GLOBAL SOLUTIONS OF THE COMPRESSIBLE EULER-POISSON EQUATIONS FOR PLASMA WITH DOPING PROFILE FOR LARGE INITIAL DATA OF SPHERICAL SYMMETRY*

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Abstract. We establish the global-in-time existence of solutions of finite relative-energy for the multidimensional compressible Euler-Poisson equations for plasma with doping profile for large initial data of spherical symmetry. Both the total initial energy and the initial mass are allowed to be *unbounded*, and the doping profile is allowed to be of large variation. This is achieved by adapting a class of degenerate density-dependent viscosity terms, so that a rigorous proof of the inviscid limit of global weak solutions of the Navier-Stokes-Poisson equations with the density-dependent viscosity terms to the corresponding global solutions of the Euler-Poisson equations for plasma with doping profile can be established. New difficulties arise when tackling the non-zero varied doping profile, which have been overcome by establishing some novel estimates for the electric field terms so that the neutrality assumption on the initial data is avoided. In particular, we prove that no concentration is formed in the inviscid limit for the finite relative-energy solutions of the compressible Euler-Poisson equations with large doping profiles in plasma physics.

Key words. Euler-Poisson equations, Navier-Stokes-Poisson equations, multi-dimension, doping profile, plasma, compressible, compactness framework, inviscid limit, large data, relative-energy, infinite mass, spherical symmetry, *a priori* estimate, higher integrability, approximate solutions

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1. Introduction. The Euler-Poisson equations have been intensively studied due to their vast relevance to modeling physical phenomena, especially in semiconductor modeling and plasma physics; see [19, 28, 36, 50, 52] and the references cited therein. In this paper, we are concerned with the global-in-time existence of solutions for the multidimensional (M-D) compressible Euler-Poisson equations (CEPEs) for

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plasma with doping profile for large initial data of spherical symmetry in $\mathbb{R}^N$:

$$\begin{cases} \partial_t\rho + \mathrm{div}\mathcal{M} = 0, \\ \partial_t\mathcal{M} + \mathrm{div}(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}) + \nabla p + \rho\nabla\Phi = \mathbf{0}, \\ \Delta\Phi = d(\mathbf{x}) - \rho, \end{cases} \tag{1.1}$$

for $(t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^N$ with $N \geq 3$, where ρ is the density, p is the pressure, and $\mathcal{M} \in \mathbb{R}^N$ represents the momentum, $d(\mathbf{x})$ is the doping profile with $\lim\limits_{|\mathbf{x}|\to\infty} d(\mathbf{x}) = \rho_* > 0$ so that the background state varies with the space variable, and Φ is the self-consistent electric field potential function. When $\rho > 0$, $U = \frac{\mathcal{M}}{\rho} \in \mathbb{R}^N$ represents the velocity. For polytropic perfect gases, the constitutive pressure-density relation is determined by

$$p = p(\rho) = \kappa\rho^\gamma,$$

where $\gamma > 1$ is the adiabatic exponent and $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ (without loss of generality).

We are concerned with the Cauchy problem (1.1) with the Cauchy initial data:

$$(\rho,\mathcal{M})(0,\mathbf{x}) = (\rho_0,\mathcal{M}_0)(\mathbf{x}) \longrightarrow (\rho_*,\mathbf{0}) \qquad \text{as } |\mathbf{x}| \to \infty, \tag{1.2}$$

subject to the asymptotic condition:

$$\Phi(t,\mathbf{x}) \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty, \tag{1.3}$$

where $(\rho_*,\mathbf{0})$ is the constant far-field state for $(\rho,\mathcal{M})$. Since a global solution of CEPEs (1.1) usually contains the vacuum states $\{(t,\mathbf{x}) : \rho(t,\mathbf{x}) = 0\}$ where the fluid velocity $U(t,\mathbf{x})$ is not well-defined, we use momentum $\mathcal{M}(t,\mathbf{x})$ or $\frac{\mathcal{M}(t,\mathbf{x})}{\sqrt{\rho(t,\mathbf{x})}}$ as the physical variables, instead of $U(t,\mathbf{x})$, when the vacuum states occur. These choices of variables will be shown to be globally well-defined. It is challenging to prove the existence of global solutions of problem (1.1)–(1.3), due to the possible formation of shock waves and blowups in a finite time [15, 20, 30, 49, 50]. To solve this problem, we consider the inviscid limit of the solutions of the compressible Navier-Stokes-Poisson equations (CNSPEs) with delicately designed density-dependent viscosity terms in $\mathbb{R}^N$:

$$\begin{cases} \partial_t\rho + \mathrm{div}\mathcal{M} = 0, \\ \partial_t\mathcal{M} + \mathrm{div}\big(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}\big) + \nabla p + \rho\nabla\Phi = \varepsilon\mathrm{div}\big(\mu(\rho)D(\frac{\mathcal{M}}{\rho})\big) + \varepsilon\nabla\big(\lambda(\rho)\mathrm{div}(\frac{\mathcal{M}}{\rho})\big), \\ \Delta\Phi = d(\mathbf{x}) - \rho, \end{cases} \tag{1.4}$$

where $D(\frac{\mathcal{M}}{\rho}) = \frac{1}{2}\big(\nabla(\frac{\mathcal{M}}{\rho}) + (\nabla(\frac{\mathcal{M}}{\rho}))^\top\big)$ is the stress tensor, $\varepsilon > 0$ denotes the inverse of the Reynolds number, and the Lamé (shear and bulk) viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ are density dependent, which may vanish on the vacuum and satisfy

$$\mu(\rho) \geq 0, \quad \mu(\rho) + N\lambda(\rho) \geq 0 \qquad \text{for } \rho \geq 0.$$

Formally, for the inviscid limit $\varepsilon \to 0+$, system (1.4) of CNSPEs converges to system (1.1) of CEPEs. However, the rigourous mathematical proof of this limit has been one of the most challenging problems in compressible fluid dynamics; see Chen-Feldman [10], Dafermos [19], and the references cited therein.

System (1.1) of CEPEs is a fundamental prototype of hyperbolic balance laws arising in either the two-fluid theory in plasma physics or in the theory of self-consistent gravitational gaseous star. In the classical two-fluid model describing plasma dynamics for electron-ion fluids, the self-consistent electrostatic Newtonian potential satisfies the Poisson equation for the repulsive Coulomb interaction. It is worth noting that plasma, composed of swiftly-moving charged particles, accounts for more than ninety percent of the observable matter in the universe, including its presence within stars, intergalactic spaces, nebulas, and even neon signs. Understanding the existence or nonexistence of weak solutions, as well as the stability or instability of CEPEs for plasma are among the key challenging problems for nuclear fusion. Specifically, the acceleration of charged particles to high speeds through particle accelerators, leading to the emission of electromagnetic energy, plays a pivotal role in this intricate process.

For CEPEs (1.1) with constant doping profile, Guo [26] first constructed a global smooth irrotational solution without shock by using the dispersive Klein-Gordon effect and adapting Shatah's normal form method. It is due to the enhanced dispersive effects induced by the repulsive electrostatic interaction, which is different from the pure compressible Euler equations for neutral gas. Guo-Strauss [29] proved the existence of global smooth solutions near a given steady state of the Euler-Poisson equations, for which the steady state and the doping profile are permitted to be of large variation while the initial velocity must be small. Later on, Guo-Pausader [28] constructed global smooth irrotational solutions with small amplitude for the Euler-Poisson equations for ion dynamics. Jang [37] constructed two-dimensional (2-D) global smooth solutions for spherically symmetric flows with small perturbed initial data. A family of smooth solutions was constructed in [38]. Smooth irrotational solutions in $\mathbb{R}^2$ were constructed independently by Ionescu-Pausader [35] and Li-Wu [42]. Such a surprising and subtle dispersive property has also been identified and exploited in other two-fluid models, which leads to the persistence of global smooth solutions and absence of shock formation. Guo-Ionescu-Pausader [27] first proved the global stability of a constant neutral background in the sense that irrotational, smooth, and localized perturbations of a constant background with small amplitude lead to global smooth solutions in $\mathbb{R}^3$ for the Euler-Maxwell two-fluid systems. These results can be applied equally well to other plasma models such as the Euler-Poisson systems for two-fluids.

As indicated earlier, due to the hyperbolic structure of the Euler-Poisson equations, for large initial data, the smooth solutions of (1.1) may form shock waves and blowup in finite time; see [15, 20, 30, 49, 50]. For the compressible Euler equations, we refer to [8, 9, 12, 14, 16, 17, 21, 22, 33, 39, 41, 46, 47] and the references therein. For the global existence of solutions of the compressible Navier-Stokes equations, we refer to [23, 40, 45] for the case of constant viscosity and [44, 62] for the case of density-dependent viscosity and the reference therein. In [44, 62], the Bresch-Desjardins entropy (BD entropy) plays an essential role to obtain the derivative estimate of the density, which was first identified by Bresch-Desjardins [2] (also see [3, 4, 5, 6, 7]). The BD entropy estimate is also crucial in our analysis in this paper. The global existence of weak solutions of the compressible Navier-Stokes equations with spherical symmetry is established in [31] for $\gamma \in (1,3)$ in a finite region under the Dirichlet boundary conditions.

Regarding to the inviscid gases as viscous gases with vanishing real physical viscosity can date back to the pioneering paper by Stokes [60] and the important contributions of Rankine [57], Hugoniot [34], and Rayleigh [58] (*cf.* Dafermos [19]). The first convergence proof of the inviscid limit from the one-dimensional (1-D) barotropic

Navier-Stokes to Euler equations was rigorously provided by Gilbarg [24], in which he established the mathematical existence and vanishing viscous limit of the Navier-Stokes shock layers. For the convergence analysis confined in the framework of piecewise smooth solutions, see [25, 32, 63] and the references cited therein. Due to the lack of the classical L^∞ uniform estimate, the L^∞ compensated compactness framework fails to work directly in the inviscid limit of the compressible Navier-Stokes equation; see [21, 22, 47, 46]. An L^p compensated compactness framework was first established by LeFloch-Wesdickenberg [41] for the isentropic Euler equations with geometric effects for $\gamma \in (1, \frac{5}{3})$, and was later further developed by Chen-Perepelitsa [12] to handle all the adiabetic exponent $\gamma > 1$ with a simplified proof to prove the vanishing real physical viscosity limit of smooth solutions for the 1-D Navier-Stokes equations to the corresponding finite relative-energy solutions of the Euler equations with $\rho_* > 0$. In [13, 14], the existence theory of global finite-energy entropy solutions of the Euler equations with spherical symmetry and large initial data was established for $\rho_* = 0$ through constructing artificial viscosity approximate smooth solutions (also see [59]). Chen-Wang [16] proved the existence of global finite relative-energy solutions for the compressible Euler equations with spherical symmetry through taking a vanishing physical viscosity limit even for $\rho_* > 0$. Most recently, Chen-He-Wang-Yuan [11] established the global existence of finite-energy solutions of the M-D Euler-Poisson equations for both compressible gaseous stars and plasmas with large initial data of spherical symmetry. For further work related with CEPEs with doping profile, we refer to [1, 43, 48, 51] and the references cited therein.

In this paper, we are concerned with the spherically symmetric solutions of (1.1) for plasma with doping profile. Denote

$$\rho(t,\mathbf{x}) = \rho(t,r), \quad \mathcal{M}(t,\mathbf{x}) = m(t,r)\frac{\mathbf{x}}{r}, \quad \Phi(t,\mathbf{x}) = \Phi(t,r) \qquad \text{for } r = |\mathbf{x}|, \tag{1.5}$$

subject to the initial conditions:

$$(\rho,\mathcal{M})(0,\mathbf{x}) = (\rho_0,\mathcal{M}_0)(\mathbf{x}) = (\rho_0(r), m_0(r)\frac{\mathbf{x}}{r}) \longrightarrow (\rho_*, \mathbf{0}) \qquad \text{as } |\mathbf{x}| \longrightarrow \infty, \tag{1.6}$$

and the asymptotic condition:

$$\Phi(t,\mathbf{x}) = \Phi(t,r) \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty. \tag{1.7}$$

It is not necessary to impose the initial data for Φ, since $\Phi(0,\mathbf{x})$ can be determined by the initial density and the boundary condition (1.7).

We establish the inviscid limit of the corresponding spherically symmetric solutions of CNSPEs (1.4) with the adapted class of degenerate density-dependent viscosity terms and approximate initial data of form (1.6). For spherically symmetric solutions of form (1.5), systems (1.1) and (1.4) become

$$\begin{cases} \rho_t + m_r + \frac{N-1}{r} m = 0, \\ m_t + (\frac{m^2}{\rho} + p)_r + \frac{N-1}{r}\frac{m^2}{\rho} + \rho\Phi_r = 0, \\ \Phi_{rr} + \frac{N-1}{r}\Phi_r = d(r) - \rho, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \rho_t + m_r + \frac{N-1}{r} m = 0, \\ m_t + (\frac{m^2}{\rho} + p)_r + \frac{N-1}{r}\frac{m^2}{\rho} + \rho\Phi_r = \varepsilon\Big((\mu+\lambda)\big((\frac{m}{\rho})_r + \frac{N-1}{r}\frac{m}{\rho}\big)\Big)_r - \varepsilon\frac{N-1}{r}\mu_r\frac{m}{\rho}, \\ \Phi_{rr} + \frac{N-1}{r}\Phi_r = d(r) - \rho, \end{cases} \tag{1.9}$$

respectively. We emphasize that the differentiation with respect to r now refers to the differentiation in the radial direction.

For the potential function Φ, we need to assume the boundary condition:

$$\lim_{r\to 0+} r^{N-1}\Phi_r(t,r) = 0 \qquad \text{for all } t \geq 0.$$

This leads to

$$r^{N-1}\Phi_r(t,r) = -\int_0^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y.$$

The study of spherically symmetric solutions can date back to the 1950s and has been motivated by many important physical problems such as the stellar dynamics including gaseous stars and supernova formation [18, 54, 55]. A fundamental open problem is whether the concentration is formed at the origin for both the compressible Euler equations and the Euler-Poisson equations, which has been answered for the solutions as the inviscid limits in [11, 16] (without doping profile). In this paper, we prove that no concentration (no delta measure) is formed at the origin, for CEPEs (1.1) even with doping profile for plasma.

The main difficulties in this paper are twofolds:

(i) Since the initial data satisfy (1.2), the initial mass is allowed to be *unbounded*:

$$\int_{\mathbb{R}^N} \rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = \omega_N \int_0^\infty \rho_0(r)\, r^{N-1}\mathrm{d}r = \infty,$$

where $\omega_N := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ denotes the surface area of unit sphere in $\mathbb{R}^N$. In addition, the initial total energy is *unbounded*, while initial total relative-energy is finite. We overcome this difficulty by fully taking advantage of (3.9) via the conservation of relative mass. Note that the finite *relative-mass* depends on the upper bound of the truncated domain; see (3.9).

(ii) Since the compressible Euler-Poisson equations involve the non-zero varied doping profile, new difficulties occur due to the electric field potential terms Φ depending on the doping profile. To solve these new difficulties, we make a more careful analysis in the estimates of the electric field potential terms. In order to control these terms, our key observation is to use the boundedness of the initial total relative-energy to control the electric potential, especially to tackle the non-zero doping profile terms. We establish two delicate uniform local estimates of the density near the origin, especially Lemma 3.4, which is essential to the convergence proof of the electric field potential; see (6.13) and (6.14). We remark that, differently from [16], when proving the lower bound of the density, the higher exponent of integrability for the derivatives of the density is required; see Lemma 4.3, especially (4.15). When we prove the higher integrability of the velocity, we need the extra weight for the electric field terms; see (4.1)–(4.3). These estimates indicate that the estimates of the electric field should be much more involved in the proof of Lemma 4.5 and the follow-up lemmas than those for the case without doping profile in [11].

We remark that no imperative neutrality condition is assumed on the initial data:

$$\int_{\mathbb{R}^N} (\rho_0(\mathbf{x}) - d(\mathbf{x}))\,\mathrm{d}\mathbf{x} \neq 0$$

in our analysis. It is natural to avoid this nonphysical assumption when we take into account of the effect of the positive density at the far fields ($|\mathbf{x}| \to \infty$). This is achieved by constructing approximate solutions without imposing the extra boundary restriction: $\Phi_r(t,\mathfrak{b})=0$, where $\mathfrak{b}$ is the upper bound of $|\mathbf{x}|$ for the truncated domain. To avoid this imperative condition, we need to establish some novel estimates when performing integration by parts in the proof of BD-type entropy estimates. The boundary terms are needed to be delicately controlled; for example, see the last term of the right-hand side of (3.26) in Steps 5–6 in the proof of Lemma 3.5. The key idea is to make these terms to be controlled by the initial total finite relative-energy.

The plan of this paper is as follows: In §2, we first introduce the definitions of finite relative-energy solutions of the Cauchy problem (1.1)–(1.3) for CEPEs and then state Main Theorem I: Theorem 2.2 for the global existence of weak solutions. To prove Theorem 2.2, the global weak solutions of the Cauchy problem (1.4) and (2.6)–(2.7) for CNSPEs are firstly constructed and their inviscid limit is then analyzed, as stated in Main Theorem II: Theorem 2.4. In §3, we give the construction of global approximate smooth solutions $(\rho^{\varepsilon,\delta,\mathfrak{b}},m^{\varepsilon,\delta,\mathfrak{b}})$ and perform the basic energy estimate and the BD-type entropy estimate of $(\rho^{\varepsilon,\delta,\mathfrak{b}},m^{\varepsilon,\delta,\mathfrak{b}})$, for CNSPEs (1.9). In §4, we derive the higher integrability of $(\rho^{\varepsilon,\delta,\mathfrak{b}},m^{\varepsilon,\delta,\mathfrak{b}})$, uniformly in $\mathfrak{b}$, for the approximate smooth solutions. In §5, through taking the limit of $(\rho^{\varepsilon,\delta,\mathfrak{b}},m^{\varepsilon,\delta,\mathfrak{b}})$ as $\mathfrak{b}\to\infty$, we obtain global strong solutions $(\rho^{\varepsilon,\delta},m^{\varepsilon,\delta})$ of system (3.1) with several uniform bounds in (ε,δ), and then we pass to the limit, $\delta\to 0+$, to obtain global existence of spherically symmetric weak solutions of CNSPEs (1.9) with several desired uniform bounds and the $H^{-1}_{\rm loc}$–compactness, which are important for us to utilize the L^p compensated compactness framework in §6 to establish Theorem 2.4. In the appendix, the approximate initial data with desired estimates are constructed, which are needed for the construction of the approximate solutions in §3.

In this paper, we write $L^p(\Omega), W^{k,p}(\Omega)$, and $H^k(\Omega)$ as the usual Sobolev spaces defined on Ω for $p\in[1,\infty]$. We also write $L^p(I;r^{N-1}\mathrm{d}r)$ or $L^p([0,T)\times I;r^{N-1}\mathrm{d}r\mathrm{d}t)$ for the open interval $I\subseteq\mathbb{R}_+$ with measure $r^{N-1}\mathrm{d}r$ or $r^{N-1}\mathrm{d}r\mathrm{d}t$ respectively, and $L^p_{\rm loc}([0,\infty);r^{N-1}\mathrm{d}r)$ to denote $L^p([0,K);r^{N-1}\mathrm{d}r)$ for arbitrary fixed $K>0$.

2. Mathematical Problems and Main Theorems.

In this section, we introduce the definition of weak solutions of finite relative-energy solutions of the Cauchy problem for CEPEs (1.1) in $\mathbb{R}^{N+1}_+ := \mathbb{R}_+\times\mathbb{R}^N=(0,\infty)\times\mathbb{R}^N$ for $N\geq 3$. We assume that the initial data $(\rho_0,\mathcal{M}_0)(\mathbf{x})$ and the corresponding initial potential function $\Phi_0(\mathbf{x})$ have both finite total initial relative-energy:

$$\mathcal{E}_0 := \int_{\mathbb{R}^N}\Big(\frac{1}{2}\Big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\Big|^2+e(\rho_0,\rho_*)+\frac{1}{2}|\nabla_{\mathbf{x}}\Phi_0|^2\Big)\mathrm{d}\mathbf{x}<\infty, \tag{2.1}$$

where $e(\rho,\rho_*)$ is the relative internal energy with respect to $\rho_*>0$:

$$e(\rho,\rho_*):=e(\rho)-e(\rho_*)-e'(\rho_*)(\rho-\rho_*)=\frac{\kappa}{\gamma-1}\big(\rho^\gamma-\rho_*^\gamma-\gamma\rho_*^{\gamma-1}(\rho-\rho_*)\big), \tag{2.2}$$

where $e(\rho)=\frac{\kappa}{\gamma-1}\rho^\gamma$ denotes the gas internal energy.

We also assume that the doping profile satisfies

$$d(r)\in C([0,\infty)),\qquad \lim_{r\to\infty}d(r)=\rho_*>0, \tag{2.3}$$

$$\int_0^\infty |d(r)-\rho_*|^j r^{N-1}\mathrm{d}r<\infty \qquad \text{for } j=1,2,\tfrac{\gamma}{\gamma-1}. \tag{2.4}$$

Definition 2.1 (Weak Solutions). *A measurable vector function* $(\rho,\mathcal{M},\Phi)$ *is said to be a weak solution of finite relative-energy of the Cauchy problem* (1.1)–(1.3) *provided that*

(i) $\rho(t,\mathbf{x})\geq 0$ *a.e., and* $(\mathcal{M},\frac{\mathcal{M}}{\sqrt{\rho}})(t,\mathbf{x})=\mathbf{0}$, *a.e. on the vacuum states* $\{(t,\mathbf{x}):\rho(t,\mathbf{x})=0\}$;

(ii) *For a.e.* $t>0$, *the total relative-energy with respect to the far field state* $(\rho_*,\mathbf{0})$ *is finite*:

$$\int_{\mathbb{R}^N}\Big(\frac{1}{2}\Big|\frac{\mathcal{M}}{\sqrt{\rho}}\Big|^2+e(\rho,\rho_*)+\frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\Big)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\leq\mathcal{E}_0. \tag{2.5}$$

(iii) *For any* $\zeta(t,\mathbf{x})\in C_0^1([0,\infty)\times\mathbb{R}^N)$,

$$\int_0^\infty\int_{\mathbb{R}^N}\big(\rho\zeta_t+\mathcal{M}\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}(\rho\zeta)(0,\mathbf{x})\,\mathrm{d}\mathbf{x}=0.$$

(iv) *For any* $\Psi(t,\mathbf{x})=(\Psi_1,\cdots,\Psi_N)(t,\mathbf{x})\in(C_0^1([0,\infty)\times\mathbb{R}^N))^N$,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\Big(\mathcal{M}\cdot\partial_t\Psi+\frac{\mathcal{M}}{\sqrt{\rho}}\cdot(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla)\Psi+p(\rho)\mathrm{div}\Psi\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\\&\quad+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\Psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_0^\infty\int_{\mathbb{R}^N}\rho\nabla_{\mathbf{x}}\Phi\cdot\Psi\,\mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

(v) *For any* $\xi(\mathbf{x})\in C_0^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\nabla_{\mathbf{x}}\Phi(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^N}\big(\rho(t,\mathbf{x})-d(\mathbf{x})\big)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}\qquad\text{for a.e. } t\geq 0.$$

THEOREM 2.2 (Main Theorem I: Existence of Global Solutions of CEPEs for Plasma with Doping Profile and Large Initial Data of Spherical Symmetry). *Consider the Cauchy problem* (1.1)–(1.3) *for CEPEs with large initial data of spherical symmetry of form* (1.6)–(1.7). *Assume that the doping profile and the initial density satisfy* (2.1), (2.3)–(2.4), *and* $\rho_0-d(\mathbf{x})\in(L^{\frac{2N}{N+2}}\cap L^1)(\mathbb{R}^N)$. *Then there exists a globally defined weak solution* $(\rho,\mathcal{M},\Phi)$ *of finite relative-energy of the Cauchy problem* (1.1)–(1.3) *and* (1.6)–(1.7) *with spherical symmetry of form* (1.5) *in the sense of* Definition 2.1 *such that* $(\rho,m,\Phi)(t,r)$ *is determined by the corresponding system* (1.8) *with initial data* $(\rho_0,m_0,\Phi_0)(r)$ *given in* (1.6) *subject to the asymptotic behavior* (1.7).

To prove Theorem 2.2, we first construct global weak solutions for CNSPEs (1.4) with appropriately adapted degenerate density-dependent viscosity terms and approximate initial data:

$$(\rho,\mathcal{M},\Phi)|_{t=0}=(\rho_0^\varepsilon,\mathcal{M}_0^\varepsilon,\Phi_0^\varepsilon)(\mathbf{x})\longrightarrow(\rho_0,\mathcal{M}_0,\Phi_0)(\mathbf{x})\qquad\text{as }\varepsilon\to 0+, \tag{2.6}$$

subject to the asymptotic boundary condition:

$$\Phi^\varepsilon(t,\mathbf{x})\to 0\qquad\text{as }|\mathbf{x}|\to\infty. \tag{2.7}$$

For simplicity, we consider the viscosity terms with $(\mu,\lambda)=(\rho,0)$ in (1.4), and $\varepsilon\in(0,1]$ without loss of generality.

