Approximately Counting H-Colourings is #BIS-Hard*

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Abstract

We consider the problem of counting H-colourings from an input graph G to a target graph H. We show that if H is any fixed graph without trivial components, then the problem is as hard as the well-known problem #BIS, which is the problem of (approximately) counting independent sets in a bipartite graph. #BIS is a complete problem in a important complexity class for approximate counting, and is believed not to have an FPRAS. If this is so, then our result shows that for every graph H without trivial components, the H-colouring counting problem has no FPRAS. This problem was studied a decade ago by Goldberg, Kelk and Paterson. They were able to show that approximately sampling H-colourings is #BIS-hard, but it was not known how to get the result for approximate counting. Our solution builds on non-constructive ideas using the work of Lovász.

1 Introduction

The independent set and k-colouring models are well-known models from statistical physics which have also been thoroughly studied in computer science. A particularly interesting question from the computer science perspective is the complexity of counting and approximate counting in these models. Given an input graph G, the problem is to approximate the number of independent sets of G or the number of proper k-colourings of G. Both of these problems can be viewed as special cases of the more general problem of approximately counting H-colourings. This paper studies the complexity of the more general problem.

We begin with few relevant definitions. Let $H = (V(H), E(H))$ be a fixed graph which is allowed to have self-loops, but not parallel edges. An H-colouring of a graph $G = (V(G), E(G))$ is a homomorphism from G to H, i.e., an assignment $h : V(G) \to V(H)$ such that for every edge (u, v) in G, it holds that $(h(u), h(v))$ is also an edge of H. It is helpful to think of the vertices of H as "colours" and the edges of H as pairs of colours which are allowed to be assigned to adjacent vertices in G. Given an input graph G, we are interested in computing the number of H-colourings of G. We will refer to this problem as the #H-Col problem. Also, we denote by #H-Col(G) the number of H-colourings of G. The examples mentioned earlier correspond to H-colourings in the following way. Proper k-colourings of G correspond to H-colourings of G when H is a k-clique. Also, independent sets of G correspond to H-colourings of G when H is the connected 2-vertex graph with exactly one self-loop.

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A fully polynomial randomised approximation scheme (FPRAS) for approximately counting H-colourings is a randomised algorithm that takes as input a graph G and an accuracy parameter $\varepsilon > 0$ and outputs a number which, with probability at least $3/4$, is in the range $[\exp(-\varepsilon)\#H\text{-Col}(G), \exp(\varepsilon)\#H\text{-Col}(G)]$. The running time of the algorithm is bounded by a polynomial in $|V(G)|$ and ε^{-1}.

Our goal in this work is to quantify the computational complexity of approximately counting H-colourings. In particular, we seek to determine for which graphs H the problem $\#H$-Col admits an FPRAS. Dyer and Greenhill [3] have completely classified the computational complexity of *exactly* counting H-colourings in terms of the parameter graph H. We say that a connected graph is *trivial* if it is either a clique with self-loops on every vertex or a complete bipartite graph with no self-loops. Dyer and Greenhill showed that the problem $\#H$-Col is polynomial-time solvable when each connected component of H is trivial; otherwise it is #P-complete. The complexity of the corresponding decision problem has also been characterised. Hell and Nešetřil [8] showed that deciding whether an input graph G has an H-colouring is NP-complete unless H contains a self-loop or H is bipartite (in which case it admits a trivial polynomial-time algorithm).

A *polynomial approximate sampler* (PAS) for sampling H-colourings is a randomised algorithm that takes as input a graph G and an accuracy parameter $\varepsilon \in (0, 1]$ and gives an output such that the total variation distance between the output distribution of the algorithm and the uniform distribution on H-colourings of G is at most ε. The running time is bounded by a polynomial in $|V(G)|$ and ε^{-1}. The PAS is said to be a *fully polynomial approximate sampler* (FPAS) if the running time is bounded by a polynomial in $|V(G)|$ and $\log(\varepsilon^{-1})$. The problem of approximately sampling H-colourings has been shown in [7] to be #BIS-hard provided that H contains no trivial components[1]. More precisely, for any fixed graph H with no trivial components, the existence of a PAS for sampling H-colourings would imply that there is an FPRAS for #BIS, which is the problem of counting the independent sets of a bipartite graph. #BIS plays an important role in approximation complexity. Despite many attempts, nobody has found an FPRAS for #BIS and it is conjectured that none exists (even though it is unlikely that approximating #BIS is NP-hard). Various natural algorithms have been ruled out as candidate FPRASes for #BIS [13, 4, 6]. Moreover, Dyer et al. [1] showed that #BIS is complete under approximation-preserving (AP) reductions in a logically defined class of problems, called $\#\mathrm{RH}\Pi_1$, to which an increasing variety of problems have been shown to belong. Other typical complete problems in $\#\mathrm{RH}\Pi_1$ include counting the number of downsets in a partially ordered set [1] and computing the partition function of the ferromagnetic Ising model with local external fields [5].

Perhaps surprisingly, the hardness result of [7] for sampling H-colourings does not imply hardness for approximately counting H-colourings. This might be puzzling at first since, for the independent set and k-colouring models, approximate counting is well-known to be equivalent to approximate sampling (this equivalence has been proved in [10] for the so-called class of self-reducible problems in #P). However, for general graphs H it is only known [2] that an FPAS for sampling H-colourings implies an FPRAS for counting H-colourings (but not the reverse direction). For a thorough discussion of this point we refer the reader to [2] (where also an example of a problem in #P is given which, under usual complexity theory assumptions, admits an FPRAS but not an FPAS).

In this paper, we address the following questions.

[1]Note that the presence of trivial components may lead to artificial approximation schemes. For example [7, Section 7] gives the example in which H consists of two components — a (trivial) size-3 clique with self-loops and a connected 2-vertex graph with exactly one self-loop. There is a PAS for this example. If ε is not too small then it typically outputs an H-colouring using the trivial component. See [7, Section 7] for details.

- "Is there a graph H for which approximately counting H-colourings is substantially easier than approximately sampling H-colourings?"
- "Is there a graph H such that $\#H$-Col lies between P and the class of #BIS-hard problems?"

We present the analogue of the hardness result of [7] in the counting setting, therefore providing evidence that the answers to the previous questions are negative. To formally state the result, recall the notion of an approximation preserving reduction $\leq_{\mathsf{AP}}$ (introduced in [1]). For counting problems $\#\mathsf{A}$ and $\#\mathsf{B}$, $\#\mathsf{A} \leq_{\mathsf{AP}} \#\mathsf{B}$ implies that an FPRAS for $\#\mathsf{B}$ yields an FPRAS for $\#\mathsf{A}$. Our main result is the following.

Theorem 1. *Let H be a graph (possibly with self-loops but without parallel edges), all of whose connected components are non-trivial. Then* $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H$-$\mathsf{Col}$.

Interestingly, in the proof of Theorem 1 we use a non-constructive approach, partly inspired by tools introduced by Lovász [12] in the context of graph homomorphisms. Before delving into the technical details of the proof, in the next section we will describe, in a simplified setting, the main obstacles to proving Theorem 1 and the most interesting elements of the proof.

2 Reductions for sampling versus reductions for counting

We start by considering the closely related work [7]. The assumptions on the graph H are the same as in Theorem 1 — namely that H does not have trivial components. The proof in [7] shows that approximately *sampling* H-colourings is at least as hard as #BIS. We will first overview the approach of this paper since we will use several ingredients of their proof. We will also describe the new ingredients which will allow us to leap from the sampling setting to the counting setting.

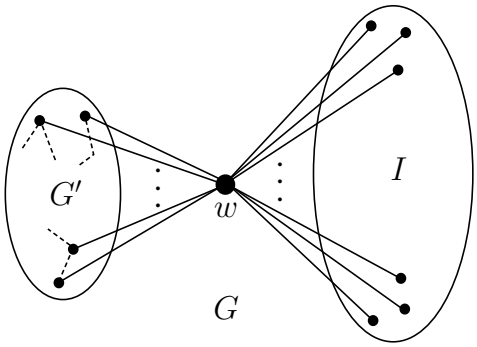
Let H be a graph for which we wish to show that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H$-$\mathsf{Col}$. To do this, it clearly suffices to find a subgraph H' of H such that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H'$-$\mathsf{Col}$ and $\#H'$-$\mathsf{Col} \leq_{\mathsf{AP}} \#H$-$\mathsf{Col}$.

We use the following definitions. The *degree* of a vertex is the number of edges incident to it, so a self-loop contributes 1 to the degree of a vertex. Given a subset S of $V(H)$, let $H[S]$ denote the subgraph of H induced by the set S. Further, let $N_{\cup}(S)$ denote the neighbourhood of S in H, i.e., the set of vertices in H which are adjacent to a vertex in S. We will also denote the vertices of H by $v_1, v_2, \ldots$.

2.1 Restricting H-colourings to induced-subgraph-colourings

To motivate how to find such a subgraph H', we give a high-level reduction scheme. This will reveal that the subgraphs induced by the neighbourhoods of maximum-degree vertices of H are natural choices for H'.[2] To see this, let's temporarily suppose that we already have a subgraph H' of H in mind, and let G' be an input for $\#H'$-Col such that $|V(G')| = n'$. We next construct an instance G of $\#H$-Col by adding to G' a special vertex w and a large independent set I with $|I| \gg n'$. We also add all edges between the special vertex w and the vertices of G' and I. Let h be a homomorphism from G to H with $h(w) = v_i$. Observe that the edges between w and the rest of G enforce that the restriction of h to G' is an $N_{\cup}(v_i)$-colouring of G'. Similarly, the restriction of h to I is a $N_{\cup}(v_i)$-colouring of I. Furthermore, there are $(\deg_H(v_i))^{|I|}$ choices for the restriction of h on the independent set I. From this, it is not hard to formally argue that the effect of the large independent set I is to enforce that in all but a negligible fraction of H-colourings of G, the special

[2] Later, we will modify this reduction scheme in a non-trivial way, but still the aspects that we highlight of the simplified reduction scheme will carry over to the modified reduction scheme.



vertex w must be assigned a colour among the vertices of H with maximum degree and thus G' is coloured using the subgraph of H induced by the neighbourhood of such a vertex.

In an ideal scenario, there is a unique vertex v in H with maximum degree and, further, the subgraph $H' = H[N_{\cup}(v)]$ of H is non-trivial and different from H. If both of these hypotheses hold, the reduction above can be used to show that $\#H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#H\text{-}\mathsf{Col}$ (since the uniqueness of v implies that $h(w) = v$ in "almost all" H-colorings) and (say, by induction) we have $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H'\text{-}\mathsf{Col}$. But what happens when these hypotheses do not hold?

A significant problem which arises at this point is the existence of multiple relevant neighbourhoods. That is, there may be several maximum-degree vertices in H, and the subgraphs induced by their neighbourhoods may not be isomorphic. It is much easier to deal with this problem in the sampling setting than in the counting setting. In the next section, we describe the difference between these settings, and we describe our approach to this (initial) hurdle.

2.2 Counting subgraph-induced-colourings: the case of multiple subgraphs

To illustrate concretely the part of the sampling argument in [7] which breaks down in the counting setting, let us consider the toy example of the graph H depicted in Figure 1 and overview how the argument in [7] works. Since H is regular, the relevant (induced) neighbourhoods of the vertices in H are given by the graphs H_1 and H_2 in Figure 1. These correspond to the neighbourhoods of the vertices v_1 and v_2, respectively (note that the remaining neighbourhoods are isomorphic to H_2). Before proceeding, we remark that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H_1\text{-}\mathsf{Col}$ and $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#H_2\text{-}\mathsf{Col}$ (see [11] where all graphs H with up to four vertices are classified; in [11, p. 314–316] H_1 corresponds to the graph n.51 while H_2 corresponds to the graph n.59).

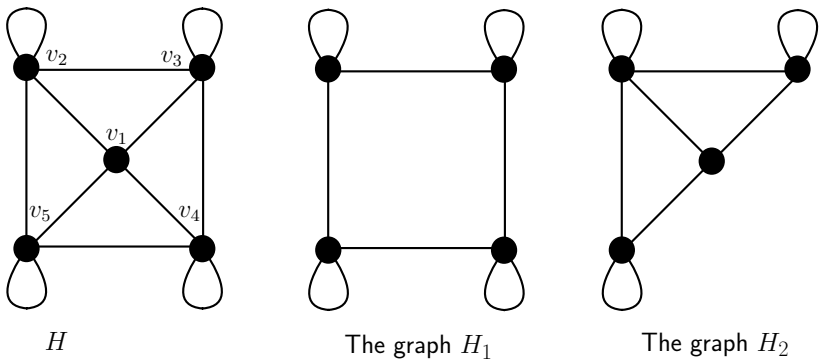


Figure 1: Example of a graph H and the subgraphs induced by the neighbourhoods of its (maximum degree) vertices.

So we already have sampling reductions from #BIS to $\#H_1$-Col and $\#H_2$-Col and we want a sampling reduction from #BIS to $\#H$-Col, e.g., an algorithm for sampling bipartite independent sets using an oracle for sampling H-colourings. Here's how it works. Let G' be an input to #BIS. Using the sampling reductions to $\#H_1$-Col and $\#H_2$-Col, construct from G' two graphs G_1 and G_2 such that an (approximately) uniform H_1-colouring of G_1 allows us to construct an (approximately) uniform independent set of G' and similarly an (approximately) uniform H_2-colouring of G_2 also allows this. Then let G be the graph obtained by taking the disjoint union of G_1 and G_2, adding a special vertex w, and adding all edges between w and G_1 and G_2 (note, there is no need for the independent set I used previously since H is regular; see the graph G in Figure 2).

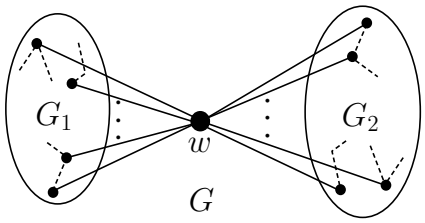


Figure 2: "Gluing" the reductions ([7])

Observe that given a random H-colouring h of G, by just revealing the colour of the special vertex w, we can generate either a random H_1-colouring of G_1 or a random H_2-colouring of G_2. In particular, if $h(w)$ is v_1, by considering the restriction of h on G_1 we obtain a random H_1-colouring of G_1. Similarly, if $h(w)$ is one of the vertices v_2, v_3, v_4, v_5, we obtain a random H_2-colouring of G_2. By our assumptions for G_1 and G_2, in each case we can then obtain a random independent set of G'.

In contrast, the aforementioned reduction scheme fails in the counting setting. Namely, considering cases for the colour $h(w)$, we obtain the following equality:

$$\#H\text{-Col}(G) = \#H_1\text{-Col}(G_1)\,\#H_1\text{-Col}(G_2) + 4\#H_2\text{-Col}(G_1)\,\#H_2\text{-Col}(G_2) \tag{1}$$

Given an approximation of $\#H$-Col(G), say Z, observe that (1) yields little information about whether Z is a good approximation for $\#H_1$-Col(G_1) or $\#H_2$-Col(G_2). This issue goes away in the sampling setting precisely because we can distinguish between the two cases by just looking at the colour of the special vertex w in the random H-colouring of G.

Thus, to proceed with the reduction in the counting setting we have to focus our attention on one of H_1 or H_2, say H_1, and somehow prove that $\#H_1\text{-Col} \leq_{\mathsf{AP}} \#H\text{-Col}$. The question which arises is how to choose between H_1 and H_2. This becomes more complicated for general graphs H since it is not hard to imagine that instead of just two graphs H_1, H_2 we will typically have a collection of graphs $H_1, \ldots, H_t$ corresponding to the induced neighbourhoods of vertices $v_1, \ldots, v_t$ of H, for some t which can be arbitrarily large (depending on the graph H). To make matters worse, apart from very basic information on the H_i's (such as connectedness or number of vertices/edges), we will not be able to control significantly their graph structure.

