Topics in the structure and classification of C^*-algebras and *-homomorphisms

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Abstract

The results in this thesis concern the structure and classification of C*-algebras, with a particular focus on maps between C*-algebras. A unifying theme is the regularity property of $\mathcal{Z}$-stability for C*-algebras (tensorially absorbing a copy of the Jiang-Su algebra $\mathcal{Z}$), both how this property gives rise to structural features, and how one can relax the assumption of $\mathcal{Z}$-stability in certain results to obtain uniqueness theorems from the weaker condition of strict comparison.

C*-algebras that are $\mathcal{Z}$-stable have nice K-theoretical properties, and this fact is used to obtain uniqueness theorems for maps into such C*-algebras. We provide a new, shorter, and self-contained proof of K-stability for $\mathcal{Z}$-stable C*-algebras, using Rørdam and Winter's picture of $\mathcal{Z}$.

A significant part of the thesis is devoted to proving uniqueness theorems for maps whose codomains are not necessarily $\mathcal{Z}$-stable. One important example is the class of unital embeddings from a separable, nuclear C*-algebra into a II_1 factor, which are well-known to be classified by traces in 2-norm by a result of Connes. We upgrade the uniqueness theorem in the norm topology, assuming in addition that the domain C*-algebra satisfies the UCT. We also prove uniqueness results for maps into ultraproducts of matrix algebras, which serve as a uniqueness counterpart to quasidiagonality. These results lie beyond the scope of the recent uniqueness theorems obtained from the abstract classification approach, as neither II_1 factors nor ultraproducts of matrices are $\mathcal{Z}$-stable.

The final part of the thesis follows a long-term strategy of proving uniqueness theorems for morphisms into C*-algebras, under the assumption of strict comparison alone, without $\mathcal{Z}$-stability. The first step of the outline is established in the thesis. In prior work, maps into $\mathcal{Z}$-stable C*-algebras are known to have a regularity property called property (SI), often obtained by extending maps from A to the larger domain $A \otimes \mathcal{Z}$. We replace the $\mathcal{Z}$-stability assumption by strict comparison and prove that such maps

also have property (SI). As a consequence, the maps can be extended to the $\mathcal{Z}$-stabilization of the domain C*-algebra. An essential ingredient is a new characterization of nuclearity in the separable setting, involving refined finite-dimensional approximations via pure states.

Contents

Chapter 1

Introduction

The study of operator algebras originated from Murray and von Neumann's effort to develop a mathematical framework for quantum mechanics. Since then, the field has developed deep connections with various areas, such as knot theory, non-commutative geometry, group theory, and dynamical systems.

Operator algebras are subalgebras of the bounded linear operators on a Hilbert space that are closed under taking adjoints. There are two main classes of operator algebras, C^*-algebras (closed in the norm topology) and von Neumann algebras (closed in the weak operator topology). The distinction between the two topologies leads to marked differences between the two classes of operator algebras. The study of C^*-algebras is sometimes referred to as noncommutative topology, since through the Gelfand transform, every commutative C^*-algebra is *-isomorphic to $C_0(X)$, the algebra of continuous functions vanishing at infinity on some locally compact Hausdorff space X. On the other hand, von Neumann algebras are measure-theoretic in nature, and every abelian von Neumann algebra is *-isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ).

The classification of amenable von Neumann algebras was essentially completed in the 1970s by combining Connes' breakthrough work on injectivity and hyperfiniteness with Murray von Neumann's foundational work on hyperfinite II_1 factors. One final case remained, which was later handled by Haagerup in [42]. Its C^*-algebraic counterpart, the Elliott classification program, was initiated by Elliott in the 1980s. In both settings, classification results have been developed through a deep understanding of morphisms, typically embeddings, between algebras. This thesis aims to generalize the classification of morphisms further and investigate their structural properties.

1.1 Classification of von Neumann algebras

The classification program for von Neumann algebras originates from the work of Murray and von Neumann on projections, which is used to subdivide von Neumann algebras into types ([77]). Two projections $p, q \in \mathcal{M}$ are *Murray-von Neumann equivalent* if there exists some $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. A projection is *infinite* if it is Murray-von Neumann equivalent to a proper sub-projection, and it is *finite* otherwise. A nonzero projection is *minimal* if it does not have any non-trivial subprojections.

We describe types for factors, which are the fundamental building blocks of von Neumann algebras. A von Neumann algebra is a *factor* if its center is trivial. For many purposes, it suffices to understand factors, as every separably acting von Neumann algebra decomposes as a direct integral of factors. Every factor $\mathcal{M}$ belongs to one of the following mutually exclusive types:

(i) type I: $\mathcal{M}$ has non-trivial minimal projections;

(ii) type II_1: $\mathcal{M}$ has no non-trivial minimal projections and all projections are finite;

(iii) type II_∞: $\mathcal{M}$ has no non-trivial minimal projections, but has nonzero finite and infinite projections;

(iv) type III: $\mathcal{M}$ is purely infinite, which means that all nonzero projections in $\mathcal{M}$ are infinite.

For instance, Type I factors admit minimal projections and are nothing but the algebras of bounded operators on a Hilbert space. Type II_1 factors have a unique faithful trace $\tau_{\mathcal{M}}$, which induces a bijection between Murray-von Neumann equivalence classes of projections and $[0, 1]$.

Murray and von Neumann constructed in [77] a type II_1 factor $\mathcal{R}$ from the CAR-algebra M_{2^∞}, which is the infinite tensor product of M_2 with a unique trace τ. The GNS-completion $\pi_\tau(M_{2^\infty})''$ with respect to τ is denoted by $\mathcal{R}$, which admits a faithful trace $\tau_{\mathcal{R}}$ extending that of M_{2^∞}. Since $M_{2^\infty} \otimes M_{2^\infty} \cong M_{2^\infty}$, it follows that $\mathcal{R}$ is *self-absorbing*, which means $\mathcal{R}\bar{\otimes}\mathcal{R} \cong \mathcal{R}$, when taking the von Neumann algebraic tensor product. The II_1 factor $\mathcal{R}$ satisfies an important internal approximation property, called hyperfiniteness. A von Neumann algebra is *hyperfinite* if all finite subsets can be approximated in the strong operator topology by finite-dimensional subalgebras. In the case of $\mathcal{R}$, such an approximation is achieved via finite tensor products of M_2. Murray and von Neumann's celebrated uniqueness theorem from [77] shows that $\mathcal{R}$

is the unique separably acting hyperfinite II_1 factor up to *-isomorphism. One way of proving their uniqueness theorem is to classify embeddings from hyperfinite and separable von Neumann algebras into finite von Neumann algebras.

However, verifying hyperfiniteness in examples is difficult in general. This problem was settled in the 1970s, when Connes' groundbreaking work provided more accessible abstract characterizations for hyperfiniteness ([17]). Connes proved that von Neumann algebras are hyperfinite if and only if they are injective in the category of von Neumann algebras and completely positive and contractive maps. Moreover, Connes showed that hyperfiniteness is equivalent to *semidiscreteness*, a finite-dimensional approximation property using completely positive maps. Injectivity and semidiscreteness are more accessible properties in practice. For example, the equivalence between the amenability of a discrete group G and the injectivity of the group von Neumann algebra $L(G)$ was known much earlier. However, it is unknown if hyperfiniteness of $L(G)$ for an amenable group G can be shown without appealing to Connes' result.

Combining Connes' structural theorem with Murray and von Neumann's uniqueness theorem, there is a unique injective separably acting II_1 factor. Moreover, Connes applied his work on type III factors, giving an almost complete classification of separable injective factors. Together with Haagerup's subsequent uniqueness theorem for III_1 factors ([42]), this gives a complete classification of separably acting injective factors.

The following feature in Connes' argument is particularly relevant to this thesis. An important step is to show that all separably acting injective II_1 factors $\mathcal{M}$ tensorially absorb $\mathcal{R}$, which means $\mathcal{M}\bar{\otimes}\mathcal{R} \cong \mathcal{M}$. Factors satisfying this condition are called *McDuff*, and this property was originally studied by McDuff in [74] using central sequence algebras.

Inspired by Murray and von Neumann's uniqueness theorem for $\mathcal{R}$, classification results were obtained in the 1970s by Elliott for approximately finite-dimensional C^*-algebras (AF-algebras), which are inductive limits of finite-dimensional C^*-algebras, using K-theoretical data ([28]). In the unital case, operator algebraic K-theory is defined using equivalence classes of unitaries and projections in matrix amplifications over a C^*-algebra. After classifying AF-algebras, Elliott launched the classification program for C^*-algebras in the 1980s, which aims to classify simple, separable, nuclear C^*-algebras. *Nuclearity* is analogous to semidiscreteness as it is a finite-dimensional approximation property by completely positive maps in the norm topology. Thus, the Elliott classification program can be considered as a C^*-analogue of the classification of separable injective factors.

Although the classification results are usually stated as theorems describing when C^*-algebras are *-isomorphic, they are typically proved by classifying embeddings of C^*-algebras. A classification of morphisms splits into two parts: an *existence* theorem and a *uniqueness* theorem. Existence theorems show that any morphism on the level of invariants can be realized by a genuine *-homomorphism, whereas a uniqueness theorem says that two morphisms are approximately unitarily equivalent if they agree on the level of invariants. Recall that unital maps $\phi, \psi : A \to B$ are *approximately unitarily equivalent*, denoted by $\approx_{au}$, if there exists a sequence of unitaries $(u_n)_n$ in B such that

$$\|u_n\phi(a)u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.1.1}$$

There is now a well-understood process for obtaining the classification of C^*-algebras from a suitable classification of embeddings. This is known as the *two-sided intertwining argument*, the idea of which appeared in Elliott's classification of AF-algebras and was abstracted in the framework pioneered by Rørdam in [89].

Thus classification of embeddings is of interest. A considerable amount of the thesis is devoted to obtaining classification results for more general classes of embeddings or more general morphisms.

1.2 Maps from C^*-algebras to von Neumann algebras

One of the motivating questions for the thesis is the classification of embeddings from C^*-algebras into von Neumann algebras. Due to the existence of decomposition theory for von Neumann algebras, we are mainly interested in maps into factors.

The starting point is the Gelfand–Naimark–Segal theorem, which shows that every C^*-algebra admits an embedding into $B(\mathcal{H})$. For the uniqueness counterpart, if $\mathcal{H}$ is finite dimensional, maps are classified up to unitary equivalence by traces. For separable and infinite dimensional $\mathcal{H}$, a deeper result is provided by Voiculescu's noncommutative Weyl–von Neumann theorem ([108]). We include the following generalization of Voiculescu's theorem by Hadwin.

Theorem 1.2.1 ([44, Theorem 7.1])**.** *Let A be a unital and separable C^*-algebra. Let $\phi, \psi : A \to B(\mathcal{H})$ be unital *-homomorphisms. Then $\phi \approx_{au} \psi$ if and only if the ranks of operators $\phi(a)$ and $\psi(a)$ agree for any $a \in A$.*

Beyond the type I case, uniqueness results for maps into von Neumann algebras up to approximate unitary equivalence in several natural topologies other than the norm topology, for instance, the weak* topology, were investigated by Ciuperca, Giordano, Ng and Niu in [16]. They proved an equivalent characterization for nuclearity, which says that for any von Neumann algebra $\mathcal{M}$ and pairs of unital maps ϕ and $\psi : A \to \mathcal{M}$, ϕ and ψ agree in W*-rank if and only if they are approximately unitarily equivalent in weak* topology. Thus, to obtain uniqueness theorems for maps into von Neumann algebras, one needs to restrict to nuclear domains.

For the classification of maps from nuclear C^*-algebras into type III factors $\mathcal{M}$, which are in particular simple purely infinite C^*-algebras with trivial KK-theory, it is folklore that the classification of such maps follows from Kirchberg and Phillips' classification theorem from the 1990s.

Theorem 1.2.2 (Kirchberg-Phillips Theorem, [59, 60])**.** *Let A, B be unital C^*-algebras, with A being separable and nuclear and B being simple and purely infinite. Let $\phi, \psi : A \to B$ be unital embeddings, then $\phi \approx_{au} \psi$ if and only if $KL(\phi) = KL(\psi)$.*

In the setting where the codomain von Neumann algebra $\mathcal{M}$ is a type II_∞ factor, the uniqueness result is obtained for full maps from separable nuclear C^*-algebras.

Theorem 1.2.3 ([64, Theorem 5.2.2], [63, Theorem 3])**.** *Let A be a separable and nuclear C^*-algebra and $\mathcal{M}$ be a separable II_∞ factor. If ϕ, ψ are unital and full *-homomorphisms, then they are approximately unitarily equivalent.*

Not much is known about uniqueness for unital embeddings into II_∞ factor. To prove such results, one needs to understand unital embeddings into II_1 factors. The classification of such maps can be traced back to Murray and von Neumann's work ([77]) on the comparison theory of projections. They showed that two projections $p, q \in \mathcal{M}$ in a II_1 factor are unitarily equivalent if and only if $\tau_{\mathcal{M}}(p) = \tau_{\mathcal{M}}(q)$, where $\tau_{\mathcal{M}}$ is the unique normal faithful trace on $\mathcal{M}$. This classification of projections can be viewed as a classification of *-homomorphisms from the algebra $\mathbb{C}$ into $\mathcal{M}$ (or from $\mathbb{C}^2$ if one prefers unital *-homomorphisms). From Murray and von Neumann's classification of projections, it follows that for any AF-algebra A, and any choice of trace τ_A on A, there exists a unital *-homomorphism $\phi\colon A \to \mathcal{M}$ with $\tau_{\mathcal{M}} \circ \phi = \tau_A$, and ϕ with this property is unique up to unitary equivalence. The existence part requires a one-sided intertwining argument, but the uniqueness counterpart is directly derived from the result of Murray and von Neumann.

It is well-known that Murray and von Neumann's argument above can be extended to the domain being any separable hyperfinite von Neumann algebra, at the cost of weakening the uniqueness result from unitary equivalence to approximate unitary equivalence in the 2-norm induced by $\tau_{\mathcal{M}}$. Combined with Connes's characterization for hyperfiniteness ([17]), one can further replace the domain by any separable nuclear C^*-algebra A, since maps from A into a finite von Neumann algebra factor through the finite part of A^{**}, which is injective and thus hyperfinite. The following theorem is also the starting point for the abstract classification approach in ([14]).

Theorem 1.2.4 (c.f. [97, Theorem 1.1])**.** *Let A be a separable, unital, nuclear C^*-algebra and $\mathcal{M}$ be a II_1 factor. Let $\phi, \psi : A \to \mathcal{M}$ be unital *-homomorphisms such that $\tau_{\mathcal{M}} \circ \phi = \tau_{\mathcal{M}} \circ \psi$. Then there exists a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in $\mathcal{M}$ such that*

$$\|u_n \phi(a) u_n^* - \psi(a)\|_{2,\tau_{\mathcal{M}}} \to 0, \quad a \in A. \tag{1.2.1}$$

While for AF-algebras, one immediately gets approximate unitary equivalence in norm for morphisms $A \to \mathcal{M}$ which agree on traces, in general, it is challenging to improve from $\|\cdot\|_{2,\tau}$-approximate unitary equivalence to $\|\cdot\|$-approximate unitary equivalence. The work of Ding and Hadwin [27] shows uniqueness of embeddings up to approximate unitary equivalence in norm, when A is approximately homogeneous ([27, Theorem 6]) and the internal inductive limit structure is used in a crucial way. Very recently, Hadwin, Li and Liu followed a similar strategy and have strengthened these results to the case of approximately subhomogeneous (ASH) C^*-algebras ([63, Theorem 4 and 5]).

Our first main theorem obtains uniqueness of maps from nuclear C^*-algebras up to approximate unitary equivalence in operator norm in the presence of Rosenberg and Schochet's universal coefficient theorem (UCT) from [95], which allows us to compute KK-theory in terms of K-theory for a large class of C^*-algebras. We require the injectivity of maps in the following uniqueness theorem, as the UCT is not known to pass to quotients for nuclear C^*-algebras.

Theorem A. *Let A be a separable, unital, nuclear C^*-algebra satisfying the UCT and $\mathcal{M}$ be a II_1 factor. Let $\phi, \psi : A \to \mathcal{M}$ be unital injective *-homomorphisms such that $\tau_{\mathcal{M}} \circ \phi = \tau_{\mathcal{M}} \circ \psi$. Then there exists a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in $\mathcal{M}$ such that*

$$\|u_n \phi(a) u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.2.2}$$

A C^*-algebra satisfies the UCT precisely when it is KK-equivalent to a commutative C^*-algebra. While it is a major open problem whether all separable nuclear C^*-algebras satisfy the UCT, all naturally occurring, concrete examples of separable nuclear C^*-algebras satisfy the UCT. Two particularly important classes of UCT C^*-algebras are inductive limits of type I algebras and those associated to amenable groupoids ([107]). Given the vast range of examples of algebras with the UCT, Theorem A is very broadly applicable. In particular, Theorem A generalizes the results from [27, 63], which cover type I domains, while our theorem covers domains that are type I at the level of KK-theory.

1.3 Regularity conditions for classifying maps

To prove Theorem A for maps from nuclear C^*-algebras to II_1 factors, we follow the strategy of the abstract classification framework pioneered by Schafhauser in [97] and further developed by Carrión, Gabe, Schafhauser, Tikuisis and White in [14]. The uniqueness theorems for maps obtained in [14] do not directly apply to maps into II_1 factors, since II_1 factors lack an important regularity property, called $\mathcal{Z}$-stability. In [14], codomains also need to be nuclear, but this is not a major obstruction. We digress slightly to provide a brief description of the Jiang-Su algebra $\mathcal{Z}$ and some of its important properties.

The Jiang-Su algebra $\mathcal{Z}$ is a unital, simple, nuclear, monotracial, and infinite-dimensional C^*-algebra, which has the same K-theory as $\mathbb{C}$. It was originally constructed in [54] as the inductive limit of dimension drop algebras, with carefully constructed connecting maps. A more detailed description of the constructions of $\mathcal{Z}$ is postponed to Section 2.7.

The Jiang-Su algebra is a central object in the Elliott classification program. It can be considered as the C^*-analogue of the hyperfinite II_1 factor $\mathcal{R}$ for various reasons that we will explain along the way. As mentioned previously, $\mathcal{R}$ is self-absorbing, and this property is also satisfied by $\mathcal{Z}$. In fact, Jiang and Su proved in [54] that $\mathcal{Z}$ satisfies a stronger property called *strong self-absorption (SSA)*, which is formally defined and further studied in [105]. As one might expect, other natural examples of strongly self-absorbing C^*-algebras include UHF-algebras of infinite type, for instance the CAR-algebra M_{2^∞} and the universal UHF-algebra $\mathcal{Q} = \otimes_n M_n$.

Analogous to the property of tensorially absorbing $\mathcal{R}$, a C^*-algebra A is *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$. As in the case of injective factors, in the separable setting, $\mathcal{Z}$-stability also has McDuff-type characterizations using central sequence algebras (Theorem 2.7.2). It is shown in [111, Corollary 3.2] that all unital strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable, which implies that $\mathcal{Z}$ is the initial object in the category of unital strongly self-absorbing C^*-algebras with unital $*$-homomorphisms. In particular, M_{2^∞} and $\mathcal{Q}$ tensorially absorb $\mathcal{Z}$. As a result, being $\mathcal{Z}$-stable is weaker than being $\mathcal{D}$-stable for any strongly self-absorbing unital C^*-algebra $\mathcal{D}$. In fact, $\mathcal{Z}$-stability is shown in

[111] to be the mildest C^*-absorption property analogous to the McDuff property for injective II_1 factors. From this point of view, $\mathcal{Z}$ is the most natural analog of $\mathcal{R}$.

C^*-algebras that are $\mathcal{Z}$-stable are relatively well-behaved, for instance, they have good K-theoretical properties. We mentioned previously that K-theory for C^*-algebras is given by equivalence classes of projections and unitaries in matrix amplification. For $\mathcal{Z}$-stable C^*-algebras, matrix amplifications are not needed to compute K-theory. In particular, there is a notion called *K_1-injectivity*, which means that every unitary in the C^*-algebra with trivial K_1-class is homotopic to the unit. K_1-injectivity for $\mathcal{Z}$-stable C^*-algebras was first proven by Jiang in his unpublished work [53].

Theorem 1.3.1 ([53, Theorem 2])**.** *If A is $\mathcal{Z}$-stable, then A is K_1-injective.*

This result was recaptured in the recent work [14] of Carrión, Gabe, Schafhauser, Tikuisis and White, where they use the K_1-injectivity result as an essential ingredient in proving the uniqueness theorem for maps. We leave other non-stable K-theoretical properties of $\mathcal{Z}$-stable C^*-algebras to Section 1.6 and Chapter 3 of the thesis.

By an elegant dichotomy theorem of Kirchberg, simple, separable, nuclear and $\mathcal{Z}$-stable C^*-algebras are either purely infinite or stably finite. This divides the Elliott classification program for C^*-algebras into two cases, which are handled separately. The purely infinite setting was settled by Kirchberg and Phillips in the early 2000s ([60]), where they show that simple, unital, separable, nuclear, purely infinite C^*-algebras satisfying the UCT are classified by K-theoretical data.

In the stably finite case, the progress was slower and more sophisticated, given that traces and the pairing map between K-theory and traces are taken into consideration. In 2015, a complete classification for finite, separable, simple and nuclear C^*-algebras was obtained, combining the work of Gong-Lin-Niu ([39, 40]), Tikuisis-White-Winter ([102]), and the large amount of work these papers are based on.

Theorem 1.3.2. *(Unital classification theorem) Let A, B be unital, finite, separable, simple, and nuclear C^*-algebras that are $\mathcal{Z}$-stable and satisfy the UCT. Then $A \cong B$ if and only if $Ell(A) \cong Ell(B)$, where the Elliott invariant Ell consists of K-theoretical and tracial data (see Section 2.4 for more details).*

The classification results before 2015 heavily used a certain internal approximation structure, known as *rational tracial topological rank*, which has been extensively developed by Lin and his collaborators, since Lin's foundational work in "tracially AF-algebras" in [66]. For a detailed account of the history, see [14, Section 1.2] for instance. The recent classification approach taken in [97, 14] is more abstract: it lifts existing classification results for maps into finite von Neumann algebras to classification results for maps into C^*-algebras (Theorem 1.2.4). We explain the main idea of the approach in the following section.

1.4 The abstract classification approach

For unital *-homomorphisms $\phi, \psi : A \to B$, where the domain C^*-algebra A is separable, approximate unitary equivalence of *-homomorphisms can be captured using ultraproduct constructions. Indeed, let $\iota : B \to B_\omega$ be the diagonal embedding into the norm ultraproduct of B, then ϕ and ψ are approximately unitarily equivalent if and only if $\iota \circ \phi, \iota \circ \psi : A \to B_\omega$ are unitarily equivalent (see Proposition 2.2.2, or the reindexing argument in [90, Lemma 2.2.5]). Thus, we study uniqueness of unital *-homomorphisms $A \to B_\omega$ up to unitary conjugacy.

For most maps in this thesis, the codomain C^*-algebras are ultrapowers of monotracial C^*-algebras. More generally, one could consider sequences of maps $A \to B_n$ with varying B_n, and we are also interested in the uniqueness of maps $A \to \prod_\omega B_n$. Moreover, we will primarily assume that B satisfies a comparison property of positive elements by traces, called *strict comparison*. This is a consequence of $\mathcal{Z}$-stability (see [91, Theorem 4.5]); however, strict comparison is more general. For instance, II_1 factors satisfy strict comparison (as a consequence of the comparison of supporting projections by the unique trace), but are never $\mathcal{Z}$-stable by [36].

In this section, we illustrate the abstract classification approach in the special case where B is the universal UHF-algebra $\mathcal{Q}$. Schafhauser's breakthrough paper [97] classifies maps $A \to \mathcal{Q}_\omega$ up to unitary equivalence by K-theory and traces. His approach captures more general results for maps into $\mathcal{Q}$-stable C^*-algebras and leads to AF-embeddability results for C^*-algebras. We state a version of the theorem with nuclear domains.

Theorem 1.4.1 ([97, Proposition 4.3])**.** *Let A be a separable, unital, nuclear C^*-algebra satisfying the UCT. Let $\phi, \psi : A \to \mathcal{Q}_\omega$ be unital and full *-homomorphisms such that $K_0(\phi) = K_0(\psi)$ and $\tau_{\mathcal{Q}_\omega} \circ \phi = \tau_{\mathcal{Q}_\omega} \circ \psi$. Then there exists a unitary $u \in \mathcal{Q}_\omega$ such that $\phi = Ad(u) \circ \psi$.*

Schafhauser proves this result using the *trace-kernel extension*, which has been studied extensively in connection with the Toms-Winter conjecture for regularity properties of C^*-algebras, for instance in the work of Matui and Sato ([72]), and later work of Kirchberg and Rørdam ([61]). For a unital C^*-algebra B with nonempty trace space, the trace-kernel extension is given by

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0, \tag{1.4.1}$$

where B^ω is the uniform 2-norm ultraproduct of B (for monotracial B, this is just the 2-norm ultraproduct) and J_B is the so-called *trace-kernel ideal*, containing "tracially small" elements of B_ω. When $B = \mathcal{Q}$, the ultrapower $\mathcal{Q}_\omega$ has strict comparison with respect to the unique trace $\tau_{\mathcal{Q}_\omega}$ given by the limit trace. The tracial ultrapower

$$\mathcal{Q}^\omega \cong (\pi_{\tau_{\mathcal{Q}}}(\mathcal{Q})'')^\omega \cong \mathcal{R}^\omega \tag{1.4.2}$$

is a finite von Neumann algebra with a unique trace $\tau_{\mathcal{Q}^\omega}$ satisfying $\tau_{\mathcal{Q}^\omega} \circ q_{\mathcal{Q}} = \tau_{\mathcal{Q}_\omega}$.

From now on, let A be a separable, unital and nuclear C^*-algebra. Take unital and full *-homomorphisms $\phi, \psi : A \to \mathcal{Q}_\omega$, which agree on K_0 and $\tau_{\mathcal{Q}_\omega}$. The proof of Theorem 1.4.1 is divided into two steps. The first step only involves the tracial data and classifies induced maps $q_{\mathcal{Q}} \circ \phi, q_{\mathcal{Q}} \circ \psi : A \to \mathcal{R}^\omega$ into the finite von Neumann algebra $\mathcal{R}^\omega$ up to unitary equivalence through Connes's theorem (Theorem 1.2.4), where nuclearity of A is essential. The unitary witnessing the equivalence can be lifted to $\mathcal{Q}_\omega$ as the unitary group of a von Neumann algebra is path connected. Conjugating by this unitary, we can assume that the difference between ϕ and ψ lives in the trace-kernel ideal.

$$\begin{array}{ccccccccc}
 & & & & A & & & & \\
 & & & & \phi\downarrow\downarrow\psi & \searrow\searrow^{q_{\mathcal{Q}}\circ\phi,\ q_{\mathcal{Q}}\circ\psi} & & & \\
0 & \longrightarrow & J_{\mathcal{Q}} & \xrightarrow{j_{\mathcal{Q}}} & \mathcal{Q}_\omega & \xrightarrow{q_{\mathcal{Q}}} & \mathcal{R}^\omega & \longrightarrow & 0 \\
 & & \| & & \downarrow\lambda & & \downarrow & & \\
0 & \longrightarrow & J_{\mathcal{Q}} & \longrightarrow & \mathcal{M}(J_{\mathcal{Q}}) & \xrightarrow{\pi} & \mathcal{C}(J_{\mathcal{Q}}) & \longrightarrow & 0
\end{array} \tag{1.4.3}$$

The second step lifts the classification back to $\mathcal{Q}_\omega$, using KK-machinery. Since KK-theory works better in the separable setting for technical reasons, we pretend that the C^*-algebras in the trace-kernel extension are separable. In the main body of the thesis, this is dealt with carefully by working with a suitable separable subextension, see Section 7.2. By the first step, the difference between ϕ and ψ lies in $J_{\mathcal{Q}}$ and thus $(\lambda\circ\phi, \lambda\circ\psi)\colon A \rightrightarrows \mathcal{M}(J_{\mathcal{Q}}) \rhd J_{\mathcal{Q}}$ forms a so-called *Cuntz pair*, where $\lambda : \mathcal{Q}_\omega \to \mathcal{M}(J_{\mathcal{Q}})$ is the canonical map into the multiplier algebra of $J_{\mathcal{Q}}$. This Cuntz pair gives a class $[\phi, \psi]$ in $KK(A, J_{\mathcal{Q}})$. The UCT assumption on A enables us to compute KK-theory from K-theoretical data, and $[\lambda \circ \phi, \lambda \circ \psi]$ turns out to be trivial.

The arguments described above can be generalized to embeddings into $\mathcal{Z}$-stable C^*-algebras. As explained in [14, Section 2], for the most of the proof, only consequences of $\mathcal{Z}$-stability are used, such as strict comparison and uniform property Gamma, with a larger set of invariants called the total invariant $\underline{KT}_u(\cdot)$. The full force of $\mathcal{Z}$-stability is needed only when proving KK-uniqueness statements, which allows us to proceed from having a trivial KK-class $[\lambda \circ \phi, \lambda \circ \psi]$ to obtain unitary equivalence between ϕ and ψ.

Consider first the case $B = \mathcal{Q}$. Dadalat and Eilers' stable uniqueness theorem provides a description for trivial KK-classes ([22]): if $[\lambda \circ \phi, \lambda \circ \psi]$ is the trivial KK-class, then the maps $\lambda \circ \phi$ and $\lambda \circ \psi$ are unitary equivalent with the witnessing unitary in the minimal unitization $J_{\mathcal{Q}}^{\dagger}$ of $J_{\mathcal{Q}}$, after adding any "absorbing" map from A to $\mathcal{M}(J_{\mathcal{Q}})$. The absorption condition is an appropriate generalization of the essential representations that appear in Voiculescu's theorem. Elliott and Kucerovsky proved in [29] a generalized Voiculescu's theorem, which can be used to show that $\lambda \circ \phi$ and $\lambda \circ \psi$ are absorbing as a consequence of fullness of both maps and strict comparison of $\mathcal{Q}$. Thus $(\lambda \circ \phi) \oplus (\lambda \circ \phi)$ is unitarily equivalent to $(\lambda \circ \psi) \oplus (\lambda \circ \psi)$, that is $(\lambda \circ \phi) \otimes 1_{M_2} \sim_u (\lambda \circ \psi) \otimes 1_{M_2}$. Since M_2 unitally embeds into $\mathcal{Q}$, this implies that

$$(\lambda \circ \phi) \otimes 1_{\mathcal{Q}} \sim_u (\lambda \circ \psi) \otimes 1_{\mathcal{Q}}. \tag{1.4.4}$$

Although $\mathcal{Q}_\omega$ does not tensorially absorb $\mathcal{Q}$, an appropriate "$\mathcal{Q}$-stability" notion is defined in this non-separable setting. By strong self-absorption of $\mathcal{Q}$, it follows that $\mathcal{Q}_\omega$ is *separably $\mathcal{Q}$-stable*, which means that for any separable C^*-subalgebra of $\mathcal{Q}_\omega$, there exists a separable $\mathcal{Q}$-stable C^*-subalgebra in $\mathcal{Q}_\omega$ containing it. Thus the tensor factor of $1_{\mathcal{Q}}$ can be removed from (1.4.4), and $\lambda \circ \phi$ and $\lambda \circ \psi$ are unitarily equivalent by some unitary in $J_{\mathcal{Q}}^{\dagger}$. Since λ restricts to the identity on $J_{\mathcal{Q}}^{\dagger}$, we obtain a genuine unitary equivalence between ϕ and ψ. More details can be found in Schafhauser's $\mathcal{Q}$-stable KK-uniqueness theorem ([97, Proposition 2.7]).

When B is $\mathcal{Z}$-stable, both its ultrapower B_ω and the ideal J_B are *separably $\mathcal{Z}$-stable*, defined analogously to separable $\mathcal{Q}$-stability (see [97, Proposition 1.12] for instance). The work of Carrión, Gabe, Schafhauser, Tikuisis and White proves the so-called $\mathcal{Z}$-stable KK-uniqueness theorem ([14]), using K_1-injectivity of $(\mathcal{C}(J_B) \cap \bar{\phi}(A)') \otimes \mathcal{Z}$, where $\pi : \mathcal{M}(J_B) \to \mathcal{C}(J_B)$ is the quotient map and $\bar{\phi} = \pi \circ \phi$ (see [14, Section 5.4] for more explanation). The algebra $(\mathcal{C}(J_B) \cap \bar{\phi}(A)') \otimes \mathcal{Z}$ is indeed K_1-injective by Jiang's theorem (Theorem 1.3.1). Recently, Farah and Szabó [30] provided a more direct proof of the $\mathcal{Z}$-stable KK-uniqueness theorem by studying multiplier algebras of $\mathcal{Z}$-stable C^*-algebras.

Theorem 1.4.2 ($\mathcal{Z}$-stable KK-uniquenees theorem, [30, Theorem 5.5], [14, Theorem 5.15])**.** *Let A be a unital and separable C^*-algebra. Let J be a stable, σ-unital and separably $\mathcal{Z}$-stable C^*-algebra. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair with ϕ and ψ unitally absorbing *-homomorphisms.*

(i) If $[\phi, \psi]_{KK(A,J)} = 0$, there exists a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $J^\dagger$ such that

$$\|u_t(\phi(a))u_t^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.4.5}$$

(ii) If $[\phi, \psi]_{KL(A,J)} = 0$, there exists a sequence $(u_n)_{n=1}^\infty$ of unitaries in $J^\dagger$ such that

$$\|u_n(\phi(a))u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.4.6}$$

Combining the $\mathcal{Z}$-stable KK-uniqueness theorem and the outline of the abstract classification approach described previously, the following is the main classification theorem obtained in [14]. In particular, the stably finite part of the unital classification theorem follows from Theorem 1.4.3 through a standard intertwining argument.

Theorem 1.4.3 ([14, Theorem 1.1])**.** *Let A be a unital, separable, nuclear C^*-algebra satisfying the UCT and B be a unital, simple, separable, nuclear $\mathcal{Z}$-stable C^*-algebra with $T(B) \neq \emptyset$. Then unital and full *-homomorphisms $A \to B_\omega$ are classified up to unitary equivalence, by enriched K-theoretical and tracial data.*

1.5 Main uniqueness theorems

In collaboration with my supervisor, Stuart White, we aim to prove uniqueness statements for more general maps where the codomains are not necessarily $\mathcal{Z}$-stable or separably $\mathcal{Z}$-stable. For instance, in the setting of Theorem A, while it remains open whether $\mathcal{R}$ is separably $\mathcal{Z}$-stable, many II_1 factors fail to be separably $\mathcal{Z}$-stable, since a separably $\mathcal{Z}$-stable II_1 factor is necessarily McDuff. Although type II_1 factors are not nuclear as C^*-algebras, the absence of nuclearity for codomain C^*-algebras is not really an issue for the abstract framework to work. When the trace space of B is nice enough, for instance, when B is monotracial, then Theorem 1.4.3 still holds, see [15] for a more detailed explanation.

Another example where our classification theorem applies is providing the uniqueness counterpart to quasidiagonality. Quasidiagonality has its origins in single operator theory, where it asks for block diagonal approximations of operators. Voiculescu provided an equivalent characterization of quasidiagonality for a separable unital nuclear C^*-algebra A ([109]). A separable, unital and nuclear C^*-algebra A is *quasidiagonal* if one of the following equivalent statements holds:

(i) there exist a unital embedding of A into $\prod_\omega M_{k_n}$ for a sequence $(k_n)_n \subseteq \mathbb{N}$;

(ii) there exist a unital embedding of A into $\mathcal{Q}_\omega$.

It is a fundamental challenge to determine whether every stably finite nuclear C^*-algebra is quasidiagonal. For simple nuclear C^*-algebras satisfying the UCT, this was settled by the quasidiagonality theorem in [102]. The uniqueness of maps into $\mathcal{Q}_\omega$ was later established by Schafhauser (Theorem 1.4.1), and we are interested in obtaining an analogous uniqueness theorem for maps into $\prod_\omega M_{k_n}$.

Take *-homomorphisms $\phi, \psi\colon A \to \prod_\omega M_{k_n}$ which agree on K-theory and traces. Since we can unitally embed each M_{k_n} into $\mathcal{Q}$, after we compose these morphisms with an embedding $\prod_\omega M_{k_n} \to \mathcal{Q}_\omega$, the maps ϕ and ψ become unitarily equivalent by Schafhauser's quasidiagonality theorem (Theorem 1.4.1). Approximating these unitaries in large matrices, it follows that we can find l_n with $k_n | l_n$ and after embedding M_{k_n} unitally in M_{l_n}, there will be unitaries in larger matrix sizes $(u_{l_n})_{n=1}^\infty$ witnessing the unitary equivalence. Sticking to the original matrix sizes $(k_n)_n$, Lin proved a uniqueness theorem in [68, Corollary 2.11] for commutative domains and generalized this further in [68, Corollary 5.9] to AH-algebras as domains. Lin's result has recently played an important role in nuclear dimension computations ([35]). His classification result requires the use of total K-theory, $\underline{K}$, which consists of K-theory and K-theory with coefficients in $\mathbb{Z}/n$ for every $n \in \mathbb{N}$. Compared to Q_ω, whose total K-theory is trivial except for the K_0-group, the total K-theory of the ultraproduct of matrices is not trivial. We generalize Lin's result further to domains satisfying the UCT.

Theorem B. *Let A be a separable, unital, nuclear C^*-algebra satisfying the UCT. Let $\phi, \psi : A \to \prod_\omega M_{k_n}$ be full unital *-homomorphisms such that $\underline{K}(\phi) = \underline{K}(\psi)$ and $\tau_\omega \circ \phi = \tau_\omega \circ \psi$. Then there exists a unitary $u \in \prod_\omega M_{k_n}$ such that $\phi = Ad(u) \circ \psi$.*

Theorem B is also not covered by Theorem 1.4.3, since $\prod_\omega M_{k_n}$ is not separably $\mathcal{Z}$-stable, because the corresponding tracial ultraproduct is not McDuff. In Theorem B, although the C^*-algebras B_n in the codomain are not constant, compared to Theorem 1.4.3, this does not cause any problem in the abstract approach. The major difficulty comes from the absence of $\mathcal{Z}$-stability.

To prove Theorems A and B, we follow the abstract classification framework described in the previous section. As mentioned previously, most of the arguments in the abstract framework follow analogously, and the remaining task is to prove a new

KK-uniqueness theorem without assuming $\mathcal{Z}$-stability. Here is our KK- (and KL-) uniqueness theorem, which will be applied when J is a suitable separable subalgebra of the trace-kernel ideal associated to the norm ultrapower of a II_1 factor, or $\prod_\omega M_{k_n}$.

Theorem C. *Let A be a unital, separable and nuclear C^*-algebra. Let J be a separable and stable C^*-algebra with real rank zero, stable rank one, strict comparison, totally ordered Murray-von Neumann semigroup $V(J)$ and $K_1(J) = 0$. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair with ϕ and ψ unitally absorbing *-homomorphisms.*

(i) If $[\phi, \psi]_{KK(A,J)} = 0$, there exists a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $J^\dagger$ such that

$$\|u_t(\phi(a))u_t^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.5.1}$$

(ii) If $[\phi, \psi]_{KL(A,J)} = 0$, there exists a sequence $(u_n)_{n=1}^\infty$ of unitaries in $J^\dagger$ such that

$$\|u_n(\phi(a))u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{1.5.2}$$

To prove Theorem C, we follow the strategy of Dadarlat and Eilers, and show bare hands that the relative commutant $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective, where we write $\bar{\phi}$ for $\pi \circ \phi$ and $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$ is the quotient map. This approach was described in [14, Section 5.4] and explicitly stated in Loreaux, Ng and Sutradhar's work [70]. Our proof of K_1-injectivity is largely inspired by the work [70] of Loreaux, Ng and Sutradhar, where they obtain the following theorem.

Theorem 1.5.1 ([70, Theorem 3.23])**.** *Let A be a unital, separable, simple and nuclear C^*-algebra. Let J be a σ-unital, simple and stable C^*-algebra with strict comparison and $T(J)$ having finitely many extreme points. Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

We generalize the theorem by removing simplicity assumptions on both A and J, and handle the complicated ideal structure in $\mathcal{M}(J)$, where J comes from the separabilization of the trace-kernel ideal. The following is our main technical theorem, from which we obtain Theorem C.

Theorem D. *Let A be a unital, separable and nuclear C^*-algebra. Let J be a separable and stable C^*-algebra which has real rank zero, stable rank one, totally ordered $V(J)$ and $K_1(J) = 0$. Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism. Then the algebra $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

With the KK-uniqueness result in place, we get the following uniqueness theorem from the abstract classification approach, which can be immediately applied to obtain Theorem A and Theorem B.

Theorem E. *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT. Let $(B_n)_{n=1}^\infty$ be a sequence of simple and unital C^*-algebras which have real rank zero, stable rank one, some tracial states τ_n, totally ordered $V(B_n)$ and $K_1(B_n) = 0$. We write B_ω for $\prod_\omega B_n$.*

Given full and unital *-homomorphisms $\phi, \psi \colon A \to B_\omega$ with $\tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi$ and $\underline{K}(\phi) = \underline{K}(\psi)$, there exists a unitary $u \in B_\omega$ with $\psi = (\mathrm{Ad}\, u) \circ \phi$.

Although strict comparison does not appear as an assumption for B_n in Theorem E, it is a consequence of B_n being simple, having real rank zero, a unique tracial state and $V(B_n)$ being totally ordered. Thus, Theorem E is a classification result for maps into C^*-algebras having strict comparison, but not necessarily being $\mathcal{Z}$-stable.

1.6 Nonstable K-theory for $\mathcal{Z}$-stable C^*-algebras

We mentioned previously that $\mathcal{Z}$-stable C^*-algebras have nice K-theoretical properties. For instance, these C^*-algebras are K_1-injective (Theorem 1.3.1), and as we described above, this plays an important role in proving uniqueness theorems for embeddings into $\mathcal{Z}$-stable C^*-algebras (Theorem 1.4.3). In this section, we include a short introduction to nonstable K-theory and collect more K-theoretical properties of $\mathcal{Z}$-stable C^*-algebras. The work described here is from my paper [50].

The nonstable K-theory of a C^*-algebra A is given by the higher homotopy groups of the unitary group $\mathcal{U}(A)$ based at 1_A. For stable C^*-algebras, K-theory can be computed without taking matrix amplifications and thus coincides with their nonstable K-theory. In many important cases, even without K-stability, nonstable K-theory and K-theory might coincide; see [85] for the example of noncommutative irrational tori.

In general, K-theory and nonstable K-theory are different, even in the case of commutative C^*-algebras. Take $C(S^3)$ as an example, where we have $K_1(C(S^3)) = \mathbb{Z}$, but $\pi_0(\mathcal{U}(C(S^3))) = 0$. Exact characterizations of when stable and non-stable K-theory coincide are not known, even at the level of the π_0-group. Blackadar raised a question in [5] of when a C^*-algebra is K_1-*injective* or K_1-*surjective*, namely when the canonical group homomorphism from $\pi_0(\mathcal{U}(A))$ to $K_1(A)$ is injective or surjective. A stronger notion called K-*stability* was defined by Thomsen in [101], which means that for any $m \in \mathbb{N}$, the canonical inclusion $\mathcal{U}(M_m \otimes A) \to \mathcal{U}_\infty(A)$ induces isomorphisms between higher homotopy groups of $\mathcal{U}(M_m \otimes A)$ and the K-theory groups of A. C^*-algebras are K-stable under certain structural conditions. It is shown in [118] that simple and purely infinite C^*-algebras are K-stable. Other results include that C^*-algebras with stable rank one are K_1-bijective ([85]).

Though tensorial absorption of $\mathcal{Z}$ is a different tensorial absorption property from stability, it does imply similar nice K-theoretical properties. For simple $\mathcal{Z}$-stable exact C^*-algebras, the dichotomy of Kirchberg ([38]) shows that they are either purely infinite or stably finite. In 2004, Rørdam proved in [91] that simple stably finite $\mathcal{Z}$-stable C^*-algebras necessarily have stable rank one, by extensively studying structural properties of $\mathcal{Z}$-stable C^*-algebras. Then combining results in [117] and [85], one can conclude that $\mathcal{Z}$-stability implies K_1-bijectivity for simple exact C^*-algebras.

The result that $\mathcal{Z}$-stable C^*-algebras are K-stable in the general setting was proven originally by Jiang in his unpublished paper [53], which in particular shows K_1-injectivity and K_1-surjectivity for $\mathcal{Z}$-stable C^*-algebras. To prove Theorem 1.4.3 in [14], K_1-injectivity for the $\mathcal{Z}$-stabilization of the relative commutant plays an important role. Since the result in [53] remains unpublished, a short and self-contained new proof for K_1-injectivity of $\mathcal{Z}$-stable C^*-algebras was provided in [14, Section 4.2].

This proof takes advantage of a modern description of $\mathcal{Z}$, given by Rørdam and Winter in [93], around ten years after its introduction (see Section 2.7 for more details). The algebra $\mathcal{Z}$ is realized as a stationary inductive limit of the so-called *generalized dimension drop algebras*, whose fibers are UHF-algebras instead of finite matrix algebras. This should be seen as an alternative definition of $\mathcal{Z}$, since many fundamental properties of $\mathcal{Z}$ can be readily obtained from this viewpoint, for instance in reproving strong self-absorption of $\mathcal{Z}$ in [98] and Winter's localization of the Elliott conjecture in [112], which is vital in the Gong-Lin-Niu theorem [39].

In the thesis and my paper [50], we recapture the rest of Jiang's K-stability results. Jiang's proof relies on explicit constructions of homomorphisms to produce inverse maps between higher homotopy groups when proving K-stability, which then implies K_1-injectivity and K_1-surjectivity of $\mathcal{Z}$-stable C^*-algebras. In comparison, we first provide a short new proof for the following fact using the Rørdam-Winter picture in the spirit of the K_1-injectivity proof of [14].

Theorem F. *If a C^*-algebra A is $\mathcal{Z}$-stable, then A is K_1-surjective.*

The extra flexibility is provided by the UHF-fibers of generalized dimension drop algebras, since M_{n^∞}-stable C^*-algebras are shown to have nice K-theoretical properties in Lemma 3.1.3 and [14, Lemma 4.9], which is not true in general when tensoring by finite matrix algebras. With K_1-injectivity and K_1-surjectivity of $\mathcal{Z}$-stable C^*-algebras, we are able to recapture K-stability in Theorem 3.3.3, without needing to construct explicit maps.

Theorem G. *If a C^*-algebra A is $\mathcal{Z}$-stable, then A is K-stable.*

The result is becoming increasingly important. For instance, in Toms' recent work [104] on higher homotopy groups of Cuntz classes for simple approximately divisible C^*-algebras of real rank zero, K_1-surjectivity plays an important role, and this will likely continue to be the case in further work in this direction.

1.7 Structural properties for maps

To prove more general uniqueness results for full maps $\phi : A \to \prod_\omega B_n$, where B_n is assumed to have the strict comparison, but is not necessarily $\mathcal{Z}$-stable, a potential strategy goes as follows. First of all, find extra room such that there is a unital embedding $\iota : \mathcal{Z} \to \prod_\omega B_n$ commuting with the image of ϕ. This induces an enlarged unital and full map $\phi \otimes \iota : A \otimes \mathcal{Z} \to \prod_\omega B_n$. Notice that if each B_n is $\mathcal{Z}$-stable, then such an augmented map comes for free, see [9, Lemma 1.22 (ii)]. Secondly, prove an analogous $\mathcal{Z}$-stable KK-uniqueness theorem (Theorem 1.4.2) with the $\mathcal{Z}$-stability assumption on the domain C^*-algebras instead of the codomain C^*-algebras. As mentioned previously, the rest of the abstract classification approach works with the assumption of strict comparison alone.

In this thesis, the first part of the "plan" has been worked out in some generality by following the strategy of Matui and Sato. In 2012, Matui and Sato made a breakthrough ([72]), where they proved that strict comparison implies $\mathcal{Z}$-stability, under the assumption that C^*-algebras have finitely many extremal traces.

Theorem 1.7.1 ([72, Theorem 1.1])**.** *Let A be a unital, separable, simple, nuclear, stably finite and non-elementary C^*-algebra with strict comparison and finitely many extremal traces. Then A is $\mathcal{Z}$-stable.*

Again, we illustrate the proof of Matui and Sato in the case where A is monotracial. In addition to the trace-kernel extension (7.2.3) associated to A and the unique trace τ_A, Matui and Sato showed that the surjection $q_A : A_\omega \to A^\omega$ induces a surjection at the level of central sequence algebras. This is now referred to as "central surjectivity":

$$0 \longrightarrow J_A \cap A' \xrightarrow{j_B} A_\omega \cap A' \xrightarrow{q_B} \mathcal{M}^\omega \cap \mathcal{M}' \longrightarrow 0, \qquad (1.7.1)$$

where $\mathcal{M} = \pi_{\tau_A}(A)'' \cong \mathcal{R}$, as A is simple, separable, nuclear, monotracial and non-elemetary. In particular, $\mathcal{M}$ satisfies the McDuff property, which means that there exists a unital embedding $M_2 \to \mathcal{M}^\omega \cap \mathcal{M}'$. This unital embedding might not lift to a unital *-homomorphism into $A_\omega \cap A'$, but does lift to a c.p.c. order zero map $\varphi : M_2 \to A_\omega \cap A'$ by a result of Loring [71].

$$\begin{array}{ccccccccc} & & & & & & M_2 & & \\ & & & & & \overset{\varphi}{\swarrow} & \downarrow & & \\ 0 & \longrightarrow & J_A \cap A' & \xrightarrow{j_A} & A_\omega \cap A' & \xrightarrow{q_A} & \mathcal{M}^\omega \cap \mathcal{M}' & \longrightarrow & 0 \end{array} \tag{1.7.2}$$

Since $q_A(1-\varphi(1)) = 0$, it follows that $1-\varphi(1) \in J_A$ is "tracially small", while $\varphi(e_{1,1})$ is "tracially large" when evaluated at the unique limit trace of A_ω. Strict comparison of A passes to strict comparison of A_ω, which then gives $1-\varphi(1) \precsim \varphi(e_{1,1})$ in A_ω. This almost provides a unital embedding of $\mathcal{Z}_{2,3}$ into $A_\omega \cap A'$, which is equivalent to the McDuff-type characterization of $\mathcal{Z}$-stability. The only remaining work is to show that the Cuntz subequivalence is witnessed by elements in the central sequence algebra. Property (SI) was introduced precisely to settle this problem, which is essentially a "small-to-large comparison" property in the central sequence algebra. In the work [72] of Matui and Sato, they showed that property (SI) can be accessed from strict comparison. The following theorem, together with the argument of central surjectivity, leads to Theorem 1.7.1.

Theorem 1.7.2 ([72, Section 3], [61, Proposition 5.10])**.** *Let A be a unital, stably finite, simple, separable, nuclear and non-elementary C^*-algebra with strict comparison. Then A has property (SI).*

The relative version of property (SI) for maps appeared in the later work of Matui and Sato ([73]) and Brown, Bosa, Sato, Tikuisis, White and Winter [9]. Analogous to property (SI) for C^*-algebras, property (SI) for a map $\phi : A \to \prod_\omega B_n$ is a "small-to-large comparison" property in the relative commutant $\phi(A)' \cap \prod_\omega B_n$. Generalizing techniques developed in [72] and [61], the following theorem is proven in [9].

