Uniqueness Results for Viscous Incompressible Fluids

Tobias Barker

St Edmund Hall

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2017

This thesis is dedicated to my cousin John Maurice Day (28th April 1988- 6th May 2008)

Acknowledgements

I would like to take this opportunity to express my sincerest gratitude and appreciation towards my supervisor Professor Gregory Seregin. The time, patience and belief he invested has enabled me to develop in a professional and personal context. The fruitful discussions we had about the Navier-Stokes equations have had a profound and lasting effect on me. I am appreciative for the support provided by my supervisor Professor Jan Kristensen. I also extend many thanks to my undergraduate tutors Professor Andrew Hodges and Professor Nicholas Woodhouse. Their brilliant tutorials and insights sparked my interest in Mathematics arising from physical problems.

The research in this thesis has been funded by an EPSRC standard research student DTG (EP/L505031/1). I am grateful for the assistance provided by support staff in OxPDE, the mathematical institute, St Edmund Hall and in the University of Oxford. A special thanks goes to my college advisor Professor Oliver Riordan, for his patience and reassurance.

It has been a privilege to have had many discussions with other researchers during the course of my DPhil. I would like to thank them all. Special thanks goes to Dallas Albritton for comments relating to Remark 6.10 (Chapter 6), Professor Joaquim Bruna for discussions related to homogeneous Besov spaces, Dr. Gabriel Koch for the invitation to the University of Sussex (1 week in September 2016), Kuijie Li for prompting me towards a sharpening of the statement of Theorem 3.1 (Chapter 3) and Professor Paolo Maremonti for the invitation to the Seconda Università degli Studi di Napoli (2 weeks in June-July 2016).

Outside of research, necessary distraction (and laughter) was provided by my friends. Thank you Ambroise, Dominik, Eliot, Ghady, Hayley, James (Petherick), James (Sayers), Jan, Julien, Matthew (Jacobs), Matthieu (Ruthven), Max, Michael, Peter, Rupert, Simon (Gladman), Simon (Robertson), Siran and Thomas (Broughton).

Lastly, words cannot describe how fortunate I am to have a family as unconditionally supportive, patient and understanding as my own. Nothing would be possible without my mother (Janet), my father (Jolyon), my girlfriend (Yasmin), my grandfather (Maurice), my stepfather (Brian), my step grandmother (Nancy), my siblings (Jessica and Madeleine) and my step siblings (Suzie and Thomas Hance). Thank you.

Abstract

First, we provide new classes of initial data, that grant short time uniqueness of the associated weak Leray-Hopf solutions of the three dimensional Navier-Stokes equations. The main novelty here is the establishment of certain continuity properties near the initial time, for weak Leray-Hopf solutions with initial data in supercritical Besov spaces. The techniques used here build upon related ideas of Calderón presented in [18].

Secondly, we prove local regularity up to the flat part of the boundary, for certain classes of solutions to the Navier-Stokes equations, provided that the velocity field belongs to $L_\infty(-1,0;L^{3,\beta}(B(1)\cap\mathbb{R}^3_+))$ with $3\leq\beta<\infty$. What enables us to build upon the work of Escauriaza, Seregin and Šverák [27] and Seregin [100] is the establishment of new scale-invariant estimates, new estimates for the pressure near the boundary and a convenient new ε-regularity criterion.

Third, we show that if a weak Leray-Hopf solution in $\mathbb{R}^3_+\times]0,\infty[$ has a finite blow-up time T, then necessarily $\lim_{t\uparrow T}\|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)}=\infty$ with $3<\beta<\infty$. The proof hinges on a rescaling procedure from Seregin's work [106], a new stability result for singular points on the boundary, suitable a priori estimates and a Liouville type theorem for parabolic operators developed by Escauriaza, Seregin and Šverák [27].

Finally, we investigate a notion of global-in-time solutions to the Navier-Stokes equations in $\mathbb{R}^3$, with solenoidal initial data in the critical Besov space $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, which has certain continuity properties with respect to weak* convergence of the initial data. Such properties are motivated by the strategy used by Seregin [106] to show that if a weak Leray-Hopf solution in $\mathbb{R}^3\times]0,\infty[$ has a finite blow-up time T, then necessarily $\lim_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}=\infty$. We prove new decomposition results for Besov spaces, which are key in the conception and existence theory of such solutions.

Contents

Chapter 1

Introduction

This thesis is centred around the investigation of the three-dimensional initial boundary value problem for the incompressible Navier-Stokes equations. The incompressible Navier-Stokes equations were introduced in the 19th century as a model governing the evolution of a viscous incompressible fluid in a domain $\Omega \subseteq \mathbb{R}^3$. According to this model, the velocity field of the fluid $v : \Omega \to \mathbb{R}^3$ and the pressure field $q : \Omega \to \mathbb{R}$ satisfy the following system of nonlinear partial differential equations equations on $\Omega \times]0, \infty[$ and the following initial and boundary conditions:

$$\partial_t v - \nu \Delta v + \nabla q + v \cdot \nabla v = f, \quad \operatorname{div} v = 0, \tag{1.0.1}$$

$$v|_{\partial\Omega}(\cdot, t) = 0, \tag{1.0.2}$$

$$v(\cdot, 0) = u_0. \tag{1.0.3}$$

In the case of $\Omega = \mathbb{R}^3$, (1.0.2) is replaced by

$$v(x, t) \to 0, \text{ as } |x| \to \infty. \tag{1.0.4}$$

In this thesis, we will consider the cases $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$. An important feature, that we will make use of throughout this thesis, is that these equations are invariant under the scaling ($\lambda > 0$):

$$v(x, t) \to v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \tag{1.0.5}$$

$$q(x, t) \to q_\lambda(x, t) = \lambda^2 q(\lambda x, \lambda^2 t), \tag{1.0.6}$$

$$f(x, t) \to f_\lambda(x, t) = \lambda^3 f(\lambda x, \lambda^2 t), \tag{1.0.7}$$

$$u_0(x) \to u_{0\lambda}(x) = \lambda u_0(\lambda x). \tag{1.0.8}$$

Many fundamental mathematical questions regarding the three-dimensional Navier-Stokes equations remain unsolved. One famous example is the sixth Millennium problem. Let us quote the sixth Millennium problem, as stated by Fefferman in [33]. This asks the proof of one of four statements (A)-(D).

> **(A) Existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$**
> Take $\nu > 0$ and let $u_0(x)$ be any smooth divergence-free vector field satisfying
> $$|\nabla^m u_0(x)| \leq C_{mK}(1+|x|)^{-K} \tag{1.0.9}$$
> on $\mathbb{R}^3$ for any $m = 0, 1 \ldots$ and $K > 0$. Take $f(x,t)$ to be identically zero. Then there exist functions
> $$(v, q) \in C^\infty(\mathbb{R}^3 \times [0, \infty[) \tag{1.0.10}$$
> that satisfy
> $$\int_{\mathbb{R}^3} |v(x,t)|^2 dx \leq C \text{ for all } t > 0 \text{ (bounded energy)}, \tag{1.0.11}$$
> and (1.0.1) and (1.0.3) on $\mathbb{R}^3 \times [0, \infty[$.
>
> **(C) Breakdown of Navier-Stokes solutions on $\mathbb{R}^3$**
> Take $\nu > 0$. Then there exists a smooth, divergence-free vector field $u_0(x)$ and a smooth force $f(x,t)$ on $\mathbb{R}^3 \times [0, \infty[$, satisfying (1.0.9),
> $$|\partial_t^l \nabla^m f(x,t)| \leq C_{mlK}(1+|x|+t)^{-K} \tag{1.0.12}$$
> (on $\mathbb{R}^3 \times [0, \infty[$ for any $l, m = 0, 1, \ldots$ and $K > 0$), for which there exist no solutions (v, q) satisfying (1.0.1), (1.0.3), (1.0.10) and (1.0.11).

Let us remark that the other two statements **(B)** and **(D)** are analogous to **(A)** and **(C)**, in the setting of spatially periodic functions.

A breakdown of Navier-Stokes solutions (statement **(C)** above) would not be consistent with physical observation and would suggest that the three-dimensional Navier-Stokes equations are not a valid physical model.

There is not a general consensus amongst researchers as to whether or not the sixth Millennium problem should be considered as the most important question regarding the Navier-Stokes equations. Ladyzhenskaya's opinion, stated in [73], was that the following question should be prioritised.

> '**Problem 1.** Do the Navier-Stokes equations together with the initial and boundary conditions actually give a deterministic description of the dynamics of an incompressible fluid?' [73]

As stated by Ladyzhenskaya in [73], providing an appropriate phase space and notion of solution, that ensures global-in-time existence and uniqueness, does not necessarily require additional smoothness properties of those solutions.

1.0.1 Weak Leray-Hopf Solutions and Global Existence

The existence of certain global-in-time solutions to the Navier-Stokes equations (1.0.1)-(1.0.4) has been known since the papers of Leray [75] and Hopf [52]. From this point onwards, we refer to the solutions described below as 'weak Leray-Hopf solutions'. This will be the main notion of solution used throughout this thesis. From now on we will always take the viscosity to be 1 and the forcing term to be zero.

Before proceeding further, let us introduce some necessary notation. For, $\Omega\subseteq\mathbb{R}^3$, the space $J(\Omega)$ is defined to be the closure of

$$C^{\infty}_{0,0}(\Omega):=\{u\in C^{\infty}_{0}(\Omega): \mathrm{div}\ u=0\}$$

with respect to the $L_2(\Omega)$ norm.
Moreover, $\overset{\circ}{J}{}^1_2(\Omega)$ is defined as the completion of the space $C^{\infty}_{0,0}(\Omega)$ with respect to the Dirichlet integral

$$\Big(\int\limits_{\Omega}|\nabla v|^2dx\Big)^{\frac{1}{2}}.$$

We also define the energy class:

$$\mathcal{L}(S):=L_{\infty}(0,S;J(\Omega))\cap L_2(0,S;\overset{\circ}{J}{}^1_2(\Omega)).$$

We define the energy norm as:

$$\|v\|_{\mathcal{L}(S)}:=\|v\|_{L_{2,\infty}(Q_S)}+\|\nabla v\|_{L_2(Q_S)}.$$

Here, $Q_S:=\Omega\times]0,S[$.

Let us now state the theorem regarding existence of global-in-time weak Leray-Hopf solutions to the Navier-Stokes equations (1.0.1)-(1.0.4).

Theorem 1.1. *Consider* $0<S\leq\infty$. *Let*

$$u_0\in J(\Omega). \tag{1.0.13}$$

There exists at least one 'weak Leray-Hopf solution' v to Navier-Stokes initial boundary value problem in $\Omega\times]0,S[$ that satisfies

$$v\in\mathcal{L}(S). \tag{1.0.14}$$

Additionally, for any $w\in L_2(\Omega)$:

$$t\rightarrow\int\limits_{\Omega}w(x)\cdot v(x,t)dx \tag{1.0.15}$$

is a continuous function on $[0,S]$ *(the semi-open interval should be taken if* $S=\infty$*). The Navier-Stokes equations are satisfied by* v *in a weak sense:*

$$\int_0^S\int_\Omega (v\cdot\partial_t w + v\otimes v : \nabla w - \nabla v : \nabla w)dxdt = 0 \tag{1.0.16}$$

for any divergence-free test function $w\in C^\infty_{0,0}(\Omega\times]0,S[)$*. The initial condition is satisfied strongly in the* $L_2(\Omega)$ *sense:*

$$\lim_{t\to 0^+}\|v(\cdot,t)-u_0\|_{L_2(\Omega)}=0. \tag{1.0.17}$$

Finally, v *satisfies the energy inequality:*

$$\|v(\cdot,t)\|^2_{L_2(\Omega)}+2\int_0^t\int_\Omega|\nabla v(x,t')|^2dxdt'\leq\|u_0\|^2_{L_2(\Omega)} \tag{1.0.18}$$

for all $t\in[0,S]$ *(the semi-open interval should be taken if* $S=\infty$*).*

There are two big open problems concerning weak Leray-Hopf solutions.

- (Uniqueness) Given any initial data $u_0\in J(\Omega)$, is there a unique global-in-time weak Leray-Hopf solution associated to u_0?
- (Regularity) Given any initial data $u_0\in J(\Omega)$, is there an associated global-in-time weak Leray-Hopf solution that is regular for all times[1]?

Non-uniqueness, for weak Leray-Hopf solutions, has been conjectured by Ladyzhenskaya in [72] and by Jia and Šverák in [57]. For a related class of solutions to the Navier-Stokes equation, defined on a 'non-cylindrical' space-time domain[2] and with non-standard boundary conditions, a non-uniqueness result was established by Ladyzhenskaya in [72]. For weak Leray-Hopf solutions defined on $\mathbb{R}^3\times]0,\infty[$, certain unverified non-uniqueness scenarios have recently been suggested by Jia and Šverák in [57]. After completion of the material of this thesis, additional numerical evidence for non-uniqueness has been supplied by Guillod and Šverák in [50].

Under certain additional assumptions on the initial data, it has been known since Leray's paper [75] that regularity implies uniqueness. This will be discussed in more detail in the subsection of this Introduction 'Regularity and Uniqueness Theory of

[1]By regular for all time, we mean $C^\infty(\Omega\times]0,\infty[)$ with every space-time derivative in $L_\infty(\epsilon,T;L_\infty(\Omega))$ for any $0<\epsilon<T<\infty$.

[2]By a non-cylindrical space-time domain, we mean that it is not of the form $\mathcal{O}\times]0,T[$, with $\mathcal{O}\subset\mathbb{R}^3$ fixed.

the Navier-Stokes Equations'. For generic initial data $u_0 \in J(\Omega)$, it is not known if regularity implies uniqueness. Indeed in the scenario suggested by Jia and Šverák in [57], the non-unique solutions are regular. The results contained in this thesis relate to the following motivating question.

> **What are sufficient conditions to grant uniqueness of weak Leray-Hopf solutions of the three-dimensional Navier-Stokes equations?**

1.1 Local Wellposedness of the Navier-Stokes Equations

In the previous subsection, we described a class of solutions of the three-dimensional Navier-Stokes equations (weak Leray-Hopf solutions), where existence of at least one global-in-time solution is guaranteed. In this subsection, we will briefly discuss classes of solutions to the Navier-Stokes equations that are locally well posed. By this, we mean that existence and uniqueness of such solutions is ensured for at least a short time.

1.1.1 Mild solutions of the Three-Dimensional Navier-Stokes Equations

Now we provide a brief digression into the framework of mild solutions to the three-dimensional Navier-Stokes equations in $\mathbb{R}^3$. Before doing so, we describe some necessary preliminaries regarding the linear Stokes system, with $F_{ij} : Q_\infty \to \mathbb{R}$ and $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ sufficiently smooth:

$$\partial_t w - \Delta w + \nabla q = -\operatorname{div} F, \;\; \operatorname{div} w = 0, \tag{1.1.1}$$

with initial condition

$$w(\cdot, 0) = u_0 \;\text{ and }\; \operatorname{div} u_0 = 0. \tag{1.1.2}$$

We also make use of the operator $\mathbb{P}$ that projects onto divergence-free vector fields. In the case of $\Omega = \mathbb{R}^3$, this may be written as

$$\mathbb{P}f := f + \nabla(-\Delta)^{-1}(\operatorname{div} f). \tag{1.1.3}$$

Using the divergence-free property of the velocity field and the initial condition, we see that

$$\mathbb{P}w = w, \;\text{ and }\; \mathbb{P}u_0 = u_0. \tag{1.1.4}$$

Furthermore, it follows from the expression (1.1.3) that

$$\mathbb{P}\nabla q = 0. \tag{1.1.5}$$

Formally, we may then apply $\mathbb{P}$ to system (1.1.1)-(1.1.2) (note that in the case of $\mathbb{R}^3$, $\mathbb{P}$ commutes with the Laplacian) to obtain:

$$\partial_t w - \Delta w + \mathbb{P}\mathrm{div}\,(F) = 0, \tag{1.1.6}$$

$$w(\cdot, 0) = u_0. \tag{1.1.7}$$

We then formally see that w satisfies the Duhamel-type integral equation

$$w(t) = S(t)u_0 - \int_0^t S(t-\tau)\mathbb{P}\mathrm{div}\, F(\cdot, \tau) ds. \tag{1.1.8}$$

Here, $S(t)u_0$ denotes the convolution of u_0 with the three-dimensional heat kernel

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp(-|x|^2/4t).$$

Consider a potential $\Phi(x,t)$ satisfying

$$\Delta\Phi(x,t) = \Gamma(x,t). \tag{1.1.9}$$

Here, $\Gamma(x,t)$ is the heat potential and we specifically take

$$\Phi(x,t) := \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{\Gamma(y,t)}{|x-y|} dy.$$

With this in hand, one may then define the Oseen tensor (we adopt the Einstein summation convention throughout this thesis):

$$K_{ijl}(x,t) := \Big(\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_l}\Big)(x,t) - \delta_{il}\Big(\frac{\partial^3 \Phi}{\partial x_j \partial x_k \partial x_k}\Big)(x,t). \tag{1.1.10}$$

From Solonnikov's paper [119], this satisfies the estimate

$$|\partial_t^m \nabla^j K(x,t)| \leq \frac{c}{(|x|^2+t)^{2+\frac{j}{2}+m}}, \quad \text{for } j, m = 0, 1 \ldots \tag{1.1.11}$$

It is then known[3] that, for sufficiently smooth F, we have:

$$-\int_0^t S(t-\tau)\mathbb{P}\mathrm{div}\, F(\cdot,\tau) ds = \int_0^t \int\limits_{\mathbb{R}^3} K_{ijl}(x-y, t-\tau) F_{jl}(y,\tau) dy d\tau. \tag{1.1.12}$$

[3]See, for example, pp.234-235 of Solonnikov's paper [119].

For the specific choice $F = u \otimes v := (u_i v_j)_{i,j=1,2,3}$, we define

$$B(u,v)_i(x,t) := \int_0^t \int_{\mathbb{R}^3} K_{ijl}(x-y,t-\tau)u_j(y,\tau)v_l(y,\tau)dyd\tau. \tag{1.1.13}$$

The definition (1.1.13) for $B(u,v)$ will be used throughout this thesis. We mention that elsewhere in the literature $B(u,v)$ is sometimes defined as

$$B(u,v) = \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla v dxdt.$$

This formal construction allows us to define mild solutions to the Navier-Stokes equations. The definition given here is essentially contained in Rusin's thesis [92][4].

Definition 1.2. *Consider $\mathcal{X} \subset \mathcal{S}'(\mathbb{R}^3)$ and $\mathcal{X}_T = \mathcal{X}_T(\mathbb{R}^3 \times]0,T[) \subset L_{2,loc}(Q_T)$ with the following properties:*

$$T' > T \Rightarrow \mathcal{X}_T \subset \mathcal{X}_{T'}, \tag{1.1.14}$$

$$u_0 \in \mathcal{X} \Rightarrow S(t)u_0 \in \mathcal{X}_T \tag{1.1.15}$$

and there exists a universal constant $C > 0$ such that for all $u_0 \in \mathcal{X}$ and $u, v \in \mathcal{X}_T$ we have

$$\|S(t)u_0\|_{\mathcal{X}_T} \leq C\|u_0\|_{\mathcal{X}} \tag{1.1.16}$$

and

$$\|B(u,v)\|_{\mathcal{X}_T} \leq C\|u\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}. \tag{1.1.17}$$

Here, B is as in (1.1.13).
Let $u_0 \in \mathcal{X}$ be divergence-free, in the sense of tempered distributions. We say that $v \in \mathcal{X}_T$ is a mild solution to the three-dimensional Navier-Stokes equations if it is a fixed point in $\mathcal{X}_T$ of the following equation:

$$v(\cdot,t) = S(t)u_0 + B(v,v)(\cdot,t). \tag{1.1.18}$$

Existence of mild solutions of the Navier-Stokes equations is proven by the method of successive iterations. This was pioneered by Fujita and Kato in [60] and there are a substantial number of papers that utilise this approach for different spaces $\mathcal{X}$ and $\mathcal{X}_T$. See for example papers of Cannone [19], Giga and Miyakawa [43], Kato [59], Kozono and Yamazaki [64], Planchon [88] and Taylor [125].

In the cited papers above, the construction of mild solutions is obtained when the solenoidal initial data belongs to critical spaces with respect to the Navier-Stokes scaling, which are defined to satisfy the property

[4]Specifically, Definition 1 p.10 of [92].

- **(critical spaces)** $u \in \mathcal{X} \subset \mathcal{S}'(\mathbb{R}^3)$ and $u_{0\lambda}(x) := \lambda u_0(\lambda x)$ $(\lambda > 0) \Rightarrow u_{0\lambda} \in \mathcal{X}$ and $\|u_{0\lambda}\|_{\mathcal{X}} = \|u_0\|_{\mathcal{X}}$.

Koch and Tataru's paper [63] provides the current widest critical spaces for the solenodial initial data, for which mild solutions to the Navier-Stokes equations can be constructed. In particular, in [63] Koch and Tataru constructed local-in-time mild solutions for solenoidal initial data in $VMO^{-1}(\mathbb{R}^3)$ and global-in-time mild solutions for solenoidal initial data with sufficiency small $BMO^{-1}(\mathbb{R}^3)$ norm. Let us mention that it has been shown by Bourgain and Pavlović in [16] that for certain critical spaces larger than $BMO^{-1}(\mathbb{R}^3)$, mild solutions to the Navier-Stokes equations are ill-posed. In particular, Bourgain and Pavlović showed that 'Initial data in the Schwartz class $\mathcal{S}$ that are arbitrarily small in $\dot{B}^{-1}_{\infty,\infty}$ can produce solutions arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$ after an arbitrarily short time.' (p.2235 of [16]). See also Yoneda's paper [130].

Unfortunately, the notion of mild solutions (as in Definition 1.2) no longer makes sense when the solenoidal initial data belongs to supercritical spaces with respect to the Navier-Stokes scaling (such as $L_2(\mathbb{R}^3)$), which are defined to satisfy the property

- **(supercritical spaces)** $u \in \mathcal{W} \subset \mathcal{S}'(\mathbb{R}^3)$ and $u_{0\lambda}(x) := \lambda u_0(\lambda x)$ $(\lambda > 0) \Rightarrow u_{0\lambda} \in \mathcal{W}$ and $\|u_{0\lambda}\|_{\mathcal{W}} = \lambda^{-\alpha}\|u_0\|_{\mathcal{W}}$ for some $\alpha > 0$.

For supercritical spaces, the associated path spaces $\mathcal{W}_T$ are supercritical path spaces, which are defined as follows

- **(supercritical path spaces)** $f(x,t) \in \mathcal{W}_T \subset \mathcal{S}'(\mathbb{R}^3 \times]0,T[)$ and $f_\lambda(x,t) := \lambda f(\lambda x, \lambda^2 t)$ $(\lambda > 0) \Rightarrow$

  $f_\lambda \in \mathcal{W}_{T/\lambda^2}$ and $\|f_\lambda\|_{\mathcal{W}_{T/\lambda^2}} = \lambda^{-\alpha}\|f\|_{\mathcal{W}_T}$ for some $\alpha > 0$.

The reason why Definition 1.2 cannot be valid for supercritical initial data and path spaces, is because the bilinear term B cannot be continuous from $\mathcal{W}_T \times \mathcal{W}_T$ to $\mathcal{W}_T$ (i.e (1.1.17) is no longer valid). This situation is in stark contrast to the case of weak Leray-Hopf solutions, which exist globally-in-time for any solenoidal data in the supercritical space $L_2(\mathbb{R}^3)$.

In the literature, mild solutions to the Navier-Stokes equations, constructed for specific $\mathcal{X}$ and $\mathcal{X}_T$, typically also belong to another path space $\mathcal{Y}_T$ with the following properties. Namely, $\mathcal{Y}_T$ satisfies (1.1.14)-(1.1.15) and is such that there can exist at most one mild solution in the class $\mathcal{X}_T \cap \mathcal{Y}_T$. In this situation, for any $u_0 \in \mathcal{X}$ we can define the 'maximal time of existence' as follows

$$T^*_{\mathcal{X}_T\cap\mathcal{Y}_T}(u_0) := \sup\{T > 0 : \exists! \text{ mild solution } v \in \mathcal{X}_T \cap \mathcal{Y}_T \text{ solving (1.1.18)}\}.$$

In such circumstances, the mild solutions to the Navier-Stokes equations are locally well posed. From the papers of Fabes, Jones and Rivière [29][5], Kato [59][6] and Cannone [19][7], we see that an example of $(\mathcal{X},\mathcal{X}_T,\mathcal{Y}_T)$ with the above properties is

$$(\mathcal{X},\mathcal{X}_T,\mathcal{Y}_T)=(L_3(\mathbb{R}^3),X_4(T),L_5(\mathbb{R}^3\times]0,T[).$$

Here,

$$X_4(T):=\{f\in\mathcal{S}^{'}(\mathbb{R}^3\times]0,T[):\|f\|_{X_4(T)}:=\sup_{0<t<T}t^{\frac{1}{8}}\|f(\cdot,t)\|_{L_4(\mathbb{R}^3)}<\infty\}.$$

Unfortunately, for arbitrary solenoidal data in $\mathcal{X}$, it is not known if the maximal time of existence is infinite or not. Any attempt to prove that it is infinite, for arbitrary data solenoidal data in $\mathcal{X}$, must make use of finer properties of the Navier-Stokes equations than those required for the construction of local-in-time mild solutions. This follows from certain toy models of the Navier-Stokes, considered by Montgomery-Smith in [84] and by Gallagher and Paicu in [38]. These toy models have an analogous notion of mild solutions and local-in-time existence theory to the Navier-Stokes equations, but have initial data with an associated finite maximal time of existence.

1.1.2 Local Wellposedness of Weak Leray-Hopf solutions

The above situation for the mild solutions contrasts with weak Leray-Hopf solutions, where (for any initial data $u_0\in J(\Omega)$) global existence is known but even short time uniqueness in their class is an open problem. The third chapter of this thesis is motivated by a very natural question, arising from the investigation of sufficient conditions for uniqueness of weak Leray-Hopf solutions. Here it is.

> **(Q) Which $\mathcal{Z}\subset\mathcal{S}^{'}(\mathbb{R}^3)$ are such that $u_0\in J(\mathbb{R}^3)\cap\mathcal{Z}$ implies uniqueness of the associated weak Leray-Hopf solutions on some time interval?**

If we consider the initial data $u_0\in J(\mathbb{R}^3)\cap\mathcal{Z}$, for a suitable critical space $\mathcal{Z}$, then one can construct a mild solution that is a weak Leray-Hopf solution on Q_T. For certain cases, the properties of the mild solution are sufficient to grant uniqueness in the weak Leray-Hopf class. This is known as 'weak-strong uniqueness'. In the third chapter, we provide positive answers to **(Q)** for new classes $\mathcal{Z}$, by proving that weak-strong uniqueness holds for these classes. Let us now state our main contributions to question **(Q)**, whose proofs will be presented in the third chapter.

[5] See Theorem (3.3) of [29].
[6] See p.476 of [59].
[7] See Lemma 1.4 of [19].

Theorem 3.1 Chapter 3. *Fix* $2 < \alpha \leq 3$.

- *For* $2 < \alpha < 3$, *take any* p *such that* $\alpha < p < \frac{\alpha}{3-\alpha}$.
- *For* $\alpha = 3$, *take any* p *such that* $3 < p < \infty$.

Consider a weak Leray-Hopf solution u *to the Navier-Stokes system on* Q_∞, *with initial data*

$$u_0 \in VMO^{-1}(\mathbb{R}^3) \cap \dot{B}_{p,p}^{-\frac{3}{\alpha}+\frac{3}{p}}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Then, there exists a $T(u_0) > 0$ *such that all weak Leray-Hopf solutions on* Q_∞, *with initial data* u_0, *coincide with* u *on* $Q_{T(u_0)} := \mathbb{R}^3 \times]0, T(u_0)[$.

Corollary 3.2 Chapter 3. *Let* $3 < p < \infty$. *Consider a weak Leray-Hopf solution* u *to the Navier-Stokes system on* Q_∞, *with initial data*

$$u_0 \in \dot{\mathbb{B}}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Then, there exists a $T(u_0) > 0$ *such that all weak Leray-Hopf solutions on* Q_∞, *with initial data* u_0, *coincide with* u *on* $Q_{T(u_0)} := \mathbb{R}^3 \times]0, T(u_0)[$.

Let us mention that in Theorem 3.1 Chapter 3, $\dot{B}_{p,p}^{-\frac{3}{\alpha}+\frac{3}{p}}(\mathbb{R}^3)$ signifies a homogeneous Besov space. Furthermore, in Corollary 3.2 Chapter 3, $\dot{\mathbb{B}}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ denotes the closure of smooth compactly supported functions in the homogeneous Besov space $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. Further explanation of the relevant function spaces will be provided in 'Chapter 2: Preliminary Material'. For a detailed discussion of previous works relating to **(Q)**, as well as the novelty of our contribution and strategy, we refer the reader to the introduction of the third chapter of this thesis.

In Chapter 3, we will also show that similar arguments used for the above theorem and corollary can be used to prove new uniqueness criteria for weak Leray-Hopf solutions in Q_T. The relevant proposition is stated below.

Proposition 3.4 Chapter 3. *Suppose* u *and* v *are weak Leray-Hopf solutions on* Q_∞ *with the same initial data* $u_0 \in J(\mathbb{R}^3)$. *Then there exists an* $\epsilon_* = \epsilon_*(p, q) > 0$ *such that if either*

- $$u \in L^{q,s}(0, T; L^{p,s}(\mathbb{R}^3)) \qquad \frac{3}{p} + \frac{2}{q} = 1 \tag{1.1.19}$$
  $$(3 < p < \infty\,,\ 2 < q < \infty \text{ and } 1 \leq s < \infty) \tag{1.1.20}$$

- *or*

$$u \in L^{q,\infty}(0,T; L^{p,\infty}(\mathbb{R}^3)) \qquad \frac{3}{p} + \frac{2}{q} = 1 \tag{1.1.21}$$

$$(3 < p < \infty\,,\ 2 < q < \infty) \tag{1.1.22}$$

with

$$\|u\|_{L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))} \leq \epsilon_*, \tag{1.1.23}$$

then $u \equiv v$ *on* $Q_T := \mathbb{R}^3\times]0,T[$.

Let us mention that $L^{p,s}$ is a Lorentz space and $L^{p,\infty}$ denotes the 'weak L_p space'. Furthermore, for $p < s \leq \infty$ and $1 \leq p < \infty$ it is the case that

$$L_p(\mathbb{R}^3) \subsetneq L^{p,s}(\mathbb{R}^3) \subsetneq L^{p,\infty}(\mathbb{R}^3). \tag{1.1.24}$$

We will define these spaces, as well as mentioning further relevant properties, in 'Chapter 2: Preliminary Material'.

From (1.1.24), we see that Proposition 3.4 Chapter 3 extends the classical uniqueness criteria of Prodi [89] and Serrin [114]. Namely, that if u is a weak Leray Hopf solution on Q_T and satisfies

$$u \in L_q(0,T; L_p(\mathbb{R}^3)) \qquad \frac{3}{p} + \frac{2}{q} = 1 \ (3 < p \leq \infty \text{ and } 2 \leq q < \infty), \tag{1.1.25}$$

then u coincides on Q_T with other any weak Leray-Hopf solution with the same initial data.

1.2 Regularity and Uniqueness Theory of the Navier-Stokes Equations

In this part of the introduction, we mention relationships between regularity theory and uniqueness statements for weak Leray-Hopf solutions of the Navier-Stokes equations. We discuss initially the setting of global-in-time weak Leray-Hopf solutions in $\mathbb{R}^3\times]0,\infty[$.

In [75], Leray considered 'turbulent solutions', which are a subclass of weak Leray-Hopf solutions. Moreover, he established a connection between uniqueness and regularity properties of turbulent solutions. In particular, he showed that the following assumptions imply uniqueness of turbulent solutions, with initial data $u_0 \in J(\mathbb{R}^3)$, on $\mathbb{R}^3\times]0,S[$. We refer to this as 'Leray's sufficient condition'.

- 1) There exists a $\delta > 0$ such that all turbulent solutions, with initial data $u_0 \in J(\mathbb{R}^3)$, coincide on $]0,\delta[$.

- 2) There exists a turbulent solution V, with initial data u_0, such that for any $0<\epsilon<S$:
$$V\in L_{\infty}(\mathbb{R}^3\times]\epsilon,S[).$$

The same statement holds for weak Leray-Hopf solutions. In the setting of weak Leray-Hopf solutions, we define $u_0\in J(\mathbb{R}^3)$ to be 'short time regular' if there exists a $\delta>0$ such that 1) and 2) hold with $S=\delta$. Let us mention that it is unknown if we can drop 1) and still get the same conclusion. Indeed, certain unverified non-uniqueness scenarios, for weak Leray-Hopf solutions, have recently been suggested by Jia and Šverák in [57]. In the scenario suggested there, the nonunique solutions satisfy 2) but not 1).

If there are two different weak Leray-Hopf solutions, with the same short time regular initial data $u_0\in J(\mathbb{R}^3)$, then Leray's sufficient condition implies the following. Namely, for any weak Leray-Hopf solution v, with initial data u_0, we have

$$T(u_0):=\sup\{\tau: v\in L_{\infty}(]\epsilon,\tau];L_{\infty}(\mathbb{R}^3))\text{ for all }0<\epsilon<\tau\}<\infty. \tag{1.2.1}$$

Leray's sufficient condition implies all weak Leray-Hopf solutions, with short time regular initial data $u_0\in J(\mathbb{R}^3)$, coincide on $\mathbb{R}^3\times]0,T(u_0)[$. Hence, $T(u_0)$ is independent of v. Furthermore, it is known that this solution belongs to $C^{\infty}(\mathbb{R}^3\times]0,T(u_0)[)$. In the literature, T is commonly referred to as a finite blow-up time.

The previous paragraph implies the following for global-in-time weak Leray-Hopf solutions, with short time regular initial data $u_0\in J(\mathbb{R}^3)$. Namely, if one can rule out the existence a finite blow-up time $T(u_0)$, then one has uniqueness and regularity of the global-in-time weak Leray-Hopf solution associated with the initial data u_0.

1.2.1 Local Interior Regularity Theory of the Navier-Stokes Equations

In the previous subsection, we have just demonstrated the importance of investigating whether or not a finite blow-up time occurs, for weak Leray-Hopf solutions on $\mathbb{R}^3\times]0,\infty[$ with short time regular initial data u_0.

The investigation of finite blow-up times, for the Navier-Stokes equations, was initiated by Leray in [75]. Furthermore, in [75] it is proven that if T is a finite blow-up time and $3<p\leq\infty$ then necessarily

$$\|v(\cdot,t)\|_{L_p(\mathbb{R}^3)}\geq\frac{c(p)}{(T-t)^{\frac{1}{2}(1-\frac{3}{p})}},\ 0\leq t<T. \tag{1.2.2}$$

Arguments from the paper of Escauriaza, Seregin and Šverák [27][8], for example,

[8]Specifically, p.226 of [27].

imply the following for a weak Leray-Hopf solution $v : \mathbb{R}^3\times]0,\infty[\rightarrow \mathbb{R}^3$, with short time regular initial data u_0 and finite blow-up time T. Namely, there exists $R := R(v,u_0,T) > 0$ such that

$$v \in L_\infty((\mathbb{R}^3 \setminus B(R))\times]T/2,T[).$$

This, along with (1.2.2), implies that if $T(u_0)$ is a finite blow-up time then there exists a $R := R(v,u_0,T) > 0$ such that

$$v \notin L_\infty(B(R)\times]T/2,T[). \tag{1.2.3}$$

This motivates the local interior regularity theory for the Navier-Stokes equations, which is centered around the investigation of sufficient conditions to give local boundedness of the velocity field v of the Navier-Stokes equations. The first sufficient conditions were provided by Serrin in [114]. Serrin proved that if $v : B(1)\times]-1,0[\rightarrow \mathbb{R}^3$ is a distributional solution to the Navier-Stokes equations such that $v \in L_\infty(-1,0;L_2(B(1))$ and $\nabla v \in L_2(-1,0;L_2(B(1))$, then the conditions

$$\|v\|_{L_{s,l}(Q(r))} < \infty \tag{1.2.4}$$

and

$$\frac{3}{s} + \frac{2}{l} < 1, \tag{1.2.5}$$

imply that v is bounded on interior sub domains of $Q := B(1)\times]-1,0[$. Here, $L_{s,l}(Q(r)) := L_l(-r^2,0;L_s(B(r)))$.

A global-in-time weak Leray-Hopf solution v, with short time regular initial data u_0 and finite blow-up time T, satisfies additional properties on $B(x_0,r)\times]T-r^2,T[$ other than $v \in L_\infty(T-r^2,T;L_2(B(x_0,r)))$ and $\nabla v \in L_2(T-r^2,T;L_2(B(x_0,r))$. In particular, such a weak Leray-Hopf solution v satisfies a local energy inequality[9] (whose notion was conceived by Scheffer in [94]) on $B(x_0,R)\times]T-r^2,T[$ for any $x_0 \in \mathbb{R}^3$, $R > 0$ and $0 < r < T^{\frac{1}{2}}$.

The additional property of a local energy inequality forms an important part of the notion of 'suitable weak solutions' to the Navier-Stokes equations, which were introduced by Caffarelli, Kohn and Nirenberg in [17] and are the modern setting for the local interior regularity theory of the Navier-Stokes equations. Let us now state the definition of suitable weak solutions of the Navier-Stokes equations, which is taken from Seregin's book [111][10].

[9] We refer the reader to (1.2.10), for a definition of what is meant by 'local energy inequality'.

[10] Specifically, Definition 6.1 p.133 of [111].

Definition 1.3. *Consider any domain $\mathcal{O} \subseteq \mathbb{R}^3$. Let $-\infty < S_1 < S_2 < \infty$. We say that the pair $v : \mathcal{O}\times]S_1, S_2[\to \mathbb{R}^3$ and $q : \mathcal{O}\times]S_1, S_2[\to \mathbb{R}$ are a suitable weak solution to the Navier-Stokes equations in $\mathcal{O}\times]S_1, S_2[$ if:*

$$v \in L_\infty(S_1, S_2; L_2(\mathcal{O})), \tag{1.2.6}$$

$$\nabla v \in L_2(S_1, S_2; L_2(\mathcal{O})), \tag{1.2.7}$$

$$q \in L_{\frac{3}{2}}(\mathcal{O}\times]S_1, S_2[) \tag{1.2.8}$$

and

$$\int\limits_{S_1}^{S_2}\int\limits_{\mathcal{O}}(-v\cdot\partial_t\varphi - v\otimes v : \nabla\varphi + \nabla v : \nabla\varphi - q\,div\,\varphi)dyd\tau = 0 \tag{1.2.9}$$

for any $\varphi \in C_0^\infty(\mathcal{O}\times]S_1, S_2[)$. Additionally, it is required that the following local energy inequality holds for all positive $\varphi \in C_0^\infty(\mathcal{O}\times]S_1, \infty[)$ and almost every time $t \in]S_1, S_2[$:

$$\int\limits_{\mathcal{O}}\varphi|v(x,t)|^2dx + 2\int\limits_0^t\int\limits_{\mathcal{O}}\varphi|\nabla v|^2dxdt'$$

$$\leq \int\limits_0^t\int\limits_{\mathcal{O}}|v|^2(\partial_t\varphi + \Delta\varphi) + v\cdot\nabla\varphi(|v|^2 + 2q)dxdt'. \tag{1.2.10}$$

For a suitable weak solution (v, q), points of $\mathcal{O}\times]S_1, S_2[$ can be classified as either singular points or regular points. In particular,

$z_0 = (x_0, t_0) \in \mathcal{O}\times]S_1, S_2[$ is a singular point of $v \iff$

$v \notin L_\infty((B(x_0, r)\times]t_0 - r^2, t_0[) \cap \mathcal{O}\times]S_1, S_2[)$ for all sufficiently small $r > 0$.

Moreover, $z_0 \in \mathcal{O}\times]S_1, S_2[$ is defined to be a regular point of v if it is not a singular point of v. Let us now state the main question of interest in the local interior regularity theory for suitable weak solutions of the Navier-Stokes equations.

> **Consider a suitable weak solution to the Navier-Stokes equations (v, q) on $\mathcal{O}\times]S_1, S_2[$ and a space-time point $z_0 \in \mathcal{O}\times]S_1, S_2[$. What assumptions, in a local parabolic neighbourhood of $z_0 := (x_0, t_0)$, ensure that z_0 is a regular point?**

Notice that if (v,q) is a suitable weak solution on $\mathcal{O}\times]S_1,S_2[$, then under the rescaling $(\lambda>0)$

$$v(x,t)\rightarrow v_\lambda(x,t)=\lambda v(\lambda x,\lambda^2 t), \tag{1.2.11}$$

$$q(x,t)\rightarrow q_\lambda(x,t)=\lambda^2 q(\lambda x,\lambda^2 t) \tag{1.2.12}$$

(v_λ,q_λ) is a suitable weak solution on $\frac{\mathcal{O}}{\lambda}\times]\frac{S_1}{\lambda^2},\frac{S_2}{\lambda^2}[$. The property of a space-time point z_0 being a regular point for (v,q) is invariant under the above scaling. As argued by Seregin and Šverák in [102], it is natural to consider criteria for regular points that are invariant under the above scaling as well. This leads to the study of scale-invariant quantities in the local regularity theory, which are defined to satisfy

$$F(v,q,z_0,r)=F(v_\lambda,q_\lambda,x_0/\lambda,t_0/\lambda^2,r/\lambda). \tag{1.2.13}$$

An example of a scale-invariant quantity is

$$E(v,r):=\frac{1}{r}\int\limits_{-r^2}^{0}\int\limits_{B(r)}|\nabla v|^2 dyd\tau. \tag{1.2.14}$$

The investigation of regularity at space-time points and scale-invariant quantities can be divided into two cases.

(1) Smallness assumptions for a scale-invariant quantity, in a local neighbourhood of the space-time point.

(2) The assumption that some scale-invariant quantity is bounded, in a local neighbourhood of the space-time point.

A celebrated result falling under (1), that gives sufficent conditions for space-time points to be regular, was proven by Caffarelli, Kohn and Nirenberg in [17]. Namely, it was shown that if (v,q) is a suitable weak solution to the Navier-Stokes system near z_0, then there exists an $\epsilon_*>0$ such that

$$\limsup_{r\rightarrow 0^+}\frac{1}{r}\int\limits_{t_0-r^2}^{t_0}\int\limits_{B_r(x_0)}|\nabla u|^2 dyd\tau\leq\epsilon_*\Rightarrow z_0=(x_0,t_0)\text{ is a regular point of }v.$$

The famous consequence of this is that the set of interior singular points $S\subset\mathcal{O}\times]S_1,S_2[$ of a suitable weak solution (v,q) on $\mathcal{O}\times]S_1,S_2[$ satisfies $\mathcal{P}_1(S)=0$. Here,

$$Q^*_r(z=(x,t)):=B_r(x)\times]t-r^2,t+r^2[, \tag{1.2.15}$$

$$\mathcal{P}_{1,\delta}(S) := \inf\left\{\sum_{i=1}^{\infty} r_i : S \subset \cup_{i=1}^{\infty} Q^*_{r_i}(z_i) \text{ and each } r_i < \delta\right\} \tag{1.2.16}$$

and

$$\mathcal{P}_1(S) := \lim_{\delta\to 0^+} \mathcal{P}_{1,\delta}(S). \tag{1.2.17}$$

We mention that Theorem B in Caffarelli, Kohn and Nirenberg's paper [17] gives that $\mathcal{P}_1(S) = 0$, for a suitable weak solution (v, q) on $\mathcal{O}\times]S_1, S_2[$ with divergence-free forcing term $f \in L_{\frac{5}{2}+\delta}(\mathcal{O}\times]S_1, S_2[)$ $(\delta > 0)$. Subsequently, Ladyzhenskaya and Seregin [74] and Kukavica [69]-[71] showed that $\mathcal{P}_1(S) = 0$ holds for more general forcing terms than those considered in [17].

Let us now discuss (2), namely the case that scale-invariant quantities are bounded. Let us mention that an example falling under (2), that is sufficient to prove $(0, 0)$ is a regular point, is the Ladyzhenskaya-Prodi-Serrin condition:

$$\|v\|_{L_{s,l}(Q(r))} < \infty. \tag{1.2.18}$$

Here,

$$\frac{3}{s} + \frac{2}{l} = 1 \text{ and } s > 3. \tag{1.2.19}$$

Under the weaker assumption that v is a distributional solution to the Navier-Stokes equations with $v \in L_\infty(-1, 0; L_2(B))$ and $\nabla v \in L_2(-1, 0; L_2(B))$, local regularity in this case was proven by Struwe in [122]. Unfortunately, it is not known if any weak Leray-Hopf solution on $\mathbb{R}^3\times]0, \infty[$, with smooth solenoidal data, satisfies (1.2.18)-(1.2.19). At the time of writing, it is only known that any weak Leray-Hopf solution on $\mathbb{R}^3\times]0, \infty[$ satisfies

$$v \in L_l(0, T; L_s(\mathbb{R}^3)) \text{ with } \frac{3}{s} + \frac{2}{l} = \frac{3}{2}. \tag{1.2.20}$$

Classifying (1.2.18)-(1.2.19) as following under category (2) is misleading. As stated on p.2110 of Seregin's paper [103], the assumptions (1.2.18)-(1.2.19) imply that

$$\lim_{R\to 0^+} \|v\|_{L_{s,l}(Q(R))} = 0. \tag{1.2.21}$$

As mentioned on p.214 of Escauriaza, Seregin and Šverák's paper [27], for the case $s = 3$ no such shrinking occurs. Regularity in this case was proven in [27] by contradiction, involving a rescaling procedure producing a nontrivial ancient solution and a Liouville theorem based on backward uniqueness and unique continuation for parabolic equations. Chapter 4 of the author's thesis will contain further discussion of endpoint cases of this type, as well as containing new results in this direction.

At the time of writing, it is unknown whether or not suitable weak solutions of the Navier-Stokes equations can possess a singular point, with a bounded scale-invariant quantity in a neighbourhood of it. We refer to such singular points as 'Type I singularities'[11]. Let us mention that in the literature, a Type I singularity (x_0, t_0) is sometimes defined to be a singular point of v with

$$\sup_{(x,t)\in B(x_0,r)\times]t_0-r^2,t_0[} (t_0-t)^{\frac{1}{2}}|v(x,t)| < \infty \tag{1.2.22}$$

for some $r > 0$. This is contained in the definition given at the beginning of this paragraph. For the special case of axi-symmetric solutions to the Navier-Stokes equations, it is known the assumption (1.2.22) implies that (x_0, t_0) is a regular point. See Chen et al. [22]-[23] and work of Seregin and Šverák [102], for example. Let us mention that $\frac{1}{(t_0-t)^{\frac{1}{2}}}$ is the blow-up rate for backward self similar solutions to the Navier-Stokes equations $v : \mathbb{R}^3\times]-\infty, t_0[\to \mathbb{R}^3$ and $q : \mathbb{R}^3\times]-\infty, t_0[\to \mathbb{R}$, which are of the form

$$(v(x,t), q(x,t)) = \Big(\lambda(t)v(\lambda(t)x), \lambda^2(t)q(\lambda(t)x)\Big) \tag{1.2.23}$$

with

$$\lambda(t) := \frac{1}{(2a(t_0-t))^{\frac{1}{2}}} \quad \text{and} \quad a > 0. \tag{1.2.24}$$

The existence of backward self similar solutions has been ruled out by Nečas, R$\mathring{u}$žička and Šverák in [85] and by Tsai in [126].

An example of a strategy for investigating (potential) singular points for the Navier-Stokes equations, with a certain bounded scale-invariant quantity in a neighbourhood of it, is the use of 'blow-up procedures' for the Navier-Stokes equations. As discussed in Koch et al. [62] (pp.83-84 of [62]), blow-up procedures involve taking a suitable rescaling of the equation (of the form (1.2.11)-(1.2.12)) and passing to the limit[12] to obtain an 'ancient solution'. As stated by Seregin in [103]: 'A particular selection of rescaling parameters depends upon a problem under consideration.' (p.2112 of [103]). Moreover different rescaling procedures, for the Navier-Stokes equations, can produce ancient solutions with quite different properties. For example, compare the papers of Escauriaza, Seregin and Šverák [27] and Koch et al. [62]. In the blow-up procedure, the bounded scale-invariant quantities of the original solution (v, q) (in the parabolic neighbourhood of the singular point) are transferred to the ancient solution. This additional information for the ancient solution can sometimes be very useful in

[11] Singular points that are not 'Type I' are referred to as 'Type II'.

[12] As mentioned in Koch et al. [62] (p.83 of [62]), this can formally be thought of as 'zooming in' on the singular point in question.

ruling out the singular point for the original velocity field. In relation to this, a general framework for studying 'Type I singularities' of the Navier-Stokes equations was established by Koch et al. in [62] and by Seregin and Šverák in [108]. The framework established in [62] and [108] shows that if a certain Liouville type conjecture can be verified, then one can rule out Type I singularities.

1.2.2 Local Boundary Regularity Theory for the Navier-Stokes Equations

In the previous subsection, we discussed the local interior regularity theory for the Navier-Stokes equations. Chapter 4 of this thesis will mostly be concerned with the case of local boundary regularity theory for the Navier-Stokes equations. We now motivate the local boundary regularity theory for the Navier-Stokes equations, as well as explaining some key differences of the local boundary regularity theory in contrast to the case of local interior regularity theory.

For global-in-time weak Leray-Hopf solutions of the initial boundary value problem on $\mathbb{R}^3_+\times]0,\infty[$, a version of Leray's sufficient condition also applies. Moreover, definitions of short time regular initial data and blow-up times extend naturally to this case as well. In analogy with whole-space case, proving local boundedness[13] at all space-time points $z_0=(x_0,t_0)\in(\mathbb{R}^3_+\cup\partial\mathbb{R}^3_+)\times]0,\infty[$ of a given global-in-time weak Leray-Hopf solution v of the initial boundary value problem, with short time regular initial data $u_0\in J(\mathbb{R}^3_+)$, implies the uniqueness of v in the weak Leray-Hopf class[14]. Here we define,

$$\partial\mathbb{R}^3_+:=\{x=(x_i)_{i=1,2,3}:x=(x_1,x_2,0)\}.$$

In Seregin and Šverák's paper [105], it is stated that, unlike parabolic systems with smooth coefficients, there is a distinction between local interior regularity properties and local boundary regularity properties for both the linear Stokes equations and the Navier-Stokes equations. For the linear Stokes system, this was first demonstrated from constructions given by Kang in [58]. This construction was simplified by Seregin and Šverák in [105]. Furthermore, the construction in [105] gave a distributional solution (v,q), with $v\in L_\infty(-1,0;L_2(B^+(0,1))$ and $\nabla v\in L_2(-1,0;L_2(B^+(0,1))$ ($B^+(0,1):=B(0,1)\cap\mathbb{R}^3_+$), to the Navier-Stokes equations in $B^+(0,1)\times]-1,0[$:

$$\partial_t v-\Delta v+\nabla q+v\cdot\nabla v=f,\ \mathrm{div}\, v=0, \tag{1.2.25}$$

[13] By this we mean that there exists an $r>0$ such that $v\in L_\infty(B(x_0,r)\cap\mathbb{R}^3_+\times]t_0-r^2,t_0[)$.

[14] For further explanation of this and relevant references, we refer the reader to Chapter 5 (specifically, p.128 and p.156).

$$v(x_1, x_2, 0, t) = 0. \tag{1.2.26}$$

The important feature of the constructed solution in [105] is that it is bounded in $B^+(0,1)\times]-1,0[$ but its spatial gradient is unbounded in any parabolic neighbourhood of the space-time origin.

Contrastingly, it has been known since the work of Serrin in [113] that for distributional solutions to the Navier-Stokes equations, with v in $L_\infty(-1,0;L_2(B))$ and $\nabla v \in L_2(-1,0;L_2(B))$, the following implication holds. Specifically, boundedness of v on $B(1)\times]-1,0[$ implies boundedness of its higher spatial derivatives.

In general, it is not at all clear if interior regularity criteria for the Navier-Stokes equations, have analogous forms up to the flat part of the boundary. The main difficulty in the local boundary regularity theory for the Navier-Stokes equations, compared to its interior counterpart, is the treatment of the pressure near the flat part of the boundary.

A general setting for the local boundary regularity theory for the Navier-Stokes equations was suggested by Seregin in [98]. Before stating this, we introduce the following notation. For $x_0 = (x_{0i})_{i=1,2,3}$:

$$B^+(x_0, R) = \{x \in B(x_0, R) : x = (x_1, x_2, x_3),\ x_3 > x_{03}\},$$

$$B(\theta) = B(0,\theta),\ \ B = B(1),\ \ B^+(\theta) = B^+(0,\theta),\ \ B^+ = B^+(1),$$

$$\Gamma(x_0, R) = \{x \in B(x_0, R) : x_3 = x_{03}\},\ \ \Gamma(\theta) = \Gamma(0,\theta),\ \ \Gamma = \Gamma(1),$$

$$Q^+(z_0, R) = B^+(x_0, R)\times]t_0 - R^2, t_0[,$$

$$Q^+(\theta) = Q^+(0,\theta) \text{ and } Q^+ = Q^+(1).$$

The definition we give below is essentially contained in Seregin's paper [98], specifically Definition 2.1 there.

Definition 1.4. *Let $-\infty < S_1 < S_2 < \infty$. We say that the pair $v : B^+(x_0, R)\times]S_1, S_2[\to \mathbb{R}^3$ and $q : B^+(x_0, R)\times]S_1, S_2[\to \mathbb{R}$ are a suitable weak solution to the Navier-Stokes equations in $B^+(x_0, R)\times]S_1, S_2[$ near $\Gamma(x_0, R) \times [S_1, S_2]$ if:*

$$v \in L_\infty(S_1, S_2; L_2(B^+(x_0, R))), \tag{1.2.27}$$

$$\nabla v \in L_2(S_1, S_2; L_2(B^+(x_0, R))), \tag{1.2.28}$$

$$\partial_t v \in L_{\frac{3}{2}}(]S_1, S_2[; L_{\frac{9}{8}}(B^+(x_0, R))), \tag{1.2.29}$$

$$\nabla^2 v \in L_{\frac{3}{2}}(]S_1, S_2[; L_{\frac{9}{8}}(B^+(x_0, R))), \tag{1.2.30}$$

$$\nabla q \in L_{\frac{3}{2}}(]S_1, S_2[; L_{\frac{9}{8}}(B^+(x_0, R))), \tag{1.2.31}$$

$$\int\limits_{S_1}^{S_2} \int\limits_{B^+(x_0,R)} (-v \cdot \partial_t \varphi - v \otimes v : \nabla \varphi + \nabla v : \nabla \varphi - q div \varphi) dy d\tau = 0 \tag{1.2.32}$$

for any $\varphi \in C_0^\infty(B^+(x_0, R) \times]S_1, S_2[)$ *and*

$$v(x,t) = 0 \text{ for } x_3 = x_{03} \text{ and almost every } S_1 < t < S_2. \tag{1.2.33}$$

Additionally, it is required that the following local energy inequality holds for all positive $\varphi \in C_0^\infty(B(x_0, R) \times]S_1, \infty[)$ *and almost every time* $t \in]S_1, S_2[$*:*

$$\int\limits_{B^+(x_0,R)} \varphi |v(x,t)|^2 dx + 2 \int\limits_0^t \int\limits_{B^+(x_0,R)} \varphi |\nabla v|^2 dx dt'$$

$$\leq \int\limits_0^t \int\limits_{B^+(x_0,R)} |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) dx dt'. \tag{1.2.34}$$

For a suitable weak solution (v, q) in $B^+(x_0, R) \times]S_1, S_2[$ near $\Gamma(x_0, x_0) \times]S_1, S_2[$, we have the following classification of points in $(B^+(x_0, R) \cup \Gamma(x_0, R)) \times]S_1, S_2[$. In particular

$(y_0, s_0) \in (B^+(x_0, R) \cup \Gamma(x_0, R)) \times]S_1, S_2[$ is a singular point of v $\iff$

$v \notin L_\infty((B^+(y_0, r) \times]s_0 - r^2, s_0[) \cap (B^+(x_0, R) \times]S_1, S_2[)$ for all sufficiently small $r > 0$.

Moreover, $(y_0, s_0) \in (B^+(x_0, R) \cup \Gamma(x_0, R)) \times]S_1, S_2[$ is defined to be a regular point of v if it is not a singular point of v. The local boundary regularity theory of the Navier-Stokes equations is centred around the following motivating question.

Consider a suitable weak solution (v, q)**, to the Navier-Stokes equations, in** $B^+(x_0, R) \times]t_0 - R^2, t_0[$ **near** $\Gamma(x_0, R) \times [t_0 - R^2, t_0]$**. What assumptions ensure that** $z_0 = (x_0, t_0)$ **is a regular point?**

The first contribution to this question is due to Seregin in [98]. There it was shown that there exists a universal constant ϵ_* such that

$$\limsup_{r \to 0^+} \int\limits_{t_0 - r^2}^{t_0} \int\limits_{B^+(x_0,r)} |\nabla v(x,t)|^2 dx dt < \epsilon_* \Rightarrow z_0 \text{ is a regular point of } v.$$

In particular, this extended the interior regularity criteria of Caffarelli, Kohn and Nirenberg [17] to cases involving a flat part of the boundary. Moreover, this regularity criterion implies that the set of singular points $E \subset \Gamma(x_0, R) \times [t_0 - R^2, t_0]$ of (v, q), that also lie on the flat part of the boundary, satisfies $\mathcal{P}_1(E) = 0$.

As mentioned in the previous subsection, in Escauriaza, Seregin and Šverák's paper [27] it was shown that if (v, q) is a suitable weak solution in $B \times]-1, 0[$ then

$$v \in L_\infty(-1, 0; L_3(B)) \Rightarrow v \text{ is Hölder continuous in } \bar{Q}(1/2).$$

Later in Seregin's paper [100], it was shown that this regularity criterion also applied up to the flat part of the boundary. In particular, in [100] it was proven that if (v, q) is a suitable weak solution in $B^+ \times]-1, 0[$ near $\Gamma(0, 1) \times [0, 1]$, then

$$v \in L_\infty(-1, 0; L_3(B^+)) \Rightarrow v \text{ is Hölder continuous in } \bar{Q}^+(1/2).$$

Let us mention that throughout this thesis, we define a function f to be Hölder continuous on the closure of $\Omega \times]S_1, S_2[$ if there exists an $0 < \alpha < 1$ and a constant C such that the following holds. Namely, for any $z_0 = (x_0, t_0)$ and $z_{00} = (x_{00}, t_{00})$ in the closure of $\Omega \times]S_1, S_2[$,

$$|f(x_0, t_0) - f(x_{00}, t_{00})| \leq C(|x_0 - x_{00}|^\alpha + |t_0 - t_{00}|^{\frac{\alpha}{2}}).$$

The main aim of Chapter 4 is to extend the $L_\infty(-1, 0; L_3(B^+))$ regularity criterion, for suitable weak solutions in $B^+(1) \times]-1, 0[$ near $\Gamma(0, 1) \times [-1, 0]$. This is achieved by the following theorem.

Theorem 4.5 Chapter 4. *Assume (v, q) is a suitable weak solution of the Navier-Stokes equations in $B^+(1) \times]-1, 0[$ near $\Gamma(0, 1) \times [-1, 0]$. Assume, in addition, that there exists $3 \leq \beta < \infty$ such that*

$$v \in L_\infty(-1, 0; L^{3,\beta}(B^+)). \tag{1.2.35}$$

Then v is Hölder continuous in the closure of the set $Q^+(1/2)$.

In the above theorem $L^{3,\beta}$ represents a Lorentz space, which satisfies:

$$L^3(\Omega) \subsetneq L^{3,\beta}(\Omega),$$

for $3 < \beta < \infty$ and $\Omega \subseteq \mathbb{R}^3$. We will provide a further description of Lorentz spaces in 'Chapter 2: Preliminary Material'.

As previously mentioned, it was shown by Leray in [75] that if T is a finite blow-up time and $3 < p \leq \infty$ then necessarily

$$\|v(\cdot,t)\|_{L_p(\mathbb{R}^3)} \geq \frac{c(p)}{(T-t)^{\frac{1}{2}(1-\frac{3}{p})}},\ 0 \leq t < T. \tag{1.2.36}$$

As will be discussed in Chapter 5, an analogous necessary condition to (1.2.36) is not possible for the $L_3(\mathbb{R}^3)$, $L_3(\mathbb{R}^3_+)$, $L^{3,\beta}(\mathbb{R}^3)$ and $L^{3,\beta}(\mathbb{R}^3_+)$ norms. Theorem 4.5 of Chapter 4 has the following implication. Specifically, if $v : \mathbb{R}^3_+ \times]0,\infty[\to \mathbb{R}^3$ is a global-in-time weak Leray-Hopf solution, with short time regular initial data and finite blow-up time T, then necessarily

$$\limsup_{t\uparrow T} \|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)} = \infty,$$

for any $3 \leq \beta < \infty$. The main result of Chapter 5 gives a more precise necessary condition in the following way.

Theorem 5.4 Chapter 5. *Let $v : \mathbb{R}^3_+ \times]0,\infty[\to \mathbb{R}^3$ be a global-in-time weak Leray-Hopf solution of the Navier-Stokes equations. Furthermore, assume that its initial datum is short time regular. Suppose that v has a finite blow-up time T. Then it necessarily holds that*

$$\lim_{t\uparrow T} \|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)} = \infty,$$

for any $3 \leq \beta < \infty$.

For a detailed discussion of the previous literature related to the above theorems, as well as the difficulties that we overcome in proving the above results, we refer the reader to Chapters 4 and 5.

1.3 Potential Blow-up of Scale-Invariant Norms and Global Solutions to the Navier-Stokes equations

The proof of Theorem 5.4 (Chapter 5) proceeds by contradiction. Namely, we assume that exists $t_k \uparrow T$ with

$$M := \sup_k \|v(\cdot,t_k)\|_{L^{3,\beta}(\mathbb{R}^3_+)} < \infty. \tag{1.3.1}$$

The aim of the proof of Theorem 5.4 (Chapter 5) is to show that (1.3.1) implies that T is not a finite blow-up time. The next step of the proof is to perform the rescaling

$$v^{(k)}(y,s) = \lambda_k v(x,t), \quad q^{(k)}(y,s) = \lambda_k^2 q(x,t), \quad u_0^{(k)}(y) = \lambda_k v(\lambda_k y, t_k), \tag{1.3.2}$$

$$x = \lambda_k y, \qquad t = T + \lambda_k^2 s \quad \text{and} \qquad \lambda_k = \sqrt{\frac{T - t_k}{2}}. \tag{1.3.3}$$

This gives that $(v^{(k)}, q^{(k)})$ are solutions to the Navier-Stokes equations on $\mathbb{R}^3_+ \times]-2, 0[$ with

$$\sup_k \|u_0^{(k)}\|_{L^{3,\beta}(\mathbb{R}^3_+)} = \|v(\cdot, t_k)\|_{L^{3,\beta}(\mathbb{R}^3_+)} = M. \tag{1.3.4}$$

The final step of the proof is to obtain a nontrivial ancient solution to the Navier-Stokes equations and to then attempt to apply a Liouville type theorem to it, based on backward uniqueness and unique continuation for parabolic operators developed by Escauriaza, Seregin and Šverák in [27].

Suppose $v : \mathbb{R}^3 \times]0, \infty[\to \mathbb{R}^3$ is a global-in-time weak Leray-Hopf solution, with short time regular initial datum and finite blow-up time T. The same strategy and rescaling procedure, as that used in proving Theorem 5.4 (Chapter 5), has been previously used by Seregin in [104] and [106] to show

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \infty \text{ and } \lim_{t \uparrow T} \|v(\cdot, t)\|_{L_3(\mathbb{R}^3)} = \infty.$$

It can also be applied to other critical spaces[15] $\mathcal{X}(\mathbb{R}^3)$ in attempting to show

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_{\mathcal{X}(\mathbb{R}^3)} = \infty.$$

As mentioned by Seregin and Šverák in [112] for $\mathcal{X} = L_3(\mathbb{R}^3)$, in order to obtain a nontrivial ancient solution from the sequence of rescaled solutions, it is advantageous to have a notion of solution to the Navier-Stokes equations with initial data in $\mathcal{X}$ (which we denote $\mathcal{N}(\mathcal{X})$), that has the following properties.

- **1) Global existence** For any $u_0 \in \mathcal{X}$ there should exist a global-in-time solution in $\mathcal{N}(\mathcal{X})$.

- **2) Weak* stability**
$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0 \quad \text{in } \mathcal{X} \Rightarrow$$
$v^{(k)}(\cdot, u_0^{(k)}) \in \mathcal{N}(\mathcal{X})$ converges up to subsequence (in sense of distributions) to $v(\cdot, u_0) \in \mathcal{N}(\mathcal{X})$.

We refer to such a notion of solution as a 'global $\mathcal{X}$ solution' of the Navier-Stokes equations.

[15] Recall that $\mathcal{X}(\mathbb{R}^3)$ is a critical space, with respect to Navier-Stokes scaling, if $u_0 \in \mathcal{X}(\mathbb{R}^3)$ implies the following. Namely, $u_{0\lambda}(x) := \lambda u_0(\lambda x) \in \mathcal{X}(\mathbb{R}^3)$ and $\|u_0\|_{\mathcal{X}(\mathbb{R}^3)} = \|u_{0\lambda}\|_{\mathcal{X}(\mathbb{R}^3)}$ for all $\lambda > 0$.

The main goal of Chapter 6 is to establish existence of global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, for any divergence-free $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, that satisfies **1)-2)**. Here $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ denotes a critical homogeneous Besov space, whose definition will be provided in 'Chapter 2: Preliminary Material'. Any global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution, with initial data u_0 has the following structure. Namely,

$$v(x,t) = V(x,t) + u(x,t), \tag{1.3.5}$$

where

$$V(x,t) := \int\limits_{\mathbb{R}^3} \Gamma(x-y,t)u_0(y)dy$$

and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp(-|x|^2/4t).$$

Moreover,

$$u \in \mathcal{L}(T) := L_\infty(0,T; J(\mathbb{R}^3)) \cap L_2(0,T; \overset{\circ}{J}{}^1_2(\mathbb{R}^3))$$

for any $T > 0$ and satisfies the energy inequality

$$\|u(\cdot,t)\|^2_{L_2} + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u(x,t')|^2 dxdt' \leq 2\int_0^t \int_{\mathbb{R}^3} (V \otimes u + V \otimes V) : \nabla u dxdt'. \tag{1.3.6}$$

The significance of the critical homogeneous Besov space $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ is as follows. Namely, $\mathcal{X} = \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ is the widest critical space for which a global $\mathcal{X}$ solution, having the structure (1.3.5)-(1.3.6), is possible. For further explanation of this and full details of our definition of global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, we refer the reader to Chapter 6. The main results of Chapter 6, regarding global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, are as follows.