At this point, we employ a non-constructive approach using a tool from [12]: for arbitrary non-isomorphic graphs H_1, H_2 there exists a (fixed) graph J depending only on H_1 and H_2 such that $\#H_1\text{-Col}(J) \neq \#H_2\text{-Col}(J)$. We extend this to an arbitrary collection of pairwise non-isomorphic graphs $H_1, \ldots, H_t$ as follows: we prove the existence of a graph J so that for some $i^* \in \{1, \ldots, t\}$ it holds that $\#H_{i^*}\text{-Col}(J) > \#H_i\text{-Col}(J)$ for all $i \neq i^*$. Intuitively, the graph J will be used to "select" the subgraph H_{i^*}. Note that we will not require any further knowledge about what H_{i^*} or

J is, freeing us from the cumbersome (and perhaps difficult) task of looking into the finer details of the structure of the graphs $H_1, \ldots, H_t$. With the graph J in hand, we then take sufficiently many disjoint copies of J and we connect them to the special vertex w. This ensures that, in a typical H-colouring, the vertex w gets coloured with the vertex v_i.

To utilise the above, we will further need to ensure that $\#\text{BIS} \leq_{\mathsf{AP}} \#H_i\text{-}\mathsf{Col}$ for every i. If we could ensure that the H_i's are proper subgraphs of H and non-trivial, then using the arguments above, we could complete the proof using induction. However, this is clearly not possible in general since for example, as we noted earlier, there may exist a vertex v in H such that $N_{\cup}(v) = V(H)$. Dealing with such cases is the bulk of the work in the sampling setting of [7] and these cases cause even more problems for us. To deal with them, we need a further non-constructive argument — one that turns out to be more technical than, and substantially different from the ideas in Lovász [12].

In the next section, we introduce the concepts that we will use to formalise the intuitive arguments that we have given above. These concepts will be critical in our argument.

3 Proof Outline

Since we are interested in instances of #BIS, which are bipartite graphs, we will often need to consider H-colourings of bipartite graphs. We will assume that every bipartite graph G comes with a (fixed) proper 2-colouring of its parts with colours $\{L, R\}$. We will use $L(G)$ and $R(G)$ to denote the vertices of G coloured with L and R respectively. We will frequently refer to a bipartite graph as a 2-coloured graph to emphasise the proper 2-colouring of its vertices. In Section 9 we will define the bipartite double cover $\mathsf{Bip}(H)$ of a graph H. This is a bipartite graph which can be viewed as the tensor product[3] of H with a single edge. To simplify matters, for bipartite graphs H, it will be convenient to restrict our attention to colour-preserving homomorphisms which are formally defined as follows, where $h(S)$ denotes the image of a set $S \subseteq V(G)$ under a homomorphism h from G to H, i.e., $h(S) = \{h(v) \mid v \in S\}$.

Definition 2. *Let G and H be 2-coloured bipartite graphs. A homomorphism h from G to H is colour-preserving if h preserves the colouring of the parts of G, i.e., $h(L(G)) \subseteq L(H)$ and $h(R(G)) \subseteq R(H)$.*

As we will see in Section 9, the H-colourings of a connected bipartite graph G correspond to the colour-preserving homomorphisms from G to $\mathsf{Bip}(H)$. By restricting our attention to colour-preserving homomorphisms, we obtain the following version of the $\#H\text{-}\mathsf{Col}$ problem.

Parameter. A bipartite graph H with vertex partition $(L(H), R(H))$.
Name. $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$
Instance. A bipartite graph G with vertex partition $(L(G), R(G))$.
Output. The number of colour-preserving homomorphisms from G to H, which we denote $\#\mathsf{Fixed}H\text{-}\mathsf{Col}(G)$.

Before justifying our interest in the problem $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$, let us introduce one more piece of terminology. A bipartite graph H will be called *full* if there exist vertices $u \in L(H)$ and $v \in R(H)$ such that u is adjacent to every vertex in $R(H)$ and v is adjacent to every vertex in $L(H)$. In this case, vertices u and v are also called full. Note that if H is full, then H is also connected.

[3]The tensor product of graphs H_1 and H_2 has vertex set $V(H_1) \times V(H_2)$. Vertices (u_1, u_2) and (v_1, v_2) are adjacent in the tensor product if and only if $(u_1, v_1) \in E(H_1)$ and $(u_2, v_2) \in E(H_2)$.

The following lemma allows us to restrict our attention to the #Fixed$H$-Col problem. The proof of Lemma 3 is along the lines described in Sections 2.1 and 2.2, albeit with some modifications to account for technical details. We thus defer the proof to Section 9.

Lemma 3 (Analogue of Lemma 7 in [7])**.** *Let H be a graph with no trivial components. Then, there exists a 2-coloured graph H', which is full but not trivial, such that*

$$\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#H\text{-}\mathsf{Col}.$$

Thus, our main result will be easy to conclude from the following central lemma, which will be proved in Section 8. In the upcoming Section 5, we give an overview of the ingredients in the proof.

Lemma 4. *Let H be a 2-coloured graph which is full but not trivial. Then* #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

Assuming Lemmas 3 and 4, the proof of Theorem 1 is immediate.

Theorem 1. *Let H be a graph (possibly with self-loops but without parallel edges), all of whose connected components are non-trivial. Then* #BIS $\leq_{\mathsf{AP}}$ #H-Col.

Proof. Just combine Lemmas 3 and 4. □

4 Tools from graph homomorphisms

In this section, we present some tools from graph-homomorphism theory which are central for our non-constructive arguments.

Recall that an isomorphism from a graph H_1 to a graph H_2 is an injective homomorphism h from H_1 to H_2 whose inverse h^{-1} is also a homomorphism from H_2 to H_1. Similarly, a colour-preserving isomorphism from a 2-coloured bipartite graph H_1 to a 2-coloured bipartite graph H_2 is a colour-preserving homomorphism h (see Definition 2), from H_1 to H_2 whose inverse is also a colour-preserving homomorphism from H_2 to H_1. We will use $H_1 \cong H_2$ to denote that H_1 and H_2 are isomorphic and $H_1 \cong_c H_2$ to denote that H_1 and H_2 are isomorphic under a colour-preserving isomorphism. The following lemma has its origins in [12, Theorem (3.6)]. For completeness, in Section 7, we present a proof for the present setting which is a close adaptation of [9, Proof of Theorem 2.11].

Lemma 5. *Let H_1 and H_2 be 2-coloured bipartite graphs such that $H_1 \ncong_c H_2$. Then, there exists a 2-coloured bipartite graph J such that* #FixedH_1-Col$(J) \neq$ #FixedH_2-Col(J)*. Moreover, $|V(J)| \leq \max\{|V(H_1)|, |V(H_2)|\}$.*

Lemma 5 can be viewed as a way of obtaining a gadget J to distinguish between the graphs H_1 and H_2. In fact, we will need to be able to pick out a particular graph H_{i^*} from a whole set of non-isomorphic graphs. For a positive integer k, let $[k] = \{1, \ldots, k\}$. In Section 7 we give an inductive proof, powering up the effect of Lemma 5 to obtain the following.

Lemma 6. *Let $H_1, \ldots, H_k$ be 2-coloured bipartite graphs such that $H_i \ncong_c H_j$ for all $\{i, j\} \in \binom{[k]}{2}$. There exists a 2-coloured graph J and an integer $i^* \in [k]$ such that* #FixedH_{i^*}-Col$(J) >$ #FixedH_i-Col(J) *for all $i \in [k] \backslash \{i^*\}$.*

5 Overview of proof of Lemma 4

Lemma 4 is our central lemma. It will be proved in Section 8. Here, we give an overview of the ingredients in the proof. Thus, for the purpose of this section, we will assume that H is a (2-coloured) bipartite graph which is full (thus, also connected) but not trivial. Our goal will be to show #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

Let (V_L, V_R) denote the vertex partition of H and let F_L, F_R be the subsets of full vertices in V_L, V_R, respectively (i.e., every vertex in F_L is connected to every vertex in V_R and every vertex in F_R is connected to every vertex in V_L). Since H is full, we have $F_L, F_R \neq \emptyset$.

Recall that, for a subset S of $V(H)$, $H[S]$ is the subgraph of H induced by the set S and that $N_\cup(S)$ denotes the neighbourhood of S in H, i.e., the set of vertices in H which are adjacent to a vertex in S. We will also use $N_\cap(S)$ to denote the *joint* neighbourhood of S in H, i.e., the set of vertices in H which are adjacent to *every* vertex in S.

5.1 An inductive approach using maximal bicliques of H

The proof of Lemma 4 will be by induction on the number of vertices of H. Our goal will be to find a subgraph H' of H (which will also be 2-coloured, full but not trivial) with $|V(H')| < |V(H)|$ such that #FixedH'-Col $\leq_{\mathsf{AP}}$ #Fixed$H$-Col. If we find such an H', we will finish using the inductive hypothesis that #BIS $\leq_{\mathsf{AP}}$ #FixedH'-Col. When we are not able to find such a subgraph H', we will use an alternative method to show that #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

To select H', we consider the set $\mathcal{C}$ of bicliques in H.

Definition 7. *A* biclique *in a graph H with vertex partition (V_L, V_R) is a pair (S_L, S_R) such that $S_L \subseteq V_L$, $S_R \subseteq V_R$ and $S_L \times S_R \subseteq E(H)$. Given a fixed H, we use $\mathcal{C}$ to denote the set of all bicliques in H. A biclique $(S_L, S_R) \in \mathcal{C}$ is said to be* maximal *if it is inclusion maximal, in the sense that there is no other biclique $(S'_L, S'_R) \in \mathcal{C}$ with $S_L \subseteq S'_L$ and $S_R \subseteq S'_R$.*

Note that $(F_L, V_R), (V_L, F_R)$ are maximal bicliques in H. For lack of better terminology, we will refer to these two special bicliques as the *extremal* bicliques.

Our interest in maximal bicliques is justified by the following simple claim which will allow us to carry out our inductive step.

Lemma 8. *Let (S_L, S_R) be a maximal biclique which is not extremal, i.e, $S_L \neq F_L$, $S_R \neq F_R$. We have that $S_L \neq V_L$, $S_R \neq V_R$, $N_\cup(S_L) = V_R$ and $N_\cup(S_R) = V_L$. Let $H_1 = H[S_L \cup V_R]$ and $H_2 = H[V_L \cup S_R]$. Then, for $i \in \{1, 2\}$, we have that H_i is full but not trivial and further satisfies $|V(H_i)| < |V(H)|$.*

Proof. Since (S_L, S_R) is a maximal biclique, we have $F_L \subseteq S_L$ and $F_R \subseteq S_R$. It follows that $N_\cup(S_L) = V_R$ and $N_\cup(S_R) = V_L$. It is easy to see that both H_1, H_2 are connected, bipartite and full. We also have that $|V(H_1)|, |V(H_2)| < |V(H)|$. If, for example, $|V(H_1)| = |V(H)|$ we must have that $S_L = V_L$. Since (S_L, S_R) is a maximal biclique, this would imply that $S_R = F_R$. Similarly, we obtain that H_1, H_2 are not trivial. If, for example, H_1 is trivial, then we would have that $S_L = F_L$. □

5.2 The basic gadget

In this section, we discuss a gadget that is used in [7]. While the gadget in [7] will not work for us, it will nevertheless help to motivate our later selection of a more elaborate gadget for our needs. The gadget used in [7] is a complete bipartite graph $K_{a,b}$ with a vertices on the left and b vertices

on the right. The integers a and b should be thought of as sufficiently large numbers which may depend on the size of the input to #Fixed$H$-Col. Roughly speaking, we will be interested in the colours appearing on the left and right of $K_{a,b}$ in a typical colour-preserving homomorphism from $K_{a,b}$ to H.

To make this precise, for a colour-preserving homomorphism $h : K_{a,b} \to H$, the *phase* of h is the pair $\big(h(L(K_{a,b})), h(R(K_{a,b}))\big)$, i.e., the subsets of V_L and V_R appearing on the left and right of $K_{a,b}$ under the homomorphism h, respectively. Since $K_{a,b}$ is a complete bipartite graph, we have that a phase is a biclique of H, i.e., an element of $\mathcal{C}$. Let $(S_L, S_R) \in \mathcal{C}$. For convenience, we will refer to the total number of colour-preserving homomorphisms whose phase equals (S_L, S_R) as the contribution of the phase/biclique (S_L, S_R) to the gadget. Our induction step crucially depends on analysing the *dominant phases* of the gadget, i.e., the phases with the largest contribution.

It is not hard to see that the contribution of a phase/biclique (S_L, S_R) to the gadget $K_{a,b}$ is roughly equal to[4]

$$|S_L|^a |S_R|^b.$$

Thus, the dominant phases are determined by the ratio a/b. Rather than restricting ourselves to integers a and b it will be convenient to consider positive *real* numbers $\alpha, \beta > 0$ and the corresponding phases with dominant contribution.

Definition 9 (The set of dominating bicliques $\mathcal{C}_{\alpha,\beta}$)**.** *Let α and β be positive real numbers. Define $\mathcal{C}_{\alpha,\beta}$ to be the following subset of $\mathcal{C}$:*

$$\mathcal{C}_{\alpha,\beta} := \big\{(S_L, S_R) \in \mathcal{C} : (S_L, S_R) = \arg \max_{(S'_L, S'_R) \in \mathcal{C}} |S'_L|^\alpha |S'_R|^\beta\big\}.$$

Note that for positive α, β the bicliques in $\mathcal{C}_{\alpha,\beta}$ are in fact maximal.

For the purpose of the following discussion and to avoid delving into (at this point) unnecessary technical details, we will assume for now that α and β are rationals. In particular, there exists an integer Q so that $a = Q\alpha$ and $b = Q\beta$ are integers and $\alpha/\beta = a/b$.[5]

5.3 A reduction scheme

The structure of our reduction scheme expands on the work of [7]. The following are implicit in [7]:

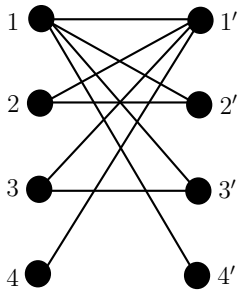
1. if $|\mathcal{C}_{\alpha,\beta}| = 2$ and $\mathcal{C}_{\alpha,\beta}$ consists only of the extremal bicliques then #BIS reduces to #Fixed$H$-Col.
2. if $|\mathcal{C}_{\alpha,\beta}| = 1$ and the unique dominating biclique in $\mathcal{C}_{\alpha,\beta}$ is *not* an extremal biclique, then #FixedH'-Col reduces to #Fixed$H$-Col for some non-trivial proper subgraph H' of H.

Unfortunately, it is not hard to come up with examples of graphs H such that, for every choice of α and β, we do not fall into one of the cases 1 and 2, see Figure 3 for an explicit such example. It will nevertheless be useful to see how the gadget $K_{a,b}$ is used, so we give a quick overview of the reductions which yield Items 1 and 2 (since these are only implicit in [7]).

For Item 1, let G' be a (2-coloured) bipartite graph which is an input to #BIS. To construct an instance of #Fixed$H$-Col, replace each vertex of G' with a distinct copy of $K_{a,b}$. Further, for each edge (u, v) in G' with $u \in L(G')$ and $v \in R(G')$ add all edges between the right part of u's copy of

[4]The quantity $|S_L|^a$ approximates the number of surjective mappings from $L(K_{a,b})$ to S_L. Similarly, $|S_R|^b$ approximates the number of surjective mappings from $R(K_{a,b})$ to S_R. We will use fairly standard techniques to make sure that the relative error of the approximation (see the relevant Lemma 17) is sufficiently small.

[5]For irrational α and β, we will later use Dirichlet's approximation theorem, cf. Lemma 16, to obtain integers a and b such that the ratio α/β is approximately equal to a/b (to any desired polynomial precision).



H

Figure 3: A graph H where for any choice of the parameters α, β, the set of dominating bicliques $\mathcal{C}_{\alpha,\beta}$ consists either of a single extremal biclique or of the two extremal and two non-extremal bicliques. Namely, when $\alpha > \beta$ (resp. $\alpha < \beta$) we have that the unique element of $\mathcal{C}_{\alpha,\beta}$ is the extremal biclique $(\{1,2,3,4\},\{1'\})$ (resp. $(\{1\},\{1',2',3',4'\})$); when $\alpha = \beta$ we have that $\mathcal{C}_{\alpha,\beta}$ consists of the bicliques $(\{1\},\{1',2',3',4'\})$, $(\{1,2,3,4\},\{1'\})$, $(\{1,2\},\{1',2'\})$ and $(\{1,3\},\{1',3'\})$.