DEFINITION 2.3. *A pair* $(\rho^\varepsilon,\mathcal{M}^\varepsilon,\Phi^\varepsilon)$ *is said to be a weak solution of the Cauchy problem* (1.4) *and* (2.6)–(2.7) *with* $(\mu,\lambda)=(\rho,0)$ *if the following conditions hold*:

(i) $\rho^\varepsilon(t,\mathbf{x}) \ge 0$ *a.e.*, $(\mathcal{M}^\varepsilon, \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}})(t,\mathbf{x}) = \mathbf{0}$ *a.e. on the vacuum states* $\{(t,\mathbf{x}) : \rho^\varepsilon(t,\mathbf{x}) = 0\}$,

$$\begin{aligned}
&\rho^\varepsilon \in L^\infty(0,T;L^\gamma_{\mathrm{loc}}(\mathbb{R}^N)), \quad \nabla\sqrt{\rho^\varepsilon} \in L^\infty(0,T;L^2(\mathbb{R}^N)),\\
&\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}} \in L^\infty(0,T;L^2(\mathbb{R}^N)), \quad \Phi^\varepsilon \in L^\infty(0,T;L^{\frac{2N}{N-2}}(\mathbb{R}^N)),\\
&\nabla\Phi^\varepsilon \in L^\infty(0,T;L^2(\mathbb{R}^N)).
\end{aligned}$$

(ii) *For any* $t_2 \ge t_1 \ge 0$ *and* $\zeta(t,\mathbf{x}) \in C^1_0([0,\infty)\times\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\rho^\varepsilon\zeta)(t_2,\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^N} (\rho^\varepsilon\zeta)(t_1,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (\rho^\varepsilon\zeta_t + \mathcal{M}^\varepsilon\cdot\nabla\zeta)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned}$$

(iii) *For any* $\Psi(t,\mathbf{x}) = (\Psi_1,\cdots,\Psi_N)(t,\mathbf{x}) \in (C^1_0([0,\infty)\times\mathbb{R}^N))^N$,

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N} \Big(\mathcal{M}^\varepsilon\cdot\partial_t\Psi + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla)\Psi + p(\rho^\varepsilon)\mathrm{div}\Psi\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\qquad + \int_{\mathbb{R}^N} \mathcal{M}^\varepsilon_0(\mathbf{x})\cdot\Psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&\quad = \int_0^\infty\int_{\mathbb{R}^N} (\rho^\varepsilon\nabla_{\mathbf{x}}\Phi^\varepsilon\cdot\Psi)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\qquad - \varepsilon\int_0^\infty\int_{\mathbb{R}^N} \Big(\frac{1}{2}\mathcal{M}^\varepsilon\cdot(\Delta\Psi + \nabla\mathrm{div}\Psi)\\
&\qquad\qquad + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla)\Psi + \nabla\sqrt{\rho^\varepsilon}\cdot(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla)\Psi\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned}$$

(iv) *For any* $\xi(\mathbf{x}) \in C^1_0(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla^\varepsilon_{\mathbf{x}}\Phi(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}\xi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \big(\rho^\varepsilon(t,\mathbf{x}) - d(\mathbf{x})\big)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \textit{for a.e. } t \ge 0.$$

We now consider spherical symmetric solutions of form (1.5). Then systems (1.1) and (1.4) become systems (1.8) and (1.9), respectively. A pair of functions $(\eta(\rho,m), q(\rho,m))$, or $(\eta(\rho,u), q(\rho,u))$ for $m = \rho u$, is called an entropy-entropy flux pair for the first two equations of system (1.8) (with $N = 1$ and $\Phi = 0$) if (η, q) is a solution of

$$\nabla q(\rho,m) = \nabla\eta(\rho,m)\nabla\begin{pmatrix} m\\ \frac{m^2}{\rho} + p(\rho)\end{pmatrix},$$

which implies that, for any smooth solution $(\rho(t,r), m(t,r))$ of (1.8), the following equation holds:

$$\partial_t\eta(\rho(t,r),m(t,r)) + \partial_r q(\rho(t,r),m(t,r)) = 0.$$

Moreover, $\eta(\rho,m)$ is called a weak entropy if

$$\eta|_{\rho=0} = 0 \qquad \text{for any fixed } u = \frac{m}{\rho}.$$

We denote $u = \frac{m}{\rho}$ and m alternatively when $\rho > 0$. We also call $\eta(\rho, m)$ a convex entropy if the Hessian $\nabla^2\eta(\rho, m) \geq 0$ in the region under consideration. From [47], it is well-known that any weak entropy pair (η, q) can be represented by

$$\begin{cases} \eta^{\psi}(\rho, m) = \eta(\rho, \rho u) = \int_{\mathbb{R}} \chi(\rho; s-u)\psi(s)\,\mathrm{d}s, \\ q^{\psi}(\rho, m) = \eta(\rho, \rho u) = \int_{\mathbb{R}} (\theta s + (1-\theta)u)\chi(\rho; s-u)\psi(s)\,\mathrm{d}s, \end{cases} \tag{2.8}$$

where the kernel is $\chi(\rho; s-u) = [\rho^{2\theta} - (s-u)^2]_+^{\lambda}$ with $\lambda = \frac{3-\gamma}{2(\gamma-1)} > -\frac{1}{2}$ and $\theta = \frac{\gamma-1}{2}$. In particular, $\psi(s) := \frac{1}{2}s^2$, the entropy pair is the mechanical energy-energy flux pair:

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + e(\rho), \quad q^*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m e'(\rho).$$

Since $(\rho, m)(t, r) \to (\rho_*, 0)$ with $\rho_* > 0$ as $r \to \infty$, We use the relative mechanical energy

$$\bar{\eta}^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + e(\rho, \rho_*)$$

with $e(\rho, \rho_*)$ is defined in (2.2) satisfying

$$e(\rho, \rho_*) \geq C_\gamma \rho(\rho^\theta - \rho_*^\theta)^2$$

for some constant $C_\gamma > 0$.

THEOREM 2.4 (Main Theorem II: Existence and Inviscid Limit for CNSPEs). *Consider CNSPEs* (1.4) *with* $N \geq 3$ *and the spherically symmetric approximate initial data* (2.6) *satisfying that, as* $\varepsilon \to 0+$,

$$(\rho_0^\varepsilon, m_0^\varepsilon)(r) \to (\rho_0, m_0)(r) \quad \text{in } L^p_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r) \times L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r) \tag{2.9}$$

with $p = \max\{\gamma, \frac{2N}{N+2}\}$ *and*

$$\mathcal{E}_0^\varepsilon = \omega_N \int_0^\infty \Big(\bar{\eta}^*(\rho_0^\varepsilon, m_0^\varepsilon) + \frac{1}{2}|\Phi_{0r}^\varepsilon(r)|^2\Big)\, r^{N-1}\mathrm{d}r \to \mathcal{E}_0, \tag{2.10}$$

$$\mathcal{E}_1^\varepsilon = \varepsilon^2 \int_0^\infty |(\sqrt{\rho_0^\varepsilon})_r|^2\, r^{N-1}\mathrm{d}r \to 0, \tag{2.11}$$

and there exists a constant $C > 0$ *independent of* $\varepsilon \in (0, 1]$ *such that*

$$\mathcal{E}_0^\varepsilon + \mathcal{E}_1^\varepsilon \leq C(\mathcal{E}_0 + 1) \tag{2.12}$$

for $\mathcal{E}_0$ *defined in* (2.1)*. Then the following statements hold*:

Part 1. Existence of Solutions for CNSPEs (1.4):

(i) *For each* $\varepsilon > 0$*, there exists a globally-defined spherically symmetric solution*

$$\begin{aligned}(\rho^\varepsilon, \mathcal{M}^\varepsilon, \Phi^\varepsilon)(t, \mathbf{x}) &= (\rho^\varepsilon(t, r), m^\varepsilon(t, r)\frac{\mathbf{x}}{r}, \Phi^\varepsilon(t, r)) \\ &= (\rho^\varepsilon(t, r), \rho^\varepsilon(t, r)u^\varepsilon(t, r)\frac{\mathbf{x}}{r}, \Phi^\varepsilon(t, r))\end{aligned}$$

of the Cauchy problem (1.4) *and* (2.6)–(2.7) *in the sense of Definition* 2.3 *with*

$$u^\varepsilon(t, r) = \begin{cases} \frac{m^\varepsilon(t,r)}{\rho^\varepsilon(t,r)} & \text{a.e. on } \{(t, r) : \rho^\varepsilon(t, r) \neq 0\}, \\ 0 & \text{a.e. on } \{(t, r) : \rho^\varepsilon(t, r) = 0\}. \end{cases}$$

In addition, $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)(t,r)$ *satisfies the following uniform estimates*: *For* $t > 0$,

$$\begin{aligned}\int_0^\infty \bar{\eta}^*(\rho^\varepsilon, m^\varepsilon)(t,r)\, r^{N-1}\mathrm{d}r + \varepsilon \int_{\mathbb{R}^2_+} \rho^\varepsilon(s,r)|u^\varepsilon(s,r)|^2\, r^{N-3}\mathrm{d}r\mathrm{d}s \\ + \frac{1}{2}\int_0^\infty |\Phi^\varepsilon_{0r}|^2\, r^{N-1}\mathrm{d}r \le \frac{\mathcal{E}^\varepsilon_0}{\omega_N} \le C(\mathcal{E}_0+1),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\varepsilon^2 \int_0^\infty |(\sqrt{\rho^\varepsilon(t,r)})_r|^2\, r^{N-1}\mathrm{d}r \\ + \varepsilon \int_{\mathbb{R}^2_+} |((\rho^\varepsilon(s,r))^{\frac{\gamma}{2}})_r|^2\, r^{N-1}\mathrm{d}r\mathrm{d}s \le C(\mathcal{E}_0+1),\end{aligned} \tag{2.14}$$

and, for any given $T \in (0,\infty)$ *and any compactly supported subset* $[r_1, r_2] \Subset (0,\infty)$,

$$\int_0^T \int_{r_1}^{r_2} (\rho^\varepsilon(t,r))^{\gamma+1}\mathrm{d}r\mathrm{d}t \le C(r_1, r_2, T, \mathcal{E}_0), \tag{2.15}$$

$$\int_0^T \int_{r_1}^{r_2} \Big(\rho^\varepsilon(t,r)|u^\varepsilon(t,r)|^3 + (\rho^\varepsilon(t,r))^{\gamma+\theta}\Big)\,\mathrm{d}r\mathrm{d}t \le C(r_1, r_2, T, \mathcal{E}_0), \tag{2.16}$$

$$\int_0^D \rho^\varepsilon(t,r) r \Big(\int_0^r \rho^\varepsilon(t,y) y^{N-1}\,\mathrm{d}y\Big)\mathrm{d}r \le C(D, \mathcal{E}_0), \tag{2.17}$$

where $C > 0$ *and* $C(r_1, r_2, T, \mathcal{E}_0) > 0$ *are two constants independent of* ε, *but may depend on* (γ, N) *and* $(r_1, r_2, T, \mathcal{E}_0)$ *respectively,* $D > 0$ *and* $\mathbb{R}^2_+ =: \{(t,r) : t \in (0,\infty), r \in (0,\infty)\}$.

(ii) *The following energy inequality holds*:

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(\frac{1}{2}\Big|\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 + e(\rho^\varepsilon, \rho_*) + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi^\varepsilon|^2\big)(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\quad \le \int_{\mathbb{R}^N} \big(\frac{1}{2}\Big|\frac{\mathcal{M}^\varepsilon_0}{\sqrt{\rho^\varepsilon_0}}\Big|^2 + e(\rho^\varepsilon_0, \rho_*) + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi^\varepsilon_0|^2\big)(\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \textit{for } t \ge 0.\end{aligned} \tag{2.18}$$

(iii) *Let* (η, q) *be an entropy pair defined by* (2.8) *for a smooth compactly supported function* $\psi(s)$ *on* $\mathbb{R}$. *Then, for* $\varepsilon \in (0,1]$,

$$\partial_t \eta^\psi(\rho^\varepsilon, m^\varepsilon) + \partial_r q^\psi(\rho^\varepsilon, m^\varepsilon) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+), \tag{2.19}$$

where $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+)$ *denotes* $H^{-1}((0,T)\times\Omega)$ *for any* $T > 0$ *and open subset* $\Omega \Subset \mathbb{R}_+$.

Part 2. Inviscid Limit and Existence of Global Solutions of CEPEs (1.1): *For the global weak solutions* $(\rho^\varepsilon, \mathcal{M}^\varepsilon, \Phi^\varepsilon)$ *of CNPSEs* (1.4) *obtained in* Part 1, *there exist a subsequence* (*still denoted*) $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)$ *and a vector function* (ρ, m, Φ) *such* *that, as* $\varepsilon \to 0+$,

$$
\begin{aligned}
&(\rho^\varepsilon, m^\varepsilon) \to (\rho, m)(t,r) \qquad \textit{in } (L^p_{\mathrm{loc}} \times L^q_{\mathrm{loc}})([0,\infty); r^{N-1}\mathrm{d}r) \\
&\qquad\qquad\qquad\qquad\qquad \textit{for } p \in [1, \gamma+1) \textit{ and } q \in [1, \frac{3(\gamma+1)}{\gamma+3}) \\
&\Phi^\varepsilon \rightharpoonup \Phi \qquad \textit{weakly in } L^2(0,T; H^1_{\mathrm{loc}}(\mathbb{R}^N)), \\
&\Phi^\varepsilon_r(t,r) r^{N-1} = -\int_0^r \big(\rho^\varepsilon(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y \\
&\qquad\qquad \to \Phi_r(t,r) r^{N-1} = -\int_0^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y \qquad \textit{a.e. } (t,r) \in \mathbb{R}^2_+, \\
&\int_0^T \int_0^{r_2} |(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}})(t,r) - (\frac{m}{\sqrt{\rho}})(t,r)|^2\, r^{N-1}\mathrm{d}r\mathrm{d}t \to 0 \qquad \textit{for any fixed } T, r_2 \in (0,\infty),
\end{aligned}
$$

and $(\rho, \mathcal{M}, \Phi)(t,\mathbf{x}) := (\rho(t,r), m(t,r)\frac{\mathbf{x}}{r}, \Phi(t,r))$ *is a global spherically symmetric solution with finite relative-energy of problem* (1.1)–(1.3) *in the sense of* Definition 2.1.

3. Constructions of Approximate Solutions and Uniform Estimates.

In this section, we construct appropriate approximate solutions and obtain some uniform estimates of the approximate solutions of the Cauchy problem for CNSPEs (1.9) that corresponds to the initial conditions (2.6)–(2.7).

3.1. Constructions of Approximate Solutions.

In this subsection, we consider the approximate CNSPEs in the truncated domains:

$$
\begin{cases}
\rho_t + (\rho u)_r + \frac{N-1}{r}\rho u = 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_r + \frac{N-1}{r}\rho u^2 - \frac{\rho}{r^{N-1}} \int_\delta^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y \\
\qquad\qquad\qquad = \varepsilon\big((\mu+\lambda)(u_r + \frac{N-1}{r}u)\big)_r - \varepsilon\frac{N-1}{r}\mu_r u,
\end{cases} \tag{3.1}
$$

where $r \in [\delta, \mathfrak{b}]$ and $t > 0$, with $\delta \in (0,1]$ and $\mathfrak{b} \geq 1 + \delta^{-1}$. We consider (3.1) with the following initial data:

$$
(\rho, u)(0,r) = (\rho_0^{\varepsilon,\delta,\mathfrak{b}}, u_0^{\varepsilon,\delta,\mathfrak{b}})(r) \qquad \text{for } r \in [\delta, \mathfrak{b}], \tag{3.2}
$$

and the boundary condition:

$$
u(t,\delta) = u(t,\mathfrak{b}) = 0 \qquad \text{for } t > 0. \tag{3.3}
$$

We take

$$
\mu(\rho) = \rho + \delta\rho^\alpha, \quad \lambda(\rho) = \delta(\alpha-1)\rho^\alpha \tag{3.4}
$$

with $\alpha = \frac{2N-1}{2N}$ so that $\mu(\rho)$ and $\lambda(\rho)$ satisfy the following BD relation:

$$
\mu(\rho) + \lambda(\rho) = \rho\mu'(\rho). \tag{3.5}
$$

We first fix parameters $\varepsilon > 0, \delta > 0$, and $\mathfrak{b} \geq 1 + \delta^{-1}$, and assume that $(\rho_0^{\varepsilon,\delta,\mathfrak{b}}, u_0^{\varepsilon,\delta,\mathfrak{b}})$ are smooth functions such that

$$
0 < (\beta\varepsilon)^{\frac{1}{4}} \leq \rho_0^{\varepsilon,\delta,\mathfrak{b}} \leq (\beta\varepsilon)^{-\frac{1}{2}} < \infty \tag{3.6}
$$

for a given small constant $0 < \beta \ll 1$. The approximate initial data functions in (3.2) is constructed in Appendix A, which meet all the properties in Lemma A.1.

The existence of global smooth solutions of the initial-boundary value problem (3.1)–(3.3) can be achieved by following similar arguments as in §3 and §4.1 in Guo-Jiu-Xin [31]; see also [40]. Notice that the lower and upper bounds of $\rho^{\varepsilon,\delta,\mathfrak{b}}$ in [31] depend on parameters $(\varepsilon,\delta,\mathfrak{b})$. Therefore, the main strategy of this section is to obtain some uniform estimates of $(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})$ independent of $(\delta,\mathfrak{b})$ such that we can take both limits $\mathfrak{b}\to\infty$ and $\delta\to 0+$ to obtain the global existence of weak solutions of (1.4) and (2.6)–(2.7); see §5.

Since $\rho^{\varepsilon,\delta,\mathfrak{b}}$ is positive, we may use $u^{\varepsilon,\delta,\mathfrak{b}}$ or $m^{\varepsilon,\delta,\mathfrak{b}}$ alternatively , and ignore the superscripts of solution $(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})(t,r)$ and the approximate initial data $(\rho_0^{\varepsilon,\delta,\mathfrak{b}},u_0^{\varepsilon,\delta,\mathfrak{b}})$ for simplicity. When the initial data functions are involved, we keep the superscripts. After solving (3.1)–(3.3), we can define the potential function Φ as the solution of the Poisson equation:

$$-\Delta\Phi=(\rho-d(\mathbf{x}))\mathbf{I}_\Omega,\qquad \lim_{|\mathbf{x}|\to\infty}\Phi=0,\tag{3.7}$$

with $\Omega=\{\mathbf{x}\in\mathbb{R}^N:\delta\le|\mathbf{x}|\le\mathfrak{b}\}$. We can obtain that $\Phi(t,\mathbf{x})=\Phi(t,r)$ with

$$\Phi_r(t,r)=\begin{cases}0 & \text{for } r\in[0,\delta],\\ -\frac{1}{r^{N-1}}\int_\delta^r\big(\rho(t,y)-d(y)\big)\,y^{N-1}\mathrm{d}y & \text{for } r\in[\delta,\mathfrak{b}],\\ -\frac{1}{r^{N-1}}\int_\delta^{\mathfrak{b}}\big(\rho(t,y)-d(y)\big)\,y^{N-1}\mathrm{d}y & \text{for } r\in[\mathfrak{b},\infty).\end{cases}\tag{3.8}$$

Using the conservation law of mass, we obtain

$$-\int_\delta^{\mathfrak{b}}\big(\rho(t,y)-d(y)\big)\,y^{N-1}\mathrm{d}y=-\int_\delta^{\mathfrak{b}}\big(\rho_0(y)-d(y)\big)\,y^{N-1}\mathrm{d}y:=\frac{M_{\mathfrak{b}}}{\omega_N}.\tag{3.9}$$

3.2. Uniform Estimates of the Approximate Solutions.

The main goal of this section is to obtain several uniform estimates that are independent of $(\varepsilon,\delta,\mathfrak{b})$ so that the two limits $\mathfrak{b}\to\infty$ and $\delta\to 0+$ can be taken in order.

LEMMA 3.1 (**Basic Energy Estimate).** *The smooth solution of* (3.1)–(3.3) *satisfies the following basic energy estimate*:

$$\begin{aligned}&\int_\delta^{\mathfrak{b}}\Big(\frac12\rho u^2+e(\rho,\rho_*)+\frac12|\Phi_r|^2\Big)(t,r)\,r^{N-1}\mathrm{d}r\\ &\quad+\varepsilon\int_0^t\int_\delta^{\mathfrak{b}}\rho\Big(u_r^2+\frac{N-1}{r^2}u^2\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\ &\quad+\varepsilon\delta\int_0^t\int_\delta^{\mathfrak{b}}\rho^\alpha\Big(\alpha u_r^2+2(\alpha-1)(N-1)\frac{uu_r}{r}\\ &\quad+\big(1+(N-1)(\alpha-1)\big)(N-1)\frac{u^2}{r^2}\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\ &=\int_\delta^{\mathfrak{b}}\Big(\frac12\rho_0u_0^2+e(\rho_0,\rho_*)+\frac12|\Phi_{0r}|^2\Big)(r)\,r^{N-1}\mathrm{d}r=:\frac{\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}}{\omega_N}.\end{aligned}$$

In particular, there exists a constant $c_N>0$ depending only on N such that, for the *smooth solution of* (3.1)–(3.3),

$$
\begin{aligned}
&\int_\delta^{\mathfrak{b}} \Big(\frac{1}{2}\rho u^2 + e(\rho,\rho_*) + \frac{1}{2}|\Phi_r|^2\Big)(t,r)\, r^{N-1}\mathrm{d}r\\
&\quad + c_N\varepsilon\delta \int_0^t\int_\delta^{\mathfrak{b}} \Big(\rho^\alpha u_r^2 + \frac{\rho^\alpha u^2}{r^2}\Big)(s,r)\, r^{N-1}\mathrm{d}r\mathrm{d}s\\
&\quad + \varepsilon\int_0^t\int_\delta^{\mathfrak{b}} \big(\rho u_r^2 + \frac{\rho u^2}{r^2}\big)(s,r) r^{N-1}\,\mathrm{d}r\mathrm{d}s \le \frac{\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}}{\omega_N} \le C(\mathcal{E}_0+1) \qquad \textit{for any } t>0
\end{aligned}
$$

for some constant $C>0$ *independent of* $(\varepsilon,\delta,\mathfrak{b})$.