Theorem 1.7.3 ([9, Lemma 4.4])**.** *Let $(B_n)_n$ be a sequence of simple, separable, unital, finite, $\mathcal{Z}$-stable C^*-algebras with $QT(B_n) = T(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$. We write B_ω for $\prod_\omega B_n$. Let A be a separable, unital and nuclear C^*-algebra. Then every unital and full map $\phi : A \to B_\omega$ has property (SI).*

In Theorem 1.7.3, the hypothesis that the codomains B_n are $\mathcal{Z}$-stable, rather than the consequence that they have strict comparison, is crucially used to obtain a technical property of representations of the domain C^*-algebra. Thus, for any A that satisfies this technical property, the proof of Theorem 1.7.3 works only assuming strict comparison of each B_n (see Section 9.1 for a more detailed explanation). By studying the structure of nuclear C^*-algebras without the technical property, we generalize Theorem 1.7.3, with only the strict comparison assumption on B_n.

Theorem H. *Let $(B_n)_n$ be a sequence of simple, separable, unital, finite C^*-algebras with strict comparison and $QT(B_n) = T(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$. We write B_ω for $\prod_\omega B_n$. Let A be a separable, unital and nuclear C^*-algebra. Then every unital and full *-homomorphism $\varphi : A \to B_\omega$ has property (SI).*

As a consequence of our property (SI) result, through the relative version of central surjectivity, which is set out in [9, Lemma 3.10], there is a unital copy of $\mathcal{Z}$ in B_ω commuting with the image of $\phi : A \to \prod_\omega B_n$.

Theorem I. *Let B_n be a sequence of simple, unital, monotracial and non-elementary C^*-algebras with strict comparison and $QT(B_n) = T(B_n)$ for all $n \in \mathbb{N}$. We write B_ω for $\prod_\omega B_n$. Let A be a separable, unital and nuclear C^*-algebra and $\phi : A \to B_\omega$ a unital and full *-homomorphism. Then there exists a unital embedding $\mathcal{Z} \to B_\omega \cap \phi(A)'$.*

We mainly focus on the monotracial case in the thesis. More general results for property (SI) of nuclear maps with exact domains, and unital embeddings of $\mathcal{Z}$ into the relative commutant $B_\omega \cap \phi(A)'$ will appear in the upcoming paper [49] of mine.

Lastly, we describe the main ingredient of the proof of Theorem H. The major input is an equivalent characterization of nuclearity for separable C^*-algebras. The technical property on the domain C^*-algebra induced by $\mathcal{Z}$-stability is used in [9], together with nuclearity, to obtain a refined completely positive finite-dimensional approximation property, which we single out in Chapter 8 and name the *Matui-Sato approximation property (MSAP)*. For instance, it was shown by Matui and Sato in [72] that simple, separable, nuclear and non-elementary C^*-algebras satisfy the MSAP. Moreover, we observe that matrix algebras or commutative C^*-algebras also satisfy the MSAP. More generally, we prove that in the separable setting, the MSAP is equivalent to nuclearity.

Theorem J. *A separable C^*-algebra A is nuclear if and only if it has the MSAP*

1.8 The structure of the thesis

The results in the thesis, unless otherwise stated, are included in my single-authored paper [50], the upcoming paper [49], and the joint work with Stuart White [51].

Some basic preliminaries are included in Chapter 2, covering topics such as K-theory, Cuntz semigroups, multiplier algebra constructions and representations theory of C^*-algebras. The main body of the thesis can be divided into three parts:

(i) K-stability: In Chapter 3, the role played by matrix amplifications in computing K-theory is explored, under the $\mathcal{Z}$-stability assumption. We provide a new proof of Jiang's result, that $\mathcal{Z}$-stable C^*-algebras are K-stable (Theorem G). The results in this chapter are covered in my paper [50];

(ii) Classification of morphisms: The main results will be contained in the upcoming paper [51] with Stuart White. In Chapter 4, a brief introduction to KK-theory and KL-theory is given, followed by Dadarlat and Eilers' stable uniqueness theorem. We explain in some detail the connection between KK-uniqueness theorems and K_1-injectivity of relative commutants in corona algebras.

In Chapter 5, we introduce a notion called *relative pure largeness* for ideals in multiplier algebras. The existence of such ideals gives the appropriate sufficient condition for proving K_1-injectivity of the relative commutants in the following chapter. In the multiplier algebra of a large class of real rank zero and stable rank one C^*-algebras, we show the existence of relative purely large ideals.

Chapter 6 is devoted to the main proof of our K_1-injectivity result (Theorem D). Due to the technicality involved in the proof, we provide an outline for the proof in Section 6.2 to explain the main strategy. At the end of the chapter, we conclude with our main KK- and KL-uniqueness theorem (Theorem C).

We follow the abstract classification approach in Chapter 7 to obtain the uniqueness theorem (Theorem E) from the KL-uniqueness theorem. Finally, the uniqueness results for maps into II_1 factors (Theorem A) and ultraproduct of matrices (Theorem B) are derived as direct consequences of Theorem E.

(iii) Structural properties for morphisms: The main results will be included in the upcoming paper [49] of me. In Chapter 8, we define the Matui-Sato approximation property (MSAP). The equivalence between the MSAP and nuclearity is proved in Section 8.3 (Theorem J), through the decomposition theory for C^*-algebras.

In Chapter 9, using the equivalence between the MSAP and nuclearity obtained in Chapter 8, we prove the property (SI) result for unital and full maps (Theorem H). Lastly, we conclude with a short proof of Theorem I.

Chapter 2

Preliminaries

In this chapter, we establish notation and background knowledge required for future chapters. Throughout the thesis, we assume that the reader is familiar with the basics of C^*-algebras, as can be found in [76].

2.1 Inductive limits

Inductive limits for both C^*-algebras and abelian groups are of great importance. Many of the central objects considered in this thesis are constructed as inductive limits, including UHF-algebras and the Jiang-Su algebra $\mathcal{Z}$. Understanding the inductive limits of abelian groups allows us to calculate the K-theory of inductive limit C^*-algebras.

There are two important categories. One of them is the category of all C^*-algebras, denoted by C^*-**Alg**, where the morphisms are *-homomorphisms. The second is the category of abelian groups, denoted by **Ab**, with morphisms being group homomorphisms. We define inductive limits first in the abstract setting of category theory.

An *inductive sequence* in a category $\mathcal{C}$ is a sequence of objects $(A_n)_{n=1}^{\infty}$ in $\mathcal{C}$ with a sequence of morphisms $\varphi_n : A_n \to A_{n+1}$. For $m > n$, we have morphisms $\varphi_{n,m} = \varphi_{m-1} \circ \cdots \circ \varphi_n : A_n \to A_m$. Together with the φ_n's, they are called *connecting maps.*

Definition 2.1.1 (cf. [92, Definition 6.2.2])**.** An *inductive limit* of an inductive sequence is an object A in $\mathcal{C}$ with a sequence of morphisms $\varphi^{(n)} : A_n \to A$ such that the following two conditions hold:

(i) The following diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow^{\varphi^{(n)}} & \downarrow{\scriptstyle \varphi^{(n+1)}} \\ & & A \end{array} \tag{2.1.1}$$

commutes for any $n \in \mathbb{N}$;

(ii) (Universality) If B is an object in $\mathcal{C}$ and there are morphisms $\psi^{(n)} : A_n \to B$ such that the following diagram commutes for each $n \in \mathbb{N}$:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow^{\psi^{(n)}} & \downarrow{\scriptstyle \psi^{(n+1)}} \\ & & B \end{array} , \tag{2.1.2}$$

then there is a unique morphism $\psi : A \to B$ such that the following diagram commutes for each $n \in \mathbb{N}$:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi^{(n)}} & A \\ & \searrow^{\psi^{(n)}} & \downarrow{\scriptstyle \psi} \\ & & B \end{array} . \tag{2.1.3}$$

Notice that once inductive limits exist, they are unique up to unique isomorphism in the category, which is guaranteed by universality in the previous definition. We denote the inductive limit by $\varinjlim(A_n, \varphi_n)$.

In the category C^***-Alg**, inductive limits always exist and can be described explicitly; see [92, Proposition 6.2.4] for more details. We will focus on an important special case, where the connecting maps φ_n are injective. Let A be a C^*-algebra and $(A_n)_{n\in\mathbb{N}}$ be a sequence of increasing C^*-subalgebras in A with $A = \overline{\bigcup_n A_n}$. Let $\varphi_n : A_n \to A_{n+1}$ be the inclusion maps. Then $A = \varinjlim(A_n, \varphi_n)$ by a standard definition checking. This special case will be important for us, since most C^*-algebraic inductive limits discussed later will have injective connecting maps, for example, $\mathcal{K}$, $M_{2^\infty}, M_{3^\infty}$ and different realizations of $\mathcal{Z}$.

Example 2.1.2. The C^*-algebra $\mathcal{K}$ of compact operators on a separable infinite-dimensional Hilbert space can be realized as an inductive limit. Simply take $A_n =$

$M_n(\mathbb{C})$ and connecting maps φ_n mapping $n \times n$ matrices into the upper left corner of $(n+1) \times (n+1)$ matrices, with last row and column taking zero values.

Example 2.1.3. *Uniformly hyperfinite (UHF) algebras* were firstly studied and classified by Glimm in [37]. They are inductive limits of increasing sequences of matrix algebras $(M_{i_n}(\mathbb{C}))_{n\in\mathbb{N}}$ with $i_n | i_{n+1}$, and unital injective maps $\varphi_n : M_{i_n}(\mathbb{C}) \to M_{i_{n+1}}(\mathbb{C})$ given by

$$a \mapsto \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}. \tag{2.1.4}$$

Such maps can also be considered in terms of tensor products, where $\varphi_n : M_{i_n}(\mathbb{C}) \to M_{i_{n+1}}(\mathbb{C})$ is defined by $a \mapsto a \otimes 1_{i_{n+1}/i_n}$.

These C^*-algebras are classified by *supernatural numbers*, which are of the form $\mathfrak{p} = \prod_{p \text{ prime}} p^{k_p}$ with $k_p \in \mathbb{N} \cup \{\infty\}$ and are of *infinite type* if $k_p \in \{0, \infty\}$ for all prime numbers p. To each UHF algebra $A = \varinjlim(M_{i_n}(\mathbb{C}), \varphi_n)$, we associate a unique supernatural number $\mathfrak{p}_A$ by taking

$$k_p = \sup\{k \in \mathbb{N} : p^k | i_m \text{ for some } m\}. \tag{2.1.5}$$

A UHF algebra A is of *infinite type* if $\mathfrak{p}_A$ is of infinite type. For example, we get the CAR-algebra M_{2^∞} when $\mathfrak{p}_A = 2^\infty$ and the UHF-algebra M_{3^∞} when $\mathfrak{p}_A = 3^\infty$. These UHF-algebras will be important in Chapter 3.

In the category **Ab**, the inductive limit of an increasing sequence of abelian groups can be explicitly constructed. For an inductive sequence of abelian groups $(G_n)_{n\in\mathbb{N}}$ and group homomorphisms $\varphi_n : G_n \to G_{n+1}$, define the following group with the pointwise operations:

$$\mathcal{G} = \left\{ (g_n)_{n=1}^\infty \in \prod_{n=1}^\infty G_n : \exists N \in \mathbb{N} \text{ such that } g_{k+1} = \varphi_k(g_k) \text{ for any } k \geq N \right\}.$$

Then the inductive limit is $G = \varinjlim(G_n, \varphi_n) = \mathcal{G}/H$, where

$$H = \{(g_n)_{n=1}^\infty \in \mathcal{G} : \exists N \in \mathbb{N} \text{ such that } g_k = e_k \text{ for any } k \geq N\}, \tag{2.1.6}$$

and e_k is the identity of G_k. The connecting maps $\varphi^{(n)} : G_n \to G$ are defined by $\varphi^{(n)}(g_n) = [(e_1, \cdots, e_{n-1}, g_n, \varphi_n(g_n), \cdots)]$. Thus we can check that $G = \bigcup_{n=1}^\infty \varphi^{(n)}(G_n)$.

Example 2.1.4. Consider the following inductive sequence,

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots.$$

Then $G = \varinjlim(\mathbb{Z}, \varphi_2)$ is isomorphic to $\mathbb{Z}[\frac{1}{2}]$, the group of dyadic rationals.

2.2 The ultraproduct construction

In this section, we introduce the norm ultraproduct of C^*-algebras and collect several useful facts about them. Fix a free ultrafilter ω on the natural numbers and let $(B_n)_n$ be a sequence of C^*-algebras. Then the product $\prod_{n=1}^{\infty} B_n$ is given by

$$\prod_{n=1}^{\infty} B_n := \left\{ (b_n)_{n=1}^{\infty} : b_n \in B_n,\ \sup_n \|b_n\| < \infty \right\}. \tag{2.2.1}$$

Take c_ω to be the closed two-sided ideal of $\prod_{n=1}^{\infty} B_n$ given by

$$c_\omega((B_n)_n) := \left\{ (b_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} B_n : \lim_{n\to\omega} \|b_n\| = 0 \right\}. \tag{2.2.2}$$

Then the *ultraproduct* of the sequence $(B_n)_{n=1}^{\infty}$ is defined by

$$\prod_{n\to\omega} B_n := \prod_{n=1}^{\infty} B_n \Big/ c_\omega((B_n)_n). \tag{2.2.3}$$

We typically denote elements in $\prod_\omega B_n$ using representing sequences $(b_n)_{n=1}^{\infty}$ from $\prod_n B_n$. When $B_n = A$ for every $n \in \mathbb{N}$, we denote the product algebra by $\ell^\infty(A)$ and the *ultrapower* by A_ω.

The advantage of working with ultraproducts is that ultraproducts provide powerful tools to amalgamate approximate statements into an exact statement. This is usually made possible through the Kirchberg's ϵ-test.

Lemma 2.2.1 (Kirchberg's ϵ-test, cf. [61, Lemma 3.1])**.** *Let $(X_n)_n$ be a sequence of sets and for each $k \in \mathbb{N}$, let $\left(f_n^{(k)}\right)_n$ be a sequence of functions $f_n^{(k)} : X_n \to [0,\infty)$. For each $k \in \mathbb{N}$, define a new function $f_\omega^{(k)} : \prod_{n=1}^{\infty} X_n \to [0,\infty]$ by*

$$f_\omega^{(k)}((s_n)_n) = \lim_{n\to\omega} f_n^{(k)}(s_n), \quad (s_n)_n \in \prod_{n=1}^{\infty} X_n. \tag{2.2.4}$$

Suppose that for each $m \in \mathbb{N}$ and $\epsilon > 0$, there exists $s = (s_n)_n \in \prod_n X_n$ such that $f_\omega^{(k)}(s) < \epsilon$ for $k = 1, \cdots, m$. It follows that there exists $t = (t_n)_n \in \prod_n X_n$ such that $f_\omega^{(k)}(t) = 0$ for every $k \in \mathbb{N}$.

One standard application of the Kirchberg ϵ-test is that for a unital C^*-algebra A, unitaries in A_ω can be lifted to a sequence of unitaries in $\ell^\infty(A)$, see [90, Lemma 6.2.4] for instance. Based on this lifting result, one can prove the following.

Proposition 2.2.2 ([90, Lemma 6.2.5]). *Let A, B be C^*-algebras with A separable and B unital. Let $\phi, \psi : A \to B$ be unital *-homomorphisms. Then the following statements are equivalent:*

(i) ϕ and ϕ are approximately unitarily equivalent;

(ii) $\iota \circ \phi$ and $\iota \circ \psi$ are unitarily equivalent, where $\iota : B \to B_\omega$ is the inclusion map;

(iii) $\iota \circ \phi$ and $\iota \circ \psi$ are approximately unitarily equivalent.

2.3 K-theory

The K-theory of a C^*-algebra A consists of a pair of abelian groups, $K_0(A)$ and $K_1(A)$, which are built from equivalence classes of projections and unitaries in matrix amplifications of A, respectively. K-theory contains much information about C^*-algebras and is a key ingredient in the Elliott classification program. For a more detailed and thorough introduction to K-theory, readers are referred to [92].

2.3.1 Properties of unitaries

Since K_1-theoretical properties will be a central topic, we collect various results describing the properties of unitaries. For a unital C^*-algebra A, we will denote the group of unitaries in A by $\mathcal{U}(A)$, which is naturally a topological group.

Definition 2.3.1. Let A be a unital C^*-algebra and $u, v \in \mathcal{U}(A)$. Then u and v are *homotopic*, denoted by $u \sim_h v$, if there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = u$ and $u_1 = v$.

The set of unitaries homotopic to 1_A is denoted by $\mathcal{U}_0(A)$. The following lemma shows that two unitaries $u, v \in \mathcal{U}(A)$ are homotopic if their norm difference is not the largest possible value 2, as $\|u - v\| \leq 2$ always holds.

Lemma 2.3.2 (cf. [92, Lemma 2.1.3]). *If $u \in \mathcal{U}(A)$ with the spectrum $sp(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$. Thus if $\|u - v\| < 2$, then $u \sim_h v$.*

Sketch Proof. For the first part of the proof, an elementary functional calculus argument shows that if $\mathrm{sp}(u) \neq \mathbb{T}$, then $u = \exp(ia)$ for a self-adjoint element $a \in A$. Thus a unitary path witnessing $u \sim_h 1$ is given by $u_t = \exp(ita)$.

If $\|u - v\| < 2$, then $\|v^*u - 1\| = \|v^*(u - v)\| < 2$, which means $-2 \notin \mathrm{sp}(v^*u - 1)$ and thus $-1 \notin \mathrm{sp}(v^*u)$. By the first part, we have $v^*u \in \mathcal{U}_0(A)$ and thus $v \sim_h u$. $\square$

Since unitaries in $M_n(\mathbb{C})$ have finite spectrums, Lemma 2.3.2 shows that $\mathcal{U}(M_n(\mathbb{C}))$ is path connected. Applying this fact, we obtain the following Whitehead lemma.

Lemma 2.3.3 (cf. [92, Lemma 2.1.5]). *Let A be a unital C^*-algebra and $u, v \in \mathcal{U}(A)$. Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathcal{U}(M_2(A)). \tag{2.3.1}$$

In particular,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathcal{U}(M_2(A)). \tag{2.3.2}$$

Lastly, we include the following lifting lemma, which is true since all unitaries $u \in \mathcal{U}_0(A)$ can be written as $u = \exp(ih_1) \cdots \exp(ih_n)$ for self-adjoint elements $h_1, \cdots, h_n \in A$; see [92, Proposition 2.1.6] for a detailed proof.

Lemma 2.3.4 (cf. [92, |lemma 2.1.7]). *Let A, B be unital C^*-algebras and $\varphi : A \to B$ be a surjective $*$-homomorphism, then $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$.*

2.3.2 Functors K_0 and K_1

For convenience, we describe K-theory for unital C^*-algebras and only include results for non-unital C^*-algebras when they will be needed later.

For a unital C^*-algebra A, we denote by $\mathcal{P}_n(A)$ the set of projections in $M_n(A)$ and $\mathcal{P}_\infty(A) = \bigcup_n \mathcal{P}_n(A)$. An equivalence relation $\sim$ can be defined on $\mathcal{P}_\infty(A)$: projections $p \in M_m(A)$ and $q \in M_n(A)$ are *Murray von Neumann equivalent*, denoted by $p \sim_0 q$, if there exists $v \in M_{n\times m}(A)$ such that $p = v^*v$ and $q = vv^*$. Then it can be shown that the *Murray-von Neumann semigroup* $V(A) = \mathcal{P}_\infty(A)/{\sim_0}$ is indeed an abelian semigroup with addition defined by $[p] + [q] = [p \oplus q]$. The group $K_0(A)$ is defined to be the Grothendieck completion of $V(A)$, which produces a group from an abelian semigroup; see [92, Section 3.1]. In the non-unital case, the definition of $K_0(A)$ can be found in [92, Section 4.1], which is defined in a way that K_0, when considered as a functor, has desirable properties.

We continue with the definition of $K_1(A)$. We denote the unit of $M_n(A)$ by 1_n, the set of unitaries in $M_n(A)$ by $\mathcal{U}_n(A)$ and $\mathcal{U}_\infty(A) = \bigcup_n \mathcal{U}_n(A)$. Define an equivalnce relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows: for any $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, then $u \sim_1 v$ if there exists $k \geq \max\{n, m\}$ such that $\mathrm{diag}(u, 1_{k-n}) \sim_h \mathrm{diag}(v, 1_{k-m})$ in $\mathcal{U}_k(A)$. Denote the equivalence classes by $[\cdot]_1$. Then

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(A)\}, \tag{2.3.3}$$

with the well-defined addition $[u]_1 + [v]_1 = [\text{diag}(u, v)]_1$, is an abelian group. The unit of the group is $[1]_1$ and the inverse of $[u]_1$ is $[u^*]_1$. For a non-unital C^*-algebra A, we denote by $A^\dagger$ to be its minimal proper unitization and define $K_1(A) = K_1(A^\dagger)$.

Following the definitions of K_0 and K_1, it is standard to check that they are both covariant functors from C^***-Alg** to **Ab**; see [92, Proposition 3.2.4, Proposition 8.2.2].

Moreover, they have nice properties as functors. Recall that a covariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is *half exact* if every short exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{2.3.4}$$

in $\mathcal{C}$ induces an exact sequence

$$F(I) \xrightarrow{F(\varphi)} F(A) \xrightarrow{F(\psi)} F(B), \tag{2.3.5}$$

or equivalently, $\text{Im}(F(\varphi)) = \ker(F(\psi))$. The functor F is *split exact* if every split exact sequence in $\mathcal{C}$ induces a split exact sequence in $\mathcal{D}$.

The next proposition describes how K_0 and K_1 interact with inductive limits; see Section 2.1 for the detailed definition of inductive limits in the language of category theory. A functor $F : \mathcal{C} \to \mathcal{D}$ is *continuous* if for an inductive limit $A = \varinjlim(A_n, \varphi_n)$ in $\mathcal{C}$, we have $F(A) \cong \varinjlim(F(A_n), F(\varphi_n))$.

Proposition 2.3.5 (cf. [92, Proposition 4.3.2, 4.3.3, 8.2.4, 8.2.5, 6.3.2, 8.2.7])**.** *The functors K_0 and K_1 are half exact, split exact and continuous.*

Using continuity, K-theory of UHF-algebras is computed from the K-theory of matrix algebras. For instance, $K_0(M_{2^\infty}) = \mathbb{Z}[\frac{1}{2}]$ by Example 2.1.4 and $K_1(M_{2^\infty}) = 0$.

For C^*-algebras A and B, $*$-homomorphisms $\varphi, \psi : A \to B$ are *homotopic*, denoted by $\varphi \sim_h \psi$, if there is a path of $*$-homomorphisms $\varphi_t : A \to B$, where $t \in [0, 1]$, such that $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $t \mapsto \varphi_t(a)$ is norm continuous for each $a \in A$. Then C^*-algebras A, B are *homotopy equivalent* if there are $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h id_A$ and $\varphi \circ \psi \sim_h id_B$.

Proposition 2.3.6 (cf. [92, Proposition 3.2.6])**.** *The functors K_0 and K_1 are homotopy invariant, i.e. if $\varphi, \psi : A \to B$ are homotopic, then $K_i(\varphi) = K_i(\psi)$. In particular, if A and B are homotopy equivalent, then $K_i(A) \cong K_i(B)$ for $i \in \{0, 1\}$.*

This allows us to calculate K-theory of the following example.

Example 2.3.7. Consider the *cone* of a C^*-algebra A defined by

$$CA = \{f \in C([0,1], A) : f(0) = 0\}.$$

Then CA is homotopy equivalent to the zero C^*-algebra. To see this, take a norm continuous path of $*$-homomorphisms $\varphi_t : CA \to CA$, where $\varphi_t(f)(s) = f(st)$ for $f \in CA$, $s, t \in [0,1]$. Then $\varphi_0 = 0$ and $\varphi_1 = id_{CA}$, which means the zero maps $\varphi : CA \to 0$ and $\psi : 0 \to CA$ satisfy $\psi \circ \varphi = \varphi_0 \sim_h \varphi_1 = id_{CA}$ and $\psi \circ \varphi = id_0$. Since K_0 and K_1 are homotopy invariant, we have $K_i(CA) = 0$ for $i \in \{0, 1\}$.

2.3.3 Bott Periodicity and Homotopy groups

It is known that for any short exact sequence of C^*-algebras,

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0, \tag{2.3.6}$$

there is an associated six-term exact sequence of abelian groups:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\varphi)} & K_0(A) & \xrightarrow{K_0(\psi)} & K_0(B) \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\ K_1(B) & \xleftarrow[K_1(\psi)]{} & K_1(A) & \xleftarrow[K_1(\varphi)]{} & K_1(I). \end{array} \tag{2.3.7}$$

Notice that exactness at $K_0(A)$ and $K_1(A)$ follows from half exactness of K_0 and K_1 given by Theorem 2.3.5. The six-term exact sequence is very useful for calculating the K-theory of non-simple C^*-algebras by putting them in short exact sequences as in (2.3.6), where the K-theory of ideals and quotients might be known. We will also define higher K-functors K_n for C^*-algebras and state the remarkable Bott periodicity, which says $K_{n+2}(A) \cong K_n(A)$ for all integers $n \geq 0$.

Starting from a short exact sequence (2.3.6), one can define an *index map* δ_1 : $K_1(B) \to K_0(I)$, explicit descriptions of which can be found in [92, Section 9.1] for instance. Moreover, the diagram (2.3.7) is exact at $K_1(B)$ and $K_0(I)$ with δ_1. Take the following short exact sequence,

$$0 \longrightarrow SA \xrightarrow{\varphi} CA \xrightarrow{\psi} A \longrightarrow 0, \tag{2.3.8}$$

where ψ is the evaluation map at 1. Since $K_0(CA) = K_1(CA) = 0$ by Example 2.3.7, then the corresponding $\delta_1 : K_1(A) \to K_0(SA)$ gives an isomorphism.

Theorem 2.3.8 (cf. [92, Theorem 10.1.3])**.** *For any C^*-algebra A, the map δ_1 : $K_1(A) \to K_0(SA)$ is an isomorphism.*

Now we start to define higher K-functors. For each $*$-homomorphism $\varphi : A \to B$, there is an associated $*$-homomorphism $S\varphi : SA \to SB$ defined by $(S\varphi(f))(t) = (\varphi \circ f)(t)$, for $t \in [0,1]$. One can easily check that S is an exact functor from C^*-**Alg** into itself. Then for $n \geq 2$, define inductively a *higher K-functor* $K_n = K_{n-1} \circ S : C^*\text{-}\mathbf{Alg} \to \mathbf{Ab}$. Since S is an exact functor and K_1 is half exact, all the K_n are also half exact. We denote the n-th iteration of S on A by $S^n A$. For each $n \geq 1$, since S is exact and hence so is S^n, we obtain the following short exact sequence from (2.3.6),

$$0 \longrightarrow S^n I \xrightarrow{S^n \varphi} S^n A \xrightarrow{S^n \psi} S^n B \longrightarrow 0. \tag{2.3.9}$$

Now we define the *higher index map* $\delta_{n+1} : K_{n+1}(B) \to K_n(I)$ to be the index map from $K_{n+1}(B) = K_1(S^n(B))$ to $K_n(I) = K_1(S^{n-1}(I)) \cong K_0(S^n(I))$, which is induced by the short exact sequence (2.3.9). By standard diagram checking, these higher index maps fit into the long exact sequence:

$$\cdots \xrightarrow{K_{n+1}(\psi)} K_{n+1}(B) \xrightarrow{\delta_{n+1}} K_n(I) \xrightarrow{K_n(\varphi)} K_n(A) \xrightarrow{K_n(\psi)} K_n(B) \xrightarrow{\delta_n} \cdots$$

$$\cdots \xrightarrow{\delta_1} K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B).$$

Although we have higher K-functors and the long exact sequence, all K-theoretic data is contained in K_0 and K_1. This fact was first proved by Atiyah and is analogous to the fundamental theorem of R. Bott [10].

To establish this, it suffices to find an isomorphism from $K_0(A)$ to $K_1(SA)$. For a unital C^*-algebra A, elements in $K_1(SA)$ can be thought of as homotopy equivalence classes of continuous loops $f : \mathbb{T} \to \mathcal{U}_n(A)$ with $f(1) \sim_h 1_n$ for some $n \in \mathbb{N}$. For each projection $p \in \mathcal{P}_\infty(A)$, we define the *projection loop* $f_p : \mathbb{T} \to \mathcal{U}_\infty((SA)^\dagger)$, $z \mapsto zp + (1-p)$. This induces a group homomorphism $\beta_A : K_0(A) \to K_1(SA)$, given by $\beta_A([p]_0) = [f_p]_1$ for $p \in \mathcal{P}_\infty(A)$. It is known as the *Bott map* and serves as the required isomorphism between $K_0(A)$ and $K_1(SA)$.

Theorem 2.3.9 (cf. [92, Corollary 11.3.1]). *The Bott map $\beta_A : K_0(A) \to K_1(SA)$ is an isomorphism for every C^*-algebra A. Thus, $K_{n+2}(A) \cong K_n(A)$ for $n \geq 0$.*

The exponential map, defined by $\delta_0 = \delta_2 \circ \beta_B$, makes the diagram (2.3.7) exact at $K_1(I)$ and $K_0(B)$. We provide an explicit description of δ_0 below in the unital case.

Theorem 2.3.10 (cf. [92, Proposition 12.2.2]). *Let A be a unital C^*-algebra and $\delta_0 : K_0(B) \to K_1(I)$ be the exponential map for (2.3.6). Then $\delta_0([p]_0)$ for some $p \in \mathcal{P}_n(B)$ is calculated as follows: let a be a self-adjoint element in $M_n(A)$ with $\psi(a) = p$, then $\varphi^\dagger(u) = \exp(2\pi i a)$ for a unique unitary $u \in \mathcal{U}_n(I^\dagger)$ and take $\delta_0([p]_0) = -[u]_1$.*

2.3.4 Homotopy groups and K-theory

This section presents a result linking operator algebraic K-theory with homotopy groups of unitary groups of C^*-algebras. We first recall the definition of the higher homotopy groups of a topological space.

Fix a point x_0 in a topological space X and for each $n \geq 0$, fix a basepoint s_n of the n-dimension sphere S^n. Then the *n-th homotopy group* $\pi_n(X, x_0)$ is the set of homotopy equivalence classes of continuous functions from S^n to X, mapping s_n to x_0. When the fixed point in X is clear from the context, we use the notation $\pi_n(X)$. Notice that when $n = 0$, then S^0 consists of two points, meaning that $\pi_0(X)$ is the set of path-connected components of X.

For $n \geq 1$, group operations on $\pi_n(X)$ are defined by roughly "pasting" together two continuous basepoint-preserving maps, which is a higher-dimensional generalization of the group operation of the fundamental group $\pi_1(X)$. Then π_n is a functor for $n \geq 1$ from the category of topological spaces to the category of groups. More details can be found in Chapter 4 of [45], for example.

We are particularly interested in the case where $X = G$ is a topological group, for instance, the unitary group $\mathcal{U}(A)$ of a C^*-algebra A. In this case, there is a well-defined group operation on $\pi_n(G)$ for $n \geq 0$: for $[f], [g] \in \pi_n(G)$, we define $[f] \cdot [g] = [f \cdot g]$, where $(f \cdot g)(x) = f(x)g(x)$ for any $x \in S^n$. This group operation makes $\pi_n(G)$ into a group and coincides with the group operation defined above on $\pi_n(G)$ for $n \geq 1$. Since we will mainly work with topological groups, we will primarily use this form of group operation in our arguments.

As an important application of Bott periodicity, K-theory of C^*-algebras can be expressed in terms of higher homotopy groups of unitary groups of C^*-algebras after matrix amplifications. In order to cover both unital and non-unital cases, we need to generalize the notion of the unitary group. For any C^*-algebra A, we denote the minimal proper unitization of A by $A^\dagger$, where $A^\dagger = A \oplus \mathbb{C}$ if A is unital. The canonical maps associated to $A^\dagger$ are $\iota : A \to A^\dagger$ and $q : A^\dagger \to \mathbb{C}$.

For any C^*-algebra A, we define the *generalized unitary group* of A to be

$$\mathcal{V}(A) = \{u \in \mathcal{U}(A^\dagger) : q(u) = 1\}. \tag{2.3.10}$$

Notice that this is a topological group and $\mathcal{V}(A) \cong \mathcal{U}(A)$ as topological groups when A is unital. We use the notation $\mathcal{V}_m(A)$ for $\mathcal{V}(M_m \otimes A)$, where $m \geq 1$.

When defining the K_1-group of a unital C^*-algebra A, we previously viewed $\mathcal{U}_\infty(A)$ as the union of groups $\mathcal{U}_m(A)$ for $m \in \mathbb{N}$. For the purpose of this section, we equip

$\mathcal{U}_\infty(A)$ with a metric as follows: for $u, v \in \mathcal{U}_\infty(A)$, there exist $u', v' \in \mathcal{U}_k(A)$ for some k such that $u = \varphi^{(k)}(u')$ and $v = \varphi^{(k)}(v')$; then define a metric $d(u, v) = \|u' - v'\|_k$ on $\mathcal{U}_\infty(A)$. The metric is well-defined since the connecting maps $\mathcal{U}_n(A) \to \mathcal{U}_{n+1}(A)$ are metric-preserving and $\mathcal{U}_\infty(A)$ is a topological group with the topology induced by the metric described above.

For a general C^*-algebra A, we similarly take the inductive limit $\mathcal{V}_\infty(A)$ of $\mathcal{V}_m(A)$ under metric-preserving connecting maps. Consequently,

$$\mathcal{V}_\infty(A) = \{u \in \mathcal{U}_\infty(A^\dagger) : q(u) = 1\}, \tag{2.3.11}$$

where q is the matrix amplification of $q : A^\dagger \to \mathbb{C}$.

Then we have the following theorem, where we always choose $1_{A^\dagger}$ as the fixed point in $\mathcal{V}_\infty(A)$ when referring to its higher homotopy groups.

Theorem 2.3.11 (cf. [92, Prop 11.4.1])**.** *Let A be a C^*-algebra. Then for any $n \geq 0$,*

$$\pi_n(\mathcal{V}_\infty(A)) = \begin{cases} K_0(A) & \text{if } n \text{ is odd,} \\ K_1(A) & \text{if } n \text{ is even.} \end{cases} \tag{2.3.12}$$

In Chapter 3, we will provide a new proof for Theorem G, which says that under the regularity condition of $\mathcal{Z}$-stability, one can calculate the homotopy groups for $\mathcal{U}_\infty(A)$ by only calculating those of $\mathcal{U}(A)$. As a consequence, the computation of K-theory needs no matrix amplifications.

2.4 Traces and the Elliott invariant

Since the main focus of this thesis is on unital C^*-algebras, by a *trace* we will always mean a state $\tau : A \to \mathbb{C}$ satisfying the tracial condition $\tau(ab) = \tau(ba)$ for any $a, b \in A$. We denote the set of all traces on a unital C^*-algebra A by $T(A)$. This is a convex subset in the state space A^* of A and is compact in the weak* topology by unitality of A. If $A = C(X)$ for some compact Hausdorff space X, then every tracial state is given by $\tau : A \to \mathbb{C}$, $f \mapsto \int f d\mu$ for some Borel probability measure μ on X. Thus, traces are considered as non-commutative measures on A.

Notice that $A \to T(A)$ is a contravariant functor from C^*-**Alg** to the category of convex topological spaces. Since $T(A)$ is part of Elliott's invariant Ell, to make the entire invariant covariant, one can instead work with Aff $T(A)$, consisting of continuous real-valued affine functions on $T(A)$. By Kadison duality studied in [55], Aff $T(A)$ is canonically dual to $T(A)$. Indeed, any unital *-homomorphism $\phi : A \to$

B induces a unital positive linear map $\mathrm{Aff}\, T(\phi) : \mathrm{Aff}\, T(A) \to \mathrm{Aff}\, T(B)$ given by $\mathrm{Aff}\, T(\phi)(f)(\tau) = f(\tau \circ \phi)$ for any $f \in \mathrm{Aff}\, T(A)$ and $\tau \in T(B)$.

Each trace τ on A induces a map $\hat{\tau} : K_0(A) \to \mathbb{R}$. For any p, q in $M_n(A)$ for some $n \in \mathbb{N}$, define $\hat{\tau}([p]_0 - [q]_0) = (\tau \otimes \mathrm{Tr}_{M_n})(p - q)$, where Tr_{M_n} is the non-normalized trace for M_n. For instance, for a II_1 factor $\mathcal{M}$, the unique trace $\tau_{\mathcal{M}}$ induces an isomorphism $K_0(\mathcal{M}) \to \mathbb{R}$. The relation between $K_0(A)$ and $T(A)$ is thus encoded in the natural *paring map* $\rho_A : K_0(A) \to \mathrm{Aff}\, T(A)$ given by

$$\rho_A([p]_0 - [q]_0)(\tau) = \hat{\tau}([p]_0 - [q]_0). \tag{2.4.1}$$

Putting these ingredients together, the *Elliott invariant* appearing in the initial classification theorem (Theorem 1.3.2) of a unital C^*-algebra A is given by

$$\mathrm{Ell}(A) \coloneqq (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), \mathrm{Aff}\, T(A), \rho_A). \tag{2.4.2}$$

The positive cone $K_0(A)_+$ has been included in the Elliott invariant since its first appearance in Elliott's classification of AF-algebras in [28]. Let A be a C^*-algebra satisfying the assumptions of the stably finite part of the unital classification theorem (Theorem 1.3.2), then the order on $K_0(A)$ is induced by the pairing map. Thus, in the statement of the unital classification theorem, instead of using the Elliott invariant, one can work with the following invariant

$$KT_u(A) \coloneqq (K_0(A), [1_A]_0, K_1(A), \mathrm{Aff}\, T(A), \rho_A), \tag{2.4.3}$$

which does not explicitly keeps track of the order on $K_0(A)$.

Lastly, we include the range of invariant for the unital classification theorem in the stably finite setting (Theorem 1.3.2). We say that a tuple $(G_0, G_0^+, u, G_1, T, r)$ is a *weakly unperforated Elliott invariant* if it satisfies the following conditions:

(i) (G_0, G_0^+) is an ordered abelian group;

(ii) u is an *order unit* of (G_0, G_0^+), which means that for any $x \in G_0$, there exists $k \in \mathbb{N}$ such that $-ku \leq x \leq ku$. It follows that $\{g \in G_0^+ : g \leq u\}$ is a *scale* of (G_0, G_0^+), which is an upward directed, hereditary and full subset of (G_0, G_0^+);

(iii) G_1 is a countable abelian group;

(iv) T is a metrizable Choquet simplex;

(v) $r : T \to S(G_0)$ is a surjective affine map from T to the state space $S(G_0)$ of G_0;

(vi) the tuple is *weakly unperforated*, which means that for any $x \in G_0$, we have $x \in G_0^+$ if and only if $r(\tau)(x) > 0$, for all $\tau \in T$.

It is well-known that the Elliott invariant for any C^*-algebra covered by the unital classification theorem (Theorem 1.3.2) is a weakly unperforated Elliott invariant. Moreover, it is shown by the work of Gong, Lin and Niu that any weakly unperforated Elliott invariant can be realized as the Elliott invariant of a C*-algebra covered by Theorem 1.3.2 (see [39] for more details).

Theorem 2.4.1 ([39, Section 13])**.** *Let $(G_0, G_0^+, u, G_1, T, r)$ be a weakly unperforated Elliott invariant with $T \neq \emptyset$. Then there exists a unital, finite, separable, simple, and nuclear C^*-algebra A that is $\mathcal{Z}$-stable and satisfies the UCT, such that*

$$\mathrm{Ell}(A) \cong (G_0, G_0^+, u, G_1, T, r). \tag{2.4.4}$$

2.5 Total K-theory

Total K-theory consists of K-theory, K-theory with coefficients in $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$, and the natural connecting maps between these groups, which are called *Bockstein maps*. Total K-theory was first introduced by Schochet in [99]. The first explicit use of total K-theory in the classification program was in the work [23] of Dadarlat and Loring, as an obstruction in the classification of $A\mathcal{D}$-algebras. We follow the introduction of total K-theory in [9].

For $n \geq 2$ and $i \in \{0, 1\}$, we define the group $K_i(A; \mathbb{Z}/n\mathbb{Z})$ of A to be

$$K_i(A; \mathbb{Z}/n\mathbb{Z}) \coloneqq K_{1-i}(A \otimes \mathbb{I}_n), \tag{2.5.1}$$

where $\mathbb{I}_n$ is the dimension drop algebra

$$\mathbb{I}_n \coloneqq \{f \in C([0,1], M_n) : f(0) \in \mathbb{C}1_{M_n},\ f(1) = 0\}. \tag{2.5.2}$$

As noted in [9, Footnote 68], one can define $K_i(A; \mathbb{Z}/n\mathbb{Z}) \coloneqq K_i(A \otimes C_n)$ using any separable nuclear C^*-algebras C_n satisfying the UCT and having $K_*(C_n) = (\mathbb{Z}/n\mathbb{Z}, 0)$. In particular, we take C_n to be the suspension of $\mathbb{I}_n$ with $K_*(\mathbb{I}_n) = (0, \mathbb{Z}/n\mathbb{Z})$.

By the definition of $\mathbb{I}_n$, there is a short exact sequence

$$0 \longrightarrow C_0(0,1) \otimes M_n \longrightarrow \mathbb{I}_n \longrightarrow \mathbb{C} \longrightarrow 0. \tag{2.5.3}$$

This induces the 6-term short exact sequence after tensoring A,

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mu^{(n)}_{0,A}} & K_0(A;\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\nu^{(n)}_{0,A}} & K_1(A) \\ {\scriptstyle \times n}\uparrow & & & & \downarrow{\scriptstyle \times n} \\ K_0(A) & \xleftarrow[\nu^{(n)}_{1,A}]{} & K_1(A;\mathbb{Z}/n\mathbb{Z}) & \xleftarrow[\mu^{(n)}_{1,A}]{} & K_1(A), \end{array} \tag{2.5.4}$$

and we obtain natural Bockstein maps

$$\mu^{(n)}_{i,A}\colon K_i(A) \to K_i(A;\mathbb{Z}/n\mathbb{Z}) \text{ and } \nu^{(n)}_{i,A}\colon K_i(A;\mathbb{Z}/n\mathbb{Z}) \to K_{1-i}(A). \tag{2.5.5}$$

As noted in [99, Proposition 1.8], the 6-term exact sequence (2.5.4) collapses to the short exact sequence for $i \in \{0, 1\}$,

$$0 \longrightarrow K_i(A) \otimes \mathbb{Z}/n\mathbb{Z} \to K_i(A;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{Tor}(K_{1-i}(A), \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0, \tag{2.5.6}$$

where $\mathrm{Tor}(K_{1-i}(A), \mathbb{Z}/n\mathbb{Z})$ consists of those elements $x \in K_{1-i}(A)$ with $nx = 0$. The sequence splits unnatually (see [99, Proposition 2.4]).

The short exact sequence (2.5.6) allows us to compute $K_i(\cdot, \mathbb{Z}/n\mathbb{Z})$ for basic examples. Since the total K-theory for both $\mathcal{R}$ and $\mathcal{Q}$ vanishes except for the K_0-group, it explains that in Theorem A and Theorem 1.4.1, total K-theory is not needed for the classification.

Example 2.5.1. For a II_1 factor $\mathcal{M}$, we have $K_1(\mathcal{M}) = 0$ and $K_0(\mathcal{M}) = \mathbb{R}$. Thus, according to the short exact sequence (2.5.6), the total K-theory groups are all trivial, except for $K_0(\mathcal{M})$. Similarly, all K-theory groups with coefficients are trivial for the universal UHF-algebra $\mathcal{Q}$, except that $K_0(\mathcal{Q}) = \mathbb{Q}$. For a matrix algebra M_m, the K_1-groups with coefficients are all trivial. However, $K_0(M_n) = \mathbb{Z}$ and $K_0(M_m;\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$.

For $n, m \geq 2$, the remaining Bockstein maps

$$\kappa^{(nm,n)}_{i,A}\colon K_i(A;\mathbb{Z}/n\mathbb{Z}) \to K_i(A;\mathbb{Z}/nm\mathbb{Z}) \quad \text{and} \tag{2.5.7}$$

$$\kappa^{(n,nm)}_{i,A}\colon K_i(A;\mathbb{Z}/nm\mathbb{Z}) \to K_i(A;\mathbb{Z}/n\mathbb{Z}) \tag{2.5.8}$$

are induced by the canonical inclusions $\mathbb{I}_n \hookrightarrow \mathbb{I}_{nm}$ and $\mathbb{I}_{nm} \hookrightarrow \mathbb{I}_n \otimes M_m$ respectively. Bockstein maps fit into several commuting diagrams and exact sequences, see [14, Section 2.3] for a more detailed exposition. We give the definition of total K-theory.

Definition 2.5.2 (cf. [14, Definition 2.14])**.** Let A be a C^*-algebra. The *total K-theory* of A, $\underline{K}(A)$, is the collection of abelian groups $K_i(A)$ and $K_i(A;\mathbb{Z}/n\mathbb{Z})$ for $i = 0, 1$ and $n \geq 2$, together with the Bockstein maps

$$\big\{\mu_{i,A}^{(n)}, \nu_{i,A}^{(n)}, \kappa_{i,A}^{(n,nm)}, \kappa_{i,A}^{(nm,n)} : n, m \geq 2, i \in \{0,1\}\big\}. \tag{2.5.9}$$

A Λ-*morphism* $\underline{\alpha}\colon \underline{K}(A) \to \underline{K}(B)$ consists of homomorphisms

$$\alpha_i\colon K_i(A) \to K_i(B) \quad \text{ and } \quad \alpha_i^{(n)}\colon K_i(A;\mathbb{Z}/n\mathbb{Z}) \to K_i(B;\mathbb{Z}/n\mathbb{Z}) \tag{2.5.10}$$

for all $i \in \{0, 1\}$ and $n \geq 2$ that intertwine all the Bockstein operations. The collection of these Λ-morphisms is written as $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

In later classifications of morphisms into ultraproducts, we need to take appropriate separabilizations without losing too much total K-theoretical information. Thus, the following lemma is needed.

Lemma 2.5.3 ([14, Appendix B.4])**.** *If A is a separable C^*-algebra, then*

$$\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)) \cong \lim_{I_0 \text{ sep}} \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(I_0)), \tag{2.5.11}$$

where the inductive limit is indexed over separable subalgebras of I, directed by inclusions.

We end this section with the following lemma, involving computations of Bockstein maps. The lemma will be applied to an appropriate separabilization of the trace-kernel ideal in the proof of Theorem 7.2.1. For an abelian group G, we write $\operatorname{Tor}(G)$ for the subgroup of elements in G of finite order. A group G is *uniquely divisible* if for any $g \in G$ and $n \in \mathbb{Z}$ with $n \neq 0$, there exists a unique $h \in G$ such that $g = h^n$.

Lemma 2.5.4 ([14, Lemma 2.16])**.** *Let A be a C^*-algebra and let*

$$0 \longrightarrow I \xrightarrow{\ j\ } E \xrightarrow{\ q\ } D \longrightarrow 0 \tag{2.5.12}$$

be an extension of C^*-algebras such that $K_1(D) = 0$ and $K_0(D)$ is uniquely divisible. There is a natural isomorphism

$$\ker \operatorname{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(j)\big) \to \operatorname{Hom}\big(K_1(A)/\operatorname{Tor}\big(K_1(A)\big), \ker K_1(j)\big). \tag{2.5.13}$$

2.6 Completely positive maps and nuclearity

Completely positive maps are an important collection of maps between C^*-algebras. These maps share some nice properties with *-homomorphisms, for instance, they preserve positivity after matrix amplifications, even though they are generally not multiplicative. For an element $a \in M_n(A)$, we denote by $a_{i,j}$ the (i,j) entry of a and use the notation $a = [a_{i,j}]_{i,j}$. For C^*-algebras A and B, a linear map $\phi : A \to B$ is *completely positive* if for every n, the map

$$\phi^{(n)} : M_n(A) \to M_n(B), \quad [a_{i,j}]_{i,j} \mapsto [\phi(a_{i,j})]_{i,j} \tag{2.6.1}$$

is positive. We use c.p. to abbreviate "completely positive" for maps, u.c.p. for "unital completely positive" and c.p.c. for "completely positive contractive".

Example 2.6.1. The following maps are c.p., see [13, Example 1.5.2] for instance:

(i) A *-homomorphism is c.p.c. since its matrix amplications are *-homomorphisms;

(ii) More generally, a map ϕ of the form $\phi(\cdot) = V^*\pi(\cdot)V$ is c.p., where π is a *-homomorphism and V is an operator;

(iii) Any positive linear functional is c.p.

Actually, all completely positive maps into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ are of the form (ii) and this is known as the Stinespring's theorem. We include the following theorem for the unital case, and the non-unital case follows from [13, Remark 1.5.4].

Theorem 2.6.2 (Stinespring's Theorem, c.f. [13, Theorem 1.5.3])**.** *Let A be a unital C^*-algebra and $\phi : A \to B(\mathcal{H})$ be a c.p. map. Then there exist a Hilbert space $\hat{\mathcal{H}}$, a *-homomorphism $\pi : A \to B(\hat{\mathcal{H}})$ and an operator $V : \mathcal{H} \to \hat{\mathcal{H}}$ such that*

$$\phi(a) = V^*\pi(a)V, \quad a \in A. \tag{2.6.2}$$

In particular, $\|\phi\| = \|\phi(1)\| = \|V^*V\|$.

When either the domain or codomain C^*-algebra is a matrix algebra, we have the following useful characterization of when the map is completely positive.

Proposition 2.6.3 (c.f. [13, Proposition 1.5.12, Proposition 1.5.14])**.** *Let A be a C^*-algebra. Then the following statements are true:*

(i) A map $\phi : A \to M_n(\mathbb{C})$ is c.p. if and only if the linear functional $\hat{\phi} : M_n(A) \to \mathbb{C}$ defined by

$$\hat{\phi}([a_{i,j}]_{i,j}) = \sum_{i,j=1}^{n} \phi(a_{i,j})_{i,j} \tag{2.6.3}$$

is positive. Moreover, $\phi \mapsto \hat{\phi}$ is a bijective correspondence between c.p. maps from A to $M_n(\mathbb{C})$ and positive linear functionals on $M_n(A)$.

(ii) A map $\phi : M_n(\mathbb{C}) \to A$ is c.p. if and only if $[\phi(e_{i,j})]_{i,j}$ is positive in $M_n(A)$, where $(e_{i,j})_{i,j}$ is the matrix unit of $M_n(\mathbb{C})$. Moreover, $\phi \mapsto [\phi(e_{i,j})]_{i,j}$ is a bijective correspondence between c.p. maps from $M_n(\mathbb{C})$ to A and $M_n(A)_+$.

Now we recall the definition of nuclear C^*-algebras, based on the nuclearity of the identity map on C^*-algebras.