Theorem 6.3 Chapter 6. *Let $u_0^{(k)} \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ and let $v^{(k)}$ be a sequence of a global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions to the Cauchy problem for the Navier-Stokes system with initial data $u_0^{(k)}$. Then there exists a subsequence, still denoted $v^{(k)}$, that converges to a global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution v to the Cauchy problem for the Navier-Stokes system with initial data u_0, in the sense of distributions.*

Corollary 6.4 Chapter 6. *Given any divergence-free $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, there exists at least one global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$-solution to the Navier-Stokes system in $\mathbb{R}^3 \times]0,\infty[$.*

In the remainder of Chapter 6, we prove a conditional uniqueness result for global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$-solutions.

Proposition 6.6 Chapter 6. *Let* $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ *be divergence free in the sense of tempered distributions. There exists an* $\varepsilon > 0$ *such that if*

$$\sup_{0<t<T} t^{\frac{1}{8}} \|S(t)u_0\|_{L_4} < \varepsilon \tag{1.3.7}$$

then all global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ *solutions, with initial data* $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$*, coincide on* $Q_T := \mathbb{R}^3 \times]0, T[$.

Unfortunately, the condition (1.3.7) does not hold for arbitrary divergence-free initial data in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$. In particular, for large minus one homogeneous solenoidal initial data. For this case, the conjectures by Jia and Šverák in [56] and [57] suggest that there may be non-uniqueness for global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, even for a short time.

For a detailed discussion of the previous literature related to the above results, as well as the difficulties that we overcome, we refer the reader to Chapter 6.

Chapter 2

Preliminary Material

2.1 Notation

In this section, we will introduce notation that will be repeatedly used throughout the rest of this thesis. We adopt the usual summation convention throughout this thesis.

We define:

$$]a, b[:= \{t : a < t < b\},$$

$$[a, b] := \{t : a \leq t \leq b\},$$

$$[a, b[:= \{t : a \leq t < b\}$$

and

$$]a, b] := \{t : a < t \leq b\}.$$

For arbitrary vectors $a = (a_i)$, $b = (b_i)$ in $\mathbb{R}^n$ and for arbitrary matrices $F = (F_{ij})$, $G = (G_{ij})$ in $\mathbb{M}^n$ we put

$$a \cdot b = a_i b_i,\ |a| = \sqrt{a \cdot a},$$

$$a \otimes b = (a_i b_j) \in \mathbb{M}^n,$$

$$FG = (F_{ik} G_{kj}) \in \mathbb{M}^n,\ \ F^T = (F_{ji}) \in \mathbb{M}^n,$$

$$F : G = F_{ij} G_{ij} \ \text{ and } \ |F| = \sqrt{F : F}.$$

For spatial domains and space-time domains, we will make use of the following notation:

$$B(x_0, R) = \{x \in \mathbb{R}^3 : |x - x_0| < R\},$$

$$\bar{B}(x_0, R) = \{x \in \mathbb{R}^3 : |x - x_0| \leq R\},$$

$$B(\theta) = B(0, \theta),\ \ B = B(1),$$

$$\bar{B}(\theta) = \bar{B}(0,\theta), \;\; \bar{B} = \bar{B}(1),$$

$$Q(z_0,R) = B(x_0,R)\times]t_0 - R^2, t_0[, \;\; z_0 = (x_0,t_0),$$

$$\bar{Q}(z_0,R) = \bar{B}(x_0,R)\times[t_0 - R^2, t_0], \;\; z_0 = (x_0,t_0),$$

$$Q(\theta) = Q(0,\theta), \;\; Q = Q(1),$$

$$\bar{Q}(\theta) = \bar{Q}(0,\theta), \;\; \bar{Q} = \bar{Q}(1) \text{ and}$$

$$Q_{a,b} := \mathbb{R}^3\times]a,b[.$$

Here $-\infty \leq a < b \leq \infty$. In the special cases where $a = 0$, we write $Q_b := Q_{0,b}$.

Setting $x' = (x_1,x_2) \in \mathbb{R}^2$ and $x_0 = (x_{01},x_{02},x_{03})$, we introduce the following definitions:

$$x_0^* := (x_{01},x_{02},-x_{03}), \;\; B^+(x_0,R) = \{x \in B(x_0,R) : x = (x',x_3), \; x_3 > x_{03}\},$$

$$\bar{B}^+(x_0,R) = \{x \in \bar{B}(x_0,R) : x = (x',x_3), \; x_3 > x_{03}\},$$

$$B^+(\theta) = B^+(0,\theta), \;\; B^+ = B^+(1), \;\; \bar{B}^+(\theta) = \bar{B}^+(0,\theta), \;\; \bar{B}^+ = \bar{B}^+(1),$$

$$\Gamma(x_0,R) = \{x \in B(x_0,R) : x_3 = x_{03}\}, \;\; \Gamma(\theta) = \Gamma(0,\theta), \;\; \Gamma = \Gamma(1),$$

$$Q^+(z_0,R) = B^+(x_0,R)\times]t_0 - R^2, t_0[,$$

$$\bar{Q}^+(z_0,R) = \bar{B}^+(x_0,R)\times[t_0 - R^2, t_0],$$

$$Q^+(\theta) = Q^+(0,\theta), \;\; Q^+ = Q^+(1), \;\; \bar{Q}^+(\theta) = \bar{Q}^+(0,\theta), \;\; \bar{Q}^+ = \bar{Q}^+(1),$$

$$\mathbb{R}^3_+ := \{(x',x_3) \in \mathbb{R}^3_+ : x_3 > 0\}, \;\; \mathbb{R}^3_{+\delta} := \{(x',x_3) \in \mathbb{R}^3 : x_3 > \delta\}$$

$$\text{and } \; Q^+_{a,b} := \mathbb{R}^3_+\times]a,b[.$$

Here, $-\infty \leq a < b \leq \infty$. In the special cases where $a = 0$, we write $Q^+_b := Q^+_{0,b}$.

For $\Omega \subseteq \mathbb{R}^3$, we define the indicator function $\chi_\Omega : \mathbb{R}^3 \to \mathbb{R}$ as being equal to 1 on Ω and 0 on $\mathbb{R}^3 \setminus \Omega$.

$$\chi_\Omega(x) := \begin{cases} 1 & \text{if } \; x \in \Omega, \\ 0 & \text{if } \; x \in \mathbb{R}^3 \setminus \Omega. \end{cases}$$

For $\Omega \subseteq \mathbb{R}^3$, mean values of integrable functions are denoted as follows

$$[p]_\Omega = \frac{1}{|\Omega|}\int\limits_\Omega p(x)dx.$$

For, $\Omega \subseteq \mathbb{R}^3$, the space $[C^\infty_{0,0}(\Omega)]^{L_s(\Omega)}$ is defined to be the closure of

$$C^\infty_{0,0}(\Omega) := \{u \in C^\infty_0(\Omega) : \text{div } u = 0\}$$

with respect to the $L_s(\Omega)$ norm. With this notation, we define

$$J(\Omega) := [C^\infty_{0,0}(\Omega)]^{L_2(\Omega)}.$$

Moreover, $\overset{\circ}{J}{}^1_2(\Omega)$ is defined as the completion of the space $C^\infty_{0,0}(\Omega)$ with respect to the Dirichlet integral

$$\Big(\int\limits_\Omega |\nabla v|^2 dx\Big)^{\frac{1}{2}}.$$

If X is a Banach space with norm $\|\cdot\|_X$, then $L_s(a,b;X)$, with $a<b$ and $s\in[1,\infty[$, will denote the usual Banach space of strongly measurable X-valued functions $f(t)$ on $]a,b[$ such that

$$\|f\|_{L_s(a,b;X)} := \left(\int\limits_a^b \|f(t)\|^s_X dt\right)^{\frac{1}{s}} < +\infty.$$

The usual modification is made if $s=\infty$. For $\Omega\subseteq\mathbb{R}^3$, $-\infty\leq a<b\leq\infty$ and $1\leq s,l\leq\infty$, we define

$$L_{s,l}(\Omega\times]a,b[) := L_l(a,b;L_s(\Omega)).$$

For $m=n$, we define $L_n(\Omega\times]a,b[) := L_{n,n}(\Omega\times]a,b[)$. Let $C([a,b];X)$ denote the space of continuous X valued functions on $[a,b]$ with usual norm. In addition, let $C_w([a,b];X)$ denote the space of X valued functions, which are continuous from $[a,b]$ to the weak topology of X.

We also define the energy class:

$$\mathcal{L}(S) := L_\infty(0,S;J(\Omega))\cap L_2(0,S;\overset{\circ}{J}{}^1_2(\Omega)).$$

We define the energy norm as:

$$\|v\|_{\mathcal{L}(S)} := \|v\|_{L_{2,\infty}(\Omega\times]0,S[)} + \|\nabla v\|_{L_2(\Omega\times]0,S[)}.$$

For $\Omega\subseteq\mathbb{R}^3$, $-\infty\leq a<b\leq\infty$ and $1\leq m,n\leq\infty$, we define the following Sobolev spaces with mixed norms:

$$W^{1,0}_{m,n}(\Omega\times]a,b[) = \{v\in L_{m,n}(\Omega\times]a,b[) : \|v\|_{W^{1,0}_{m,n}(\Omega\times]a,b[)} := \|v\|_{L_{m,n}(\Omega\times]a,b[)}+$$

$$+\|\nabla v\|_{L_{m,n}(\Omega\times]a,b[)} < \infty\}$$

and

$$W^{2,1}_{m,n}(\Omega\times]a,b[) = \{v\in L_{m,n}(\Omega\times]a,b[) : \|v\|_{W^{2,1}_{m,n}(\Omega\times]a,b[)} := \|v\|_{L_{m,n}(\Omega\times]a,b[)}+$$

$$+\|\nabla v\|_{L_{m,n}(\Omega\times]a,b[)} + \|\nabla^2 v\|_{L_{m,n}(\Omega\times]a,b[)} + \|\partial_t v\|_{L_{m,n}(\Omega\times]a,b[)} < \infty\}.$$

For $m = n$, we define $W^{1,0}_n(\Omega\times]a,b[) := W^{1,0}_{n,n}(\Omega\times]a,b[)$ and $W^{2,1}_n(\Omega\times]a,b[) := W^{2,1}_{n,n}(\Omega\times]a,b[)$. Throughout this thesis, we will also make use of the following fact. Namely if Ω is a bounded domain with sufficiently smooth boundary, $1 < n \leq \infty$, $1 \leq m \leq \infty$ and $-\infty < a < b < \infty$, then we have the compact embedding:

$$W^{2,1}_{m,n}(\Omega\times]a,b[) \hookrightarrow C([a,b]; L_m(\Omega)).$$

A proof of this can be inferred from Theorem 1 of Simon's paper [115], for example.

For $\Omega \subseteq \mathbb{R}^3$ and $-\infty < S_1 < S_2 < \infty$, we define a function f to be Hölder continuous on the closure of $\Omega\times]S_1,S_2[$ if there exists an $0 < \alpha < 1$ and a constant C such that the following holds. Namely, for any $z_0 = (x_0,t_0)$ and $z_{00} = (x_{00},t_{00})$ in the closure of $\Omega\times]S_1,S_2[$,

$$|f(x_0,t_0) - f(x_{00},t_{00})| \leq C(|x_0 - x_{00}|^\alpha + |t_0 - t_{00}|^{\frac{\alpha}{2}}).$$

Throughout this thesis we will use the notation

$$S(t)u_0(x) := \int\limits_{\mathbb{R}^3} \Gamma(x-y,t)u_0(y)dy,$$

where Γ is the three-dimensional heat kernel

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}}\exp(-|x|^2/4t).$$

2.2 Lorentz Spaces

2.2.1 Definition and Properties of Lorentz Spaces

Given a measurable subset $\Omega \subseteq \mathbb{R}^n$, let us define the Lorentz spaces. For a measurable function $f : \Omega \to \mathbb{R}$ define:

$$d_{f,\Omega}(\alpha) := \mu(\{x \in \Omega : |f(x)| > \alpha\}), \tag{2.2.1}$$

where μ denotes the Lebesgue measure. The Lorentz space $L^{p,q}(\Omega)$, with $p \in [1,\infty[$, $q \in [1,\infty]$, is the set of all measurable functions g on Ω such that the quasinorm $\|g\|_{L^{p,q}(\Omega)}$ is finite. Here:

$$\|g\|_{L^{p,q}(\Omega)} := \Big(p\int\limits_0^\infty \alpha^q d_{g,\Omega}(\alpha)^{\frac{q}{p}}\frac{d\alpha}{\alpha}\Big)^{\frac{1}{q}}, \tag{2.2.2}$$

$$\|g\|_{L^{p,\infty}(\Omega)} := \sup_{\alpha>0} \alpha d_{g,\Omega}(\alpha)^{\frac{1}{p}}. \tag{2.2.3}$$

It is known there exists a norm, which is equivalent to the quasinorm defined above, for which $L^{p,q}(\Omega)$ is a Banach space. For $p \in [1,\infty[$ and $1 \leq q_1 < q_2 \leq \infty$, we have the following continuous embeddings

$$L^{p,q_1}(\Omega) \hookrightarrow L^{p,q_2}(\Omega) \tag{2.2.4}$$

and the inclusion is known to be strict.

Let X be a Banach space with norm $\|\cdot\|_X$, $a < b$, $p \in [1,\infty[$ and $q \in [1,\infty]$. Then $L^{p,q}(a,b;X)$ will denote the space of strongly measurable X-valued functions $f(t)$ on $]a,b[$ such that

$$\|f\|_{L^{p,q}(a,b;X)} := \|\|f(t)\|_X\|_{L^{p,q}(a,b)} < \infty. \tag{2.2.5}$$

In particular, if $1 \leq q_1 < q_2 \leq \infty$, we have the following continuous embeddings

$$L^{p,q_1}(a,b;X) \hookrightarrow L^{p,q_2}(a,b;X) \tag{2.2.6}$$

and the inclusion is known to be strict.

Let us recall a known proposition known as 'O'Neil's convolution inequality' (Theorem 2.6 of O'Neil's paper [86]), which will be used in proving Proposition 3.4 in Chapter 3.

Proposition 2.1. *Suppose* $1 \leq p_1, p_2, q_1, q_2, r \leq \infty$ *are such that*

$$\frac{1}{r} + 1 = \frac{1}{p_1} + \frac{1}{p_2} \tag{2.2.7}$$

and

$$\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}. \tag{2.2.8}$$

Suppose that

$$f \in L^{p_1,q_1}(\mathbb{R}^n) \textit{ and } g \in L^{p_2,q_2}(\mathbb{R}^n). \tag{2.2.9}$$

Then

$$f \star g \in L^{r,s}(\mathbb{R}^n) \text{ with} \tag{2.2.10}$$

$$\|f \star g\|_{L^{r,s}(\mathbb{R}^n)} \leq 3r\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^n)}. \tag{2.2.11}$$

In Chapters 4 and 5 of this thesis, we will use an inequality that we will refer to as 'Hunt's inequality'. The statement below and proof can be found in Hunt's paper [53] (Theorem 4.5, p.271 of [53]).

Proposition 2.2. *Suppose that* $0 < p, q, r \le \infty$ *and* $0 < s_1, s_2 \le \infty$. *Furthermore, suppose that* p, q, r, s_1 *and* s_2 *satisfy the following relations:*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

and

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}.$$

Then the assumption that $f \in L^{p,s_1}(\Omega)$ *and* $g \in L^{q,s_2}(\Omega)$ *implies that* $fg \in L^{r,s}(\Omega)$, *with the estimate*

$$\|fg\|_{L^{r,s}(\Omega)} \le C(p, q, s_1, s_2)\|f\|_{L^{p,s_1}(\Omega)}\|g\|_{L^{q,s_2}(\Omega)}. \tag{2.2.12}$$

In Chapter 4, we will make extensive use of the following known interpolative inequality for Lorentz spaces.

Proposition 2.3. *Suppose that* $1 \le p < r < q \le \infty$ *and* $1 \le s \le \infty$. *Furthermore, let* $0 < \theta < 1$ *be such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Then the assumption that $f \in L^{p,\infty}(\Omega) \cap L^{q,\infty}(\Omega)$ *implies* $f \in L^{r,1}(\Omega)$ *with the estimate*

$$\|f\|_{L^{r,1}(\Omega)} \le C(p, q, r)\|f\|^{\theta}_{L^{p,\infty}(\Omega)}\|f\|^{1-\theta}_{L^{q,\infty}(\Omega)}. \tag{2.2.13}$$

We will also make use of a known theorem, which we will refer to as 'the generalized Marcinkiewicz theorem'. See [14] (Theorem 5.3.2 there).

Theorem 2.4. *Let* $\Omega \subseteq \mathbb{R}^3$. *Suppose* $p_0 \neq p_1$, $q_0 \neq q_1$ *and* $\theta \in]0, 1[$ *satisfy the following relations*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$
$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Suppose that $r_1, r_2, s_1, s_2 \ge 1$ *and that* T *is a bounded linear operator of the following Lorentz spaces* ($i = 0, 1$)*:*

$$T : L^{p_i,r_i}(\Omega) \to L^{q_i,s_i}(\Omega).$$

Under these assumptions, it holds true that, for any $r \in]0, \infty]$,

$$T : L^{p,r}(\Omega) \to L^{q,r}(\Omega)$$

is a bounded linear operator.

2.2.2 Decomposition of Lorentz Spaces

In this subsection, we state and prove a simple (but important) fact about decompositions of Lorentz spaces. This will be formulated as a lemma and will be used in proving Theorem 5.4 (Chapter 5). An analogous statement is Lemma II.I proven by Calderón in [18].

Lemma 2.5. *Take $1 < t < r < s \leq \infty$, and suppose that $g \in L^{r,\infty}(\Omega)$. For any $N > 0$, we let $g_{N_-} := g\chi_{|g|\leq N}$ and $g_{N_+} := g - g_{N_-}$. Then*

$$\|g_{N_-}\|^s_{L_s(\Omega)} \leq \frac{s}{s-r} N^{s-r} \|g\|^r_{L^{r,\infty}(\Omega)} \tag{2.2.14}$$

if $s < \infty$, $\|g_{N_-}\|_{L_\infty(\Omega)} \leq N$ and

$$\|g_{N_+}\|^t_{L_t(\Omega)} \leq \frac{r}{r-t} N^{t-r} \|g\|^r_{L^{r,\infty}(\Omega)}. \tag{2.2.15}$$

Suppose further that $g \in L^{r,p}(\mathbb{R}^3_+)$ with $1 \leq p \leq \infty$, $1 < r < \infty$ and

$$\int\limits_{\mathbb{R}^3_+} g(x) \cdot \nabla\varphi(x) dx = 0 \quad \textit{for all } \varphi \in C_0^\infty(\mathbb{R}^3_+). \tag{2.2.16}$$

Then $g = g_1 + g_2$ where $g_1 \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_s(\mathbb{R}^3_+)}$, $g_2 \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_t(\mathbb{R}^3_+)}$,

$$\|g_1\|_{L_s(\mathbb{R}^3_+)} \leq C(s, r, p, \|g\|_{L^{r,p}(\mathbb{R}^3_+)}) \tag{2.2.17}$$

and

$$\|g_2\|_{L_t(\mathbb{R}^3_+)} \leq C(r, t, p, \|g\|_{L^{r,p}(\mathbb{R}^3_+)}). \tag{2.2.18}$$

Proof. Proof of decomposition (2.2.14)-(2.2.15) can be found in the book of Grafakos [48][1].

Now consider $g \in L^{r,p}(\mathbb{R}^3_+)$ satisfying (2.2.16). We can use the Helmoltz-Weyl decomposition[2] to deduce that $g_{1_-} = g_1 + \nabla q_1$, where g_1 belongs to the required space. Furthermore, we have that

$$\|g_1\|_{L_s(\mathbb{R}^3_+)} \leq c\|g_{1_-}\|_{L_s(\mathbb{R}^3_+)}, \quad \|\nabla q_1\|_{L_s(\mathbb{R}^3_+)} \leq c\|g_{1_-}\|_{L_s(\mathbb{R}^3_+)}$$

and

$$\int\limits_{\mathbb{R}^3_+} \nabla q_1 \cdot \nabla\varphi dx = \int\limits_{\mathbb{R}^3_+} g_{1_-} \cdot \nabla\varphi dx \;\; \forall \varphi \in C_0^\infty(\mathbb{R}^3).$$

[1] Specifically, p.22 of [48].

[2] See Theorem III.1.2 p.152 of Galdi's book [36], for example.

We can treat the second counterpart g_{1_+} in a similar way. From the above and (2.2.16), we deduce that

$$\int\limits_{\mathbb{R}^3_+} \nabla(q_1+q_2)\cdot\nabla\varphi dx = 0 \quad \forall\varphi\in C_0^\infty(\mathbb{R}^3).$$

Using properties of harmonic functions and the above global integrability of ∇q_1 and ∇q_2, we conclude that $\nabla(q_1+q_2)=0$. The estimates (2.2.17) and (2.2.18) can now be derived using $\|g_1\|_{L_s(\mathbb{R}^3_+)}\le c\|g_{1-}\|_{L_s(\mathbb{R}^3_+)}$, $\|g_1\|_{L_t(\mathbb{R}^3_+)}\le c\|g_{1-}\|_{L_t(\mathbb{R}^3_+)}$, the estimates (2.2.14)-(2.2.15) and the continuous embedding $L^{r,p}(\mathbb{R}^3_+)\hookrightarrow L^{r,\infty}(\mathbb{R}^3_+)$. □

2.3 Besov Spaces and BMO^{-1}

2.3.1 Definition and Properties of Besov Spaces and BMO^{-1}

The definitions we use are contained in the book of Bahouri, Chemin and Danchin [3]. For a tempered distribution f, let

$$\mathcal{F}(f)(\xi) := \int\limits_{\mathbb{R}^3} \exp(-ix\cdot\xi)f(x)dx$$

denote its Fourier transform. Let C be the annulus

$$\{\xi\in\mathbb{R}^3 : 3/4\le|\xi|\le 8/3\}.$$

Let $\chi\in C_0^\infty(B(4/3))$ and $\varphi\in C_0^\infty(C)$ be such that

$$\forall\xi\in\mathbb{R}^3,\ 0\le\chi(\xi),\varphi(\xi)\le 1, \tag{2.3.19}$$

$$\forall\xi\in\mathbb{R}^3,\ \chi(\xi)+\sum_{j\ge 0}\varphi(2^{-j}\xi)=1 \tag{2.3.20}$$

and

$$\forall\xi\in\mathbb{R}^3\setminus\{0\},\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1. \tag{2.3.21}$$

For a being a tempered distribution, let us define for $j\in\mathbb{Z}$:

$$\dot{\Delta}_j a := \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\mathcal{F}(a)) \text{ and } \dot{S}_j a := \mathcal{F}^{-1}(\chi(2^{-j}\xi)\mathcal{F}(a)). \tag{2.3.22}$$

Now we are in a position to define the homogeneous Besov spaces on $\mathbb{R}^3$. Let $s\in\mathbb{R}$ and $(p,q)\in[1,\infty]\times[1,\infty]$. Then $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is the subspace of tempered distributions such that

$$\lim_{j\to-\infty}\|\dot{S}_j u\|_{L_\infty(\mathbb{R}^3)}=0 \tag{2.3.23}$$

and

$$\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\dot{\Delta}_j u\|^q_{L_p(\mathbb{R}^3)}\Big)^{\frac{1}{q}}. \qquad (2.3.24)$$

Remark 2.6. *This definition provides a Banach space if $s < \frac{3}{p}$. See Theorem 2.27 p.67 of Bahouri, Chemin and Danchin's book [3], for example.*

Remark 2.7. *It is known that if $1 \le q_1 \le q_2 \le \infty$, $1 \le p_1 \le p_2 \le \infty$ and $s \in \mathbb{R}$, then*

$$\dot{B}^s_{p_1,q_1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s-3(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}(\mathbb{R}^3).$$

See Proposition 2.2 p.64 of Bahouri, Chemin and Danchin's book [3], for example.

Remark 2.8. *It is known[3] that for $s = -2s_1 < 0$ and $p, q \in [1,\infty]$, the norm can be characterised by the heat flow. Namely there exists a $C > 1$ such that for all $u \in \dot{B}^{-2s_1}_{p,q}(\mathbb{R}^3)$:*

$$C^{-1}\|u\|_{\dot{B}^{-2s_1}_{p,q}(\mathbb{R}^3)} \le \|\|t^{s_1}S(t)u\|_{L_p(\mathbb{R}^3)}\|_{L_q(\frac{dt}{t})} \le C\|u\|_{\dot{B}^{-2s_1}_{p,q}(\mathbb{R}^3)}.$$

Here,

$$S(t)u(x) := \Gamma(\cdot,t)\star u$$

where $\Gamma(x,t)$ is the kernel for the heat flow in $\mathbb{R}^3$.

We will also need the following proposition, whose statement and proof can be found in Bahouri, Chemin and Danchin's book [3] (Proposition 2.22 there). In the proposition below we use the notation

$$\mathcal{S}'_h := \{\text{ tempered distributions } u \text{ such that } \lim_{j\to-\infty}\|S_j u\|_{L_\infty(\mathbb{R}^3)} = 0\}. \qquad (2.3.25)$$

Proposition 2.9. *A constant C exists with the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in]0,1[$, then we have, for any $p \in [1,\infty]$ and any $u \in \mathcal{S}'_h$,*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}(\mathbb{R}^3)} \le \frac{C}{s_2-s_1}\Big(\frac{1}{\theta}+\frac{1}{1-\theta}\Big)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^3)}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}(\mathbb{R}^3)}. \qquad (2.3.26)$$

Finally, $BMO^{-1}(\mathbb{R}^3)$ is the space of all tempered distributions such that the following norm is finite:

$$\|u\|_{BMO^{-1}(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3, R>0}\frac{1}{\mu(B(0,R))}\int\limits_0^{R^2}\int\limits_{B(x,R)}|S(t)u|^2 dydt. \qquad (2.3.27)$$

Here, μ denotes the Lebesgue measure. We define $VMO^{-1}(\mathbb{R}^3)$ to be the closure of test functions $C^\infty_0(\mathbb{R}^3)$ with respect to the norm (2.3.27).

[3]See Theorem 2.3.4 p.72 of Bahouri, Chemin and Danchin's book [3], for example.

2.3.2 Decomposition of Homogeneous Besov Spaces

In this subsection, we state and prove certain decompositions for homogeneous Besov spaces, which will play a crucial role in the proofs of Lemma 3.3 (Chapter 3) and Lemma 6.5 (Chapter 6). We first take note of a useful lemma presented in Bahouri, Chemin and Danchin's book [3] (specifically, Lemma 2.23 and Remark 2.24 in [3]).

Lemma 2.10. *Let C' be an annulus and let $(u^{(j)})_{j\in\mathbb{Z}}$ be a sequence of functions such that*

$$\mathit{Supp}\,\mathcal{F}(u^{(j)}) \subset 2^j C' \tag{2.3.28}$$

and

$$\Big(\sum_{j\in\mathbb{Z}} 2^{jsr}\|u^{(j)}\|_{L_p}^r\Big)^{\frac{1}{r}} < \infty, \tag{2.3.29}$$

where $s\in\mathbb{R}$, $1\leq p\leq\infty$ and $1\leq r\leq\infty$. Moreover, assume in addition that

$$s < \frac{3}{p}. \tag{2.3.30}$$

Then the following holds true. The series

$$\sum_{j\in\mathbb{Z}} u^{(j)}$$

converges (in the sense of tempered distributions) to some $u\in\dot{B}^s_{p,r}(\mathbb{R}^3)$, which satisfies the following estimate:

$$\|u\|_{\dot{B}^s_{p,r}} \leq C_s\Big(\sum_{j\in\mathbb{Z}} 2^{jsr}\|u^{(j)}\|_{L_p}^r\Big)^{\frac{1}{r}}. \tag{2.3.31}$$

Now, we can state the first proposition regarding decomposition of homogeneous Besov spaces. In the context of Lebesgue spaces, an analogous statement is Lemma II.I proven by Calderón in [18]. Note that decompositions of a similar type can be obtained abstractly from real interpolation theory, applied to homogeneous Besov spaces. See Chapter 6 of Bergh and Löfström's book [14], for example.

Proposition 2.11. *For $i=1,2,3$ let $p_i\in]1,\infty[$, $s_i\in\mathbb{R}$ and $\theta\in]0,1[$ be such that $s_1<s_0<s_2$ and $p_2<p_0<p_1$. In addition, assume the following relations hold:*

$$s_1(1-\theta)+\theta s_2 = s_0, \tag{2.3.32}$$

$$\frac{1-\theta}{p_1}+\frac{\theta}{p_2} = \frac{1}{p_0} \tag{2.3.33}$$

and
$$s_i < \frac{3}{p_i}. \tag{2.3.34}$$
Suppose that $u_0 \in \dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^3)$ *is a divergence-free tempered distribution. Then for all* $\epsilon > 0$, *there exist divergence-free tempered distributions* $u^{1,\epsilon} \in \dot{B}^{s_1}_{p_1,p_1}(\mathbb{R}^3)$, $u^{2,\epsilon} \in \dot{B}^{s_2}_{p_2,p_2}(\mathbb{R}^3)$ *such that*
$$u_0 = u^{1,\epsilon} + u^{2,\epsilon}, \tag{2.3.35}$$
$$\|u^{1,\epsilon}\|^{p_1}_{\dot{B}^{s_1}_{p_1,p_1}} \leq \epsilon^{p_1-p_0}\|u_0\|^{p_0}_{\dot{B}^{s_0}_{p_0,p_0}} \tag{2.3.36}$$
and
$$\|u^{2,\epsilon}\|^{p_2}_{\dot{B}^{s_2}_{p_2,p_2}} \leq C(s_1, s_2, p_0, p_1, p_2, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{p_2-p_0}\|u_0\|^{p_0}_{\dot{B}^{s_0}_{p_0,p_0}}. \tag{2.3.37}$$
Proof. Denote,
$$f^{(j)} := \dot{\Delta}_j u_0,\;\; f^{(j)N}_{-} := f^{(j)}\chi_{|f^{(j)}|\leq N} \;\text{ and }\; f^{(j)N}_{+} := f^{(j)}(1-\chi_{|f^{(j)}|\leq N}).$$
It is easily verified that the following holds:
$$\|f^{(j)N}_{-}\|^{p_1}_{L_{p_1}} \leq N^{p_1-p_0}\|f^{(j)}\|^{p_0}_{L_{p_0}} \;\text{ and }\; \|f^{(j)N}_{+}\|^{p_2}_{L_{p_2}} \leq N^{p_2-p_0}\|f^{(j)}\|^{p_0}_{L_{p_0}}.$$
Thus, we may write
$$2^{p_1s_1j}\|f^{(j)N}_{-}\|^{p_1}_{L_{p_1}} \leq N^{p_1-p_0}2^{(p_1s_1-p_0s_0)j}2^{p_0s_0j}\|f^{(j)}\|^{p_0}_{L_{p_0}} \tag{2.3.38}$$
and
$$2^{p_2s_2j}\|f^{(j)N}_{+}\|^{p_2}_{L_{p_2}} \leq N^{p_2-p_0}2^{(p_2s_2-p_0s_0)j}2^{p_0s_0j}\|f^{(j)}\|^{p_0}_{L_{p_0}}. \tag{2.3.39}$$
Define
$$N(j,\epsilon,s_0,s_1,p_0,p_1) := \epsilon 2^{\frac{(p_0s_0-p_1s_1)j}{p_1-p_0}}.$$
For the sake of brevity we will write $N(j,\epsilon)$. From (2.3.38), we deduce that
$$2^{p_1s_1j}\|f^{(j)N(j,\epsilon)}_{-}\|^{p_1}_{L_{p_1}} \leq \epsilon^{p_1-p_0}2^{p_0s_0j}\|f^{(j)}\|^{p_0}_{L_{p_0}}. \tag{2.3.40}$$
Using the relations of the Besov indices given by (2.3.32)-(2.3.33), we can infer that
$$N(j,\epsilon)^{p_2-p_0}2^{(p_2s_2-p_0s_0)j} = \epsilon^{p_2-p_0}.$$
The crucial point is that this is independent of j. Thus, we infer
$$2^{p_2s_2j}\|f^{(j)N(j,\epsilon)}_{+}\|^{p_2}_{L_{p_2}} \leq \epsilon^{p_2-p_0}2^{p_0s_0j}\|f^{(j)}\|^{p_0}_{L_{p_0}}. \tag{2.3.41}$$

Next, it is known that for any $u_0 \in \dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^3)$ we have that $\sum_{j=-m}^{m} \dot{\Delta}_j u_0$ converges to u_0 in the sense of tempered distributions. Furthermore, we have that $\dot{\Delta}_j \dot{\Delta}_{j'} u_0 = 0$ if $|j - j'| > 1$. Combing these two facts allows us to observe that

$$\dot{\Delta}_j u_0 = \sum_{|m-j|\leq 1} \dot{\Delta}_m f^{(j)} = \sum_{|m-j|\leq 1} \dot{\Delta}_m f_-^{(j)N(j,\epsilon)} + \sum_{|m-j|\leq 1} \dot{\Delta}_m f_+^{(j)N(j,\epsilon)}. \tag{2.3.42}$$

Recall from Chapter 1 that the Leray projector $\mathbb{P}$, which projects onto divergence-free vector fields, is defined as

$$\mathbb{P}f := f + \nabla(-\Delta)^{-1}(\operatorname{div} f).$$

This can also be written as (for $j = 1, 2, 3$)

$$\mathbb{P}f_j = (\delta_{jk} - \mathcal{R}_j \mathcal{R}_k) f_k.$$

Here $\mathcal{R}_i$ are Riesz transforms given by

$$\mathcal{R}_i g(x) := cP.V. \int \frac{y_i}{|y|^4} g(x-y)dy$$

and the summation convention is adopted. Since u_0 is a divergence-free tempered distribution in $\dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^3)$, we have that $\mathbb{P}(\dot{\Delta}_j u_0) = \dot{\Delta}_j u_0$. From this and (2.3.42) we deduce that

$$\dot{\Delta}_j u_0 = \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m f_-^{(j)N(j,\epsilon)}) + \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m f_+^{(j)N(j,\epsilon)}). \tag{2.3.43}$$

Define

$$u_j^{1,\epsilon} := \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m f_-^{(j)N(j,\epsilon)}) \tag{2.3.44}$$

and

$$u_j^{2,\epsilon} := \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m f_+^{(j)N(j,\epsilon)}). \tag{2.3.45}$$

It is clear that

$$\operatorname{Supp} \mathcal{F}(u_j^{1,\epsilon}),\ \operatorname{Supp} \mathcal{F}(u_j^{2,\epsilon}) \subset 2^j C'. \tag{2.3.46}$$

Here, C' is the annulus defined by $C' := \{\xi \in \mathbb{R}^3 : 3/8 \leq |\xi| \leq 16/3\}$. Using (2.3.40)-(2.3.41) and the continuity of the Riesz transforms on $L_p(\mathbb{R}^3)$ $(1 < p < \infty)$, we obtain the following estimates:

$$\begin{aligned} 2^{p_1 s_1 j} \|u_j^{1,\epsilon}\|_{L_{p_1}}^{p_1} &\leq \lambda_1(p_1, \|\mathcal{F}^{-1}\varphi\|_{L_1}) 2^{p_1 s_1 j} \|f_-^{(j)N(j,\epsilon)}\|_{L_{p_1}}^{p_1} \\ &\leq \lambda_1(p_1, \|\mathcal{F}^{-1}\varphi\|_{L_1}) \epsilon^{p_1 - p_0} 2^{p_0 s_0 j} \|f^{(j)}\|_{L_{p_0}}^{p_0}, \end{aligned} \tag{2.3.47}$$

$$\begin{aligned}2^{p_2 s_2 j}\|u_j^{2,\epsilon}\|_{L_{p_2}}^{p_2} &\leq \lambda_2(p_2, \|\mathcal{F}^{-1}\varphi\|_{L_1}) 2^{p_2 s_2 j}\|f_-^{(j)N(j,\epsilon)}\|_{L_{p_2}}^{p_2} \\ &\leq \lambda_2(p_2, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{p_2-p_0} 2^{p_0 s_0 j}\|f^{(j)}\|_{L_{p_0}}^{p_0}. \qquad (2.3.48)\end{aligned}$$

Note that for $i = 1$ or $i = 2$, $\lambda_i(p_i, \|\mathcal{F}^{-1}\varphi\|_{L_1})$ denotes a constant depending on p_i and $\|\mathcal{F}^{-1}\varphi\|_{L_1}$.

It is then the case that (2.3.46)-(2.3.48) allow us to apply the results of Lemma 2.10. This gives us the desired decomposition with the choice

$$u^{1,\epsilon} = \sum_{j\in\mathbb{Z}} u_j^{1,\epsilon}$$

and

$$u^{2,\epsilon} = \sum_{j\in\mathbb{Z}} u_j^{2,\epsilon}.$$

□

As a consequence of the previous proposition, we obtain the following corollary, which will play a central role in proving Lemma 3.3 (Chapter 3). Note that in the corollary below and throughout this thesis, for $p_0 > 3$ we define

$$s_{p_0} := -1 + \frac{3}{p_0} < 0.$$

Moreover, for $2 < \alpha \leq 3$ and $p_1 > \alpha$ we define

$$s_{p_1,\alpha} := -\frac{3}{\alpha} + \frac{3}{p_1} < 0.$$

Corollary 2.12. *Fix* $2 < \alpha \leq 3$.

- *For* $2 < \alpha < 3$, *take* p *such that* $\alpha < p < \frac{\alpha}{3-\alpha}$.
- *For* $\alpha = 3$, *take* p *such that* $3 < p < \infty$.

For p *and* α *satisfying these conditions, suppose that*

$$u_0 \in \dot{B}_{p,p}^{s_{p,\alpha}}(\mathbb{R}^3) \cap L_2(\mathbb{R}^3) \qquad (2.3.49)$$

and div $u_0 = 0$ *in the sense of distributions.*
Then the above assumptions imply that there exists p_0 *with* $\max(p, 4) < p_0 < \infty$ *and* $\delta > 0$ *such that for any* $\epsilon > 0$ *there exist weakly divergence-free functions* $u^{1,\epsilon} \in \dot{B}_{p_0,p_0}^{s_{p_0}+\delta}(\mathbb{R}^3) \cap L_2(\mathbb{R}^3)$ *and* $u^{2,\epsilon} \in L_2(\mathbb{R}^3)$ *such that*

$$u_0 = u^{1,\epsilon} + u^{2,\epsilon}, \qquad (2.3.50)$$

$$\|u^{1,\epsilon}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}} \leq \epsilon^{p_0-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}}, \tag{2.3.51}$$

$$\|u^{2,\epsilon}\|^2_{L_2} \leq C(\alpha, p, p_0, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{2.3.52}$$

and

$$\|u^{1,\epsilon}\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{L_2}. \tag{2.3.53}$$

Proof. **First case:** $2 < \alpha < 3$ **and** $\alpha < p < \frac{\alpha}{3-\alpha}$

Under this condition, we can find p_0 with $\max(4, p) < p_0 < \infty$ such that

$$\theta := \frac{\frac{1}{p} - \frac{1}{p_0}}{\frac{1}{2} - \frac{1}{p_0}} > \frac{6}{\alpha} - 2. \tag{2.3.54}$$

Clearly, $0 < \theta < 1$ and moreover

$$\frac{1-\theta}{p_0} + \frac{\theta}{2} = \frac{1}{p}. \tag{2.3.55}$$

Define

$$\delta := \frac{1 - \frac{3}{\alpha} + \frac{\theta}{2}}{1-\theta}. \tag{2.3.56}$$

From (2.3.54), we see that $\delta > 0$. One can also see that we have the following relation:

$$(1-\theta)(s_{p_0} + \delta) = s_{p,\alpha}. \tag{2.3.57}$$

The above relations allow us to apply Proposition 2.11 to obtain the following decomposition: (we note that $\dot{B}^0_{2,2}(\mathbb{R}^3)$ coincides with $L_2(\mathbb{R}^3)$ with equivalent norms)

$$u_0 = u^{1,\epsilon} + u^{2,\epsilon}, \tag{2.3.58}$$

$$\|u^{1,\epsilon}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}} \leq \epsilon^{p_0-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{2.3.59}$$

and

$$\|u^{2,\epsilon}\|^2_{L_2} \leq C(\alpha, p, p_0, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}}. \tag{2.3.60}$$

Furthermore, $u^{1,\epsilon}$ and $u^{2,\epsilon}$ are divergence-free tempered distributions. For $j \in \mathbb{Z}$ and $m \in \mathbb{Z}$, it can be seen that

$$\|\mathbb{P}\left(\dot{\Delta}_m\left((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \leq N(j,\epsilon)}\right)\right)\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}))\|\dot{\Delta}_j u_0\|_{L_2}. \tag{2.3.61}$$

It is known that $u_0 \in L_2$ implies

$$\|u_0\|^2_{L_2} = \sum_{j\in\mathbb{Z}} \|\dot{\Delta}_j u_0\|^2_{L_2}.$$

Using this, (2.3.61) and the definition of $u^{1,\epsilon}$ from Proposition 2.11, we can infer that

$$\|u^{1,\epsilon}\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{L_2}.$$

Thus, we have established the decomposition of the corollary.

Second case: $\alpha = 3$ **and** $3 < p < \infty$

In the second case, we choose any p_0 such that $\max(4, p) < p_0 < \infty$. With this p_0 we choose θ such that

$$\frac{1-\theta}{p_0} + \frac{\theta}{2} = \frac{1}{p}.$$

If we define

$$\delta := \frac{\theta}{2(1-\theta)} > 0, \tag{2.3.62}$$

we see that

$$(s_{p_0} + \delta)(1-\theta) = s_p. \tag{2.3.63}$$

These relations allow us to obtain the decomposition of the corollary, by means of identical arguments to those presented in the the first case of this proof.

□

Next we state a proposition, which refines the decompositions of homogeneous Besov spaces given by the previous corollary. The proposition below will play an important role in proving Lemma 6.5 (Chapter 6).

Proposition 2.13. *Suppose that* $3 < p < \infty$*,*

$$g \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \tag{2.3.64}$$

and div $g = 0$ *in sense of tempered distributions. Then the above assumptions imply that there exist* p_2*,* δ_2*,* γ_1 *and* γ_2 *with*

$$p < p_2 < \infty, \;\; 0 < \delta_2 < -s_{p_2}, \;\; \gamma_1 := \gamma_1(p) > 0 \;\; and \;\; \gamma_2 := \gamma_2(p) > 0$$

such that for any $N > 0$ *there exist divergence-free tempered distributions*

$$\bar{g}^N \in \dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \;\; and \;\; \widetilde{g}^N \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \tag{2.3.65}$$

with the following properties. Namely,

$$g = \bar{g}^N + \widetilde{g}^N, \tag{2.3.66}$$

$$\|\bar{g}^N\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}} \leq N^{\gamma_1} C(p, \|g\|_{\dot{B}^{s_p}_{p,\infty}}) \tag{2.3.67}$$

and

$$\|\widetilde{g}^N\|_{L_2} \leq N^{-\gamma_2} C(p, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.68}$$

Furthermore,

$$\|\bar{g}^N\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \|g\|_{\dot{B}^{s_p}_{p,\infty}}) \tag{2.3.69}$$

and

$$\|\widetilde{g}^N\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.70}$$

Unlike the decompositions in Proposition 2.11 and Corollary 2.12, it is not clear if related decompositions are obtainable by the known real interpolation theory of homogeneous Besov spaces.

In order to prove Proposition 2.13, we must first state and prove two lemmas. Here is the first.

Lemma 2.14. *Let $3 < p < \infty$ and suppose that u_0 is a divergence-free tempered distribution belonging to $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$. Then there exist $p < p_0 < \infty$ and $0 < \delta < -s_{p_0}$ such that for any $N > 0$ there exist divergence-free tempered distributions*

$$u^{1,N} \in \dot{B}^{s_{p_0}+\delta}_{p_0,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \;\; and \;\; u^{2,N} \in \dot{B}^{0}_{2,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$$

with

$$u_0 = u^{1,N} + u^{2,N}, \tag{2.3.71}$$

$$\|u^{1,N}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,\infty}} \leq N^{p_0-p}\|u_0\|^p_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.72}$$

$$\|u^{2,N}\|^2_{\dot{B}^0_{2,\infty}} \leq C(p,p_0,\delta,\|\mathcal{F}^{-1}\varphi\|_{L_1})N^{2-p}\|u_0\|^p_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.73}$$

$$\|u^{1,N}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}} \tag{2.3.74}$$

and

$$\|u^{2,N}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.75}$$

Proof. If $p < p_0 < \infty$, there exists a $\theta \in]0,1[$ such that

$$\frac{1-\theta}{p_0} + \frac{\theta}{2} = \frac{1}{p}. \tag{2.3.76}$$

If we define

$$\delta := \frac{\theta}{2(1-\theta)} > 0, \tag{2.3.77}$$

we see that

$$(s_{p_0}+\delta)(1-\theta) = s_p. \tag{2.3.78}$$

Denote,

$$f^{(j)} := \dot{\Delta}_j u,\;\; f^{(j)M}_{-} := f^{(j)}\chi_{|f^{(j)}|\leq M} \text{ and } f^{(j)M}_{+} := f^{(j)}(1-\chi_{|f^{(j)}|\leq M}).$$

Define

$$M(j,N,p,p_0,\delta) := N2^{\frac{(ps_p-p_0(s_{p_0}+\delta))j}{p_0-p}}.$$

For the sake of brevity we will write $M(j,N)$. Using the relations of the Besov indices given by (2.3.76)-(2.3.78), one can use identical arguments as those utilised in proving Proposition 2.11 to obtain

$$2^{p_0(s_{p_0}+\delta)j}\|f_-^{(j)M(j,N)}\|_{L_{p_0}}^{p_0} \le N^{p_0-p}2^{ps_pj}\|f^{(j)}\|_{L_p}^p, \tag{2.3.79}$$

$$2^{s_pj}\|f_-^{(j)M(j,N)}\|_{L_p} \le 2^{s_pj}\|f^{(j)}\|_{L_p}, \tag{2.3.80}$$

$$\|f_+^{(j)M(j,N)}\|_{L_2}^2 \le N^{2-p}2^{ps_pj}\|f^{(j)}\|_{L_p}^p \tag{2.3.81}$$

and

$$2^{s_pj}\|f_+^{(j)M(j,N)}\|_{L_p} \le 2^{s_pj}\|f^{(j)}\|_{L_p}. \tag{2.3.82}$$

For any $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ we have that $\sum_{j=-m}^{m}\dot{\Delta}_j u_0$ converges to u_0 in the sense of tempered distributions. Using the same arguments as in Proposition 2.11, we see that

$$\dot{\Delta}_j u_0 = \sum_{|m-j|\le 1}\mathbb{P}(\dot{\Delta}_m f_-^{(j)M(j,N)}) + \sum_{|m-j|\le 1}\mathbb{P}(\dot{\Delta}_m f_+^{(j)M(j,N)}). \tag{2.3.83}$$

Define

$$u_j^{1,N} := \sum_{|m-j|\le 1}\mathbb{P}(\dot{\Delta}_m f_-^{(j)M(j,N)}) \tag{2.3.84}$$

and

$$u_j^{2,N} := \sum_{|m-j|\le 1}\mathbb{P}(\dot{\Delta}_m f_+^{(j)M(j,N)}). \tag{2.3.85}$$

It is clear that

$$\mathrm{Supp}\,\mathcal{F}(u_j^{1,N}),\ \ \mathrm{Supp}\,\mathcal{F}(u_j^{2,N}) \subset 2^jC'. \tag{2.3.86}$$

Here, C' is the annulus defined by $C' := \{\xi \in \mathbb{R}^3 : 3/8 \le |\xi| \le 16/3\}$. Using (2.3.79)-(2.3.82) and the continuity of the Leray projector $\mathbb{P}$ on $L_p(\mathbb{R}^3)$ $(1 < p < \infty)$, we can obtain the following estimates:

$$\begin{aligned}2^{p_0(s_{p_0}+\delta)j}\|u_j^{1,N}\|_{L_{p_0}}^{p_0} &\le \lambda_1(p_0,\|\mathcal{F}^{-1}\varphi\|_{L_1})2^{p_0(s_{p_0}+\delta)j}\|f_-^{(j)M(j,N)}\|_{L_{p_0}}^{p_0}\\ &\le \lambda_1(p_0,\|\mathcal{F}^{-1}\varphi\|_{L_1})N^{p_0-p}2^{ps_pj}\|f^{(j)}\|_{L_p}^p,\end{aligned} \tag{2.3.87}$$

$$\begin{aligned}2^{s_pj}\|u_j^{1,N}\|_{L_p} &\le \lambda_2(p,\|\mathcal{F}^{-1}\varphi\|_{L_1})2^{s_pj}\|f_-^{(j)M(j,N)}\|_{L_p}\\ &\le \lambda_2(p,\|\mathcal{F}^{-1}\varphi\|_{L_1})2^{s_pj}\|f^{(j)}\|_{L_p},\end{aligned} \tag{2.3.88}$$

$$\begin{aligned}\|u_j^{2,N}\|_{L_2}^2 &\le \lambda_3(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|f_+^{(j)M(j,N)}\|_{L_2}^2\\ &\le \lambda_3(\|\mathcal{F}^{-1}\varphi\|_{L_1})N^{2-p}2^{ps_pj}\|f^{(j)}\|_{L_p}^p\end{aligned} \tag{2.3.89}$$

and

$$2^{s_p j}\|u_j^{2,N}\|_{L_p} \leq \lambda_4(p, \|\mathcal{F}^{-1}\varphi\|_{L_1}) 2^{s_p j}\|f_+^{(j)M(j,N)}\|_{L_p}$$
$$\leq \lambda_4(p, \|\mathcal{F}^{-1}\varphi\|_{L_1}) 2^{s_p j}\|f^{(j)}\|_{L_p}. \tag{2.3.90}$$

Note that $\lambda_1(p_0, \|\mathcal{F}^{-1}\varphi\|_{L_1})$ denotes a constant depending on p_0 and $\|\mathcal{F}^{-1}\varphi\|_{L_1}$. For $i = 2$ and 4, $\lambda_i(p, \|\mathcal{F}^{-1}\varphi\|_{L_1})$ denotes a constant depending on p and $\|\mathcal{F}^{-1}\varphi\|_{L_1}$. Additionally, $\lambda_3(\|\mathcal{F}^{-1}\varphi\|_{L_1})$ denotes a constant depending on $\|\mathcal{F}^{-1}\varphi\|_{L_1}$.

It is then the case that (2.3.86)-(2.3.90) allow us to apply the results of Lemma 2.10. This allows us to achieve the desired decomposition with the choice

$$u^{1,N} = \sum_{j\in\mathbb{Z}} u_j^{1,N},$$

$$u^{2,N} = \sum_{j\in\mathbb{Z}} u_j^{2,N}.$$

□

Lemma 2.15. *For $2 < \alpha < 3$, take p such that $3 < p < \frac{\alpha}{3-\alpha}$. Suppose that the divergence-free tempered distribution u_0 is such that*

$$u_0 \in \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3). \tag{2.3.91}$$

Then the above assumptions imply that there exist $p < p_1 < \infty$ and $0 < \delta_1 < -s_{p_1}$ such that for any $\epsilon > 0$ there exist divergence-free tempered distributions $U^{1,\epsilon} \in \dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $U^{2,\epsilon} \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ with

$$u_0 = U^{1,\epsilon} + U^{2,\epsilon}, \tag{2.3.92}$$

$$\|U^{1,\epsilon}\|^{p_1}_{\dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}} \leq \epsilon^{p_1-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}}, \tag{2.3.93}$$

$$\|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.94}$$

$$\|U^{2,\epsilon}\|^2_{L_2} \leq C(p, p_1, \alpha, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{2.3.95}$$

and

$$\|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.96}$$

Proof. As a result of the assumptions on p and α, we can find $p < p_1 < \infty$ such that

$$\theta := \frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{2} - \frac{1}{p_1}} > \frac{6}{\alpha} - 2. \tag{2.3.97}$$

Clearly, $0 < \theta < 1$ and moreover

$$\frac{1-\theta}{p_1} + \frac{\theta}{2} = \frac{1}{p}. \tag{2.3.98}$$

Define

$$\delta_1 := \frac{1 - \frac{3}{\alpha} + \frac{\theta}{2}}{1-\theta}. \tag{2.3.99}$$

From (2.3.97), we see that $\delta_1 > 0$. One can also see that we have the following relation:

$$(1-\theta)(s_{p_1} + \delta_1) = s_{p,\alpha}. \tag{2.3.100}$$

The above relations allow us to apply Proposition 2.11 to obtain the following decomposition: (we note that $\dot{B}^0_{2,2}(\mathbb{R}^3)$ coincides with $L_2(\mathbb{R}^3)$ with equivalent norms)

$$u_0 = U^{1,\epsilon} + U^{2,\epsilon}, \tag{2.3.101}$$

$$\|U^{1,\epsilon}\|^{p_1}_{\dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}} \leq \epsilon^{p_1-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{2.3.102}$$

and

$$\|U^{2,\epsilon}\|^2_{L_2} \leq C(p, p_1, \alpha, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}}. \tag{2.3.103}$$

Here, $U^{1,\epsilon}$ and $U^{2,\epsilon}$ are divergence-free tempered distributions. For $j \in \mathbb{Z}$ and $m \in \mathbb{Z}$, it can be seen that

$$\|\mathbb{P}(\dot{\Delta}_m((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \leq N(j,\epsilon)}))\|_{L_p} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|\dot{\Delta}_j u_0\|_{L_p} \tag{2.3.104}$$

and

$$\|\mathbb{P}(\dot{\Delta}_m((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \geq N(j,\epsilon)}))\|_{L_p} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|\dot{\Delta}_j u_0\|_{L_p}. \tag{2.3.105}$$

We see that the definitions of $U^{1,\epsilon}$ and $U^{2,\epsilon}$ used in Proposition 2.11 are of the following form:

$$U^{1,\epsilon} := \sum_j \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \leq N(j,\epsilon)}))$$

and

$$U^{2,\epsilon} := \sum_j \sum_{|m-j|\leq 1} \mathbb{P}(\dot{\Delta}_m((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \geq N(j,\epsilon)})).$$

Using this, along with (2.3.104)-(2.3.105), we can infer that

$$\|U^{1,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}$$

and

$$\|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}.$$

$\square$

Proof of Proposition 2.13

Proof. Applying Lemma 2.14, we see that there exist $p < p_0 < \infty$ and $0 < \delta < -s_{p_0}$ such that for any $N > 0$ there exist divergence-free tempered distributions

$$u^{1,N} \in \dot{B}^{s_{p_0}+\delta}_{p_0,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \text{ and } u^{2,N} \in \dot{B}^{0}_{2,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$$

with

$$g = u^{1,N} + u^{2,N}, \tag{2.3.106}$$

$$\|u^{1,N}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,\infty}} \leq N^{p_0-p}\|g\|^p_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.107}$$

$$\|u^{1,N}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.108}$$

$$\|u^{2,N}\|^2_{\dot{B}^0_{2,\infty}} \leq C(p, p_0, \delta, \|\mathcal{F}^{-1}\varphi\|_{L_1})N^{2-p}\|g\|^p_{\dot{B}^{s_p}_{p,\infty}} \tag{2.3.109}$$

and

$$\|u^{2,N}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.110}$$

Since $3 < p < \infty$, it is clear that there exists an $\alpha := \alpha(p)$ such that $2 < \alpha < 3$ and

$$3 < p < \frac{\alpha}{3-\alpha}. \tag{2.3.111}$$

With these p and α, we may apply Proposition 2.9 with $s_1 = -\frac{3}{2} + \frac{3}{p}$, $s_2 = -1 + \frac{3}{p}$ and $\theta = 6\Big(\frac{1}{\alpha} - \frac{1}{3}\Big)$. In particular this gives for any $f \in \mathcal{S}'_h$:

$$\|f\|_{\dot{B}^{s_{p,\alpha}}_{p,1}} \leq c(p,\alpha)\|f\|^{6(\frac{1}{\alpha}-\frac{1}{3})}_{\dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,\infty}}\|f\|^{6(\frac{1}{2}-\frac{1}{\alpha})}_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}}. \tag{2.3.112}$$

From Remark 2.7, we see that $\dot{B}^{s_{p,\alpha}}_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)$ and $\dot{B}^0_{2,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$. Thus, we have the inclusion

$$\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap \dot{B}^0_{2,\infty}(\mathbb{R}^3) \subset \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3). \tag{2.3.113}$$

Now (2.3.112)-(2.3.113), together with (2.3.109)-(2.3.110), imply that $u^{2,N} \in \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)$ and there exists a $\beta(p) > 0$ such that

$$\|u^{2,N}\|_{\dot{B}^{s_{p,\alpha}}_{p,p}} \leq N^{-\beta(p)}C(p, p_0, \delta, \|\mathcal{F}^{-1}\varphi\|_{L_1}, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.114}$$

Noting (2.3.111), along with (2.3.110) and (2.3.114), we may now apply Lemma 2.15. In particular, there exist $p < p_1 < \infty$ and $0 < \delta_1 < -s_{p_1}$ such that for any $\lambda > 0$ there exist divergence-free tempered distributions

$$U^{1,\lambda} \in \dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \text{ and } U^{2,\lambda} \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$$

satisfying the following properties. Specifically,

$$u^{2,N} = U^{1,\lambda} + U^{2,\lambda}, \tag{2.3.115}$$

$$\|U^{1,\lambda}\|^{p_1}_{\dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}} \leq \lambda^{p_1-p}\|u^{2,N}\|^p_{\dot{B}^{s_p,\alpha}_{p,p}}$$
$$\leq \lambda^{p_1-p}N^{-\beta(p)p}C(p,p_0,\delta,\|\mathcal{F}^{-1}\varphi\|_{L_1},\|g\|_{\dot{B}^{s_p}_{p,\infty}}), \tag{2.3.116}$$

$$\|U^{1,\lambda}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|u^{2,N}\|_{\dot{B}^{s_p}_{p,\infty}}$$
$$\leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.117}$$

$$\|U^{2,\lambda}\|^2_{L_2} \leq C(p,p_1,\|\mathcal{F}^{-1}\varphi\|_{L_1})\lambda^{2-p}\|u^{2,N}\|^p_{\dot{B}^{s_p,\alpha}_{p,p}}$$
$$\leq \lambda^{2-p}N^{-\beta(p)p}C(p,p_0,p_1,\delta,\|\mathcal{F}^{-1}\varphi\|_{L_1},\|g\|_{\dot{B}^{s_p}_{p,\infty}}) \tag{2.3.118}$$

and

$$\|U^{2,\lambda}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|u^{2,N}\|_{\dot{B}^{s_p}_{p,\infty}}$$
$$\leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.119}$$

Taking $\lambda = N^\kappa$ gives that

$$u_0 = u^{1,N} + U^{1,N^\kappa} + U^{2,N^\kappa} \ \text{ with}$$

$$u^{1,N} \in \dot{B}^{s_{p_0}+\delta}_{p_0,\infty}(\mathbb{R}^3)\cap\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3),\ \ U^{1,N^\kappa} \in \dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}(\mathbb{R}^3)\cap\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$$

and

$$U^{2,N^\kappa} \in L_2(\mathbb{R}^3)\cap\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3).$$

Furthermore,

$$\|u^{1,N}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,\infty}} \leq N^{p_0-p}\|g\|^p_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.120}$$

$$\|u^{1,N}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.121}$$

$$\|U^{1,N^k}\|^{p_1}_{\dot{B}^{s_{p_1}+\delta_1}_{p_1,p_1}} \leq N^{\kappa(p_1-p)-\beta(p)p}C(p,p_0,\delta,\|\mathcal{F}^{-1}\varphi\|_{L_1},\|g\|_{\dot{B}^{s_p}_{p,\infty}}), \tag{2.3.122}$$

$$\|U^{1,N^\kappa}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}, \tag{2.3.123}$$

$$\|U^{2,N^\kappa}\|^2_{L_2} \leq N^{\kappa(2-p)-\beta(p)p}C(p,p_0,p_1,\delta,\|\mathcal{F}^{-1}\varphi\|_{L_1},\|g\|_{\dot{B}^{s_p}_{p,\infty}}) \tag{2.3.124}$$

and

$$\|U^{2,N^\kappa}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1},p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.125}$$

Let $p_2 = 2\max(p_0, p_1)$ and $\delta_2 = \min(\delta, \delta_1)/2$. From Remark 2.7 and (2.3.120)-(2.3.121), we have that $u^{1,N} \in \dot{B}^{s_{p_2}}_{p_2,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_{p_2}+\delta}_{p_2,\infty}(\mathbb{R}^3)$ with estimates

$$\|u^{1,N}\|_{\dot{B}^{s_{p_2}+\delta}_{p_2,\infty}} \leq c(p_2) N^{\frac{p_0-p}{p_0}} \|g\|^{\frac{p}{p_0}}_{\dot{B}^{s_p}_{p,\infty}} \tag{2.3.126}$$

and

$$\|u^{1,N}\|_{\dot{B}^{s_{p_2}}_{p_2,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p, p_2)\|g\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.127}$$

We may now apply Proposition 2.9 with $s_1 = s_{p_2}$, $s_2 = s_{p_2} + \delta$ and $\theta = 1 - \frac{\delta_2}{\delta} \in]0, 1[$. In particular this gives, for any $f \in \mathcal{S}'_h$:

$$\|f\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,1}} \leq c(p_2, \delta, \delta_2)\|f\|^{1-\frac{\delta_2}{\delta}}_{\dot{B}^{s_{p_2}}_{p_2,\infty}} \|f\|^{\frac{\delta_2}{\delta}}_{\dot{B}^{s_{p_2}+\delta}_{p_2,\infty}}. \tag{2.3.128}$$

From Remark 2.7, we see that $\dot{B}^{s_{p_2}+\delta_2}_{p_2,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3)$. This fact, together with (2.3.126)-(2.3.128), implies that

$$\|u^{1,N}\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}} \leq N^{\frac{\delta_2(p_0-p)}{\delta p_0}} C(p_2, p, \delta, \delta_2, \|\mathcal{F}^{-1}\varphi\|_{L_1}, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.129}$$

Using identical reasoning, it can also be inferred that

$$\|U^{1,N^k}\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}} \leq N^{\frac{\delta_2(\kappa(p_1-p)-\beta(p)p)}{\delta_1 p_1}} C(p, p_0, p_2, p_1, \delta, \delta_1, \delta_2, \|\mathcal{F}^{-1}\varphi\|_{L_1}, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.130}$$

The choice

$$\kappa = \frac{1}{p_1 - p}\Big(p\beta(p) + \frac{\delta_1 p_1(p_0 - p)}{\delta p_0}\Big)$$

implies

$$\|u^{1,N} + U^{1,N^\kappa}\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}} \leq N^{\frac{\delta_2(p_0-p)}{\delta p_0}} C(p, p_0, p_2, p_1, \delta, \delta_1, \delta_2, \|\mathcal{F}^{-1}\varphi\|_{L_1}, \|g\|_{\dot{B}^{s_p}_{p,\infty}}). \tag{2.3.131}$$

It is also the case that

$$\|u^{1,N} + U^{1,N^\kappa}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}, p)\|g\|_{\dot{B}^{s_p}_{p,\infty}}. \tag{2.3.132}$$

Defining $\bar{g}^N := u^{1,N} + U^{1,N^\kappa}$ and $\widetilde{g}^N := U^{2,N^\kappa}$ gives the desired decomposition in Proposition 2.13. □

2.4 Regularity Results for the Stokes Equations

2.4.1 Local Regularity Theory for the Linear Stokes System

In Chapters 4 and 5, we will make use of local interior and local boundary regularity theory results for the linear Stokes system. The first proposition below regards local interior regularity results and is taken from Seregin's book [111][4]. A proof is also presented in [111].

Proposition 2.16. *Let m, n and s be such that $1 < m, n < \infty$ and $m \leq s < \infty$. Suppose $u \in W^{1,0}_{m,n}(Q)$, $p \in L_{m,n}(Q)$ and $f \in L_{s,n}(Q)$. In addition, suppose that*

$$\partial_t u - \Delta u + \nabla p = f,$$

$$div\, u = 0,$$

in the sense of distributions in $Q := B\times]-1, 0[$. Then, we conclude that $u \in W^{2,1}_{s,n}(Q(1/2))$ and $p \in W^{1,0}_{s,n}(Q(1/2))$. Furthermore, the estimate

$$\|\partial_t u\|_{L_{s,n}(Q(1/2))} + \|\nabla^2 u\|_{L_{s,n}(Q(1/2))} + \|\nabla p\|_{L_{s,n}(Q(1/2))}$$

$$\leq c(s, n, m)(\|f\|_{L_{s,n}(Q)} + \|u\|_{L_{m,n}(Q)} + \|\nabla u\|_{L_{m,n}(Q)} + \|p\|_{L_{m,n}(Q)})$$

holds.

The next proposition concerns the local boundary regularity theory for the linear Stokes system. The proposition below is from Seregin's paper [95][5].

Proposition 2.17. *Let m, n and s be such that $1 < m < \infty$, $1 < n \leq 2$ and $m \leq s < \infty$. Suppose $u \in W^{2,1}_{m,n}(Q^+)$, $p \in W^{1,0}_{m,n}(Q^+)$ and $f \in L_{s,n}(Q^+)$. In addition, suppose that*

$$\partial_t u - \Delta u + \nabla p = f,$$

$$div\, u = 0,$$

in $Q^+ := B^+\times]-1, 0[$ and suppose u satisfies the boundary condition

$$u = 0 \quad on \quad \Gamma(1)\times]-1, 0[.$$

Then, we conclude that $u \in W^{2,1}_{s,n}(Q^+(1/2))$ and $p \in W^{1,0}_{s,n}(Q^+(1/2))$. Furthermore, the estimate

$$\|\partial_t u\|_{L_{s,n}(Q^+(1/2))} + \|\nabla^2 u\|_{L_{s,n}(Q^+(1/2))} + \|\nabla p\|_{L_{s,n}(Q^+(1/2))}$$

$$\leq c(s, n, m)(\|f\|_{L_{s,n}(Q^+)} + \|u\|_{L_{m,n}(Q^+)} + \|\nabla u\|_{L_{m,n}(Q^+)} + \|p\|_{L_{m,n}(Q^+)})$$

holds.

