Obstructions to arithmetic properties of rational points on varieties



Francesca Balestrieri

Lady Margaret Hall

University of Oxford

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Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. The content of Chapter 3 is taken from [Bal16], while the content of Chapter 4 has been submitted for publication. This thesis has not been submitted for a degree at another university.

Francesca Balestrieri
Oxford, 12 May 2017

A codalunga.

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Abstract

The aim of this thesis is to develop further the theory of obstruction sets, which is used in the study of the arithmetic behaviour of rational points on nice varieties over number fields.

Let X be a nice variety over a number field k. In the first part of this thesis, we characterise in pure descent-type terms some inequivalent obstruction sets refining the inclusion $X(\mathbf{A}_k)^{\text{ét Br}} \subset X(\mathbf{A}_k)^{\text{Br}_1}$. More precisely, we apply ideas from papers by Skorobogatov and Demarche to new cases, by proving a comparison theorem for obstruction sets. We then show that descent under unipotent linear algebraic groups over k does not give any obstruction at all. This allow us, among other things, to characterise the étale-Brauer set in terms of descent under (non-necessarily connected) reductive linear algebraic groups and to characterise descent under solvable linear algebraic groups in terms of a Brauer-type set.

The second main part is motivated by the need of finding new obstruction sets, as the current finest one − the étale-Brauer set − is not sufficiently small to explain all the violations of the Hasse principle (there is a counterexample by Poonen for which there are no rational points but the étale-Brauer set is non-empty). We iterate the algebraic étale-Brauer set for any nice variety X over a number field k with $\pi_1^{\text{ét}}(\overline{X})$ finite and we show that the iterated set coincides with the original algebraic étale-Brauer set. This provides some evidence towards the conjectures by Colliot-Thélène on the arithmetic of rational points on nice geometrically rationally connected varieties over k and by Skorobogatov on the arithmetic of rational points on K3 surfaces over k; moreover, it gives a partial answer to an algebraic analogue of a question by Poonen about iterating the descent set.

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Chapter 1

Introduction

Throughout this thesis, a nice variety over a field K is a smooth, projective, geometrically integral variety over K.

1.1 General introduction

The general topic of this thesis is a classical one in arithmetic geometry, namely, the study of arithmetic properties of rational points on varieties over number fields via obstruction sets.

Throughout this thesis, we let k be a number field and $\overline{k} \subset \mathbf{C}$ be a fixed algebraic closure of k (with a fixed embedding into $\mathbf{C}$). Let $\mathbf{Sch}_k$ denote the category of schemes over $\operatorname{Spec} k$ ("over k", for short).

Definition 1.1.1. Let $X \in Obj(\mathbf{Sch}_k)$. Let K be any k-algebra. The set of K-rational points of X is defined by $X(K) := \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} K, X)$.

The set $X(k)$ will be, implicitly, our main object of interest: throughout, the underlying aim is to study some of the arithmetic properties of $X(k)$, and we will focus in particular on two qualitative properties: existence and density properties. We will not, in this thesis, discuss some of the more refined and

quantitative arithmetic behaviours of the set of rational points. Moreover, our tools will mainly be geometric; similar questions as the ones posed in this introduction could, perhaps, be tackled with other, more analytical tools as well (see [Bro15] for some examples), but we will not do so here.

Existence properties: the Hasse principle. Let X be a variety over k (that is, in this thesis, a separated scheme of finite type over k). Throughout this section, we will take X to be also smooth and geometrically integral over k. The most naive qualitative question we can ask about $X(k)$ is the following.

Question 1.1.2. Is $X(k) = \emptyset$?

Despite its apparent simplicity, this question is already extremely hard to tackle in full generality: it is deeply connected to Hilbert's tenth problem and thus it is not even clear whether the problem of determining in finite time the emptiness of $X(k)$ is a decidable problem in general.

Let Ω_k denote the set of places of k and let k_v denote the completion of k at the place $v \in \Omega_k$. Let $\mathcal{X}$ be a separated model for X of finite type over the ring of integers $\mathcal{O}_k$ (that is, a separated scheme of finite type over $\mathrm{Spec}(\mathcal{O}_k)$ with generic fibre X). The set of adelic points on X is defined by

$$X(\mathbf{A}_k) := \varinjlim_{S} \left[\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_{k_v}) \right],$$

where the limit is over all finite subsets $S \subset \Omega_k$ containing the infinite places, ordered by inclusion; the definition of the set of adelic points does not depend on the choice of the model $\mathcal{X}$, as, if $\mathcal{X}'$ is another separated model for X of finite type over $\mathcal{O}_k$, then $\mathcal{X}(\mathcal{O}_{k_v}) = \mathcal{X}'(\mathcal{O}_{k_v})$ for all but finitely many places. We can put a topology on $X(\mathbf{A}_k)$, namely the adelic topology, with an open

basis given by sets of the form

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v),$$

where S ranges over all finite subsets of Ω_k containing the infinite places and where $U_v \subset X(k_v)$ is open in the local topology for $v \in S$. We refer the reader to [Con12] for a more detailed exposition.

Remark 1.1.3. Checking whether $X(\mathbf{A}_k)$ is empty or not can be effectively in a finite number of steps by using a combination of the Weil conjectures and of Hensel's lifting lemma.

A sensible way to approach Question 1.1.2 is by establishing appropriate "local-to-global principles", that is, to relate the arithmetic behaviour of $X(k)$ to that of $X(\mathbf{A}_k)$. Indeed, there is a diagonal embedding

$$X(k) \hookrightarrow X(\mathbf{A}_k),$$

meaning that $X(\mathbf{A}_k) = \emptyset$ implies $X(k) = \emptyset$.

Definition 1.1.4. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k. We say that $\{X_\omega\}_\omega$ satisfies the Hasse principle (or local-to-global principle) if, for all $X \in \{X_\omega\}_\omega$, we have that $X(k) = \emptyset$ if and only if $X(\mathbf{A}_k) = \emptyset$.

Remark 1.1.5. In other words, when the Hasse principle holds, the set of adelic points $X(\mathbf{A}_k)$ is a perfect approximation for the set $X(k)$ with respect to Question 1.1.2.

Example 1.1.6. Smooth projective quadrics over a number field k satisfy the Hasse principle; this is the content of the famous Hasse-Minkowski theorem. There are other interesting families of varieties which satisfy the Hasse principle − an incomplete list can be found in [Sko01, §5.1].

Density properties: weak and strong approximation. As mentioned, we will be interested not only in existence properties, but also in density properties of rational points. Let $X(k_\Omega) := \prod_{v\in\Omega_k} X(k_v)$. There are natural topologies on the sets $X(\mathbf{A}_k)$ and $X(k_\Omega)$: on the first set, we have the adelic topology, while on $X(k_\Omega)$, we can put the standard product topology. We remark that when X is proper (for example, when X is projective), we have $X(\mathbf{A}_k) = X(k_\Omega)$ and the adelic and product topologies coincide. (In most of this thesis, our varieties will be proper, but not always.) In general, it makes thus sense to ask the following.

Question 1.1.7. Is $X(k)$ is dense, with respect to the appropriate topology, in $X(\mathbf{A}_k)$ or $X(k_\Omega)$?

Definition 1.1.8. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k. We say that $\{X_\omega\}_\omega$ satisfies strong approximation if $\overline{X(k)} = X(\mathbf{A}_k)$ (i.e. the image of the diagonal embedding $X(k) \hookrightarrow X(\mathbf{A}_k)$ is dense with respect to the adelic topology) for all $X \in \{X_\omega\}_\omega$.

Definition 1.1.9. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k. We say that $\{X_\omega\}_\omega$ satisfies weak approximation if $\overline{X(k)} = X(k_\Omega)$ (i.e. the image of the diagonal embedding $X(k) \hookrightarrow X(k_\Omega)$ is dense with respect to the product topology) for all $X \in \{X_\omega\}_\omega$.

Example 1.1.10. The family of k-rational varieties (i.e. varieties birationally equivalent to $\mathbf{P}^n_k$ for some $n \in \mathbf{N}$) satiesfies weak approximation (cf. [Sko01, Thm 5.1.2]). The family of smooth, projective, geometrically integral intersections of two quadrics with a k-point in $\mathbf{P}^n_k$ satisfies strong approximation (equivalently, weak approximation) whenever $n \geq 6$. A theorem by Minchev (cf. [Min89]) tells us that strong approximation can never hold for varieties with non-trivial geometric étale fundamental group.

Remark 1.1.11. From the definitions, we get the following chain of implications: strong approximation $\Rightarrow$ weak approximation $\Rightarrow$ Hasse principle.

Obstruction sets. So far, we have seen that the sets $X(\mathbf{A}_k)$ and $X(k_\Omega)$ are sensible first objects to study in order to understand some of the arithmetic properties of $X(k)$. In general, however, these sets are too big to capture whichever arithmetic property of $X(k)$ we are trying to study. Indeed, the literature abounds of violations of the Hasse principle, i.e. cases in which $X(k) = \emptyset$ but $X(\mathbf{A}_k) \neq \emptyset$, violations of weak approximation, i.e. cases in which $\overline{X(k)} \subsetneq X(k_\Omega)$, and violations of strong approximation, i.e. cases in which $\overline{X(k)} \subsetneq X(\mathbf{A}_k)$; an example of a violation of the Hasse principle (and, thus, of weak and strong approximation) can be found in [BBM$^+$16]. Hence, we need better approximations of $X(k)$. The concept of "obstruction sets" is introduced to deal with this problem: an obstruction set S is a "useful" set such that

$$X(k) \subset S,$$

where the "usefulness" is loosely taken to mean the following: S should depend at least on X in a good − e.g. functorial − way, and S should be such that we retain some of the advantages of working with local objects. The idea, then, is to use obstruction sets to obtain refinements of the Hasse principle and of weak and strong approximation.

Definition 1.1.12. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k and let $S := \{S_\omega \subset X_\omega(\mathbf{A}_k)\}_\omega$ be a family of obstruction sets. We say that the S-Hasse principle holds for the family $\{X_\omega\}_\omega$ if $S_\omega \neq \emptyset$ implies that $X_\omega(k) \neq \emptyset$, for all ω (i.e. if the Hasse principle holds with each "$X_\omega(\mathbf{A}_k)$" replaced by "S_ω").

Definition 1.1.13. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k and let $S := \{S_\omega \subset X_\omega(\mathbf{A}_k)\}_\omega$ be a family of obstruction sets. We say that S-strong approximation holds for the family $\{X_\omega\}_\omega$ if $\overline{X_\omega(k)} = S_\omega$, for all ω (i.e. if strong approximation holds with each "$X_\omega(\mathbf{A}_k)$" replaced by "S_ω").

Definition 1.1.14. Let $\{X_\omega\}_\omega$ be a family of varieties over a number field k and let $S := \{S_\omega \subset X_\omega(k_\Omega)\}_\omega$ be a family of obstruction sets. We say that S-weak approximation holds for the family $\{X_\omega\}_\omega$ if $\overline{X_\omega(k)} = S_\omega$, for all ω (i.e. if weak approximation holds with each "$X_\omega(k_\Omega)$" replaced by "S_ω").

In an analogous way as before, given an obstruction set S we can talk of an S-violation of the Hasse principle if $X(k) = \emptyset$ but $S \neq \emptyset$, or of an S-violation of weak (respectively, strong) approximation if $\overline{X(k)} \subset S \subsetneq X(k_\Omega)$ (respectively, if $\overline{X(k)} \subset S \subsetneq X(\mathbf{A}_k)$). Hence, if we are interested in any of the aforementioned arithmetic properties of the rational points of a family of varieties $\{X_\omega\}_\omega$ over k, then S should ideally give us a measure of how far our family of varieties is from satisfying the wanted property.

Remark 1.1.15. With these definitions in place, we can also think of using obstruction sets to classifiy families of varieties based on the arithmetic behaviour of their rational points. Unfortunately, we currently do not know much about this classification. For example, for curves, the few available results are either conditional on important conjectures (see, for example, Manin's result on the behaviour of rational points on genus 1 curves [Sko01, p. 114], conditional on the finiteness of the Tate-Shafarevich group of the Jacobian) or of a heuristic nature (see [Poo05]); for higher dimensional varieties, we know even less − even for the 'simplest' surfaces, namely geometrically rational surfaces (i.e. surfaces which, over $\overline{k}$, become birationally equivalent to $\mathbf{P}^2_{\overline{k}}$), we only have conjectures about the arithmetic behaviour of rational points. We will come

back to these and other conjectures in Chapter 4.

1.2 Aim of this thesis

The main aim of this thesis is to develop the theory of obstruction sets further. We will focus on two aspects: comparing obstruction sets and iterating obstruction sets.

1.2.1 Comparing obstruction sets

A general way to construct useful obstruction sets is the following (for more details, see Chapter 2): we start with a variety X over k and a functor $\mathcal{F} : \mathbf{Sch}_k^{op} \to \mathbf{Sets}$ from the opposite category of schemes over k to the category of sets; for any k-algebra L and for any $\alpha \in \mathcal{F}(X)$, pulling back by points in $X(L)$ yields an evaluation map $\mathrm{ev}_\alpha : X(L) \to \mathcal{F}(L)$; by considering the commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbf{A}_k) \\ {\scriptstyle \mathrm{ev}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\alpha} \\ \mathcal{F}(k) & \longrightarrow & \mathcal{F}(\mathbf{A}_k) \end{array} \tag{1.2.1}$$

for each $\alpha \in \mathcal{F}(X)$, we can define the obstruction set

$$X(\mathbf{A}_k)^{\mathcal{F}} := \bigcap_{\alpha \in \mathcal{F}(X)} \{(x_v) \in X(\mathbf{A}_k) : \mathrm{ev}_\alpha((x_v)) \in \mathrm{im}(\mathcal{F}(k) \to \mathcal{F}(\mathbf{A}_k))\}.$$

Historically, Manin ([Man71]) was the first to exploit this construction by noting that, thanks to class field theory, the choice $\mathcal{F} = \mathrm{Br}$ (the Brauer group functor) was a useful one; his work started a systematic and fruitful approach to the study of rational points on varieties. There are other interesting choices for $\mathcal{F}$: for example, if G is a linear algebraic group over k, one can define

obstruction sets based on the functor $\mathcal{F} = H^1_{\text{ét}}(-, G)$ (see [CTS80] and [CTS87] for the case when G is abelian, and [HS02] for the general case). Combinations of the above two constructions have also been used, especially in more recent works (e.g. [Sko99], [Poo10], [CTPS16]).

It is not clear, in general, whether and how obstruction sets coming from different choices of $\mathcal{F}$ are related. Hence, the first aspect that we propose to study in this thesis concerns the interrelation between differently defined obstruction sets, that is, obstruction sets coming from functors (usually related to cohomological objects) which, *a priori*, might have no connection to each other. Specifically, if X is a smooth, projective, geometrically integral variety over a number field k, we will relate obstruction sets associated to subgroups of the Brauer group $\operatorname{Br} X$ to obstruction sets associated to the first cohomology set $H^1_{\text{ét}}(X, G)$ classifying torsors over X under certain linear algebraic groups G over k. Having a dictionary between obstruction sets is useful for a multitude of reasons: for instance, sometimes a characterisation of a same obstruction set might be considerably easier to use in practice than others; moreover, as we will see below, characterisations of obstruction sets which are purely in terms of torsors are important when trying to use techniques to construct potentially finer obstruction sets from old ones.

1.2.2 Iterating obstruction sets

The second aspect concerns the need for finding a general way to construct new, finer obstruction sets. The problem of constructing finer obstruction sets is a pressing one, as currently there are many counterexamples to the Hasse principle which are not explained by the available obstruction sets. To be more precise, if X is a smooth and geometrically integral variety over a number field

k, we classically have the following chain of (sometimes strict) containments:

$$X(k) \subset X(\mathbf{A}_k)^{\text{ét Br}} \subset X(\mathbf{A}_k)^{\text{Br}} \subset X(\mathbf{A}_k),$$

where the obstruction sets $X(\mathbf{A}_k)^{\text{ét Br}}$ and $X(\mathbf{A}_k)^{\text{Br}}$ (called, respectively, the *étale-Brauer set* and the *Brauer-Manin set*) will be defined in Chapter 2; the obstruction set $X(\mathbf{A}_k)^{\text{ét Br}}$ is currently the finest general obstruction set at our disposal. The problem is that the étale-Brauer set is not enough to explain all the violations of the Hasse principle, as the following example illustrates.

Example 1.2.1. In [Poo10], Poonen constructs a smooth, projective, geometrically integral variety X over any number field k such that $X(\mathbf{A}_k)^{\text{ét Br}} \neq \emptyset$ and $X(k) = \emptyset$. We remark that in his unpublished paper [Sch], Schlank gives an obstruction set finer than the étale-Brauer set which explains the counterexample by Poonen, by considering a more relaxed notion of "torsors" and by considering descent for open varieties. However, Schlank's construction is quite *ad hoc* and not straightforwardly generalisable; in this thesis, we are more concerned with the task of finding *general* ways to construct obstruction sets finer than the étale-Brauer set.

One direction to construct potentially new obstruction sets has been recently explored by Harpaz and Schlank in their technical work [HS13]: by using the étale homotopy type defined by Artin and Mazur, Harpaz and Schlank construct the following homotopy obstruction sets (for the notation, see [HS13]):

$$\begin{array}{ccccccccc} & & X(\mathbf{A}_k)^{\mathbf{Z}^h} & \longrightarrow \cdots \longrightarrow & X(\mathbf{A}_k)^{\mathbf{Z}^{h,1}} & \longrightarrow & X(\mathbf{A}_k)^{\mathbf{Z}^{h,0}} & \longrightarrow & X(\mathbf{A}_k) \\ & & \uparrow & & \uparrow & & \uparrow & & \\ X(k) & \longrightarrow & X(\mathbf{A}_k)^{h} & \longrightarrow \cdots \longrightarrow & X(\mathbf{A}_k)^{h,1} & \longrightarrow & X(\mathbf{A}_k)^{h,0} & & \end{array}$$

where all the arrows are inclusions of sets. It turns out, however, that these homotopy obstruction sets coincide with already known classical obstruction sets, at least when X is smooth and geometrically integral over k (see the main theorem in [HS13]): $X(\mathbf{A}_k)^h = X(\mathbf{A}_k)^{h,n} = X(\mathbf{A}_k)^{\text{ét Br}}$ for $n \geq 2$, $X(\mathbf{A}_k)^{\mathbf{Z}h} = X(\mathbf{A}_k)^{\mathbf{Z}h,n} = X(\mathbf{A}_k)^{\text{Br}}$ for $n \geq 2$, $X(\mathbf{A}_k)^{h,1} = X(\mathbf{A}_k)^{\mathcal{F}_k}$, and $X(\mathbf{A}_k)^{h,1} = X(\mathbf{A}_k)^{\mathcal{F}_k^{\text{Ab}}}$. Using their topological tools, however, Harpaz and Schlank manage to prove new properties of the already known obstruction sets (cf. the "Applications" section in [HS13]).

A different direction than the one explored by Harpaz and Schlank is one which was already implicit in a result by Harari (cf. [Har02]), which tells us, roughly, that $X(\mathbf{A}_k)^{\text{Br}}$ is the same as the set which we would get by considering the intersection of the images of the adelic points of all torsors $Y \to X$ under connected linear algebraic groups over k, and in later results by Demarche and Skorobogatov (cf. [Dem09b] and [Sko09]), which tell us, roughly, that $X(\mathbf{A}_k)^{\text{ét Br}}$ is the same as the set which we would get by considering the intersection of the images of the adelic points of all torsors $Y \to X$ under all linear algebraic groups over k. Hence, we could think of applying further constructions of obstruction sets to each of the torsors $Y \to X$, in the hope of getting a finer obstruction set. This inspires the following notion. Let X be a smooth, projective, geometrically integral variety over a number field k. By "iterating" an obstruction set $X(\mathbf{A}_k)^S$, we will mean the following two-step process:

1. characterise $X(\mathbf{A}_k)^S$ in terms of torsors only, say

$$X(\mathbf{A}_k)^S = \bigcap_{G \in \mathcal{A}} \bigcap_{[f:Y \to X] \in H^1_{\text{ét}}(X,G)} \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k)),$$

for some family $\mathcal{A}$ (containing the trivial group) of linear algebraic groups

over k;

2. define the "iterated" set by

$$\mathrm{Iter}_{\mathcal{A}}(X(\mathbf{A}_k)^S) := \bigcap_{G\in\mathcal{A}} \bigcap_{[f:Y\to X]\in H^1_{\text{ét}}(X,G)} \bigcup_{[\sigma]\in H^1_{\text{ét}}(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k)^S).$$

Remark 1.2.2. There is some ambiguity in how one can define iteration: in fact, while X is proper and hence it does not matter whether we consider the adelic or product topology on it, the torsors $Y \to X$ might not be proper, and hence considering $Y(\mathbf{A}_k)$ or $Y(k_\Omega)$ (with the appropriate topologies) might make a difference; for example, if one is interested in studying strong approximation, then one should iterate by using the adelic topology, while if one is more interested in studying weak approximation, then iterating with respect to the product topology could be more convenient. In this thesis, we will consider the iteration always with respect to the adelic topology, but the reader should keep in mind that one can iterate also by considering the product topology.

By construction, $X(k) \subset \mathrm{Iter}_{\mathcal{A}}(X(\mathbf{A}_k)^S) \subset X(\mathbf{A}_k)^S$, so it natural to ask whether $\mathrm{Iter}_{\mathcal{A}}(X(\mathbf{A}_k)^S) = X(\mathbf{A}_k)^S$ or not. We will investigate this question in Chapter 4, in the case when $X(\mathbf{A}_k)^S$ is the algebraic étale-Brauer set (see Chapter 2 for the definition) and where X satisfies some geometrical constraints. Ideally, one would like to answer the above question for the case when $X(\mathbf{A}_k)^S$ is the étale-Brauer set, but the current tools at our disposal do not seem to be enough to tackle the question yet.

1.3 Structure of the thesis and main results

The rest of this thesis is organised as follows. After a review of the classical theory of obstruction sets in Chapter 2, we enter the core of the thesis, con-

sisting of Chapters 3 and 4. In Chapter 3, we enlarge the dictionary between obstruction sets coming from different functors. Let $\mathcal{L}_k$ be the (k-equivalence classes of) all linear algebraic groups over k, let $\mathcal{F}_k \subset \mathcal{L}_k$ and $\mathcal{F}_k^{\mathrm{Sol}} \subset \mathcal{L}_k$ be, respectively, the subsets of finite linear algebraic groups over k and finite solvable linear algebraic groups over k, and, for $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_k$, let $\mathrm{Ext}(\mathcal{A}, \mathcal{B}) \subset \mathcal{L}_k$ be the set of (k-equivalence classes of) linear algebraic groups $G \in \mathcal{L}_k$ which are extensions of some $B \in \mathcal{B}$ by some $A \in \mathcal{A}$. For $\mathcal{O} : \mathbf{Sch}_k^{opp} \to \mathbf{Sets}$ any functor and $\mathcal{A} \subset \mathcal{L}_k$, we let

$$X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} := \bigcap_{G \in \mathcal{A}} \bigcap_{[f:Y \to X] \in H^1_{\text{ét}}(X,G)} \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k)^{\mathcal{O}}).$$

Recall that, for any $\mathcal{A} \subset \mathcal{L}_k$,

$$X(\mathbf{A}_k)^{\mathcal{A}} := \bigcap_{G \in \mathcal{A}} \bigcap_{[f:Y \to X] \in H^1_{\text{ét}}(X,G)} \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k)).$$

The main results are the following.