Definition 2.6.4. A contractive map $\theta : A \to B$ is *nuclear* if there exist c.p.c. maps $\phi_\lambda : A \to M_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : M_{n_\lambda}(\mathbb{C}) \to B$ such that $\psi_\lambda \circ \phi_\lambda \to \theta$ in the point-norm topology:

$$\|(\psi_\lambda \circ \phi_\lambda)(a) - \theta(a)\| \to 0, \quad a \in A. \tag{2.6.4}$$

A C^*-algebra A is *nuclear* if id_A is a nuclear map.

There are straightforward reformulations for contractive nuclear maps. In particular, the first characterization shows that nuclearity is a local property and there is little difference between the separable and non-separable settings.

Lemma 2.6.5 (c.f. [13, Exercise 2.1.1, Exercise 2.1.2])**.** *Let A, B be C^*-algebras and $\theta : A \to B$ be a contractive map. Then the following statements are equivalent:*

(i) θ is nuclear;

(ii) for any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there exist $n \in \mathbb{N}$, c.p.c. maps $\phi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to B$ such that for any $a \in \mathcal{F}$,

$$\|\theta(a) - (\psi \circ \phi)(a)\| < \epsilon; \tag{2.6.5}$$

(iii) there exist c.p.c. maps $\phi_\lambda : A \to F_\lambda$ and $\psi_\lambda : F_\lambda \to B$ such that F_λ are finite dimensional algebras and $\psi_\lambda \circ \phi_\lambda \to \theta$ in the point-norm topology.

Sketch Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious. The implication (ii) $\Rightarrow$ (i) is true by indexing over a net of $(\mathcal{F}, \epsilon)$ for all finite subsets $\mathcal{F}$ in A and $\epsilon > 0$. For (iii) $\Rightarrow$ (i), we embed finite dimensional algebras F_λ into matrices $M_{n_\lambda}(\mathbb{C})$ with big enough size to obtain c.p.c. maps from A to $M_{n_\lambda}(\mathbb{C})$. Composing ψ_λ with conditional expectations $M_{n_\lambda}(\mathbb{C}) \to F_\lambda$ gives c.p.c. maps from $M_{n_\lambda}(\mathbb{C})$ to A. □

By Lemma 2.6.5, all finite-dimensional C^*-algebras are nuclear. Commutative C^*-algebras are also fundamental examples of nuclear C^*-algebras. The proof involves a partition of unity argument on a locally compact Hausdorff space, see [13, proposition 2.4.2] for instance.

Example 2.6.6 (c.f. [13, proposition 2.4.2])**.** Commutative C^*-algebras are nuclear.

Many C^*-algebras that arise naturally are nuclear. We collect a list of the operations under which nuclearity is preserved.

Theorem 2.6.7 (c.f. [6, Theorem 15.8.2])**.** *The class of nuclear C^*-algebras includes all type I C^*-algebras and is closed under stable isomorphism, taking inductive limits, ideals, quotients, extensions and tensor products.*

Among these operations, the proof of nuclearity passing to quotients is particularly hard. The only known proof uses the consequence of Connes' theorem, that a C^*-algebra A is nuclear if and only if A^{**} is semidiscrete. If $I \lhd A$ is an ideal and A is nuclear, then $A^{**} \cong I^{**} \oplus (A/I)^{**}$ is semidiscrete. This implies that $(A/I)^{**}$ is semidiscrete and thus hyperfinite by Connes' theorem. Lastly, hyperfiniteness of $(A/I)^{**}$ provides finite dimensional approximations for A/I.

We include a discussion regarding the uniform norm control for c.p.. maps ϕ_λ and ψ_λ in Definition 2.6.4. It turns out that if a bounded map is approximated in the point-norm topology by maps factoring through matrix algebras via c.p. maps, then we can obtain norm control on both c.p. maps. The key idea can be found in [13, Lemma 2.2.5, Proposition 2.2.6] and an explicit statement appeared in [33, Lemma 2.3], for instance. We include a proof of the following lemma for completeness.

Lemma 2.6.8. *Let A, B be C^*-algebras and $\theta : A \to B$ be a map with bounded norm. Suppose that there exist c.p. maps $\phi_\lambda : A \to M_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : M_{n_\lambda}(\mathbb{C}) \to B$ such that*

$$\|(\psi_\lambda \circ \phi_\lambda)(a) - \theta(a)\| \to 0, \quad a \in A. \tag{2.6.6}$$

Then there exists a net $(m_i)_i$ of natural numbers, c.p.c. maps $\alpha_i : A \to M_{m_i}(\mathbb{C})$ and c.p. maps $\beta_i : M_{m_i}(\mathbb{C}) \to B$ such that $\|\beta_i\| \leq \|\theta\|$ and

$$\|(\beta_i \circ \alpha_i)(a) - \theta(a)\| \to 0, \quad a \in A. \tag{2.6.7}$$

Proof. Suppose that θ is approximated in the point-norm topology by compositions of c.p. maps $\phi_\lambda : A \to M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \to B$. We first consider the case where A is unital. For each λ, the element $x_\lambda := \phi_\lambda(1_A)$ generates a hereditary C^*-subalgebra of

M_{n_λ}, which is a matrix algebra M_{m_λ} of possibly smaller size. Take $\tilde{\phi}_\lambda : A \to M_{m_\lambda}$ to be the corestriction of ϕ_λ to M_{m_λ} and $\tilde{\psi}_\lambda : M_{m_\lambda} \to B$ the restriction of ψ_λ to M_{m_λ}, we have $\psi_\lambda \circ \phi_\lambda = \tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda$. Since $\tilde{\phi}_\lambda(1_A) = x_\lambda$ generates M_{m_λ} and thus is invertible, $\alpha_\lambda : A \to M_{m_\lambda}$ defined by

$$\alpha_\lambda(a) \coloneqq x_\lambda^{-1/2} \tilde{\phi}_\lambda(a) x_\lambda^{-1/2}, \quad a \in A \tag{2.6.8}$$

is a u.c.p. map and thus contractive. Take c.p. maps $\beta_\lambda : M_{m_\lambda} \to B$ defined by

$$\beta_\lambda(b) \coloneqq \tilde{\psi}_\lambda(x_\lambda^{1/2} b x_\lambda^{1/2}), \quad b \in M_{m_\lambda}. \tag{2.6.9}$$

Then $\beta_\lambda \circ \alpha_\lambda = \tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda = \psi_\lambda \circ \phi_\lambda$ converges to θ in the point norm topology and

$$\|\beta_\lambda\| = \|\beta_\lambda(1)\| = \|\beta_\lambda \circ \alpha_\lambda(1_A)\| \to \|\theta(1_A)\| = \|\theta\|. \tag{2.6.10}$$

in particular. By perturbing β_λ slightly, we can make $\|\beta_\lambda\| = \|\theta\|$ and maps $\beta_\lambda \circ \alpha_\lambda$ still approximate θ in the point-norm topology. This concludes the unital case.

If A is non-unital, take an approximate unit $(e_i)_i$ of A consisting of positive contractions. For each i, define a map $\theta_i : A^\dagger \to B$ by $\theta_i(a+\mu 1) = \theta(e_i a e_i + \mu e_i^2)$. Then θ is approximated by $\theta_i|_A$ in the point-norm topology and $\|\theta_i\| = \|\theta_i(1)\| = \|\theta(e_i^2)\| \leq \|\theta\|$. Define c.p. maps $\phi_{\lambda,i} : A^\dagger \to M_{n_\lambda}$ by $\phi_{\lambda,i}(a + \mu 1) = \phi_\lambda(e_i a e_i + \mu e_i^2)$. For each i, the map θ_i is further approximated by $\psi_\lambda \circ \phi_{\lambda,i}$ in the point-norm topology. By what we have proven for the unital case, there exists $(m_{\lambda,i})_{\lambda,i}$ of natural numbers, c.c.p. maps $\alpha_{\lambda,i} : A^\dagger \to M_{m_{\lambda,i}}$ and maps $\beta_{\lambda,i} : M_{m_{\lambda,i}} \to B$ such that $\|\beta_{\lambda,i}\| \leq \|\theta_i\| \leq \|\theta\|$ and $\beta_{\lambda,i} \circ \alpha_{\lambda,i} = \psi_\lambda \circ \phi_{\lambda,i}$. Then $\alpha_{\lambda,i}|_A$ and $\beta_{\lambda,i}$ gives the desired approximation for θ. □

The following equivalent characterization of nuclearity is a direct consequence, where the condition that the c.p. maps used for approximations are contractive can be dropped.

Corollary 2.6.9. *A C^*-algebra A is nuclear if and only if id_A is approximated in the point-norm topology by maps factoring through matrix algebras by c.p. maps, which means there exist c.p. maps $\phi_\lambda : A \to M_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : M_{n_\lambda}(\mathbb{C}) \to B$ such that*

$$\|(\psi_\lambda \circ \phi_\lambda)(a) - a\| \to 0, \quad a \in A. \tag{2.6.11}$$

Lastly, we focus on a special class of orthogonal-preserving completely positive maps. Let A, B be C^*-algebras and $\varphi : A \to B$ be a c.p. map. We say that φ *has order zero* if for any positive elements a, b in A with $ab = 0$, we have $\varphi(ab) = 0$. Such maps are introduced by Winter and Zacharias in [114, Definition 2.3]. Since then, c.p. order zero maps frequently play an important role in the classification program for C^*-algebras, for instance, in the definition of nuclear dimension (see [115]) and in the structure theory of dimension drop algebras (see [93, Proposition 2.5]). The following theorem captures the structure theory for c.p. order zero maps.

Theorem 2.6.10 ([114, Theorem 3.3])**.** *Let A, B be C^*-algebras with A unital, and $\varphi : A \to B$ be a c.p. order zero map. Take $h = \varphi(1_A)$ and $C = C^*(\varphi(A))$. Then there exists a *-homomorphism $\pi_\varphi : A \to C^{**} \cap \{h\}'$ such that*

$$\pi_\varphi(a)h = \varphi(a), \quad a \in A. \tag{2.6.12}$$

In particular, φ is *-preserving and $\varphi(a)\varphi(b) = \varphi(ab)h$ for any $a, b \in A$.

2.7 Jiang-Su algebra and $\mathcal{Z}$-stability

The Jiang-Su algebra $\mathcal{Z}$ plays a crucial role in the Elliott classification program and serves as the C^*-algebraic analog of $\mathcal{R}$, and the stably finite analog of $\mathcal{O}_\infty$. It was initially constructed in [54] as an inductive limit of the so-called dimension drop algebras, together with delicately defined unital connecting maps. For coprime integers p, q, the *prime dimension drop algebra* is defined to be

$$\mathcal{Z}_{p,q} = \{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q\}. \tag{2.7.1}$$

The K-theory for such $\mathcal{Z}_{p,q}$ can be calculated by the six-term exact sequence in Section 2.3.3, and gives $K_0(\mathcal{Z}_{p,q}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}_{p,q}) = 0$, where the equivalence class of the unit of $\mathcal{Z}_{p,q}$ generates the K_0-group. Thus by continuity of K-theory in Proposition 2.3.5, we have $K_0(\mathcal{Z}) = \mathbb{Z}$ and $K_1(\mathcal{Z}) = 0$. The unital connecting maps are defined explicitly to be approximately trace-collapsing so that the resulting inductive limit is simple and monotracial. Jiang and Su also classified simple inductive limits of dimension drop algebras in [54, Theorem 6.2], which ensures that $\mathcal{Z}$ is independent of the multitude of choices made throughout the construction.

Since $\mathcal{Z}$ shares the same K-theory and trace space as $\mathbb{C}$, by the Künneth formula, the K-theory and traces of A and $A \otimes \mathcal{Z}$ are the same for a simple, separable, unital and nuclear C^*-algebra A. Thus to give a complete classification using only the Elliott invariant, it will be necessary to restrict to the class of C^*-algebras that tensorially absorb $\mathcal{Z}$.

Definition 2.7.1. A C^*-algebra A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z} \cong A$.

In many examples, $\mathcal{Z}$-stability is difficult to verify. The following equivalent characterizations for unital $\mathcal{Z}$-stable C^*-algebras are useful in practice. A C^*-algebra A embeds into its ultrapower A_ω diagonally. The *central sequence algebra* of A is defined to be the relative commutant $A_\omega \cap A'$ of A in A_ω.

Theorem 2.7.2 ([105, Theorem 2.2]). *Let A be a unital and separable C^*-algebra. The following statements are equivalent:*

(i) A is $\mathcal{Z}$-stable;

(ii) there exists a unital *-homomorphism $\mathcal{Z} \hookrightarrow A_\omega \cap A'$;

(iii) there exists a unital *-homomorphism $\mathcal{Z}_{n,n+1} \to A_\omega \cap A'$ for some $n \in \mathbb{N}$;

(iv) there exists a unital *-homomorphism $\mathcal{Z}_{n,n+1} \to A_\omega \cap A'$ for any $n \in \mathbb{N}$.

We also include the following criterion for obtaining unital embeddings of $\mathcal{Z}_{n,n+1}$. This criterion is used in Matui and Sato's argument to obtain $\mathcal{Z}$-stability.

Lemma 2.7.3 ([93, Proposition 5.1]). *Let A be a unital C^*-algebra. If there are elements $v, s_1, \cdots, s_n \in A$ such that*

$$s_1^* s_1 = s_i s_i^*, \quad s_i^* s_i \perp s_j^* s_j, \quad v^* v = 1_A - \sum_{k=1}^{n} s_k^* s_k, \quad vv^* s_1^* s_1 = vv^*, \tag{2.7.2}$$

for all i and j with $i \neq j$, then there exists a unital *-homomorphism $\mathcal{Z}_{n,n+1} \to A$.

Among different constructions of $\mathcal{Z}$, the Rørdam-Winter picture of $\mathcal{Z}$ introduced in [93], is particularly useful and will be needed in Chapter 3. The building blocks are replaced by the so-called generalized dimension drop algebras. For infinite coprime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$, the *generalized prime dimension drop algebra* $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is defined to be

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \big\{ f \in C([0,1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) : f(0) \in M_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}, f(1) \in 1_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \big\}. \tag{2.7.3}$$

These algebras have some properties that dimension drop algebras lack. For instance, it is shown in [47] that $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is $\mathcal{Z}$-stable, since all fibers tensorially absorb $\mathcal{Z}$. Instead of explicitly defining connecting maps, this approach imposes an abstract condition for the trace-collapsing nature of connecting maps. A unital endomorphism φ on a unital C^*-algebra A is said to be *trace-collapsing* if $\tau \circ \varphi = \tau' \circ \varphi$ for any $\tau, \tau' \in T(A)$. The following theorem provides an alternative picture of $\mathcal{Z}$.

Theorem 2.7.4 ([93, Theorem 3.4]). *Let $\mathfrak{p}, \mathfrak{q}$ be infinite coprime supernatural numbers. Then there exists a trace-collapsing unital endomorphism φ on $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. For any such φ, the Jiang-Su algebra $\mathcal{Z}$ is *-isomorphic to the stationary inductive limit of the sequence*

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\varphi} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\varphi} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\varphi} \cdots . \tag{2.7.4}$$

As noted in the introduction, this viewpoint of $\mathcal{Z}$ was used in the alternative proof of K_1-injectivity of unital $\mathcal{Z}$-stable C^*-algebras in [14, Theorem 4.8]. The result quickly extends to the non-unital setting, which recaptures Jiang's K_1-injectivity result in [53, Theorem 2]. Similarly, we provide a new proof of K_1-surjectivity and K-stability for $\mathcal{Z}$-stable C^*-algebras using this picture.

2.8 Cuntz semigroup and strict comparison

The Cuntz semigroup is an invariant for C^*-algebras, which resembles the Murray-von Neumann semigroup consisting of equivalence classes of projections, as defined in Section 2.3.2. Instead of equivalence classes of projections, the Cuntz semigroup is constructed from Cuntz equivalence classes of positive elements.

Since every C^*-algebra contains an abundance of positive elements, the Cuntz semigroup has the potential to carry a wealth of information about the algebra. This makes the Cuntz semigroup particularly useful in cases where K-theory provides limited insight, such as for projectionless C^*-algebras like the Jiang-Su algebra $\mathcal{Z}$. Moreover, the Cuntz semigroup encodes the ideal structure of C^*-algebras, whereas a simple C^*-algebra and a non-simple C^*-algebra could share the same K-theory. On the other hand, the Cuntz semigroup is notoriously difficult to compute due to its intricate structure.

In most situations, extracting partial structural information of the C^*-algebra from properties of the Cuntz semigroup has proven to be highly useful. For instance, the strict comparison property is equivalent to a property of the Cuntz semigroup called almost unperforation, which will be explained in Section 2.8.2. Most of the theorems in this section are collected from [34] and [88], unless otherwise stated.

2.8.1 Cuntz comparison of positive elements

To begin with, we define the Cuntz equivalence relation for positive elements. It is straightforward to verify that this is indeed an equivalence relation. For elements $a, b \in A$ and $\epsilon > 0$, we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. Moreover, we denote by A_+ the set of positive elements in A.

Definition 2.8.1. Let A be a C^*-algebra and $a, b \in A_+$. We say that a is *Cuntz subequivalent* to b in A, denoted by $a \precsim b$, if for any $\epsilon > 0$, there exists $r \in A$ such that $a \approx_\epsilon rbr^*$. They are *Cuntz equivalent*, denoted by $a \sim b$, if $a \precsim b$ and $b \precsim a$.

To provide some intuition for the Cuntz equivalence relation, we illustrate what the concept means for commutative C^*-algebras. For a function $f \in C_0(X)$, where X is a locally compact Hausdorff space, we denote its *open support* by

$$\mathrm{supp}_o(f) = \{x \in X : f(x) \neq 0\}. \tag{2.8.1}$$

The following proposition shows that the Cuntz subequivalence for positive functions is equivalent to the containment of open supports.

Proposition 2.8.2. *Let X be a locally compact Hausdorff space and $f, g \in C_0(X)$ be positive functions. Then $f \precsim g$ in $C_0(X)$ if and only if $supp_o(f) \subseteq supp_o(g)$.*

Proof. Suppose that $\mathrm{supp}_o(f) \subseteq \mathrm{supp}_o(g)$. For $\epsilon > 0$, consider the compact subset $V = \{x \in X : f(x) \geq \epsilon/2\}$ of X. Then $g(x) > 0$ for any $x \in V$ and by compactness of V, there exists $\delta > 0$ such that $g(V) > \delta$. Taking the open subset $U = \{x \in X : g(x) \geq \delta/2\}$ of X containing V, there exists a positive function $h \in C_0(X)$ such that $h|_V = 1$ and $h|_{U^c} = 0$ by Urysohn's lemma. Define a positive function $r \in C_0(X)$ by

$$r(x) = \begin{cases} \frac{f(x)h(x)}{g(x)} & x \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8.2}$$

Then $f \approx_\epsilon r^{1/2} g r^{1/2}$ and thus $f \precsim g$. The other implication is straightforward. □

Combining Proposition 2.8.2 and standard functional calculus arguments, the first part of the following lemma is easily derived. For any element $a \in A$, we denote by $\mathrm{sp}(a)$ the spectrum of a. When a is a positive element, then $\mathrm{sp}(a) \subseteq [0, \infty)$.

Lemma 2.8.3. *Let A be a C^*-algebra and $a \in A_+$. If f and g are positive continuous functions on $sp(a)$ and $f \precsim g$ in $C_0(sp(a))$, then $f(a) \precsim g(a)$ in A. Thus for any $b \in A$, we have $bb^* \sim b^*b$.*

Proof. For the second statement, since $f(x) = x$ and $g(x) = x^2$ are positive continuous functions on the spectrum of any positive elements with the same open supports, then

$$bb^* \sim (bb^*)^2 = b(b^*b)b^* \precsim b^*b. \tag{2.8.3}$$

One can prove $b^*b \precsim bb^*$ in a similar fashion. □

In what follows, we collect some technical results regarding Cuntz comparison. Recall that a C^*-subalgebra B of A is said to be *hereditary* if for $a \in A_+$ and $b \in B_+$, then $a \leq b$ implies that $a \in B$. Given $a \in A_+$, then $\overline{aAa}$ is the minimal hereditary C^*-subalgebra of A containing a, with an approximate unit $(a^{1/n})_n$.

Lemma 2.8.4 (c.f. [34, Proposition 2.4]). *Let A be a C^*-algebra and $a, b \in A_+$. Then $a \in \overline{bAb}$ implies $a \precsim b$. In particular, if $a \leq b$ then $a \precsim b$.*

Given any $\epsilon > 0$, we define a continuous positive function $(x - \epsilon)_+$ on $[0, \infty)$ by $(x-\epsilon)_+(t) = \max\{t-\epsilon, 0\}$. For any $a \in A_+$, we use the notation $(a-\epsilon)_+$ for $(x-\epsilon)_+(a)$, which is usually referred to as the ϵ-*cut down* of a. Then $(a - \epsilon)_+ \leq a$, which implies in particular that $(a - \epsilon)_+ \precsim a$ by Lemma 2.8.4. The following important lemma shows that in general if two positive elements are ϵ-close, then an ϵ-cut down of one element is Cuntz-dominated by the other.

Lemma 2.8.5 (c.f. [34, Lemma 2.5]). *Let A be a C^*-algebra and $a, b \in A_+$. If $a \approx_\epsilon b$ for some $\epsilon > 0$, then there exists a contraction $r \in A$ such that $(a - \epsilon)_+ = rbr^*$. In particular $(a - \epsilon)_+ \precsim b$.*

The following technical result provides equivalent characterizations of Cuntz subequivalence, which offers extra flexibility by allowing small cut-downs.

Lemma 2.8.6 ([88, Proposition 2.4]). *Let A be a C^*-algebra and $a, b \in A_+$. Each of the following statements is equivalent to $a \precsim b$:*

(i) For every $\epsilon > 0$, we have $(a - \epsilon)_+ \precsim b$;

(ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that $(a - \epsilon)_+ \precsim (b - \delta)_+$;

(iii) For every $\epsilon > 0$, there exists $\delta > 0$ and $x \in A$ such that $(a - \epsilon)_+ = x^*x$ and xx^* is in the hereditary C^*-subalgebra generated by $(b - \delta)_+$;

(iv) For every $\epsilon > 0$, there exists $\delta > 0$ and $r \in A$ such that $(a - \epsilon)_+ = r(b - \delta)_+r^*$.

We also observe that Cuntz subequivalence between projections coincides with Murray von Neumann subequivalence. For two projections p and q in A, recall that p is *Murray von Neumann subequivalent* to q, denoted by $p \leq_0 q$, if there exists $v \in A$ such that $p = v^*v$ and $vv^* \leq q$.

Lemma 2.8.7 (c.f. [34, Lemma 2.8]). *Let A be a C^*-algebra and $p, q \in A$ be projections. Then $p \leq_0 q$ if and only $p \precsim q$.*

However, it is worth noticing that the Cuntz equivalence between projections does not necessarily imply the Murray von Neumann equivalence, since $p \leq_0 q$ and $q \leq_0 p$ do not always imply $p \sim_0 q$. For instance, all non-zero projections in a simple purely infinite C^*-algebra are Cuntz equivalent, but the C^*-algebra might have a non-zero K_0-group. If the C^*-algebra is stably finite, then $p \leq_0 q$ and $q \leq_0 p$ imply $p \sim_0 q$ (see [34, Footnote 2] for instance).

In general, elements witnessing Cuntz subequivalences might have arbitrary norms. However, we can obtain some norm control using functional calculus techniques. For $\epsilon > 0$, we define a positive-valued continuous function on $\mathbb{R}_{\geq 0}$ by

$$f_\epsilon(t) = \begin{cases} t/\epsilon & 0 \leq t \leq \epsilon, \\ 1 & t \geq \epsilon. \end{cases} \tag{2.8.4}$$

For any positive element $a \in A$, then $f_\epsilon(a)(a-\epsilon)_+ = (a-\epsilon)_+$. The technique for the following lemma is standard, appearing in [70] for instance, and we include a short proof for completeness.

Lemma 2.8.8. *Let A be a C^*-algebra, $a, b, c \in A$ be positive contractions and $\epsilon > 0$. If $a \precsim b$ and $cb = b$, then there exists a contraction $x \in A$ such that $a \approx_\epsilon xcx^*$. Thus if $a \precsim (b-\delta)_+$ for some $\delta > 0$, there exists a contraction $y \in A$ such that $a \approx_\epsilon y f_\delta(b) y^*$.*

Proof. If $a \precsim b$, then in particular $(a-\epsilon/2)_+ \precsim b$ and there exists some $r \in A$ such that $rbr^* \approx_{\epsilon/2} (a-\epsilon/2)_+$. Take $x = rb^{1/2}$, which is a contraction in A since

$$\|xx^*\| = \|rbr^*\| \leq \|(a-\epsilon/2)_+\| + \epsilon/2 \leq 1. \tag{2.8.5}$$

Moreover, since $cb = b$, then $xcx^* = rbr^* \approx_{\epsilon/2} (a-\epsilon/2)_+ \approx_{\epsilon/2} a$. Then the second statement follows since $a \precsim (b-\delta)_+$ for some $\delta > 0$ and $f_\delta(b)(b-\delta)_+ = (b-\delta)_+$. □

2.8.2 Cuntz semigroup and strict comparison

Now we give the definition of the Cuntz semigroup of C^*-algebras.

Definition 2.8.9. Let A be a C^*-algebra. The *Cuntz semigroup* of A is defined as

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+ / \sim, \tag{2.8.6}$$

where $\sim$ is the Cuntz equivalence relation. For $a \in (A \otimes \mathcal{K})_+$, we denote by $[a]$ its Cuntz equivalence class in $\mathrm{Cu}(A)$.

Remark 2.8.10. Computing the Cuntz semigroups even for commutative C^*-algebras is difficult in general. In Proposition 2.8.2, we characterize the Cuntz-subequivalence of continuous functions on X, but the Cuntz-subequivalence for elements in matrix amplifications is much more complicated. There has only been a partial computation of the Cuntz semigroup of $C(X)$ with finite-dimensional spectrum in [86] (the computation is complete when $\dim(X) \leq 3$).

There is a natural partial order on $\mathrm{Cu}(A)$ defined by setting $[a] \leq [b]$ if $a \precsim b$ in $A \otimes \mathcal{K}$. We can define an addition on $\mathrm{Cu}(A)$ by $[a] + [b] = [a \oplus b]$, with which $\mathrm{Cu}(A)$ is an abelian monoid, with $[0]$ being the neutral element. Moreover, the addition and the order are compatible in $\mathrm{Cu}(A)$, in the sense that $[a_1] \leq [b_1]$ and $[a_2] \leq [b_2]$ implies $[a_1] + [a_2] \leq [b_1] + [b_2]$. Thus $\mathrm{Cu}(A)$ is an abelian positively ordered monoid.

By Lemma 2.8.7, there is a semigroup map from the Murray von Neumann semigroup $V(A)$ to $\mathrm{Cu}(A)$. Recall that an *order-embedding* $\iota : S \to T$ between ordered sets is a map satisfying $\iota(s) \leq \iota(s')$ if and only if $s \leq s'$.

Lemma 2.8.11 (cf. [34, Corollary 3.4])**.** *Let A be a C^*-alegbra. Then there is a natural semigroup map $\iota : V(A) \to \mathrm{Cu}(A)$. If A is stably finite, then ι is an order embedding.*

In addition to the ordered semigroup structure for Cuntz semigroups, they have more interesting internal structure. For instance, the following was shown in [18].

Theorem 2.8.12 (cf. [34, Theorem 3.8])**.** *Let A be a C^*-algebra. Then every increasing sequence in $\mathrm{Cu}(A)$ has a supremum.*

As mentioned previously, the Cuntz semigroup encodes the ideal structure of C^*-algebras. Let A be a C^*-algebra, then an *ideal* $\mathcal{I}$ in $\mathrm{Cu}(A)$ is a submonoid in $\mathrm{Cu}(A)$, which is closed under suprema of increasing sequences and is *hereditary*, in the sense that $a \leq b$ and $b \in \mathcal{I}$ implies that $a \in \mathcal{I}$.

Lemma 2.8.13 (cf. [34, Lemma 5.2])**.** *Let $I \lhd A$ be an ideal in a C^*-algebra A, with the canonical inclusion $\iota : I \to A$. For positive elements $a, b \in I$, it follows that $a \precsim b$ in I if and only if $a \precsim b$ in A. Moreover,*

$$\mathrm{Cu}(\iota) : \mathrm{Cu}(I) \to \mathrm{Cu}(A) \tag{2.8.7}$$

is an order embedding, and its image is an ideal in $\mathrm{Cu}(A)$.

We introduce the following important regularity property for Cuntz semigroups.

Definition 2.8.14. Let A be a C^*-algebra. Then $\mathrm{Cu}(A)$ is *almost unperforated* if for any $x, y \in \mathrm{Cu}(A)$ satisfying $(n+1)x \le ny$ for some $n \in \mathbb{N}$, then $x \le y$.

We get the following easy consequence of Lemma 2.8.13.

Lemma 2.8.15. *Let $I \lhd A$ be an ideal of a C^*-algebra A. If $\mathrm{Cu}(A)$ is almost unperforated, then $\mathrm{Cu}(I)$ is almost unperforated.*

The following theorem was proved by Rørdam, and provides a nice obstruction when searching for examples of C^*-algebras that are not $\mathcal{Z}$-stable, for instance, the counterexample of Elliott's initial classification conjecture constructed by Toms in [103].

Theorem 2.8.16 ([91, Theorem 4.5])**.** *Let A be a $\mathcal{Z}$-stable C^*-algebra. Then $\mathrm{Cu}(A)$ is almost unperforated.*

For a unital simple C^*-algebra A, the Cuntz semigroup being almost unperforated is equivalent to saying that the order on $\mathrm{Cu}(A)$ is largely determined by functionals on $\mathrm{Cu}(A)$. A *functional* is an order-preserving, additive and lower-semicontinuous map $f : \mathrm{Cu}(A) \to [0, \infty]$ such that $f([1_A]) = 1$ and $f([0]) = 0$.

Such functionals are in one-to-one correspondence with *quasitraces* on A, see [34, Theorem 6.9] for instance. A *quasitrace* τ on A is a function $\tau : A \to \mathbb{C}$ satisfying

(i) $\tau(ab) = \tau(ba)$ for any $a, b \in A$;

(ii) τ is linear on commutative C^*-subalgebras of A;

(iii) $\tau(a + ib) = \tau(a) + i\tau(b)$ for $a, b \in A_{sa}$.

It is a major open problem whether every quasitrace is a trace. By a deep result of Haagerup in [43], this is the case for exact C^*-algebras. In particular, quasitraces are traces for nuclear C^*-algebras, which are the classes of interest in this thesis.

For any quasitrace τ on A, one can associate a functional $d_\tau : \mathrm{Cu}(A) \to [0, \infty]$ by

$$d_\tau([a]) = \lim_{n \to \infty} \tau(a^{1/n}) \tag{2.8.8}$$

for any $a \in (A \otimes \mathcal{K})_+$. Now we give the definition of strict comparison for simple and unital C^*-algebras.

Definition 2.8.17. Let A be a unital, simple and separable C^*-algebra. We say that A has *strict comparison (of positive elements by quasitraces)* if whenever $a, b \in (A \otimes \mathcal{K})_+$ are nonzero and satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in QT(A)$, then $a \precsim b$.

By the correspondence between quasitraces and functionals on $\mathrm{Cu}(A)$ mentioned above, we get the following theorem.

Theorem 2.8.18 (cf. [79, Remark 2.4])**.** *Let A be a unital and simple C^*-algebra. Then A has strict comparison if and only if* $\mathrm{Cu}(A)$ *is almost unperforated.*

In particular, Theorem 2.8.16 asserts that $\mathcal{Z}$-stable C^*-algebras have strict comparison. This provides an answer to part of the Toms-Winter conjecture on the equivalence of three seemingly unrelated regularity properties.

Conjecture 2.8.19 ([106], [113, Conjecture 5.2])**.** *Let A be a simple, separable, unital, nuclear and non-elementary C^*-algebra. The following are equivalent:*

(i) A has finite nuclear dimension;

(ii) A is $\mathcal{Z}$-stable;

(iii) A has strict comparison.

The nuclear dimension is the noncommutative covering dimension defined in [115]. Most of the implications are proven, see [14, Example 1.1.4] for a list of known results. The only remaining implication is from strict comparison to $\mathcal{Z}$-stability. A first breakthrough on this implication appeared in the work of Matui and Sato in 2012, through the so-called *property (SI)*, which will be the main topic of Chapter 9.

Lastly, we include a result regarding traces for ultraproducts. In general, an ultraproduct of a sequence of unital and monotracial C^*-algebras is not necessarily monotracial. However, this is the case if we assume strict comparison. A *limit trace* τ on $\prod_\omega A_n$ is given by a sequence of traces τ_n on A_n for each $n \in \mathbb{N}$, where $\tau((a_n)_n) = \lim_\omega \tau_n(a_n)$ for any representing sequence $(a_n)_n$. The set of limit traces on $\prod_\omega A_n$ is denoted by $T_\omega(\prod_\omega A_n)$.

Theorem 2.8.20 ([78, Theorem 1.2])**.** *Let $(A_n)_n$ be a sequence of unital and simple C^*-algebras with strict comparison. Then $T(\prod_\omega A_n) = \overline{T_\omega(\prod_\omega A_n)}^{w^*}$.*

2.9 Multiplier algebras

By a *unitization* of a C^*-algebra A, we mean a unital C^*-algebra that contains A as an essential ideal, that is to say, A intersects non-trivially with every nonzero ideal of the unitization. We denote the minimal unitization of A by $\tilde{A}$, and in particular, $A \cong \tilde{A}$ when A is unital. In the commutative setting, the minimal unitization of

$C_0(X)$ for a locally compact Hausdorff space X is $C(X^*)$, where X^* is the one-point compactification of X.

In this section, we focus on the multiplier algebra $\mathcal{M}(A)$, which serves as the maximal unitization of A. There are various ways to view a multiplier algebra, and we explore some of these equivalent definitions and their connections. After illustrating several examples and properties, we recall an important topology on the multiplier algebra, the so-called *strict topology*, which is the appropriate topology under which elements of $\mathcal{M}(A)$ can be approximated by elements from A. More details on the results presented in this section can be found in [110, Chapter 2] for instance.

2.9.1 Multiplier algebras and examples

The first characterization of the multiplier algebra is given in terms of multipliers. To define a C^*-algebra containing A as an ideal, we start with any faithful representation $\pi : A \to B(\mathcal{H})$, providing space around A, and A can be considered as a C^*-subalgebra of $B(\mathcal{H})$. Then an element $x \in B(\mathcal{H})$ is called a *multiplier* of A if $xA \subseteq A$ and $Ax \subseteq A$. In particular, when π is *non-degenerate*, meaning that $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$, we define the *multiplier algebra* of A as the set of multipliers,

$$\mathcal{M}(A) = \{x \in B(\mathcal{H}) : xA \subseteq A, Ax \subseteq A\}. \tag{2.9.1}$$

Proposition 2.9.1 (cf. [110, Proposition 2.2.5])**.** *Let A be a C^*-algebra, and let $\pi : A \to B(\mathcal{H})$ be a faithful and non-degenerate representation, then $\mathcal{M}(A)$ is a unital C^*-algebra containing A as an essential ideal.*

Remark 2.9.2. When A is a unital C^*-algebra, any faithful and non-degenerate representation $\pi : A \to B(\mathcal{H})$ is unital. Considering A as a subalgebra of $B(\mathcal{H})$, we have $1_A = \mathrm{id}_{\mathcal{H}}$. Thus $x \in B(\mathcal{H})$ is a multiplier if and only if $x \in A$, which implies that $\mathcal{M}(A) = A$ when A is unital.

The definition of $\mathcal{M}(A)$ is independent of the choice of faithful and non-degenerate representations, since they are all *-isomorphic to the multiplier algebra defined through an alternative approach that does not involve a specific representation of A.

A *double centralizer* for a C^*-algebra A is a pair (L, R) of maps $L, R : A \to A$ satisfying $R(x)y = xL(y)$ for all $x, y \in A$. For a double centralizer (L, R), the maps L and R are shown to be linear and bounded with the same operator norm (see [110, Proposition 2.2.8]). Define $\mathcal{M}'(A)$ as the set of double centralizers of A, which forms a unital C^*-algebra with the unit $(\mathrm{id}_A, \mathrm{id}_A)$, and norm and operations defined in [110, Proposition 2.2.9] for instance.

The multiplier algebra $\mathcal{M}(A)$ defined in (2.9.1), constructed via a faithful and non-degenerate representation, is *-isomorphic to $\mathcal{M}'(A)$. The *-isomorphism can be described explicitly. For any $x \in \mathcal{M}(A)$, consider the maps

$$L_x : a \mapsto xa, \quad R_x : a \mapsto ax. \tag{2.9.2}$$

Since $\mathcal{M}(A)$ contains A as an ideal, L_x, R_x map elements of A back into A. Moreover, the pair (L_x, R_x) forms a double centralizer of A, since for any $a, b \in A$, we have $R_x(a)b = axb = aL_x(b)$. The following proposition provides an explicit isomorphism between $\mathcal{M}(A)$ and $\mathcal{M}'(A)$.

Proposition 2.9.3 (cf. [110, Proposition 2.2.11])**.** *Let A be a C^*-algebra. The map*

$$\Phi : \mathcal{M}(A) \to \mathcal{M}'(A), \quad x \mapsto (L_x, R_x) \tag{2.9.3}$$

is a *-isomorphism. Thus $\mathcal{M}(A)$ is independent of the chosen representation of A.

From now on, we denote the multiplier algebra of A by $\mathcal{M}(A)$ and use both the multiplier picture and the double centralizer definition interchangeably. In the double centralizer definition, the algebra A embeds into $\mathcal{M}(A)$ via the map $a \mapsto (L_a, R_a)$.

The following universal property shows that $\mathcal{M}(A)$ is the largest unitization of A.

Proposition 2.9.4 (cf. [110, Proposition 2.2.14])**.** *If A is an ideal of a C^*-algebra B, then there exists a unique *-homomorphism $\lambda : B \to \mathcal{M}(A)$ such that $\lambda|_A = id_A$, i.e. the following diagram commutes,*

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow & \downarrow{\scriptstyle \lambda} \\ & & \mathcal{M}(A). \end{array} \tag{2.9.4}$$

Moreover, λ is injective if and only if A is essential in B.

Sketch of the proof. We prove the existence of the map. Given $b \in B$, define $\lambda(b) = (L_b, R_b)$, which is a double centralizer since A is an ideal in B. Moreover, $\lambda|_A$ gives the embedding of A into $\mathcal{M}(A)$, which implies $\lambda|_A = \mathrm{id}_A$. $\square$

We now proceed with two basic examples of multiplier algebras.

Example 2.9.5. Consider $A = \mathcal{K}(\mathcal{H}) \subseteq B(\mathcal{H})$, where the inclusion map is non-degenerate. Then every element in $B(\mathcal{H})$ is a multiplier and thus $\mathcal{M}(\mathcal{K}) = B(\mathcal{H})$. More generally, for any C^*-algebra A, there is a copy of $B(\mathcal{H})$ in $\mathcal{M}(A \otimes \mathcal{K})$, since $(\mathrm{id}_A \otimes L_x, \mathrm{id}_A \otimes R_x)$ is a double centralizer of $A \otimes \mathcal{K}$, for any $x \in \mathcal{M}(\mathcal{K}) \cong B(\mathcal{H})$.

This motivates another way to approach the multiplier algebra. Hilbert bimodules over C^*-algebras generalize Hilbert spaces over $\mathbb{C}$, and the multiplier algebra can be viewed as the C^*-algebra of the so-called adjointable operators on A, when considered as a Hilbert A-bimodule. This viewpoint is particularly useful in connection with KK-theory. More details can be found in [110, Chapter 15] or [52] for instance.

Lastly, we consider the multiplier algebra of non-unital commutative C^*-algebras.

Example 2.9.6. By [76, Example 3.1.3] for example, if X is a locally compact Hausdorff space, then $\mathcal{M}(C_0(X)) = C_b(X)$, which is the algebra of bounded continuous functions on X. Since $C_b(X)$ is a unital and commutative C^*-algebra, there exists some compact Hausdorff space βX such that $C_b(X) = C(\beta X)$.

By the universality of the multiplier algebra, any unital C^*-algebra containing $C_0(X)$ as an essential ideal can be considered as a C^*-subalgebra of $C(\beta X)$, which is commutative and of the form $C(Y)$ for some compact Hausdorff space Y. Moreover, $C_0(X) \subseteq C(Y)$ is an essential ideal if and only if Y is a *compactification* of X, which means that X embeds as a dense subspace of the compact space Y (see [110, Exercise 2.A] for instance). In particular, βX is a compactification of X with the universal property that βX surjects onto any compactification of X. Thus βX is the Stone-Čech compatification of X. Taking multiplier algebras can be regarded as the non-commutative analogue of taking maximal compactifications.

2.9.2 Strict topology on multiplier algebras

We turn to the definition of strict topology on the multiplier algebra, which is useful especially when computing $\mathcal{M}(A)$ from A.

Definition 2.9.7. Let A be a C^*-algebra. The *strict topology* on $\mathcal{M}(A)$ is the locally convex topology generated by the seminorms $x \mapsto \|xa\|$, $x \mapsto \|ax\|$ for all $a \in A$.

For any net $(x_\lambda)_\lambda \in \mathcal{M}(A)$, we have $(x_\lambda)_\lambda$ converge strictly to x in $\mathcal{M}(A)$ if and only if $(x_\lambda a)_\lambda$ and $(ax_\lambda)_\lambda$ converge to xa and ax in norm respectively. Thus the strict topology is weaker than the norm topology, since the norm is submultiplicative. In the commutative setting, strict convergence implies localized uniform convergence.

Indeed, for a locally compact Hausdorff space X, a net $(f_\lambda)_\lambda \subseteq C_b(X)$ converging in strict topology implies the uniform convergence of the net on compact subsets of X.

If $(e_\lambda)_\lambda$ is an approximate unit in A, then $(e_\lambda)_\lambda$ converges strictly to $1_{\mathcal{M}(A)}$, since both $(e_\lambda a)_\lambda$ and $(ae_\lambda)_\lambda$ converge in norm to a. This argument of approximation in the strict topology can be generalized to all elements in $\mathcal{M}(A)$. For any $x \in \mathcal{M}(A)$, one has that $(xe_\lambda)_\lambda$ converges to x strictly. Thus we obtain the following theorem.

Proposition 2.9.8. *Let A be a C^*-algebra, then $\mathcal{M}(A)$ is the strict closure of A.*

Using the observation above, we compute the multiplier algebra of $M_k \otimes A$.

Example 2.9.9. For a C^*-algebra A, we show that $\mathcal{M}(M_k \otimes A) \cong M_k \otimes \mathcal{M}(A)$ for any $k \in \mathbb{N}$. Take any faithful and non-degenerate $\pi : A \to B(\mathcal{H})$, consider

$$\pi \otimes \mathrm{id}_{M_k} : A \otimes M_k \to M_k \otimes B(\mathcal{H}) \cong B(\mathcal{H} \otimes \mathbb{C}^k), \tag{2.9.5}$$

which is again faithful and non-degenerate. Thus $\mathcal{M}(M_k \otimes A)$ is regarded as a C^*-subalgebra of $M_k \otimes B(\mathcal{H})$ and elements are of the form $x = (x_{i,j})_{i,j}$ for $x_{i,j} \in B(\mathcal{H})$. It is straightforward to check that $M_k \otimes \mathcal{M}(A)$ is contained in $\mathcal{M}(M_k \otimes A)$.

By Theorem 2.9.8, there exists a net $(x_\lambda = (x_{i,j,\lambda})_{i,j})_\lambda \subseteq M_k \otimes A$ converging to x in the strict topology. In particular, for any $a \in A$, then $x_\lambda(\mathrm{id}_{M_k} \otimes a)$ converges to $x(\mathrm{id}_{M_k} \otimes a)$ in norm, which implies that $x_{i,j,\lambda}a$ converges to $x_{i,j}a$ in norm for any $1 \leq i, j \leq k$. Arguing similarly for multiplying a on the left, we have $x_{i,j} \in \mathcal{M}(A)$.

Given a net $(x_\lambda)_\lambda$ in the multiplier algebra, checking strict convergence by the definition without knowing the limit could be difficult. It suffices to show that the net is Cauchy in the strict topology.

Proposition 2.9.10. *Let A be a C^*-algebra, then $\mathcal{M}(A)$ is strictly complete.*

Proof. Let $(x_\lambda)_\lambda$ be a net in $\mathcal{M}(A)$ that is Cauchy in the strict topology. Then $(x_\lambda a)_\lambda$ and $(ax_\lambda)_\lambda$ are Cauchy nets, or equivalently convergent nets, in the norm topology of A, for any $a \in A$. Then define $L, R : A \to A$ by

$$L(a) = \lim x_\lambda a, \;\; R(a) = \lim a x_\lambda, \quad a \in A. \tag{2.9.6}$$

Then (L, R) is a double centralizer of A since for any $a, b \in A$,

$$R(a)b = \lim a x_\lambda b = aL(b). \tag{2.9.7}$$

Suppose that (L, R) corresponds to an element $x \in \mathcal{M}(A)$, meaning that $L = L_x$ and $R = R_x$. Then $(x_\lambda)_\lambda$ converges to x strictly by definition. □

In particular, in Chapter 6, we are interested in elements in the multiplier algebra which are of "diagonal" form with respect to an approximate unit of A. Recall that an approximate unit $(e_n)_n$ is said to be *almost idempotent* if $e_{n+1}e_n = e_n$ for any $n \in \mathbb{N}$.

Lemma 2.9.11. *Let A be a non-unital C^*-algebra with an almost idempotent approximate unit $(e_n)_n$. Let $(n_{k,1})_k$ and $(n_{k,2})_k$ be strictly increasing sequences of natural numbers and let $(x_k)_k$ be a bounded sequence with $x_k \in \overline{(e_{n_{k,1}} - e_{n_{k-1,1}})A(e_{n_{k,2}} - e_{n_{k-1,2}})}$. Then $x = \sum_{k=1}^{\infty} x_k \in \mathcal{M}(A)$ and if $(x_k)_k$ is bounded by some M, then $\|x\| \leq 2M$.*

Proof. We suppose that $(x_k)_k$ is norm bounded by M. Notice that for any partial sum up to K, we have

$$\left\| \sum_{k=1}^{K} x_k \right\| \leq \left\| \sum_{k=1 \text{ even}}^{K} x_k \right\| + \left\| \sum_{k=1 \text{ odd}}^{K} x_k \right\|. \tag{2.9.8}$$

Since $(e_n)_n$ is almost idempotent, it follows that $x_k^* x_{k'} = 0$ for $|k - k'| > 1$. Thus, the sum of even terms and the sum of odd terms are both norm bounded by M.

We show that the series $\sum_{k=1}^{\infty} x_k$ is Cauchy in the strict topology, then by Proposition 2.9.10, the series converges strictly and thus $x \in \mathcal{M}(A)$. Fix $a \in A$, $\epsilon > 0$. Since $(e_n)_n$ is an approximate unit for A, there exists n_0 such that $a \approx_{\epsilon/2M} e_{n_0}a$. Because the sequence $(n_{k,1})_k$ is strictly increasing, there exists k_0 big enough such that $x_{k_0}e_{n_0} = 0$. Take $m' > m > k_0$, then

$$\left\| \sum_{k=m+1}^{m'} x_k a \right\| \leq \left\| \sum_{k=m+1}^{m'} x_k(e_{n_0}a) \right\| + \left\| \sum_{k=m+1}^{m'} x_k(e_{n_0}a - a) \right\| \tag{2.9.9}$$

$$\leq \left\| \sum_{k=m+1}^{m'} x_k \right\| \|e_{n_0}a - a\| < \left\| \sum_{k=m+1}^{m'} x_k \right\| \cdot \epsilon/2M \tag{2.9.10}$$

$$\leq \left\| \sum_{k=m+1 \text{ even}}^{m'} x_k \right\| \cdot \epsilon/2M + \left\| \sum_{k=m+1 \text{ odd}}^{m'} x_k \right\| \cdot \epsilon/2M \leq \epsilon, \tag{2.9.11}$$

where the last estimation holds for similar reasons as for the norm bound of (2.9.8). Similarly, we can prove Cauchyness when x_k acts on the right. Thus $x = \sum_{k=1}^{\infty} x_k \in \mathcal{M}(A)$. Moreover, x has norm at most $2M$ since it is the strict limit of a sequence of elements in A of norm at most $2M$. □

We recall the notion of strict continuity for functions. For a locally compact Hausdorff space X and a C^*-algebra A, then $f : X \to \mathcal{M}(A)$ is *strictly continuous* if for any $a \in A$, maps $x \mapsto f(x)a$ and $x \mapsto af(x)$ are both continuous. We use the notation $C_b(X, \mathcal{M}(A)_\beta)$ for the set of bounded and strictly continuous functions from

X to $\mathcal{M}(A)$, which is a C^*-algebra under pointwise operation and supremum norm (see [3, Lemma 3.2] for instance). Moreover, we have the following identification.

Theorem 2.9.12 ([3, Corollary 3.4])**.** *Let A be a C^*-algebra and let X be a locally compact Hausdorff space, then*

$$\mathcal{M}(C_0(X, A)) \cong C_b(X, \mathcal{M}(A)_\beta). \tag{2.9.12}$$

If X is a compact Hausdorff space, any strictly continuous function f from X to $\mathcal{M}(A)$ is automatically bounded. Indeed, since $x \mapsto f(x)a$ is continuous, by compactness of X, it follows that $\sup_{x\in X} \|f(x)a\| < \infty$ for any $a \in A$. By the uniform boundedness principle, we have $\|f\|_{\sup} < \infty$. Thus, when X is compact, we will denote all strictly continuous and thus bounded functions by $C(X, \mathcal{M}(A)_\beta)$.

Lastly, we include the following theorem, which shows that the unitary group of the multiplier algebra of a stable C^*-algebra is path connected both in the norm topology and the strict topology.

Theorem 2.9.13 ([58, Section 1], [75, Theorem 2.5])**.** *Let A be a stable C^*-algebra.*

(i) $\mathcal{U}(\mathcal{M}(A))$ is path connected in the strict topology.;

(ii) If A is additionally σ-unital, then $\mathcal{U}(\mathcal{M}(A))$ is path connected in norm.

Notice that the second statement implies the first statement, and the proof of the first statement is much more elementary.

2.10 Representation theory for C^*-algebras

2.10.1 Pure states and irreducible representations

Recall that a *representation* of a C^*-algebra A is a *-homomorphism $\phi : A \to B(\mathcal{H}_\phi)$, where $\mathcal{H}_\phi$ is a Hilbert space. Two representations $\phi : A \to B(\mathcal{H}_\phi)$ and $\psi : A \to B(\mathcal{H}_\psi)$ on A are said to be *unitarily equivalent* if there exists a unitary isomorphism $u : \mathcal{H}_\psi \to \mathcal{H}_\phi$ such that

$$\phi(a) = u\psi(a)u^*, \quad a \in A. \tag{2.10.1}$$

We denote by $[\phi]$ the unitary equivalence class of ϕ. It is straightforward to check that unitary equivalence is an equivalence relation. The *direct sum* $\phi \oplus \psi : A \to B(\mathcal{H}_\phi \oplus \mathcal{H}_\psi)$ is defined by

$$(\phi \oplus \psi)(a)(\zeta, \xi) = (\phi(a)\zeta, \psi(a)\xi). \tag{2.10.2}$$

We recall several concepts for representations. A closed subspace $\mathcal{K}$ of $\mathcal{H}_\varphi$ is *invariant* under φ if $\varphi(a)\zeta \in \mathcal{K}$ for any $a \in A$ and $\zeta \in \mathcal{K}$.