[4]In particular, Proposition 6.7 p.85 in [111].
[5]See Proposition 2 p.2824 of [95].

2.4.2 Maximal Regularity Estimates for the Linear Stokes System

In this subsection, we consider the linear Stokes system for $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$:

$$\partial_t u - \Delta u + \nabla p = f, \ \operatorname{div} u = 0 \ \text{in} \ \Omega\times]0,T[, \tag{2.4.133}$$

$$u|_{\partial\Omega}(\cdot,t) = 0 \ \text{and} \tag{2.4.134}$$

$$u(\cdot,0) = 0. \tag{2.4.135}$$

In Chapters 5 and 6, we will utilise maximal regularity estimates for the linear Stokes system, which were developed by Solonnikov in [120][6] for equal space and time exponents and subsequently by Giga and Sohr in [44] for unequal space-time exponents. The proposition we will use in Chapters 5 and 6 is stated below and is taken from Giga and Sohr's paper [44] (specifically, Theorem 2.8 p.82 of [44]).

Proposition 2.18. *Assume that $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$. Let T, s and q be such that $0 < T < \infty$, $1 < q < \infty$ and $1 < s < \infty$. Assume in addition that*

$$f \in L_s\left(0,T; [C^\infty_{0,0}(\Omega)]^{L_q(\Omega)}\right). \tag{2.4.136}$$

Then there exists a unique solution $(u,\nabla p)$ of (2.4.133)-(2.4.135) satisfying

$$(u,\nabla p) \in W^{2,1}_{q,s}(\Omega\times]0,T[) \times L_{q,s}(\Omega\times]0,T[) \tag{2.4.137}$$

and

$$\|\partial_t u\|_{L_{q,s}(\Omega\times]0,T[)} + \|\nabla^2 u\|_{L_{q,s}(\Omega\times]0,T[)} + \|\nabla p\|_{L_{q,s}(\Omega\times]0,T[)} \leq c(q,s,\Omega)\|f\|_{L_{q,s}(\Omega\times]0,T[)}. \tag{2.4.138}$$

Remark 2.19. *For $g \in L_{q,s}(\Omega\times]0,T[)$ $(1 < q,s < \infty)$, we can apply the Helmoltz-Weyl decomposition[7] to deduce the following. Namely,*

$$g = G + \nabla H, \ \ G \in L_s\left(0,T; [C^\infty_{0,0}(\Omega)]^{L_q(\Omega)}\right) \tag{2.4.139}$$

and

$$\nabla H \in L_{q,s}(\Omega\times]0,T[) \ \ \textit{with} \ \ \|\nabla H\|_{L_{q,s}(\Omega\times]0,T[)} \leq C(\Omega,q)\|g\|_{L_{q,s}(\Omega\times]0,T[)}. \tag{2.4.140}$$

Consequently, for $f = g - \nabla H$ the conclusions of Proposition 2.18 hold.

[6] Specifically, Theorem 3.I p.169 of [120].

[7] See Theorem III.1.2 p.152 of Galdi's book [36], for example.

Chapter 3

Uniqueness Results for Weak Leray-Hopf Solutions of the Navier-Stokes System with Initial Values in Critical Spaces

3.1 Introduction

In 'Chapter 1: Introduction', the definition of weak Leray-Hopf solutions was stated (see (1.0.13)-(1.0.18) in Chapter 1). Furthermore, it was mentioned that the main goal of this thesis is to determine sufficient conditions to grant uniqueness of weak Leray-Hopf solutions of the Navier-Stokes system. This chapter will be motivated by the following closely related question.

> **(Q) Which $\mathcal{Z}\subset\mathcal{S}^{'}(\mathbb{R}^3)$ are such that $u_0\in J(\mathbb{R}^3)\cap\mathcal{Z}$ implies uniqueness of the associated weak Leray-Hopf solutions on some time interval?**

The results and text base of this chapter are based on

> Barker, T. Uniqueness Results for Weak Leray-Hopf Solutions of the Navier-Stokes System with Initial Values in Critical Spaces. J. Math. Fluid Mech. (2017). doi:10.1007/s00021-017-0315-8.

Unless mentioned otherwise, the notation and definitions used in this chapter are contained in 'Chapter 2: Preliminary Material'.

It was shown by Leray in [75] that if $\mathcal{Z}$ is the class of 'semi-regular initial values', we have short time uniqueness in the slightly narrower class of 'turbulent solutions'. In [75], Leray defines $u_0\in J(\mathbb{R}^3)$ to be a semi-regular initial value if there exists a global-in-time turbulent solution $V(u_0)$ satisfying

$$\int_0^T \|V(\cdot,\tau)\|^2_{L_\infty(\mathbb{R}^3)} d\tau < \infty \quad \text{for some } T > 0. \tag{3.1.1}$$

Let u_0 be a semi-regular initial value and $V(u_0)$ satisfy (3.1.1). Let us briefly outline Leray's proof for showing any other turbulent solution $u(u_0)$, with the same initial data, coincides with $V(u_0)$ on $Q_T := \mathbb{R}^3\times]0,T[$. Leray then showed that $w := u(u_0) - V(u_0)$ satisfies the following inequality for $0 < t < T$:

$$\|w(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 dy d\tau \leq 2\int_0^t \int_{\mathbb{R}^3} V\otimes w : \nabla w dy d\tau$$

$$\leq \int_0^t \|V(\cdot,\tau)\|^2_{L_\infty(\mathbb{R}^3)} \|w(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)} d\tau + \int_0^t \int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy d\tau$$

Here, the Hölder and Young inequalities have been used in the last line. This then implies that for $0 < t < T$ we have

$$\|w(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} \leq \int_0^t \|V(\cdot,\tau)\|^2_{L_\infty(\mathbb{R}^3)} \|w(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)} d\tau. \tag{3.1.2}$$

From (3.1.2), Leray concludes that $w = 0$ in $\mathbb{R}^3\times]0,T[$ by using that w is zero at the initial time, the fact that $V(u_0)$ satisfies (3.1.1) and Gronwall's lemma.

Similar arguments to the above proof have been frequently applied in the literature to show that **(Q)** is true for other classes $\mathcal{Z}$. From now on, we will refer to such arguments as 'Leray's approach'. The main aim of Leray's approach is to show that for certain $\mathcal{Z}$ and any $u_0 \in \mathcal{Z} \cap J(\mathbb{R}^3)$, one can construct a weak Leray Hopf solution $V(u_0)$ belonging to a path space $\mathcal{X}_T$ having certain features. Specifically, $\mathcal{X}_T$ has the property that any weak Leray-Hopf solution (with arbitrary $u_0 \in J(\mathbb{R}^3)$ initial data) in $\mathcal{X}_T$ is unique amongst all weak Leray-Hopf solutions with the same initial data.

In the literature, construction of such a $V(u_0)$ is typically achieved by means of Banach contraction arguments, where possible. When this is the case, showing the mild solution $V(u_0)$ coincides with all weak Leray-Hopf solutions on some time interval is frequently referred to as 'weak-strong uniqueness'.

As previously demonstrated for the case of the semi-regular initial values, a crucial step in Leray's approach is the establishment of appropriate estimates of the trilinear form $F : \mathcal{L}(T) \times \mathcal{L}(T) \times \mathcal{X}_T \times]0, T[\to \mathbb{R}$ given by:

$$F(a, b, c, t) := \int_0^t \int_{\mathbb{R}^3} (a \otimes c) : \nabla b dy d\tau. \tag{3.1.3}$$

As previously stated by Germain in [41], these estimates of this trilinear form typically play two roles. The first is to provide a rigorous proof of the energy inequality for $w := V(u_0) - u(u_0)$, where $u(u_0)$ is another weak Leray-Hopf solution with the same initial data. The second is to allow the applicability of Gronwall's lemma to infer $w \equiv 0$ on Q_T.

There are a vast number of papers related to question **(Q)**, many of which apply Leray's approach. We now give some incomplete references, which are directly concerned with this question and closely related to this Chapter. It was proven by Leray in [75] that for $\mathcal{Z} = \overset{\circ}{J}{}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\mathcal{Z} = L_p(\mathbb{R}^3)$ $(3 < p \leq \infty)$, we have short time uniqueness in the slightly narrower class of 'turbulent solutions'. The same conclusion was shown to hold by Fabes, Jones and Rivière in [29] for the weak Leray-Hopf class. It was later shown by Kato in [59] that $\mathcal{Z} = L_3(\mathbb{R}^3)$ was sufficient for short time uniqueness of weak Leray-Hopf solutions. An alternative proof can also be found in the paper of Robinson and Sadowski [91]. At the start of the 21st Century, Gallagher and Planchon [37] provided a positive answer for question **(Q)** for the homogeneous Besov spaces

$$\mathcal{Z} = \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$$

with $p, q < \infty$ and

$$\frac{3}{p} + \frac{2}{q} \geq 1.$$

An incomplete selection of further results in this direction can be found in the papers of Chemin [20], Dong and Zhang [25], Dubois [26], Germain [41] and the book of Lemarié-Rieusset [76], for example. A more complete history regarding question **(Q)** can be found in Germain's paper [41].

The estimates of the trilinear form (3.1.3) needed for Leray's approach appear to be restrictive, with regards to the spaces $\mathcal{Z}$ and $\mathcal{X}_T$ that can be considered. Consequently, **(Q)** has remained open for the Besov spaces

$$\mathcal{Z} = \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$$

with $3 < p \leq \infty,\ 1 \leq q \leq \infty$ and

$$\frac{3}{p} + \frac{2}{q} < 1.$$

The obstacle of using Leray's approach for this case, has been explicitly noted by Gallagher and Planchon [37] and Germain [41].

> 'It does not seem possible to improve on the continuity (of the trilinear term) without using in a much deeper way that not only u and $V(u_0)$ are in the Leray class $\mathcal{L}$ but also solutions of the equation.'[37]

For analogous Besov spaces on bounded domains, question **(Q)** has also been considered recently by various subsets of Farwig, Giga and Hsu in [30]-[32]. There, a restricted version of **(Q)** is shown to hold. Namely, the authors prove uniqueness within the subclass of 'well-chosen weak solutions', which describe weak Leray-Hopf solutions constructed by concrete approximation procedures. Furthermore, in [30]-[32] it is explicitly mentioned that a complete answer to **(Q)** for these cases is 'out of reach'.

In this chapter, we provide a positive answer to **(Q)** for $\mathcal{Z} = \dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$, with any $3 < p < \infty$. Here, $\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ is the closure of smooth compactly supported functions in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ and is such that

$$\dot{B}^{-1+\frac{3}{p}}_{p,p}(\mathbb{R}^3) \hookrightarrow \dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3).$$

In fact this is a corollary of our main theorem, which provides a positive answer to **(Q)** for other classes of $\mathcal{Z}$. Recall from 'Chapter 2: Preliminary Material' that for $p_0 > 3$ we define

$$s_{p_0} := -1 + \frac{3}{p_0} < 0.$$

Moreover, for $2 < \alpha \leq 3$ and $p_1 > \alpha$, we define

$$s_{p_1,\alpha} := -\frac{3}{\alpha} + \frac{3}{p_1} < 0.$$

Now, we state the main theorem of this chapter.

Theorem 3.1. *Fix* $2 < \alpha \leq 3$.

- *For* $2 < \alpha < 3$, *take any* p *such that* $\alpha < p < \frac{\alpha}{3-\alpha}$.
- *For* $\alpha = 3$, *take any* p *such that* $3 < p < \infty$.

Consider a weak Leray-Hopf solution u *to the Navier-Stokes system on* Q_∞, *with initial data*

$$u_0 \in VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Then, there exists a $T(u_0) > 0$ *such that all weak Leray-Hopf solutions on* Q_∞, *with initial data* u_0, *coincide with* u *on* $Q_{T(u_0)} := \mathbb{R}^3 \times]0, T(u_0)[$.

Let us remark that previous results of this type are given by Chemin in [20] and by Dong and Zhang in [25] respectively, with the additional assumption that u_0 belongs to a nonhomogeneous Sobolev space $H^s(\mathbb{R}^3)$, with $s > 0$. By comparison the assumptions of Theorem 3.1 are weaker. This follows because of the following embeddings. Namely, for $s > 0$ there exists $2 < \alpha \leq 3$ such that for $p \geq \alpha$:

$$H^s(\mathbb{R}^3) \hookrightarrow L_\alpha(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3).$$

Corollary 3.2. *Let $3 < p < \infty$. Consider a weak Leray-Hopf solution u to the Navier-Stokes system on Q_∞, with initial data*

$$u_0 \in \dot{\mathbb{B}}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Then, there exists a $T(u_0) > 0$ such that all weak Leray-Hopf solutions on Q_∞, with initial data u_0, coincide with u on $Q_{T(u_0)} := \mathbb{R}^3 \times]0, T(u_0)[$.

Our main tool to prove Theorem 3.1 is the new observation that weak Leray-Hopf solutions, with this class of initial data, have stronger continuity properties near $t = 0$ than general members of the energy class $\mathcal{L}$. In [21], Chemin and Planchon obtained a similar property for mild solutions with initial data in $\dot{B}^{-\frac{1}{4}}_{4,4}(\mathbb{R}^3)$. Recently, in case of 'global weak L_3 solutions' with $L_3(\mathbb{R}^3)$ initial data, properties of this type were established by Seregin and Šverák in [112]. See also [6] for the case of $L^{3,\infty}$ initial data, in the context of 'global weak $L^{3,\infty}(\mathbb{R}^3)$ solutions'. Here is our main lemma.

Lemma 3.3. *Take α and p as in Theorem 3.1. Assume that*

$$u_0 \in J(\mathbb{R}^3) \cap \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3).$$

Then for any weak Leray-Hopf u solution on $Q_T := \mathbb{R}^3 \times]0, T[$, with initial data u_0, we infer the following. There exists

$$\beta(p, \alpha) > 0$$

and

$$\gamma(\|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)}, p, \alpha) > 0$$

such that for $t \leq \min(1, \gamma, T)$:

$$\|u(\cdot, t) - S(t)u_0\|^2_{L_2} \leq t^\beta c(p, \alpha, \|u_0\|_{L_2(\mathbb{R}^3)}, \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)}). \qquad (3.1.4)$$

This then allows us to apply a less restrictive version of Leray's approach. Namely, we show that for any initial data in this specific class, there exists a weak Leray-Hopf solution $V(u_0)$ on Q_T, which belongs to a path space $\mathcal{X}_T$ with the following property. Namely, $\mathcal{X}_T$ grants uniqueness for weak Leray-Hopf solutions with the same initial data in this specific class (rather than for arbitrary initial data in $J(\mathbb{R}^3)$, as required in Leray's approach). This strategy has been previously used by Dong and Zhang in [25]. However, in [25] an additional restriction is imposed, requiring that the initial data has positive Sobolev regularity.

Remarks

1. In [107], Seregin discusses the following open question:

   (Q.1) Suppose that $u_0^{(k)}(\in J(\mathbb{R}^3)) \rightharpoonup 0$ in $L_2(\mathbb{R}^3)$. Let $v^{(k)}$ be a global-in-time weak Leray-Hopf solution, associated with the initial value $u_0^{(k)}$. Does there exist a subsequence of $v^{(k)}$ that converges to zero in the distributional sense?

   In [107], it was shown that (Q.1) holds true under the following additional restrictions. Namely

$$\sup_k \|u_0^{(k)}\|_{L_s(\mathbb{R}^3)} < \infty \quad \text{for some } 3 < s \leq \infty \tag{3.1.5}$$

   and that $v^{(k)}$ and its associated pressure $q^{(k)}$ satisfy the local energy inequality:

$$\int\limits_{\mathbb{R}^3} \varphi(x,t)|v^{(k)}(x,t)|^2 dx + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} \varphi|\nabla v^{(k)}|^2 dxds$$

$$\leq \int\limits_0^t \int\limits_{\mathbb{R}^3} |v^{(k)}|^2(\partial_t\varphi + \Delta\varphi) + v^{(k)}\cdot\nabla\varphi(|v^{(k)}|^2 + 2q^{(k)})dxds \tag{3.1.6}$$

   for all nonnegative functions $\varphi \in C_0^\infty(Q_\infty)$. Subsequently, Seregin and Šverák showed in [112] that the same conclusion holds with (3.1.5) replaced by weaker assumption that

$$\sup_k \|u_0^{(k)}\|_{L_3(\mathbb{R}^3)} < \infty. \tag{3.1.7}$$

   Lemma 3.3 has the consequence that **(Q.1)** still holds true, if (3.1.5) is replaced by the assumption that $u_0^{(k)}$ is bounded in the supercritical Besov spaces

$\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)$[1]. Consequently, as the following continuous embedding (recall $\alpha \leq p$ and $2 \leq \alpha \leq 3$)

$$L_\alpha(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)$$

holds, we see that this improves the previous assumptions under which **(Q.1)** holds true.

2. In the papers of Serrin [114] and Prodi [89], it was shown that if u is a weak Leray-Hopf solution on Q_T and satisfies

$$u \in L_q(0,T;L_p(\mathbb{R}^3)) \qquad \frac{3}{p}+\frac{2}{q}=1 \ (3<p\leq\infty \text{ and } 2\leq q<\infty), \tag{3.1.8}$$

then u coincides on Q_T with other any weak Leray-Hopf solution with the same initial data. The same conclusion for the endpoint case $u \in L_\infty(0,T;L_3(\mathbb{R}^3))$ appeared to be much more challenging and was settled by Kozono and Sohr in [68]. As a consequence of Theorem 3.1, we are able to extend the uniqueness criterion (3.1.8) for weak Leray-Hopf solutions. Let us state this as a proposition.

Proposition 3.4. *Suppose u and v are weak Leray-Hopf solutions on Q_∞ with the same initial data $u_0 \in J(\mathbb{R}^3)$. Then there exists a $\epsilon_* = \epsilon_*(p,q) > 0$ such that if either*

- $$u \in L^{q,s}(0,T;L^{p,s}(\mathbb{R}^3)) \qquad \frac{3}{p}+\frac{2}{q}=1 \tag{3.1.9}$$
  $$(3<p<\infty\,,\ 2<q<\infty \textit{ and } 1\leq s<\infty) \tag{3.1.10}$$

- *or*
  $$u \in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3)) \qquad \frac{3}{p}+\frac{2}{q}=1 \tag{3.1.11}$$
  $$(3<p<\infty\,,\ 2<q<\infty) \tag{3.1.12}$$

  with
  $$\|u\|_{L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))} \leq \epsilon_*, \tag{3.1.13}$$

  then $u \equiv v$ on $Q_T := \mathbb{R}^3\times]0,T[$.

[1] with p and α as in Theorem 3.1

It has previously been known that if either (3.1.9)-(3.1.10) or (3.1.11)-(3.1.13) are satisfied for the velocity field u of the Navier-Stokes equations (with ϵ_* sufficiently small), then regularity of u can be inferred. In [123], Takahashi considered distributional solutions u of the Navier-Stokes equations, with $u \in W^{1,0}_2(Q(1)) \cap L_{2,\infty}(Q(1))$. Theorem 3.2 of [123] showed that there exists a universal constant $\epsilon_* > 0$ such that if

$$\|u\|_{L^{q,\infty}(-1,0;L_p(B(1)))} \le \epsilon_*, \text{ with } \frac{3}{p} + \frac{2}{q} = 1 \text{ and } p > 3,$$

then u is bounded on $Q(1/2)$. For sufficently small ϵ_*, it was shown by Sohr in [117] (see also the paper of Bosia, Pata and Robinson [15]) that if u is a weak Leray-Hopf solution on Q_∞ satisfying either (3.1.9)-(3.1.10) or (3.1.11)-(3.1.13), then u is regular[2] on Q_T. To the best of the author's knowledge, it was not previously known whether these conditions on u were sufficient to grant uniqueness on Q_T, amongst all weak Leray-Hopf solutions with the same initial value.

3.2 Some Estimates Near the Initial Time for Weak Leray-Hopf Solutions

3.2.1 Construction of Mild Solutions with Subcritical Besov Initial Data

Let $\delta > 0$ be such that $s_{p_0} + \delta < 0$ and define the space

$$X_{p_0,\delta}(T) := \{f \in \mathcal{S}'(\mathbb{R}^3\times]0,T[) : \sup_{0<t<T} t^{-\frac{s_{p_0}}{2}-\frac{\delta}{2}} \|f(\cdot,t)\|_{L_{p_0}(\mathbb{R}^3)} < \infty\}.$$

From Remarks 2.7 and 2.8 in 'Chapter 2: Preliminary Material', we observe that

$$u_0 \in \dot{B}^{s_{p_0}+\delta}_{p_0,p_0}(\mathbb{R}^3) \Rightarrow \|S(t)u_0\|_{X_{p_0,\delta}(T)} \le C\|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,\infty}} \le C\|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}. \tag{3.2.1}$$

In this subsection, we construct mild solutions with weakly divergence-free initial data in $\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}(\mathbb{R}^3) \cap J(\mathbb{R}^3)$.

As was done in 'Chapter 1: Introduction' (specifically, (1.1.13)), we define

$$B(u,v)_i(x,t) := \int_0^t \int_{\mathbb{R}^3} K_{ijl}(x-y,t-\tau)u_j(y,\tau)v_l(y,\tau)dyd\tau. \tag{3.2.2}$$

[2] By regular on Q_T, we mean $C^\infty(\mathbb{R}^3\times]0,T[)$ with every space-time derivative in $L_\infty(\epsilon,T;L_\infty(\mathbb{R}^3))$ for any $0 < \epsilon < T$.

Here, K_{ijl} is given by (1.1.10) in Chapter 1. It what follows, we will use the notation $\pi_{u\otimes u} := \mathcal{R}_i\mathcal{R}_j(u_iu_j)$. Here $\mathcal{R}_i$ are Riesz transforms given by

$$\mathcal{R}_i g(x) := cP.V. \int \frac{y_i}{|y|^4} g(x-y)dy$$

and the summation convention is adopted.

In this subsection, we will make use of an ε-regularity criteria developed by Caffarelli, Kohn and Nirenberg in [17] (Proposition 1 p.775 of [17]). The statement below is taken from Seregin's book [111] (Lemma 6.1 p.134 of [111]).

Lemma 3.5. *Let (v,q) be a suitable weak solution of the Navier-Stokes equations in $B(x_0,R)\times]t_0-R^2,t_0[$. Then there exist universal constants ε_{CKN} and $c_{0k}>0$ (with $k=0,1\dots$) with the following property. Assume that (v,q) satisfy*

$$\frac{1}{R^2}\int\limits_{t_0-R^2}^{t}\int\limits_{B(x_0,R)}(|v|^3+|q|^{\frac{3}{2}})dxdt \le \varepsilon_{CKN}. \tag{3.2.3}$$

Then for any $k=0,1\dots$, $\nabla^k v$ is Hölder continuous in the closure of $B(x_0,R/2)\times]t_0-(R/2)^2,t_0[$ and the following bound is valid:

$$\sup_{(x,t)\in B(x_0,R/2)\times]t_0-(R/2)^2,t_0[}|\nabla^k v(x,t)| < \frac{c_{0k}}{R^{k+1}}. \tag{3.2.4}$$

Let us mention that the proof of (3.2.4) for the cases $k=1,\dots$ can be found in the paper of Nečas, Růžička and Šverák [85] (specifically, Proposition 2.1 in [85]).

Now we state and prove the theorem regarding construction of mild solutions with weakly divergence-free initial data in $\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$.

Theorem 3.6. *Consider p_0 and δ such that $4<p_0<\infty$, $\delta>0$ and $s_{p_0}+\delta<0$. Suppose that $u_0\in\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$. There exists a constant $c=c(p_0)$ such that if*

$$4cT^{\frac{\delta}{2}}\|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}<1, \tag{3.2.5}$$

then there exists a $v\in X_{p_0,\delta}(T)$, which solves the Navier-Stokes equations in the sense of distributions and satisfies the following properties. The first property is that v solves the following integral equation:

$$v(x,t) := S(t)u_0 + B(v,v)(x,t) \tag{3.2.6}$$

in Q_T, along with the estimate

$$\|v\|_{X_{p_0,\delta}(T)} < 2\|S(t)u_0\|_{X_{p_0,\delta}(T)} := 2M^{(0)}. \tag{3.2.7}$$

The second property is (recalling that by assumption $u_0 \in J(\mathbb{R}^3)$):

$$v \in W^{1,0}_2(Q_T) \cap C([0,T]; J(\mathbb{R}^3)) \cap L_4(Q_T). \tag{3.2.8}$$

Moreover, the following estimate is valid for $0 < t \leq T$:

$$\|B(v,v)(\cdot,t)\|^2_{L_2} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla B(v,v)|^2 dy d\tau$$

$$\leq Ct^{\frac{1}{p_0-2}+2\delta\theta_0}(2M^{(0)})^{4\theta_0} \left(Ct^{\frac{1}{2\theta_0-1}\left(\frac{1}{p_0-2}+2\delta\theta_0\right)}(2M^{(0)})^{\frac{4\theta_0}{2\theta_0-1}} + \|u_0\|^2_{L_2}\right)^{2(1-\theta_0)}. \tag{3.2.9}$$

Here,

$$\frac{1}{4} = \frac{\theta_0}{p_0} + \frac{1-\theta_0}{2} \tag{3.2.10}$$

and $C = C(p_0, \delta, \theta_0)$.
The associated pressure $\pi_{v\otimes v}$ satisfies (here, $\lambda \in]0,T[$):

$$\pi_{v\otimes v} \in L_2(Q_T) \cap L_{\frac{p_0}{2},\infty}(Q_{\lambda,T})). \tag{3.2.11}$$

The final property is that for $\lambda \in]0,T[$ and $k = 0, 1\ldots$, $l = 0, 1\ldots$:

$$\sup_{(x,t)\in Q_{\lambda,T}} |\partial^l_t \nabla^k v| + |\partial^l_t \nabla^k \pi_{v\otimes v}| \leq c(p_0, \delta, \lambda, \|u_0\|_{B^{s_{p_0}+\delta}_{p_0,p_0}}, k, l). \tag{3.2.12}$$

Proof. Recall Young's convolution inequality:

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} \tag{3.2.13}$$

where $1 \leq p, q < \infty$, $1 \leq r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

Applying this and the pointwise estimate (1.1.11) in Chapter 1 gives the following:

$$\|K(\cdot, t-\tau) \star (r \otimes r)(\cdot,\tau)\|_{L_{p_0}} \leq \|K(\cdot,t-\tau)\|_{L_{(p_0)'}} \|r \otimes r\|_{L_{\frac{p_0}{2}}}$$

$$\leq C(t-\tau)^{-(1+\frac{s_{p_0}}{2})}\|r\|^2_{L_{p_0}} \leq C(t-\tau)^{-(1+\frac{s_{p_0}}{2})}\frac{\|r\|^2_{X_{p_0,\delta}(T)}}{\tau^{-s_{p_0}-\delta}}. \tag{3.2.14}$$

Here, $\frac{1}{(p_0)'} := 1 - \frac{1}{p_0}$ and we recall that $s_{p_0} := -1 + \frac{3}{p_0}$. Using (3.2.14), together with the definition of $B(r,r)$ given by (3.2.2), one can show that

$$\|B(r,r)\|_{X_{p_0,\delta}(T)} \leq CT^{\frac{\delta}{2}}\|r\|^2_{X_{p_0,\delta}(T)}. \tag{3.2.15}$$

Let us briefly describe the method of successive approximations. For $n = 1, 2, \cdots$ let $v^{(0)} = S(t)u_0$,

$$v^{(n+1)} = v^{(0)} + B(v^{(n)}, v^{(n)}).$$

Moreover for $n = 0, 1, \ldots$ define:

$$M^{(n)} := \|v^{(n)}\|_{X_{p_0,\delta}(T)}. \tag{3.2.16}$$

Then using (3.2.15), we have the following iterative relation:

$$M^{(n+1)} \leq M^{(0)} + CT^{\frac{\delta}{2}}(M^{(n)})^2. \tag{3.2.17}$$

If

$$4CT^{\frac{\delta}{2}}M^{(0)} < 1, \tag{3.2.18}$$

then one can show that for $n = 1, 2\ldots$ we have

$$M^{(n)} < 2M^{(0)}. \tag{3.2.19}$$

Step 2: Establishing Energy Bounds

Let $r \in X_{p_0,\delta}(T) \cap L_{2,\infty}(Q_T)$. Recalling (3.2.10), we note that by interpolation $(0 \leq \tau \leq T)$:

$$\|r(\cdot,\tau)\|_{L_4} \leq \|r(\cdot,\tau)\|_{L_{p_0}}^{\theta_0}\|r(\cdot,\tau)\|_{L_2}^{1-\theta_0} \leq \|r\|_{X_{p_0,\delta}(T)}^{\theta_0}\|r(\cdot,\tau)\|_{L_2}^{1-\theta_0}\tau^{\theta_0(\frac{s_{p_0}}{2}+\frac{\delta}{2})}. \tag{3.2.20}$$

From (3.2.10) it follows that

$$\frac{-\theta_0 s_{p_0}}{2} = \frac{1-3/p_0}{4(1-2/p_0)} = \frac{1}{4} - \frac{1}{4(p_0-2)} < \frac{1}{4}.$$

From this, we conclude that for $0 \leq t \leq T$:

$$\|r\|_{L_4(Q_t)} \leq Ct^{\frac{\theta_0\delta}{2}+\frac{1}{4(p_0-2)}}\|r\|_{X_{p_0,\delta}(T)}^{\theta_0}\|r\|_{L_{2,\infty}(Q_t)}^{1-\theta_0}. \tag{3.2.21}$$

Define $R := B(r,r)$ and $\pi_{r\otimes r} := \mathcal{R}_i\mathcal{R}_j(r_ir_j)$, where $\mathcal{R}_i$ denotes a Riesz transform and repeated indices are summed. One can readily show that on Q_T, $(R, \pi_{r\otimes r})$ are solutions to

$$\partial_t R - \Delta R + \operatorname{div} r\otimes r = -\nabla\pi_{r\otimes r}, \tag{3.2.22}$$

$$\operatorname{div} R = 0, \tag{3.2.23}$$

$$R(\cdot,0) = 0. \tag{3.2.24}$$

We can also infer that $R \in W_2^{1,0}(Q_T) \cap C([0,T]; J(\mathbb{R}^3))$, along with the estimate

$$\|R(\cdot,t)\|_{L_2}^2 + \int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla R(x,s)|^2dxds \leq \|r\otimes r\|_{L_2(Q_t)}^2$$

$$\leq c\|r\|^4_{L_4(Q_t)} \leq Ct^{2\theta_0\delta+\frac{1}{(p_0-2)}}\|r\|^{4\theta_0}_{X_{p_0,\delta}(T)}\|r\|^{4(1-\theta_0)}_{L_{2,\infty}(Q_t)}. \tag{3.2.25}$$

Using that the associated pressure is a composition of Riesz transforms acting on $r\otimes r$, together with the known fact that Calderón-Zygmund singular integral operators are continuous on L_p for $1 < p < \infty$, we deduce the estimates

$$\|\pi_{r\otimes r}\|_{L_2(Q_T)} \leq C\|r\|^2_{L_4(Q_T)} \tag{3.2.26}$$

and

$$\|\pi_{r\otimes r}\|_{L_{\frac{p_0}{2},\infty}(Q_{\lambda,T})} \leq C(\lambda, T, p_0, \delta)\|r\|^2_{X_{p_0,\delta}(T)}. \tag{3.2.27}$$

Substituting $r = v^{(0)} \in X_{p_0,\delta}(T) \cap L_{2,\infty}(Q_T)$ therefore gives that

$$v^{(1)} \in W^{1,0}_2(Q_T) \cap C([0,T]; J(\mathbb{R}^3)) \cap L_4(Q_T) \cap X_{p_0,\delta}(T).$$

Using that $v^{(n)} \in X_{p_0,\delta}(T)$ for all $n \in \mathbb{N}$, we may argue by induction to conclude that

$$v^{(n)} \in W^{1,0}_2(Q_T) \cap C([0,T]; J(\mathbb{R}^3)) \cap L_4(Q_T) \cap X_{p_0,\delta}(T)$$

for all $n \in \mathbb{N}$.

For $n = 0, 1, \dots$ and $t \in [0,T]$ define:

$$E^{(n)}(t) := \|v^{(n)}\|_{L_\infty(0,t;L_2)} + \|\nabla v^{(n)}\|_{L_2(Q_t)}. \tag{3.2.28}$$

Clearly, by the assumptions on the initial data $E^{(0)}(t) \leq \|u_0\|_{L_2} < \infty$. Then applying (3.2.25) with $r = v^{(n)}$, gives the following iterative relations:

$$E^{(n+1)}(t) \leq E^{(0)}(t) + Ct^{\theta_0\delta+\frac{1}{2(p_0-2)}}(M^{(n)})^{2\theta_0}(E^{(n)}(t))^{2(1-\theta_0)}. \tag{3.2.29}$$

From (3.2.10), it is clear that $2(1-\theta_0) < 1$. Hence by Young's inequality:

$$E^{(n+1)}(t) \leq Ct^{\frac{1}{2\theta_0-1}(\theta_0\delta+\frac{1}{2(p_0-2)})}(M^{(n)})^{\frac{2\theta_0}{2\theta_0-1}} + E^{(0)} + \frac{1}{2}E^{(n)}(t). \tag{3.2.30}$$

Observing (3.2.19) and (3.2.30), it is possible to show that for $t \in [0,T]$:

$$\begin{aligned} E^{(n+1)}(t) &\leq 2Ct^{\frac{1}{2\theta_0-1}(\theta_0\delta+\frac{1}{2(p_0-2)})}(2M^{(0)})^{\frac{2\theta_0}{2\theta_0-1}} + 2E^{(0)}(t) \\ &\leq 2Ct^{\frac{1}{2\theta_0-1}(\theta_0\delta+\frac{1}{2(p_0-2)})}(2M^{(0)})^{\frac{2\theta_0}{2\theta_0-1}} + 2\|u_0\|_{L_2}. \end{aligned} \tag{3.2.31}$$

Step 3: Convergence and Showing Energy Inequalities

Using (3.2.17)-(3.2.19), one can argue along the same lines as in Kato's paper [59] to deduce that there exists a $v \in X_{p_0,\delta}(T)$ such that

$$\lim_{n\to 0} \|v^{(n)} - v\|_{X_{p_0,\delta}(T)} = 0. \tag{3.2.32}$$

Furthermore v satisfies the integral equation (3.2.6) as well as the Navier-Stokes equations, in the sense of distributions. From $u_0 \in J(\mathbb{R}^3) \cap \dot{B}^{s_{p_0}+\delta}_{p_0,p_0}(\mathbb{R}^3)$, (3.2.31), (3.2.32) and estimates analogous to (3.2.21) and (3.2.25), applied to $v^{(m)} - v^{(n)}$, we have the following:

$$v^{(n)} \to v \text{ in } C([0,T]; J(\mathbb{R}^3)) \cap W^{1,0}_2(Q_T), \tag{3.2.33}$$

$$v(\cdot, 0) = u_0, \tag{3.2.34}$$

$$v^{(n)} \to v \text{ in } L_4(Q_T), \tag{3.2.35}$$

$$\pi_{v^{(n+1)}\otimes v^{(n+1)}} \to \pi_{v\otimes v} \text{ in } L_2(Q_T) \text{ and } L_{\frac{p_0}{2},\infty}(Q_{\lambda,T}) \tag{3.2.36}$$

where $0 < \lambda < T$. Using this, along with (3.2.25) and (3.2.31), we infer the estimate (3.2.9).

Step 4: Estimate of Higher Derivatives

First note that from the definition of $X_{p_0,\delta}(T)$, we have the estimate:

$$\|v\|_{L_{p_0,\infty}(Q_{\lambda,T})} \le \lambda^{\frac{1}{2}(s_{p_0}+\delta)} \|v\|_{X_{p_0,\delta}(T)}. \tag{3.2.37}$$

Since $\pi_{v\otimes v}$ is a composition of Riesz transforms acting on $v \otimes v$, we deduce from (3.2.37) that

$$\|\pi_{v\otimes v}\|_{L_{\frac{p_0}{2},\infty}(Q_{\lambda,T})} \le C\lambda^{s_{p_0}+\delta} \|v\|^2_{X_{p_0,\delta}(T)}. \tag{3.2.38}$$

One can obtain the conclusion (3.2.12) by using (3.2.8), (3.2.11) and (3.2.37)-(3.2.38), along with arguments contained in Seregin's book [111][3]. As the context here is different, the author finds it instructive to provide details of certain parts of the arguments used in [111]. Let us do this now.

Using (3.2.8), (3.2.11) and a mollification argument[4], one can infer that $(v, \pi_{v\otimes v})$ satisfies the local energy equality. If $(x,t) \in Q_{\lambda,T}$, then for $0 < r^2 < \frac{\lambda}{2}$ we can apply Hölder's inequality and (3.2.37)-(3.2.38) to infer that

$$\frac{1}{r^2} \int\limits_{t-r^2}^{t} \int\limits_{B(x,r)} (|v|^3 + |\pi_{v\otimes v}|^{\frac{3}{2}}) dxdt \le C\lambda^{\frac{3}{2}(s_{p_0}+\delta)} r^{3(1-\frac{3}{p_0})} \|u_0\|^3_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}. \tag{3.2.39}$$

[3] In particular, pp.160-162 of [111].

[4] The required mollification argument can be found in Seregin's book [111] (specifically Proposition 3.9 pp.160-161), for example.

Clearly, there exists $r_0^2(\lambda, \varepsilon_{CKN}, \|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}) < \frac{\lambda}{2}$ such that

$$\frac{1}{r_0^2}\int\limits_{t-r_0^2}^{t}\int\limits_{B(x,r_0)}(|v|^3+|\pi_{v\otimes v}|^{\frac{3}{2}})dxdt \leq \varepsilon_{CKN}.$$

From Lemma 3.5, there exist universal constants $c_{0k} > 0$ such that (for (x,t) and r_0 as above) we have that for $k = 0, 1\ldots$

$$|\nabla^k v(x,t)| \leq \frac{c_{0k}}{r_0^{k+1}} = C(k, \lambda, \|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}).$$

Thus,

$$\sup_{(x,t)\in Q_{\lambda,T}} |\nabla^k v| \leq C(k, \lambda, \|u_0\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}}). \tag{3.2.40}$$

Using (3.2.40) and the singular integral representation of $\pi_{v\otimes v}$, one can deduce (3.2.12). The arguments showing this are identical to those presented in Seregin's book [111] (specifically, the proof of Proposition 3.9, pp.161-162).

□

3.2.2 Proof of Lemma 3.3

The proof of Lemma 3.3 is achieved by careful analysis of certain decompositions of weak Leray-Hopf solutions, with initial data in the class given in Lemma 3.3. A key part of this involves decompositions of the initial data (Corollary 2.12 in Chapter 2), together with properties of mild solutions, whose initial data belong to a subcritical homogeneous Besov space (Theorem 3.6). Our methods naturally build upon the work of Calderón [18]. In [18], splitting arguments involving Lebesgue spaces are used to show existence of global-in-time solutions of the Navier-Stokes equations, with solenodial initial data belonging to L_p $(2 < p < 3)$. Related splitting arguments have also previously been used by Jia and Šverák in [54], in order to show estimates near the initial time for Lemarié-Rieusset local energy solutions of the Navier-Stokes equations with L_3 initial data.

Before proceeding, we state a lemma that is a consequence of Hölder's inequality, Lebesgue interpolation and the Sobolev embedding theorem. The lemma below has been stated and used in the papers of Prodi [89] and Serrin [114].

Lemma 3.7. *Let* $p \in]3, \infty]$ *and*

$$\frac{3}{p} + \frac{2}{r} = 1. \tag{3.2.41}$$

Suppose that $w \in L_{p,r}(Q_T)$, $v \in L_{2,\infty}(Q_T)$ and $\nabla v \in L_2(Q_T)$. Then for $t \in]0,T[$:

$$\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla v||v||w|dxdt' \leq C\int\limits_0^t \|w\|_{L_p}^r \|v\|_{L_2}^2 dt' + \frac{1}{2}\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla v|^2 dxdt'. \tag{3.2.42}$$

In proving Lemma 3.3 and Theorem 3.1, we will make use of the following lemma, whose statement and proof can be found in in Lemarié-Rieusset's book [76][5].

Lemma 3.8. *Let $T \in]0,\infty]$. Suppose $v^{(1)}$ and $v^{(2)}$ belong to $C_w([0,T]; J(\mathbb{R}^3)) \cap L_2(0,T; \dot{H}^1(\mathbb{R}^3))$. In addition, suppose that there exists distributions $q^{(i)} \in \mathcal{D}'(\mathbb{R}^3\times]0,T[)$ such that*

$$\partial_t v^{(i)} - \Delta v^{(i)} + \nabla q^{(i)} + v^{(i)}\cdot\nabla v^{(i)} = 0, \tag{3.2.43}$$

$$div\, v^{(i)} = 0, \tag{3.2.44}$$

in the sense of distributions on $\mathbb{R}^3\times]0,T[$.

Assume also that

$$v^{(2)} \in L_q(0,T; L_p(\mathbb{R}^3)), \tag{3.2.45}$$

with

$$\frac{3}{p} + \frac{2}{q} = 1 \;\; (3 \leq p \leq \infty \text{ and } 2 \leq q \leq \infty). \tag{3.2.46}$$

Then for any δ and t with $0 \leq \delta \leq t \leq T$, we have the equality:

$$\int\limits_{\mathbb{R}^3} v^{(1)}(y,t)\cdot v^{(2)}(y,t)dy = \int\limits_{\mathbb{R}^3} v^{(1)}(y,\delta)\cdot v^{(2)}(y,\delta)dy - 2\int\limits_\delta^t \int\limits_{\mathbb{R}^3} \nabla v^{(1)} : \nabla v^{(2)} dyd\tau$$

$$-\int\limits_\delta^t \int\limits_{\mathbb{R}^3} v^{(2)} \otimes (v^{(1)} - v^{(2)}) : \nabla(v^{(1)} - v^{(2)})dyd\tau. \tag{3.2.47}$$

Proof of Lemma 3.3. Throughout this subsection, $u_0 \in \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap L_2(\mathbb{R}^3)$. Here, p and α satisfy the assumptions of Theorem 3.1. We will write $u_0 = u^{1,N} + u^{2,N}$. Here the decomposition has been performed according to Corollary 2.12 in Chapter 2 (specifically, (2.3.50)-(2.3.53)), with $\epsilon = N > 0$. Thus,

$$\|u^{1,N}\|^{p_0}_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}} \leq N^{p_0-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{3.2.48}$$

$$\|u^{2,N}\|^2_{L_2} \leq C(p, p_0, \alpha, \|\mathcal{F}^{-1}\varphi\|_{L_1})N^{2-p}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}} \tag{3.2.49}$$

[5] Specifically, Proposition 14.3 in [76]

and

$$\|u^{1,N}\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{L_2}. \tag{3.2.50}$$

Throughout this section we will let w^N be the mild solution from Theorem 3.6 generated by the initial data $u^{1,N}$. Recall from Theorem 3.6 that w^N is defined on Q_{T_N}, where

$$4c(p_0)T_N^{\frac{\delta}{2}}\|u^{1,N}\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,p_0}} < 1. \tag{3.2.51}$$

In accordance with this and (3.2.48), we will take

$$T_N := \frac{1}{\left(8c(p_0)N^{\frac{p_0-p}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}}\right)^{\frac{2}{\delta}}} \tag{3.2.52}$$

The two main estimates we will use are as follows. Using (3.2.1), (3.2.7) and (3.2.48), we have

$$\|w^N\|_{X_{p_0,\delta}(T_N)} < CN^{\frac{p_0-p}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}}. \tag{3.2.53}$$

The second property, from Theorem 3.6, is (recalling that by assumption $u^{1,N} \in L_2(\mathbb{R}^3)$):

$$w^N \in W^{1,0}_2(Q_{T_N}) \cap C([0,T_N]; J(\mathbb{R}^3)) \cap L_4(Q_{T_N}). \tag{3.2.54}$$

Consequently, it can be shown that w^N satisfies the energy equality:

$$\|w^N(\cdot,s)\|^2_{L_2} + 2\int\limits_{\tau}^{s}\int\limits_{\mathbb{R}^3}|\nabla w^N(y,s')|^2 dyds' = \|w^N(\cdot,\tau)\|^2_{L_2} \tag{3.2.55}$$

for $0 \leq \tau \leq s \leq T_N$. Moreover, using (3.2.9), (3.2.48) and (3.2.50), the following estimate is valid for $0 \leq s \leq T_N$:

$$\|w^N(\cdot,s) - S(s)u^{1,N}\|^2_{L_2} + \int\limits_{0}^{s}\int\limits_{\mathbb{R}^3}|\nabla w^N(\cdot,\tau) - \nabla S(\tau)u^{1,N}|^2 dyd\tau$$

$$\leq Cs^{\frac{1}{p_0-2}+2\delta\theta_0}(N^{\frac{p_0-p}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}})^{4\theta_0}$$

$$\times\left(s^{\frac{1}{2\theta_0-1}\left(\frac{1}{p_0-2}+2\delta\theta_0\right)}(N^{\frac{p_0-p}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}})^{\frac{4\theta_0}{2\theta_0-1}} + \|u_0\|^2_{L_2}\right)^{2(1-\theta_0)}. \tag{3.2.56}$$

Here,

$$\frac{1}{4} = \frac{\theta_0}{p_0} + \frac{1-\theta_0}{2} \tag{3.2.57}$$

and $C = C(p_0,\delta,\theta_0)$. Let

$$S_N := \min(T, T_N).$$

Define on Q_{S_N}, $v^N(x,t) := u(x,t) - w^N(x,t)$. Clearly we have :

$$v^N \in L_\infty(0,S_N;L_2) \cap C_w([0,S_N];J(\mathbb{R}^3)) \cap W^{1,0}_2(Q_{S_N}) \tag{3.2.58}$$

and

$$\lim_{\tau\to 0} \|v^N(\cdot,\tau) - u^{2,N}\|_{L_2} = 0. \tag{3.2.59}$$

Moreover, v^N satisfies the following equations

$$\partial_t v^N + v^N\cdot\nabla v^N + w^N\cdot\nabla v^N + v^N\cdot\nabla w^N - \Delta v^N = -\nabla p^N, \qquad \operatorname{div} v^N = 0 \tag{3.2.60}$$

in Q_{S_N}, with the initial condition

$$v^N(\cdot,0) = u^{2,N}(\cdot). \tag{3.2.61}$$

From (3.2.53) and the definition of $X_{p_0,\delta}(T_N)$, we see that for $0 < s \leq T_N$:

$$\int\limits_0^s \|w^N(\cdot,\tau)\|^{r_0}_{L_{p_0}} d\tau \leq C(\delta,p_0) s^{\frac{\delta r_0}{2}} (N^{\frac{p_0-p}{p_0}} \|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}})^{r_0}. \tag{3.2.62}$$

Here, $r_0 \in]2,\infty[$ is such that

$$\frac{3}{p_0} + \frac{2}{r_0} = 1.$$

Applying Lemma 3.8 with $v^{(1)} = u$ and $v^{(2)} = w^N$ implies that the following equality holds for $0 \leq s \leq S_N$:

$$\int\limits_{\mathbb{R}^3} w^N(y,s)\cdot u(y,s)dy = \int\limits_{\mathbb{R}^3} u^{1,N}(y)\cdot u_0(y)dy - 2\int\limits_0^s\int\limits_{\mathbb{R}^3} \nabla w^N : \nabla u dy d\tau$$

$$-\int\limits_0^s\int\limits_{\mathbb{R}^3} w^N\otimes v^N : \nabla v^N dy d\tau. \tag{3.2.63}$$

This, along with (3.2.55) and the fact that the energy inequality holds for the weak Leray-Hopf solution u, implies that the following energy inequality holds for $s\in[0,S_N]$:

$$\|v^N(\cdot,s)\|^2_{L_2} + 2\int\limits_0^s\int\limits_{\mathbb{R}^3} |\nabla v^N(x,\tau)|^2 dx d\tau \leq \|u^{2,N}\|^2_{L_2}$$

$$+2\int\limits_0^s\int\limits_{\mathbb{R}^3} w^N\otimes v^N : \nabla v^N dx d\tau. \tag{3.2.64}$$

We may then apply Lemma 3.7 to infer that for $s \in [0, S_N]$:

$$\|v^N(\cdot, s)\|_{L_2}^2 + \int\limits_0^s \int\limits_{\mathbb{R}^3} |\nabla v^N(x,\tau)|^2 dx d\tau \leq \|u^{2,N}\|_{L_2}^2$$

$$+C \int\limits_0^s \|v^N(\cdot,\tau)\|_{L_2}^2 \|w^N(\cdot,\tau)\|_{L_{p_0}}^{r_0} d\tau. \quad (3.2.65)$$

By an application of Gronwall's lemma, we see that for $s \in [0, S_N]$:

$$\|v^N(\cdot, s)\|_{L_2}^2 \leq C\|u^{2,N}\|_{L_2}^2 \exp\Big(C \int\limits_0^s \|w^N(\cdot,\tau)\|_{L_{p_0}}^{r_0} d\tau\Big). \quad (3.2.66)$$

We may then use (3.2.49) and (3.2.62) to infer for $s \in [0, S_N]$:

$$\|v^N(\cdot, s)\|_{L_2}^2 \leq C(p_0, p, \alpha) N^{2-p} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^p \exp\Big(C(\delta, p_0) s^{\frac{\delta r_0}{2}} \Big(N^{\frac{p_0-p}{p_0}} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^{\frac{p}{p_0}}\Big)^{r_0}\Big). \quad (3.2.67)$$

Clearly, for $s \in [0, S_N]$ we have

$$\|u(\cdot,s) - S(s)u_0\|_{L_2}^2 \leq C(\|v^N(\cdot,s)\|_{L_2}^2 + \|S(s)u^{2,N}\|_{L_2}^2 + \|w^N(\cdot,s) - S(s)u^{1,N}\|_{L_2}^2)$$

$$\leq C(\|v^N(\cdot,s)\|_{L_2}^2 + \|u^{2,N}\|_{L_2}^2 + \|w^N(\cdot,s) - S(s)u^{1,N}\|_{L_2}^2).$$

Thus, using this together with (3.2.49), (3.2.56) and (3.2.67), we infer that for $s \in [0, S_N]$:

$$\|u(\cdot,s) - S(s)u_0\|_{L_2}^2$$

$$\leq C(p_0, p, \alpha) N^{2-p} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^p \Big(\exp\Big(C(\delta, p_0) s^{\frac{\delta r_0}{2}} \Big(N^{\frac{p_0-p}{p_0}} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^{\frac{p}{p_0}}\Big)^{r_0}\Big) + 1\Big)$$

$$+\Big(C s^{\frac{1}{p_0-2}+2\delta\theta_0} \Big(N^{\frac{p_0-p}{p_0}} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^{\frac{p}{p_0}}\Big)^{4\theta_0}$$

$$\times \Big(s^{\frac{1}{2\theta_0-1}\left(\frac{1}{p_0-2}+2\delta\theta_0\right)} \Big(N^{\frac{p_0-p}{p_0}} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^{\frac{p}{p_0}}\Big)^{\frac{4\theta_0}{2\theta_0-1}} + \|u_0\|_{L_2}^2\Big)^{2(1-\theta_0)}\Big). \quad (3.2.68)$$

Take $N = t^{-\gamma}$, where $0 < t \leq \min(1, T)$ and $\gamma > 0$. Observing (3.2.52), we see that in order to also satisfy $t \in]0, T_N]$ (and hence $t \in]0, \min(1, S_N)]$) , we should take

$$0 < \gamma < \frac{\delta p_0}{2(p_0 - p)} \quad (3.2.69)$$

and we should consider the additional restriction

$$0 < t < \min\Big(1, T, \Big(\frac{1}{8c}\|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}^{-\frac{p}{p_0}}\Big)^{\frac{2p_0}{\delta p_0 - 2\gamma(p_0-p)}}\Big). \quad (3.2.70)$$

Assuming these restrictions, we see that according to (3.2.68), we then have:

$$\|u(\cdot,t)-S(t)u_0\|_{L_2}^2$$
$$\leq C(p_0,p,\alpha)t^{\gamma(p-2)}\|u_0\|^p_{\dot{B}^{s_{p,\alpha}}_{p,p}}\Big(\exp\Big(C(\delta,p_0)t^{\frac{\delta r_0}{2}}\Big(t^{\frac{-\gamma(p_0-p)}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}}\Big)^{r_0}\Big)+1\Big)$$
$$+\Big(Ct^{\frac{1}{p_0-2}+2\delta\theta_0}\Big(t^{\frac{-\gamma(p_0-p)}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}}\Big)^{4\theta_0}$$
$$\times\Bigg(t^{\frac{1}{2\theta_0-1}\left(\frac{1}{p_0-2}+2\delta\theta_0\right)}\Big(t^{\frac{-\gamma(p_0-p)}{p_0}}\|u_0\|^{\frac{p}{p_0}}_{\dot{B}^{s_{p,\alpha}}_{p,p}}\Big)^{\frac{4\theta_0}{2\theta_0-1}}+\|u_0\|^2_{L_2}\Bigg)^{2(1-\theta_0)}\Big). \tag{3.2.71}$$

Recalling (3.2.69)-(3.2.70), one can now recover the conclusions of Lemma 3.3. □

3.3 Short Time Uniqueness of Weak Leray-Hopf Solutions for Initial Values in VMO^{-1}

3.3.1 Construction of Strong Solutions

The approach we will take to prove Theorem 3.1 is as follows. We construct a weak Leray-Hopf solution, with initial data $u_0\in J(\mathbb{R}^3)\cap\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)\cap VMO^{-1}(\mathbb{R}^3)$, by perturbation methods. We refer to this constructed solution as the 'strong solution'. Then, Lemma 3.3 plays a crucial role in showing that the strong solution has good enough properties to coincide with all weak Leray-Hopf solutions, with the same initial data, on some time interval $]0,T(u_0)[$. With this in mind, we now state the relevant theorem related to the construction of this 'strong solution'.

Let us introduce the necessary preliminaries. The path space $\mathcal{P}_T$ for the mild solutions constructed by Koch and Tataru in [63] is defined to be

$$\mathcal{P}_T:=\{u\in\mathcal{S}^{'}(\mathbb{R}^3\times\mathbb{R}_+):\|u\|_{\mathcal{E}_T}<\infty\}. \tag{3.3.1}$$

Here,

$$\|u\|_{\mathcal{E}_T}:=\sup_{0<t<T}\sqrt{t}\|u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}$$
$$+\sup_{(x,t)\in\mathbb{R}^3\times]0,T[}\Big(\frac{1}{\mu(B(0,\sqrt{t}))}\int\limits_0^t\int\limits_{|y-x|<\sqrt{t}}|u|^2dyds\Big)^{\frac{1}{2}}. \tag{3.3.2}$$

In (3.3.2) and throughout this thesis, μ denotes the Lebesgue measure. It can be shown[6] that the following holds true for any tempered distribution f. Namely,

$$\sup_{0<t<T}\sqrt{t}\|S(t)f\|_{L_\infty(\mathbb{R}^3)}\leq C\sup_{(x,t)\in\mathbb{R}^3\times]0,T[}\Big(\frac{1}{\mu(B(0,\sqrt{t}))}\int\limits_0^t\int\limits_{|y-x|<\sqrt{t}}|S(\tau)f|^2dyd\tau\Big)^{\frac{1}{2}},\tag{3.3.3}$$

with C being a universal constant independent of T. From this and (2.3.27) in Chapter 2, we see that for $0<T\leq\infty$

$$u_0\in BMO^{-1}(\mathbb{R}^3)\Rightarrow\|S(t)u_0\|_{\mathcal{E}_T}\leq C\|u_0\|_{BMO^{-1}}.\tag{3.3.4}$$

For $\varphi\in C_0^\infty(\mathbb{R}^3)$, it can be verified that

$$\|S(t)\varphi\|_{\mathcal{E}_T}\leq C\sqrt{T}\|\varphi\|_{L_\infty(\mathbb{R}^3)}.\tag{3.3.5}$$

Since $C_0^\infty(\mathbb{R}^3)$ is dense in $VMO^{-1}(\mathbb{R}^3)$, we can see from (3.3.4)-(3.3.5) that for $u_0\in VMO^{-1}(\mathbb{R}^3)$

$$\lim_{T\to 0^+}\|S(t)u_0\|_{\mathcal{E}_T}=0.\tag{3.3.6}$$

Recalling the definition of $B(f,g)$ given by (3.2.2), it was shown in [63] that there exists a universal constant C such that for all f, $g\in\mathcal{E}_T$

$$\|B(f,g)\|_{\mathcal{E}_T}\leq C\|f\|_{\mathcal{E}_T}\|g\|_{\mathcal{E}_T}.\tag{3.3.7}$$

Here is the theorem related to the construction of the 'strong solution'. The proof relies on ideas contained in the paper of Lemarié-Rieusset and Prioux [77]. Since the proof is not explicitly contained in [77], we find it beneficial to provide a sketch of certain parts of the proof.

Theorem 3.9. *Suppose that $u_0\in VMO^{-1}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$. There exists a universal constant $\epsilon_0>0$ such that if*

$$\|S(t)u_0\|_{\mathcal{E}_T}<\epsilon_0,\tag{3.3.8}$$

then there exists a $v\in\mathcal{E}_T$, which solves the Navier-Stokes equations in the sense of distributions and satisfies the following properties. The first property is that v solves the following integral equation:

$$v(x,t):=S(t)u_0+B(v,v)(x,t)\tag{3.3.9}$$

[6] A proof can be found in Lemarié-Rieusset's book [76], specifically Lemma 16.1 there.

in Q_T, along with the estimate

$$\|v\|_{\mathcal{E}_T} < 2\|S(t)u_0\|_{\mathcal{E}(T)}. \tag{3.3.10}$$

The second property is that v is a weak Leray-Hopf solution on Q_T.

If $\pi_{v\otimes v}$ is the associated pressure we have (here, $\lambda \in]0,T[$ and $p \in]2,\infty[$):

$$\pi_{v\otimes v} \in L_{\frac{5}{3}}(Q_T) \cap L_{\frac{p}{2},\infty}(Q_{\lambda,T})). \tag{3.3.11}$$

The final property is that for $\lambda \in]0,T[$ and $k = 0,1\ldots$, $l = 0,1\ldots$:

$$\sup_{(x,t)\in Q_{\lambda,T}} |\partial_t^l \nabla^k v| + |\partial_t^l \nabla^k \pi_{v\otimes v}| \leq c(p_0, \lambda, \|u_0\|_{BMO^{-1}}, \|u_0\|_{L_2}, k, l). \tag{3.3.12}$$

Proof. **Step 1: The Mollified Integral Equation**

Let $\omega \in C_0^\infty(B(1))$ be a non-negative function such that

$$\int\limits_{\mathbb{R}^3} \omega(x)dx = 1.$$

Moreover, denote

$$\omega_\epsilon(x) := \frac{1}{\epsilon^3}\omega\Big(\frac{x}{\epsilon}\Big).$$

Applying Young's convolution inequality and the pointwise estimate (1.1.11) in Chapter 1 gives the following:

$$\|K(\cdot,t-\tau)\star(f\otimes(\omega_\epsilon\star g))(\cdot,\tau)\|_{L_2} \leq \|K(\cdot,t-\tau)\|_{L_1}\|f\otimes(\omega_\epsilon\star g)\|_{L_2}$$

$$\leq C(t-\tau)^{-\frac{1}{2}}\|f\otimes(\omega_\epsilon\star g)\|_{L_2} \leq C(t-\tau)^{-\frac{1}{2}}\frac{\|f\|_{\mathcal{E}_T}\|g\|_{L_\infty(0,T;L_2)}}{\tau^{\frac{1}{2}}}. \tag{3.3.13}$$

One can then show that

$$\|B(f,(\omega_\epsilon\star g))\|_{L_\infty(0,T;L_2)} \leq C\|f\|_{\mathcal{E}_T}\|g\|_{L_\infty(0,T;L_2)}. \tag{3.3.14}$$

From the paper of Lemarié-Rieusset and Prioux (specifically, p.289 of [77]), it is seen that $\mathcal{E}_T$ is preserved by the operation of mollification:

$$\|\omega_\epsilon\star g\|_{\mathcal{E}_T} \leq C'\|g\|_{\mathcal{E}_T}. \tag{3.3.15}$$

Here, C' is independent of T and ϵ. Using this and (3.3.7), we obtain:

$$\|B(f,(\omega_\epsilon\star g))\|_{\mathcal{E}_T} \leq C\|f\|_{\mathcal{E}_T}\|g\|_{\mathcal{E}_T}. \tag{3.3.16}$$

We briefly describe successive approximations. For $n = 1,2,\cdots$ let $v^{(0)} = S(t)u_0$,

$$v^{(n+1)} = v^{(0)} + B(v^{(n)},(\omega_\epsilon\star v^{(n)})).$$

Moreover for $n = 0, 1, \dots$ define:

$$M^{(n)} := \|v^{(n)}\|_{\mathcal{E}_T} \tag{3.3.17}$$

and

$$K^{(n)} := \|v^{(n)}\|_{L_\infty(0,T;L_2)}. \tag{3.3.18}$$

Then using (3.3.14) and (3.3.16), we have the following iterative relations:

$$M^{(n+1)} \leq M^{(0)} + C(M^{(n)})^2 \tag{3.3.19}$$

and

$$K^{(n+1)} \leq K^{(0)} + CM^{(n)}K^{(n)}. \tag{3.3.20}$$

If

$$4CM^{(0)} < 1, \tag{3.3.21}$$

then one can show that for $n = 1, 2 \cdots$ we have

$$M^{(n)} < 2M^{(0)} \tag{3.3.22}$$

and

$$K^{(n)} < 2K^{(0)}. \tag{3.3.23}$$

Using (3.3.19)-(3.3.23) and arguing as in Kato's paper [59], we see that there exists a $v^\epsilon \in L_\infty(0, T; L_2) \cap \mathcal{E}_T$ such that

$$\lim_{n\to\infty} \|v^n - v^\epsilon\|_{\mathcal{E}_T} = 0, \tag{3.3.24}$$

$$\lim_{n\to\infty} \|v^n - v^\epsilon\|_{L_\infty(0,T;L_2)} = 0 \tag{3.3.25}$$

and v^ϵ solves the integral equation

$$v^\epsilon(x, t) = S(t)u_0 + B(v^\epsilon, (\omega_\epsilon \star v^\epsilon))(x, t) \tag{3.3.26}$$

in Q_T. Define

$$\pi_{v^\epsilon\otimes(\omega_\epsilon\star v^\epsilon)} := \mathcal{R}_i\mathcal{R}_j(v_i^\epsilon(\omega_\epsilon \star v^\epsilon)_j),$$

where $\mathcal{R}_i$ denotes the Riesz transform and repeated indices are summed. One can readily show that on Q_T, $(v^\epsilon, \pi_{v^\epsilon\otimes(\omega_\epsilon\star v^\epsilon)})$ are solutions to the mollified Navier-Stokes system:

$$\partial_t v^\epsilon - \Delta v^\epsilon + \operatorname{div}(v^\epsilon \otimes (\omega_\epsilon \star v^\epsilon)) = -\nabla\pi_{v^\epsilon\otimes(\omega_\epsilon\star v^\epsilon)}, \tag{3.3.27}$$

$$\operatorname{div} v^\epsilon = 0, \tag{3.3.28}$$

$$v^\epsilon(\cdot,0)=u_0(\cdot). \tag{3.3.29}$$

We can also infer that $v^\epsilon\in W^{1,0}_2(Q_T)\cap C([0,T];J(\mathbb{R}^3))$, along with the energy equality

$$\|v^\epsilon(\cdot,t)\|^2_{L_2}+\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla v^\epsilon(x,s)|^2dxds=\|u_0\|^2_{L_2}. \tag{3.3.30}$$

Step 2: Passing to the Limit as $\epsilon\to 0$

Since $u_0\in VMO^{-1}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$, it is known from [77] (specifically, Theorem 3.5 there) that there exists a $v\in\mathcal{E}_T$ such that

$$\lim_{\epsilon\to 0}\|v^\epsilon-v\|_{\mathcal{E}_T}=0 \tag{3.3.31}$$

with v satisfying the integral equation (3.3.9) in Q_T. Using arguments from Leray's paper [75] and (3.3.27)-(3.3.30), we see that v^ϵ will converge to a weak Leray-Hopf solution on Q_T with initial data u_0. Thus, $v\in\mathcal{E}_T$ is a weak Leray-Hopf solution.

Since v is a weak Leray-Hopf solution on Q_T, Lebesgue interpolation and the Sobolev inequality imply that

$$v\in L_{\frac{5}{3}}(Q_T). \tag{3.3.32}$$

Using (3.3.10), $v\in L_{2,\infty}(Q_T)$ and Lebesgue interpolation also gives that

$$v\in L_{p,\infty}(Q_{\lambda,T})\ \text{ for }\ 2\leq p\leq\infty\ \text{ and }\ 0<\lambda<T. \tag{3.3.33}$$

We deduce (3.3.11) by using (3.3.32)-(3.3.33), along with the fact that $\pi_{v\otimes v}$ is a composition of Riesz transforms acting on $v\otimes v$.

Using (3.3.10) and the singular integral representation of $\pi_{v\otimes v}$, one can deduce (3.3.12). The arguments showing this are identical to those presented in Seregin's book [111] (specifically, the proof of Proposition 3.9, pp.160-162). □

3.3.2 Proof of Theorem 3.1

Now we demonstrate that the decay estimate of Lemma 3.3, together with properties of the strong solution in Theorem 3.9, imply weak-strong uniqueness. Let us mention that Dong and Zhang [25] have previously made the connection between decay estimates of the same type as in Lemma 3.3 and weak-strong uniqueness for the Navier-Stokes equations. However, in Dong and Zhang's paper [25], an additional restriction is imposed, requiring that the initial data has positive Sobolev regularity. The arguments we present below are based upon those found in Dong and Zhang's paper [25].