Proof. Multiplying $(3.1)_2$ by $r^{N-1}u$ and then integrating by parts, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} \frac{1}{2}\rho u^2\, r^{N-1}\mathrm{d}r + \int_\delta^{\mathfrak{b}} p_r u\, r^{N-1}\mathrm{d}r\\
&= -\varepsilon\int_\delta^{\mathfrak{b}} \Big((\mu+\lambda)(u_r + \frac{N-1}{r}u)(r^{N-1}u)_r - (N-1)\mu(r^{N-2}u^2)_r\Big)\,\mathrm{d}r\\
&\quad + \int_\delta^{\mathfrak{b}} \rho u \int_\delta^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y\mathrm{d}r.
\end{aligned}
\tag{3.10}
$$

For the second term on the left-hand side (LHS) of (3.10), one has

$$
\begin{aligned}
\int_\delta^{\mathfrak{b}} p_r u\, r^{N-1}\mathrm{d}r &= \frac{\kappa\gamma}{\gamma-1}\int_\delta^{\mathfrak{b}} \rho u(\rho^{\gamma-1})_r\, r^{N-1}\mathrm{d}r = -\frac{\kappa\gamma}{\gamma-1}\int_\delta^{\mathfrak{b}} (r^{N-1}\rho u)_r \rho^{\gamma-1}\,\mathrm{d}r\\
&= \frac{\kappa}{\gamma-1}\int_\delta^{\mathfrak{b}} (\rho^\gamma)_t\, r^{N-1}\mathrm{d}r\\
&= \frac{\kappa}{\gamma-1}\int_\delta^{\mathfrak{b}} \Big(\rho^\gamma - \rho_*^\gamma - \gamma\rho_*{}^{\gamma-1}(\rho-\rho_*)\Big)_t\, r^{N-1}\mathrm{d}r\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} e(\rho,\rho_*)(t,r)\, r^{N-1}\mathrm{d}r.
\end{aligned}
$$

For the viscous term, a direct calculation shows that

$$
\begin{aligned}
&(\mu+\lambda)(u_r + \frac{N-1}{r})(r^{N-1}u)_r - (N-1)\lambda(r^{N-2}u^2)_r\\
&= \delta\rho^\alpha\Big\{\alpha r^{N-1}u_r^2 + 2(\alpha-1)(N-1)r^{N-2}uu_r + \big(N-1+(\alpha-1)(N-1)^2\big)r^{N-3}u^2\Big\}\\
&\quad + \rho\big(r^{N-1}u_r^2 + (N-1)r^{N-3}u^2\big).
\end{aligned}
\tag{3.11}
$$

For the first term on the right-hand side (RHS) of (3.11), noting that $\alpha\in(\frac{N-1}{N},1)$, we calculate its discriminant:

$$\begin{aligned}\Delta &= 4(\alpha-1)^2(N-1)^2 - 4\alpha\big(N-1+(\alpha-1)(N-1)^2\big)\\ &= 4(N-1)^2\big((\alpha-1)^2-\alpha(\alpha-1)\big) - 4(N-1)\alpha\\ &= 4(N-1)^2(1-\alpha) - 4(N-1)\alpha\\ &= 4(N-1)^2\big(1-\frac{N}{N-1}\alpha\big) < 0 \qquad \text{if } \alpha\in(\frac{N-1}{N},1),\end{aligned}$$

so that there exists a constant $c_N>0$ such that

$$\begin{aligned}&(\mu+\lambda)\big(u_r+\frac{N-1}{r}\big)(r^{N-1}u)_r-(N-1)\mu(r^{N-2}u^2)_r\\ &\ge c_N\delta\rho^\alpha\big(r^{N-1}u_r^2+r^{N-3}u^2\big)+\rho\big(r^{N-1}u_r^2+r^{N-3}u^2\big).\end{aligned}$$

For the last term on RHS of (3.10), a direct calculation yields

$$\begin{aligned}&\int_\delta^{\mathfrak{b}}\rho u\int_\delta^r\big(\rho(t,y)-d(y)\big)\,y^{N-1}\mathrm{d}y\mathrm{d}r\\ &=-\int_\delta^{\mathfrak{b}}\rho u\Phi_r(t,r)\,r^{N-1}\mathrm{d}r=\int_\delta^{\mathfrak{b}}(\rho u r^{N-1})_r\Phi\,\mathrm{d}r\\ &=-\int_\delta^{\mathfrak{b}}(r^{N-1}\rho)_t\Phi\,\mathrm{d}r=-\int_\delta^{\mathfrak{b}}\big(r^{N-1}(\rho-d(r))\big)_t\Phi\,\mathrm{d}r\\ &=\int_\delta^{\mathfrak{b}}\big(\Phi_{rr}+\frac{N-1}{r}\Phi_r\big)_t\Phi\,r^{N-1}\mathrm{d}r=\int_\delta^{\mathfrak{b}}(r^{N-1}\Phi_{rt})_r\Phi\,\mathrm{d}r\\ &=-\int_\delta^{\mathfrak{b}}\Phi_{rt}\Phi_r\,r^{N-1}\mathrm{d}r+(r^{N-1}\Phi_{rt}\Phi)\Big|_\delta^{\mathfrak{b}}=-\int_\delta^{\mathfrak{b}}\big(\frac{|\Phi_r|^2}{2}\big)_t\,r^{N-1}\mathrm{d}r\\ &=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}}|\Phi_r|^2\,r^{N-1}\mathrm{d}r,\end{aligned}$$

where we have used (3.8) and

$$\Phi_{rt}(t,\delta)=\Phi_{rt}(t,\mathfrak{b})=0 \qquad \text{for any } t\ge 0. \tag{3.12}$$

Combining (3.10)–(3.12), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\Big)\,r^{N-1}\mathrm{d}r\Big\}\\ &\quad+\varepsilon\int_\delta^{\mathfrak{b}}\big(c_N\delta\rho^\alpha(r^{N-1}u_r^2+r^{N-3}u^2)+\rho(r^{N-1}u_r^2+r^{N-3}u^2)\big)\mathrm{d}r\le 0.\end{aligned}\tag{3.13}$$

Integrating (3.13), we have

$$\begin{aligned}&\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\Big)\,r^{N-1}\mathrm{d}r\\ &\quad+\varepsilon\int_0^t\int_\delta^{\mathfrak{b}}\big(\rho(r^{N-1}u_r^2+r^{N-3}u^2)+c_N\delta\rho^\alpha(r^{N-1}u_r^2+r^{N-3}u^2)\big)\,\mathrm{d}r\mathrm{d}s\\ &\le\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho_0u_0^2+e(\rho_0,\rho_*)+\frac{1}{2}|\Phi_{0r}|^2\Big)\,r^{N-1}\mathrm{d}r.\end{aligned}$$

Then we conclude Lemma 3.1. □

Remark 3.2. Using (3.9), we can rewrite the electric field term as

$$
\begin{aligned}
\|\nabla\Phi\|^2_{L^2(\mathbb{R}^N)} &= \omega_N \int_0^\infty |\Phi_r|^2\, r^{N-1}\mathrm{d}r \\
&= \omega_N\Big(\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \int_{\mathfrak{b}}^\infty |\Phi_r|^2\, r^{N-1}\mathrm{d}r\Big) \\
&= \omega_N\Big(\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \int_{\mathfrak{b}}^\infty \frac{M_{\mathfrak{b}}^2}{\omega_N^2 r^{N-1}}\,\mathrm{d}r\Big) \\
&= \omega_N\Big(\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \frac{1}{(N-2)\mathfrak{b}^{N-2}}\big(\frac{M_{\mathfrak{b}}}{\omega_N}\big)^2\Big).
\end{aligned}
\tag{3.14}
$$

On one hand, it follows from (3.14) that this term has a good sign, which is important such that this makes the following integral also nonnegative, via integration by parts. On the other hand, we also have the following equality:

$$
\begin{aligned}
&\int_\delta^{\mathfrak{b}} \Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big)(\rho-d(r))r\,\mathrm{d}r \\
&= -\int_\delta^{\mathfrak{b}} \big(\rho-d(r)\big)\Phi_r\, r^N\mathrm{d}r = \int_\delta^{\mathfrak{b}} \big(\Phi_{rr}+\frac{N-1}{r}\Phi_r\big)\Phi_r\, r^N\mathrm{d}r \\
&= \int_\delta^{\mathfrak{b}} (r^{N-1}\Phi_r)_r\Phi_r\, r\mathrm{d}r = \int_\delta^{\mathfrak{b}} \frac{1}{r^{N-2}}\big(\frac{1}{2}|r^{N-1}\Phi_r|^2\big)_r\,\mathrm{d}r \\
&= -\int_\delta^{\mathfrak{b}} \frac{1}{2}(r^{-N+2})_r|r^{N-1}\Phi_r|^2\,\mathrm{d}r + \frac{1}{2r^{N-2}}\big|r^{N-1}\Phi_r\big|^2\Big|_\delta^{\mathfrak{b}} \\
&= \frac{N-2}{2}\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \frac{1}{2\mathfrak{b}^{N-2}}|\mathfrak{b}^{N-1}\Phi_r(t,\mathfrak{b})|^2 \\
&= \frac{N-2}{2}\int_\delta^{\mathfrak{b}} r^{N-1}|\Phi_r|^2\mathrm{d}r + \frac{1}{2\mathfrak{b}^{N-2}}\big(\frac{M_{\mathfrak{b}}}{\omega_N}\big)^2 \\
&= \frac{N-2}{2}\Big(\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \frac{1}{(N-2)\mathfrak{b}^{N-2}}\big(\frac{M_{\mathfrak{b}}}{\omega_N}\big)^2\Big) \\
&= \frac{N-2}{2\omega_N}\|\nabla\Phi\|^2_{L^2(\mathbb{R}^N)} \geq 0.
\end{aligned}
$$

Now, we prove two crucial uniform local estimates of the density near the origin, especially Lemma 3.4 below, which will be used in the convergence proof of the electric field potential.

LEMMA 3.3 (**Uniform Local Estimate for the Density near the Origin I).** *The smooth solution of* (3.1)–(3.3) *satisfies*

$$
\int_\delta^D \frac{1}{r^{N-1}}\Big|\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big|^2\,\mathrm{d}r \leq C(D,\mathcal{E}_0)
\tag{3.15}
$$

for any $D\in(\delta,\mathfrak{b})$.

Proof. First, notice that there exists $C_1>0$ such that $|d(y)|\leq C_1$ for any $y\in[0,\infty)$.

Furthermore, we have

$$
\begin{aligned}
&\Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)y^{N-1}\,\mathrm{d}y\Big)^2 \\
&= \Big(\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big)^2 - 2\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\int_\delta^r d(y)y^{N-1}\,\mathrm{d}y \\
&\quad + \Big(\int_\delta^r d(y)y^{N-1}\,\mathrm{d}y\Big)^2 \\
&\ge \frac{1}{2}\Big(\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big)^2 - \Big(\int_\delta^r d(y)y^{N-1}\,\mathrm{d}y\Big)^2.
\end{aligned}
\tag{3.16}
$$

Then, from (3.16), we obtain

$$
\Big|\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big|^2 \le 2\Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)y^{N-1}\,\mathrm{d}y\Big)^2 + 2\Big(\int_\delta^r d(y)y^{N-1}\,\mathrm{d}y\Big)^2.
\tag{3.17}
$$

Using Lemma 3.1 and (3.17), we have

$$
\begin{aligned}
&\int_\delta^D \frac{1}{r^{N-1}}\Big|\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big|^2\,\mathrm{d}r \\
&\le 2\int_\delta^D \frac{1}{r^{N-1}}\Big|\int_\delta^r \big(\rho(t,y)-d(y)\big)y^{N-1}\,\mathrm{d}y\Big|^2\,\mathrm{d}r \\
&\quad + 2\int_\delta^D \frac{1}{r^{N-1}}\Big(\int_\delta^r d(y)y^{N-1}\,\mathrm{d}y\Big)^2\,\mathrm{d}r \\
&\le 2\int_\delta^D |\Phi_r|^2 r^{N-1}\,\mathrm{d}r + 2C_1^2\int_\delta^D \frac{1}{r^{N-1}}\Big(\int_\delta^r y^{N-1}\,\mathrm{d}y\Big)^2\,\mathrm{d}r \\
&\le 2C(\mathcal{E}_0) + \frac{2C_1^2}{N^2}\int_\delta^D \frac{1}{r^{N-1}}r^{2N}\,\mathrm{d}r \le C(D,\mathcal{E}_0).
\end{aligned}
$$

This completes the proof. □

Lemma 3.4 (**Uniform Local Estimate of the Density near the Origin II).** *The smooth solution of* (3.1)–(3.3) *satisfies*

$$
\int_\delta^D \Big(\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big)\rho(t,r)r\,\mathrm{d}r \le C(D,\mathcal{E}_0)
\tag{3.18}
$$

for any $D\in(\delta,\mathfrak{b})$.

Proof. Set $h(r) := \Big(\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big)^2$. A simple calculation yields

$$
h'(r) = 2\rho(t,r)r^{N-1}\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y = 2r^{N-2}\rho(t,r)r\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y.
$$

Then it follows from (3.15) that

$$
\begin{aligned}
\int_\delta^D \Big(\int_\delta^r \rho(t,y)y^{N-1}\,\mathrm{d}y\Big)\rho(t,r)r\,\mathrm{d}r &= \frac{1}{2}\int_\delta^D \frac{1}{r^{N-2}}h'(r)\,\mathrm{d}r \\
&= \frac{1}{2}\frac{h(D)}{D^{N-2}} + \frac{N-2}{2}\int_\delta^D \frac{1}{r^{N-1}}h(r)\,\mathrm{d}r \\
&\le \frac{1}{2}\frac{h(D)}{D^{N-2}} + C(D,\mathcal{E}_0) \le C(D,\mathcal{E}_0),
\end{aligned}
$$

where we have used the fact that

$$h(D) \le \Big(\int_\delta^D \rho^\gamma y^{N-1}\,\mathrm{d}y \Big)^{\frac{2}{\gamma}} \Big(\int_\delta^D y^{N-1}\,\mathrm{d}y \Big)^{\frac{2}{\gamma'}} \le C(D, \mathcal{E}_0).$$

This completes the proof. $\square$

LEMMA 3.5 (**BD Entropy Estimate).** *The smooth solution of* (3.1)–(3.3) *satisfies*

$$\begin{aligned} \varepsilon^2 \int_\delta^{\mathfrak{b}} \Big((1 + \alpha\delta\rho^{\alpha-1} + \alpha^2\delta^2\rho^{2(\alpha-1)}) \frac{\rho_r^2}{\rho} \Big)(t,r)\, r^{N-1}\mathrm{d}r \\ + \varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big((1+\alpha\delta\rho^{\alpha-1})\rho^{\gamma-2}\rho_r^2 \big)(s,r)\, r^{N-1}\mathrm{d}r\mathrm{d}s \le C(\mathcal{E}_0 + 1), \end{aligned} \tag{3.19}$$

where we have used the following relation:

(3.20)

$$\begin{aligned} &\sup_{0<\varepsilon,\delta\le 1} \sup_{\mathfrak{b} \ge 1+\delta^{-1}} \big(\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}} + \mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}} \big) \\ &\quad + CT \int_\delta^{\mathfrak{b}} \big(|\rho_* - d(r)| + |\rho_* - d(r)|^2 + |\rho_* - d(r)|^{\frac{\gamma}{\gamma-1}} \big)\, r^{N-1}\mathrm{d}r \le C(\mathcal{E}_0 + 1) \end{aligned}$$

with

$$\mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}} := \varepsilon^2 \int_\delta^{\mathfrak{b}} \big(1 + 2\alpha\delta\rho_0^{\alpha-1} + \alpha^2\delta^2\rho_0^{2\alpha-2} \big) |(\sqrt{\rho_0})_r|^2\, r^{N-1}\mathrm{d}r. \tag{3.21}$$

Proof. We divide the proof into seven steps:

1. It is convenient to deal with (3.1) in the Lagrangian coordinates. For simplicity, we assume that

$$L_{\mathfrak{b}} := \int_\delta^{\mathfrak{b}} \rho_0(r)\, r^{N-1}\mathrm{d}r.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\delta^{\mathfrak{b}} \rho(t,r)\, r^{N-1}\mathrm{d}r = -\int_\delta^{\mathfrak{b}} (r^{N-1}\rho u)_r(t,r)\,\mathrm{d}r = 0,$$

we have

$$\int_\delta^{\mathfrak{b}} \rho(t,r)\, r^{N-1}\mathrm{d}r = \int_\delta^{\mathfrak{b}} \rho_0(r)\, r^{N-1}\mathrm{d}r = L_{\mathfrak{b}} \qquad \text{for all } t > 0.$$

For $r \in [\delta, \mathfrak{b}]$ and $t \in [0,T]$, we define the Lagrangian coordinates transformation:

$$x = \int_\delta^r \rho(t,y)\, y^{N-1}\mathrm{d}y, \qquad \tau = t,$$

which translates domain $[0,T] \times [\delta, \mathfrak{b}]$ into $[0,T] \times [0, L_{\mathfrak{b}}]$ and satisfies

$$\begin{cases} \frac{\partial x}{\partial r} = \rho r^{N-1}, \quad \frac{\partial x}{\partial t} = -\rho u r^{N-1}, \quad \frac{\partial \tau}{\partial r} = 0, \quad \frac{\partial \tau}{\partial t} = 1, \\ \frac{\partial r}{\partial x} = \frac{1}{\rho r^{N-1}}, \quad \frac{\partial r}{\partial \tau} = u, \quad \frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial t}{\partial x} = 0. \end{cases}$$

Performing the Lagrange transformation to (3.1), we have

$$\begin{cases} \rho_\tau + \rho^2 (r^{N-1}u)_x = 0, \\ u_\tau + r^{N-1}p_x = \varepsilon r^{N-1}\big(\rho(\mu+\lambda)(r^{N-1}u)_x\big)_x - \varepsilon(N-1)r^{N-2}\mu_x u \\ \qquad\qquad + \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\, y^{N-1}\mathrm{d}y\Big), \end{cases}$$

and the boundary conditions (3.3) become

$$u(\tau,0) = u(\tau, L_{\mathfrak{b}}) = 0 \qquad \text{for } \tau > 0.$$

2. Multiplying $(3.2)_1$ by $\mu'(\rho)$ and using (3.5), we have

$$\mu_\tau + \rho(\mu+\lambda)(r^{N-1}u)_x = 0. \tag{3.22}$$

Then we substitute (3.22) into $(3.2)_2$ to obtain

$$u_\tau + r^{N-1}p_x = -\varepsilon r^{N-1}\mu_{x\tau} - \varepsilon(N-1)r^{N-2}\mu_x u + \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\, y^{N-1}\mathrm{d}y\Big). \tag{3.23}$$

Noting that $u = \frac{\partial r}{\partial \tau}$, we have

$$\varepsilon(N-1)r^{N-2}\mu_x u = \varepsilon(N-1)r^{N-2}\mu_x r_\tau = \varepsilon(r^{N-1})_\tau \mu_x,$$

which, together with (3.23), yields

$$(u + \varepsilon r^{N-1}\mu_x)_\tau + r^{N-1}p_x = \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\, y^{N-1}\mathrm{d}y\Big). \tag{3.24}$$

3. Multiplying (3.24) by $u + \varepsilon r^{N-1}\mu_x$, we see that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{L_{\mathfrak{b}}} \frac{1}{2}\big|u + \varepsilon r^{N-1}\mu_x\big|^2\,\mathrm{d}x + \int_0^{L_{\mathfrak{b}}} p_x\big(u + \varepsilon r^{N-1}\mu_x\big)\, r^{N-1}\mathrm{d}x \\ &= \int_0^{L_{\mathfrak{b}}} \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\, y^{N-1}\mathrm{d}y\Big)\big(u + \varepsilon r^{N-1}\mu_x\big)\,\mathrm{d}x. \end{aligned} \tag{3.25}$$

Integrating (3.25) over $[0,\tau]$ and pulling it back to the Eulerian coordinates, we have

$$\begin{aligned} &\int_\delta^{\mathfrak{b}} \frac{1}{2}\rho\Big|u + \varepsilon\frac{\mu_r}{\rho}\Big|^2 r^{N-1}\mathrm{d}r + \varepsilon\int_0^t\int_\delta^{\mathfrak{b}} \frac{p_r\mu_r}{\rho}\, r^{N-1}\mathrm{d}r\mathrm{d}s + \int_0^t\int_\delta^{\mathfrak{b}} u p_r\, r^{N-1}\mathrm{d}r\mathrm{d}s \\ &= \int_\delta^{\mathfrak{b}} \frac{1}{2}\rho_0\Big|u_0 + \varepsilon\frac{\mu_{0r}}{\rho_0}\Big|^2 r^{N-1}\mathrm{d}r + \int_0^t\int_\delta^{\mathfrak{b}} \rho u\Big(\int_\delta^r \big(\rho(s,y) - d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}s \\ &\quad + \varepsilon\int_0^t\int_\delta^{\mathfrak{b}} \mu_r\Big(\int_\delta^r \big(\rho(s,y) - d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}s. \end{aligned} \tag{3.26}$$

4. A direct calculation shows that

$$\int_0^t\int_\delta^{\mathfrak{b}} u p_r\, r^{N-1}\mathrm{d}r\mathrm{d}s = \int_\delta^{\mathfrak{b}} e(\rho,\rho_*)\, r^{N-1}\mathrm{d}r - \int_\delta^{\mathfrak{b}} e(\rho_0,\rho_*)\, r^{N-1}\mathrm{d}r, \tag{3.27}$$

$$\begin{aligned} &\int_0^t\int_\delta^{\mathfrak{b}} \rho u\Big(\int_\delta^r \big(\rho(s,y) - d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}s \\ &= -\frac{1}{2}\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1}\mathrm{d}r + \frac{1}{2}\int_\delta^{\mathfrak{b}} |\Phi_{0r}|^2\, r^{N-1}\mathrm{d}r. \end{aligned} \tag{3.28}$$

5. For the last term of RHS of (3.26), we integrate by parts to obtain

$$
\begin{aligned}
(3.29)\qquad & \varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \mu_r \Big(\int_\delta^r \big(\rho(s,y) - d(y)\big)\, y^{N-1} \mathrm{d}y \Big) \mathrm{d}r \mathrm{d}s \\
&= \varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\mu(\rho) - \mu(\rho_*)\big)_r \Big(\int_\delta^r \big(\rho(s,y) - d(y)\big)\, y^{N-1} \mathrm{d}y \Big) \mathrm{d}r \mathrm{d}s \\
&= -\varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\mu(\rho) - \mu(\rho_*)\big) \big(\rho(s,r) - d(r)\big)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\quad + \varepsilon \int_0^t \big(\mu(\rho(s,\mathfrak{b})) - \mu(\rho_*)\big) \Big(\int_\delta^{\mathfrak{b}} \big(\rho(s,y) - d(y)\big)\, y^{N-1} \mathrm{d}y \Big) \mathrm{d}s \\
&= -\varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\mu(\rho) - \mu(\rho_*)\big) \big(\rho(s,r) - \rho_*\big)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\quad - \varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\mu(\rho) - \mu(\rho_*)\big) \big(\rho_* - d(r)\big)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\quad + \varepsilon \int_0^t \big(\mu(\rho(s,\mathfrak{b})) - \mu(\rho_*)\big) \Big(\int_\delta^{\mathfrak{b}} \big(\rho(s,y) - d(y)\big)\, y^{N-1} \mathrm{d}y \Big) \mathrm{d}s \\
&\le -\varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\rho_* - d(r)\big) \big(\mu(\rho(s,r)) - \mu(\rho_*)\big)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\quad + \Big| \frac{M_{\mathfrak{b}}}{\omega_N} \Big| \int_0^t \varepsilon \big|\mu(\rho(s,b)) - \mu(\rho_*)\big| \,\mathrm{d}s.
\end{aligned}
$$

The first term of RHS of (3.29) becomes

(3.30)

$$
\begin{aligned}
& -\varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big(\rho_* - d(r)\big) \big(\mu(\rho(s,r)) - \mu(\rho_*)\big)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\le C\varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big|\rho_* - d(r)\big|\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\quad + \varepsilon \int_0^t \int_\delta^{\mathfrak{b}} \big|\rho_* - d(r)\big| \mathbf{1}_{\{\rho(s,r) \ge 2\rho_*\}} \big|\mu(\rho(s,r)) - \mu(\rho_*)\big|\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\le CT \int_\delta^{\mathfrak{b}} \big|\rho_* - d(r)\big|\, r^{N-1} \mathrm{d}r \\
&\quad + C \int_0^T \int_\delta^{\mathfrak{b}} \Big(\big|\mu(\rho(s,r)) - \mu(\rho_*)\big|^\gamma + \big|\rho_* - d(r)\big|^{\frac{\gamma}{\gamma-1}} \Big) \mathbf{1}_{\{\rho(s,r) \ge 2\rho_*\}}\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\le CT \int_\delta^{\mathfrak{b}} \Big(\big|\rho_* - d(r)\big| + \big|\rho_* - d(r)\big|^{\frac{\gamma}{\gamma-1}} \Big) r^{N-1} \mathrm{d}r + C \int_0^T \int_\delta^{\mathfrak{b}} e(\rho, \rho_*)\, r^{N-1} \mathrm{d}r \mathrm{d}s \\
&\le CT \int_\delta^{\mathfrak{b}} \Big(\big|\rho_* - d(r)\big| + \big|\rho_* - d(r)\big|^{\frac{\gamma}{\gamma-1}} \Big) r^{N-1} \mathrm{d}r + C\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}} \le C(1 + \mathcal{E}_0),
\end{aligned}
$$

where $\mathbf{1}_{\{\rho(s,r) \ge 2\rho_*\}}$ is the indicator function of set $\{\rho(s,r) \ge 2\rho_*\}$.