Theorem (Theorem 3.5.1)**.** *Let $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$. Let X be a smooth, projective, geometrically integral variety over k. Let $\mathcal{O} : \boldsymbol{Sch}_k^{opp} \to \boldsymbol{Sets}$ be any functor such that $X(\mathbf{A}_k)^{\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1}$ functorially, and let $\mathcal{S} \subset \mathcal{L}_k$ be subject to Conditions 1, 2, and 3 in §3.5 (for example, one can take $\mathcal{S}$ to be the set of all connected linear algebraic groups over k, up to k-isomorphism, or the set of all tori over k, up to k-isomorphism). Then $X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S})}$.*

Let $\mathcal{U}_k \subset \mathcal{L}_k$ be the set of (k-equivalence classes of) unipotent linear algebraic groups over k.

Theorem (Theorem 3.8.1)**.** *Let X be a smooth, projective, geometrically integral variety over k. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$. Then $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)^{\mathcal{B}}$.*

In particular, as a corollary of the above, we obtain among other things a translation in Brauer-type terms of the descent set associated to all solvable linear connected groups (cf. Corollary 3.9.5) and we obtain a translation of the algebraic étale-Brauer set in pure descent-type terms (cf. Theorem 3.7.8).

In Chapter 4, we give more details on the concept of 'iterating' obstruction sets. We then ask an "algebraic" analogue of a question by Poonen, that is, we ask whether, for a smooth, projective, geometrically integral variety X over a number field k iterating the algebraic étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}_1}$ (defined in Chapter 2) yields a finer obstruction set. Let $\mathcal{T}_k \subset \mathcal{L}_k$ be the set of (k-equivalence classes of) algebraic tori over k. We proceed to prove the following theorem (the notation will be explained in Chapter 4).

Theorem (Theorem 4.3.1)**.** *Let X be a smooth, projective, geometrically integral variety over k such that $\pi_1^{\text{ét}}(\overline{X})$ is finite. Then* $\text{Iter}_{\text{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{ét Br}_1}$.

This theorem applies to interesting families of varieties − geometrically rational varieties, K3 surfaces, and Enriques surfaces, to name a few − and corroborates some conjectures on the arithmetic behaviout of rational points.

Chapter 2

Obstruction sets: background material

2.1 A functorial approach to constructing obstructions

Let k be a number field and fix $X \in \mathbf{Sch}_k$. Let $\spadesuit \in \{\mathbf{A}_k, k_\Omega\}$. We denote by $\mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})$ the category whose objects are functors $\mathcal{F} : \mathbf{Sch}_k^{op} \to \mathbf{Sets}$ and whose morphisms are natural transformations. We define a map

$$Obs_{X/k,\spadesuit} : Obj(\mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})) \to Obj(\mathbf{Sets})$$

in the following way. Let $\mathcal{F} \in \mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})$. For any k-algebra L, let $X(L) := \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, L, X)$. Given any $\alpha \in \mathcal{F}(X)$, we obtain an evaluation map

$$\mathrm{ev}_\alpha : X(L) \to \mathcal{F}(\mathrm{Spec}\, L) =: \mathcal{F}(L)$$

as follows: for any $x \in X(L)$, viewed as a morphism $x : \mathrm{Spec}L \to X$ in $\mathbf{Sch}_k$, we set $\mathrm{ev}_\alpha(x) := (\mathcal{F}(x))(\alpha)$. By considering the diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\spadesuit) \\ {\scriptstyle \mathrm{ev}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\alpha} \\ \mathcal{F}(k) & \longrightarrow & \mathcal{F}(\spadesuit) \end{array} \tag{2.1.1}$$

for each $\alpha \in \mathcal{F}(X)$, we define the set

$$\begin{aligned} Obs_{X/k,\spadesuit}(\mathcal{F}) &:= X(\spadesuit)^{\mathcal{F}} \\ &:= \textstyle\bigcap_{\alpha\in\mathcal{F}(X)}\{(x_v) \in X(\spadesuit) : \mathrm{ev}_\alpha((x_v)) \in \mathrm{im}(\mathcal{F}(k) \to \mathcal{F}(\spadesuit))\}. \end{aligned}$$

By the commutativity of (2.1.1), it is clear that $X(k) \subset X(\spadesuit)^{\mathcal{F}} \subset X(\spadesuit)$.

Lemma 2.1.1. *Let $\mathcal{F} \in \mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})$. The map $\mathcal{G} : \mathbf{Sch}_k \to \mathbf{Set}$ given by $X \mapsto X(\spadesuit)^{\mathcal{F}}$ is (covariantly) functorial.*

Proof. Let $\phi \in \mathrm{Hom}_{\mathbf{Sch}_k}(Y, X)$, let $(y_v) \in Y(\spadesuit)^{\mathcal{F}}$, and set $(x_v) := \mathcal{G}(\phi)((y_v)) := \phi((y_v)) \in X(\spadesuit)$. For any $\alpha \in \mathcal{F}(X)$, we have $\mathrm{ev}_\alpha((x_v)) = \mathrm{ev}_{\mathcal{F}(\phi)(\alpha)}((y_v)) \in \mathrm{Im}(\mathcal{F}(k) \to \mathcal{F}(\spadesuit))$, as $(y_v) \in Y(\spadesuit)^{\mathcal{F}}$ and $\mathcal{F}(\phi)(\alpha) \in \mathcal{F}(Y)$. This means that $(x_v) \in X(\spadesuit)^{\mathcal{F}}$. The other properties of functoriality are easy. □

Remark 2.1.2. Let $\mathcal{F} \in \mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})$. Since our aim is to define potentially better approximations of $X(k)$ than $X(\spadesuit)$, for $X(\spadesuit)^{\mathcal{F}} := Obs_{X/k,\spadesuit}(\mathcal{F})$ to be a "useful" obstruction set we would like

- $\mathcal{F}(k) \to \mathcal{F}(\spadesuit)$ not to be surjective, since otherwise we would always have $X(\spadesuit)^{\mathcal{F}} = X(\spadesuit)$, which is not great for our purposes, and
- $\mathrm{im}(\mathcal{F}(k) \to \mathcal{F}(\spadesuit))$ to be easy to characterise or compute.

In the rest of this chapter, we list some of the standard choices of the functor $\mathcal{F}$ satisfying the above properties.

2.2 The Brauer-Manin obstruction

Definition 2.2.1. The Brauer functor $\mathrm{Br} : \mathbf{Sch}_k^{opp} \to \mathbf{Ab}$ is defined by the assignment $X \mapsto H^2_{\text{ét}}(X, \mathbf{G}_m)$. Sometimes, when we refer to the Brauer functor, we mean the composition forgetful $\circ$ Br : $\mathbf{Sch}_k^{opp} \to \mathbf{Sets}$. For $X \in \mathbf{Sch}_k$, the (cohomological) Brauer group of X is $\mathrm{Br}\, X$.

Remark 2.2.2. There are other definitions of the Brauer group of a scheme X. Recall that an Azumaya algebra $\mathcal{A}$ on X is a $\mathcal{O}_X$-algebra that is coherent and locally free as a $\mathcal{O}_X$-module and such that, for each $x \in X$, the fibre $\mathcal{A} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a central simple algebra over $\kappa(x)$ (the residue field at x). We say that two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ on X are (Morita)-equivalent, and we write it $\mathcal{A} \sim_{\mathrm{Br}} \mathcal{A}'$, if there exist locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}')$. We can define the Azumaya Brauer group of X by

$$\mathrm{Br}^{Az} X := \{\text{Azumaya algebras on } X\}/\sim_{\mathrm{Br}} .$$

The group operation is given by tensoring over $\mathcal{O}_X$, the inverse of any $[\mathcal{A}] \in \mathrm{Br}^{Az} X$ is the equivalence class of its opposite algebra $[\mathcal{A}^{opp}] \in \mathrm{Br}^{Az} X$, and the identity is given by $[\mathcal{O}_X] \in \mathrm{Br}^{Az} X$. There is a natural inclusion $\mathrm{Br}^{Az} X \hookrightarrow \mathrm{Br}\, X$ (cf. [Mil80, Thm IV.2.5]). If we impose further conditions on X, we can say more: a result by Gabber (cf. [dJ]) states that, if X is a scheme endowed with an ample invertible sheaf, then the above natural inclusion gives rise to an isomorphism $\mathrm{Br}^{Az} X \cong (\mathrm{Br}\, X)_{tors}$. As a consequence of this, using the facts that, when X is moreover regular and quasi-compact, there is an inclusion $\mathrm{Br}\, X \hookrightarrow \mathrm{Br}\, k(X)$ (cf. [Mil80, Example III.2.22]), and that $\mathrm{Br}\, k(X)$ is torsion, we can conclude that, whenever X is a regular quasi-projective variety over k, we have $\mathrm{Br}^{Az} X \cong \mathrm{Br}\, X$.

Let X be a smooth, geometrical integral variety over a number field k.

Definition 2.2.3. The Brauer-Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ of X is the obstruction set associated to Br (i.e. $X(\mathbf{A}_k)^{\mathrm{Br}} := Obs_{X/k,\mathbf{A}_k}(\mathrm{Br})$).

For $X(\mathbf{A}_k)^{\mathrm{Br}}$ to be of any use, we should characterise $\mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, \mathbf{A}_k)$ in more concrete terms. It turns out that we can do so using class field theory (and this was a great intuition by Manin [Man71], whence the name "Brauer-*Manin* set"). Indeed, consider the fundamental exact sequence of class field theory (cf. [Mil13]):

$$0 \to \mathrm{Br}\, k \to \bigoplus_{v\in\Omega_k} \mathrm{Br}\, k_v \xrightarrow{\sum_{v\in\Omega_k} \mathrm{inv}_v} \mathbf{Q}/\mathbf{Z} \to 0, \tag{2.2.1}$$

where, for each $v \in \Omega_k$, the map $\mathrm{inv}_v : \mathrm{Br}\, k_v \to \mathbf{Q}/\mathbf{Z}$ is the standard invariant map coming from local class field theory; note that, by [Česl5, Thm 2.13], the natural homomorphism $\mathrm{Br}(\mathbf{A}_k) \to \bigoplus_{v\in\Omega_k} \mathrm{Br}\, k_v$ is an isomorphism. Above the exact sequence (2.2.1), for any $\alpha \in \mathrm{Br}\, X$ one can construct the commutative diagram

$$\begin{array}{ccccccccc} & & X(k) & \longrightarrow & X(\mathbf{A}_k) & & & & \\ & & \downarrow{\scriptstyle \mathrm{ev}_\alpha} & & \downarrow{\scriptstyle \mathrm{ev}_\alpha} & & & & \\ 0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \bigoplus_{v\in\Omega_k} \mathrm{Br}\, k_v & \xrightarrow{\sum_{v\in\Omega_k} \mathrm{inv}_v} & \mathbf{Q}/\mathbf{Z} & \longrightarrow & 0, \end{array}$$

where the right vertical evaluation map lands indeed in the direct sum by a standard spread-out argument: if $(x_v) \in X(\mathbf{A}_k)$, then there exists a finite subset $S \subset \Omega_k$ such that X has an integral model $\mathcal{X}$ over $\mathcal{O}_{k,S}$, such that α spreads out to some $\mathcal{A} \in \mathrm{Br}\, \mathcal{X}$, and such that $x_v \in \mathcal{X}(\mathcal{O}_v)$ for all $v \notin S$; then, by functoriality, $\mathrm{ev}_\alpha(x_v) = \mathrm{ev}_{\mathcal{A}}(x_v) \in \mathrm{Br}\, \mathcal{O}_v$ for all $v \notin S$; but $\mathrm{Br}\, \mathcal{O}_v = 0$ (this

follows since the Brauer group of a finite field is trivial).

Hence, by an easy diagram chasing, we see that

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcap_{\alpha \in \mathrm{Br}\, X} \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v\, \alpha(x_v) = 0 \right\}. \tag{2.2.2}$$

Remark 2.2.4. The Brauer-Manin pairing

$$\langle -, - \rangle_{\mathrm{BM}} : X(\mathbf{A}_k) \times \mathrm{Br}\, X \to \mathbf{Q}/\mathbf{Z}$$

is given by $((x_v), \alpha) \mapsto \sum_{v \in \Omega_k} \mathrm{inv}_v\, \mathrm{ev}_\alpha(x_v)$. For any $\alpha \in \mathrm{Br}\, X$, it is clear that $\langle -, \alpha \rangle_{\mathrm{BM}} = \mathrm{inv}_v \circ \mathrm{ev}_\alpha$. Hence,

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcap_{\alpha \in \mathrm{Br}\, X} \{ (x_v) \in X(\mathbf{A}_k) : \langle (x_v), \alpha \rangle_{\mathrm{BM}} = 0 \}$$

can be viewed as the set of adelic points on X which are orthogonal (with respect to the Brauer-Manin pairing) to $\mathrm{Br}\, X$.

Remark 2.2.5. If we endow $\mathbf{Q}/\mathbf{Z}$ with the discrete topology, then $\langle -, \alpha \rangle_{\mathrm{BM}} : X(\mathbf{A}_k) \to \mathbf{Q}/\mathbf{Z}$ is continuous. This implies that $X(\mathbf{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbf{A}_k)$ (with respect to the restricted product topology), as

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcap_{\alpha \in \mathrm{Br}\, X} \langle -, \alpha \rangle_{\mathrm{BM}}^{-1}(\{0\})$$

and arbitrary intersections of closed sets are closed. This is very useful if we are interested in density properties of rational points (e.g. strong approximation).

Fix an algebraic closure $\overline{k}$ of k, and let $\overline{X} := X \times_k \overline{k}$ denote the base change of X to $\overline{k}$. There is a filtration on $\mathrm{Br}\, X$ given by

$$\mathrm{Br}_0\, X \subset \mathrm{Br}_1\, X \subset \mathrm{Br}\, X,$$

where

- $\mathrm{Br}_0 X := \mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, X)$ is the set of constant elements. By the fundamental sequence of class field theory, the elements of $\mathrm{Br}_0 X$ can never give an obstruction, in the sense that $X(\mathbf{A}_k)^{\mathrm{Br}_0 X} = X(\mathbf{A}_k)$.
- $\mathrm{Br}_1 X := \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$ is the algebraic Brauer group of X. We remark that $\mathrm{Br}_1 : \mathbf{Sch}_k^{opp} \to \mathbf{Sets}$ is a subfunctor of Br.

Definition 2.2.6. The algebraic Brauer-Manin set for X is defined by

$$X(\mathbf{A}_k)^{\mathrm{Br}_1} = \bigcap_{\alpha \in \mathrm{Br}_1 X} \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v\, \alpha(x_v) = 0 \right\}.$$

Note that

$$\overline{X(k)} \subset X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1} \subset X(\mathbf{A}_k).$$

Remark 2.2.7. There is also a Brauer-Manin set for the product topology, defined by using the unramified Brauer group $\mathrm{Br}_{nr}(X)$; this set is called the product Brauer-Manin set and is denoted by $X(k_\Omega)^{\mathrm{Br}_{nr}}$; see [Sko01, §5.2] for more details. In this thesis, we will only consider the adelic Brauer-Manin set.

2.3 The G-descent obstruction

2.3.1 Linear algebraic groups

Let S be a scheme.

Definition 2.3.1. A group scheme over S is a group object in $\mathbf{Sch}_S$.

Another way to characterise algebraic group schemes and some of their properties is by using the functor of points and Yoneda's lemma: if $G \in \mathbf{Sch}_S$, making G a group scheme over S is equivalent to equipping $G(T)$ with a group

structure for all $T \in \mathbf{Sch}_S$ such that, for any $\phi \in \mathrm{Hom}_{\mathbf{Sch}_S}(T, T')$, the induced map $G(\phi) : G(T) \to G(T')$ of sets is actually in $\mathrm{Hom}_{\mathbf{Groups}}(G(T), G(T'))$.

Definition 2.3.2. An algebraic group over k is a group object in $\mathbf{Var}_k$. A linear algebraic group over k is a group object in $\mathbf{AffVar}_k$.

Remark 2.3.3. In older literature (e.g. [Bor91]), some of the above definitions are slightly different (and, in some sense, less flexible): for example, algebraic groups over k are defined as algebraic varieties over $\overline{k}$ together with a group structure defined over k.

Definition 2.3.4. We let $\mathcal{L}_k := \{G : G \text{ is a linear algebraic group over } k\}/\sim$ be the set of k-isomorphism classes of linear algebraic groups over k.

2.3.2 Torsors

Let S be a scheme and G be an fppf S-group scheme. For convenience, we assume that G is *affine* over S – this has the advantage of making the fppf descent effective, meaning that we can work with torsors as schemes, and not just as sheaves. Moreover, in this thesis we will only be interested in linear algebraic groups (which are affine).

Definition 2.3.5. A (right) G-torsor over S is an fppf S-scheme $f : Y \to S$ equipped with a (right) action of G compatible with f such that the map $Y \times_S G \to Y \times_S Y$ given by $(y, g) \mapsto (y, y \cdot g)$ is an isomorphism.

Example 2.3.6. The torsor $G \to S$ with the (right) action of G is the trivial torsor.

Remark 2.3.7. Since G is affine over S, we have that the isomorphism classes of G-torsors over S are classified by the pointed set – the class of the trivial

torsor being the marked point – $H^1(S,G) := \check{H}^1_{\mathrm{fppf}}(S,G)$ (cf. [Sko01, §2.2]), which is an abelian group if G is commutative.

We will make extensive use of the following operation on torsors.

Definition 2.3.8. Let T be a (right) S-torsor under the S-group scheme G and let Z be an affine S-scheme equipped with a left G-action. The contracted product $T \times_S^G Z$ (or simply $T \times^G Z$) is defined as the quotient of $T \times_S Z$ by the action of G given by $(t,z) \mapsto (t \cdot g^{-1}, g \cdot z)$. (If T is a left G-torsor over S and if Z is an affine S-scheme equipped with a right G-action, we can define $Z \times_S^G T$ analogously.)

Remark 2.3.9. The contracted product $T \times^G X$ exists as an affine S-scheme (cf. [Sko01, Lemma 2.2.3]), by an application of fppf descent.

The main example of contracted product, for our purposes, is the following.

Example 2.3.10. Let K be a field with $\operatorname{char} K = 0$. Let G be a linear algebraic group over K, and let S be a K-scheme. Then $G_S := G \times_K S$ is an fppf S-group scheme, affine over S. Let $f : Z \to S$ be a (right) G_S-torsor over S. Let $T \to \operatorname{Spec} K$ be a left G-torsor, with class $[\tau] = [T] \in H^1(K,G)$, for some 1-cocycle $\tau \in Z^1(k,G)$. We define the twist Z^τ of Z by τ by

$$Z^\tau := Z \times_S^{G_S} T_S = Z \times_{\operatorname{Spec} K}^G T.$$

Let $f^\tau : Z^\tau \to S$ be the structure morphism. Then $f^\tau : Z^\tau \to S$ is a (right) $(G^\tau)_S$-torsor, where G^τ is an inner form of G defined by the quotient of $\overline{G} := G \times_K \operatorname{Spec} \overline{K}$ by the "twisted" action $\mathrm{Gal}(\overline{K}/K) \times G(\overline{K}) \to G(\overline{K})$, $(\gamma, g) \mapsto \tau(\gamma)(\gamma \cdot g)$.

Remark 2.3.11. We will often make the following abuse of notation: if B is

a scheme, S is a B-scheme, and G is a B-group scheme, then we will talk of a "G-torsor over S" instead of a "G_S-torsor over S".

2.3.3 The G-descent obstruction

Let k be a number field, G a linear algebraic group over k, and X a smooth, geometrically integral variety over k. Note that $H^1(-, G) \in \mathrm{Fun}(\mathbf{Sch}_k^{op}, \mathbf{Sets})$.

Definition 2.3.12. The G-descent set $X(\mathbf{A}_k)^{H^1(X,G)}$ of X is the obstruction set associated to $H^1(-, G)$ (i.e. $X(\mathbf{A}_k)^{H^1(X,G)} := Obs_{X/k,\mathbf{A}_k}(H^1(-, G))$).

By [Čes15, Thm 2.18], the natural map $H^1(\mathbf{A}_k, G) \to \prod_{v\in\Omega_k} H^1(k_v, G)$ is injective. Hence, the image of the natural map $H^1(k, G) \to H^1(\mathbf{A}_k, G)$ is the same as the image of the natural map $H^1(k, G) \to \prod_{v\in\Omega_k} H^1(k_v, G)$.

For each $[Y] := [f : Y \to X] \in H^1(X, G)$, we can characterise the set

$$X(\mathbf{A}_k)^f = \{(x_v) \in X(\mathbf{A}_k) : \mathrm{ev}_{[Y]}((x_v)) \in \mathrm{im}(H^1(k, G) \to \prod_{v\in\Omega_k} H^1(k_v, G))\}$$

more explicitly as $X(\mathbf{A}_k)^f = \bigcup_{[\tau]\in H^1(k,G)} f^\tau(Y^\tau(\mathbf{A}_k))$. Hence,

$$X(\mathbf{A}_k)^{H^1(X,G)} = \bigcap_{[f:Y\to X]\in H^1(X,G)} \bigcup_{[\tau]\in H^1(k,G)} f^\tau(Y^\tau(\mathbf{A}_k)).$$

Remark 2.3.13. One can show that $f^\tau(Y^\tau(\mathbf{A}_k)) = X(\mathbf{A}_k) \cap (\cup_{v\in\Omega_k} f^\tau(Y^\tau(k_v)))$ and that, if X is proper, then $f^\tau(Y^\tau(\mathbf{A}_k))$ is closed in $X(\mathbf{A}_k)$. Moreover, when X is proper, there are only finitely many $\tau \in H^1(k, G)$ such that $Y^\tau(\mathbf{A}_k) \neq \emptyset$ (cf. [Sko01, Prop. 5.3.2]). For similar statements in the case when X is not proper, see the recent preprint [CDX16].