$K_{a,b}$ and the left part of v's copy of $K_{a,b}$. In the final graph, say G, by scaling a, b to be much larger than the size of G' (while keeping fixed the ratio $a/b = \alpha/\beta$), the phases of the gadgets $K_{a,b}$ in "almost all" colour-preserving homomorphisms from G to H are elements of $\mathcal{C}_{\alpha,\beta}$ and in particular are extremal bicliques. It then remains to observe that independent sets of G' are encoded by those homomorphisms where

- the phase of a gadget corresponding to a vertex in $L(G')$ is (F_L, V_R) if the vertex is in the independent set and (V_L, F_R) otherwise, and
- the phase of a gadget corresponding to a vertex in $R(G')$ is (V_L, F_R) if the vertex is in the independent set and (F_L, V_R) otherwise.

For Item 2, the use of the gadget $K_{a,b}$ is depicted in Figure 4. Namely, for a 2-coloured bipartite graph G', consider the graph G obtained by adding all edges between the left part of G' and the right part of $K_{a,b}$. We will typically denote the graph obtained by this construction as $K_{a,b}(G')$. For the following discussion, let us set $G := K_{a,b}(G')$. In the setting of Item 2, we have that $\mathcal{C}_{\alpha,\beta}$ consists of a unique maximal biclique (S_L, S_R) which is not extremal. Once again, by making a, b large relative to the size of G' (while maintaining the ratio $a/b = \alpha/\beta$), the phase of the gadget $K_{a,b}$ in "almost all" homomorphisms h of the graph G will be the dominating biclique (S_L, S_R). Let us consider such a homomorphism h whose restriction on $K_{a,b}$ has as its phase the maximal biclique (S_L, S_R). The edges between $R(K_{a,b})$ and $L(G')$ enforce that $h(L(G')) \subseteq N_\cap(S_R) = S_L$, where in the latter equality we used that (S_L, S_R) is a maximal biclique. It follows that $h(R(G')) \subseteq N_\cup(S_L) = V_R$ (see Lemma 8 for the latter equality). It follows that the restriction of h on G' is an H_1-colouring of the graph G', where $H_1 = H[S_L \cup V_R]$ is the same graph as in Lemma 8. Viewing G' as an instance of $\#H_1$-Col and G as an instance of $\#H$-Col, one obtains $\#H_1\text{-Col} \leq_{\mathsf{AP}} \#H\text{-Col}$. Since H_1 is full, not trivial and has fewer vertices than H, one can use the inductive hypothesis to conclude $\#\text{BIS} \leq_{\mathsf{AP}} \#H_1\text{-Col}$.

We remark here that using the non-constructive approach of Lemma 6 and along the lines we described in Section 2.2, we will be able to remove the restriction that $|\mathcal{C}_{\alpha,\beta}| = 1$ as long as $\mathcal{C}_{\alpha,\beta}$ does not include an extremal biclique of H.

In view of Items 1 and 2, the scheme pursued in [7] (and which we will also follow to a certain

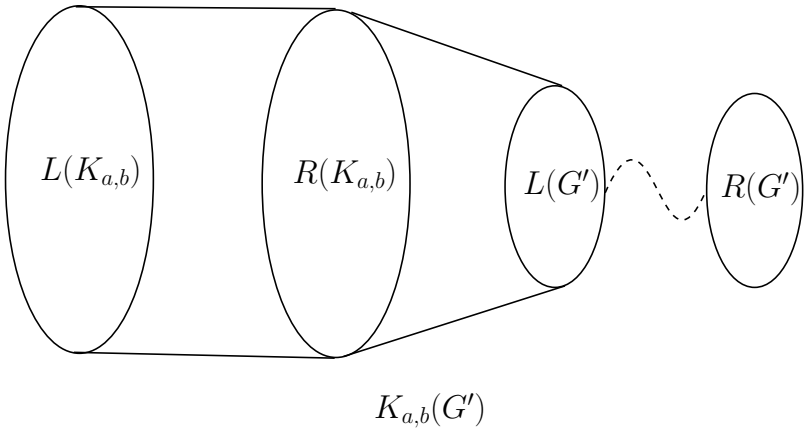


Figure 4: The reduction for Item 2

extent) is to fix $0 < \alpha, \beta < 1$ such that

$$|F_L|^\alpha |V_R|^\beta = |V_L|^\alpha |F_R|^\beta, \tag{2}$$

so that the contribution of the extremal bicliques (F_L, V_R) and (V_L, F_R) to the gadget $K_{a,b}$ is equal. This has the beneficial effect that $\mathcal{C}_{\alpha,\beta}$ includes either none or both of the extremal bicliques. The only very bad scenario remaining is when $\mathcal{C}_{\alpha,\beta}$ includes both the extremal bicliques as well as (at least) one non-extremal biclique, since then not only $|\mathcal{C}_{\alpha,\beta}| > 1$ (which is already a problem for the approach implicit in [7]) but also the coexistence of extremal and non-extremal bicliques in $\mathcal{C}_{\alpha,\beta}$ impedes the non-constructive approach of Lemma 6. While for the sampling problem studied in [7] the coexistence of extremal and non-extremal bicliques was recoverable by "gluing" the reductions together (as we explained in a simplified setting in Section 2.2), this is no longer the case in the counting setting. More precisely, for the counting problem we will have to understand for which graphs H the coexistence of extremal and non-extremal bicliques occurs and consider more elaborate gadgets in the reduction to overcome this coexistence.

5.4 A non-constructive gadget

The key idea is to introduce another non-constructive argument (in addition to the approach suggested by Lemma 6) by viewing the construction of $K_{a,b}(G')$, cf. Figure 4, as a gadget parameterised by the graph G'. To emphasize that G' is no longer an input graph, let us switch notation from G' to Γ, i.e., Γ is a 2-coloured graph and $K_{a,b}(\Gamma)$ is the graph in Figure 4 where G' is replaced by the graph Γ. We will choose a, b sufficiently large so that the graph Γ is "small" relative to the graph $K_{a,b}$, so its effect on the dominant phases will be of second order. We stress here that we will never try to specify Γ explicitly; all we need is the existence of a helpful Γ. In the following, we expand on this point and set up some relevant quantities for the proof.

As for the basic gadget, we define the phase of a colour-preserving homomorphism $h : K_{a,b}(\Gamma) \to H$ as the pair $\big(h(L(K_{a,b})), h(R(K_{a,b}))\big)$. Note that the phase of h is determined by its restriction on $K_{a,b}$ (but not on Γ) and, thus, as before the phases are supported on bicliques of H. We once again set $a = Q\alpha, b = Q\beta$ and let Q be a large integer relative to the size of Γ. (Again, we will later use Dirichilet's approximation theorem for the case in which α and β are not rational, but we do not worry about this here.) With this setup, the phases with the dominant contribution in $K_{a,b}(\Gamma)$ are related to those in $K_{a,b}$ and in particular we will make a and b sufficiently large to ensure that

they are a subset of $\mathcal{C}_{\alpha,\beta}$. Note however that the graph Γ has the effect of reweighting each phase contribution in $K_{a,b}(\Gamma)$ relative to the one in $K_{a,b}$.

To understand the reweighted contribution, let us consider a homomorphism $h : K_{a,b}(\Gamma) \to H$ whose phase is a biclique $(S_L, S_R) \in \mathcal{C}$. For such h, the edges between $R(K_{a,b})$ and $L(\Gamma)$ enforce that $h(L(\Gamma)) \subseteq N_\cap(S_R)$ and thus[6] we obtain that $h(R(\Gamma)) \subseteq N_\cup(N_\cap(S_R))$; it follows that the restriction of h to Γ is supported by vertices in $H[N_\cap(S_R) \cup N_\cup(N_\cap(S_R))]$. It is useful to see what happens when the phase (S_L, S_R) of the homomorphism is a maximal biclique (say in $\mathcal{C}_{\alpha,\beta}$): then, $N_\cap(S_R) = S_L$ and $N_\cup(S_L) = V_R$ (from Lemma 8). Thus, in the case where the phase of h corresponds to a maximal biclique, the restriction of h to Γ is supported by vertices in $H[S_L \cup V_R]$.

It will be useful to distill the following definitions from the above remarks.

Definition 10 (The graph H_{S_L,S_R})**.** *Let (S_L, S_R) be a biclique in H, i.e., $(S_L, S_R) \in \mathcal{C}$. Define H_{S_L,S_R} to be the (bipartite) graph*

$$H[N_\cap(S_R) \cup N_\cup(N_\cap(S_R))],$$

whose 2-colouring is naturally induced by the 2-colouring of H. Note that when (S_L, S_R) is a maximal biclique, we have that $H_{S_L,S_R} = H[S_L \cup V_R]$.

Definition 11 (The parameter $\zeta(S_L, S_R, \Gamma)$)**.** *Let Γ be a 2-coloured graph and let (S_L, S_R) be a biclique in H, i.e., $(S_L, S_R) \in \mathcal{C}$. We will denote*

$$\zeta(S_L, S_R, \Gamma) := \#\mathsf{Fixed}H_{S_L,S_R}\text{-}\mathsf{Col}(\Gamma), \tag{3}$$

where H_{S_L,S_R} is as in Definition 10.

Utilising the above definitions and the remarks earlier, we obtain that the contribution of the biclique (S_L, S_R) to the gadget $K_{a,b}(\Gamma)$ is roughly equal to

$$\zeta(S_L, S_R, \Gamma)|S_L|^a|S_R|^b.$$

Similarly to the outline for the simple gadget, the guiding principle will be to choose a, b, and Γ appropriately so that the dominant phases are supported either on (both of) the extremal bicliques or on the non-extremal bicliques (but not a combination of both). Roughly, the choice of a and b will restrict the dominant phases in $K_{a,b}(\Gamma)$ to be a subset of $\mathcal{C}_{\alpha,\beta}$, while the graph Γ will pick out either the extremal bicliques or a set of non-extremal bicliques. (In the latter case, we will further need to ensure that exactly one non-extremal biclique has significant contribution to the gadget. To do this, we will utilise Lemma 6.) When there is no such graph Γ, we will use an alternative method to find a (2-coloured) proper subgraph H' of H which is also full but not trivial such that $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}$ reduces to $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$. In other words, the non-existence of a "helpful" gadget for any choice of Γ will establish a useful property for $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$ on an arbitrary input.

To do this, we will again equalise the contribution of the extremal bicliques in the gadget $K_{a,b}(\Gamma)$, so it will be useful to explicitly write out the definition (3) for the extremal bicliques.

Definition 12 (The parameters $\zeta_1^{\text{ex}}(\Gamma)$, $\zeta_2^{\text{ex}}(\Gamma)$)**.** *Let Γ be a 2-coloured graph. Let $\zeta_1^{\text{ex}}(\Gamma)$, $\zeta_2^{\text{ex}}(\Gamma)$ be the values of $\zeta(S_L, S_R, \Gamma)$ when (S_L, S_R) is the extremal biclique $(F_L, V_R), (V_L, F_R)$ respectively.*

$$\zeta_1^{\text{ex}}(\Gamma) := \zeta(F_L, V_R, \Gamma) = |F_L|^{|L(\Gamma)|}|V_R|^{|R(\Gamma)|} \text{ and } \zeta_2^{\text{ex}}(\Gamma) := \zeta(V_L, F_R, \Gamma) = \#\mathsf{Fixed}H\text{-}\mathsf{Col}(\Gamma). \tag{4}$$

[6] Technically, to conclude that $h(R(\Gamma)) \subseteq N_\cup(N_\cap(S_R))$ from $h(L(\Gamma)) \subseteq N_\cap(S_R)$ we need that every vertex in $R(\Gamma)$ has at least one neighbour in $L(\Gamma)$. In the upcoming Lemmas 19 and 20, we address this technical point by requiring that Γ has no isolated vertices in $R(\Gamma)$.

(To see the second equality in the definition of $\zeta_1^{\text{ex}}(\Gamma)$, note that $H[F_L \cup V_R]$ is a complete bipartite graph; for the second equality in the definition of $\zeta_2^{\text{ex}}(\Gamma)$, note that $H[V_L \cup V_R] = H$. We remark here that the asymmetry in the definitions of $\zeta_1^{\text{ex}}(\Gamma)$ and $\zeta_2^{\text{ex}}(\Gamma)$ is caused by the choice of connecting the right part of $K_{a,b}$ to the left part of Γ.)

To equalise the contribution of the extremal bicliques in the final gadget we will need to slightly perturb our selection of a, b. That is, instead of setting $a = Q\alpha$ and $b = Q\beta$, we will choose $\hat{a} = Q\alpha$ and $\hat{b} = Q\beta + \gamma$ for some appropriate choice of γ (note that we only perturb the size of b). Now, for a phase (S_L, S_R) the multiplicative correction to its contribution in $K_{\hat{a},\hat{b}}(\Gamma)$ relative to the one in $K_{a,b}$ is given by

$$\zeta(S_L, S_R, \Gamma)|S_R|^{\gamma}.$$

Thus, to equalise the contribution of the extremal bicliques in $K_{\hat{a},\hat{b}}(\Gamma)$, we will need the following parameter $\gamma = \gamma(\Gamma)$ (we will drop the dependence of γ on the graph Γ when there is no danger of confusion), which is formally defined below.

Definition 13 (The parameter $\gamma(\Gamma)$)**.** *Let Γ be a 2-coloured graph. Define $\gamma(\Gamma)$ to be the unique (real) solution to the following equation:*

$$\zeta_1^{\text{ex}}(\Gamma)\,|V_R|^{\gamma(\Gamma)} = \zeta_2^{\text{ex}}(\Gamma)\,|F_R|^{\gamma(\Gamma)}. \tag{5}$$

Remark 14. *Note that $F_R \subset V_R$ so $\gamma(\Gamma)$ is well-defined for all Γ. Further, it is not hard to see that $\zeta_1^{\text{ex}}(\Gamma) \leq \zeta_2^{\text{ex}}(\Gamma)$ (since H_{F_L,V_R} is a subgraph of H_{V_L,F_R}), so $\gamma(\Gamma) \geq 0$ for all Γ.*

With these definitions, for a 2-coloured graph Γ, we define the following subset of $\mathcal{C}_{\alpha,\beta}$ (while the definition makes sense for general values of α, β, we will typically assume that α, β are defined by (2)):

Definition 15 (The set of dominating bicliques $\mathcal{C}^{\Gamma}_{\alpha,\beta}$)**.** *Let $0 < \alpha, \beta < 1$ and Γ be a 2-coloured graph. Define $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ to be the following subset of $\mathcal{C}_{\alpha,\beta}$:*

$$\mathcal{C}^{\Gamma}_{\alpha,\beta} := \left\{ (S_L, S_R) \in \mathcal{C}_{\alpha,\beta} \;\middle|\; (S_L, S_R) = \arg \max_{(S'_L,S'_R)\in\mathcal{C}_{\alpha,\beta}} \zeta(S'_L, S'_R, \Gamma)|S'_R|^{\gamma(\Gamma)} \right\}. \tag{6}$$

Note that for all $\alpha, \beta > 0$ and all 2-coloured graphs Γ, the elements of $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ are maximal bicliques of H.

It is useful to see how the above definitions degenerate in the case where Γ is the empty graph. Then, we have that $\zeta(S_L, S_R, \Gamma) = 1$ for all $(S_L, S_R) \in \mathcal{C}$, and in particular we have that $\zeta_1^{\text{ex}}(\Gamma) = \zeta_2^{\text{ex}}(\Gamma) = 1$. It follows from (5) that $\gamma(\Gamma) = 0$. It follows that when Γ is the empty graph, $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ is identical to $\mathcal{C}_{\alpha,\beta}$.

In the next section, we give an outline for the cases which arise in the proof of Lemma 4 and give examples for each case.

5.5 The cases in the proof of Lemma 4 (overview with examples)

Consider the set of maximal bicliques $\mathcal{C}_{\alpha,\beta}$ where $0 < \alpha, \beta < 1$ are as in (2). For the discussion in this section we may assume that $\mathcal{C}_{\alpha,\beta}$ includes both extremal bicliques and at least one non-extremal biclique. Let

$$\left(S_L^{(1)}, S_R^{(1)}\right), \ldots, \left(S_L^{(t)}, S_R^{(t)}\right)$$

be an enumeration of the non-extremal bicliques in $\mathcal{C}_{\alpha,\beta}$. Recall that all elements of $\mathcal{C}_{\alpha,\beta}$ are maximal bicliques of H. For convenience, in this section let H_i denote the subgraph $H[S_L^{(i)} \cup$

$V_R]$ (this corresponds to the graph $H_{S_L^{(i)},S_R^{(i)}}$ in Definition 10) and set $\zeta_i(\Gamma) = \zeta(S_L^{(i)}, S_R^{(i)}, \Gamma) = \#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(\Gamma)$ (cf. Definition 11). For the extremal bicliques we will instead use the notation $H_1^{\text{ex}}, H_2^{\text{ex}}$ to denote the graphs H_{F_L,V_R}, H_{V_L,F_R} respectively. Note that H_1^{ex} is a complete bipartite graph with bipartition $\{F_L, V_R\}$ while H_2^{ex} is H itself.