6. To bound the last term of RHS of (3.29), we have to be very careful, since it involves the boundary value of the density.

$$
\begin{aligned}
&\varepsilon \int_0^t |\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s\\
&= \varepsilon \int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})\le 4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s + \varepsilon \int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})> 4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s\\
&\le CT + \varepsilon \int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})> 4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

Since

$$\int_\delta^{\mathfrak{b}} (\rho(t,r)-\rho_*)\,r^{N-1}\mathrm{d}r = \int_\delta^{\mathfrak{b}} (\rho_0(r)-\rho_*)\,r^{N-1}\mathrm{d}r,$$

then

$$\int_\delta^{\mathfrak{b}} \rho(t,r)\,r^{N-1}\mathrm{d}r = \rho_*\frac{1}{N}(\mathfrak{b}^N-\delta^N) + \int_\delta^{\mathfrak{b}} (\rho_0(r)-\rho_*)\,r^{N-1}\mathrm{d}r.$$

Therefore, there exists $r_0\in[\delta,\mathfrak{b}]$ such that, for any $t\ge 0$,

$$\rho(t,r_0(t)) = \rho_* + \frac{N}{\mathfrak{b}^N-\delta^N}\int_\delta^{\mathfrak{b}} (\rho_0(r)-\rho_*)\,r^{N-1}\mathrm{d}r \in [\frac{1}{2}\rho_*, 2\rho_*] \quad \text{if } \mathfrak{b} >> 1, \tag{3.32}$$

where we have used

$$\Big|\int_\delta^{\mathfrak{b}} (\rho_0(r)-\rho_*)\,r^{N-1}\mathrm{d}r\Big| \le C < \infty.$$

Motivated by [16], we define

$$
\begin{aligned}
&A(t) = \{r\,:\, r\in[\delta,\mathfrak{b}],\ \rho(t,r)\ge 2\rho_*\}\\
&A_1(t) = \{r\in A(t)\,:\, r\ge 1\}, \qquad A_2(t) = A(t)\backslash A_1(t).
\end{aligned}
$$

Similarly, we also define

$$
\begin{aligned}
&B(t) = \{r\,:\, r\in[\delta,\mathfrak{b}],\ 0\le\rho(t,r)\le \frac{1}{2}\rho_*\},\\
&B_1(t) = \{r\in B(t)\,:\, r\ge 1\}, \qquad B_2(t) = B(t)\backslash B_1(t).
\end{aligned}
$$

It is direct to check that

$$e(\rho,\rho_*) = \begin{cases} |\rho-\rho_*|^2 & \text{if } \rho\in[\frac{1}{2}\rho_*, 2\rho_*],\\ |\rho-\rho_*|^\gamma & \text{if } \rho\in[0,\frac{1}{2}\rho_*]\cup[2\rho_*,\infty). \end{cases} \tag{3.33}$$

Then there exists a constant $c(\rho_*)>0$ such that

$$e(\rho,\rho_*)(t,r)\ge c(\rho_*)>0 \qquad \text{for any } r\in A(t)\cup B(t),$$

which, together with (3.1), yields

$$
\begin{aligned}
\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}} \ge \int_\delta^{\mathfrak{b}} e(\rho,\rho_*)(t,r)\,r^{N-1}\mathrm{d}r &\ge c(\rho_*)\int_{A_1(t)\cup B_1(t)} r^{N-1}\mathrm{d}r\\
&\ge c(\rho_*)\big(|A_1(t)|+|B_1(t)|\big).
\end{aligned}
\tag{3.34}
$$

Hence, we obtain

$$|A(t)|+|B(t)|\le |A_1(t)|+|B_1(t)|+2\le C(\rho_*)\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big). \tag{3.35}$$

Since $\rho(t,r)$ is a continuous function of (t,r), for $s\in\{s\,:\,\rho(s,\mathfrak{b})>4\rho_*\}$, it follows from (3.32)–(3.34) that there exists $\tilde r_0(s)\in[1,\mathfrak{b}]$ such that

$$\begin{cases}|\mathfrak{b}-\tilde r_0(s)|\le C(\rho_*)\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big),\\ \rho(s,r)\ge 2\rho_* \qquad \text{for all } r\in[\tilde r_0(s),\mathfrak{b}],\\ \rho(s,\tilde r_0(s))=2\rho_*.\end{cases} \tag{3.36}$$

Thus, using (3.36), we have

$$\begin{aligned}
&\varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s\\
&\le \varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho(s,\tilde r_0(s)))|\,\mathrm{d}s+C(\rho_*)T\\
&\le \varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}\int_{\tilde r_0(s)}^{\mathfrak{b}}|\mu_r|\,\mathrm{d}r\mathrm{d}s+C(\rho_*)T\\
&\le \varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}\int_{\tilde r_0(s)}^{\mathfrak{b}}|\rho_r|\big(1+\alpha\delta\rho^{\alpha-1}\big)\,\mathrm{d}r\,\mathrm{d}s+C(\rho_*)T\\
&\le C(\rho_*)\varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}\int_{\tilde r_0(s)}^{\mathfrak{b}}|\rho_r|\mathrm{d}r\mathrm{d}s+C(\rho_*)T\\
&\le C(\rho_*)\varepsilon\int_0^t \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}\Big(\int_{\tilde r_0(s)}^{\mathfrak{b}}\rho^{\gamma-2}|\rho_r|^2\,\mathrm{d}r\Big)^{\frac12}\Big(\int_{\tilde r_0(s)}^{\mathfrak{b}}\rho^{2-\gamma}\,\mathrm{d}r\Big)^{\frac12}\mathrm{d}s+C(\rho_*)T.
\end{aligned} \tag{3.37}$$

For $\gamma\in(1,2]$,

$$\rho^{2-\gamma}\le C(1+\rho^\gamma),$$

which, together with (3.36), yields

$$\begin{aligned}
\int_{\tilde r_0(s)}^{\mathfrak{b}}\rho^{2-\gamma}\,\mathrm{d}r&\le C\int_{\tilde r_0(s)}^{\mathfrak{b}}\big(1+\rho^\gamma\big)\,\mathrm{d}r\le C\int_{\tilde r_0(s)}^{\mathfrak{b}}\big(1+e(\rho,\rho_*)\big)\,\mathrm{d}r\\
&\le C(\rho_*)\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big)+C\int_{\tilde r_0(s)}^{\mathfrak{b}}e(\rho,\rho_*)\,r^{N-1}\mathrm{d}r\\
&\le C(\rho_*)\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big).
\end{aligned} \tag{3.38}$$

For $\gamma\in(2,\infty)$, it follows from (3.36) that

$$\int_{\tilde r_0(s)}^{\mathfrak{b}}\rho^{2-\gamma}\,\mathrm{d}r\le C(\rho_*)|\mathfrak{b}-\tilde r_0(s)|\le C(\rho_*)\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big). \tag{3.39}$$

Combining (3.37)–(3.39), we obtain

$$
\begin{aligned}
(3.40)\qquad & \varepsilon\int_0^T \mathbf{1}_{\{\rho(s,\mathfrak{b})>4\rho_*\}}|\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s\\
&\le \frac{\kappa\gamma\varepsilon}{8}\int_0^t\int_{\tilde r_0(s)}^{\mathfrak{b}}\rho^{\gamma-2}|\rho_r|^2\,\mathrm{d}r\mathrm{d}s + C(\rho_*)T\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big)\\
&\le \frac{\kappa\gamma\varepsilon}{8}\int_0^t\int_{\delta}^{\mathfrak{b}} r^{N-1}\rho^{\gamma-2}|\rho_r|^2\,\mathrm{d}r\mathrm{d}s + C(\rho_*)T\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big),
\end{aligned}
$$

which, together with (3.31), implies that

$$
(3.41)\qquad \varepsilon\int_0^t |\mu(\rho(s,\mathfrak{b}))-\mu(\rho_*)|\,\mathrm{d}s \le \frac{\kappa\gamma\varepsilon}{8}\int_0^T\int_\delta^{\mathfrak{b}}\rho^{\gamma-2}|\rho_r|^2\,r^{N-1}\mathrm{d}r\mathrm{d}s + C(\rho_*)T\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big).
$$

7. Substituting (3.30) and (3.41) into (3.29), we have

$$
\begin{aligned}
(3.42)\qquad & \varepsilon\int_0^t\int_\delta^{\mathfrak{b}} \mu_r\Big(\int_\delta^r \big(\rho(s,y)-d(y)\big)\,y^{N-1}\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}s\\
&\le \frac{\kappa\gamma\varepsilon}{8}\int_0^T\int_\delta^{\mathfrak{b}}\rho^{\gamma-2}|\rho_r|^2\,r^{N-1}\mathrm{d}r\mathrm{d}s + CT\int_\delta^{\mathfrak{b}}\big(|\rho_*-d(r)|+|\rho_*-d(r)|^2\big)\,r^{N-1}\mathrm{d}r\\
&\quad + C(\rho_*)T\big(1+\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}\big).
\end{aligned}
$$

Combining (3.42) with (3.26)–(3.28), we obtain

$$
\begin{aligned}
&\int_\delta^{\mathfrak{b}}\frac12\rho|u+\varepsilon\frac{\mu_r}{\rho}|^2\,r^{N-1}\mathrm{d}r + \frac{\kappa\gamma\varepsilon}{2}\int_0^t\int_\delta^{\mathfrak{b}}\big(\rho^{\gamma-2}\rho_r^2+\alpha\delta\rho^{\alpha+\gamma-3}\rho_r^2\big)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\
&\quad + \int_\delta^{\mathfrak{b}} e(\rho,\rho_*)\,r^{N-1}\mathrm{d}r + \int_\delta^{\mathfrak{b}}\frac12|\Phi_r|^2\,r^{N-1}\mathrm{d}r\\
&\le C(\rho_*,T)\big(\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}+\mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}}\big) + CT\int_\delta^{\mathfrak{b}}\big(|\rho_*-d(r)|+|\rho_*-d(r)|^2\big)\,\mathrm{d}r\\
&\le C\big(1+\mathcal{E}_0\big).
\end{aligned}
$$

This completes the proof. □

As shown in [12], the higher integrabilities of the density and the velocity are important for the L^p compensated compactness framework. We now prove the higher integrability of the density.

Lemma 3.6 (**Higher Integrability on the Density).** *For given* r_1 *and* r_2 *with* $\delta<r_1<r_2<\mathfrak{b}$, *any smooth solution of problem* (3.1)–(3.3) *satisfies*

$$
(3.43)\qquad \int_0^T\int_{\mathfrak{K}}\rho^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t \le C(r_1,r_2,T,\mathcal{E}_0),
$$

where $\mathfrak{K}$ *is any compact subset of* $[r_1,r_2]$.

Proof. Let $w(r)$ be a smooth compact support function such that $\operatorname{supp} w\subseteq(r_1,r_2)$

and $w(r)=1$ for $r\in\mathfrak{K}$. Multiplying $(3.1)_2$ by $w(r)$, we obtain

$$
\begin{aligned}
&(\rho uw)_t+\big((\rho u^2+p)w\big)_r+\frac{N-1}{r}\rho u^2 w\\
&=\varepsilon\big((\rho+\alpha\delta\rho^\alpha)(u_r+\frac{N-1}{r}u)w\big)_r-\varepsilon\frac{N-1}{r}u(\rho+\delta\rho^\alpha)_r w\\
&\quad+\big(\rho u^2+p-\varepsilon(\rho+\alpha\delta\rho^\alpha)(u_r+\frac{N-1}{r}u)\big)w_r-\rho w\Phi_r.
\end{aligned}
\tag{3.44}
$$

Integrating (3.44) over $[r_1,r)$, we have

$$
\begin{aligned}
(\rho u^2+p)w=&\,\varepsilon(\rho+\alpha\delta\rho^\alpha)(u_r+\frac{N-1}{r}u)w-\varepsilon\int_{r_1}^{r}\frac{N-1}{y}u(\rho+\delta\rho^\alpha)_y w\,\mathrm{d}y\\
&-\Big(\int_{r_1}^{r}\rho uw\mathrm{d}y\Big)_t-\int_{r_1}^{r}\frac{N-1}{y}\rho u^2 w\,\mathrm{d}y\\
&+\int_{r_1}^{r}\Big(\rho u^2+p-\varepsilon(\rho+\alpha\delta\rho^\alpha)(u_y+\frac{N-1}{y}u)\Big)w_y\,\mathrm{d}y\\
&-\int_{r_1}^{r}\rho w\Phi_y\,\mathrm{d}y.
\end{aligned}
$$

Then we multiply the above equality by ρw to obtain

$$
\begin{aligned}
&(\rho^2u^2+\rho p)w^2\\
&=\varepsilon\rho(\rho+\alpha\delta\rho^\alpha)(u_r+\frac{N-1}{r}u)w^2-\varepsilon\rho w\int_{r_1}^{r}\frac{N-1}{y}u(\rho+\alpha\delta\rho^\alpha)_y w\,\mathrm{d}y\\
&\quad-\rho w\Big(\int_{r_1}^{r}\rho uw\,\mathrm{d}y\Big)_t-\rho w\int_{r_1}^{r}\frac{N-1}{y}\rho u^2 w\,\mathrm{d}y\\
&+\rho w\int_{r_1}^{r}\Big(\rho u^2+p-\varepsilon(\rho+\alpha\delta\rho^\alpha)(u_y+\frac{N-1}{y}u)\Big)w_y\,\mathrm{d}y\\
&\quad-\rho w\int_{r_1}^{r}\rho w\Phi_y\,\mathrm{d}y.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\rho w\Big(\int_{r_1}^{r}\rho uw\,\mathrm{d}y\Big)_t=&-\rho^2u^2w^2+\Big(\rho w\int_{r_1}^{r}\rho uw\mathrm{d}y\Big)_t+\Big(\rho uw\int_{r_1}^{r}\rho uw\,\mathrm{d}y\Big)_r\\
&-\rho uw_r\int_{r_1}^{r}\rho uw\,\mathrm{d}y+\frac{N-1}{r}\rho uw\int_{r_1}^{r}\rho uw\,\mathrm{d}y,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\rho p w^2 ={}& \varepsilon\rho(\rho+\alpha\delta\rho^\alpha)(u_r+\frac{N-1}{r}u)w^2-\varepsilon\rho w\int_{r_1}^{r}\frac{N-1}{y}u(\rho+\alpha\delta\rho^\alpha)_y w\,\mathrm{d}y\\
&-\Big(\rho w\int_{r_1}^{r}\rho u w\,\mathrm{d}y\Big)_t-\Big(\rho u w\int_{r_1}^{r}\rho u w\,\mathrm{d}y\Big)_r+\rho u w_r\int_{r_1}^{r}\rho u w\,\mathrm{d}y\\
&-\frac{N-1}{r}\rho u w\int_{r_1}^{r}\rho u w\,\mathrm{d}y-\rho w\int_{r_1}^{r}\frac{N-1}{y}\rho u^2 w\,\mathrm{d}y\\
&+\rho w\int_{r_1}^{r}\Big(\rho u^2+p-\varepsilon(\rho+\alpha\delta\rho^\alpha)\big(u_y+\frac{N-1}{y}u\big)\Big)w_y\,\mathrm{d}y\\
&-\rho w\int_{r_1}^{r}\rho w\Phi_y\,\mathrm{d}y:=\sum_{i=1}^{9}J_i.
\end{aligned}
\tag{3.45}
$$

To estimate RHS of (3.45), it follows first from Lemma 3.1 that

$$\int_{r_1}^{r_2}\rho^\gamma\, r^{N-1}\mathrm{d}r\le C(r_2,\mathcal{E}_0),\tag{3.46}$$

$$\int_{r_1}^{r_2}\rho\,\mathrm{d}r\le\frac{C}{r_1^{N-1}}\int_{r_1}^{r_2}\rho\, r^{N-1}\mathrm{d}r\le C\int_{r_1}^{r_2}\big(\rho^\gamma+1\big)\,r^{N-1}\mathrm{d}r\le C(r_1,r_2,\mathcal{E}_0),\tag{3.47}$$

$$\int_{r_1}^{r_2}\rho u^2\,\mathrm{d}r\le\frac{C}{r_1^{N-1}}\int_{r_1}^{r_2}\rho u^2\, r^{N-1}\mathrm{d}r\le C(r_1,\mathcal{E}_0).\tag{3.48}$$

We also obtain

$$
\begin{aligned}
&\Big|\int_0^T\int_{r_1}^{r_2}\rho w\Big(\int_{r_1}^{r}\rho w\Phi_y\,\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big|\\
&\le\Big|\int_0^T\int_{r_1}^{r_2}\rho w\Big(\int_{r_1}^{r}\rho w\big(\int_{\delta}^{y}(\rho-d(z))\,z^{N-1}\mathrm{d}z\big)\frac{\mathrm{d}y}{y^{N-1}}\Big)\,\mathrm{d}r\mathrm{d}t\Big|\le C(r_1,r_2,T,\mathcal{E}_0).
\end{aligned}
\tag{3.49}
$$

Now it follows from (3.47)–(3.48) that

$$\Big|\int_0^T\int_{r_1}^{r_2}\big(J_3+\cdots+J_7\big)\,\mathrm{d}r\mathrm{d}t\Big|\le C(r_1,r_2,T,\mathcal{E}_0).\tag{3.50}$$

Next, we estimate J_8. Note that

$$\Big|\int_0^T\int_{r_1}^{r_2}\rho w\Big(\int_{r_1}^{r}(\rho u^2+p)w_y\,\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big|\le C(r_1,r_2,T,\mathcal{E}_0),$$

and

$$
\begin{aligned}
&\varepsilon\Big|\int_0^T\int_{r_1}^{r_2}\rho w\Big(\int_{r_1}^{r}(\rho+\alpha\delta\rho^\alpha)(u_y+\frac{N-1}{y}u)w_y\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big|\\
&\le C(r_1,r_2,\mathcal{E}_0)\varepsilon\int_0^T\int_{r_1}^{r_2}\frac{1}{y}\big|(\rho+\alpha\delta\rho^\alpha)\big(yu_y+(N-1)u\big)w_y\big|\mathrm{d}y\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)\Big(\varepsilon\int_0^T\int_{r_1}^{r_2}(\rho+\alpha\delta\rho^\alpha)\big(yu_y+(N-1)u\big)^2\,\mathrm{d}y\mathrm{d}t\\
&\quad+\int_0^T\int_{r_1}^{r_2}\varepsilon(\rho+\alpha\delta\rho^\alpha)\,\mathrm{d}y\mathrm{d}t\Big)\\
&\le C(r_1,r_2,T,\mathcal{E}_0),
\end{aligned}
$$

which yield

$$
\Big|\int_0^T\int_{r_1}^{r_2}J_8\,\mathrm{d}r\mathrm{d}t\Big|\le C(r_1,r_2,T,\mathcal{E}_0). \tag{3.51}
$$

For J_2, since

$$
\begin{aligned}
&\Big|\int_{r_1}^{r}\frac{N-1}{y}u(\rho+\delta\rho^\alpha)_y w\mathrm{d}y\Big|=\Big|\frac{N-1}{r}\big((\rho+\delta\rho^\alpha)uw\big)(t,r)\Big|\\
&\quad+(N-1)\Big|\int_{r_1}^{r}\Big(-(\rho+\delta\rho^\alpha)uw+(\rho+\delta\rho^\alpha)u_yw+(\rho+\delta\rho^\alpha)uw_y\Big)(t,y)\,\frac{\mathrm{d}y}{y}\Big|\\
&\le\Big|\frac{N-1}{r}\big((\rho+\delta\rho^\alpha)uw\big)(t,r)\Big|+C(r_1)\int_{r_1}^{r_2}r^{N-1}(\rho+\delta\rho^\alpha)u_r^2\,\mathrm{d}r+C(r_1,r_2,\mathcal{E}_0),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\Big|\int_0^T\int_{r_1}^{r_2}J_2\,\mathrm{d}r\mathrm{d}t\Big|\\
&=\Big|\int_0^T\int_{r_1}^{r_2}\varepsilon\rho w\Big(\int_{r_1}^{r}\frac{N-1}{y}u(\rho+\delta\rho^\alpha)_y w\,\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big|\\
&\le C(r_1,r_2,T,\mathcal{E}_0)+C(r_1,r_2,T,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\varepsilon r^2(\rho+\delta\rho^\alpha)u_r^2\,\mathrm{d}r\mathrm{d}t\\
&\quad+\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^2(\rho+\delta\rho^\alpha)w^2\,\mathrm{d}r\mathrm{d}t\\
&\le C(r_1,r_2,T,\mathcal{E}_0)+\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t,
\end{aligned} \tag{3.52}
$$

since $\alpha\in(0,1)$.

Finally, we estimate J_1. It is clear that

$$\begin{aligned}(3.53)\qquad &\Big|\int_0^T\int_{r_1}^{r_2} J_1 \mathrm{d}r\mathrm{d}t\Big|\\
&=\varepsilon\Big|\int_0^T\int_{r_1}^{r_2}(\rho^2+\alpha\delta\rho^{\alpha+1})\big(u_r+\frac{N-1}{r}u\big)w^2\mathrm{d}r\mathrm{d}t\Big|\\
&\le\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^2\big|u_r+\frac{N-1}{r}u\big|w^2\,\mathrm{d}r\mathrm{d}t\\
&\quad+\alpha\delta\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^{\alpha+1}\big|u_r+\frac{N-1}{r}u\big|w^2\,\mathrm{d}r\mathrm{d}t\\
&\le\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^2(\rho+\alpha\delta\rho^{\alpha})w^2\,\mathrm{d}r\mathrm{d}t\\
&\quad+C\varepsilon\int_0^T\int_{r_1}^{r_2}(\rho+\alpha\delta\rho^{\alpha})\big(u_r+\frac{N-1}{r}u\big)^2w^2\,\mathrm{d}r\mathrm{d}t\\
&\le C(r_1,r_2,T,\mathcal{E}_0)+\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t.\end{aligned}$$

To close the estimate, we still need to bound the last term on RHS of (3.52)–(3.53). We first consider the case: $\gamma\in(1,2]$. Then we have

$$\begin{aligned}(3.54)\qquad &\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t\\
&\le\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^{\gamma}\,\mathrm{d}r\sup_{r\in[r_1,r_2]}\big(\rho^{3-\gamma}w^2\big)\,\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\varepsilon\sup_{r\in[r_1,r_2]}\big(\rho^{3-\gamma}w^2\big)\,\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\varepsilon\int_{r_1}^{r_2}\big|\big(\rho^{3-\gamma}w^2\big)_r(t,r)\big|\,\mathrm{d}r\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\big(\varepsilon\rho^{2-\gamma}|\rho_r|w^2+\varepsilon\rho^{3-\gamma}w|w_r|\big)\,\mathrm{d}r\mathrm{d}t.\end{aligned}$$

We now estimate each term of RHS of (3.54). A direct calculation shows that

$$\begin{aligned}(3.55)\qquad &C(r_1,r_2,\mathcal{E}_0)\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^{2-\gamma}|\rho_r|w^2\,\mathrm{d}r\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\varepsilon\rho^{\gamma-2}\rho_r^2\,\mathrm{d}r\mathrm{d}t+\frac{\varepsilon}{2}\int_0^T\int_{r_1}^{r_2}\rho^{3(2-\gamma)}w^2\,\mathrm{d}r\mathrm{d}t\\
&\le C(r_1,r_2,\mathcal{E}_0)+\frac{\varepsilon}{2}\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}
&C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\varepsilon\rho^{3-\gamma}w|w_r|\,\mathrm{d}r\mathrm{d}t \\
&\quad\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\varepsilon\sup_r\,(\rho w)(t,r)\int_{r_1}^{r_2}\rho^{2-\gamma}|w_r|\,\mathrm{d}r\mathrm{d}t \\
&\quad\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\varepsilon\sup_r\,(\rho w)(t,r)\,\mathrm{d}t \\
&\quad\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\varepsilon\int_{r_1}^{r_2}\big(|\rho_r|w+\rho|w_r|\big)\,\mathrm{d}r\mathrm{d}t \\
&\quad\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\varepsilon\rho^{\gamma-2}\rho_r^2\,\mathrm{d}r\mathrm{d}t+C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\rho^{2-\gamma}w\,\mathrm{d}r\mathrm{d}t \\
&\quad\le C(r_1,r_2,\mathcal{E}_0).
\end{aligned}\tag{3.56}$$

Combining (3.55)–(3.56), we obtain

$$\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,\mathcal{E}_0)\qquad\text{for }\gamma\in(1,2].\tag{3.57}$$

For the case: $\gamma\in[2,3]$, we have

$$\begin{aligned}
&\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t \\
&\le\varepsilon\int_0^T\sup_{r\in[r_1,r_2]}(\rho^2w)\int_{r_1}^{r_2}\rho w\,\mathrm{d}r\mathrm{d}t \\
&\le C(r_1,r_2,\mathcal{E}_0)\varepsilon\int_0^T\int_{r_1}^{r_2}\big(\rho|\rho_r|w+\rho^2|w_r|\big)\,\mathrm{d}r\mathrm{d}t \\
&\le C(r_1,r_2,\mathcal{E}_0)\int_0^T\int_{r_1}^{r_2}\big(\varepsilon^2\rho^{\gamma-2}|\rho_r|^2w+\rho^2|w_r|+\rho^{4-\gamma}w\big)\,\mathrm{d}r\mathrm{d}t \\
&\le C(r_1,r_2,\mathcal{E}_0).
\end{aligned}\tag{3.58}$$

For the case: $\gamma\in(3,\infty)$, we obtain

$$\varepsilon\int_0^T\int_{r_1}^{r_2}\rho^3w^2\,\mathrm{d}r\mathrm{d}t\le C\int_0^T\int_{r_1}^{r_2}(\rho+\rho^\gamma)\,\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,\mathcal{E}_0,T).\tag{3.59}$$

Now substituting (3.57)–(3.59) into (3.53), we have

$$\Big|\int_0^T\int_{r_1}^{r_2}\big(J_1+J_2\big)\,\mathrm{d}r\mathrm{d}t\Big|\le C(r_1,r_2,T,\mathcal{E}_0).\tag{3.60}$$

Integrating (3.45) over $[0,T]\times[r_1,r_2]$ and then using (3.50)–(3.51) and (3.60), we conclude (3.43). ☐

4. Uniform Higher Integrability of the Approximate Solutions. To employ the compensated compactness framework [12], we need additional uniform integrability of the velocity for the approximate solutions. We first define

$$\mathcal{M}_1:=\mathcal{E}_0+\rho_*+\rho_*^{-1}+\delta^{-1}+\varepsilon^{-1}+\sup_{\mathfrak{b}\ge1+\delta^{-1}}\mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}}<\infty,\quad\mathcal{M}_2:=\mathcal{M}_1+\sup_{\mathfrak{b}\ge1+\delta^{-1}}\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}},\tag{4.1}$$

where

$$\mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}} := \int_\delta^{\mathfrak{b}} \rho_0\big(u_0^8 + \big|\frac{\mu_{0r}}{\rho_0}\big|^8\big)\, r^{N-1}\mathrm{d}r, \tag{4.2}$$

$$\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}} := \int_\delta^{\mathfrak{b}} \big(\frac{1}{2}\rho_0 u_0^2 + e(\rho_0,\rho_*) + \frac{1}{2}|\Phi_{0r}|^2\big)\, r^{2(N-1)+\vartheta}\mathrm{d}r, \tag{4.3}$$

for some $\vartheta \in (0,1)$. It follows from Lemma A.1 that $\mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}}$ and $\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}}$ are uniformly bounded with respect to $\mathfrak{b}$. However, the upper bounds may depend on (ε,δ), such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite for any fixed (ε,δ), independent of $\mathfrak{b} > 0$.