Proof. Let us now consider any other weak Leray-Hopf solution u, defined on Q_∞ and with initial data $u_0 \in J(\mathbb{R}^3) \cap \dot{B}^{s_p,\alpha}_{p,p}(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$. Let $\widehat{T}(u_0)$ be such that

$$\|S(t)u_0\|_{\mathcal{E}_{\widehat{T}(u_0)}} < \epsilon_0,$$

where ϵ_0 is from (3.3.8) of Theorem 3.9. Consider $0 < T < \widehat{T}(u_0)$, where T is to be determined. Let $v : Q_T \to \mathbb{R}^3$ be as in Theorem 3.9. From (3.3.10), we have

$$\|v\|_{\mathcal{E}_T} < 2\|S(t)u_0\|_{\mathcal{E}_T} \le 2\|S(t)u_0\|_{\mathcal{E}_{\widehat{T}(u_0)}} < 2\epsilon_0. \tag{3.3.34}$$

We define

$$w = u - v \in W^{1,0}_2(Q_T) \cap C_w([0,T]; J(\mathbb{R}^3)). \tag{3.3.35}$$

Moreover, w satisfies the following equations

$$\partial_t w + w\cdot\nabla w + v\cdot\nabla w + w\cdot\nabla v - \Delta w = -\nabla q, \qquad \operatorname{div} w = 0 \tag{3.3.36}$$

in Q_T, with the initial condition satisfied in the strong L_2 sense:

$$\lim_{t\to 0^+} \|w(\cdot,0)\|_{L_2} = 0. \tag{3.3.37}$$

From the definition of $\mathcal{E}_T$, we have that $v \in L_\infty(Q_{\delta,T})$ for $0 < \delta < s \le T$.

Applying Lemma 3.8 with $v^{(1)} = u$ and $v^{(2)} = v$ implies that the following equality holds for $t \in [\delta, T]$:

$$\int\limits_{\mathbb{R}^3} u(y,t)\cdot v(y,t)dy = \int\limits_{\mathbb{R}^3} u(y,\delta)\cdot v(y,\delta)dy - 2\int\limits_\delta^t\int\limits_{\mathbb{R}^3} \nabla u : \nabla v dyd\tau$$
$$-\int\limits_\delta^t\int\limits_{\mathbb{R}^3} v\otimes w : \nabla w dyd\tau. \tag{3.3.38}$$

Observing Lemma 3.7 and (3.3.10), we see that

$$\int\limits_\delta^t\int\limits_{\mathbb{R}^3} |v||w||\nabla w| dyd\tau \le C\int\limits_\delta^t \|v(\cdot,\tau)\|^2_{L_\infty}\|w(\cdot,\tau)\|^2_{L_2}d\tau$$
$$+\frac{1}{2}\int\limits_\delta^t\int\limits_{\mathbb{R}^3} |\nabla w|^2 dyd\tau$$
$$\le C\|v\|^2_{\mathcal{E}_T}\int\limits_\delta^t \frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau}d\tau + \frac{1}{2}\int\limits_\delta^t\int\limits_{\mathbb{R}^3} |\nabla w|^2 dyd\tau. \tag{3.3.39}$$

The main point[7] now is that Lemma 3.3, applied to both u and v, implies the following for $w = u - v$. Namely, there exist

$$\beta(p, \alpha) > 0$$

and

$$\gamma(\|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)}, p, \alpha) > 0$$

such that for $0 < t < \min(1, T, \gamma)$:

$$\|w(\cdot,t)\|^2_{L_2} \leq t^\beta c(p, \alpha, \|u_0\|_{L_2(\mathbb{R}^3)}, \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)}). \tag{3.3.40}$$

Hence,

$$\sup_{0<t<T} \frac{\|w(\cdot,t)\|^2_{L_2}}{t^\beta} < \infty. \tag{3.3.41}$$

This, together with (3.3.39), allows us to take $\delta \to 0$ in (3.3.38). In doing so, one obtains that for any $t \in [0, T]$ we have

$$\int\limits_{\mathbb{R}^3} u(y,t)\cdot v(y,t)dy = \|u_0\|^2_{L_2} - 2\int\limits_0^t \int\limits_{\mathbb{R}^3} \nabla u : \nabla v dy d\tau - \int\limits_0^t \int\limits_{\mathbb{R}^3} v \otimes w : \nabla w dy d\tau. \tag{3.3.42}$$

From this and the fact that u and v satisfy the energy inequality (1.0.18), one can deduce that, for any $t \in [0, T]$,

$$\|w(\cdot,t)\|^2_{L_2} + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla w|^2 dy d\tau \leq 2\int\limits_0^t \int\limits_{\mathbb{R}^3} v \otimes w : \nabla w dy d\tau. \tag{3.3.43}$$

Observing (3.3.39) and (3.3.41), one concludes that for $t \in [0, T]$:

$$\begin{aligned}\|w(\cdot,t)\|^2_{L_2} &\leq C\|v\|^2_{\mathcal{E}_T} \int\limits_0^t \frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau} d\tau \\ &\leq \frac{C}{\beta} t^\beta \|v\|^2_{\mathcal{E}_T} \sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau^\beta}\Big).\end{aligned} \tag{3.3.44}$$

From the above and (3.3.34), it can be inferred that

$$\sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau^\beta}\Big) \leq \frac{C'}{\beta}\|S(t)u_0\|^2_{\mathcal{E}_T} \sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau^\beta}\Big). \tag{3.3.45}$$

Using (3.3.6), we see that we can choose $0 < T = T(u_0) < \widehat{T}(u_0)$ such that

$$\|S(t)u_0\|_{\mathcal{E}_T} \leq \min\Big(\frac{\beta}{2C'}\Big)^{\frac{1}{2}}.$$

With this choice of $T(u_0)$, it immediately follows that $w = 0$ on $Q_{T(u_0)}$. $\square$

[7] This observation has been previously made on p.2420 of Dong and Zhang's paper [25], in the context of weak Leray-Hopf solutions with initial data in $VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \cap H^\beta(\mathbb{R}^3)$ ($\beta > 0$).

3.3.3 Proof of Corollary 3.2

Proof. Since $3 < p < \infty$, it is clear that there exists an $\alpha := \alpha(p)$ such that $2 < \alpha < 3$ and

$$\alpha < p < \frac{\alpha}{3-\alpha}. \tag{3.3.46}$$

With this p and α, we may apply Proposition 2.9 (Chapter 2) with $s_1 = -\frac{3}{2} + \frac{3}{p}$, $s_2 = -1 + \frac{3}{p}$ and $\theta = 6\Big(\frac{1}{\alpha} - \frac{1}{3}\Big)$. In particular this gives for any $u_0 \in \mathcal{S}_h'$:

$$\|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,1}} \leq c(p,\alpha)\|u_0\|^{6(\frac{1}{\alpha}-\frac{1}{3})}_{\dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,\infty}}\|u_0\|^{6(\frac{1}{2}-\frac{1}{\alpha})}_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}}.$$

Here, $\mathcal{S}_h'$ is given by (2.3.25) in Chapter 2. From Remark 2.7 in 'Chapter 2: Preliminary Material', we see that $\dot{B}^{s_{p,\alpha}}_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3)$ and $L_2(\mathbb{R}^3) \hookrightarrow \dot{B}^{0}_{2,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$. Thus, we have the inclusion

$$\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \subset \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap J(\mathbb{R}^3). \tag{3.3.47}$$

Using this, along with the inclusion $\dot{\mathbb{B}}^{s_p}_{p,\infty}(\mathbb{R}^3) \hookrightarrow VMO^{-1}(\mathbb{R}^3)$, it can be deduced that

$$\dot{\mathbb{B}}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \subset VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap J(\mathbb{R}^3). \tag{3.3.48}$$

From (3.3.46) and (3.3.48), we infer that the conclusions of Corollary 3.2 are an immediate consequence of Theorem 3.1. $\square$

Remark 3.10. *From (3.3.47), we see that the conclusions of Lemma 3.3 apply if* $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap J(\mathbb{R}^3)$. *Using this we claim that the assumptions of Corollary 3.2 can be weakened. Namely, there exists a small* $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(p)$ *such that for all* $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \cap J(\mathbb{R}^3)$ *with*

$$\sup_{0<t<T} t^{\frac{-s_p}{2}}\|S(t)u_0\|_{L_p} < \tilde{\epsilon}_0, \tag{3.3.49}$$

we have the following implication. Specifically, all weak Leray-Hopf solutions on Q_∞*, with initial data* u_0*, coincide on* Q_T*.*

Indeed taking any fixed u_0 *in this class and taking* $\tilde{\epsilon}_0$ *sufficiently small, we may argue*[8] *in an identical way as in the proof of Theorem 3.6 (setting* $\delta = 0$*). Consequently, there exists a weak Leray-Hopf solution* v *on* Q_T *such that*

$$\sup_{0<t<T} t^{\frac{-s_p}{2}}\|v(\cdot,t)\|_{L_p} < 2\tilde{\epsilon}_0. \tag{3.3.50}$$

Using this and Lemma 3.3, we may argue in a similar way to the proof of Theorem 3.1 to obtain the desired conclusion.

[8]For this case, we also refer the reader to arguments presented in Cannone's paper [19].

3.4 Uniqueness Criterion for Weak Leray-Hopf Solutions

Now let us state two known facts, which will be used in the proof of Proposition 3.4. The first fact is that if v is a weak Leray-Hopf solution on Q_∞, with initial data $u_0 \in J(\mathbb{R}^3)$, then v satisfies the integral equation in Q_∞:

$$v(x,t) = S(t)u_0 + B(v,v)(x,t). \tag{3.4.1}$$

The second fact is as follows. Consider $3 < p < \infty$ and $2 < q < \infty$ such that $3/p + 2/q = 1$. Then there exists a constant $C = C(p,q)$ such that for all $f, g \in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))$

$$\|B(f,g)\|_{L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))} \leq C\|f\|_{L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))}\|g\|_{L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))}. \tag{3.4.2}$$

These statements and their corresponding proofs, can be found in Lemarié-Rieusset's book [76][9] and the paper of Lemarié-Rieusset and Prioux [77][10], for example.

3.4.1 Proof of Proposition 3.4

Let us first state and prove a simple lemma, which we will make use of in proving Proposition 3.4.

Lemma 3.11. *Let $f :]0,T[\to]0,\infty[$ be a function satisfying the following property. Suppose that there exists a $C \geq 1$ such that for any $0 < t_0 \leq t_1 < T$ we have*

$$f(t_1) \leq Cf(t_0). \tag{3.4.3}$$

In addition, assume that for some $1 \leq r < \infty$:

$$f \in L^{r,\infty}(0,T). \tag{3.4.4}$$

Then one can conclude that for all $t \in]0,T[$:

$$f(t) \leq \frac{2C\|f\|_{L^{r,\infty}(0,T)}}{t^{\frac{1}{r}}}. \tag{3.4.5}$$

Proof. It suffices to prove that if f satisfies the hypothesis of Lemma 3.11, along with the additional constraint

$$\|f\|_{L^{r,\infty}(0,T)} = \frac{1}{2}, \tag{3.4.6}$$

[9]Specifically, Theorem 11.2 of [76].

[10]Specifically, Lemma 6.1 of [77].

then we must necessarily have that for any $0 < t < T$

$$f(t) \leq \frac{C}{t^{\frac{1}{r}}}. \tag{3.4.7}$$

The assumption (3.4.6), together with the the definition of the $L^{r,\infty}(0,T)$ norm in Chapter 2 (specifically (2.2.3) with $p = r$), implies that

$$\sup_{\alpha>0} \alpha^r \mu(\{s \in]0,T[\text{ such that } f(s) > \alpha\}) < 1. \tag{3.4.8}$$

Fixing $t \in]0,T[$ and setting $\alpha = \frac{1}{t^{\frac{1}{r}}}$, we see that

$$\mu(\{s \in]0,T[\text{ such that } f(s) > 1/t^{\frac{1}{r}}\}) < t. \tag{3.4.9}$$

For $0 < t_0 \leq t$, we have

$$f(t) \leq Cf(t_0). \tag{3.4.10}$$

This, together with (3.4.9), implies (3.4.7). □

Proof of Proposition 3.4

Case 1: u satisfies (3.1.11)-(3.1.13)

In this case, we see that the facts mentioned in the previous paragraph imply

$$\|S(t)u_0\|_{L^{q,\infty}(0,T;L^{p,\infty})} \leq \epsilon_* + C\epsilon_*^2. \tag{3.4.11}$$

For $0 < t_0 < t_1$, we may apply O'Neil's convolution inequality (Proposition 2.1 in 'Chapter 2: Preliminary Material'), to infer that $S(t_1)u_0 = S(t_1 - t_0)(S(t_0)u_0)$ satisfies

$$\|S(t_1)u_0\|_{L^{p,\infty}} \leq 3p\|S(t_0)u_0\|_{L^{p,\infty}}. \tag{3.4.12}$$

This, in conjunction with (3.4.11), allows us to apply Lemma 3.11 to obtain that for $0 < t < T$

$$\|S(t)u_0\|_{L^{p,\infty}} \leq \frac{6p(\epsilon_* + C\epsilon_*^2)}{t^{\frac{1}{q}}}. \tag{3.4.13}$$

A further application of O'Neil's convolution inequality gives

$$\|S(t)u_0\|_{L_{2p}} \leq \frac{C'(p)\|S(t/2)u_0\|_{L^{p,\infty}}}{t^{\frac{3}{4p}}}. \tag{3.4.14}$$

This and (3.4.13) implies that for $0 < t < T$ we have

$$\sup_{0<t<T} t^{\frac{-s_{2p}}{2}}\|S(t)u_0\|_{L_{2p}} \leq C''(p,q)(\epsilon_* + \epsilon_*^2). \tag{3.4.15}$$

Recalling that $u_0 \in J(\mathbb{R}^3)$ ($\subset \mathcal{S}_h^{'}$, where $\mathcal{S}_h^{'}$ is given by (2.3.25) in Chapter 2), we see that by Young's convolution inequality

$$t^{\frac{-s_{2p}}{2}}\|S(t)u_0\|_{L_{2p}} \leq \frac{C'''(p)\|u_0\|_{L_2}}{t^{\frac{1}{4}}}.$$

From this and (3.4.15), we deduce that

$$u_0 \in J(\mathbb{R}^3) \cap \dot{B}^{s_{2p}}_{2p,\infty}(\mathbb{R}^3). \tag{3.4.16}$$

Using (3.4.15)-(3.4.16) and Remark 3.10, once reaches the desired conclusion provided ϵ_* is sufficiently small.

Case 2: u satisfies (3.1.9)-(3.1.10)

The assumptions (3.1.9)-(3.1.10) imply that $u \in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))$ with

$$\lim_{S\to 0}\|u\|_{L^{q,\infty}(0,S;L^{p,\infty})} = 0. \tag{3.4.17}$$

From Sohr's paper [117] (specifically, Theorem 1.1 of [117]), we have that

$$u \in L_\infty(\mathbb{R}^3\times]T',T[) \text{ for any } 0 < T' < T. \tag{3.4.18}$$

Now (3.4.17), allows us to reduce to case 1 on some time interval. This observation, combined with (3.4.18), enables us to deduce that u is unique on Q_T amongst all other weak Leray-Hopf solutions with the same initial value. $\square$

Chapter 4

Local Regularity for the Navier-Stokes Equations in Critical Lorentz Spaces

4.1 Introduction

In ‘Chapter 1: Introduction’, we discussed that for weak Leray-Hopf solutions, with certain classes of initial data, ruling out singular points[1] implies uniqueness. In this chapter, we will state and prove new criteria that are sufficient to rule out interior singular points and singular points on the flat part of the boundary. The results and text base of this chapter are based on

> Barker, T. Local boundary regularity for the Navier-Stokes equations in nonendpoint borderline Lorentz spaces. J Math Sci (2017). doi:10.1007/s10958-017-3424-2

Unless otherwise mentioned, the notation and definitions used in this chapter are contained in ‘Chapter 2: Preliminary Material’.

First let us recall from Chapter 1 the definition of suitable weak solutions to the Navier-Stokes equations, which is the modern setting of the local interior regularity theory. This definition can be found in Seregin’s book [111][2], for example.

Definition 4.1. *Consider any domain $\mathcal{O}\subseteq\mathbb{R}^3$. Let $-\infty<S_1<S_2<\infty$. We say that the pair $v:\mathcal{O}\times]S_1,S_2[\rightarrow\mathbb{R}^3$ and $q:\mathcal{O}\times]S_1,S_2[\rightarrow\mathbb{R}$ are a suitable weak solution to the Navier-Stokes equations in $\mathcal{O}\times]S_1,S_2[$ if:*

$$v\in L_\infty(S_1,S_2;L_2(\mathcal{O})), \tag{4.1.1}$$

[1]Recall we say that $z_0=(x_0,t_0)$ is a singular point of v if $v\notin L_\infty(B(x_0,r)\times]t_0-r^2,t_0[)$ for any $r>0$. Any point which is not singular is defined to be a regular point of v.

[2]Specifically, Definition 6.1 p.133 of [111].

$$\nabla v \in L_2(S_1, S_2; L_2(\mathcal{O})), \tag{4.1.2}$$

$$q \in L_{\frac{3}{2}}(\mathcal{O}\times]S_1, S_2[) \tag{4.1.3}$$

and

$$\int\limits_{S_1}^{S_2}\int\limits_{\mathcal{O}}(-v\cdot\partial_t\varphi - v\otimes v:\nabla\varphi + \nabla v:\nabla\varphi - q\,div\,\varphi)dyd\tau = 0 \tag{4.1.4}$$

for any $\varphi \in C_0^\infty(\mathcal{O}\times]S_1, S_2[)$. *Additionally, it is required that the following local energy inequality holds for all positive* $\varphi \in C_0^\infty(\mathcal{O}\times]S_1, \infty[)$ *and almost every time* $t \in]S_1, S_2[$*:*

$$\int\limits_{\mathcal{O}}\varphi|v(x,t)|^2dx + 2\int\limits_0^t\int\limits_{\mathcal{O}}\varphi|\nabla v|^2dxdt'$$
$$\leq \int\limits_0^t\int\limits_{\mathcal{O}}|v|^2(\partial_t\varphi + \Delta\varphi) + v\cdot\nabla\varphi(|v|^2 + 2q)dxdt'. \tag{4.1.5}$$

For a suitable weak solution (v, q) as in the above definition, points of $\mathcal{O}\times]S_1, S_2[$ can be classified as either singular points or regular points. In particular,

$z_0 = (x_0, t_0) \in \mathcal{O}\times]S_1, S_2[$ is a singular point of $v \iff$

$v \notin L_\infty((B(x_0, r)\times]t_0 - r^2, t_0[) \cap (\mathcal{O}\times]S_1, S_2[))$ for all sufficiently small $r > 0$.

Moreover, $z_0 \in \mathcal{O}\times]S_1, S_2[$ is defined to be a regular point of v if it is not a singular point of v.

In Chapter 1, we had mentioned that the work of Struwe in [122] implies that a suitable weak solution $v : Q(1) \to \mathbb{R}^3$ is bounded on interior subdomains of $B(1)\times]-1, 0[$ under the assumption that it satisfies the Ladyzhenskaya-Prodi-Serrin condition:

$$v \in L_{s,l}(Q(1)) := L_l(-1, 0; L_s(B)) \text{ with}$$
$$\frac{3}{s} + \frac{2}{l} = 1 \text{ and } s > 3. \tag{4.1.6}$$

On p.2110 of Seregin's paper [103], it was mentioned that the assumption on the space-time norm in (4.1.6) implies that for any point $z_0 = (x_0, t_0) \in Q(1)$ and $\epsilon > 0$ there exists an $r > 0$ such that

$$\|v\|_{L_l(t_0 - r^2, t_0; L_s(B(x_0)))} < \epsilon.$$

As mentioned on p.214 of Escauriaza, Seregin and Šverák's paper [27], such a shrinking property does not hold for the case $s = 3$. It was first established by Kozono and

Sohr in [68] that if a weak Leray-Hopf solution v belongs to $L_\infty(0,T;L_3(\mathbb{R}^3))$, then it is unique on $]0,T[$ amongst all other weak Leray-Hopf solutions with the same initial data. Regularity of a suitable weak solution v belonging to the endpoint case $s=3$ of the Ladyzhenskaya-Prodi-Serrin criterion ($v\in L_\infty(-1,0;L_3(B))$) was proven by Escauriaza, Seregin and Šverák in [27] by reductio ad absurdum. There, a suitable rescaling and limiting procedure was performed and gives a nontrivial ancient solution to the Navier-Stokes equations in $\mathbb{R}^3\times]-\infty,0[$. The contradiction is then obtained in [27], by proving a Liouville type theorem involving backward uniqueness and unique continuation for a certain class of parabolic operators. As discussed by Seregin and Šverák in [102] (p.173 of [102]), the proof in [27] takes advantage of some implicit smallness properties of $L_\infty(-1,0;L_3(B))$. Namely, that if $v\in L_\infty(-1,0;L_3(B))$ is a suitable weak solution, then at each time $t\in[-1,0]$ we have $v(\cdot,t)\in L_3(B)$ and moreover, for every $x_0\in B$:

$$\lim_{r\to 0^+}\|v(\cdot,t)\|_{L_3(B(x_0,r)\cap B)}=0. \tag{4.1.7}$$

The shrinking property (4.1.7) is used in the proof by contradiction by Escauriaza, Seregin and Šverák in showing that the nontrivial ancient solution vanishes at the last moment of time. This is crucial for the subsequent application of a Liouville type theorem based on backward uniqueness of parabolic operators.

Consider a normed space $(\mathcal{X}(B),\|\cdot\|_{\mathcal{X}(B)})$, consisting of measurable functions $u:B\to\mathbb{R}^3$ such that

$$u\in\mathcal{X}(B)\Rightarrow u_{0\lambda}(x):=\lambda u_0(\lambda x)\in\mathcal{X}(B(1/\lambda))\text{ and }\|u_{0\lambda}\|_{\mathcal{X}(B(1/\lambda))}=\|u_0\|_{\mathcal{X}(B)}.$$

Such a space $\mathcal{X}(B)$ will be referred to as critical. For such a space $\mathcal{X}(B)$, it is not difficult to show that $v:B(1)\times]-1,0[\to\mathbb{R}^3$ and $v_\lambda(x,t):=\lambda v(\lambda x,\lambda^2 t)$ ($\lambda>0$) satisfy

$$\|v_\lambda\|_{L_\infty(-\lambda^2,0;\mathcal{X}(B(1/\lambda)))}=\|v\|_{L_\infty(-1,0;\mathcal{X}(B(1))}.$$

Thus, $F(v):=\|v\|_{L_\infty(-1,0;\mathcal{X}(B))}$ is a scale-invariant quantity for the velocity field.

A natural question is whether Escauriaza, Seregin and Šverák's local interior regularity criterion can be extended. This motivates the following question:

> **What are the critical spaces $\mathcal{X}(B)$, with $L_3(B)\subsetneq\mathcal{X}(B)$ and $L_3(B)\hookrightarrow\mathcal{X}(B)$, such that the following holds:**
> **(v,q) is a suitable weak solution to the Navier-Stokes system on $B\times]-1,0[$ with $v\in L_\infty(-1,0;\mathcal{X}(B))\Rightarrow(0,0)$ is a regular point of v?**

In Phuc's paper [87], the following is proven. We will sometimes refer to this as 'Phuc's Theorem'.

Theorem 1.5 of [87]. *Let (v, q) be a suitable weak solution to the Navier-Stokes equations on $B\times]-1,0[$. Then*

$$v \in L_\infty(-1,0;L^{3,\beta}(B)) \;\; (3 \le \beta < \infty) \;\; \textit{and} \;\; q \in L_2(-1,0;L_1(B)) \tag{4.1.8}$$

$\Rightarrow$ *v is Hölder continuous on the closure of $Q(1/2)$.*

As mentioned in 'Chapter 2: Preliminary Material', we have that $L^3 = L^{3,3} \hookrightarrow L^{3,q_1} \hookrightarrow L^{3,q_2}$ for $3 < q_1 < q_2 \le \infty$ and these inclusions are known to be strict. It is stated by Phuc in [87] (specifically p.743) that 'Lorentz spaces can be used to capture logarithmic singularities.'. In [87], Phuc then provides evidence for this observation by means of the following example. Namely, for any $1 > q > 0, \beta > 3$ we have

$$|x|^{-1}|\log(|x|^{-1})|^{-q}\chi_{|x|<1}(x) \in L^{3,\beta}(\mathbb{R}^3) \text{ if and only if } \beta > \frac{1}{q}. \tag{4.1.9}$$

The proof of Theorem 1.5 of [87] has the same structure as Escauriaza, Seregin and Šverák's proof of the $L_\infty(-1,0;L_3(B))$ interior regularity criterion. In particular Phuc's proof is by contradiction and a rescaling procedure produces a nontrivial ancient solution. Furthermore, Phuc applies a Liouville type theorem to this ancient solution, based on backward uniqueness and unique continuation of certain classes of parabolic operators. The main features of Phuc's proof that enable an extension of Escauriaza, Seregin and Šverák's interior regularity criterion are as follows.

Phuc's first observation is that the rescaled solutions

$$(v^{(k)}(y,s), q^{(k)}(y,s)) := (\lambda_k v(\lambda_k x, \lambda_k^2 t), \lambda_k^2 q(\lambda_k x, \lambda_k^2 t)) \tag{4.1.10}$$

used in the contradiction argument, satisfy the following estimates on $\Omega := \omega\times]a,b[$ ($\omega \subset \mathbb{R}^3$ and $-\infty < a < b \le 0$) for $k \ge k_0(\Omega)$. Namely,

$$\|v^{(k)}\|_{L_\infty(a,b;L^{3,\beta}(\omega))} \le \|v\|_{L_\infty(-1,0;L^{3,\beta}(B))} \tag{4.1.11}$$

and

$$\begin{aligned}\|v^{(k)}\|_{L_\infty(a,b;L_2(\omega))} + \|\nabla v^{(k)}\|_{L_2(a,b;L_2(\omega))} + \|q^{(k)}\|_{L_2(a,b;L_{\frac{6}{5}}(\omega))} \le \\ \le C(\Omega, \|q\|_{L_2(-1,0;L_1(B))}, \|v\|_{L_\infty(-1,0;L^{3,\beta}(B))}).\end{aligned} \tag{4.1.12}$$

These estimates play a crucial role in providing sufficient compactness for the rescaled solutions to converge to a nontrivial ancient solution to the Navier-Stokes equations.

The second key observation in Phuc's paper [87], that enables the extension of Escauriaza, Seregin and Šverák's interior regularity criterion, is the following ε-regularity criterion.

Proposition 3.2 of [87]. *There exist universal constants* ϵ_1 *and* C_k*,* $k = 0, 1, \dots$ *such that the following holds. Suppose that* (v, q) *is a suitable weak solution to the Navier-Stokes equations on* $B(1)\times]-1,0[$ *and satisfies the smallness condition*

$$\int_{-1}^{0} \|v(\cdot, s)\|^4_{L_{\frac{12}{5}}(B)} + \|q(\cdot, s)\|^2_{L_{\frac{6}{5}}(B)} ds < \epsilon_1. \tag{4.1.13}$$

Then $\nabla^k v$ *is Hölder continuous in the closure of* $Q(1/2) := B(1/2)\times]-1/4, 0[$ *and satisfies*

$$\sup_{(x,t)\in\bar{Q}(1/2)} |\nabla^k u(x,t)| \leq C_k. \tag{4.1.14}$$

The role of this ε-regularity criterion, in the proof of Phuc's theorem, is to obtain sufficient decay of the nontrivial ancient solution $(v_\infty, q_\infty) \in L_\infty(0, \infty; L^{3,\beta}(\mathbb{R}^3)) \times L_\infty(0, \infty; L^{\frac{3}{2},\frac{\beta}{2}}(\mathbb{R}^3))$ in order to apply backward uniqueness and unique continuation theorems for parabolic operators, developed by Escauriaza, Seregin and Šverák in [27].

Recall from the previous paragraphs that the additional assumption on the pressure, namely

$$q \in L_2(-1, 0; L_1(B)), \tag{4.1.15}$$

plays a key role in the proof of Phuc's Theorem. In particular this assumption appears in the estimate (4.1.12). This condition on the pressure appears restrictive with regards to the range of situations for which Phuc's Theorem applies. For example, consider the case of a weak Leray-Hopf solution $v : B(1)\times]0,1[\to \mathbb{R}^3$ to the Navier-Stokes system. For that case, it is known[3] that there is an associated pressure q such that for any $0 < \delta < 1$, $\nabla q \in L_{\frac{3}{2}}(\delta, 1; L_{\frac{9}{8}}(B(1)))$. However, to the best of the author's knowledge, it is unknown if conditions of the type (4.1.15) hold for the pressure q.

The first result of this chapter aims to prove that an analogous interior regularity result to Phuc's theorem holds for a suitable weak solution (v, q), without the additional restriction (4.1.15) on the time integrability of the pressure.

Theorem 4.2. *Assume* (v, q) *is a suitable weak solution of the Navier-Stokes system in* $Q(1) := B(1)\times]-1, 0[$*. Assume, in addition, that there exists* $3 \leq \beta < \infty$ *such that*

$$v \in L_\infty(-1, 0; L^{3,\beta}(B)). \tag{4.1.16}$$

Then v *is Hölder continuous in the closure of the set* $Q(1/2) := B(1/2)\times]-1/4, 0[$.

[3]See, for example, Theorem 1.1 of [74]

The contradiction argument used in this chapter to prove Theorem 4.2 is in the same spirit as that used in Escauriaza, Seregin and Šverák's paper [27] and subsequently in Phuc's paper [87]. What enables us to extend these results is the development of new estimates involving the following scale-invariant quantities:

$$A(z_0,r;v) := \sup_{t_0-r^2\leq t\leq t_0} r^{-1}\int\limits_{B(x_0,r)} |v(x,t)|^2 dx, \tag{4.1.17}$$

$$B(z_0,r;v) := r^{-1}\int\limits_{t_0-r^2}^{t_0}\int\limits_{B(x_0,r)} |\nabla v(x,t)|^2 dxdt, \tag{4.1.18}$$

$$C_\infty(z_0,r;v) := r^{-2}\int\limits_{t_0-r^2}^{t_0} \|v\|^3_{L^{3,\infty}(B(x_0,r))} dt, \tag{4.1.19}$$

$$D_\infty(z_0,r;q) := r^{-2}\int\limits_{t_0-r^2}^{t_0} \|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(x_0,r))} dt. \tag{4.1.20}$$

These new estimates are stated below as a lemma.

Lemma 4.3. *Let (v,q) be a suitable weak solution in $Q(z_0,1)$. Then for $0<r<1$ the following holds (c is some universal constant):*

$$A(z_0,r/2;v)+B(z_0,r/2;v)\leq c(C_\infty(z_0,r;v)^{\frac{4}{3}}+C_\infty(z_0,r;v)^{\frac{2}{3}}+D_\infty(z_0,r;q)^{\frac{4}{3}}). \tag{4.1.21}$$

The role of the above lemma[4] is to provide convergence of the rescaled solutions to a nontrivial ancient solution, under less restrictive assumptions on the pressure than in Phuc's theorem.

After completion of the proof of Theorem 4.2, the author of this thesis became aware that Theorem 4.2 had been previously proven by Wang and Zhang in [129]. Their proof is completely different to the proof provided by the author. Moreover, the main observation in our proof of Theorem 4.2 (Lemma 4.3) plays an important role in the main result of this chapter concerning an analogous local boundary regularity criterion up to the flat part of the boundary. Contrastingly, it is not obvious if the method used by Wang and Zhang in [129] can be used to prove the analogous local boundary regularity criterion.

[4]More than a year after the contents of this Chapter was uploaded onto arXiv as the preprint [5], Guevara and Phuc uploaded a preprint onto arXiv [49] containing improvements in relation to Lemma 4.3.

At the time of writing it is not known if the assumption $v \in L_\infty(-1,0; L^{3,\infty}(B))$ guarantees regularity of a suitable weak solution (v, q). Here, $L^{3,\infty}(B)$ is the critical space 'weak L_3' containing the function $|x|^{-1}$. It is known that if the velocity field has sufficiently small $L_\infty(L^{3,\infty})$ norm, then it is regular. See, for example, the papers of Kozono [67], Kim and Kozono [66] and Tsai and Luo [79]. Under certain additional assumptions, regularity has been proven without smallness of the $L_\infty(-1,0; L^{3,\infty}(B))$ norm of the velocity field. For suitable weak solutions on $B\times]-1,0[$ that are axisymmetric, it is known that

> $\sup_{(x,t)\in B\times]-1,0[} |x||v(x,t)| < \infty \Rightarrow v$ is Hölder continuous on the closure of $Q(1/2)$.

See the papers of Koch et al [62] and Seregin and Šverák [102]. Let us also mention that a year after this work was uploaded to arXiv as the preprint [5], Choe, Wolf and Yang proved in [24] that a weak Leray-Hopf solution v on $\mathbb{R}^3\times]0,T[$, that satisfies $v \in L_\infty(0,T; L^{3,\infty}(\mathbb{R}^3))$ and certain additional restrictions concerning $v(\cdot, T)$, is regular at time T[5].

The main contribution of this chapter is to show that an analogous condition to Theorem 4.2 is sufficient to provide regularity of the Navier-Stokes equations up to the flat part of the boundary. In Chapter 1, we described the general setting of local boundary regularity theory for the Navier-Stokes equations, suggested by Seregin in [98]. The definition below is essentially contained in Seregin's paper [98], specifically Definition 2.1 there.

Definition 4.4. *Let $-\infty < S_1 < S_2 < \infty$. We say that the pair $v : B^+(x_0,R)\times]S_1,S_2[\to \mathbb{R}^3$ and $q : B^+(x_0,R)\times]S_1,S_2[\to \mathbb{R}$ are a suitable weak solution to the Navier-Stokes equations in $B^+(x_0,R)\times]S_1,S_2[$ near $\Gamma(x_0,R)\times[S_1,S_2]$ if:*

$$v \in L_\infty(S_1,S_2; L_2(B^+(x_0,R))), \tag{4.1.22}$$

$$\nabla v \in L_2(S_1,S_2; L_2(B^+(x_0,R))), \tag{4.1.23}$$

$$\partial_t v \in L_{\frac{3}{2}}(]S_1,S_2[; L_{\frac{9}{8}}(B^+(x_0,R))), \tag{4.1.24}$$

$$\nabla^2 v \in L_{\frac{3}{2}}(]S_1,S_2[; L_{\frac{9}{8}}(B^+(x_0,R))), \tag{4.1.25}$$

$$\nabla q \in L_{\frac{3}{2}}(]S_1,S_2[; L_{\frac{9}{8}}(B^+(x_0,R))), \tag{4.1.26}$$

[5]By regular at time T, we mean that there exists an $\epsilon > 0$ such that $v \in L_\infty(T-\epsilon,T; L_\infty(\mathbb{R}^3))$.

$$\int_{S_1}^{S_2} \int_{B^+(x_0,R)} (-v \cdot \partial_t \varphi - v \otimes v : \nabla \varphi + \nabla v : \nabla \varphi - q div \varphi) dy d\tau = 0 \tag{4.1.27}$$

for any $\varphi \in C_0^\infty(B^+(x_0, R)\times]S_1, S_2[)$ *and*

$$v(x,t) = 0 \text{ for } x_3 = x_{03} \text{ and almost every } S_1 < t < S_2. \tag{4.1.28}$$

Additionally, it is required that the following local energy inequality holds for all positive $\varphi \in C_0^\infty(B(x_0, R)\times]S_1, \infty[)$ *and almost every time* $t \in]S_1, S_2[$*:*

$$\int_{B^+(x_0,R)} \varphi |v(x,t)|^2 dx + 2 \int_0^t \int_{B^+(x_0,R)} \varphi |\nabla v|^2 dx dt'$$

$$\leq \int_0^t \int_{B^+(x_0,R)} |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) dx dt'. \tag{4.1.29}$$

Recall that in Chapter 1 we mentioned that there are differences in the local interior and boundary smoothing effects for the Stokes and Navier-Stokes equations. Furthermore, we had discussed in Chapter 1 that interior regularity criteria for the Navier-Stokes equations are not automatically guaranteed to imply the analogous regularity criteria up to the flat part of the boundary. Differences can also be seen from a technical perspective as follows. In many interior regularity proofs in the literature, involving a suitable weak solution (v, q) on $B\times] - 1, 0[$, the pressure is decomposed on $B(r)\times] - 1, 0[$ $(0 < r < 1)$ in the following way:

$$q(x,t) = -\Delta^{-1}(\text{div}\,(\chi_{B(r)} v \otimes v)) + (q(x,t) + \Delta^{-1}(\chi_{B(r)} \text{div}\, v \otimes v))$$

$$:= q^{(1)}(x,t) + q^{(2)}(x,t). \tag{4.1.30}$$

Here, $\chi_{B(r)} : \mathbb{R}^3 \to \mathbb{R}$ is equal to 1 on $B(r)$ and zero elsewhere. Note that

$$q^{(1)}(x,t) := -\Delta^{-1}(\text{div}\,(\chi_{B(r)} v \otimes v)) = \mathcal{R}_i \mathcal{R}_j (\chi_{B(r)} v_i v_j), \tag{4.1.31}$$

where $\mathcal{R}_i$ are Riesz transforms given by

$$\mathcal{R}_i g(x) := cP.V. \int \frac{y_i}{|y|^4} g(x-y) dy.$$

In proofs of many interior regularity criteria, the first term of the pressure decomposition (4.1.30) is estimated using (4.1.31) and continuity estimates for Calderón-Zygmund singular integral operators. A key point in many proofs of interior regularity statements is that the second term of the pressure decomposition (4.1.30) is a harmonic function in $B(r)$ and satisfies estimates of the following form (where $k = 0, 1, \dots$ and $0 < r_1 < r$):

$$\|\nabla^k q^{(2)}(\cdot, t)\|_{L_\infty(B(r_1))} \leq C(k, r_1, r)\|q^{(2)}(\cdot, t)\|_{L_1(B(r))}. \tag{4.1.32}$$

In the case of proving boundary regularity criteria, for a suitable weak solution (v, q) in $B^+(0, 1)\times] - 1, 0[$ near $\Gamma(0, 1) \times [0, 1]$, the analogous pressure decomposition has the form

$$\begin{aligned} q(x, t) = -\Delta^{-1}(\mathrm{div}\,(\chi_{B^+(r)} v \otimes v)) + (q(x, t) + \Delta^{-1}(\chi_{B^+(r)} \mathrm{div}\, v \otimes v)) \\ := q^{(1)}(x, t) + q^{(2)}(x, t). \end{aligned} \tag{4.1.33}$$

Unfortunately, it appears difficult to estimate the harmonic term $q^{(2)}(x, t)$ near the flat part of the boundary. This is because $q^{(2)}(x, t)$ may not have zero trace on the flat part of the boundary, hence an estimate of the type (4.1.32) may no longer hold.

In Seregin's paper [98], sufficient conditions for regularity of a suitable weak solution (v, q) in $B^+(0, 1)\times] - 1, 0[$ near $\Gamma(0, 1) \times [-1, 0]$ are proven by estimating the pressure near the flat part of the boundary using a different decomposition to (4.1.33). Related pressure estimates near the flat part of the boundary, together with certain arguments from Escauriaza, Seregin and Šverák's paper [27], have been used by Seregin in [100] in proving the following theorem.

Theorem 1.1 of [100]. *Assume (v, q) is a suitable weak solution of the Navier-Stokes system in Q^+ near $\Gamma(0, 1) \times [-1, 0]$. Assume, in addition, that*

$$v \in L_\infty(-1, 0; L_3(B^+)). \tag{4.1.34}$$

Then v is Hölder continuous in the closure of the set $Q^+(1/2)$.

We also mention that Mikhailov and Shilkin subsequently proved regularity up to the curved boundary in [83], under analogous assumptions to Theorem 1.1 of [100].

The main aim of this chapter is to show that the same conditions to Theorem 4.2 imply regularity up to the flat part of the boundary. This is given by the following theorem.

Theorem 4.5. *Assume (v, q) is a suitable weak solution of the Navier-Stokes system in Q^+ near $\Gamma(0, 1) \times [-1, 0]$. Assume, in addition, that there exists $3 \leq \beta < \infty$ such that*

$$v \in L_\infty(-1, 0; L^{3,\beta}(B^+)). \tag{4.1.35}$$

Then v is Hölder continuous in the closure of the set $Q^+(1/2)$.

Recall from Chapter 2, we have the strict continuous embeddings $L^3(\Omega) = L^{3,3}(\Omega) \hookrightarrow L^{3,q_1}(\Omega) \hookrightarrow L^{3,q_2}(\Omega)$ for $3 < q_1 < q_2 \le \infty$. In this way Theorem 4.5 gives a strengthening of the previous result obtained in [100].

As in Escauriaza, Seregin and Šverák's work and Theorem 1.1 of [100], the proof of Theorem 4.5 is also by reductio ab adsurdum. The contradiction assumption, along with the interior regularity result of this chapter (Theorem 4.2), implies that there exists a singular point lying on the flat part of the boundary. An appropriate rescaling procedure then produces a nontrivial ancient solution, which is ruled out by a Liouville type theorem based on backward uniqueness and unique continuation. Our investigation is in the same spirit as that of Seregin's paper [100] and utilises certain arguments presented there. Let us now describe the two main difficulties encountered, as well as our observations that enable us to overcome them.

In order to provide sufficient compactness of the rescaled solutions to obtain a nontrivial ancient solution, certain bounds are needed for the pressure of the rescaled solutions. In the proof of the corresponding interior result (Theorem 4.2), the required estimates for the pressure are obtained by means of the aforementioned pressure decomposition (4.1.30). This is also used in Phuc's paper [87]. Unfortunately, such a decomposition does not seem to work well for estimating the pressure near the flat part of the boundary. In order to obtain the required estimates for the pressure of the rescaled solutions near the flat part of the boundary, we use a boundary analogue of the scale-invariant estimates in Lemma 4.3, together with arguments used by Seregin in [100]. The pressure estimates obtained generalize those obtained by Seregin in [100][6].

The second difficulty, associated with proving Theorem 4.5, is showing that the nontrivial ancient solution obtained has sufficient decay to apply a Liouville theorem based on backward uniqueness for parabolic operators. Recall that for the corresponding interior cases (Theorem 1.5 of [87] and Theorem 4.2) sufficient decay of the nontrivial ancient solution $(v_\infty, q_\infty) \in L_\infty(-\infty, 0; L^{3,\beta}(\mathbb{R}^3)) \times L_\infty(-\infty, 0; L^{\frac{3}{2},\frac{\beta}{2}}(\mathbb{R}^3))$ is achieved by using an ε-regularity criterion developed by Phuc (see Proposition 3.2 of [87], as well as (4.1.13)). Unfortunately, for the nontrivial ancient solution (v_∞, q_∞) obtained in the contradiction proof Theorem 4.5, we do not know if q_∞ belongs to the space $L_\infty(-\infty, 0; L^{\frac{3}{2},\frac{\beta}{2}}(\mathbb{R}^3_+))$. We only know that q_∞ has the same integrability, on compact space time subsets of $\mathbb{R}^3_+ \times]-\infty, 0[$, as the original pressure. For $\beta = 3$ this difficulty has been previously noted by Seregin in [100]. This creates major difficulties with the ε-regularity criterion of Phuc in [87], used in the proof of the interior

[6] Specifically, Propositions 2.5-2.6 of [100].

regularity result for Lorentz spaces. The reason for this is that Phuc's ε-regularity criterion requires the limiting pressure to have more local integrability in time than that assumed for the original pressure in Theorem 4.5 (see (4.1.13)). In order to overcome this difficulty, we develop a convenient ε-regularity criterion for interior regularity of suitable weak solutions. This will be applicable to the ancient solution obtained in the contradiction argument of Theorem 4.5 and will provide sufficient decay of the ancient solution to apply backward uniqueness arguments.

Theorem 4.6. *Let (v, q) be a suitable weak solution in $Q := B(1)\times]-1, 0[$. Then there exist universal constants ϵ_0 and c_{0k} (with $k = 1, 2\dots$) with the following property. Assume*

$$\int\limits_{-1}^{0} \|v(\cdot, s)\|^3_{L^{3,\infty}(B)} + \|q(\cdot, s)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B)} ds < \epsilon_0, \tag{4.1.36}$$

then for any natural number k, $\nabla^{k-1}v$ is Hölder continuous in $\bar{Q}(1/8)$ and the following bound is valid:

$$\sup_{(x,t)\in\bar{Q}(1/8)} |\nabla^{k-1}v(x, t)| < c_{0k}. \tag{4.1.37}$$

The proof of this ε-regularity criterion makes use the scale-invariant estimates in Lemma 4.3. This ε-regularity criterion may be of independent interest, as it provides a strengthening of the statements given by Ladyzhenskaya and Seregin in [74] and by Lin in [78]. Let us mention that over a year after the contents of this chapter was uploaded onto arXiv as the preprint [5], Guevara and Phuc uploaded a preprint onto arXiv [49] that implies improvements of the above ε-regularity criterion. In particular, Proposition 3.2 p.9 of [49] implies the following. Namely, there exists a universal constant ϵ_1 such that if

$$\int\limits_{-1}^{0} \|v(\cdot, s)\|^3_{L_{\frac{36}{13}}(B)} + \|q(\cdot, s)\|^{\frac{3}{2}}_{L_{\frac{18}{13}}(B)} ds < \epsilon_1,$$

then for any natural number k, $\nabla^{k-1}v$ is Hölder continuous in $\bar{Q}(1/8)$ and the estimate (4.1.37) holds true for v.

4.2 Local Interior Regularity in Critical Lorentz spaces

4.2.1 Scale-Invariant Estimates of Suitable Weak Solutions

Recall the definitions of the relevant scale-invariant functionals: $(0 < r < 1)$

$$A(z_0, r; v) := \sup_{t_0 - r^2 \leq t \leq t_0} r^{-1} \int_{B(x_0,r)} |v(x,t)|^2 dx, \tag{4.2.1}$$

$$B(z_0, r; v) := r^{-1} \int_{t_0 - r^2}^{t_0} \int_{B(x_0,r)} |\nabla v(x,t)|^2 dx dt, \tag{4.2.2}$$

$$C_\infty(z_0, r; v) := r^{-2} \int_{t_0 - r^2}^{t_0} \|v\|^3_{L^{3,\infty}(B(x_0,r))} dt \tag{4.2.3}$$

and

$$D_\infty(z_0, r; q) := r^{-2} \int_{t_0 - r^2}^{t_0} \|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(x_0,r))} dt. \tag{4.2.4}$$

Now let us state a lemma, involving estimates for the above scale-invariant quantities, which we will make use of in proving in Theorem 4.2 and Theorem 4.5.

Lemma 4.7. *Let (v, q) be a suitable weak solution in $Q(z_0, 1)$. Then for $0 < r < 1$ the following holds (c is some universal constant):*

$$A(z_0, r/2; v) + B(z_0, r/2; v) \leq c(C_\infty(z_0, r; v)^{\frac{4}{3}} + C_\infty(z_0, r; v)^{\frac{2}{3}} + D_\infty(z_0, r; q)^{\frac{4}{3}}). \tag{4.2.5}$$

First, we begin with stating a well known algebraic lemma, whose proof can be found in Giusti's book [47].

Lemma 4.8. *Let $I(s)$ be a bounded nonnegative function for $s \in [R_1, R_2]$. Assume that for every $s, \rho \in [R_1, R_2]$ and $s < \rho$ we have*

$$I(s) \leq [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho)$$

where $A, B, C \geq 0$ are constants, $\alpha > \beta > 0$ and $\theta \in [0, 1[$. Then there holds

$$I(R_1) \leq c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C].$$

Proof of Lemma 4.7

Proof. Without loss of generally, consider $z_0 = (0,0)$. Let $0 < \frac{r}{2} \leq s < \rho \leq r < 1$. Let $\eta_1 \in C_0^\infty(B(\rho))$ be such that $0 \leq \eta_1 \leq 1$, $\eta_1 = 1$ on $B(s)$ and

$$|\nabla^\alpha \eta_1| \leq \frac{C}{(\rho - s)^{|\alpha|}} \tag{4.2.6}$$

for $|\alpha| \leq 2$. Let $\eta_2 \in C_0^\infty(-\rho^2, \rho^2)$ be such that $0 \leq \eta_2 \leq 1$, $\eta_2 = 1$ on $[-s^2, s^2]$ and

$$|\eta_2^{'}| \leq \frac{C}{(\rho^2 - s^2)} \leq \frac{C}{r(\rho - s)} \leq \frac{C}{(\rho - s)^2}. \tag{4.2.7}$$

Let $\varphi(x,t) := \eta_1(t)\eta_2(x)$. From (4.2.6)-(4.2.7), we infer that

$$|\nabla \varphi| \leq \frac{C}{\rho - s}, \qquad |\nabla^2 \varphi| \leq \frac{C}{(\rho - s)^2}, \qquad \text{and } |\partial_t \varphi| \leq \frac{C}{(\rho - s)^2}. \tag{4.2.8}$$

Our aim is to estimate scale-invariant quantities by substituting φ into the local energy inequality and making us of Lemma 4.8. This strategy has also been used by Phuc in [87][7].

From the local energy inequality, we have that for almost every $t \in]-1, 0[$:

$$\int\limits_B \varphi(x,t)|v(x,t)|^2 dx + 2\int\limits_{-1}^{t}\int\limits_B \varphi|\nabla v|^2 dxdt'$$

$$\leq \int\limits_{-1}^{t}\int\limits_B |v|^2(\Delta\varphi + \partial_t\varphi) + v \cdot \nabla\varphi(|v|^2 + 2q)dxdt'. \tag{4.2.9}$$

Let $I_1(s) := sA(0,s;v)$, $I_2(s) := sB(0,s;v)$ and $I(s) = I_1(s) + I_2(s)$. From (4.2.9), we deduce that

$$I(s) \leq \frac{3}{2}((1) + (2) + (3)). \tag{4.2.10}$$

Here,

$$(1) := \sup_{-s^2<t<0} \int\limits_{-1}^{t}\int\limits_B |v|^2(\Delta\varphi + \partial_t\varphi)dxdt', \tag{4.2.11}$$

$$(2) := \sup_{-s^2<t<0} \int\limits_{-1}^{t}\int\limits_B v \cdot \nabla\varphi|v|^2 dxdt' \tag{4.2.12}$$

and

$$(3) := \sup_{-s^2<t<0} \int\limits_{-1}^{t}\int\limits_B 2v \cdot \nabla\varphi q dxdt'. \tag{4.2.13}$$

[7]See Lemma 2.3 of [87].

Let us treat (1) first. Using Hunt's inequality in space (Proposition 2.2 in Chapter 2), we infer that

$$(1) \leq C \int\limits_{-\rho^2}^{0} \||v|^2\|_{L^{\frac{3}{2},\infty}(B(\rho))} \|\Delta\varphi + \partial_t \varphi\|_{L^{3,1}(B(\rho))} dt'. \tag{4.2.14}$$

From the definition of the quasinorm, we see that there exists a universal constant C such that

$$\||f|^2\|_{L^{\frac{3}{2},\infty}(\Omega)} \leq C\|f\|^2_{L^{3,\infty}(\Omega)}.$$

By using this and Hölder's inequality in time, one can infer from (4.2.14) that

$$(1) \leq c\frac{\rho}{(\rho-s)^2} \int\limits_{-\rho^2}^{0} \|v\|^2_{L^{3,\infty}(B(\rho))} dt' \leq \frac{\rho^{\frac{5}{3}}}{(\rho-s)^2} \left(\int\limits_{-\rho^2}^{0} \|v\|^3_{L^{3,\infty}(B(\rho))} dt' \right)^{\frac{2}{3}}. \tag{4.2.15}$$

Next, let us estimate (3). By Hunt's inequality, we deduce that

$$(3) \leq C \int\limits_{-\rho^2}^{0} \|v\cdot\nabla\varphi\|_{L^{3,1}(B(\rho))} \|q\|_{L^{\frac{3}{2},\infty}(B(\rho))} dt'. \tag{4.2.16}$$

From Proposition 2.3 (Chapter 2) and the Sobolev embedding theorem, we infer that

$$\|v\cdot\nabla\varphi\|_{L^{3,1}(B(\rho))} \leq C\|v\cdot\nabla\varphi\|^{\frac{1}{2}}_{L_2(B(\rho))} \|v\cdot\nabla\varphi\|^{\frac{1}{2}}_{L_6(B(\rho))} \leq$$

$$\leq C\|v\cdot\nabla\varphi\|^{\frac{1}{2}}_{L_2(B(\rho))} \|\nabla(v\cdot\nabla\varphi)\|^{\frac{1}{2}}_{L_2(B(\rho))}$$

$$\leq \frac{C\|v\|_{L_2(B(\rho))}}{(\rho-s)^{\frac{3}{2}}} + \frac{C\|v\|^{\frac{1}{2}}_{L_2(B(\rho))} \|\nabla v\|^{\frac{1}{2}}_{L_2(B(\rho))}}{\rho-s}.$$

Thus, (4.2.16) implies that

$$(3) \leq C \int\limits_{-\rho^2}^{0} \Big[\frac{\|v\|_{L_2(B(\rho))}}{(\rho-s)^{\frac{3}{2}}} + \frac{\|v\|^{\frac{1}{2}}_{L_2(B(\rho))} \|\nabla v\|^{\frac{1}{2}}_{L_2(B(\rho))}}{\rho-s}\Big] \|q\|_{L^{\frac{3}{2},\infty}(B(\rho))} dt'. \tag{4.2.17}$$

Applying Hölder's inequality in the time variable to (4.2.17), one obtains that

$$(3) \leq \frac{C}{(\rho-s)^{\frac{3}{2}}} \left(\int\limits_{-\rho^2}^{0} \|v\|^3_{L_2(B(\rho))} dt' \right)^{\frac{1}{3}} \left(\int\limits_{-\rho^2}^{0} \|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(\rho))} dt' \right)^{\frac{3}{2}}$$

$$+\frac{C}{\rho-s}\left(\int\limits_{-\rho^2}^{0}\|v\|^{\frac{3}{2}}_{L_2(B(\rho))}\|\nabla v\|^{\frac{3}{2}}_{L_2(B(\rho))}dt'\right)^{\frac{1}{3}}\left(\int\limits_{-\rho^2}^{0}\|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(\rho))}dt'\right)^{\frac{2}{3}}.$$

From this, one deduces that

$$(3)\leq\left[\frac{Cr^{\frac{2}{3}}I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{\frac{3}{2}}}+\frac{Cr^{\frac{1}{6}}}{\rho-s}I_2(\rho)^{\frac{1}{4}}I_1(\rho)^{\frac{1}{4}}\right]\times\left(\int\limits_{-r^2}^{0}\|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))}dt'\right)^{\frac{2}{3}}. \tag{4.2.18}$$

Similar reasoning gives

$$(2)\leq\left[\frac{Cr^{\frac{2}{3}}I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{\frac{3}{2}}}+\frac{Cr^{\frac{1}{6}}}{\rho-s}I_2(\rho)^{\frac{1}{4}}I_1(\rho)^{\frac{1}{4}}\right]\times\left(\int\limits_{-r^2}^{0}\|v\|^{3}_{L^{3,\infty}(B(r))}dt'\right)^{\frac{2}{3}}. \tag{4.2.19}$$

From (4.2.10), (4.2.15), (4.2.18), (4.2.19), together with Young's inequality, we deduce that

$$I(s)\leq\frac{r^{\frac{5}{3}}}{(\rho-s)^2}\left(\int\limits_{-r^2}^{0}\|v\|^{3}_{L^{3,\infty}(B(r))}dt'\right)^{\frac{2}{3}}+\frac{1}{2}I(\rho)$$

$$+\left[\frac{Cr^{\frac{4}{3}}}{(\rho-s)^3}+\frac{Cr^{\frac{1}{3}}}{(\rho-s)^2}\right]\times\left[\left(\int\limits_{-r^2}^{0}\|v\|^{3}_{L^{3,\infty}(B(r))}dt'\right)^{\frac{4}{3}}+\left(\int\limits_{-r^2}^{0}\|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))}dt'\right)^{\frac{4}{3}}\right]. \tag{4.2.20}$$

By Lemma 4.8, we obtain

$$I(r/2)\leq r^{-\frac{1}{3}}\left(\int\limits_{-r^2}^{0}\|v\|^{3}_{L^{3,\infty}(B(r))}dt'\right)^{\frac{2}{3}}$$

$$+Cr^{-\frac{5}{3}}\left[\left(\int\limits_{-r^2}^{0}\|v\|^{3}_{L^{3,\infty}(B(r))}dt'\right)^{\frac{4}{3}}+\left(\int\limits_{-r^2}^{0}\|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(r))}dt'\right)^{\frac{4}{3}}\right].$$

From here the conclusion is immediate. □

Remark 4.9. *After appropriate relabellings, an analogous estimate holds when* (v,q) *is a suitable weak solution in* $B^+(x_0,1)\times]t_0-1,t_0[$ *near* $\Gamma(x_0,1)\times[t_0-1,t_0]$.

4.2.2 Proof of Theorem 4.2

Let us proceed with a lemma. The analogous lemma (Lemma 4.1) was stated and proven by Phuc in [87]. As the proof of this statement is essentially unchanged we omit it.

Lemma 4.10. *Suppose that the pair of functions (v, q) satisfy the hypotheses of Theorem 4.2. These hypotheses imply that the inequality*

$$\|v(\cdot,t)\|_{L^{3,\beta}(B(3/4))} \leq \|v\|_{L_\infty(-(3/4)^2,0;L^{3,\beta}(B(3/4)))}, \tag{4.2.21}$$

holds for all $t \in [-(3/4)^2, 0]$ and that the function

$$t \to \int\limits_{B(3/4)} v(x,t) \cdot w(x)dx$$

is continuous on $[-(3/4)^2, 0]$ for any $w \in L^{\frac{3}{2},\frac{\beta}{\beta-1}}(B(3/4))$.

In proving Theorem 4.2, we will use the following lemma which is stated and proven in Seregin and Šverák's paper [97] (specifically Lemma 3.3 of [97]).

Lemma 4.11. *Let (v, q) be a suitable weak solution of the Navier-Stokes equations in $B(1)\times]-1, 0[$. There exists a universal constant c_0 with the following property. Assume for some $0 < R_0 < 1$:*

$$\sup_{0<R<R_0}\Big(\sup_{-R^2\leq t\leq 0}\frac{1}{R}\int\limits_{B(0,R)}|v(x,t)|^2dx\Big) < c_0. \tag{4.2.22}$$

Then, $(0, 0)$ is a regular point of v. In particular, there exists an $0 < r < 1$ such that $v \in L_\infty(Q(r))$.

Step 1: Rescaling Procedure

Next, we recap the rescaling prodecure for the proof of Theorem 4.2 used by Escauriaza, Seregin and Šverák in [27]. Suppose the conditions for (v, q) of Theorem 4.2 hold. Then by the previous Lemma, for $t \in [-1, 0]$ we have

$$v(\cdot,t) \in L^{3,\beta}(B). \tag{4.2.23}$$

As in [27], the rescaling procedure arises from assuming Theorem 4.2 is false. This implies that v has no representative that is Hölder continuous on $\bar{Q}(1/2)$. Serrin's paper [113], together with the local interior regularity theory for the Stokes system[8], implies that essentially bounded suitable weak solutions of the Navier-Stokes equations are Hölder continuous on interior subdomains. As in [27], we infer from the contradiction assumption that there exists a singular point $z_0 \in \bar{Q}(3/4)$ such that, for any $r > 0$,

$$v \notin L_\infty(Q(z_0, r) \cap Q). \tag{4.2.24}$$

Without loss of generality, we can take $z_0 = (0, 0)$.

Now we repeat arguments used by Escauriaza, Seregin and Šverák in [27] (specifically, p.223 of [27]) to show the following. Namely, there exists a universal constant $c_0 > 0$ and a sequence of numbers $R_k \in]0, 1[$ such that $R_k \to 0$ as $k \to +\infty$ and

$$\sup_{-R_k^2 \le s \le 0} \frac{1}{R_k} \int\limits_{B(0,R_k)} |v(x, s)|^2 dx \ge c_0 \tag{4.2.25}$$

for any $k = 1, 2, \ldots$. Indeed, (4.2.24) and Lemma 4.11 imply that for all $R_0 \in]0, 1[$ we have

$$\sup_{0<R<R_0} \Big(\sup_{-R^2 \le t \le 0} \frac{1}{R} \int\limits_{B(0,R)} |v(x, t)|^2 dx \Big) \ge c_0. \tag{4.2.26}$$

This then implies that there exists $R_k \to 0$ as $k \to +\infty$ such that (4.2.25) holds true.

For each $a > 0$, there exists a $k_0(a) \ge 1$ such that for all $k \ge k_0$ we have

$$Q(R_k a) \subset Q(1/4).$$

As done in Escauriaza, Seregin and Šverák's paper [27] and in Phuc's paper [87], we perform the following rescaling:

$$v^{(k)}(x, t) := R_k v(R_k x, R_k^2 t) \tag{4.2.27}$$

and

$$q^{(k)}(x, t) := R_k^2 q(R_k x, R_k^2 t). \tag{4.2.28}$$

As done by Phuc in [87], we may decompose the pressure

$$q = \tilde{q} + h, \tag{4.2.29}$$

where h is harmonic in B and $\tilde{q} := \mathcal{R}_i \mathcal{R}_j [(v_i v_j)\chi_B]$. The fact that Calderón-Zygmund singular integral operators are bounded linear operators on L_p $(1 < p < \infty)$, together with the 'generalised Marcinkiewicz theorem' (Theorem 2.4), implies the known fact

[8]See Proposition 6.7-6.8 of [111] (pp.85-87), for example.

that Calderón-Zygmund singular integral operators are bounded linear operators on $L^{p,q}$ ($p \in [1,\infty[$ and $q \in [1,\infty]$). Using this, we infer that

$$\|\tilde{q}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(\mathbb{R}^3)} \leq \|v\|^2_{L_\infty(-1,0;L^{3,\beta}(B))}. \tag{4.2.30}$$

It is also clear that for $(x,t) \in Q(2a)$ and $k \geq k_0(a)$:

$$q^{(k)} = \tilde{q}^{(k)} + h^{(k)}. \tag{4.2.31}$$

Here,

$$\tilde{q}^{(k)}(x,t) := R_k^2 \tilde{q}(R_k x, R_k^2 t) \quad \text{and} \quad h^{(k)}(x,t) := R_k^2 h(R_k x, R_k^2 t).$$

It is also the case that $(v^{(k)}, q^{(k)})$ are a suitable weak solution to the Navier-Stokes equations in $Q(2a)$. A detailed explanation of this can be found in Phuc's paper [87][9], for example.

Step 2: Estimates of Rescaled Solutions

Now, let us make more explicit the role that Lemma 4.7 plays by means of a proposition.

Proposition 4.12. *The rescaled velocity and pressure satisfy the following uniform estimates for $k \geq k_0(a)$:*

$$\|q^{(k)}\|_{L_{\frac{3}{2}}(-(2a)^2,0;L^{\frac{3}{2},\frac{\beta}{2}}(B(2a)))} \leq C(a)(\|q\|_{L_{\frac{3}{2}}(Q)} + \|v\|^2_{L_\infty(-1,0;L^{3,\beta}(B))}), \tag{4.2.32}$$

$$\int\limits_{-(2a)^2}^{0} \sup_{B(2a)} |h^{(k)}(x,t)|^{\frac{3}{2}} dt \leq C R_k (\|q\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(Q)} + \|v\|^3_{L_\infty(-1,0;L^{3,\beta}(B))}), \tag{4.2.33}$$

$$\|v^{(k)}(\cdot,t)\|_{L^{3,\beta}(B(2a))} \leq \|v\|_{L_\infty(-1,0;L^{3,\beta}(B))} \ \textit{for}\ t \in [-(2a)^2, 0] \tag{4.2.34}$$

and

$$\|v^{(k)}\|_{L_\infty(-a^2,0;L_2(B(a)))} + \|\nabla v^{(k)}\|_{L_2(-a^2,0;L_2(B(a)))} \leq C(a, \|q\|_{L_{\frac{3}{2}}(Q)}, \|v\|_{L_\infty(-1,0;L^{3,\beta}(B))}). \tag{4.2.35}$$

Proof. It is clear that (4.2.34) follows from Lemma 4.10 along with the easily seen property that $u_0 \in L^{3,\beta}(B(a))$ and $u_{0,\lambda}(x) := \lambda u_0(\lambda x)$ implies $\|u_{0,\lambda}\|_{L^{3,\beta}(B(a/\lambda))} = \|u_0\|_{L^{3,\beta}(\Omega)}$. Let us now demonstrate the role played by the new scale-invariant estimates, provided by Lemma 4.7. Indeed, Lemma 4.7 implies that

$$\|v^{(k)}\|^2_{L_\infty(-a^2,0;L_2(B(a)))} + \|\nabla v^{(k)}\|^2_{L_2(-a^2,0;L_2(B(a)))}$$

[9] Specifically, p.754-755 of [87].

$$\leq C(a)[\|v^{(k)}\|^4_{L_\infty(-(2a)^2,0;L^{3,\infty}(B(2a)))}$$
$$+\|v^{(k)}\|^2_{L_\infty(-(2a)^2,0;L^{3,\infty}(B(2a)))}+\|q^{(k)}\|^2_{L_{\frac{3}{2}}(-(2a)^2,0;L^{\frac{3}{2},\infty}(B(2a)))}]. \tag{4.2.36}$$

If we assume for the moment that the estimate (4.2.32) is valid, then (4.2.32), (4.2.34) and (4.2.36) immediately imply that (4.2.35) holds true.

