} \circ (x_{2n-1}, a)$ , with  $t \geq n - 1$ . By the lemma below,  $t = n - 1$  and  $H_n = n - 1$ . The proof for  $\text{aut}_\#(SU(n))$  is the same.

**Lemma 17.** If  $h_1 \circ \dots \circ h_{t-1} \circ (x_{2n-1}, a) \neq 0$ , where each  $h_i \in \text{Der}_\#(\Lambda V_n)$  is elementary, then  $t \leq n - 1$ .

For  $n = 2$  it is true because the sole elementary derivation is  $(x_3, 1)$ . Suppose that it is true for every  $i < n$ . Note that the monomials in  $\Lambda V_n$  are square-free, as the generators are odd-dimensional. If  $a = x_{2i_1-1} \dots x_{2i_k-1}$  in  $\Lambda V_n$ , then the derivation  $h_i$  has necessarily the form  $(x_\tau, b)$ , where the monomial  $h_{i+1} \circ \dots \circ h_{t-1}(a)$  is a multiple of  $x_\tau$ , but has no common factors with the monomial  $b$ . This implies that there is a partition of the set  $\{1, \dots, t - 1\}$  into subsets  $G_j = \{u_1^j, \dots, u_{v_j}^j\}$ , for  $j = 1, \dots, k$ , such that

$$(h_1 \circ \dots \circ h_{t-1})(a) = \prod_{j=1}^k (h_{u_1^j} \circ \dots \circ h_{u_{v_j}^j})(x_{2i_j-1})$$

and  $h_l \in \text{Der}_\#(AV_{i_j})$  if  $l \in G_j$ . By inductive hypothesis

$$t - 1 = \sum_{j=1}^k v_j \leq \sum_{j=1}^k (i_j - 1) \leq n - (k + 1)/2.$$

But if  $k = 1$  then  $(x_{2n-1}, a)$  has positive degree and  $i_1 < n$ , and if  $\sum_{j=1}^k (2i_j - 1) = 2n - 1$  then  $a$  is decomposable of length  $k \geq 3$ , so that  $t \leq n - 1$ .

**Example 18.** Let  $X$  be a 1-connected finite CW-complex with two infinite homotopy groups in dimension  $m$  and  $n > m$ . Let  $u$  the largest integer such that  $X$  rationally has  $K(\mathbb{Q}^u, m)$  as factor; if we denote the integer part by  $[]$  then  $\text{Hnil}_0(\text{aut}_1(X)) = \min(-[-n/m], u + 1)$  and  $\text{Hnil}_0(\text{aut}_\#(X)) = \min([n/m], u) + 1$ .

The generators of the minimal model of  $X$  form two rational vector spaces:  $V$  in dimension  $m$  with basis  $(x_1, \dots, x_r)$  and  $W$  in dimension  $n$  with basis  $(y_1, \dots, y_s)$ . The unique possibly nontrivial  $k$ -invariant  $k^{(n+1)}$  of  $X$  occurs if  $m$  divides  $n + 1$ . If it is trivial then  $u = r$ . This  $k$ -invariant is represented by the differential  $d(W) \subseteq AV$  [9], hence by homogeneous rational polynomials  $d(y_i) = P_i(x_1, \dots, x_r)$  of degree  $h$ ,  $i = 1, \dots, s$ . The derivations of the form  $(y_i, a)$ , where  $a$  is a polynomial in  $(x_1, \dots, x_r)$ , are cycles, and

$$\partial(x_i, 1) = (-1)^m \sum_{j=1}^s \left( y_j, \frac{\partial P_j}{\partial x_i} \right).$$

So a derivation  $\sum_{i=1}^r q_i(x_i, 1)$  is a cycle if and only if

$$\sum q_i \frac{\partial P_j}{\partial x_i} = 0$$

for every  $j$ . That means that if we complete  $q = \sum q_i x_i$  to a basis of  $V$  and rewrite the polynomials  $P_j$  with respect to this basis,  $q$  does not appear in any of them. Therefore  $X$  is rationally homotopic to  $Y \times K(\mathbb{Q}, m)$ , where  $Y$  is the realization of the algebra obtained by killing  $q$  in the model of  $X$ . It is easy to check that if such  $q$  does not exist, or equivalently  $u = 0$ , then  $\text{aut}_\#(X)_0$  is H-commutative. If  $u > 0$  then  $m$  must be odd because  $X$  is finite-dimensional. Furthermore  $u$  is the largest number of linearly independent elements  $q_{(1)}, \dots, q_{(u)}$  in  $V$  that do not appear in the polynomials  $P_j$ , after completing them to a basis of  $V$ . Put  $t = \min(u, -[-n/m] - 1)$ . Then the longest nontrivial commutator in homology is the class of

$$[(q_{(t)}, 1), \dots, [(q_{(1)}, 1), (y_i, q_{(1)}q_{(2)} \dots q_{(t)})] \dots] = (y_i, 1),$$

and  $\text{Hnil}_0(\text{aut}_1(X)) = t + 1$ . A similar computation gives the result for  $\text{aut}_\#(X)$ . Note that a derivation of the form  $(y_i, y_j)$  is not an element of  $\text{Der}_\#(AV)$ , for if it is a cycle then  $H^n(X) \neq 0$ , and  $\dim(X) \geq n$ .