Proposition 4.1. *Given r_1 and r_2 with $\delta < r_1 < r_2 < \mathfrak{b}$, the smooth solution of* (3.1)–(3.3) *satisfies that there exists $\vartheta \in (0,1)$ such that*

$$\int_0^T\int_{r_1}^{r_2} \big(\rho|u|^3 + \rho^{\gamma+\theta}\big)(t,r) r^{N-1}\mathrm{d}r\mathrm{d}t \le C(r_1,r_2,T,\mathcal{E}_0) + C(T,\mathcal{M}_2)\,\mathfrak{b}^{-\frac{\vartheta}{2}}. \tag{4.4}$$

The proof of Proposition 4.1 will be given later, after several lemmas are verified. Motivated by [16], it is important to see that the positive constant $C(T,\mathcal{M}_2)$ is independent of $\mathfrak{b}$, so that this term vanishes when $\mathfrak{b}\to\infty$ in Proposition 4.1. Note that CNSPEs with spherical symmetry is singular at $r = 0$ and $r = \infty$. In order to integrate the quantities in r from ∞, we need to know the asymptotic behavior of $\rho(t,r)$ near boundary $r = \mathfrak{b}$. Thus, we have to obtain the lower and upper bound of ρ, which should be independent of $\mathfrak{b}$. First, we have the following lemma for the upper bound of ρ (also see [16, Lemma 4.2]).

Lemma 4.2 (**Upper Bound of the Density).** *The smooth solution of* (3.1)–(3.3) *satisfies*

$$0 < \rho(t,r) \le C(\mathcal{M}_1) \qquad \text{for } t \ge 0 \text{ and } r \in [\delta,\mathfrak{b}], \tag{4.5}$$

We are now going to prove Lemma 4.3 as a preparation to estimate the lower bound of the density.

Lemma 4.3. *The smooth solution of* (3.1)–(3.3) *satisfies*

$$\int_\delta^{\mathfrak{b}} \rho|\rho^{-1-\frac{1}{2N}}\rho_r|^8\, r^{N-1}\mathrm{d}r \le C(T,\mathcal{M}_1) \qquad \text{for } t\in[0,T]. \tag{4.6}$$

Proof. We write (3.24) as

$$\varepsilon(r^{N-1}\mu_x)_\tau + r^{N-1}p_x = -u_\tau + \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\, y^{N-1}\mathrm{d}y\Big), \tag{4.7}$$

and then integrate (4.7) over $[0,\tau]$ to obtain

$$\begin{aligned}\varepsilon(r^{N-1}\mu_x)(\tau,x) &= \varepsilon(r^{N-1}\mu_x)(0,x) - \big(u(\tau,x) - u_0(x)\big)\\ &\quad - \int_0^\tau (r^{N-1}p_x)(s,x)\,\mathrm{d}s + \int_0^\tau \frac{1}{r^{N-1}}\Big(x - \int_\delta^r d(y)\,y^{N-1}\mathrm{d}y\Big)\mathrm{d}s.\end{aligned} \tag{4.8}$$

Multiplying (4.8) by $(r^{N-1}\mu_x)^{2k-1}$ and integrating the resultant equation, we have

$$
\begin{aligned}
(4.9)\qquad & \varepsilon\int_0^{L_\mathfrak{b}} |r^{N-1}\mu_x|^{2k}\,\mathrm{d}x \\
& \le \Big(\int_0^{L_\mathfrak{b}} |r^{N-1}\mu_x|^{2k}\mathrm{d}x\Big)^{\frac{2k-1}{2k}} \\
& \quad \times \Big\{\varepsilon\Big(\int_0^{L_\mathfrak{b}} |(r^{N-1}\mu_x)(0,x)|^{2k}\,\mathrm{d}x\Big)^{\frac{1}{2k}} + \|(u(\tau),u_0)\|_{L^{2k}} \\
& \qquad + C_T\Big(\int_0^T\int_0^{L_\mathfrak{b}} |r^{N-1}(\rho^\gamma)_x|^{2k}\,\mathrm{d}x\mathrm{d}s\Big)^{\frac{1}{2k}} \\
& \qquad + C_T\Big(\int_0^T\int_0^{L_\mathfrak{b}} \Big|\frac{1}{r^{N-1}}\int_\delta^r (\rho(s,y)-d(y))\mathrm{d}y\Big|^{2k}\mathrm{d}x\mathrm{d}s\Big)^{\frac{1}{2k}}\Big\}.
\end{aligned}
$$

Since

$$
|\mu_x| = |\rho_x + \delta(\rho^\alpha)_x| = |(\frac{1}{\alpha}\rho^{1-\alpha}+\delta)(\rho^\alpha)_x| \ge \delta(\rho^\alpha)_x,
$$

and $(\rho^\gamma)_x = \frac{\gamma}{\alpha}\rho^{\gamma-\alpha}(\rho^\alpha)_x$, it follows from (4.5) and (4.9) that

$$
\begin{aligned}
(4.10)\qquad & \int_0^{L_\mathfrak{b}} |r^{N-1}(\rho^\alpha)_x|^{2k}\,\mathrm{d}x \\
& \le C(\mathcal{E}_0,T,\varepsilon,\delta)\Big(\int_0^{L_\mathfrak{b}} \big(|(r^{N-1}\mu_x)(0,x)|^{2k} + |u(\tau)|^{2k} + |u_0|^{2k}\big)\mathrm{d}x \\
& \qquad + \int_0^\tau\int_0^{L_\mathfrak{b}} |r^{N-1}(\rho^\alpha)_x|^{2k}\,\mathrm{d}x\mathrm{d}s \\
& \qquad + \int_0^\tau\int_0^{L_\mathfrak{b}} \Big|\frac{1}{r^{N-1}}\int_\delta^r (\rho(s,y)-d(y))\,y^{N-1}\mathrm{d}y\Big|^{2k}\mathrm{d}x\mathrm{d}s\Big).
\end{aligned}
$$

Pulling (4.10) back to the Eulerian coordinates, we see that

$$
\begin{aligned}
(4.11)\qquad & \int_\delta^\mathfrak{b} \rho|\rho^{-1-\frac{1}{2N}}\rho_r|^{2k}\,r^{N-1}\mathrm{d}r \\
& \le C(\mathcal{E}_0,T,\varepsilon,\delta)\Big(\mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}} + \int_\delta^\mathfrak{b}\rho|u|^{2k}\,r^{N-1}\mathrm{d}r + \int_0^t\int_\delta^\mathfrak{b}\rho|\rho^{-1-\frac{1}{2N}}\rho_r|^{2k}\,r^{N-1}\mathrm{d}r\mathrm{d}s \\
& \qquad + \int_0^t\int_\delta^\mathfrak{b}\Big|\frac{1}{r^{N-1}}\int_\delta^r \big(\rho(s,y)-d(y)\big)\,y^{N-1}\mathrm{d}y\Big|^{2k}\rho\, r^{N-1}\mathrm{d}r\mathrm{d}s\Big).
\end{aligned}
$$

Now, we consider the last term of (4.11). Notice that

$$
\begin{aligned}
(4.12)\qquad & \frac{1}{r^{N-1}}\Big|\int_\delta^r \big(\rho(s,y)-d(y)\big)\,y^{N-1}\mathrm{d}y\Big| \\
& \le \frac{1}{r^{N-1}}\int_\delta^r |\rho(s,y)-\rho_*|\,y^{N-1}\mathrm{d}y + \frac{1}{r^{N-1}}\int_\delta^r \big|\rho_*-d(y)\big|\,y^{N-1}\mathrm{d}y \\
& \le \frac{1}{r^{N-1}}\int_\delta^r |\rho(s,y)-\rho_*|\mathbf{1}_{A(t)\cup B(t)}\,y^{N-1}\mathrm{d}y \\
& \quad + \frac{1}{r^{N-1}}\int_\delta^r |\rho(s,y)-\rho_*|\big(1-\mathbf{1}_{A(t)\cup B(t)}\big)\,y^{N-1}\mathrm{d}y + \|d-\rho_*\|_{L^2}r^{-\frac{N}{2}+1}.
\end{aligned}
$$

Using (3.33), we have

$$
\begin{aligned}
&\frac{1}{r^{N-1}}\int_\delta^r |\rho(s,y)-\rho_*|\mathbf{1}_{A(t)\cup B(t)}\, y^{N-1}\mathrm{d}y \\
&\le \frac{C(\rho_*)}{r^{N-1}}\int_\delta^r \mathbf{1}_{B(t)}\, y^{N-1}\mathrm{d}y + \frac{C}{r^{N-1}}\int_\delta^r \rho\mathbf{1}_{A(t)}\, y^{N-1}\mathrm{d}y \\
&\le \frac{C(\rho_*)}{r^{N-1}}\int_\delta^r e(\rho,\rho_*)\, y^{N-1}\mathrm{d}y \le C(\rho_*)\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}} r^{-N+1},
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
&\frac{1}{r^{N-1}}\int_\delta^r |\rho(s,y)-\rho_*|(1-\mathbf{1}_{A(t)\cup B(t)})\, y^{N-1}\mathrm{d}y \\
&\le \frac{1}{r^{N-1}}\Big(\int_\delta^r |\rho(s,y)-\rho_*|^2\big(1-\mathbf{1}_{A(t)\cup B(t)}\big)\, y^{N-1}\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_\delta^r y^{N-1}\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\le C(\rho_*)r^{-\frac{N}{2}+1}\Big(\int_\delta^r e(\rho,\rho_*)\, y^{N-1}\mathrm{d}y\Big)^{\frac{1}{2}} \le C(\rho_*,\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}})r^{-\frac{N}{2}+1}.
\end{aligned}
\tag{4.14}
$$

Substituting (4.13)–(4.14) into (4.12), we conclude

$$
\begin{aligned}
\frac{1}{r^{N-1}}\Big|\int_\delta^r \big(\rho(s,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big| &\le C(\rho_*,\mathcal{E}_0)(r^{-N+1}+r^{-\frac{N}{2}+1}) \\
&\le C(\rho_*,\mathcal{E}_0,\delta)r^{-\frac{N}{2}+1}.
\end{aligned}
$$

Therefore, using (4.15), we obtain

$$
\begin{aligned}
&\int_0^t\int_\delta^{\mathfrak{b}} \rho\Big|\frac{1}{r^{N-1}}\int_\delta^r \big(\rho(s,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big|^{2k} r^{N-1}\mathrm{d}r\mathrm{d}s \\
&\le C(\rho_*,\mathcal{E}_0,\delta)\int_0^t\int_\delta^{\mathfrak{b}} \rho r^{-k(N-2)}\, r^{N-1}\mathrm{d}r\mathrm{d}s \\
&= C(\rho_*,\mathcal{E}_0,\delta)\Big(\int_0^t\int_\delta^{\mathfrak{b}} 2\rho_*\mathbf{1}_{\{\rho(t,r)\le 2\rho_*\}} r^{N-1-k(N-2)}\mathrm{d}r\mathrm{d}s \\
&\qquad\qquad + \int_0^t\int_\delta^{\mathfrak{b}} e(\rho,\rho_*) r^{N-1-k(N-2)}\mathbf{1}_{\{\rho(t,r)>2\rho_*\}}\mathrm{d}r\mathrm{d}s\Big) \\
&\le C(\rho_*,\mathcal{E}_0,T,\delta)\Big(1+\int_\delta^{\mathfrak{b}} r^{N-1-k(N-2)}\mathrm{d}r\Big) \\
&\le C(\rho_*,\mathcal{E}_0,T,\delta)\Big(1+\frac{1}{k(N-2)-N}\big(\delta^{N-k(N-2)}-\mathfrak{b}^{N-k(N-2)}\big)\Big) \\
&\le C(\rho_*,\mathcal{E}_0,T,\delta),
\end{aligned}
\tag{4.15}
$$

provide that $k>\frac{N}{N-2}$, which can be satified by taking $k=4$.

In order to bound the fourth term, we multiply $(3.1)_2$ by $r^{N-1}u^{2k-1}$ to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} \frac{1}{2k} r^{N-1}\rho u^{2k}\,\mathrm{d}r - \int_\delta^{\mathfrak{b}} p(r^{N-1}u^{2k-1})_r\,\mathrm{d}r \\
&= -\varepsilon\int_\delta^{\mathfrak{b}} \Big((\mu+\lambda)(u_r+\frac{N-1}{r}u)(r^{N-1}u^{2k-1})_r-(N-1)\mu(r^{N-2}u^{2k})_r\Big)\,\mathrm{d}r \\
&\quad + \int_\delta^{\mathfrak{b}} \rho u^{2k-1}\Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\mathrm{d}r.
\end{aligned}
\tag{4.16}
$$

For the last term of RHS of (4.16), it follows from (4.15) that

$$
\begin{aligned}
(4.17)\qquad & \int_\delta^{\mathfrak{b}} \rho u^{2k-1}\Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\mathrm{d}r \\
&\le C(\rho_*,\mathcal{E}_0)\int_\delta^{\mathfrak{b}} \rho u^{2k-1} r^{\frac{N}{2}}\mathrm{d}r \\
&\le C(\rho_*,\mathcal{E}_0)\Big(\int_\delta^{\mathfrak{b}} \rho u^{2k}\, r^{N-1}\mathrm{d}r\Big)^{\frac{2k-1}{2k}}\Big(\int_\delta^{\mathfrak{b}} \rho r^{-N(k-1)+2k-1}\mathrm{d}r\Big)^{\frac{1}{2k}} \\
&\le C(\rho_*,\mathcal{E}_0,\varepsilon,\delta)\Big(\int_\delta^{\mathfrak{b}} \rho u^{2k}\, r^{N-1}\mathrm{d}r\Big)^{\frac{2k-1}{2k}}\Big(\int_\delta^{\mathfrak{b}} r^{-N(k-1)+2k-1}\mathrm{d}r\Big)^{\frac{1}{2k}} \\
&\le C(\rho_*,\mathcal{E}_0,\varepsilon,\delta)\Big(\int_\delta^{\mathfrak{b}} \rho u^{2k}\, r^{N-1}\mathrm{d}r\Big)^{\frac{2k-1}{2k}},
\end{aligned}
$$

provided that $N(k-1)>2k$, *i.e.*, $k>\frac{N}{N-2}$. A direct calculation shows that

$$
\begin{aligned}
(4.18)\qquad & (\mu+\lambda)(u_r+\frac{N-1}{r}u)(r^{N-1}u^{2k-1})_r-(N-1)\mu(r^{N-2}u^{2k})_r \\
&=\rho\big((2k-1)r^{N-1}u^{2k-2}u_r^2+(N-1)r^{N-3}u^{2k}\big) \\
&\quad+\delta\Big(\alpha(2k-1)r^{N-1}u^{2k-2}u_r^2+2k(N-1)(\alpha-1)r^{N-2}u^{2k-1}u_r \\
&\qquad\quad+(N-1)\big(\alpha(N-1)-(N-2)\big)r^{N-3}u^{2k}\Big).
\end{aligned}
$$

Since $\alpha=1-\frac{1}{2N}$, we calculate the discriminant $\triangle$ for the last term of (4.18) to have

$$
\begin{aligned}
\triangle &= 4k^2(N-1)^2(\alpha-1)^2-4\alpha(2k-1)(N-1)\big(\alpha(N-1)-(N-2)\big) \\
&=\frac{N-1}{N^2}\big(k^2(N-1)-(2k-1)(2N-1)(N+1)\big).
\end{aligned}
$$

Choosing $k=4$, we have

$$
\triangle=-\frac{N-1}{N^2}(14N^2-9N+9)<0,
$$

which, together with (4.18) yields

$$
\begin{aligned}
(4.19)\qquad & (\mu+\lambda)(u_r+\frac{N-1}{r}u)(r^{N-1}u^{2k-1})_r-(N-1)\mu(r^{N-2}u^{2k})_r \\
&\ge\rho\big((2k-1)r^{N-1}u^{2k-2}u_r^2+(N-1)r^{N-3}u^{2k}\big).
\end{aligned}
$$

For the pressure term, it follows from (4.5) that

$$\begin{aligned}
&\Big|\int_\delta^{\mathfrak{b}} p(r^{N-1}u^{2k-1})_r\,\mathrm{d}r\Big|\\
&=\Big|\int_\delta^{\mathfrak{b}} p\big((2k-1)r^{N-1}u^{2k-2}u_r+(N-1)r^{N-2}u^{2k-1}\big)\,\mathrm{d}r\Big|\\
&\le\frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}}\rho\big(r^{N-1}u^{2k-2}u_r^2+r^{N-3}u^{2k}\big)\,\mathrm{d}r+C_\varepsilon\int_\delta^{\mathfrak{b}}\rho^{2\gamma-1}u^{2k-2}\,r^{N-1}\mathrm{d}r\\
&\le\frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}}\rho\big(r^{N-1}u^{2k-2}u_r^2+r^{N-3}u^{2k}\big)\,\mathrm{d}r+C(T,\mathcal{M}_1)\int_\delta^{\mathfrak{b}}\big(\rho u^2+\rho u^{2k}\big)\,r^{N-1}\mathrm{d}r\\
&\le\frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}}\rho\big(r^{N-1}u^{2k-2}u_r^2+r^{N-3}u^{2k}\big)\,\mathrm{d}r+C(T,\mathcal{M}_1)\Big(1+\int_\delta^{\mathfrak{b}}\rho u^{2k}\,r^{N-1}\mathrm{d}r\Big).
\end{aligned}\tag{4.20}$$

Combining (4.12), (4.17), and (4.20) with (4.16), we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}}\frac{1}{2k}\rho u^{2k}\,r^{N-1}\mathrm{d}r\le C(T,\mathcal{M}_1)\Big(1+\int_\delta^{\mathfrak{b}}\rho u^{2k}\,r^{N-1}\mathrm{d}r\Big),$$

which, together with Grönwall's inequality, implies

$$\int_\delta^{\mathfrak{b}}\rho u^{2k}\,r^{N-1}\mathrm{d}r\le C(T,\mathcal{M}_1)\qquad\text{for all } t\in[0,T].\tag{4.21}$$

Substituting (4.15) and (4.21) into (4.11), we have

$$\int_\delta^{\mathfrak{b}}\rho|\rho^{-1-\frac{1}{2N}}\rho_r|^{2k}\,r^{N-1}\mathrm{d}r\le C(T,\mathcal{M}_1)\Big(1+\int_0^T\int_\delta^{\mathfrak{b}}\rho|\rho^{-1-\frac{1}{2N}}\rho_r|^{2k}\,r^{N-1}\mathrm{d}r\mathrm{d}s\Big).$$

Again, applying the Grönwall inequality, we obtain

$$\int_\delta^{\mathfrak{b}}\rho|\rho^{-1-\frac{1}{2N}}\rho_r|^{2k}\,r^{N-1}\mathrm{d}r\le C(T,\mathcal{M}_1).\tag{4.22}$$

Taking $k=4$, we conclude the proof. □

With Lemma 4.3, we obtain the following lower bound of the density.

Lemma 4.4 (**Lower Bound of the Density).** *There exists $C(T,\mathcal{M}_1)>0$ depending only on T and $\mathcal{M}_1$ such that the smooth solution of* (3.1)–(3.3) *satisfies*

$$\rho(t,r)\ge C(T,\mathcal{M}_1)^{-1}>0\qquad\textit{for } (t,r)\in[0,T]\times[\delta,\mathfrak{b}].\tag{4.23}$$

Proof. It is noted that $\rho(t,r)$ is a continuous function on $[\delta,\mathfrak{b}]$. Therefore, for any $r\in B(t)$ with $\rho(t,r)<\frac{1}{2}\rho_*$, there exists $r_0\in B(t)$ such that

$$\rho(t,r_0)=\frac{\rho_*}{2},\qquad |r-r_0|\le C(\rho_*,\mathcal{E}_0).$$

Therefore, for $0<\beta<\frac{k-N}{2kN}$ (for instance, $\beta=\frac{k-N}{4kN}$), we have

$$\begin{aligned}
\rho(t,r)^{-\beta}&=\rho(t,r_0)^{-\beta}+\beta\int_{r_0}^r\rho^{-\beta-1}\rho_y\,\mathrm{d}y\\
&\le C(\rho_*)+\beta\Big(\int_{r_0}^r\rho|\rho^{-1-\frac{1}{2N}}\rho_y|^{2k}\,\mathrm{d}y\Big)^{\frac{1}{2k}}\Big(\int_{r_0}^r\big(\rho^{-\beta+\frac{k-N}{2Nk}}\big)^{\frac{2k}{2k-1}}\mathrm{d}y\Big)^{\frac{2k-1}{2k}}\\
&\le C(\rho_*)+\beta C(T,\mathcal{M}_1),
\end{aligned}$$

where we have used (4.5), (4.6), and (3.35). Thus, we obtain

$$\rho(t,r) \geq C(T,\mathcal{M}_1)^{-1} \qquad \text{for any } r \in B(t),$$

which yields (4.23). This completes the proof. □

Using Lemmas 4.2 and 4.4, though the upper and lower bounds of density depend on ε^{-1} and δ^{-1}, we can have the following useful weighted estimate.

LEMMA 4.5. *For any constant $\vartheta \in (0,1)$, the smooth solution of (3.1)–(3.3) satisfies*

$$\begin{aligned}\int_\delta^{\mathfrak{b}} \Big(\frac{1}{2}\rho u^2 + e(\rho,\rho_*) + \frac{1}{2}|\Phi_r|^2\Big)(t,r) r^{2(N-1)+\vartheta}\,\mathrm{d}r \\ + \varepsilon \int_0^T \int_\delta^{\mathfrak{b}} \Big(r^{2(N-1)+\vartheta}\rho u_r^2 + \alpha\delta r^{2(N-1)+\vartheta}\rho^\alpha u_r^2\Big)\mathrm{d}r\mathrm{d}t \leq C(T,\mathcal{M}_2).\end{aligned} \tag{4.24}$$

Proof. Let $L>0$. Multiplying $(3.1)_2$ by $r^{N-1+L}u$ and integrating by parts, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} \frac{1}{2}\rho u^2\, r^{N-1+L}\mathrm{d}r + \int_\delta^{\mathfrak{b}} p_r u\, r^{N-1+L}\mathrm{d}r \\ &= \frac{1}{2}\int_\delta^{\mathfrak{b}} \rho u^3\, r^{N-2+L}\mathrm{d}r \\ &\quad - \varepsilon\int_\delta^{\mathfrak{b}} (\mu+\lambda)\big(u_r + \frac{N-1}{r}u\big)\big(ru_r + (N-1+L)u\big)\, r^{N-2+L}\mathrm{d}r \\ &\quad + \varepsilon(N-1)\int_\delta^{\mathfrak{b}} \mu\big(2ruu_r + (N-2+L)u^2\big)\, r^{N-3+L}\mathrm{d}r \\ &\quad + \int_\delta^{\mathfrak{b}} \rho u\Big(\int_\delta^r \big(\rho(t,z)-d(z)\big)\, z^{N-1}\mathrm{d}z\Big)\, r^L\mathrm{d}r.\end{aligned}$$

For the last term on RHS of (4.25), we notice from (3.8) that, for any $r \in [\delta,\mathfrak{b}]$,

$$r^{N-1}\Phi_{rt} = -\int_\delta^r (y^{N-1}\rho(t,y))_t\,\mathrm{d}y = \int_\delta^r (y^{N-1}\rho u)_y\,\mathrm{d}y = \rho u r^{N-1},$$

which yields

$$\Phi_{rt} = \rho u \qquad \text{for any } t \in [\delta,\mathfrak{b}]. \tag{4.25}$$

Using (4.25), we have

$$\begin{aligned}&\int_\delta^{\mathfrak{b}} \rho u\Big(\int_\delta^r \big(\rho(t,y)-d(y)\big)\, y^{N-1}\mathrm{d}y\Big)\, r^L\mathrm{d}r \\ &= -\int_\delta^{\mathfrak{b}} \Phi_{rt}\Phi_r\, r^{N-1+L}\mathrm{d}r = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} |\Phi_r|^2\, r^{N-1+L}\mathrm{d}r.\end{aligned}$$

The other terms can be dealt by similar arguments as in [16]. Then we conclude

(4.26)
$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\Big)\,r^{N-1+L}\mathrm{d}r+\frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}}\big(\rho u_r^2+\alpha\delta\rho^\alpha u_r^2\big)\,r^{N-1+L}\mathrm{d}r\\
&\le C(T,M_1)\Big(\int_\delta^{\mathfrak{b}}\big(\rho u^2+e(\rho,\rho_*)\big)\,r^{N-2+L}\mathrm{d}r\\
&\quad+\Big(\int_\delta^{\mathfrak{b}}\rho u^2\,r^{N-2+L}\mathrm{d}r\Big)^{\frac{4}{3}}+\int_\delta^{\mathfrak{b}}\rho u_r^2\,r^{N-1}\mathrm{d}r\Big).
\end{aligned}$$

Taking $L=1$ in (4.26) and integrating it over $[0,t]$, it follows from from (3.1) that

$$\begin{aligned}
&\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\Big)\,r^{N}\mathrm{d}r+\frac{\varepsilon}{2}\int_0^t\int_\delta^{\mathfrak{b}}\big(\rho u_r^2+\alpha\delta\rho^\alpha u_r^2\big)\,r^{N}\mathrm{d}r\mathrm{d}s\\
&\le\int_\delta^{\mathfrak{b}}\Big(\frac{1}{2}\rho_0 u_0^2+e(\rho_0,\rho_*)+\frac{1}{2}|\Phi_{0r}|^2\Big)\,r^{N}\mathrm{d}r+C(T,\mathcal{M}_2)\le C(T,\mathcal{M}_2).
\end{aligned}$$

Then, taking $L=2,3,\cdots,N-1$ in (4.26) step by step, we have

$$\begin{aligned}
&\int_\delta^{\mathfrak{b}}\big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\big)\,r^{2N-2}\mathrm{d}r+\frac{\varepsilon}{2}\int_0^t\int_\delta^{\mathfrak{b}}(\rho+\alpha\delta\rho^\alpha)u_r^2\,r^{2N-2}\mathrm{d}r\mathrm{d}s\\
&\le\int_\delta^{\mathfrak{b}}\big(\frac{1}{2}\rho_0 u_0^2+e(\rho_0,\rho_*)+\frac{1}{2}|\Phi_{0r}|^2\big)\,r^{2N-2}\mathrm{d}r+C(T,\mathcal{M}_2)\le C(T,\mathcal{M}_2)
\end{aligned}\tag{4.27}$$

for any $t\in[0,T]$.