There are three complementary cases to consider for the proof of Lemma 4. Recall by construction that for every 2-coloured graph Γ, the extremal bicliques have equal contribution in the graph $K_{\hat{a},\hat{b}}(\Gamma)$ (where $\hat{a} = Qa$, $\hat{b} = Qb + \gamma(\Gamma)$ for some large integer Q). The three cases are as follows.

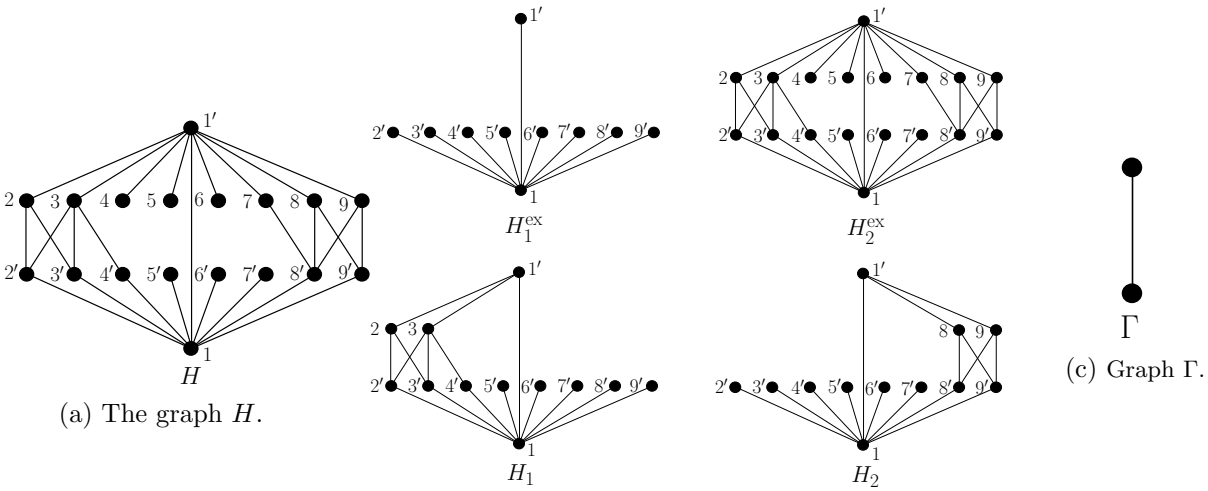
Case 1: There exists $i \in [t]$ and a 2-coloured graph Γ such that the biclique $\left(S_L^{(i)}, S_R^{(i)}\right)$ dominates over the extremal bicliques in the gadget $K_{\hat{a},\hat{b}}(\Gamma)$.

Case 2: There exists $i \in [t]$ such that for every 2-coloured graph Γ the biclique $\left(S_L^{(i)}, S_R^{(i)}\right)$ has the same contribution as the extremal bicliques in the gadget $K_{\hat{a},\hat{b}}(\Gamma)$.

Case 3: For all $i \in [t]$ and every 2-coloured graph Γ, the contribution of the biclique $\left(S_L^{(i)}, S_R^{(i)}\right)$ is less than or equal to the contribution of the extremal bicliques in the gadget $K_{\hat{a},\hat{b}}(\Gamma)$. Further, for all $i \in [t]$ there exists a 2-coloured graph Γ_i such that the biclique $\left(S_L^{(i)}, S_R^{(i)}\right)$ is dominated by the extremal bicliques in the gadget $K_{\hat{a},\hat{b}}(\Gamma_i)$.

We next give an example for each case and overview how our proof works.

Example of Case 1: An example of Case 1 is depicted in Figure 5. Let us first see why



(a) The graph H.

(b) The graphs $H_1^{\text{ex}}, H_2^{\text{ex}}$ (top) and H_1, H_2 (bottom).

(c) Graph Γ.

Figure 5: An example of a graph H which falls into Case 1 of our analysis.

the example is in Case 1. The full vertices of H, as depicted in Figure 5, are vertices 1 and $1'$ and the extremal bicliques of H are $(\{1\}, [9'])$ and $([9], \{1'\})$. Thus, the α, β pairs which equalise the contribution of the extremal bicliques in $K_{a,b}$ satisfy $\alpha = \beta$. The dominating bicliques $\mathcal{C}_{\alpha,\beta}$ for $\alpha = \beta$ are the extremal bicliques and the two bicliques $(S_L^{(1)}, S_R^{(1)}) = (\{1,2,3\}, \{1',2',3'\})$ and $(S_L^{(2)}, S_R^{(2)}) = (\{1,8,9\}, \{1',8',9'\})$.

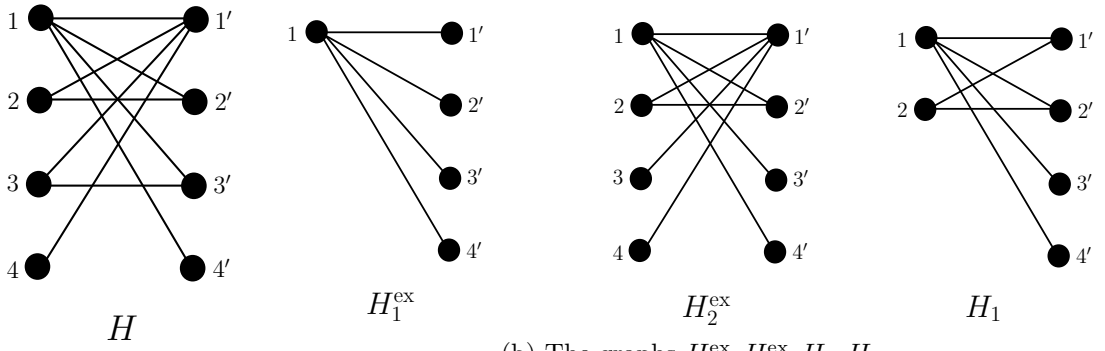
For the graph Γ in Figure 5c, it holds that $\#\mathsf{Fixed}\widehat{H}\text{-}\mathsf{Col}(\Gamma) = E(\widehat{H})$ where $\widehat{H}$ is any of the graphs $H_1^{\text{ex}}, H_2^{\text{ex}}, H_1, H_2$. Thus,

$$\#\mathsf{Fixed}H_1^{\text{ex}}\text{-}\mathsf{Col}(\Gamma) = 9,\ \#\mathsf{Fixed}H_2^{\text{ex}}\text{-}\mathsf{Col}(\Gamma) = 27,\ \#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(\Gamma) = 16,\ \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(\Gamma) = 15.$$

It follows that $\gamma(\Gamma) = 1/2$ (cf. Definition 13) and $\zeta_1^{\text{ex}}(\Gamma)|V_R|^\gamma = \zeta_2^{\text{ex}}(\Gamma)|F_R|^\gamma = 27$. For the non-extremal elements of $\mathcal{C}_{\alpha,\beta}$ it holds that $\zeta_1(\Gamma)|S_R^{(1)}|^\gamma = 16 \times \sqrt{3} > 27, \quad \zeta_2(\Gamma)|S_R^{(2)}|^\gamma = 15 \times \sqrt{3} < 27$. Thus the only dominating biclique in $\mathcal{C}_{\alpha,\beta}^\Gamma$ for the gadget $K_{\hat{a},\hat{b}}(\Gamma)$ is $(\{1,2,3\},\{1',2',3'\})$.

Now, in the general setting of Case 1, we have that, in the gadget $K_{\hat{a},\hat{b}}(\Gamma)$, the extremal bicliques are not dominating (since they are dominated by the biclique $(S_L^{(i)}, S_R^{(i)})$). Note however that there may still be more than one element in $\mathcal{C}_{\alpha,\beta}^\Gamma$, unlike the example in Figure 5. To pick out only one biclique from the set $\mathcal{C}_{\alpha,\beta}^\Gamma$ we further apply Lemma 6 on the graphs H_i corresponding to bicliques in $\mathcal{C}_{\alpha,\beta}^\Gamma$. This yields a graph J which "prefers" a particular graph, say, H_j. Then, using an argument analogous to the one in Section 2.2 (i.e., by pasting sufficiently many disjoint copies of J in $K_{\hat{a},\hat{b}}(\Gamma)$), one can show that $\#\mathsf{Fixed}H_j\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$. By induction, we have $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H_j\text{-}\mathsf{Col}$ and hence $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$ as well.

Example of Case 2: An example of Case 2 is depicted in Figure 6. Note that this is the



(a) The graph H.

(b) The graphs $H_1^{\text{ex}}, H_2^{\text{ex}}, H_1, H_2$.

Figure 6: An example of a graph H which falls into Case 2 of our analysis.

same graph that we encountered earlier in Figure 3. The α, β pairs which equalise the contribution of the extremal bicliques in $K_{a,b}$ satisfy $\alpha = \beta$, in which case

$$\mathcal{C}_{\alpha,\beta} = \{(\{1\},[4']),([4],\{1'\}),(\{1,2\},\{1',2'\}),(\{1,3\},\{1',3'\})\}.$$

To see why H falls into case 2 of our analysis, let us first prove the following equality which holds for every 2-coloured graph Γ:

$$(\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(\Gamma))^2 = \#\mathsf{Fixed}H_1^{\text{ex}}\text{-}\mathsf{Col}(\Gamma)\,\#\mathsf{Fixed}H_2^{\text{ex}}\text{-}\mathsf{Col}(\Gamma). \tag{7}$$

An analogous equality holds if we replace H_1 with H_2 (since $H_1 \cong_c H_2$). The fastest way to derive (7) is to observe that the tensor product of H_1 with itself is isomorphic to the tensor product of H_1^{ex} and H_2^{ex}. An alternative way to deduce the equality using tensor products of smaller graphs is as follows. Let P_n denote the (2-coloured) path with n vertices, whose 2-colouring is induced by colouring one of its degree-one vertices with the colour R. Then it is not hard to see that $H_1^{\text{ex}} \cong_c P_3 \times P_3$, $H_2^{\text{ex}} \cong_c P_4 \times P_4$, $H_1 \cong_c P_3 \times P_4$ (where $\times$ denotes graph tensor product). Thus, for every 2-coloured graph Γ it holds that

$$\#\mathsf{Fixed}H_1^{\text{ex}}\text{-}\mathsf{Col}(\Gamma) = (\#\mathsf{Fixed}P_3\text{-}\mathsf{Col}(\Gamma))^2, \quad \#\mathsf{Fixed}H_2^{\text{ex}}\text{-}\mathsf{Col}(\Gamma) = (\#\mathsf{Fixed}P_4\text{-}\mathsf{Col}(\Gamma))^2,$$
$$\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(\Gamma) = \#\mathsf{Fixed}P_3\text{-}\mathsf{Col}(\Gamma)\,\#\mathsf{Fixed}P_4\text{-}\mathsf{Col}(\Gamma),$$

and (7) follows. (Note, by Lemma 5, (7) implies that $H_1 \times H_1$ is isomorphic to $H_1^{\text{ex}} \times H_2^{\text{ex}}$.)

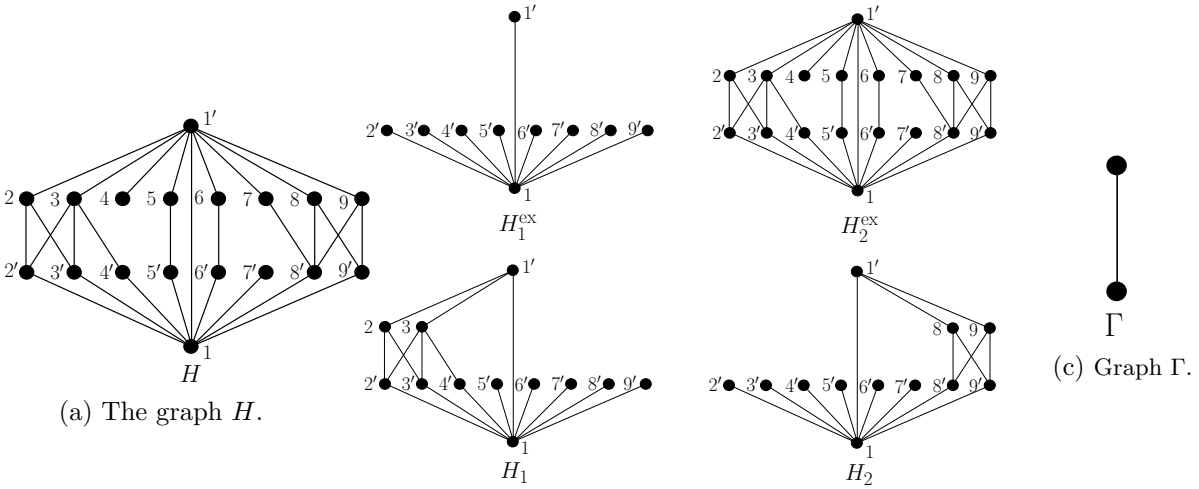
Using (7), we next show that, for $\alpha = \beta$, for all 2-coloured graphs Γ, the set $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ consists of the extremal bicliques and the non-extremal bicliques $(S^{(1)}_L, S^{(1)}_R) = (\{1,2\},\{1',2'\})$, $(S^{(2)}_L, S^{(2)}_R) = (\{1,3\},\{1',3'\})$. Note that $\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(\Gamma) = \zeta_1(\Gamma)$ and

$$\#\mathsf{Fixed}H^{\mathrm{ex}}_1\text{-}\mathsf{Col}(\Gamma)\,\#\mathsf{Fixed}H^{\mathrm{ex}}_2\text{-}\mathsf{Col}(\Gamma) = 4^{|R(\Gamma)|}\#\mathsf{Fixed}H\text{-}\mathsf{Col}(\Gamma) = 4^{|R(\Gamma)|}\zeta^{\mathrm{ex}}_2(\Gamma).$$

Since $\gamma(\Gamma)$ by definition satisfies $\zeta^{\mathrm{ex}}_2(\Gamma) = \zeta^{\mathrm{ex}}_1(\Gamma)4^{\gamma} = 4^{|R(\Gamma)|}4^{\gamma}$, we obtain from (7) that $\zeta_1(\Gamma) = 4^{|R(\Gamma)|}2^{\gamma}$. So, since $|S^{(1)}_R| = 2$, we have $\zeta_1(\Gamma)|S^{(1)}_R|^{\gamma} = \zeta_1(\Gamma)2^{\gamma} = 4^{|R(\Gamma)|}4^{\gamma} = \zeta^{\mathrm{ex}}_2(\Gamma)$. Analogously, $\zeta_1(\Gamma)||S^{(2)}_R|^{\gamma} = \zeta^{\mathrm{ex}}_2(\Gamma)$. It follows that for all 2-coloured graphs Γ, $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ consists of the extremal bicliques and the non-extremal bicliques $(S^{(1)}_L, S^{(1)}_R)$, $(S^{(2)}_L, S^{(2)}_R)$.

Now, in the general setting of Case 2, the non-existence of a gadget $K_{\hat{a},\hat{b}}(\Gamma)$ that distinguishes between $(S^{(i)}_L, S^{(i)}_R)$ and the extremal bicliques for any choice of the graph Γ allows us to quantitatively relate $\#\mathsf{Fixed}H\text{-}\mathsf{Col}(\Gamma)$ and $\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(\Gamma)$ on every "input" Γ, i.e., obtain an equality analogous to (7). It thus follows that $\#\mathsf{Fixed}H_i\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$ and we thus obtain that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}(\Gamma)$ as in Case 1.

Example of Case 3: An example of Case 3 is depicted in Figure 7.



(a) The graph H.

(b) The graphs $H^{\mathrm{ex}}_1, H^{\mathrm{ex}}_2$ (top) and H_1, H_2 (bottom).