Now let us prove that the estimates (4.2.32)-(4.2.33) hold. The arguments we use here are essentially contained in Phuc's paper [87][10]. As there are some minor differences, we provide full details for completeness. As argued by Phuc in [87], Lemma 4.10 and (4.2.30) imply that for $t\in]-(2a)^2,0[$:

$$\|\tilde{q}^{(k)}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(2a))}\leq \|\tilde{q}(\cdot,R_k^2t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(1/2))}\leq C\|v\|^2_{L_\infty(-1,0;L^{3,\beta}(B))}. \tag{4.2.37}$$

Since $h(\cdot,t)$ is a harmonic function in B for almost every $t\in]-1,0[$, it satisfies the mean value property. Hence, for almost every $t\in]-1,0[$ we have

$$\sup_{B(1/2)}|h(\cdot,t)|\leq \|h(\cdot,t)\|_{L_1(B)}. \tag{4.2.38}$$

Using this, along with Hölder's inequality, we infer

$$\|h\|_{L_{\frac{3}{2}}(-1,0;L_\infty(B(1/2)))}\leq C\|h\|_{L_{\frac{3}{2}}(-1,0;L_1(B(1)))}\leq C(\|\tilde{q}\|_{L_{1,\frac{3}{2}}(Q)}+\|q\|_{L_{\frac{3}{2}}(Q)}). \tag{4.2.39}$$

By Hunt's inequality (Proposition 2.2 in Chapter 2), we have

$$\|\tilde{q}(\cdot,t)\|_{L_1(B)}\leq c\|\chi_B\|_{L^{3,\frac{\beta}{\beta-2}}(B)}\|\tilde{q}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B)}\leq c\|\tilde{q}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B)}.$$

The above and (4.2.30) imply that

$$\|\tilde{q}\|_{L_{1,\frac{3}{2}}(Q)}\leq c\|\tilde{q}\|_{L_{\frac{3}{2}}(-1,0;L^{\frac{3}{2},\frac{\beta}{2}}(B))}\leq c\|v\|^2_{L_\infty(-1,0;L^{3,\beta}(B))}. \tag{4.2.40}$$

From (4.2.39)-(4.2.40), it can be deduced that

$$\int\limits_{-(2a)^2}^{0}\sup_{B(2a)}|h^{(k)}(x,t)|^{\frac{3}{2}}dt\leq R_k\int\limits_{-1/4}^{0}\sup_{B(2a)}|h(R_kx,s)|^{\frac{3}{2}}ds$$

$$\leq R_k\|h\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(-1,0;L_\infty(B(1/2)))}\leq CR_k(\|q\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(Q)}+\|v\|^3_{L_\infty(-1,0;L^{3,\beta}(B))}). \tag{4.2.41}$$

This establishes (4.2.33).

Note that

$$\|h^{(k)}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(2a))}\leq \sup_{x\in B(2a)}|h^{(k)}(x,t)|\|\chi_{B(2a)}\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(2a))}\leq a^2\sup_{x\in B(2a)}|h^{(k)}(x,t)|. \tag{4.2.42}$$

This, (4.2.37) and (4.2.41) immediately give (4.2.32).

□

[10] Specifically, p.754 of [87].

Step 3: Passing to the Limit

Using the estimates of the rescaled solutions provided by Proposition 4.12, we may then apply identical arguments to those presented in Phuc's paper [87][11] to obtain limit functions (v_∞, q_∞), arising from the rescaled solutions, with the following properties.

- 1) $(v_\infty, q_\infty) \in L_\infty(-\infty, 0; L^{3,\beta}(\mathbb{R}^3)) \times L_\infty(-\infty, 0; L^{\frac{3}{2},\frac{\beta}{2}}(\mathbb{R}^3))$.
- 2) (v_∞, q_∞) form a suitable weak solution to the Navier-Stokes equations in $Q(a) := B(a)\times]-a^2, 0[$ for any $a > 0$.
- 3) $v_\infty \in C([a, b]; L_s(\omega))$ for any $s \in]1, 3[$, any bounded set $\omega \subset \mathbb{R}^3$ and any $-\infty < a < b < \infty$.
- 4) For almost every $x \in \mathbb{R}^3$, $v_\infty(x, 0) = 0$.
- 5) The function v_∞ is nontrivial. In particular, there exists a universal constant c_0 such that
$$\sup_{t\in[-1,0]} \int_B |v_\infty(x,t)|^2 dx \geq c_0.$$

Step 4[12]: Obtaining a Contradiction

The final part of the proof of Theorem 4.2 involves showing that properties 1)-4) of (v_∞, q_∞) allow us to apply backward and uniqueness theorems for parabolic operators, developed by Escauriaza, Seregin and Šverák in [27], to the equation satisfied by the vorticity $\omega_\infty = \nabla \wedge v_\infty$. From this it can be shown that $v_\infty = 0$ in $\mathbb{R}^3 \times [-1, 0]$, which contradicts property 5).

4.3 Local Regularity Near the Boundary for the Navier-Stokes Equations in Critical Lorentz Spaces

4.3.1 Pressure Estimates

Before stating and proving a certain lemma, let us introduce some notation.

[11]Specifically, arguments from Proposition 4.2 in [87].

[12]We do not present full details of Step 4, since the required arguments are identical to those found in pp.756-757 of Phuc's paper [87].

Take $\beta \in [3, \infty]$ and define the following scale-invariant quantities

$$C_\beta(z_0, r; v) := \frac{1}{r^2} \int\limits_{t_0 - r^2}^{t_0} \|v(\cdot, t)\|^3_{L^{3,\beta}(B(x_0,r))} dt \tag{4.3.1}$$

$$D_\beta(z_0, r; q) := \frac{1}{r^2} \int\limits_{t_0 - r^2}^{t_0} \|q(\cdot, t) - [q]_{B(x_0,r)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0,r))} dt \tag{4.3.2}$$

The following generalises Lemma 2.1 of Seregin's paper [100], which established that under the assumptions of the lemma below, (4.3.3) holds true for $\beta = 3$.

Lemma 4.13. *Let $v \in L_3(Q(z_0, R))$ and $q \in L_{\frac{3}{2}}(Q(z_0, R))$ satisfy the Navier-Stokes equations in the sense of distributions. Then, for $0 < r \le \rho \le R$, we have*

$$D_\beta(z_0, r; q) \le c\left[\left(\frac{r}{\rho}\right)^{\frac{5}{2}} D_\beta(z_0, \rho; q) + \left(\frac{\rho}{r}\right)^2 C_\beta(z_0, \rho; v)\right]. \tag{4.3.3}$$

Let us remark that the proof below is essentially contained in Seregin's paper [96][13]. In the setting of Lorentz spaces there are some minor differences, hence we provide full details for the convenience of the reader.

Proof. Without loss of generality take $z_0 = 0$.

Case 1: $0 < r \le \frac{\rho}{2} \le \rho \le R$

We utilise the same pressure decomposition as used on p.223 of Seregin's paper [100]. Namely, we write

$$q = q^{(1)} + q^{(2)},$$

where

$$q^{(1)}(\cdot, t) := \mathcal{R}_i \mathcal{R}_j(\chi_{B(\rho)} v_i v_j(\cdot, t)). \tag{4.3.4}$$

Here $\mathcal{R}_i$ is a Riesz operator and we adopt the summation convention. It is not difficult to notice that in $B(\rho)$:

$$\Delta q^{(2)}(\cdot, t) = 0. \tag{4.3.5}$$

For $q^{(1)}$, we can use the well known fact that Calderón-Zygmund singular integral operators are bounded linear operators on $L^{p,q}$ (with $p \in]1, \infty[$ and $q \in [1, \infty]$), to obtain that

$$\|q^{(1)}(\cdot, t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))} \le c\|v \otimes v(\cdot, t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))} \le c\|v(\cdot, t)\|^3_{L^{3,\beta}(B(\rho))}. \tag{4.3.6}$$

[13] In particular, Lemma 3.1 in Seregin's paper [96].

Consider a measurable function $f : B(r) \to \mathbb{R}$. Using Hunt's inequality (Proposition 2.2 in Chapter 2), we obtain that

$$|[f]_{B(r)}| \leq \frac{c}{r^3} \int\limits_{B(r)} |f(y)|dy \leq \frac{c\|f\|_{L^{p,s}(B(r))}\|\chi_{B(r)}\|_{L^{p',s'}(B(r))}}{r^3} \leq$$

$$\leq cr^{-\frac{3}{p}}\|f\|_{L^{p,s}(B(r))}.$$

Here, $1 \leq s \leq \infty$, $1 < p < \infty$, $\frac{1}{p'} = 1 - \frac{1}{p}$ and $\frac{1}{s'} = 1 - \frac{1}{s}$. From the above we infer,

$$\|[f]_{B(r)}\|_{L^{p,s}(B(r))} \leq cr^{-\frac{3}{p}}\|f\|_{L^{p,s}(B(r))}\|\chi_{B(r)}\|_{L^{p,s}(B(r))}$$

$$\leq c\|f\|_{L^{p,s}(B(r))}. \tag{4.3.7}$$

Substituting $f = q^{(1)}(\cdot, t)$, $p = \frac{3}{2}$ and $s = \frac{\beta}{2}$ into (4.3.7) then gives

$$\|[q^{(1)}]_{B(r)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))} \leq c\|q^{(1)}(\cdot, t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}. \tag{4.3.8}$$

One can then easily obtain

$$D_\beta(0, r; q^{(1)}) \leq \frac{c}{r^2}\left(\int\limits_{-r^2}^{0} \|q^{(1)}(\cdot, t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))} dt\right)$$

$$\leq \frac{c}{r^2}\left(\int\limits_{-\rho^2}^{0} \|q^{(1)}(\cdot, t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))} dt\right). \tag{4.3.9}$$

Using (4.3.6) and (4.3.9), one can infer the following

$$D_\beta(0, r, q) \leq c\left[\left(\frac{\rho}{r}\right)^2 C_\beta(0, \rho, v) + D_\beta(0, r, q^{(2)})\right]. \tag{4.3.10}$$

Identical arguments to those used by Seregin in [100] (specifically, p.224 of [100]) imply that the harmonic function $q^{(2)}$ satisfies

$$\sup_{x\in B(r)} |q^{(2)}(x, t) - [q^{(2)}]_{B(r)}(t)| \leq \frac{cr}{\rho^4} \int\limits_{B(\rho)} |q^{(2)}(x, t) - [q^{(2)}]_{B(\rho)}(t)|dx. \tag{4.3.11}$$

The proof of this estimate uses the fact that any harmonic function g in $B(\rho)$ satisfies the Caccioppoli estimate:

$$\sup_{x\in B(\rho/2)} |\nabla g(x)| \leq \frac{c}{\rho^4} \int\limits_{B(\rho)} |g(x)|dx. \tag{4.3.12}$$

Applying this to $g = q^{(2)} - [q^{(2)}]_{B(\rho)}$ then gives

$$\sup_{x\in B(\rho/2)} |\nabla q^{(2)}(x,t)| = \sup_{x\in B(\rho/2)} |\nabla (q^{(2)} - [q^{(2)}]_{B(\rho)})(x,t)|$$
$$\leq \frac{c}{\rho^4} \int\limits_{B(\rho)} |q^{(2)}(x,t) - [q^{(2)}]_{B(\rho)}(t)| dx. \tag{4.3.13}$$

From this, one can then deduce (4.3.11).

From (4.3.11) and Hunt's inequality, we obtain that

$$\sup_{x\in B(r)} |q^{(2)}(x,t) - [q^{(2)}]_{B(r)}(t)| \leq \frac{cr}{\rho^4} \int\limits_{B(\rho)} |q^{(2)}(x,t) - [q^{(2)}]_{B(\rho)}(t)| dx$$
$$\leq \frac{cr}{\rho^4} \|q^{(2)}(\cdot,t) - [q^{(2)}]_{B(\rho)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))} \|\chi_{B(\rho)}\|_{L^{3,\frac{\beta}{\beta-2}}(B(\rho))}$$
$$\leq \frac{cr}{\rho^3} \|q^{(2)}(\cdot,t) - [q^{(2)}]_{B(\rho)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))}. \tag{4.3.14}$$

From (4.3.14) and Hunt's inequality, we see that

$$\|q^{(2)}(\cdot,t) - [q^{(2)}]_{B(r)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}$$
$$\leq \Big(\sup_{x\in B(r)} |q^{(2)}(x,t) - [q^{(2)}]_{B(r)}(t)|\Big)^{\frac{3}{2}} \|\chi_{B(r)}\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}$$
$$\leq cr^3 \Big(\sup_{x\in B(r)} |q^{(2)}(x,t) - [q^{(2)}]_{B(r)}(t)|\Big)^{\frac{3}{2}}$$
$$\leq c\Big(\frac{r}{\rho}\Big)^3 \Big(\frac{r}{\rho}\Big)^{\frac{3}{2}} \|q^{(2)}(\cdot,t) - [q^{(2)}]_{B(\rho)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))}. \tag{4.3.15}$$

From this, it can be deduced that

$$D_\beta(0,r,q^{(2)}) \leq \Big(\frac{r}{\rho}\Big)^{\frac{5}{2}} D_\beta(0,\rho,q^{(2)}). \tag{4.3.16}$$

This, together with (4.3.10), implies

$$D_\beta(0,r,q) \leq c\left[\Big(\frac{\rho}{r}\Big)^2 C_\beta(0,\rho,v) + \Big(\frac{r}{\rho}\Big)^{\frac{5}{2}} D_\beta(0,\rho,q^{(2)})\right]$$
$$\leq c\left[\Big(\frac{\rho}{r}\Big)^2 C_\beta(0,\rho,v) + \Big(\frac{r}{\rho}\Big)^{\frac{5}{2}} D_\beta(0,\rho,q) + \Big(\frac{r}{\rho}\Big)^{\frac{5}{2}} D_\beta(0,\rho,q^{(1)})\right]. \tag{4.3.17}$$

The same arguments as the ones used in deriving (4.3.8) imply that

$$\|[q^{(1)}]_{B(\rho)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))} \leq c\|q^{(1)}(\cdot,t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(\rho))}. \tag{4.3.18}$$

It then follows from this and (4.3.6) that

$$D_\beta(0,\rho,q^{(1)}) \leq c\,C_\beta(0,\rho,v). \tag{4.3.19}$$

The conclusion then follows immediately from (4.3.17)-(4.3.19) and the fact that $r \leq \rho$.

Case 2: $\frac{\rho}{2} \leq r \leq \rho$

Under the assumptions of this case, one can verify that

$$D_\beta(0,r,q) \leq cD_\beta(0,\rho,q) + cI. \tag{4.3.20}$$

Here,

$$I := \frac{1}{r^2}\int\limits_{-r^2}^{0} \|[q]_{B(\rho)}(t) - [q]_{B(r)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))} dt. \tag{4.3.21}$$

Using Hunt's inequality, one obtains the following estimate:

$$\begin{aligned}
|[q]_{B(\rho)}(t) - [q]_{B(r)}(t)| &\leq \frac{c}{r^3}\int\limits_{B(r)} |q(x,t) - [q]_{B(\rho)}(t)|dx \\
&\leq \frac{c}{r^3}\|q(\cdot,t) - [q]_{B(\rho)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}\|\chi_{B(r)}\|_{L^{3,\frac{\beta}{\beta-2}}(B(r))} \\
&\leq \frac{c}{r^2}\|q(\cdot,t) - [q]_{B(\rho)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}.
\end{aligned} \tag{4.3.22}$$

Thus,

$$\begin{aligned}
I &\leq \frac{c}{r^5}\int\limits_{-r^2}^{0} \|q(\cdot,t) - [q]_{B(\rho)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))}\|\chi_{B(r)}\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))} dt \\
&\leq \frac{c}{r^2}\int\limits_{-r^2}^{0} \|q(\cdot,t) - [q]_{B(\rho)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(r))} dt \leq cD_\beta(0,\rho,q).
\end{aligned} \tag{4.3.23}$$

Using this, together with (4.3.20), the conclusion readily follows. □

Lemma 4.13 enables us to prove a certain generalisation to Proposition 2.5 in [100]. Namely the following result holds.

Proposition 4.14. *Assume that all the criteria of Lemma 4.13 are fulfilled. Assume, in addition, that there exists a* $\beta \in [3,\infty[$ *such that*

$$\|v\|_{L_\infty(t_0-R^2,t_0;L^{3,\beta}(B(x_0,R)))} \leq L < \infty. \tag{4.3.24}$$

Then, for any $\gamma \in]0,1[$*, there exists a constant* $c_1(L,\gamma)$ *such that, for* $0 < r \leq R$*, we have*

$$D_\beta(z_0,r;q) \leq c_1\left[\left(\frac{r}{R}\right)^{\frac{5}{2}\gamma} D_\beta(z_0,R;q) + 1\right]. \tag{4.3.25}$$

Proof. The proof involves using Lemma 4.13 and identical iteration arguments to those presented by Seregin in [100][14]. We provide the details for completeness.

Clearly from Lemma 4.13, we have

$$D_{\beta}(z_0,\tau^{k+1}R;q)\leq c\tau^{\frac{5}{2}(1-\gamma)}\tau^{\frac{5}{2}\gamma}D_{\beta}(z_0,\tau^{k}R;q)+\frac{cL^3}{\tau^2} \tag{4.3.26}$$

for any $0<\tau<1$. It then follows from an induction argument that

$$D_{\beta}(z_0,\tau^{k+1}R;q)\leq (c\tau^{\frac{5}{2}(1-\gamma)}\tau^{\frac{5}{2}\gamma})^{k+1}D_{\beta}(z_0,R;q)+\frac{cL^3}{\tau^2}\sum_{i=0}^{k}(c\tau^{\frac{5}{2}(1-\gamma)}\tau^{\frac{5}{2}\gamma})^{i}. \tag{4.3.27}$$

Fix $0<\tau<1$ such that $c\tau^{\frac{5}{2}(1-\gamma)}<1$. Then one can deduce from (4.3.27) that for $k=0,1\dots$

$$D_{\beta}(z_0,\tau^{k}R;q)\leq (\tau^{\frac{5}{2}\gamma})^{k}D_{\beta}(z_0,R;q)+\frac{cL^3}{\tau^2}\frac{1}{1-c\tau^{\frac{5}{2}(1-\gamma)}}. \tag{4.3.28}$$

For $0<r\leq R$, we see that there exists a $k\in\mathbb{N}\cup\{0\}$ such that $\tau^{k+1}R<r\leq\tau^{k}R$. Using similar arguments to case 2 of Proposition 4.13, we obtain

$$D_{\beta}(z_0,r;q)\leq\frac{c}{\tau^2}D_{\beta}(z_0,\tau^{k}R;q).$$

From this and (4.3.28), we infer

$$D_{\beta}(z_0,r;q)\leq\frac{c}{\tau^2}\Big(\frac{r}{\tau R}\Big)^{\frac{5}{2}\gamma}D_{\beta}(z_0,R;q)+\frac{cL^3}{\tau^4}\frac{1}{1-c\tau^{\frac{5}{2}(1-\gamma)}}.$$

□

Before proceeding further, let us introduce some further notation. Namely, we define the following scale-invariant quantities over half-balls:

$$A^{+}(z_0,r;v):=\sup_{t_0-r^2\leq t\leq t_0} r^{-1}\int\limits_{B^{+}(x_0,r)}|v(x,t)|^2dx, \tag{4.3.29}$$

$$B^{+}(z_0,r;v):=r^{-1}\int\limits_{t_0-r^2}^{t_0}\int\limits_{B^{+}(x_0,r)}|\nabla v(x,t)|^2dxdt, \tag{4.3.30}$$

$$C^{+}_{\infty}(z_0,r;v):=r^{-2}\int\limits_{t_0-r^2}^{t_0}\|v(\cdot,t)\|^3_{L^{3,\infty}(B^{+}(x_0,r))}dxdt \tag{4.3.31}$$

[14]Specifically, the iteration arguments are taken from Proposition 2.5 of [100].

and

$$D_1^+(z_0,r;q) := r^{-\frac{3}{2}} \int\limits_{t_0-r^2}^{t_0} \Big(\int\limits_{B^+(x_0,r)} |\nabla q|^{\frac{9}{8}} dx \Big)^{\frac{4}{3}} dt. \tag{4.3.32}$$

The following lemma was proven in [98]. We state it without proof.

Lemma 4.15. *Consider a pair of functions (v,q) that are a suitable weak solution to the Navier-Stokes equations in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times[t_0-R^2,t_0]$. Then for any $0<r\le\rho\le R$, we have*

$$D_1^+(z_0,r;q) \le c\Big(\frac{r}{\rho}\Big)^2 [D_1^+(z_0,\rho;q) + (B^+(z_0,\rho;v))^{\frac{3}{4}}]$$

$$+c\left(\frac{\rho}{r}\right)^{\frac{3}{2}} (A^+(z_0,\rho;v))^{\frac{1}{2}} B^+(z_0,\rho;v). \tag{4.3.33}$$

Next we state a new estimate. The statement and proof is very similar to Lemma 4.7, and is a key observation that allows us to work in the context of critical Lorentz spaces.

Lemma 4.16. *Let (v,q) be a suitable weak solution in $Q^+(z_0,1)$ near $\Gamma(x_0,1)\times[t_0-1,t_0]$. Then for $0<r<1$ the following holds (c is some universal constant):*

$$A^+(z_0,r/2;v) + B^+(z_0,r/2;v) \le c[C_\infty^+(z_0,r;v)^{\frac{4}{3}} + C_\infty^+(z_0,r;v)^{\frac{2}{3}}$$

$$+D_1^+(z_0,r;q)^{\frac{2}{3}} C_\infty^+(z_0,r;v)^{\frac{1}{3}}]. \tag{4.3.34}$$

Proof. The set up is the same as in Lemma 4.7; we recall it here for the convenience of the reader. Without loss of generality, consider $z_0=(0,0)$. Let $0<\frac{r}{2}\le s<\rho\le r<1$. Let $\eta_1\in C_0^\infty(B(\rho))$ be such that $0\le\eta_1\le 1$, $\eta_1=1$ on $B(s)$ and

$$|\nabla^\alpha \eta_1| \le \frac{C}{(\rho-s)^{|\alpha|}} \tag{4.3.35}$$

for $|\alpha|\le 2$. Let $\eta_2\in C_0^\infty(-\rho^2,\rho^2)$ be such that $0\le\eta_2\le 1$, $\eta_2=1$ on $[-s^2,s^2]$ and

$$|\eta_2'| \le \frac{C}{(\rho^2-s^2)} \le \frac{C}{r(\rho-s)} \le \frac{C}{(\rho-s)^2}. \tag{4.3.36}$$

Let $\varphi(x,t) := \eta_1(t)\eta_2(x)$. From (4.3.35)-(4.3.36), we infer that

$$|\nabla\varphi| \le \frac{C}{\rho-s}, \qquad |\nabla^2\varphi| \le \frac{C}{(\rho-s)^2}, \qquad \text{and } \ |\partial_t\varphi| \le \frac{C}{(\rho-s)^2}. \tag{4.3.37}$$

As was the case with Lemma 4.7, our aim is to estimate scale-invariant quantities by substituting φ into the local energy inequality and making use of Lemma 4.8. Such an approach has been previously utilised by Phuc in [87][15].

From the local energy inequality, we have that for almost every $t \in]-1,0[$:

$$\int\limits_{B^+} \varphi(x,t)|v(x,t)|^2 dx + 2\int\limits_{-1}^{t}\int\limits_{B^+} \varphi|\nabla v|^2 dxdt'$$

$$\leq \int\limits_{-1}^{t}\int\limits_{B^+} |v|^2(\Delta\varphi+\partial_t\varphi) + v\cdot\nabla\varphi(|v|^2+2q)dxdt'. \tag{4.3.38}$$

Let $I_1(s) := sA^+(0,s;v)$, $I_2(s) := sB^+(0,s;v)$ and $I(s) = I_1(s)+I_2(s)$. From (4.3.38), we deduce that

$$I(s) \leq \frac{3}{2}((1)+(2)+(3)). \tag{4.3.39}$$

Here,

$$(1) := \sup_{-s^2<t<0}\int\limits_{-1}^{t}\int\limits_{B^+} |v|^2(\Delta\varphi+\partial_t\varphi)dxdt', \tag{4.3.40}$$

$$(2) := \sup_{-s^2<t<0}\int\limits_{-1}^{t}\int\limits_{B^+} v\cdot\nabla\varphi|v|^2 dxdt' \tag{4.3.41}$$

and

$$(3) := \sup_{-s^2<t<0}\int\limits_{-1}^{t}\int\limits_{B^+} 2v\cdot\nabla\varphi q dxdt'. \tag{4.3.42}$$

By identical reasoning to the proof of Lemma 4.7, we have

$$(1) \leq c\frac{\rho}{(\rho-s)^2}\int\limits_{-\rho^2}^{0}\|v\|^2_{L^{3,\infty}(B^+(\rho))}dt' \leq \frac{\rho^{\frac{5}{3}}}{(\rho-s)^2}\left(\int\limits_{-\rho^2}^{0}\|v\|^3_{L^{3,\infty}(B^+(\rho))}dt'\right)^{\frac{2}{3}} \tag{4.3.43}$$

and

$$(2) \leq \left[\frac{Cr^{\frac{2}{3}}I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{\frac{3}{2}}} + \frac{Cr^{\frac{1}{6}}}{\rho-s}I_2(\rho)^{\frac{1}{4}}I_1(\rho)^{\frac{1}{4}}\right] \times \left(\int\limits_{-r^2}^{0}\|v\|^3_{L^{3,\infty}(B^+(r))}dt'\right)^{\frac{2}{3}}. \tag{4.3.44}$$

The main difference compared to the proof of Lemma 4.7 is the estimation of

$$(3) := \sup_{-s^2<t<0}\int\limits_{-1}^{t}\int\limits_{B^+} 2v\cdot\nabla\varphi q dxdt'.$$

[15] See Lemma 3.2 of [87].

Using that v is divergence-free, with zero trace on $\Gamma(0,1)$, we can write:

$$(3) := \sup_{-s^2<t<0} \int_{-1}^{t} \int_{B^+} 2v \cdot \nabla\varphi(q - [q]_{B^+(r)}) dx dt'.$$

From Hunt's inequality (Proposition 2.2 in Chapter 2), one can infer that

$$\|q(\cdot,t')-[q]_{B^+(r)}(t')\|_{L^{\frac{3}{2},1}(B^+(r))} \leq C\|q(\cdot,t')-[q]_{B^+(r)}(t')\|_{L_{\frac{7}{4}}(B^+(r))}\|\chi_{B^+(r)}\|_{L^{\frac{21}{2},\frac{7}{3}}(B^+(r))} \leq$$

$$\leq cr^{\frac{2}{7}}\|q(\cdot,t') - [q]_{B^+(r)}(t')\|_{L_{\frac{7}{4}}(B^+(r))}. \tag{4.3.45}$$

Noting that

$$\frac{7}{4} < \frac{3\cdot\frac{9}{8}}{3-\frac{9}{8}} = \frac{9}{5},$$

the following Poincaré inequality holds:

$$\|q(\cdot,t') - [q]_{B^+(r)}(t')\|_{L_{\frac{7}{4}}(B^+(r))} \leq cr^{\frac{1}{21}}\|\nabla q(\cdot,t')\|_{L_{\frac{9}{8}}(B^+(r))}. \tag{4.3.46}$$

Using (4.3.45), (4.3.46) and Hunt's inequality, we deduce that

$$|(3)| \leq \frac{Cr^{\frac{1}{3}}}{\rho - s} \int_{-r^2}^{0} \|\nabla q(\cdot,t')\|_{L_{\frac{9}{8}}(B^+(r))}\|v(\cdot,t')\|_{L^{3,\infty}(B^+(r))} dt'. \tag{4.3.47}$$

Thus, by using Hölder's inequality in time we obtain:

$$|(3)| \leq \frac{Cr^{\frac{1}{3}}}{\rho - s}\left(\int_{-r^2}^{0} \|\nabla q\|^{\frac{3}{2}}_{L_{\frac{9}{8}}(B^+(r))} dt'\right)^{\frac{2}{3}}\left(\int_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt'\right)^{\frac{1}{3}}. \tag{4.3.48}$$

Using (4.3.43)-(4.3.44) and (4.3.48), together with by Young's inequality, we infer that

$$I(s) \leq \frac{r^{\frac{5}{3}}}{(\rho-s)^2}\left(\int_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt'\right)^{\frac{2}{3}} + \frac{1}{2}I(\rho)$$

$$+\left[\frac{Cr^{\frac{4}{3}}}{(\rho-s)^3} + \frac{Cr^{\frac{1}{3}}}{(\rho-s)^2}\right] \times \left(\int_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt'\right)^{\frac{4}{3}}$$

$$+\frac{Cr^{\frac{1}{3}}}{\rho - s}\left(\int_{-r^2}^{0} \|\nabla q\|^{\frac{3}{2}}_{L_{\frac{9}{8}}(B^+(r))} dt'\right)^{\frac{2}{3}}\left(\int_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt'\right)^{\frac{1}{3}}. \tag{4.3.49}$$

By Lemma 4.8, we obtain

$$I(r/2) \leq Cr^{-\frac{1}{3}} \left(\int\limits_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt' \right)^{\frac{2}{3}} + Cr^{-\frac{5}{3}} \left(\int\limits_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt' \right)^{\frac{4}{3}}$$

$$+Cr^{-\frac{2}{3}} \left(\int\limits_{-r^2}^{0} \|\nabla q\|^{\frac{3}{2}}_{L_{\frac{9}{8}}(B^+(r))} dt' \right)^{\frac{2}{3}} \left(\int\limits_{-r^2}^{0} \|v\|^3_{L^{3,\infty}(B^+(r))} dt' \right)^{\frac{1}{3}}.$$

From here the conclusion is immediate. □

The new scale-invariant estimates given by the previous lemma, together with arguments presented by Seregin in [100][16], enable us to prove a new decay estimate for the pressure near the flat part of the boundary. These estimates will play a key role in the proof of Theorem 4.5.

Proposition 4.17. *Let $\beta \in [3, \infty[$. Assume that (v, q) is a suitable weak solution in $Q^+(z_0, R)$ near $\Gamma(x_0, R) \times [t_0 - R^2, t_0]$. Suppose, in addition, that*

$$\|v\|_{L_\infty(t_0-R^2,t_0;L^{3,\beta}(B^+(x_0,R)))} \leq L < \infty. \tag{4.3.50}$$

Then, for any $\gamma \in]0, 1[$, there exists a constant c_2 depending on γ and L only such that, for $0 < r \leq R$, we have

$$D_1^+(z_0, r; q) \leq c_2 \left[\left(\frac{r}{R}\right)^{2\gamma} D_1^+(z_0, R; q) + 1 \right]. \tag{4.3.51}$$

Proof. Let $\rho \leq R/2$. Then by Lemma 4.16 and (4.3.50) we have

$$B^+(z_0, \rho; v) \leq c[L^4 + L^2 + (D_1^+(z_0, 2\rho; q))^{\frac{2}{3}} L]. \tag{4.3.52}$$

Using Hunt's inequality (Proposition 2.2 in Chapter 2), we deduce that the following inequality holds for $0 \leq s \leq R$:

$$\int\limits_{B^+(x_0,s)} |v(x,t)|^2 dx \leq C\|\chi_{B^+(0,s)}\|_{L^{3,1}(B^+(0,s))} \||v|^2(\cdot,t)\|_{L^{\frac{3}{2},\infty}(B^+(x_0,s))}$$

$$\leq Cs\|v(\cdot,t)\|^2_{L^{3,\infty}(B^+(x_0,s))}.$$

Thus, the following estimate holds:

$$A^+(z_0, s; v) \leq CL^2. \tag{4.3.53}$$

[16] In particular, p.225 of [100].

With the estimates (4.3.52)-(4.3.53) in hand, we may complete the proof by using arguments essentially due to Seregin [100][17]. We provide these arguments for the sake of completeness.

From (4.3.52)-(4.3.53), along with Lemma 4.15, we see that for any $0 < r \leq \rho \leq R/2$:

$$D_1^+(z_0, r; q) \leq c\Big(\frac{r}{\rho}\Big)^2 \Big[D_1^+(z_0, \rho; q) + (L^4 + L^2 + (D_1^+(z_0, 2\rho; q))^{\frac{2}{3}} L)^{\frac{3}{4}}\Big] +$$

$$+ \Big(\frac{\rho}{r}\Big)^{\frac{3}{2}} L \Big[L^4 + L^2 + (D_1^+(z_0, 2\rho; q))^{\frac{2}{3}} L\Big].$$

Then by Young's inequality, one obtains that for $0 < r \leq \rho \leq R/2$:

$$D_1^+(z_0, r; q) \leq c(L) \Big[\Big(\frac{r}{\rho}\Big)^2 (D_1^+(z_0, 2\rho; q) + 1) + \Big(\frac{\rho}{r}\Big)^{\frac{17}{2}}\Big].$$

Clearly this implies that the following estimate holds for $0 < r \leq \rho \leq R$:

$$D_1^+(z_0, r; q) \leq c(L) \Big[\Big(\frac{r}{\rho}\Big)^2 (D_1^+(z_0, \rho; q) + 1) + \Big(\frac{\rho}{r}\Big)^{\frac{17}{2}}\Big].$$

The conclusion now follows by identical reasoning as Proposition 4.14. □

4.3.2 ε-Regularity Criterion in Weak Lebesgue Spaces

Before proceeding with the main part of the proof of Theorem 4.5, let us first state and prove a new ε-regularity criterion. This will play an important role in the proof of Theorem 4.5. It is a generalisation of the basic ε-regularity criterion found in the paper of Lin [78] and the paper of Ladyzhenskaya and Seregin [74].

Let us recall some notation:

$$A(z_0, r; v) := \sup_{t_0 - r^2 \leq t \leq t_0} r^{-1} \int\limits_{B(x_0, r)} |v(x,t)|^2 dx, \tag{4.3.54}$$

$$B(z_0, r; v) := r^{-1} \int\limits_{t_0 - r^2}^{t_0} \int\limits_{B(x_0, r)} |\nabla v(x,t)|^2 dx dt, \tag{4.3.55}$$

$$C(z_0, r; v) := r^{-2} \int\limits_{t_0 - r^2}^{t_0} \int\limits_{B(x_0, r)} |v|^3 dx dt, \tag{4.3.56}$$

[17] Namely, p.225 in [100].

$$D(z_0, r; q) := r^{-2} \int\limits_{t_0-r^2}^{t_0} \int\limits_{B(x_0,r)} |q|^{\frac{3}{2}} dxdt, \tag{4.3.57}$$

$$C_\infty(z_0, r; v) := r^{-2} \int\limits_{t_0-r^2}^{t_0} \|v\|^3_{L^{3,\infty}(B(x_0,r))} dt \tag{4.3.58}$$

and

$$D_\infty(z_0, r; q) := r^{-2} \int\limits_{t_0-r^2}^{t_0} \|q\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B(x_0,r))} dt. \tag{4.3.59}$$

Now we state our new ε-regularity criterion, here it is.

Theorem 4.18. *Let (v, q) be a suitable weak solution in Q. Then there exist universal constants ϵ_0 and c_{0k} (with $k = 1, 2 \ldots$) with the following property. Assume*

$$C_\infty(0, 1; v) + D_\infty(0, 1; q) < \epsilon_0. \tag{4.3.60}$$

Then for any natural number k, $\nabla^{k-1}v$ is Hölder continuous in $\bar{Q}(1/8)$ and the following bound is valid:

$$\max_{\bar{Q}(1/8)} |\nabla^{k-1}v(x, t)| < c_{ok}. \tag{4.3.61}$$

Proof. From Lemma 4.7 and (4.3.60) it follows that

$$A(0, 1/2; v) + B(0, 1/2; v) \leq c \left(\epsilon_0 + \epsilon_0^2\right)^{\frac{2}{3}}. \tag{4.3.62}$$

Here c will always denote some universal constant. By Proposition 2.3 and the Sobolev embedding theorem, one can show that

$$\|v(\cdot, t)\|^3_{L_3(B(1/2))} \leq \|v(\cdot, t)\|^{\frac{3}{2}}_{L_6(B(1/2))} \|v(\cdot, t)\|^{\frac{3}{2}}_{L_2(B(1/2))}$$

$$\leq \|\nabla v(\cdot, t)\|^{\frac{3}{2}}_{L_2(B(1/2))} \|v(\cdot, t)\|^{\frac{3}{2}}_{L_2(B(1/2))} + \|v(\cdot, t)\|^3_{L_2(B(1/2))}.$$

This inequality, combined with an application of the Hölder inequality in time variables, implies the following estimate

$$C(0, 1/2, v) \leq c[A(0, 1/2; v)^{\frac{3}{4}} B(0, 1/2; v)^{\frac{3}{4}} + A(0, 1/2; v)^{\frac{3}{2}}].$$

Thus, by (4.3.62) we have

$$C(0, 1/2, v) \leq c \left(\epsilon_0 + \epsilon_0^2\right). \tag{4.3.63}$$

For similar reasons one sees that

$$\|\mathrm{div}\,(v\otimes v)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}\leq c[A(0,1/2;v)^{\frac{1}{2}}B(0,1/2;v)^{\frac{1}{2}}+A(0,1/2;v)^{\frac{1}{3}}B(0,1/2;v)^{\frac{2}{3}}].$$

Thus,

$$\|\mathrm{div}\,(v\otimes v)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}\leq \left(\epsilon_0+\epsilon_0^2\right)^{\frac{2}{3}}. \tag{4.3.64}$$

By Hölder's inequality in space and time variables, it is obvious that

$$\begin{aligned}\|v\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}&\leq c(A(0,1/2;v)^{\frac{1}{2}}+B(0,1/2;v)^{\frac{1}{2}})\\ &\leq c\left(\epsilon_0+\epsilon_0^2\right)^{\frac{1}{3}}\end{aligned} \tag{4.3.65}$$

Using Hunt's inequality (Proposition 2.2 in Chapter 2), we infer that

$$\int\limits_{B(1/2)}|q(x,t)|^{\frac{9}{8}}dx\leq c\||q(\cdot,t)|^{\frac{9}{8}}\|_{L^{\frac{4}{3},\infty}(B(1/2))}\|\chi_{B(1/2)}\|_{L^{4,1}(B(1/2))}=c\|q(\cdot,t)\|^{\frac{9}{8}}_{L^{\frac{3}{2},\infty}(B(1/2))}.$$

Hence,

$$\|q\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}\leq c\epsilon_0^{\frac{2}{3}}. \tag{4.3.66}$$

Local interior regularity theory for the Stokes system (Proposition 2.16 in 'Chapter 2: Preliminary Material') gives

$$\begin{aligned}&\|\partial_t v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/4))}+\|\nabla^2 v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/4))}\\ &\quad+\|\nabla q\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/4))}\\ &\leq c[\|\mathrm{div}\,((v\otimes v)\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}+\|v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}\\ &\quad+\|\nabla v\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}+\|q\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/2))}].\end{aligned} \tag{4.3.67}$$

Using this, together with (4.3.64)-(4.3.66), we obtain that

$$\|\nabla q\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/4))}\leq c[\left(\epsilon_0+\epsilon_0^2\right)^{\frac{1}{3}}+\left(\epsilon_0+\epsilon_0^2\right)^{\frac{2}{3}}]. \tag{4.3.68}$$

Noting that

$$\frac{3}{2}<\frac{3\cdot\frac{9}{8}}{3-\frac{9}{8}}=\frac{9}{5},$$

the following Poincaré inequality holds:

$$\|q(\cdot,t)-[q]_{B(1/4)}(t)\|_{L_{\frac{3}{2}}(B(1/4))}\leq c\|\nabla q(\cdot,t)\|_{L_{\frac{9}{8}}(B(1/4))}. \tag{4.3.69}$$

This, together with (4.3.68), implies

$$\|q-[q]_{B(1/4)}\|_{L_{\frac{3}{2}}(Q(1/4))}\leq c\|\nabla q\|_{L_{\frac{9}{8},\frac{3}{2}}(Q(1/4))}\leq c[\left(\epsilon_0+\epsilon_0^2\right)^{\frac{1}{3}}+\left(\epsilon_0+\epsilon_0^2\right)^{\frac{2}{3}}]. \tag{4.3.70}$$

But by Hunt's inequality

$$|[q(\cdot,t)]_{B(1/4)}| \leq c\|q(\cdot,t)\|_{L^{\frac{3}{2},\infty}(B)}\|\chi_{B(3/4)}\|_{L^{3,1}(B)} \leq c\|q(\cdot,t)\|_{L^{\frac{3}{2},\infty}(B)}.$$

Therefore, we can deduce that

$$\|q\|_{L_{\frac{3}{2}}(Q(1/4))} \leq c\big[\big(\epsilon_0+\epsilon_0^2\big)^{\frac{1}{3}}+\big(\epsilon_0+\epsilon_0^2\big)^{\frac{2}{3}}\big]. \tag{4.3.71}$$

This, along with (4.3.63), gives

$$C(0,1/4;v)+D(0,1/4;v) \leq c\big[\big(\epsilon_0+\epsilon_0^2\big)^{\frac{1}{2}}+\big(\epsilon_0+\epsilon_0^2\big)\big].$$

Choosing ϵ_0 sufficiently small, gives the conclusion by Lemma 3.5 (Chapter 3). $\square$

Remark 4.19. *With certain adjustments to the proof, namely local boundary regularity for the Stokes system and the boundary analogue of Lemma 3.5 (see [98] and [101]), we have the following near the flat part of the boundary.*
Let (v,q) be a suitable weak solution of the Navier-Stokes equations in $Q^+(1)$ near $\Gamma(0,1)\times[-1,0]$. Then there exist universal constants ϵ_0 and c_0 such that if

$$C^+_\infty(0,1;v)+D^+_\infty(0,1;q)<\epsilon_0,$$

then v is Hölder continuous in the closure of $Q^+(1/2)$ and the following is valid:

$$\sup_{Q^+(1/2)}|v(x,t)| \leq c_0.$$

4.3.3 Proof of Theorem 4.5

In proving Theorem 4.5, we will use the following lemma which is stated and proven in Seregin's paper [100] (specifically Lemma 3.3 of [100]).

Lemma 4.20. *Let (v,q) be a suitable weak solution to the Navier-Stokes equations in $Q^+(1)$ near $\Gamma(0,1)\times[-1,0]$. There exists a universal constant $\epsilon_3>0$ with the following property. Assume that for some $0<R_0<1$, v satisfies*

$$\sup_{0<R\leq R_0}\frac{1}{R^2}\int\limits_{Q^+(R)}|v|^3dz<\epsilon_3. \tag{4.3.72}$$

Then, $(0,0)$ is a regular point of v. In particular, there exists $r\in]0,1[$ with $v\in L_\infty(Q^+(r))$.

We will also make use of the following proposition.

Proposition 4.21. *Suppose (v,q) satisfy the hypotheses of Theorem 4.5. Then the inequality*

$$\|v(\cdot,t)\|_{L^{3,\beta}(B^+)} \le \|v\|_{L_\infty(-1,0;L^{3,\beta}(B^+))} \tag{4.3.73}$$

holds for all $t\in[-1,0]$, and the function

$$t \to \int\limits_{B^+} v(x,t)\cdot w(x)dx$$

is continuous on $[-1,0]$ for any $w\in L^{\frac{3}{2},\frac{\beta}{\beta-1}}(B^+)$.

Proof. The proof of Proposition 4.21 is essentially contained in Phuc's paper [87][18]. We recall the details for completeness.

Let us recall the known fact[19] that $C_0^\infty(B^+)$ is dense in $L^{s,p}(B^+)$, for $1<s,\, p<\infty$. Also, recall[20] that

$$(L^{s,p}(B^+))' = L^{s',p'}(B^+), \qquad s'=\frac{s}{s-1}, \qquad p'=\frac{p}{p-1}.$$

Furthermore, for any $S\in(L^{s,p}(B^+))'$ there exists a $f\in L^{s',p'}(B^+)$ such that the following holds for all $g\in L^{s,p}(B^+)$. Namely,

$$S(g)=\int\limits_{B^+} fgdx.$$

Conversely, if $F\in L^{s',p'}(B^+)$ and $g\in L^{s,p}(B^+)$ then $T_F\in(L^{s,p}(B^+))'$, where

$$T_F(g)=\int\limits_{B^+} Fgdx.$$

The hypotheses of Theorem 4.5 imply that

$$v\in W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+)\hookrightarrow C([-1,0];L_{\frac{9}{8}}(B^+)).$$

Hence, for any $\varphi(x)\in C_0^\infty(B^+)$ we have that

$$t\to\int\limits_{B^+} v(x,t)\cdot\varphi(x)dx\in C([-1,0]). \tag{4.3.74}$$

[18] In particular, Lemma 4.1 in [87].
[19] See Lemma 3.3 in [80], for example.
[20] See section 1.4.3 of Grafakos' book [48], for example.

The assumption of Theorem 4.5 ($v \in L_\infty(-1,0;L^{3,\beta}(B^+))$) implies by Hunt's inequality[21] that for almost every $t \in [-1,0]$ the following estimate holds for any $\varphi \in C_0^\infty(B^+)$:

$$|\int\limits_{B^+} v(x,t)\cdot\varphi(x)dx| \leq \|v\|_{L_\infty(-1,0;L^{3,\beta}(B^+))}\|\varphi\|_{L^{\frac{3}{2},\frac{\beta}{\beta-1}}(B^+)}. \tag{4.3.75}$$

We now use the same arguments as in Phuc's paper (specifically, p.752 of [87]) to show that (4.3.74)-(4.3.75) imply the desired conclusion. Using (4.3.74), we see that the above estimate in fact holds for every $t \in [-1,0]$. This implies, from the aforementioned facts regarding Lorentz spaces, that for every $t \in [-1,0]$,

$$\|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)} \leq \|v\|_{L_\infty(-1,0;L^{3,\beta}(B^+))}. \tag{4.3.76}$$

The remaining conclusion of the proposition follows from (4.3.74)-(4.3.76), together with the previously mentioned facts concerning Lorentz spaces. $\square$

Step 1: Rescaling Procedure

From the above proposition, we have that, for every $t \in [-1,0]$,

$$\|v(\cdot,t)\|_{L^{3,\beta}(B^+(1))} \leq L := \|v\|_{L_\infty(-1,0;L^{3,\beta}(B^+(1)))} < \infty. \tag{4.3.77}$$

Let us describe the rescaling procedure taken from Seregin's paper [100]. As in [100], the rescaling procedure arises from assuming Theorem 4.5 is false. This implies that v has no representative that is Hölder continuous on the closure of $Q^+(1/2)$. From the local boundary regularity theory for the linear Stokes system (Proposition 3 of [95]), we see that boundedness of v on $Q^+(3/4)$ would imply that v is Hölder continuous on $Q^+(1/2)$. As in [100], we infer from the contradiction assumption that there exists a singular point z_0 in the closure of $Q^+(3/4)$. From our previously established interior regularity result (Theorem 4.2), it must be the case that z_0 lies on the boundary $\bar{\Gamma}(0,3/4)$. By using a suitable translation and rescaling, we may assume without loss of generality that $z_0 = 0$.

We now repeat arguments used by Seregin in [100] (specifically, p.230 of [100]) to show that there exists a decreasing sequence $R_k < \frac{1}{2}$ tending to zero, together with a universal constant ϵ_3, such that for, $k = 1,2\ldots,$

$$\frac{1}{R_k^2}\int\limits_{Q^+(R_k)} |v|^3 dz \geq \epsilon_3. \tag{4.3.78}$$

[21]For Hunt's inequality, we refer the reader to Proposition 2.2 in Chapter 2.

Lemma 4.20, together with the fact that $(0,0)$ is a singular point of v, implies that for all $R_0 \in]0,1[$ we have

$$\sup_{0<R<R_0} \frac{1}{R^2} \int\limits_{Q^+(R)} |v(x,t)|^3 dz \geq \epsilon_3. \tag{4.3.79}$$

This then implies that there exists $R_k \to 0$ as $k \to +\infty$ such that (4.3.78) holds true.

Extending the functions (v,q) outside $Q^+(1)$ to zero, for $(y,s) \in \mathbb{R}^3_+ \times]-\infty,0[$, we perform the same rescaling as in Seregin's paper [100]:

$$v^{(k)}(y,s) = R_k v(R_k y, R_k^2 s), \quad q^{(k)}(y,s) = R_k^2 q(R_k y, R_k^2 s).$$

Step 2: Passage to the Limit and Properties of the Limit Functions

Next, we establish convergence properties of the rescaled functions, along with properties of the limit velocity. The new scale-invariant estimates (Lemma 4.16) and the new pressure estimates (Proposition 4.17), near the flat part of the boundary, play an important role in the proof. With these new estimates in hand, the remaining parts of the proof are essentially contained in Seregin's paper [100][22] and Phuc's paper [87][23]. We provide full details for completeness.

Proposition 4.22. *There exists a subsequence, still denoted by, $(v^{(k)}, q^{(k)})$, along with a pair of functions (v_∞, q_∞), such that*

$$v^{(k)} \overset{*}{\rightharpoonup} v_\infty \ \ in\ L_\infty(-\infty,0; L^{3,\beta}(\mathbb{R}^3_+)) \ \ and \ \ div\, v_\infty = 0 \ \ in\ \mathbb{R}^3_+ \times]-\infty,0[. \tag{4.3.80}$$

For any $a > 0$, we have that

$$v^{(k)} \overset{*}{\rightharpoonup} v_\infty \ \ in\ \ L_{2,\infty}(Q^+(a)), \tag{4.3.81}$$

$$\nabla v^{(k)} \rightharpoonup \nabla v_\infty \ \ in\ \ L_2(Q^+(a)) \tag{4.3.82}$$

and

$$v^{(k)} \rightharpoonup v_\infty \ \ in\ \ L_4(Q^+(a)). \tag{4.3.83}$$

Additionally,

$$v^{(k)} \rightharpoonup v_\infty \ \ in\ \ W^{2,1}_{\frac{4}{3}}(Q^+(a)) \ \ and \ \ W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(a)). \tag{4.3.84}$$

Moreover,

$$q^{(k)} \rightharpoonup q_\infty \ \ in\ \ W^{1,0}_{\frac{4}{3}}(Q^+(a)) \ \ and \ \ W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(a)). \tag{4.3.85}$$

[22] Specifically, arguments found on pp.230-233 of [100].
[23] Specifically, arguments found on p.755 and p.757 of [87].

One also has that

$$v^{(k)} \to v_\infty \ \ in\ \ C([-a^2,0];L_s(B^+(a)))\ \ (s\in]1,3[) \tag{4.3.86}$$

and

$$v_\infty(x,0)=0 \tag{4.3.87}$$

for almost every $x\in\mathbb{R}^3_+$. *Furthermore,*

$$v^{(k)} \to v_\infty \ \ in\ \ L_3(Q^+(a)) \tag{4.3.88}$$

and v_∞ *satisfies the lower bound*

$$\int\limits_{Q^+} |v_\infty|^3 dz \geq \epsilon_3. \tag{4.3.89}$$

Finally, for any $a>0$, (v_∞, q_∞) *forms a suitable weak solution to the Navier-Stokes equations in* $Q^+(a)$ *near* $\Gamma(a)\times[-a^2,0]$.

Proof. Fix $a>0$ and let $k(a)$ be such that

$$R_k a < \frac{1}{4} \ \text{ for all } \ k\geq k(a). \tag{4.3.90}$$

From Proposition 4.21, we see that for all $t\in[-(2a)^2,0]$ and $k\geq k(a)$, we have

$$v^{(k)}(\cdot,t)\in L^{3,\beta}(\mathbb{R}^3_+) \ \text{ and} \tag{4.3.91}$$

$$\|v^{(k)}(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)} \leq \|v^{(k)}\|_{L_\infty(-(2a)^2,0;L^{3,\beta}(\mathbb{R}^3_+))} \leq L. \tag{4.3.92}$$

Using these two facts, we can deduce that (4.3.80) holds true for an appropriate subsequence. Additionally, weak lower semicontinuity implies that

$$\|v_\infty\|_{L_\infty(-\infty,0;L^{3,\beta}(\mathbb{R}^3_+))} \leq \limsup_{k\to\infty} \|v^{(k)}\|_{L_\infty(-\infty,0;L^{3,\beta}(\mathbb{R}^3_+))} = L<\infty. \tag{4.3.93}$$

Using Hunt's inequality (Proposition 2.2 in Chapter 2), we deduce that the following inequality holds for $k\geq k(a)$ and $t\in[-(2a)^2,0]$:

$$\int\limits_{B^+(2a)} |v^{(k)}(x,t)|^2 dx \leq C\|\chi_{B^+(2a)}\|_{L^{3,1}(B^+(2a))} \||v^{(k)}|^2(\cdot,t)\|_{L^{\frac{3}{2},\infty}(B^+(2a))}$$

$$\leq Ca\|v^{(k)}(\cdot,t)\|^2_{L^{3,\infty}(B^+(2a))}.$$

This and (4.3.92) imply that the following estimate holds:

$$A^+(0,2a;v^{(k)})\leq CL^2. \tag{4.3.94}$$

From Lemma 4.16, we infer that

$$B^+(0,2a;v^{(k)}) + D_1^+(0,2a,q^{(k)}) = B^+(0,2aR_k;v) + D_1^+(0,2aR_k,q)$$
$$\leq c_0(L)[1 + D_1^+(0,4aR_k,q)]. \tag{4.3.95}$$

Using Proposition 4.17, with $\gamma = 1/2$, one obtains

$$D_1^+(0,4aR_k,q) \leq c_1(L)[4aR_k D_1^+(0,1,q) + 1] \leq c_1(L)[D_1^+(0,1,q) + 1].$$

Thus,

$$B^+(0,2a;v^{(k)}) + D_1^+(0,2a,q^{(k)}) \leq c_2(L)[D_1^+(0,1,q) + 1]. \tag{4.3.96}$$

From Proposition 2.3 in Chapter 2, we deduce that

$$\|v^{(k)}(\cdot,t)\|_{L_4(B^+(2a))} \leq C\|v^{(k)}(\cdot,t)\|^{\frac{1}{2}}_{L^{3,\beta}(B^+(2a))}\|v^{(k)}(\cdot,t)\|^{\frac{1}{2}}_{L_6(B^+(2a))}$$
$$\leq CL^{\frac{1}{2}}\|v^{(k)}(\cdot,t)\|^{\frac{1}{2}}_{L_6(B^+(2a))}.$$

From this, the Sobolev embedding theorem, (4.3.94) and (4.3.96), it follows that

$$\|v^{(k)}\|_{L_4(Q^+(2a))} \leq c_0(L,a)(D_1^+(0,1,q) + 1)^{\frac{1}{4}}. \tag{4.3.97}$$

Taking into account (4.3.94) and (4.3.96)-(4.3.97), we can deduce that (4.3.81)-(4.3.83) hold true, for an appropriate subsequence.

Let us now focus on proving (4.3.84)-(4.3.85). Observe that

$$\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{4}{3}}(Q^+(2a))} \leq \|v^{(k)}\|_{L_4(Q^+(2a))}\|\nabla v^{(k)}\|_{L_2(Q^+(2a))}. \tag{4.3.98}$$

This and (4.3.96)-(4.3.97) implies that

$$\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{4}{3}}(Q^+(2a))} \leq c_1(L,a)(D_1^+(0,1,q) + 1)^{\frac{3}{4}}. \tag{4.3.99}$$

Using interpolation of Lebesgue spaces, the Sobolev inequality and Hölder's inequality, one can show that

$$\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))} \leq C_2(a)[\|v^{(k)}\|_{L_{2,\infty}(Q^+(2a))}\|\nabla v^{(k)}\|_{L_2(Q^+(2a))}$$
$$+\|v^{(k)}\|^{\frac{2}{3}}_{L_{2,\infty}(Q^+(2a))}\|\nabla v^{(k)}\|^{\frac{4}{3}}_{L_2(Q^+(2a))}]. \tag{4.3.100}$$

From this, (4.3.94) and (4.3.96), it follows that

$$\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))} \leq C_3(a,L)(D_1^+(0,1,q) + 1)^{\frac{2}{3}}. \tag{4.3.101}$$

Now we use identical arguments to those used in Seregin's paper [100] to show (4.3.84)-(4.3.85). Let us recall those arguments now. The local boundary regularity theory for the linear Stokes system (Proposition 2.17 in 'Chapter 2: Preliminary Material') implies that

$$\|v^{(k)}\|_{W^{2,1}_{\frac{4}{3}}(Q^+(a))} + \|\nabla q^{(k)}\|_{L_{\frac{4}{3}}(Q^+(a))} \leq C(a)[\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{4}{3}}(Q^+(2a))} + \|v^{(k)}\|_{W^{1,0}_{\frac{4}{3}}(Q^+(2a))}$$

$$+\|q^{(k)} - [q^{(k)}]_{B^+(2a)}\|_{L_{\frac{4}{3}}(Q^+(2a))}] \tag{4.3.102}$$

and

$$\|v^{(k)}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} + \|\nabla q^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} \leq C(a)[\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))}$$

$$+\|v^{(k)}\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))} + \|q^{(k)} - [q^{(k)}]_{B^+(2a)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))}]. \tag{4.3.103}$$

Using (4.3.94), (4.3.96), (4.3.99) and (4.3.101), we then deduce that

$$\|v^{(k)}\|_{W^{2,1}_{\frac{4}{3}}(Q^+(a))} + \|\nabla q^{(k)}\|_{L_{\frac{4}{3}}(Q^+(a))} \leq c_4(a, L, D_1^+(0,1,q)) \tag{4.3.104}$$

and

$$\|v^{(k)}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} + \|\nabla q^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(a))} \leq c_5(a, L, D_1^+(0,1,q)). \tag{4.3.105}$$

Having obtained (4.3.104)-(4.3.105), (4.3.84)-(4.3.85) follow, for an appropriate subsequence.

Let us now prove (4.3.86)-(4.3.87), by using arguments essentially contained in Phuc's paper [87] (specifically, p.755 and p.757 of [87]). From (4.3.104) and compactness of the embedding[24]

$$W^{2,1}_{\frac{4}{3}}(Q^+(a)) \hookrightarrow C([-a^2,0]; L_{\frac{4}{3}}(B^+(a))),$$

we conclude the following. Namely, by taking further subsequences and using further Cantor diagonalisation if necessary, we see that, for all $a > 0$,

$$v^{(k)} \to v_\infty \text{ in } \; C([-a^2,0]; L_{\frac{4}{3}}(B^+(a))). \tag{4.3.106}$$

For $s \in]4/3, 3[$ there exists $\theta = \theta(s) \in]0,1[$ such that

$$\frac{1}{s} = \frac{3\theta}{4} + \frac{1-\theta}{3}.$$

[24]Such compact embeddings were discussed in 'Chapter 2: Preliminary Material'.

Using Proposition 2.3 in Chapter 2 and (4.3.92), we can argue as in [87] to conclude that for $k, m \geq k(a)$ and $t, t' \in [-a^2, 0]$:

$$\|v^{(k)}(\cdot,t) - v^{(m)}(\cdot,t')\|_{L_s(B^+(a))}$$
$$\leq C\|v^{(k)}(\cdot,t) - v^{(m)}(\cdot,t')\|^{\theta}_{L_{\frac{4}{3}}(B^+(a))}\|v^{(k)}(\cdot,t) - v^{(m)}(\cdot,t')\|^{1-\theta}_{L^{3,\beta}(B^+(a))}$$
$$\leq C(2L)^{1-\theta}\|v^{(k)}(\cdot,t) - v^{(m)}(\cdot,t')\|^{\theta}_{L_{\frac{4}{3}}(B^+(a))}. \tag{4.3.107}$$

From this, along with (4.3.106), we infer that, for $s \in]4/3, 3[$,

$$v^{(k)} \to v_\infty \text{ in } C([-a^2, 0]; L_s(B^+(a))).$$

Fix $y \in \mathbb{R}^3_+$ and let a be such that $B^+(y,1) \subset B^+(a)$. Using the same reasoning as used by Phuc in [87][25], we see that

$$\int\limits_{B^+(y,1)} |v_\infty(x,0)|dx \leq \int\limits_{B^+(y,1)} |v_\infty(x,0) - v^{(k)}(x,0)|dx + \int\limits_{B^+(y,1)} |v^{(k)}(x,0)|dx$$
$$\leq \|v_\infty - v^{(k)}\|_{C([-a^2,0];L_1(B^+(a)))} + \|\chi_{B^+(y,1)}\|_{L^{\frac{3}{2},\frac{\beta}{\beta-1}}(B(y,1))}\|v^{(k)}(\cdot,0)\|_{L^{3,\beta}(B(y,1))}$$
$$\leq \|v_\infty - v^{(k)}\|_{C([-a^2,0];L_1(B^+(a)))} + c\|v(\cdot,0)\|_{L^{3,\beta}(B^+(R_k y,R_k))}. \tag{4.3.108}$$

Recalling Proposition 4.21, we have that $v(\cdot,\tau) \in L^{3,\beta}(B^+(1))$ for all $\tau \in [-1,0]$. This, in conjunction with the fact that continuous functions with compact support in $B^+(1)$ are dense[26] in $L^{3,\beta}(B^+(1))$ for $\beta < \infty$, gives that

$$\lim_{k\to 0} \|v(\cdot,0)\|_{L^{3,\beta}(B^+(R_k y,R_k))} = 0.$$

This, together with (4.3.106)-(4.3.108), implies that

$$\int\limits_{B^+(y,1)} |v_\infty(x,0)|dx = 0. \tag{4.3.109}$$

Next we prove (4.3.88)-(4.3.89) by using arguments from Seregin's paper [100][27]. Furthermore, we will prove that for all $a > 0$, (v_∞, q_∞) form a suitable weak solution to the Navier-Stokes equations in $Q^+(a)$ near $\Gamma(a) \times [-a^2, 0]$. First, the proof of (4.3.88) follows from (4.3.83), (4.3.86) and the interpolation of Lebesgue spaces. Second, (4.3.78) implies by inverse scaling that

$$\int\limits_{Q^+} |v^{(k)}|^3 dz \geq \epsilon_3. \tag{4.3.110}$$

[25] Specifically, p.757 of [87].
[26] See Lemma 3.3 of [80], for example.
[27] Specifically, pp.231-232 of [100].

The above and (4.3.88) immediately give (4.3.89). Finally, we note that for all $k \geq k(a)$, $(v^{(k)}, q^{(k)})$ are a suitable weak solution of the Navier-Stokes system in $Q^+(a)$ near $\Gamma(a) \times [-a^2, 0]$. From this and the convergence properties (4.3.81)-(4.3.82), (4.3.84)-(4.3.85) and (4.3.88), we deduce that (v_∞, q_∞) are a suitable weak solution of the Navier-Stokes system in $Q^+(a)$ near $\Gamma(a) \times [-a^2, 0]$. $\square$

It is necessary to introduce the following notation. For $\beta \in [3, \infty]$, define:

$$D^+_\beta(z_0, r; q) := \frac{1}{r^2} \int\limits_{t_0 - r^2}^{t_0} \|q(\cdot, t) - [q]_{B^+(x_0,r)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B^+(x_0,r))} dt. \tag{4.3.111}$$

Next, we establish properties of the limit pressure, which will later be used in proving certain decay properties of the limit velocity. The new interior pressure estimates (Proposition 4.14) and the new pressure estimates near the flat part of the boundary (Proposition 4.17) play an important role in the proof of these properties. With these estimates in hand, the remaining parts of the proof of these properties are essentially contained in Seregin's paper [100][28]. In the setting of Lorentz spaces there are some minor differences, hence we provide full details for the convenience of the reader.

Lemma 4.23. *There exists a positive constant $c_5(L, D^+_1(0,1,q))$ with the following property. Fix $h > 0$ and $T > 0$ arbitrarily, then*

$$D_\beta(e_0, 2h; q_\infty) \leq c_5 \tag{4.3.112}$$

for any $e_0 = (y_0, s_0) \in \mathbb{R}^3_{+3h} \times] - T, 0[$.

Proof. Let a be sufficiently large such that $Q(e_0, 2h) \subset Q^+(a)$. From Proposition 4.22 and the Poincaré inequality we have that there exists $C_a(t) \in L_{\frac{3}{2}}(-a^2, 0)$ such that

$$q^{(k)} - [q^{(k)}]_{B^+(a)}(t) \rightharpoonup q_\infty + C_a(t) \qquad \text{in } L_{\frac{3}{2}}(-a^2, 0; L^{\frac{3}{2},\frac{\beta}{2}}(B^+(a))). \tag{4.3.113}$$

Since $B(y_0, 2h) \subset B^+(a)$, we deduce from the above that

$$[q^{(k)} - [q^{(k)}]_{B^+(a)}]_{B(y_0,2h)} \rightharpoonup [q_\infty + C_a(t)]_{B(y_0,2h)} \qquad \text{in } L_{\frac{3}{2}}(-a^2, 0; L^{\frac{3}{2},\frac{\beta}{2}}(B^+(a))). \tag{4.3.114}$$

It follows from (4.3.113)-(4.3.114) that

$$q^{(k)} - [q^{(k)}]_{B(y_0,2h)}(t) \rightharpoonup q_\infty - [q_\infty]_{B(y_0,2h)}(t) \qquad \text{in } L_{\frac{3}{2}}(-a^2, 0; L^{\frac{3}{2},\frac{\beta}{2}}(B^+(a))).$$

[28] Namely, Lemma 4.1 in [100].

Thus, by weak lower semicontinuity we have that:

$$D_\beta(e_0, 2h; q_\infty) \le \limsup_{k\to\infty} D_\beta(e_0, 2h; q^{(k)}). \tag{4.3.115}$$

As observed by Seregin in [100][29] for the case $\beta = 3$, (4.3.115) implies that it is sufficient to prove the following bound

$$D_\beta(e_0, 2h; q^{(k)}) \le c_5(L, D_1^+(0, 1, q)) \tag{4.3.116}$$

for all k sufficiently large such that

$$x_0^{(k)} := y_0 R_k \in B^+(1/4), \quad t_0^{(k)} := s_0 R_k^2 > -(1/4)^2. \tag{4.3.117}$$

By inverse scaling, we have

$$D_\beta(e_0, 2h; q^{(k)}) = D_\beta(z_0^k, 2hR_k; q), \quad z_0^{(k)} = (x_0^{(k)}, t_0^{(k)}). \tag{4.3.118}$$

As observed by Seregin on p.233 of [100], we have that $d_k = x_{03}^{(k)} = \mathrm{dist}(x_0^{(k)}, \Gamma) \ge 3hR_k$, along with the inclusions

$$Q(z_0^{(k)}, 2hR_k) \subset Q(z_0^{(k)}, d_k) \subset Q^+(1/2).$$

Thus, from Proposition 4.14 (with $\gamma = 1/2$) we see that

$$\begin{aligned} D_\beta(z_0^{(k)}, 2hR_k; q) &\le c(L)\left[\left(\frac{2hR_k}{d_k}\right)^{\frac{5}{4}} D_\beta(z_0^{(k)}, d_k; q) + 1\right] \\ &\le c(L)\left[D_\beta(z_0^{(k)}, d_k; q) + 1\right]. \end{aligned} \tag{4.3.119}$$

Let $\bar{x}_0^{(k)} = (x_{01}^{(k)}, x_{02}^{(k)}, 0)$ and also $\bar{z}_0^{(k)} = (\bar{x}_0^{(k)}, t_0^{(k)})$. As noticed by Seregin on p.233 of [100], we have that

$$Q(z_0^{(k)}, d_k) \subset Q^+(\bar{z}_0^{(k)}, 2d_k).$$

Consequently,

$$D_\beta(z_0^{(k)}, d_k, q) \le cD_\beta^+(\bar{z}_0^{(k)}, 2d_k, q) + cI. \tag{4.3.120}$$

Here,

$$I := \frac{1}{d_k^2}\int\limits_{-d_k{}^2}^{0} \|[q]_{B(x_0^{(k)}, d_k)}(t) - [q]_{B^+(\bar{x}_0^{(k)}, 2d_k)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2}, \frac{\beta}{2}}(B(x_0^{(k)}, d_k))} dt. \tag{4.3.121}$$

[29]See p.233 of [100].