Finally, taking $L=N-1+\vartheta$ with $\vartheta\in(0,1)$ in (4.26) and integrating it over $[0,t]$, we conclude from (4.27) that

$$\begin{aligned}
&\int_\delta^{\mathfrak{b}}\big(\frac{1}{2}\rho u^2+e(\rho,\rho_*)+\frac{1}{2}|\Phi_r|^2\big)\,r^{2N-2+\vartheta}\mathrm{d}r+\varepsilon\int_0^t\int_\delta^{\mathfrak{b}}(\rho+\alpha\delta\rho^\alpha)u_r^2\,r^{2N-2+\vartheta}\mathrm{d}r\mathrm{d}s\\
&\le\int_\delta^{\mathfrak{b}}\big(\frac{1}{2}\rho_0 u_0^2+e(\rho_0,\rho_*)+\frac{1}{2}|\Phi_{0r}|^2\big)\,r^{2N-2+\vartheta}\mathrm{d}r+C(T,\mathcal{M}_2)\le C(T,\mathcal{M}_2)
\end{aligned}$$

for any $t\in[0,T]$. Then the proof is completed. □

Using Lemmas 3.1, 3.5, 4.2, and 4.4–4.5, by similar arguments as in [16], we obtain the following decay estimates.

LEMMA 4.6 ([16], Lemma 4.7). *The smooth solution of* (3.1)–(3.3) *satisfies that, for $r\in[1,\mathfrak{b}]$,*

$$\begin{aligned}
&|(\rho-\rho_*)(t,r)|\le C(T,\mathcal{M}_2)\,r^{-\frac{3}{4}N+\frac{3}{4}-\frac{\vartheta}{4}},\\
&\int_0^T\big(|u(t,r)|+|u(t,r)|^3\big)\,r^{N-1}\mathrm{d}t\le C(T,\mathcal{M}_2)\,r^{-\frac{\vartheta}{2}}.
\end{aligned}$$

Taking $\psi(s)=\frac{1}{2}s|s|$ in (2.8), then the corresponding entropy and entropy flux pair can be represented as

$$\begin{cases}\eta^\star(\rho,\rho u)=\frac{1}{2}\rho\int_{-1}^{1}(u+\rho^\theta s)|u+\rho^\theta s|[1-s^2]_+^\lambda\,\mathrm{d}s,\\
q^\star(\rho,\rho u)=\frac{1}{2}\rho\int_{-1}^{1}(u+\theta\rho^\theta s)(u+\rho^\theta s)|u+\rho^\theta s|[1-s^2]_+^\lambda\,\mathrm{d}s,\end{cases}$$

where $\theta = \frac{\gamma-1}{2}$. Then it is direct to show that

$$|\eta^\star(\rho,\rho u)| \lesssim \rho|u|^2 + \rho^\gamma, \qquad q^\star(\rho,\rho u) \gtrsim \rho|u|^3 + \rho^{\gamma+\theta}. \tag{4.28}$$

Since

$$\begin{cases}\partial_\rho \eta^\star = \int_{-1}^1 \big(-\frac{1}{2}u + (\theta+\frac{1}{2})\rho^\theta s\big)\,|u+\rho^\theta s|[1-s^2]_+^\lambda \,\mathrm{d}s,\\ \partial_m \eta^\star = \int_{-1}^1 |u+\rho^\theta s|[1-s^2]_+^\lambda \,\mathrm{d}s,\end{cases}$$

then for some constant $C = C(\gamma) > 0$, it is direct to see that,

$$\begin{aligned}&|\eta_m^\star| \le C\big(|u| + \rho^\theta\big), \qquad |\eta_\rho^\star| \le C\big(|u|^2 + \rho^{2\theta}\big),\\ &\eta_\rho^\star(\rho,0) = 0, \quad \eta_m^\star(\rho,0) = 2\rho^\theta \int_0^1 s[1-s^2]^\lambda \mathrm{d}s.\end{aligned}$$

Define the relative-entropy pair:

$$\begin{cases}\tilde\eta(\rho,\rho u) = \eta^\star(\rho,\rho u) - \eta^\star(\rho_*,0) - \eta_m^\star(\rho_*,0)\rho u,\\ \tilde q(\rho,\rho u) = q^\star(\rho,\rho u) - q^\star(\rho_*,0) - \eta_m^\star(\rho_*,0)\big(\rho u^2 + p(\rho) - p(\rho_*)\big).\end{cases}$$

Now, we recall a key lemma established in [16], which will be used to prove Proposition 4.1.

Lemma 4.7 ([16], Lemma 4.8). *There exists $C_\gamma(\rho_*) > 0$ depending only on γ such that*

$$-\tilde q(\rho,\rho u) + \rho u \partial_\rho \tilde\eta(\rho,\rho u) + \rho u^2 \partial_m \tilde\eta(\rho,\rho u) \le C_\gamma(\rho_*)\big(\rho u^2 + e(\rho,\rho_*)\big). \tag{4.29}$$

Now, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1: A direct calculation shows that

(4.30)

$$\begin{aligned}&(r^{N-1}\tilde\eta)_t + (r^{N-1}\tilde q)_r + (N-1)r^{N-2}\big(-\tilde q + \rho u\partial_\rho\tilde\eta + \rho u^2\partial_m\tilde\eta\big)\\ &= \varepsilon r^{N-1}\partial_m\tilde\eta\Big\{\big((\rho+\alpha\delta\rho^\alpha)(u_r + \frac{N-1}{r}u)\big)_r - \frac{N-1}{r}(\rho+\delta\rho^\alpha)_r u\Big\} + \partial_m\tilde\eta\rho r^{N-1}\Phi_r.\end{aligned}$$

Let $y \in [\mathfrak{b}-1,\mathfrak{b}]$ and $r \in [r_1,r_2]$. Integrating (4.30) over $[r,y]$ and then over

$[0,T]\times[\mathfrak{b}-1,\mathfrak{b}]\times[r_1,r_2]$, we have

$$
\begin{aligned}
(4.31)\qquad &\int_0^T\int_{r_1}^{r_2}\tilde q(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&=(N-1)\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\big(-\tilde q+\rho u\partial_\rho\tilde\eta+\rho u^2\partial_m\tilde\eta\big)\,z^{N-2}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\quad+\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\big(\tilde\eta(t,z)-\tilde\eta(0,z)\big)\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\\
&\quad+(r_2-r_1)\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\tilde q(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\quad-\varepsilon\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\partial_m\tilde\eta\Big\{\big((\rho+\alpha\delta\rho^\alpha)(u_z+\frac{N-1}{z}u)\big)_z\\
&\quad-\frac{N-1}{z}(\rho+\delta\rho^\alpha)_z u\Big\}\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\quad-\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\partial_m\tilde\eta\rho\Phi_z\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t:=\sum_{i=1}^5K_i.
\end{aligned}
$$

It follows from (4.29) that

$$
K_1\le C(\rho_*)\frac{r_2}{r_1}\int_0^T\int_{r_1}^{\mathfrak{b}}\big(\rho u^2+e(\rho,\rho_*)\big)(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t\le C(\rho_*)\frac{r_2}{r_1}(\mathcal{E}_0+1).
$$

Similar to the argument in [11, Proposition 4.1], we can obtain

$$
|\tilde\eta(\rho,\rho u)|\le C_\gamma\big(\frac{m^2}{\rho}+e(\rho,\rho_*)\big),
$$

and then

$$
|K_2|\le C_\gamma r_2(\mathcal{E}_0+1).
$$

For the third term on RHS of (4.31), since

$$
|q^\star(\rho,\rho u)-q^\star(\rho,0)|\le C_\gamma\big(\rho|u|^3+\rho^{1+2\theta}|u|\big),
$$

we have

$$
(4.32)\qquad \tilde q(\rho,\rho u)\le C(T,\mathcal{M}_2)\big(|\rho-\rho_*|^2+|u|^3+|u|\big).
$$

It follows from (4.32) and Lemma 4.6 that

$$
(4.33)\quad |K_3|\le C(T,\mathcal{M}_2)r_2\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_0^T\big(|\rho-\rho_*|^2+|u|^3+|u|\big)(t,y)\,y^{N-1}\mathrm{d}y\mathrm{d}t\le C(T,\mathcal{M}_2)\,\mathfrak{b}^{-\frac{\vartheta}{2}}
$$

To bound the viscous term, we regard $\tilde\eta_m(\rho,\rho u)$ as a function of (ρ,u), then

$$
(4.34)\qquad \begin{cases}|\partial_m\tilde\eta(\rho,\rho u)|\le C_\gamma\big(|u|+|\rho^\theta-\rho_*^\theta|\big),\\ |\partial_{mu}\tilde\eta(\rho,\rho u)|+\rho^{1-\theta}|\partial_{m\rho}\tilde\eta(\rho,\rho u)|\le C_\gamma,\end{cases}
$$

It follows from integration by parts that

$$\begin{aligned} K_4 \le\, & \varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \int_{r_1}^{r_2} \Big| \int_r^y \partial_m \tilde{\eta} \Big\{ \big((\rho+\alpha\delta\rho^\alpha)u_z\big)_z + (\rho+\alpha\delta\rho^\alpha)(\frac{N-1}{z}u)_z \\ & \qquad\qquad + (\alpha-1)\delta(\rho^\alpha)_z \frac{N-1}{z}u \Big\}\, z^{N-1}\mathrm{d}z \Big|\, \mathrm{d}r\mathrm{d}y\mathrm{d}t \\ \le\, & \varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \int_{r_1}^{r_2} \int_r^y (\rho+\alpha\delta\rho^\alpha)|u_z(z^{N-1}\partial_m\tilde{\eta})_z|\, \mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t \\ & + \varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \int_{r_1}^{r_2} \int_r^y \delta\rho^\alpha \Big|(z^{N-1}\partial_m\tilde{\eta})_z \frac{u}{z}\Big|\, \mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t \\ & + \varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \int_{r_1}^{r_2} \int_r^y (\rho+\delta\rho^\alpha)|\partial_m\tilde{\eta}(\frac{u}{z})_z|\, z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t \\ & + C\varepsilon \int_0^T \int_{r_1}^{r_2} \big(r|(\partial_m\tilde{\eta}(\rho+\delta\rho^\alpha)u_r)(t,r)| + \delta|\rho^\alpha u \partial_m\tilde{\eta}(t,r)|\big)\, r^{N-2}\mathrm{d}r\mathrm{d}t \\ & + C\varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \big(y(\rho+\delta\rho^\alpha)|\partial_m\tilde{\eta}u_y| + \delta|(\rho^\alpha u\partial_m\tilde{\eta})(t,y)|\big)\, y^{N-2}\mathrm{d}y\mathrm{d}t. \end{aligned} \tag{4.35}$$

In order to estimate the terms on RHS of (4.35), we notice that

$$e(\rho,\rho_*)\,\mathbf{1}_{B(t)}(r) \ge \frac{1}{C(\rho_*)} \qquad \text{for } r\in B(t).$$

Then

$$\begin{aligned} & \int_{r_1}^{\mathfrak{b}} \rho^\alpha(\rho^\theta-\rho_*^\theta)^2\, r^{N-1}\mathrm{d}r \\ & \le C(\rho_*)\int_{r_1}^{\mathfrak{b}} \mathbf{1}_{B^c(t)}(r)\rho(\rho^\theta-\rho_*^\theta)^2\, r^{N-1}\mathrm{d}r + \int_{r_1}^{\mathfrak{b}} \mathbf{1}_{B(t)}(r)\rho^\alpha(\rho^\theta-\rho_*^\theta)^2(t,r)\, r^{N-1}\mathrm{d}r \\ & \le C(\rho_*)\int_{r_1}^{\mathfrak{b}} e(\rho,\rho_*)\, r^{N-1}\mathrm{d}r + C(\rho_*)\int_{r_1}^{\mathfrak{b}} \mathbf{1}_{B(t)}(r)\, r^{N-1}\mathrm{d}r \\ & \le C(\rho_*)\int_{r_1}^{\mathfrak{b}} e(\rho,\rho_*) r^{N-1}\mathrm{d}r + C(\rho_*)\int_{r_1}^{\mathfrak{b}} e(\rho,\rho_*)\, r^{N-1}\mathrm{d}r \\ & \le C(\rho_*,\mathcal{E}_0). \end{aligned} \tag{4.36}$$

Using (4.34) and (4.36), the first to third terms on RHS of (4.35) can be bounded by

$$\begin{aligned} & C(\frac{1}{r_1})\varepsilon \int_0^T \int_{\mathfrak{b}-1}^{\mathfrak{b}} \int_{r_1}^{r_2} \int_r^y \Big\{ \rho u_z^2 + \rho^{\gamma-2}\rho_z^2 + \delta\rho^\alpha u_z^2 + \delta\rho^{\gamma+\alpha-3}\rho_z^2 + \delta z^{-2}\rho^\alpha u^2 \\ & \qquad\qquad + \rho u^2 + e(\rho,\rho_*) + \delta z^{-2}\rho^\alpha(\rho^\theta-\rho_*^\theta)^2 \Big\}\, z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t \\ & \le C(\rho_*,\mathcal{E}_0,r_1,r_2,T). \end{aligned}$$

The fourth term on RHS of (4.35) is bounded by

$$\begin{aligned}
&C(r_2,\frac{1}{r_1})\int_0^T\int_{r_1}^{r_2}\Big\{\varepsilon\rho u_r^2+\varepsilon\delta\rho^\alpha u_r^2+\varepsilon\delta r^{-2}\rho^\alpha u^2\\
&\qquad\qquad\qquad+\rho u^2+e(\rho,\rho_*)+\rho^\alpha(\rho^\theta-\rho_*^\theta)^2\Big\}\,r^{N-1}\mathrm{d}r\mathrm{d}t\\
&\le C(\rho_*,\mathcal{E}_0,r_1,r_2,T).
\end{aligned}$$

The last term of (4.35) is bounded by

$$\begin{aligned}
&r_2\varepsilon\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}(\rho+\delta\rho^\alpha)|u_y|\big(|u|+|\rho^\theta-\rho_*^\theta|\big)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\le Cr_2\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\big(\varepsilon(\rho+\delta\rho^\alpha)u_y^2+(\rho u^2+e(\rho,\rho_*))\big)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\quad+C(T,\mathcal{M}_2)\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_0^T|u(t,y)|^2\,y^{N-1}\mathrm{d}t\mathrm{d}y\\
&\le C(\rho_*,\mathcal{E}_0,r_1,r_2,T)+C(T,\mathcal{M}_2)\mathfrak{b}^{-\frac{\vartheta}{2}}.
\end{aligned}\tag{4.37}$$

Finally, for K_5, it follows from (4.15) that

$$|\Phi_r|=\frac{1}{r^{N-1}}\Big|\int_\delta^r\big(\rho(t,y)-d(y)\big)\,y^{N-1}\mathrm{d}y\Big|\le C(\rho_*,\mathcal{E}_0)\big(r^{-N+1}+r^{-\frac{N}{2}+1}\big).\tag{4.38}$$

Using (4.38), we have

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\partial_m\tilde\eta\,\rho\,\Phi_z z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\Big|\\
&\le\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y|z^{N-1}\Phi_z|\rho\big(|u|+|\rho^\theta-\rho_*^\theta|\big)\,\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\le\int_0^T\int_{\mathfrak{b}-1}^{\mathfrak{b}}\int_{r_1}^{r_2}\int_r^y\big(\rho u^2+|\Phi_z|^2\rho+\rho|\rho^\theta-\rho_*^\theta|^2\big)\,z^{N-1}\mathrm{d}z\mathrm{d}r\mathrm{d}y\mathrm{d}t\\
&\le Cr_2\int_0^T\int_{r_1}^{\mathfrak{b}}\big(\rho u^2+e(\rho,\rho_*)\big)\,y^{N-1}\mathrm{d}y\mathrm{d}t+Cr_2\int_0^T\int_{r_1}^{\mathfrak{b}}|\Phi_y|^2\rho\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\le C(T,r_2,\mathcal{E}_0)+Cr_2\int_0^T\int_{r_1}^{\mathfrak{b}}|\Phi_y|^2\rho_*\mathbf{1}_{\{\rho(t,y)\le2\rho_*\}}\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\quad+Cr_2\int_0^T\int_{r_1}^{\mathfrak{b}}|\Phi_y|^2\rho\mathbf{1}_{A(t)}\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\le C(r_2,T,\mathcal{E}_0)+C(\rho_*,\mathcal{E}_0,r_1)\int_0^T\int_{r_1}^{\mathfrak{b}}\rho\mathbf{1}_{\{\rho>2\rho_*\}}\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\le C(r_2,T,\mathcal{E}_0)+C(\rho_*,\mathcal{E}_0,r_1)\int_0^T\int_{r_1}^{\mathfrak{b}}e(\rho,\rho_*)\,y^{N-1}\mathrm{d}y\mathrm{d}t\\
&\le C(\rho_*,\mathcal{E}_0,r_1,r_2,T).
\end{aligned}\tag{4.39}$$

Combining all the estimates, we obtain

$$\int_0^T\int_{r_1}^{r_2}\tilde q(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\le C(\rho_*,r_1,r_2,T,\mathcal{E}_0)+C(T,\mathcal{M}_2)\,\mathfrak{b}^{-\frac{\vartheta}{2}}.\tag{4.40}$$

Then we conclude (4.4) from (3.1), (4.28), and (4.40). $\square$

Now, we prove the following lemma, which is needed when passing to the limit $\mathfrak{b} \to \infty$.

Lemma 4.8. *The smooth solution of* (3.1)–(3.3) *satisfies that, for any* $t \in [0, T]$,

$$\|u_r(t)\|_{L^2}^2 + \int_0^T \|u_t(t)\|_{L^2}^2 + \|u_{rr}(t)\|_{L^2}^2 \mathrm{d}t \leq C(T, \|u_{0r}\|_{L^2}, \mathcal{M}_2). \tag{4.41}$$

Proof. It follows from $(3.1)_1$ that

$$-\varepsilon((\mu+\lambda)u_r)_r + \rho u_t = \mathcal{H}, \tag{4.42}$$

where $\mathcal{H} := -\rho u u_r - p_r + \varepsilon(\mu+\lambda)(\frac{N-1}{r}u)_r + \varepsilon\frac{N-1}{r}u\lambda_r - \rho\Phi_r$. Multiplying (4.42) by u_t and integrating it over $[\delta, \mathfrak{b}]$, we have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}} (\mu+\lambda)|u_r|^2 \,\mathrm{d}r + \int_\delta^{\mathfrak{b}} \rho u_t^2 \,\mathrm{d}r = \frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}} (\mu+\lambda)_t |u_r|^2 \,\mathrm{d}r + \int_\delta^{\mathfrak{b}} \mathcal{H} u_t \,\mathrm{d}r. \tag{4.43}$$

Using (3.1), (3.19), (4.5), (4.23), and the Sobolev inequalities, we obtain

$$\|u_r\|_{L^\infty} \leq C\big(\|u_r\|_{L^2} + \|u_r\|_{L^2}^{\frac12}\|u_{rr}\|_{L^2}^{\frac12}\big),$$

and

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\delta^{\mathfrak{b}} (\mu+\lambda)_t |u_r|^2 \,\mathrm{d}r \\
&\leq C(T, \mathcal{M}_2)\int_\delta^{\mathfrak{b}} \big(|\rho_r u| + |u_r| + |u|\big)|u_r|^2 \,\mathrm{d}r \\
&\leq C(T, \mathcal{M}_2)\Big(\|u\|_{L^2}^{\frac12}\|\rho_r\|_{L^2}\big(\|u_r\|_{L^2}^2\|u_{rr}\|_{L^2}^{\frac12} + \|u_r\|_{L^2}^{\frac52}\big) \\
&\qquad + \|u\|_{L^2}^{\frac12}\|u_r\|_{L^2}^{\frac52} + \|u_r\|_{L^2}^2\big(\|u_r\|_{L^2} + \|u_r\|_{L^2}^{\frac12}\|u_{rr}\|_{L^2}^{\frac12}\big)\Big) \\
&\leq C(T, \mathcal{M}_2)\Big(\big(\|u_r\|_{L^2}^2 + \|u_r\|_{L^2}^{\frac52}\big)\|u_{rr}\|_{L^2}^{\frac12} + \|u_r\|_{L^2}^3 + 1\Big).
\end{aligned} \tag{4.44}$$

Notice that

$$\int_\delta^{\mathfrak{b}} \rho^{-1}|\rho\Phi_r|^2 \,\mathrm{d}r \leq C(T, \mathcal{M}_2). \tag{4.45}$$

Then, using (4.45), we obtain

$$\begin{aligned}
\Big|\int_\delta^{\mathfrak{b}} \mathcal{H} u_t \mathrm{d}r\Big| &\leq \frac18\int_\delta^{\mathfrak{b}} \rho|u_t|^2 \,\mathrm{d}r + C\int_\delta^{\mathfrak{b}} \rho^{-1}|\mathcal{H}|^2 \,\mathrm{d}r \\
&\leq C(T, \mathcal{M}_2)\big((\|u\|_{L^\infty}^2 + 1)\|(\rho_r, u_r)\|_{L^2}^2 + \|u\|_{L^2}^2\big) + \frac18\int_\delta^{\mathfrak{b}} \rho|u_t|^2 \,\mathrm{d}r \\
&\leq \frac18\int_\delta^{\mathfrak{b}} \rho|u_t|^2 \,\mathrm{d}r + C(T, \mathcal{M}_2)\big(\|u_r\|_{L^2}^3 + 1\big).
\end{aligned} \tag{4.46}$$

To close estimate (4.46), we use (3.19), (4.5), (4.23), and (4.42) to obtain

$$
\begin{aligned}
(4.47)\qquad \|u_{rr}\|_{L^2}^2 &\le C(T,\mathcal{M}_2)\Big(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\rho_r\|_{L^2}^2\|u_r\|_{L^2}\|u_{rr}\|_{L^2}+\|\mathcal{H}\|_{L^2}^2\Big)\\
&\le C(T,\mathcal{M}_2)\Big(\|\sqrt{\rho}u_t\|_{L^2}^2+\|u_r\|_{L^2}\|u_{rr}\|_{L^2}+\|u_r\|_{L^2}^3+1\Big)\\
&\le C(T,\mathcal{M}_2)\Big(\|\sqrt{\rho}u_t\|_{L^2}^2+\|u_r\|_{L^2}^3+1\Big).
\end{aligned}
$$

Using (4.43)–(4.47), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\delta^{\mathfrak{b}}(\mu+\lambda)|u_r|^2\,\mathrm{d}r+\int_\delta^{\mathfrak{b}}\rho u_t^2\,\mathrm{d}r\le C(T,\mathcal{M}_2)\Big(1+\|u_r\|_{L^2}^2\int_\delta^{\mathfrak{b}}(\mu+\lambda)|u_r|^2\,\mathrm{d}r\Big).
$$

Applying the Grönwall inequality, we obtain

$$
\int_\delta^{\mathfrak{b}}(\mu+\lambda)|u_r|^2\,\mathrm{d}r+\int_0^t\int_\delta^{\mathfrak{b}}\rho u_t^2\,\mathrm{d}r\mathrm{d}s\le C(T,||u_{0r}||_{L^2},\mathcal{M}_2).
$$

together with (4.47), yields (4.41). ☐

5. Limits of the Approximate solutions for the CNSPEs. In this section, we first pass to the limit, $\mathfrak{b}\to\infty$, to obtain global strong solutions $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ of CNSPEs with required uniform bounds. Then, taking the limit, $\delta\to0+$, we obtain global spherically symmetric solutions of CNSPEs (1.4) with some desired uniform bounds on $[0,T]\times[0,\infty)$, which are important for us to utilize the compensated compactness framework in §6.