(c) Graph Γ.

Figure 7: An example of a graph H which falls into Case 3 of our analysis.

The only difference between H here and the graph H in Figure 5 is that here there are the extra edges $(5,5')$ and $(6,6')$. In particular, only the graph H^{ex}_2 changes. The dominating bicliques $\mathcal{C}_{\alpha,\beta}$ for $\alpha = \beta$ are once again the extremal bicliques and the two bicliques $(S^{(1)}_L, S^{(1)}_R) = (\{1,2,3\},\{1',2',3'\})$, $(S^{(2)}_L, S^{(2)}_R) = (\{1,8,9\},\{1',8',9'\})$.

For the graph Γ in Figure 7c, it holds that $\#\mathsf{Fixed}\widehat{H}\text{-}\mathsf{Col}(\Gamma) = E(\widehat{H})$ where $\widehat{H}$ is any of the graphs $H^{\mathrm{ex}}_1, H^{\mathrm{ex}}_2, H_1, H_2$. Thus,

$$\#\mathsf{Fixed}H^{\mathrm{ex}}_1\text{-}\mathsf{Col}(\Gamma) = 9,\ \#\mathsf{Fixed}H^{\mathrm{ex}}_2\text{-}\mathsf{Col}(\Gamma) = 29,\ \#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(\Gamma) = 16,\ \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(\Gamma) = 15.$$

From $\zeta^{\mathrm{ex}}_1(\Gamma)|V_R|^{\gamma} = \zeta^{\mathrm{ex}}_2(\Gamma)|F_R|^{\gamma} = 29$ (cf. Definition 13), we obtain that $\gamma(\Gamma)$ satisfies $9^{\gamma} = 29/9$. For the non-extremal elements of $\mathcal{C}_{\alpha,\beta}$ it holds that $\zeta_1(\Gamma)|S^{(1)}_R|^{\gamma} = 16\sqrt{29}/3 < 29$, $\zeta_2(\Gamma)|S^{(2)}_R|^{\gamma} = 15\sqrt{29}/3 < 29$. Thus the dominating bicliques in $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ for the gadget $K_{\hat{a},\hat{b}}(\Gamma)$ are the extremal bicliques.

Now consider the general setting of Case 3. Recall from the statement of Case 3 that for all $i \in [t]$ there is a 2-coloured graph Γ_i such that the biclique $\left(S_L^{(i)}, S_R^{(i)}\right)$ is dominated by the extremal bicliques in the gadget $K_{\hat{a},\hat{b}}(\Gamma_i)$. Consider the graph Γ which is the disjoint union of the Γ_i's. It is not hard then to show that in the gadget $K_{\hat{a},\hat{b}}(\Gamma)$ the extremal bicliques are dominating over all non-extremal bicliques. One can then use the reduction for Item 1 in Section 5.3 to show that #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col (instead of using the gadget $K_{a,b}$ as discussed there, we instead use the gadget $K_{\hat{a},\hat{b}}(\Gamma)$).

6 Technical lemmas

We next state a few technical approximation results which will be used in the course of the proofs. The first will be used in the reductions to approximate real numbers with appropriate integers.

Lemma 16 (Dirichlet's Approximation Theorem, cf. [14, p. 34])**.** *Let $\alpha_1, \ldots, \alpha_d > 0$ be real numbers and N be a natural number. Then, there exist positive integers $p_1, \ldots, p_d, q$ with $q \leq N$ such that $|q\alpha_i - p_i| \leq 1/N^{1/d}$ for every $i \in [d]$.*

We will also need the following estimate for the number of *surjective* mappings from a set of size n to a set of size k. We will denote this number by $T(n,k)$. In terms of the Stirling number of the second kind $S(n,k)$, it holds that

$$T(n,k) = S(n,k)k!.$$

It is well known that for fixed k, the asymptotic order of $S(n,k)$ is $k^n/k!$ and thus $T(n,k)$ is $(1+o(1))k^n$. We will need the following more precise estimate.

Lemma 17 (Lemma 18 in [1])**.** *Let n, k be positive integers. Denote by $T(n,k)$ be the number of* surjective *mappings from a set of size n to a set of size k, i.e., T equals the cardinality of the set $\{h : [n] \to [k] \mid h([n]) = [k]\}$. Then, for all $n \geq 2k \ln k$,*

$$(1 - 2k/n)k^n \leq T(n,k) \leq k^n.$$

Proof. It is shown in [1] that

$$\left(1 - e^{-n/(2k)}\right)k^n \leq T(n,k) \leq k^n.$$

The result follows from $1/x > \exp(-x)$ applied to $x = n/(2k)$. □

Finally, we will use the following bound to bound expressions of the form $c^{\pm 1/n}$, where c is a constant independent of n.

Lemma 18. *Let n, K be positive integers with $n \geq K$. Suppose that x is a real number such that $1 \leq x \leq K$ and let z be a real number such that $|z| \leq 1/n$. Then*

$$|x^z - 1| \leq 2K/n.$$

Proof. Let $f(z) = x^z$. By the mean value theorem on f, there exists ρ satisfying $|\rho| \leq |z| \leq 1/n$ such that $f(z) - f(0) = zf'(\rho)$. Using that $x \geq 1$, we thus obtain

$$|x^z - 1| = |z|\,|f'(\rho)| = |z|\,|x^\rho \ln x| = |z|\,x^\rho \ln x \leq (2 \ln x)/n \leq 2K/n.$$

where in the middle inequality we used that $|z| \leq 1/n$ and $x^\rho \leq 2$, while in the last inequality we used that $\ln x \leq \ln K < K$. We argue briefly for the validity of $x^\rho \leq 2$: since f is increasing (from $x \geq 1$), we have $x^\rho \leq x^{|\rho|} \leq x^{1/n}$. We also have $2^n \geq n \geq K \geq x$, which yields $x^{1/n} \leq 2$. □

We conclude this section with a piece of notation. For non-negative real numbers X, Z and $\varepsilon > 0$ we will write $X = (1 \pm \varepsilon)Z$ to denote that

$$(1-\varepsilon)Z \leq X \leq (1+\varepsilon)Z.$$

It is not hard to see that if $X_1 = (1 \pm \varepsilon_1)Z_1$ and $X_2 = (1 \pm \varepsilon_2)Z_2$ for $0 < \varepsilon_1, \varepsilon_2 < 1$, then $X_1 X_2 = (1 \pm \varepsilon)Z_1 Z_2$ where $\varepsilon = \frac{3}{2}(\varepsilon_1 + \varepsilon_2)$.

7 Proof of graph homomorphism lemma

In ths section, we prove the graph-homomorphism lemmas from Section 4. First, the proof of the following lemma is a close adaptation of [9, Proof of Theorem 2.11].

Lemma 5. *Let H_1 and H_2 be 2-coloured bipartite graphs such that $H_1 \not\cong_c H_2$. Then, there exists a 2-coloured bipartite graph J such that $\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J) \neq \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J)$. Moreover, $|V(J)| \leq \max\{|V(H_1)|, |V(H_2)|\}$.*

Proof. For the sake of contradiction, we may assume that for all 2-coloured graphs J with $|V(J)| \leq \max\{|V(H_1)|, |V(H_2)|\}$ it holds that

$$\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J). \tag{8}$$

For 2-coloured graphs H, J denote by $\#\mathsf{InjFixed}H\text{-}\mathsf{Col}(J)$ the number of *injective* colour-preserving homomorphisms from J to H. Assuming that (8) holds for all 2-coloured graphs J with $|V(J)| \leq \max\{|V(H_1)|, |V(H_2)|\}$, we will further show that for all such 2-coloured graphs J it holds that

$$\#\mathsf{InjFixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{InjFixed}H_2\text{-}\mathsf{Col}(J). \tag{9}$$

Plugging $J = H_1$ and $J = H_2$ into (9) yields the existence of a colour-preserving isomorphism between H_1 and H_2, i.e., $H_1 \cong_c H_2$, contradicting the assumption in the statement of the lemma.

To prove (9), we proceed by induction on $|V(J)|$. When $|V(J)| = 1$, for every 2-coloured graph H we (trivially) have that $\#\mathsf{Fixed}H\text{-}\mathsf{Col}(J) = \#\mathsf{InjFixed}H\text{-}\mathsf{Col}(J)$ from where we obtain $\#\mathsf{InjFixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{InjFixed}H_2\text{-}\mathsf{Col}(J)$. So assume that:

$$\begin{gathered}\text{for all 2-coloured graphs } J' \text{ with } |V(J')| < |V(J)|,\\ \text{it holds that } \#\mathsf{InjFixed}H_1\text{-}\mathsf{Col}(J') = \#\mathsf{InjFixed}H_2\text{-}\mathsf{Col}(J').\end{gathered} \tag{10}$$

We will show that $\#\mathsf{InjFixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{InjFixed}H_2\text{-}\mathsf{Col}(J)$ as well. Let Θ_L, Θ_R be partitions of $L(J), R(J)$, respectively. We denote by $J/(\Theta_L, \Theta_R)$ the 2-coloured graph which is obtained by contracting every part in the partition Θ_L and every part in the partition Θ_R. For 2-coloured graphs H, J it is straightforward to check that

$$\#\mathsf{Fixed}H\text{-}\mathsf{Col}(J) = \sum_{\Theta_L} \sum_{\Theta_R} \#\mathsf{InjFixed}H\text{-}\mathsf{Col}\big(J/(\Theta_L, \Theta_R)\big), \tag{11}$$

where in the latter sum Θ_L, Θ_R range over all partitions of $L(J)$ and $R(J)$. Note that $J/(\Theta_L, \Theta_R)$ has fewer vertices than J, with the single exception of the case that Θ_L, Θ_R consist only of singleton sets where we have that $J/(\Theta_L, \Theta_R) = J$. Applying (11) for $H = H_1$ and $H = H_2$, from (10) and the assumption $\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J)$, we obtain $\#\mathsf{InjFixed}H_1\text{-}\mathsf{Col}(J) = \#\mathsf{InjFixed}H_2\text{-}\mathsf{Col}(J)$, as desired. □

We will also use the following generalisation of Lemma 5.

Lemma 6. *Let $H_1, \dots, H_k$ be 2-coloured bipartite graphs such that $H_i \not\cong_c H_j$ for all $\{i,j\} \in \binom{[k]}{2}$. There exists a 2-coloured graph J and an integer $i^* \in [k]$ such that $\#\mathsf{Fixed}H_{i^*}\text{-}\mathsf{Col}(J) > \#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J)$ for all $i \in [k]\backslash\{i^*\}$.*

Proof. The proof is by induction on k. For $k = 2$, the claim is immediate from Lemma 5. Assume that the claim holds for $k-1$, we next show it for k. By applying the inductive hypothesis to the graphs $H_2, \dots, H_k$ (and renaming if necessary), we may assume that J_2 is a bipartite graph such that

$$\#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J_2) > \#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2) \text{ for all } i > 2. \tag{12}$$

We may assume that

$$\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J_2) = \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J_2) =: M. \tag{13}$$

(Otherwise we may choose $J = J_2$ and $i^* = \arg\max_{i\in\{1,2\}} \#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2)$.) By Lemma 5 (and renaming H_1, H_2 if necessary; this is justified by (13)), we may assume that J' is a bipartite graph such that

$$\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J') > \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J'). \tag{14}$$

Let J be the disjoint union of J' and t copies of J_2, where $t := \lceil CM \rceil$ where M is as in (13) and

$$C := \max_{i\in[k]\backslash\{1,2\}} \frac{\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J')}{\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J')}.$$

For every $i \in [k]$, we have that

$$\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J) = \big(\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2)\big)^t \, \#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J').$$

From (13) and (14), we have that $\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J) > \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J)$. Further, for $i \notin \{1,2\}$, we have

$$\begin{aligned}
\frac{\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J)}{\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J)} &= \frac{\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J')}{\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J')}\Big(\frac{\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J_2)}{\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2)}\Big)^t \\
&\geq \frac{1}{C}\Big(1 + \frac{1}{\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2)}\Big)^t \\
&> \frac{t}{C\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J_2)} \geq 1,
\end{aligned}$$

where we used (12) to deduce $\#\mathsf{Fixed}H_1\text{-}\mathsf{Col}(J') \geq \#\mathsf{Fixed}H_2\text{-}\mathsf{Col}(J') + 1$, the inequality $(1+x)^t \geq 1 + tx > tx$ (for $t \geq 1$ and $x \geq 0$) as well as the choice of t (cf. the definitions of C, M). □

8 Proof of Lemma 4

The following two lemmas will be important for the proof of Lemma 4.

Lemma 19. *Let H be a 2-coloured graph which is full but not trivial. Let $0 < \alpha, \beta < 1$ be as in (2). Suppose that Γ is a 2-coloured graph with no isolated vertices in $R(\Gamma)$. Assume further that $(F_L, V_R), (V_L, F_R) \notin \mathcal{C}^{\Gamma}_{\alpha,\beta}$. Then a 2-coloured, full but not trivial graph H' can be specified such that $|V(H')| < |V(H)|$ and*

$$\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}.$$

Proof. For $(S_L, S_R) \in \mathcal{C}^{\Gamma}_{\alpha,\beta}$, recall that (S_L, S_R) is a maximal biclique. From the assumptions, (S_L, S_R) is not extremal, so by Lemma 8 the graph H_{S_L,S_R} is full but not trivial and has fewer vertices than H (to see this, recall the remark in Definition 10 that for a maximal biclique (S_L, S_R) we have that $H_{S_L,S_R} = H[S_L \cup V_R]$; thus H_{S_L,S_R} corresponds to the graph H_1 in Lemma 8). We remark for later use that for a maximal biclique (S_L, S_R) it holds that $R(H_{S_L,S_R}) = V_R$.

We will find a biclique (S_L, S_R) in $\mathcal{C}^{\Gamma}_{\alpha,\beta}$ such that $\#\mathsf{Fixed}H_{S_L,S_R}\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$. To find such a biclique, we will use Lemma 6. Specifically, let $\mathcal{H}$ be the set consisting of the (labelled) graphs H_{S_L,S_R} when (S_L, S_R) ranges over maximal bicliques in $\mathcal{C}^{\Gamma}_{\alpha,\beta}$. Further, let $\mathcal{H}_1, \ldots, \mathcal{H}_k$ be the equivalence classes of $\mathcal{H}$ under the (colour-preserving isomorphism) relation $\cong_c$ which was formally defined in Section 4. For $i \in [k]$, let H_i be a representative of the class $\mathcal{H}_i$. By Lemma 6, there exists a gadget J and $i \in [k]$ such that for every $t \in [k] \setminus \{i\}$, it holds that $\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J) > \#\mathsf{Fixed}H_t\text{-}\mathsf{Col}(J)$. We will show that $H' = H_i$ satisfies the statement of the lemma. For future use, let $\mathcal{C}^* \subseteq \mathcal{C}^{\Gamma}_{\alpha,\beta}$ be the subset of bicliques (S_L, S_R) whose corresponding graph H_{S_L,S_R} is isomorphic to H' under a colour-preserving isomorphism. Formally,

$$\mathcal{C}^* := \left\{(S_L, S_R) \in \mathcal{C}^{\Gamma}_{\alpha,\beta} \,|\, H_{S_L,S_R} \cong_c H'\right\}. \tag{15}$$

For technical convenience we assume that the graph J has no isolated vertices in $R(J)$. This is without loss of generality. To see this, suppose that $I \subseteq R(J)$ is the set of isolated vertices in $R(J)$. Let J' be the 2-coloured graph which is obtained from J by removing the vertices in I; clearly, J' has no isolated vertices in $R(J)$. Observe that for each $t \in [k]$ it holds that $R(H_t) = V_R$ since by definition H_t is the graph H_{S_L,S_R} for some maximal biclique (S_L, S_R). Thus, for every $t \in [k]$ it holds that $\#\mathsf{Fixed}H_t\text{-}\mathsf{Col}(J) = |V_R|^{|I|}\#\mathsf{Fixed}H_t\text{-}\mathsf{Col}(J')$. Hence, by the choice of J, we have that $\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J') > \#\mathsf{Fixed}H_t\text{-}\mathsf{Col}(J')$ for every $t \neq i$.

Having defined the graph H', we next give the details of the reduction. Let $0 < \varepsilon < 1$ and G' be an input to $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}$. Our goal is to approximate $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G')$ within a multiplicative factor of $(1 \pm \varepsilon)$ using an oracle for $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$. Using standard arguments,[7] we may assume w.l.o.g. that G' is connected.