Remark 2.3.14. Let $x \in X(k)$ and let $f : Y \to X$ be a G-torsor over X. We have that $\mathrm{ev}_{[Y]}(x) = 0$ if and only if $Y_x(k) \neq \emptyset$, where Y_x is the fibre of

x under f. More generally, if $\tau \in Z^1(k,G)$, then $\mathrm{ev}_{[Y]}(x) = [\tau]$ if and only if $\mathrm{ev}_{[Y^\tau]}(x) = 0$. Hence, using the above observations, we have a partition $X(k) = \coprod_{[\tau]\in H^1(k,G)}\{x \in X(k) : \mathrm{ev}_{[Y]}(x) = [\tau]\}$. As shown in [Sko01, (2.12)], one can rewrite the above partition also as

$$X(k) = \coprod_{[\tau]\in H^1(k,G)} f^\tau(Y^\tau(k)).$$

2.4 The étale-Brauer obstruction

One can combine the constructions in the previous sections by first considering a descent obstruction set and then applying the Brauer-Manin construction to the torsors.

Let $\mathcal{F}_k \subset \mathcal{L}_k$ be the set of all (k-isomorphism classes of) finite linear algebraic groups over k. Let X be a smooth, geometrically integral variety over k.

Definition 2.4.1. The étale-Brauer set of X is defined by

$$X(\mathbf{A}_k)^{\text{ét Br}} := \bigcap_{F\in\mathcal{F}_k} \bigcap_{[f:Y\to X]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Y^\tau(\mathbf{A}_k)^{\mathrm{Br}}).$$

Similarly, we define the algebraic étale-Brauer set of X by

$$X(\mathbf{A}_k)^{\text{ét Br}_1} := \bigcap_{F\in\mathcal{F}_k} \bigcap_{[f:Y\to X]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Y^\tau(\mathbf{A}_k)^{\mathrm{Br}_1}).$$

Remark 2.4.2. It is clear, from the definitions, that $X(\mathbf{A}_k)^{\text{ét Br}} \subset X(\mathbf{A}_k)^{\mathrm{Br}}$ and $X(\mathbf{A}_k)^{\text{ét Br}_1} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1}$.

Chapter 3

A dictionary between obstruction types

3.1 Disclaimer

Much of the content of this chapter is taken from the paper [Bal16].

3.2 Specific background and notation

Before explaining the motivation and giving the statements of the main results of this chapter, we remind the reader of some definitions and properties of algebraic groups. Let K be a field with $\operatorname{char} K = 0$. Let $\mathbf{AlgGrp}_K$ be the category of algebraic groups over K. Let $G \in \mathbf{AlgGrp}_K$. Recall that a (closed) K-subgroup H of G, written $H \leq_K G$, is an $H \in \mathbf{AlgGrp}_K$ that is a K-closed subvariety of G, with respect to the Zariski topology, and whose multiplication and identity morphisms are induced from those of G. If $G \in \mathbf{AlgGrp}_K$ and $H \leq_K G$ is a normal K-subgroup, then G/H exists and is an algebraic K-group ([DGA70, VI.A, Thms 3.2 and 5.2]), which is linear if G is linear.

Example 3.2.1. Let $G \in \mathbf{AlgGrp}_K$. We denote by G^0 the connected component of the identity of G, i.e. the unique connected component of G that contains the identity. This is a (closed) normal K-subgroup of G of finite index, whose cosets are the connected (equivalently, irreducible) components of G (see [Bor91, (1.2)]). Note that if G is a linear algebraic K-group, then so is G^0.

Example 3.2.2. Let $G \in \mathbf{AlgGrp}_K$. The K-radical $R(G)$ of G is the largest connected solvable normal (closed) K-subgroup of G. The set $R_u(G)$ of all unipotent elements of $R(G)$ is the unipotent K-radical of G, which is the largest connected unipotent normal (closed) K-subgroup of G. Note that $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$.

Since we will work with short exact sequences of linear algebraic groups, we recall a few further facts. Let $1 \to A \to B \to C \to 1$ be a short exact sequence in $\mathbf{AlgGrp}_K$. Then B is affine if and only if A and C are affine ([DGA70, VI.B, 9.2(viii)]). Hence, since $\operatorname{char} K = 0$, it follows that B is linear if and only if A and C are linear. Let G and G' be K-groups and let $\phi : G \to G'$ be a morphism of K-groups. Then $\phi(G)$ is a (closed) normal K-subgroup of G' ([Spr98, Prop. 2.2.5]) and, similarly, $\ker(\phi)$ is a (closed) K-subgroup of G.

Fix an algebraic closure $\overline{K}$ of K.

Definition 3.2.3. A linear algebraic group G over K is:

- finite (over K) if $\mathcal{O}_G(U)$ is finitely generated as a K-vector space for each open affine $U \subset G$. This is equivalent, using the fact that $\operatorname{char} K = 0$, to requiring that $G(\overline{K})$ be finite and discrete (see [Mil15, Chap 2, (2.12)]).
- connected if its underlying topological space is connected, with respect to the Zariski topology.

- solvable if its derived series $G \supset \mathcal{D}G \supset \mathcal{D}^2G \supset ...$ has $\mathcal{D}^iG$ trivial for some $i \in \mathbf{Z}_{\geq 0}$. If $G \in \mathbf{AlgGrp}_K$ is connected and linear, then its commutator $\mathcal{D}G$ is a connected linear K-subgroup of G ([Spr98, Cor. 2.2.8]).

- reductive if its geometric unipotent radical is trivial, i.e. if $R_u(G^0_{\overline{K}}) = R_u(G^0)_{\overline{K}}$ is trivial (since K is perfect, reductive groups are the same as pseudo-reductive groups, i.e. linear algebraic groups whose unipotent K-radical is trivial). Note that we don't require G to be connected.

- semisimple if its geometric radical is trivial, i.e. if $R(G^0_{\overline{K}}) = R(G^0)_{\overline{K}}$ is trivial (since K is perfect, semisimple groups are the same as pseudo-semisimple groups, i.e. linear algebraic groups whose K-radical is trivial). Again, note that we don't require G to be connected.

- of multiplicative type if $G_{\overline{K}}$ is $\overline{K}$-isomorphic to a closed subgroup of $\mathbf{G}^N_{m,\overline{K}}$, for some $N \in \mathbf{Z}_{\geq 0}$.

- a torus if $G_{\overline{K}}$ is $\overline{K}$-isomorphic to $\mathbf{G}^N_{m,\overline{K}}$, for some $N \in \mathbf{Z}_{\geq 0}$.

- unipotent if $G_{\overline{K}}$ admits a (finite) composition series over $\overline{K}$ such that each successive quotient is isomorphic (over $\overline{K}$) to a closed subgroup of $\mathbf{G}_{a,\overline{K}}$; equivalently (cf. [DGA70, XVII, Thm 3.5]), if G admits a (finite) central series over K such that each successive quotient is isomorphic (over K) to $\mathbf{G}_a$ (here we use the fact that $\operatorname{char} K = 0$); equivalently, if $G = R_u(G)$. In particular, if G is unipotent, then G is connected. Note that "being unipotent" is a geometric property, i.e. is stable under base-extending K (cf. [DGA70, XVII, Prop. 2.2(i)]).

We define the following sets:

$$\begin{aligned}
\mathcal{L}_K &= \{G : G \text{ is a linear algebraic } K\text{-group}\}/\sim,\\
\mathcal{F}_K &= \{[G] \in \mathcal{L}_K : G \text{ is finite}\},\\
\mathcal{C}_K &= \{[G] \in \mathcal{L}_K : G \text{ is connected}\},\\
\mathcal{A}b_K &= \{[G] \in \mathcal{L}_K : G \text{ is commutative}\},\\
\mathrm{Sol}_K &= \{[G] \in \mathcal{L}_K : G \text{ is solvable}\},\\
\mathcal{M}_K &= \{[G] \in \mathcal{L}_K : G \text{ is of multiplicative type}\},\\
\mathcal{T}_K &= \{[G] \in \mathcal{L}_K : G \text{ is a torus}\},\\
\mathcal{R}_K &= \{[G] \in \mathcal{L}_K : G \text{ is reductive}\},\\
\mathcal{RR}_K &= \{[G] \in \mathcal{R}_K : H \leq_K G \Rightarrow [H] \in \mathcal{R}_K\},\\
\mathcal{U}_K &= \{[G] \in \mathcal{L}_K : G \text{ is unipotent}\},
\end{aligned}$$

where $G_1 \sim G_2$ if and only if G_1 is K-isomorphic to G_2 as an algebraic K-group.

Notation. With abuse of notation, we often write $G \in \mathcal{S}$ to mean $[G] \in \mathcal{S}$, when $\mathcal{S} \subset \mathcal{L}_K$. Moreover, for any $\mathcal{S} \subset \mathcal{L}_K$, we write $\mathcal{S}^{\mathrm{Sol}}$ for $\mathcal{S} \cap \mathrm{Sol}_K$, $\mathcal{S}^{\mathrm{Ab}}$ for $\mathcal{S} \cap \mathcal{A}b_K$, and so on.

Recall that, if A, B, and G are linear algebraic groups over K, we say that G is an extension of A by B if G fits into a short exact sequence $1 \to B \to G \to A \to 1$ in $\mathbf{AlgGrp}_K$.

Definition 3.2.4. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_K$. We define

$$\mathrm{Ext}(\mathcal{A}, \mathcal{B}) = \{G \in \mathcal{L}_K : G \text{ is an extension of } A \text{ by } B, \text{ for } A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Remark 3.2.5. If $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, then, by definition, G fits into a short exact sequence $1 \to B \to G \to A \to 1$, for some $B \in \mathcal{B}$ and $A \in \mathcal{A}$. If

$(f, Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})(X)$, then f naturally decomposes into $(Y, B) \in \mathcal{B}(Z)$ and $(Z, A) \in \mathcal{A}(X)$, where $Z := Y/B$ is the push-forward of $f : Y \to X$ along the morphism $G \to A$ (see [Sko01, §2.2] for more details).

It might be worth mentioning that some of the sets defined above can be recast in terms of extensions of linear algebraic groups. For example, it is easy to check that $\mathcal{L}_K = \mathrm{Ext}(\mathcal{F}_K, \mathcal{C}_K)$; perhaps less obviously, $\mathcal{RR}_K = \mathrm{Ext}(\mathcal{F}_K, \mathcal{T}_K)$ (cf. Lemma 3.7.9).

3.3 Motivation

As we have seen in Chapter 2, one can construct obstruction sets $X(\mathbf{A}_k)^{\mathcal{F}}$ by

(i) taking $\mathcal{F}$ to be the G-descent functor $H^1(-, G)$, where $G \in \mathcal{L}_k$. More generally, if $\mathcal{S} \subset \mathcal{L}_k$, we can also construct the obstruction set

$$X(\mathbf{A}_k)^{\mathcal{S}} := \bigcap_{G \in \mathcal{S}} X(\mathbf{A}_k)^{H^1(X,G)}.$$

(ii) taking $\mathcal{F}$ to be the Brauer functor Br or any of its (non-trivial) subfunctors (e.g. the algebraic Brauer functor Br_1);

(iii) combining (i) and (ii): if $\mathcal{S} \subset \mathcal{L}_k$ and $\mathcal{O}$ is a (non-trivial) subfunctor of Br, then we can construct an obstruction set $X(\mathbf{A}_k)^{\mathcal{S},\mathcal{O}}$.

The above give rise to different approaches to defining obstruction sets, a potential dichotomy being

$$\left\{\begin{array}{c}\text{Descent-type obstructions,}\\ \text{defined by using}\\ \text{no subfunctors of Br, as in (i).}\end{array}\right\} \text{ vs } \left\{\begin{array}{c}\text{Brauer-type obstructions,}\\ \text{defined by using subfunctors}\\ \text{of Br, as in (ii) and (iii).}\end{array}\right\}.$$

An interesting task, then, is to try to reconcile these different approaches, that is, to provide a *translation* between descent-type and Brauer-type obstruction sets. Now, any linear algebraic group can be decomposed into simpler building blocks: finite étale algebraic groups, connected linear algebraic groups, reductive linear algebraic groups, solvable linear algebraic groups, semisimple linear algebraic groups, linear algebraic tori, and unipotent linear algebraic groups. Hence, a natural question is the following.

Question 3.3.1. If $\mathcal{S} \subset \mathcal{L}_k$ is an interesting subset of $\mathcal{L}_k$ (e.g. if $\mathcal{S}$ is one of the above "building blocks"), is there a translation of $X(\mathbf{A}_k)^{\mathcal{S}}$ into a Brauer-type obstruction set? Conversely, given any natural enough Brauer-type obstruction set, is there a translation of it into a descent-type obstruction set?

3.3.1 Literature review

The main results in the literature tackling Question 3.3.1 are the following. Let X be a nice variety over k.

- Consider $\mathcal{M}_k \subset \mathcal{L}_k$. Then $X(\mathbf{A}_k)^{\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathcal{M}_k}$ (see [Sko99, Thm 3]; a less general result, requiring $\operatorname{Pic} \overline{X}$ to be torsion-free, had been proven by Colliot-Thélène and Sansuc in [CTS87]).

- A result by Harari (cf. [Har02, Thm 2 (ii)]), together with a result of Gabber (cf. [dJ]) and Proposition 5.3.4 in [Sko01], implies that $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathcal{C}_k}$: Harari shows that $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathcal{C}_k}$, while Gabber shows that $\operatorname{Br} X = \operatorname{Br}^{Az} X$; since $X(\mathbf{A}_k)^{\mathrm{Br}^{Az}} = \bigcap_{n \geq 1} X(\mathbf{A}_k)^{\mathrm{PGL}_n}$ (cf. Proposition 5.3.4 in [Sko01]) and the linear algebraic groups PGL_n, for $n \geq 1$, are connected, the result follows.

- The articles by Demarche (cf. [Dem09b]) and Skorobogatov (cf. [Sko09]) show that $X(\mathbf{A}_k)^{\text{ét Br}} = X(\mathbf{A}_k)^{\mathcal{L}_k}$, for X nice over k.

Hence, the above results answer Question 3.3.1 for all linear algebraic groups of multiplicative type (symmetrically, for the algebraic Brauer-Manin set), for all connected linear algebraic groups (symmetrically, for the Brauer-Manin set), and for all linear algebraic groups (symmetrically, for the étale-Brauer set).

3.3.2 Main results of this chapter

The aim of this chapter is to answer Question 3.3.1 in some other cases: for unipotent, reductive, and solvable linear algebraic groups, and for the Brauer-type obstruction sets $X(\mathbf{A}_k)^{\mathrm{Sol,Br}}$, $X(\mathbf{A}_k)^{\text{ét Br}_1}$, and $X(\mathbf{A}_k)^{\mathrm{Sol,Br}_1}$.

The main results are the following.

Theorem (Theorem 3.5.1)**.** *Let $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$. Let X be a nice variety over k. Let $\mathcal{O} : \boldsymbol{Sch}_k^{opp} \to \boldsymbol{Sets}$ be any functor such that $X(\mathbf{A}_k)^{\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1}$ functorially, and let $\mathcal{S}_k \subset \mathcal{L}_k$ be subject to Conditions 1, 2, and 3 in §3.5. Then $X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$.*

Remark 3.3.2. Examples of $\mathcal{S}_k$ satisfying Conditions 1, 2, and 3 in §3.5 include $\mathcal{S}_k = \mathcal{C}_k$ or $\mathcal{S}_k = \mathcal{T}_k$ (see §3.7 for more details).

Theorem (Theorem 3.8.1)**.** *Let X be a nice variety over k. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$. Then $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)^{\mathcal{B}}$.*

From these theorems, we obtain, among other things, that

- $X(\mathbf{A}_k)^{\mathrm{Sol,Br}} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}},\mathcal{C}_k)}$ (Theorem 3.7.1),
- $X(\mathbf{A}_k)^{\text{ét Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}$ (Corollary 3.7.10),
- $X(\mathbf{A}_k)^{\mathcal{U}_k} = X(\mathbf{A}_k)$ (Corollary 3.9.1),
- $X(\mathbf{A}_k)^{\mathcal{L}_k} = X(\mathbf{A}_k)^{\mathcal{R}_k}$ (Corollary 3.9.3),
- $X(\mathbf{A}_k)^{\mathrm{Sol,Br}_1} = X(\mathbf{A}_k)^{\mathrm{Sol}_k}$ (Corollary 3.9.5).

3.4 Extensions of linear algebraic groups

In this section, we describe some properties of $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ for the special cases when $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}, \mathcal{F}_K^{\mathrm{Ab}}\}$.

In general, we say that some $\mathcal{S} \subset \mathcal{L}_K$ is closed

(E1) under taking direct products if $S_1, S_2 \in \mathcal{S}$ implies that $S_1 \times_K S_2 \in \mathcal{S}$;

(E2) under taking (closed) K-subgroups if $S \in \mathcal{S}$ and $H \leq_K S$ imply that $H \in \mathcal{S}$;

(E3) under taking K-twists if, whenever $S \in \mathcal{S}$ and $S' \in \mathcal{L}_K$ are such that $S'_{\overline{K}} \cong S_{\overline{K}}$ as a $\overline{K}$-groups, then we have that $S' \in \mathcal{S}$;

(E4) under "base changing/restricting" if $S \in \mathcal{S}$ and L/K a finite field extension imply that $\mathfrak{R}_{L/K}(S_L) \in \mathcal{S}$, where $\mathfrak{R}_{L/K}$ denotes the Weil restriction.

Remark 3.4.1. The sets $\mathcal{F}_K$, $\mathcal{F}_K^{\mathrm{Sol}}$, and $\mathcal{F}_K^{\mathrm{Ab}}$ satisfy all the properties above. Note, however, that $\mathcal{F}_K^{\mathrm{Ab}}$ is not closed under extensions, while both $\mathcal{F}_K$ and $\mathcal{F}_K^{\mathrm{Sol}}$ are.

Lemma 3.4.2. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking closed K-subgroups, then so is* $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$*, for* $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}, \mathcal{F}_K^{\mathrm{Ab}}\}$.

Proof. Let $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, and say it fits into the short exact sequence $1 \to B \xrightarrow{\beta} G \xrightarrow{\alpha} A \to 1$, for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $H \leq_K G$. Consider the short exact sequence

$$1 \to \beta(B) \cap H \xrightarrow{\beta'} H \xrightarrow{\alpha'} \alpha(H) \to 1,$$

where α' and β' are the obvious maps. This is indeed exact at the ends, and to see exactness at the middle, just notice that $\beta(B) \cap H \cong \ker(\alpha) \cap H \cong \ker(\alpha')$.

Since α is a K-homomorphism, it follows that $\alpha(H) \leq_K A$, and hence that $\alpha(H) \in \mathcal{A}$ (cf. Remark 3.4.1). Moreover, by assumption, $\mathcal{B}$ is closed under taking K-subgroups, meaning that $\beta(B) \cap H \in \mathcal{B}$. Hence, $H \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$. □

Lemma 3.4.3. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking direct products, then so is* $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$*, for $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}, \mathcal{F}_K^{\mathrm{Ab}}\}$.*

Proof. We omit the proof, as it is very easy. □

Lemma 3.4.4. *Let $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}, \mathcal{F}_K^{\mathrm{Ab}}\}$. Suppose* $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ *is closed under taking direct products and closed K-subgroups. Let $G, G', G'' \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, and suppose $\phi' : G' \to G$ and $\phi'' : G'' \to G$ are morphisms of algebraic K-groups. Then $G' \times_G G'' \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, where the fibred product is with respect to ϕ and ϕ'.*

Proof. The statement follows from the assumption that $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ is closed under taking (closed) K-subgroups and direct products, since $G' \times_G G'' \cong \ker(\phi : G' \times_K G'' \to G)$ with $\phi((g', g'')) = \phi'(g')\phi''(g'')^{-1}$ by Yoneda's lemma, and so $G' \times_G G''$ is a closed K-subgroup of $G' \times_K G''$. □

Let L/K be a finite field extension. Recall that, for any quasi-projective L-scheme W, the Weil restriction $\mathfrak{R}_{L/K}(W)$ exists and represents the contravariant functor $T \mapsto W(T \times_K L)$ from $\mathbf{Sch}_K$ to $\mathbf{Set}$. We list here some of the properties of $\mathfrak{R}_{L/K}$ (see [Vos11, Chap 1, §3.12]).

(W1) If X is a K-variety, there is a functorial bijection

$$\mathrm{Hom}_K(X, \mathfrak{R}_{L/K}(X \times_K L)) \cong \mathrm{Hom}_L(X \times_K L, X \times_K L).$$

This gives, by considering $\mathrm{id}_{X \times_K L} : X \times_K L \to X \times_K L$ on the right-hand side, the canonical embedding $\iota : X \to \mathfrak{R}_{L/K}(X \times_K L)$.

(W2) If X is affine (respectively, smooth) variety over L, then $\mathfrak{R}_{L/K}(X)$ is also affine (respectively, smooth). If G is an L-group, then $\mathfrak{R}_{L/K}(G)$ is a K-group. Moreover, $\mathfrak{R}_{L/K}(G)$ is connected if and only if G is connected, and if G is commutative, then so is $\mathfrak{R}_{L/K}(G)$.

(W3) Given a short exact sequence $1 \to G' \to G \to G'' \to 1$ of algebraic L-groups,

$$1 \to \mathfrak{R}_{L/K}(G') \to \mathfrak{R}_{L/K}(G) \to \mathfrak{R}_{L/K}(G'') \to 1$$

is a short exact sequence of algebraic K-groups. In other words, the functor $\mathfrak{R}_{L/K}(-)$ preserves short exactness.