5.1. Passage the limit: $\mathfrak{b}\to\infty$. In this subsection, we fix parameters (ε,δ) and denote the solution of (3.1)–(3.3) as $(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})$. Similar to [11, 16] (see Appendix for details), we can construct that there exist sequences of smooth approximate initial data functions $(\rho_0^{\varepsilon,\delta,\mathfrak{b}},u_0^{\varepsilon,\delta,\mathfrak{b}})$ and $(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta})$ satisfying (3.6) and

$$
(5.1)\quad
\begin{cases}
(\rho_0^{\varepsilon,\delta,\mathfrak{b}},m_0^{\varepsilon,\delta,\mathfrak{b}})(r)\to(\rho_0^{\varepsilon,\delta},m_0^{\varepsilon,\delta})(r) & \text{in } L^p_{\mathrm{loc}}([\delta,\infty))\times L^1_{\mathrm{loc}}([\delta,\infty)) \text{ as } \mathfrak{b}\to\infty,\\
(\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}},\mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}})\to(\mathcal{E}_0^{\varepsilon,\delta},\mathcal{E}_1^{\varepsilon,\delta}) & \text{as } \mathfrak{b}\to\infty,\\
\mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}}+\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}}+\|u_{0r}^{\varepsilon,\delta,\mathfrak{b}}\|_{L^2} & \text{is uniformly bounded with respect to } \mathfrak{b},
\end{cases}
$$

where $p=\max\{\gamma,\frac{2N}{N+2}\}$,

$$
(5.2)\quad
\begin{aligned}
\mathcal{E}_0^{\varepsilon,\delta}&:=\int_\delta^\infty\big(\bar{\eta}^*(\rho_0^{\varepsilon,\delta},m_0^{\varepsilon,\delta})+|\Phi_{0r}^{\varepsilon,\delta}|^2\big)\,r^{N-1}\mathrm{d}r<\infty,\\
\mathcal{E}_1^{\varepsilon,\delta}&:=\varepsilon^2\int_\delta^\infty\big(1+2\alpha\delta(\rho_0^{\varepsilon,\delta})^{\alpha-1}+\alpha^2\delta^2(\rho_0^{\varepsilon,\delta})^{2\alpha-2}\big)\big|(\sqrt{\rho_0^{\varepsilon,\delta}})_r\big|^2\,r^{N-1}\mathrm{d}r<\infty.
\end{aligned}
$$

From (3.1), (3.19), (4.5), (4.23), and (4.41), there exists a constant $\tilde{C}>0$ that may depend on (ε,δ,T), but is independent of b, so that

$$
(5.3)\qquad 0<\tilde{C}^{-1}\le\rho^{\varepsilon,\delta,\mathfrak{b}}(t,r)\le\tilde{C},
$$

$$
\begin{aligned}
(5.4)\qquad &\sup_{t\in[0,T]}\Big(\big\|(\rho^{\varepsilon,\delta,\mathfrak{b}}-\rho_*,u^{\varepsilon,\delta,\mathfrak{b}})(t)\big\|_{H^1([\delta,\mathfrak{b}])}^2+\big\|\rho_t^{\varepsilon,\delta,\mathfrak{b}}(t)\big\|_{L^2([\delta,\mathfrak{b}])}^2\Big)\\
&\quad+\int_0^T\big\|(u_t^{\varepsilon,\delta,\mathfrak{b}},u_{rr}^{\varepsilon,\delta,\mathfrak{b}})(t)\big\|_{L^2([\delta,\mathfrak{b}])}^2\mathrm{d}t\le\tilde{C}.
\end{aligned}
$$

We extend $\rho^{\varepsilon,\delta,\mathfrak{b}}(t,r)$ and $u^{\varepsilon,\delta,\mathfrak{b}}(t,r)$ to $[0,T]\times[\delta,\infty)$ by defining $\rho^{\varepsilon,\delta,\mathfrak{b}}(t,r)=\rho_*$ and $u^{\varepsilon,\delta,\mathfrak{b}}(t,r)=0$ for all $(t,r)\in[0,T]\times(\mathfrak{b},\infty)$. Then it follows from (5.4) and the Aubin-Lions lemma that

$$(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})\qquad\text{is compact in } C([0,T];L^p_{\mathrm{loc}}[\delta,\infty)) \text{ with } p\in[1,\infty).$$

To be more precise, we obtain

LEMMA 5.1. *There exist functions* $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})(t,r)$ *such that, as* $\mathfrak{b}\to\infty$ (*up to a subsequence*),

$$(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})\to(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})\qquad\textit{strongly in } C([0,T];L^p_{\mathrm{loc}}[\delta,\infty)) \textit{ for all } p\in[1,\infty).$$

In particular, as $\mathfrak{b}\to\infty$ (*up to a subsequence*),

$$(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})\to(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})\qquad\textit{a.e. } (t,r)\in[0,T]\times[\delta,\infty).$$

Using Lemma 5.1, it is straightforward to show that $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ is a global weak solution of the initial-boundary value problem (IBVP) of CNSPEs (3.1):

$$\begin{cases}(\rho,u)(0,r)=(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta})(r) & \text{for } r\in[\delta,\infty),\\ u|_{r=\delta}=0 & \text{for } t\geq 0.\end{cases}\tag{5.5}$$

Moreover, it yields from (5.3)–(5.4) and the lower semicontinuity of approximate solutions that

$$0<\tilde{C}^{-1}\leq\rho^{\varepsilon,\delta}(t,r)\leq\tilde{C},\tag{5.6}$$

$$\begin{aligned}&\sup_{t\in[0,T]}\big(\|(\rho^{\varepsilon,\delta}-\rho_*,u^{\varepsilon,\delta})(t)\|^2_{H^1([\delta,\infty))}+\|\rho_t^{\varepsilon,\delta}(t)\|^2_{L^2([\delta,\infty))}\big)\\&\quad+\int_0^T\|(u_t^{\varepsilon,\delta},u_{rr}^{\varepsilon,\delta})(t)\|^2_{L^2([\delta,\infty))}\mathrm{d}t\leq\tilde{C}.\end{aligned}\tag{5.7}$$

These facts show that the weak solution $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},\Phi^{\varepsilon,\delta})$ of (5.5) is indeed a strong solution. The uniqueness of the strong solution $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},\Phi^{\varepsilon,\delta})$ is guaranteed by properties (5.6)–(5.7) and the L^2–energy basic estimate. This indicates that the whole sequence $(\rho^{\varepsilon,\delta,\mathfrak{b}},u^{\varepsilon,\delta,\mathfrak{b}})$ converges to $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ as $\mathfrak{b}\to\infty$.

Then it is straightforward to show that

$$(\rho^{\varepsilon,\delta},\mathcal{M}^{\varepsilon,\delta},\Phi^{\varepsilon,\delta})(t,\mathbf{x})=(\rho^{\varepsilon,\delta}(t,r),m^{\varepsilon,\delta}(t,r)\,\frac{\mathbf{x}}{r},\Phi^{\varepsilon,\delta}(t,r))$$

with $\rho^{\varepsilon,\delta}(t,\mathbf{x})>0$ is a strong solution of the initial-boundary problem of system (1.4) with (μ,λ) determined by (3.4) for $(t,\mathbf{x})\in[0,\infty)\times\big(\mathbb{R}^N\backslash B_\delta(\mathbf{0})\big)$ with initial-boundary data as follows:

$$\begin{cases}(\rho^{\varepsilon,\delta},\mathcal{M}^{\varepsilon,\delta})(0,\mathbf{x})=(\rho_0^{\varepsilon,\delta}(r),m_0^{\varepsilon,\delta}(r)\,\frac{\mathbf{x}}{r}),\\ \mathcal{M}^{\varepsilon,\delta}(t,\mathbf{x})|_{\mathbf{x}\in\partial B_\delta(\mathbf{0})}=\mathbf{0}.\end{cases}$$

From Lemma 5.1, Lemma 3.4, (3.1), (3.19), (3.30), (3.43), (4.4)–(4.5), (5.1), Fatou's lemma, and the lower semicontinuity, we have

PROPOSITION 5.2. *Under assumption* (5.1), *for any fixed* (ε,δ), *there exists a unique strong solution* $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},\Phi^{\varepsilon,\delta})$ *of IBVP* (5.5). *Moreover,* $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ *satisfies* (5.6) *and, for* $t>0$,

$$\begin{aligned}
&\int_\delta^\infty\Big(\frac12\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^2+e(\rho^{\varepsilon,\delta},\rho_*)+\frac12|\Phi_r^{\varepsilon,\delta}|^2\Big)(t,r)\,r^{N-1}\mathrm{d}r\\
&\quad+\varepsilon\int_0^T\int_\delta^\infty\Big(\rho^{\varepsilon,\delta}|u_r^{\varepsilon,\delta}|^2+\rho^{\varepsilon,\delta}\frac{|u^{\varepsilon,\delta}|^2}{r^2}\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\\
&\quad+c_N\varepsilon\delta\int_0^T\int_\delta^\infty\Big((\rho^{\varepsilon,\delta})^\alpha\big(|u_r^{\varepsilon,\delta}|^2+\frac{|u^{\varepsilon,\delta}|^2}{r^2}\big)\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\le\mathcal{E}_0^{\varepsilon,\delta}\le C(\mathcal{E}_0+1),
\end{aligned}$$

$$\begin{aligned}
&\varepsilon^2\int_\delta^\infty\Big(\big|(\sqrt{\rho^{\varepsilon,\delta}})_r\big|^2+\delta(\rho^{\varepsilon,\delta})^{\alpha-2}|\rho_r^{\varepsilon,\delta}|^2+\delta^2(\rho^{\varepsilon,\delta})^{2\alpha-3}|\rho_r^{\varepsilon,\delta}|^2\Big)(t,r)\,r^{N-1}\mathrm{d}r\\
&\quad+\varepsilon\int_0^T\int_\delta^\infty\Big(\big|\big((\rho^{\varepsilon,\delta})^{\frac{\gamma}{2}}\big)_r\big|^2+\delta(\rho^{\varepsilon,\delta})^{\gamma+\alpha-3}|\rho_r^{\varepsilon,\delta}|^2\Big)(s,r)\,r^{N-1}\mathrm{d}r\mathrm{d}s\le C(\mathcal{E}_0+1),
\end{aligned}$$

$$\int_0^T\int_{r_1}^{r_2}(\rho^{\varepsilon,\delta})^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,T,\mathcal{E}_0),$$

$$\int_0^T\int_{r_1}^{r_2}\big(\rho^{\varepsilon,\delta}|u^{\varepsilon,\delta}|^3+(\rho^{\varepsilon,\delta})^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,T,\mathcal{E}_0),$$

$$\int_\delta^D\Big(\int_\delta^r\rho^{\varepsilon,\delta}(y)y^{N-1}\,\mathrm{d}y\Big)\rho^{\varepsilon,\delta}(r)\,r\mathrm{d}r\le C(D,\mathcal{E}_0),$$

for any fixed $T,D>0$ *and any compact subset* $[r_1,r_2]$ *of* $[\delta,\infty)$.

5.2. Passage the limit: $\delta\to0+$.

In this subsection, for fixed $\varepsilon>0$, we consider the limit: $\delta\to0+$ to obtain the weak solution of CNSPEs. It follows from Lemma A.1 in the appendix that

$$\begin{cases}(\rho_0^{\varepsilon,\delta},m_0^{\varepsilon,\delta})(r)\to(\rho_0^\varepsilon,m_0^\varepsilon)(r), & \text{in } L^p_{\mathrm{loc}}([0,\infty);r^{N-1}\mathrm{d}r)\times L^1_{\mathrm{loc}}([0,\infty);r^{N-1}\mathrm{d}r)\\ & \quad\text{as }\delta\to0+,\\ (\mathcal{E}_0^{\varepsilon,\delta},\mathcal{E}_1^{\varepsilon,\delta})\to(\mathcal{E}_0^\varepsilon,\mathcal{E}_1^\varepsilon) & \text{as }\delta\to0+,\end{cases}\tag{5.8}$$

where $p=\max\{\gamma,\frac{2N}{N+2}\}$. To take the limit, we need to be careful, since the weak solution may contain the vacuum state. Here, we consider the limit process: $\delta\to0+$, by adopting similar compactness arguments as in [53, 31]. It is noted that our solution $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ can be extended as the zero extension of $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})$ outside $[0,T]\times[\delta,\infty)$. In the following lemma, we collect some results that are used for the limits for self-containedness.

LEMMA 5.3. *For fixed* ε, *as* $\delta\to0+$ (*up to a subsequence*), *the extended solution sequence* $(\rho^{\varepsilon,\delta},m^{\varepsilon,\delta})$ *satisfies the following*:

(i) *There exists a function* $\rho^\varepsilon(t,r)$ *such that*

$$(\sqrt{\rho^{\varepsilon,\delta}},\rho^{\varepsilon,\delta})\to(\sqrt{\rho^\varepsilon},\rho^\varepsilon)\quad a.e.\ and\ strongly\ in\ C(0,T;L^q_{\mathrm{loc}})\tag{5.9}$$

for any $q\in[1,\infty)$, *where* L^q_{loc} *stands for* $L^q(\mathcal{O})$ *for any* $\mathcal{O}\Subset(0,\infty)$.

(ii) *The pressure function sequence* $p(\rho^{\varepsilon,\delta})$ *is uniformly bounded in* $L^\infty(0,T;L^q_{\mathrm{loc}})$ *for all* $q\in[1,\infty]$, *and*

$$p(\rho^{\varepsilon,\delta})\to p(\rho^\varepsilon)\qquad \textit{strongly in } L^q(0,T;L^q_{\mathrm{loc}}) \textit{ for all } q\in[1,\infty). \tag{5.10}$$

(iii) *The momentum function sequence* $m^{\varepsilon,\delta}$ *converges strongly in* $L^2(0,T;L^q_{\mathrm{loc}})$ *to some function* $m^\varepsilon(t,r)$ *for all* $q\in[1,\infty)$, *which yields*

$$m^{\varepsilon,\delta}(t,r)=(\rho^{\varepsilon,\delta}u^{\varepsilon,\delta})(t,r)\to m^\varepsilon(t,r)\qquad \textit{a.e. in } [0,T]\times(0,\infty).$$

In Lemma 5.3, the convergence results in (i)–(iii) are from Lemma 5.3, Corollary 5.4, and Lemma 5.5 in [16].

The proof of the following lemma is similar to that in [11, Lemma 4.4].

LEMMA 5.4. $m^\varepsilon(t,r)=0$ *a.e. on* $\{(t,r):\rho^\varepsilon(t,r)=0\}$. *Moreover, there exists a function* $u^\varepsilon(t,r)$ *such that* $m^\varepsilon(t,r)=\rho^\varepsilon(t,r)u^\varepsilon(t,r)$ *a.e.*, $u^\varepsilon(t,r)=0$ *a.e. on* $\{(t,r):\rho^\varepsilon(t,r)=0\}$, *and*

$$\begin{aligned} &m^{\varepsilon,\delta}\to m^\varepsilon && \textit{strongly in } L^2(0,T;L^p_{\mathrm{loc}}) \textit{ for } p\in[1,\infty),\\ &\frac{m^{\varepsilon,\delta}}{\sqrt{\rho^{\varepsilon,\delta}}}\to\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}=\sqrt{\rho^\varepsilon}u^\varepsilon && \textit{strongly in } L^2(0,T;L^2_{\mathrm{loc}}). \end{aligned}$$

Then we can obtain the following theorem for the global weak solutions of CNSPEs:

THEOREM 5.5. *As* $\delta\to0+$. *Let* $(\rho^\varepsilon,m^\varepsilon)$ *be the limit of* $(\rho^{\varepsilon,\delta},m^{\varepsilon,\delta})$, *and* $(\rho^\varepsilon_0,m^\varepsilon_0)$ *be the initial data satisfying* (2.9)–(2.12). *For each fixed* $\varepsilon>0$, *there exists a global spherical symmetry weak solution*

$$(\rho^\varepsilon,\mathcal{M}^\varepsilon,\Phi^\varepsilon)(t,\mathbf{x}):=\Big(\rho^\varepsilon(t,r),m^\varepsilon(t,r)\frac{\mathbf{x}}{r},\Phi^\varepsilon(t,r)\Big)$$

of CNSPEs (1.4) *in the sense of Definition* 2.3. *Moreover,*

$$(\rho^\varepsilon,m^\varepsilon,\Phi^\varepsilon)(t,r)=(\rho^\varepsilon(t,r),\rho^\varepsilon(t,r)u^\varepsilon(t,r),\Phi^\varepsilon(t,r)),$$

with $u^\varepsilon(t,r):=\frac{m^\varepsilon(t,r)}{\rho^\varepsilon(t,r)}$ *a.e. on* $\{(t,r):\rho^\varepsilon(t,r)\neq0\}$ *and* $u^\varepsilon(t,r):=0$ *a.e. on*

$\{(t,r)\,:\,\rho^\varepsilon(t,r)=0$ *or* $r=0\}$*, satisfies the following bounds*:

$$\rho^\varepsilon(t,r)\ge 0 \ \ a.e., \tag{5.11}$$

$$\Big(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big)(t,r)=\sqrt{\rho^\varepsilon(t,r)}\,u^\varepsilon(t,r)=0 \ \ a.e. \ \ on \ \{(t,r)\,:\,\rho^\varepsilon(t,r)=0\}, \tag{5.12}$$

$$\begin{aligned}&\int_0^\infty \Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2+e(\rho^\varepsilon,\rho_*)+\frac{1}{2}|\Phi^\varepsilon_r|^2\Big)(t,r)\,r^{N-1}\mathrm{d}r\\ &\qquad+\varepsilon\int_{\mathbb{R}^2_+}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2(s,r)\,r^{N-3}\mathrm{d}r\mathrm{d}s\le \mathcal{E}_0^\varepsilon\le\mathcal{E}_0+1 \ \ for\ t>0,\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\varepsilon^2\int_0^\infty \big|\big(\sqrt{\rho^\varepsilon(t,r)}\big)_r\big|^2\,r^{N-1}\mathrm{d}r\\ &\qquad+\varepsilon\int_{\mathbb{R}^2_+}\big|\big((\rho^\varepsilon(s,r))^{\frac{\gamma}{2}}\big)_r\big|^2\,r^{N-1}\mathrm{d}r\mathrm{d}s\le C(\mathcal{E}_0+1) \quad for\ t>0,\end{aligned} \tag{5.14}$$

$$\int_0^T\int_{r_1}^{r_2}(\rho^\varepsilon)^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,T,\mathcal{E}_0), \tag{5.15}$$

$$\int_0^T\int_{r_1}^{r_2}\big(\rho^\varepsilon|u^\varepsilon|^3+(\rho^\varepsilon)^{\gamma+\theta}\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\le C(r_1,r_2,T,\mathcal{E}_0), \tag{5.16}$$

$$\int_0^D\Big(\int_0^r\rho^\varepsilon(t,y)y^{N-1}\,\mathrm{d}y\Big)\rho^\varepsilon(t,r)\,r\mathrm{d}r\le C(D,\mathcal{E}_0), \tag{5.17}$$

for any fixed $T,D>0$ *and any compact subset* $[r_1,r_2]\Subset(0,\infty)$*. In addition, the following energy inequality holds*:

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(\frac{1}{2}\Big|\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2+e(\rho^\varepsilon,\rho_*)+\frac{1}{2}|\nabla_{\mathbf{x}}\Phi^\varepsilon|^2\Big)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\le\int_{\mathbb{R}^N}\Big(\frac{1}{2}\Big|\frac{\mathcal{M}_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}}\Big|^2+e(\rho_0^\varepsilon,\rho_*)+\frac{1}{2}|\nabla_{\mathbf{x}}\Phi_0^\varepsilon|^2\Big)(\mathbf{x})\,\mathrm{d}\mathbf{x}\qquad for\ t\ge 0.\end{aligned}$$

For any entropy pair (η,q) *defined in* (2.8) *for any smooth compactly supported function* $\psi(s)$ *on* $\mathbb{R}$*,*

$$\partial_t\eta(\rho^\varepsilon,m^\varepsilon)+\partial_r q(\rho^\varepsilon,m^\varepsilon)\qquad is\ compact\ in\ H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+).$$

Proof. We divide the proof into four steps:

1. It follows from the lower semicontinuity, Fatou's lemma, and Proposition 5.2 that, *under assumption* (5.8)*, for any fixed* ε *and* $T>0$*, the limit functions* $(\rho^\varepsilon,m^\varepsilon)=$

$(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ *satisfy*

$$
\begin{aligned}
&\rho^\varepsilon(t,r) \ge 0 \quad \textit{a.e.},\\
&u^\varepsilon(t,r) = 0,\ \big(\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big)(t,r) = \sqrt{\rho^\varepsilon}(t,r)u^\varepsilon(t,r) = 0 \qquad \textit{a.e. on } \{(t,r)\,:\,\rho^\varepsilon(t,r)=0\},\\
&\int_0^\infty \Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 + e(\rho^\varepsilon,\rho_*) + \frac{1}{2}|\Phi_r^\varepsilon|^2\Big)(t,r)\, r^{N-1}\mathrm{d}r\\
&\qquad + \varepsilon\int_{\mathbb{R}^2_+}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2(s,r)\, r^{N-3}\mathrm{d}r\mathrm{d}s \le \mathcal{E}_0^\varepsilon \le \mathcal{E}_0 + 1 \quad \textit{ for } t \ge 0,\\
&\varepsilon^2\int_0^\infty \big|(\sqrt{\rho^\varepsilon(t,r)})_r\big|^2\, r^{N-1}\mathrm{d}r\\
&\qquad + \varepsilon\int_{\mathbb{R}^2_+}\big|\big((\rho^\varepsilon(s,r))^{\frac{\gamma}{2}}\big)_r\big|^2\, r^{N-1}\mathrm{d}r\mathrm{d}s \le C(\mathcal{E}_0+1) \textit{ for } t \ge 0,\\
&\int_0^T\int_{r_1}^{r_2} (\rho^\varepsilon)^{\gamma+1}(t,r)\,\mathrm{d}r\mathrm{d}t \le C(r_1,r_2,T,\mathcal{E}_0),\\
&\int_0^T\int_{r_1}^{r_2} \big(\rho^\varepsilon|u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta}\big)(t,r)\, r^{N-1}\mathrm{d}r\mathrm{d}t \le C(r_1,r_2,T,\mathcal{E}_0),\\
&\int_0^D \Big(\int_0^r \rho^\varepsilon(t,y)y^{N-1}\,\mathrm{d}y\Big)\rho^\varepsilon(t,r)\, r\mathrm{d}r \le C(D,\mathcal{E}_0).
\end{aligned}
$$

where $[r_1,r_2] \Subset (0,\infty)$, $D > 0$.

2. Similar to [11, Lemma 4.6], we can obtain the convergence of the potential function $\Phi^{\varepsilon,\delta}$: *For fixed* $\varepsilon > 0$, *there exists a function* $\Phi^\varepsilon(t,\mathbf{x}) = \Phi^\varepsilon(t,r)$ *such that, as* $\delta \to 0+$ (*up to a subsequence*),

(5.18) $\Phi^{\varepsilon,\delta} \rightharpoonup \Phi^\varepsilon$ *weak-$*$ in* $L^\infty(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N))$ *and weakly in* $L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N))$,

(5.19)
$$
\begin{aligned}
\Phi_r^{\varepsilon,\delta}(t,r)r^{N-1} &\to \Phi_r^\varepsilon(t,r)r^{N-1}\\
&= -\int_0^r \big(\rho^\varepsilon(t,z) - d(z)\big)\, z^{N-1}dz \quad \textit{ in } C_{\mathrm{loc}}([0,T]\times[0,\infty)),
\end{aligned}
$$

(5.20) $\|\Phi^\varepsilon(t)\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + \|\nabla\Phi^\varepsilon(t)\|_{L^2(\mathbb{R}^N)} \le C(\mathcal{E}_0) \qquad$ *for* $t \ge 0$.