The Dichotomy Theorem [9] tells that the finite 1-connected CW-complexes divide rationally into two classes: those *elliptic*, as the spheres, for which  $\sum_{i=2}^\infty \text{rank}(\pi_i(X))$  is finite and those *hyperbolic*, as wedges of spheres, for which the series  $\sum_{i=2}^\infty \text{rank}(\pi_i(X))$

grows exponentially. If  $X$  is elliptic, then its minimal model has a finite number of generators, hence  $\text{aut}_1(X)_0$ ,  $\text{aut}_\#(X)_0$  and  $\text{aut}_\star(X)_0$  are homotopy nilpotent. We give an example of a hyperbolic space  $X$  such that  $\text{aut}_1(X)_0$  is not homotopy nilpotent. We do not know if this happens for every hyperbolic space. Our example is complemented by [11, Proposition 3].

**Example 19.** If  $X$  is a wedge of copies of  $S^{2n+1}$ , then  $\text{aut}_1(X)_0$  is not homotopy nilpotent.

For the sake of simplicity we prove the statement only for a wedge of two spheres.

The rational graded Lie algebra  $L = \pi_*(\Omega(S^{2n+1} \vee S^{2n+1}))_0$  is free on two generators  $a, b$  of degree  $2n$ . Let  $\text{Der}(L)$  be the rational graded Lie algebra of its derivations and  $\text{Ad}(L)$  the ideal of the inner derivations of the form  $I_x = [x, -]$ ,  $x \in L$ . Then [12] there is a graded Lie algebra isomorphism

$$\pi_*(\text{aut}_1(S^{2n+1} \vee S^{2n+1}))_0 \cong \text{Der}(L)/\text{Ad}(L).$$

By freeness a derivation is uniquely determined by its values on  $a$  and  $b$ . Furthermore we can identify the derivations of  $L$  with some particular derivations of its universal enveloping algebra  $UL$ , the free graded associative algebra on  $a$  and  $b$ . Consider the derivations  $X$  and  $Y$  determined, for  $x, y \in L$  of positive degree, by  $X(a) = [x, a]$ ,  $X(b) = 0$ ;  $Y(a) = 0$ ,  $Y(b) = [y, b]$ . We claim that the iterated product  $(\text{ad } Y)^m(X) = [\dots Y, [Y, X] \dots]$  is not an inner derivation. The derivation  $U = [Y, X] + I_{X(y)}$  is given by  $U(a) = [X(y) + Y(x), a]$ ,  $U(b) = 0$ . Now  $U$ , or equivalently  $[Y, X]$ , is inner if and only if  $X(y) + Y(x) = 0$ , because  $[-, a]$  and  $[-, b]$  are injective on  $L$  in degree  $\geq 2n$  (look at  $UL$ ). Choose  $y = [a, b]$  and  $x$  of degree  $\geq 2n$  such that

the first monomial of  $x$  in alphabetical order has the form  $a^i b^j at$ ,  
where  $i, j \geq 1$  and  $t$  is a generic monomial. (\*)

For example take  $x = [[a, b], [[a, b], b]]$ . What is the first monomial of  $X(y) + Y(x)$ ? By definition  $X(y) = X([a, b]) = [[x, a], b] = xab - axb - bxa + bax$ . The last two terms begin with  $b$  and have no influence. Note that  $b$  appears first at position  $i+1$  in the first monomial of  $xab$ , whilst it appears first at position  $i+2$  in the first monomial of  $-axb$ . By Leibniz's formula  $Y(x)$  has the form  $a^i Y(b) b^{j-1} at + a^i b Y(b) b^{j-2} at + \dots$ , so that the only monomial where  $b$  appears first at position  $i+2$  is  $a^{i+1} b^{j+1} at$ ; but here  $a$  reappears first at position  $i+j+3$ , whilst in the first monomial of  $-axb$ , that is  $-a^{i+1} b^j at$ ,  $a$  reappears first at position  $i+j+2$ . Hence the latter is the very first monomial, and is not cancelled by any other. So  $X(y) + Y(x) \neq 0$ , and  $[Y, X]$  is not inner. Now set  $X_1 = X$ ,  $x = x_1$ , and define by recursion  $x_{m+1} = X_m(y) + Y(x_m)$ ;  $X_{m+1}(a) = [x_{m+1}, a]$ ,  $X_{m+1}(b) = 0$ . Each  $x_m$  satisfies hypothesis (\*), so by induction  $(\text{ad } Y)^m(X)$  is not an inner derivation. Therefore the Lie algebra  $\pi_*(\text{aut}_1(S^{2n+1} \vee S^{2n+1}))_0$  is not nilpotent. By Theorem 3 we are done.

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