Definition 2.10.1. Let $\phi : A \to B(\mathcal{H}_\varphi)$ be a nonzero representation on A, then ϕ is

(i) *cyclic* if there exists a vector $\zeta \in \mathcal{H}_\phi$ such that $\overline{\text{span}}\{\phi(a)\zeta : a \in A\} = \mathcal{H}_\phi$, where we call ζ *a cyclic vector* of ϕ;

(ii) *irreducible* if there is no closed invariant subspace apart from $\{0\}$ and $\mathcal{H}_\varphi$.

A direct observation is that irreducible representations are cyclic. Take any irreducible representation $\varphi : A \to B(\mathcal{H}_\varphi)$, since $\overline{\text{span}}\{\varphi(a)\zeta : a \in A\}$ is a closed invariant subspace for any $\zeta \in \mathcal{H}_\varphi$, then it is either $\mathcal{H}_\varphi$ or $\{0\}$. Since φ is nonzero, then $\overline{\text{span}}\{\varphi(a)\xi : a \in A\} = \mathcal{H}_\varphi$ for some $\xi \in \mathcal{H}_\varphi$.

The *spectrum* $\hat{A}$ of A is the set of unitary equivalence classes of irreducible representations of A. Ideals of irreducible representations are called *primitive ideals*, and the *primitive ideal space*, denoted by $\text{Prim}(A)$, is the set of primitive ideals in A. We endow $\text{Prim}(A)$ with the hull-kernel topology, where the open sets in $\text{Prim}(A)$ are precisely of the form $\mathcal{O}_J = \{I \in \text{Prim}(A), I \not\subseteq J\}$ for some ideal J in A. Since there is a canonical surjection $\hat{A} \to \text{Prim}(A)$, $\phi \mapsto \ker(\phi)$, we can pull the hull-kernel topology of $\text{Prim}(A)$ back to a topology on $\hat{A}$. From now on, we denote by $\hat{A}$ and $\text{Prim}(A)$ the spectrum and the primitive ideal space endowed with the topologies described above respectively.

The representation theory of a C^*-algebra is relevant to its state space. Recall that a *state* on A is a positive linear functional $\rho : A \to \mathbb{C}$ of norm one. We denote the set of states on a C^*-algebra A by $S(A)$, which is a convex and compact subset of A^* in the weak* topology. A *pure state* ρ on A is an extreme point of $S(A)$, which means that if $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ for some $\rho_1, \rho_2 \in S(A)$ and $\lambda \in (0,1)$, then $\rho = \rho_1 = \rho_2$. We denote the set of pure states on A by $PS(A)$. For every $\rho \in S(A)$, the GNS-construction produces a representation ϕ_ρ of A, which is called the *GNS-representation* of ρ. It turns out that pure states are precisely those whose GNS-representations are irreducible.

Proposition 2.10.2 (cf. [83, Lemma A.12])**.** *Let ρ be a state on A. Then the GNS-representation ϕ_ρ is irreducible if and only if ρ is a pure state.*

Moreover, every irreducible representation is unitarily equivalent to the GNS-representation of a pure state, see [83, Proposition A.6]. Thus, there is a canonical surjection $\Lambda : PS(A) \to \hat{A}$, $\rho \mapsto [\phi_\rho]$.

Theorem 2.10.3 (cf. [83, Theorem A.38])**.** *Let A be a C^*-algebra and give $PS(A)$ the weak*-topology. Then the canonical map $\Lambda : PS(A) \to \hat{A}$, $\rho \mapsto [\phi_\rho]$, is a continuous open surjection.*

The following corollary is a consequence of Theorem 2.10.3. Recall that pure states ρ and λ on A are *inequivalent* if their GNS-representations are not unitarily equivalent. The following corollary says that if the unitary equivalence class $[\phi_\rho]$ of a pure state ρ is a limit point in $\hat{A}$, there is some flexibility in obtaining an inequivalent pure state λ such that the evaluations of ρ and λ on a finite subset in A are very close. The corollary will be crucially used in Chapter 8.

Corollary 2.10.4. *Let A be a C^*-algebra with Hausdorff spectrum $\hat{A}$. Let ρ be a pure state on A such that $[\phi_\rho]$ is a limit point in $\hat{A}$. For any finite subset $\mathcal{F} \subseteq A$, any finite subset $\mathcal{G} \in \hat{A}$ not containing $[\phi_\rho]$ and $\epsilon > 0$, there exists a pure state λ on A such that $[\phi_\lambda] \notin \mathcal{G}$, $[\phi_\lambda] \neq [\phi_\rho]$ and $|\lambda(a) - \rho(a)| < \epsilon$ for any $a \in \mathcal{F}$.*

Proof. Fix finite subsets $\mathcal{F} \subseteq A$, $\mathcal{G} \in \hat{A}$ and $\epsilon > 0$. Take an open neighborhood of ρ

$$U = \{\mu \in PS(A) : |\mu(a) - \rho(a)| < \epsilon \text{ for any } a \in \mathcal{F}\} \tag{2.10.3}$$

in $PS(A)$ equipped with the weak* topology. By Theorem 2.10.3, we have that $\Lambda(U)$ is open in $\hat{A}$ containing $[\phi_\rho]$. Since $\hat{A}$ is Hausdorff, then $\mathcal{G}$ is closed and thus $\Lambda(U) \setminus \mathcal{G}$ is an open neighborhood of $[\phi_\rho]$ in $\hat{A}$. As $[\phi_\rho]$ is a limit point in $\hat{A}$, there exists a pure state $\lambda \in U$ such that $\Lambda(\lambda) \notin \mathcal{G}$ and $\Lambda(\lambda) \neq [\phi_\rho]$. This concludes the proof. ☐

The next lemma allows us to switch between equivalent pure states by conjugating elements and allowing perturbations. It will be frequently used in Chapter 8.

Lemma 2.10.5. *Let ρ and λ be equivalent pure states on A. Then for any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there exists $x \in A$ such that $\lambda(a) \approx_\epsilon \rho(xax^*)$.*

Proof. Without loss of generality, we can assume that the elements in $\mathcal{F}$ are contractive. Suppose that $(\phi_\rho, \mathcal{H}_\rho, \zeta_\rho)$ and $(\phi_\lambda, \mathcal{H}_\lambda, \zeta_\lambda)$ are irreducible representations of A induced by ρ and λ respectively. This means that for any $a \in A$,

$$\rho(a) = \langle \phi_\rho(a)\zeta_\rho, \zeta_\rho\rangle \text{ and } \lambda(a) = \langle \phi_\lambda(a)\zeta_\lambda, \zeta_\lambda\rangle. \tag{2.10.4}$$

Since ρ and λ are equivalent, there exists a unitary isomorphism $u : \mathcal{H}_\rho \to \mathcal{H}_\lambda$ such that $u^*\phi_\lambda(a)u = \phi_\rho(a)$ for any $a \in A$. Then for any $a \in A$,

$$\lambda(a) = \langle u\phi_\rho(a)u^*\zeta_\lambda, \zeta_\lambda\rangle = \langle \phi_\rho(a)(u^*\zeta_\lambda), (u^*\zeta_\lambda)\rangle. \tag{2.10.5}$$

Because $u^*\zeta_\lambda \in \mathcal{H}_\rho = \overline{\phi_\rho(A)\zeta_\rho}$, there exists $x \in A$ such that $u^*\zeta_\lambda \approx_{\epsilon/2} \phi_\rho(x)\zeta_\rho$. Then

$$\lambda(a) \approx_\epsilon \langle \phi_\rho(a)\phi_\rho(x)\zeta_\rho, \phi_\rho(x)\zeta_\rho \rangle \tag{2.10.6}$$

$$= \langle \phi_\rho(x^*ax)\zeta_\rho, \zeta_\rho \rangle = \rho(x^*ax), \quad a \in \mathcal{F}. \tag{2.10.7}$$

This concludes the proof. □

We proceed with several fundamental examples of pure states.

Example 2.10.6. Take $A = \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then every irreducible representation is unitarily equivalent to the identity representation $\mathrm{id} : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$, see [76, Example 5.1.1] for example. Thus, pure states on M_n or $\mathcal{K}$ are equivalent to the $(1,1)$ corner evaluation map $\rho_{1,1}$.

Example 2.10.7. Take $A = C_0(X)$ for some locally compact Hausdorff space X, then every irreducible representation of A is 1-dimensional, given by the point evaluation $\rho_x : C_0(X) \to \mathbb{C}$, $f \mapsto f(x)$, at some $x \in X$ (see [83, Example A.16] for example). Pure states on $C_0(X)$ are also given by point evaluations ρ_x, and two pure states are equivalent if and only if they are point evaluations at the same point in X.

We collect several results for pure states on tensor products.

Lemma 2.10.8 (cf. [83, Proposition B.35, Lemma B.36])**.** *Let A, B be C^*-algebras.*

(i) Suppose further that A is nuclear. If $\rho \in PS(A)$ and $\lambda \in PS(B)$, then $\rho \otimes \lambda$ given by $(\rho \otimes \lambda)(a \otimes b) = \rho(a)\lambda(b)$ is a pure state on $A \otimes B$;

(ii) If $\rho \in PS(A \otimes \mathcal{K})$, then there exists $\lambda \in PS(A)$ such that ρ and $\lambda \otimes \rho_{1,1}$ are equivalent, where $\rho_{1,1}$ is the evaluation map at the $(1,1)$ corner of $\mathcal{K}$;

Pure states extend canonically from ideals or quotients to extensions.

Lemma 2.10.9 (cf. [24])**.** *Let $I \lhd A$ be an ideal in A and let $q : A \to A/I$ be the quotient map. Then the following statements are true:*

(i) If $\rho \in PS(I)$, then ρ extends canonically to a pure state of A. If $\rho, \lambda \in PS(I)$ are inequivalent, then they extend canonically to inequivalent pure states on A;

(ii) If $\rho \in PS(A/I)$, then $\rho \circ q \in PS(A)$. If $\rho, \lambda \in PS(A/I)$ are inequivalent, then $\rho \circ q$ and $\lambda \circ q$ are inequivalent pure states on A.

We include the Akemann-Anderson-Pedersen Excision Theorem for pure states.

Theorem 2.10.10 (cf. [2])**.** *Let A be a C^*-algebra and λ be a pure state. There exists a net of contractive positive elements $(e_i)_{i \in I}$ in A such that for any $a \in A$,*

$$\lim_{i \to \infty} \|e_i a e_i - \lambda(a) e_i^2\| = 0. \tag{2.10.8}$$

2.10.2 C^*-algebras with Hausdorff spectrum

In this section, we study the class of C^*-algebras with Hausdorff spectrum. For a C^*-algebra A, if $\hat{A}$ is Hausdorff, then the canonical map $\hat{A} \to \mathrm{Prim}(A)$ is a homeomorphism. Thus, we identify $\hat{A}$ and $\mathrm{Prim}(A)$. For $t \in \hat{A} = \mathrm{Prim}(A)$, we denote by $J_{\{t\}}$ the primitive ideal corresponding to t. Then we have the following lemma.

Lemma 2.10.11 (cf. [83, Lemma 5.1])**.** *Let A be a C^*-algebra with Hausdorff spectrum and $t \in \hat{A}$. Then $A(t) := A/J_{\{t\}}$ is a simple C^*-algebra with a unique irreducible representation up to unitary equivalence.*

In the separable setting, the quotient $A(t)$ is *-isomorphic to $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, see the work [94] of Rosenberg. Outside of the separable setting, Akemann and Weaver constructed counterexamples in [1] to Naimark's conjecture: if A is a C^*-algebra with a unique irreducible representation up to unitary equivalence, then A is *-isomorphic to $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

For each $t \in \hat{A}$ and $a \in A$, we denote by $a(t)$ the image of a in the quotient $A(t) = A/J_{\{t\}}$ and we think of a as the field of elements $\{a(t)\}_t$.

Lemma 2.10.12 (cf. [83, Lemma 5.2])**.** *Let A be a C^*-algebra with Hausdorff spectrum.*

(i) If $a, b \in A$ and $a(t) = b(t)$ for every $t \in \hat{A}$, then $a = b$;

(ii) For each a, the function $t \mapsto \|a(t)\|$ is continuous on $\hat{A}$, vanishes at infinity, and has sup-norm equal to $\|a\|$.

For any C^*-algebra A, the Dauns-Hofmann Theorem says that A is a module over the algebra of continuous functions on $\mathrm{Prim}(A)$.

Theorem 2.10.13 (Dauns-Hofmann, cf. [83, Theorem A.34])**.** *Let A be a C^*-algebra. For $P \in \mathrm{Prim}(A)$, let $\pi_P : A \to A/P$ be the quotient map. There is a *-isomorphism ϕ of $C_b(\mathrm{Prim}A)$ onto the center $Z\mathcal{M}(A)$ of $\mathcal{M}(A)$ such that for all $f \in C_b(\mathrm{Prim}A)$ and $a \in A$,*

$$\pi_P(\phi(f)a) = f(P)\pi_P(a), \quad P \in \mathrm{Prim}(A). \tag{2.10.9}$$

We write $f \cdot a$ for $\phi(f)a$.

In the case that $\hat{A}$ is Hausdorff, by Dauns-Hofmann theorem, $C_b(\hat{A})$ is identified with $Z\mathcal{M}(A)$, which restricts to an action of $C_0(\hat{A})$ on A given by the formula

$$(f \cdot a)(t) = f(t)a(t), \quad f \in C_0(\hat{A}), a \in A, t \in \hat{A}. \tag{2.10.10}$$

It is direct to check that the $C_0(\hat{A})$-action is nondegenerate, which means that

$$A = \overline{C_0(\hat{A}) \cdot A} = \overline{\operatorname{span}}\{f \cdot a : f \in C_0(\hat{A}), a \in A\}. \tag{2.10.11}$$

Locally, we also have such nondegenerate actions. For any closed subset $F \subseteq \hat{A} \cong \operatorname{Prim}(A)$, take the ideal $J_F = \bigcap\{J_{\{t\}} : t \in F\}$ corresponding to F and denote by $\pi_F : A \to A/J_F$ the quotient map. Then the following lemma is a consequence of the Dauns-Hofman theorem in the Hausdorff setting.

Lemma 2.10.14 (cf. [83, Corollary 5.11]). *Let A be a C^*-algebra with Hausdorff spectrum. Let $F \subseteq \hat{A}$ be a closed subset and $U \subseteq F$ be an open subset in $\hat{A}$. Then*

(i) $J_F = \overline{C_0(\hat{A} \setminus F) \cdot A} = \overline{\operatorname{span}}\{f \cdot a : f \in C_0(\hat{A} \setminus F), a \in A\}$.

(ii) The quotient map $\pi_F : A \to A/J_F$ is a *-isomorphism on $\overline{C_0(U) \cdot A} \subseteq A$.

2.11 Quasicentral approximate units

Quasicentral approximate units are an important tool in the study of extensions. If I is an ideal in a C^*-algebra A, then a *quasicentral approximate unit of I* is an approximate unit $(e_i)_{i \in I}$ of I such that for any $a \in A$, we have $\|[e_i, a]\| \to 0$ as $i \to \infty$. A sequential approximate unit $(e_n)_n$ is *almost idempotent* if $e_{n+1}e_n = e_n$ for all $n \in \mathbb{N}$. Every σ-unital C^*-algebra has an almost idempotent approximate unit (see [7, II.4.2.5]).

Lemma 2.11.1 (cf. [25, Theorem I.9.16]). *Let $I \lhd A$ be an ideal in a C^*-algebra A and I is σ-unital. There exists an almost idempotent quasicentral approximate unit $(e_n)_n$ of I.*

In Lemma 2.11.1, since I has an almost idempotent approximate unit $(h_n)_n$, we can find $(e_n)_n$ in the convex hull of $(h_n)_n$ as described in [7, II.4.3.2]. In general, one cannot arrange the quasicentral approximate unit to consist of projections, since the existence of such an approximate unit is equivalent to the quasidiagonality of the extension of I in A, see [100, Section 2] for a more detailed exposition, for instance.

The quasicentral approximate unit provides an approximate decomposition of elements in the extension. The unital case of the following proposition is settled in [13, Proposition 1.2.2], and we provide a similar proof for the non-unital case. For elements $a, b \in A$ and $\epsilon > 0$, we denote $\|a - b\| < \epsilon$ by $a \approx_\epsilon b$.

Proposition 2.11.2. *Let A be a C^*-algebra and $I \lhd A$ be an ideal. Let $(e_j)_{j\in J}$ be a quasicentral approximate unit for I consisting of contractions. For every $\epsilon > 0$ and elements $a, b \in A$ such that $a - b \in I$, there exists $i \in I$ such that*

$$a \approx_\epsilon e_i^{1/2} a e_i^{1/2} + (1_{\mathcal{M}(A)} - e_i)^{1/2} b (1_{\mathcal{M}(A)} - e_i)^{1/2}. \tag{2.11.1}$$

Proof. Fix $\epsilon > 0$, elements $a, b \in A$ with $a - b \in I$. Take a quasicentral approximate unit $(e_j)_{j\in J}$ for I consisting of contractions. Then there exists $i \in J$ such that

(i) $\|[a, e_i^{1/2}]\| < \epsilon/8$;

(ii) $\|[a, (1_{\mathcal{M}(A)} - e_i)^{1/2}]\| < \epsilon/8$;

(iii) $\|(1_{\mathcal{M}(A)} - e_i)^{1/2}(a - b)(1_{\mathcal{M}(A)} - e_i)^{1/2}\| < \epsilon/8$.

Thus we have

$$\begin{aligned} a &= ae_i + a(1_{\mathcal{M}(A)} - e_i) && (2.11.2)\\ &\overset{(i),(ii)}{\approx_{\epsilon/4}} e_i^{1/2} a e_i^{1/2} + (1_{\mathcal{M}(A)} - e_i)^{1/2} a (1_{\mathcal{M}(A)} - e_i)^{1/2} && (2.11.3)\\ &\overset{(iii)}{\approx_{\epsilon/8}} e_i^{1/2} a e_i^{1/2} + (1_{\mathcal{M}(A)} - e_i)^{1/2} b (1_{\mathcal{M}(A)} - e_i)^{1/2}, && (2.11.4) \end{aligned}$$

which provides the desired approximation for a. □

The following corollary follows immediately.

Corollary 2.11.3. *Let A be a C^*-algebra and $I \lhd A$ be an ideal. Let $q : A \to A/I$ be the quotient map. If $a, b \in A$ satisfy $q(a) \approx_\epsilon q(b)$, then there exists $e \in I$ such that*

$$a \approx_\epsilon e^{1/2} a e^{1/2} + (1_{\mathcal{M}(A)} - e)^{1/2} b (1_{\mathcal{M}(A)} - e)^{1/2}. \tag{2.11.5}$$

Proof. Since $q(a) \approx_\epsilon q(b)$, there exists $c \in I$ such that $a \approx_\epsilon b + c$. By Proposition 2.11.2, for any $\delta > 0$, there exists a contraction $e \in I$ such that

$$b + c \approx_\delta e^{1/2}(b + c)e^{1/2} + (1_{\mathcal{M}(A)} - e)^{1/2} b (1_{\mathcal{M}(A)} - e)^{1/2}. \tag{2.11.6}$$

Taking δ small enough and combining with $a \approx_\epsilon b + c$, we have

$$a \approx_\epsilon e^{1/2} a e^{1/2} + (1_{\mathcal{M}(A)} - e)^{1/2} b (1_{\mathcal{M}(A)} - e)^{1/2}. \tag{2.11.7}$$

This concludes the proof. □

Chapter 3

K-stability of $\mathcal{Z}$-stable C^*-algebras

In this chapter, we provide a shorter and self-contained new proof of K-stability of $\mathcal{Z}$-stable C^*-algebras, using the Rørdam-Winter picture of the Jiang-Su algebra introduced in Section 2.7. The chapter covers alternative proofs for Theorem F and Theorem G, which are originally shown in the unpublished paper [53] of Jiang.

In his work, Theorem G is proven by explicitly constructing maps between higher homotopy groups, and Theorem F follows as a consequence. In comparison, we prove theorems in the reverse order. Theorem F is proven first by taking advantage of the extra flexibility in the UHF-fibers of generalized dimension drop algebras. Combining K_1-injectivity result of $\mathcal{Z}$-stable C^*-algebras in [14], we get Theorem G by a standard homological algebra argument.

Main results of the chapter, unless otherwise stated explicitly, are based on my single-authored paper [50].

3.1 K_1-bijectivity of UHF-stable C^*-algebras

For a unital C^*-algebra A, the set of unitaries is denoted by $\mathcal{U}(A)$ and the set of unitaries that are homotopic to 1_A is denoted by $\mathcal{U}_0(A)$. For $n \geq 2$, the set of unitaries in $M_n(A)$ is denoted by $\mathcal{U}_n(A)$. If A is a non-unital C^*-algebra, we denote by $A^\dagger$ the minimal unitization of A, with the canonical maps $\iota : A \to A^\dagger$ and $q : A^\dagger \to \mathbb{C}$. We recall the definitions of K_1-injectivity and K_1-surjectivity.

Definition 3.1.1. A unital C^*-algebra A is called K_1-*surjective (K_1-injective)* if the group homomorphism $\mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$, $[u] \mapsto [u]_1$, is surjective (injective).

In the non-unital case, a C^*-algebra A is called K_1-*surjective (K_1-injective)* if $A^\dagger$ is K_1-surjective (K_1-injective).

Every C^*-algebra tensorially absorbing UHF-algebras of infinite type is K_1-injective and K_1-surjective, which was originally proved in the work [101] of Thomsen. One can also derive the fact as a consequence of Jiang's theorem in [53] and the fact that UHF-algebras of infinite type are $\mathcal{Z}$-stable. We include here more direct and elementary proofs of this fact. The following lemma is proven in [14] as an ingredient for the proof of K_1-injectivity of $\mathcal{Z}$-stable C^*-algebras.

Lemma 3.1.2 ([14, Lemma 4.9])**.** *If A is a unital C^*-algebra and $n \geq 2$, then $A \otimes M_{n^\infty}$ is K_1-injective and $K_0(A \otimes M_{n^\infty})$ is generated by*

$$\{[p]_0 : p \text{ is a projection in } A \otimes M_{n^\infty}\}. \tag{3.1.1}$$

Proof. Take $B = A \otimes M_{n^\infty}$ and suppose that $u \in \mathcal{U}(B)$ satisfies $[u]_1 = 0$. Then there exists $m \geq 1$ such that $u \oplus 1_B^{\oplus m-1} \sim_h 1_B^{\oplus m}$ in $\mathcal{U}_m(B)$. By enlarging m if necessary, we may assume that $m = n^k$ for some $k \in \mathbb{N}$. By Lemma 2.3.3, then

$$u \otimes 1_{M_m} = u^{\oplus m} = (u \oplus 1_B^{\oplus m-1}) \cdots (1_B^{\oplus m-1} \oplus u) \sim_h 1_{B \otimes M_m} \tag{3.1.2}$$

in $\mathcal{U}(B \otimes M_m)$. Thus $u \otimes 1_{M_{n^\infty}}$ is homotopic to the identity in the unitary group of $B \otimes M_{n^\infty}$. Since M_{n^∞} is strongly self-absorbing, there exists a *-isomorphism $\theta : B \to B \otimes M_{n^\infty}$ that is approximately unitarily equivalent to $\mathrm{id}_B \otimes 1_{M_{n^\infty}}$. Then there is a unitary $v \in B \otimes M_{n^\infty}$ such that

$$\|\theta(u) - v(u \otimes 1_{M_{n^\infty}})v^*\| < 1. \tag{3.1.3}$$

By Lemma 2.3.2, we have $\theta(u) \sim_h v(u \otimes 1_{M_{n^\infty}})v^* \sim_h 1_{B \otimes M_{n^\infty}}$ in $\mathcal{U}(B \otimes M_{n^\infty})$. Applying θ^{-1} shows that $u \sim_h 1_B$ in $\mathcal{U}(B)$. The proof for K_0 is similar. $\square$

The analogous statement for K_1-surjectivity is proved using similar techniques in my work [50], taking advantage of the extra space provided by M_{n^∞}.

Lemma 3.1.3. *If A is a unital C^*-algebra and $n \geq 2$, then $A \otimes M_{n^\infty}$ is K_1-surjective.*

Proof. Take $B = A \otimes M_{n^\infty}$ and $[u]_1 \in K_1(B)$, where $u \in \mathcal{U}_m(B)$ for some $m \in \mathbb{N}$. By enlarging m if necessary, we may assume that $m = n^k$ for some $k \in \mathbb{N}$. Since there exists a *-isomorphism between M_{n^∞} and $M_{n^\infty} \otimes M_{n^k}$ that is approximately unitarily equivalent to the first factor embedding, there exists a *-isomorphism $\theta : B \to B \otimes M_{n^k}$ such that θ is approximately unitarily equivalent to $\mathrm{id}_B \otimes 1_{M_{n^k}}$.

Take $v = \theta^{-1}(u) \in \mathcal{U}(B)$, then there exists $w \in \mathcal{U}_{n^k}(B)$ such that

$$\|u - w(v \otimes 1_{M_{n^k}})w^*\| = \|\theta(v) - w(v \otimes 1_{M_{n^k}})w^*\| < 1. \tag{3.1.4}$$

By Lemma 2.3.2, the unitary u is homotopic to $w(v \otimes 1_{M_{n^k}})w^*$ in $\mathcal{U}_{n^k}(B)$ and thus $[u]_1 = [w(v \otimes 1_{M_{n^k}})w^*]_1 = [v \otimes 1_{M_{n^k}}]_1 = [v^{\oplus n^k}]_1 \in K_1(B)$ by Lemma 2.3.3. $\square$

Both lemmas above can be generalized to non-unital UHF-absorbing C^*-algebras by an application of the following standard arguments. Since the lemma will also be applied to show K_1-bijectivity of non-unital $\mathcal{Z}$-stable C^*-algebras later, we include the proof for completeness.

Lemma 3.1.4. *Let A, B be C^*-algebras with A non-unital and B unital. Then*

(i) If $A^\dagger \otimes B$ is K_1-injective, then $A \otimes B$ is K_1-surjective;

(ii) If $A^\dagger \otimes B$ is K_1-surjective and B is K_1-injective, then $A \otimes B$ is K_1-surjective,

where $\otimes$ is the minimal tensor product.

Proof. By [26, Lemma 1], for example, we have the split exact sequence

$$0 \longrightarrow A \otimes B \xrightarrow{\iota\otimes \mathrm{id}_B} A^\dagger \otimes B \xrightarrow{q\otimes \mathrm{id}_B} B \longrightarrow 0, \tag{3.1.5}$$

which induces a split short exact sequence for K_1-groups by split-exactness of K_1 mentioned in Proposition 2.3.5. The map $(\iota\otimes \mathrm{id}_B)^\sim$ is injective, since it sends $1_{(A\otimes B)^\sim}$ to $1_{A^\dagger\otimes B}$ and is the identity map when restricted to $A \otimes B$. Thus $(A \otimes B)^\sim$ can be considered as a C^*-subalgebra of $A^\dagger \otimes B$ by identifying $1_{(A\otimes B)^\sim}$ with $1_{A^\dagger} \otimes 1_B$. For any $u \in \mathcal{U}(A^\dagger \otimes B)$, by short exactness of the sequence, it belongs to $\mathcal{U}((A \otimes B)^\sim)$ if and only if $(q \otimes \mathrm{id_B})(u) \in \mathbb{C}1_B$.

To prove statement (i), take any $u \in \mathcal{U}((A \otimes B)^\sim)$ with $[u]_1 = 0 \in K_1(A \otimes B)$. Accordingly when we regard u as an element of $A^\dagger\otimes B$, we have $[u]_1 = 0 \in K_1(A^\dagger\otimes B)$ and $(q \otimes \mathrm{id}_B)(u) = \lambda 1_B$ for some $\lambda \in \mathbb{T}$. Since $A^\dagger \otimes B$ is K_1-injective, there is a path of unitaries $(u_t)_{t\in[0,1]} \subseteq \mathcal{U}(A^\dagger \otimes B)$ such that $u_0 = 1_{A^\dagger\otimes B}$ and $u_1 = u$. For $t \in [0, 1]$, take $v_t = (1_{A^\dagger} \otimes (q \otimes \mathrm{id}_B)(u_t))^* u_t$, then $(v_t)_{t\in[0,1]}$ is a path of unitaries in $\mathcal{U}(A^\dagger \otimes B)$ with $v_0 = 1_{A^\dagger\otimes B}$ and $v_1 = \bar{\lambda} u$. Since $(q \otimes \mathrm{id}_B)(v_t) = 1_B$ for any $t \in [0, 1]$, it follows that $(v_t)_{t\in[0,1]} \subseteq \mathcal{U}((A \otimes B)^\sim)$, which implies $1_{(A\otimes B)^\sim} \sim_h \bar{\lambda} u \sim_h u$ in $(A \otimes B)^\sim$.

To prove statement (ii), take any $x \in K_1(A\otimes B) \subseteq K_1(A^\dagger\otimes B)$. Then there exists $v_0 \in \mathcal{U}(A^\dagger\otimes B)$ such that $x = [v_0]_1$ since $A^\dagger\otimes B$ is assumed to be K_1-surjective. Then

$$[(q \otimes \mathrm{id}_B)(v_0)]_1 = K_1(q \otimes \mathrm{id}_B)(x) = 0 \in K_1(B), \tag{3.1.6}$$

which implies $(q \otimes \mathrm{id}_B)(v_0) \in \mathcal{U}_0(B)$ by K_1-injectivity of B. Therefore by Lemma 2.3.4, there exists $v_1 \in \mathcal{U}_0(A^\dagger \otimes B)$ with $(q \otimes \mathrm{id}_B)(v_0) = (q \otimes \mathrm{id}_B)(v_1)$. We take $u = v_1^* v_0 \in \mathcal{U}(A^\dagger \otimes B)$ so that $[u]_1 = [v_1^* v_0]_1 = [v_0]_1 = x$ and $(q\otimes \mathrm{id}_B)(u) = 1_B$, which implies that $u \in \mathcal{U}((A \otimes B)^\sim)$. □

An immediate consequence is the following corollary.

Corollary 3.1.5. *If A is a C^*-algebra and $n \geq 2$, then $A \otimes M_{n^\infty}$ is K_1-bijective.*

Proof. In the unital case, the result is Lemma 3.1.2 and Lemma 3.1.3. If A is non-unital, since $A^\dagger \otimes M_{n^\infty}$ is K_1-bijective and M_{n^∞} is unital and K_1-injective, by Lemma 3.1.4, we get K_1-bijectivity of $A \otimes M_{n^\infty}$. □

3.2 K_1-bijectivity of $\mathcal{Z}$-stable C^*-algebras

By Theorem 2.7.4, the Jiang-Su algebra $\mathcal{Z}$ can be realized as a stationary inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$. Thus for a unital C^*-algebra A, to show that $A \otimes \mathcal{Z}$ is K_1-injective or K_1-surjective, it suffices to show the property for $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. Indeed, both properties are preserved by taking inductive limits, which is shown in the following lemma.

Lemma 3.2.1. *Let $A = \varinjlim(A_n, \varphi_n)$ be an inductive limit of unital C^*-algebras A_n and unital *-homomorphisms φ_n. If A_n are all K_1-surjective (or K_1-injective), then A is K_1-surjective (or K_1-injective).*

Proof. We denote by $\varphi^{(n)} : A_n \to A$ the induced unital maps of the inductive system. By continuity of K_1, we have $K_1(A) = \varinjlim(K_1(A_n), K_1(\varphi_n))$ with the induced maps $K_1(\varphi^{(n)})$. By the description of inductive limits in **Ab** in Section 2.1, for any $g \in K_1(A)$, there exists $m \in \mathbb{N}$ and $h \in K_1(A_m)$ such that $g = K_1(\varphi^{(m)})(h)$. Since A_m is K_1-surjective, there is $u \in \mathcal{U}(A_m)$ with $h = [u]_1$. Thus $g = K_1(\varphi^{(m)})[u]_1 = [\varphi^{(m)}(u)]_1$ where $\varphi^{(m)}(u) \in \mathcal{U}(A)$. The proof for K_1-injectivity is similar. □

In the rest of the section, we prove the main result of the section, that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-bijective for a unital C^*-algebra A. We view $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ as a C^*-subalgebra of $C([0,1], A \otimes M_{6^\infty})$ and we have the short exact sequence

$$0 \to A \otimes SM_{6^\infty} \xrightarrow{\iota} A \otimes \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{q} (A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty}) \to 0, \tag{3.2.1}$$

where $q(f) = (f(0), f(1))$ for any $f \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ and the *suspension* of the C^*-algebra M_{6^∞} is denoted by $SM_{6^\infty} = C_0((0,1)) \otimes M_{6^\infty}$.

Passing the short exact sequence (3.2.1) to the six-term exact sequence as explained in Section 2.3.3, it is shown in [14] that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-injective and we include below a sketch of their proof.

Theorem 3.2.2 ([14, Theorem 4.8])**.** *If A is a unital C^*-algebra, then $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-injective.*

Sketch Proof. Take $u \in \mathcal{U}(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$ such that $[u]_1 = 0$ in $K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$. Then $[q(u)]_1 = 0$ and thus $q(u)$ is homotopic to the identity of $(A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})$ since the algebra is K_1-injective by Lemma 3.1.2. Then Lemma 2.3.4 shows the existence of some $v \in \mathcal{U}_0(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$ such that $q(u) = q(v)$. After replacing u with uv^*, we may assume that $q(u) = 1_{(A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})}$ and thus $u \in (A \otimes SM_{6^\infty})^\sim$.

Since $[u]_1 = 0$ in $K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$, the six-term exact sequence gives an element $x \in K_0((A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty}))$ such that $[u]_1 = \delta_0(x)$, where δ_0 is the exponential map. Lemma 3.2.2 shows the existence of projections $p_1, \cdots, p_n, q_1, \cdots, q_n$ such that $x = \sum([p_i]_0 - [q_i]_0)$. By Theorem 2.3.10, in $K_1(A \otimes SM_{6^\infty})$, we have

$$[u]_1 = [e^{-2\pi i h_1} \cdots e^{-2\pi i h_n} e^{2\pi i k_1} \cdots e^{2\pi i k_n}]_1, \tag{3.2.2}$$

where h_i, k_i are positive contractions in $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ lifting p_i and q_i respectively. Replacing u with $ue^{2\pi i h_1} \cdots e^{2\pi i h_n} e^{-2\pi i k_1} \cdots e^{-2\pi i k_n}$, we may assume $[u]_1 = 0$ in $K_1(A \otimes SM_{6^\infty})$. Then Corollary 3.1.5 implies that u is homotopic to the identity of the unitary group of $(A \otimes SM_{6^\infty})^\sim$ and thus in $\mathcal{U}(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$. □

We proved K_1-surjectivity of $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ in [50].

Theorem 3.2.3. *If A is a unital C^*-algebra, then $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-surjective.*

Proof. Consider the short exact sequence (3.2.1). For any $x \in K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$, there exists a unitary $u \in \mathcal{U}_m(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$ for some $m \in \mathbb{N}$ such that $x = [u]_1$. Then $u(0) \in \mathcal{U}_m(A \otimes M_{2^\infty})$ and $u(1) \in \mathcal{U}_m(A \otimes M_{3^\infty})$. By Theorem 3.1.3, both $A \otimes M_{2^\infty}$ and $A \otimes M_{3^\infty}$ are K_1-surjective. Then there exists $w_0 \in \mathcal{U}(A \otimes M_{2^\infty})$ and $w_1 \in \mathcal{U}(A \otimes M_{3^\infty})$ such that $[u(0)]_1 = [w_0]_1 \in K_1(A \otimes M_{2^\infty})$ and $[u(1)]_1 = [w_1]_1 \in K_1(A \otimes M_{3^\infty})$.

Since u is a continuous path in $\mathcal{U}_m(A \otimes M_{6^\infty})$ connecting $u(0)$ and $u(1)$, we have

$$[w_0]_1 = [u(0)]_1 = [u(1)]_1 = [w_1]_1 \in K_1(A \otimes M_{6^\infty}), \tag{3.2.3}$$

where we regard $A \otimes M_{2^\infty}$ and $A \otimes M_{3^\infty}$ as C^*-subalgebras of $A \otimes M_{6^\infty}$. Since $A \otimes M_{6^\infty}$ is unital and UHF-absorbing, it is K_1-injective by Lemma 3.1.2. Thus there exists a unitary path $w \in C([0,1]) \otimes A \otimes M_{6^\infty}$ with $w(0) = w_0 \in A \otimes M_{2^\infty}$ and $w(1) = w_1 \in A \otimes M_{3^\infty}$. Thus $w \in \mathcal{U}(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$ with $q(w) = (w_0, w_1)$ and we have

$$[q(u)]_1 = [q(w)]_1 \in K_1(A \otimes M_{2^\infty}) \oplus K_1(A \otimes M_{3^\infty}). \tag{3.2.4}$$

Then $[u(w \oplus 1_{m-1})^*]_1 \in \ker(K_1(q)) = \operatorname{Im}(K_1(\iota))$ by half-exactness of K_1; see Proposition 2.3.5. Since $A \otimes SM_{6^\infty}$ is K_1-surjective by Corollary 3.1.5, there exists

$$z \in \mathcal{U}((A \otimes SM_{6^\infty})^\sim) \subseteq A \otimes \mathcal{Z}_{2^\infty,3^\infty} \tag{3.2.5}$$

such that $[u(w \oplus 1_{m-1})^*]_1 = K_1(\iota)([z]_1)$. Thus $[u]_1 = [zw]_1 \in K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty})$. □

Remark 3.2.4. For infinite coprime supernatural numbers $\mathfrak{p}, \mathfrak{q}$, the algebra $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is $\mathcal{Z}$-stable by [46, Theorem 4.6], since the fibers of $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ are $\mathcal{Z}$-stable. A direct application of Jiang's result shows that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-surjective, whereas in our approach, Lemma 3.2.3 is the last ingredient needed to reprove K_1-surjectivity of $\mathcal{Z}$-stable C^*-algebras.

Combining the results above, we deduce the following theorem for all C^*-algebras.

Theorem 3.2.5. *If A is a C^*-algebra, then $A \otimes \mathcal{Z}$ is K_1-bijective.*

Proof. In the unital case, this follows from Theorem 3.2.2, Theorem 3.2.3 and Lemma 3.2.1, which allows us to pass K_1-bijectivity through inductive limits. For non-unital A, since $\mathcal{Z}$ is K_1-injective for instance by Theorem 3.2.2, $A \otimes \mathcal{Z}$ is K_1-bijective by applying Lemma 3.1.4. □

All we need in the proof is that $\mathcal{Z}$ is an inductive limit of generalized dimension drop algebras, regardless of the specific connecting maps.

3.3 Homotopy groups for $\mathcal{Z}$-stable C^*-algebras

In this section, we prove K-stability for both unital and non-unital $\mathcal{Z}$-stable C^*-algebras. For a C^*-algebra A, we denote the minimal proper unitization by $A^\dagger$. We use the notation $\mathcal{V}(A)$ for the generalized unitary group of A and $\mathcal{V}_m(A)$ for $\mathcal{V}(M_m \otimes A)$, where $m \geq 1$. By the definition of $\mathcal{V}_m(A)$, there are inclusions $\iota_m : \mathcal{V}_m(A) \to \mathcal{U}_m(A^\dagger)$. Moreover, $\mathcal{V}_\infty(A)$ is defined to be inductive limit of $\mathcal{V}_m(A)$ with metric preserving embeddings $i_{m,m'} : \mathcal{V}_m(A) \to \mathcal{V}_{m'}(A)$, $u \mapsto u \oplus 1_{A^\dagger}$.

It is standard to check that maps $\pi_n(\mathcal{V}_m(A)) \to \pi_n(\mathcal{U}_m(A^\dagger))$ induced by inclusions $\iota_m : \mathcal{V}_m(A) \to \mathcal{U}_m(A^\dagger)$ are isomorphisms for $n \geq 1$ and $m \in \mathbb{N} \cup \{\infty\}$. We include a proof for the case $n = 0$ and $m = 1$, and other cases follow similarly.

Lemma 3.3.1. *The induced map $\pi_0(\iota_1) : \pi_0(\mathcal{V}(A)) \to \pi_0(\mathcal{U}(A^\dagger))$ is an isomorphism.*

Proof. For any $u \in \mathcal{U}(A^\dagger)$, we have $\lambda = q(u) \in \mathbb{T}$. Let $(\lambda_t)_{t\in[0,1]} \subseteq \mathbb{T}$ be a continuous path with $\lambda_0 = 1$ and $\lambda_1 = \bar{\lambda}$, then $(\lambda_t u)_{t\in[0,1]} \subseteq \mathbb{T}$ is a norm-continuous path in $\mathcal{U}(A^\dagger)$ connecting u and $\bar{\lambda} u \in \mathcal{V}(A)$. Thus the map $\pi_0(\iota_1)$ is surjective.

Let $u_0, u_1 \in \mathcal{V}(A)$ be connected by a homotopy $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A^\dagger)$. Then $(\overline{q(u_t)}u_t)_{t\in[0,1]}$ is a continuous path in $\mathcal{V}(A)$ connecting u_0 and u_1, and thus $\pi_0(\iota_1)$ is injective. □

The definition for K-stability was given by Thomsen using quasi-unitaries in [101]. Using [101, Lemma 1.2], it is shown that $a \in A$ is a quasi-unitary if and only if $1-a \in \mathcal{U}(A^\dagger)$. Thus $\mathcal{V}(A)$ is homeomorphic to the topological group of quasi-unitaries in A, and the following definition for K-stability indeed coincides with Thomsen's.

Definition 3.3.2. ([101, Definition 3.1]) A C^*-algebra A is called *K-stable* if for any $m \geq 1$ and $n \geq 0$, the induced maps $\pi_n(i_{m,m+1})$ are isomorphisms.

A K-stable C^*-algebra is K_1-bijective, since in particular $\pi_0(i_{m,m+1})$ are isomorphisms, which implies that $\pi_0(\mathcal{U}(A^\dagger)) \cong \pi_0(\mathcal{V}(A)) \cong K_1(A)$.

Now we prove the main theorem of the section.

Theorem 3.3.3 ([53, Theorem 2.8 and Theorem 3])**.** *If a C^*-algebra A is $\mathcal{Z}$-stable, then it is K-stable. Moreover, in this case, for any integer $n \geq 0$, the following isomorphisms are induced by the canonical embedding $i_{1,\infty} : \mathcal{V}(A) \to \mathcal{V}_\infty(A)$:*

$$\pi_n(\mathcal{V}(A)) \cong \begin{cases} K_0(A) & \text{if } n \text{ is odd,} \\ K_1(A) & \text{if } n \text{ is even.} \end{cases} \tag{3.3.1}$$

Proof. When A is $\mathcal{Z}$-stable, then A is K_1-injective and K_1-surjective by Theorem 3.2.5, which implies $\pi_0(\mathcal{U}(A^\dagger)) \cong K_1(A^\dagger)$. In addition, by Theorem 2.3.11, we have $K_1(A^\dagger) \cong \pi_0(\mathcal{U}_\infty(A^\dagger))$. Since $\pi_0(\iota_0)$ and $\pi_0(\iota_\infty)$ are isomorphisms, the map $i_{1,\infty}$ induces an isomorphism through

$$\pi_0(\mathcal{V}(A)) \cong \pi_0(\mathcal{U}(A^\dagger)) \cong \pi_0(\mathcal{U}_\infty(A^\dagger)) \cong \pi_0(\mathcal{V}_\infty(A)) = K_1(A). \tag{3.3.2}$$

Since $\mathcal{Z}$-stability of A implies $\mathcal{Z}$-stability of $M_m \otimes A$, the maps $i_{m,\infty}$ induce $\pi_0(\mathcal{V}_m(A)) \cong K_1(A)$ for $m \geq 1$ by the argument above and thus all the induced maps $\pi_0(i_{m,m+1})$ are isomorphisms.

For $n \geq 1$, there is a canonical identification between $\pi_n(\mathcal{V}(A))$ and $\pi_0(\mathcal{V}(S^n A))$ by [101, Lemma 2.3], where $S^n A = C_0((0,1)^n) \otimes A$ is the n^{th} suspension of A. When A is $\mathcal{Z}$-stable, then $S^n(M_m(A))$ is $\mathcal{Z}$-stable for any $m, n \geq 1$. Thus by (3.3.2), the maps $i_{m,\infty}$ induce isomorphisms for $m \geq 1$:

$$\pi_n(\mathcal{V}_m(A)) \cong \pi_0(\mathcal{V}(S^n(M_m(A)))) \cong K_1((S^n(M_m(A))) \cong K_{n+1}(A). \tag{3.3.3}$$

Thus $\pi_n(i_{m,m+1})$ are isomorphisms for $m \geq 1$ and $n \geq 0$, which implies K-stability of A. Further by Bott periodicity of K-theory, see Theorem 2.3.9 for instance, for $m \geq 1$ and $n \geq 0$,

$$\pi_n(\mathcal{V}_m(A)) \cong \begin{cases} K_0(A) & \text{if } n \text{ is odd,} \\ K_1(A) & \text{if } n \text{ is even.} \end{cases} \tag{3.3.4}$$

In particular, we get (3.3.1). □

Chapter 4

KK-theory and uniqueness theorems

The target of this chapter is to obtain a uniqueness theorem for unital maps using the machinery of KK- and KL-theory. We start with an introduction to KK-theory and provide one of the concrete descriptions of the object, the Cuntz-Thomsen picture. The definition for KL-theory is provided, which can be considered as the Hausdorffized version of KK-theory, and is the machinery used in classification results later.

The stable uniqueness theorem of Dadarlat and Eilers in [22] shows that the KK-class of a Cuntz pair $[\phi, \psi]$ of morphisms is trivial if and only if, after adding on an "absorbing" map, the maps are properly asymptotically unitarily equivalent. In Section 4.2, we include the definition of absorbing maps. In the case $J = \mathcal{K}$, absorption of maps $A \to \mathcal{M}(J)$ is fully characterized by Voiculescu's theorem (Theorem 4.2.7). To verify absorption for more general maps, Elliott and Kucerovsky proved the generalized Voiculescu's theorem for nuclear domains (Theorem 4.2.12).

In the last section, we describe how to go from the stable uniqueness theorem to KK-uniqueness theorems (for instance, Theorem C and Theorem 1.4.2). We provide a proof of the KK-uniqueness theorem, under the assumption that a relative commutant is K_1-injective. The argument is similar to that of [14, Theorem 5.15], but we are able to avoid some technicalities.

4.1 Basics of KK- and KL-theory

Kasparov introduced KK-theory in [58] as an additive bivariant functor on C^*-algebras. One can make sense of $KK(A, J)$ for A and J not separable, by taking the inductive limit of separable C^*-subalgebras (see [14, Appendix 2]). However, we adhere to the setting where A is separable and J is σ-unital, and we always apply appropriate separabilizations before using KK-theory.

There are various equivalent ways of defining KK-theory, see [52] for a more detailed expository. Among them, the Cuntz-Thomsen picture, which is originally due to Cuntz in [20], suits classification of morphisms especially well, and is the primary approach to KK-theory used in [22] and [14], for instance. In this picture, KK-theory is defined as homotopy equivalence classes of the so-called Cuntz pairs.

Definition 4.1.1. Let A be a separable C^*-algebra and J be a σ-unital and stable C^*-algebra. An (A, J)-*Cuntz pair* is a pair of $*$-homomorphisms $\phi, \psi : A \to \mathcal{M}(J \otimes \mathcal{K})$ such that $\phi(a) - \psi(a) \in J \otimes \mathcal{K}$ for all $a \in A$. We will denote such a Cuntz pair by

$$(\phi, \psi) : A \rightrightarrows \mathcal{M}(J \otimes \mathcal{K}) \rhd J \otimes \mathcal{K}. \tag{4.1.1}$$

Two Cuntz pairs $(\phi_0, \psi_0), (\phi_1, \psi_1) : A \rightrightarrows \mathcal{M}(J \otimes \mathcal{K}) \rhd J \otimes \mathcal{K}$ are *homotopic* if there exists a Cuntz pair

$$(\Phi, \Psi) : A \rightrightarrows \mathcal{M}(C[0,1] \otimes J \otimes \mathcal{K}) \cong C_b([0,1], \mathcal{M}(J \otimes \mathcal{K})_\beta) \rhd C[0,1] \otimes J \otimes \mathcal{K}, \tag{4.1.2}$$

such that the evaluations at 0 and 1 give (ϕ_0, ψ_0) and (ϕ_1, ψ_1) respectively.

In the Cuntz-Thomsen picture, $KK(A, J)$ is defined to be the group of homotopy equivalence classes of (A, J)-Cuntz pairs, where we denote by $[\phi, \psi]_{KK(A,J)}$ the equivalence class of the Cuntz pair $(\phi, \psi) : A \rightrightarrows \mathcal{M}(J \otimes \mathcal{K}) \rhd J \otimes \mathcal{K}$ in $KK(A, J)$.

The addition operation on $KK(A, J)$ is an orthogonal direct sum witnessed by isometries in $\mathcal{M}(J \otimes \mathcal{K})$. Since $\mathcal{M}(J \otimes \mathcal{K})$ contains a copy of $B(H)$ as explained in Example 2.9.5, there exists isometries $s_1, s_2 \in \mathcal{M}(J \otimes \mathcal{K})$ such that $s_1 s_1^* + s_2 s_2^* = 1$. Given *-homomorphisms $\phi_1, \phi_2 : A \to \mathcal{M}(J \otimes \mathcal{K})$, we define the *Cuntz sum* of ϕ_1 and ϕ_2 to be $\phi_1 \oplus_{s_1,s_2} \phi_2 : A \to \mathcal{M}(J \otimes \mathcal{K})$, where

$$(\phi_1 \oplus_{s_1,s_2} \phi_2)(a) = s_1 \phi_1(a) s_1^* + s_2 \phi_2(a) s_2^*, \quad \forall a \in A. \tag{4.1.3}$$

For a different choice of isometries t_1 and t_2 with $t_1 t_1^* + t_2 t_2^* = 1$, the unitary $u = t_1 s_1^* + t_2 s_2^* \in \mathcal{M}(J \otimes \mathcal{K})$ conjugates Cuntz sums. Thus the Cuntz sum is independent of the choice of isometries up to unitary equivalence. The addition on $KK(A, J)$ is defined by,

$$[\phi_1, \psi_1]_{KK(A,J)} + [\phi_2, \psi_2]_{KK(A,J)} = [\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2]_{KK(A,J)}. \tag{4.1.4}$$

The KK-equivalence class of the sum is independent of the choice of isometries, since the unitary group of $\mathcal{M}(J \otimes \mathcal{K})$ is strictly path connected by Proposition 2.9.13, and hence two different sums are homotopic. Thus we omit s_1 and s_2 in the notation.