Lemma 3.4.5. *Let $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}, \mathcal{F}_K^{\mathrm{Ab}}\}$. If $\mathcal{B} \subset \mathcal{L}_K$ is closed under "base changing/restricting" and under taking (closed) K-subgroups, then* $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ *is closed under K-twists.*

Proof. We need to show that if $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$ and $\tilde{G} \in \mathcal{L}_K$ is such that $\tilde{G}_{\overline{K}} \cong G_{\overline{K}}$ (over $\overline{K}$), then $\tilde{G} \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$. Consider the short exact sequence

$$1 \to B \xrightarrow{b} G \xrightarrow{a} A \to 1$$

in $\mathbf{AlgGrp}_K$. Let $\tilde{G}$ be a K-twist of G, so that $\tilde{G}_{\overline{K}} \cong G_{\overline{K}}$ over $\overline{K}$. In particular, there is a finite field extension L/K such that $G_L \cong \tilde{G}_L$. Let $\phi : G_L \xrightarrow{\sim} \tilde{G}_L$ be an L-isomorphism in $\mathbf{AlgGrp}_L$, with $\varphi : \tilde{G}_L \xrightarrow{\sim} G_L$ as inverse. Since group schemes represent functors of fppf group sheaves, short exact sequences of group schemes are stable under base change. This gives a short exact sequence $1 \to B_L \xrightarrow{b_L} G_L \xrightarrow{a_L} A_L \to 1$, which, using the L-isomorphisms ϕ and

φ above, in turn gives the short exact sequence in $\mathbf{AlgGrp}_L$

$$1 \to B_L \xrightarrow{\phi \circ b_L} \tilde{G}_L \xrightarrow{a_L \circ \varphi} A_L \to 1. \tag{3.4.1}$$

Applying $\mathfrak{R}_{L/K}(-)$ to (3.4.1) gives a short exact sequence in $\mathbf{AlgGrp}_K$

$$1 \to \mathfrak{R}_{L/K}(B_L) \xrightarrow{\beta} \mathfrak{R}_{L/K}(\tilde{G}_L) \xrightarrow{\alpha} \mathfrak{R}_{L/K}(A_L) \to 1.$$

Now, by assumption $\mathfrak{R}_{L/K}(B_L) \in \mathcal{B}$. Moreover, $\mathfrak{R}_{L/K}(A_L) \in \mathcal{A}$ – note that $\mathfrak{R}_{L/K}(A_L)_{\overline{K}} = \prod_{i \in I} A_{\overline{K}}$, for some finite indexing set I – and $\iota : \tilde{G} \to \mathfrak{R}_{L/K}(\tilde{G}_L)$ is an embedding. Hence, we can consider the short exact sequence

$$1 \to \mathfrak{R}_{L/K}(B_L) \cap \iota(\tilde{G}) \xrightarrow{\beta'} \iota(\tilde{G}) \cong \tilde{G} \xrightarrow{\alpha'} \alpha(\iota(\tilde{G})) \to 1,$$

where α' and β' are the obvious maps. Since $\mathcal{B}$ is closed under taking (closed) K-subgroups, and since $\alpha(\iota(\tilde{G})) \leq_K \mathfrak{R}_{L/K}(A_L)$, meaning that $\alpha(\iota(\tilde{G})) \in \mathcal{A}$ (cf. Remark 3.4.1), we have that $\tilde{G} \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, as required. □

Let G be an abstract group. By the constant K-group $\tilde{G}$ induced by G we mean the disjoint union of copies of $\operatorname{Spec} K$, indexed by G, together with the obvious group structure induced by G. If W is a K-scheme, by a (left) action of the abstract group G on W we mean a group homomorphism $\phi : G \to \mathrm{Aut}_K(W)$. It is not too hard to see that the group homomorphisms $\phi : G \to \mathrm{Aut}_K(W)$ defining an action of G on W correspond precisely to the K-morphisms $\tilde{\phi} : \tilde{G} \times W \to W$ defining an action of $\tilde{G}$ on W; we will use these two notions of "action" interchangeably. For this reason, with abuse of notation, we will denote by "G" both the abstract group G and the constant algebraic group over K induced by G.

Lemma 3.4.6. *Let $\mathcal{A} \in \{\mathcal{F}_K, \mathcal{F}_K^{\mathrm{Sol}}\}$. Suppose that $\mathcal{B} \subset \mathcal{L}_K$ is closed under direct products and that $1 \in \mathcal{B}$. Let $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$. Let F be a finite (solvable, if $\mathcal{A} = \mathcal{F}_K^{\mathrm{Sol}}$) abstract group and, by abusing notation, let $F \in \mathcal{A}$ be the induced constant algebraic group over K. Let F act on $G^{|F|}$ (the direct product of G with itself $|F|$ times) by permuting the coordinates, and let $\phi : F \to \mathrm{Aut}_K(G^{|F|})$ be the corresponding group homomorphism coming from this action. Then $G^{|F|} \rtimes_\phi F \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$.*

Proof. Let $\phi : F \to \mathrm{Aut}_K(G^{|F|})$ denote the morphism induced by the action of F (with F seen as a subgroup of the symmetric group $S_{|F|}$, by Cayley's theorem). Since $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, we have that G fits, say, into the short exact sequence

$$1 \to B \xrightarrow{b} G \xrightarrow{a} A \to 1,$$

where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. By Lemma 3.4.3, we know that $G^{|F|} \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$ and that it fits into the exact sequence

$$1 \to B^{|F|} \xrightarrow{b^{|F|}:=(b,\dots,b)} G^{|F|} \xrightarrow{a^{|F|}:=(a,\dots,a)} A^{|F|} \to 1,$$

with $A^{|F|} \in \mathcal{A}$ and $B^{|F|} \in \mathcal{B}$. Since $a^{|F|} : G^{|F|} \to A^{|F|}$ is surjective and its kernel $\ker(a^{|F|}) = (\ker(a), ..., \ker(a)) \cong B^{|F|}$ is ϕ-characteristic (i.e. $\phi(\ker(a^{|F|})) = \ker(a^{|F|})$) as ϕ just acts by permuting the coordinates, there is a $\varphi : F \to \mathrm{Aut}_K(A)$ making the following diagram commute

$$\begin{array}{ccc} G^{|F|} & \xrightarrow{\phi} & G^{|F|} \\ {\scriptstyle a^{|F|}}\big\downarrow & & \big\downarrow{\scriptstyle a^{|F|}} \\ A^{|F|} & \xrightarrow{\varphi} & A^{|F|} \end{array}$$

In particular, this means that, for any $f \in F$ and $g \in G^{|F|}$, we have

$$a^{|F|}(\phi_f(g)) = \varphi_f(a^{|F|}(g)), \tag{3.4.2}$$

where $\phi_f := \phi(f)$ and $\varphi_f := \varphi(f)$.

Since, by assumption, $1 \in \mathcal{B}$, we have that F is in $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$, as it fits into the short exact sequence $1 \to 1 \to F \xrightarrow{\mathrm{id}} F \to 1$. Now consider the sequence

$$1 \to \underbrace{B^{|F|} \rtimes_\phi 1}_{\cong B^{|F|}} \xrightarrow{(b^{|F|},1)} G^{|F|} \rtimes_\phi F \xrightarrow{(a^{|F|},\mathrm{id})} A^{|F|} \rtimes_\varphi F \to 1,$$

where $B^{|F|} \rtimes_\phi 1 \cong B^{|F|} \in \mathcal{B}$ and $A^{|F|} \rtimes_\varphi F \in \mathcal{A}$, since $\mathcal{A}$ is closed under extensions (cf. Remark 3.4.1). Clearly, $(b^{|F|}, 1)$ is a K-group homomorphism. One can check that $(a^{|F|}, \mathrm{id})$ is also a K-group homomorphism, given our choice of φ and (3.4.2). Finally, it is clear that the sequence is exact, since the maps come from two short exact sequences. □

3.5 A comparison theorem

Let k be a number field and X a nice variety over k. Let $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$ and define

$$X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} := \bigcap_{F \in \mathcal{A}} \bigcap_{[Z] \in H^1(X,F)} \bigcup_{[\tau] \in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathcal{O}}), \tag{3.5.1}$$

where $\mathcal{O} : \mathbf{Sch}_k^{opp} \to \mathbf{Sets}$ is any functor such that $X(\mathbf{A}_k)^{\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1}$ functorially. Suppose that there exists some $\mathcal{S}_k \subset \mathcal{L}_k$ such that the following three conditions hold.

Condition 1. $\mathcal{S}_k$ is contained in $\mathcal{C}_k$, is closed under k-twists, and is such that, for any nice variety W over k, we have $W(\mathbf{A}_k)^{\mathcal{O}} \subset W(\mathbf{A}_k)^{\mathcal{S}_k}$.

Condition 2. For any nice variety W' over k, we have $W'(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} \subset W'(\mathbf{A}_k)^{\mathcal{O}}$.

Condition 3. $\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)$ is closed under taking closed k-subgroups, under taking k-twists, under taking direct products, and is such that conclusion of Lemma 3.4.6 (with $\mathcal{B} = \mathcal{S}_k$) holds.

Theorem 3.5.1. *With $k, X, \mathcal{A}, \mathcal{O}$, and $\mathcal{S}_k$ as above, we have*

$$X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}. \tag{3.5.2}$$

In other words, Theorem 3.5.1 gives us a dictionary between the mixed-type obstruction set $X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$ and the descent-type obstruction set $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$.

3.6 Proof of Theorem 3.5.1

Since Theorem 3.5.1 is a generalisation of the main theorems in [Dem09b] and [Sko09], the reader will find that many statements in this section are close variations of statements in [Dem09b] and [Sko09]; nonetheless, we will often try to sketch the proofs of these results, for the reader's convenience. We remark, however, that we use a *finesse* − Proposition 3.6.12 − not present (and not needed) in [Dem09b] and [Sko09], which gives us flexibility in the applications of Theorem 3.5.1 (see §3.7).

In this section, if $\mathcal{A} \subset \mathcal{L}_k$, the notation "$(f, Z, F) \in \mathcal{A}(X)$" will mean "$f : Z \to X$ is an F-torsor over X, for some $F \in \mathcal{A}$". Similarly, "$(Z, F) \in \mathcal{A}(X)$" will mean "$Z \to X$ is an F-torsor over X, for some $F \in \mathcal{A}$".

3.6.1 Proof that $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} \supset X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$.

All the results in this subsection just require $\mathcal{S}_k$ to satisfy Conditions 1 and 3 (without Lemma 3.4.6).

Lemma 3.6.1. *Let X be a smooth, projective variety over a number field k. Let $(f, Z, F) \in \mathcal{A}(X)$. Then Z is smooth and projective.*

Proof. This follows as f is finite and étale. □

Definition 3.6.2. Let $(Y, G), (Z, H) \in \mathcal{L}_k(X)$. An X-torsor morphism $f = (\phi, \psi) : (Y, G) \to (Z, H)$ is a pair (ϕ, ψ) where $\phi : Y \to Z$ is an X-morphism and $\psi : G \to H$ is a homomorphism, compatible in the sense that the diagram

$$\begin{array}{ccc} Y \times_X G_X & \xrightarrow{\phi \times \psi} & Z \times_X H_X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\phi} & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes, where the vertical maps are the action maps. We say that the X-torsor morphism $f = (\phi, \psi)$ is surjective if ϕ (equivalently, ψ) is surjective.

Remark 3.6.3. If $f = (\phi, \psi) : (Y, G) \to (Z, H)$ is surjective, then $Y \to Z$ is a $(\ker \psi)$-torsor over Z.

Proposition 3.6.4. *Let X be a nice variety over a number field k. Let $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$ and let $(f, Z, F) \in \mathcal{A}(X)$. Then there exist:*

(i) an $F' \in \mathcal{A}$,

(ii) a $(X', F') \in \mathcal{A}(X)$ with X' geometrically integral,

(iii) a 1-cocycle $\sigma \in Z^1(k, F)$,

(iv) a morphism of X-torsors $(\phi, \psi) : (X', F') \to (Z^\sigma, F^\sigma)$

such that (P_v) lifts to some $(Q_v) \in X'(\mathbf{A}_k)^{\mathcal{O}}$.

Proof. By [Sto07, Proposition 5.17] (which works for $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$), we have

$$X(\mathbf{A}_k)^{\mathcal{A}} = \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,F)} f^\sigma(Z^\sigma(\mathbf{A}_k)^{\mathcal{A}}).$$

Since $(x_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathcal{A}}$, there exists some $\sigma \in Z^1(k, F)$ such that (x_v) lifts to some $(y_v) \in Z^\sigma(\mathbf{A}_k)^{\mathcal{A}}$. Let $\Omega_{k,\mathbf{C}}$ denote the set of complex places of k and let $Z^\sigma(\mathbf{A}_k^{nc})$ denote the restricted product $\prod'_{v \in (\Omega_k \setminus \Omega_{k,\mathbf{C}})} Z^\sigma(k_v)$, where the restriction is the usual adelic one. Let $Z^\sigma = Z_1 \coprod ... \coprod Z_n$ be the decomposition of Z^σ into its k-connected components. By [Sto07, Proposition 5.11], we have that

$$Z^\sigma(\mathbf{A}_k^{nc})^{\mathcal{A}} = Z_1(\mathbf{A}_k^{nc})^{\mathcal{A}} \coprod ... \coprod Z_n(\mathbf{A}_k^{nc})^{\mathcal{A}}.$$

Now, since (x_v) lifts to some point $(y_v) \in Z^\sigma(\mathbf{A}_k)^{\mathcal{A}}$, we can assume that $y_v \in Z_i(k_v)$ for all non-complex places v, for some Z_i a k-connected component of Z^σ; hence, $Z_i(\mathbf{A}_k) \neq \emptyset$. Since Z_i is connected and $Z_i(\mathbf{A}_k) \neq \emptyset$, by [Sto07, Lemma 5.5] we can deduce that Z_i is geometrically connected, and thus that $Z_i \to X$ is a torsor under the stabiliser $F_i := \mathrm{Stab}_{F^\sigma}(Z_i) \subset F^\sigma$ of Z_i. Since $(x_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$, there exists a $\tau \in Z^1(k, F_i)$ such that (x_v) lifts to a point in $Z_i^\tau(\mathbf{A}_k)^{\mathcal{O}}$. Let $[\xi] \in H^1(k, F^\sigma)$ be the image of $[\tau]$ under the morphism induced by $F_i \to F^\sigma$. Then we have an X-torsors morphism $(Z_i^\tau, F_i^\tau) \to ((Z^\sigma)^\xi, (F^\sigma)^\xi)$. By letting $X' := Z_i^\tau$ and $F' := F_i^\tau$ and by letting $\nu := t_\sigma(\xi)$ be the image of ξ under the standard bijection $t_\sigma : Z^1(k, F^\sigma) \to Z^1(k, F)$ (cf. [Ser01, Proposition 35bis, §I.5.3]), we get the required result. □

The next proposition, which allows us to lift cocycles, is essentially due

to Demarche ([Dem09b, Prop. 4]), with the caveat that, while he requires $(P_v) \in X(\mathbf{A}_k)^{\text{ét Br}}$, we take $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$.

Proposition 3.6.5 (Demarche, [Dem09b] Prop. 4)**.** *Let X be a nice variety over a number field k. Let $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$ and let $(f, Y, G) \in \mathcal{L}_k(X)$. Let*

$$1 \to H \to G \to F \to 1$$

be a short exact sequence in $\mathcal{L}_k$, with $H \in \mathcal{C}_k$ and $F \in \mathcal{A}$. We denote the pushforward of $f : Y \to X$ under the morphism $G \to F$ by $(Z, F) \in \mathcal{A}(X)$. Let $\sigma \in Z^1(k, F)$ be a cocycle coming from Proposition 3.6.4 applied to the torsor $Z \to X$ and the point (P_v). Then the class $[\sigma] \in H^1(k, F)$ lifts to a class $[\tau] \in H^1(k, G)$.

Remark 3.6.6. Let $H \in \mathcal{C}_k$. Since H contains a k-point (the identity), it is also geometrically connected (see [Gro65, 4.5.14]).

Proof. In the proof of [Dem09b, Prop. 4], the only place where Demarche uses the fact that $(P_v) \in X(\mathbf{A}_k)^{\text{ét Br}}$ is in a passage of the proof of "Le lemme 7 implique la proposition 4". In this passage, Demarche wants to prove the existence of some torsor of type λ'. To do so, he appeals to [Sko01, Cor. 6.1.3 (1)]. Indeed, he knows that (P_v) lifts to some $(Q_v) \in X'(\mathbf{A}_k)^{\text{Br}}$ for X' as in the conclusion of Proposition 3.6.4; this means in particular that $X'(\mathbf{A}_k)^{\text{Br}_1} \neq \emptyset$. But X' is proper and geometrically integral (and, in particular, $\overline{k}[X']^\times = \overline{k}^\times$), and $\emptyset \neq X'(\mathbf{A}_k)^{\text{Br}_1} = \bigcap_\lambda X'(\mathbf{A}_k)^{\text{Br}_\lambda}$ (see [Sko01] for the definitions). Hence, since $X'(\mathbf{A}_k)^{\text{Br}_{\lambda'}} \neq \emptyset$, the required torsor of type λ' exists by Corollary 6.1.3 (1) in [Sko01].

In our case, $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$. By Proposition 3.6.4, (P_v) lifts to some $(Q_v) \in X'(\mathbf{A}_k)^{\mathcal{O}}$ for X' as in the conclusion of Proposition 3.6.4. Hence, $\emptyset \neq X'(\mathbf{A}_k)^{\mathcal{O}} \subset X'(\mathbf{A}_k)^{\text{Br}_1}$ (where the inclusion holds by assumption on $\mathcal{O}$),

and since X' is proper and geometrically integral, we can appeal to [Sko01, Cor. 6.1.3 (1)] as well to conclude that the relevant torsor of type λ' exists. All the other parts of the proof of [Dem09b, Prop. 4] are unchanged. □

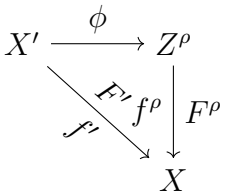
We have an analogue of [Dem09b, Thm 1].

Theorem 3.6.7. *Let X be nice over k. Then $X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$.*

Proof. Let $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$. We fix some $G \in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)$ and some $(g, Y, G) \in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)(X)$. Then G fits, say, into the short exact sequence

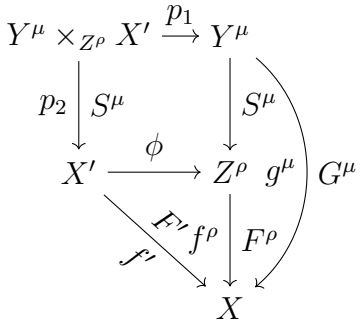
$$1 \to S \to G \to F \to 1,$$

for some $F \in \mathcal{A}$ and $S \in \mathcal{S}_k$. Using the push-forward of g along $G \to F$, we decompose $Y \to X$ into $(Y, S) \in \mathcal{S}_k(Z)$ and $(Z, F) \in \mathcal{A}(X)$. By applying Proposition 3.6.4 to (Z, F) and (P_v), we get an $F' \in \mathcal{A}$, an $(X', F') \in \mathcal{A}(X)$ with X' geometrically integral, a 1-cocycle $\rho \in Z^1(k, F)$, and an X-torsor morphism $\phi : X' \to Z'$ such that (P_v) lifts to a point $(Q_v) \in X'(\mathbf{A}_k)^{\mathcal{O}}$. Hence, we have the commutative triangle



We now apply Proposition 3.6.5 to the X-torsor (g, Y, G), the short exact sequence $1 \to S \to G \to F \to 1$ (note here that $S \in \mathcal{C}_k$, as $\mathcal{S}_k \subset \mathcal{C}_k$ by Condition 1), and $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$ to conclude that $[\rho] \in H^1(k, F)$ lifts to some $[\mu] \in H^1(k, G)$. It follows that the G^μ-torsor $g^\mu : Y^\mu \to X$ can be decomposed naturally into the S^μ-torsor $Y^\mu \to Z^\rho$ and the F^ρ-torsor $Z^\rho \to X$.

Now consider the fibred product $Y^\mu \times_{Z^\rho} X'$. This is naturally a S^μ-torsor over X', and we remark that $(p_2, Y^\mu \times_{Z^\rho} X', S^\mu) \in \mathcal{S}_k(X')$ as $\mathcal{S}_k$ is closed under k-twists. The following diagram summarises the constructions done so far:



Note that, since X is smooth and projective and $X' \to X$ is finite and étale, then X' is also smooth and projective; moreover, since X' is geometrically connected, it follows that X' is nice. Hence, we can use Condition 1 to deduce that

$$X'(\mathbf{A}_k)^{\mathcal{O}} \subset \bigcup_{[\nu] \in H^1(k, S^\mu)} p_2^\nu((Y^\mu \times_{Z^\rho} X')^\nu(\mathbf{A}_k)).$$

In particular, there is some $\nu \in Z^1(k, S^\mu)$ such that (Q_v) lifts to $(R_v) \in (Y^\mu \times_{Z^\rho} X')^\nu(\mathbf{A}_k)$. Then, arguing in an identical manner as in [Dem09b, Thm 1], we can conclude that

$$(P_v) \in \bigcup_{[\tau] \in H^1(k, G)} g^\tau(Y^\tau(\mathbf{A}_k)).$$

Since $(P_v) \in X(\mathbf{A}_k)^{\mathcal{A}, \mathcal{O}}$, $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$, and (g, Y, G) were arbitrary, we get that

$$X(\mathbf{A}_k)^{\mathcal{A}, \mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)},$$

as required. □

3.6.2 Proof that $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} \subset X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$.

In this subsection, we need to assume that $\mathcal{S}_k$ satisfies Conditions 1, 2, 3 in their fullness.

Proposition 3.6.8 (Based on [Sto07], Prop. 5.17)**.** *Let X be a smooth, proper variety over a number field k. Let $(Y,F) \in \mathcal{A}(X)$. For any $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$, there exists a twist $(Y',F') \in \mathcal{A}(X)$ of (Y,F) with the following property. For any surjective X-torsor morphism $(\phi,\psi) : (Z,G) \to (Y',F')$, where $(Z,G) \in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)(X)$, there exists a twist $Z' \to Y'$ of $(Z, \ker\psi) \in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)(Y')$ such that (P_v) lifts to a point in $Z'(\mathbf{A}_k)$.*

Proof of Proposition 3.6.8. We follow closely the first part of the proof of [Sto07, Prop. 5.17], modifying it accordingly when needed.

Let $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$ and let $(Y,F) \in \mathcal{A}(X)$. Since X is proper, by [Sko01, Prop. 5.3.2] there are only finitely many twists (Y^σ, F^σ) of (Y,F) such that $Y^\sigma(\mathbf{A}_k) \neq \emptyset$. Moreover, since $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$ and $\mathcal{A} \subset \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)$, there is at least a $[\sigma] \in H^1(k,F)$ such that (P_v) lifts to some $(Q_v) \in Y^\sigma(\mathbf{A})$. Let $Y_1,...,Y_s$ be the finitely many twists of (Y,F) such that (P_v) lifts to some adelic point in them. Let $\tau(j) \subset \{1,...,s\}$ be the set of indices i such that, for every $(Z,G) \in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)(X)$ mapping to $Y_j \to X$ as an X-torsor morphism, there is a twist Z^ξ that lifts (P_v) and induces a twist of Y_j that is isomorphic to Y_i. Following Stoll, one can show that

(i) $\tau(j) \neq \emptyset$ (using Condition 3);

(ii) if $i \in \tau(j)$, then $\tau(i) \subset \tau(j)$;

(iii) for some j, we have $j \in \tau(j)$.

We can take Y' to be Y_j, where $j \in \tau(j)$. □

Lemma 3.6.9 (Skorobogatov, [Sko09] Cor. 2.7)**.** *Let Y be a proper variety over k, and let $(g, Z, G) \in \mathcal{L}_k(Y)$. Then the set $Y(\mathbf{A}_k)^g$ is closed in $Y(\mathbf{A}_k)$.*

The following proposition is essentially due to Skorobogatov.