Suppose that $n' = |V(G')|$ and let $n = \lceil \kappa n'/\varepsilon \rceil$ where κ is a sufficiently large constant depending only on H, Γ. Note that by letting κ be large, we also ensure that n is sufficiently large both in absolute terms as well as relatively to n' and $1/\varepsilon$. For the time being, let us assume that κ is such that $\alpha n \geq 1$, $\beta n \geq 1$, $n \geq 2\gamma$, $2\max\{|V_L|, |V_R|\}/n \leq \varepsilon/10^4$. Later, we will refine appropriately the choice of κ.

By Lemma 16 applied to the numbers αn^3 and $n^2(\beta n + \gamma)$, there exists a positive integer $Q \leq n^2$ and positive integers a, b such that

$$|Q\alpha n^3 - a| \leq 1/n, \quad |Q(\beta n^3 + \gamma n^2) - b| \leq 1/n. \tag{16}$$

Note that since $1 \leq Q \leq n^2$ and α, β, γ are constants, for all sufficiently large n, it holds that $n^2 \leq a, b \leq 2n^5$. The choice of a, b ensures that for $(S_L, S_R) \in \mathcal{C}$, it holds that

$$\begin{aligned}\zeta(S_L, S_R, \Gamma)^{Qn^2}|S_L|^a|S_R|^b &= \left(1 \pm \varepsilon/10^3\right)\left(\zeta(S_L, S_R, \Gamma)\,|S_L|^{\alpha n}|S_R|^{\beta n+\gamma}\right)^{Qn^2} \\ &= \left(1 \pm \varepsilon/10^3\right)\left(|S_L|^{\alpha}|S_R|^{\beta}\right)^{Qn^3}\left(\zeta(S_L, S_R, \Gamma)\,|S_R|^{\gamma}\right)^{Qn^2}, \end{aligned} \tag{17}$$

where in the first equality we used Lemma 18 for $x = |S_L|$ and $K = |V_L|$ to bound $|S_L|^{a-Q\alpha n^3} = 1 \pm 2|V_L|/n = 1 \pm \varepsilon/10^4$ and similarly $|S_R|^{b-Q(\beta n^3+\gamma n^2)} = 1 \pm \varepsilon/10^4$, while the second equality is a mere rearrangement of terms.

[7] If $G'_1, \ldots, G'_l$ are the connected components of G' then it holds that $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G') = \prod_{t=1}^{l} \#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G'_l)$.

We next construct an instance G of #Fixed$H$-Col. Recall that $K_{a,b}$ is the complete bipartite graph with a, b vertices on the left and right, respectively. Let $\overline{G}$ be the disjoint union of the graphs G', $K_{a,b}$, Qn^2 disjoint copies of the graph Γ and $\lceil n^{3/2} \rceil$ copies of the gadget J. We denote by Γ_r the r-th copy of Γ and by J_r the r-th copy of J. To construct G, attach on $\overline{G}$ all edges in $R(K_{a,b}) \times L(G')$, for $r = 1, \ldots, Qn^2$ add all edges in $R(K_{a,b}) \times L(\Gamma_r)$ and, finally, for $r = 1, \ldots, \lceil n^{3/2} \rceil$ add all edges in $R(K_{a,b}) \times L(J_r)$. Note that G has size polynomial in $n, 1/\varepsilon$.

For the remainder of the proof, for 2-coloured graphs $\hat{G}, \hat{H}$, we will use for convenience the shorthand $Z_{\hat{G}}(\hat{H})$ to denote #Fixed$\hat{H}$-Col$(\hat{G})$, i.e., $Z_{\hat{G}}(\hat{H})$ denotes the number of colour-preserving homomorphisms from $\hat{G}$ to $\hat{H}$. It will also be helpful to denote

$$M_K := \max_{(S_L,S_R)\in\mathcal{C}} |S_L|^\alpha |S_R|^\beta, \quad M_\Gamma := \max_{(S_L,S_R)\in\mathcal{C}_{\alpha,\beta}} \zeta(S_L, S_R, \Gamma)|S_R|^\gamma, \quad M_J := Z_J(H').$$

Denote further by c^* the cardinality of the set $\mathcal{C}^*$, i.e., $c^* := |\mathcal{C}^*|$. We will prove the following:

$$Z_G(H) = \big(1 \pm \varepsilon/10\big) c^* M_K^{Qn^3} M_\Gamma^{Qn^2} M_J^{\lceil n^{3/2} \rceil} Z_{G'}(H'). \tag{18}$$

Note that we can approximate M_K, M_Γ to any desired accuracy (polynomial in n and ε) since Γ is fixed graph and α, β, γ are constants which are efficiently approximable using (2) and (5). Since J is a fixed graph, M_J and c^* can be computed exactly by brute force (which only takes constant time). Thus, if we can approximate $Z_G(H)$ within a multiplicative factor of $(1 \pm \varepsilon/10)$, equation (18) yields an approximation of $Z_{G'}(H')$ within a multiplicative factor of $(1 \pm \varepsilon)$ (with room to spare).

It remains to argue for the validity of (18). Let Σ be the set of all colour-preserving homomorphisms from G to H. For subsets $S_L \subseteq V_L$ and $S_R \subseteq V_R$, let $\Sigma(S_L, S_R)$ be the following subset of Σ:

$$\Sigma(S_L, S_R) = \{h \in \Sigma \mid h(L(K_{a,b})) = S_L,\, h(R(K_{a,b})) = S_R\}. \tag{19}$$

Since $K_{a,b}$ is a complete bipartite graph, note that $\Sigma(S_L, S_R)$ is non-empty iff $S_L \times S_R \subseteq E(H)$, i.e., (S_L, S_R) is a biclique in H. It follows that the collection of sets $\Sigma(S_L, S_R)$ where (S_L, S_R) ranges over all bicliques in H is a partition of the set Σ, so

$$Z_G(H) = |\Sigma| = \sum_{(S_L,S_R)\in\mathcal{C}} |\Sigma(S_L, S_R)|. \tag{20}$$

Fix $(S_L, S_R) \in \mathcal{C}$ and let $h \in \Sigma(S_L, S_R)$. Note that for every copy Γ_r of Γ we have that $h(L(\Gamma_r)) \subseteq N_\cap(S_R)$ since by construction $R(K_{a,b}) \times L(\Gamma_r) \subseteq E(G)$. Using the assumption in the lemma that Γ has no isolated vertices in $R(\Gamma)$, we thus obtain that for every copy Γ_r of Γ it holds that $h(R(\Gamma_r)) \subseteq N_\cup(N_\cap(S_R))$. Analogously, since G' is connected and J has no isolated vertices in $R(J)$, for G' and every copy J_r of J, it holds that $h(L(G')), h(L(J_r)) \subseteq N_\cap(S_R)$ and $h(R(G')), h(R(J_r)) \subseteq N_\cup(N_\cap(S_R))$. It follows that

$$|\Sigma(S_L, S_R)| = T(a, |S_L|)\, T(b, |S_R|) \left(Z_\Gamma\big(H_{S_L,S_R}\big)\right)^{Qn^2} \left(Z_J(H_{S_L,S_R})\right)^{\lceil n^{3/2} \rceil} Z_{G'}\big(H_{S_L,S_R}\big), \tag{21}$$

where, recall, $T(m, q)$ is the number of surjective mappings from $[m]$ to $[q]$. Applying Lemma 17 (and using that $2|S_L|/a, 2|S_R|/b \leq 2\max\{|V_L|, |V_R|\}/n \leq \varepsilon/10^4$), we have that

$$T(a, |S_L|) = (1 \pm \varepsilon/10^4)|S_L|^a, \quad T(b, |S_R|) = (1 \pm \varepsilon/10^4)|S_R|^b,$$

and thus

$$T(a, |S_L|)T(b, |S_R|) = (1 \pm \varepsilon/10^3)|S_L|^a |S_R|^b.$$

Observe also that $Z_\Gamma\big(H_{S_L,S_R}\big) = \zeta(S_L,S_R,\Gamma)$, cf. Definition 11. Thus, (21) yields

$$\begin{aligned}|\Sigma(S_L,S_R)| &= \big(1\pm\varepsilon/10^3\big)|S_L|^a|S_R|^b\,\zeta(S_L,S_R,\Gamma)^{Qn^2}\\ &\qquad\qquad \big(Z_J(H_{S_L,S_R})\big)^{\lceil n^{3/2}\rceil} Z_{G'}\big(H_{S_L,S_R}\big)\\ &= \big(1\pm\varepsilon/10^2\big)\,\big(|S_L|^\alpha|S_R|^\beta\big)^{Qn^3}\big(\zeta(S_L,S_R,\Gamma)\,|S_R|^\gamma\big)^{Qn^2}\\ &\qquad\qquad \big(Z_J(H_{S_L,S_R})\big)^{\lceil n^{3/2}\rceil} Z_{G'}\big(H_{S_L,S_R}\big),\end{aligned} \tag{22}$$

where for the second equality we used the approximation in (17).

We next use (22) to find the terms in (20) with significant contribution. Observe that for every biclique (S_L,S_R) we have the bounds

$$|S_L|^\alpha|S_R|^\beta \le M_K,\quad \zeta(S_L,S_R,\Gamma)\,|S_R|^\gamma \le Z_\Gamma(H)\,|V_R|^\gamma,\quad Z_J(H_{S_L,S_R}) \le Z_J(H),$$

i.e., the bases of the powers in the r.h.s. of (22) are bounded by absolute constants. Further,

$$Z_{G'}\big(H_{S_L,S_R}\big) \le Z_{G'}\big(H\big) \le |V(H)|^{n'} \le |V(H)|^n.$$

Using these observations, we will consider the factors in the r.h.s. of (22) in decreasing order of magnitude (which can easily be read off from the exponents) and argue that only bicliques in $\mathcal{C}^*$ have significant contribution.

From the definition of $\mathcal{C}^*$ (see (15)), we have that a biclique $(S_L,S_R)\in\mathcal{C}^*$ contributes $(1\pm\varepsilon/10^2)W$ to the sum in (20), where

$$W := M_K^{Qn^3} M_\Gamma^{Qn^2} M_J^{\lceil n^{3/2}\rceil} Z_{G'}(H').$$

Thus, the total contribution from bicliques in $\mathcal{C}^*$ is $(1\pm\varepsilon/10^2)c^*W$, that is,

$$\sum_{(S_L,S_R)\in\mathcal{C}^*} |\Sigma(S_L,S_R)| = (1\pm\varepsilon/10^2)c^*W. \tag{23}$$

From the definition of $\mathcal{C}_{\alpha,\beta}$, we have that for $(S_L,S_R)\notin\mathcal{C}_{\alpha,\beta}$, it holds that

$$M_K > |S_L|^\alpha|S_R|^\beta,$$

so that terms in (20) such that $(S_L,S_R)\notin\mathcal{C}_{\alpha,\beta}$ are smaller than W by a multiplicative factor of $\exp(-\Omega(n^3))$. From the definition of $\mathcal{C}^\Gamma_{\alpha,\beta}$, for $(S_L,S_R)\in\mathcal{C}_{\alpha,\beta}\backslash\mathcal{C}^\Gamma_{\alpha,\beta}$, it holds that

$$M_\Gamma > \zeta(S_L,S_R,\Gamma)|S_R|^\gamma,$$

so that terms in (20) such that $(S_L,S_R)\in\mathcal{C}_{\alpha,\beta}\backslash\mathcal{C}^\Gamma_{\alpha,\beta}$ are smaller than W by a multiplicative factor of $\exp(-\Omega(n^2))$. Finally, from the definition of $\mathcal{C}^*$ (see (15)), for $(S_L,S_R)\in\mathcal{C}^\Gamma_{\alpha,\beta}\backslash\mathcal{C}^*$ we have

$$M_J > Z_J(H_{S_L,S_R})$$

so that terms in (20) such that $(S_L,S_R)\in\mathcal{C}^\Gamma_{\alpha,\beta}\backslash\mathcal{C}^*$ are smaller than W by a multiplicative factor of $\exp(-\Omega(n^{3/2}))$.

Combining the above estimates, we have that for every biclique $(S_L,S_R)\in\mathcal{C}\backslash\mathcal{C}^*$, it holds that

$$|\Sigma(S_L,S_R)| \le \exp\big(-\Omega(n^{3/2})\big)W \le \varepsilon W/(10^2|\mathcal{C}|).$$

(Note that in the combined inequality we assumed that the constant κ in the definition of n is sufficiently large.) It follows that the aggregate contribution from bicliques which are not in $\mathcal{C}^*$ to the sum (20) is at most $\varepsilon W/10^2$, that is,

$$\sum_{(S_L,S_R)\in\mathcal{C}\setminus\mathcal{C}^*} |\Sigma(S_L,S_R)| \leq \varepsilon W/10^2. \tag{24}$$

Plugging the bounds (23) and (24) in the sum (20) yields (18), as desired. This completes the proof of Lemma 19. □

Lemma 20. *Let H be a 2-coloured graph which is full but not trivial. Let $0<\alpha,\beta<1$ be as in (2). Suppose that Γ is a 2-coloured graph such that Γ has no isolated vertices in $R(\Gamma)$. Further, assume that*

$$\mathcal{C}^{\Gamma}_{\alpha,\beta} = \big\{(F_L,V_R),(V_L,F_R)\big\}.$$

Then, #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

Proof. Let $0<\varepsilon<1$ and G' be an input to #BIS. Our goal is to approximate #BIS(G') within a multiplicative factor of $(1\pm\varepsilon)$ using an oracle for #Fixed$H$-Col.

Suppose that $n'=|V(G')|$ and let $n=\lceil \kappa n'^2/\varepsilon\rceil$ where κ is a sufficiently large constant depending only on H and Γ. Similarly to the proof of Lemma 19, we will use κ to ensure that n is sufficiently large both in absolute terms as well as relatively to n'^2 and $1/\varepsilon$. We will assume that κ is such that $\alpha n\geq 1$, $\beta n\geq 1$, $n\geq 2\gamma$, $2\max\{|V_L|,|V_R|\}/n\leq \varepsilon/(10^4 n')$ and, later, we will further refine the choice of κ.

Analogously to the proof of Lemma 19, we apply Lemma 16 to the numbers αn^3 and $n^2(\beta n+\gamma)$. This yields positive numbers a,b,Q with $Q\leq n^2$ such that

$$|Q\alpha n^3 - a|\leq 1/n, \quad |Qn^2(\beta n+\gamma)-b|\leq 1/n,$$

and, once again, note that $n^2\leq a,b\leq 2n^5$. The choice of a,b yields the following analogue of (17) for $(S_L,S_R)\in\mathcal{C}$:

$$\zeta(S_L,S_R,\Gamma)^{Qn^2}|S_L|^a|S_R|^b = \big(1\pm\varepsilon/(10^3n')\big)\big(|S_L|^\alpha|S_R|^\beta\big)^{Qn^3}\big(\zeta(S_L,S_R,\Gamma)\,|S_R|^\gamma\big)^{Qn^2}, \tag{25}$$

(Note that the r.h.s. in (25) differs from the respective one in (17) only in the error term, which can be accounted by the slightly different choice of n.)

To construct an instance G of #Fixed$H$-Col, we first construct a bipartite graph J as follows. Let $K_{a,b}$ be the complete bipartite graph with a,b vertices on the left and right, respectively. Let $\overline{K}$ be the disjoint union of the graphs $K_{a,b}$ and Qn^2 disjoint copies of the graph Γ. We denote by Γ_r the r-th copy of Γ. Finally, the graph J is obtained by attaching on $\overline{K}$ for $r=1,\dots,Qn^2$ all edges in $R(K_{a,b})\times L(\Gamma_r)$.