Then $KK(A, J)$ is an abelian group with the addition given by the Cuntz sum and the zero element is given by $[\phi, \phi]$ for any *-homomorphism $\phi : A \to \mathcal{M}(J \otimes \mathcal{K})$. An important property of KK-theory is that it is a bifunctor, which is contravariant in the first coordinate and covariant in the second coordinate, see [52] and [6] for details.

For a *-homomorphism $\varphi : A \to J$, we get a map $\varphi_p : A \to \mathcal{M}(J \otimes \mathcal{K})$ defined by $\varphi_p(a) = \varphi(a) \otimes p$ where $p \in \mathcal{K}$ is a rank one projection. Then $(\varphi_p, 0)$ forms a Cuntz pair and we denote the corresponding equivalence class in $KK(A, J)$ by $[\varphi]$, which is independent of the choice of the rank one projection.

While KK-theory allows asymptotic classification of maps, the full force of KK-theory is not always needed. To classify morphisms up to approximate unitary equivalence, a weaker machinery called KL-theory was defined by Rørdam in [89]. It is shown that two approximately unitarily equivalent *-homomorphisms from A to J have the same class in $KL(A, J)$, while they might differ in $KK(A, J)$. The group $KL(A, J)$ is defined to be the quotient of $KK(A, J)$ by the closure of 0 in several equivalent topologies. We use the following definition appearing in [21].

Definition 4.1.2. Let A and J be C^*-algebra with A separable. Define

$$Z_{KK(A,J)} = \left\{ KK(A, \mathrm{ev}_\infty)(\kappa) : \begin{array}{c} \kappa \in KK(A, C(\overline{\mathbb{N}}, J)) \text{ and} \\ KK(A, \mathrm{ev}_n)(\kappa) = 0, \forall n \in \mathbb{N} \end{array} \right\} \subseteq KK(A, J), \tag{4.1.5}$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the one-point compatification of $\mathbb{N}$ and ev_n is the evaluation function at $n \in \overline{\mathbb{N}}$ with values in J. Then we define

$$KL(A, J) = KK(A, J)/Z_{KK(A,J)}. \tag{4.1.6}$$

With this definition, KL-theory inherits the bifunctor structure from KK-theory.

4.2 Absorption for maps

The notion of absorbing maps coming out of Voiculescu's non-commutative Weyl–von Neumann Theorem has been critical in the theory of extensions and has become increasingly important in the classification of C^*-algebras. By Voiculescu's theorem, essential extensions by $\mathcal{K}$ are absorbing, and essential trivial extensions by $\mathcal{K}$ are absorbing in the class of trivial extensions. Moreover, essential trivial extensions give the zero element in the extension theory.

We start with several definitions for extensions. For a C^*-algebra J, the *Corona algebra* of J is $\mathcal{C}(J) = \mathcal{M}(J)/J$, with the quotient map $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$. For an element $a \in \mathcal{M}(J)$, we denote $\pi(a)$ by $\bar{a}$ and likewise, we denote $\pi \circ \varphi$ by $\bar{\varphi}$.

Recall that an *extension* of C^*-algebras is a short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} E \xrightarrow{q} A \longrightarrow 0. \tag{4.2.1}$$

By Proposition 2.9.4, there is a canonical map $\phi : E \to \mathcal{M}(J)$. We define the *Busby invariant* of the extension to be the map $\theta : A \to \mathcal{C}(J)$, defined by $\theta(a) = \pi \circ \phi(e)$, where $e \in E$ is an arbitrary lift of a. It is standard to check that θ is well-defined. Moreover, two extensions have the same Busby invariant if and only if they are strongly isomorphic (c.f. [110, Corollary 3.2.12]). All the properties of interest are preserved under strong isomorphism and thus we refer to extensions by their Busby invariants. We say that an extension $\theta : A \to \mathcal{C}(J)$ is

(i) *unital* if A is unital and θ is unital;

(ii) *essential* if θ is injective;

(iii) *trivial* if there exists a *-homomorphism $\varphi : A \to \mathcal{M}(J)$ such that $\theta = \bar{\varphi}$.

For a stable C^*-algebra J and two extensions $\theta, \beta : A \to \mathcal{C}(J)$, the direct sum $\theta \oplus \beta$ can be defined in a similar way as the Cuntz sum, using isometries coming from $\mathcal{M}(J)$. By a similar argument, the sum is independent of the choice of isometries up to unitary equivalence by a unitary in $\mathcal{M}(J)$. We will give definition of absorption for both general extension and for trivial extension, which are *-homomorphisms into multiplier algebras. The relation between the two notions is later described in Corollary 4.2.5.

Definition 4.2.1. Let A be a (unital) separable C^*-algebra and J be a σ-unital and stable C^*-algebra. A (unital) extension $\theta : A \to \mathcal{C}(J)$ is *(unitally) absorbing* if for every (unital) trivial extension $\beta : A \to \mathcal{C}(J)$, there is a unitary $u \in \mathcal{M}(J)$ such that

$$\mathrm{Ad}(\bar{u}) \circ (\theta \oplus \beta) = \theta. \tag{4.2.2}$$

The most important examples of absorbing extensions, particularly for our applications, are trivial extensions $\phi : A \to \mathcal{M}(J)$.

Definition 4.2.2. Let A be a (unital) separable C^*-algebra and J be a σ-unital and stable C^*-algebra. A (unital) *-homomorphism $\phi : A \to \mathcal{M}(J)$ is *(unitally) absorbing* if for every (unital) *-homomorphism $\psi : A \to \mathcal{M}(J)$, then $\phi \oplus \psi$ and ψ

are *approximately unitarily equivalent modulo* J, i.e. there is a sequence of unitaries $(u_n)_n \in \mathcal{M}(J)$ such that for any $a \in A$,

(i) $u_n(\phi(a) \oplus \psi(a))u_n^* - \phi(a) \in J$;

(ii) $\lim_{t\to\infty} \|u_n(\phi(a) \oplus \psi(a))u_n^* - \phi(a)\| = 0$.

From the definition, it is straightforward to check that any two (unitally) absorbing *-homomorphisms ϕ, ψ are approximately unitarily equivalent modulo J, since they are both approximately unitarily equivalent to $\phi \oplus \psi$ modulo J. It turns out that they are equivalent in a stronger sense of asymptotical unitary equivalence as a direct consequence of the following theorem. The technique used in the proof originates from [22, Lemma 2.3], and an explicit proof for absorbing maps can be found in [14, Proposition 5.9, Corollary 5.10]. Similar proof also works for unitally absorbing maps verbatim.

Theorem 4.2.3 ([14, Proposition 5.9, Corollary 5.10])**.** *Let A be a (unital) separable C^*-algebra and J be a σ-unital and stable C^*-algebra. Then the following statements are equivalent for a (unital) *-homomorphism $\phi : A \to \mathcal{M}(J)$,*

(i) ϕ is (unitally) absorbing;

(ii) for every (unital) *-homomorphism $\psi : A \to \mathcal{M}(J)$, we have $\phi \oplus \psi$ is unitarily equivalent to ϕ modulo J, i.e. there exists a unitary $u \in \mathcal{M}(J)$ such that

$$u(\phi \oplus \psi)(a)u^* - \phi(a) \in J, \quad a \in A. \tag{4.2.3}$$

(iii) for every (unital) *-homomorphism $\psi : A \to \mathcal{M}(J)$, we have $\phi \oplus \psi$ is asymptotically unitarily equivalent to ϕ modulo J, i.e. there exists a continuous path of unitaries $(u_t)_{t\geq 0} \subseteq \mathcal{M}(J)$ such that

(1) $\lim_{t\to\infty} \|u_t(\phi \oplus \psi)(a)u_t^* - \psi(a)\| = 0$, for all $a \in A$;

(2) $u_t(\phi \oplus \psi)(a)u_t^* - \psi(a) \in J$, for all $a \in A$ and $t \geq 0$.

Corollary 4.2.4. *Let A be a (unital) separable C^*-algebra and J be a σ-unital and stable C^*-algebra. If $\phi, \psi : A \to \mathcal{M}(J)$ are both (unitally) absorbing, then ϕ and ψ are asymptotically unitarily equivalent modulo J.*

The theorem also shows that the two definitions of absorption for trivial extensions are equivalent. Thus we will use two notions of absorption for trivial extensions interchangeably.

Corollary 4.2.5 (c.f. [14, Corollary 5.10]). *Let A be a (unital) separable C^*-algebra and J be a σ-unital and stable C^*-algebra. Then a *-homomorphism $\phi : A \to \mathcal{M}(J)$ is (unitally) absorbing if and only if $\bar{\phi} : A \to \mathcal{C}(J)$ is (unitally) absorbing.*

We have defined both absorption and unital absorption. Let us now turn to the interplay between these two notions. Notice that an absorbing map is never unital. Indeed, for a unital C^*-algebra A and a unital map $\phi : A \to \mathcal{M}(J)$, it follows that $\phi \oplus 0$ and ϕ are not unitarily equivalent modulo J, since $(\phi \oplus 0)(1) = s_1 s_1^*$ is not equal to $\phi(1) = 1_{\mathcal{M}(J)}$ modulo J. The connection between unitally absorbing maps and absorbing maps, as we shall see in Proposition 4.2.9 for instance, allows us to transfer results from one class to the other class. The following lemma is contained in [101, Theorem 2.5]. In fact, Thomsen defined a representation to be absorbing if its forced unitization is unitally absorbing as defined in Definition 4.2.2.

Lemma 4.2.6 ([101, Theorem 2.5]). *Let A be a separable C^*-algebra and J be a σ-unital C^*-algebra. Then a *-homomorphism $\phi : A \to \mathcal{M}(J)$ is absorbing if and only if the forced unitization $\phi^\dagger : A^\dagger \to \mathcal{M}(J)$ is unitally absorbing.*

In the fundamental case $J = \mathcal{K}$, absorption for maps is fully characterized by Voiculescu's non-commutative Weyl–von Neumann Theorem.

Theorem 4.2.7 (c.f. [24, Theorem II.5.8]). *Let A be a (unital) separable C^*-algebra. The following are equivalent for an (unital) extension $\theta : A \to \mathcal{C}(\mathcal{K})$,*

(i) θ is (unitally) absorbing;

(ii) θ is an essential extension.

In addition, if θ is a trivial extension, with a lifting map $\phi : A \to \mathcal{M}(\mathcal{K})$, then the two statements above are furthermore equivalent to

(iii) ϕ is faithful and $\phi(A) \cap \mathcal{K} = \{0\}$.

The equivalence between (*ii*) and (*iii*) holds for relatively trivial reasons. A trivial extension $\theta : A \to \mathcal{C}(J)$ is essential if and only if the lifting *-homomorphism $\phi : A \to \mathcal{M}(J)$ is injective and $\phi(A) \cap J = \{0\}$.

The following theorem shows the existence of (unitally) absorbing maps when $J = \mathcal{K}$, since every C^*-algebra admits injective representations whose image intersects $\mathcal{K}$ trivially, by taking infinite direct sum of any faithful representation for instance. In general, the existence of absorbing maps was shown by Thomsen in [101].

Theorem 4.2.8 ([101, Theorem 2.4, Theorem 2.7])**.** *Let A be a (unital) C^*-algebra and J be a σ-unital and stable C^*-algebra. Then there exists an (unitally) absorbing *-homomorphism $\phi : A \to \mathcal{M}(J)$.*

For a general σ-unital and stable C^*-algebra J, the characterization of absorption is not easy. In the case $J = \mathcal{K}$, absorption is equivalent to essentialness, which turns out to be necessary in the general setting. We include a short proof of this fact.

Proposition 4.2.9. *Let A be a (unital) C^*-algebra and J be a σ-unital and stable C^*-algebra. If $\phi : A \to \mathcal{M}(J)$ is (unitally) absorbing, then $\bar{\phi}$ is essential, or equivalently, ϕ is injective and $\phi(A) \cap J = \{0\}$.*

Proof. Without loss of generality, we assume that ϕ is unitally absorbing, since if not, it suffices to look at the forced unionization by Lemma 4.2.6. Take a unital faithful representation $\varphi : A \to B(\mathcal{H})$ with $\varphi(A) \cap \mathcal{K} = \{0\}$. The unital *-homomorphism

$$\psi : A \to B(\mathcal{H}) \otimes \mathcal{M}(J) \subseteq \mathcal{M}(J \otimes \mathcal{K}), \quad a \mapsto \varphi(a) \otimes 1_{\mathcal{M}(J)} \tag{4.2.4}$$

is injective and $\psi(A) \cap J = \{0\}$, which implies that $\bar{\psi}$ is injective. Since ϕ is unitally absorbing, it follows that $\bar{\phi} \oplus \bar{\psi}$ and $\bar{\phi}$ are unitarily equivalent. Thus $\bar{\phi}$ is injective as the direct sum of extensions with one of them being injective is again injective. □

Essential extensions are not necessarily absorbing. For instance, when A is nuclear and separable, every essential (unital) extension $\phi : A \to \mathcal{M}(J)$ is (unitally) absorbing if and only if $J = \mathcal{K}$ or J is simple and purely infinite (see [32, Theorem 3.16]).

In order to check whether a map is unitally absorbing, Elliott and Kucerovsky provided a characterization in [29], in the presence of nuclearity, in terms of a notion called pure largeness. This result has since played an important role in the abstract classification approach (see [97] and [14]).

More recently, a new definition of pure largeness for extensions is given in [11] and will be further explored in the follow-up paper of [14]. It is shown to be equivalent to the definition given by Elliott and Kucerovsky, when the ideal J is σ-unital and stable. For the purpose of classifying maps, we are mainly interested in J with these standing assumptions. Thus we will use the new definition of purely largeness.

Definition 4.2.10. An extension of J,

$$0 \longrightarrow J \longrightarrow E \longrightarrow D \longrightarrow 0 \tag{4.2.5}$$

is *purely large* if for each $a \in J_+$ and $b \in E_+ \setminus J$, we have $a \precsim b$ in E.

The equivalence between the two different definitions is given by the theorem below. To state the theorem, recall that a C^*-subalgebra $B \subseteq A$ is said to be *full* in A if B generates A as a two-sided ideal.

Proposition 4.2.11 (c.f. [11, Theorem 4.14])**.** *Let J be a σ-unital and stable C^*-algebra, then an extension of J,*

$$0 \longrightarrow J \longrightarrow E \longrightarrow D \longrightarrow 0 \tag{4.2.6}$$

is purely large if and only it is purely large in the sense of Elliott and Kuverovsky: for any $e \in E_+ \setminus J$, there is a σ-unital and stable C^*-subalgebra in $\overline{eJe^*}$ that is full in J.

Notice that a purely large extension is essential using a standard characterization of essentialness (see [110, Proposition 3.2.15]). Moreover, combining the result of Elliott–Kuverovsky ([29]) and Gabe ([31]), one gets characterizations for absorption, with the unital and non-unital case separated. They proved the result for nuclearly absorbing maps, but for simplicity, we will instead assume that our domain C^*-algebra is nuclear.

Theorem 4.2.12 ([29, 31])**.** *Let A be a separable and nuclear C^*-algebra and J be a σ-unital and stable C^*-algebra. Let $\phi : A \to \mathcal{C}(J)$ be the Busby map of an extension.*

(i) If ϕ is unital, ϕ is unitally absorbing if and only if the extension is purely large;

(ii) If ϕ is non-unital, ϕ is absorbing if and only if the extension is purely large and absorbs the zero extension.

In practice, Kucerovsky and Ng's corona factorization property in [62] provides a provides a clean characterization for pure largeness. A σ-unital and stable C^*-algebra J has the *corona factorization property* if every full projection in $\mathcal{M}(J)$ is properly infinite. This is a relatively mild regularity condition and in particular, C^*-algebras whose Cuntz semigroup is almost unperforated have this property.

Proposition 4.2.13 ([79, Proposition 2.17])**.** *Let A be a C^*-algebra. If* $\mathrm{Cu}(A)$ *is almost unperforated, then A has the corona factorization property.*

Corona factorization property allows one to obtain pure largeness for maps that are full or unitizably full (see [62, Theorem 2.1]). Combining with Theorem 4.2.12, the following theorem gives an accessible verification for absorption. Recall that a *-homomorphism $\varphi : A \to B$ is *full* if for any nonzero $a \in A$, we have $\varphi(a)$ is full in B. The map is *unitizably full* if $\varphi^\dagger$ is full.

Theorem 4.2.14 ([62, Theorem 2.1]). *Let A be a separable and nuclear C^*-algebra and J be a σ-unital and stable C^*-algebra with the corona factorization property. Let $\phi : A \to \mathcal{C}(J)$ be a *-homomorphism.*

(i) If ϕ is unital, then ϕ is unitally absorbing if and only if ϕ is full;

(ii) If ϕ is non-unital, then ϕ is absorbing if and only if ϕ is unitizably full.

In particular, both (i) and (ii) hold when J has strict comparison.

4.3 Towards KK-uniqueness

A breakthrough in KK-theory is the stable uniqueness theorem, which was proved independently by Lin in [67], and Dadarlat and Eilers in [22, Theorem 3.8]. In the work of Dadarlat and Eilers, they show that a Cuntz pair $[\phi, \psi]$ is the zero element of $KK(A, J)$ if and only if ϕ and ψ are *stably properly asymptotically unitarily equivalent* modulo J. Part of the definition is given by the following.

Definition 4.3.1. Let A be a separable C^*-algebra and J be a σ-unital and stable C^*-algebra. Then *-homomorphisms $\varphi, \psi : A \to \mathcal{M}(J)$ are *asymptotically unitarily equivalent modulo J* if there is a continuous path of unitaries $(u_t)_{t\geq 0} \subseteq \mathcal{M}(J)$ with

(i) $\lim_{t\to\infty} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$, for all $a \in A$;

(ii) $u_t\varphi(a)u_t^* - \psi(a) \in J$ for all $a \in A$ and $t \geq 0$.

If in addition the path of unitaries $(u_t)_{t\geq 0}$ can be found in $J^\dagger$, then φ and ψ are *properly asymptotically unitarily equivalent modulo J*

The term "proper" refers to the fact that the unitaries witnessing the equivalence can be chosen in the minimal unitization of J. This is exactly the feature that allows KK-theory to work nicely in the abstract classification framework developed in [97] and described briefly in Section 1.4. To classify $\phi, \psi : A \to B_\omega$, we form a Cuntz pair into the multiplier algebra of the trace-kernel ideal. If the maps are only asymptotically unitarily equivalent, the witnessing unitaries are a priori found in the multiplier algebra, which is usually much bigger than B_ω. The equivalence relation being "proper" gives unitaries in the smallestpossible algebra contained in B_ω, and thus an asymptotic unitary equivalence between the original pair of maps ϕ, ψ. This point of view will become clear in the main proof of the classification theorem (Theorem E).

The word "stable" reflects that the equivalence is only satisfied after adding an absorbing map to both ϕ and ψ. We make this more precise by the following result. We will mainly focus on unital maps and unitally absorbing maps.

Theorem 4.3.2 ([22, Theorem 3.8]). *Let A be a separable unital C^*-algebra, and let J be a separable stable C^*-algebra. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair. Then $[\phi, \psi] = 0$ in $KK(A, J)$ if and only if for any unitally absorbing map $\theta : A \to \mathcal{M}(J)$, $\phi \oplus \theta$ and $\psi \oplus \theta$ are proper asymptotically unitarily equivalent modulo J.*

It is natural to consider whether one could get proper asymptotic unitary equivalence directly, without needing to add on an absorbing map. Dadarlat and Eilers proved the following results when $J = \mathcal{K}$.

Theorem 4.3.3 ([22, Theorem 3.12]). *Let A be a unital separable C^*-algebra. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(\mathcal{K}) \rhd \mathcal{K}$ be a Cuntz pair. Suppose that ϕ and ψ are unitally absorbing. Then $[\phi, \psi] = 0$ in $KK(A, \mathbb{C})$ if and only if ϕ and ψ are properly asymptotically unitarily equivalent modulo J.*

Their proof is based on the observation that for the path of unitaries $(u_t)_{t\geq 0}$ in Corollary 4.2.4 witnessing the asymptotic unitary equivalence, $\pi(u_0)$ commutes with $\bar{\phi}(A)$, where $\pi_{\mathcal{K}} : \mathcal{M}(\mathcal{K}) \to \mathcal{C}(\mathcal{K})$ is the quotient map and $\bar{\phi} = \pi_{\mathcal{K}} \circ \phi$. If $[\phi, \psi] = 0$ in $KK(A, \mathbb{C})$, by passing through the Paschke duality $KK(A, \mathbb{C}) \cong K_1(\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)')$, we get that $\pi_{\mathcal{K}}(u_0)$ has the trivial K_1-class in $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$. As we will show later in Theorem 6.1.4, the relative commutant $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$ is K_1-injective by the result of Paschke ([81]) and thus, $\pi_{\mathcal{K}}(u_0)$ is homotopic to the identity in $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$. The rest of the proof is based on [22, Theorem 3.12] or the following lemma derived from [14, Lemma 5.16] (taking E to be $\mathcal{M}(J)$ and I to be J), extracting the main argument of Dadarlat and Eilers. We denote by $\pi_J : \mathcal{M}(J) \to \mathcal{C}(J)$ the quotient map. For any element $a \in \mathcal{M}(J)$, we write $\bar{a}$ for $\pi_J(a)$. Similarly, for any map $\phi : A \to \mathcal{M}(J)$, we write $\bar{\phi}$ for $\pi_J \circ \phi$.

Lemma 4.3.4 ([22, Theorem 3.12], [14, Lemma 5.16]). *Let A be a unital separable C^*-algebra and J be a separable stable C^*-algebra. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair. Suppose that ϕ and ψ are unitally absorbing and they are asymptotically unitarily equivalent modulo J.*

If the path of unitaries $(u_t)_{t\geq 0}$ witnessing the equivalence can be chosen such that $\pi_J(u_0)$ is homotopic to the identity in $\mathcal{C}(J) \cap \bar{\phi}(A)'$, then $(u_t)_{t\geq 0}$ can be chosen in $J^\dagger$.

The argument of Dadarlat and Eilers for proving Theorem 4.3.3 can be made to work much more generally, as mentioned in [14, Section 5.4], and explicitly stated and proved in [70, Theorem 2.5]. The statement of [70, Theorem 2.5] is correct, although in the proof, there is a subtle but minor gap when referring to the result of Thomsen in [101, Theorem 3.2]. Thomsen's Paschke duality as established in [101] is proved when ϕ is absorbing, while the proof of [70, Theorem 2.5] goes with unitally absorbing maps. Thus, for completeness, we include a proof of the KK-uniqueness theorem following the argument of [14, Theorem 5.15].

Theorem 4.3.5. *Let A be a separable and unital C^*-algebra and J be a separable and stable C^*-algebra. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair with ϕ and ψ unitally absorbing. If $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective, then the following statements hold:*

(i) If $[\phi, \psi]_{KK(A,J)} = 0$, there exists a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $J^\dagger$ such that

$$\|u_t(\phi(a))u_t^* - \psi(a)\| \to 0, \quad a \in A. \tag{4.3.1}$$

(ii) If $[\phi, \psi]_{KL(A,J)} = 0$, there exists a sequence $(u_n)_{n=1}^\infty$ of unitaries in $J^\dagger$ such that

$$\|u_n(\phi(a))u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{4.3.2}$$

We follow the proof of [14, Theorem 5.15]. It is worth noting that although the main focus of [14] is the classification of unital maps, due to the highly non-unital nature of KK-theory, the deunitization trick is needed in proving the existence theorem (see remarks preceding [14, Definition 5.12] and [14, Proposition 7.11]). Thus, the statement of [14, Theorem 5.15] is for absorbing representations, while a forced unitization is needed eventually in the proof. Given that we are only interested in the uniqueness theorem of maps, we can avoid the deunitization and forced unitization techniques in [14] and provide a more direct proof for Theorem 4.3.5. We include the following reformulation of Dadarlat and Eilers' result in terms of Cuntz pairs, since the original theorem is stated in Kasparov's picture.

Lemma 4.3.6 ([22, Lemma 3.5])**.** *Let A be a separable unital C^*-algebra and J be a separable and stable C^*-algebra. Let $\phi : A \to \mathcal{M}(J)$ be a unital *-homomorphism and u_1, u_2 are unitaries in $\mathcal{M}(J)$ such that $\bar{u}_1, \bar{u}_2 \in \mathcal{C}(J) \cap \bar{\phi}(A)'$.*

If $[\phi, \mathrm{Ad}(u_1)\phi] = [\phi, \mathrm{Ad}(u_2)\phi] \in KK(A, J)$, there exists a unital *-homomorphism $\theta : A \to \mathcal{M}(J)$ and a norm-continuous path $(v_t)_{t\in[0,1]}$ of unitaries in $\mathcal{M}(J)$ such that $v_0 = u_1 \oplus 1$, $v_1 = u_2 \oplus 1$ and $(\phi \oplus \theta, \mathrm{Ad}(v_t)(\phi \oplus \theta))$ are Cuntz pairs.

We can think of v_t as unitaries in $M_2(J)$ since J is stable. Now we are ready to prove the main KK-uniqueness theorem.

Proof of Theorem 4.3.5. Let ϕ, ψ be unitally absorbing maps with $[\phi, \psi] = 0$ in $KK(A, J)$. By Theorem 4.2.4, it follows that ϕ and ψ are asymptotically unitarily equivalent witnessed by a norm-continuous path $(u_t)_{t\geq 0} \subseteq \mathcal{M}(J)$ of unitaries. Then $(\phi, \mathrm{Ad}(u_0)\phi)$ is a Cuntz pair. By [22, Lemma 3.1],

$$[\phi, \mathrm{Ad}(u_0)\phi] = [\phi, \psi] = 0 = [\phi, \phi] \in KK(A, J). \tag{4.3.3}$$

By Lemma 4.3.4, it suffices to show that $\bar{u}_0$ is in the path connected component of the identity in $\mathcal{C}(J) \cap \bar{\phi}(A)'$. Since $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is assumed to be K_1-injective, it suffices to show that $[\bar{u}_0]_1 = 0$ in $K_1(\mathcal{C}(J) \cap \bar{\phi}(A)')$.

Since $[\phi, \phi] = [\phi, \mathrm{Ad}(u_0)\phi]$, by Lemma 4.3.6, we have $u_0 \oplus 1 \sim_h 1 \oplus 1$ in $M_2(\mathcal{M}(J)) \cap (\phi \oplus \theta)(A)')$. It follows that $\bar{u}_0 \oplus 1 \sim_h 1 \oplus 1$ in $M_2(\mathcal{C}(J)) \cap (\bar{\phi} \oplus \bar{\theta})(A)')$. Adding another copy of the identity, we get $\bar{u}_0 \oplus 1 \oplus 1 \sim_h 1 \oplus 1 \oplus 1$ in $M_3(\mathcal{C}(J)) \cap (\bar{\phi} \oplus \bar{\theta} \oplus \bar{\phi})(A)')$. Since ϕ is absorbing, it follows that $\bar{\phi} \oplus \bar{\theta}$ is unitarily equivalent to $\bar{\phi}$. Thus, $\bar{u}_0 \oplus 1 \sim_h 1 \oplus 1$ in $M_2(\mathcal{C}(J)) \cap (\bar{\phi} \oplus \bar{\phi})(A)') \cong M_2(\mathcal{C}(J) \cap \bar{\phi}(A)')$. This implies that $[\bar{u}_0]_1 = 0$ in $K_1(\mathcal{C}(J) \cap \bar{\phi}(A)')$ and concludes the proof. □

Chapter 5

Ideals in multiplier algebras

In Chapter 5, we define a notion called relative pure largeness for ideals, resembling the pure largeness of extensions in Chapter 4. This turns out to be the abstract condition needed to prove K_1-injectivity of the relative commutant in the next chapter.

We are motivated by the work [70] of Loreaux, Ng, and Sutradhar, where they studied ideals between J and $\mathcal{M}(J)$, for a simple, stable and separable C^*-algebra J with unique trace and strict comparison. The unique ideal they found has the nice property that every positive element not in the ideal Cuntz dominates all positive elements in J.

In the second half of the chapter, we show the existence of such proper closed ideals in $\mathcal{M}(J)$ containing J, under regularity assumptions such as real rank zero and stable rank one (Theorem 5.2.1). The existence of relatively purely large ideals will be an important ingredient in the proof of the main K_1-injectivity theorem (Theorem 6.1.8).

5.1 Relative pure largeness

We define the abstract property needed to prove the main K_1-injectivity theorem in Chapter 6, by means of a relative version of pure largeness given in Definition 4.2.10.

Definition 5.1.1. Let J be an ideal of I. An extension by I,

$$0 \longrightarrow I \longrightarrow E \longrightarrow D \longrightarrow 0 \tag{5.1.1}$$

is *purely large relative to* J if for each $a \in J_+$ and $b \in E_+ \setminus I$, we have $a \precsim b$ in E.

As we shall illustrate further in Chapter 6, the central ingredient to prove K_1-injectivity of the relative commutant for unitally absorbing maps, and thus obtaining a suitable KK-uniqueness theorem for the framework of Theorem A and Theorem

B, is the following: identifying an ideal $\mathcal{I}$ in $\mathcal{M}(J)$ containing J, such that the corresponding extension of $\mathcal{I}$,

$$0 \longrightarrow \mathcal{I} \stackrel{\iota}{\longrightarrow} \mathcal{M}(J) \stackrel{q}{\longrightarrow} \mathcal{M}(J)/\mathcal{I} \longrightarrow 0 \tag{5.1.2}$$

is purely large relative to J. That is to say, any positive element in $\mathcal{M}(J)$ that is not in $\mathcal{I}$ Cuntz-dominates any positive element in J.

The following equivalent characterization of such an ideal $\mathcal{I}$ allows us to push the elements witnessing the Cuntz subequivalence into J. Moreover, we get norm control for the witnessing elements. The lemma below is an analogue of [11, Proposition 4.10] for relatively purely large extensions with almost the same proof. The technique can be traced back to the work of Elliott and Kucerovsky ([29, Lemma 7]).

Lemma 5.1.2. *Let $\mathcal{I}$ be an ideal in $\mathcal{M}(J)$ containing J. The following are equivalent:*

(i) the extension (5.1.2) *of $\mathcal{I}$ in $\mathcal{M}(J)$ is purely large relative to J;*

(ii) for any non-zero positive elements $b \in \mathcal{M}(J) \setminus \mathcal{I}$, $a \in J$, and $\epsilon > 0$, there exists $x \in J$ with $\|x\|^2 \leq \|a\|/\|q(b)\|$, such that $a \approx_\epsilon xbx^*$.

Proof. We prove the nontrivial implication from (i) to (ii).

Fix positive elements $b \in \mathcal{M}(J) \setminus \mathcal{I}$ and $a \in J$, then $a \precsim b$ as the extension is purely large relative to J. Dividing b by $\|q(b)\|$ and dividing a and ϵ by $\|a\|$ whenever necessary, we can assume that $\|a\| = 1$ and $\|q(b)\| = 1$.

Consider first the case $\|b\| = 1$, then $f_\delta(b) \approx_{\epsilon/2} b$ for $\delta = 1 - \epsilon/2$, where f_δ is the positive norm 1 funncion defined in (6.3.1). Since $\|q(b)\| = 1$, we have

$$\|q((b-\delta)_+)\| = \|(q(b)-\delta)_+\| > 0. \tag{5.1.3}$$

This implies that $(b-\delta)_+ \in \mathcal{M}(J)_+ \setminus \mathcal{I}$ and thus $a \precsim (b-\delta)_+$ by the relative pure largeness of the extension. By Lemma 2.8.8, there exists a contraction $x \in \mathcal{M}(J)$ such that

$$a \approx_{\epsilon/3} xf_\delta(b)x^* \approx_{\epsilon/3} xbx^*. \tag{5.1.4}$$

Let $e \in J$ be a positive contraction such that $a \approx_{\epsilon/3} eae$, then

$$a \approx_{\epsilon/3} eae \approx_{2\epsilon/3} exbx^*e, \tag{5.1.5}$$

where the witnessing element ex is a contraction in J.

For the general case where $\|b\| > \|q(b)\| = 1$, consider the continuous function

$$h : [0, \|b\|] \to [0,1], \quad t \mapsto \min\{t, 1\}. \tag{5.1.6}$$

then $\|h(b)\| = 1$ and $\|q(h(b))\| = \|h(q(b))\| = 1$. By what we have proven previously, there exists a contraction $y \in J$ such that $a \approx_\epsilon yh(b)y^*$. Moreover, we have $h(b) = g(b)bg(b)$ where $g : [0, \|b\|] \to [0, 1]$ is defined by

$$g(t) = \begin{cases} 1 & 0 \leq t \leq 1, \\ 1/t^2 & 1 \leq t \leq \|b\|. \end{cases} \tag{5.1.7}$$

Taking $x = yg(b)$, which is a contraction in J, we then have $a \approx_\epsilon xbx^*$. □

This notion of relatively purely large ideal is abstracted from the study of ideals between J and $\mathcal{M}(J)$ when J is assumed to be monotracial, simple, stable and have strict comparison appearing in the work of Loreaux, Ng and Sutradhar. In the stably unital case, where $J = A \otimes \mathcal{K}$ for a unital C^*-algebra A, the ideals between $\mathcal{M}(J)$ and J are understood for C^*-algebras with finitely many extremal traces by the work of Rørdam in [87]. For C^*-algebras J that are not stably unital, but have a unique trace τ_J, the existence of such relatively purely large ideals is a direct consequence of the strict comparison results for multiplier algebras in [57, Theorem 5.3].

In the work [57] of Kaftal, Ng, and Zhang, since J is stable and thus non-unital, they consider the lower semicontinuous densely defined tracial weight on J ([57, Section 2.2]). By [80, Proposition 5.2], these tracial weights on J can be uniquely extended to lower semicontinuous tracial weights on J^{**}. We call the restriction of these tracial weights to $\mathcal{M}(J)$ the *extended traces from J*. The multiplier algebra $\mathcal{M}(J)$ is said to have *strict comparison with respect to the traces extended from J* (see [57]), if for any nonzero $a, b \in \mathcal{M}(J)_+$, we have $a \precsim b$ if the following are true:

(i) a is in the ideal generated by b in $\mathcal{M}(J)$;

(ii) $d_\tau(a) < d_\tau(b)$ for all traces extended from J with $d_\tau(b) < \infty$.

Theorem 5.1.3. *Let J be a separable, simple and stable C^*-algebra with a unique trace and strict comparison, then there exists a closed proper ideal $\mathcal{I} \subseteq \mathcal{M}(J)$ containing J such that the corresponding extension is purely large relative to J.*

Proof. We denote by τ_J the unique trace of J, which extends naturally to $\mathcal{M}(J)$. Consider the closed proper ideal in $\mathcal{M}(J)$,

$$\mathcal{I} = \overline{\{x \in \mathcal{M}(J) : \tau_J(x^*x) < \infty\}}, \tag{5.1.8}$$

which contains J as an ideal. Take nonzero elements $b \in \mathcal{M}(J)_+ \backslash \mathcal{I}$ and $a \in J_+$. Then a is in the ideal generated by b in $\mathcal{M}(J)$, since any non-trivial ideal in $\mathcal{M}(J)$ intersects

J nontrivally and thus generates J by simplicity. Moreover, by the definition of $\mathcal{I}$, we have $d_{\tau_J}(b) \geq \tau_J(b) = \infty$, which means that the condition (ii) above is trivially satisfied. Thus $a \precsim b$ by strict comparison of $\mathcal{M}(J)$ shown in [57, Theorem 5.3]. $\square$

The existence of such purely large ideals, together with the main result of Chapter 6, will allow us to recapture the result of Loreaux, Ng, and Sutradhar in the unique trace case ([70, Theorem 3.23]). In the rest of the chapter, we show the existence of ideals in $\mathcal{M}(J)$ that are relatively purely large, with the help of certain regularity conditions on J, such as real rank zero and stable rank one.

5.2 Purely large ideals in multiplier algebras

Let J be a C^*-algebra. In this section, we show the existence of ideals $\mathcal{I}$ in $\mathcal{M}(J)$ such that the extension of $\mathcal{I}$ in $\mathcal{M}(J)$ is purely large with respect to J, whenever J satisfies the following list of properties:

(i) separable and stable;

(ii) *real rank zero*, which means that the set of invertible self-adjoint elements in the minimal unitization $\tilde{J}$ of J is dense in the set of self-adjoint elements of $\tilde{J}$;

(iii) *stable rank one*, which means the set of invertible elements is dense in $\tilde{J}$;

(iv) $K_1(J) = 0$;

(v) the Murray-von Neumann semigroup $V(J)$ is totally ordered.

Notice that when J is stable, the semigroup $V(J)$ is identified with the dimension range $\mathcal{D}(J)$, consisting of Murray von Neumann equivalence classes of projections in J. We will mainly work with the dimension range in this chapter, since relevant results in the literature are formulated in terms of $\mathcal{D}(J)$. On the other hand, theorems will be stated in terms of $V(J)$.

Despite the seemingly restrictive assumptions, this class of C^*-algebras covers two particular examples of interest: appropriate separabilizations of the trace-kernel ideals of $\prod_\omega M_{n_k}$ and $\mathcal{R}_\omega$. Although these separabilizations of the trace-kernel ideal will have strict comparison inherited from strict comparison of $\prod_\omega M_{n_k}$ and $\mathcal{R}_\omega$, they are highly non-simple and thus will have many tracial weights. Thus, the following main theorem of the section covers situations outside of the setting of Theorem 5.1.3 and the work of Loreaux, Ng, and Sutradhar.

Theorem 5.2.1. *Let J be a separable and stable C^*-algebra with real rank zero, stable rank one, $K_1(J) = 0$, and totally ordered $V(J)$. Then there exists a unique maximal proper ideal $\mathcal{I}$ of $\mathcal{M}(J)$ containing J such that for any $b \in \mathcal{M}(J)_+$, we have $b \notin \mathcal{I}$ if and only if $b \sim 1_{\mathcal{M}(J)}$. In particular, $\mathcal{I}$ is purely large in $\mathcal{M}(J)$ with respect to J and the quotient $\mathcal{M}(J)/\mathcal{I}$ is simple and purely infinite.*

We start with a brief exposition of how the assumptions are used in the proof:

(i) Stability of J is used to identify $\mathcal{D}(J) \cong V(J)$ and $\mathcal{D}(\mathcal{M}(J)) \cong V(\mathcal{M}(J))$;

(ii) when J is separable with real rank zero, the lattice of closed ideals in $\mathcal{M}(J)$ is in one-to-one correspondence with ordered ideals in $\mathcal{D}(\mathcal{M}(J))$, see Theorem 5.2.4;

(iii) when J is separable, has real rank zero and cancellation (a consequence of stable rank one), a slight adaptation of Goodearl's theory describes $\mathcal{D}(\mathcal{M}(J))$ in terms of intervals in $\mathcal{D}(J)$, see Theorem 5.2.3;

(iv) total ordering of $\mathcal{D}(J)$ shows that proper intervals in $\mathcal{D}(J)$ are bounded above by some projection in J, see Lemma 5.2.5;

(v) when J is separable, has real rank zero, stable rank one and $K_1(J) = 0$, the multiplier algebra $\mathcal{M}(J)$ has real rank zero by Lin's result ([56, Lemma 2.3]). This is used in the proof of Theorem 5.2.1, to allow approximations of positive elements in $\mathcal{M}(J)$ by positive linear combinations of projections.

5.2.1 Goodearl's theory for $\mathcal{D}(\mathcal{M}(J))$

When J is stable, both $\mathcal{D}(J)$ and $\mathcal{D}(\mathcal{M}(J))$ are positively pre-ordered monoids, by identifying them with $V(J)$ and $V(\mathcal{M}(J))$ respectively.

Definition 5.2.2. Let J be a C^*-algebra. A non-empty subset $I \subseteq \mathcal{D}(J)$ is

(i) *hereditary* if for any $x, y \in \mathcal{D}(J)$, such that $x \in I$ and $y \leq x$, we have $y \in I$;

(ii) *upward-directed* if for any $x, y \in I$, there exists some $z \in I$ such that $x, y \leq z$;

(iii) an *interval* if it is hereditary and upward-directed.

An interval I is *countably generated* if there exists a countable cofinal subset of I, i.e. there exists a sequence $(x_n)_n \subseteq I$ such that for any $x \in I$, we have $x \leq x_n$ for some n. The set of countably generated intervals in $\mathcal{D}(J)$ is denoted by $\Lambda_\sigma(\mathcal{D}(J))$.

Since intervals are upward directed, countably generated intervals always admit a countable cofinal increasing subset.

For a stable C^*-algebra J with real rank zero, an additive structure and a partial order can be defined on $\Lambda_\sigma(\mathcal{D}(J))$ to make it an ordered monoid. For intervals I_1 and I_2 in $\Lambda_\sigma(\mathcal{D}(J))$, define the sum to be the set

$$I_1 + I_2 = \{x + y : x \in I_1, y \in I_2\}, \tag{5.2.1}$$

which is a non-empty upward-directed subset of $\mathcal{D}(J)$. Since J has real rank zero, it follows that $\mathcal{D}(J)$ has the *Riesz decomposition property* by [116, Theorem 1.1], which means that for $x, y, z \in \mathcal{D}(J)$ with $x \leq y + z$, there exists $y', z' \in \mathcal{D}(J)$ such that $x = y' + z'$, $y' \leq y$ and $z' \leq z$. This implies that the sum of intervals is hereditary and is thus also an interval. Moreover, if I_1 and I_2 are countably generated by $(x_n)_n$ and $(y_n)_n$ respectively, then I_1+I_2 is an interval in $\mathcal{D}(J)$ countably generated by $(x_n+y_n)_n$. A partial order on $\Lambda_\sigma(\mathcal{D}(J))$ is given by the *algebraic order*, where $I_1 \leq_{\text{alg}} I_2$ if there exists $I' \in \Lambda_\sigma(\mathcal{D}(A))$ such that $I_2 = I_1 + I'$. This equips $\Lambda_\sigma(\mathcal{D}(J))$ with the structure of an ordered monoid.

Goodearl initiated the study of K-theory of multiplier algebras of C^*-algebras J of real rank zero and stable rank one in [41]. For such C^*-algebras, there is an order-preserving bijective monoid homomorphism between $\mathcal{D}(\mathcal{M}(J))$ and $\Lambda_\sigma(\mathcal{D}(J))$. In fact, only a consequence of stable rank one, called cancellation of projections, is really used. Recall that a C^*-algebra has *cancellation of projections* if for any projections $p, q, e, f \in A$ satisfying $pe = 0$, $qf = 0$, $e \sim f$ and $p + e \sim q + f$, we have $p \sim q$.

Theorem 5.2.3 ([41, Theorem 1.10]). *Let J be a separable and stable C^*-algebra with real rank zero and cancellation of projections. There exists a well-defined ordered monoid isomorphism $\theta : \mathcal{D}(\mathcal{M}(J)) \to \Lambda_\sigma(\mathcal{D}(J))$ given by*

$$[e] \mapsto I_e \coloneqq \{[p] \in \mathcal{D}(J) : p \in eJe\}, \tag{5.2.2}$$

for any projection $e \in \mathcal{M}(J)$. Thus $\mathcal{D}(\mathcal{M}(J))$ is a partially ordered monoid.

Proof. It is shown in [41, Lemma 1.6, Proposition 1.7] that θ is an injective order-preserving monoid homomorphism. For surjectivity, take any interval $I \in \Lambda_\sigma(\mathcal{D}(J))$, then $I + \mathcal{D}(J) = \mathcal{D}(J)$ by stability of J. By [41, Proposition 1.8, Remark 1.9], there exist orthogonal projections $e, f \in \mathcal{M}(J)$ such that $I_e = I$, $I_f = \mathcal{D}(J)$ and $e + f = 1_{\mathcal{M}(J)}$.

There exists a map $\sigma : \Lambda_\sigma(\mathcal{D}(J)) \to \mathcal{D}(\mathcal{M}(J))$, given by bijectivity of θ. It is automatically a monoid homomorphism and satisfies $\sigma \circ \theta = \mathrm{id}_{\mathcal{D}(\mathcal{M}(J))}$ and $\theta \circ \sigma = \mathrm{id}_{\Lambda_\sigma(\mathcal{D}(J))}$. The only non-obvious part is that σ is order-preserving. Take intervals in $\mathcal{D}(J)$ and by surjectivity of θ, we assume that the intervals taken are I_e, I_f for projections e, f in $\mathcal{M}(J)$. Suppose that $I_e \leq_{\mathrm{alg}} I_f$, then there exists I such that $I_e + I = I_f$. By surjectivity of θ again, we have $I = I_{e'}$ for some projection $e' \in \mathcal{M}(J)$. Thus $\theta([e \oplus e']) = \theta([f])$ and by injectivity of θ, $[e + e'] = [f]$ and thus $[e] \leq [f]$. □

5.2.2 Purely large ideals

When J has real rank zero, the ideal lattice of $\mathcal{M}(J)$ can be described in terms of ideals in $\mathcal{D}(\mathcal{M}(J))$. A non-empty subset $I \subseteq \mathcal{D}(\mathcal{M}(J))$ is an *ordered ideal* if it is hereditary and closed under addition.

Theorem 5.2.4 ([116, Theorem 2.3])**.** *Let J be a separable and stable C^*-algebra with real rank zero. The lattice of closed ideals in $\mathcal{M}(J)$ is isomorphic to the lattice of ordered ideals in $\mathcal{D}(\mathcal{M}(J))$. For an ordered ideal $\mathcal{L} \subseteq \mathcal{D}(\mathcal{M}(J))$,*

$$\mathcal{I}_{\mathcal{L}} = \overline{\mathrm{Ideal}}\{e \in \mathcal{P}(\mathcal{M}(J)) : [e] \in \mathcal{L}\}, \tag{5.2.3}$$

is the corresponding closed ideal in $\mathcal{M}(J)$ and conversely, $\mathcal{L}$ is recovered from $\mathcal{I}$ by

$$\mathcal{L}_{\mathcal{I}} = \{[e] \in \mathcal{D}(\mathcal{M}(J)) : e \in \mathcal{I}\}. \tag{5.2.4}$$

In particular, J is the closed ideal in $\mathcal{M}(J)$ generated by projections in J.

Using correspondences in Theorem 5.2.4 and Theorem 5.2.3, we aim to find a closed ideal in $\mathcal{M}(J)$ containing J which is purely large in $\mathcal{M}(J)$ with respect to J. Take

$$\mathcal{L}_0 \coloneqq \mathcal{D}(\mathcal{M}(J)) \setminus \{[1_{\mathcal{M}(J)}]\}. \tag{5.2.5}$$

If $\mathcal{D}(J)$ is totally ordered, we show that $\mathcal{I}_{\mathcal{L}_0}$ is the desired ideal.

Lemma 5.2.5. *Let J be a separable C^*-algebra with an approximate unit $(p_n)_n$ of increasing projections and $\mathcal{D}(J)$ totally ordered. Then for any $I \in \Lambda_\sigma(\mathcal{D}(J))$ with $I \neq \mathcal{D}(J)$, there exists n_0 such that $I \subseteq I_{p_{n_0}}$.*

Proof. Suppose that for any n, there exists $y_n \in I$ such that $y_n \notin I_{p_n}$, which means $[p_n] \leq y_n$ by total ordering of $\mathcal{D}(J)$. We will show that $\mathcal{D}(J) \subseteq I$ and thus $\mathcal{D}(J) = I$.

Take any projection $p \in J$, then $[p] \leq [p_m]$ for some m since $(p_n)_n$ is an approximate unit of J. Then

$$[p] \leq [p_m] \leq y_m, \tag{5.2.6}$$

and thus $[p] \in I$ since I is hereditary. □

We now show that $\mathcal{I}_{\mathcal{L}_0}$ is a closed ideal in $\mathcal{M}(J)$ containing all projections that are not Murray-von Neumann equivalent to $1_{\mathcal{M}(J)}$.

Theorem 5.2.6. *Let J be a separable and stable C^*-algebra with real rank zero and stable rank one such that $\mathcal{D}(J)$ is totally ordered. Then there exists a unique maximal proper closed ideal $\mathcal{I}_0$ of $\mathcal{M}(J)$ containing J such that for any projection e in $\mathcal{M}(J)$, we have that $e \in \mathcal{I}_0$ if and only $[e] \neq [1_{\mathcal{M}(J)}]$ in $\mathcal{D}(\mathcal{M}(J))$.*

Proof. Take $\mathcal{L}_0 = \mathcal{D}(\mathcal{M}(J)) \setminus \{[1_{\mathcal{M}(J)}]\}$. We first show that $\mathcal{L}_0$ is hereditary. Take projections p, q in $\mathcal{M}(J)$ such that $[p] \leq [q]$ and $[q] \in \mathcal{L}_0$. Applying the ordered monoid isomorphism θ gives $I_p \leq_{\text{alg}} I_q$ and $I_q \neq \mathcal{D}(J)$. This implies that $I_p \neq \mathcal{D}(J)$ and thus $[p] \neq [1_{\mathcal{M}(J)}]$. Thus, $\mathcal{L}_0$ is a proper hereditary subset of $\mathcal{D}(\mathcal{M}(J))$ containing all equivalence classes of projections in J.

We show that $\mathcal{L}_0$ is closed under addition. Take projections e, f in $\mathcal{M}(J)$ with $[e], [f] \in \mathcal{L}_0$. Since J has real rank zero and stable rank one, by Theorem 5.2.3, there exists an ordered monoid isomorphism $\theta : \mathcal{D}(\mathcal{M}(J)) \to \Lambda_\sigma(\mathcal{D}(J))$, $p \mapsto I_p$. By injectivity of θ, we have $I_e, I_f \neq \mathcal{D}(J)$. Again by real rank zero of J, there exists an approximate unit $(p_n)_n$ of increasing projections. By Lemma 5.2.5 and total ordering of $\mathcal{D}(J)$, there exists n_1, n_2 such that $I_e \subseteq I_{p_{n_1}}$ and $I_f \subseteq I_{p_{n_2}}$. This implies that $I_e + I_f \subseteq I_{p_{n_1}} + I_{p_{n_2}} \subseteq I_{p_{n_0}}$ for some n_0, which is a proper subset of $\mathcal{D}(J)$. Since θ is injective, then $[e] + [f] \neq [1_{\mathcal{M}(J)}]$ in $\mathcal{D}(\mathcal{M}(J))$.

Using the correspondences in Theorem 5.2.4, we take the closed ideal in $\mathcal{M}(J)$ to be

$$\mathcal{I}_{\mathcal{L}_0} := \overline{\text{Ideal}}\,\{e \in \mathcal{M}(J) : [e] \in \mathcal{L}_0\}. \tag{5.2.7}$$

The closed ideal $\mathcal{I}_{\mathcal{L}_0}$ contains J since it contains all projections in J, and is proper since $1_{\mathcal{M}(J)}$ is not contained in $\mathcal{I}_{\mathcal{L}_0}$. Moreover, through the correspondence in Theorem 5.2.4, $\mathcal{I}_{\mathcal{L}_0}$ is the unique maximal ideal satisfying the property that any projection outside of $\mathcal{I}_{\mathcal{L}_0}$ is Murray von Neumann equivalent to $1_{\mathcal{M}(J)}$. $\square$

The defining property of $\mathcal{I}_{\mathcal{L}_0}$ can be extended to give the comparison of positive elements in $\mathcal{M}(J)$, which is necessary for relative pure largeness of $\mathcal{I}_{\mathcal{L}_0}$ in $\mathcal{M}(J)$. The condition $K_1(J) = 0$ is needed to ensure that $\mathcal{M}(J)$ also has real rank zero.