Using Hunt's inequality (Proposition 2.2 in Chapter 2), one obtains the following estimate:

$$|[q]_{B(x_0^{(k)},d_k)}(t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)|\leq\frac{c}{d_k^3}\int\limits_{B(x_0^{(k)},d_k)}|q(x,t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)|dx$$

$$\leq\frac{c}{d_k^3}\|q(\cdot,t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0^{(k)},d_k))}\|\chi_{B(x_0^{(k)},d_k)}\|_{L^{3,\frac{\beta}{\beta-2}}(B(x_0^{(k)},d_k))}$$

$$\leq\frac{c}{d_k^2}\|q(\cdot,t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0^{(k)},d_k))}. \tag{4.3.122}$$

Thus,

$$I\leq\frac{c}{d_k^5}\int\limits_{-d_k^2}^{0}\|q(\cdot,t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0^{(k)},d_k))}\|\chi_{B(x_0^{(k)},d_k)}\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0^{(k)},d_k))}dt$$

$$\leq\frac{c}{d_k^2}\int\limits_{-d_k^2}^{0}\|q(\cdot,t)-[q]_{B^+(\bar{x}_0^{(k)},2d_k)}(t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0^{(k)},d_k))}dt. \tag{4.3.123}$$

Using this, together with (4.3.120), it readily follows that

$$D_\beta(z_0^{(k)},d_k;q)\leq cD_\beta^+(\bar{z}_0^{(k)},2d_k;q). \tag{4.3.124}$$

Note that we have the following Poincaré inequality:

$$D_\beta^+(\bar{z}_0^{(k)},2d_k;q)\leq D_1^+(\bar{z}_0^{(k)},2d_k;q).$$

Using this, (4.3.119) and (4.3.124), we infer that

$$D_\beta(z_0^{(k)},2hR_k;q)\leq c(L)[D_1^+(\bar{z}_0^{(k)},2d_k;q)+1]. \tag{4.3.125}$$

As observed by Seregin on p.233 of [100], we have the inclusions

$$Q^+(\bar{z}_0^{(k)},2d_k)\subset Q^+(\bar{z}_0^{(k)},1/2)\subset Q^+(3/4).$$

Then one can utilise Proposition 4.17, with $\gamma=1/2$, to infer that

$$D_1^+(\bar{z}_0^{(k)},2d_k;q)\leq c(L)[4d_kD_1^+(\bar{z}_0^{(k)},1/2;q)+1]$$

$$\leq c(L)[D_1^+(0,1;q)+1]. \tag{4.3.126}$$

Thus putting everything together gives

$$D_\beta(e_0,2h;q^{(k)})\leq c(L)[D_1^+(0,1;q)+1].$$

□

Step 3: Decay Properties of the Limit Function

Fix $h \in]0,1[$ arbitrarily. We will demonstrate that Proposition 4.22 and Lemma 4.23 imply the following. Namely, there exists an $R_1 > 1$ and a $\rho(L, D_1^+(0,1,q)) \in]0,1[$ such that

$$C_\infty(z_0, h\rho; v_\infty) + D_\infty(z_0, h\rho; q_\infty) < \epsilon_0, \tag{4.3.127}$$

for all $z_0 \in (\mathbb{R}^3_{+3h}) \times]-100, 0[$ with $|x_0| > R_1$. This allows us to apply the new ε-regularity criterion in weak Lebesgue spaces (Theorem 4.18). Thus, we obtain that for any $z_0 \in (\mathbb{R}^3_{3h+} \setminus B(R_1)) \times]-100, 0[$ and $k = 0, 1 \ldots$

$$|\nabla^k v_\infty(z_0)| \leq \frac{c_{0k}}{(h\rho)^{k+1}}. \tag{4.3.128}$$

As we will see, the decay property (4.3.128) will play a crucial role in showing that the limit function v_∞ is zero.

Now, we explain why (4.3.127) can be deduced from Proposition 4.22 and Lemma 4.23. The reasoning here is essentially contained in Seregin's paper[30] [100]. In the setting of Lorentz spaces there are some minor differences, hence we present the arguments fully for the convenience of the reader.

Consider an arbitrary point $z_0 = (x_0, t_0) \in \mathbb{R}^3_{+3h} \times]-100, 0[$. In the ball $B(x_0, 2h) \subset \mathbb{R}^3_{+h}$, we decompose the pressure in the same way as done by Seregin on p.234 of [100]. Specifically,

$$q_\infty = q_\infty^{(1)} + q_\infty^{(2)}$$

with

$$q_\infty^{(1)} := \mathcal{R}_i \mathcal{R}_j((v_\infty)_i (v_\infty)_j \chi_{B(x_0,2h)}). \tag{4.3.129}$$

One then has that for almost every t:

$$\Delta q_\infty^{(2)}(\cdot, t) = 0 \ \text{ in } \ B(x_0, 2h).$$

Thus, for the same reasons as previously stated we have

$$\|q_\infty^{(1)}(\cdot, t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0,2h))} \leq c\|v_\infty(\cdot, t)\|^2_{L^{3,\beta}(B(x_0,2h))}. \tag{4.3.130}$$

The reasoning used by Seregin on p.224 of [100] implies that the harmonic part $q_\infty^{(2)}(\cdot, t)$ satisfies

$$\sup_{x \in B(x_0,h)} |\nabla q_\infty^{(2)}(x, t)| \leq \frac{c}{h^4} \int\limits_{B(x_0,2h)} |q_\infty^{(2)}(\cdot, t) - [q_\infty^{(2)}]_{B(x_0,2h)}(t)| dx.$$

[30] Specficially, pp.234-235 of [100].

By Hunt's inequality (Proposition 2.2 in Chapter 2), we obtain

$$\sup_{x\in B(x_0,h)} |\nabla q^{(2)}_\infty(x,t)| \leq \frac{c}{h^3}\|q^{(2)}_\infty(\cdot,t)-[q^{(2)}_\infty]_{B(x_0,2h)}(t)\|_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0,2h))}. \tag{4.3.131}$$

For any $0<\rho<1$ we have

$$\sup_{x\in B(x_0,\rho h)} |q^{(2)}_\infty(x,t)-[q^{(2)}_\infty]_{B(x_0,\rho h)}| \leq c(\rho h)\sup_{x\in B(x_0,h)} |\nabla q^{(2)}_\infty(x,t)|. \tag{4.3.132}$$

For any $0<\rho<1$, we can use (4.3.130)-(4.3.132) to infer

$$\begin{aligned} D_\beta(z_0,h\rho;q_\infty) &\leq c[D_\beta(z_0,h\rho;q^{(1)}_\infty)+D_\beta(z_0,h\rho;q^{(2)}_\infty)] \\ &\leq \frac{c}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0} \|q^{(1)}_\infty(\cdot,t)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\frac{\beta}{2}}(B(x_0,2h))}dt \\ &+c(h\rho)^{\frac{5}{2}}\int\limits_{t_0-(h\rho)^2}^{t_0}\sup_{x\in B(x_0,h)}|\nabla q^{(2)}_\infty(x,t)|^{\frac{3}{2}}dt \\ \leq c&\left[\frac{1}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0}\|v_\infty(\cdot,t)\|^3_{L^{3,\beta}(B(x_0,2h))}dt+\rho^{\frac{5}{2}}D_\beta(z_0,2h,q_\infty)\right]. \end{aligned}$$

From this, Lemma 4.23 ($c_5=c_5(L,D^+_1(0,1,q))$) and the well known embeddings for Lorentz spaces we obtain:

$$\begin{aligned} &C_\infty(z_0,h\rho;v_\infty)+D_\infty(z_0,h\rho;q_\infty) \\ &\leq C_\beta(z_0,h\rho;v_\infty)+D_\beta(z_0,h\rho;q_\infty) \\ &\leq c\left[\frac{1}{(h\rho)^2}\int\limits_{t_0-4h^2}^{t_0}\|v_\infty(\cdot,t)\|^3_{L^{3,\beta}(B(x_0,h))}dt+\rho^{\frac{5}{2}}c_5\right] \end{aligned} \tag{4.3.133}$$

for any $z_0\in(\mathbb{R}^3_{+3h})\times]-100,0[$. As done by Seregin in p.235 of [100] (for the case $\beta=3$), we next fix $\rho(L,D^+_1(0,1;q))\in]0,1[$ such that

$$c\rho^{\frac{5}{2}}c_5<\frac{\epsilon_0}{2}.$$

Since $v_\infty\in L_\infty(-\infty,0;L^{3,\beta}(\mathbb{R}^3_+))$ and $\beta\in]3,\infty[$ we have that, for almost every t,

$$\lim_{R\to\infty}\|v_\infty(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+\setminus B(R))}=0.$$

This follows from the known fact[31] that continuous functions, with compact support in $\mathbb{R}^3_+$, are dense in $L^{3,\beta}(\mathbb{R}^3_+)$ for $1 \leq \beta < \infty$. By the dominated convergence theorem, we can find $R_1 > 100$ such that

$$\frac{c}{(h\rho)^2} \int\limits_{-200}^{0} \|v_\infty(\cdot, t)\|^3_{L^{3,\beta}(\mathbb{R}^3_+ \setminus B(R_1/2))} dt < \frac{\epsilon_0}{2}.$$

Using this and (4.3.133), we can infer (4.3.127).

Step 4: Showing that the Limit Function is Zero

Now we show that the limit function v_∞ must in fact be zero. From (4.3.89), we see that showing this proves Theorem 4.5 by contradiction.

Recall that for any $z_0 \in (\mathbb{R}^3_{3h+} \setminus B(R_1)) \times]-100, 0[$ and $k = 0, 1 \ldots$

$$|\nabla^k v_\infty(z_0)| \leq \frac{c_{0k}}{(h\rho)^{k+1}}. \tag{4.3.134}$$

Since $v_\infty \in L_\infty(-\infty, 0; L^{3,\beta}(\mathbb{R}^3_+))$, we can also apply the interior regularity result of Theorem 4.2 to obtain boundedness of $\nabla^k v_\infty$ on the set $(\mathbb{R}^3_{+3h} \cap B(R_1)) \times]-100, 0[$. Hence, we have that

$$\nabla^k v_\infty \in L_\infty(\mathbb{R}^3_{+3h} \times]-100, 0[) \text{ for any } \ h \in]0, 1[\text{ and } k = 0, 1 \ldots. \tag{4.3.135}$$

Define the vorticity $\omega_\infty = \nabla \wedge v_\infty$. Using (4.3.135), we may apply identical arguments to those used by Seregin in [100][32] in order to show that $\omega_\infty = 0$ on $\mathbb{R}^3_+ \times]-100, 0[$. Let us recall those arguments now.

On the set $(\mathbb{R}^3_{+3h}) \times]-100, 0[$, we have that there exists $M > 0$ such that

$$|\omega_\infty| \leq M$$

and

$$|\partial_t \omega_\infty - \Delta \omega_\infty| \leq M(|\omega_\infty| + |\nabla \omega_\infty|).$$

Furthermore, from (4.3.109) we see that

$$\omega_\infty(\cdot, 0) = 0 \ \text{ in } \ \mathbb{R}^3_+.$$

By applying a backward uniqueness theorem (Theorem 5.1 in Escauriaza, Seregin and Šverák's paper [27]), one deduces that

$$\omega_\infty = 0 \ \text{ in } \ (\mathbb{R}^3_{+3h}) \times]-100, 0[.$$

[31] See Lemma 3.3 of [80], for example.

[32] Specifically, arguments on p.235 of [100].

Since $h \in]0,1[$ is arbitrary, we infer that

$$\omega_\infty = 0 \ \text{ in } \ \mathbb{R}^3_+ \times]-100,0[.$$

Hence, for almost every $t \in]-100,0[$, v_∞ is a harmonic function satisfying the boundary condition $v_\infty(x_1,x_2,0,t) = 0$. Moreover, for almost every $t \in]-100,0[$ the $L^{3,\beta}$ norm of v_∞ over $\mathbb{R}^3_+$ is finite. Define

$$\widetilde{v_\infty}(x_1,x_2,x_3,t) := \begin{cases} v_\infty(x_1,x_2,x_3,t) & \text{if } \ 0 \leq x_3, \\ -v_\infty(x_1,x_2,-x_3,t) & \text{if } \ x_3 < 0. \end{cases}$$

The above facts about v_∞ imply that for almost every $t \in]-100,0[$, $\widetilde{v_\infty}$ is a harmonic function[33] in $\mathbb{R}^3$ with finite $L^{3,\beta}(\mathbb{R}^3)$ norm. Thus, for the same $t \in]-100,0[$, $\widetilde{v_\infty}$ satisfies the mean value property. This, together with Hunt's inequality (Proposition 2.2 in Chapter 2), implies that for almost every $t \in]-100,0[$, the following estimate holds for arbitrary $x_0 \in \mathbb{R}^3$ and $R > 0$. Specifically,

$$|\widetilde{v_\infty}(x_0,t)| \leq \frac{c}{R^3} \int_{B(x_0,R)} |\widetilde{v_\infty}(y,t)| dy \leq$$

$$\leq \frac{c \|\chi_{B(x_0,R)}\|_{L^{\frac{3}{2},\frac{\beta}{\beta-1}}(\mathbb{R}^3)}}{R^3} \|\widetilde{v_\infty}(\cdot,t)\|_{L^{3,\beta}(B(x_0,R)} \leq \frac{c\|\widetilde{v_\infty}(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3)}}{R^2} \tag{4.3.136}$$

By taking $R \to \infty$ in the above estimate, one deduces (for the same t) that $v_\infty(\cdot,t) = 0$ in $\mathbb{R}^3_+$. This contradicts (4.3.89). $\Box$

[33]See Lemma 9.12 of Gilbarg and Trudinger's book [46], for example.

Chapter 5

A Necessary Condition of Potential Blow-up for the Navier-Stokes System in Half-Space

5.1 Introduction

In subsection 1.2 of 'Chapter 1: Introduction', we explained that for certain classes of initial data (specifically, 'short time regular' initial data), non-uniqueness of the associated weak Leray-Hopf solutions implies the existence of a finite blow-up time T.

In this chapter, we will consider weak Leray-Hopf solutions in the half-space (namely, $v:\mathbb{R}^3_+\times]0,\infty[\to\mathbb{R}^3$), with short time regular initial data. The focus of this chapter will be to prove new necessary conditions for the existence of a finite blow-up time T. The results of this chapter are mostly based on

> Barker, T.; Seregin, G. A necessary condition of potential blowup for the Navier-Stokes system in half-space. Math. Ann. (2016).
> doi:10.1007/s00208-016-1488-9.
> Received: 4 December 2015. Published online: 31 October 2016.

Unless otherwise mentioned, the notation and definitions used in this chapter are contained in 'Chapter 2: Preliminary Material'.

Before further discussion, we find it appropriate to recall from 'Chapter 1: Introduction' the definitions of 'weak Leray-Hopf solution', 'short time regular initial data' and 'finite blow-up time'.

Now let us recall the definition of weak Leray-Hopf solutions. Throughout this chapter, we only consider the cases $\Omega=\mathbb{R}^3$ or $\Omega=\mathbb{R}^3_+$.

Definition 5.1. *Consider $0 < S \leq \infty$. Let*

$$u_0 \in J(\Omega). \tag{5.1.1}$$

A 'weak Leray-Hopf solution' v to Navier-Stokes initial boundary value problem in $\Omega\times]0,S[$, with initial data u_0, is defined to satisfy the following properties.

- *The function v belongs to $\mathcal{L}(S)$.*
- *For any $w \in L_2(\Omega)$:*

$$t \to \int\limits_{\Omega} w(x)\cdot v(x,t)dx \tag{5.1.2}$$

  is a continuous function on $[0,S]$ (the semi-open interval should be taken if $S=\infty$).

- *The Navier-Stokes equations are satisfied by v in a weak sense:*

$$\int\limits_0^S\int\limits_{\Omega}(v\cdot\partial_t w + v\otimes v : \nabla w - \nabla v : \nabla w)dxdt = 0 \tag{5.1.3}$$

  for any divergence-free test function $w \in C^{\infty}_{0,0}(\Omega\times]0,S[)$.

- *The initial condition is satisfied strongly in the $L_2(\Omega)$ sense:*

$$\lim_{t\to 0^+}\|v(\cdot,t)-u_0\|_{L_2(\Omega)} = 0. \tag{5.1.4}$$

- *Finally, v satisfies the energy inequality:*

$$\|v(\cdot,t)\|^2_{L_2(\Omega)} + 2\int\limits_0^t\int\limits_{\Omega}|\nabla v(x,t')|^2dxdt' \leq \|u_0\|^2_{L_2(\Omega)} \tag{5.1.5}$$

  for all $t \in [0,S]$ (the semi-open interval should be taken if $S=\infty$).

In this chapter, we will always consider short time regular initial data for weak Leray-Hopf solutions. Now, we recall the definition of short time regular initial data.

Definition 5.2. *We define $u_0 \in J(\Omega)$ to be a 'short time regular' initial data if the following properties hold.*

- *There exists a $\delta > 0$ such that all global in time weak Leray-Hopf solutions, with initial data u_0, coincide on $\Omega\times]0,\delta[$.*

- *This unique Leray-Hopf solution satisfies*

$$v \in L_\infty(]\epsilon, \delta]; L_\infty(\Omega))$$

for any $0 < \epsilon < \delta$.

Finally, let us define the notion of a 'finite blow-up time'.

Definition 5.3. *Let v be a weak Leray-Hopf solution, with short time regular initial data u_0. Suppose that there exists an $S > 0$ and an ϵ such that $0 < \epsilon < S$ and*

$$v \notin L_\infty(]\epsilon, S[; L_\infty(\Omega)). \tag{5.1.6}$$

Then we define the finite blow-up time $T < \infty$ as

$$T := \sup\{\tau : v \in L_\infty(]\epsilon, \tau]; L_\infty(\Omega)) \ \forall \ 0 < \epsilon < \tau\}. \tag{5.1.7}$$

Furthermore, we define the 'interval of regularity' as $]0, T[$.

We recall the important fact that all global-in-time weak Leray-Hopf solutions, with short time regular initial data u_0 and finite blow-up time $T(u_0)$, coincide[1] on $\Omega\times]0, T(u_0)[$ and belong to[2] $C^\infty(\Omega\times]0, T(u_0)[)$.

The first necessary conditions for a finite blow-up time T were provided by Leray in [75], for the slightly narrower class of 'turbulent solutions' defined on $\mathbb{R}^3\times]0, \infty[$. Leray's paper [75] implies that if $v : \mathbb{R}^3\times]0, \infty[\to \mathbb{R}^3$ is a turbulent solution, with finite blow-up time T, then v necessarily satisfies

$$\|\nabla v(\cdot, t)\|_{L_2(\mathbb{R}^3)} \geq \frac{c}{(T-t)^{\frac{1}{4}}} \tag{5.1.8}$$

and

$$\|v(\cdot, t)\|_{L_s(\mathbb{R}^3)} \geq \frac{c(s)}{(T-t)^{\frac{s-3}{2s}}}, \ 0 \leq t < T, \ 3 < s \leq \infty. \tag{5.1.9}$$

Leray's proof is centred around showing that initial data $\dot{H}^1(\mathbb{R}^3) \cap J(\mathbb{R}^3)$ or $L_s(\mathbb{R}^3) \cap J(\mathbb{R}^3)$ $(3 < s \leq \infty)$ is short time regular, with the corresponding intervals of regularity bounded below by:

- $C/\|\nabla u_0\|^4_{L_2(\mathbb{R}^3)}$,
- $C/\|u_0\|^2_{L_\infty(\mathbb{R}^3)}$ and

[1] A proof can be found in Sohr's book [118]. Specifically, Theorems 1.4.1 (p.272) and 1.5.1 (p.276).
[2] A proof can be found in Galdi's monograph [35]. Specifically, Theorem 5.2 (p.53).

- $C(s)/\|u_0\|_{L_s(\mathbb{R}^3)}^{\frac{2s}{s-3}}$ $(3 < s < \infty)$.

For $3 < s < \infty$, the condition (5.1.9) was later extended by Giga [42] to the case of bounded domains with smooth boundary. Ukai [127] showed that for $3 < s < \infty$, (5.1.9) is valid also for $\Omega = \mathbb{R}^3_+$. See also Robinson and Sadowski's paper [91] for a proof of (5.1.9) when Ω is the three-dimensional torus. In the literature there are many other results providing necessary conditions for the existence of a positive finite blow-up time T of the Navier-Stokes equations. See work by Bae et al. [2], Benameur and Lofti [13], McCormick et al. [82] and Robinson et al. [90], for example.

For the interesting marginal case of (5.1.9) ($s = 3$), one might conjecture that there is a necessary condition for the $L_3(\Omega)$ norm that takes the same form as (5.1.8)-(5.1.9). In particular, one may hope that there exists a universal function $f : (0,\infty) \to (0,\infty)$ such that the following holds.

- 1) $\lim_{s\to 0^+} f(s) = \infty$.

- 2) If $v : \Omega\times]0,\infty[\to \mathbb{R}^3$ is a weak Leray-Hopf solution, with short time regular initial data and finite blow-up time T, then v necessarily satisfies[3]

$$\|v(\cdot,t)\|_{L_3(\Omega)} \geq f(T-t) \tag{5.1.10}$$

  for all $t \in [0,T[$.

We will now show that if there exists a function $f : (0,\infty) \to (0,\infty)$ such that 1) and 2) hold true, then any weak Leray-Hopf solution $v : \Omega\times]0,\infty[\to \mathbb{R}^3$, with short time regular initial data in $L_3(\Omega)$, must be regular for all times. The argument demonstrating this below is similar to that presented in a talk by Seregin [109], concerning his paper [106].

The hypotheses 1) and 2) imply that if u_0 is any short time regular initial data with

$$\|u_0\|_{L_3(\Omega)} < f(\mathcal{T}), \tag{5.1.11}$$

then the following conclusions hold true.

- All global-in-time weak Leray-Hopf solutions, with initial data u_0, coincide on $\Omega\times]0,\mathcal{T}[$.

[3]In (5.1.10), $v(\cdot,0) \notin L_3(\Omega)$ is interpreted as meaning $\|v(\cdot,0)\|_{L_3(\Omega)} = \infty$.

- Any global-in-time weak Leray-Hopf solution v, with initial data u_0, satisfies

$$v \in L_\infty(]\epsilon, \mathcal{T}]; L_\infty(\Omega)) \tag{5.1.12}$$

for all $0 < \epsilon < \mathcal{T}$.

Consider a global-in-time weak Leray-Hopf solution $v : \Omega\times]0,\infty[\to \mathbb{R}^3$, with initial data u_0. Now we perform the rescaling ($\lambda > 0$):

$$(v_\lambda, q_\lambda)(x,t) := (\lambda v, \lambda^2 q)(\lambda x, \lambda^2 t) \ \text{ and } \ u_0^\lambda(x) := \lambda u_0(\lambda x).$$

It is obvious that v_λ is a weak Leray-Hopf solution on $\Omega\times]0,\infty[$ with initial data u_0^λ. Since

$$\|u_0^\lambda\|_{L_3} = \|u_0\|_{L_3} < f(\mathcal{T}),$$

our original hypotheses 1) and 2) give that

$$v_\lambda \in L_\infty(]\epsilon, \mathcal{T}]; L_\infty(\Omega)) \ \text{ for all } \ 0 < \epsilon < \mathcal{T}$$

$$\Rightarrow v \in L_\infty(]\epsilon, \mathcal{T}\lambda^2]; L_\infty(\Omega)) \ \text{ for any } \ \lambda > 0 \text{ and } 0 < \epsilon < \mathcal{T}\lambda^2.$$

Hence, hypotheses 1) and 2) imply that for any short time regular initial data in $L_3(\Omega)$, the associated Leray-Hopf solutions do not possess a finite blow-up time. Let us also mention that the above arguments also apply with $L_3(\Omega)$ replaced by any other critical space $(\mathcal{X}(\Omega), \|\cdot\|_{\mathcal{X}(\Omega)})$[4]. Hence, for any critical space $\mathcal{X}$ and sufficiently regular initial data, if a finite blow-up time T exists then there cannot be a necessary condition of the form (5.1.10).

For any $u_0 \in L_3(\mathbb{R}^3) \cap J(\mathbb{R}^3)$, it was shown by Kato in [59] that u_0 is short time regular. Kato also showed the length of the interval of regularity is bounded below by $L(u_0)$, where

$$\sup_{0<t<L(u_0)} t^{\frac{1}{5}} \|S(t)u_0\|_{L_5(\mathbb{R}^3)} = \epsilon. \tag{5.1.13}$$

Here, ϵ is some universal constant and

$$S(t)u_0(x) := \int\limits_{\mathbb{R}^3} \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp(-|x-y|^2/4t) u_0(y) dy.$$

Unfortunately, this lower bound does not depend solely on $\|u_0\|_{L_3(\mathbb{R}^3)}$. Moreover, for initial data $u_0 \in J(\mathbb{R}^3) \cap L_3(\mathbb{R}^3)$, it is not currently known if the length of the interval

[4]Recall that we say that $(\mathcal{X}(\Omega), \|\cdot\|_{\mathcal{X}(\Omega)})$ is a critical space if the following holds for any $\lambda > 0$. Namely, $u_0(x) \in \mathcal{X}(\Omega) \Rightarrow u_{0\lambda} := \lambda u_0(\lambda x) \in \mathcal{X}(\Omega)$ and $\|u_{0\lambda}\|_{\mathcal{X}(\Omega)} = \|u_0\|_{\mathcal{X}(\Omega)}$.

of regularity has a lower bound dependent only on $\|u_0\|_{L_3(\mathbb{R}^3)}$. Consider a weak Leray-Hopf solution v with finite blow-up time T. At first sight, it seems viable that the following scenario may occur for a weak Leray-Hopf solution $v:\mathbb{R}^3\times]0,\infty[\to\mathbb{R}^3$, with short time regular initial data and a finite blow-up time T. Namely, that $\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}$ remains bounded as we approach T, with the intervals of regularity associated with $\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}$ shrinking to zero in length as t approaches T.

The first necessary condition, ruling out this type of phenomenon, was given for the critical norm $L_3(\mathbb{R}^3)$ in the paper of Escauriaza, Seregin and Šverák in [27]. Their paper implied that if $v:\mathbb{R}^3\times]0,\infty[\to\mathbb{R}^3$ has short time regular initial data and a finite blow-up time T, then necessarily

$$\limsup_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}=\infty. \tag{5.1.14}$$

Recent progress has been made at establishing the validity of the necessary condition

$$\limsup_{t\uparrow T}\|v(\cdot,t)\|_{\mathcal{X}(\mathbb{R}^3)}=\infty, \tag{5.1.15}$$

for other critical norms $\|\cdot\|_{\mathcal{X}(\mathbb{R}^3)}$.

In [87], Phuc showed that condition (5.1.15) remains to be true when $\mathcal{X}=L^{3,\beta}(\mathbb{R}^3)$ with β finite. Recently, Gallagher, Koch and Planchon proved in [40] that (5.1.15) holds for $\mathcal{X}=\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ $(3<p,q<\infty)$ in the framework of "strong" solutions. Their proof is based upon a strategy first developed for dispersive equations, namely the method of critical elements. Other necessary conditions of the type (5.1.15) have been obtained very recently in papers by Wang and Zhang [129] and Choe, Yang and Wolf [24].

For the case of a weak Leray-Hopf solution in the half-space ($v:\mathbb{R}^3_+\times]0,\infty[\to\mathbb{R}^3$), with short time regular initial data and a finite blow-up time T, Seregin's work [100] implies that

$$\limsup_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3_+)}=\infty. \tag{5.1.16}$$

Mikhailov and Shilkin showed in [83] that this necessary condition also holds for a sufficiently smooth bounded spatial domain. Let us also mention that the results of Chapter 4 of this thesis imply that the following refinement of the necessary condition (5.1.16) holds true. Specifically,

$$\limsup_{t\uparrow T}\|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)}=\infty. \tag{5.1.17}$$

Here, $3\leq\beta<\infty$.

Notice that the necessary condition (5.1.14) implies that if $v:\mathbb{R}^3\times]0,\infty[\to\mathbb{R}^3$ is a weak Leray-Hopf solution, with short time regular initial data and a finite blow-up time T, then there exists a sequence $t_k\uparrow T$ such that

$$\lim_{k\to\infty}\|v(\cdot,t_k)\|_{L_3(\mathbb{R}^3)}=\infty. \tag{5.1.18}$$

It is therefore very natural to ask whether or not $\lim_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}=\infty$. For the critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ ($\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L_3(\mathbb{R}^3)$), it was shown by Seregin in [104] that

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}=\infty. \tag{5.1.19}$$

This was extended to the case of the $L_3(\mathbb{R}^3)$ norm by Seregin in [106], by means of the following theorem[5].

Theorem 1.1 of [106]. *Consider a weak Leray-Hopf solution $v:\mathbb{R}^3\times]0,\infty[\to\mathbb{R}^3$, with short time regular initial data and a finite blow-up time T. Then necessarily*

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}=\infty. \tag{5.1.20}$$

Seregin's proof of the above theorem proceeds by contradiction. A suitable rescaling procedure, arising from the contradiction assumption, produces a sequence of solutions $(v^{(k)},q^{(k)})$ to the Navier-Stokes equations on $\mathbb{R}^3\times]-S,0[$ such that

- $\sup_k\|v^{(k)}(\cdot,-S)\|_{L_3(\mathbb{R}^3)}=M<\infty$,
- for $k=1,2\ldots$ $v^{(k)}$ has a singular point[6] at $(x,t)=(0,0)$,
- $v^{(k)}$ coincides with a Lemarié-Rieusset local energy solution of the Navier-Stokes equations on $\mathbb{R}^3\times]-S,0[$.

We mention that Lemarié-Rieusset local energy solutions of the Navier-Stokes equations were conceived by Lemarié-Rieusset in [76] and provide a global-in-time solution for the Navier-Stokes equations with certain classes of non-energy divergence free initial data[7]. Specifically, $u_0\in L_{2,loc}(\mathbb{R}^3)$ such that

$$\lim_{R\to\infty}\sup_{x_0\in\mathbb{R}^3:|x_0|\geq R}\int\limits_{B(x_0,1)}|u_0(x)|^2dx=0.$$

[5] Strictly speaking, Theorem 1.1 of [106] considers smooth compactly supported initial data. Identical arguments apply for the case of short time regular initial data.

[6] Recall that this means that $v^{(k)}\notin L_\infty(B(r)\times]-r^2,0[)$ for all sufficiently small $r>0$.

[7] By non-energy initial data, we mean $u_0\notin L_2(\mathbb{R}^3)$.

The next part of Seregin's proof involves showing that $v^{(k)}$ tends to a nontrivial ancient solution (v_∞, q_∞) to the Navier-Stokes equations with the following property. Namely, (v_∞, q_∞) coincide with a Lemarié-Rieusset local energy solution with $L_3(\mathbb{R}^3)$ initial data, on $\mathbb{R}^3\times]-S,0[$. Seregin's proof then crucially exploits known decay properties of Lemarié-Rieusset local energy solutions to infer that there exists an $R>0$ such that for $k=1,2\ldots$:

$$\sup_{(\mathbb{R}^3\setminus B(R))\times]-S/2,0[} |\nabla^{k-1}v_\infty| \leq C(k,S). \tag{5.1.21}$$

This decay property plays the very important role of allowing applicability of a Liouville type theorem to v_∞, based on backward uniqueness and unique continuation of parabolic operators developed by Escauriaza, Seregin and Šverák in [27]. This then shows that $v_\infty = 0$, which contradicts the nontriviality of v_∞.

Let us now state the main result of this chapter, which represents a refinement of the necessary condition (5.1.17) implied by the results of Chapter 4.

Theorem 5.4. *Let $v:\mathbb{R}^3_+\times]0,\infty[\to\mathbb{R}^3$ be a global-in-time weak Leray-Hopf solution of the Navier-Stokes equations, with short time regular initial data and a finite blow-up time T. Then it necessarily holds that*

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{L^{3,\beta}(\mathbb{R}^3_+)}=\infty,$$

for any $3\leq\beta<\infty$.

As will be discussed in the subsection of this chapter called 'Rescaling. Scenario I', a weak Leray-Hopf solution $v:\mathbb{R}^3_+\times]0,\infty[\to\mathbb{R}^3$, with short time regular initial data and a finite blow-up time T, will possess a singular point (x_1,x_2,x_3,T) at time T. In proving Theorem 5.4, we will focus on the case that this singular point lies on the boundary (i.e $x_3=0$). We will explain later in this chapter that the case $x_3>0$ can be treated using similar arguments, along with certain simplifications.

As in the proof of Theorem 1.1 in Seregin's paper [106], the proof of Theorem 5.4 proceeds by contradiction. Similarly, the contradiction argument and a rescaling procedure produces a sequence of solutions $(v^{(k)},q^{(k)})$ with the following properties. Namely,

$$\partial_t v^{(k)}-\Delta v^{(k)}+v^{(k)}\cdot\nabla v^{(k)}+\nabla q^{(k)}=0 \text{ on } \mathbb{R}^3_+\times]-2,0[,$$

$$v^{(k)}(x_1,x_2,0,t)=0,\ \mathrm{div}\, v^{(k)}=0,$$

$$\sup_k\|v^{(k)}(\cdot,-2)\|_{L^{3,\beta}(\mathbb{R}^3_+)}=M<\infty,$$

for all $k = 1, 2 \dots v^{(k)}$ has a singular point at $(x, t) = (0, 0)$.

The main difficulty, compared to the case of $\Omega = \mathbb{R}^3$ of Theorem 1.1 in [106], is that there is currently no such analogue of Lemarié-Rieusset solutions in the case of the half-space. In fact, this is an interesting open problem.

The key observation of this chapter, which allows us to overcome this difficulty, is showing that $(v^{(k)}, q^{(k)})$ satisfy the following a priori estimates. Namely,

1. $v^{(k)} = \sum_{i=1}^{3} v^{(k),i}$, $q^{(k)} = \sum_{i=1}^{3} q^{(k),i}$ for $(x, t) \in \mathbb{R}^3_+ \times]-3/2, 0[$,
2. $\partial_t v^{(k),i} - \Delta v^{(k),i} + \nabla q^{(k),i} = g^{(k),i}$ on $\mathbb{R}^3_+ \times]-2, 0[$,
3. $v^{(k),i}(x_1, x_2, 0, t) = 0$, $v^{(k),i}(x, -2) = 0$, $\operatorname{div} v^{(k),i} = 0$,
4. $\|\partial_t v^{(k),1}\|_{L_2(-2,0;L_2(\mathbb{R}^3_+))} + \|\nabla^2 v^{(k),1}\|_{L_2(-2,0;L_2(\mathbb{R}^3_+))} + \|\nabla q^{(k),1}\|_{L_2(-2,0;L_2(\mathbb{R}^3_+))} \leq C(M)$,
5. $\|\partial_t v^{(k),2}\|_{L_{\frac{3}{2}}(-2,0;L_{\frac{9}{8}}(\mathbb{R}^3_+))} + \|\nabla^2 v^{(k),2}\|_{L_{\frac{3}{2}}(-2,0;L_{\frac{9}{8}}(\mathbb{R}^3_+))} + \|\nabla q^{(k),2}\|_{L_{\frac{3}{2}}(-2,0;L_{\frac{9}{8}}(\mathbb{R}^3_+))}$ $\leq C(M)$,
6. $\|\partial_t v^{(k),3}\|_{L_{\frac{10}{3}}(-2,0;L_{\frac{10}{3}}(\mathbb{R}^3_+))} + \|\nabla^2 v^{(k),3}\|_{L_{\frac{10}{3}}(-2,0;L_{\frac{10}{3}}(\mathbb{R}^3_+))} + \|\nabla q^{(k),3}\|_{L_{\frac{10}{3}}(-2,0;L_{\frac{10}{3}}(\mathbb{R}^3_+))}$ $\leq C(M)$.

The a priori estimates 1.-6. are useful in showing that $(v^{(k)}, q^{(k)})$ converges to an ancient solution (v_∞, q_∞) of the Navier-Stokes equations. In order to show the non-triviality of this ancient solution, we prove a new lemma concerning stability properties of singular points of the Navier-Stokes equations, that lie on the flat part of the boundary.

The key point in proving Theorem 5.4 is that the limit function (v_∞, q_∞) have a similar decomposition and associated estimates to those for $(v^{(k)}, q^{(k)})$ described above. This, in conjunction with the ε-regularity theory, implies that v_∞ has sufficient decay to apply a Liouville type theorem based on backward uniqueness and unique continuation of certain classes of parabolic operators. This then allows us to achieve that v_∞ must be trivial, giving a contradiction.

5.2 Preliminaries

5.2.1 Suitable Weak Solutions and Scale-Invariant Quantities

In this chapter, we will use the definition of a suitable weak solution to the Navier-Stokes equations in $B^+(x_0, R) \times]S_1, S_2[$ near $\Gamma(x_0, R) \times [S_1, S_2]$ that was given in

'Chapter 1: Introduction' (specifically, Definition 1.4). Definition 1.4 in Chapter 1 is essentially contained in Seregin's paper [98], specifically Definition 2.1 there. We will also use the definition of regular and singular points that was presented in Chapter 1[8].

For a suitable weak solution (v, q) to the Navier-Stokes equations in $Q^+(z_0, R)$ near $\Gamma(x_0, R) \times [t_0 - R^2, t_0]$, we define the following scale-invariant quantities $(0 < r < R)$:

$$A^+(z_0, r; v) := \sup_{t_0 - r^2 \leq t \leq t_0} r^{-1} \int\limits_{B^+(x_0, r)} |v(x, t)|^2 dx, \tag{5.2.1}$$

$$B^+(z_0, r; v) := r^{-1} \int\limits_{t_0 - r^2}^{t_0} \int\limits_{B^+(x_0, r)} |\nabla v(x, t)|^2 dx dt, \tag{5.2.2}$$

$$C^+(z_0, r; v) := r^{-2} \int\limits_{t_0 - r^2}^{t_0} \int\limits_{B^+(x_0, r)} |v(x, t)|^3 dx dt, \tag{5.2.3}$$

$$D_1^+(z_0, r; q) := r^{-\frac{3}{2}} \int\limits_{t_0 - r^2}^{t_0} \Big(\int\limits_{B^+(x_0, r)} |\nabla q|^{\frac{9}{8}} dx \Big)^{\frac{4}{3}} dt. \tag{5.2.4}$$

5.2.2 Green's Function for the Stokes System in the Half-Space

Before further discussion, we introduce the following notation. Recall that for $y = (y_1, y_2, y_3)$, we define $y^* = (y_1, y_2, -y_3)$.

Let $w_0 : \mathbb{R}^3_+ \to \mathbb{R}^3$ be divergence free and such that $w_0(x_1, x_2, 0) = 0$. Consider the following Stokes problem for $w : \mathbb{R}^3_+ \times]0, \infty[\to \mathbb{R}^3$ and $q : \mathbb{R}^3_+ \times]0, \infty[\to \mathbb{R}$. Specifically,

$$\partial_t w - \Delta w + \nabla q = 0, \qquad \operatorname{div} w = 0$$

in $\mathbb{R}^3_+ \times]0, \infty[$,

$$w(x_1, x_2, 0, t) = 0$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [0, \infty]$ and

$$w(\cdot, 0) = w_0$$

in $\mathbb{R}^3_+$. Under certain restrictions on w_0, Solonnikov showed in [121] that $w(x, t)$ can be expressed as

$$w_i(x, t) = \int\limits_{\mathbb{R}^3_+} G_{ij}(x, y, t) w_{0j}(y) dy \quad \text{for } \; i \in \{1, 2, 3\}.$$

[8] See p.14 and p.20.

Moreover, Solonnikov showed that the Green function G for the Stokes system in the half-space has the form

$$G := G^1 + G^2, \tag{5.2.5}$$

with

$$G^1_{ij}(x,y,t) := \delta_{ij}\Big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big), \tag{5.2.6}$$

$$G^2_{i\beta}(x,y,t) := 4\frac{\partial}{\partial x_\beta}\int\limits_0^{x_3}\int\limits_{\mathbb{R}^2}\frac{\partial E}{\partial x_i}(x-z)\Gamma(z-y^*,t)dz, \quad G^2_{i3}(x,y,t) = 0, \tag{5.2.7}$$

$$\Gamma(x,t) := \frac{1}{(4\pi t)^{\frac{3}{2}}}\exp(-|x|^2/4t) \quad \text{and} \quad E(x) := -\frac{1}{4\pi|x|}. \tag{5.2.8}$$

Additionally in [121], Solonnikov showed that G^2 satisfies the estimate

$$\begin{aligned}\Big|\frac{\partial^s\partial^{|\alpha|+|\gamma|}G^2}{\partial t^s\partial x^\alpha\partial y^\gamma}(x,y,t)\Big| &\leq c(\alpha,\gamma)t^{-s-\frac{\gamma_3}{2}}(t+x_3^2)^{-\frac{\alpha_3}{2}}\\ &\times(|x-y^*|^2+t)^{-\frac{3+|\alpha'|+|\gamma'|}{2}}\exp\Big(-\frac{cy_3^2}{t}\Big).\end{aligned} \tag{5.2.9}$$

Here, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha' = (\alpha_1,\alpha_2)$, $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ and $\gamma' = (\gamma_1,\gamma_2)$. We will frequently refer to the estimate (5.2.9) as the 'Solonnikov estimate'.

5.3 Stability Properties of Singular Points on the Flat Part of the Boundary

Let us now state and prove some new stability properties of singular points on the flat part of the boundary, for sequences of suitable weak solutions to the Navier-Stokes equations. Such properties will be useful in the proof of Theorem 5.4. Specifically, these properties will be used to show that the ancient solution, obtained from the contradiction assumption and a rescaling procedure, is nonzero.

Previously such stability properties have been established for interior singular points by Rusin and Šverák in [93]. In the paper of Rusin and Šverák [93] and the paper of Jia and Šverák [54], this plays an important role in the proof of existence of minimal blow-up $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $L_3(\mathbb{R}^3)$ initial data for the three-dimensional Navier-Stokes equations. It is probable that our proposition could be useful for proving the existence of minimal blow-up initial data for the Navier-Stokes equations in the half space. This topic will not be pursued further in this thesis.

Let us now state the relevant proposition.

Proposition 5.5. *Suppose $(v^{(k)}, q^{(k)})$ are suitable weak solutions to the Navier-Stokes equations in $Q^+(1)$ near $\Gamma(0,1)\times[-1,0]$. Suppose that there exists a finite $M>0$ such that*

$$\sup_k \Big(\|v^{(k)}\|_{L_{2,\infty}(Q^+(1))} + \|\nabla v^{(k)}\|_{L_2(Q^+(1))} + \|q^{(k)}\|_{L_{\frac{3}{2}}(Q^+(1))}\Big) = M < \infty. \tag{5.3.1}$$

Furthermore assume that

- $$\lim_{k\to\infty} \|v^{(k)} - v\|_{L_3(Q^+(1))} = 0 \tag{5.3.2}$$
- $$q^{(k)} \rightharpoonup q \text{ in } L_{\frac{3}{2}}(Q^+(1)) \tag{5.3.3}$$
- $(0,0)$ *is a singular point of* $v^{(k)}$ *for all* $k=1,2,\ldots$.

Then the above assumptions imply that, for any λ with $0<\lambda<1$, (v,q) is a suitable weak solution to the Navier-Stokes equations in $Q^+(\lambda)$ near $\Gamma(0,\lambda)\times[-\lambda^2,0]$ with $(0,0)$ being a singular point of v.

We will use the following lemma, stated and proven by Seregin in [98] (specifically Lemma 7.2 in [98]).

Lemma 5.6. *Consider a pair of functions (v,q) that are a suitable weak solution to the Navier-Stokes equations in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times[t_0-R^2,t_0]$. Then for any $0<r\leq\rho\leq R$, we have*

$$D_1^+(z_0,r;q) \leq c\Big(\frac{r}{\rho}\Big)^2 [D_1^+(z_0,\rho;q) + (B^+(z_0,\rho;v))^{\frac{3}{4}}]$$

$$+c\left(\frac{\rho}{r}\right)^{\frac{3}{2}} (A^+(z_0,\rho;v))^{\frac{1}{2}} B^+(z_0,\rho;v). \tag{5.3.4}$$

Proof of Proposition 5.5

Proof. Showing that the assumptions (5.3.1)-(5.3.3) imply that (v,q) is a suitable weak solution to the Navier-Stokes equations in $Q^+(\lambda)$ near $\Gamma(0,\lambda)\times[-\lambda^2,0]$, for any $0<\lambda<1$, can be verified using the same arguments as in Proposition 4.22 of the previous chapter.

We will focus on showing that $(0,0)$ is a singular point of (v,q). As done by Rusin and Šverák in [93][9] for the case of interior singular points, we assume for contradiction

[9] In particular, pp.882-883 of [93].

that $(0,0)$ is a regular point of (v,q). We now use the same arguments as in Rusin and Šverák's paper [93] (p.883 of [93]) to show that the contradiction assumption and (5.3.2) implies that

$$\frac{1}{r^2}\int\limits_{-r^2}^{0}\int\limits_{B^+(r)}|v^{(k)}|^3dxdt$$

is small for sufficiently small r and sufficiently large k.

If $(0,0)$ is a regular point of (v,q) then there exists an $\epsilon_0>0$ such that

$$v\in L_\infty(Q^+(\epsilon_0)). \tag{5.3.5}$$

Thus for any $r<\epsilon_0$ we have

$$\frac{1}{r^2}\int\limits_{-r^2}^{0}\int\limits_{B^+(r)}|v|^3dxdt\le r^3\|v\|^3_{L_\infty(Q^+(\epsilon_0))}. \tag{5.3.6}$$

Let $0<\varepsilon<1/2$ be arbitrary. Define

$$R_0:=\min\Big\{\frac{\epsilon_0}{2},\frac{\varepsilon^{\frac{1}{3}}}{2(\|v\|_{L_\infty(Q^+(\epsilon_0))}+1)}\Big\}. \tag{5.3.7}$$

Clearly, $R_0<2^{-\frac{4}{3}}$. Moreover, (5.3.6) implies that

$$\frac{1}{R^2}\int\limits_{-R^2}^{0}\int\limits_{B^+(R)}|v|^3dxdt<\frac{\varepsilon}{8} \tag{5.3.8}$$

for any $0<R\le R_0$. The assumption (5.3.2) implies that for all $0<R\le R_0$ there exists $K_\varepsilon :]0,R_0]\to\mathbb{N}\cup\{0\}$ such that

$$\int\limits_{Q^+(1)}|v^{(k)}-v|^3dxdt<\frac{\varepsilon R^2}{8}\ \text{ for all }\ k\ge K_\varepsilon(R). \tag{5.3.9}$$

So from (5.3.8) and (5.3.9), we see that for $0<R\le R_0$ we have

$$\frac{1}{R^2}\int\limits_{-R^2}^{0}\int\limits_{B^+(R)}|v^{(k)}|^3dxdt<\varepsilon\ \text{ for all }\ k\ge K_\varepsilon(R). \tag{5.3.10}$$

Our aim now is to show that (5.3.10) implies that for all ε_1, there exists an r' with $0<r'<R_0$ such that

$$C^+(0,r';v^{(k)})+D_1^+(0,r';q^{(k)})<\varepsilon_1 \tag{5.3.11}$$

for all k sufficiently large. The arguments we use to show this are contained in Seregin's paper [100] (specifically, Lemma 3.3 there). However as the context is different, the author finds it instructive to provide all the details.

The local boundary regularity theory for the linear Stokes system (Proposition 2.17 in Chapter 2), along with (5.3.1), implies

$$\|\partial_t v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1/2))}+\|\nabla^2 v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1/2))}+\|\nabla q^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1/2))}$$
$$\leq c\Big\{\|v^{(k)}\cdot\nabla v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1))}+\|\nabla v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1))}+\|v^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1))}$$
$$+\|q^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+(1))}\Big\}\leq C(M). \tag{5.3.12}$$

Thus,

$$D_1^+(0,R_0;q^{(k)})\leq C(M)R_0^{-\frac{3}{2}}. \tag{5.3.13}$$

From the fact that $(v^{(k)},q^{(k)})$ satisfies a local energy inequality, one can infer[10] that for $R\in]0,R_0]$:

$$A^+(0,R/2;v^{(k)})+B^+(0,R/2;v^{(k)})$$
$$\leq c[(C^+(0,R;v^{(k)}))^{\frac{2}{3}}+(C^+(0,R;v^{(k)}))^{\frac{1}{3}}(D_1^+(0,R;q^{(k)}))^{\frac{2}{3}}+C^+(0,R;v^{(k)})]. \tag{5.3.14}$$

Using this, together with (5.3.10), we obtain:

$$A^+(0,R/2;v^{(k)})+B^+(0,R/2;v^{(k)})\leq c[\varepsilon^{\frac{2}{3}}+\varepsilon^{\frac{1}{3}}(D_1^+(0,R;q^{(k)}))^{\frac{2}{3}}+\varepsilon] \tag{5.3.15}$$

for $R\in]0,R_0]$ and $k\geq K_\varepsilon(R)$. Using Lemma 5.6, we have that the following holds for $0<\tau<1$ and $R\in]0,R_0]$. Namely,

$$D_1^+(0,\tau R/2;q^{(k)})\leq c\{\tau^2[D_1^+(0,R/2;q^{(k)})+(B^+(0,R/2;v^{(k)}))^{\frac{3}{4}}]$$
$$+\tau^{-\frac{3}{2}}(A^+(0,R/2;v^{(k)}))^{\frac{1}{2}}B^+(0,R/2;v^{(k)})\}. \tag{5.3.16}$$

From this and (5.3.15), one can deduce the following estimate for $R\in]0,R_0]$ and $k\geq K_\varepsilon(R)$. Specifically,

$$D_1^+(0,\tau R/2;q^{(k)})\leq c\Big\{\tau^2 D_1^+(0,R/2;q^{(k)})$$
$$+\tau^2\Big(\varepsilon^{\frac{2}{3}}+\varepsilon^{\frac{1}{3}}(D_1^+(0,R;q^{(k)}))^{\frac{2}{3}}+\varepsilon\Big)^{\frac{3}{4}}+\tau^{-\frac{3}{2}}\Big(\varepsilon^{\frac{2}{3}}+\varepsilon^{\frac{1}{3}}(D_1^+(0,R;q^{(k)}))^{\frac{2}{3}}+\varepsilon\Big)^{\frac{3}{2}}\Big\}. \tag{5.3.17}$$

The above, combined with Young's inequality and the fact that $0<\varepsilon<1$ and $0<\tau<1$, implies that

$$D_1^+(0,\tau R/2;q^{(k)})\leq c\Big\{\Big(\tau^2+\frac{\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}\Big)D_1^+(0,R;q^{(k)})+\frac{\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}\Big\} \tag{5.3.18}$$

[10] See Lemma 3.3 (specifically, (3.15)) of Seregin's paper [100].

for $R \in]0, R_0]$ and $k \geq K_\varepsilon(R)$. Let us now fix τ such that

$$0 < \tau < \min\Big\{1, \frac{1}{4c}\Big\}. \tag{5.3.19}$$

With this constraint on τ, we now choose ε such that

$$\varepsilon < \frac{\tau^5}{16} \Rightarrow \frac{\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}} < \frac{\tau}{4}. \tag{5.3.20}$$

With the choices (5.3.19)-(5.3.20), we obtain for $R \in]0, R_0]$ that

$$D_1^+(0, \tau R/2; q^{(k)}) \leq \frac{\tau}{2} D_1^+(0, R; q^{(k)}) + \frac{c\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}} \text{ for } k \geq K_\varepsilon(R). \tag{5.3.21}$$

Using this and (5.3.10), we see that the following iterative relations hold for $i = 0, \ldots, j$ and $k \geq \max_{i=1,2\ldots,j+1}\{K_\varepsilon((\tau/2)^i R_0)\}$. Namely,

$$C^+\Big(0, \Big(\frac{\tau}{2}\Big)^{i+1} R_0; v^{(k)}\Big) < \varepsilon \tag{5.3.22}$$

and

$$D_1^+\Big(0, \Big(\frac{\tau}{2}\Big)^{i+1} R_0; q^{(k)}\Big) \leq \frac{\tau}{2} D_1^+\Big(0, \Big(\frac{\tau}{2}\Big)^{i} R_0; q^{(k)}\Big) + \frac{c\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}. \tag{5.3.23}$$

Iterating (5.3.23), it can be inferred that for $i = 1, \ldots, j$

$$D_1^+\Big(0, \Big(\frac{\tau}{2}\Big)^{i+1} R_0; q^{(k)}\Big) \leq \Big(\frac{\tau}{2}\Big)^{i+1} D_1^+(0, R_0; q^{(k)}) + \frac{c\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}} \frac{2}{2-\tau} \tag{5.3.24}$$

provided that $k \geq \max_{i=1,2\ldots,j+1}\{K_\varepsilon((\tau/2)^i R_0)\}$. Using (5.3.22), (5.3.24) and (5.3.13), it can be inferred that for $k \geq \max_{i=1,2\ldots,j+1}\{K_\varepsilon((\tau/2)^i R_0)\}$ one has the bound

$$C^+\Big(0, \Big(\frac{\tau}{2}\Big)^{j+1} R_0; v^{(k)}\Big) + D_1^+\Big(0, \Big(\frac{\tau}{2}\Big)^{j+1} R_0; q^{(k)}\Big)$$
$$\leq \varepsilon + \Big(\frac{\tau}{2}\Big)^{j+1} \frac{C(M)}{R_0^{\frac{3}{2}}} + \frac{c\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}} \frac{2}{2-\tau}. \tag{5.3.25}$$

Fix ε_1 and consider the following additional constraints on ε. Namely, $\varepsilon < \varepsilon_1$ and

$$\frac{c\varepsilon^{\frac{1}{2}}}{\tau^{\frac{3}{2}}} \frac{2}{2-\tau} < \frac{\varepsilon_1}{2}.$$

Since $\tau < 1$, we may fix a sufficiently large j such that

$$\Big(\frac{\tau}{2}\Big)^{j+1} \frac{C(M)}{R_0^{\frac{3}{2}}} < \frac{\varepsilon_1}{2}.$$

These choices, together with (5.3.25), imply that

$$C^{+}\Big(0,\Big(\frac{\tau}{2}\Big)^{j+1}R_0;v^{(k)}\Big)+D_1^{+}\Big(0,\Big(\frac{\tau}{2}\Big)^{j+1}R_0;q^{(k)}\Big)\leq 2\varepsilon_1 \tag{5.3.26}$$

for $k\geq \max_{i=1,2\dots,j+1}\{K_{\varepsilon}((\tau/2)^{i}R_0)\}$.

For ε_1 sufficently small, (5.3.26), together with the ε-regularity theory for the Navier-Stokes equation up to the boundary[11], implies that for all $k\geq \max_{i=1,2\dots,j+1}\{K_{\varepsilon}((\tau/2)^{i}R_0)\}$:

$$v^{(k)}\in L_{\infty}\Big(Q^{+}\Big(0,\frac{\tau^{j+1}R_0}{2^{j+2}}\Big)\Big). \tag{5.3.27}$$

This contradicts the fact that $(0,0)$ is a singular point of $v^{(k)}$ for all $k=1,2\dots$. $\square$

5.4 A Priori Estimates

Let us consider a sufficiently smooth solution (v,q) to the Navier-Stokes system in the space-time strip $Q^{+}_{-2,0}=\mathbb{R}^3_{+}\times]-2,0[$ to the following initial boundary value problem:

$$\partial_t v+v\cdot\nabla v-\Delta v+\nabla q=0,\qquad \operatorname{div} v=0 \tag{5.4.1}$$

in $Q^{+}_{-2,0}$,

$$v(x_1,x_2,0,t)=0 \tag{5.4.2}$$

for $(x_1,x_2,t)\in\mathbb{R}^2\times[-2,0]$ and

$$v(\cdot,-2)=u_0(\cdot)\in L^{3,\beta}(\mathbb{R}^3_{+})\cap J(\mathbb{R}^3)$$

for $3\leq\beta<\infty$. Suppose that

$$\|u_0\|_{L^{3,\beta}(\mathbb{R}^3_{+})}=M<\infty. \tag{5.4.3}$$

Lemma 2.5 from Chapter 2, along with the continuous embedding $L^{3,\beta}(\mathbb{R}^3_{+})\hookrightarrow L^{3,\infty}(\mathbb{R}^3_{+})$, implies that we have the following decomposition for u_0. Namely,

$$u_0:=u_0^1+u_0^2,$$

where

$$u_0^1\in[C^{\infty}_{0,0}(\mathbb{R}^3_{+})]^{L_{\frac{10}{3}}(\mathbb{R}^3_{+})},\ u_0^2\in[C^{\infty}_{0,0}(\mathbb{R}^3_{+})]^{L_2(\mathbb{R}^3_{+})} \tag{5.4.4}$$

and

$$\|u_0^1\|_{L_{\frac{10}{3}}(\mathbb{R}^3_{+})}+\|u_0^2\|_{L_2(\mathbb{R}^3_{+})}\leq c(M). \tag{5.4.5}$$

[11] See Lemma 3.1-3.2 in Seregin's paper [100], for example.

Next we make use of ideas related to those in Calderón's paper [18], though with slightly different decompositions. Specifically, we then decompose v into two parts such that $v = u^1 + u^2$, where u^1 and p^1 solve the following linear problem. Namely,

$$\partial_t u^1 - \Delta u^1 + \nabla p^1 = 0, \qquad \operatorname{div} u^1 = 0 \tag{5.4.6}$$

in $Q^+_{-2,0} := \mathbb{R}^3_+ \times]-2,0[$,

$$u^1(x_1, x_2, 0, t) = 0 \tag{5.4.7}$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [-2, 0]$ and

$$u^1(\cdot, -2) = u^1_0 \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_{\frac{10}{3}}}. \tag{5.4.8}$$

Below we give some relevant estimates for the linear problem. From Ukai's paper [127] (Theorem 3.1 there), we obtain

$$\|u^1(\cdot, t)\|_{L_l(\mathbb{R}^3_+)} \leq \frac{c(M)}{(t+2)^{\frac{3}{2}(\frac{3}{10}-\frac{1}{l})}} \text{ and } \|\nabla u^1(\cdot, t)\|_{L_l(\mathbb{R}^3_+)} \leq \frac{c(M)}{(t+2)^{\frac{1}{2}+\frac{3}{2}(\frac{3}{10}-\frac{1}{l})}} \tag{5.4.9}$$

for $10/3 \leq l < \infty$. In the subsection of this chapter called 'Preliminaries', we discussed Solonnikov estimates for the Green function in the half-space. These estimates imply that

$$\|\partial^m_t \nabla^k u^1(\cdot, t)\|_{L_\infty(\mathbb{R}^3_+)} \leq \frac{c(M)}{(t+2)^{m+\frac{k}{2}+\frac{9}{20}}}. \tag{5.4.10}$$

From (5.4.9), we have that

$$\|u^1(\cdot, t)\|_{L_5(\mathbb{R}^3_+)} \leq \frac{c(M)}{(t+2)^{\frac{3}{20}}}, \quad \|u^1(\cdot, t)\|_{L_4(\mathbb{R}^3_+)} \leq \frac{c(M)}{(t+2)^{\frac{3}{40}}}.$$

It is then seen that:

$$\|u^1\|_{L_5(Q^+_{-2,0})} + \|u^1\|_{L_4(Q^+_{-2,0})} + \sup_{t \in]-2,0[} \|u^1(\cdot, t)\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)} \leq c(M). \tag{5.4.11}$$

The second piece in the decomposition of v is u^2. This satisfies the nonlinear system

$$\partial_t u^2 + v \cdot \nabla v - \Delta u^2 + \nabla p^2 = 0, \qquad \operatorname{div} u^2 = 0 \tag{5.4.12}$$

in $Q^+_{-2,0}$, the boundary conditions

$$u^2(x_1, x_2, 0, t) = 0 \tag{5.4.13}$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [-2, 0]$ and the initial condition

$$u^2(\cdot, -2) = u^2_0 \in J(\mathbb{R}^3_+) \tag{5.4.14}$$

in $\mathbb{R}^3_+$.

The standard energy approach to the second system gives

$$\partial_t\|u^2(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)}+2\|\nabla u^2(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)}$$
$$=2\int\limits_{\mathbb{R}^3_+} v\otimes v:\nabla u^2dxds=I_1+I_2,$$

where

$$I_1=2\int\limits_{\mathbb{R}^3_+} u^1\otimes u^1:\nabla u^2dx,\quad I_2=2\int\limits_{\mathbb{R}^3_+} u^1\otimes u^2:\nabla u^2dx.$$

Next, let us consequently evaluate terms on the right-hand side of the energy identity. For the first term, we have by Hölder's inequality that

$$|I_1|\leq c\|u^1(\cdot,t)\|^2_{L_4(\mathbb{R}^3_+)}\|\nabla u^2(\cdot,t)\|_{L_2(\mathbb{R}^3_+)}.$$

For the second term, Hölder's inequality implies that

$$|I_2|\leq c\|u^1(\cdot,t)\otimes u^2(\cdot,t)\|_{L_2(\mathbb{R}^3_+)}\|\nabla u^2(\cdot,t)\|_{L_2(\mathbb{R}^3_+)}$$
$$\leq c\|u^1(\cdot,t)\|_{L_5(\mathbb{R}^3_+)}\|u^2(\cdot,t)\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)}\|\nabla u^2(\cdot,t)\|_{L_2(\mathbb{R}^3_+)}. \tag{5.4.15}$$

Utilising interpolation of Lebesgue spaces and the Sobolev inequality, we find that

$$\|u^2(\cdot,t)\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)}\leq\|u^2(\cdot,t)\|^{\frac25}_{L_2(\mathbb{R}^3_+)}\|u^2(\cdot,t)\|^{\frac35}_{L_6(\mathbb{R}^3_+)}$$
$$\leq c\|u^2(\cdot,t)\|^{\frac25}_{L_2(\mathbb{R}^3_+)}\|\nabla u^2(\cdot,t)\|^{\frac35}_{L_2(\mathbb{R}^3_+)}. \tag{5.4.16}$$

Using this and (5.4.15), we find

$$|I_2|\leq c\|u^1(\cdot,t)\|_{L_5(\mathbb{R}^3_+)}\|u^2(\cdot,t)\|^{\frac25}_{L_2(\mathbb{R}^3_+)}\|\nabla u^2(\cdot,t)\|^{\frac85}_{L_2(\mathbb{R}^3_+)}.$$

Letting

$$y(t):=\|u^2(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)}$$

and using the Young inequality, we deduce that

$$y'(t)+\|\nabla u^2(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)}\leq c\|u^1(\cdot,t)\|^5_{L_5(\mathbb{R}^3_+)}y(t)+c\|u^1(\cdot,t)\|^4_{L_4(\mathbb{R}^3_+)}. \tag{5.4.17}$$

Next, elementary arguments lead to the inequality

$$\Big(y(t)\exp\Big(-c\int\limits_{-2}^{t}\|u^1(\cdot,s)\|^5_{L_5(\mathbb{R}^3_+)}ds\Big)\Big)'$$

$$\leq c\exp\Big(-c\int\limits_{-2}^{t}\|u^1(\cdot,s)\|^5_{L_5(\mathbb{R}^3_+)}ds\Big)\|u^1(\cdot,t)\|^4_{L_4(\mathbb{R}^3_+)}.$$

So,

$$y(t)\leq c\int\limits_{-2}^{t}\exp\Big(c\int\limits_{\tau}^{t}\|u^1(\cdot,s)\|^5_{L_5(\mathbb{R}^3_+)}ds\Big)\|u^1(\cdot,\tau)\|^4_{L_4(\mathbb{R}^3_+)}d\tau$$

$$+\|u^2_0\|^2_{L_2(\mathbb{R}^3_+)}\exp\Big(c\int\limits_{-2}^{t}\|u^1(\cdot,s)\|^5_{L_5(\mathbb{R}^3_+)}ds\Big).$$

Using this, together with (5.4.5), (5.4.11) and (5.4.17), it is easily seen that

$$\sup_{-2<t<0}y(t)\leq C(M),\qquad \|\nabla u^2\|_{L_2(Q^+_{-2,0})}\leq C(M).\tag{5.4.18}$$

Using these estimates, together with interpolation of Lebesgue spaces, the Sobolev inequality and Hölder's inequality, one can deduce that

$$\|u^2\|_{L_s(Q^+_{-2,0})}\leq C(s,M)\tag{5.4.19}$$

for any $s\in[2,10/3]$. Moreover,

$$\|v\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}\leq C(M).\tag{5.4.20}$$

Then $f=v\cdot\nabla v$ can be decomposed as $f=f^1+f^2+f^3$. Here,

$$f^1:=u^2\cdot\nabla u^2,\tag{5.4.21}$$

$$f^2:=u^1\cdot\nabla u^1+u^2\cdot\nabla u^1\tag{5.4.22}$$

and

$$f^3:=u^1\cdot\nabla u^2.\tag{5.4.23}$$

Using (5.4.18), together with interpolation of Lebesgue spaces, the Sobolev inequality and Hölder's inequality, one can verify that

$$\|f^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-2,0})}\leq C(M).\tag{5.4.24}$$

Using (5.4.9)-(5.4.10) and (5.4.19), we deduce that

$$\|f^2\|_{L_{\frac{10}{3}}(Q^+_{-7/4,0})}\leq C(M).\tag{5.4.25}$$

Finally, using (5.4.10) and (5.4.18), one can infer that

$$\|f^3\|_{L_2(Q^+_{-7/4,0})}\leq C(M).\tag{5.4.26}$$

Putting everything together we have

$$\|f^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-7/4,0})}+\|f^2\|_{L_{\frac{10}{3}}(Q^+_{-7/4,0})}+\|f^3\|_{L_2(Q^+_{-7/4,0})}+\|v\|_{L_{\frac{10}{3}}(Q^+_{-7/4,0})}\leq C(M). \quad (5.4.27)$$

Let us fix a smooth cut-off function $\chi(t)$ such that

$$\chi(t):=\begin{cases} 1 & \text{if } \quad -3/2<t<1,\\ 0 & \text{if } \quad -2<t<-7/4.\end{cases}$$

Proposition 2.18 and Remark 2.19 in 'Chapter 2: Preliminary Material', together with uniqueness results for the linear Stokes system[12] imply that we may split χv and χq in the following way. Namely, $\chi v=v^1+v^2+v^3$ and $\chi q=q^1+q^2+q^3$ for $(x,t)\in Q^+_{-2,0}$. Here,

$$\partial_t v^i(x,t)-\Delta v^i(x,t)+\nabla q^i(x,t)=g^i(x,t) \text{ for } (x,t)\in Q^+_{-2,0},$$

$$g^i(x,t):=-\chi(t)f^i(x,t)+\delta_{i2}\chi'(t)v(x,t),$$

$$\operatorname{div} v^i=0,$$

$$v^i(x_1,x_2,0,t)=0 \text{ for all } (x_1,x_2,t)\in\mathbb{R}^2\times[-2,0],$$

$$v^i(x,-2)=0.$$

$$(v^i,\nabla q^i)\in W^{2,1}_{s_i,q_i}(Q^+_{-2,0})\times L_{s_i,q_i}(Q^+_{-2,0}),$$

$$(s_1,q_1)=(9/8,3/2),\ (s_2,q_2)=(10/3,10/3) \text{ and } (s_3,l_3)=(2,2).$$

Using (5.4.20) and (5.4.24)-(5.4.26), it follows from the maximal regularity estimates in Proposition 2.18 that

$$\|\partial_t v^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-2,0})}+\|\nabla^2 v^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-2,0})}+\|\nabla q^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-2,0})}$$
$$\leq c\|g^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-2,0})}\leq C_1(M), \quad (5.4.28)$$

$$\|\partial_t v^2\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}+\|\nabla^2 v^2\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}+\|\nabla q^2\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}$$
$$\leq c\|g^2\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}\leq C_2(M) \quad (5.4.29)$$

and

$$\|\partial_t v^3\|_{L_2(Q^+_{-2,0})}+\|\nabla^2 v^3\|_{L_2(Q^+_{-2,0})}+\|\nabla q^3\|_{L_2(Q^+_{-2,0})}\leq$$
$$\leq c\|g^3\|_{L_2(Q^+_{-2,0})}\leq C_3(M). \quad (5.4.30)$$

[12]See, for example, Lemma 2.4.2 pp.234-235 in Sohr's book [117].