Proposition 3.6.10 (Skorobogatov, [Sko09] Prop. 2.3)**.** *Let X be a variety over k. Let $(Y, F) \in \mathcal{A}(X)$ and $(Z, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(Y)$. Then there exist a $(V, G') \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(X)$ and a surjective X-torsor morphism $(\theta, \psi) : (V, G') \to (Y, F)$ such that there is a surjective Y-torsor morphism $(\Theta, \Psi) : (V, \ker \psi) \to (Z, G)$ with $\ker \Psi \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$.*

Proof. As in the proof of [Sko09, Prop. 2.3], let $V := \mathfrak{R}_{Y/X}(Z) \times_X Y$, where $\mathfrak{R}_{Y/X}(Z)$ is the Weil restriction (which exists since $Y \to X$ is finite étale, cf. [Bos90, §7.6]), and let $\theta : V \to Y$ be the second projection. Consider the group $G^m \rtimes_\rho F(\overline{k})$, where $m = |F(\overline{k})|$ and where ρ is the action of the constant group $F(\overline{k})$ on G^m by permutation of coordinates. Since $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$, by Condition 3 we also have that $G^m \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$. Moreover, since $F(\overline{k}) \in \mathcal{A}$ is a constant group acting on G^m by permuting the coordinates, and since we are assuming Condition 3, by Lemma 3.4.6 we know that $G^m \rtimes_\rho F(\overline{k}) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$. Now consider the k-twist of $G^m \rtimes_\rho F(\overline{k})$ given by $G' := \mathfrak{R}_{F/\operatorname{Spec} k}(G_F) \rtimes_\phi F$, where ϕ is the action induced by ρ (and where $\mathfrak{R}_{F/\operatorname{Spec} k}(G_F)$ exists, again cf. [Bos90, §7.6]). Since, by Condition 3, $\mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$ is closed under taking k-twists, it follows that $G' \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$. Following the proof of [Sko09, Prop. 2.3], one can show that $V \to X$ is a G'-torsor, that θ and the natural projection $\psi : G' \to F$ together give a surjective X-torsor morphism $(\theta, \psi) : (V, G') \to (Y, F)$, and that there exists a surjective Y-torsor morphism $(\Theta, \Psi) : (V, \ker \psi) \to (Z, G)$. Moreover, since $\mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$ is closed under taking (closed) k-subgroups (by Condition 3), we also have that $\ker \psi \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$. □

We can now prove that $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$ is "well-behaved" with respect to torsors under groups in $\mathcal{A}$. Our proof is almost verbatim that of [Sko09, Thm 1.1].

Theorem 3.6.11. *Let X be a nice variety over k. For any $F \in \mathcal{A}$ and any $[Z] \in H^1(X, F)$, we have*

$$X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} = \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}).$$

Proof. The inclusion "$\supset$" follows by pulling back torsors. Indeed, suppose that

$$(P_v) \in \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}),$$

say $(P_v) = f^\tau((Q_v))$ for some $(Q_v) \in Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$ and some $[\tau] \in H^1(k, F)$. Then, if $(g, Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(X)$, by pulling back $(g, Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(X)$ along $f^\tau : Z^\tau \to X$ we obtain the torsor $(p_1, Z^\tau \times_X Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(Z^\tau)$. Since $(Q_v) \in Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$ and $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$, there is some $[\mu] \in H^1(k, G)$ such that (Q_v) lifts to some $(R_v) \in (Z^\tau \times_X Y)^\mu(\mathbf{A}_k) = (Z^\tau \times_X Y^\mu)(\mathbf{A}_k)$. By the commutativity of obvious pullback diagram, we conclude that $(P_v) \in g^\mu(Y^\mu(\mathbf{A}_k))$. Since $(g, Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(X)$ was arbitrary, it follows that $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$.

For the other inclusion, let $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$. Let $(\tilde{f}, \tilde{Z}, \tilde{F}) \in \mathcal{A}(X)$ be the twist of $(f, Z, F) \in \mathcal{A}(X)$ coming from Proposition 3.6.8. It is clear that we only need to show that (P_v) lifts to a point in $\tilde{Z}(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$. Suppose not, for a contradiction. Then

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{(g',W',G')\in \mathrm{Ext}(\mathcal{A},\mathcal{S}_k)(\tilde{Z})} \tilde{Z}(\mathbf{A}_k) \backslash \tilde{Z}(\mathbf{A}_k)^{g'},$$

where the cover on the right hand side is an open cover since each $\tilde{Z}(\mathbf{A}_k)^{g'}$ is closed in $\tilde{Z}(\mathbf{A}_k)$, by Lemma 3.6.9. Using compactness, we have

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{i=1}^{n} \tilde{Z}(\mathbf{A}_k) \backslash \tilde{Z}(\mathbf{A}_k)^{g_i},$$

for some $(g_i, W_i, G_i) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(\tilde{Z})$. Let $(g, W, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(\tilde{Z})$ denote the fibred product of the g_i over $\tilde{Z}$; here $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$, since $\mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)$ is closed under finite fibred products. It is clear from the construction that $\tilde{f}^{-1}((P_v)) \cap \tilde{Z}(\mathbf{A}_k)^g = \emptyset$. By Proposition 3.6.10, there is a torsor $(V, L) \in \mathrm{Ext}(\mathcal{A}, \mathcal{S}_k)(X)$ and a surjective X-torsor morphism $(\theta, \psi) : (V, L) \to (\tilde{Z}, \tilde{F})$ such that there exists a surjective $\tilde{Z}$-torsor morphism $(V, \ker \psi) \to (W, G)$. To get the required contradiction, we can then argue identically as in the proof of [Sko09, Thm 1.1]. □

Proposition 3.6.12. *Let X be a nice variety over k. If $X(\mathbf{A}_k)^{\mathcal{A}} \neq \emptyset$, then*

$$X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} = \bigcap_{F \in \mathcal{A}} \bigcap_{\substack{[Z] \in H^1(X,F) \\ Z \text{nice}}} \bigcup_{[\tau] \in H^1(k,F)} f^{\tau}(Z^{\tau}(\mathbf{A}_k)^{\mathcal{O}}),$$

that is, we can restrict our attention to nice *X-torsors $[Z] \in H^1(X, F)$ only.*

Proof. The inclusion "$\subset$" is clear, so we just prove the opposite inclusion.

Let $(P_v) \in \bigcap_{F \in \mathcal{A}} \bigcap_{[Z] \in H^1(X,F)\text{nice}} \bigcup_{[\tau] \in H^1(k,F)} f^{\tau}(Z^{\tau}(\mathbf{A}_k)^{\mathcal{O}})$. Fix $F \in \mathcal{A}$ and $[f : Z \to X] \in H^1(X, F)$. We need to show that $(P_v) \in \bigcup_{[\tau] \in H^1(k,F)} f^{\tau}(Z^{\tau}(\mathbf{A}_k)^{\mathcal{O}})$. Consider $(\overline{Z}, \overline{F}) \in \mathcal{A}_{\overline{k}}(\overline{X})$. Let $\overline{Z}_0$ be a connected component of $\overline{Z}$, and let $\overline{F}_0 \leq_{\overline{k}} \overline{F}$ be its stabiliser. Then we have an $\overline{X}$-torsor morphism $(\overline{Z}_0, \overline{F}_0) \to (\overline{Z}, \overline{F})$, given by inclusion.

By the notion of cofinal coverings in [Sto07] and by [Sto07, Lemma 5.7], using our assumption that $X(\mathbf{A}_k)^{\mathcal{A}} \neq \emptyset$, we know that there exist a $F' \in \mathcal{A}$ and a $(f', W, F') \in \mathcal{A}(X)$ with W geometrically connected such that there is

an $\overline{X}$-torsor morphism $(\overline{W}, \overline{F'}) \to (\overline{Z}_0, \overline{F}_0)$. Hence, by composition, we get an $\overline{X}$-torsor morphism

$$(\overline{W}, \overline{F'}) \to (\overline{Z}_0, \overline{F}_0) \to (\overline{Z}, \overline{F}),$$

with W geometrically connected. We can then apply [Sto07, Lemma 5.6] to conclude that there exists a twist $(f^\tau, Z^\tau, F^\tau) \in \mathcal{A}(X)$ of (f, Z, F) such that there is an X-torsor morphism

$$(W, F') \to (Z^\tau, F^\tau),$$

with W geometrically connected. Since W is also smooth and projective (cf. Lemma 3.6.1), then W is nice, and the same is true for all its twists.

Without loss of generality (twisting if necessary), we can assume that (P_v) lifts to some point in $W(\mathbf{A}_k)^{\mathcal{O}}$. We can then use [Sto07, Lemma 5.6] and the functoriality of $\mathcal{O}$ to deduce that $(P_v) \in \bigcup_{\tau \in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathcal{O}})$. Since F and f were arbitrary, we conclude that

$$\bigcap_{F \in \mathcal{A}} \bigcap_{[Z] \in H^1(X,F) \text{ nice}} \bigcup_{[\tau] \in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathcal{O}}) \subset X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}.$$

Hence, $\bigcap_{F \in \mathcal{A}} \bigcap_{[Z] \in H^1(X,F) \text{ nice}} \bigcup_{[\tau] \in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathcal{O}}) = X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$, meaning that we can restrict our attention to nice torsors only. □

Remark 3.6.13. The statements and proofs of [Sto07, Lemma 5.5], [Sto07, Lemma 5.6], and [Sto07, Lemma 5.7], as appearing in [Sto07], are not completely correct (among other things, there are some problems with the case concerning abelian covers). The statements are, however, correct for the cases we care about, i.e. for $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$, even though some of the proofs need fixing (e.g. one can prove the finite and solvable cases of [Sto07, Lemma 5.7]

by using [Sto07, Prop. 5.17], [Sto07, Lemma 5.10], and [Sto07, Prop. 5.11]).

Proof of Theorem 3.5.1. By Theorem 3.6.7, $X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}} \subset X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}$.

If $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} = \emptyset$, then trivially $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} \subset X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}$, and we are done. So suppose $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} \neq \emptyset$. In particular, this implies that $X(\mathbf{A}_k)^{\mathcal{A}} \neq \emptyset$. By Theorem 3.6.11, for any $[Z] \in H^1(X,F)$ and any $F \in \mathcal{A}$, we have

$$X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} = \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}).$$

Hence,

$$\begin{aligned} X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)} &= \textstyle\bigcap_{F\in\mathcal{A}} \bigcap_{[Z]\in H^1(X,F) \text{ nice}} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{S}_k)}) \\ &\subset \textstyle\bigcap_{F\in\mathcal{A}} \bigcap_{[Z]\in H^1(X,F) \text{ nice}} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathcal{O}}) \\ &= X(\mathbf{A}_k)^{\mathcal{A},\mathcal{O}}, \end{aligned}$$

where the inclusion in the second line comes from Condition 2, and the equality in the bottom line comes from Proposition 3.6.12 (and the fact that $X(\mathbf{A}_k)^{\mathcal{A}} \neq \emptyset$). □

Remark 3.6.14. Proposition 3.6.12 makes Theorem 3.5.1 robust enough for applications. Indeed, we usually only have Condition 2 as we have stated it, and not a more general version as, for example, is the case for the proof of $X(\mathbf{A}_k)^{\mathrm{Desc}} \subset X(\mathbf{A}_k)^{\text{ét Br}}$ in [Sko09]: there, as a corollary to a result by Gabber, one has that $W(\mathbf{A}_k)^{\mathcal{L}_k} \subset W(\mathbf{A}_k)^{\text{ét Br}}$ for *all* smooth, projective varieties over k (cf. [Sko09, Lemma 2.8]). Proposition 3.6.12 tells us that, in fact, our version of Condition 2 is sufficient.

3.7 Some applications of Theorem 3.5.1

3.7.1 The étale-Brauer set and its variations

Let X be a nice variety over k. Consider the "solvable" étale-Brauer set

$$X(\mathbf{A}_k)^{\mathrm{Sol,Br}} := \bigcap_{F\in\mathcal{F}_k^{\mathrm{Sol}}} \bigcap_{[Z]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Br}}).$$

In our terminology, $\mathcal{O} = \mathrm{Br}$. As we have seen in the introduction, for any nice variety W over k we have that $W(\mathbf{A}_k)^{\mathrm{Br}} = W(\mathbf{A}_k)^{\mathcal{C}_k}$. Hence, in our setting, we can take $\mathcal{S}_k = \mathcal{C}_k$. As Condition 1 is clear, we only need to check that the "solvable version" of Conditions 2 and 3 hold. The latter condition is easy to see by the closure properties of $\mathcal{F}_k^{\mathrm{Sol}}$, and the former one holds since $\mathcal{C}_k \subset \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)$. Hence, by applying the "solvable version" of Theorem 3.5.1, we can deduce the following.

Theorem 3.7.1. *Let X be nice over k. Then $X(\mathbf{A}_k)^{\mathrm{Sol,Br}} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)}$.*

□

3.7.2 The algebraic étale-Brauer set and its variations

Let X be a nice variety over k and let $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$. Consider

$$X(\mathbf{A}_k)^{\mathcal{A},\mathrm{Br}_1} := \bigcap_{F\in\mathcal{A}} \bigcap_{[Z]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbf{A}_k)^{\mathrm{Br}_1}).$$

In our terminology, $\mathcal{O} = \mathrm{Br}_1$.

Theorem 3.7.2 ([Sko01], Thm 6.1.1)**.** *Let Z be a variety over a number field*

k such that $\overline{k}[Z]^\times = \overline{k}^\times$. *Then*

$$Z(\mathbf{A}_k)^{\mathrm{Br}_1} = \bigcap_{\lambda:\hat{S}\hookrightarrow \mathrm{Pic}(\overline{Z})} \bigcup_{type(Y,f)=\lambda} f(Y(\mathbf{A}_k)), \tag{3.7.1}$$

where $\hat{S}$ *ranges over the finitely generated* $\mathrm{Gal}(\overline{k}/k)$*-submodules of* $\mathrm{Pic}(\overline{X})$.

Remark 3.7.3. We recall that the type λ of a torsor $(f, Y, S) \in \mathcal{M}_k(Z)$ is defined by $\chi([Y]) =: \lambda$, where χ is the map in the fundamental exact sequence due to Colliot-Thélène and Sansuc (see [Sko01])

$$0 \to H^1(k, S) \to H^1(Z, S) \xrightarrow{\chi} \mathrm{Hom}_{\mathrm{Gal}_k}(\hat{S}, \mathrm{Pic}\,\overline{Z}) \xrightarrow{\partial} H^2(k, S) \to H^2(Z, S),$$

where $\hat{S} := \mathrm{Hom}_{\overline{k}-groups}(\overline{S}, \mathbf{G}_{m,\overline{k}})$. We will say more about the type of a torsor in Chapter 4.

In particular, Theorem 3.7.2 tells us that, for any nice variety W over k, we have $W(\mathbf{A}_k)^{\mathcal{M}_k} = W(\mathbf{A}_k)^{\mathrm{Br}_1}$. A sensible idea would be to take $\mathcal{S}_k = \mathcal{M}_k$, but unfortunately $\mathcal{M}_k \not\subset \mathcal{C}_k$, so Condition 1 fails. We take instead $\mathcal{S}_k = \mathcal{T}_k \subset \mathcal{C}_k$. Since our aim is to apply Theorem 3.5.1, we check that Conditions 1, 2, and 3 hold for our choice of $\mathcal{S}_k$.

Condition 1. This holds since $\mathcal{T}_k \subset \mathcal{C}_k$, the k-twist of an algebraic torus is again an algebraic torus, and since Proposition 3.7.2 (which tells us that $W(\mathbf{A}_k)^{\mathrm{Br}_1} \subset W(\mathbf{A}_k)^{\mathcal{T}_k}$) holds.

Condition 2. As we have seen, $W(\mathbf{A}_k)^{\mathcal{M}_k} \subset W(\mathbf{A}_k)^{\mathrm{Br}_1}$. But, by definition of groups of multiplicative type,

$$\mathcal{M}_k = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathcal{A}b_k \subset \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) \subset \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k),$$

where, in the first inclusion, we have used the fact that $\mathcal{T}_k \subset \mathcal{A}b_k \subset \mathrm{Sol}_k$ and that Sol_k is closed under taking quotients. Hence, Condition 2 holds for $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$.

Condition 3. Notice that $\mathcal{T}_k$ is clearly closed under taking finite direct products and that it contains the trivial group. By Lemma 3.4.3, we have that $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k)$ is closed under taking direct products. Moreover, the hypotheses of Lemma 3.4.6 hold (with $\mathcal{B} = \mathcal{T}_k$), meaning that the conclusion of Lemma 3.4.6 also holds. Hence, it remains to check that $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k)$ is closed under taking closed k-subgroups and k-twists.

Lemma 3.7.4. *We have* $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{A}, \mathcal{M}_k)$.

Proof. We first prove that $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$. The inclusion "$\subset$" is clear. For the other inclusion, let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ fit into the short exact sequence $1 \to M \to G \to F \to 1$, with $F \in \mathcal{F}_k$ and $M \in \mathcal{M}_k$. But M contains a maximal subtorus T (cf. [DGA70, IV, §1.3]), which is a normal subgroup of G (since T is characteristic in M) and with $G/T \in \mathcal{F}_k$. Hence, G fits into the short exact sequence

$$1 \to T \to G \to G/T \to 1,$$

with $T \in \mathcal{T}_k$ and $G/T \in \mathcal{F}_k$. We now prove that $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k)$. By the closure properties of solvable groups, we have that $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathrm{Sol}_k$ and $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k) \cap \mathrm{Sol}_k$. But we have just seen that $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$; by intersecting both sides with Sol_k, the result follows. □

Lemma 3.7.5. $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k)$ *is closed under* k*-subgroups.*

Proof. Let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and let $H \leq_k G$ be a k-subgroup of G. Suppose that G fits into a short exact sequence $1 \to T \to G \to F \to 1$, where $T \in \mathcal{T}_k$

and $F \in \mathcal{F}_k$. Then $H \cap T \in \mathcal{M}_k$ and $H/(T \cap H) \in \mathcal{F}_k$, so H fits into the short exact sequence

$$1 \to H \cap T \to H \to H/(H \cap T) \to 1,$$

that is, $H \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$. By Lemma 3.7.4, we deduce that $H \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. An analogous proof also gives the result for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$, once we notice that $G \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$ implies, since solvable groups are closed under extensions, that G is solvable, meaning that H is also solvable and thus that $H/(T \cap H) \in \mathcal{F}_k^{\mathrm{Sol}}$. □

Lemma 3.7.6. *$\mathcal{M}_k$ is closed under taking k-subgroups and under "base changing/restricting".*

Proof. The first statement is clear, by definition of groups of multiplicative type. For the second statement, let $M \in \mathcal{M}_k$ and let l/k be a finite extension. Then, since "being a group of multiplicative type" is a geometric condition, $M_l \in \mathcal{M}_l$. Moreover, since $\mathcal{M}_l = \mathrm{Ext}(\mathcal{F}_l, \mathcal{T}_l) \cap \mathcal{A}b_l$ and since the Weil restriction preserves tori, commutative groups, finite groups, and short exact sequences, it follows that $\mathfrak{R}_{l/k}(M_l) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathcal{A}b_k = \mathcal{M}_k$. □

Lemma 3.7.7. $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k)$ *is closed under k-twists.*

Proof. By Lemma 3.7.6, we can apply Lemma 3.4.5 with $\mathcal{B} = \mathcal{M}_k$ to conclude that $\mathrm{Ext}(\mathcal{A}, \mathcal{M}_k)$ is closed under taking k-twists. By Lemma 3.7.4, we have that $\mathrm{Ext}(\mathcal{A}, \mathcal{M}_k) = \mathrm{Ext}(\mathcal{A}, \mathcal{T}_k)$, hence the result. □

Hence, Condition 3 holds for $\mathcal{A} \in \{\mathcal{F}_k, \mathcal{F}_k^{\mathrm{Sol}}\}$. Since all the hypotheses in the statement of Theorem 3.5.1 are satisfied, we have the following.

Theorem 3.7.8. *Let X be nice over k. Then $X(\mathbf{A}_k)^{\mathcal{A},\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{A},\mathcal{T}_k)}$.*

We can characterise the sets $\mathrm{Ext}(\mathcal{A}, \mathcal{T}_k) \subset \mathcal{L}_k$ in a slightly different way.

Lemma 3.7.9. $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathcal{RR}_k$. *In particular,* $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathcal{RR}_k^{Sol}$.

Proof. We follow the proof of [Mey10, Lemma 2.2]. We first prove that if $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, then $G \in \mathcal{R}_k$; since $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking k-subgroups, it follows that $G \in \mathcal{RR}_k$. So let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ with short exact sequence $1 \to T \to G \to F \to 1$. Then G^0/T is trivial, since finite and connected, meaning that $T = G^0$. It follows that $R_u(G) = R_u(G^0) = R_u(T)$ is trivial. Hence, $G \in \mathcal{R}_k$, as required. For the converse implication, suppose that $G \in \mathcal{L}_k \backslash \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. Then clearly $G^0 \notin \mathcal{T}_k$. If $G^0 \notin \mathcal{R}_k$, then we are done. So suppose that $G^0 \in \mathcal{R}_k$. Its commutator $\mathcal{D}G^0$ is a connected linear k-subgroup, which is reductive and semisimple (since G^0 is reductive) ([Mil, Thm 15.1]), and non-trivial, since otherwise $G^0 \in \mathcal{T}_k$, a contradiction. Since $\mathcal{D}G^0$ is connected and semisimple, it contains a k-subgroup of type A_1 ([Spr98, Thm 7.2.4]). But groups of type A_1 all have non-reductive k-subgroups, so we are done. The last claim is clear, since $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathrm{Sol}_k$. □

Corollary 3.7.10. *Let X be nice over k. Then*

$$X(\mathbf{A}_k)^{\text{ét Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)} = X(\mathbf{A}_k)^{\mathcal{RR}_k},$$
$$X(\mathbf{A}_k)^{\mathrm{Sol,Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k)} = X(\mathbf{A}_k)^{\mathcal{RR}_k^{Sol}}.$$

Proof. Clear from Theorem 3.7.8, Lemma 3.7.4, and Lemma 3.7.9. □

3.8 Unipotent linear algebraic groups give no obstruction

In this section, we show that unipotent linear algebraic groups over k usually give no obstructions whatsoever − this is used in the next section, where we give a translation of descent-type obstruction sets under unipotent, reductive,

and solvable linear algebraic groups in terms of Brauer-type obstruction sets.

Theorem 3.8.1. *Let X be a nice variety over k. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$. Then $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)^{\mathcal{B}}$.*

Before we can prove the above theorem, we need to recall some preliminary results. Let $\mathcal{A} \subset \mathcal{B}$ be as in the statement of Theorem 3.8.1 and let $G \in \mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$, say fitting into the short exact sequence

$$1 \to U_G \to G \to A_G \to 1,$$

with $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. We want to show that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets, for $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$.

Since we are dealing with pointed sets and *not* necessarily with groups, showing that $\ker(H^1(K, G) \to H^1(K, A_G))$ is trivial is not enough to conclude injectivity. Fortunately, U_G is a normal K-subgroup of G, meaning that we can use the following result.