Proof of Theorem 5.2.1. Take $\mathcal{I}_{\mathcal{L}_0}$ to be the ideal defined in (5.2.7). If $b \sim 1_{\mathcal{M}(A)}$, then $1_{\mathcal{M}(A)} \notin \mathcal{I}_{\mathcal{L}_0}$ implies $b \notin \mathcal{I}_{\mathcal{L}_0}$. For the other direction, take any $b \notin \mathcal{I}_{\mathcal{L}_0}$. Since J is separable with real rank zero, stable rank one and $K_1(J) = 0$, by [65, Corollary 12], the multiplier algebra $\mathcal{M}(J)$ also has real rank zero. Then by [56, Lemma 2.3], for any

$b \in \mathcal{M}(J)_+$ and $\epsilon > 0$, there exist mutually orthogonal projection $f_1, \cdots, f_l \in \mathcal{M}(J)$ and $\alpha_1, \cdots, \alpha_l \in \mathbb{R}_{>0}$, such that

$$0 \leq \sum_{i=1}^{l} \alpha_i f_i \leq b \quad \text{and} \quad \left\| \sum_{i=1}^{l} \alpha_i f_i - b \right\| < \epsilon. \tag{5.2.8}$$

Since $b \notin \mathcal{I}_{\mathcal{L}_0}$, there exists $\epsilon > 0$ such that the corresponding sum $b' = \sum_{i=1}^{l} \alpha_i f_i$ is not in $\mathcal{I}_{\mathcal{L}_0}$. Consider the projection $e = \sum_{i=1}^{l} f_i$, which satisfies $e \sim b'$ and thus is not in $\mathcal{I}_{\mathcal{L}_0}$. By Theorem 5.2.6, e is Murray-von Neumann equivalent to $1_{\mathcal{M}(J)}$ and thus

$$1_{\mathcal{M}(J)} \sim e \sim b' \precsim b. \tag{5.2.9}$$

In conclusion, every positive element in $\mathcal{M}(J) \setminus \mathcal{I}_{\mathcal{L}_0}$ is Cuntz equivalent to $1_{\mathcal{M}(J)}$, which implies that $\mathcal{I}_{\mathcal{L}_0}$ is relatively purely large in $\mathcal{M}(J)$ and $\mathcal{M}(J)/\mathcal{I}_{\mathcal{L}_0}$ is simple and purely infinite. □

Chapter 6

K_1-injectivity of relative commutants

In this chapter, we prove our main K_1-injectivity theorem (Theorem D) for the relative commutant, thus derive the KK- and KL-uniqueness theorem (Theorem C).

We begin by reviewing a collection of historical results on K_1-injectivity for relative commutants of the form $\bar{\phi}(A) \cap \mathcal{C}(J)$, as presented in Theorem 4.3.5. In the 1980s, Paschke established K_1-injectivity in the case $J = \mathcal{K}$ (Theorem 6.1.4). More recently, his strategy was generalized by Loreaux, Ng and Sutradhar in [69] and [70], where they proved K_1-injectivity under the assumption that both A and J are simple, and that J satisfies certain regularity assumptions, such as strict comparison.

Our approach is heavily inspired by the work of Loreaux, Ng and Sutradhar, and we push their methods further by removing the simplicity assumptions on both A and J. Due to the technical complexity involved in the proof, we provide an outline of the main strategy in Section 6.2, while the proof is presented in Section 6.3. The results in this chapter will appear in an upcoming paper by White and me ([51]).

We denote the quotient map from the multiplier algebra to the corona algebra by $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$. For any $a \in \mathcal{M}(J)$, we write $\bar{a}$ for $\pi(a)$. Likewise, for any $*$-homomorphism φ into $\mathcal{M}(J)$, we write $\bar{\varphi}$ for $\pi \circ \varphi$.

6.1 Summary of previous results

We include a summary of known results for relative commutants associated with unitally absorbing *-homomorphisms. Recall that a unital C^*-algebra is *properly infinite* if the unit is Murray-von Neumann equivalent to two mutually orthogonal subprojections. We provide a short proof of the following folklore result.

Proposition 6.1.1. *Let A be a unital and separable C^*-algebra and J be a σ-unital and stable C^*-algebra. Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism.*

Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is properly infinite.

Proof. As J is stable, the multiplier algebra $\mathcal{M}(J)$ contains a copy of $B(\mathcal{H})$. Thus we can pick a pair of isometries $s_1, s_2 \in \mathcal{M}(J)$ with $s_1 s_1^* + s_2 s_2^* = 1$ that implement the Cuntz sum. Since φ is unitally absorbing, there exists a unitary $u \in \mathcal{M}(J)$ such that

$$u(\phi(a) \oplus_{s_1,s_2} \phi(a))u^* - \phi(a) \in J, \quad a \in A. \tag{6.1.1}$$

In particular, both $\bar{u}\bar{s}_1$ and $\bar{u}\bar{s}_2$ are isometries in $\mathcal{C}(J)$, which commute with $\bar{\phi}(A)$ and have orthogonal range projections. □

Cuntz showed in the 80s that properly infinite C^*-algebras are K_1-surjective ([19]). The analogous question for K_1-injectivity for properly infinite C^*-algebras remains an open problem.

Question 6.1.2. *Are all properly infinite unital C^*-algebras K_1-injective?*

This question was first formally asked by Blanchard, Rohde, and Rørdam in [8], where they also developed a list of equivalent characterizations of K_1-injectivity for unital properly infinite C^*-algebras.

Proposition 6.1.3 ([8, Proposition 5.1, Proposition 5.2])**.** *Let A be a unital and properly infinite C^*-algebra. The following conditions are equivalent:*

(i) A is K_1-injective, i.e. the natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ is injective;

(ii) the natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective;

(iii) the natural map $\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A)$ is injective for any $n \in \mathbb{N}$;

(iv) let p and q be projections in A such that $p \sim q$ and p, q, $1-p$, $1-q$ are properly infinite and full. Then $p \sim_h q$;

(v) let p and q be properly infinite, full projections in A. There exists properly infinite and full projections $p_0, q_0 \in A$ such that $p_0 \leq q$, $q_0 \leq q$ and $p_0 \sim_h q_0$.

Blanchard, Rohde, and Rørdam further showed in [8, Theorem 5.5] that it suffices to answer Question 6.1.2 in particular cases, for instance for the full unital free product $\mathcal{O}_2 * \mathcal{O}_2$. Many examples of properly infinite C^*-algebras have been shown to be K_1-injective, for instance multiplier algebras of stable C^*-algebras, as mentioned in Theorem 2.9.13.

We will consider the class of C^*-algebras $\mathcal{C}(J) \cap \bar{\phi}(A)'$ associated to unitally absorbing *-homomorphims $\phi : A \to \mathcal{M}(J)$. These relative commutants are not only natural test cases for Question 6.1.2, but their K_1-injectivity results also lead to KK-uniqueness theorems as explained in Chapter 4. The first result in this direction was proved by Paschke in the 1980s, in the case $J = \mathcal{K}$.

Theorem 6.1.4 ([81, Lemma 3]). *Let A be a unital and separable C^*-algebra and $\phi : A \to \mathcal{M}(\mathcal{K})$ be unitally absorbing. Then $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$ is K_1-injective.*

Proof. The algebra $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$ is properly infinite by Proposition 6.1.1. Fix a unitary $u \in \mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$ such that in $\mathcal{U}_2(\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)')$,

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1.2}$$

Then by Proposition 6.1.3, it suffices to show $u \sim_h 1$ in $\mathcal{C}(\mathcal{K}) \cap \bar{\phi}(A)'$.

Consider a unitally absorbing map $\psi : C^*(\bar{\phi}(A), u) \to \mathcal{M}(\mathcal{K})$, the existence of which is guaranteed by Theorem 4.2.8. Then ψ is injective, and the same holds when restricting to $\bar{\phi}(A)$ in particular. Thus $\psi|_{\bar{\phi}(A)} : \bar{\phi}(A) \to \mathcal{M}(\mathcal{K})$ is unitally absorbing by Voiculescu's theorem (Theorem 4.2.7). On the other hand, since $\phi : A \to \mathcal{M}(\mathcal{K})$ is unitally absorbing, ϕ is injective and $\phi(A) \cap \mathcal{K} = \{0\}$ by Proposition 4.2.9. This induces a map

$$\varphi : \bar{\phi}(A) \to \mathcal{M}(\mathcal{K}), \quad \bar{\phi}(a) \mapsto \phi(a), \tag{6.1.3}$$

which is also unitally absorbing since $\bar{\varphi}$ is essential. Then $\psi|_{\bar{\phi}(A)}$ and φ are unitarily equivalent modulo J since they are both unitally absorbing. Conjugating by a unitary if necessary, we can assume that $\psi(\bar{\phi}(a)) = \varphi(\bar{\phi}(a)) = \phi(a)$ for any $a \in A$.

Since u commutes with $\bar{\phi}(A)$, it follows that $\psi(u)$ belongs to the unitary group of the von Neumann algebra $\psi(\bar{\phi}(A))' \subseteq B(\mathcal{H})$. Because unitary groups of von Neumann algebras are path-connected, it follows that $\psi(u) \sim_h 1$ in $\psi(\bar{\phi}(A))' \subseteq B(\mathcal{H})$. Thus $\bar{\psi}(u) \sim_h 1$ in $\bar{\phi}(A)' \cap \mathcal{C}(\mathcal{K})$ by passing to the quotient. Then in $\mathcal{U}_2(\bar{\phi}(A)' \cap \mathcal{C}(\mathcal{K}))$, by the assumption (6.1.2),

$$\begin{pmatrix} u & 0 \\ 0 & \bar{\psi}(u) \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1.4}$$

Take the inclusion $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(\mathcal{K})$, which is essential and thus is unitally absorbing by Voiculescu's theorem (Theorem 4.2.7). Take s_1, s_2 to be the pair of isometries that implement the Cuntz sum. Then there exists a unitary $v \in \mathcal{M}(\mathcal{K})$ such that $\iota = \mathrm{Ad}(\bar{v}) \circ (\iota \oplus \bar{\psi})$. In particular,

$$u = \mathrm{Ad}(\bar{v})(u \oplus \bar{\psi}(u)), \tag{6.1.5}$$

$$\bar{\phi}(a) = \mathrm{Ad}(\bar{v})(\bar{\phi}(a) \oplus \bar{\phi}(a)), \quad a \in A. \tag{6.1.6}$$

Consider the following *-isomorphism given by the isometries s_1, s_2:

$$\Phi : M_2 \otimes \mathcal{C}(\mathcal{K}) \to \mathcal{C}(\mathcal{K}), \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} s_1 & s_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}^*. \tag{6.1.7}$$

In particular, we have $\Phi(\mathrm{diag}(a, b)) = a \oplus b$ for any $a, b \in \mathcal{C}(\mathcal{K})$. Then (6.1.5) and (6.1.6) show that

$$\mathrm{Ad}(\bar{v}) \circ \Phi\Big(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \Big) = \mathrm{Ad}(\bar{v})(u \oplus \bar{\psi}(u)) = u, \quad \text{and} \tag{6.1.8}$$

$$\mathrm{Ad}(\bar{v}) \circ \Phi\Big(\begin{pmatrix} \bar{\phi}(a) & 0 \\ 0 & \bar{\phi}(a) \end{pmatrix} \Big) = \mathrm{Ad}(\bar{v})(\bar{\phi}(a) \oplus \bar{\phi}(a)) = \bar{\phi}(a). \tag{6.1.9}$$

Since ϕ is unital, by applying the map $\mathrm{Ad}(\bar{v}) \circ \Phi$ to the homotopy in (6.1.4), we have $u \sim_h 1$ in $\mathcal{C}(\mathcal{K})$. It remains to show that for any $w \in \mathcal{U}_2(\bar{\phi}(A)' \cap \mathcal{C}(\mathcal{K}))$, we have $\mathrm{Ad}(\bar{v}) \circ \Phi(w) \in \bar{\phi}(A)' \cap \mathcal{C}(\mathcal{K})$. This follows from the following computation:

$$\begin{aligned} & (\mathrm{Ad}(\bar{v}) \circ \Phi(w))\bar{\phi}(a) && (6.1.10) \\ \overset{(6.1.9)}{=} & \bar{v}\Phi(w)\Phi\Big(\begin{pmatrix} \bar{\phi}(a) & 0 \\ 0 & \bar{\phi}(a) \end{pmatrix} \Big)\bar{v}^* && (6.1.11) \\ = & \bar{v}\Phi\Big(\begin{pmatrix} \bar{\phi}(a) & 0 \\ 0 & \bar{\phi}(a) \end{pmatrix} \Big)\Phi(w)\bar{v}^* && (6.1.12) \\ \overset{(6.1.9)}{=} & \bar{\phi}(a)(\mathrm{Ad}(\bar{v}) \circ \Phi(w)). \end{aligned}$$

□

In the general setting, partial results on K_1-injectivity of the algebra $\mathcal{C}(J) \cap \bar{\phi}(A)'$ have been obtained in [69] and [70] by Loreaux, Ng and Sutradhar recently. Their proofs follow a strategy similar to that of Theorem 6.1.4. However, there is a notable difference between the case when $J = \mathcal{K}$ and more general choices of J. While the inclusion map $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(\mathcal{K})$ in the proof of Theorem 6.1.4 is always unitally absorbing by Voiculescu's theorem (Theorem 4.2.7), such essential extensions need not to be unitally absorbing for general J. Under the assumptions that A is nuclear and J has the corona factorization property, the inclusion maps ι are unitally absorbing if and only if they are full, as a consequence of Elliott-Kucerovsky's generalized Voiculescu's theorem (Theorem 4.2.14) and the following standard lemma.

Lemma 6.1.5. *Let A be a nuclear C^*-algebra and $\phi : A \to \mathcal{M}(J)$ be a *-homomorphism. Let u be a unitary in $\mathcal{C}(J) \cap \bar{\phi}(A)'$. Then $C^*(\bar{\phi}(A), u)$ is a nuclear C^*-algebra.*

Proof. Consider $*$-homomorphisms $\psi_1 : C(\sigma(u)) \to C^*(\bar{\phi}(A), u)$, $f \mapsto f(u)$ and the inclusion map $\psi_2 : \bar{\phi}(A) \to C^*(\bar{\phi}(A), u)$. Then ψ_1, ψ_2 have commuting ranges since $u \in \mathcal{C}(J) \cap \bar{\phi}(A)'$ and thus the product map

$$\psi_1 \times \psi_2 : C(\sigma(u)) \odot \bar{\phi}(A) \to C^*(\bar{\phi}(A), u), \quad f \otimes \bar{\phi}(a) \mapsto \psi_1(f)\psi_2(\bar{\phi}(a)), \tag{6.1.13}$$

is a $*$-homomorphism whose image is dense in $C^*(\bar{\phi}(A), u)$ by [13, Proposition 3.1.17].

By the universal property of the maximal tensor product and nuclearity of $C(\sigma(u))$, there is a unique surjective $*$-homomorphism $q : C(\sigma(u)) \otimes \bar{\phi}(A) \to C^*(\bar{\phi}(A), u)$ extending $\psi_1 \times \psi_2$. As a result, the algebra $C^*(\bar{\phi}(A), u)$ is nuclear since it is a quotient of the nuclear C^*-algebra $C(\sigma(u)) \otimes \bar{\phi}(A)$. □

To establish K_1-injectivity, Loreaux and Ng observed in [69] that it suffices to show: any unitary in $\bar{\phi}(A)' \cap \mathcal{C}(J)$ is homotopic in $\bar{\phi}(A)' \cap \mathcal{C}(J)$ to a unitary u, such that the inclusion $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(J)$ is unitally absorbing. This idea was made clearer in their subsequent work [70] with Sutradha. The following lemma is a minor adaptation of [70, Theorem 2.9], which generalizes Theorem 6.1.4 by exploiting Kucerovsky-Ng's generalized Voiculescu's theorem (Theorem 4.2.14).

Lemma 6.1.6. *Let A, J be separable C^*-algebras with A unital and nuclear, and J stable and having the corona factorization property. Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism. Suppose that every unitary in $\bar{\phi}(A)' \cap \mathcal{C}(J)$ is homotopic through unitaries in $\bar{\phi}(A)' \cap \mathcal{C}(J)$ to a unitary u such that the inclusion $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(J)$ is full. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

Proof. Fix a unitary $u_0 \in \mathcal{C}(J) \cap \bar{\phi}(A)'$ such that in $\mathcal{U}_2(\mathcal{C}(J) \cap \bar{\phi}(A)')$, we have

$$\begin{pmatrix} u_0 & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1.14}$$

By Theorem 6.1.3, it suffices to show that $u_0 \sim_h 1$ in $\mathcal{C}(J) \cap \bar{\phi}(A)'$. By our assumption, the unitary u_0 is homotopic in $\mathcal{C}(J) \cap \bar{\phi}(A)'$ to a unitary u such that the inclusion $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(J)$ is full. In particular, u also satisfies (6.1.14) and it suffices to show that $u \sim_h 1$ in $\mathcal{C}(J) \cap \bar{\phi}(A)'$. Since $C^*(\bar{\phi}(A), u)$ is nuclear by Lemma 6.1.5 and J has the corona factorization property, the inclusion ι is unitally absorbing as a consequence of Kucerovsky-Ng's theorem (Theorem 4.2.14).

Take a unitally absorbing map $\psi : C^*(\bar{\phi}(A), u) \to \mathcal{M}(J)$, which exists by Theorem 4.2.8. Then ϕ is full and the same holds when restricting to $\bar{\phi}(A)$, which implies that $\psi|_{\bar{\phi}(A)} : \bar{\phi}(A) \to \mathcal{M}(J)$ is unitally absorbing by Kucerovsky-Ng's theorem (Theorem

4.2.14). Since $\phi : A \to \mathcal{M}(J)$ is unitally absorbing, it follows that ϕ is full and essential by Proposition 4.2.9. This induces a map

$$\varphi : \bar{\phi}(A) \to \mathcal{M}(J), \quad \bar{\phi}(a) \mapsto \phi(a), \tag{6.1.15}$$

which is full and thus unitally absorbing again by Kucerovsky-Ng's theorem (Theorem 4.2.14). Then $\psi|_{\bar{\phi}(A)}$ and φ are unitarily equivalent modulo J since they are unitally absorbing. Conjugating by a unitary, we can assume that $\psi(\bar{\phi}(a)) = \varphi(\bar{\phi}(a)) = \phi(a)$ for any $a \in A$.

Next we show $\bar{\psi}(u) \sim_h 1$ in $\bar{\phi}(A)' \cap \mathcal{C}(J)$. Take a unital trivial essential extension $\theta_1 : C^*(\bar{\phi}(A), u) \to B(\mathcal{H})$. Since u commutes with $\bar{\phi}(A)$, it follows that $\theta_1(u)$ belongs to the unitary group of the von Neumann algebra $\theta_1(\bar{\phi}(A))' \subseteq B(\mathcal{H})$, which implies that $\theta_1(u) \sim_h 1$ in $\theta_1(\bar{\phi}(A))'$. Consider the unital *-homomorphism

$$\theta : C^*(\bar{\phi}(A), u) \to B(\mathcal{H}) \otimes \mathcal{M}(J) \subseteq \mathcal{M}(J \otimes \mathcal{K}), \quad a \mapsto \sigma_1(a) \otimes 1_{\mathcal{M}(J)}, \tag{6.1.16}$$

which is purely large by [29, Theorem 17 (iii)] and thus unitally absorbing by Theorem 4.2.12. Moreover, $\theta(u)$ is homotopic to 1 in $\sigma(\bar{\phi}(A))' \cap \mathcal{M}(J \otimes \mathcal{K})$ by a continuous path of unitaries $(v_t)_{t\in[0,1]}$. Since the maps ψ and θ are unitally absorbing, there exists a unitary $w \in \mathcal{M}(J)$ such that $\bar{\psi} = \mathrm{Ad}(\bar{w}) \circ \bar{\theta}$. Take $u_t = \mathrm{Ad}(\bar{w})\bar{v}_t$, then $(u_t)_{t\in[0,1]}$ is a continuous path of unitaries in $\mathrm{Ad}(\bar{w}) \circ \bar{\theta}(\bar{\phi}(A))' = \bar{\psi}(\bar{\phi}(A))' = (\bar{\phi}(A))' \cap \mathcal{C}(J)$ with

$$u_0 = \mathrm{Ad}(\bar{w})\bar{v}_0 = \mathrm{Ad}(\bar{w})\bar{\theta}(u) = \bar{\psi}(u), \quad u_1 = 1, \tag{6.1.17}$$

which means $\bar{\psi}(u) \sim_h 1$ in $\bar{\phi}(A)' \cap \mathcal{C}(J)$. Then in $\mathcal{U}_2(\bar{\phi}(A)')$, by our assumption,

$$\begin{pmatrix} u & 0 \\ 0 & \bar{\psi}(u) \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1.18}$$

The rest of the proof is the same as that of Theorem 6.1.4. □

When A is a simple C^*-algebra, the following lemma provides an approach to verify that the inclusion map ι associated to a unitary in $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is full. A unitary $u \in B$ is *strongly full* if any nonzero element in $C^*(u)$ is full in B.

Lemma 6.1.7 ([69, Lemma 2.6])**.** *Let A, J be separable C^*-algebras with A unital, simple and nuclear, and J stable with the corona factorization property. Let $\phi : A \to \mathcal{M}(J)$ be a unital *-homomorphism. If u is a unitary in $\bar{\phi}(A)' \cap \mathcal{C}(J)$ and is strongly full in $\mathcal{C}(J)$, then the inclusion $\iota : C^*(\bar{\phi}(A), u) \to \mathcal{C}(J)$ is full.*

Proof. Fix a nonzero positive element $a \in C^*(\bar{\phi}(A), u)$, which we will show to be full in $\mathcal{C}(J)$. Take $q : C(\sigma(u)) \otimes \bar{\phi}(A) \to C^*(\bar{\phi}(A), u)$ to be the quotient map appearing in the proof of Lemma 6.1.5. There is a positive element $a' \in C(\sigma(u)) \otimes \bar{\phi}(A)$ such that $q(a') = a$. Take $D = \overline{a'(C(\sigma(u)) \otimes \bar{\phi}(A))a'}$, which is a hereditary C^*-subalgebra of $C(\sigma(u)) \otimes \bar{\phi}(A)$ and is nonzero since $a' \in D$. By Kirchberg's slice lemma ([90, Lemma 4.1.9]), there exists a nonzero element $z \in C(\sigma(u)) \otimes \bar{\phi}(A)$ such that z^*z is an elementary tensor $g \otimes b$ for some nonzero $g \in C(\sigma(u))$ and $b \in \bar{\phi}(A)$, and $zz^* \in D$. This implies that

$$g \otimes b = z^*z \sim zz^* \precsim a' \tag{6.1.19}$$

in $C(\sigma(u)) \otimes \bar{\phi}(A)$ and by applying q, we have $g(u)b \precsim a$ in $C^*(\bar{\phi}(A), u)$.

We show that $g(u)b$ is a full element in $\mathcal{C}(J)$, which implies that a is full and the map ι is full. Since A is simple, it follows that $\bar{\phi}(A)$ is simple and there exist $x_1, \cdots, x_m \in \bar{\phi}(A)$ such that $\sum_{j=1}^m x_j b x_j^* = 1$. Since u commutes with $\bar{\phi}(A)$, we have

$$\sum_{j=1}^m x_j g(u) b x_j^* = \left(\sum_{j=1}^m x_j b x_j^*\right) g(u) = g(u). \tag{6.1.20}$$

By our assumption, the unitary u is strongly full in $\mathcal{C}(J)$, which implies that $g(u)$ and thus $g(u)b$ are full elements in $\mathcal{C}(J)$. □

The tools above provide the architecture for the K_1-injectivity results of Loreaux, Ng and Sutradhar in [69] and [70]. The remaining challenge is to find conditions on A, J such that one can homotope unitaries in $\mathcal{C}(J) \cap \bar{\phi}(A)'$ to those that are strongly full in $\mathcal{C}(J)$. They achieve this by assuming that J has strict comparison and finitely many extremal traces. The following is their main K_1-injectivity theorem and we will explain their strategy of proof in more detail in the next section.

Theorem 6.1.8 ([70, Theorem 3.23])**.** *Let A be a unital, separable, simple and nuclear C^*-algebra. Let J be a σ-unital, simple and stable C^*-algebra with strict comparison and $T(J)$ having finitely many extreme points. Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

6.2 An overview of the main theorem

6.2.1 Main theorems and applications

In this chapter, we aim to prove our main K_1-injectivity result (Theorem D). The result will appear in the forthcoming paper [51] of White and me.

Theorem 6.2.1. *Let A be a separable, unital and nuclear C^*-algebra and let J be a separable and stable C^*-algebra with the corona factorization property. Moreover, suppose there exists a closed proper ideal $\mathcal{I} \lhd \mathcal{M}(J)$ containing J such that the extension of $\mathcal{I}$ in $\mathcal{M}(J)$ is purely large relative to J (see Definition 5.1.1). Let $\phi : A \to \mathcal{M}(J)$ be a unitally absorbing *-homomorphism. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

As a reminder, our main situation of interest is when J is a suitable separabilization of trace-kernel ideals of $\prod_\omega M_{n_k}$ or $\mathcal{R}_\omega$. Theorem 5.2.1 provides the ideal with the required relative pure largeness, and thus we get the following direct corollary of Theorem 6.2.1.

Corollary 6.2.2. *Let A be a unital, separable and nuclear C^*-algebra. Let J be a separable and stable C^*-algebra which has real rank zero, stable rank one, $K_1(J) = 0$, totally ordered $V(J)$ and satisfies the corona factorization property. Let $\phi : A \to \mathcal{M}(J)$ be a unital and full $*$-homomorphism. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

In the case where J is simple and has a unique trace with respect to which it has strict comparison, Theorem 6.2.1 extends the theorem of Loreaux, Ng and Sutradhar (Theorem 6.1.8) by removing the simplicity assumption on the domain A. The required ideals with relative pure largeness are recorded in Theorem 5.1.3.

Corollary 6.2.3. *Let A be a separable, unital and nuclear C^*-algebra. Let J be a simple, separable and stable C^*-algebra with a unique trace with respect to which J has strict comparison. Let $\phi : A \to \mathcal{M}(J)$ be a unital and full $*$-homomorphism. Then $\mathcal{C}(J) \cap \bar{\phi}(A)'$ is K_1-injective.*

Using the sufficient condition of K_1-injectivity shown in Theorem 6.1.6, to prove Theorem 6.2.1, it is sufficient to prove the following technical lemma.

Lemma 6.2.4. *Let A, J be separable C^*-algebras with A unital and B stable. Suppose that there is a proper ideal $\mathcal{I} \lhd \mathcal{M}(J)$ containing J such that the extension of $\mathcal{I}$ is purely large relative to J. Let $\phi : A \to \mathcal{M}(J)$ be a unital and full *-homomorphism. Then for any unitary $\bar{u} \in \mathcal{C}(J) \cap \bar{\phi}(A)'$, there exists a unitary $\bar{v} \in \mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$ such that the inclusion $\iota : C^*(\bar{\phi}(A), \bar{v}\bar{u}) \to \mathcal{C}(J)$ is full.*

We will include an outline of the proof of Lemma 6.2.4, collecting several useful lemmas along the way in the next section. The proof of Lemma 6.2.4 is technically involved and inspired by the proof of [70]. For this reason, we give an outline of the key steps in the following section. The complete proof for the main lemma is found in Section 6.3.

6.2.2 Overall plan for the proof

The following outline is based on our deconstruction of the arguments of [70]. Throughout the rest of the chapter, we fix A, J to be C^*-algebras and ϕ to be a map that satisfies the assumptions of Lemma 6.2.4. For the ease of notation, we denote the unitaries in the corona algebra by $\bar{u}$, with a contraction $u \in \mathcal{M}(J)$ lifting $\bar{u}$. As suggested in Lemma 6.2.4, we would hope to construct a unitary $\bar{v} \in \mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$ such that $\iota : C^*(\bar{\phi}(A), \bar{v}\bar{u}) \to \mathcal{C}(J)$ is full. Let us start by explaining how to construct unitaries in $\mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$.

For the case of interest explained in Section 5.2, where J is a σ-unital C^*-algebra, with real rank zero, stable rank one and $K_1(J) = 0$, there is a general description of unitaries in $\mathcal{U}_0(\mathcal{C}(J))$, as a consequence of Lin's result in [65, Theorem 11]. For every unitary in $\mathcal{U}_0(\mathcal{C}(J))$, there exists an approximate unit $(p_n)_n$ for J consisting of projections and a sequence of scalars $(\alpha_n)_n$ in $\mathbb{T}$ such that the unitary is the image of the quotient map $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$ of the element

$$v_\alpha = \sum_{n=1}^{\infty} \alpha_n (p_n - p_{n-1}) \tag{6.2.1}$$

with the convention that $p_0 = 0$. By Lemma 2.9.11, the series converges in the strict topology and v_α is a contraction since it is a sum of multiples of pairwise orthogonal projections with coefficients in $\mathbb{T}$. We will call such a series a *diagonal series*, as the terms are pairwise orthogonal.

In general, there is no reason why such a $\bar{v}_\alpha$ should commute with $\bar{\phi}(A)$. Ideally, if we could take the approximate unit of projections $(p_n)_n$ to be *quasicentral* with respect to $\phi(A)$, which means that $\|[p_n, \phi(a)]\| \to 0$ as $n \to \infty$ for any $a \in A$, then $\bar{v}_\alpha$ would commute with $\bar{\phi}(A)$. However, such a quasicentral approximate unit consisting of projections might not exist. Instead, a standard technique is to work with an *almost idempotent* and *quasicentral* approximate unit $(e_n)_n$ of J with respect to $\phi(A)$, which means that

- $\|[e_n, \phi(a)]\| \to \infty$ as $n \to \infty$ for every $a \in A$;
- $e_n e_{n-1} = e_{n-1}$ for $n \in \mathbb{N}$,

with the convention that $e_0 = 0$. Such approximate units are guaranteed to exist by Lemma 2.11.1. The corresponding series given by a sequence $\alpha = (\alpha_n)_n$ in $\mathbb{T}$,

$$v_\alpha = \sum_{n=1}^{\infty} \alpha_n (e_n - e_{n-1}) \tag{6.2.2}$$

converges in the strict topology by Lemma 2.9.11. This is no longer a diagonal series, but is a *bidiagonal series*: namely, strictly convergent sums $\sum_n a_n$, where $a_n a_{n'} = 0$ for $|n - n'| > 1$. The element v_α is not necessarily a unitary in $\mathcal{M}(J)$, but we can make $\bar{v}_\alpha$ a unitary in the corona algebra by imposing extra conditions on $(\alpha_n)_n$. These are made more precise and collected in the following proposition, which essentially combines the results of [70, Lemma 3.2, Lemma 3.3].

Proposition 6.2.5. *Let J be a separable C^*-algebra, and let S be a separable C^*-subalgebra of $\mathcal{M}(J)$. Then there exists an almost idempotent approximate unit $(e_n)_n$ for J such that for any bounded sequence $(\alpha_n)_{n=1}^\infty$ in $\mathbb{C}$, the strictly convergent series*

$$v_\alpha := \sum_{n=1}^{\infty} \alpha_n (e_n - e_{n-1}), \tag{6.2.3}$$

is an element in $\mathcal{M}(J)$, whose image $\bar{v}_\alpha$ in $\mathcal{C}(J)$ lies in the relative commutant $\mathcal{C}(J) \cap \bar{S}'$. Moreover, if $(\alpha_n)_n$ is a sequence in $\mathbb{T}$ such that $\lim_{n\to\infty} |\alpha_n - \alpha_{n+1}| = 0$, then $\bar{v}_\alpha$ is a unitary in $\mathcal{C}(J) \cap \bar{S}'$.

Proof. Fix $(x_k)_k$ to be a countable dense subset of S. Take an almost idempotent approximate unit $(e_n)_n$ of J that is quasicentral with respect to S, which exists by appealing to Lemma 2.11.1. Passing to a subsequence if necessary, we can assume that $\|[e_n, x_k]\| < 1/2^{n+k}$ for any $n, k \in \mathbb{N}$. Then $\bar{v}_\alpha$ is a unitary in $\mathcal{C}(J) \cap \bar{S}'$ following standard computations in [70, Lemma 3.3] and [70, Lemma 3.2]. □

We will fix such an approximate unit $(e_n)_{n=1}^\infty$ as in Proposition 6.2.5 for $S = \phi(A)$, and use the notation v_α for the element given by (6.2.3) for a sequence $\alpha = (\alpha_n)_n$ in $\mathbb{T}$. The goal is to show that we can find a sequence α such that $\bar{v}_\alpha$ lies in $\mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$ and the inclusion $\iota : C^*(\bar{\phi}(A), \bar{v}_\alpha \bar{u}) \to \mathcal{C}(J)$ is full. The first part of this is straightforward to achieve: In [70], Loreaux, Ng and Sutradhar called a sequence $\alpha = (\alpha_n)_n$ in $\mathbb{T}$ *unit oscillating* when adjacent terms of the sequence are interpolated by the arc between them that never cross the unit. The following proposition captures the essential idea of Loreaux, Ng and Sutradhar.

Proposition 6.2.6. *Let A, J be separable C^*-algebras with A unital and J σ-unital. Let $\phi : A \to \mathcal{M}(J)$ be a unital *-homomorphism and take $S = \phi(A)$. Let $(e_n)_n$ be the approximate unit given by Proposition 6.2.5. If $(\theta_n)_n$ is a sequence in $[0, 2\pi)$ with $\lim_{n\to\infty} |\theta_{n+1} - \theta_n| = 0$, then $\bar{v}_\alpha$ is a unitary in $\mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$, where $\alpha = (e^{i\theta_n})_n$.*

Proof. For each $t \in [0,1]$, define $\alpha_n(t) := e^{it\theta_n}$ and take the sequence $\alpha(t) = (\alpha_n(t))_n$ in $\mathbb{T}$. Since $\lim_{n\to\infty} |\theta_{n+1} - \theta_n| = 0$, it follows that $\lim_{n\to\infty} |\alpha_{n+1}(t) - \alpha_n(t)| = 0$ for each $t \in [0,1]$. By the choice of $(e_n)_n$ and Proposition 6.2.5, we have that $\bar{v}_{\alpha(t)}$ is a unitary in $\mathcal{C}(J) \cap \bar{\phi}(A)'$ for each $t \in [0,1]$. Moreover, $|\alpha_n(s) - \alpha_n(t)| < 2\pi|s-t|$ for any $n \in \mathbb{N}$ and thus $t \mapsto \bar{v}_{\alpha_t}$ is a continuous path of unitaries, by Lemma 2.9.11, connecting $1_{\mathcal{C}(J)}$ and $\bar{v}_{\alpha(1)} = \bar{v}_\alpha$. $\square$

The main challenge is to arrange for the inclusion map ι to be full. In [70], taking advantage of simplicity of A through Lemma 6.1.7 and using a density argument, the authors showed that it is sufficient to verify fullness of $h_k(\bar{v}_\alpha \bar{u})$ in $\mathcal{C}(J)$ for a dense collection of nonzero positive contractions in $C(\mathbb{T})$. Similarly, to prove that ι is full, we will only need to verify the fullness of $h_k(\bar{v_\alpha}\bar{u})\bar{\phi}(a_k)$ in $\mathcal{C}(J)$ for countably many positive functions $(h_k)_k$ in $C(\mathbb{T})$ and positive elements $(a_k)_k$ in A. The density argument in the following lemma is standard.

Lemma 6.2.7. *Let A, J be separable C^*-algebras with A unital and J σ-unital. Let $\phi : A \to \mathcal{M}(J)$ be a unital $*$-homomorphism. There exist sequences $(h_k)_k$ and $(a_k)_k$ of norm one positive functions in $C(\mathbb{T})$ and nonzero positive contractions in A respectively, such that if $\bar{u} \in \mathcal{C}(J) \cap \bar{\phi}(A)'$ is a unitary with $h_k(\bar{u})\bar{\phi}(a_k)$ being full in $\mathcal{C}(J)$ for each k, then the inclusion map $\iota : C^*(\bar{\phi}(A), \bar{u}) \to \mathcal{C}(J)$ is full.*

Proof. Take countable dense subsets C in $C(\mathbb{T})_+$ and D in A_+ respectively and let

$$E := \{(f - 1/n)_+ : f \in C, n \in \mathbb{N}, (f-1/n)_+ \text{ nonzero}\} \subseteq C(\mathbb{T})_+, \tag{6.2.4}$$

$$F := \{(a - 1/n)_+ : a \in D, n \in \mathbb{N}, (a-1/n)_+ \text{ nonzero}\} \subseteq A_+. \tag{6.2.5}$$

Then for any nonzero positive function $g \in C(\mathbb{T})$, there exists $f \in E$ such that $f \precsim g$ in $C(\mathbb{T})$. Similarly, this holds for positive elements in A with respect to F. Suppose that $\bar{u} \in \mathcal{C}(J) \cap \bar{\phi}(A)'$ is a unitary such that $f(\bar{u})\bar{\phi}(a)$ is full for any $f \in E$ and $a \in F$.

For any positive $b \in C^*(\bar{\phi}(A), \bar{u})$, by the argument in Lemma 6.1.7, there exist a nonzero positive function $g \in C(\mathbb{T})$ and positive element $d \in A$ with $0 \leq g(\bar{u})\bar{\phi}(d) \leq b$. By the choice of E and F, there exists $f \in E$ and $a \in F$ such that $f \precsim g$ in $C(\mathbb{T})$ and $a \precsim d$ in A. Then by commutativity of $\bar{u}$ and $\bar{\phi}(A)$ in $\mathcal{C}(J)$, we have

$$f(\bar{u})\bar{\phi}(a) \precsim g(\bar{u})\bar{\phi}(a) \precsim g(\bar{u})\bar{\phi}(d) \leq b. \tag{6.2.6}$$

Then b is full in $\mathcal{C}(J)$ since $f(\bar{u})\bar{\phi}(a)$ is full. Since both E and F are countable, the elements in $E \times F$ can be enumerated into sequences $(h_k, a_k)_k$. $\square$

Let us now describe how to construct α such that $h_k(\bar{v}_\alpha \bar{u})\bar{\phi}(a_k)$ is full in $\mathcal{C}(J)$ for each k. We would like to prove a stronger condition and obtain fullness for lifts of these elements in $\mathcal{M}(J)$, since it is easier to manipulate elements in $\mathcal{M}(J)$ compared to $\mathcal{C}(J)$. However, while $v_\alpha u$ is a lift of the unitary $\bar{v}_\alpha \bar{u}$, it is not necessarily a unitary itself, so one cannot make sense of the functional calculus $h(v_\alpha u)$ for $h \in C(\mathbb{T})$. For this reason, it is useful to work with *Laurent polynomials* $\hat{h}$:

$$\hat{h}(z) = \sum_{s=0}^{S_1} \beta_s z^s + \sum_{s=1}^{S_2} \beta_{-s} \bar{z}^s, \tag{6.2.7}$$

where $S_1, S_2 \in \mathbb{N}$ and $\beta_s \in \mathbb{C}$, as there is no ambiguity regarding the definition of

$$\hat{h}(a) = \sum_{n=0}^{N_1} \beta_n a^n + \sum_{n=1}^{N_2} \beta_{-n} (a^*)^n \tag{6.2.8}$$

for an element a in any C^*-algebra. Since Laurent polynomials, when restricted to $\mathbb{T}$, are uniformly dense in $C(\mathbb{T})$, we approximate each of the h_k given by Lemma 6.2.7 by a Laurent polynomial $\hat{h}_k$ up to some tolerance to be specified later. In this way, it will suffice to construct α such that $\hat{h}_k(v_\alpha u)\phi(a_k)$ is full in $\mathcal{M}(J)$ for each $k \in \mathbb{N}$.

Fixing k for a moment, one can find a finite collection $\mu_1, \ldots, \mu_r \in \mathbb{T}$ so that the sum of the translates $\sum_{i=1}^{r} h_k(\mu_i z) > \gamma > 0$ for all $z \in \mathbb{T}$. Since $a_k \neq 0$ and ϕ is full, we have that $\bar{\phi}(a_k)$ is full in $\mathcal{C}(J)$ and thus $\sum_{i=1}^{r} h_k(\mu_i \bar{u})\bar{\phi}(a_k) \geq \gamma\bar{\phi}(a_k)$ is full in $\mathcal{C}(J)$. Hence at least one of the lifts of $h_k(\mu_i \bar{u})\bar{\phi}(a_k)$ does not lie in the ideal $\mathcal{I}$. Take λ_k to be the coefficient among $\mu_1, \cdots, \mu_r$ such that the lift of $h_k(\lambda_k \bar{u})\bar{\phi}(a_k)$ does not lie in $\mathcal{I}$.

Next, we use the abstract hypothesis on $\mathcal{I}$ that the extension of $\mathcal{I}$ in $\mathcal{M}(J)$ is purely large relative to J. This ensures that any positive lift of $h_k(\lambda_k \bar{u})\bar{\phi}(a_k)$ Cuntz-dominates all the elements in J and such positive lift can be approximated by $\hat{h}_k(v_\alpha u)\phi(a_k)$. Given starting indices n and m, and any $b \in J_+$, we can find some $n' > n$, and contractions $x \in \overline{bJ(e_{m'} - e_m)}$ for some $m' > m$ with

$$x\hat{h}_k(v_\alpha u)\phi(a_k)x^* \approx_\epsilon b, \tag{6.2.9}$$

for any sequence $\alpha = (\alpha_n)_n$ which takes the constant value λ_k on the block of indices from n to n'. See Lemma 6.3.1 for the details of this part of the proof.

This procedure allows one to find $n_1 < n'_1 < n_2 < n'_2 < \ldots$ with $n_{r+1} - n'_r \to \infty$ as $r \to \infty$. They divide the index set into infinitely many *blocks* $B_r = \{n_r, n_r{+}1, \ldots, n'_r\}$ on which the α_n will take specified constant values in $\mathbb{T}$, coming from the construction of the previous paragraph. The r-th block and the $(r+1)$-th block are separated by *gaps* of length $n_{r+1} - n'_r$ tending to infinity. Hence it is possible to fill in the values of α_n in the gaps so that the condition of Proposition 6.2.6 is satisfied and $\bar{v}_\alpha$ is indeed a unitary in the connected component of the identity in $\mathcal{C}(J) \cap \bar{\phi}(A)'$. Moreover, we will do this in such a way that each value λ_k will appear in infinitely many blocks B_r, enabling us to approximate the sequence $(e_n - e_{n-1})_n$, and hence approximate $1_{\mathcal{M}(J)}$ by combining infinitely many elements x with orthogonal supports from (6.2.9). This will enable us to ensure that each $\hat{h}_k(v_\alpha u)\phi(a_k)$ is full in $\mathcal{M}(J)$ for every $k \in \mathbb{N}$. See the proof of Lemma 6.2.4 for more details in the next section.

The following technical lemma for Laurent polynomials and almost idempotent approximate units is included. It is proven in [69] when the $(e_n)_n$ are projections, and as remarked in [70, Lemma 3.15], the proof works similarly for almost idempotent $(e_n)_n$. The proof involves standard computations taking advantage of the approximate unit, and it suffices to verify the approximations for monomials. For elements $a, b \in A$ and $\epsilon > 0$, we denote $\|a - b\| < \epsilon$ by $a \approx_\epsilon b$.

Lemma 6.2.8 ([69, Lemma 4.2, 4.3]). *Let J be a σ-unital C^*-algebra with an almost idempotent approximate unit $(e_n)_n$ and u a contraction in $\mathcal{M}(J)$. For any Laurent polynomial $\hat{h}$, $\epsilon > 0$, integers $n_0, m_0 > 0$, $b \in J$, $a \in \mathcal{M}(J)$, the following properties hold:*

(i) There exists $m_1 \geq m_0 \in \mathbb{N}$ such that

$$(1 - e_{m_1})\hat{h}(v_\alpha u) \approx_\epsilon (1 - e_{m_1})\hat{h}(v_\beta u) \tag{6.2.10}$$

for any sequences $\alpha = (\alpha_n), \beta = (\beta_n) \subseteq \mathbb{T}$ with $\alpha_n = \beta_n$ for $n \geq n_0$.

(ii) There exists $n_1 \in \mathbb{N}$ such that

$$\hat{h}(v_\alpha u)b \approx_\epsilon \hat{h}(v_\beta u)b \tag{6.2.11}$$

for any sequences $\alpha = (\alpha_n), \beta = (\beta_n) \subseteq \mathbb{T}$ with $\alpha_n = \beta_n$ for $n \leq n_1$.

(iii) There exists $m_2 \geq m_0$ such that for any sequences $\alpha = (\alpha_n) \subseteq \mathbb{T}$, we have

$$\|e_{m_0}\hat{h}(v_\alpha u)a(1 - e_{m_2})\| < \epsilon \text{ and } \|(1 - e_{m_2})\hat{h}(v_\alpha u)ae_{m_0}\| < \epsilon. \tag{6.2.12}$$

6.3 Proof of the main lemma

The following lemma provides the inductive step of the proof of Lemma 6.2.4. Recall that for $\epsilon > 0$, we define a positive-valued continuous function on $[0,\infty)$,

$$f_\epsilon(t) = \begin{cases} t/\epsilon & 0 \le t \le \epsilon, \\ 1 & t \ge \epsilon. \end{cases} \tag{6.3.1}$$

For completeness, we state all the assumptions involved.

Lemma 6.3.1. *Let A be a unital and separable C^*-algebra and let J be a stable C^*-algebra with an almost idempotent approximate unit $(e_m)_m$. Suppose there exists a proper ideal $\mathcal{I} \subseteq \mathcal{M}(J)$ containing J such that the extension of $\mathcal{I}$ in $\mathcal{M}(J)$ is relatively purely large with respect to J. Let $\phi : A \to \mathcal{M}(J)$ be a unital full *-homomorphism. Let $\bar{u}$ be a unitary in $\mathcal{C}(J) \cap \bar{\phi}(A)'$ with a contractive lifting $u \in \mathcal{M}(J)$.*

Let $a \in A_+$ be a nonzero positive contraction and let $h \in C(\mathbb{T})_+$ be of norm one. There exists $\delta > 0$ and $\lambda \in \mathbb{T}$ such that for any $m_0, n_0 \in \mathbb{N}$, any positive contraction $b_0 \in J$, $\epsilon > 0$ and Laurent polynomial $\hat{h}$ with $\|f_{1/3} \circ h - \hat{h}|_{\mathbb{T}}\|_\infty < \epsilon$, there exist $m_2 > m_1 > m_0$, $n_1 > n_0$ and a contraction $x \in \overline{b_0 J(e_{m_2} - e_{m_1})}$ such that

$$x\hat{h}(v_\alpha u) f_\delta(\phi(a)) x^* \approx_\epsilon b_0, \tag{6.3.2}$$

whenever $\alpha = (\alpha_n)_n$ is a sequence in $\mathbb{T}$ with $\alpha_n = \lambda$ for $n_0 \le n \le n_1$.

Proof. Fix $a \in A$ to be a non-zero positive contraction and $h \in C(\mathbb{T})_+$ with norm one. Since $a \neq 0$ and φ is full, there exists $\delta > 0$ such that $(\varphi(a)-\delta)_+ = \varphi((a-\delta)_+)$ is full in $\mathcal{M}(J)$. As $h \in C(\mathbb{T})$ is positive and has norm one, by choosing a neighborhood in $\mathbb{T}$ where the value of h is strictly bigger than $1/3$ and rotating it by sufficiently many points on $\mathbb{T}$, we can find $\mu_1, \cdots, \mu_s \in \mathbb{T}$ such that

$$0 < \kappa \le \sum_{i=1}^{s} (h - 1/3)_+(\mu_i t), \quad t \in \mathbb{T}, \tag{6.3.3}$$

for some constant $\kappa > 0$. Since $\bar{u} \in \mathcal{C}(J)$ commutes with $\bar{\phi}(A)$, it follows that

$$0 \le \kappa \cdot (\bar{\phi}(a) - \delta)_+ \le \sum_{i=1}^{s} (h - 1/3)_+(\mu_i \bar{u})(\bar{\phi}(a) - \delta)_+. \tag{6.3.4}$$

Since $(\phi(a) - \delta)_+ \in \mathcal{M}(J)$ is a full element, the sum on the right-hand side of (6.3.4) is full in $\mathcal{C}(J)$, and so does not lie in the proper ideal $\pi(\mathcal{I}) \lhd \mathcal{C}(J)$, where $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$ is the quotient map. Thus there exists $i_0 \in \{1, \ldots, s\}$ such that

$$\bar{d} \coloneqq (h - 1/3)_+(\mu_{i_0}\bar{u})(\bar{\phi}(a) - \delta)_+ \notin \pi(\mathcal{I}). \tag{6.3.5}$$

Fix $\lambda \coloneqq \mu_{i_0}$, and write

$$\bar{c} \coloneqq (f_{1/3} \circ h)(\lambda \bar{u}) f_\delta(\bar{\phi}(a)), \tag{6.3.6}$$

where $f_{1/3}$ and f_δ are the positive functions of norm one defined in (6.3.1). Then $\bar{c}$ and $\bar{d}$ are contractions in $\mathcal{C}(J)_+$ satisfying $\bar{c}\bar{d} = \bar{d}$ since $\bar{u} \in \bar{\phi}(A)'$. Moreover, applying the canonical map $\mathcal{C}(J) \to \mathcal{M}(J)/\mathcal{I}$, the image of $\bar{c}$ in $\mathcal{M}(J)/\mathcal{I}$ acts as the unit on the image of $\bar{d}$ (which is non-zero as $\bar{d} \notin \pi(\mathcal{I})$). Accordingly,

$$\|\bar{c} + \mathcal{I}\| = 1 \text{ in } \mathcal{M}(J)/\mathcal{I}. \tag{6.3.7}$$

Let $m_0, n_0 \in \mathbb{N}$, $\epsilon > 0$, $b_0 \in J$ be a positive contraction and $\hat{h}$ be a Laurent polynomial with $\|f_{1/3} \circ h - \hat{h}|_{\mathbb{T}}\|_\infty < \epsilon$. Let $c \in \mathcal{M}(J)$ be a contractive positive lift of $\bar{c}$. For any sequence $\alpha = (\alpha_n)_n$ with $\alpha_n = \lambda$ for all $n \geq n_0$, we have $\bar{v}_\alpha = \lambda 1_{\mathcal{C}(J)}$ and thus

$$\pi\big(\hat{h}(v_\alpha u)\big) = \hat{h}(\bar{v}_\alpha \bar{u}) = \hat{h}(\lambda \bar{u}) = \pi\big(\hat{h}(\lambda u)\big). \tag{6.3.8}$$

Take $0 < \mu < \epsilon - \|f_{1/3} \circ h - \hat{h}|_{\mathbb{T}}\|_\infty$, then by the choice of μ and $\hat{h}$,

$$\pi(c) = \bar{c} \approx_{\epsilon-\mu} \hat{h}(\lambda \bar{u}) f_\delta(\bar{\phi}(a)) \overset{(6.3.8)}{=} \pi(\hat{h}(v_\alpha u) f_\delta(\phi(a))). \tag{6.3.9}$$

As $(e_m)_m$ is an almost idempotent approximate unit of J, we can approximately lift (6.3.9) to elements in the multiplier algebra. By applying Lemma 6.2.8 (i), there exists $m_1 > m_0$ such that

$$(1 - e_{m_1}) c (1 - e_{m_1}) \approx_{\epsilon-\mu} (1 - e_{m_1}) \hat{h}(v_\alpha u) f_\delta(\phi(a))(1 - e_{m_1}), \tag{6.3.10}$$

for any sequence $\alpha = (\alpha_n)_n$ with $\alpha_n = \lambda$ for all $n \geq n_0$.