5.5 Passage to the Limit

Suppose that we have a sequence of sufficiently smooth functions $v^{(k)}$ and $q^{(k)}$, which are defined in the domain $Q^+_{-2,0} = \mathbb{R}^3_+ \times]-2,0[$ and are solutions to the following initial boundary value problem. Specifically,

$$\partial_t v^{(k)} + v^{(k)} \cdot \nabla v^{(k)} - \Delta v^{(k)} + \nabla q^{(k)} = 0 \tag{5.5.1}$$

in $Q^+_{-2,0}$,

$$\operatorname{div} v^{(k)} = 0$$

in $Q^+_{-2,0}$,

$$v^{(k)}(x_1, x_2, 0, t) = 0 \tag{5.5.2}$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [-2, 0]$ and

$$v^{(k)}(\cdot, -2) = u_0^{(k)}(\cdot) \in L^{3,\beta}(\mathbb{R}^3_+). \tag{5.5.3}$$

Let

$$M := \sup_k \|u_0^{(k)}\|_{L^{3,\beta}(\mathbb{R}^3_+)} < \infty.$$

We decompose $u_0^{(k)}$ according to Lemma 2.5 in Chapter 2. Namely,

$$u_0^{(k)} := u_0^{1(k)} + u_0^{2(k)}, \tag{5.5.4}$$

where

$$u_0^{1(k)} \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_{\frac{10}{3}}(\mathbb{R}^3_+)},\ u_0^{2(k)} \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_2(\mathbb{R}^3_+)} \tag{5.5.5}$$

and

$$\|u_0^{1(k)}\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)} + \|u_0^{2(k)}\|_{L_2(\mathbb{R}^3_+)} \leq c(M). \tag{5.5.6}$$

In order to show compactness properties of $(v^{(k)}, q^{(k)})$, as well as decay properties of the limit function (v_∞, q_∞), we will utilise the estimates (5.4.28)-(5.4.30).

Proposition 5.7. *There exists a subsequence of $v^{(k)}$, which we still denote as $v^{(k)}$, with the following properties. Namely,*

$$v^{(k)} \rightharpoonup v_\infty \tag{5.5.7}$$

in $L_{\frac{10}{3}}(Q^+_{-2,0})$,

$$v^{(k)} \overset{*}{\rightharpoonup} v_\infty \tag{5.5.8}$$

in $L_\infty(]-2+\delta, 0[; L_2(B^+(R)))$ for any $R > 0$ and any $0 < \delta < 2$,

$$\nabla v^{(k)} \rightharpoonup \nabla v_\infty \tag{5.5.9}$$

in $L_2(B^+(R)\times]-2+\delta,0[)$ *for any* $R>0$ *and any* $0<\delta<2$,

$$v^{(k)}\to v_\infty \tag{5.5.10}$$

in $L_3(B^+(R)\times]-3/2,0[)$ *for any* $R>0$ *and*

$$q^{(k)}\rightharpoonup q_\infty \tag{5.5.11}$$

in $L_{\frac{3}{2}}(B^+(R)\times]-3/2,0[)$ *for any* $R>0$.

The functions (v_∞,q_∞) *satisfy (5.4.1) in* $Q^+_{-3/2,0}$ *and (5.4.2) for* $(x_1,x_2,t)\in\mathbb{R}^2\times]-3/2,0[$. *For* v_∞, *we have that, for any* $\phi\in C_0^\infty(\mathbb{R}^3)$,

$$t\to\int\limits_{\mathbb{R}^3_+}v_\infty(x,t)\cdot\phi(x)dx\in C([-3/2,0]). \tag{5.5.12}$$

For the pressure q_∞, *we have that*

$$q_\infty=\sum_{i=1}^3 q^i_\infty\ \ \textit{in}\ \mathbb{R}^3_+\times]-3/2,0[,$$

with the estimates

$$\|\nabla q^1_\infty\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-3/2,0})}+\|\nabla q^2_\infty\|_{L_{\frac{10}{3}}(Q^+_{-3/2,0})}+\|\nabla q^3_\infty\|_{L_2(Q^+_{-3/2,0})}<\infty. \tag{5.5.13}$$

For any $R>0$, *the limit pair* (v_∞,q_∞) *satisfy the local energy inequality*

$$\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}\varphi(x,t)|v_\infty(x,t)|^2dx+2\int\limits_{-3/2}^t\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}\varphi|\nabla v_\infty|^2dxds$$

$$\leq\int\limits_{-3/2}^t\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}|v_\infty|^2(\partial_t\varphi+\Delta\varphi)+v_\infty\cdot\nabla\varphi(|v_\infty|^2+2q_\infty)dxds \tag{5.5.14}$$

for all $-3/2\leq t\leq 0$, *for all* $x_0\in\mathbb{R}^3$ *and for all positive* $\varphi\in C_0^\infty(B(x_0,R)\times]-3/2,\infty[)$.

Proof. Obviously, (5.5.7) follows from (5.4.20). Moreover, the limit function obeys the estimate

$$\|v_\infty\|_{L_{\frac{10}{3}}(Q^+_{-2,0})}<\infty. \tag{5.5.15}$$

Obviously, (5.4.10) and (5.4.18) imply (5.5.8)-(5.5.9). From (5.4.28)–(5.4.30), we may split $v^{(k)}=v^{(k),1}+v^{(k),2}+v^{(k),3}$ such that for $(x,t)\in Q^+_{-3/2,0}$ we have the following uniform bounds:

$$\sup_k\big(\|v^{(k),1}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+_{-3/2,0})}+\|v^{(k),2}\|_{W^{2,1}_{\frac{10}{3}}(Q^+_{-3/2,0})}+\|v^{(k),3}\|_{W^{2,1}_2(Q^+_{-3/2,0})}\big)\leq C(M). \tag{5.5.16}$$

As discussed in 'Chapter 2: Preliminary Material', one has the compact embedding

$$W^{2,1}_{m,n}(B^+(a)\times]-3/2,0]) \hookrightarrow C([-3/2,0];L_m(B^+(a))$$

for $1<n\leq\infty$ and $1\leq m\leq\infty$. From this and (5.5.16), we infer that there exists a subsequence (denoted here by the original sequence) such that the following holds for any $a>0$:

$$v^{(k),1}\to v^1_\infty \text{ in } C([-3/2,0];L_{\frac{9}{8}}(B^+(a))), \tag{5.5.17}$$

$$v^{(k),2}\to v^2_\infty \text{ in } C([-3/2,0];L_{\frac{10}{3}}(B^+(a))) \tag{5.5.18}$$

and

$$v^{(k),3}\to v^3_\infty \text{ in } C([-3/2,0];L_2(B^+(a))). \tag{5.5.19}$$

The above facts clearly imply that (5.5.12) holds true. Using (5.5.17)-(5.5.19), along with the fact $v^{(k)}$ is bounded in $L_{\frac{10}{3}}(Q^+_{-2,0})$, we can deduce (5.5.10).

We now treat the pressure $q^{(k)}$, using the decomposition of the previous section

$$q^{(k)}=\sum_{i=1}^{3}q^{(k),i} \text{ in } \mathbb{R}^3_+\times]-3/2,0[.$$

Using (5.4.28)-(5.4.30) and the Poincaré inequality, we can show (5.5.11) and (5.5.13).

The functions $(v^{(k)},q^{(k)})$ satisfy the local energy inequality. Specifically,

$$\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}\varphi(x,t)|v^{(k)}(x,t)|^2dx+2\int\limits_{-3/2}^{t}\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}\varphi|\nabla v^{(k)}|^2dxds$$

$$\leq\int\limits_{-3/2}^{t}\int\limits_{B(x_0,R)\cap\mathbb{R}^3_+}|v^{(k)}|^2(\partial_t\varphi+\Delta\varphi)+v^{(k)}\cdot\nabla\varphi(|v^{(k)}|^2+2q^{(k)})dxds$$

for all $-3/2\leq t\leq 0$, for all $x_0\in\mathbb{R}^3$ and for all positive $\varphi\in C^\infty_0(B(x_0,R)\times]-3/2,\infty[)$. Using this, together with (5.5.8)-(5.5.11) and (5.5.17)-(5.5.19), we can obtain (5.5.14) by passing to the limit. □

The next proposition regards decay properties of the limit functions (v_∞,q_∞) from the previous proposition. Similar arguments to those used in the proof of the next proposition are contained in Seregin's paper [100][13], where decay properties are shown for certain classes of weak Leray-Hopf solutions $v:\mathbb{R}^3_+\times]0,\infty[\to\mathbb{R}^3$.

The proof of this proposition will make use of the ε-regularity criteria stated in Chapter 3 (Lemma 3.5 Chapter 3), as well as the following lemma due to Seregin in [99] (Theorem 1.2 in [99]).

[13]In particular p.237 of [100].

Lemma 5.8. *Let (v, q) be a suitable weak solution in $B^+(x_0, R)\times]t_0 - R^2, t_0[$ near $\Gamma(x_0, R)\times]t_0 - R^2, t_0[$. Then there exist universal constants ε and $c_0 > 0$ with the following property. Assume that (v, q) satisfy*

$$\frac{1}{R^2} \int\limits_{t_0 - R^2}^{t} \int\limits_{B^+(x_0,R)} (|v|^3 + |q|^{\frac{3}{2}}) dx dt \leq \varepsilon. \tag{5.5.20}$$

Then v is Hölder continuous in the closure of $B^+(x_0, R/2)\times]t_0 - (R/2)^2, t_0[$ and the following bound is valid:

$$\sup_{(x,t)\in B^+(x_0,R/2)\times]t_0-(R/2)^2,t_0[} |v(x, t)| < \frac{c_0}{R}. \tag{5.5.21}$$

Now we state the proposition regarding decay properties of the limit functions (v_∞, q_∞).

Proposition 5.9. *Let (v_∞, q_∞) be the limit functions from Proposition 5.7. There exists a number $R_1 > 0$ such that*

$$\sup_{(x,t)\in(\mathbb{R}^3_+\setminus B^+(R_1))\times]-4/3,0[} |v_\infty(x, t)| \leq c \tag{5.5.22}$$

for some universal constant c. Moreover, given $\delta > 0$ and $k = 1, 2, \ldots,$

$$\sup_{(x,t)\in(\mathbb{R}^3_{+\delta}\setminus B^+(2R_1))\times]-5/4,0[} |\nabla^k v_\infty(x, t)| \leq c_1(k, \delta, R_1). \tag{5.5.23}$$

Here, $\mathbb{R}^3_{+\delta} := \mathbb{R}^3_+ \cap \{x_3 > \delta\}$.

Proof. From (5.5.7) and (5.5.13), we infer that

$$\int\limits_{-3/2}^{0} \Big(\int\limits_{\mathbb{R}^3_+\setminus B^+(R)} |v_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{10}} dt + \int\limits_{-3/2}^{0} \Big(\int\limits_{\mathbb{R}^3_+\setminus B^+(R)} |\nabla q^1_\infty|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt$$

$$+ \int\limits_{-3/2}^{0} \Big(\int\limits_{\mathbb{R}^3_+\setminus B^+(R)} |\nabla q^2_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{20}} dt + \int\limits_{-3/2}^{0} \Big(\int\limits_{\mathbb{R}^3_+\setminus B^+(R)} |\nabla q^3_\infty|^2 dx\Big)^{\frac{3}{4}} dt \to 0$$

as $R \to \infty$.

Using this, together with the Poincaré inequality, we infer the following. Namely, given any $\varepsilon > 0$, there exists a positive number $R_1 > 0$ such that

$$\frac{1}{r^2} \int\limits_{Q(z_0,r)} (|v_\infty|^3 + |q_\infty - [q_\infty]_{B(x_0,r)}|^{\frac{3}{2}}) dx dt$$

$$\leq \frac{1}{r^2}\int\limits_{Q(z_0,r)} |v_\infty|^3 dxdt + \frac{c}{r^2}\sum_{i=1}^{3}\int\limits_{Q(z_0,r)} |q^i_\infty - [q^i_\infty]_{B(x_0,r)}|^{\frac{3}{2}} dxdt$$

$$\leq cr^{-\frac{17}{20}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B(x_0,r)} |v_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{10}} dt + cr^{-\frac{3}{2}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B(x_0,r)} |\nabla q^1_\infty|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt$$

$$+cr^{\frac{23}{20}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B(x_0,r)} |\nabla q^2_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{20}} dt + cr^{\frac{1}{4}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B(x_0,r)} |\nabla q^3_\infty|^{2} dx\Big)^{\frac{3}{4}} dt < \varepsilon$$

with $r = 1/100$ and $Q(z_0,r) \subset (\mathbb{R}^3_+ \setminus B^+(R_1/2))\times]-3/2,0[$.

The same can be done for boundary points:

$$\frac{1}{\rho^2}\int\limits_{Q^+(z_0,\rho)} (|v_\infty|^3 + |q_\infty - [q_\infty]_{B^+(x_0,\rho)}|^{\frac{3}{2}}) dxdt \leq \frac{1}{\rho^2}\int\limits_{Q^+(z_0,\rho)} |v_\infty|^3 dxdt +$$

$$+\frac{c}{\rho^2}\sum_{i=1}^{3}\int\limits_{Q^+(z_0,\rho)} |q^i_\infty - [q^i_\infty]_{B^+(x_0,\rho)}|^{\frac{3}{2}} dxdt$$

$$\leq c\rho^{-\frac{17}{20}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B^+(x_0,\rho)} |v_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{10}} dt + c\rho^{-\frac{3}{2}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B^+(x_0,\rho)} |\nabla q^1_\infty|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt$$

$$+c\rho^{\frac{23}{20}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B^+(x_0,\rho)} |\nabla q^2_\infty|^{\frac{10}{3}} dx\Big)^{\frac{9}{20}} dt + c\rho^{\frac{1}{4}}\int\limits_{-3/2}^{0}\Big(\int\limits_{B^+(x_0,\rho)} |\nabla q^3_\infty|^{2} dx\Big)^{\frac{3}{4}} dt < \varepsilon$$

with $\varrho = 1/100$, $Q^+(z_0,\varrho) := B^+(x_0,\varrho)\times]t_0-\varrho^2, t_0[\subset (\mathbb{R}^3_+ \setminus B^+(R_1/2))\times]-3/2,0[$ and $x_{03} = 0$. Using Lemma 3.5 in Chapter 3 and Lemma 5.8, we can show the validity of (5.5.22) in $(\mathbb{R}^3_+ \setminus B^+(R_1))\times]-4/3,0[$.

The second statement of the proposition can be deduced from the local regularity theory for the heat equation and bootstrap arguments applied to the vorticity equation and is described in detail in the paper of Serrin [114]. □

Next, we show additional smoothness properties of the limit function, away from a set of times of measure zero. Related observations have been made by Leray in [75]. In the proposition below and throughout this chapter, $\mu(\Sigma)$ denotes the Lebesgue measure of Σ.

Proposition 5.10. *Let (v_∞, q_∞) be the limit functions from Proposition 5.7. There exists a set $\Sigma \subset]-3/2,0[$ with the following properties.*

- $\mu(\Sigma) = 0$.
- *For all* $t_0 \in]-3/2, 0[\setminus\Sigma$ *there exists a* $\lambda(t_0) > 0$ *such that the following holds true for any* $\delta > 0$, $R > 0$, $0 < \varepsilon < \lambda(t_0)$ *and* $k = 0, 1 \ldots$. *Specifically,*

$$\nabla^k v_\infty \in L_\infty((\mathbb{R}^3_{+\delta} \cap B^+(R)) \times]t_0 + \varepsilon, t_0 + \lambda(t_0)[). \tag{5.5.24}$$

Proof. **Step 1: Treatment of $u^{1,(k)}$**

Let $u_0^{1(k)}$ and $u_0^{2(k)}$ be as in (5.5.4)-(5.5.6). We use the same decomposition of $v^{(k)}$ as was used in the section 'A Priori Estimates', in particular $v^{(k)} := u^{1,(k)} + u^{2,(k)}$. Here, $u^{1,(k)}$ satisfies (5.4.6)-(5.4.8), with $u^{1,(k)}$ in place of u^1 and $u_0^{1(k)}$ in place of u_0^1. Moreover, $u^{2,(k)}$ satisfies (5.4.12)-(5.4.14), with $u^{2,(k)}$ in place of u^2 and $u_0^{2(k)}$ in place of u_0^2.

The Solonnikov estimates imply the following inequalities for $m, l = 0, 1 \ldots$

$$\|\partial_t^m \nabla^l u^{1,(k)}(\cdot, t)\|_{L_\infty(\mathbb{R}^3_+)} \leq \frac{c}{(t+2)^{\frac{9}{20}+\frac{l}{2}+m}} \|u_0^{1,(k)}\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)} \leq \frac{cM}{(t+2)^{\frac{9}{20}+\frac{l}{2}+m}}. \tag{5.5.25}$$

Recall also that for $l = 0, 1$

$$\|\nabla^l u^{1,(k)}(\cdot, t)\|_{L_4(\mathbb{R}^3_+)} \leq \frac{c}{(t+2)^{\frac{3}{40}+\frac{l}{2}}} \|u_0^{1,(k)}\|_{L_{\frac{10}{3}}(\mathbb{R}^3_+)} \leq \frac{cM}{(t+2)^{\frac{3}{40}+\frac{l}{2}}}. \tag{5.5.26}$$

From (5.5.25), we infer the following convergence properties for $m, l = 0, 1, 2, \ldots$ and an appropriate subsequence of $u^{1,(k)}$ (which we will also label as $u^{1,(k)}$). In particular, we have

$$\partial_t^m \nabla^l u^{1,(k)} \to \partial_t^m \nabla^l u^1_\infty \quad \text{in} \quad C([-7/4, 0]; L_2(B^+(a)), \tag{5.5.27}$$

along with the estimate

$$\|\partial_t^m \nabla^l u^1_\infty\|_{L_\infty(Q^+_{-7/4,0})} \leq c(M, m, l). \tag{5.5.28}$$

Step 2: Convergence of $u^{2,(k)}$

Recall that $u^{2,(k)}$ satisfies

$$\partial_t u^{2,(k)} + u^{2,(k)} \cdot \nabla u^{2,(k)} - \Delta u^{2,(k)} = -\nabla p^{2,(k)} + f^{2,(k)} \text{ in } Q^+_{-2,0},$$

$$\operatorname{div} u^{2,(k)} = 0,$$

$$u^{2,(k)}(x_1, x_2, 0, t) = 0 \text{ for } (x_1, x_2, t) \in \mathbb{R}^2 \times [-2, 0] \text{ and}$$

$$u^{2,(k)}(\cdot, -2) = u_0^{2,(k)} \in [C^\infty_{0,0}(\mathbb{R}^3_+)]^{L_2(\mathbb{R}^3_+)}.$$

Here,

$$f^{2,(k)} := \mathrm{div}\,(u^{1,(k)} \otimes u^{1,(k)} + u^{2,(k)} \otimes u^{1,(k)} + u^{1,(k)} \otimes u^{2,(k)}). \tag{5.5.29}$$

Recall from (5.4.18)-(5.4.19) that we have

$$\|u^{2,(k)}\|_{L_{2,\infty}(Q^+_{-2,0})} \leq C(M), \quad \|\nabla u^{2,(k)}\|_{L_2(Q^+_{-2,0})} \leq C(M) \tag{5.5.30}$$

and

$$\|u^{2,(k)}\|_{L_s(Q^+_{-2,0})} \leq C(s, M) \tag{5.5.31}$$

with any $s \in [2, 10/3]$. These facts, along with (5.5.25)-(5.5.26), imply that

$$\|f^{2,(k)}\|_{L_2(Q^+_{-7/4,0})} \leq C(M). \tag{5.5.32}$$

Repeating arguments from Proposition 5.7, we claim (up to subsequence):

$$u^{2,(k)} \stackrel{*}{\rightharpoonup} u^2_\infty \tag{5.5.33}$$

in $L_{2,\infty}(Q^+_{-2,0})$,

$$\nabla u^{2,(k)} \rightharpoonup \nabla u^2_\infty \tag{5.5.34}$$

in $L_2(Q^+_{-2,0})$,

$$u^{2,(k)} \rightharpoonup u^2_\infty \tag{5.5.35}$$

in $L_{\frac{10}{3}}(Q^+_{-2,0})$,

$$p^{2,(k)} \rightharpoonup \widetilde{Q}_\infty \tag{5.5.36}$$

in $L_{\frac{3}{2}}(B(R)\times]-3/2, 0[)$ for any $R > 0$,

$$u^{2,(k)} \to u^2_\infty \tag{5.5.37}$$

in $L_3(B(R)\times]-3/2, 0[)$ for any $R > 0$,

$$f^{2,(n)} \rightharpoonup f^2_\infty \tag{5.5.38}$$

in $L_2(Q^+_{-7/4,0})$. Upon passage to the limit, we also obtain that for any $w \in C^\infty_0(\mathbb{R}^3)$ the function

$$t :\to \int\limits_{\mathbb{R}^3_+} u^2_\infty(x,t) \cdot w(x)dx \tag{5.5.39}$$

is in $C([-3/2, 0])$. This and (5.5.33) imply that for all $t \in [-3/2, 0]$ we have that $u^2_\infty(\cdot, t) \in J(\mathbb{R}^3)$. Moreover for any $w \in L_2(\mathbb{R}^3)$, the function

$$t :\to \int\limits_{\mathbb{R}^3_+} u^2_\infty(x,t) \cdot w(x)dx \tag{5.5.40}$$

is in $C([-3/2,0])$. The functions $(u^2_\infty, \widetilde{Q}_\infty)$ satisfy the non linear system

$$\partial_t u^2_\infty + u^2_\infty \cdot \nabla u^2_\infty - \Delta u^2_\infty + \nabla \widetilde{Q}_\infty = f^2_\infty, \qquad \operatorname{div} u^2_\infty = 0 \tag{5.5.41}$$

in $Q^+_{-3/2,0}$ and the boundary condition

$$u^2_\infty(x_1, x_2, 0, t) = 0 \tag{5.5.42}$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [-3/2, 0]$. For the pressure $\widetilde{Q}_\infty$, we have that

$$\widetilde{Q}_\infty = Q^1_\infty + Q^2_\infty,$$

with the estimates

$$\|\nabla Q^1_\infty\|_{L_{\frac{9}{8},\frac{3}{2}}(Q^+_{-3/2,0})} + \|\nabla Q^2_\infty\|_{L_2(Q^+_{-3/2,0})} < \infty. \tag{5.5.43}$$

Finally, for any $R > 0$ the limit pair $(u^2_\infty, \widetilde{Q}_\infty)$ satisfy the local energy inequality

$$\int\limits_{\mathbb{R}^3_+ \cap B(x_0,R)} \varphi(x,t)|u^2_\infty(x,t)|^2 dx + 2 \int\limits_{-3/2}^{t} \int\limits_{\mathbb{R}^3_+ \cap B(x_0,R)} \varphi |\nabla u^2_\infty|^2 dxds$$

$$\leq \int\limits_{-3/2}^{t} \int\limits_{\mathbb{R}^3_+ \cap B(x_0,R)} |u^2_\infty|^2 (\partial_t \varphi + \Delta \varphi) + u^2_\infty \cdot \nabla \varphi (|u^2_\infty|^2 + 2\widetilde{Q}_\infty) + f^2_\infty \cdot u^2_\infty \varphi dxds \tag{5.5.44}$$

for all $-3/2 \leq t \leq 0$, for all $x_0 \in \mathbb{R}^3$ and for all positive $\varphi \in C^\infty_0(B(x_0,R)\times]-3/2,\infty[)$.

Step 3: Properties of u^2_∞

Let us mention that for every $R > 0$, Q^1_∞ and Q^2_∞ satisfies the Poincaré inequalities

$$\int\limits_{-3/2}^{0} \int\limits_{B^+(x_0,R)} |Q^1_\infty - [Q^1_\infty]_{B^+(x_0,R)}|^{\frac{3}{2}} dxdt \leq cR^{\frac{1}{2}} \int\limits_{-3/2}^{0} \Big(\int\limits_{B^+(x_0,R)} |\nabla Q^1_\infty|^{\frac{9}{8}} dx \Big)^{\frac{4}{3}} dt \tag{5.5.45}$$

and

$$\int\limits_{-3/2}^{0} \int\limits_{B^+(x_0,R)} |Q^2_\infty - [Q^2_\infty]_{B(x_0,R)}|^2 dxdt \leq cR^2 \int\limits_{-3/2}^{0} \int\limits_{B^+(x_0,R)} |\nabla Q^2_\infty|^2 dxdt. \tag{5.5.46}$$

Let $\varphi(x,t) = \varphi_1(t)\varphi_R(x)$. Here, $\varphi_1 \in C^\infty_0(-3/2,\infty)$ and $\varphi_R \in C^\infty_0(B(R))$ are positive functions. Moreover, $\varphi_R = 1$ on $B(R/2)$, $0 \leq \varphi_R \leq 1$,

$$|\nabla \varphi_R| \leq c/R \text{ and } |\nabla^2 \varphi_R| \leq c/R^2.$$

Clearly, we may substitute $\varphi_1(t)\varphi_R(x)$ into the local energy inequality (5.5.44). Then (5.5.33)-(5.5.43) and (5.5.45)-(5.5.46) imply that we can take $R\to\infty$ in the local energy inequality with the test function $\varphi_1(t)\varphi_R(x)$. In doing this we obtain:

$$\int\limits_{\mathbb{R}^3_+}\varphi_1(t)|u^2_\infty(x,t)|^2dx+2\int\limits_{-3/2}^{t}\int\limits_{\mathbb{R}^3_+}\varphi_1|\nabla u^2_\infty|^2dxds$$

$$\leq\int\limits_{-3/2}^{t}\int\limits_{\mathbb{R}^3_+}|u^2_\infty|^2\partial_t\varphi_1+f^2_\infty\cdot u^2_\infty\varphi_1dxds \tag{5.5.47}$$

for all $-3/2<t<0$ and all positive $\varphi_1\in C^\infty_0(]-3/2,\infty[)$. By the Lebesgue differentiation theorem, there exists a set $\Sigma\subset]-3/2,0[$, with $\mu(\Sigma)=0$, such that for all $t_0\in]-3/2,0[\backslash\Sigma$:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{t_0}^{t_0+\varepsilon}\|u^2_\infty(\cdot,s)\|^2_{L_2(\mathbb{R}^3_+)}ds=\|u^2_\infty(\cdot,t_0)\|^2_{L_2(\mathbb{R}^3_+)} \tag{5.5.48}$$

and

$$\|\nabla u^2_\infty(\cdot,t_0)\|_{L_2(\mathbb{R}^3_+)}<\infty. \tag{5.5.49}$$

For $t_0\in]-3/2,0[\backslash\Sigma$ and $\varepsilon>0$ let

$$\varphi_{\varepsilon,t_0}(s):=\begin{cases}0 & \text{if } s\leq t_0,\\ (s-t_0)/\varepsilon & \text{if } t_0\leq s\leq\varepsilon+t_0,\\ 1 & \text{if } \varepsilon+t_0\leq s.\end{cases}$$

By taking suitable approximations of $\varphi_{\varepsilon,t_0}$, it can be shown that $\varphi_1=\varphi_{\varepsilon,t_0}$ is admissible in (5.5.47). With the choice $\varphi_1=\varphi_{\varepsilon,t_0}$ in (5.5.47), we may use (5.5.48) to obtain the following limit as ε tends to zero. Namely, for any $t_0\in]-3/2,0[\backslash\Sigma$ and $t_0<t\leq 0$:

$$\|u^2_\infty(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)}+\int\limits_{t_0}^{t}\int\limits_{\mathbb{R}^3_+}|\nabla u^2_\infty(x,s)|^2dxds\leq\|u^2_\infty(\cdot,t_0)\|^2_{L_2(\mathbb{R}^3_+)}+\int\limits_{t_0}^{t}\int\limits_{\mathbb{R}^3_+}f^2_\infty\cdot u^2_\infty dxds. \tag{5.5.50}$$

This and (5.5.40) imply

$$\lim_{t\to t_0^+}\|u^2_\infty(\cdot,t)-u^2_\infty(\cdot,t_0)\|_{L_2(\mathbb{R}^3_+)}=0. \tag{5.5.51}$$

Taking into account (5.5.40)-(5.5.42) and (5.5.50)-(5.5.51), together with short time unique solvability results[14] for weak Leray-Hopf solutions to the Navier-Stokes system in unbounded domains with smooth boundary, we can conclude the following. Namely, we can find a number $\lambda=\lambda(t_0,u^2_\infty,f^2_\infty)>0$ such that

$$\nabla u^2_\infty \text{ and } u^2_\infty\in L_\infty(]t_0,t_0+\lambda[;L_2(\mathbb{R}^3_+))\Rightarrow u^2_\infty\in L_\infty(]t_0,t_0+\lambda[;L_6(\mathbb{R}^3_+)). \quad (5.5.52)$$

Step 4: Spatial Smoothing for v_∞

Using (5.5.28) and (5.5.52), we conclude that for all $t_0\in]-3/2,0[\setminus\Sigma$ there exists a $\lambda=\lambda(t_0,u^2_\infty,f^2_\infty)>0$ such that

$$v_\infty\in L_\infty(]t_0,t_0+\lambda[;L_6(B^+(R))) \quad (5.5.53)$$

for all $R>0$. The conclusion (5.5.24) then follows from the local regularity theory for the heat equation and bootstrap arguments applied to the vorticity equation and is described in detail in the paper of Serrin [114].

□

5.6 Rescaling. Scenario I

Let us go back to our original problem

$$\partial_t v+v\cdot\nabla v-\Delta v+\nabla q=0, \qquad \operatorname{div} v=0 \quad (5.6.1)$$

in $\mathbb{R}^3_+\times]0,\infty[$, the boundary conditions

$$v(x_1,x_2,0,t)=0 \quad (5.6.2)$$

for $(x_1,x_2,t)\in\mathbb{R}^2\times[0,\infty[$, and the initial conditions

$$v(\cdot,0)=v_0(\cdot) \quad (5.6.3)$$

in $\mathbb{R}^3_+$. Recall that we assume $u_0\in J(\mathbb{R}^3_+)$ is a short time regular initial data in the sense of Definition 5.2. We also assume that $0<T(u_0)<\infty$ is a blow-up time. Recall from the introduction of this chapter that all global weak Leray-Hopf solutions, with short time regular initial data u_0 and finite blow-up time $T(u_0)$, coincide on $\mathbb{R}^3_+\times]0,T(u_0)[$ and belong to $C^\infty(\mathbb{R}^3_+\times]0,T(u_0)[)$. Throughout, we will refer to the unique solution on $\mathbb{R}^3_+\times]0,T(u_0)[$ as v.

[14] See Theorem $2'$ p.672 of Heywood's paper [51] and Theorem 1.5.1 p.276 of Sohr's book [118], for example.

We will prove Theorem 5.4 by reductio ad absurdum. In particular, we will suppose that there exists a sequence $t_k \uparrow T$ such that

$$M := \sup_k \|v(\cdot, t_k)\|_{L^{3,\beta}(\mathbb{R}^3_+)} < \infty. \tag{5.6.4}$$

The aim is to show that (5.6.4) implies that T is NOT a blow-up time.

From pp.236-237 of Seregin's paper [100], we deduce that for every $\epsilon > 0$ there exists an $R_1(\epsilon) > 0$ such that

$$\sup_{x \in \mathbb{R}^3_+ \setminus B^+(R_1), \epsilon \leq t \leq T} |v(x,t)| < \infty. \tag{5.6.5}$$

As previously mentioned, the papers of Giga [42] and Ukai [127] imply that for
$0 < t < T$ and $3 < s < \infty$

$$\|v(\cdot,t)\|_{L_s(\mathbb{R}^3_+)} \geq \frac{c(s)}{(T-t)^{\frac{s-3}{2s}}}. \tag{5.6.6}$$

The fact that $v \in L_\infty(0,T; J(\mathbb{R}^3_+))$, together with (5.6.6), implies that

$$v \notin L_\infty(\lambda, T; L_\infty(\mathbb{R}^3_+)) \text{ for any } 0 < \lambda < T. \tag{5.6.7}$$

From (5.6.5), (5.6.7) and the definition of blow-up time T, one can infer that there exists a singular point $x_0 = (x_{01}, x_{02}, x_{03}) \in \mathbb{R}^3_+$ at $t = T$. In particular,

$$v \notin L_\infty(B(x_0, r) \cap \mathbb{R}^3_+) \times]T - r^2, T[) \tag{5.6.8}$$

for any positive r. Without loss of generality, we may assume that $(x_{01}, x_{02}) = (0,0)$. One should then consider the two cases,

$$x_{03} = 0$$

and

$$x_{03} > 0.$$

Now let us focus on the first case, which will be referred to as 'Scenario I'. The case $x_{03} > 0$ will be referred to as 'Scenario II'.

In this scenario, our rescaling will be similar to that used by Seregin in [106]. Namely

$$v^{(k)}(y,s) = \lambda_k v(x,t) \quad \text{and} \quad q^{(k)}(y,s) = \lambda_k^2 q(x,t).$$

Here,

$$x = \lambda_k y, \qquad t = T + \lambda_k^2 s \quad \text{and} \quad \lambda_k = \sqrt{\frac{T - t_k}{2}}.$$

So, sufficiently smooth solutions $v^{(k)}$ and $q^{(k)}$ satisfy (5.5.1)-(5.5.3) with

$$u_0^{(k)}(y) = \lambda_k v(\lambda_k y, t_k).$$

Hence,

$$\sup_k \|u_0^{(k)}\|_{L^{3,\beta}(\mathbb{R}^3_+)} = M.$$

So, all assumptions and statements of Propositions 5.7-5.10 hold for $v^{(k)}$ and $q^{(k)}$ and for their limits v_∞ and q_∞.

Now, we shall show that

$$v_\infty(x, 0) = 0. \tag{5.6.9}$$

The arguments used to show this are essentially contained in Phuc's paper [87] (p.752 and p.757 of [87]), see also Proposition 4.21 and Proposition 4.22 in Chapter 4.

First examine the profile. We mention that our assumption that $v : \mathbb{R}^3_+ \times]0, \infty[\to \mathbb{R}^3$ is a weak Leray-Hopf solution implies that v can be adjusted on some set of time of Lebesgue measure zero such that $v(\cdot, t) \in J(\mathbb{R}^3_+)$. Moreover, for any $\varphi(x) \in C_0^\infty(\mathbb{R}^3_+)$ the following functional is continuous in time on $[0, \infty[$:

$$t \to \int\limits_{\mathbb{R}^3_+} v(x, t) \cdot \varphi(x) dx.$$

Our assumption that $\|v(\cdot, t_k)\|_{L^{3,\beta}(\mathbb{R}^3_+)} \leq M$ implies by Hunt's inequality (see Proposition 2.2 of Chapter 2) that for $\varphi \in C_0^\infty(\mathbb{R}^3)$:

$$|\int\limits_{\mathbb{R}^3_+} v(x, t_k) \cdot \varphi(x) dx| \leq M \|\varphi\|_{L^{\frac{3}{2},\beta'}(\mathbb{R}^3_+)}.$$

Hence, the weak continuity implies

$$|\int\limits_{\mathbb{R}^3_+} v(x, T) \cdot \varphi(x) dx| \leq M \|\varphi\|_{L^{\frac{3}{2},\beta'}(\mathbb{R}^3_+)}. \tag{5.6.10}$$

Using the same arguments as on p.752 of Phuc's paper [87] (see also Proposition 4.21 in Chapter 4), we see that (5.6.10) implies that

$$\|v(\cdot, T)\|_{L^{3,\beta}(\mathbb{R}^3_+)} \leq M. \tag{5.6.11}$$

Now we use arguments on p.757 of Phuc's paper [87] (see also Proposition 4.22 in Chapter 4) to show that (5.5.17)-(5.5.19) and (5.6.11) imply that $v_\infty(x, 0) = 0$. It follows from (5.5.17)-(5.5.19) that

$$\int\limits_{B^+(a)} |v^{(k)}(x, 0)| dx \to \int\limits_{B^+(a)} |v_\infty(x, 0)| dx.$$

Using Hunt's inequality (see Proposition 2.2 of Chapter 2) and scaling properties of Lorentz spaces, we see that

$$\frac{1}{a^2}\int\limits_{B^+(a)} |v^{(k)}(x,0)|dx \leq \frac{c}{a^2}\|v^{(k)}(\cdot,0)\|_{L^{3,\beta}(B^+(a))}\|\chi_{B^+(a)}\|_{L^{\frac{3}{2},\beta'}(B^+(a))}$$

$$\leq c\|v^{(k)}(\cdot,0)\|_{L^{3,\beta}(B^+(a))} = c\|v(\cdot,T)\|_{L^{3,\beta}(B^+(\lambda_k a))}.$$

Now,

$$d_{v(\cdot,T),B^+(\lambda_k a)}(\alpha) \leq \mu(B^+(\lambda_k a)) \to 0 \ \text{ as } \ k \to \infty$$

and

$$\|v(\cdot,T)\|_{L^{3,\beta}(\mathbb{R}^3_+)} = \Big(3\int\limits_0^\infty \alpha^\beta d_{v(\cdot,T),\mathbb{R}^3_+}(\alpha)^{\frac{\beta}{3}}\frac{d\alpha}{\alpha}\Big)^{\frac{1}{\beta}} < \infty.$$

We may therefore apply the dominated convergence theorem to deduce that

$$\|v(\cdot,T)\|_{L^{3,\beta}(B^+(\lambda_k a))} = \Big(3\int\limits_0^\infty \alpha^\beta d_{v(\cdot,T),B^+(\lambda_k a)}(\alpha)^{\frac{\beta}{3}}\frac{d\alpha}{\alpha}\Big)^{\frac{1}{\beta}} \to 0.$$

The above facts imply that for any $a > 0$

$$\int\limits_{B^+(a)} |v_\infty(x,0)|dx = 0.$$

Now, we need to show that the limit function v_∞ is not identically zero. Recall that all assumptions and statements of Propositions 5.7-5.10 hold for $(v^{(k)}, q^{(k)})$ and for their limits (v_∞, q_∞). Also recall that $(0,0)$ is a singular point for the each $v^{(k)}$. From these facts one infers that the assumptions of Proposition 5.5 are satisfied. Thus, for any $r > 0$ we have

$$v_\infty \notin L_\infty(Q^+(r)). \tag{5.6.12}$$

Next, we follow arguments of Escauriaza, Seregin and Šverák's paper [27] concerned with backward uniqueness and unique continuation for the heat operator with lower order terms. By Proposition 5.9, we have

$$|\partial_t\omega_\infty - \Delta\omega_\infty| \leq c(|\omega_\infty| + |\nabla\omega_\infty|),\ |\omega_\infty| \leq c$$

in $\{x \in \mathbb{R}^3 :\ x_3 > 2R_1\}\times]-5/4, 0[$, where $\omega_\infty = \nabla \wedge v_\infty$. Next recall (5.6.9), which implies

$$\omega_\infty(x,0) = 0$$

for $x \in \{\mathbb{R}^3 : \ x_3 > 2R_1\}$. Thus, we can apply arguments from Escauriaza, Seregin and Šverák's paper[15] [27] that are related with backward uniqueness for the heat operator with lower order terms to obtain

$$\omega_\infty(x,t) = 0 \tag{5.6.13}$$

for all $(x,t) \in \{x \in \mathbb{R}^3 : \ x_3 > 2R_1\}\times]-5/4,0[$. From Proposition 5.10, there exists a set $\Sigma^{'} \subset]-5/4,0[$ with the following properties[16].

- $\mu(\Sigma^{'}) = 0$.
- For all $t_0 \in]-5/4,0[\setminus\Sigma^{'}$ there exists a $\lambda(t_0) > 0$ such that the following holds true for any $\delta > 0$, $R > 0$, $0 < \varepsilon < \lambda(t_0)$ and $k = 0, 1 \ldots$. Specifically,

$$\nabla^k v_\infty \in L_\infty((\mathbb{R}^3_{+\delta} \cap B^+(R))\times]t_0+\varepsilon, t_0+\lambda(t_0)[). \tag{5.6.14}$$

This allows one to apply unique continuation through spatial boundaries (Theorem 4.1 of Escauriaza, Seregin and Šverák's paper [27]) to conclude that (5.6.13) is valid in $\mathbb{R}^3_+\times]t_0, t_0+\lambda(t_0)[$. Using this, along with properties of v_∞ from Proposition 5.7, we infer that for almost every $t \in]t_0, t_0+\lambda(t_0)[$

$$\Delta v_\infty(x,t) = 0,$$

$$v_\infty(x_1,x_2,0,t) = 0 \text{ if } (x_1,x_2) \in \mathbb{R}^2 \text{ and}$$

$$v_\infty(x,t) \in L_{\frac{10}{3}}(\mathbb{R}^3_+).$$

The above three facts imply that

$$v_\infty(x,t) = 0 \ \text{ for } \ \text{almost } \ \text{every } \ t \in]t_0, t_0+\lambda(t_0)[. \tag{5.6.15}$$

Recall, from (5.5.12) that we have continuity on $[-5/4,0]$ for the following functional in time (here $\varphi \in C_0^\infty(\mathbb{R}^3)$):

$$t \to \int\limits_{\mathbb{R}^3_+} \varphi(x)\cdot v_\infty(x,t)dx. \tag{5.6.16}$$

The above and (5.6.15), gives that $v_\infty = 0$ in $\mathbb{R}^3_+\times]-5/4,0[$.

The latter contradicts with (5.6.12), thus T is not a blow-up time. $\square$

[15] Specifically, Theorem 5.1 of [27].

[16] Note that $\mu(\Sigma^{'})$ denotes the Lebesgue measure of $\Sigma^{'}$.

5.7 Rescaling. Scenario II

Recall that 'Scenario II' refers to the case $x_{03} > 0$ for the singular point at blow-up time T. Here, the scaling is $x = x_0 + \lambda_k y$. So, we replace $\mathbb{R}^3_+$ with
$\mathbb{R}^3_{+h} = \{y = (y_1, y_2, y_3) \in \mathbb{R} : \ y_3 > h\}$ with $h = h_k = -x_{03}/\lambda_k$.

In this case, sufficiently smooth functions $(v^{(k)}, q^{(k)})$ satisfy the following initial boundary value problem. Namely

$$\partial_t v^{(k)} + v^{(k)} \cdot \nabla v^{(k)} - \Delta v^{(k)} + \nabla q^{(k)} = 0, \qquad \operatorname{div} v^{(k)} = 0$$

in $\mathbb{R}^3_{+h_k} \times]-2, 0[$,

$$v^{(k)}(x_1, x_2, h_k, t) = 0$$

for $(x_1, x_2, t) \in \mathbb{R}^2 \times [-2, 0]$,

$$v^{(k)}(\cdot, -2) = u_0^{(k)}(\cdot) \in L^{3,\beta}(\mathbb{R}^3_{+h_k})$$

and

$$\sup_k \|u_0^{(k)}\|_{L^{3,\beta}(\mathbb{R}^3_{+h_k})} \le M.$$

We can use estimates of Section 5.4 'A Priori Estimates' in the domains $\mathbb{R}^3_{+h_k}$, with constants independent of k.

Proposition 5.11. *There exist subsequences, still denoted in the same way, with the following properties:*

$$v^{(k)} \rightharpoonup v_\infty \tag{5.7.1}$$

in $L_{\frac{10}{3}}(\mathbb{R}^3_{+h} \times]-2, 0[)$ *for any* $h > -\infty$ *with* $v_\infty \in L_{\frac{10}{3}}(Q_{-2,0})$ *and* $Q_{-2,0} := \mathbb{R}^3 \times]-2, 0[$,

$$v^{(k)} \overset{*}{\rightharpoonup} v_\infty \tag{5.7.2}$$

in $L_\infty(]-2+\delta, 0[; L_2(B(R)))$ *for any* $R > 0$ *and any* $0 < \delta < 2$,

$$\nabla v^{(k)} \rightharpoonup \nabla v_\infty \tag{5.7.3}$$

in $L_2(B(R) \times]-2+\delta, 0[)$ *for any* $R > 0$ *and any* $0 < \delta < 2$,

$$v^{(k)} \to v_\infty \tag{5.7.4}$$

in $L_3(B(R) \times]-3/2, 0[)$ *for any* $R > 0$ *and*

$$q^{(k)} \rightharpoonup q_\infty \tag{5.7.5}$$

in $L_{\frac{3}{2}}(B(R)\times]-3/2,0[)$ *for any* $R>0$.

Functions v_∞ *and* q_∞ *satisfy the Navier-Stokes system in* $Q_{-3/2,0}:=\mathbb{R}^3\times]-3/2,0[$. *For* v_∞, *we have that, for any* $\phi\in C_0^\infty(\mathbb{R}^3)$,

$$t\to\int\limits_{\mathbb{R}^3} v_\infty(x,t)\cdot\phi(x)dx\in C([-3/2,0]). \tag{5.7.6}$$

For the pressure q_∞, *the following decomposition is valid:*

$$q_\infty=\sum_{i=1}^3 q_\infty^i \ \ in\ \mathbb{R}^3\times]-3/2,0[.$$

The pieces of the above decomposition satisfy the estimates

$$\|\nabla q_\infty^1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_{-3/2,0})}+\|\nabla q_\infty^2\|_{L_{\frac{10}{3}}(Q_{-3/2,0})}+\|\nabla q_\infty^3\|_{L_2(Q_{-3/2,0})}<\infty. \tag{5.7.7}$$

Moreover, for any $R>0$, *the limit pair* (v_∞,q_∞) *satisfy the local energy inequality*

$$\int\limits_{B(x_0,R)}\varphi(x,t)|v_\infty(x,t)|^2dx+2\int\limits_{-3/2}^{t}\int\limits_{B(x_0,R)}\varphi|\nabla v_\infty|^2dxds$$

$$\leq\int\limits_{-3/2}^{t}\int\limits_{B(x_0,R)}|v_\infty|^2(\partial_t\varphi+\Delta\varphi)+v_\infty\cdot\nabla\varphi(|v_\infty|^2+2q_\infty)dxds \tag{5.7.8}$$

for all $-3/2\leq t\leq 0$, *for all* $x_0\in\mathbb{R}^3$ *and for all positive* $\varphi\in C_0^\infty(B(x_0,R)\times]-3/2,\infty[)$.

Proposition 5.12. *Let* (v_∞,q_∞) *be the limit functions from Proposition 5.11. There exists a number* $R_1>0$ *such that*

$$\sup_{(x,t)\in(\mathbb{R}^3\setminus B(R_1))\times]-4/3,0[}|v_\infty(x,t)|\leq c \tag{5.7.9}$$

for some universal constant c. *Moreover, for* $k=1,2,\ldots$:

$$\sup_{(x,t)\in(\mathbb{R}^3\setminus B(2R_1))\times]-5/4,0[}|\nabla^k v_\infty(x,t)|\leq c_1(k,R_1). \tag{5.7.10}$$

Proposition 5.13. *Let* (v_∞,q_∞) *be the limit functions from Proposition 5.11. There exists a set* $\Sigma\subset]-3/2,0[$ *with the following properties.*

- $\mu(\Sigma)=0$.

- *For all* $t_0 \in]-3/2, 0[\setminus\Sigma$ *there exists a* $\lambda(t_0) > 0$ *such that the following holds true for any* $R > 0$*,* $0 < \varepsilon < \lambda(t_0)$ *and* $k = 0, 1 \dots$*. Specifically,*

$$\nabla^k v_\infty \in L_\infty(B(R) \times]t_0 + \varepsilon, t_0 + \lambda(t_0)[). \tag{5.7.11}$$

The proof of Propositions 5.11- 5.13 goes along the same lines as the proof of Propositions 5.7-5.10. Scenario II can then be ruled out by using similar reasoning as was used in Scenario I.

□

Chapter 6

Existence and Weak* Stability for the Navier-Stokes System with Initial Values in Critical Besov Spaces

6.1 Introduction

In this chapter, we consider the Cauchy problem for the Navier-Stokes system in the space-time domain $Q_\infty = \mathbb{R}^3 \times]0, \infty[$ for vector-valued function $v = (v_1, v_2, v_3) = (v_i)$ and scalar function q. In particular, we consider (v, q), which satisfy the equations

$$\partial_t v + v \cdot \nabla v - \Delta v + \nabla q = 0, \qquad \operatorname{div} v = 0 \tag{6.1.1}$$

in Q_∞, the decay condition

$$v(x, t) \to 0 \tag{6.1.2}$$

as $|x| \to \infty$ for all $t \in [0, \infty[$, and the initial condition

$$v(\cdot, 0) = u_0(\cdot). \tag{6.1.3}$$

In this chapter, we will be interested in developing notions of global-in-time solutions to the Navier-Stokes equations, with solenoidal initial data u_0 belonging to $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, that satisfy certain special properties. We will refer to such a notion of solution as 'global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions' of the Navier-Stokes equations. The results and text base of this chapter closely follow

Barker, T.; Seregin, G. On global solutions to the Navier-Stokes system with large $L^{3,\infty}$ initial data. March, 2016. arXiv:1603.03211v1.

Barker, T. Existence and Weak* Stability for the Navier-Stokes System with Initial Values in Critical Besov Spaces. March. 2017.
arXiv:1703.06841.

Unless otherwise mentioned, the notation and definitions used in this chapter are contained in 'Chapter 2: Preliminary Material'.

In a recent paper by Seregin and Šverák [112], a notion of global-in-time solution to the Navier-Stokes equation was developed with L_3 solenoidal initial data. In [112], this notion of solution is referred to as a 'global L_3 solution'. A key feature of global L_3 solutions is as follows. Suppose $v^{(k)}(\cdot, u_0^{(k)})$ are global L_3 solutions and $u_0^{(k)} \rightharpoonup u_0$ in $L_3(\mathbb{R}^3)$. Then $v^{(k)}(\cdot, u_0^{(k)})$ converges, up to subsequence, to a global L_3 solution $v(\cdot, u_0)$. Throughout this chapter, we will often write $V(x,t) := S(t)u_0(x)$. In [112], any global weak L_3 solution of the Navier-Stokes equation, with initial data $u_0 \in L_3(\mathbb{R}^3)$, has the following structure. Specifically,

$$v(x,t) = V(x,t) + u(x,t), \tag{6.1.4}$$

where u is globally in the energy class in $\mathbb{R}^3\times]0,T[$ for any finite $T > 0$ and satisfies the global energy inequality

$$\|u(\cdot,t)\|_{L_2}^2 + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u(x,t')|^2 dxdt' \leq 2\int_0^t \int_{\mathbb{R}^3} (V\otimes u + V\otimes V) : \nabla u dxdt'. \tag{6.1.5}$$

A natural question concerns whether an analogous notion of of global-in-time solutions (which we will refer to as 'global $\mathcal{X}$ solutions' or sometimes $\mathcal{N}(\mathcal{X})$) with initial data u_0 in other critical spaces $\mathcal{X}$, that satisfy the properties

- **1) Global existence** for any $u_0 \in \mathcal{X}$ there exists a global-in-time solution in $\mathcal{N}(\mathcal{X})$;

- **2) Weak* stability**
$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0 \quad \text{in } \mathcal{X} \Rightarrow$$
$v^{(k)}(\cdot, u_0^{(k)}) \in \mathcal{N}(\mathcal{X})$ converges up to subsequence (in sense of distributions) to $v(\cdot, u_0) \in \mathcal{N}(\mathcal{X})$.

As mentioned by Seregin and Šverák in [112] for the case $\mathcal{X} = L_3(\mathbb{R}^3)$, such properties are useful if one wants to show that critical norms of the Navier-Stokes equations tend to infinity at a potential blow-up time. Let us now describe this in more detail. Suppose one wanted to prove that if v is a weak Leray-Hopf solution on $\mathbb{R}^3\times]0,\infty[$, with sufficiently regular initial data as well as a finite blow-up time T, then necessarily

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{\mathcal{X}}=\infty. \tag{6.1.6}$$

Here, $\mathcal{X}$ is a critical space. One such strategy for showing this, given in Seregin's paper [104] and subsequently used in Seregin's paper [106], the previous chapter and in Albritton's paper [1], is to assume for contradiction that there exists $t_k\uparrow T$ with

$$M:=\sup_k\|v(\cdot,t_k)\|_{\mathcal{X}}<\infty. \tag{6.1.7}$$

The next step is to perform the rescaling

$$v^{(k)}(y,s)=\lambda_k v(x,t),\quad q^{(k)}(y,s)=\lambda_k^2 q(x,t),\quad u_0^{(k)}(y)=\lambda_k v(\lambda_k y,t_k), \tag{6.1.8}$$

$$x=\lambda_k y,\qquad t=T+\lambda_k^2 s,\ \text{ and }\ \lambda_k=\sqrt{\frac{T-t_k}{2}}. \tag{6.1.9}$$

This gives that $(v^{(k)},q^{(k)})$ are solutions to the Navier-Stokes equations on $\mathbb{R}^3\times]-2,0[$ with

$$\sup_k(\|u_0^{(k)}\|_{\mathcal{X}}=\|v(\cdot,t_k)\|_{\mathcal{X}})=M. \tag{6.1.10}$$

The final part of the strategy is to obtain a nontrivial ancient solutions to the Navier-Stokes equations and to then attempt to apply a Liouville type theorem to it, based on backward uniqueness and unique continuation for parabolic operators developed by Escauriaza, Seregin and Šverák in [27].

The main motivation for considering 'global $\mathcal{X}$ solutions', with properties **1) Global existence- 2) Weak* stability**, is that if such solutions exist then they are particularly useful in obtaining a nontrivial ancient solution in the above strategy. Indeed the property **1) Global existence** of global $\mathcal{X}$ solutions, together with the assumption that it coincides with $v^{(k)}$ on $]-2,0[$, implies that the global $\mathcal{X}$ solution is unbounded in a parabolic neighbourhood $Q(r)$ of the space-time origin. This, along with property **2) Weak* stability** and the stability of singular points[1], allows one to obtain a nontrivial ancient solution[2].

In Seregin and Šverák's paper [112], the crucial estimate

$$\|u(\cdot,t)\|_{L_2}^2+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla u(x,t')|^2dxdt'\leq t^{\frac{1}{2}}C(\|u_0\|_{L_3(\mathbb{R}^3)}). \tag{6.1.11}$$

[1]See Lemma 2.1 in [93] for the stability of interior singular points statement. See Proposition 2.4 of the previous Chapter for the stability of boundary singular points.

[2]The convergence properties sufficient to use stability of singular points arguments are more restrictive than those formally stated in **2) Weak* stability**.

is proven for u, with u and V as in (6.1.4)-(6.1.5). This estimate plays a central role in [112] in the proof of continuity of global weak L_3 solutions, with respect to the weak convergence of the initial data. Let us briefly outline how (6.1.11) is established. It is well known that for $u_0 \in L_3$

$$\|V\|_{L_5(\mathbb{R}^3\times]0,\infty[)} \leq c\|u_0\|_{L_3(\mathbb{R}^3)} \tag{6.1.12}$$

and

$$\|V(\cdot,t)\|_{L_4(\mathbb{R}^3)} \leq \frac{c\|u_0\|_{L_3(\mathbb{R}^3)}}{t^{\frac{1}{8}}}. \tag{6.1.13}$$

In [112], (6.1.5) and (6.1.12)-(6.1.13) are used to show that for $t \in]0,\infty[$:

$$\|u(\cdot,t)\|^2_{L_2} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u(x,t')|^2 dxdt'$$

$$\leq C\int\limits_0^t \|V(\cdot,t')\|^5_{L_5(\mathbb{R}^3)}\|u(\cdot,t')\|^2_{L_2(\mathbb{R}^3)}dt' + Ct^{\frac{1}{2}}\|u_0\|^4_{L_3(\mathbb{R}^3)}. \tag{6.1.14}$$

This, along with (6.1.12), implies (6.1.11) by means of Gronwall's Lemma.

Suppose one attempts constructs a global $\mathcal{X}$ solution to the Navier-Stokes equation, with $\mathcal{X}$ being a critical space, having the structure (6.1.4)-(6.1.5). To ensure the finiteness of the right-hand side of the energy inequality (6.1.5), one should have that for any $u_0 \in \mathcal{X}$ that

$$S(t)u_0 \in L_{4,loc}(0,\infty; L_4(\mathbb{R}^3)).$$

Hence, it is natural to consider $\mathcal{X}$ such that for any $0 < T < \infty$ and $0 < \epsilon < T$:

$$\Big(\int\limits_\epsilon^T \|S(t)u_0\|^4_{L_4}dt\Big)^{\frac{1}{4}} \leq c(T,\epsilon)\|u_0\|_{\mathcal{X}}. \tag{6.1.15}$$

This implies that

$$\|S(1)u_0\|_{L_4} \leq c\|u_0\|_{\mathcal{X}}. \tag{6.1.16}$$

For $\lambda > 0$ and $u_{0\lambda}(x) := \lambda u_0(\lambda x)$, it is clear that $S(1)u_{0\lambda}(x) = \lambda S(\lambda^2)u_0(\lambda x)$. Hence, (6.1.16) implies that for any $\lambda > 0$

$$\lambda^{\frac{1}{4}}\|S(\lambda^2)u_0\|_{L_4} = \|S(1)u_{0\lambda}\|_{L_4} \leq C\|u_{0\lambda}\|_{\mathcal{X}} = C\|u_0\|_{\mathcal{X}}. \tag{6.1.17}$$

One can now deduce that,

$$\sup_{s>0} s^{\frac{1}{8}}\|S(s)u_0\|_{L_4} \leq C\|u_0\|_{\mathcal{X}}. \tag{6.1.18}$$

From (6.1.18) and the heat flow characterisation of homogeneous Besov spaces with negative upper index (Remark 2.8 in Chapter 2), we infer that $\mathcal{X} \hookrightarrow \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$. Thus, if one seeks a global $\mathcal{X}$ solution, satisfying the requirements **1) Global existence- 2) Weak* Stability** and having the structure (6.1.4)-(6.1.5), $\mathcal{X} = \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ is the widest critical space for such a possibility.

The aim of this chapter is to show that, for any divergence-free initial data in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$, there exists 'global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution' to the Navier-Stokes equations (6.1.1)-(6.1.3), which satisfies the requirements **1) Global existence- 2) Weak* Stability**. The difference compared to L_3 initial data is that if u_0 belongs to the larger scale-invariant space $\dot{B}^{-\frac{1}{4}}_{4,\infty}$, then V does not necessarily belong to $L_5(\mathbb{R}^3\times]0,T[)$. Indeed, as demonstrated by Cannone in [19], for the solenoidal initial data

$$u_0(x) = \Big(0, \frac{x_3}{|x|^2}, -\frac{x_2}{|x|^2}\Big) \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3) \setminus L_3(\mathbb{R}^3), \tag{6.1.19}$$

one has

$$\|S(t)u_0\|_{L_5(\mathbb{R}^3)} = \frac{\|S(1)u_0\|_{L_5(\mathbb{R}^3)}}{t^{\frac{1}{5}}}. \tag{6.1.20}$$

These properties of V create a serious obstacle when one attempts to apply the arguments of Seregin and Šverák in [112] (outlined on the previous page) to the case of solenoidal $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ initial data.

To the best of the author's knowledge, the literature did not previously contain a notion of solutions to the Navier-Stokes equations with arbitrary $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solenodial initial data. Note that under certain smallness conditions on the $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ data, solutions were constructed by Planchon [88] and Cannone [19], by means of the Banach contraction principle. Let us mention that the recent preprint of Bradshaw and Tsai [12] implies that if $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ is discretely self similar[3], then there exists a discretely self similar solution to the Navier-Stokes equation. These solutions will also belong to our class of global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions.

Let us now give the definition of 'global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions'.

Definition 6.1. *Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ be solenoidal, in the sense of tempered distributions, and let $V = S(t)u_0$. We say that v is a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution to the Navier-Stokes initial value problem in $Q_T := \mathbb{R}^3\times]0,T[$ (with $0 < T < \infty$) if*

$$v = V + u, \tag{6.1.21}$$

[3]The discretely self similar solutions in Bradshaw and Tsai's paper [12] are shown to exist for any discretely self similar data in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ with $3 < p < 6$.

with $u\in L_\infty(0,T;J(\mathbb{R}^3))\cap L_2(0,T;\overset{\circ}{J}{}^1_2(\mathbb{R}^3))$ *and there exists an* $\alpha>0$ *such that*

$$\sup_{0<t<T}\frac{\|u(\cdot,t)\|^2_{L_2}}{t^\alpha}<\infty. \tag{6.1.22}$$

Additionally, it is required that there exists a $q\in L_{\frac{3}{2},\mathrm{loc}}(Q_T)$ *such that* (u,q) *satisfy the perturbed Navier-Stokes system:*

$$\partial_t u+v\cdot\nabla v-\Delta u=-\nabla q,\quad div\, u=0 \tag{6.1.23}$$

in the sense of distributions in Q_T*. Furthermore, it is required that for any* $w\in L_2$*:*

$$t\rightarrow\int\limits_{\mathbb{R}^3} w(x)\cdot u(x,t)dx \tag{6.1.24}$$

is a continuous function on $[0,T]$*. Moreover,* u *satisfies the energy inequality:*

$$\|u(\cdot,t)\|^2_{L_2}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla u(x,t')|^2dxdt'$$

$$\leq 2\int_0^t\int_{\mathbb{R}^3}(V\otimes u+V\otimes V):\nabla u dxdt' \tag{6.1.25}$$

for all $t\in[0,T]$*.*

Finally, it is required that (v,q) *satisfy the local energy inequality. Namely, for almost every* $t\in]0,T[$ *the following inequality holds for all nonnegative functions* $\varphi\in C_0^\infty(Q_\infty)$*:*

$$\int\limits_{\mathbb{R}^3}\varphi(x,t)|v(x,t)|^2dx+2\int\limits_0^t\int\limits_{\mathbb{R}^3}\varphi|\nabla v|^2dxdt^{'}$$

$$\leq\int\limits_0^t\int\limits_{\mathbb{R}^3}|v|^2(\partial_t\varphi+\Delta\varphi)+v\cdot\nabla\varphi(|v|^2+2q)dxdt^{'}. \tag{6.1.26}$$

We define v *to be a global* $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ *solution if it is a weak* $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ *solution in* Q_T *for any* $0<T<\infty$*.*

Remark 6.2. *The role of the requirement (6.1.22) is to ensure that the right-hand side of the energy inequality (6.1.25) is finite. See Lemma 6.8 for more details. This therefore ensures that the function* u *satisfies the initial condition in the strong* L_2*-sense. Namely,* $u(\cdot,t)\to 0$ *in* L_2*.*

Next us state the main results of this chapter.

Theorem 6.3. *Let $u_0^{(k)} \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ and let $v^{(k)}$ be a sequence of a global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions to the Cauchy problem for the Navier-Stokes system with initial data $u_0^{(k)}$. Then there exists a subsequence (still denoted $v^{(k)}$) that converges, in the sense of distributions, to a global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution v of the Cauchy problem for the Navier-Stokes system, with initial data u_0.*

Corollary 6.4. *There exists at least one global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$-solution to the Cauchy problem (6.1.1)-(6.1.3).*

Let us state two observations that enable us to show the existence of global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions and that the property **2) Weak* stability** holds true for them.

The first observation regards a new decomposition result for homogeneous Besov spaces (Proposition 2.13, Chapter 2). The novelty of Proposition 2.13 is that it is not clear if related decompositions are obtainable by the known real interpolation theory of homogeneous Besov spaces.

The second main observation of this chapter makes use of Proposition 2.13, together with arguments related to those contained in Chapter 3, to obtain improved decay properties of $u(x,t) := v(x,t) - V(x,t)$ near the initial time. We state this observation as a lemma below. Analogous properties have also been exploited by Dong and Zhang in [25] and subsequently in Chapter 3, to establish weak-strong uniqueness results for the Navier-Stokes equations. Related properties have also been utilised by Jia and Šverák in [54], in showing existence of minimal blow-up data in L_3.

Lemma 6.5. *Let u, v and u_0 be as in Definition 6.1. Let γ_1, γ_2, δ_2 and p_2 be as in Proposition 2.13 in Chapter 2. Then there exists a $\beta(\gamma_1, \gamma_2, \delta_2) > 0$ such that the following estimate is valid for any $0 < t < T$:*

$$\|u(\cdot,t)\|^2_{L_2} + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' \le C(T, \|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, \delta_2) t^{\beta}. \tag{6.1.27}$$

We will also prove some conditional uniqueness and regularity statements for global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions.