Proposition 3.8.2 ([Ser01], §I.5.5, Cor. 2)**.** *Let $[\tau] \in H^1(K, G)$. Then the elements of $H^1(K, G)$ with the same image in $H^1(K, A_G)$ as $[\tau]$ are in bijection with the elements of the quotient of $H^1(K, U_G^\tau)$ by the action of the group $H^0(K, A_G^\tau)$.* □

Corollary 3.8.3. *The map of pointed sets $H^1(K, G) \to H^1(K, A_G)$ is injective.*

Proof. Since $\mathcal{U}_K$ is closed under K-twists (as "being unipotent" is a geometric property), then $U_G^\tau \in \mathcal{U}_K$. But $H^1(K, U^\tau)$ is trivial (cf. [Ser01, §III.2.1, Prop. 6]), meaning that the quotient of $H^1(K, U_G^\tau)$ by the action of the group $H^0(K, A_G^\tau)$ is also trivial. The result then follows by Proposition 3.8.2. □

The surjectivity of $H^1(K,G) \to H^1(K,A_G)$ follows from a "vanishing" theorem for unipotent K-liens by Douai (see [Bor93, Cor. 4.2]), which requires some knowledge of non-abelian second Galois cohomology. We refer the reader to [FSS98, (1.2) and (1.5)] and to [Bor93] for more details on the construction of K-liens and on the theory of non-abelian cohomology.

Finally, since $H^1(K,G) \to H^1(K,A_G)$ preserves base points, we have the following.

Corollary 3.8.4. *For $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$,*

$$H^1(K,G) \cong H^1(K,A_G) \tag{3.8.1}$$

as pointed sets. □

Proof of Theorem 3.8.1. Let $G \in \mathcal{B}$. Since $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A},\mathcal{U}_k)$, G fits into a short exact sequence

$$1 \to U_G \to G \to A_G \to 1,$$

where $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. Fix $[g : Y \to X] \in H^1(X,G)$. We can push forward Y along $p : G \to A_G$ to obtain $[a_g : Z \to X] \in H^1(X,A_G)$.

The inclusion $X(\mathbf{A}_k)^g \subset X(\mathbf{A}_k)^{a_g}$ is easy. For the opposite inclusion, let $(x_v) \in X(\mathbf{A}_k)^{a_g}$, so that $[Z]((x_v)) \in \mathrm{im}(H^1(k,A_G) \to \prod_{v\in\Omega_k} H^1(k_v,A_G))$. Our aim is to show that $[Y]((x_v)) \in \mathrm{im}(H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G))$, as, using the functorial description of obstructions (cf. Chapter 2), this would indeed imply that $(x_v) \in X(\mathbf{A}_k)^g$.

By Corollary 3.8.4, for $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$, we have that

$$p_* : H^1(K,G) \xrightarrow{\sim} H^1(K,A_G) \tag{3.8.2}$$

is an isomorphism of pointed sets, where p_* is the map induced by the quotient

$p : G \to A_G$.

Now, for each place $v \in \Omega_k$, there is a commutative diagram

$$\begin{array}{ccccc} H^1(X,G) & \xrightarrow{x_v^*} & H^1(k_v,G) & \xleftarrow{\mathrm{res}_v} & H^1(k,G) \\ \downarrow p_*^X & & p_*^{-1}\uparrow\downarrow p_* & & p_*^{-1}\uparrow\downarrow p_* \\ H^1(X,A_G) & \xrightarrow{x_v^*} & H^1(k_v,A_G) & \xleftarrow{\mathrm{res}_v} & H^1(k,A_G). \end{array}$$

From the left square, we deduce that

$$p_*(x_v^*([Y])) = x_v^*(p_*^X([Y])) = x_v^*([Z]) = \mathrm{res}_v\,\alpha,$$

where we have used the fact that, by construction, $p_*^X([Y]) = [Z]$, and that there exists some $\alpha \in H^1(k, A_G)$ such that $x_v^*([Z]) = \mathrm{res}_v\,\alpha$ for all $v \in \Omega_k$, since by assumption $x_v^*([Z]) \in \mathrm{im}(H^1(k,A_G) \to \prod_{v\in\Omega_k} H^1(k_v,A_G))$. Moreover, from the right square, we have that

$$\mathrm{res}_v\,\alpha = p_*\,\mathrm{res}_v(p_*^{-1}(\alpha)).$$

Hence, by letting $\beta := p_*^{-1}(\alpha) \in H^1(k,G)$, we deduce that

$$p_*(x_v^*([Y])) = p_*(\mathrm{res}_v\,\beta).$$

By applying p_*^{-1}, we thus obtain that $x_v^*([Y]) = \mathrm{res}_v\,\beta$ for all $v \in \Omega_k$, that is, $(x_v) \in X(\mathbf{A}_k)^g$. Hence, $X(\mathbf{A}_k)^{a_g} \subset X(\mathbf{A}_k)^g$, and so

$$X(\mathbf{A}_k)^{a_g} = X(\mathbf{A}_k)^g. \tag{3.8.3}$$

Intersecting over all torsors in $H^1(X,G)$ and all $G \in \mathcal{B}$ gives

$$\bigcap_{G\in\mathcal{B}} \bigcap_{[Y]\in H^1(X,G)} X(\mathbf{A}_k)^{a_g} = \bigcap_{G\in\mathcal{B}} \bigcap_{[Y]\in H^1(X,G)} X(\mathbf{A}_k)^{g}. \tag{3.8.4}$$

For any $G \in \mathcal{A} \subset \mathcal{B}$, we have $G = A_G$ and $g = a_g$. Hence, the left-hand side of (3.8.4) can be rewritten as

$$\begin{aligned}\bigcap_{G\in\mathcal{B}} \bigcap_{\substack{[g:Y\to X]\\ \in H^1(X,G)}} X(\mathbf{A}_k)^{a_g} &= X(\mathbf{A}_k)^{\mathcal{A}} \cap \underbrace{\left(\bigcap_{G\in\mathcal{B}\setminus\mathcal{A}} \bigcap_{\substack{[a_g:Z\to X]\\ \in H^1(X,A_G)}} X(\mathbf{A}_k)^{a_g} \right)}_{\supset X(\mathbf{A}_k)^{\mathcal{A}}} \\ &= X(\mathbf{A}_k)^{\mathcal{A}}.\end{aligned}$$

Hence, since the right-hand side of (3.8.4) is, by definition, $X(\mathbf{A}_k)^{\mathcal{B}}$, we get that $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)^{\mathcal{B}}$, as required. □

Remark 3.8.5. In Theorem 3.8.1, we use $\mathcal{U}_k$ just to guarantee that $H^1(K,G) \cong H^1(K,A_G)$ as pointed sets for $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$ (cf. (3.8.2)). In theory, we can get a similar result as Theorem 3.8.1 if we replace $\mathcal{U}_k$ with any other $\mathcal{S} \subset \mathcal{L}_k$ such that (3.8.2) holds for $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$: for example, when k is a totally imaginary number field, a good candidate for such an $\mathcal{S}$ is the set of semisimple simply connected linear algebraic groups over k – see [Ser01, §3.1] and [Bor93, Cor. 5.1].

Example 3.8.6. Any $G \in \mathcal{A}b_k$ can be written as $G = U \times M$, where $U \in \mathcal{U}_k$ and $M \in \mathcal{M}_k$ (cf. [Mil12, 1.21]). Hence, we have the inclusions $\mathcal{M}_k \subset \mathcal{A}b_k \subset \mathrm{Ext}(\mathcal{M}_k, \mathcal{U}_k)$, implying that $X(\mathbf{A}_k)^{\mathcal{M}_k} = X(\mathbf{A}_k)^{\mathcal{A}b_k}$.

3.9 Translating the descent-type obstructions under $\mathcal{U}_k$, $\mathcal{R}_k$, and $\mathcal{S}ol_k$

As corollaries of Theorem 3.8.1 and of Theorem 3.5.1, we obtain the following translations of the descent-type sets $X(\mathbf{A}_k)^{\mathcal{U}_k}$, $X(\mathbf{A}_k)^{\mathcal{R}_k}$, and $X(\mathbf{A}_k)^{\mathcal{S}ol_k}$ in terms of Brauer-type obstruction sets.

Corollary 3.9.1. *Let X be a nice variety over k. Then $X(\mathbf{A}_k)^{\mathcal{U}_k} = X(\mathbf{A}_k)$.*

Proof. Let $\mathcal{A} = \{1\}$ and $\mathcal{B} = \mathcal{U}_k$. Clearly $\mathcal{U}_k \subset \mathrm{Ext}(\{1\}, \mathcal{U}_k) = \mathcal{U}_k$, and so the hypotheses of Theorem 3.8.1 are satisfied. Therefore, we can conclude that $X(\mathbf{A}_k)^{\mathcal{U}_k} = X(\mathbf{A}_k)^{\{1\}} = X(\mathbf{A}_k)$, as required. □

Remark 3.9.2. Since, when $K \in \{k, k_v \text{ for all } v \in \Omega_k\}$, the pointed set $H^1(K, U)$ is trivial for all $U \in \mathcal{U}_k$ (cf. [Ser01, §III.2.1, Prop. 6]), we have that, in the proof of Corollary 3.9.1, (3.8.2) holds without having to resort to non-abelian second Galois cohomology.

Since linear algebraic unipotent groups over k give no obstruction, and since reductive groups are, in a sense, the "opposite" of unipotent groups, the following shouldn't be too surprising.

Corollary 3.9.3. *Let X be a nice variety over k. Then $X(\mathbf{A}_k)^{\mathcal{L}_k} = X(\mathbf{A}_k)^{\mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{R}_k$ and $\mathcal{B} = \mathcal{L}_k$. It is clear that $\mathcal{L}_k = \mathrm{Ext}(\mathcal{R}_k, R_u(\mathcal{L}_k)) \subset \mathrm{Ext}(\mathcal{R}_k, \mathcal{U}_k)$. This means that the hypotheses of Theorem 3.8.1 are satisfied. Therefore, we can conclude that $X(\mathbf{A}_k)^{\mathcal{L}_k} = X(\mathbf{A}_k)^{\mathcal{R}_k}$, as required. □

The above also suggests the following proposition.

Proposition 3.9.4. *Let $\mathcal{B} \subset \mathcal{L}_k$ be such that $B/R_u(B) \in \mathcal{B}$, for any $B \in \mathcal{B}$. Let X be a nice variety over k. Then $X(\mathbf{A}_k)^{\mathcal{B}} = X(\mathbf{A}_k)^{\mathcal{B} \cap \mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{B} \cap \mathcal{R}_k \subset \mathcal{B}$. By assumption $B/R_u(B) \in \mathcal{B} \cap \mathcal{R}_k$ for any $B \in \mathcal{B}$, so we can apply Theorem 3.8.1 to conclude the result. □

Corollary 3.9.5. *$X(\mathbf{A}_k)^{Sol_k} = X(\mathbf{A}_k)^{Sol_k \cap \mathcal{R}_k}$. In particular,*

$$X(\mathbf{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} = X(\mathbf{A}_k)^{Sol_k}.$$

Proof. By Proposition 3.9.4, the first statement is clear, since solvable linear algebraic groups are closed under quotients. Hence, we just need to prove the last statement. We show that $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Sol}_k \cap \mathcal{R}_k$. Let $G \in \mathrm{Sol}_k \cap \mathcal{R}_k$. Then G^0 is reductive, connected, and solvable. Hence, $G^0 \in \mathcal{T}_k$. Moreover, since $G \in \mathrm{Sol}_k$, then $G/G^0 \in \mathcal{F}_k^{\mathrm{Sol}}$. Conversely, let $G \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$. Then $G \in \mathrm{Sol}_k$, as solvable groups are closed under extensions, and $G \in \mathcal{R}_k$ by Lemma 3.7.9. Hence, $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Sol}_k \cap \mathcal{R}_k$. From Corollary 3.7.10 and the fact that $X(\mathbf{A}_k)^{\mathrm{Sol}_k} = X(\mathbf{A}_k)^{\mathrm{Sol}_k \cap \mathcal{R}_k}$, we can then deduce the required result. □

3.10 A summary of the obstructions

Let X be a nice variety over k. We give here a summary of the relations between the various obstruction sets seen so far. These relations can be summarised by Figure 3.1 below (the notation "$X(\mathbf{A}_k)^{\mathcal{C}_k=\mathrm{Br}}$" means "$X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathcal{C}_k}$", and so on).

The rest of this chapter is devoted to reminding the reader of why all the inclusions in Figure 3.1 can be strict.

Proposition 3.10.1. *In general, $X(k) \subsetneq X(\mathbf{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét Br}}$.*

Proof. Poonen [Poo10], using fibrations over Châtelet surfaces, has constructed examples over any number field k of nice varieties X over k with $X(\mathbf{A}_k)^{\text{ét Br}} \neq \emptyset$

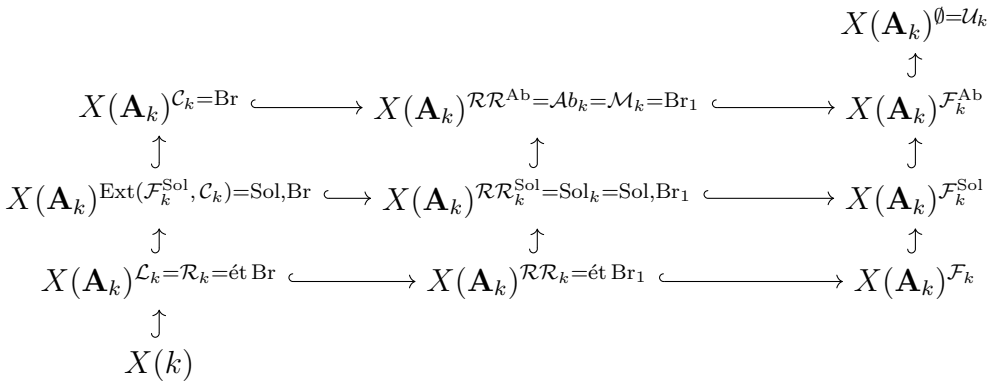


Figure 3.1:

but $X(k) = \emptyset$. □

Proposition 3.10.2. *In general,* $X(\mathbf{A}_k)^{\mathcal{R}\mathcal{R}^{Ab}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1} \subsetneq X(\mathbf{A}_k)^{\emptyset/\mathcal{U}_k}$.

Proof. See [BSD75] for an example over $k = \mathbf{Q}$. □

Remark 3.10.3. More strongly, in general we have $X(\mathbf{A}_k)^{\mathcal{T}_k} \subsetneq X(\mathbf{A}_k)^{\emptyset/\mathcal{U}_k}$. Indeed, let X be the (nice) degree 4 del Pezzo surface over $k = \mathbf{Q}$ in [BSD75]. We know that $X(\mathbf{A}_k)^{\mathrm{Br}_1} = \emptyset$ but $X(\mathbf{A}_k) \neq \emptyset$. Moreover, since X is geometrically rational, $\pi_1(\overline{X}) = 0$, meaning that there are no non-trivial finite torsors over X. In particular, $X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)} = X(\mathbf{A}_k)^{\mathcal{T}_k}$. Hence, $\emptyset = X(\mathbf{A}_k)^{\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathcal{M}_k} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)\cap\mathcal{A}b_k} = X(\mathbf{A}_k)^{\mathcal{T}_k} \neq X(\mathbf{A}_k)$.

Proposition 3.10.4. *In general,*

(a) $X(\mathbf{A}_k)^{\mathrm{Ab,Br}_1} \subsetneq X(\mathbf{A}_k)^{\mathcal{R}\mathcal{R}^{Ab}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$;

(b) $X(\mathbf{A}_k)^{\mathrm{Ab,Br}} \subsetneq X(\mathbf{A}_k)^{\mathcal{C}_k/\mathrm{Br}}$;

(c) the column in Figure 3.1 with endpoints given by $X(\mathbf{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét Br}_1}$ *and* $X(\mathbf{A}_k)^{\mathcal{R}\mathcal{R}^{Ab}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$ *can be strictly contained in the column with endpoints given by* $X(\mathbf{A}_k)^{\mathcal{F}_k}$ *and* $X(\mathbf{A}_k)^{\mathcal{F}_k^{Ab}}$;

(d) $X(\mathbf{A}_k)^{\mathcal{C}_k/\mathrm{Br}} \not\subset X(\mathbf{A}_k)^{\mathrm{Ab},\mathrm{Br}_1}$;

(e) $X(\mathbf{A}_k)^{\mathcal{F}_k^{Sol}} \not\subset X(\mathbf{A}_k)^{\mathcal{RR}^{Ab}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$.

Proof. Let $Y = Y_{a,b,c}$ and $X = Y_{a,b,c}/\langle\iota\rangle$ be, respectively, the (nice) K3 and Enriques surfaces over $k = \mathbf{Q}$ from [VAV11]. Then $(f, Y, F) \in \mathcal{F}_k^{\mathrm{Ab}}(X)$, where $F = \mathbf{Z}/2\mathbf{Z}$.

(a), (b), (d). Using Kummer theory and following [VAV11], it is not difficult to show that $\bigcup_{[\tau]\in H^1(X,F)} f^\tau(Y^\tau(\mathbf{A}_k)^{\mathrm{Br}_1})) = \emptyset$, meaning, in particular, that $X(\mathbf{A}_k)^{\mathrm{Ab},\mathrm{Br}_1} = \emptyset$. Moreover, in [BBM+16], the authors have shown that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$.

(c), (e). Note that $Y_{a,b,c}(\mathbf{A}_k)^{\mathcal{F}_k} = Y_{a,b,c}(\mathbf{A}_k) \neq \emptyset$ and $Y_{a,b,c}(\mathbf{A}_k)^{\text{ét Br}_1} = Y_{a,b,c}(\mathbf{A}_k)^{\mathrm{Br}_1} = \emptyset$. □

Proposition 3.10.5. *In general,*

(a) $X(\mathbf{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} \subsetneq X(\mathbf{A}_k)^{\mathcal{F}_k^{Ab}}$;

(b) $X(\mathbf{A}_k)^{\mathcal{RR}^{Ab}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1} \not\subset X(\mathbf{A}_k)^{\mathcal{F}_k}$.

Proof. (a), (b). Let X be the (nice) bielliptic surface over $k = \mathbf{Q}$ from Skorobogatov's counterexample (see [Sko01], Chapter 8). Then $X(\mathbf{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} = \emptyset$ (cf. [HS02, §5.1]), but $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. □

Proposition 3.10.6. *In general,*

(a) the column in Figure 3.1 starting with $X(\mathbf{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét Br}}$ *can be strictly contained in the column starting with* $X(\mathbf{A}_k)^{\mathcal{RR}_k/\text{ét Br}_1}$;

(b) $X(\mathbf{A}_k)^{\mathcal{RR}_k/\text{ét Br}_1} \not\subset X(\mathbf{A}_k)^{\mathcal{C}_k/\mathrm{Br}}$.

Proof. (a), (b). In [HVA13], the authors have constructed a (nice) K3 surface X over $k = \mathbf{Q}$ such that $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}_1} \neq \emptyset$. Now, since any K3

surface is simply connected, we have that $\pi_1(\overline{X}) = 0$. In particular, we have that $X(\mathbf{A}_k)^{\text{ét Br}} = X(\mathbf{A}_k)^{\text{Br}}$ and $X(\mathbf{A}_k)^{\text{ét Br}_1} = X(\mathbf{A}_k)^{\text{Br}_1}$. □

Proposition 3.10.7. *The row in Figure 3.1 starting with* $X(\mathbf{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét Br}}$ *can be strictly contained in the row starting with* $X(\mathbf{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}},\mathcal{C}_k)/\text{Sol,Br}}$.

Proof. Let G be a finite, perfect (i.e. with $G = \mathcal{D}G$), non-abelian, simple group; in particular, G is not solvable. Let X and k be as in the conclusion of [Har00, Cor. 6.1]. Since $G = \pi(\overline{X})$ and G is simple, it follows that $X(\mathbf{A}_k)^{\mathcal{F}_k^{\text{Sol}}} = X(\mathbf{A}_k)$, as there are no non-trivial finite solvable covers of X. This implies that $X(\mathbf{A}_k)^{\text{Sol,Br}_1} = X(\mathbf{A}_k)^{\text{Br}_1}$ and $X(\mathbf{A}_k)^{\text{Sol,Br}} = X(\mathbf{A}_k)^{\text{Br}}$. Since G is perfect, $\text{Br}(\overline{X}) = 0$ (cf. the remark after [Har00, Cor. 6.1]), meaning that $X(\mathbf{A}_k)^{\text{Br}} = X(\mathbf{A}_k)^{\text{Br}_1}$ and that $\text{Br}(X)/\text{Br}k$ is finite. By taking k large enough, we can assume that $\text{Br}(X)/\text{Br}k$ is trivial and, thus, that $X(\mathbf{A}_k)^{\text{Br}} = X(\mathbf{A}_k)$. Hence,

$$X(\mathbf{A}_k)^{\text{Sol,Br}_1} = X(\mathbf{A}_k)^{\text{Br}_1} = X(\mathbf{A}_k)^{\text{Sol,Br}} = X(\mathbf{A}_k)^{\text{Br}} = X(\mathbf{A}_k)^{\mathcal{F}_k^{\text{Sol}}} = X(\mathbf{A}_k).$$

But, by construction of X, we have $X(\mathbf{A}_k)^{\mathcal{F}_k} \subsetneq X(\mathbf{A}_k)$. Since $X(\mathbf{A}_k)^{\text{ét Br}} \subset X(\mathbf{A}_k)^{\text{ét Br}_1} \subset X(\mathbf{A}_k)^{\mathcal{F}_k}$, the result follows. □

We conclude with an open question.

Question 3.10.8. What are, if they exist, the translations of $X(\mathbf{A}_k)^{\text{Ab,Br}}$ and $X(\mathbf{A}_k)^{\text{Ab,Br}_1}$ in pure "descent-type" terms? Is $X(\mathbf{A}_k)^{\text{Ab,Br}} = X(\mathbf{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Ab}},\mathcal{C}_k)}$? Is $X(\mathbf{A}_k)^{\text{Ab,Br}_1} = X(\mathbf{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Ab}},\mathcal{T}_k)}$?

Chapter 4

Iterating the algebraic étale-Brauer set

4.1 Notation

For convenience, we briefly remind the reader of the notation that we will use in this chapter. For any smooth variety X over k, the Brauer group of X is $\operatorname{Br} X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ and the Brauer-Manin set of X is

$$X(\mathbf{A}_k)^{\mathrm{Br}} := \bigcap_{\alpha \in \operatorname{Br} X} \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \operatorname{inv}_v(\alpha(x_v)) = 0 \right\},$$

where the $\operatorname{inv}_v : \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ are the local invariant maps coming from class field theory. The algebraic Brauer group of X is $\operatorname{Br}_1(X) := \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$, where $\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ and where $\operatorname{Br} X \to \operatorname{Br} \overline{X}$ is the canonical map induced by the natural morphism $\overline{X} \to X$. We define the algebraic Brauer-Manin set $X(\mathbf{A}_k)^{\mathrm{Br}_1}$ by restricting the intersection in the definition of the Brauer-Manin set to the elements in $\operatorname{Br}_1 X$. One can show that both $X(\mathbf{A}_k)^{\mathrm{Br}}$ and $X(\mathbf{A}_k)^{\mathrm{Br}_1}$ are closed in $X(\mathbf{A}_k)$ and that $\overline{X(k)} \subset X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{Br}_1}$

(see e.g. [Sko01, §5.2]).