Note that as $(1 - e_{m_1}) c (1 - e_{m_1})$ is a lift of $\bar{c}$ in $\mathcal{M}(J)$, (6.3.7) implies that

$$\|q((1 - e_{m_1}) c (1 - e_{m_1}))\| = \|\bar{c} + \mathcal{I}\| = 1 \text{ in } \mathcal{M}(J)/\mathcal{I}, \tag{6.3.11}$$

where $q : \mathcal{M}(J) \to \mathcal{M}(J)/\mathcal{I}$ is the quotient map. As $\mathcal{I}$ is relatively purely large in $\mathcal{M}(J)$ with respect to J, Lemma 5.1.2 provides a contraction $y \in J$ with

$$y(1 - e_{m_1}) c (1 - e_{m_1}) y^* \approx_{\mu/2} b_0. \tag{6.3.12}$$

Combining this with (6.3.10), we have

$$y(1 - e_{m_1}) \hat{h}(v_\alpha u) f_\delta(\phi(a))(1 - e_{m_1}) y^* \approx_{\epsilon-\mu/2} b_0, \tag{6.3.13}$$

for all sequences (α_n) with $\alpha_n = \lambda$ when $n \geq n_0$. Since $b_0^{1/\ell} b_0 \to b_0$ when $\ell \to \infty$ and $(e_m)_m$ is an approximate unit of J, there exists $\ell_0 \in \mathbb{N}$ and $m_2 > m_1$ such that when replacing $y(1 - e_{m_1})$ in (6.3.13) by the contraction

$$x \coloneqq b_0^{1/\ell_0} y(e_{m_2} - e_{m_1}) \in \overline{b_0 B(e_{m_2} - e_{m_1})}, \tag{6.3.14}$$

the approximation (6.3.13) still holds, namely,

$$x\hat{h}(v_\alpha u)f_\delta(\phi(a))x^* \approx_{\epsilon-\mu/2} b_0, \tag{6.3.15}$$

for all sequences (α_n) with $\alpha_n = \lambda$ when $n \geq n_0$. Lastly, since $x \in J$, by Lemma 6.2.8 (i), there exists $n_1 > n_0$ such that

$$x\hat{h}(v_\alpha u) \approx_{\mu/2} x\hat{h}(v_\beta u), \tag{6.3.16}$$

whenever the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ in $\mathbb{T}$ have $\alpha_n = \beta_n$ for all $n \leq n_1$. Combining this with (6.3.15), we obtain

$$x\hat{h}(v_\alpha u)f_\delta(\phi(a))x^* \approx_\epsilon b_0, \tag{6.3.17}$$

for any $\alpha = (\alpha_n)_n$ in $\mathbb{T}$ with $\alpha_n = \lambda$ for $n_0 \leq n \leq n_1$. □

We can now inductively apply Lemma 6.3.1 to prove the main technical lemma (Lemma 6.2.4).

Proof of Lemma 6.2.4. Fix a unitary $\bar{u} \in \mathcal{C}(J) \cap \bar{\phi}(A)'$ with a contractive lift u in $\mathcal{M}(J)$. Let $(e_n)_{n=1}^\infty$ be an almost idempotent approximate unit of J quasicentral for $\bar{\phi}(a)$ given by Proposition 6.2.5, and write $e_0 = 0$. Fix $0 < \epsilon < 1/12$.

Let $(h_k)_{k=1}^\infty$ and $(a_k)_{k=1}^\infty$ be sequences of norm one positive functions in $C(\mathbb{T})$ and non-zero positive contractions in A respectively given by Lemma 6.2.7. For each k, we fix a Laurent polynomial $\hat{h}_k$ so that $\|h_k - \hat{h}_k\|_\infty < \epsilon$, and take $\delta_k > 0$ and $\lambda_k \in \mathbb{T}$ satisfying the conditions mentioned in Lemma 6.3.1. Finally, fix a bijection $\theta\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $m \in \mathbb{N}$ and $\ell_1 < \ell_2$, then $\theta(k, \ell_1) < \theta(k, \ell_2)$. In the following proof, we perform an induction on r and $\theta(k, \ell) = r$ means that the r-th inductive step is the ℓ-th time we use the pair (h_k, a_k) to approximate $e_\ell - e_{\ell-1}$, see (ii) for details.

Combining Lemma 6.2.8 and Lemma 6.3.1, it is possible to inductively construct

- contractions $x_r \in J$ for every $r \in \mathbb{N}$;
- increasing sequences of natural numbers $n_1 < n_1' < n_2 < n_2' < \ldots$ and $m_1 < m_1' < m_2 < m_2' < \ldots$ such that $n_r \geq n_{r-1}' + r$,

with the following properties for each $r \in \mathbb{N}$ and $k, \ell \in \mathbb{N}$ satisfying $\theta(k, \ell) = r$:

(i) $x_r \in \overline{(e_\ell - e_{\ell-1})J(e_{m_r'} - e_{m_r})}$;

(ii) for any sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$ in $\mathbb{T}$ with $\alpha_n = \lambda_m$ for $n_r \leq n \leq n'_r$, one has

$$x_r \hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k)) x_r^* \approx_\epsilon e_\ell - e_{\ell-1}; \tag{6.3.18}$$

(iii) for any sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$ in $\mathbb{T}$, the element $\hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k))$ is approximately block diagonal and its "off-diagonal" terms are norm small, in the sense that

$$\left\| e_{m'_{r-1}+1} \hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k))(1 - e_{m_r - 1}) \right\| < \frac{\epsilon}{\ell \cdot 2^{\ell+1}} \text{ and} \tag{6.3.19}$$

$$\left\| (1 - e_{m_r - 1}) \hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k)) e_{m'_{r-1}+1} \right\| < \frac{\epsilon}{\ell \cdot 2^{\ell+1}}. \tag{6.3.20}$$

Indeed, assume these objects have been constructed for the first $r-1$ inductive steps (in the case $r = 1$, we can formally commence with $n'_0 = m'_0 = 0$). Suppose that k, ℓ are natural numbers such that $\theta(k, \ell) = r$. Then Lemma 6.2.8 (iii) enables us to find some $\tilde{m}_r > m'_{r-1}$ such that for any $m \geq \tilde{m}_r$ and $\alpha = (\alpha_n)_n$ in $\mathbb{T}$, we have

$$\left\| e_{m'_{r-1}+1} \hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k))(1 - e_{\tilde{m}_r}) \right\| < \frac{\epsilon}{\ell \cdot 2^{\ell+1}}, \tag{6.3.21}$$

$$\left\| (1 - e_{\tilde{m}_r}) \hat{h}_k(v_\alpha u) f_{\delta_k}(\phi(a_k)) e_{m'_{r-1}+1} \right\| < \frac{\epsilon}{\ell \cdot 2^{\ell+1}}. \tag{6.3.22}$$

Now we can apply Lemma 6.3.1 to $\tilde{m}_r$ and some $n_r := n'_{r-1} + r + 1$ as starting indices, and find $m'_r > m_r > \tilde{m}_r$ (so that (iii) holds by the choice of $\tilde{m}_r$), together with $n'_r > n_r$ and x_r such that (i) and (ii) hold.

For each $r \in \mathbb{N}$, suppose that k, ℓ are the natural numbers that $\theta(k, \ell) = r$. Then define $\beta_n = \lambda_k$ for those $n_r \leq n \leq n'_r$ lying in the r-th block. Since the gap length $n_r - n'_{r-1} > r$, we can fill in the values of β_n in the gap so that they slowly move from the value in the $(r-1)$-th block to the value in the r-th block without winding around the circle. More precisely, suppose the constant value in the $(r-1)$-th block and r-th block are μ and ν respectively. Take $\theta_{n'_{r-1}} = \log \mu$ and $\theta_{n_r} = \log \nu$ in $[0, 2\pi)$. For $n'_{r-1} \leq n \leq n_r$, define

$$\theta_n = \log \mu + \frac{n - n'_{r-1}}{n_r - n'_{r-1}} (\log \nu - \log \mu), \tag{6.3.23}$$

and $\beta_n = e^{\theta_n}$. Since $n_r - n'_{r-1} > r$, then $|\beta_n - \beta_{n+1}| < 2\pi/r$ for $n'_{r-1} \leq n \leq n_r$. By Proposition 6.2.6, we have constructed a sequence $\beta = (\beta_n)_n$ in $\mathbb{T}$ such that $\bar{v}_\beta$ is a unitary in $\mathcal{U}_0(\mathcal{C}(J) \cap \bar{\phi}(A)')$.

It remains to show that the inclusion $\iota : C^*(\bar{\phi}(A), \bar{v}_\beta u) \to \mathcal{C}(J)$ is full. By our choices of $(h_k)_k$ and $(a_k)_k$ from Lemma 6.2.7, it suffices to show that for each $k \in \mathbb{N}$, the element $h_k(\bar{v}_\beta \bar{u}) \bar{\phi}(a_k)$ is full in $\mathcal{C}(J)$.

Fix some $k \in \mathbb{N}$. Since x_r is contractive for any $r \in \mathbb{N}$ and satisfies (i), the bidiagonal sum

$$x \coloneqq \sum_{\ell=1}^{\infty} x_{\theta(k,\ell)} \tag{6.3.24}$$

is strictly convergent in $\mathcal{M}(J)$ with $\|x\| \leq 2$ by Lemma 2.9.11. We will first estimate

$$x\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^* = \sum_{\ell_1,\ell_2=1}^{\infty} x_{\theta(k,\ell_1)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell_2)}. \tag{6.3.25}$$

Firstly, we show that the sum of the "off-diagonal" terms, namely the terms with $\ell_1 \neq \ell_2$, is norm convergent with small norm. Suppose $\ell_1 < \ell_2$, then $r_1 \coloneqq \theta(k,\ell_1) < r_2 \coloneqq \theta(k,\ell_2)$ and by construction,

$$m_{r_1} < m'_{r_1} < m'_{r_2-1} + 1 \leq m_{r_2} < m'_{r_2}. \tag{6.3.26}$$

Since x_{r_1} satisfies (i) and $(e_n)_n$ is almost idempotent, it follows that $x_{r_1} e_{m'_{r_2-1}+1} = x_{r_1}$. Likewise, x_{r_2} satisfies (i) and $(e_{m'_{r_2}} - e_{m_{r_2}})(1 - e_{m_{r_2}-1}) = e_{m'_{r_2}} - e_{m_{r_2}}$, then $x_{r_2}(1 - e_{m_{r_2}-1}) = x_{r_2}$. Putting these together, we have

$$x_{\theta(k,\ell_1)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell_2)} \tag{6.3.27}$$

$$= x_{r_1}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{r_2} \tag{6.3.28}$$

$$= x_{r_1} e_{m'_{r_2-1}+1}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))(1 - e_{m_{r_2}-1})x^*_{r_2}. \tag{6.3.29}$$

Then by property (iii), we have

$$\left\|x_{\theta(k,\ell_1)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell_2)}\right\| < \frac{\epsilon}{\ell_2 \cdot 2^{\ell_2+1}}. \tag{6.3.30}$$

Similar calculations work in an identical fashion when $\ell_1 > \ell_2$ and we get

$$\left\|x_{r_1} e_{m'_{r_2-1}+1}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))(1 - e_{m_{r_2}-1})x^*_{r_2}\right\| < \frac{\epsilon}{\ell_1 \cdot 2^{\ell_1+1}}. \tag{6.3.31}$$

Summing up all of the "off-diagonal" terms gives us

$$\left\|\sum_{\ell_1\neq\ell_2} x_{\theta(k,\ell_1)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell_2)}\right\| < 2\sum_{\ell_2=1}^{\infty}\sum_{\ell_1<\ell_2} \frac{\epsilon}{\ell_2 \cdot 2^{\ell_2+1}} < \epsilon. \tag{6.3.32}$$

For the "diagonal" terms in (6.3.25), condition (ii) and the definition of β gives

$$x_{\theta(k,\ell)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell)} \approx_\epsilon e_\ell - e_{\ell-1}, \tag{6.3.33}$$

for every $\ell \in \mathbb{N}$. Then Lemma 2.9.11 gives

$$\sum_{\ell=1}^{\infty} x_{\theta(k,\ell)}\hat{h}_k(v_\beta u) f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell)} \approx_{2\epsilon} \sum_{\ell=1}^{\infty}(e_\ell - e_{\ell-1}) = 1_{\mathcal{M}(J)}, \tag{6.3.34}$$

where the series converges strictly since they are bidiagonal. Combining the estimates (6.3.32) for the “off-diagonal” terms and (6.3.34) for these “diagonal” terms gives

$$x\hat{h}_k(v_\beta u)f_{\delta_k}(\phi(a_k))x^* \overset{(6.3.32)}{\approx_\epsilon} \sum_{\ell=1}^{\infty} x_{\theta(k,\ell)}\hat{h}_k(v_\beta u)f_{\delta_k}(\phi(a_k))x^*_{\theta(k,\ell)} \overset{(6.3.34)}{\approx_{2\epsilon}} 1_{\mathcal{M}(J)}. \quad (6.3.35)$$

Now apply the quotient map $\pi : \mathcal{M}(J) \to \mathcal{C}(J)$ to the estimate (6.3.35) to obtain

$$\bar{x}\hat{h}_k(\bar{v}_\beta \bar{u})f_{\delta_k}(\bar{\phi}(a_k))\bar{x}^* \approx_{3\epsilon} 1_{\mathcal{C}(J)}. \quad (6.3.36)$$

Since $\|h_k - \hat{h}_k\|_\infty < \epsilon$ and $\|\bar{x}\| \leq \|x\| \leq 2$, we have

$$\bar{x}h_k(\bar{v}_\beta \bar{u})f_{\delta_k}(\bar{\phi}(a_k))\bar{x}^* \approx_{7\epsilon} 1_{\mathcal{C}(J)}, \quad (6.3.37)$$

which implies that $h_k(\bar{v}_\beta\bar{u})f_{\delta_k}(\phi(a_k))$ is full in $\mathcal{C}(J)$ provided that we chose $\epsilon < 1/7$. Notice that $f_{\delta_k}(\bar{\phi}(a_k))$ is Cuntz-equivalent to $\bar{\phi}(a_k)$ in $\bar{\phi}(A)$. Because $\bar{v}_\beta\bar{u} \in \bar{\phi}(A)' \cap \mathcal{C}(J)$, it commutes with elements witnessing $f_{\delta_k}(\bar{\phi}(a_k)) \sim \bar{\phi}(a_k)$ in $\bar{\phi}(A)$, and thus $h_k(\bar{v}_\beta\bar{u})f_{\delta_k}(\bar{\phi}(a_k))$ is Cuntz equivalent to $h_k(\bar{v}_\beta\bar{u})\bar{\phi}(a_k)$ in $\mathcal{C}(J)$. As a result, the latter is full in $\mathcal{C}(J)$. $\square$

As explained in Section 4.3, to obtain a KK- and KL-uniqueness theorem, it suffices to show K_1-injectivity of $\bar{\phi}(A)' \cap \mathcal{C}(J)$ corresponding to any unitally absorbing $\phi : A \to \mathcal{M}(J)$. Hence, combining Theorem 4.3.5 and Corollary 6.2.2, we get the following uniqueness theorem (Theorem C).

Theorem 6.3.2. *Let A be a unital and separable C^*-algebra. Let J be a separable stable C^*-algebra with real rank zero, stable rank one, strict comparison, $K_1(J) = 0$, and the totally ordered Murray-von Neumann semigroup $V(J)$. Let $(\phi, \psi)\colon A \rightrightarrows \mathcal{M}(J) \rhd J$ be a Cuntz pair with ϕ and ψ unitally absorbing *-homomorphisms.*

(i) If $[\phi, \psi]_{KK(A,J)} = 0$, there exists a norm-continuous path $(u_t)_{t\geq 0}$ of unitaries in $J^\dagger$ such that

$$\|u_t(\phi(a))u_t^* - \psi(a)\| \to 0, \quad a \in A. \quad (6.3.38)$$

(ii) If $[\phi, \psi]_{KL(A,J)} = 0$, there exists a sequence $(u_n)_{n=1}^\infty$ of unitaries in $J^\dagger$ such that

$$\|u_n(\phi(a))u_n^* - \psi(a)\| \to 0, \quad a \in A. \quad (6.3.39)$$

In the next chapter, the KK and KL-uniqueness theorem will be used to obtain uniqueness results for maps (Theorem A and Theorem B), through the abstract classification framework.

Chapter 7

New classification results for maps

In Chapter 7, we prove our main uniqueness theorem for maps via the abstract classification approach described in Section 1.4. The first section presents the standard separabilization argument, which detects properties locally in separable C^*-subalgebras of non-separable ambient C^*-algebras. This allows us to deduce properties for separable C^*-subalgebras of either ultrapowers B_ω or trace-kernel ideals J_B.

The second part of the chapter builds on the KK-uniqueness theorem proved in Chapter 6 and establishes the main uniqueness result (Theorem E). As direct corollaries, we obtain uniqueness theorems for maps into II_1 factors (Theorem A) and ultraproducts of matrices (Theorem B). The results in this chapter will appear in a forthcoming joint paper [51] by White and me.

7.1 Separabilization arguments

Throughout the thesis, several non-separable C^*-algebras are of interest, for example, the ultraproduct constructions including $\mathcal{R}_\omega$ and $\prod_\omega M_{n_k}$. For technical reasons, $KK(A,B)$ works better when A is separable and B is σ-unital. However, later in this chapter, we need to work with $KK(A,J_B)$, where J_B is the trace-kernel ideal described in Section 1.4, and is highly non-separable.

Thus, we include a general method for constructing separable subalgebras, which inherit prescribed properties of the ambient C^*-algebra. This method was initially developed by Blackadar in [7, Section II.8.5] and is now a standard technique. Recently, the separabilization argument has been heavily used in the work of Schafhauser ([97]) and later work of Carrión, Gabe, Schafhauser, Tikuisis and White ([14]), where they develop the abstract classification approach. Similar arguments are needed in the proof of the main classification theorem for maps (Theorem E) in Section 7.2.

Definition 7.1.1 ([7, Section II.8.5])**.** A property (P) of C^*-algebras is called *separably inheritable* if the following statements are true:

(i) if A is a C^*-algebra satisfying (P) and A_0 is a separable C^*-subalgebra of A, there exists a separable C^*-subalgebra B of A which satisfies (P) and $A_0 \subseteq B$;

(ii) if $A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$ is an inductive limit of separable C^*-algebras with injective connecting maps, if each A_n satisfies (P), then $\varinjlim A_n$ satisfies (P).

Many important properties of C^*-algebras are separably inheritable, and we list some that will be relevant later. See Section II.8.5 of [7] for more examples of separably inheritable properties.

Lemma 7.1.2. *The following properties are separably inheritable:*

(i) real rank zero;

(ii) stable rank one;

(iii) $K_i(\cdot; \mathbb{Z}/n\mathbb{Z}) = 0$ *for some* $i \in \{0, 1\}$ *and* $n \in \mathbb{N}$*;*

(iv) total ordering of the Murray-von Neumann semigroup $V(\cdot)$*;*

(v) almost unperforation of the Cuntz semigroup $Cu(\cdot)$*.*

(vi) unique divisibility of $K_0(\cdot)$*.*

Proof. Properties (i) and (ii) are separably inheritable by [7, II.8.5.5]. We include a standard argument for property (iv) being separably inheritable and the other properties can be proven in an almost identical way.

Suppose that $V(A)$ is totally ordered for a C^*-algebra A and A_0 is a separable C^*-subalgebra of A. Then $V(A_0)$ is a countable semigroup and there are countably many pairs $([p_n], [q_n])$ of equivalence classes of projections in $M_\infty(A_0)$. Since $V(A)$ is totally ordered, we can find partial isometries $v_n \in M_\infty(A)$, which witness either $[p_n] \leq [q_n]$ or $[q_n] \leq [p_n]$ in $V(A)$. Let A_1 be the C^*-subalgebra of A generated by A_0 and entries of each v_n. Then A_1 is a separable C^*-subalgebra of A, containing A_0 and for projections $p, q \in M_\infty(A_0)$, either $[p] \leq [q]$ or $[q] \leq [p]$ in $V(A_1)$. Continue the construction inductively and take $B = \varinjlim A_n$, which is separable, contains A_0 and $V(B)$ is totally ordered. It is obvious that the property (iv) is preserved by the inductive limit of separable C^*-algebras with injective connecting maps. □

The following lemma is standard and useful in applications.

Lemma 7.1.3 ([7, II.8.5.3])**.** *Let $(P_n)_n$ be a sequence of separably inheritable properties. Then the meet of $(P_n)_n$ is separably inheritable.*

$\mathcal{Z}$-stability or stability are not the right notions outside of the separable setting. For instance, it is shown by [36] that the ultrapower of a $\mathcal{Z}$-stable C^*-algebra cannot be $\mathcal{Z}$-stable. The following definition provides the right solution by checking these properties locally in separable C^*-subalgebras.

Definition 7.1.4 (cf. [97, Definition 1.4])**.** Let (P) be a property of separable C^*-algebras. A C^*-algebra A *separably satisfies* (P) if for any separable C^*-subalgebra A_0 of A, there exists a separable C^*-subalgebra B of A which satisfies (P) and $A_0 \subseteq B$.

Note that for a separable C^*-algebra A, it separably satisfies (P) if and only if it satisfies (P). Note also that if (P) is a property for separable C^*-algebras, preserved under sequential inductive limits of separable C^*-algebras with injective connecting maps, then separably (P) is a separably inheritable property by Definition 7.1.1. For instance, stability is preserved under the sequential inductive limit of separable C^*-algebras, see [48, Corollary 2.3] for instance. This leads to the following conclusion.

Lemma 7.1.5. *Separable stability is a separably inheritable property.*

For maps from separable C^*-algebras to non-separable ones, for instance, ultrapowers, the following lemma allows us to corestrict to a separable codomain.

Proposition 7.1.6 ([97, Proposition 1.9])**.** *Suppose A and B are C^*-algebras such that A is separable and B is unital. If $\phi : A \to B$ is a full and nuclear *-homomorphism, there is a separable, unital C^*-subalgebra B_0 of B such that $\phi(A) \subseteq B_0$ and the corestriction of ϕ to B_0 is full and nuclear.*

Lastly, we include a technical result, which will later provide separabilizations for trace-kernel extensions defined in Section 7.2.

Proposition 7.1.7 ([97, Proposition 1.6])**.** *Consider an extension*

$$0 \longrightarrow I \xrightarrow{\ j\ } E \xrightarrow{\ q\ } D \longrightarrow 0 \tag{7.1.1}$$

of C^*-algebras and suppose for each $X \in \{I, E, D\}$, (P_X) is separably inheritable and X satisfies (P_X). If for each $X \in \{I, E, D\}$, a separable C^*-subalgebra X_0 of X is *given, then for each $X \in \{I, E, D\}$, there is a separable C^*-subalgebra $\hat{X}$ of X which satisfies (P_X), contains X_0 and such that there is a homomorphism*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{I} & \longrightarrow & \hat{E} & \longrightarrow & \hat{D} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{j} & E & \xrightarrow{q} & D & \longrightarrow & 0 \end{array}$$

of extensions where the vertical arrows are the inclusion maps.

7.2 From KK-uniqueness to classification

The trace-kernel extension will provide the abstract framework for uniqueness theorems for maps through the KL-uniqueness theorem (Theorem 6.3.2). Let $(B_n)_n$ be a sequence of C^*-algebras. We denote by $\prod_{n=1}^{\infty} B_n$ the C^*-algebra of all bounded sequences in B_n's and recall the definition for ultraproduct of $(B_n)_n$ in Section 2.2,

$$\prod_{n\to\omega} B_n = \prod_{n=1}^{\infty} B_n \Big/ \left\{ (x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} B_n : \lim_{n\to\omega} \|x_n\| = 0 \right\}. \tag{7.2.1}$$

We will often write B_ω for the ultraproduct $\prod_{n\to\omega} B_n$. As is standard we will typically denote elements in B_ω using representing sequences $(x_n)_n$ from $\prod_{n=1}^{\infty} B_n$.

Suppose additionally that each B_n is equipped with a unique tracial state τ_n, inducing the trace seminorm $\|x\|_{2,\tau_n} \coloneqq \tau_n(x^*x)^{1/2}$ on each of B_n. We obtain a trace τ_{B_ω} on B_ω, which is well-defined on representative sequences $(x_n)_n$ by $\tau_{B_\omega}((x_n)_n) = \lim_{n\to\omega} \tau_n(x_n)$. This induces a seminorm on B_ω, given by $\|x\|_{2,\omega} \coloneqq \tau_{B_\omega}(x^*x)^{1/2}$ for $x \in B_\omega$. The *trace-kernel ideal* J_B of B_ω is given by

$$J_B \coloneqq \{x \in B_\omega : \tau_{B_\omega}(x^*x) = 0\}, \tag{7.2.2}$$

and is an ideal in B_ω. Define the *trace-kernel quotient* to be $B^\omega \coloneqq B_\omega / J_B$ and note that τ_{B_ω} descends to a faithful trace τ_{B^ω} on B^ω, which means that $\tau_{B_\omega} = \tau_{B^\omega} \circ q_B$. Then the *trace-kernel extension* is given by

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0. \tag{7.2.3}$$

With this setup, we now state our main uniqueness theorem (Theorem E).

Theorem 7.2.1. *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT. Let $(B_n)_{n=1}^{\infty}$ be a sequence of unital and simple C^*-algebras which have real rank* *zero, stable rank one, a unique tracial state τ_n, totally ordered $V(B_n)$ and $K_1(B_n) = 0$. We write B_ω for $\prod_\omega B_n$.*

Given full and unital *-homomorphisms $\phi, \psi\colon A \to B_\omega$ with $\tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi$ and $\underline{K}(\phi) = \underline{K}(\psi)$, there exists a unitary $u \in B_\omega$ with $\psi = \operatorname{Ad} u \circ \phi$.

Although the list of assumptions on B_n seems quite restrictive, Theorem 7.2.1 covers two cases of interest, when taking $B_n = M_{k_n}$ or $B_n = \mathcal{R}$ for $n \in \mathbb{N}$. To prove this main theorem, we collect below properties of the trace-kernel extension under the assumptions on B_n in Theorem 7.2.1.

Proposition 7.2.2. *Let $(B_n)_{n=1}^\infty$ be a sequence of unital C^*-algebras such that each B_n has a unique trace τ_n. Then*

(i) B^ω is a finite von Neumann factor with the unique trace τ_{B^ω}. This is type I_m if and only if $\{n \in \mathbb{N} : B_n \cong M_m\} \in \omega$, and in this case $B^\omega = B_\omega \cong M_m$.

Suppose additionally that each B_n has real rank zero, stable rank one, a unique tracial state τ_n, totally ordered $V(B_n)$ and $K_1(B_n) = 0$. Then

(ii) τ_n is the unique quasitrace on B_n, with respect to which B_n has strict comparison, for every $n \in \mathbb{N}$;

(iii) B_ω has real rank zero, stable rank one, a unique trace τ_{B_ω} with respect to which B_ω has strict comparison, totally ordered $V(B_\omega)$ and $K_1(B_\omega) = 0$;

(iv) J_B has real rank zero, stable rank one, almost unperforated $Cu(J_B)$, totally ordered $V(J_B)$, $K_1(J_B) = 0$, and J_B is separably stable.

Proof. The argument for (i) is standard, see [97, Proposition 3.2].

For (ii), we show first that $d_{\tau_n}(p) \leq d_{\tau_n}(q)$ if and only if $p \leq_0 q$ for any projections $p, q \in M_m(B_n)$. For the non-trivial "only if" direction, suppose that p is not subequivalent to q. Since B_n has stable rank one and $V(B_n)$ is totally ordered, it follows that $q \leq_0 p$. There exists $v \in M_m(B_n)$ such that $q = v^*v$ and $vv^* \leq p$, $vv^* \neq p$. Since B_n is simple, then $d_{\tau_n}(p - vv^*) > 0$, which implies that $d_{\tau_n}(p) > d_{\tau_n}(q)$.

Now we show that B_n has strict comparison with respect to τ_n. Take positive elements $a, b \in B_n \otimes \mathcal{K}$ with $d_{\tau_n}(a) < d_{\tau_n}(b)$. Since B_n has real rank zero, by [4, Theorem 5.7], it follows that $[a] = \sup_n [p_n]$ and $[b] = \sup_n [q_n]$ for sequences of projections $(p_n)_n$ and $(q_n)_n$ in $B_n \otimes \mathcal{K}$. By lower-semicontinuity of d_{τ_n}, there exists n_0 such that for any $k \geq n_0$, $d_{\tau_n}(p_k) \leq d_{\tau_n}(a) < d_{\tau_n}(q_k) \leq d_{\tau_n}(b)$. Thus, $p_k \leq_0 q_k$ for any $k \geq n_0$ and this implies $a \precsim b$ in $B_n \otimes \mathcal{K}$.

Lastly, we show that τ_n is the unique quasitrace by showing that there is a unique functional on $V(B_n)$, and thus a unique functional on $\mathrm{Cu}(B_n)$ as a consequence of real rank zero. Suppose that there are functionals φ_1, φ_2 such that $\varphi_1(1) = \varphi_2(1)$ and $\varphi_1(p) < \varphi_2(p)$ for some projection $p \in B_n \otimes \mathcal{K}$. There exists $m, n \in \mathbb{N}$ such that

$$n\varphi_1(p) = \varphi_1(p^{\oplus n}) < m = \varphi_1(1^{\oplus m}) = \varphi_2(1^{\oplus m}) < \varphi_2(p) = n\varphi_2(p^{\oplus n}). \tag{7.2.4}$$

By total ordering of $V(B_n)$, this implies that $p^{\oplus n} \lneqq_0 1^{\oplus m}$ and $1^{\oplus m} \lneqq_0 p^{\oplus n}$. By total ordering of $V(B_n)$ and simplicity of B_n, we have $n\varphi_1(p) < n\varphi_1(p)$, which is a contradiction. Thus τ_n is also the unique quasitrace.

For (iii), B_ω has real rank zero and stable rank one, as ultraproducts preserve both properties, see [96, Proposition 3.2] for instance. Following the proof of [96, Proposition 3.2 (2)], we similarly have $K_1(B_\omega) = 0$. As each B_n has strict comparison by the unique trace τ_n, it follows that τ_{B_ω} is the unique trace on B_ω by Theorem 2.8.20. Moreover, B_ω has strict comparison by the trace τ_{B_ω} by [9, Lemma 1.23].

Total ordering of $V(B_n)$ implies that $V(B_\omega)$ is totally ordered. Indeed, take projections $p, q \in M_m(B_\omega)$ for some $m \in \mathbb{N}$, which can be represented by sequences of projections $(p_n)_n$ and $(q_n)_n$, where each $p_n, q_n \in M_m(B_n)$. Since $V(B_n)$ is totally ordered and ω is an ultrafilter, either $\{n \in \mathbb{N} : [p_n] \leq [q_n] \text{ in } V(B_n)\} \in \omega$ or $\{n \in \mathbb{N} : [q_n] \leq [p_n] \text{ in } V(B_n)\} \in \omega$. Thus either $[p] \leq [q]$ or $[q] \leq [p]$ in $V(B_\omega)$.

For (iv), we only consider the case where B^ω is a II_1 factor, so that J_B is not trivial. Then J_B has real rank zero and stable rank one, as both properties pass to ideals by [12, Corollary 2.8] and [84, Theorem 4.3]. Since B_ω has strict comparison by τ_{B_ω}, the Cuntz semigroup $\mathrm{Cu}(B_\omega)$ is almost unperforated by Lemma 2.8.18. By Lemma 2.8.13, $\mathrm{Cu}(J_B)$ is almost unperforated. Similarly, by Lemma 2.8.13, Murray von Neumann subequivalence of projections holds in J_B if and only if it holds in B_ω. Thus, $V(B_\omega)$ being totally ordered implies that $V(J_B)$ is totally ordered.

Again by strict comparison of B_ω with respect to τ_{B_ω}, following the argument in [14, Lemma 6.11], J_B is separably stable by verifying that J_B satisfies the Hjelmborg–Rørdam criterion (see [14, Proposition 6.10]). Lastly we show that $K_1(J_B) = 0$. Consider the following exact sequence induced by the trace-kernel extension,

$$K_0(B_\omega) \xrightarrow{K_0(q_B)} K_0(B^\omega) \xrightarrow{\partial_0} K_1(J_B) \xrightarrow{K_1(j_B)} K_1(B_\omega). \tag{7.2.5}$$

Since $K_1(B_\omega) = 0$, it suffices to show that $K_0(q_B)$ is surjective. As B^ω is a II_1 factor, τ_{B^ω} induces an isomorphism $\hat{\tau}_{B^\omega} : K_0(B^\omega) \to \mathbb{R}$. On the other hand, B_ω has a unique trace τ_{B_ω}, which induces the pairing map $\hat{\tau}_{B_\omega} : K_0(B_\omega) \to \mathrm{Aff}\, T(B_\omega) \cong \mathbb{R}$. Since B_ω has real rank zero, the image of $\hat{\tau}_{B_\omega}$ is dense in $\mathbb{R}$. By strict comparison of B_ω

and [14, Proposition 4.20], the image of $\hat{\tau}_{B_\omega}$ is closed in $\mathbb{R}$ and thus $\hat{\tau}_{B_\omega}$ is surjective. Now we have $\hat{\tau}_{B_\omega} = \hat{\tau}_{B^\omega} \circ K_0(q_B)$, which implies that $K_0(q_B)$ is surjective and hence $K_1(J_B) = 0$. □

We are now in position to prove Theorem 7.2.1 following the proof of Schafhauser's uniqueness theorem ([97, Proposition 4.3]), using our KL-uniqueness theorem (Theorem 6.3.2) in place of the $\mathcal{Q}$-stable KK-uniqueness theorem ([97, Proposition 2.7]). As mentioned in Chapter 4, we no longer need to deunitize the unital map.

Proof of Theorem 7.2.1. Fix full and unital *-homomorphisms $\phi, \psi\colon A \to B_\omega$ with $\tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi$ and $\underline{K}(\phi) = \underline{K}(\psi)$. By Proposition 7.2.2 (i), B^ω is a finite von Neumann factor. In the case that B^ω is type I_m for some m, then $B_\omega = B^\omega \cong M_m$. Since maps ϕ and ψ are full, they are both injective and thus A is a finite-dimensional C^*-algebra. The uniqueness theorem then follows from the classification of embeddings of finite-dimensional C^*-algebras to matrix algebras up to unitary equivalence. In particular, this classification result only needs $\tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi$. The rest of the proof is devoted to the case when B^ω is a II_1 factor. Then we have $K_1(B^\omega) = 0$ and $K_0(B^\omega) \cong \mathbb{R}$, which implies that $K_i(B^\omega; \mathbb{Z}/n\mathbb{Z}) = 0$ for $i \in \{0, 1\}$ and $n \geq 2$ by Example 2.5.1.

By our assumption,

$$\tau_{B^\omega} \circ q_B \circ \phi = \tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi = \tau_{B^\omega} \circ q_B \circ \psi. \tag{7.2.6}$$

Then the classification of maps from nuclear C^*-algebras into finite von Neumann algebras, see Theorem 1.2.4, shows that $q_B \circ \phi, q_B \circ \psi\colon A \to B^\omega$ are unitarily equivalent. As the unitary group of a II_1 factor is path-connected in the norm topology, unitaries in B^ω lift to B_ω, and we can replace ψ by a unitary conjugate so that $q_B \circ \phi = q_B \circ \psi$.

Since A is separable, $K_*(A)$ is countable and thus, by Lemma 2.5.3,

$$\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B_\omega)) \cong \varinjlim_{I \text{ sep}} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)), \tag{7.2.7}$$

with the limit taken over all separable C^*-subalgebras $I \subseteq B_\omega$. Hence there exists a separable C^*-subalgebra I_0 in B_ω such that the corestrictions of ϕ and ψ to I_0 agree on total K-theory. By Lemma 7.1.6, there exists a separable and unital C^*-subalgebra B_0 in B_ω such that $\phi(A) \cup \psi(A) \subseteq B_0$ and the corestrictions of ϕ and ψ to B_0 are full. Take the unital and separable C^*-subalgebra E_0 generated by I_0 and B_0 in B_ω. Then $\phi(A) \cup \psi(A) \subseteq E_0$ and the corestrictions of ϕ and ψ to E_0 are full and agree on total K-theory.

By Proposition 7.2.2, J_B has real rank zero, stable rank one, $K_1(J_B) = 0$, $\mathrm{Cu}(J_B)$ is almost unperforated, $K_0(J_B)$ is totally ordered and J_B is separably stable. Moreover, $K_1(B^\omega)$ and $K_i(B^\omega; \mathbb{Z}/n\mathbb{Z})$ vanish for $i \in \{0,1\}$ and $n \geq 2$. Moreover, since B^ω is a II_1 factor, it follows that $K_0(B^\omega) \cong \mathbb{R}$, which is uniquely divisible. We have shown that all these properties are separably inheritable in Lemma 7.1.2 and Lemma 7.1.5. Then by Proposition 7.1.7 for separabilization of extensions, there exists a separable subalgebra $J \subseteq J_B$ and unital separable C^*-subalgebras $E \subseteq B_\omega$ and $D \subseteq B^\omega$ such that:

(i) there is a homomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \xrightarrow{\hat{\jmath}} & E & \xrightarrow{\hat{q}} & D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_B & \xrightarrow{j_B} & B_\omega & \xrightarrow{q_B} & B^\omega & \longrightarrow & 0 \end{array} \tag{7.2.8}$$

of extensions where the vertical arrows are the inclusion maps;

(ii) $\phi(A) \cup \psi(A) \subseteq E_0 \subseteq E$, and the corestrictions $\hat{\phi}, \hat{\psi}\colon A \to E$ of ϕ and ψ to E are unital and full and satisfy $\underline{K}(\hat{\phi}) = \underline{K}(\hat{\psi})$;

(iii) $K_1(J)$, $K_1(D)$ and $K_i(D; \mathbb{Z}/n\mathbb{Z})$ vanish for $i \in \{0,1\}$ and $n \geq 2$, and $K_0(D)$ is uniquely divisible;

(iv) J is stable, has real rank zero and stable rank one, $K_0(J)$ is totally ordered and $\mathrm{Cu}(J)$ is almost unperforated.

Let $\lambda\colon E \to \mathcal{M}(J)$ be the canonical unital *-homomorphism provided by Proposition 2.9.4. Since $q \circ \phi = q_B \circ \psi$, we have $q \circ \hat{\phi} = q_B \circ \hat{\psi}$ and hence

$$(\lambda \circ \hat{\phi}, \lambda \circ \hat{\psi})\colon A \rightrightarrows \mathcal{M}(J) \rhd J \tag{7.2.9}$$

is a Cuntz pair inducing a class $\kappa \in KL(A, J)$ which satisfies

$$KL(A, \hat{\jmath})(\kappa) = [\hat{\phi}]_{KL(A,E)} - [\hat{\psi}]_{KL(A,E)} \tag{7.2.10}$$

by the computation from [14, Proposition 5.7]. Since A satisfies the UCT, Dadarlat and Loring's universal multi-coefficient theorem (cf. [14, Theorem 8.5]) gives an isomorphism

$$\Gamma^{(I)}\colon KL(A, I) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(I)), \tag{7.2.11}$$

which is natural in I for any separable C^*-algebra I. Naturality and the fact that

$$\Gamma^{(E)}([\hat{\phi}]_{KL(A,E)}) = \underline{K}(\hat{\phi}) = \underline{K}(\hat{\psi}) = \Gamma^{(E)}([\hat{\psi}]_{KL(A,E)}) \tag{7.2.12}$$

gives $\underline{K}(\hat{j}) \circ \Gamma^{(J)}(\kappa) = \Gamma^{(E)} \circ KL(A, \hat{j})(\kappa) = 0$. Since $K_1(I)$, $K_1(D)$ and $K_i(D; \mathbb{Z}/n\mathbb{Z})$ vanish for $i \in \{0, 1\}$ and $n \geq 2$, and $K_0(D)$ is uniquely divisible, six-term exact sequence computations for total K-theory show that $\underline{K}(\hat{j})$ is injective, see Theorem 2.5.4. Thus $\Gamma^{(J)}(\kappa) = 0$, which implies that $\kappa = [\lambda \circ \hat{\phi}, \lambda \circ \hat{\psi}]_{KL(A,J)} = 0$.

Since $\mathrm{Cu}(J)$ is almost unperforated by condition (iv) above, it follows that J has the corona factorization property by Proposition 4.2.13. Condition (ii) shows that maps $\lambda \circ \hat{\phi}$ and $\lambda \circ \hat{\psi}$ from the nuclear C^*-algebra A to $\mathcal{M}(J)$ are unital and full. Thus $\lambda \circ \hat{\phi}$ and $\lambda \circ \hat{\psi}$ are unitally absorbing by Elliott-Kucerovsky's generalized Weyl-von Neumann theorem (Theorem 4.2.12) and Kucerovsky-Ng's theorem (Theorem 4.2.14). By our KL-uniqueness theorem (Theorem 6.3.2), there exists a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in $J^{\dagger}$ such that

$$\|u_n(\lambda \circ \hat{\phi}(a))u_n^* - \lambda \circ \hat{\psi}(a)\| \to 0, \quad a \in A. \tag{7.2.13}$$

Since λ restricts to the identify map on J, it follows that

$$\|u_n\hat{\phi}(a)u_n^* - \hat{\psi}(a)\| \to 0, \quad a \in A. \tag{7.2.14}$$

Thus the unital *-homomorphisms $\phi, \psi : A \to B_\omega$ are approximately unitarily equivalent. By Kirchberg's ϵ-test, they are unitarily equivalent, see Proposition 2.2.2. □

Since II_1 factors or matrices satisfy the conditions in Theorem 7.2.1, the following theorems (Theorem A and Theorem B) follow immediately.

Theorem 7.2.3. *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT and let $\mathcal{M}$ be a II_1 factor. Let $\phi, \psi : A \to \mathcal{M}$ be unital faithful *-homomorphisms such that $\tau_{\mathcal{M}} \circ \phi = \tau_{\mathcal{M}} \circ \psi$. Then there exists a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in $\mathcal{M}$ such that*

$$\|u_n\phi(a)u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{7.2.15}$$

Notice that only traces are needed to classify maps into II_1 factors, since the total K-theory is trivial for II_1 factors, except for the K_0-group. The unique trace $\tau_{\mathcal{M}}$ induces an isomorphism between $K_0(\mathcal{M})$ and $\mathbb{R}$. Thus, two maps into $\mathcal{M}$ agreeing on $\tau_{\mathcal{M}}$ also agree on the level of K_0.

Theorem 7.2.4. *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT. Let $\phi, \psi : A \to \prod_\omega M_{k_n}$ be full and unital *-homomorphisms such that $\underline{K}(\phi) = \underline{K}(\psi)$ and $\tau_\omega \circ \phi = \tau_\omega \circ \psi$. Then there exists a unitary $u \in \prod_\omega M_{k_n}$ such that $\phi = \mathrm{Ad}(u) \circ \psi$.*

Chapter 8

An equivalent characterization of nuclearity

In this chapter, we first define the Matui–Sato approximation property (MSAP), which first appeared in the work of Matui and Sato, in their proof that strict comparison implies $\mathcal{Z}$-stability when the trace space is sufficiently well-behaved. The MSAP can be considered a refined version of nuclearity and arises naturally in the case of matrix algebras or commutative C^*-algebras. We include the result of Matui and Sato that unital, simple and nuclear C^*-algebras have the MSAP. In the non-simple case, the presence of the MSAP for nuclear C^*-algebras is only partially understood. We show in Section 8.2 that the MSAP is preserved under several operations, such as extensions and matrix amplifications.

The equivalence between the MSAP and nuclearity for separable C^*-algebras (Theorem J) is proved in Section 8.3, using the decomposition theory of C^*-algebras. It is sufficient to show the MSAP for type I C^*-algebras, which can be further decomposed into continuous trace C^*-algebras. The latter class of C^*-algebras is locally stably isomorphic to commutative C^*-algebras, and the local approximations can be glued together to obtain the MSAP.

The results of the section will be contained in my upcoming paper [49].

8.1 The Matui-Sato approximation property

We give the definition for the following finite-dimensional approximation property using pure states and call it the *Matui-Sato approximation property.* Such approximations first appeared in the work [72, Lemma 3.1] of Matui and Sato for simple C^*-algebras, where only a single pure state was needed for the approximation (see

Theorem 8.1.6). To obtain approximations for non-simple nuclear C^*-algebras, multiple inequivalent pure states are needed. This point will be made clearer in Example 8.1.5 for commutative C^*-algebras.

Definition 8.1.1. A C^*-algebra A is said to satisfy the *Matui-Sato approximation property (MSAP)* if for any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there exist $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ on A and elements c_i, $d_{i,\ell} \in A$ for $1 \leq i \leq N$, $1 \leq \ell \leq L$ such that for any $a \in \mathcal{F}$,

$$a \approx_\epsilon \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* a d_{j,\ell}) \right) c_i^* c_j. \tag{8.1.1}$$

The first observation is that the MSAP implies nuclearity, since the MSAP provides a special form of c.p. approximations through finite-dimensional C^*-algebras.

Proposition 8.1.2. *If a C^*-algebra A satisfies the MSAP, then A is nuclear.*

Proof. For a C^*-algebra satisfying the MSAP, we show that id_A can be approximated in the point-norm topology by completely positive maps factoring through matrix algebras. Then the result follows from Lemma 2.6.8.

Fix any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, since A satisfies the MSAP, there exists $L, N \in \mathbb{N}$, pure states $\rho_1, \cdots, \rho_L$ and c_i, $d_{i,\ell} \in A$ for $1 \leq i \leq N$, $1 \leq \ell \leq L$ such that (8.1.1) holds for any $a \in \mathcal{F}$. Then define maps $\phi : A \to M_N$ and $\psi : M_N \to A$ in the following way:

$$\phi(a) = \left(\sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* a d_{j,\ell}) \right)_{i,j}, \quad \psi(b) = \sum_{i,j=1}^{N} b_{ij} c_i^* c_j, \tag{8.1.2}$$

where $b = (b_{ij})_{i,j} \in M_N$. By Proposition 2.6.3, both ϕ and ψ are completely positive. Moreover, having the approximation (8.1.1) is equivalent to saying $a \approx_\epsilon \psi \circ \phi(a)$ for any $a \in \mathcal{F}$. □

In practice, several approximations might need to be added together and rearranged into the form in (8.1.1) to obtain the MSAP. We include the following technical lemma to avoid repetitive use of the same technical manipulation.

Lemma 8.1.3. *Let A be a C^*-algebra. Then A has the MSAP if and only if for any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there exist $M \in \mathbb{N}$, $L_m, N_m \in \mathbb{N}$ for $1 \leq m \leq M$, pairwise inequivalent pure states $\rho_{m,\ell}$ on A, and elements $c_{m,i}, d_{m,i,\ell} \in A$ for $1 \leq m \leq M$, $1 \leq \ell \leq L_m$ and $1 \leq i \leq N_m$ such that for any $a \in \mathcal{F}$,*

$$a \approx_\epsilon \sum_{m=1}^{M} \sum_{i,j=1}^{N_m} \left(\sum_{\ell=1}^{L_m} \rho_{m,\ell}(d_{m,i,\ell}^* a d_{m,j,\ell}) \right) c_{m,i}^* c_{m,j}. \tag{8.1.3}$$

Proof. The "only if" direction holds by taking $M = 1$. For the "if" direction, we rearrange M pieces of approximations in (8.1.3) into the form of (8.1.1):

$$a \approx_\epsilon \sum_{(m,i),(k,j)\in\mathcal{S}} \left(\sum_{(n,\ell)\in\mathcal{C}} \rho_{n,\ell}(d^*_{(m,i),(n,\ell)} a d_{(k,j),(n,\ell)}) \right) c^*_{(m,i)} c_{(k,j)}. \tag{8.1.4}$$

where $\mathcal{S} = \{(m,i) : 1 \le m \le M, 1 \le i \le N_m\}$, $\mathcal{C} = \{(n,\ell) : 1 \le n \le M, 1 \le \ell \le L_m\}$, and $d_{(m,i),(n,\ell)} = d_{m,i,\ell}$ if $m = n$ and 0 otherwise. □

The MSAP can be spotted in the following two fundamental examples. First of all, for matrix algebras, there is a uniform and exact decomposition in the form of (8.1.1) for every element, involving a single pure state. For a matrix algebra M_n, we denote by $e_{i,j}$ the matrix unit with the value 1 in the (i,j) entry and 0 elsewhere.

Example 8.1.4. Let $A = M_n$ for some $n \in \mathbb{N}$. We fix matrix units $(e_{i,j})_{i,j}$ for M_n. As explained in Example 2.10.6, every pure state on A is equivalent to $\rho_{1,1} : A \to \mathbb{C}$, $(a_{i,j})_{i,j} \mapsto a_{1,1}$, which evaluates the $(1,1)$ entry of the matrix. For any $a = (a_{i,j})_{i,j} \in A$, we have

$$a = \sum_{i,j=1}^{N} a_{i,j} e_{i,j} = \sum_{i,j=1}^{N} \rho_{1,1}(e_{1,i} a e_{j,1}) e_{i,1} e_{1,j}. \tag{8.1.5}$$

Hence, $A = M_n$ satisfies the MSAP. Moreover, c.p. maps produced by the approximation from (8.1.2) are exactly $\phi = \psi = \mathrm{id}_{M_n}$.

For commutative C^*-algebras, the MSAP holds as a reformulation of the standard partition of unity argument used when proving nuclearity, see [13, Proposition 2.4.2]. This example also illustrates the need to allow multiple inequivalent pure states in Definition 8.1.1 to approximate elements in a non-simple C^*-algebra.

Example 8.1.5. Let $A = C_0(X)$ for a locally compact Hausdorff space X. Fix a finite subset $\mathcal{F} \subseteq C(X)$ and $\epsilon > 0$, and take

$$K = \{x \in X : |f(x)| \ge \epsilon \text{ for some } f \in \mathcal{F}\}, \tag{8.1.6}$$

which is compact in X as it is a finite union of compact sets. Since X is Hausdorff, it follows that K is closed and $U_0 = X \setminus K$ is open. By compactness of K, there exists a finite open covering $U_1, \cdots, U_N$ of K such that $|f(x) - f(y)| < \epsilon$ for any $f \in \mathcal{F}$ and $x, y \in U_i$ for each $i \in \mathbb{N}$. Without loss of generality, we can assume that each U_i is not included in the union of U_j's with $j \ne i$. Take distinct $x_i \in U_i$ for $i \in \{1, \cdots, N\}$ and take a partition of unity $g_0, g_1, \cdots, g_N$ subordinate to $U_0, U_1, \cdots, U_N$, which are positive functions taking values in $[0,1]$ such that $\sum_{i=0}^N g_i = 1$ and each g_i is supported on U_i. Then for any $f \in \mathcal{F}$,

$$f \approx_\epsilon \sum_{i=1}^N f(x_i) g_i = \sum_{i=1}^N \rho_{x_i}(f) = \sum_{i,j=1}^N \left(\sum_{\ell=1}^N \rho_{x_\ell}(\delta_{\ell,i}^* f \delta_{\ell,j}) \right) g_i^{1/2} g_j^{1/2}, \tag{8.1.7}$$

where ρ_{x_ℓ} are point evaluations at x_ℓ and $\delta_{\ell,i}$ takes constant value 1 if $\ell = i$ and 0 otherwise. Since pure states of $C_0(X)$ are point evaluations (see Section 2.10) and x_ℓ are chosen to be distinct, then ρ_{x_ℓ} are inequivalent pure states. Thus $C_0(X)$ satisfies the MSAP.