Let Σ_J be the set of all colour-preserving homomorphisms from J to H. For subsets $S_L\subseteq V_L$ and $S_R\subseteq V_R$, let $\Sigma_J(S_L,S_R)$ be the following subset of Σ_J:

$$\Sigma_J(S_L,S_R) = \{h\in\Sigma_J \mid h(L(K_{a,b}))=S_L,\ h(R(K_{a,b}))=S_R\}.$$

Since $K_{a,b}$ is a complete bipartite graph, note that $\Sigma_J(S_L,S_R)$ is non-empty iff $S_L\times S_R\subseteq E(H)$, i.e., (S_L,S_R) is a biclique in H. It follows that the collection of sets $\Sigma_J(S_L,S_R)$ where (S_L,S_R) ranges over all bicliques in H is a partition of the set Σ_J. Fix $(S_L,S_R)\in\mathcal{C}$ and let $h\in\Sigma(S_L,S_R)$. Note

that for every copy Γ_r of Γ we have that $h(L(\Gamma_r)) \subseteq N_\cap(S_R)$ and thus $h(R(\Gamma_r)) \subseteq N_\cup(N_\cap(S_R))$ (since by the assumptions in the lemma Γ has no isolated vertices in $R(\Gamma)$). It follows that

$$|\Sigma_J(S_L,S_R)| = T(a,|S_L|)\,T(b,|S_R|)\left(\zeta(S_L,S_R,\Gamma)\right)^{Qn^2},$$

where, recall that for positive integers m,q, $T(m,q)$ is the number of surjective mappings from $[m]$ to $[q]$ and, from Definition 11, $\zeta(S_L,S_R,\Gamma) = \#\mathsf{Fixed}H_{S_L,S_R}\text{-}\mathsf{Col}(\Gamma)$. Applying Lemma 17 and the approximation in (25) yields that for every $(S_L,S_R)\in\mathcal{C}$, it holds that

$$|\Sigma_J(S_L,S_R)| = \left(1\pm\varepsilon/(10^2n')\right)\left(|S_L|^\alpha|S_R|^\beta\right)^{Qn^3}\left(\zeta(S_L,S_R,\Gamma)\,|S_R|^\gamma\right)^{Qn^2}. \tag{26}$$

Let us also denote

$$M_K := \max_{(S_L,S_R)\in\mathcal{C}} |S_L|^\alpha|S_R|^\beta, \quad M_\Gamma := \max_{(S_L,S_R)\in\mathcal{C}_{\alpha,\beta}} \zeta(S_L,S_R,\Gamma)|S_R|^\gamma.$$

From (26) and the assumption that the extremal bicliques are elements of both $\mathcal{C}_{\alpha,\beta}$ and $\mathcal{C}^\Gamma_{\alpha,\beta}$, we obtain

$$|\Sigma_J(F_L,V_R)| = (1\pm\varepsilon/(10^2n'))M_K^{Qn^3}M_\Gamma^{Qn^2},\ |\Sigma_J(V_L,F_R)| = (1\pm\varepsilon/(10^2n'))M_K^{Qn^3}M_\Gamma^{Qn^2}. \tag{27}$$

Now consider a non-extremal biclique (S_L,S_R), i.e., $(S_L,S_R)\neq(F_L,V_R),(V_L,F_R)$. From the definition of $\mathcal{C}_{\alpha,\beta}$ (cf. Definition 9), if $(S_L,S_R)\in\mathcal{C}\backslash\mathcal{C}_{\alpha,\beta}$, we have $M_K > |S_L|^\alpha|S_R|^\beta$ and thus

$$|\Sigma_J(S_L,S_R)| \le \exp(-\Omega(n^3))M_K^{Qn^3}M_\Gamma^{Qn^2}.$$

If $(S_L,S_R)\in\mathcal{C}_{\alpha,\beta}$ but $(S_L,S_R)\notin\mathcal{C}^\Gamma_{\alpha,\beta}$, from the definition of $\mathcal{C}^\Gamma_{\alpha,\beta}$ (cf. Definition 15) we have $M_\Gamma > \zeta(S_L,S_R,\Gamma)|S_R|^\gamma$ so that

$$|\Sigma_J(S_L,S_R)| \le \exp(-\Omega(n^2))M_K^{Qn^3}M_\Gamma^{Qn^2}.$$

Combining the above bounds, it follows that for every non-extremal biclique $(S_L,S_R)\in\mathcal{C}$ it holds that $|\Sigma_J(S_L,S_R)| \le \exp(-\Omega(n^2))M_K^{Qn^3}M_\Gamma^{Qn^2}$, i.e.,

$$M_3 \le \exp(-\Omega(n^2))M_K^{Qn^3}M_\Gamma^{Qn^2}, \text{ where } M_3 = \max_{\substack{(S_L,S_R)\in\mathcal{C};\\ (S_L,S_R)\neq(F_L,V_R),(V_L,F_R)}} |\Sigma_J(S_L,S_R)|. \tag{28}$$

With the gadget J at hand, we now give the reduction from #BIS to $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$. First, we construct an instance G of $\#\mathsf{Fixed}H\text{-}\mathsf{Col}$ as follows. For every vertex u in G' take a disjoint copy of J, which we shall denote by J^u. We will further denote by $K^u_{a,b}$ the copy of $K_{a,b}$ in J^u. For every edge $(u,v)\in E(G')$ with $u\in L(G')$ and $v\in R(G')$ add all edges between $R(K^u_{a,b})$ and $L(K^v_{a,b})$. Denote by G the final graph. The construction ensures that G is a bipartite graph (the 2-colouring of G is naturally induced by the 2-colourings of the disjoint copies of J).

We will show that

$$\#\mathsf{Fixed}H\text{-}\mathsf{Col}(G) = \left(1\pm\varepsilon/10\right)\left(M_K^{Qn^3}M_\Gamma^{Qn^2}\right)^{n'}\#\mathrm{BIS}(G'). \tag{29}$$

Note that M_K, M_Γ can be computed to any desired polynomial precision (in $n, 1/\varepsilon$), so if we can approximate $\#\mathsf{Fixed}H\text{-}\mathsf{Col}(G)$ within a multiplicative factor of $1\pm\varepsilon/10$, equation (29) yields an approximation of $\#\mathrm{BIS}(G')$ within a multiplicative factor of $1\pm\varepsilon$.

We next argue for the validity of (29). Let h be a colour-preserving homomorphism from G to H. For a vertex u in G', we call (S_L, S_R) the *phase* of u under h iff the restriction of h to the graph J^u is a homomorphism in $\Sigma_J(S_L, S_R)$. We will denote by h^u the *phase* of u under h. Finally, we let $\mathcal{Y}(h)$ to be the vector $\{h^u\}_{u\in V(G')}$, i.e., the collection of phases under h.

Let Σ be the set of colour-preserving homomorphisms from G to H and for a given phase vector $Y \in \mathcal{C}^{n'}$, let $\Sigma(Y)$ be the set of homomorphisms h such that $\mathcal{Y}(h) = Y$, i.e.,

$$\Sigma(Y) = \{h \in \Sigma \mid \mathcal{Y}(h) = Y\}.$$

Note that the $\Sigma(Y)$ form a partition of Σ, so we have that

$$|\Sigma| = \sum_{Y\in\mathcal{C}^{n'}} |\Sigma(Y)|.$$

Note also the following upper bound on $|\Sigma(Y)|$:

$$|\Sigma(Y)| \le \prod_{u\in V(G')} |\Sigma_J(Y_u)|. \tag{30}$$

We call a phase vector $Y \in \mathcal{C}^{n'}$ *bad* if there exists a vertex $u \in G'$ whose phase Y_u is not an extremal biclique. Otherwise, Y will be called *good*. Using the bound (30) in combination with (27) and (28) (which hold for every copy of the gadget J), for a bad phase vector Y we have that

$$|\Sigma(Y)| \le \Big(1 \pm \frac{\varepsilon}{10^2 n'}\Big)^{n'-1} \big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'-1} M_3 \le 2\big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'-1} M_3. \tag{31}$$

Let Σ' be the set of colour-preserving homomorphisms from G to H whose phase vector is bad. Since there are less than $|\mathcal{C}|^{n'}$ choices for the bad phase vector Y, using (31), we have the crude bound

$$|\Sigma'| \le 2|\mathcal{C}|^{n'} \big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'-1} M_3 < \exp(-\Omega(n^2))\big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'}. \tag{32}$$

Note that in the latter inequality we used that M_3 is smaller from $M_K^{Qn^3} M_\Gamma^{Qn^2}$ by a factor of $\exp(-\Omega(n^2))$, cf. (28). By letting the constant κ to be sufficiently large, (32) yields that

$$|\Sigma'| \le (\varepsilon/20)\big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'}. \tag{33}$$

Let $Y \in \mathcal{C}^{n'}$ be a good phase vector. We call Y *permissible* if for every edge $(u, v) \in E(G)$ we have either $Y_u \neq (F_L, V_R)$ or $Y_v \neq (V_L, F_R)$. Note that if Y is not permissible we have that $|\Sigma(Y)| = 0$ (since H is not trivial). If Y is permissible, we have from (27) applied to every copy of J that

$$|\Sigma(Y)| = \Big(1 \pm \frac{\varepsilon}{10^2 n'}\Big)^{n'} \big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'} = \Big(1 \pm \frac{\varepsilon}{20}\Big) \big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'}. \tag{34}$$

Now observe that permissible phase vectors Y are in one-to-one correspondence with independent sets of G': a vertex $u \in L(G')$ is in the independent set if $Y_u = (F_L, V_R)$; a vertex $v \in R(G')$ is in the independent set if $Y_v = (V_L, F_R)$. It follows that the total number of homomorphisms with permissible phase vectors is

$$(1 \pm \varepsilon/20)\big(M_K^{Qn^3} M_\Gamma^{Qn^2}\big)^{n'} \#\mathrm{BIS}(G'). \tag{35}$$

Since $\#\mathrm{BIS}(G') \ge 1$, combining (33) and (35) yields (29), as wanted. This completes the proof of Lemma 20. □

We are now ready to prove Lemma 4.

Lemma 4. *Let H be a 2-coloured graph which is full but not trivial. Then* #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

Proof. We will prove the lemma using induction on the number $|V(H)|$ of vertices of H. Since H is full but not trivial, the base case is the case in which H is a path on four vertices. In this case, it is known that #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col (see for example [1]).

We now do the inductive case. Roughly, we will show that we can choose Γ so that either the assumptions of Lemma 19 or Lemma 20 hold.

Consider the set of maximal bicliques $\mathcal{C}_{\alpha,\beta}$ where $0 < \alpha, \beta < 1$ are as in (2). For the rest of the proof we may assume that $\mathcal{C}_{\alpha,\beta}$ includes both extremal bicliques and at least one non-extremal biclique. Otherwise, observe that if an extremal biclique is not in $\mathcal{C}_{\alpha,\beta}$, then $\mathcal{C}_{\alpha,\beta}$ does not in fact include either of the extremal bicliques (this is enforced by (2), i.e., the selection of α, β). It then follows by Lemma 19 (where Γ is the empty graph) that there is a full but not trivial graph H' such that $|V(H')| < |V(H)|$ and #FixedH'-Col $\leq_{\mathsf{AP}}$ #Fixed$H$-Col. By induction, we have that #BIS $\leq_{\mathsf{AP}}$ #FixedH'-Col, so we obtain that #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col. Similarly, if $\mathcal{C}_{\alpha,\beta}$ includes both extremal bicliques but no other biclique, by Lemma 20 (where Γ is the empty graph), we have that #BIS $\leq_{\mathsf{AP}}$ #Fixed$H$-Col.

Let

$$(S_L^{(1)}, S_R^{(1)}), \ldots, (S_L^{(t)}, S_R^{(t)})$$

be an enumeration of the non-extremal bicliques in $\mathcal{C}_{\alpha,\beta}$. Recall that all elements of $\mathcal{C}_{\alpha,\beta}$ are maximal bicliques of H. Recall the definition of $\zeta_1^{\text{ex}}(\Gamma)$, and $\zeta_2^{\text{ex}}(\Gamma)$ from Definition 12. We consider three cases, which correspond (in order) to the 3 cases in the overview.

Case I. There exists $i \in [t]$ and a 2-coloured graph Γ such that

$$\ln \frac{|V_R|}{|F_R|} \ln \frac{\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma\big)}{\zeta_1^{\text{ex}}(\Gamma)} > \ln \frac{|V_R|}{\big|S_R^{(i)}\big|} \ln \frac{\zeta_2^{\text{ex}}(\Gamma)}{\zeta_1^{\text{ex}}(\Gamma)}. \tag{36}$$

We claim that the extremal bicliques are not in $\mathcal{C}_{\alpha,\beta}^{\Gamma}$. To see this, from (5), we have that $\gamma(\Gamma) = \ln \frac{\zeta_2^{\text{ex}}(\Gamma)}{\zeta_1^{\text{ex}}(\Gamma)} / \ln \frac{|V_R|}{|F_R|}$, so (36) yields that $\ln \frac{\zeta\left(S_L^{(i)}, S_R^{(i)}, \Gamma\right)}{\zeta_1^{\text{ex}}(\Gamma)} / \ln \frac{|V_R|}{\left|S_R^{(i)}\right|} > \gamma(\Gamma)$ which can be massaged into

$$\zeta(S_L^{(i)}, S_R^{(i)}, \Gamma)\, |S_R^{(i)}|^{\gamma(\Gamma)} > \zeta_1^{\text{ex}}(\Gamma) |V_R|^{\gamma(\Gamma)} = \zeta_2^{\text{ex}}(\Gamma)\, |F_R|^{\gamma(\Gamma)}, \tag{37}$$

where in the latter equality we used the definition of $\gamma(\Gamma)$, see (5). From (37), we obtain that the extremal bicliques are not in $\mathcal{C}_{\alpha,\beta}^{\Gamma}$, as wanted.

To apply Lemma 19 using the graph Γ, we need to ensure that Γ has no isolated vertices in $R(\Gamma)$ (see also footnote 6). Let I be the set of isolated vertices in $R(\Gamma)$ and let Γ' be the 2-coloured graph obtained by removing the set I from Γ (and which otherwise inherits the 2-colouring of Γ), so that Γ' has no isolated vertices in $R(\Gamma')$. From Definition 11, we have that $\zeta(S_L^{(i)}, S_R^{(i)}, \Gamma) =$ #Fixed$H_{S_L^{(i)}, S_R^{(i)}}$-Col(Γ), where $H_{S_L^{(i)}, S_R^{(i)}} = H[S_L^{(i)} \cup V_R]$ since $(S_L^{(i)}, S_R^{(i)})$ is a maximal biclique, see also Definition 10 for details. The observation which is important for us is that $R(H_{S_L^{(i)}, S_R^{(i)}}) = V_R$. We thus have

$$\zeta(S_L^{(i)}, S_R^{(i)}, \Gamma) = |V_R|^{|I|} \zeta(S_L^{(i)}, S_R^{(i)}, \Gamma'),$$

and analogously (observing that $R(H_{S_L, S_R}) = V_R$ for every maximal biclique and hence also for the extremal ones)

$$\zeta_1^{\text{ex}}(\Gamma) = |V_R|^{|I|}\, \zeta_1^{\text{ex}}(\Gamma'), \quad \zeta_2^{\text{ex}}(\Gamma) = |V_R|^{|I|}\, \zeta_2^{\text{ex}}(\Gamma').$$

It follows that (36) also holds if we replace Γ with Γ'.

From the argument above, we may assume w.l.o.g. that Γ has no isolated vertices. From Lemma 19, it then follows that there is a full but not trivial graph H' such that $|V(H')| < |V(H)|$ and $\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$. By induction, we have that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H'\text{-}\mathsf{Col}$, so we obtain that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$. This completes the argument in the first case.

Case II. There exists $i \in [t]$ such that for all 2-coloured graphs Γ it holds that

$$\ln \frac{|V_R|}{|F_R|} \ln \frac{\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma\big)}{\zeta_1^{\mathrm{ex}}(\Gamma)} = \ln \frac{|V_R|}{|S_R^{(i)}|} \ln \frac{\zeta_2^{\mathrm{ex}}(\Gamma)}{\zeta_1^{\mathrm{ex}}(\Gamma)}. \tag{38}$$

We will show that $\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$ where $H' = H_{S_L^{(i)}, S_R^{(i)}} = H[S_L^{(i)} \cup V_R]$. To see this, set $c_1 := \ln \frac{|V_R|}{|F_R|}$, $c_2 := \ln \frac{|V_R|}{|S_R^{(i)}|}$ and $c := c_2/c_1$ (note that $0 < c < 1$). Then, rewriting (38), for every 2-coloured graph Γ it holds that

$$\zeta(S_L^{(i)}, S_R^{(i)}, \Gamma) = \frac{\big(\zeta_2^{\mathrm{ex}}(\Gamma)\big)^c}{(\zeta_1^{\mathrm{ex}}(\Gamma))^{c-1}}, \text{ so that } \#\mathsf{Fixed}H'\text{-}\mathsf{Col}(\Gamma) = \frac{\big(\#\mathsf{Fixed}H\text{-}\mathsf{Col}(\Gamma)\big)^c}{|F_L|^{(c-1)|L(\Gamma)|}|V_R|^{(c-1)|R(\Gamma)|}}, \tag{39}$$

cf. Definitions 10, 11 and 12. Equation (39) trivially implies that $\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$. By Lemma 8, we have that H' is full but not trivial and further satisfies $|V(H')| < |V(H)|$, so by induction we have that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H'\text{-}\mathsf{Col}$. It follows that $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$.