Let $\mathcal{L}_k := \{G : G \text{ is a linear algebraic } k\text{-group}\}/\sim$, where $G_1 \sim G_2$ if and only if G_1 and G_2 are k-isomorphic as k-groups. We will abuse notation and write $G \in \mathcal{L}_k$ also to mean a representative of the k-isomorphism class of G. For any $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_k$, we let

$$\mathrm{Ext}(\mathcal{A}, \mathcal{B}) = \{G \in \mathcal{L}_k : G \text{ is an extension of } A \text{ by } B, \text{ for some } A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}/\sim .$$

For any $\mathcal{S} \subset \mathcal{L}_k$, the $\mathcal{S}$-descent set is

$$X(\mathbf{A}_k)^{\mathcal{S}} := \bigcap_{G \in \mathcal{S}} \bigcap_{[f:Y\to X] \in H^1_{\text{ét}}(X,G)} \bigcup_{[\tau] \in H^1_{\text{ét}}(k,G)} f^\tau(Y^\tau(\mathbf{A}_k));$$

when $\mathcal{S} = \emptyset$, we define $X(\mathbf{A}_k)^{\emptyset} := X(\mathbf{A}_k)$, while when $\mathcal{S} = \mathcal{L}_k$, the $\mathcal{L}_k$-descent set is just called descent set. For any $\mathcal{S} \subset \mathcal{L}_k$, the set $X(\mathbf{A}_k)^{\mathcal{S}}$ is closed in $X(\mathbf{A}_k)$ and contains the adelic closure of $X(k)$ (see [CDX16, Prop. 6.4]). Let $\mathcal{F}_k := \{G \in \mathcal{L}_k : G \text{ is finite}\}/\sim$.

The étale-Brauer set of X is

$$X(\mathbf{A}_k)^{\text{ét Br}} := \bigcap_{F \in \mathcal{F}_k} \bigcap_{[f:Y\to X] \in H^1_{\text{ét}}(X,F)} \bigcup_{[\tau] \in H^1_{\text{ét}}(k,F)} f^\tau(Y^\tau(\mathbf{A}_k)^{\mathrm{Br}}).$$

Similarly, we can define the algebraic étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}_1}$ by replacing "Br" with "Br_1" in the definition above. Both $X(\mathbf{A}_k)^{\text{ét Br}}$ and $X(\mathbf{A}_k)^{\text{ét Br}_1}$ are closed in $X(\mathbf{A}_k)$ (see the discussion after [CDX16, Prop. 6.6]), and they both contain the adelic closure of $X(k)$. Finally, for any $\mathcal{S}, \mathcal{S}' \subset \mathcal{L}_k$ and any $\star \in \{\emptyset, \mathrm{Br}, \mathrm{Br}_1, \text{ét Br}, \text{ét Br}_1, \mathcal{S}'\}$, we define

$$\mathrm{Iter}_{\mathcal{S}}(X(\mathbf{A}_k)^{\star}) := \bigcap_{G \in \mathcal{S}} \bigcap_{[f:Y\to X] \in H^1_{\text{ét}}(X,G)} \bigcup_{[\tau] \in H^1_{\text{ét}}(k,G)} f^\tau(Y^\tau(\mathbf{A}_k)^{\star}).$$

4.1.1 The étale fundamental group

Since the main theorem of this chapter will feature the étale fundamental group, we briefly recall here its definition. Let S be a scheme. We consider the category $\mathbf{F\acute{E}t}_S$ whose objects are pairs (f, T) where T is a scheme and $f : T \to S$ is a finite étale morphism and with

$$\begin{aligned} \mathrm{Mor}_{\mathbf{F\acute{E}t}_S}((f,T),(f',T')) &= \mathrm{Mor}_{\mathbf{Sch}_S}(T,T') \\ &= \{\text{finite étale } g : T \to T' | f' \circ g = f\}. \end{aligned}$$

Let s' be a geometric point of S. We consider the fibre functor

$$F_{s'} : \mathbf{F\acute{E}t}_S \to \mathbf{FSets}$$

given by $F_{s'}((f, T)) := f^{-1}(s')$, where **FSets** is the category of finite sets.

Definition 4.1.1. For S a connected scheme and s' a geometric point of S, we define the étale fundamental group of S with base point s' by $\pi_1^{\text{ét}}(S, s') := \mathrm{Aut}(F_{s'})$.

Remark 4.1.2. We will sometimes write $\pi_1^{\text{ét}}(S)$ for the étale fundamental group of S, omitting the base point.

4.2 Motivation

4.2.1 Some conjectures

The aim of this chapter is to give some evidence and partial answers to various conjectures and open questions about the arithmetic behaviour of rational points on certain classes of varieties over k. More specifically, the conjectures that we are interested in are the following.

Definition 4.2.1. A variety X over k is geometrically rationally connected if any two general points $x_1, x_2 \in \overline{X}$ can be joined by a chain of $\overline{k}$-rational curves.

Example 4.2.2. Examples of geometrically rationally connected varieties include geometrically unirational varieties and Fano varieties.

Conjecture 4.2.3 (Colliot-Thélène, [CT03, p.174])**.** Let X be a nice geometrically rationally connected variety over k. Then $\overline{X(k)} = X(\mathbf{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction is the only one for strong (equivalently, weak) approximation.

Definition 4.2.4. A smooth, projective, geometrically integral surface X over k is a K3 surface if it has trivial canonical bundle $\mathscr{K}_X$ and $H^1(X, \mathscr{O}_X) = 0$.

Definition 4.2.5. A smooth, projective, geometrically integral surface X over k is an Enriques surface if $\mathscr{K}_X \neq 0$, $\mathscr{K}_X^2 = 0$, and $H^1(X, \mathscr{O}_X) = 0$.

Remark 4.2.6. Any Enriques surface X over k arises as some quotient of a K3 surface Y by a fixed point-free involution ι, and the $\mathbf{Z}/2\mathbf{Z}$-torsor given by this quotient $Y \to X = Y/\iota$ is the only non-trivial geometrically connected étale cover of X (see [CV89, Prop. 1.3.1] and [Bea96, Prop. VIII.17]).

Conjecture 4.2.7 (Skorobogatov)**.** Let X be a nice K3 surface over k. Then $\overline{X(k)} = X(\mathbf{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction is the only one for strong (equivalently, weak) approximation.

Conjecture 4.2.8. Let X be a nice Enriques surface over k. Then $\overline{X(k)} = X(\mathbf{A}_k)^{\text{ét Br}}$. In other words, the étale-Brauer obstruction is the only one for strong (equivalently, weak) approximation.

Remark 4.2.9. In general, K3 surfaces over k do not satisfy $\overline{X(k)} = X(\mathbf{A}_k)$; see [HVA13] for an example over $k = \mathbf{Q}$ violating the Hasse principle. Similarly,

in [BBM+16], the authors have constructed an Enriques surface X over $k = \mathbf{Q}$ such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$; this implies that, for Enriques surfaces, $\overline{X(k)} = X(\mathbf{A}_k)^{\mathrm{Br}}$ does not hold in general.

4.2.2 Iteration

Another source of motivation for the content in this chapter is the following: for any nice variety X over k, the étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}}$ is currently the smallest general obstruction set known. Unfortunately, the étale-Brauer set is not small enough to explain all the failures of the Hasse principle: see e.g. [Poo10] for a counterexample. We thus want a way to construct obstruction sets smaller than $X(\mathbf{A}_k)^{\text{ét Br}}$. A possible strategy is to mimick the construction of the étale-Brauer set itself: for any nice variety X over k, the results in [Dem09b] and [Sko09] imply that $X(\mathbf{A}_k)^{\text{ét Br}} = X(\mathbf{A}_k)^{\mathcal{L}_k}$; if $\mathcal{S} \subset \mathcal{L}_k$ contains the trivial group, then the obstruction set $\mathrm{Iter}_{\mathcal{S}}(X(\mathbf{A}_k)^{\mathcal{L}_k})$ is certainly *potentially* smaller than $X(\mathbf{A}_k)^{\text{ét Br}}$. It turns out, however, that for certain choices of $\mathcal{S}$, the set $\mathrm{Iter}_{\mathcal{S}}(X(\mathbf{A}_k)^{\mathcal{L}_k})$ is the same as the original étale-Brauer set: this is the case, for example, when $\mathcal{S} = \mathcal{F}_k$ (cf. [Sko09, Thm 1.1]). It is natural to ask about the case when $\mathcal{S}$ is, in some sense, maximal, i.e. when $\mathcal{S} = \mathcal{L}_k$; in this case, we can think of $\mathrm{Iter}_{\mathcal{L}_k}(X(\mathbf{A}_k)^{\mathcal{L}_k})$ as an "iteration" of the descent set.

Question 4.2.10 (Poonen)**.** Let X be a nice variety over k. Is $\mathrm{Iter}_{\mathcal{L}_k}(X(\mathbf{A}_k)^{\mathcal{L}_k}) = X(\mathbf{A}_k)^{\text{ét Br}}$?

Remark 4.2.11. In [CDX16, Thm 7.5], the authors show that $Y(\mathbf{A}_k)^{\mathcal{L}_k} = Y(\mathbf{A}_k)^{\text{ét Br}}$ for any smooth, quasi-projective, geometrically connected variety Y over k, thus removing the properness condition from the earlier results in [Dem09b] and [Sko09]. As a consequence, we have that $\mathrm{Iter}_{\mathcal{L}_k}(X(\mathbf{A}_k)^{\mathcal{L}_k}) =$

$\mathrm{Iter}_{\mathcal{L}_k}(X(\mathbf{A}_k)^{\text{ét Br}})$ for any nice variety X over k.

We focus on a question similar to Question 4.2.10: we want to iterate the algebraic étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}_1}$. To make sense of this, we first need an analogue of the result $X(\mathbf{A}_k)^{\text{ét Br}} = X(\mathbf{A}_k)^{\mathcal{L}_k}$ for $X(\mathbf{A}_k)^{\text{ét Br}_1}$. Such an analogue is given by [Bal16, Thm 5.8] (cf. Theorem 3.7.8 in Chapter 3): if X is a nice variety over k, then

$$X(\mathbf{A}_k)^{\text{ét Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)},$$

where $\mathcal{T}_k := \{G \in \mathcal{L}_k : G \text{ is a torus}\}/\sim$.

Question 4.2.12. Let X be a nice variety over k. Is $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{ét Br}_1}$?

Remark 4.2.13. By putting together the results in [CDX16] and [Bal16], we have $Y(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)} = Y(\mathbf{A}_k)^{\text{ét Br}_1}$ for any smooth, quasi-projective, geometrically connected variety Y over k. From this, we can easily deduce that $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}) = \mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1})$ for any nice variety X over k.

4.3 Main result

Motivated by the above conjectures and questions, our main theorem is the following.

Theorem 4.3.1 (Main Theorem)**.** *Let X be a nice variety over k such that $\pi_1^{\text{ét}}(\overline{X})$ is finite. Then* $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{ét Br}_1}$.

Some comments:

- Let X be a nice geometrically rationally connected variety over k. Then $\pi_1^{\text{ét}}(\overline{X}) = 0$, meaning that Theorem 4.3.1 holds. (Compare this with the assertion in [CTS77b] that "la descente sur une variété rationnelle, lisse et complète, est une opération en un coup.") If moreover $\operatorname{Br}\overline{X} = 0$, as is the case when $\dim X = 2$ or $H^3_{\text{ét}}(\overline{X}, \mathbf{Z}_\ell(1))_{\text{tors}} = 0$ for all primes ℓ (cf. [CTS13, Lemma 1.3]), then $X(\mathbf{A}_k)^{\text{ét Br}_1} = X(\mathbf{A}_k)^{\text{Br}_1} = X(\mathbf{A}_k)^{\text{Br}}$. Hence, in this case, Theorem 4.3.1 tells us that $\text{Iter}_{\mathcal{T}_k}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{Br}}$, thus giving some evidence for Conjecture 4.2.3.

- Let X be a nice K3 surface over k. Then $\pi_1^{\text{ét}}(\overline{X}) = 0$, and thus the hypotheses of Theorem 4.3.1 are satisfied. When, moreover, $\operatorname{Br}\overline{X} = 0$ (see e.g. the comment after [SZ12, Prop. 5.1]), then Theorem 4.3.1 yields $\text{Iter}_{\mathcal{T}_k}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{Br}}$, thus giving some evidence for Conjecture 4.2.7.

- Let X be a nice Enriques surface over k. Then $\pi_1^{\text{ét}}(\overline{X}) \cong \mathbf{Z}/2\mathbf{Z}$, and so the hypotheses of Theorem 4.3.1 are satisfied. When $X(\mathbf{A}_k)^{\text{ét Br}} = X(\mathbf{A}_k)^{\text{ét Br}_1}$ (e.g. [VAV11]), then $\text{Iter}_{\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) = X(\mathbf{A}_k)^{\text{ét Br}}$, which is some evidence for Conjecture 4.2.8. Theorem 4.3.1 also applies to some higher-dimensional analogues of Enriques surfaces (cf. [BNWS11, §2]).

- Theorem 4.3.1 gives a positive answer to Question 4.2.12, assuming the finiteness of $\pi_1^{\text{ét}}(\overline{X})$; it would be interesting to see whether this condition can be weakened or removed (a possible weakening could be that of considering nice varieties X over k such that $\operatorname{Pic}\overline{Y}$ is finitely generated as a $\mathbf{Z}$-module for any finite cover $Y \to X$).

- In the literature, there are several conditional proofs of the existence of nice varieties X over k with $\pi_1^{\text{ét}}(\overline{X}) = 0$, $X(k) = \emptyset$, and $X(\mathbf{A}_k)^{\text{Br}} \neq \emptyset$:

see [SW95] for an example conditional on Lang's conjectures, [Poo01] for one conditional on the existence of a complete interesection satisfying certain properties, and [Sme17, Thm 4.1] for one conditional on the *abc* conjecture. Theorem 4.3.1 would apply to these examples.

4.4 Some properties of universal torsors

Let X be a variety over k with $\overline{k}[X]^\times = \overline{k}^\times$ and with $\operatorname{Pic}\overline{X}$ finitely generated as a $\mathbf{Z}$-module. Let $\mathcal{M}_k := \{G \in \mathcal{L}_k : G \text{ is of multiplicative type}\}/\sim$. For any $S \in \mathcal{M}_k$, let $\widehat{S} := \operatorname{Hom}_{\mathbf{GrpSch}_{\overline{k}}}(\overline{S}, \mathbf{G}_{m,\overline{k}})$ denotes the Cartier dual.

Definition 4.4.1 ([Sko01, Lemma 2.3.1])**.** Let $S \in \mathcal{M}_k$ and $Y \to X$ be an S-torsor. The type of the S-torsor $Y \to X$ is a homomorphism $\lambda_Y : \widehat{S} \to \operatorname{Pic}\overline{X}$ defined in the following way. Any character $\chi \in \widehat{S}$ induces a morphism $\chi_* : H^1(\overline{X}, \overline{S}) \to \operatorname{Pic}\overline{X}$; we define $\lambda_Y(\chi) := \chi_*[\overline{Y} \to \overline{X}]$, where $\overline{Y} \to \overline{X}$ is the $\overline{S}$-torsor over $\overline{X}$ coming from the S-torsor $Y \to X$ by base-change to $\overline{k}$.

Definition 4.4.2. An S-torsor $Y \to X$ is a universal torsor for X if its type $\lambda_Y : \widehat{S} \to \operatorname{Pic}\overline{X}$ is an isomorphism.

As explained in [Sko01, §2.3], universal torsors do not always exist over k, as a universal torsor over $\overline{k}$ might not descend to k. The following proposition, proven in [Sko99] by Skorobogatov (who extended earlier results from [CTS76], [CTS77b], [CTS77a], and [CTS87]), gives sufficient conditions for the existence of universal torsors with non-empty sets of adelic points.

Proposition 4.4.3 ([Sko01, Cor. 6.1.3(1)])**.** *Let X be a variety over k such that $\overline{k}[X]^\times = \overline{k}^\times$, $\operatorname{Pic}\overline{X}$ is finitely generated as a $\mathbf{Z}$-module, and $X(\mathbf{A}_k)^{\mathrm{Br}_1} \neq \emptyset$. Then there exists a universal torsor $W \to X$ with an adelic point.*

When universal torsors exist, they have some desirable properties. The following lemmas (which can be deduced from [CTS76], [CTS77b], [CTS77a], [CTS87, §2.1, Prop. 2.1.1 and Thm 2.1.2]) give some of these properties.

Lemma 4.4.4. *Let X be a nice variety over k with $\operatorname{Pic}\overline{X}$ torsion-free. Suppose that there exists a universal torsor $W \to X$ under S (a torus). Then W is geometrically connected, $\overline{k}[W]^\times = \overline{k}^\times$, and $\operatorname{Pic}\overline{W} = 0$.*

Lemma 4.4.5. *Let W be a smooth, geometrically integral variety over k such that $\overline{k}[W]^\times = \overline{k}^\times$ and $\operatorname{Pic}\overline{W} = 0$. Then $\operatorname{Br}_1(W) = \operatorname{Br} k$. In particular, $W(\mathbf{A}_k)^{\mathrm{Br}_1} = W(\mathbf{A}_k)$.*

Finally, universal torsors satisfy the following universal property: let X be a variety over k such that $\operatorname{Pic}\overline{X}$ is finitely generated as a $\mathbf{Z}$-module and $\overline{k}[X]^\times = \overline{k}^\times$; given a universal torsor $W \to X$ and any other torsor $Y \to X$ under some $M \in \mathcal{M}_k$, there is a $[\sigma] \in H^1_{\text{ét}}(k, M)$ such that there exists a map $W \to Y^\sigma$ of X-torsors (see the discussion after [Sko01, Defn 2.3.3]).

Remark 4.4.6. Using universal torsors, one can easily prove results such as the following: if X is a nice variety over k with $\operatorname{Pic}\overline{X}$ finitely generated as a $\mathbf{Z}$-module, then $\operatorname{Iter}_{\mathcal{M}_k}(X(\mathbf{A}_k)^{\mathrm{Br}_1}) = X(\mathbf{A}_k)^{\mathrm{Br}_1}$ (compare this with [CDX16, Cor. 4.2]).

Lemma 4.4.7. *Let W be a smooth, geometrically integral variety over k with $\overline{k}[W]^\times = \overline{k}^\times$ and $\pi_1^{\text{ét}}(\overline{W}) = 0$. Let $F \in \mathcal{F}_k$ and let $f : U \to W$ be an F-torsor over W. Then $\bigcup_{[\tau] \in H^1_{\text{ét}}(k,F)} f^\tau(U^\tau(\mathbf{A}_k)) = W(\mathbf{A}_k)$.*

Proof. Recall from e.g. [Sko01, §5.3] that, since W is geometrically integral and smooth,

$$\bigcup_{[\tau]} f^\tau(U^\tau(\mathbf{A}_k)) = \left\{ (x_v) \in W(\mathbf{A}_k) : \mathrm{ev}_{[U]}((x_v)) \in \operatorname{im}\left(H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F) \right) \right\},$$

where "$\mathrm{ev}_{[U]}((x_v))$" is the evalutation of $[f : U \to W] \in H^1_{\text{ét}}(W, F)$ at (x_v) and the map $H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F)$ is the canonical restriction map. Since W is geometrically integral, we have the exact sequence of fundamental groups (omitting base-points)

$$1 \to \pi_1^{\text{ét}}(\overline{W}) \to \pi_1^{\text{ét}}(W) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

From the hypothesis that $\pi_1^{\text{ét}}(\overline{W}) = 0$, we deduce that $\pi_1^{\text{ét}}(W) \cong \mathrm{Gal}(\overline{k}/k)$. Since $F \in \mathcal{F}_k$, using the Grothendieck-Galois theory we have that $H^1_{\text{ét}}(W, F) = H^1(\pi_1^{\text{ét}}(W), F(\overline{k}))$, where the action of $\pi_1^{\text{ét}}(W)$ on $F(\overline{k})$ is via $\mathrm{Gal}(\overline{k}/k)$. By using [Ser01, §5.8(a)], we deduce that $H^1_{\text{ét}}(k, F) = H^1_{\text{ét}}(W, F)$, which implies that the F-torsor $U \to W$ comes from some F-torsor $V \to \operatorname{Spec} k$. Then, for each place $v \in \Omega_k$, the evaluation map $\mathrm{ev}_{[U]}(x_v)$ is defined by the following commutative diagram of pullbacks:

$$\begin{array}{ccccc} V \times \operatorname{Spec} k_v & \longrightarrow & U = V \times W & \longrightarrow & V \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ \operatorname{Spec} k_v & \xrightarrow[x_v]{} & W & \xrightarrow[p]{} & \operatorname{Spec} k, \end{array}$$

i.e. $\mathrm{ev}_{[U]}(x_v) = [V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v]$. Note that the composition $p \circ x_v : \operatorname{Spec} k_v \to \operatorname{Spec} k$ is the canonical morphism induced by the inclusion $k \subset k_v$. Since the pullback of a pullback is again a pullback, we have that the F-torsor $V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v$ is the pullback of the F-torsor $V \to \operatorname{Spec} k$ along the canonical morphism $p \circ x_v : \operatorname{Spec} k_v \to \operatorname{Spec} k$, meaning that $[V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v] = \mathrm{res}_v([V])$. In other words, $\mathrm{ev}_{[U]}(x_v) = \mathrm{res}_v([V])$. By considering all the places $v \in \Omega_k$, we get that $\mathrm{ev}_{[U]}((x_v))$ is the image of $[V]$ under the canonical restriction map $H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F)$, as required. □

Lemma 4.4.8. *Let W be a smooth, geometrically integral variety over k such*

that $W(\mathbf{A}_k) = W(\mathbf{A}_k)^{\mathcal{F}_k} \neq \emptyset$. *Let* $(x_v) \in W(\mathbf{A}_k)$ *and let* $f : R \to W$ *be an* F*-torsor, for some* $F \in \mathcal{F}_k$. *Then there exists an* $F' \in \mathcal{F}_k$, *a 1-cocycle* $\sigma \in Z^1(k, F)$, *an* F'*-torsor* $R' \to W$ *over* W, *a morphism* $p : F' \to F^\sigma$, *and a morphism of* W*-torsors* $R' \to R^\sigma$ *such that the following diagram commutes*

$$\begin{array}{ccc} R' & \longrightarrow & R^\sigma \\ & \searrow^{f'} & \downarrow F^\sigma \\ & & W \end{array}$$

and such that R' *is geometrically connected and* (x_v) *lifts to a point in* $R'(\mathbf{A}_k)$.