Similar to [11, Lemma 4.8], we can also obtain the following energy inequality: *For* $\gamma > 1$, *the following energy inequality holds*:

$$
\begin{aligned}
&\int_0^\infty \Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 + e(\rho^\varepsilon,\rho_*)\Big)(t,r)\, r^{N-1}\mathrm{d}r + \frac{1}{2}\int_0^\infty |\Phi^\varepsilon(t,r)|^2\, r^{N-1}\mathrm{d}r\\
&\le \int_0^\infty \Big(\frac{1}{2}\Big|\frac{m_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}}\Big|^2 + e(\rho_0^\varepsilon,\rho_*)\Big)(r)\, r^{N-1}\mathrm{d}r + \frac{1}{2}\int_0^\infty |\Phi_0^\varepsilon(r)|^2\, r^{N-1}\mathrm{d}r.
\end{aligned}
$$

3. Now, we prove that

$$
(\rho^\varepsilon, \mathcal{M}^\varepsilon, \Phi^\varepsilon)(t,\mathbf{x}) = (\rho^\varepsilon(t,r), m^\varepsilon(t,r)\frac{\mathbf{x}}{r}, \Phi^\varepsilon(t,r))
$$

is a global weak solution of the Cauchy problem (1.4) and (2.6) defined in $\mathbb{R}^N$ in the sense of Definition 2.3, following the arguments as for Lemmas 4.9–4.10 in [11]. More precisely, *the function sequence* $(\rho^\varepsilon, \mathcal{M}^\varepsilon, \Phi^\varepsilon)(t,\mathbf{x})$ *satisfies the following properties:*

(i) *Let* $0 \le t_1 < t_2 \le T$*, and let* $\zeta(t,\mathbf{x}) \in C^1([0,T]\times\mathbb{R}^N)$ *be any smooth function with compact support. Then*

$$\begin{aligned}&\int_{\mathbb{R}^N} \rho^\varepsilon(t_2,\mathbf{x})\zeta(t_2,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^N} \rho^\varepsilon(t_1,\mathbf{x})\zeta(t_1,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \big(\rho^\varepsilon\zeta_t + \mathcal{M}^\varepsilon\cdot\nabla\zeta\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

(ii) *Let* $\Psi(t,\mathbf{x}) \in \big(C_0^2([0,\infty)\times\mathbb{R}^N)\big)^N$ *be any smooth function with compact support* supp $\Psi \Subset [0,T)\times\mathbb{R}^N$*. Then, it has*

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}} \Big\{\mathcal{M}^\varepsilon\cdot\partial_t\Psi + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\big)\Psi + p(\rho^\varepsilon)\,\mathrm{div}\,\Psi\Big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\quad - \int_{\mathbb{R}_+^{N+1}} \rho^\varepsilon\nabla_{\mathbf{x}}\Phi^\varepsilon\cdot\Psi d\mathbf{x}dt + \int_{\mathbb{R}^N} \mathcal{M}_0^\varepsilon(\mathbf{x})\cdot\Psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= -\varepsilon\int_{\mathbb{R}_+^{N+1}} \Big\{\frac{1}{2}\mathcal{M}^\varepsilon\cdot\big(\Delta\Psi + \nabla\mathrm{div}\,\Psi\big) + \frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla\big)\\ &\quad + \nabla\sqrt{\rho^\varepsilon}\cdot\big(\frac{\mathcal{M}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\big)\Big\}\Psi\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &= \sqrt{\varepsilon}\int_{\mathbb{R}_+^{N+1}} \sqrt{\rho^\varepsilon}\Big\{\mathrm{V}^\varepsilon\frac{\mathbf{x}\otimes\mathbf{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big(I_{N\times N} - \frac{\mathbf{x}\otimes\mathbf{x}}{r^2}\big)\Big\} : \nabla\Psi\,\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

with $\mathrm{V}^\varepsilon(t,\mathbf{x}) \in L^2(0,T;L^2(\mathbb{R}^N))$ *as a function satisfying*

$$\int_0^T\int_{\mathbb{R}^N} |\mathrm{V}^\varepsilon(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t \le C\mathcal{E}_0$$

for some $C > 0$*, independent of* $T > 0$*.*

(iii) *It follows from* (5.19) *that* Φ^ε *satisfies the Poisson equation in the classical sense except for the origin:*

$$\Delta\Phi^\varepsilon = d(\mathbf{x}) - \rho^\varepsilon(t,\mathbf{x}) \qquad \text{for } t \ge 0, \quad \mathbf{x}\in\mathbb{R}^N\setminus\{\mathbf{0}\}.$$

Moreover, for any smooth function $\xi(\mathbf{x}) \in C_0^1(\mathbb{R}^N)$ *with compact support, it has*

$$\int_{\mathbb{R}^N} \nabla\Phi^\varepsilon(t,\mathbf{x})\cdot\nabla\xi(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^N} \big(\rho(t,\mathbf{x}) - d(\mathbf{x})\big)\xi(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

4. We can also prove the H_{loc}^{-1}–compactness of weak entropy dissipation measures: *Let* (η, q) *be a weak entropy pair defined in* (2.8) *for any smooth compact supported function* $\psi(s)$ *on* $\mathbb{R}$*. Then, it has*

$$\partial_t\eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in } H_{\mathrm{loc}}^{-1}(\mathbb{R}_+^2).$$

The proof is similar with [11, Lemma 4.12]. Hence, we omit the proof for brevity. We also refer to [56, 61] as an introduction to the method of compensated compactness, especially div-curl lemma.

In summary, Theorem 5.5 is proved by combining Steps 1–4 above together. □

6. Proof of the Main Theorems. In this section, we complete the proof of Main Theorem II: Theorem 2.4. And then we prove Main Theorem I: Theorem 2.2. Since the proof is similar to that in [11], we sketch it briefly for self-containedness. The proof is divided into four steps.

1. The uniform estimates and compactness properties obtained in Theorem 5.5 imply that the weak solutions $(\rho^\varepsilon, m^\varepsilon)$ of CNSPEs (1.9) satisfy the compensated compactness framework in Chen-Perepelitsa [12] for the general case $\gamma > 1$; also see LeFloch-Westdickenberg [41] for $\gamma \in (1, \frac{5}{3}]$. Then the compactness theorem in [12] shows that there exists $(\rho, m)(t,r)$ such that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \qquad a.e.\ \ (t,r) \in \mathbb{R}^2_+ \ \text{ as } \varepsilon \to 0+ \text{ (up to a subsequence)}. \tag{6.1}$$

By adopting similar arguments as in the proof of [11, Lemma 4.4], we obtain that $m(t,r) = 0$ *a.e.* on $\{(t,r) : \rho(t,r) = 0\}$. By denoting $u(t,r) := \frac{m(t,r)}{\rho(t,r)}$ *a.e.* on $\{(t,r) : \rho(t,r) \neq 0\}$, and $u(t,r) := 0$ *a.e.* on $\{(t,r) : \rho(t,r) = 0 \text{ or } r = 0\}$, we can define the limit velocity $u(t,r)$. Then we obtain that

$$m(t,r) = \rho(t,r)u(t,r).$$

Similarly, we can define $(\frac{m}{\sqrt{\rho}})(t,r) := \sqrt{\rho(t,r)}u(t,r)$, which is zero a.e. on the vacuum states $\{(t,r) \ : \ \rho(t,r) = 0\}$. Moreover, one can obtain that, as $\varepsilon \to 0+$,

$$\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}} \equiv \sqrt{\rho^\varepsilon}u^\varepsilon \to \frac{m}{\sqrt{\rho}} \equiv \sqrt{\rho}u \qquad \text{strongly in } \ L^2([0,T]\times[0,r_2], r^{N-1}\mathrm{d}r\mathrm{d}t). \tag{6.2}$$

Notice that $|m|^{\frac{3(\gamma+1)}{\gamma+3}} \leq C\big(\frac{|m|^3}{\rho^2} + \rho^{\gamma+1}\big)$, which, together with (5.15)–(5.16), yields that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2_+) \times L^q_{\mathrm{loc}}(\mathbb{R}^2_+) \text{ as } \varepsilon \to 0+ \tag{6.3}$$

for $p \in [1, \gamma+1)$ and $q \in [1, \frac{3(\gamma+1)}{\gamma+3})$, where $L^q_{\mathrm{loc}}(\mathbb{R}^2_+)$ represents $L^q([0,T]\times\mathcal{O})$ for any $T > 0$ and $\mathcal{O} \Subset (0,\infty)$.

From (6.2)–(6.3), we also obtain the convergence of the relative mechanical energy as $\varepsilon \to 0+$:

$$\bar{\eta}^*(\rho^\varepsilon, m^\varepsilon) \to \bar{\eta}^*(\rho, m) \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+).$$

By passing the limit in (5.13) and noticing that $\bar{\eta}^*(\rho, m)$ is a convex function, we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^\infty \big(\bar{\eta}^*(\rho, m) + \frac{1}{2}|\Phi_r|^2\big)(t,r)\, r^{N-1}\mathrm{d}r\mathrm{d}t \\
&\quad \leq (t_2 - t_1)\int_0^\infty \big(\bar{\eta}^*(\rho_0, m_0) + \frac{1}{2}|\Phi_{0r}|^2\big)(r)\, r^{N-1}\mathrm{d}r,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\int_0^\infty \big(\bar{\eta}^*(\rho, m) + \frac{1}{2}|\Phi_r|^2\big)(t,r)\, r^{N-1}\mathrm{d}r \\
&\quad \leq \int_0^\infty \big(\bar{\eta}^*(\rho_0, m_0) + \frac{1}{2}|\Phi_{0r}|^2\big)(r)\, r^{N-1}\mathrm{d}r \qquad \text{for } a.e.\ t \geq 0.
\end{aligned} \tag{6.4}$$

This indicates that no concentration (Dirac mass) is formed in the density ρ at the origin $r = 0$.

2. For the convergence of the electric potential function $\Phi^\varepsilon(t, r)$, by similar calculation in [11], as $\varepsilon \to 0+$ (up to a subsequence), we obtain

$$\begin{aligned}(6.5)\qquad \Phi^\varepsilon_r(t,r)r^{N-1} &= -\int_0^r \big(\rho^\varepsilon(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y\\ &\to -\int_0^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y \quad \text{a.e. } (t,r)\in\mathbb{R}^2_+.\end{aligned}$$

Then (5.20) implies that there exists a function $\Phi(t, \mathbf{x}) = \Phi(t, r)$ such that, as $\varepsilon \to 0+$ (up to a subsequence),

$$\begin{aligned}&\Phi^\varepsilon \rightharpoonup \Phi \text{ weak-}* \text{ in } L^\infty(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N)) \text{ and weakly in } L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N)),\\ &\|\Phi(t)\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + \|\nabla\Phi(t)\|_{L^2(\mathbb{R}^N)} \le C(\mathcal{E}_0) \qquad a.e.\ t\ge 0.\end{aligned}$$

It follows from (6.5) and the uniqueness of the limit that

$$\Phi_r(t,r)r^{N-1} = -\int_0^r \big(\rho(t,y) - d(y)\big)\, y^{N-1}\mathrm{d}y \qquad a.e.\ (t,r)\in\mathbb{R}^2_+.$$

3. Define the functions:

$$(\rho,\mathcal{M},\Phi)(t,\mathbf{x}) := (\rho(t,r), m(t,r)\frac{\mathbf{x}}{r}, \Phi(t,r)) = (\rho(t,r), \rho(t,r)u(t,r)\frac{\mathbf{x}}{r}, \Phi(t,r)).$$

Then we obtain that, for $a.e.\ t \ge 0$,

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(\frac12\Big|\frac{\mathcal{M}}{\sqrt{\rho}}\Big|^2 + e(\rho,\rho_*) + \frac12|\nabla_{\mathbf{x}}\Phi|^2\Big)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\le \int_{\mathbb{R}^N} \Big(\frac12\Big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\Big|^2 + e(\rho_0,\rho_*) + \frac12|\nabla_{\mathbf{x}}\Phi_0|^2\Big)(\mathbf{x})\,\mathrm{d}\mathbf{x},\end{aligned}$$

This implies (2.5) in Definition 2.1. From (6.4), we know that $\frac{\mathcal{M}}{\sqrt{\rho}} = \sqrt{\rho}u\,\frac{\mathbf{x}}{r}$ is well-defined and in L^2 for $a.e.\ t > 0$.

4. Using Theorem 5.5 and applying the similar argument as in [11, Section 5], we can finally conclude that $(\rho,\mathcal{M},\Phi)(t,\mathbf{x})$ is a global weak solution of the Cauchy problem of CEPEs (1.1) in $\mathbb{R}^N$ and (1.5)–(1.6) in the sense of Definition 2.1.

For the sake of completeness, we only consider the momentum equations. Let $\psi = (\psi_1,\cdots,\psi_N) \in \big(C^2_0(\mathbb{R}\times\mathbb{R}^N)\big)^N$ be a smooth function with compact support, and let $\chi_\sigma(r) \in C^\infty(\mathbb{R})$ be a cut-off function satisfying given $\sigma \in (0, 1]$,

$$(6.6)\qquad \chi_\sigma(r) = 0 \ \text{ for } r\le\sigma, \quad \chi_\sigma(r) = 1 \ \text{ for } r\ge 2\sigma, \quad |\chi'_\sigma(r)| \le \frac{C}{\sigma}, \quad |\chi''_\sigma(r)| \le \frac{C}{\sigma^2}.$$

Without loss of generality, we assume that $\operatorname{supp}\psi \subset [-T,T]\times B_D(\mathbf{0})$, for some $D > 0$. Denote $\Psi_\sigma = \psi\chi_\sigma$. Then, performing similar argument as in [11, Section 5], one can obtain

$$\int_{\mathbb{R}^{N+1}_+}\Big\{\mathcal{M}\cdot\partial_t\Psi_\sigma+\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\Psi_\sigma+p(\rho)\,\mathrm{div}\Psi_\sigma-\rho\nabla\Phi\cdot\Psi_\sigma\Big\}\mathrm{d}\mathbf{x}\mathrm{d}t \tag{6.7}$$
$$+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}=0.$$

Notice that, for any $T>0$ and $D>0$,

$$\int_0^T\int_0^D\big(\frac{m^2}{\rho}+\rho^\gamma\big)(t,r)\,r^{N-1}\mathrm{d}r\mathrm{d}t\le C(D,T,E_0), \tag{6.8}$$

which leads to

$$\lim_{\sigma\to0}\Big\{\int_{\mathbb{R}^{N+1}_+}\mathcal{M}\cdot\partial_t\Psi_\sigma\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\Psi_\sigma(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\Big\} \tag{6.9}$$
$$=\int_{\mathbb{R}^{N+1}_+}\mathcal{M}\cdot\partial_t\psi\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^N}\mathcal{M}_0(\mathbf{x})\cdot\psi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Using Lebesgue's Dominated Convergence Theorem, one can also prove that

$$\lim_{\sigma\to0}\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\Psi_\sigma+p(\rho)\,\mathrm{div}\Psi_\sigma\Big\}\mathrm{d}\mathbf{x}\mathrm{d} \tag{6.10}$$
$$=\int_{\mathbb{R}^{N+1}_+}\Big\{\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\big(\frac{\mathcal{M}}{\sqrt{\rho}}\cdot\nabla\big)\psi+p(\rho)\,\mathrm{div}\psi\Big\}\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Define

$$\varphi(t,r):=\int_{\partial B_1(\mathbf{0})}\omega\cdot\psi(t,r\omega)\,\mathrm{d}\omega=\frac{1}{r^{N-1}}\int_{\partial B_r(\mathbf{0})}\omega\cdot\psi(t,\mathbf{y})\,\mathrm{d}S_{\mathbf{y}} \tag{6.11}$$
$$=\frac{1}{r^{N-1}}\int_{B_r(\mathbf{0})}\mathrm{div}\,\psi(t,\mathbf{y})\,\mathrm{d}\mathbf{y},$$

which implies

$$|\varphi(t,r)r^{-1}|\le C(\|\psi\|_{C^1}); \tag{6.12}$$

also see [40, 59]. Using Theorem 5.5, we can obtain that

$$\rho(t,r)r\int_0^r\rho(t,y)y^{N-1}\,\mathrm{d}y\in L^1((0,D)\times(0,T)). \tag{6.13}$$

Using the Lebesgue's Dominated Convergence Theorem, (6.12), (6.13), we obtain that

$$\lim_{\sigma\to0+}\int_{\mathbb{R}^{N+1}_+}\rho\nabla\Phi\cdot\Psi_\sigma\,\mathrm{d}\mathbf{x}\mathrm{d}t=\lim_{\sigma\to0+}\int_{\mathbb{R}^2_+}\rho\Phi_r\varphi(t,r)\chi_\sigma(r)\omega_Nr^{N-1}\,\mathrm{d}r\mathrm{d}t \tag{6.14}$$
$$=-\lim_{\sigma\to0+}\int_{\mathbb{R}^2_+}\rho(t,r)r\Big(\int_0^r(\rho(t,z)-d(z))\omega_Nz^{N-1}dz\Big)(\varphi(t,r)r^{-1})\chi_\sigma(r)\,\mathrm{d}r\mathrm{d}t$$
$$=-\int_{\mathbb{R}^2_+}\rho(t,r)r\Big(\int_0^r(\rho(t,z)-d(z))\omega_Nz^{N-1}\mathrm{d}z\Big)(\varphi(t,r)r^{-1})\,\mathrm{d}r\mathrm{d}t$$
$$=\int_{\mathbb{R}^{N+1}_+}\rho\nabla\Phi\cdot\Psi\,\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Substituting (6.9)–(6.10) into (6.7), we conclude that $(\rho, \mathcal{M}, \Phi)$ is a global weak solution of the Cauchy problem of CEPEs (1.1) in $\mathbb{R}^N$ and (1.5)–(1.6) in the sense of Definition 2.1. The proof of Theorem 2.2 is completed. ☐

Appendix A. Construction of Approximate Initial Data. For completeness, in this appendix, we present the construction of the approximate initial data functions with desired estimates and regularity. From (1.2), there exists a positive constant $\hat{r} \gg 1$ so that

$$0 < \frac{1}{2}\rho_* \le \rho_0(r) \le \frac{3}{2}\rho_* \qquad \text{for } r \ge \hat{r}.$$

The density function $\rho_0(r)$ is cut off as follows:

$$\breve{\rho}_0^\varepsilon(r) = \begin{cases} (\beta\varepsilon)^{\frac{1}{4}} & \text{if } \rho_0(r) \le (\beta\varepsilon)^{\frac{1}{4}}, \\ \rho_0(r) & \text{if } (\beta\varepsilon)^{\frac{1}{4}} \le \rho_0(r) \le (\beta\varepsilon)^{-\frac{1}{2}}, \\ (\beta\varepsilon)^{-\frac{1}{2}} & \text{if } \rho_0(r) \ge (\beta\varepsilon)^{-\frac{1}{2}}, \end{cases}$$

where $\varepsilon \in (0,1]$, and $0 < \beta \ll 1$ is a fixed small positive constant, which is to guarantee $(\beta\varepsilon)^{\frac{1}{4}} \ll (\beta\varepsilon)^{-\frac{1}{2}}$ for all $\varepsilon \in (0,1]$. It is direct to check that

$$\text{(A.1)} \qquad \breve{\rho}_0^\varepsilon(r) \le \rho_0(r) + 1, \qquad \breve{\rho}_0^\varepsilon(r) \to \rho_0(r) \ \text{ as } \varepsilon \to 0+ \ \ a.e. \ r \in \mathbb{R}_+.$$

To keep the L^p mollification properties, it is more convenient to mollify the initial data in the original coordinate $\mathbb{R}^N$; therefore we do not distinguish functions $(\rho_0, m_0)(r)$ from $(\rho_0, m_0)(\mathbf{x}) = (\rho_0, m_0)(|\mathbf{x}|)$ for simplicity.

It yields from (2.2)–(2.5) that $\rho_0(\mathbf{x}) \in L^\gamma_{\rm loc}(\mathbb{R}^N)$. We also assume that $\rho_0(\mathbf{x}) - d(\mathbf{x}) \in (L^{\frac{2N}{N+2}} \cap L^1)(\mathbb{R}^N)$. Denote

$$\Delta\Phi_0 = -(\rho_0 - d(\mathbf{x})),$$

which is well-defined under the above assumption. Using the convexity of $e(\rho, \rho_*)$, we have

$$\text{(A.2)} \qquad 0 \le e(\breve{\rho}_0^\varepsilon(\mathbf{x}), \rho_*) \le e(\rho_0(\mathbf{x}), \rho_*).$$

Using (2.5), (A.1)–(A.2) and the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{\varepsilon\to 0+} \int_{\mathcal{O}} \Big(|\breve{\rho}_0^\varepsilon(\mathbf{x}) - \rho_0(\mathbf{x})|^\gamma + |\sqrt{\breve{\rho}_0^\varepsilon(\mathbf{x})} - \sqrt{\rho_0(\mathbf{x})}|^{2\gamma} \Big)\, \mathrm{d}\mathbf{x} = 0 \qquad \text{for any } \mathcal{O} \Subset \mathbb{R}^N.$$

In the following, we take the following cut-off density function for $\breve{\rho}_0^\varepsilon(\mathbf{x})$ at the far-field:

$$\hat{\rho}_0^\varepsilon(\mathbf{x}) = \begin{cases} \breve{\rho}_0^\varepsilon(\mathbf{x}) & \text{if } |\mathbf{x}| \le (\beta\varepsilon)^{-\frac{1}{2N}}, \\ \rho_* & \text{if } |\mathbf{x}| > (\beta\varepsilon)^{-\frac{1}{2N}}, \end{cases}$$

since a better decay property for approximate initial data is needed. Here, we further take β small enough such that $|\mathbf{x}| \ge (\beta\varepsilon)^{-\frac{1}{2N}} \ge \hat{r} + 2$ for all $\varepsilon \in (0,1]$. It is obvious that $\hat{\rho}_0^\varepsilon(\mathbf{x})$ is not a smooth function. Therefore, we need to mollify $\hat{\rho}_0^\varepsilon(\mathbf{x})$. Let $\mathfrak{J}(\mathbf{x})$

be the standard mollifier and $\mathfrak{J}_\sigma(\mathbf{x}) := \frac{1}{\sigma^N}\mathfrak{J}(\frac{\mathbf{x}}{\sigma})$ for $\sigma \in (0,1)$. Here, we take $\sigma = \varepsilon^{\frac{1}{4}}$ and define $\rho_0^\varepsilon(\mathbf{x})$ as follows:

$$\rho_0^\varepsilon(\mathbf{x}) := \Big(\int_{\mathbb{R}^N} \sqrt{\hat{\rho}_0^\varepsilon(\mathbf{x}-\mathbf{y})}\,\mathfrak{J}_\sigma(\mathbf{y})\mathrm{d}\mathbf{y} \Big)^2.$$

Then $\rho_0^\varepsilon(\mathbf{x})$ is still a spherically symmetric function, *i.e.*, $\rho_0^\varepsilon(\mathbf{x}) = \rho_0^\varepsilon(|\mathbf{x}|)$. By similar arguments as in [11, 16], we can obtain the following Lemma (the details are omitted here for simplicity of presentation).

Lemma A.1. *The following three statements hold*:

(i) *As* $\varepsilon \to 0+$,

$$(\mathcal{E}_0^\varepsilon, \mathcal{E}_1^\varepsilon) \to (\mathcal{E}_0, 0),$$
$$(\rho_0^\varepsilon, m_0^\varepsilon)(r) \to (\rho_0, m_0)(r) \quad \textit{in } L^p_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r) \times L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r).$$

where $\mathcal{E}_0^\varepsilon, \mathcal{E}_1^\varepsilon$, *and* $\mathcal{E}_0$ *are defined in* (2.10), (2.11), *and* (2.1), *separately, and* $p \in \{\gamma, \frac{2N}{N+2}\}$.

(ii) *For any fixed* $\varepsilon \in (0,1]$, *as* $\delta \to 0+$,

$$(\mathcal{E}_0^{\varepsilon,\delta}, \mathcal{E}_1^{\varepsilon,\delta}) \to (\mathcal{E}_0^\varepsilon, \mathcal{E}_1^\varepsilon),$$
$$(\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r) \to (\rho_0^\varepsilon, m_0^\varepsilon)(r) \quad \textit{in } L^p_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r) \times L^1_{\mathrm{loc}}([0,\infty); r^{N-1}\mathrm{d}r),$$

where $\mathcal{E}_0^{\varepsilon,\delta}$ *and* $\mathcal{E}_1^{\varepsilon,\delta}$ *are defined in* (5.2), *and* $p \in \{\gamma, \frac{2N}{N+2}\}$.

(iii) *For any fixed* (ε, δ), *as* $\mathfrak{b} \to \infty$,

$$(\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}, \mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}}) \to (\mathcal{E}_0^{\varepsilon,\delta}, \mathcal{E}_1^{\varepsilon,\delta}),$$
$$(\rho_0^{\varepsilon,\delta,\mathfrak{b}}, m_0^{\varepsilon,\delta,\mathfrak{b}})(r) \to (\rho_0^{\varepsilon,\delta}, m_0^{\varepsilon,\delta})(r)$$
$$\textit{in } L^p_{\mathrm{loc}}((\delta,\infty); r^{N-1}\mathrm{d}r) \times L^1_{\mathrm{loc}}((\delta,\infty); r^{N-1}\mathrm{d}r),$$

where $\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}, \mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}}, \mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}}$, *and* $\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}}$ *are defined in* Lemmas 3.1–3.5 *and* (4.3), *and* $p \in \{\gamma, \frac{2N}{N+2}\}$. *In addition, the upper bounds of* $\mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}}, \mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}}, \mathcal{E}_2^{\varepsilon,\delta,\mathfrak{b}}$, *and* $\mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}}$ *are independent of* $\mathfrak{b}$ *(but may depend on* ε, δ*), and*

$$\begin{aligned} \mathcal{E}_0^{\varepsilon,\delta,\mathfrak{b}} + \mathcal{E}_1^{\varepsilon,\delta,\mathfrak{b}} &\le C(\mathcal{E}_0 + 1), \\ \mathcal{E}_3^{\varepsilon,\delta,\mathfrak{b}} &:= \int_\delta^{\mathfrak{b}} \bar{\eta}^*(\rho_0^{\varepsilon,\delta,\mathfrak{b}}, m_0^{\varepsilon,\delta,\mathfrak{b}})\, r^{2(N-1)+\vartheta}\mathrm{d}r \\ &\le C\mathcal{E}_0\big(\delta^{-N+1-\vartheta} + \varepsilon^{-\frac{N-1+\vartheta}{2N}}\big), \end{aligned}$$

for some $C > 0$ *independent of* $(\varepsilon, \delta, \mathfrak{b})$, *where* $\vartheta \in (0,1)$ *is arbitrary fixed constant.*

Conflict of Interest: The authors declare that they have no conflict of interest. The authors also declare that this manuscript has not been previously published, and will not be submitted elsewhere before your decision.

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