Proposition 6.6. *Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ be divergence-free in the sense of tempered distributions. There exists an $\varepsilon > 0$ such that if*

$$\sup_{0<t<T} t^{\frac{1}{8}} \|S(t)u_0\|_{L_4} < \varepsilon \tag{6.1.28}$$

then all global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, with initial data $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$, coincide on Q_T.

In order to prove Proposition 6.6, we show that under assumption (6.1.28) we can find $T(u_0)$ such that there exists a mild solution on $\mathbb{R}^3\times]0,T(u_0)[$ and that this is also a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution on $\mathbb{R}^3\times]0,T(u_0)[$. We then show that certain properties of the mild solution are enough to guarantee that it coincides with all global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions on $\mathbb{R}^3\times]0,T(u_0)[$, with the same initial data. This strategy was also used in Chapter 3.

Unfortunately, the assumptions of Proposition 6.6 do not hold for arbitrary divergence-free initial data in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ and in particular, for large minus one homogeneous solenoidal initial data. For this case, the conjectures by Jia and Šverák in [56] and [57] suggest that there may be non-uniqueness for global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, even for a short time.

This contrasts drastically with the global L_3 solutions to the Navier-Stokes equations, which were constructed by Seregin and Šverák in [112] for any L_3 solenoidal initial data. In [112], short time uniqueness and regularity results are shown to hold for arbitrary L_3 solenoidal initial data. A key feature in showing this is that for any L_3 solenoidal initial data, there exists a local-in-time mild solution to the Navier-Stokes equations. This was proven by Kato in [59], whose proof crucially makes use of the fact that

$$\lim_{T\to 0}\sup_{0<t<T} t^{\frac{1}{5}}\|S(t)u_0\|_{L_5(\mathbb{R}^3)} = 0, \tag{6.1.29}$$

for any $u_0 \in L_3$. Such a property follows from the density of test functions in L_3.

Although local-in-time mild solutions to the Navier-Stokes equations have been constructed for wide classes of initial data[4], it is not known if they can be constructed for arbitrary $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solenoidal initial data. The main obstacle is that, unlike L_3, test functions are not dense in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$.

6.2 Weak* Stability of Global $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ Solutions

6.2.1 A Priori Estimates

Through this chapter, we will make use of the following estimates for any tempered distribution in $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$.

Proposition 6.7. *Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$. Then we have*

$$\sup_{0<t<\infty} t^{\frac{1}{8}}\|S(t)u_0\|_{L_4} \leq C\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}. \tag{6.2.1}$$

[4] See, for example, the papers of Cannone [19], Fujita and Kato [34], Giga and Miyakawa [43], Koch and Tataru [63], Kozono and Yamazaki [64], Planchon [88] and Taylor [125].

Moreover for $4 \leq r \leq \infty$, $m, k \in \mathbb{N}$*:*

$$\|\partial_t^m \nabla^k S(t) u_0\|_{L_r} \leq \frac{C(m,k,r)\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}}{t^{m+\frac{k}{2}+\frac{1}{2}(1-\frac{3}{r})}}. \tag{6.2.2}$$

Proof. Remark 2.8 in Chapter 2 immediately implies (6.2.1). Once (6.2.1) is established, (6.2.2) follows from applying Young's inequality for convolutions. □

The estimates from the above proposition will be useful in proving the following lemma, which we state and prove below.

Lemma 6.8. *Assume that* $u \in L_\infty(0,T; J(\mathbb{R}^3)) \cap L_2(0,T; \overset{\circ}{J}{}^1_2(\mathbb{R}^3))$ *and that there exists an* $\alpha > 0$ *such that*

$$\frac{\|u(\cdot,t)\|^2_{L_2}}{t^\alpha} \in L_\infty(0,T). \tag{6.2.3}$$

Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ *be divergence free and let* $V(x,t) := S(t)u_0$*. Then*

$$V \otimes V \in L_2(Q_T), \tag{6.2.4}$$

$$V \otimes u + u \otimes V \in L_{\frac{3}{2}}(Q_T), \tag{6.2.5}$$

$$u \otimes u \in L_{\frac{3}{2}}(Q_T), \tag{6.2.6}$$

$$V \cdot \nabla V \in L_{2,\frac{5}{4}}(Q_T), \tag{6.2.7}$$

$$V \cdot \nabla u + u \cdot \nabla V \in L_{\frac{3}{2},\frac{6}{5}}(Q_T), \tag{6.2.8}$$

and

$$V \otimes u : \nabla u \in L_1(Q_T). \tag{6.2.9}$$

Proof. Let us first focus on showing (6.2.4)-(6.2.6). By Hölder's inequality and Proposition 6.7:

$$\|V \otimes V\|_{L_2} \leq \|V\|^2_{L_4} \leq \frac{c}{t^{\frac{1}{4}}}\|u_0\|^2_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}.$$

From here, (6.2.4) is easily established. Hölder's inequality and Proposition 6.7 also imply that

$$\|u \otimes V\|_{L_{\frac{3}{2}}} \leq \|V\|_{L_6}\|u\|_{L_2} \leq \frac{c\|u\|_{L_{2,\infty}(Q_T)}}{t^{\frac{1}{4}}}\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}.$$

From this, it is immediate that $u \otimes V + V \otimes u \in L_{\frac{3}{2}}(Q_T)$. We can show (6.2.6) by using that $u \in L_\infty(0,T; J(\mathbb{R}^3)) \cap L_2(0,T; \overset{\circ}{J}{}^1_2(\mathbb{R}^3))$, together with Hölder's inequality and Sobolev embeddings.

Now, we prove (6.2.7)-(6.2.9). By Hölder's inequality and Proposition 6.7:

$$\|V\cdot\nabla V\|_{L_2}\leq\|V\|_{L_4}\|\nabla V\|_{L_4}\leq\frac{c}{t^{\frac34}}\|u_0\|^2_{\dot{B}^{-\frac14}_{4,\infty}}.$$

From here, (6.2.7) is easily established. Hölder's inequality and Proposition 6.7 also imply that

$$\|u\cdot\nabla V\|_{L_{\frac32}}\leq\|\nabla V\|_{L_6}\|u\|_{L_2}\leq\frac{c\|u\|_{L_{2,\infty}(Q_T)}}{t^{\frac34}}\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}}.$$

From this, it is immediate that $u\cdot\nabla V\in L_{\frac32,\frac65}(Q_T)$. Using Hölder's inequality once more, one can verify that

$$\int\limits_0^T\|V\cdot\nabla u\|^{\frac65}_{L_{\frac32}}dt\leq\left(\int\limits_0^T\|\nabla u\|^2_{L_2}dt\right)^{\frac35}\left(\int\limits_0^T\|V\|^3_{L_6}dt\right)^{\frac25}.$$

The desired conclusion is reached by noting that Proposition 6.7 gives:

$$\|V\|^3_{L_6}\leq\frac{c}{t^{\frac34}}\|u_0\|^3_{\dot{B}^{-\frac14}_{4,\infty}}.$$

The proof of the last estimate demonstrates why it is nontrivial to prove energy estimate for u. From Proposition 6.7, (6.2.3) and the Hölder inequality, it is possible to deduce that

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}|V\otimes u:\nabla u|dxd\tau\leq\left(\int\limits_0^T\int\limits_{\mathbb{R}^3}|\nabla u|^2dxd\tau\right)^{\frac12}\left(\int\limits_0^T\int\limits_{\mathbb{R}^3}|V\otimes u|^2dxd\tau\right)^{\frac12}$$

$$\leq\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}}\left(\int\limits_0^T\int\limits_{\mathbb{R}^3}|\nabla u|^2dxd\tau\right)^{\frac12}\left(\int\limits_0^T\int\limits_{\mathbb{R}^3}\frac{1}{\tau^{1-\alpha}}\operatorname*{ess\,sup}_{0<s<T}\Big(\frac{\|u(\cdot,s)\|^2_{L_2}}{s^\alpha}\Big)dxd\tau\right)^{\frac12}.$$

□

The next statement is a consequence of Lemma 6.8, Proposition 2.18 in Chapter 2, Remark 2.19 in Chapter 2 and uniqueness results for the linear Stokes system[5].

Lemma 6.9. *Let v be a weak $\dot{B}^{-\frac14}_{4,\infty}$-solution in Q_T, with functions u and q as in Definition 6.1. Then*

$$(u,\nabla q)=\sum_{i=1}^2(u^i,\nabla p^i)\ \ in\ Q_T\tag{6.2.10}$$

[5]See, for example, Lemma 2.4.2 pp.234-235 in Sohr's book [117].

such that:

$$(u^i, \nabla p^i) \in W^{2,1}_{s_i,l_i}(Q_T) \times L_{s_i,l_i}(Q_T) \tag{6.2.11}$$

and

$$(s_1, l_1) = (9/8, 3/2), (s_2, l_2) = (2, 5/4), (s_3, l_3) = (3/2, 6/5). \tag{6.2.12}$$

In addition (u^i, p^i) satisfy the following:

$$\partial_t u^1 - \Delta u^1 + \nabla p^1 = -u \cdot \nabla u, \tag{6.2.13}$$

$$\partial_t u^2 - \Delta u^2 + \nabla p^2 = -V \cdot \nabla V \tag{6.2.14}$$

$$\partial_t u^3 - \Delta u^3 + \nabla p^3 = -V \cdot \nabla u - u \cdot \nabla V \tag{6.2.15}$$

in Q_T,

$$\operatorname{div} u^i = 0 \ \text{ in } Q_T \ \text{ for } i = 1, 2, 3 \ \text{ and} \tag{6.2.16}$$

$$u^i(\cdot, 0) = 0, \ \text{ for } i = 1, 2, 3. \tag{6.2.17}$$

Remark 6.10. *Let v be a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$-solution in Q_T, with functions u and q as in Definition 6.1. Consider the Riesz transforms $\mathcal{R}$, which are given by*

$$\mathcal{R}_i g(x) := cP.V. \int \frac{y_i}{|y|^4} g(x-y) dy.$$

Let

$$\widetilde{q} = q_1 + q_2 + q_3, \tag{6.2.18}$$

$$q_1 = \mathcal{R}_i \mathcal{R}_j (u_i u_j), \tag{6.2.19}$$

$$q_2 = \mathcal{R}_i \mathcal{R}_j (V_i V_j) \tag{6.2.20}$$

and

$$q_3 = \mathcal{R}_i \mathcal{R}_j (V_i u_j + u_i V_j). \tag{6.2.21}$$

Using Lemma 6.8, together with the known continuity of the Riesz transforms on Lebesgue spaces, we infer that

$$q_1 \in L_{\frac{3}{2}}(Q_T), \ \ q_2 \in L_2(Q_T), \ \ q_3 \in L_{\frac{3}{2}}(Q_T) \tag{6.2.22}$$

and

$$\nabla q_i \in L_{s_i,l_i}(Q_T). \tag{6.2.23}$$

Here,

$$(s_1, l_1) = (9/8, 3/2), (s_2, l_2) = (2, 5/4), (s_3, l_3) = (3/2, 6/5). \tag{6.2.24}$$

One can show that $\nabla(q-\widetilde{q})(\cdot,t)$ is a harmonic function for almost every $t\in]0,T[$. Moreover, from Lemma 6.9 we infer that

$$\nabla(q-\widetilde{q})\in\sum_{i=1}^{3}L_{s_i,l_i}(Q_T)\ \textit{for}\ i=1,2,3.$$

These two facts, together with (6.2.22) and properties of q from Definition 6.1, imply that there exists a $C\in L_{\frac{3}{2},loc}(Q_T)$ such that

$$q=\tilde{q}+C(t)\ \textit{ in }\ Q_T. \tag{6.2.25}$$

Thus, without loss of generality, we can take the pressure of a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution in Q_T to be equal to $\widetilde{q}$. This representative of the pressure will be chosen throughout the rest of this chapter.

Before the next lemma let us introduce some notation. Recall from 'Chapter 2: Preliminary Material' that for $p_0>3$, we define

$$s_{p_0}:=-1+\frac{3}{p_0}<0.$$

Let u, v and u_0 be as in Definition 6.1. Let $u_0=\bar{u}_0^N+\widetilde{u}_0^N$ denote the splitting from Proposition 2.13 in Chapter 2. In particular, $4<p_2<\infty$, $0<\delta_2<-s_{p_2}$, $\gamma_1>0$ and $\gamma_2>0$ are such that for any $N>0$ there exists weakly divergence-free functions

$$\bar{u}_0^N\in\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3)\cap\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)\ \text{ and }\ \widetilde{u}_0^N\in L_2(\mathbb{R}^3)\cap\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$$

satisfying the following properties. Namely,

$$u_0=\bar{u}_0^N+\widetilde{u}_0^N, \tag{6.2.26}$$

$$\|\bar{u}_0^N\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}}\leq N^{\gamma_1}C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}), \tag{6.2.27}$$

$$\|\widetilde{u}_0^N\|_{L_2}\leq N^{-\gamma_2}C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}), \tag{6.2.28}$$

$$\|\bar{u}_0^N\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}\leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}) \tag{6.2.29}$$

and

$$\|\widetilde{u}_0^N\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}\leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}). \tag{6.2.30}$$

Let us define the following:

$$\bar{V}^N(\cdot,t):=S(t)\bar{u}_0^N(\cdot,t) \tag{6.2.31}$$

and

$$\tilde{V}^N(\cdot,t) := S(t)\tilde{u}_0^N(\cdot,t). \tag{6.2.32}$$

Inspired by decompositions of the Navier-Stokes equations performed by Calderón in [18], we define

$$w^N(x,t) := u(x,t) + \tilde{V}^N(x,t). \tag{6.2.33}$$

Lemma 6.11. *In the above notation, we have the following global energy inequality*

$$\|w^N(\cdot,t)\|_{L_2}^2 + 2\int\limits_0^t\int\limits_{\mathbb{R}^3} |\nabla w^N(x,t')|^2 dxdt'$$

$$\leq \|\tilde{u}_0^N\|_{L_2}^2 + 2\int_0^t\int\limits_{\mathbb{R}^3} (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N dxdt' \tag{6.2.34}$$

that is valid for positive N and $t \in [0,T]$.

Proof. The first stage is showing that w^N satisfies the local energy inequality. Let us briefly sketch how this can be done. Let $\varphi \in C_0^\infty(Q_\infty)$ be a positive function. Observe that the assumptions in Definition 6.1 imply that the following function

$$t \to \int\limits_{\mathbb{R}^3} w^N(x,t)\cdot \bar{V}^N(x,t)\varphi(x,t)dx \tag{6.2.35}$$

is continuous for all $t \in [0,T]$. It is not so difficult to show that this term has the following expression:

$$\int\limits_{\mathbb{R}^3} w^N(x,t)\cdot \bar{V}^N(x,t)\varphi(x,t)dx = \int\limits_0^t\int\limits_{\mathbb{R}^3} (w^N\cdot \bar{V}^N)(\Delta\varphi + \partial_t\varphi)dxdt'$$

$$-2\int\limits_0^t\int\limits_{\mathbb{R}^3} \nabla w^N : \nabla \bar{V}^N \varphi dxdt' + \int\limits_0^t\int\limits_{\mathbb{R}^3} \bar{V}^N \cdot \nabla\varphi q dxdt'$$

$$+\frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^3} (|v|^2 - |w^N|^2)v\cdot\nabla\varphi dxdt'$$

$$-\int\limits_0^t\int\limits_{\mathbb{R}^3} (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \varphi dxdt'$$

$$-\int\limits_0^t\int\limits_{\mathbb{R}^3} (\bar{V}^N \otimes \bar{V}^N + \bar{V}^N \otimes w^N) : (w^N \otimes \nabla\varphi)dxdt'. \tag{6.2.36}$$

It is also readily shown that

$$\int\limits_{\mathbb{R}^3}|\bar V^N(x,t)|^2\varphi(x,t)dx=\int\limits_0^t\int\limits_{\mathbb{R}^3}|\bar V^N(x,t')|^2(\Delta\varphi(x,t')+\partial_t\varphi(x,t'))dxdt'-$$

$$-2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla\bar V^N|^2\varphi dxdt'. \tag{6.2.37}$$

Using the local energy inequality (6.1.26), together with (6.2.36)-(6.2.37), we obtain that for almost every $t\in]0,T[$ and for all nonnegative functions $\varphi\in C_0^\infty(Q_\infty)$:

$$\int\limits_{\mathbb{R}^3}\varphi(x,t)|w^N(x,t)|^2dx+2\int\limits_0^t\int\limits_{\mathbb{R}^3}\varphi|\nabla w^N|^2dxdt'$$

$$\leq\int\limits_0^t\int\limits_{\mathbb{R}^3}|w^N|^2(\partial_t\varphi+\Delta\varphi)+2qw^N\cdot\nabla\varphi+|w^N|^2v\cdot\nabla\varphi dxdt'$$

$$+2\int\limits_0^t\int\limits_{\mathbb{R}^3}(\bar V^N\otimes\bar V^N+\bar V^N\otimes w^N):(\nabla w^N\varphi+w^N\otimes\nabla\varphi)dxdt' \tag{6.2.38}$$

In the next part of the proof, let $\varphi(x,t)=\varphi_1(t)\varphi_R(x)$. Here, $\varphi_1\in C_0^\infty(0,\infty)$ and $\varphi_R\in C_0^\infty(B(2R))$ are nonnegative functions. Moreover, $\varphi_R=1$ on $B(R)$, $0\leq\varphi_R\leq 1$,

$$|\nabla\varphi_R|\leq c/R \text{ and } |\nabla^2\varphi_R|\leq c/R^2.$$

Since $\widetilde u_0^N\in J(\mathbb{R}^3)$, it is obvious that $\tilde V^N(\cdot,t):=S(t)\widetilde u_0^N(\cdot,t)$ satisfies the following properties. In particular, for any $T>0$

$$\tilde V^N\in C([0,T];J(\mathbb{R}^3))\cap L_2(0,T;\overset{\circ}{J}{}^1_2(\mathbb{R}^3)), \tag{6.2.39}$$

$\tilde V^N$ satisfies the energy equality

$$\|\tilde V^N(\cdot,t)\|^2_{L_2}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla\tilde V^N|^2dxdt'=\|\widetilde u_0^N\|^2_{L_2} \tag{6.2.40}$$

for any $t>0$ and

$$\lim_{t\to0^+}\|\tilde V^N(\cdot,t)-\widetilde u_0^N\|_{L_2}=0. \tag{6.2.41}$$

Hence, we have

$$w^N\in C_w([0,T];J(\mathbb{R}^3))\cap L_2(0,T;\overset{\circ}{J}{}^1_2(\mathbb{R}^3)). \tag{6.2.42}$$

Using Hölder's inequality and Sobolev embeddings, this implies that

$$w^N \in L_p(Q_T) \quad \text{for} \quad 2 \le p \le 10/3. \tag{6.2.43}$$

Using Proposition 6.7 and Young's inequality for convolutions, we deduce that for $2 \le p \le \infty$, $4 \le q \le \infty$:

$$\|\tilde{V}^N(\cdot,t)\|_{L_p} \le \frac{C(p)}{t^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}}\|\widetilde{u}_0^N\|_{L_2} \tag{6.2.44}$$

and

$$\|\bar{V}^N(\cdot,t)\|_{L_q} \le \frac{C(q)}{t^{\frac{3}{2}(\frac{1}{4}-\frac{1}{q})+\frac{1}{8}}}\|\bar{u}_0^N\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}. \tag{6.2.45}$$

Using (6.2.42)-(6.2.43) and (6.2.45), it is obvious that the following limits hold true:

$$\lim_{R\to\infty}\int\limits_{\mathbb{R}^3}\varphi_R(x)\varphi_1(t)|w^N(x,t)|^2dx + 2\int\limits_0^t\int\limits_{\mathbb{R}^3}\varphi_R\varphi_1|\nabla w^N|^2dxdt'$$

$$= \int\limits_{\mathbb{R}^3}\varphi_1(t)|w^N(x,t)|^2dx + 2\int\limits_0^t\int\limits_{\mathbb{R}^3}\varphi_1|\nabla w^N|^2dxdt',$$

$$\lim_{R\to\infty}\int\limits_0^t\int\limits_{\mathbb{R}^3}(|w^N|^2\partial_t\varphi_1\varphi_R + 2(\bar{V}^N\otimes w^N + \bar{V}^N\otimes\bar{V}^N):\nabla w^N\varphi_1\varphi_R)dxdt'$$

$$= \int\limits_0^t\int\limits_{\mathbb{R}^3}(|w^N|^2\partial_t\varphi_1 + 2(\bar{V}^N\otimes w^N + \bar{V}^N\otimes\bar{V}^N):\nabla w^N\varphi_1)dxdt'$$

and

$$\lim_{R\to\infty}\int\limits_0^t\int\limits_{\mathbb{R}^3}(|w^N|^2\varphi_1\Delta\varphi_R + \varphi_1|w^N|^2v\cdot\nabla\varphi_R +$$

$$+2\varphi_1(\bar{V}^N\otimes w^N + \bar{V}^N\otimes\bar{V}^N):(w^N\otimes\nabla\varphi_R))dxdt' = 0.$$

Let us focus on the term containing the pressure, namely

$$\int\limits_0^t\int\limits_{B(2R)\setminus B(R)} qw^N\cdot\nabla\varphi_R\varphi_1dxdt'.$$

From Remark 6.10, we see that

$$q = q_1 + q_2 + q_3, \tag{6.2.46}$$

$$q_1 \in L_{\frac{3}{2}}(Q_T),\ \ q_2 \in L_2(Q_T)\ \text{ and }\ q_3 \in L_{\frac{3}{2}}(Q_T). \tag{6.2.47}$$

From this and (6.2.43), we infer that

$$\lim_{R\to\infty}\int_0^t\int_{B(2R)\setminus B(R)} qw^N\cdot\nabla\varphi_R\varphi_1 dxdt^{'}=0.$$

Thus, putting everything together, we get that for almost every $t\in]0,T[$ and any positive function $\phi_1\in C_0^\infty(0,\infty)$:

$$\int_{\mathbb{R}^3}\varphi_1(t)|w^N(x,t)|^2dx+2\int_0^t\int_{\mathbb{R}^3}\varphi_1(t')|\nabla w^N|^2dxdt^{'}$$

$$\leq\int_0^t\int_{\mathbb{R}^3}|w^N|^2\partial_t\varphi_1+2(\bar{V}^N\otimes w^N+\bar{V}^N\otimes\bar{V}^N):\nabla w^N\varphi_1dxdt' \tag{6.2.48}$$

From Remark 6.2 and (6.2.41), we see that

$$\lim_{t\to 0}\|w^N(\cdot,t)-\tilde{u}_0^N(\cdot)\|_{L_2}=0. \tag{6.2.49}$$

For $\bar{V}^N$, we have

$$\|\bar{V}^N(\cdot,t)\|_{L_4}\leq\frac{1}{t^{\frac18}}\|\bar{u}_0^N\|_{\dot{B}^{-\frac14}_{4,\infty}}\leq\frac{1}{t^{\frac18}}C(\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}}). \tag{6.2.50}$$

Using that $\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3)\hookrightarrow\dot{B}^{-1+\delta_2}_{\infty,\infty}(\mathbb{R}^3)$ (Remark 2.7 in Chapter 2), together with the heat flow characterisation of homogeneous Besov spaces with negative upper index (Remark 2.8 in Chapter 2), we infer that

$$\|\bar{V}^N(\cdot,t)\|_{L_\infty}\leq\frac{1}{t^{\frac12-\frac{\delta_2}{2}}}\|\bar{u}_0^N\|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}}\leq\frac{N^{\gamma_1}}{t^{\frac12-\frac{\delta_2}{2}}}C(\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}}). \tag{6.2.51}$$

Thus, we have the following estimates:

$$\int_0^t\int_{\mathbb{R}^3}|\bar{V}^N\otimes w^N:\nabla w^N|dxdt'$$

$$\leq N^{\gamma_1}C(\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}})\left(\int_0^t\int_{\mathbb{R}^3}|\nabla w^N|^2dxdt'\right)^{\frac12}\left(\int_0^t\frac{\|w^N(\cdot,\tau)\|^2_{L_2}}{\tau^{1-\delta_2}}d\tau\right)^{\frac12} \tag{6.2.52}$$

and

$$\int_0^t\int_{\mathbb{R}^3}|\bar{V}^N\otimes\bar{V}^N:\nabla w^N|dxdt'\leq Ct^{\frac14}C(\|u_0\|_{\dot{B}^{-\frac14}_{4,\infty}})\left(\int_0^t\int_{\mathbb{R}^3}|\nabla w^N|^2dxdt'\right)^{\frac12}. \tag{6.2.53}$$

Let

$$\varphi_\varepsilon(s) := \begin{cases} 0 & \text{if } 0 \leq s \leq \varepsilon/2, \\ 2(s-(\varepsilon/2))/\varepsilon & \text{if } \varepsilon/2 \leq s \leq \varepsilon, \\ 1 & \text{if } \varepsilon \leq s. \end{cases}$$

Using (6.2.52)-(6.2.53) and by taking suitable approximations of φ_ε, it can be shown that $\varphi_1 = \varphi_\varepsilon$ is admissible in (6.2.48). From this we obtain that the following inequality is valid for almost every $t \in]0,T[$ and any $\varepsilon > 0$:

$$\int\limits_{\mathbb{R}^3} |w^N(x,t)|^2 dx + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} \varphi_\varepsilon(t')|\nabla w^N|^2 dxdt'$$

$$\leq \int\limits_0^t \int\limits_{\mathbb{R}^3} |w^N|^2 \partial_t \varphi_\varepsilon + 2(\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \varphi_\varepsilon dxdt'. \tag{6.2.54}$$

Using (6.2.42), (6.2.49) and (6.2.52)-(6.2.53), we can let ε tend to zero in (6.2.54) to obtain that (6.2.34) holds for almost every $t \in]0,T[$. By taking into account (6.2.42) and (6.2.52)-(6.2.53), we conclude further that (6.2.34) holds for every $t \in [0,T]$.

□

Proof of Lemma 6.5

Our aim is to prove the decay estimates of Lemma 6.5 by utilising the splittings (6.2.26)-(6.2.33), together with the previous lemma. Such ideas have also been exploited in Chapter 3 of this thesis. Related splitting arguments have also previously been used by Jia and Šverák in [54], in order to show estimates near the initial time for Lemarié-Rieusset local energy solutions of the Navier-Stokes equations with L_3 initial data.

Proof. For $t \in [0,T]$, we denote

$$y_N(t) := \|w^N(\cdot,t)\|_{L_2}^2.$$

First observe that $u = w^N - \tilde{V}^N$. Thus using (6.2.40) we deduce that

$$\|u(\cdot,t)\|_{L_2}^2 + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u|^2 dxdt'$$

$$\leq 2\|\tilde{u}_0^N\|_{L_2}^2 + 2y_N(t) + 2\int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla w^N|^2 dxdt'$$

for $t \in [0,T]$. By (6.2.28):

$$\|\tilde{u}_0^N\|_{L_2}^2 \leq N^{-2\gamma_2} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}). \tag{6.2.55}$$

Using (6.2.34), estimates (6.2.52)-(6.2.53), (6.2.55) and Young's inequality, we obtain the estimate

$$y_N(t) + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla w^N|^2 dx dt' \leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}) N^{2\gamma_1} \left(\int\limits_0^t \frac{y_N(\tau)}{\tau^{1-\delta_2}} d\tau \right)$$

$$+ C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}})(N^{-2\gamma_2} + t^{\frac{1}{2}}) \tag{6.2.56}$$

is valid for $t \in [0,T]$. Using Gronwall's Lemma, it is possible to infer that for $t \in [0,T]$:

$$y_N(t) \leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}})(N^{-2\gamma_2} + t^{\frac{1}{2}}) \exp\Big(N^{2\gamma_1} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, \delta_2) t^{\delta_2} \Big). \tag{6.2.57}$$

Hence,

$$\|u(\cdot,t)\|_{L_2}^2 + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u|^2 dx dt' \leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, \delta_2)(N^{-2\gamma_2} + t^{\frac{1}{2}})$$

$$\times \Big(t^{\delta_2} N^{2\gamma_1} \exp\Big(N^{2\gamma_1} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, \delta_2) t^{\delta_2} \Big) + 1 \Big). \tag{6.2.58}$$

The conclusion is then easily reached by taking $N = t^{-\kappa}$ with

$$0 < \kappa < \frac{\delta_2}{2\gamma_1}.$$

□

6.2.2 Proof of Weak* Stability and Existence of Global $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ Solutions

Proof of Theorem 6.3

Proof. We have

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0 \ \text{ in } \ \dot{B}^{-\frac{1}{4}}_{4,\infty}$$

and

$$M := \sup_k \|u_0^{(k)}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} < \infty.$$

Let us define

$$V^{(k)}(\cdot,t) := S(t) u_0^{(k)}(\cdot,t) \ \text{ and } \ V(\cdot,t) := S(t) u_0(\cdot,t).$$

It is clear that that $V^{(k)}$ converges to V on Q_∞ in the sense of tempered distributions. From Proposition 6.7, we see that

$$\|V^{(k)}(\cdot,t)\|_{L_4} \leq \frac{CM}{t^{\frac{1}{8}}}, \tag{6.2.59}$$

$$\|\partial_t^m \nabla^l V^{(k)}(\cdot,t)\|_{L_r} \leq \frac{C(m,l,r)M}{t^{m+\frac{k}{2}+\frac{1}{2}(1-\frac{3}{r})}} \ \text{ for } \ 4 \leq r. \tag{6.2.60}$$

For $T<\infty$, $s\in]1,\infty[$ and any $n\in\mathbb{N}$, we have the compact embedding[6]

$$W^{2,1}_s(B(n)\times]0,T[) \hookrightarrow C([0,T];L_s(B(n))).$$

From the above compact embedding and (6.2.60), one immediately infers that there exists a subsequence of $V^{(k)}$ (which we will still denote as $V^{(k)}$) such that

$$\partial_t^m \nabla^l V^{(k)} \to \partial_t^m \nabla^l V \ \text{ in } \ C([1/n,n];L_s(B(n))) \tag{6.2.61}$$

for every $(l,m,s,n)\in\mathbb{N}^4$.

Lemma 6.5 implies that, for any $0<T<\infty$:

$$\sup_{0<t<T} \|u^{(k)}(\cdot,t)\|_{L_2}^2 + \int\limits_0^T \int\limits_{\mathbb{R}^3} |\nabla u^{(k)}|^2 dx dt' \leq f_0(M,T,\beta,\delta_2). \tag{6.2.62}$$

By Hölder's inequality and the Sobolev inequality, this implies that

$$\|u^{(k)}\|_{L_p(Q_T)} \leq f_1(M,T,\beta,\delta_2) \quad \text{for any} \quad 2\leq p\leq 10/3. \tag{6.2.63}$$

By means of a Cantor diagonalisation argument, we can find a subsequence (which will still be labelled as $u^{(k)}$) such that, for any finite $T>0$:

$$u^{(k)} \stackrel{*}{\rightharpoonup} u \ \text{ in } \ L_{2,\infty}(Q_T) \tag{6.2.64}$$

and

$$\nabla u^{(k)} \rightharpoonup \nabla u \ \text{ in } \ L_2(Q_T). \tag{6.2.65}$$

Using (6.2.64), together with Lemma 6.5 (specifically, (6.1.27)), we also get that, for $0<t<1$:

$$\|u\|_{L_{2,\infty}(Q_t)} \leq c(M,\delta_2)t^{\frac{\beta}{2}}. \tag{6.2.66}$$

From (6.2.62), it is easily inferred that

$$\|u^{(k)}\cdot\nabla u^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_T)} \leq f_2(M,T,\beta,\delta_2). \tag{6.2.67}$$

[6]Embeddings of this type were discussed in 'Chapter 2: Preliminary Material'.

By the same reasoning as in Lemma 6.8, we obtain:

$$\|V^{(k)}\otimes V^{(k)}\|_{L_2(Q_T)}\leq f_3(M,T), \tag{6.2.68}$$

$$\|V^{(k)}\otimes u^{(k)}+u^{(k)}\otimes V^{(k)}\|_{L_{\frac{3}{2}}(Q_T)}\leq f_4(M,T,\beta,\delta_2), \tag{6.2.69}$$

$$\|u^{(k)}\otimes u^{(k)}\|_{L_{\frac{3}{2}}(Q_T)}\leq f_5(M,T,\beta,\delta_2), \tag{6.2.70}$$

$$\|V^{(k)}\cdot\nabla V^{(k)}\|_{L_{2,\frac{5}{4}}(Q_T)}\leq f_6(M,T) \tag{6.2.71}$$

and

$$\|V^{(k)}\cdot\nabla u^{(k)}+u^{(k)}\cdot\nabla V^{(k)}\|_{L_{\frac{3}{2},\frac{6}{5}}(Q_T)}\leq f_7(M,T,\beta,\delta_2). \tag{6.2.72}$$

Split

$$u^{(k)}=\sum_{i=1}^{3}u^{(k),i}\ \text{ and }\ q^{(k)}=\sum_{i=1}^{3}q_i^{(k)}$$

according to Lemma 6.9 and Remark 6.10 (namely (6.2.10) and (6.2.18)). Taking into account (6.2.68)-(6.2.70) and using the continuity of Riesz transforms on Lebesgue spaces, we deduce that, for any finite $T>0$,

$$\|q_1^{(k)}\|_{L_{\frac{3}{2}}(Q_T)}\leq Cf_5(M,T,\beta,\delta_2),\ \ \|q_2^{(k)}\|_{L_2(Q_T)}\leq Cf_3(M,T) \tag{6.2.73}$$

and

$$\|q_3^{(k)}\|_{L_{\frac{3}{2}}(Q_T)}\leq Cf_4(M,T,\beta,\delta_2). \tag{6.2.74}$$

In order to establish certain compactness properties of $u^{(k)}$, we now utilise the maximal regularity estimates for the linear Stokes system stated in 'Chapter 2: Preliminary Material'. Related ideas have been previously exploited in Chapter 4 and in Seregin and Šverák's paper [112] regarding 'global L_3 solutions'. By the maximal regularity estimates for the Stokes system[7], along with (6.2.67) and (6.2.71)-(6.2.72), we obtain:

$$\|u^{(k),1}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_T)}+\|\nabla q_1^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_T)}\leq Cf_2(M,T,\beta,\delta_2), \tag{6.2.75}$$

$$\|u^{(k),2}\|_{W^{2,1}_{2,\frac{5}{4}}(Q_T)}+\|\nabla q_2^{(k)}\|_{L_{2,\frac{5}{4}}(Q_T)}\leq Cf_6(M,T), \tag{6.2.76}$$

$$\|u^{(k),3}\|_{W^{2,1}_{\frac{3}{2},\frac{6}{5}}(Q_T)}+\|\nabla q_3^{(k)}\|_{L_{\frac{3}{2},\frac{6}{5}}(Q_T)}\leq Cf_7(M,T,\beta,\delta_2). \tag{6.2.77}$$

From (6.2.73)-(6.2.74), one can deduce that there exists a subsequence of $q^{(k)}$ (which we will still label as $q^{(k)}$) such that

$$q^{(k)}\rightharpoonup q\ \text{ in }\ L_{\frac{3}{2}}(B(R)\times]0,T[)\ \text{ for any }\ R>0\ \text{ and }\ 0<T<\infty. \tag{6.2.78}$$

[7]See Proposition 2.18 and Remark 2.19 in Chapter 2.

Recall that in 'Chapter 2: Preliminary Material' we mentioned the compact embedding[8]

$$W^{2,1}_{m,n}(\Omega\times]a,b[)\hookrightarrow C([a,b];L_m(\Omega)).$$

Using this, together with (6.2.75)-(6.2.77), we infer that that we have the following convergence for a certain subsequence (which will still be denoted as $u^{(k)}$):

$$u^{(k)}\to u\ \text{ in }\ C([0,n];L_{\frac{9}{8}}(B(n))\ \text{ for any }\ n\in\mathbb{N}. \tag{6.2.79}$$

Using this and (6.2.63), one can conclude that

$$u^{(k)}\to u\ \text{ in }\ L_s(B(n)\times]0,n[)\ \text{ for any }\ s\in]1,10/3[\ \text{ and }\ n\in\mathbb{N}. \tag{6.2.80}$$

Using (6.2.64) and (6.2.79), it is possible to show that, for any $f\in L_2$:

$$\int\limits_{\mathbb{R}^3} u^{(k)}(x,t)\cdot f(x)dx\to\int\limits_{\mathbb{R}^3} u(x,t)\cdot f(x)dx\ \text{ in }\ C([0,n])\ \text{ for any } n\in\mathbb{N}. \tag{6.2.81}$$

Using (6.2.81), along with (6.2.66), we establish that

$$\lim_{t\to 0}\|u(\cdot,t)\|_{L_2}=0. \tag{6.2.82}$$

Taking into account (6.2.66), along with the fact that $u\in L_{2,\infty}(Q_T)$ for any $0<T<\infty$, we see that

$$\sup_{0<t<T}\frac{\|u(\cdot,t)\|^2_{L_2}}{t^\beta}<\infty\ \text{ for any } 0<T<\infty. \tag{6.2.83}$$

It can be deduced from (6.2.61), (6.2.64)-(6.2.65), (6.2.78) and (6.2.80) that $(v:=u+V,q)$ is a distributional solution to the Navier-Stokes equations and (u,q) satisfies (6.1.23) in the sense of distributions. All that remains to be shown is the local energy inequality (6.1.26) for (v,q) and the energy inequality (6.1.25) for u. Using the assumption that $(v^{(k)},q^{(k)})$ satisfies the local energy inequality, together with the properties (6.2.61), (6.2.64)-(6.2.65), (6.2.78) and (6.2.80)-(6.2.81), we can verify that (v,q) satisfy the local energy inequality.

Let us focus on showing the global energy inequality (6.1.25) for u. Using the fact that (v,q) satisfy the local energy inequality, together with (6.2.64)-(6.2.65), we may use identical reasoning to Lemma 6.11 to infer that for an arbitrary positive function $\phi_1(t)\in C^\infty_0(0,\infty)$:

$$\int\limits_{\mathbb{R}^3}\varphi_1(t)|u(x,t)|^2dx+2\int\limits_0^t\int\limits_{\mathbb{R}^3}\varphi_1(t)|\nabla u|^2dxdt'$$

[8] Here, $1\leq m\leq\infty$, $1<n\leq\infty$, $-\infty<a<b<\infty$ and Ω is a bounded domain with sufficiently smooth boundary.

$$\leq \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \varphi_1 + 2(V \otimes u + V \otimes V) : \nabla u \varphi_1 dx dt'. \qquad (6.2.84)$$

Using (6.2.65) and (6.2.83), we may apply the results of Lemma 6.8 to deduce that

$$(V \otimes u + V \otimes V) : \nabla u \in L_1(Q_T)$$

for any positive finite T. Taking these facts and (6.2.82) into account, the conclusion is reached in a similar way to the final steps of the proof of Lemma 6.11. □

Let us now state and prove a proposition, which will be useful in proving Corollary 6.4.

Proposition 6.12. *Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ be divergence-free, in the sense of tempered distributions. Then there exists a weakly divergence-free sequence $u_0^{(k)} \in L_3(\mathbb{R}^3)$ such that*

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0$$

in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$.

Proof. Let $\dot{\Delta}_j$ and the homogeneous Besov spaces be as defined in 'Chapter 2: Preliminary Material'. For any $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, we have that $u_{0,<|k|} := \sum_{j=-k}^{k} \dot{\Delta}_j u_0$ converges to u_0 in the sense of tempered distributions[9]. Furthermore, we have that $\dot{\Delta}_j \dot{\Delta}_{j'} u_0 = 0$ if $|j - j'| > 1$. Thus,

$$\|u_{0,<|k|}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \qquad (6.2.85)$$

and

$$\dot{\Delta}_N u_{0,<|k|} = 0 \quad \text{if} \quad |N| > k+1. \qquad (6.2.86)$$

It then follows from Bahouri, Chemin and Danchin's book [3][10] that there exists a Schwartz function $g^{(k)}$, whose Fourier transform is supported away from the origin, such that

$$\|g^{(k)} - u_{0,<|k|}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} < \frac{1}{k}. \qquad (6.2.87)$$

We now make use of the operator $\mathbb{P}$ that projects onto divergence-free vector fields. In the case of $\Omega = \mathbb{R}^3$ this may be written as

$$\mathbb{P}f := f + \nabla(-\Delta)^{-1}(\operatorname{div} f). \qquad (6.2.88)$$

[9]See Section 2 of Bahouri, Chemin and Danchin's book [3], for example.

[10]Specifically, Remark 2.28 p.69 of [3].

This can also be written as (for $j = 1, 2, 3$)

$$\mathbb{P}f_j = (\delta_{jk} - \mathcal{R}_j\mathcal{R}_k)f_k.$$

Here $\mathcal{R}_i$ are Riesz transforms given by

$$\mathcal{R}_i g(x) := cP.V. \int \frac{y_i}{|y|^4} g(x-y)dy$$

and the summation convention is adopted.

Define $u_0^{(k)}$ to be the Leray Projector $\mathbb{P}$ applied to $g^{(k)}$. Then we can show, for any $j \in \mathbb{Z}$, that

$$\dot{\Delta}_j u_0^{(k)} = \dot{\Delta}_j(\mathbb{P}g^{(k)}) = \mathbb{P}(\Delta_j g^{(k)}).$$

Using the continuity of Riesz transforms on $L_p(\mathbb{R}^3)$ $(1 < p < \infty)$, we deduce further that

$$\|\dot{\Delta}_j u_0^{(k)}\|_{L_4(\mathbb{R}^3)} \leq \hat{C}\|\dot{\Delta}_j g^{(k)}\|_{L_4(\mathbb{R}^3)}. \tag{6.2.89}$$

Here, $\hat{C}$ is a universal constant. Using (6.2.85), (6.2.87) and (6.2.89), we see that

$$\|u_0^{(k)}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \leq \hat{C}\|g^{(k)}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})(1 + \|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}). \tag{6.2.90}$$

Obviously since $g^{(k)}$ is a Schwartz function, we have that $u_0^{(k)}$ is a weakly divergence-free function in $L_3(\mathbb{R}^3)$. Since u_0 is divergence-free in the sense of tempered distributions, we have that

$$\mathbb{P}u_{0,<|k|} = u_{0,<|k|}.$$

Thus,

$$\|u_0^{(k)} - u_{0,<|k|}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} = \|\mathbb{P}(g^{(k)} - u_{0,<|k|})\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} < \frac{C}{k}. \tag{6.2.91}$$

From Bahouri, Chemin and Danchin's book [3][11], we have

$$| < u_0^{(k)} - u_{0<|k|}, \varphi > | \leq \|u_0^{(k)} - u_{0,<|k|}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \|\varphi\|_{\dot{B}^{\frac{1}{4}}_{\frac{4}{3},1}}$$

for any Schwartz function φ. Using this, along with (6.2.91), we deduce that

$$| < u_0^{(k)} - u_0, \varphi > | \leq | < u_0^{(k)} - u_{0<|k|}, \varphi > | + | < u_0 - u_{0<|k|}, \varphi > | \leq$$

$$\leq \frac{C}{k}\|\varphi\|_{\dot{B}^{\frac{1}{4}}_{\frac{4}{3},1}} + | < u_0 - u_{0<|k|}, \varphi > |.$$

From this and (6.2.90), we deduce that $u_0^{(k)}$ satisfies the desired properties of the proposition.

□

[11]Specifically, Proposition 2.29 p.70 of [3].

Proof of Corollary 6.4

Proof. From Proposition 6.12 that there exists a solenoidal sequence $u_0^{(k)} \in L_3(\mathbb{R}^3)$ such that

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0 \text{ in } \dot{B}^{-\frac{1}{4}}_{4,\infty}.$$

The work of Seregin and Šverák [112] implies that for any k there exists a global L_3 solution $v^{(k)}$, which satisfies

$$\sup_{0<t<\infty} \frac{\|v^{(k)} - S(t)u_0^{(k)}\|^2_{L_2}}{t^{\frac{1}{2}}} < \infty.$$

It is also the case that $v^{(k)}$ is a global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution. Corollary 6.4 now follows from Theorem 6.3. $\square$

6.3 Uniqueness Results

6.3.1 Construction of Strong Solutions

As was done in 'Chapter 1: Introduction' (specifically, (1.1.13)), we define

$$B(u,v)_i(x,t) := \int_0^t \int_{\mathbb{R}^3} K_{ijl}(x-y,t-\tau)u_j(y,\tau)v_l(y,\tau)dyd\tau. \tag{6.3.1}$$

Here, K_{ijl} is given by (1.1.10) in Chapter 1. In what follows, we will use the notation $\pi_{u\otimes u} := \mathcal{R}_i\mathcal{R}_j(u_iu_j)$, where $\mathcal{R}$ is a Riesz transform and the summation convention is adopted. We also define the path space:

$$X_4(T) := \{f \in \mathcal{S}'(\mathbb{R}^3\times]0,T[) : \|f\|_{X_4(T)} := \sup_{0<t<T} t^{\frac{1}{8}}\|f(\cdot,t)\|_{L_4(\mathbb{R}^3)} < \infty\}.$$

Proposition 6.13. *Suppose that $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$ is divergence free, in the sense of tempered distributions. There exists a universal constant ε_3 such that if*

$$\|S(t)u_0\|_{X_4(T)} < \varepsilon_3, \tag{6.3.2}$$

then there exists a $\widetilde{v} \in X_4(T)$, which solves the Navier-Stokes equations (6.1.1)-(6.1.3) in the sense of distributions and satisfies the following properties.

1. *The function $\widetilde{v}$ solves the following integral equation:*

$$\widetilde{v}(x,t) := S(t)u_0 + B(\widetilde{v},\widetilde{v})(x,t) \tag{6.3.3}$$

in Q_T, along with the estimate

$$\|\widetilde{v}\|_{X_4(T)} < 2\|V\|_{X_4(T)} := 2M^{(0)}. \tag{6.3.4}$$

2. *The pressure associated to $\widetilde{v}$ ($\pi_{\widetilde{v}\otimes\widetilde{v}}$) satisfies :*

$$\pi_{\widetilde{v}\otimes\widetilde{v}} \in L_2(Q_T)\cap L_{2,\infty}(Q_{\lambda,T}))\ \ for\ \ 0<\lambda<T. \tag{6.3.5}$$

3. *The function $\widetilde{v}$ is a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution to the Navier-Stokes initial value problem in $Q_T := \mathbb{R}^3\times]0,T[$.*

4. *For $\lambda\in]0,T[$ and $k=0,1\ldots$, $l=0,1\ldots$ the following estimate is valid:*

$$\sup_{(x,t)\in Q_{\lambda,T}} |\partial_t^l\nabla^k\widetilde{v}| + |\partial_t^l\nabla^k\pi_{\widetilde{v}\otimes\widetilde{v}}| \le c(\lambda, \|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, k, l). \tag{6.3.6}$$

Proof. Properties 1.-2. can be established by using arguments presented in Cannone's paper [19], for example. Hence, let us focus on proving property 3. It is clear that $\widetilde{u} := B(\widetilde{v},\widetilde{v})$ satisfies the system:

$$\partial_t\widetilde{u} - \Delta\widetilde{u} + \nabla\pi_{\widetilde{v}\otimes\widetilde{v}} = -\mathrm{div}\,\widetilde{v}\otimes\widetilde{v},\ \ \mathrm{div}\,\widetilde{u} = 0 \tag{6.3.7}$$

in Q_T and

$$\widetilde{u}(\cdot,0) = 0. \tag{6.3.8}$$

It is not difficult to show that $\widetilde{v}\otimes\widetilde{v}\in L_2(Q_T)$, together with properties of the Oseen tensor K, imply that

$$\widetilde{u} := B(\widetilde{v},\widetilde{v}) \in L_2(0,T; \overset{\circ}{J}{}^1_2(\mathbb{R}^3))\cap C([0,T];J(\mathbb{R}^3)) \tag{6.3.9}$$

and for $t\in[0,T]$:

$$\|\widetilde{u}(\cdot,t)\|^2_{L_2} + 2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla\widetilde{u}(x,t')|^2dxdt' = 2\int\limits_0^t\int\limits_{\mathbb{R}^3}\widetilde{v}\otimes\widetilde{v} : \nabla\widetilde{u}dxdt' =$$

$$= 2\int\limits_0^t\int\limits_{\mathbb{R}^3}(V\otimes\widetilde{u} + V\otimes V) : \nabla\widetilde{u}dxdt'. \tag{6.3.10}$$

Hence, $\widetilde{u}$ satisfies the energy inequality (on Q_T) present in the definition of a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution (in fact, in this case it is an equality). Using (6.3.10), together with Hölder's and Young's inequality, one can show that, for $0<t<T$,

$$\|\widetilde{u}(\cdot,t)\|^2_{L_2} + \int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla\widetilde{u}(x,t')|^2dxdt' \le \int\limits_0^t\int\limits_{\mathbb{R}^3}|\widetilde{v}\otimes\widetilde{v}|^2dxdt' \le 32t^{\frac{1}{2}}\|V\|^4_{X_4(T)}. \tag{6.3.11}$$

Thus,

$$\sup_{0<t<T} \frac{\|\widetilde{u}(\cdot,t)\|_{L_2}^2}{t^{\frac{1}{2}}} < \infty. \tag{6.3.12}$$

The final part of showing that $\widetilde{v}$ is a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution on Q_T (property 3.) involves verifying that $(\widetilde{v}, \pi_{\widetilde{v}\otimes\widetilde{v}})$ satisfies the local energy inequality (6.1.26). This follows from (6.3.4)-(6.3.5),

$$\widetilde{v} \in L_{2,\infty}(B(R)\times]\varepsilon, R[) \text{ for any } 0 < \varepsilon < R < \infty,$$

$$\nabla\widetilde{v} \in L_2(B(R)\times]\varepsilon, R[) \text{ for any } 0 < \varepsilon < R < \infty$$

and a mollification argument[12].

The proof of the estimate (6.3.6) for higher derivatives uses that $(\widetilde{v}, \pi_{\widetilde{v}\otimes\widetilde{v}})$ satisfies the local energy inequality (6.1.26), the properties (6.3.4)-(6.3.5) and arguments contained in Seregin's book [111][13]. For further discussion, we refer the reader to the proof of Theorem 3.6 in Chapter 3, in particular 'Step 4: Estimate of Higher Derivatives'.

□

6.3.2 Proof of Proposition 6.6

The proof of Proposition 6.6 is based on ideas used by Dong and Zhang in [25] and subsequently in Chapter 3. Namely, we show the decay estimates provided by Lemma 6.5, together with properties of the strong solution in Proposition 6.13, imply weak-strong uniqueness for weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions.

Proof. Suppose,

$$\sup_{0<t<T} t^{\frac{1}{8}}\|S(t)u_0\|_{L_4} := \|V\|_{X_4(T)} < \varepsilon.$$

Furthermore, suppose $\varepsilon < \varepsilon_3$, where ε_3 is as in the statement of Proposition 6.13. Then by Proposition 6.13 there exists a weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution $\widetilde{v} := \widetilde{u} + V$ on Q_T, which satisfies

$$\|\widetilde{v}\|_{X_4(T)} < 2\|V\|_{X_4(T)}. \tag{6.3.13}$$

Let us now consider any global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solution $u := v + V$, with the same initial data $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$. We define the difference of these solutions as

$$w = v - \widetilde{v} = u - \widetilde{u} \in L_2(0,T; \overset{\circ}{J}{}^1_2(\mathbb{R}^3)) \cap C_w([0,T]; J(\mathbb{R}^3)). \tag{6.3.14}$$

[12]The required mollification argument can be found in Seregin's book [111] (specifically Proposition 3.9 pp.160-161), for example.

[13]In particular, pp.160-162 of [111].

Moreover, w satisfies the following equations

$$\partial_t w + w\cdot\nabla w + \widetilde{v}\cdot\nabla w + w\cdot\nabla\widetilde{v} - \Delta w + \nabla q = 0, \qquad \operatorname{div} w = 0 \tag{6.3.15}$$

in Q_T. Moreover, w satisfies its initial condition in the strong L_2 sense:

$$\lim_{t\to 0^+}\|w(\cdot,0)\|_{L_2} = 0. \tag{6.3.16}$$

Using minor adjustments to arguments presented in Lemarié-Rieusset's book [76] (specifically, Proposition 14.3 of [76]), one can deduce that, for $t\in[\delta,T]$:

$$\int\limits_{\mathbb{R}^3} u(y,t)\cdot\widetilde{u}(y,t)dy = \int\limits_{\mathbb{R}^3} u(y,\delta)\cdot\widetilde{u}(y,\delta)dy - 2\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}\nabla u:\nabla\widetilde{u}dyd\tau$$

$$+\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3} V\otimes V:\nabla(u+\widetilde{u})dyd\tau - \int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}\widetilde{v}\otimes w:\nabla w dyd\tau$$

$$+\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3} V\otimes\widetilde{u}:\nabla\widetilde{u}dyd\tau + \int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3} V\otimes u:\nabla u dyd\tau. \tag{6.3.17}$$

Using Lemma 3.7 in Chapter 3 and (6.3.13), one infers that

$$\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|\widetilde{v}||w||\nabla w|dyd\tau \leq C\int\limits_{\delta}^{t}\|\widetilde{v}(\cdot,\tau)\|^8_{L_4}\|w(\cdot,\tau)\|^2_{L_2}d\tau + \frac{1}{2}\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|\nabla w|^2dyd\tau$$

$$\leq C\|V\|^8_{X_4(T)}\int\limits_{\delta}^{t}\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau}d\tau + \frac{1}{2}\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|\nabla w|^2dyd\tau, \tag{6.3.18}$$

$$\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|V||u||\nabla u|dyd\tau \leq C\|V\|^8_{X_4(T)}\int\limits_{\delta}^{t}\frac{\|u(\cdot,\tau)\|^2_{L_2}}{\tau}d\tau + \frac{1}{2}\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|\nabla u|^2dyd\tau \tag{6.3.19}$$

and

$$\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|V||\widetilde{u}||\nabla\widetilde{u}|dyd\tau \leq C\|V\|^8_{X_4(T)}\int\limits_{\delta}^{t}\frac{\|\widetilde{u}(\cdot,\tau)\|^2_{L_2}}{\tau}d\tau + \frac{1}{2}\int\limits_{\delta}^{t}\int\limits_{\mathbb{R}^3}|\nabla\widetilde{u}|^2dyd\tau. \tag{6.3.20}$$

We now apply Lemma 6.5 to both u and $\tilde{u}$, which implies that there exists $\beta(\gamma_1,\gamma_2,\delta_2)>0$ a such that the following estimates are valid for $0<t<T$. Specifically,

$$\max\{\|u(\cdot,t)\|^2_{L_2}, \|\widetilde{u}(\cdot,t)\|^2_{L_2}, \|w(\cdot,t)\|^2_{L_2}\} \leq t^{\beta}c(T,\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}},\delta_2). \tag{6.3.21}$$

Hence,

$$\sup_{0<t<T}\frac{\|w(\cdot,t)\|^2_{L_2}}{t^\beta}<\infty. \qquad (6.3.22)$$

Then (6.3.18)-(6.3.21) allow us to take $\delta\to 0$ in (6.3.17) to deduce the following identity, for any $t\in[0,T]$. Specifically,

$$\int\limits_{\mathbb{R}^3} u(y,t)\cdot\widetilde{u}(y,t)dy=-2\int\limits_0^t\int\limits_{\mathbb{R}^3}\nabla u:\nabla\widetilde{u}dyd\tau$$

$$+\int\limits_0^t\int\limits_{\mathbb{R}^3}V\otimes V:\nabla(u+\widetilde{u})dyd\tau-\int\limits_0^t\int\limits_{\mathbb{R}^3}\widetilde{v}\otimes w:\nabla wdyd\tau$$

$$+\int\limits_0^t\int\limits_{\mathbb{R}^3}V\otimes\widetilde{u}:\nabla\widetilde{u}dyd\tau+\int\limits_0^t\int\limits_{\mathbb{R}^3}V\otimes u:\nabla udyd\tau. \qquad (6.3.23)$$

Recall that u and $\widetilde{u}$ satisfy the following energy inequalities, for $0<t<T$:

$$\|u(\cdot,t)\|^2_{L_2}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla u(x,t')|^2dxdt'$$

$$\leq 2\int_0^t\int_{\mathbb{R}^3}(V\otimes u+V\otimes V):\nabla udxdt' \qquad (6.3.24)$$

and

$$\|\widetilde{u}(\cdot,t)\|^2_{L_2}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla \widetilde{u}(x,t')|^2dxdt'$$

$$\leq 2\int_0^t\int_{\mathbb{R}^3}(V\otimes \widetilde{u}+V\otimes V):\nabla\widetilde{u}dxdt' \qquad (6.3.25)$$

From this and (6.3.23), we deduce that, for any $t\in[0,T]$

$$\|w(\cdot,t)\|^2_{L_2}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla w|^2dyd\tau\leq 2\int\limits_0^t\int\limits_{\mathbb{R}^3}\widetilde{v}\otimes w:\nabla wdyd\tau. \qquad (6.3.26)$$

Using (6.3.18) and (6.3.22), it can be shown that, for any $t\in[0,T]$:

$$\|w(\cdot,t)\|^2_{L_2}\leq C'\|V\|^8_{X_4(T)}\int\limits_0^t\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau}d\tau$$

$$\leq\frac{C'}{\beta}t^\beta\|V\|^8_{X_4(T)}\sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|^2_{L_2}}{\tau^\beta}\Big). \qquad (6.3.27)$$

Thus,

$$\sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|_{L_2}^2}{\tau^\beta}\Big) \le \frac{C'}{\beta}\|V\|_{X_4(T)}^8 \sup_{0<\tau<T}\Big(\frac{\|w(\cdot,\tau)\|_{L_2}^2}{\tau^\beta}\Big). \tag{6.3.28}$$

If we assume further that

$$\varepsilon \le \min\Big(\Big(\frac{\beta}{2C'}\Big)^{\frac{1}{8}}, \varepsilon_3\Big),$$

then

$$\|V\|_{X_4(T)} \le \Big(\frac{\beta}{2C'}\Big)^{\frac{1}{8}}.$$

Observing (6.3.28), it then immediately follows that $w = 0$ on Q_T.

$\square$

Chapter 7

Conclusions and Open Problems

As stated in 'Chapter 1: Introduction', the main contributions in this thesis are motivated by the following question.

> **What are sufficient conditions to grant uniqueness of weak Leray-Hopf solutions of the three-dimensional Navier-Stokes equations?**

7.1 Short Time Uniqueness of Weak Leray-Hopf Solutions

In Chapter 3, we contribute to the above question by showing that weak Leray-Hopf solutions are unique for a short time interval, provided that the initial data u_0 satisfies

$$u_0 \in VMO^{-1}(\mathbb{R}^3) \cap \dot{B}_{p,p}^{-\frac{3}{\alpha}+\frac{3}{p}}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \tag{7.1.1}$$

with

$$\alpha < p < \frac{\alpha}{3-\alpha} \quad \text{and} \quad 2 < \alpha < 3. \tag{7.1.2}$$

The main observation, which enables us to obtain this results, is as follows. Namely the establishment of improved estimates near the initial time, for weak Leray-Hopf solutions with initial data in supercritical homogeneous Besov spaces. The techniques used here build upon related ideas of Calderón presented in [18].

In Chapter 3, we observed that the conclusions in the paper of Koch and Tataru [63], together with arguments of Lemarié-Rieusset and Prioux [77], can be used to show the following. Specifically, if $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$ then for all $\varepsilon > 0$ there exists a $T(u_0, \varepsilon)$ and a weak Leray-Hopf solution v such that

$$v \in C^\infty(\mathbb{R}^3 \times]0, T(u_0,\varepsilon)[) \quad \text{and} \quad \sup_{0<t<T(u_0,\varepsilon)} t^{\frac{1}{2}} \|v(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} < \varepsilon. \tag{7.1.3}$$

Consequently, we formulate the following plausible conjecture.

Conjecture: For any $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$, there exists a $T(u_0)$ such that all associated weak Leray-Hopf solutions coincide on $\mathbb{R}^3\times]0, T(u_0)[$.

One can establish the inclusion

$$J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \subset J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \cap \dot{B}_{p,p}^{-\frac{3}{\alpha}+\frac{3}{p}}(\mathbb{R}^3)$$

with $p = \frac{\alpha}{3-\alpha}$ and $2 < \alpha < 3$. Unfortunately, the initial data belonging to $J(\mathbb{R}^3) \cap \dot{B}_{p,p}^{-\frac{3}{\alpha}+\frac{3}{p}}(\mathbb{R}^3)$ (with $p = \frac{\alpha}{3-\alpha}$ and $2 < \alpha < 3$) represents a borderline case, where the techniques used in proving Lemma 3.3 (Chapter 3) breakdown. Hence, it is no longer clear if an improved decay property holds near the initial time for weak Leray-Hopf solutions, with initial data $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$.

7.2 Local Boundary Regularity Criteria and Blow-Up of Scale-Invariant Norms

In Chapter 4, we proved that if (v, q) is a suitable weak solution of the Navier-Stokes equations in $B^+(1)\times]-1, 0[$ near $\Gamma(0, 1) \times [0, 1]$, then

> $v \in L_\infty(-1, 0; L^{3,\beta}(B^+(1))) \Rightarrow v$ is Hölder continuous in $\bar{B}^+(1/2) \times [-1/2, 0]$ provided that $3 \leq \beta < \infty$.

What enables us to build upon the work of Escauriaza, Seregin and Šverák [27] and Seregin [100] is the establishment of new scale-invariant estimates, new estimates for the pressure near the boundary and a convenient new ε-regularity criterion.

In Chapter 5 of this thesis, it was shown that a weak Leray-Hopf solution v in $\mathbb{R}^3_+\times]0, \infty[$, with short time regular initial data and a finite blow-up time T, necessarily satisfies

$$\lim_{t\uparrow T} \|v(\cdot, t)\|_{L^{3,\beta}(\mathbb{R}^3_+)} = \infty \quad \text{with} \quad 3 < \beta < \infty. \tag{7.2.1}$$

The proof hinges on a rescaling procedure from Seregin's work [106], a new stability result for singular points on the boundary, suitable a priori estimates and a Liouville type theorem for parabolic operators developed by Escauriaza, Seregin and Šverák [27].

The methods we present in Chapter 5 for proving (7.2.1) can be used to prove that for a weak Leray-Hopf solution v in $\mathbb{R}^3\times]0, \infty[$, with short time regular initial data and a finite blow-up time T, other scale-invariant norms of v tend to infinity as time approaches T from below. Indeed the methods of Chapter 5, together with techniques from Chapters 2 and 3, have been used by the author of this thesis [8] and by Albritton [1] to prove that v necessarily satisfies

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)}=\infty \quad\text{for}\quad 3<p,q<\infty. \tag{7.2.2}$$

In the recent preprint [11], the author of this thesis has obtained a further improvement to the necessary condition (7.2.2). In particular, in [11] it is shown that

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)}=\infty \quad\text{for}\quad 3<p<\infty, \tag{7.2.3}$$

provided $v(\cdot,T)\in J(\mathbb{R}^3)$ satisfies the following additional assumption. Namely

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\int\limits_{\mathbb{R}^3} v(y,T)\cdot\varphi(y/\lambda)dy=0 \quad\text{for all}\quad \varphi\in C_0^\infty(\mathbb{R}^3). \tag{7.2.4}$$

In the proofs of the main results in Chapters 4 and 5, a suitable rescaling procedure (arising from the contradiction assumption) produces a nontrivial ancient solution. In both Chapters 4 and 5, we show that this nontrivial ancient solution vanishes at the last moment of time. This is crucially used in the application of a Liouville type theorem, based on backward uniqueness and unique continuation statements for parabolic operators, developed by Escauriaza, Seregin and Šverák in [27]. In Chapters 4 and 5, a key shrinking property of the $L^{3,\beta}(\Omega)$ norm is used to show that the ancient solution vanishes at the last moment of time. Specifically, $3<\beta<\infty$ and $f\in L^{3,\beta}(\Omega)$ implies that, for every $x_0\in\Omega$,

$$\lim_{r\to 0^+}\|f\|_{L^{3,\beta}(\Omega\cap B(x_0,r))}=0. \tag{7.2.5}$$

Unfortunately, (7.2.5) no longer holds for $\beta=\infty$, as can be easily seen for $|x|^{-1}\in L^{3,\infty}(\mathbb{R}^3)$. Consequently, we cannot obtain an ancient solution vanishing at the last moment of time and the following problem remains open.

Open Problem: Consider a weak Leray-Hopf solution v, with short-time regular initial data. Does the assumption $v\in L_\infty(0,S;L^{3,\infty}(\mathbb{R}^3))$ imply that v is regular (and hence unique) on $\mathbb{R}^3\times]0,S[$?

7.3 Global Weak Solutions of the Navier-Stokes Equations with Initial Data in Scale-Invariant Spaces

In Chapter 6, we prove existence of 'global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions' to the Navier-Stokes equations in $\mathbb{R}^3\times]0,\infty[$, with solenodial initial data in the scale-invariant homogeneous Besov space $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$, which have certain continuity properties with respect to weak* convergence of the initial data. Such properties are motivated by Seregin's strategy in [106] for proving that a weak Leray-Hopf solution v in $\mathbb{R}^3\times]0,\infty[$, with short time regular initial data and a finite blow-up time T, necessarily satisfies

$$\lim_{t\uparrow T}\|v(\cdot,t)\|_{L_3(\mathbb{R}^3)}=\infty.$$

It is natural to ask if an analogous notion of global weak solution exists when the initial data belongs to larger scale-invariant spaces than $\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)$. Such a notion of solution can be shown to exist for solenoidal initial data belonging to $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$, with $4<p<\infty$. This will be proven in a forthcoming work with Albritton, by exploiting techniques from Chapters 2 and 6 of this thesis. For the wider scale-invariant space $BMO^{-1}(\mathbb{R}^3)$, our approach in Chapter 6 breaks down. The reason why our techniques no longer apply here is that $BMO^{-1}(\mathbb{R}^3)$ is an L_∞ based space. Hence, it cannot be decomposed by the methods we developed for homogeneous Besov spaces in Chapter 2. Related difficulties have been noted by Tsai and Bradshaw [12], in the context of existence theory of forward self similar solutions of the Navier-Stokes equations with large data.

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