Proof. By assumption, $W(\mathbf{A}_k) = W(\mathbf{A}_k)^{\mathcal{F}_k}$. Modifying the proof of [Sto07, Prop. 5.17] by using [CDX16, Prop. 6.3] yields

$$W(\mathbf{A}_k) = \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,F)} f^\sigma(R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}).$$

Since $(x_v) \in W(\mathbf{A}_k)$, there exists some $\sigma \in Z^1(k, F)$ such that (x_v) lifts to some $(y_v) \in R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}$. Let $\Omega_{k,\mathbf{C}}$ denote the set of complex places of k and let $R^\sigma(\mathbf{A}_k^{nc})$ denote the restricted product $\prod'_{v \in (\Omega_k \setminus \Omega_{k,\mathbf{C}})} R^\sigma(k_v)$, where the restriction is the usual adelic one. Let $R^\sigma = R_1 \coprod ... \coprod R_n$ be the decomposition of R^σ into its k-connected components. By [Sto07, Prop. 5.11] (also, see [HS13, Remark 9.114]), we have that

$$R^\sigma(\mathbf{A}_k^{nc})^{\mathcal{F}_k} = R_1(\mathbf{A}_k^{nc})^{\mathcal{F}_k} \coprod ... \coprod R_n(\mathbf{A}_k^{nc})^{\mathcal{F}_k}.$$

Now, since (x_v) lifts to some point $(y_v) \in R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}$, we can assume that $y_v \in R'(k_v)$ for all non-complex places v, where R' is a k-connected component of R^σ; hence, $R'(\mathbf{A}_k) \neq \emptyset$. Since R' is connected and $R'(\mathbf{A}_k) \neq \emptyset$, by [Sto07,

Lemma 5.5] we can deduce that R' is geometrically connected, and thus that $R' \to W$ is a torsor under the stabiliser $F' \subset F^\sigma$ of R'. In particular, $R' \to W$ is surjective, so we can change (y_v) at the complex places if necessary to obtain an adelic point $(y_v) \in R'(\mathbf{A}_k)$ above (x_v). □

Remark 4.4.9. Instead of the above two lemmas, in the proof of the main theorem we could also use the following lemma.

Lemma 4.4.10. *Let W be a smooth and geometrically integral variety over k with $\pi_1^{\text{ét}}(\overline{W}) = 0$. Let $F \in \mathcal{F}_k$ and let $U \to W$ be a torsor under F. Then there exists some $\sigma \in H^1_{\text{ét}}(k, F)$ such that $U^\sigma \to W$ is a trivial torsor under F^σ.*

Proof. Since W is geometrically integral, we have the exact sequence of fundamental groups (omitting base-points)

$$1 \to \pi_1^{\text{ét}}(\overline{W}) \to \pi_1^{\text{ét}}(W) \to \text{Gal}(\overline{k}/k) \to 1.$$

From the hypothesis that $\pi_1^{\text{ét}}(\overline{W}) = 0$, we deduce that $\pi_1^{\text{ét}}(W) \cong \text{Gal}(\overline{k}/k)$. Since $F \in \mathcal{F}_k$, using the Grothendieck-Galois theory we have that $H^1_{\text{ét}}(W, F) = H^1(\pi_1^{\text{ét}}(W), F(\overline{k}))$, where the action of $\pi_1^{\text{ét}}(W)$ on $F(\overline{k})$ is via $\text{Gal}(\overline{k}/k)$. By using [Ser01, §5.8(a)], we deduce that $H^1_{\text{ét}}(k, F) = H^1_{\text{ét}}(W, F)$. Hence, $U \to W$ is the pullback of some F-torsor $V \to \operatorname{Spec} k$ under the structure morphism $W \to \operatorname{Spec} k$. Let $\sigma = [V \to \operatorname{Spec} k] \in H^1_{\text{ét}}(k, F)$. Since $V^\sigma \to \operatorname{Spec} k$ is a trivial F^σ-torsor (that is, it admits a section), by the universal property of pullbacks it follows that the F^σ-torsor $U^\sigma \to W$ also admits a section, as required. □

4.5 Proof of the main theorem

Lemma 4.5.1. *Let X be a nice variety over k with $\pi_1^{\text{ét}}(\overline{X})$ finite. Then* $\operatorname{Pic}\overline{X}$ *is finitely generated as a* $\mathbf{Z}$*-module.*

Proof. Let $r \in \mathbf{N}$, and consider the Kummer sequence

$$0 \to \boldsymbol{\mu}_{r,\overline{k}} \to \mathbf{G}_{m,\overline{k}} \xrightarrow{t \mapsto t^r} \mathbf{G}_{m,\overline{k}} \to 0.$$

Passing to cohomology and identifying (non-canonically) $\boldsymbol{\mu}_{r,\overline{k}}$ with $\mathbf{Z}/r\mathbf{Z}$, we obtain an isomorphism $H^1_{\text{ét}}(\overline{X}, \mathbf{Z}/r\mathbf{Z}) \cong (\operatorname{Pic}\overline{X})[r]$, where we have used the fact that $H^0(\overline{X}, \mathbf{G}_m) = \overline{k}[X]^\times = \overline{k}^\times$ is divisible. Further, $H^1_{\text{ét}}(\overline{X}, \mathbf{Z}/r\mathbf{Z}) \cong \operatorname{Hom}(\pi_1^{\text{ét}}(\overline{X}), \mathbf{Z}/r\mathbf{Z})$ (cf. [Fu11, Prop. 5.7.20]), and hence

$$\operatorname{Hom}(\pi_1^{\text{ét}}(\overline{X}), \mathbf{Z}/r\mathbf{Z}) \cong (\operatorname{Pic}\overline{X})[r].$$

Since $\pi_1^{\text{ét}}(\overline{X})$ is finite, say with $|\pi_1^{\text{ét}}(\overline{X})| = d$, it follows that $(\operatorname{Pic}\overline{X})[r] = (\operatorname{Pic}^0\overline{X})[r] = 0$ for any $r \in \mathbf{N}$ with $\gcd(r, d) = 1$. Since $\overline{X}$ is proper, $\operatorname{Pic}^0\overline{X}$ is an abelian variety over $\overline{k}$; if $\operatorname{Pic}^0\overline{X} \neq 0$, then $\operatorname{Pic}^0\overline{X}[r] \cong (\mathbf{Z}/r\mathbf{Z})^{2\dim \operatorname{Pic}^0\overline{X}}$ is non-trivial for all $r \in \mathbf{N}$, a contradiction to the fact that $(\operatorname{Pic}^0\overline{X})[r] = 0$ when $\gcd(r, d) = 1$. Hence, $\operatorname{Pic}^0\overline{X} = 0$, which implies that $\operatorname{Pic}\overline{X} = \operatorname{NS}\overline{X}$ is finitely generated as a $\mathbf{Z}$-module. □

Proposition 4.5.2. *Let Y be a smooth and geometrically connected variety over k with $\pi_1^{\text{ét}}(\overline{Y}) = 0$. Let $W \to Y$ be a torsor under some connected linear algebraic group T over k. Then $\pi_1^{\text{ét}}(\overline{W})$ is abelian.*

Proof. Since $\overline{k}$ is an algebraically closed field of characteristic 0 and since the étale fundamental group does not change under base-change over extensions $K/\overline{k}$ of algebraically closed fields (cf. [Sza09, Second proof of Cor. 5.7.6

and Rmk 5.7.8] together with [Gro71, XII] and [Org03]), there is a "Lefschetz principle" and we can work over $\mathbf{C}$ instead of $\overline{k}$. Let $\mathbf{LFT_C}$ and $\mathbf{AN_C}$ denote, respectively, the category of schemes locally of finite type over $\mathbf{C}$ and the category of complex analytic spaces. The analytification functor $(-)^{\mathrm{an}} : \mathbf{LFT_C} \to \mathbf{AN_C}$ (cf. [Gro71, XII]) induces an equivalence of categories from the category of finite étale covers of $X \in \mathbf{LFT_C}$ to the category of finite étale covers of $X^{\mathrm{an}} \in \mathbf{AN_C}$ (cf. [Gro71, XII, Thm 5.1 "Théorème d'existence de Riemann"]); by [Gro71, XII, Cor. 5.2], if $X \in \mathbf{LFT_C}$ is connected, then (omitting base-points)

$$\pi_1^{\text{ét}}(X) \cong \widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}})}.$$

The fibration obtained by applying $(-)^{\mathrm{an}}$ to the $T_{\mathbf{C}}$-torsor $W_{\mathbf{C}} \to Y_{\mathbf{C}}$ induces the homotopy (exact) sequence

$$\pi_1^{\mathrm{top}}((T_{\mathbf{C}})^{\mathrm{an}}) \to \pi_1^{\mathrm{top}}((W_{\mathbf{C}})^{\mathrm{an}}) \to \pi_1^{\mathrm{top}}((Y_{\mathbf{C}})^{\mathrm{an}}) \to \pi_0^{\mathrm{top}}((T_{\mathbf{C}})^{\mathrm{an}}),$$

where $\pi_1^{\mathrm{top}}((T_{\mathbf{C}})^{\mathrm{an}})$ is abelian (since $(T_{\mathbf{C}})^{\mathrm{an}}$ is a topological group) and where $\pi_0^{\mathrm{top}}((T_{\mathbf{C}})^{\mathrm{an}}) = 0$ as $(T_{\mathbf{C}})^{\mathrm{an}}$ is connected (cf. [Gro71, XII, Prop. 2.4]). Since taking the profinite completion is right-exact (cf. [RZ00, Prop. 3.2.5]), we obtain the exact sequence

$$\pi_1^{\text{ét}}(T_{\mathbf{C}}) \to \pi_1^{\text{ét}}(W_{\mathbf{C}}) \to \pi_1^{\text{ét}}(Y_{\mathbf{C}}) \to 0,$$

where $\pi_1^{\text{ét}}(T_{\mathbf{C}})$ is abelian (as the profinite completion of an abelian group is abelian); since, by assumption (and by the Lefschetz principle) $\pi_1^{\text{ét}}(Y_{\mathbf{C}}) = 0$, from the above sequence we deduce that $\pi_1^{\text{ét}}(W_{\mathbf{C}})$ (and thus $\pi_1^{\text{ét}}(\overline{W})$) is a quotient of an abelian group and hence abelian, as required. □

Lemma 4.5.3. *Let W be a geometrically integral variety over k with $\overline{k}[W]^{\times} =$*

$\overline{k}^\times$ *and* $\operatorname{Pic}\overline{W}$ *torsion-free. Then* $\pi_1^{ab}(\overline{W}) = 0$*, where* π_1^{ab} *denotes the abelianised étale fundamental group.*

Proof. From e.g. [Sko01, pp. 35-36], we have that

$$\pi_1^{ab}(\overline{W}) = \varprojlim_n \operatorname{Hom}(H^1(\overline{W}, \mu_n), \overline{k}^\times).$$

For any $n \in \mathbf{N}$, the Kummer sequence yields the short exact sequence of $\operatorname{Gal}(\overline{k}/k)$-modules (cf. [Sko01, p. 36])

$$0 \to \overline{k}[W]^\times/\overline{k}[W]^{\times,n} \to H^1(\overline{W}, \mu_n) \to \operatorname{Pic}\overline{W}[n] \to 0$$

and, since $\overline{k}[W]^\times = \overline{k}^\times$ is divisible (implying that $\overline{k}[W]^\times/\overline{k}[W]^{\times,n} = 0$) and $\operatorname{Pic}\overline{W}[n] = 0$, we get that $H^1(\overline{W}, \mu_n) = 0$ for each n and thus that $\pi_1^{ab}(\overline{W}) = 0$, as required. □

Proof of Theorem 4.3.1. The inclusion $\operatorname{Iter}_{\operatorname{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) \subset X(\mathbf{A}_k)^{\text{ét Br}_1}$ holds by construction, so we just need to prove the opposite inclusion. We may assume that $X(\mathbf{A}_k)^{\text{ét Br}_1} \neq \emptyset$, since otherwise the conclusion of the theorem is trivially true as $\operatorname{Iter}_{\operatorname{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét Br}_1}) \subset X(\mathbf{A}_k)^{\text{ét Br}_1}$. Let $(x_v) \in X(\mathbf{A}_k)^{\text{ét Br}_1}$. We need to prove that, for any $G \in \operatorname{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and for any $[Z \to X] \in H^1_{\text{ét}}(X, G)$, there exists some $[\xi] \in H^1_{\text{ét}}(k, G)$ such that, for any $F' \in \mathcal{F}_k$ and for any $[U \to Z^\xi] \in H^1_{\text{ét}}(Z^\xi, F')$, there exists some $[\psi] \in H^1_{\text{ét}}(k, F')$ such that (x_v) lifts to a point in $U^\psi(\mathbf{A}_k)^{\text{Br}_1}$.

Step 1. Let $Z \to X$ be a torsor under G, for some $G \in \operatorname{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, say with G fitting into a short exact sequence

$$1 \to T \to G \to F \to 1,$$

with $T \in \mathcal{T}_k$ and $F \in \mathcal{F}_k$. Let $Y := Z/T$ and decompose the G-torsor $Z \to X$ into the T-torsor $Z \to Y$ and the F-torsor $Y \to X$. By [Dem09a, Lemme 2.2.7] (see also [CDX16, Lemma 7.1]) there exists some $[\sigma] \in H^1_{\text{ét}}(k, F)$, some $F_1 \in \mathcal{F}_k$, some F_1-torsor $Y_1 \to X$, and a X-torsor morphism $Y_1 \to Y^\sigma$ such that Y_1 is geometrically integral, and (x_v) lifts to a point in $Y_1(\mathbf{A}_k)^{\mathrm{Br}_1}$. Moreover, since X is smooth and projective and $Y_1 \to X$ is étale, it follows that Y is smooth, projective, and geometrically integral over k. By [Dem09a, Prop. 2.2.9] (see also [CDX16, Prop. 7.4]), we have that $[\sigma] \in H^1_{\text{ét}}(k, F)$ lifts to some $[\tau] \in H^1_{\text{ét}}(k, G)$, meaning that the diagram

$$\begin{array}{ccc} & & Z^\tau \\ & & \downarrow T^\tau \\ Y_1 & \to & Y^\sigma \\ & \searrow^{F_1} & \downarrow F^\sigma \\ & & X. \end{array} \qquad \circlearrowleft \tau$$

is commutative.

Step 2. Since $\pi_1^{\text{ét}}(\overline{X})$ is finite, so is $\pi_1^{\text{ét}}(\overline{Y}_1)$. We now construct a geometrically connected torsor $Y_2 \to Y_1$ under some $F_2 \in \mathcal{F}_k$ such that $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$. Let $U' \to \overline{Y}_1$ be a torsor under some $B' \in \mathcal{F}_{\overline{k}}$ with $\pi_1^{\text{ét}}(U') = 0$ and U' (geometrically) connected. We claim that, up to twisting Y_1 by some element in $H^1(k, F_1)$, there exists some $F_2 \in \mathcal{F}_k$ and some F_2-torsor $Y_2 \to Y_1$ such that the B'-torsor $U' \to \overline{Y}_1$ is obtained from the F_2-torsor $Y_2 \to Y_1$ by base-changing k to $\overline{k}$; in particular, such a Y_2 is geometrically connected and satisfies $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$. Indeed, by [HS02, Prop. 2.2 and §3.1] and [HS12, Thm 2.1 and Rmk 2.2(1)], the torsor $U' \to \overline{Y}_1$ has a k-form over Y_1 if $Y_1(\mathbf{A}_k)^{\mathcal{F}_k} \neq \emptyset$. But the latter is true, up to twisting Y_1, by [Sto07, Prop. 5.17]. Hence, $U' \to \overline{Y}_1$ has a k-form, say $Y_2 \to Y_1$ under some $F_2 \in \mathcal{F}_k$ satisfying $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$, as

claimed.

We now claim that, without loss of generality, (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$. Indeed, by [Sko09, Prop. 2.3] there exists a B-torsor $V \to X$ under some $B \in \mathcal{F}_k$ and a surjective X-torsor morphism $h : E \to Y_1$ under $\ker(B \to F_1)$; moreover, when considered as a Y_1-torsor via h, E admits a surjective Y_1-torsor morphism to Y_2. By a modification of [Sko09, Lemma 2.2] (we replace the assumption "$(x_v) \in X(\mathbf{A}_k)^{\mathrm{desc}}$" with "$(x_v) \in X(\mathbf{A}_k)^{\text{ét Br}_1}$" and then use [Sto07, Prop. 5.17] to check that the proof holds under this new assumption), there exists some $\gamma \in H^1(k, \ker(B \to F_1))$ and some point $(M_v) \in E^\gamma(\mathbf{A}_k)^{\mathrm{Br}_1}$ which lifts (x_v). Let $\tilde{\gamma}$ be the image of γ in $H^1(k, F_2)$ under the image of $\ker(B \to F_1) \to F_2$. Then $E^\gamma \to Y_1$ factors through $Y_2^{\tilde{\gamma}} \to Y_1$, implying that we can use the functoriality of Br_1 to push (M_v) to a point in $Y_2^{\tilde{\gamma}}(\mathbf{A}_k)^{\mathrm{Br}_1}$ above (x_v). Hence, without loss of generality (up to twisting everything as above if necessary), we can assume that (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$.

Let $R := Y_2 \times_{Y^\sigma} Z^\tau \to Y_2$ be the pullback of $Z^\tau \to Y^\sigma$ along $Y_2 \to Y_1 \to Y^\sigma$; this is naturally a T^τ-torsor. By Lemma 4.5.1, $\operatorname{Pic} \overline{Y}_2$ is finitely generated as a $\mathbf{Z}$-module and $(\operatorname{Pic} \overline{Y}_2)_{\mathrm{tors}} = 0$; since $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1} \neq \emptyset$, by Proposition 4.4.3 there is a universal torsor $W_2 \to Y_2$ under a torus $T_2 \in \mathcal{T}_k$ with $W_2(\mathbf{A}_k) \neq \emptyset$. Since the type $\lambda_{W_2} : \widehat{T}_2 \to \operatorname{Pic} \overline{Y}_2$ is an isomorphism, from the exact sequence of Colliot-Thélène and Sansuc (cf. [CTS87, (2.1.1)])

$$0 \to \overline{k}[W_2]^\times / \overline{k}^\times \to \widehat{T}_2 \xrightarrow{\lambda_{T_2}} \operatorname{Pic} \overline{Y}_2 \to \operatorname{Pic} \overline{W}_2 \to 0,$$

we deduce that $\operatorname{Pic} \overline{W}_2 = 0$ and $\overline{k}[W_2]^\times = \overline{k}^\times$. By the universal property of universal torsors, there is also a morphism of Y_2-torsors $W_2 \to R^\mu$, for some $[\mu] \in H^1_{\text{ét}}(k, T^\tau)$. Let $\tilde{\mu}$ be the image of μ under the map $Z^1(k, T^\tau) \to$

$Z^1(k, G^\tau)$. Then $(Z^\tau)^\mu = (Z^\tau)^{\tilde{\mu}}$. Let $t_\tau : Z^1(k, G^\tau) \to Z^1(k, G)$ be the bijection as in [Ser94, §I.5.3, Prop. 35bis], and let $\nu := t_\tau(\tilde{\mu})$. Then $(Z^\tau)^{\tilde{\mu}} = Z^\nu$, $(G^\tau)^{\tilde{\mu}} = G^\nu$, and $(T^\tau)^\mu = T^\nu$. Since (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$ and since by [Sko99, Thm 3] we have that $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1} = Y_2(\mathbf{A}_k)^{\mathcal{M}_k}$, there is some $[\lambda] \in H^1_{\text{ét}}(k, T_2)$ such that (x_v) lifts to a point in $W_2^\lambda(\mathbf{A}_k)$. Let $\tilde{\lambda}$ be the image of λ under the map $Z^1(k, T_2) \to Z^1(k, T^\nu)$ induced by the type $\lambda_{R^\mu} : \widehat{T^\nu} \to \operatorname{Pic} \overline{Y}_2$; then we get a morphism of Y_2-torsors $W^\lambda \to (R^\mu)^{\tilde{\lambda}}$. Let ω be the image of $\tilde{\lambda}$ under the morphism $H^1_{\text{ét}}(k, T^\nu) \to H^1_{\text{ét}}(k, G^\nu)$. Then $(Z^\nu)^{\tilde{\lambda}} = (Z^\nu)^\omega$. Let $t_\nu : Z^1(k, G^\nu) \to Z^1(k, G)$ be the bijection as in [Ser94, §I.5.3, Prop. 35bis], and let $\xi := t_\nu(\omega)$. Then $(Z^\nu)^\omega = Z^\xi$ and $(G^\nu)^\omega = G^\xi$. Summarising, we have the commutative diagram

$$\begin{array}{ccccc} W_2^\lambda & \longrightarrow & (R^\mu)^{\tilde{\lambda}} & \longrightarrow & Z^\xi \\ & \searrow^{T_2} & \downarrow{\scriptstyle T^\xi} & & \downarrow{\scriptstyle T^\xi} \quad \searrow^{G^\xi} \\ & & Y_2 \to Y_1 \to Y^\sigma & \xrightarrow[F^\sigma]{} & X. \end{array}$$

Step 3. Since $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$, by Proposition 4.5.2 we have that $\pi_1^{\text{ét}}(\overline{W_2^\lambda})$ is abelian; hence, since W_2^λ is geometrically connected, $\overline{k}[W_2^\lambda]^\times = \overline{k}^\times$, and $\operatorname{Pic} \overline{W_2^\lambda} = 0$, by Lemma 4.5.3 we deduce that $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$.

Let $F' \in \mathcal{F}_k$ and let $[U \to Z^\xi] \in H^1_{\text{ét}}(Z^\xi, F')$. Consider the fibred product $V := W_2^\lambda \times_{Z^\xi} U$; this is naturally an F'-torsor over W_2^λ. Since $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$, by Lemma 4.4.7 and Lemma 4.4.8 we get the existence of some $[\rho] \in H^1_{\text{ét}}(k, F')$, some $\tilde{F}' \in \mathcal{F}_k$, an $\tilde{F}'$-torsor $V_1 \to W_2^\lambda$, and a W_2^λ-torsor morphism $V_1 \to V^\rho$ with V_1 geometrically connected and with (x_v) lifting to a point in $V_1(\mathbf{A}_k)$. But $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$ implies that $\tilde{F}'$ is trivial and that V_1 is isomorphic to W_2^λ. Hence, by using $W_2^\lambda(\mathbf{A}_k) = W_2^\lambda(\mathbf{A}_k)^{\mathrm{Br}_1}$, we can easily see that the fact that (x_v) lifts to a point $(w_v) \in W_2^\lambda(\mathbf{A}_k)^{\mathrm{Br}_1}$ implies that (x_v) lifts to a point $(u_v) \in V_1(\mathbf{A}_k)^{\mathrm{Br}_1}$,

which can then be pushed by functoriality of Br_1 to a point $(u_v) \in U^\psi(\mathbf{A}_k)^{\mathrm{Br}_1}$ above (x_v), as required. $\square$

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