Case III. For all $i \in [t]$, for all 2-coloured graphs Γ, it holds that

$$\ln \frac{|V_R|}{|F_R|} \ln \frac{\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma\big)}{\zeta_1^{\mathrm{ex}}(\Gamma)} \leq \ln \frac{|V_R|}{|S_R^{(i)}|} \ln \frac{\zeta_2^{\mathrm{ex}}(\Gamma)}{\zeta_1^{\mathrm{ex}}(\Gamma)}. \tag{40}$$

and, for all $i \in [t]$, the inequality in (40) is strict for at least one graph Γ_i. Using an identical argument to the one in Case I, we may assume that Γ_i has no isolated vertices in $R(\Gamma_i)$.

Let Γ^* be the disjoint union of the graphs Γ_i for $i = 1, \ldots, t$ (the 2-colouring of Γ^* is induced in the natural way by the 2-colourings of its components). For every $i \in [t]$, we have

$$\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma^*\big) = \prod_{j=1}^{t} \zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma_j\big), \quad \zeta_1^{\mathrm{ex}}(\Gamma^*) = \prod_{j=1}^{t} \zeta_1^{\mathrm{ex}}\big(\Gamma_j\big), \quad \zeta_2^{\mathrm{ex}}(\Gamma^*) = \prod_{j=1}^{t} \zeta_2^{\mathrm{ex}}\big(\Gamma_j\big). \tag{41}$$

Further, for every $i \in [t]$, from (40) and the choice of Γ_i, we have

$$\ln \frac{|V_R|}{|F_R|} \sum_{j=1}^{t} \ln \frac{\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma_j\big)}{\zeta_1^{\mathrm{ex}}(\Gamma_j)} < \ln \frac{|V_R|}{|S_R^{(i)}|} \sum_{j=1}^{t} \ln \frac{\zeta_2^{\mathrm{ex}}(\Gamma_j)}{\zeta_1^{\mathrm{ex}}(\Gamma_j)}. \tag{42}$$

Combining (41) and (42) we obtain

$$\ln \frac{|V_R|}{|F_R|} \ln \frac{\zeta\big(S_L^{(i)}, S_R^{(i)}, \Gamma^*\big)}{\zeta_1^{\mathrm{ex}}(\Gamma^*)} < \ln \frac{|V_R|}{|S_R^{(i)}|} \ln \frac{\zeta_2^{\mathrm{ex}}(\Gamma^*)}{\zeta_1^{\mathrm{ex}}(\Gamma^*)}. \tag{43}$$

By essentially the same calculation as in case I, inequality (43) implies that $\mathcal{C}_{\alpha,\beta}^{\Gamma^*}$ contains only the extremal bicliques; to see this note that for every $i \in [t]$, we have that the inequality (36) is reversed and thus so is the inequality in (37).

Since $\mathcal{C}_{\alpha,\beta}^{\Gamma^*}$ contains only the extremal bicliques and Γ^* has no isolated vertices in $R(\Gamma^*)$ (by construction), Lemma 20 yields $\#\mathrm{BIS} \leq_{\mathsf{AP}} \#\mathsf{Fixed}H\text{-}\mathsf{Col}$, which finishes the proof of Lemma 4. □

9 Proof of Lemma 3

We follow as closely as possible [7, Section 6]. The reduction is similar to the one in [7, Proof of Lemma 6], yet to account for the counting setting (rather than the sampling setting in [7]), we will have to deviate at a certain point.

Let H be a graph with no trivial components (note that in this section H is allowed to have self-loops but not multiple edges). Following [7], let Δ_1 be the maximum degree [8] of H and let Δ_2 be the maximum degree amongst *the neighbours of maximum degree vertices*. Formally,

$$\Delta_1 = \max_{u \in V(H)} \deg_H(u), \quad \Delta_2 = \max_{\substack{v \in V(H); \\ v \in N_\cup(u),\, \deg(u) = \Delta_1}} \deg_H(v). \tag{44}$$

As in [7], we also consider the bipartite double cover of H, which will be denoted by $\mathsf{Bip}(H)$. Formally, $\mathsf{Bip}(H)$ is a bipartite graph with vertex set $V(H) \times \{1,2\}$ (the 2-colouring of $\mathsf{Bip}(H)$ is given by the natural projection of the set $V(H) \times \{1,2\}$ onto its second component). For convenience, for $u \in V(H)$, we will denote by u_1, u_2 the two copies of the vertex u in $V(\mathsf{Bip}(H))$. The edge set of $\mathsf{Bip}(H)$ is then given by

$$E(\mathsf{Bip}(H)) = \big\{(u_1, v_2) \,|\, (u,v) \in E(H)\big\} \cup \big\{(v_1, u_2) \,|\, (u,v) \in E(H)\big\}.$$

In particular, note that for every edge (u,v) in H which is not a self-loop, there are two edges (u_1, v_2) and (v_1, u_2) in $\mathsf{Bip}(H)$, while for every self-loop (u,u) in H, there is a (single) edge (u_1, u_2) in $\mathsf{Bip}(H)$. Note that $\mathsf{Bip}(H)$ is a bipartite graph without self-loops. Also, since H has no trivial components, it is simple to see that $\mathsf{Bip}(H)$ has no trivial components as well.

For an edge $(u,v) \in E(H)$, let $H_{u,v}$ be the subgraph of $\mathsf{Bip}(H)$ induced by the vertex set $N'_\cup(u_1) \cup N'_\cup(v_2)$ where $N'_\cup(u_1), N'_\cup(v_2)$ are the neighbourhoods of u_1, v_2, respectively, in $\mathsf{Bip}(H)$. Note that the graph $H_{u,v}$ is bipartite with a 2-colouring which is naturally induced by the 2-colouring of $\mathsf{Bip}(H)$, i.e., $L(H_{u,v}) = N'_\cup(v_2)$ and $R(H_{u,v}) = N'_\cup(u_1)$. Further, $H_{u,v}$ is full since u_1 is connected to $N'_\cup(u_1)$ and v_2 is connected to $N'_\cup(v_2)$.

Let

$$\Lambda(H) := \{(u,v) \,|\, (u,v) \in E(H),\, \deg_H(u) = \Delta_1,\, \deg_H(v) = \Delta_2\}.$$

The following lemma is proved in [7].

Lemma 21. *[7, Lemma 6] Let H be a graph with no trivial components and let $(u,v) \in \Lambda(H)$. Then, $H_{u,v}$ is a full but not trivial graph.*

With this setup, we are now ready to prove Lemma 3.

Lemma 3. *Let H be a graph with no trivial components. Then, there exists a 2-coloured graph H', which is full but not trivial, such that*

$$\#\mathsf{Fixed}H'\text{-}\mathsf{Col} \leq_{\mathsf{AP}} \#H\text{-}\mathsf{Col}.$$

Proof. Let $\mathcal{H}$ be the multiset consisting of the (labelled) graphs $H_{u,v}$ when (u,v) ranges over elements of $\Lambda(H)$. (Note that there might be multiple elements of $\Lambda(H)$ which map to the same graph.) Further, let $\mathcal{H}_1, \ldots, \mathcal{H}_k$ be the equivalence classes of $\mathcal{H}$ under the (colour-preserving isomorphism) relation $\cong_c$ which was formally defined in Section 4. For $i \in [k]$, let H_i be a representantive of the class $\mathcal{H}_i$. By Lemma 6, there exists a gadget J and $i \in [k]$ such that for every $t \neq i$, it holds that

[8] Recall that a self-loop (u,u) contributes one to the degree of the vertex u.

$\#\mathsf{Fixed}H_i\text{-}\mathsf{Col}(J) > \#\mathsf{Fixed}H_t\text{-}\mathsf{Col}(J)$. We will show that $H' = H_i$ satisfies the statement of the lemma. For future use, let $\Lambda^*(H) \subseteq \Lambda(H)$ be the subset of elements (u, v) whose corresponding graph $H_{u,v}$ is isomorphic to H' under a colour-preserving isomorphism, that is,

$$\Lambda^*(H) := \big\{(u, v) \in \Lambda(H) \,|\, H_{u,v} \cong_c H'\big\}.$$

Let $0 < \varepsilon < 1$ and G' be an input to $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}$. Our goal is to approximate $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G')$ within a multiplicative factor of $(1 \pm \varepsilon)$ using an oracle for $\#H\text{-}\mathsf{Col}$. Suppose that $n' = |V(G')|$ and let $n = \lceil \kappa n'/\varepsilon \rceil$ where κ is a sufficiently large constant depending only on H.

To construct an instance G of $\#H\text{-}\mathsf{Col}$, consider first the (independent) sets of vertices A and B with

$$|A| = n^4, |B| = n^3.$$

Let $\overline{G}$ be the disjoint union of G', n^2 copies of the graph J, the independent sets A, B and two "special" vertices w_A and w_B. We denote by J_r the r-th copy of J. To obtain G, attach on $\overline{G}$ the edge (w_A, w_B), all edges in $w_A \times A$, $w_B \times B$, $w_A \times R(G')$, $w_B \times L(G')$ and for $r = 1, \ldots, n^2$ all edges in $w_A \times R(J_r)$, $w_B \times L(J_r)$. See Figure 8.

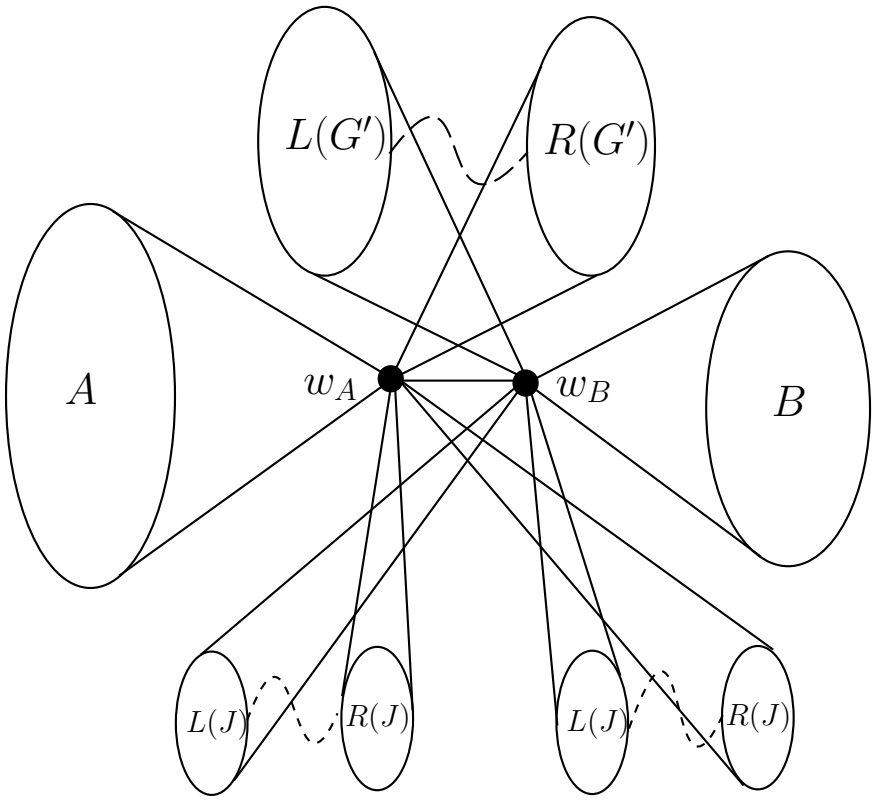


Figure 8: The graph G

Let Σ be the set of homomorphisms from G to H. Note that $|\Sigma| = \#H\text{-}\mathsf{Col}(G)$. We will show that

$$|\Sigma| = (1 \pm \varepsilon/10)\lambda^* \Delta_1^{n^4} \Delta_2^{n^3} (\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(J))^{n^2} \#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G'), \tag{45}$$

where $\lambda^* := |\Lambda^*(H)|$ and, recall, Δ_1, Δ_2 are as in (44). Note that $\Delta_1, \Delta_2, \lambda^*$ are constants and J is a fixed graph. It follows that an approximation of $|\Sigma|$ within a multiplicative factor of $1 \pm \varepsilon/2$ yields an approximation of $\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G')$ within a multiplicative factor of $1 \pm \varepsilon$.

Thus, it remains to show (45). Let h be an arbitrary homomorphism in Σ. Since $(w_A, w_B) \in E(G)$, we have that $(h(w_A), h(w_B)) \in E(H)$. For vertices $u, v \in V(H)$, let $\Sigma(u,v)$ be the subset of Σ such that $h(w_A) = u$ and $h(w_B) = v$. Formally,

$$\Sigma(u,v) := \{h \in \Sigma \mid h(w_A) = u,\, h(w_B) = v\}.$$

We have

$$|\Sigma| = \sum_{\substack{u,v \in V(H);\\ (u,v) \in E(H)}} |\Sigma(u,v)|. \tag{46}$$

Let $h \in \Sigma(u,v)$. Then, the edges $w_A \times A, w_A \times R(G')$ enforce that $h(A), h(R(G')) \subseteq N_\cup(u)$ (the neighbourhood of u in H) and similarly $h(B), h(L(G')) \subseteq N_\cup(v)$. Also, $h(R(J_r)) \subseteq N_\cup(u)$ and $h(L(J_r)) \subseteq N_\cup(v)$. Now observe that a homomorphism from the bipartite graph G to H corresponds to a colour-preserving homomorphism from G to $\mathsf{Bip}(H)$. Combining with the previous remarks, it is not hard to see that

$$|\Sigma(u,v)| = (\deg_H(u))^{|A|}(\deg_H(v))^{|B|}(\#\mathsf{Fixed}H_{u,v}\text{-}\mathsf{Col}(J))^{n^2}\#\mathsf{Fixed}H_{u,v}\text{-}\mathsf{Col}(G').$$

Note that for $(u,v) \in \Lambda^*(H)$, we have

$$|\Sigma(u,v)| = W, \text{ where } W := \Delta_1^{n^4}\Delta_2^{n^3}(\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(J))^{n^2}\#\mathsf{Fixed}H'\text{-}\mathsf{Col}(G'). \tag{47}$$

For the following estimates, observe that for every $(u,v) \in E(H)$, it holds that

$$\#\mathsf{Fixed}H_{u,v}\text{-}\mathsf{Col}(G') \leq (2|V(H)|)^{n'} \leq (2|V(H)|)^n.$$

For $u \in H$ such that $\deg_H(u) < \Delta_1$, we have for all $v \in H$ that

$$|\Sigma(u,v)| \leq \exp(-\Omega(n^4))W. \tag{48}$$

For u, v such that $\deg_H(u) = \Delta_1$ but $\deg_H(v) < \Delta_2$, we have that

$$|\Sigma(u,v)| \leq \exp(-\Omega(n^3))W. \tag{49}$$

For u, v such that $\deg_H(u) = \Delta_1$, $\deg_H(u) = \Delta_2$ but $(u,v) \notin \Lambda^*(H)$, by the choice of J we have $\#\mathsf{Fixed}H_{u,v}\text{-}\mathsf{Col}(J) < \#\mathsf{Fixed}H'\text{-}\mathsf{Col}(J)$ and hence

$$|\Sigma(u,v)| \leq \exp(-\Omega(n^2))W. \tag{50}$$

Combining the estimates (48), (49), (50) and letting κ to be a sufficiently large constant, we have that for $(u,v) \notin \Lambda^*(H)$ it holds that

$$|\Sigma(u,v)| \leq \varepsilon W/(10|E(H)|). \tag{51}$$

Plugging (47) and (51) back into (46) yields (45), as wanted. This completes the proof of Lemma 3. □

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