For general simple and nuclear C^*-algebras, it was first proven by Matui and Sato that these C^*-algebras satisfy what we call the MSAP, involving only one pure state, in [72, Lemma 3.1]. This serves as an ingredient of Matui and Sato's proof for going from strict comparison to $\mathcal{Z}$-stability under nice assumptions on the trace space.

Theorem 8.1.6 ([72, Lemma 3.1])**.** *Let A be separable, simple and non-elementary C^*-algebra. Then A has the MSAP.*

The proof of Proposition 8.1.6 uses Voiculescu's theorem (Theorem 4.2.7), and uses the fact that every simple and non-elementary C^*-algebra admits an essential representation. This proposition, together with Example 8.1.4, immediately leads to the following corollary.

Corollary 8.1.7. *All separable, simple and nuclear C^*-algebras have the MSAP.*

In the non-simple setting, the following theorem generalizes the argument of Matui and Sato, and Kirchberg and Rørdam. A technical condition, analogous to essentiality, on the representation theory of the C^*-algebra is needed to obtain the MSAP.

Theorem 8.1.8 ([9, Lemma 4.8])**.** *Let A be a separable, unital and nuclear C^*-algebra such that $\theta(A) \cap \mathcal{K} = \emptyset$, where θ is the faithful representation obtained as the direct sum of one irreducible representation from each unitary equivalence class of irreducible representations of A. Then A satisfies the MSAP.*

For example, all $\mathcal{Z}$-stable C^*-algebras satisfy the technical condition on representations since these C^*-algebras lack finite-dimensional representations, and are the major source of examples that the authors of [9] apply Theorem 8.1.8 to.

Another class of examples that satisfy the technical condition are commutative C^*-algebras $C(X)$, where X has no isolated points. Indeed, for any element $f \in$

$C(X)$ taking a nonzero value at x_0, since x_0 is a limit point, it follows that f takes nonzero values at infinitely many points in X by continuity. Thus $\theta(f)$ is not a compact operator. On the other hand, if X has an isolated point, it fails the technical hypothesis, since $\theta(f)$ is always a rank one operator if f takes a nonzero value at an isolated point and is zero elsewhere. However, we have shown in Example 8.1.5 that commutative C^*-algebras have the MSAP. Thus, one would expect that the MSAP holds for more general nuclear C^*-algebras, outside of the setting of Theorem 8.1.8.

8.2 Permanence properties

In this section, we include permanence properties of the MSAP, similar to those of nuclearity in Theorem 2.6.7. We show first that the MSAP is preserved under extensions.

Theorem 8.2.1. *Let A be a C^*-algebra and $I \lhd A$ be an ideal of A. If I and A/I satisfy the MSAP, then A satisfies the MSAP.*

Proof. Fix a finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$. We denote by q the quotient map $q : A \to A/I$. Since A/I has the MSAP, there exists $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ and elements c_i, $d_{i,l} \in A/I$ for $1 \leq i \leq N$, $1 \leq l \leq L$ such that for any $x \in \mathcal{F}$, we can approximate $q(x)$ by the following,

$$q(x) \approx_{\epsilon/2} \sum_{i,j=1}^{N} \left(\sum_{l=1}^{L} \rho_l(d_{i,l}^* q(x) d_{j,l}) \right) c_i^* c_j. \tag{8.2.1}$$

Take $a_i, b_{i,l} \in A$ such that $q(a_i) = c_i$ and $q(b_{i,l}) = d_{i,l}$. Then from (8.2.1), we get

$$q(x) \approx_{\epsilon/2} \sum_{i,j=1}^{N} \left(\sum_{l=1}^{L} (\rho_l \circ q)(b_{i,l}^* x b_{j,l}) \right) q(a_i^* a_j). \tag{8.2.2}$$

Then we have $q(x) \approx_{\epsilon/2} q(y_x)$ by taking

$$y_x \coloneqq \sum_{i,j=1}^{N} \left(\sum_{l=1}^{L} (\rho_l \circ q)(b_{i,l}^* x b_{j,l}) \right) a_i^* a_j. \tag{8.2.3}$$

By Corollary 2.11.3, there exists a contraction $e \in I$ such that

$$x \approx_{\epsilon/2} e^{1/2} x e^{1/2} + (1_{\mathcal{M}(A)} - e)^{1/2} y_x (1_{\mathcal{M}(A)} - e)^{1/2}. \tag{8.2.4}$$

Now we approximate the cut downs $e^{1/2} x e^{1/2} \in I$ in the ideal up to $\epsilon/2$, for any $x \in \mathcal{F}$, by means of the MSAP of I. There exists $R, M \in \mathbb{N}$, pairwise inequivalent pure states $\lambda_1, \cdots, \lambda_R$, and elements c_i, $d_{i,r} \in I$ for $1 \leq i \leq N$, $1 \leq r \leq R$ such that for any $x \in \mathcal{F}$, we can approximate $e^{1/2}xe^{1/2}$ by the following,

$$e^{1/2}xe^{1/2} \approx_{\epsilon/2} \sum_{i,j=1}^{M} \left(\sum_{r=1}^{R} \lambda_r(d_{i,r}^* e^{1/2}xe^{1/2}d_{j,r}) \right) c_i^* c_j. \tag{8.2.5}$$

Lastly, we add up the approximations coming from the quotient and the ideal. Since $(\rho_\ell)_{1\leq \ell \leq L}$ and $(\lambda_r)_{1\leq r\leq R}$ are pairwise inequivalent pure states on A/I and I respectively, it follows that $\tilde{\rho}_\ell := \rho_\ell \circ q$ and the canonical extensions $\tilde{\lambda}_r$ of λ_r to A are pairwise inequivalent pure states by Lemma 2.10.9. By (8.2.4), any $x \in \mathcal{F}$ is approximated up to ϵ by the sum of the approximation (8.2.5) for $e^{1/2}xe^{1/2}$, and $(1_{\mathcal{M}(A)} - e)^{1/2}y_x(1_{\mathcal{M}(A)} - e)^{1/2}$, later of which equals

$$\sum_{i,j=1}^{N} \left(\sum_{l=1}^{L} (\rho_l \circ q)(b_{i,l}^* x b_{j,l}) \right) (1_{\mathcal{M}(A)} - e)^{1/2} a_i^* a_j (1_{\mathcal{M}(A)} - e)^{1/2}, \tag{8.2.6}$$

by (8.2.3). Then the MSAP of A follows from Lemma 8.1.3. □

The MSAP is preserved under matrix amplifications, and the proof is elementary. Recall that for a matrix algebra M_n, we denote by $(e_{i,j})_{i,j}$ the matrix units of M_n and by $\rho_{1,1}$ the pure state that evaluates at the $(1,1)$ entry.

Lemma 8.2.2. *If A has the MSAP, then $A \otimes M_n$ has the MSAP for any $n \in \mathbb{N}$.*

Proof. Fix the matrix size $n \in \mathbb{N}$, $\epsilon > 0$ and a finite subset $\mathcal{F} \subseteq A \otimes M_n$, where each element $x \in \mathcal{F}$ is of the form $(x_{i,j})_{i,j}$ for $x_{i,j} \in A$. Take $\mathcal{G} = \{x_{i,j} : x \in \mathcal{F}, 1 \leq i,j \leq n\}$ to be the finite subset of A. If A has the MSAP, there exist $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ on A and elements c_m, $d_{m,\ell} \in A$ such that

$$y \approx_{\epsilon/n^2} \sum_{m,k=1}^{N} \left(\sum_{\ell=1}^{L} \rho_\ell(d_{m,\ell}^* y d_{k,\ell}) \right) c_m^* c_k, \quad y \in \mathcal{G}, \tag{8.2.7}$$

Take the pairwise inequivalent pure states $\rho_\ell \otimes \rho_{1,1}$, elements $d_{m,\ell} \otimes e_{i,1}$ and $c_m \otimes e_{1,i}$ in $A \otimes M_n$, indexed over $1 \leq \ell \leq L$ and $(i,m) \in \{1, \cdots, n\} \times \{1, \cdots, N\}$. Then any $x \in \mathcal{F}$ is approximated in the form of (8.1.1) by these pure states and elements. Thus we have shown that $A \otimes M_n$ satisfies the MSAP. □

To show the MSAP of $A \otimes \mathcal{K}$ given the MSAP of A, we prove that the MSAP is preserved under taking the closure of a directed family of hereditary subalgebras.

Lemma 8.2.3. *Let $(A_\lambda)_{\lambda \in \Lambda}$ be a directed family of hereditary C^*-subalgebras of A and $A = \overline{\cup_\lambda A_\lambda}$. If A_λ has the MSAP for every λ, then A has the MSAP.*

Proof. Fix a finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$. There exists some n_0 such that for any $x \in \mathcal{F}$, there exists $y_x \in A_{n_0}$ such that $x \approx_{\epsilon/4} y_x$. Take a contractive element $e \in A_{n_0}$ from an approximate unit of A_{n_0}, such that $y_x \approx_{\epsilon/4} ey_xe$ for any $x \in \mathcal{F}$. Then

$$x \approx_{\epsilon/4} y_x \approx_{\epsilon/4} ey_xe \approx_{\epsilon/4} exe, \quad x \in \mathcal{F}, \tag{8.2.8}$$

and $exe \in A_{n_0}$ since A_{n_0} is a hereditary subalgebra of A. By the MSAP of A_{n_0}, there exist $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ on A_{n_0} and elements c_i, $d_{i,\ell} \in A_{n_0}$ for $1 \leq i \leq N$, $1 \leq \ell \leq L$ such that for any $x \in \mathcal{F}$,

$$x \approx_{3\epsilon/4} exe \approx_{\epsilon/4} \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* exed_{j,\ell}) \right) c_i^* c_j. \tag{8.2.9}$$

Extending ρ_ℓ canonically to pairwise inequivalent pure states on A gives the proof. □

We immediately get the following result.

Corollary 8.2.4. *If A has the MSAP, then $A \otimes \mathcal{K}$ satisfies the MSAP.*

Proof. Since A has the MSAP, it follows that $M_n \otimes A$ also has the MSAP by Lemma 8.2.2. Moreover, $(M_n \otimes A)$ is a collection of hereditary C^*-subalgebras in $A \otimes \mathcal{K}$, whose union is dense. The result then follows from Lemma 8.2.3. □

Lastly, we show that the MSAP property "almost" passes through stable isomorphisms. The key fact we use in the next lemma is that pure states on the stabilization $A \otimes \mathcal{K}$ are, up to unitary equivalence, tensor products of pure states on A and $\mathcal{K}$, see Lemma 2.10.8. The best approximation obtained at this stage looks similar to the MSAP, but the differences in indices can not be fixed by merely rearranging the sum.

Lemma 8.2.5. *Let A, B be C^*-algebras such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. If A has the MSAP, then for any finite subset $\mathcal{F} \subseteq B$ and $\epsilon > 0$, there exists $L, N, K \in \mathbb{N}$, pairwise inequivalent pure states ρ_ℓ, elements $c_{i,k}$, $d_{i,\ell}$ in B, for $1 \leq \ell \leq L$, $1 \leq i \leq N$ and $1 \leq k \leq K$ such that for any $x \in \mathcal{F}$,*

$$x \approx_\epsilon \sum_{k=1}^{K} \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* ad_{j,\ell}) \right) c_{i,k}^* c_{j,k}. \tag{8.2.10}$$

Proof. If A has the MSAP, then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ satisfies the MSAP by Lemma 8.2.2. Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subseteq B$, then $x \otimes e_{1,1} \in B \otimes \mathcal{K}$ for any $x \in \mathcal{F}$. By the

MSAP of $B \otimes \mathcal{K}$, there exists $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ on $B \otimes \mathcal{K}$ and c_i, $b_{i,\ell} \in B \otimes \mathcal{K}$ such that

$$x \otimes e_{1,1} \approx_{\epsilon/2} \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \rho_\ell(b_{i,\ell}^*(x \otimes e_{1,1})b_{j,\ell}) \right) c_i^* c_j, \quad x \in \mathcal{F}. \tag{8.2.11}$$

Without loss of generality, we can assume that $c_i \in B \otimes M_K$ for some K. For each ℓ and for any $\delta > 0$, by Lemma 2.10.5 and Lemma 2.10.8 (ii), there exists $\tilde{\rho}_\ell$ on B and $a_\ell \in B \otimes \mathcal{K}$ such that ρ_ℓ is unitarily equivalent to $\tilde{\rho}_\ell \otimes \rho_{1,1}$ and for $x \in \mathcal{F}$, $1 \leq i, j \leq N$,

$$\rho_\ell(b_{i,\ell}^*(x \otimes e_{1,1})b_{j,\ell}) \approx_\delta (\tilde{\rho}_\ell \otimes \rho_{1,1})(a_\ell^* b_{i,\ell}^*(x \otimes e_{1,1})b_{j,\ell}a_\ell). \tag{8.2.12}$$

Take $d_{i,\ell}$ to be the value in the $(1,1)$ corner of $b_{i,\ell}a_\ell$, which is an element in B. Since $x \otimes e_{1,1}$ is an element in $B \otimes e_{1,1}$, we have the following for $x \in \mathcal{F}$ and $1 \leq i, j \leq N$,

$$(\tilde{\rho}_\ell \otimes \rho_{1,1})(a_\ell^* b_{i,\ell}^*(x \otimes e_{1,1})b_{j,\ell}a_\ell) = \tilde{\rho}_\ell(d_{i,\ell}^* x d_{j,\ell}). \tag{8.2.13}$$

By taking δ small enough, combining (8.2.11), (8.2.12) and (8.2.13), we have found $d_{i,\ell}$ in B for $1 \leq i \leq N$ and $1 \leq \ell \leq L$ such that

$$x \otimes e_{1,1} \approx_\epsilon \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \tilde{\rho}_\ell(d_{i,\ell}^* x d_{j,\ell}) \right) c_i^* c_j, \quad x \in \mathcal{F}. \tag{8.2.14}$$

Then the norm difference in the $(1,1)$ corner of $B \otimes \mathcal{K}$ is bounded by ϵ, which means

$$x \approx_\epsilon \sum_{k=1}^{K} \sum_{i,j=1}^{N} \left(\sum_{\ell=1}^{L} \tilde{\rho}_\ell(d_{i,\ell}^* x d_{j,\ell}) \right) c_{i,(k,1)}^* c_{j,(k,1)}, \quad x \in \mathcal{F}, \tag{8.2.15}$$

where $c_{i,(k,1)}$ is the $(k,1)$ entry of matrix $c_i \in B \otimes M_K$. Moreover, the pure states $\tilde{\rho}_\ell$ are pairwise inequivalent, since ρ_ℓ are pairwise inequivalent and ρ_ℓ is unitarily equivalent to $\tilde{\rho}_\ell \otimes \rho_{1,1}$ for each ℓ. □

8.3 From nuclearity to the MSAP

Previously, we have shown in Proposition 8.1.2 that the MSAP implies nuclearity. In this section, we show that the converse is true assuming separability, through a detailed analysis of the decomposition theory of C^*-algebras, see [82, Chapter 6].

Every C^*-algebra admits a decomposition into an ideal that behaves well (type I) and a quotient that is totally misbehaved (antiliminary). We start with definitions in terms of abelian elements. A positive element a in a C^*-algebra A is *abelian* if the hereditary C^*-subalgebra generated by a is commutative.

Definition 8.3.1. Let A be a C^*-algebra. Then

(i) A is of *type I* if every nonzero quotient of A contains a nonzero abelian element;

(ii) A is *antiliminary* if it contains no nonzero abelian elements.

For instance, the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ is of type I since every rank one projection is abelian. However $B(\mathcal{H})$ is not of type I for any infinite dimensional Hilbert space $\mathcal{H}$, since the Calkin algebra $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ does not contain any abelian element and is thus antiliminary.

Characterizations for type I and antiliminary C^*-algebras are intimately related to their representation theory. This is the case since any positive element $a \in A$ is abelian if and only if $\dim(\pi(a)) \leq 1$ for every irreducible representation $\pi : A \to B(\mathcal{H}_\pi)$ of A (see [82, Lemma 6.1.3] for instance). We include the following result.

Proposition 8.3.2 (cf. [82, Corollary 6.2.8, Theorem 6.8.7])**.** *Let A be a C^*-algebra.*

(i) A is a C^*-algebra of type I if and only if $\mathcal{K}(\mathcal{H}_\pi) \subseteq \pi(A)$ for every irreducible representation $\pi : A \to B(\mathcal{H}_\pi)$;

(ii) If A is antiliminary, then for any nonzero $a \in A$, there exists an irreducible representation $\pi : A \to B(\mathcal{H}_\pi)$ such that $\pi(a) \notin \mathcal{K}(\mathcal{H}_\pi)$.

In particular, any separable and antiliminary C^*-algebra A satisfies the technical condition in Theorem 8.1.8. Indeed, for any nonzero $a \in A$, there is an irreducible representation π such that $\pi(a)$ is not compact by Proposition 8.3.2 (ii). If θ is the faithful representation obtained as the direct sum of one irreducible representation from each unitary equivalence class of irreducible representations of A, then $\theta(a)$ is not compact since the direct sum with a non-compact operator is non-compact. Thus, we get the following as a result of Theorem 8.1.8 and Proposition 8.3.2 (ii).

Corollary 8.3.3. *Any separable, nuclear and antiliminary C^*-algebra has the MSAP.*

To prove the MSAP for all separable and nuclear C^*-algebras, we need the following decomposition of C^*-algebras into type I ideals and antiliminary quotients.

Theorem 8.3.4 (cf. [82, Theorem 6.2.7])**.** *Every C^*-algebra has a largest ideal J of type I such that A/J is antiliminary.*

Every separable and nuclear C^*-algebra A, by Theorem 8.3.4, admits a type I ideal J such that A/J is antiliminary. Moreover, both J and A/J are nuclear by Theorem 2.6.7. Hence, A/J has the MSAP by Corollary 8.3.3. Since the MSAP is preserved under extensions (Theorem 8.2.1), to show the MSAP for A, it suffices to show the MSAP for the type I ideal J. For this purpose, we decompose type I C^*-algebras further.

A *composition series* for a C^*-algebra A is a strictly increasing family of closed ideals $\{I_\alpha\}_\alpha$ indexed over ordinals $\{\alpha : 0 \leq \alpha \leq \beta\}$, where $I_0 = 0$ and $I_\beta = A$, and for each limit ordinal γ, we have $I_\gamma = \overline{\cup_{\alpha \leq \gamma} I_\alpha}$. For a separable C^*-algebra, the composition series can be at most countable.

Separable type I C^*-algebras are well known to have a composition series in which every quotient is a so-called *continuous trace C^*-algebra* (see [82, Theorem 6.8.7] for example). By Lemma 8.2.3, to prove the MSAP for separable type I C^*-algebras, it suffices to show the property for separable continuous trace C^*-algebras.

Continuous trace C^*-algebras are a class of C^*-algebras with Hausdorff spectrum $\hat{A}$, the study of which uses ideas from the theory of fiber bundles. Since $\hat{A}$ is Hausdorff, as explained in Section 2.10.2, we can identify $\mathrm{Prim}(A)$ with $\hat{A}$. It is therefore more convenient to consider $A/J_{\{t\}}$ for $t \in \mathrm{Prim}(A)$ than $\pi_t(A)$, where π_t is a representation corresponding to a point $t \in \hat{A}$. For our purpose, it is more useful to think of continuous trace algebras to be locally stably isomorphic to commutative C^*-algebras. The equivalence between this point of view and other equivalent characterizations can be found in [83, Proposition 5.15].

We recall some notation before giving the definition of continuous trace C^*-algebras. For a compact subset $K \subseteq \hat{A} = \mathrm{Prim}(A)$, we denote by J_K the intersection of the primitive ideals in K and denote by $\pi_K : A \to A/J_K$ the quotient map. If $F \subseteq K \subseteq \hat{A}$ are closed subsets, then J_K is an ideal of J_F and we denote by $\pi_{K,F} : A/J_K \to A/J_F$ the canonical quotient map. When $K = \{t\}$ is a singleton, we denote $\pi_K(a)$ by $a(t)$ for any $a \in A$.

Definition 8.3.5 (cf. [83, Proposition 5.15]). A separable C^*-algebra A is a *continuous trace C^*-algebra* if it has a Hausdorff spectrum $\hat{A}$ and is *locally stably isomorphic* to $C_0(\hat{A})$, which means that for any $t \in \hat{A}$, there exists a compact neighborhood K of t in $\hat{A}$ such that there is a *-isomorphism $\theta : A/J_K \otimes \mathcal{K} \to C(K) \otimes \mathcal{K}$ fixing the spectrum K.

Remark 8.3.6. Proposition 5.15 in [83] proves the more general result that a continuous trace C^*-algebra A, which is not necessarily separable, is locally Morita equivalent to $C_0(\hat{A})$ (defined in terms of Hilbert bimodules). In the separable setting, Morita equivalence and stable isomorphism are equivalent by the Brown-Green-Rieffel theorem ([83, Theorem 5.55]).

Now we prove the main result of this section, which states that separable continuous trace C^*-algebras have the MSAP. The main idea is to obtain local approximations from stable isomorphisms to commutative C^*-algebras, and glue the approximations globally by a partition of unity argument similar to that of Example 8.1.5.

Theorem 8.3.7. *Every separable continuous trace C^*-algebra has the MSAP.*

Proof. Let A be a separable continuous trace C^*-algebra. Thus the spectrum $\hat{A} = \mathrm{Prim}(A)$ of A is Hausdorff, second countable, and locally compact by [82, Theorem 6.1.11] for instance. Fix a finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, and take

$$K = \{t \in \hat{A} : \|x(t)\| \geq \epsilon/6 \text{ for some } x \in \mathcal{F}\}, \tag{8.3.1}$$

which is compact in $\hat{A}$ as it is a finite union of compact sets by [83, Lemma A.30]. Because $\hat{A}$ is Hausdorff, K is closed and $U_0 := X \setminus K$ is an open subset of X.

For each $t \in K$, if t is an isolated point, take $K_t = U_t = \{t\}$ to be the open compact neighborhood. In the case that t is a limit point of $\hat{A}$, by Definition 8.3.5, there exists an open subset U_t and a compact subset K_t such that $t \in U_t \subseteq K_t$ and a *-isomorphism

$$\theta_t : A/J_{K_t} \otimes \mathcal{K} \to C(K_t) \otimes \mathcal{K} \tag{8.3.2}$$

fixing the spectrum K_t. For any $x \in \mathcal{F}$, the element $y_x := \theta_t(\pi_{K_t}(x))$ lies in $C(K_t) \otimes \mathcal{K}$. Thus,

$$y_x \approx_{\epsilon/6} (1 \otimes p_{n_t}) y_x (1 \otimes p_{n_t}), \quad x \in \mathcal{F} \tag{8.3.3}$$

for some n_t, where p_{n_t} is the projection $e_{1,1} + \cdots + e_{n_t,n_t}$ in $\mathcal{K}$. There exist continuous functions $f_{x,i,j}$ on K_t for $x \in \mathcal{F}$ and $1 \leq i, j \leq n_t$ such that

$$(1 \otimes p_{n_t}) y_x (1 \otimes p_{n_t}) = \sum_{i,j=1}^{n_t} f_{x,i,j} \otimes e_{i,j}. \tag{8.3.4}$$

By continuity of $f_{x,i,j}$, there exists an open neighborhood V_t of t such that

$$\|f_{x,i,j}(s) - f_{x,i,j}(t)\| < \frac{\epsilon}{6n_t^2}, \quad s \in V_t,\ x \in \mathcal{F},\ 1 \leq i, j \leq n_t. \tag{8.3.5}$$

Thus for $s \in V_t$,

$$y_x(s) \overset{(8.3.3)}{\approx_{\epsilon/6}} (1 \otimes p_{n_t}) y_x (1 \otimes p_{n_t})(s) \tag{8.3.6}$$

$$\overset{(8.3.5)}{\approx_{\epsilon/6}} \sum_{i,j=1}^{n_t} f_{x,i,j}(t) e_{i,j} \tag{8.3.7}$$

$$= \sum_{i,j=1}^{n_t} (\lambda_t \otimes \rho_{1,1})\big((1 \otimes e_{i,1})^* y_x (1 \otimes e_{j,1})\big) e_{1,i}^* e_{1,j}, \tag{8.3.8}$$

where λ_t is the pure state on $C(K_t)$ given by the point evaluation at t. The last sum in (8.3.8) is a matrix in $A \otimes M_{n_t}$ and we take $z_{x,t}$ to be the function in $C(K_t) \otimes \mathcal{K}$ with the constant value of this matrix, which means that

$$z_{x,t} = \sum_{i,j=1}^{n_t} (\lambda_t \otimes \rho_{1,1})\big((1 \otimes e_{i,1})^* y_x (1 \otimes e_{j,1})\big)(1 \otimes e_{1,i}^*)(1 \otimes e_{1,j}). \tag{8.3.9}$$

By the approximations (8.3.6) and (8.3.7), we have the following for any $s \in V_t$,

$$(\lambda_s \otimes \mathrm{id}_{\mathcal{K}})(y_x) = y_x(s) \approx_{\epsilon/3} z_{x,t}(s) = (\lambda_s \otimes \mathrm{id}_{\mathcal{K}})(z_{x,t}). \tag{8.3.10}$$

We obtain an open covering $\{V_t\}_{t \in K}$ of K and by compactness of K, there exists a finite open covering $V_{t_1}, \cdots, V_{t_n}$ for some n. Without loss of generality, we can assume that $V_{t_1}, \cdots V_{t_m}$ are singleton sets containing isolated points $t_1, \cdots, t_m$ respectively and $V_{t_{m+1}}, \cdots V_{t_n}$ are open neighborhood around limit points $t_{m+1}, \cdots, t_n$. Moreover, we can assume that $V_{t_{m+1}}, \cdots V_{t_n}$ do not contain any of the isolated points $t_1, \cdots, t_m$. Take a partition of unity $g_0, g_1, \cdots, g_n$ subordinate to $U_0, V_{t_1}, \cdots, V_{t_n}$. Then

$$x = \sum_{k=0}^{n} g_k \cdot x \approx_{\epsilon/6} \sum_{k=1}^{n} g_k \cdot x, \quad x \in \mathcal{F} \tag{8.3.11}$$

by Theorem 2.10.12 (ii) and evaluating at each point in $\hat{A}$.

To get the MSAP for A, we claim that it is sufficient to prove the statement $(*)$: for each $1 \le k \le n$, there exists $N_k, L_k \in \mathbb{N}$, pure states $\rho_{\ell,k}$ on $A/J_{K_{t_k}}$, $d_{i,k}$, $c_{i,k,\ell}$ in $A/J_{K_{t_k}}$ for $1 \le i \le N_k$ and $1 \le \ell \le L_k$ such that for any $x \in \mathcal{F}$ and $s \in V_{t_k}$,

$$x(s) \approx_{5\epsilon/6} \sum_{\ell=1}^{L_k} \sum_{i,j=1}^{N_k} \rho_{\ell,k}(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,,\ell,k}^*(s) c_{j,\ell,k}(s), \tag{8.3.12}$$

and moreover, $\rho_{\ell,k} \circ \pi_{K_{t_k}}$ are pairwise inequivalent pure states on A for $1 \le k \le n$ and $1 \le \ell \le L_k$.

Suppose that the statement holds and take lifts $\tilde{d}_{i,k}$ and $\tilde{c}_{i,\ell,k}$ of $d_{i,k}$ and $c_{i,\ell,k}$ in A, which means that for any $1 \le k \le n$, $1 \le i \le N_k$ and $1 \le \ell \le L_k$,

$$\pi_{K_{t_k}}(\tilde{d}_{i,k}) = d_{i,k} \text{ and } \pi_{K_{t_k}}(\tilde{c}_{i,\ell,k}) = c_{i,\ell,k}. \tag{8.3.13}$$

In particular, for any $s \in V_{t_k} \subseteq K_{t_k}$, we have $\tilde{c}_{i,\ell,k}(s) = c_{i,\ell,k}(s)$. Thus,

$$\sum_{\ell=1}^{L_k} \sum_{i,j=1}^{N_k} (\rho_{\ell,k} \circ \pi_{K_{t_k}})(\tilde{d}_{i,k}^* x \tilde{d}_{j,k}) \tilde{c}_{i,\ell,k}^*(s) \tilde{c}_{j,\ell,k}(s) \tag{8.3.14}$$

$$= \sum_{\ell=1}^{L_k} \sum_{i,j=1}^{N_k} \rho_{\ell,k}(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,\ell,k}^*(s) c_{j,\ell,k}(s) \overset{(8.3.12)}{\approx_{5\epsilon/6}} x(s) \tag{8.3.15}$$

for any $1 \le k \le n$ and $s \in V_{t_k}$. Combining (8.3.11) and (8.3.14), (8.3.15) above,

$$x \approx_{\epsilon/6} \sum_{k=1}^{n} g_k \cdot x \approx_{5\epsilon/6} \sum_{k=1}^{n} \sum_{\ell=1}^{L_k} \sum_{i,j=1}^{N_k} (\rho_{\ell,k} \circ \pi_{K_{t_k}})(\tilde{d}_{i,k}^* x \tilde{d}_{j,k})(g_k^{1/2} \cdot \tilde{c}_{i,\ell,k}^*)(g_k^{1/2} \cdot \tilde{c}_{j,\ell,k}) \tag{8.3.16}$$

by Lemma 2.10.12 and evaluating the approximation at each $s \in \hat{A}$. This can be rearranged to an approximation of the form (8.1.3) (by allowing m to index over the set $\{(k,\ell) : 1 \le k \le n, 1 \le \ell \le L_k\}$). By Lemma 8.1.3, A has the MSAP.

Now we show that the statement $(*)$ holds. For $1 \le k \le m$, $K_{t_k} = V_{t_k} = \{t_k\}$ and $g_k(t_k) = 1$ and 0 elsewhere. By Lemma 2.10.11, the quotient $A/J_{\{t_k\}}$ is a simple and separable C^*-algebra with a unique irreducible representation, which means $A/J_{\{t_k\}}$ is either a matrix algebra or $\mathcal{K}$, and thus has the MSAP with the unique pure state ρ_k up to equivalence, by Example 8.1.4 and Lemma 8.2.4. There exist $N_k \in \mathbb{N}$, elements $d_{i,k}$, $c_{i,k}$ in $A/J_{\{t_k\}}$ for $1 \le i \le N_k$ such that for any $x \in \mathcal{F}$,

$$x(t_k) \approx_{5\epsilon/6} \sum_{i,j=1}^{N_k} \rho_k(d_{i,k}^* x(t_k) d_{j,k}) c_{i,k}^* c_{j,k}, \tag{8.3.17}$$

which provides the approximation (8.3.12) required as $c_{i,k}(s) = c_{i,k}$. The unitary class of the irreducible representation produced by $\rho_k \circ \pi_{K_{t_k}}$ corresponds to t_k in $\hat{A}$.

Now we approximate terms coming from limit points. For the ease of notation, we pretend for now that there is only one limit point $t_{m+1} = t$. Previously we have found $z_{x,t}$ in $C(K_t) \otimes \mathcal{K}$ for each $x \in \mathcal{F}$, see (8.3.9). By taking $b_i = \theta_t^{-1}(1 \otimes e_{i,1})$ in $A/J_{K_t} \otimes \mathcal{K}$, we have the following from (8.3.9),

$$\theta_t^{-1}(z_{x,t}) = \sum_{i,j=1}^{n_t} ((\lambda_t \otimes \rho_{1,1}) \circ \theta_t)(b_i^* \pi_{K_t}(x) b_j) b_i b_j^*. \tag{8.3.18}$$

Moreover, we can rearrange the approximation (8.3.10),

$$((\lambda_s \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_t)(\pi_{K_t}(x)) \approx_{\epsilon/3} ((\lambda_s \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_t)(\theta_t^{-1}(z_{x,t})). \tag{8.3.19}$$

Since the *-isomorphism θ_t fixes the spectrum K_t, it follows that $(\lambda_s \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_t$ is a representation on $A/J_{K_t} \otimes \mathcal{K}$ and is unitarily equivalent to $\pi_{K_t,\{s\}} \otimes \mathrm{id}_{\mathcal{K}}$, for any $s \in V_{t_k}$. Thus

$$x(s) = (\pi_{K_t,\{s\}} \otimes \mathrm{id}_{\mathcal{K}})(\pi_{K_t}(x)) \overset{(8.3.18)}{\approx_{\epsilon/3}} (\pi_{K_t,\{s\}} \otimes \mathrm{id}_{\mathcal{K}})(\theta_t^{-1}(z_{x,t})). \tag{8.3.20}$$

In particular, $(\lambda_t \otimes \rho_{1,1}) \circ \theta_t$ is equivalent to $\rho_t \otimes \rho_{1,1}$, for some pure state ρ_t on A/J_{K_t}, whose irreducible representation corresponds to t in the spectrum K_t. By Lemma 2.10.5, for any $\delta > 0$, there exists $a \in A/J_{K_t}$ such that for any $x \in \mathcal{F}$, $1 \leq i, j \leq n_t$,

$$((\lambda_t \otimes \rho_{1,1}) \circ \theta_t)(b_i^* \pi_{K_t}(x) b_j) \approx_\delta (\rho_t \otimes \rho_{1,1})(a^* b_i^* \pi_{K_t}(x) b_j a). \tag{8.3.21}$$

Take d_i to be the the value in A/J_{K_t} at the $(1,1)$ corner of $b_i a$, since $\pi_{K_t}(x)$ is an element in $(A/J_{K_t}) \otimes e_{1,1}$, we have the following for $x \in \mathcal{F}$ and $1 \leq i, j \leq n_t$,

$$(\rho_t \otimes \rho_{1,1})(a^* b_i^* \pi_{K_t}(x) b_j a) = \rho_t(d_i^* \pi_{K_t}(x) d_j). \tag{8.3.22}$$

By taking δ small enough, combining (8.3.18), (8.3.21) and (8.3.22), we have found d_i in A/J_{K_t} for $1 \leq i \leq n_t$ such that

$$\theta_t^{-1}(z_{x,t}) \approx_{\epsilon/6} \sum_{i,j=1}^{n_t} \rho_t(d_i^* \pi_{K_t}(x) d_j) b_i b_j^*. \tag{8.3.23}$$

There exists $\tilde{b}_i$ in $A/J_{K_t} \otimes M_{R_t}$ for big enough R_t such that

$$\theta_t^{-1}(z_{x,t}) \approx_{\epsilon/3} \sum_{i,j=1}^{n_t} \rho_t(d_i^* \pi_{K_t}(x) d_j) \tilde{b}_i^* \tilde{b}_j. \tag{8.3.24}$$

Combining with (8.3.20), we get the following,

$$x(s) \approx_{2\epsilon/3} \sum_{i,j=1}^{n_t} \rho_t(d_i^* \pi_{K_t}(x) d_j) \tilde{b}_i^*(s) \tilde{b}_j(s), \quad s \in V_t. \tag{8.3.25}$$

Then the norm difference at the $(1,1)$ corner of $A/J_{\{s\}} \otimes \mathcal{K}$ is bounded by $2\epsilon/3$. Thus

$$x(s) \approx_{2\epsilon/3} \sum_{r=1}^{R_t} \sum_{i,j=1}^{n_t} \rho_t(d_i^* \pi_{K_t}(x) d_j) c_{i,r}^*(s) c_{j,r}(s), \quad s \in V_t, \tag{8.3.26}$$

where $c_{i,r}$ is the $(k,1)$ entry of $\tilde{b}_i$ and is an element in A/J_{K_t}.

Repeating the argument for $t_{m+1}, \cdots, t_n$. For each $m+1 \le k \le n$, there exists R_k, $n_k \in \mathbb{N}$, a pure state ρ_k on $A/J_{K_{t_k}}$ with the irreducible representation corresponding to t_k, elements $d_{i,k}$, $c_{i,r,k}$ in $A/J_{K_{t_k}}$ such that

$$x(s) \approx_{2\epsilon/3} \sum_{r=1}^{R_k} \sum_{i,j=1}^{n_k} \rho_k(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,r,k}^*(s) c_{j,r,k}(s), \quad s \in V_{t_k}. \tag{8.3.27}$$

We start to adjust the pure states to match with approximations of the form (8.3.12). Fix $k = m+1$ and start with $r = 1$, since t_k is a limit point, by Corollary 2.10.4, there exists a pure state $\rho_{k,1}$ on $A/J_{K_{t_k}}$ such that the unitary class $[\phi_{\rho_{k,1}}]$ of the irreducible representation does not belong to $\mathcal{G}_{k,1} = \{t_1, \cdots, t_k\}$ and

$$\sum_{i,j=1}^{n_k} \rho_k(d_{i,k^*} \pi_{K_{t_k}}(x) d_{j,k}) c_{i,1,k}^* c_{j,1,k} \approx_{\epsilon/6R_k} \sum_{i,j=1}^{n_k} \rho_{k,1}(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,1,k}^* c_{j,1,k}. \tag{8.3.28}$$

Moreover, $\rho_{k,1}$ is inequivalent to $\rho_1, \cdots, \rho_m$ coming from isolated points. For $r = 2$, take $\mathcal{G}_{k,2} = \{t_1, \cdots, t_k, [\phi_{\rho_{k,1}}]\}$ and by Corollary 2.10.4 again, one can find a pure state $\rho_{k,2}$ on $A/J_{K_{t_k}}$ such that $[\phi_{\rho_{k,2}}] \notin \mathcal{G}_{k,2}$ and similar approximation as (8.3.30) holds. By the choice of $\rho_{k,1}$, it is inequivalent to $\rho_1, \cdots, \rho_m, \rho_{k,1}$. Repeating the same argument for R_k times, we have found pairwise inequivalent pure states $\rho_{k,r}$ for $1 \le r \le R_k$ such that for each r,

$$\sum_{i,j=1}^{n_k} \rho_k(d_{i,k^*} \pi_{K_{t_k}}(x) d_{j,k}) c_{i,r,k}^* c_{j,r,k} \approx_{\epsilon/6R_k} \sum_{i,j=1}^{n_k} \rho_{k,r}(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,r,k}^* c_{j,r,k}. \tag{8.3.29}$$

Taking the sum over approximations indexed by r and combining with (8.3.27), we have

$$x(s) \approx_{5\epsilon/6} \sum_{r=1}^{R_k} \sum_{i,j=1}^{n_k} \rho_{k,r}(d_{i,k}^* \pi_{K_{t_k}}(x) d_{j,k}) c_{i,r,k}^*(s) c_{j,r,k}(s), \quad s \in V_{t_k}, \tag{8.3.30}$$

which is in the required form (8.3.12) of the statement $(*)$.

For $k = m+2$, since t_k is again a limit point, Corollary 2.10.4 allows us to pick pure states that are pairwise inequivalent to $\rho_1, \cdots, \rho_m$ and $\rho_{m+1,r}$ for $1 \le r \le R_{m+1}$. Repeating the argument for $k = m+1$ concludes the proof. □

The following is an immediate consequence of Theorem 8.3.7 and Lemma 8.2.3.

Corollary 8.3.8. *Every separable type I C^*-algebra has the MSAP.*

Combining Corollary 8.3.3 and Corollary 8.3.8, we have proved that the MSAP is equivalent to nuclearity, at least for separable C^*-algebras.

Theorem 8.3.9. *A separable C^*-algebra is nuclear if and only if it has the MSAP.*

Chapter 9

Property (SI) and $\mathcal{Z}$-stability

In this chapter, we recall Matui and Sato's definition of property (SI) for C^*-algebras. Property (SI) was used in the proof that strict comparison implies $\mathcal{Z}$-stability, where the C^*-algebra only has finitely many extremal traces. We describe the strategy of Matui and Sato and point out how the MSAP was used in their proof.

Property (SI) was later generalized to maps in [9], and unital and full maps from nuclear separable C^*-algebras to ultraproducts of $\mathcal{Z}$-stable C^*-algebras are shown to have property (SI). We explain how the $\mathcal{Z}$-stability assumption is used.

Using the equivalence between nuclearity and the MSAP for separable C^*-algebras (Theorem 8.3.9), we generalize the property (SI) result to maps into C^*-algebras with strict comparison only (Theorem 9.1.7). Consequently, we obtain a unital copy of the Jiang-Su algebra $\mathcal{Z}$ into the relative commutant of the map (Theorem 9.2.2).

The results of this section will be contained in my upcoming paper [49].

9.1 Property (SI) for C^*-algebras and morphisms

Property (SI) for C^*-algebras was first defined in [72] by Matui and Sato, and later reformulated in terms of ultraproducts by Kirchberg and Rørdam in [61]. For the equivalence between their definitions, see [61, Lemma 5.2]. For a unital C^*-algebra A with a nonempty trace space, we denote by $T_\omega(A_\omega)$ the set of limit traces on A_ω,

Definition 9.1.1 ([61, Definition 2.6]). A unital and simple C^*-algebra A with $T(A) \neq \emptyset$ is said to have *property (SI)* if for all positive contractions $e, f \in A_\omega \cap A'$ with $e \in J_A$ and f having the property that there exists $\gamma > 0$ such that

$$\tau(f^n) > \gamma, \quad \text{for any } n \in \mathbb{N}, \tau \in T_\omega(A_\omega), \tag{9.1.1}$$

there exists $s \in A_\omega \cap A'$ such that $fs = s$ and $s^*s = e$.

Notice that the existence of $s \in A_\omega \cap A'$ such that $fs = s$ and $s^*s = e$ is equivalent to the existence of $s \in A_\omega$ such that $fs = s$ and $s^*xs = xe$ for any $x \in A$. Indeed, if $s \in A_\omega$ satisfies $s^*xs = xe$ for any $x \in A$, then $s^*s = e$ by taking $x = 1$ and

$$\begin{aligned} &(sx - xs)^*(sx - xs) && (9.1.2)\\ &= x^*s^*sx - x^*s^*xs - s^*x^*sx + s^*x^*xs && (9.1.3)\\ &= x^*ex - x^*xe - x^*ex + x^*xe = 0, && (9.1.4) \end{aligned}$$

which implies that s is found in $A_\omega \cap A'$. The argument can also be found in [72] or the remark after [61, Proposition 5.10], for instance.

Property (SI) can be considered as a comparison property between a "tracially small element" $e \in J_A \cap A'$ and a "tracially large element" $f \in A_\omega \cap A'$. Matui and Sato showed that for simple nuclear C^*-algebras, strict comparison implies property (SI) in the following result.

Theorem 9.1.2 ([72, Section 3], [61, Proposition 5.10]). *Let A be a unital, simple, separable, nuclear and non-elementary C^*-algebra with strict comparison and $QT(A) = T(A) \neq \emptyset$. Then A has property (SI).*

The proof for Theorem 9.1.2 can be divided into two parts, and we briefly describe the strategy. The first step is to show that simple, separable, nuclear and non-elementary C^*-algebras have the MSAP: the approximation property using pure states established in Chapter 8. Moreover, a single pure state is needed for approximations. To achieve property (SI), by the Kirchberg ϵ-test (Lemma 2.2.1) and the equivalence described directly after Definition 9.1.1, it suffices to find $s \in A_\omega$ for every finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, such that $fs \approx_\epsilon s$ and $sas^* \approx_\epsilon as$ for $a \in \mathcal{F}$. The second step uses the Akemann-Anderson-Pedersen theorem for pure states (Theorem 2.10.10) to obtain an element a_0 in A that approximately excises the single pure states appearing in the approximation for $(\mathcal{F}, \epsilon)$. Strict comparison allows one to find $r \in A_\omega$ such that $fr \approx r$ and $ra_0r^* \approx a_0s$. Combining with approximations of elements in $\mathcal{F}$ by a_0 produces $s \in A_\omega$ required, see the proof of [72, Proposition 2.2] for more details.

As explained in the introduction (Section 1.7), together with a McDuff-type property for $A^\omega \cap A'$, property (SI) allows one to access $\mathcal{Z}$-stability.

Proposition 9.1.3 ([61, Proposition 5.12]). *Let A be a separable, simple and unital C^*-algebra with $T(A) \neq \emptyset$ and property (SI). Then A is $\mathcal{Z}$-stable if and only if there exists a unital *-homomorphism $M_k \to A^\omega \cap A'$ for some $k \geq 2$.*

The generalization of property (SI) for maps $A \to B_\omega$ appeared in the subsequent work of Matui and Sato ([73, Lemma 3.2]) and is explicitly stated in [9, Definition 4.2]. For a sequence of unital C^*-algebras $(B_n)_n$, we denote the ultraproduct of these C^*-algebras by B_ω for ease of notation.

Definition 9.1.4 ([9, Definition 4.2])**.** Let $(B_n)_n$ be a sequence of separable, unital and finite C^*-algebras with $T(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Let A be a separable and unital C^*-algebra and $\phi : A \to B_\omega$ be a unital $*$-homomorphism.

The map ϕ has *property (SI)* if for any positive contractions $e, f \in B_\omega \cap \phi(A)'$ with $e \in J_B$ and f having the property that for any $a \in A_+ \setminus \{0\}$, there exists $\gamma_a > 0$ with $\tau(\phi(a)f^n) > \gamma_a$ for any $\tau \in T_\omega(B_\omega)$ and $n \in \mathbb{N}$, there exists $s \in B_\omega \cap \phi(A)'$ such that $s^*s = e$ and $fs = s$.

In particular, when A is a simple C^*-algebra and $B_n = A$ for all $n \in \mathbb{N}$, then property (SI) of maps recovers property (SI) for simple C^*-algebras (Definition 9.1.1). The only difference between A having property (SI) and the diagonal embedding $\iota : A \to A_\omega$ having property (SI) is the "tracially large" condition with respect to the ι. We show in the following lemma that for a simple C^*-algebra A, an element in $A_\omega \cap A'$ being "tracially large" and "tracially large" with respect to ι are equivalent.

Lemma 9.1.5. *Let A be a unital and simple C^*-algebra with $T(A) \neq \emptyset$. Then A has property (SI) if and only if the diagonal embedding $\iota_A : A \to A_\omega$ has property (SI).*

Proof. The "only if" implication is straightforward, and we prove the "if" direction. Take positive contractions $e, f \in A_\omega \cap A'$ with $e \in J_A$ and $\tau(f^n) > \gamma$ for some $\gamma > 0$ and any $n \in \mathbb{N}$, $\tau \in T_\omega(A_\omega)$. For any nonzero contraction $a \in A_+$, since A is simple, there exists $x_1, \cdots, x_n$ in A such that $\sum_{i=1}^n x_i a x_i^* = 1_A$. Thus for $\tau \in T_\omega(B_\omega)$, $n \in \mathbb{N}$,

$$\gamma < \tau(f^n) = \tau(\iota_A(1)f^n) \tag{9.1.5}$$

$$= \tau\left(\sum_{i=1}^n \iota_A(x_i)\iota_A(a)\iota_A(x_i)^* f^n\right) \tag{9.1.6}$$

$$\leq \sum_{i=1}^n \|\iota_A(x_i)\|^2 \tau(\iota_A(a)f^n). \tag{9.1.7}$$

Thus $\tau(\iota_A(a)f^n)$ is uniformly bounded from below for any $n \in \mathbb{N}$ and $\tau \in T_\omega(B_\omega)$. By our assumption, there exists $s \in A_\omega \cap \phi(A)'$ such that $s^*s = e$ and $fs = s$, which implies property (SI) for A. $\square$

Similarly, one would expect to obtain property (SI) for maps with appropriate comparison properties, for instance, when the codomain C^*-algebra has strict comparison. We include the following theorem from [9, Lemma 4.4], and restrict to the setting of unital *-homomorphisms.

Theorem 9.1.6 ([9, Lemma 4.4])**.** *Let $(B_n)_n$ be a sequence of simple, separable, unital, finite and $\mathcal{Z}$-stable C^*-algebras with $QT(B_n) = T(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Let A be a separable, unital and nuclear C^*-algebra. Then every unital and full map $\phi : A \to B_\omega$ has property (SI).*

As commented in [9, Remark 4.5 (i)], a more general result is proven in [9, Section 4]. The $\mathcal{Z}$-stability assumption on B_n is only essentially used to provide the space for an augmented map $\tilde{\phi} : A \otimes \mathcal{Z} \to B_\omega$, and in particular $A \otimes \mathcal{Z}$ satisfies the technical condition that appeared in Theorem 8.1.8: $\theta(A)$ contains no compact operators, where θ is the faithful representation of A obtained by taking the direct sum of one representation from each unitary equivalence class of irreducible representations of A. Then Theorem 8.1.8 provides approximations (MSAP) needed to prove property (SI) of ϕ. Thus, if A is a separable, unital and nuclear C^*-algebra such that $\theta(A) \cap \mathcal{K} = \emptyset$, which consequently has the MSAP, then the proof of Theorem 9.1.6 works when each B_n is assumed to have strict comparison of positive elements, instead of $\mathcal{Z}$-stability.

The main theorem we proved in chapter 8 (Theorem 8.3.9) shows that every nuclear and separable C^*-algebra has the MSAP. As a result, we get a generalization of Theorem 9.1.6, only requiring each B_n to have strict comparison.

Theorem 9.1.7. *Let $(B_n)_n$ be a sequence of separable, unital and finite C^*-algebras with strict comparison and $QT(B_n) = T(B_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Let A be a separable, unital and nuclear C^*-algebra. Then every unital and full *-homomorphism $\varphi : A \to B_\omega$ has property (SI).*

The proof of the theorem follows the route of [9, Section 4], which generalizes the proof of Theorem 9.1.2. We provide the proof for completeness in the next section.

9.2 Main theorem and applications

Following the strategy described after Theorem 9.1.2, Theorem 8.3.9 has provided the analogous approximations on the domain C^*-algebra A needed in the first step. However, multiple pure states are needed for non-simple C^*-algebras, compared to single pure state approximations for simple C^*-algebras. Thus, the following theorem proven in [9] is needed, which excises multiple pairwise inequivalent pure states simultaneously by approximately mutually orthogonal positive elements.

Lemma 9.2.1 ([9, Lemma 4.7])**.** *Let A be a C^*-algebra and $\rho_1, \cdots, \rho_n$ be pairwise inequivalent pure states. For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$, there exist positive contractions $a_1, \cdots, a_n \in A$ such that*

(i) $\rho_i(a_i) = 1$, *for* $i = 1, \cdots, n$;

(ii) $a_i x a_i \approx_\epsilon \rho_i(x) a_i^2$ *for* $i = 1, \cdots, n$ *and* $x \in \mathcal{F}$;

(iii) $a_i x a_j \approx_\epsilon 0$ *for* $i \neq j$ *and* $x \in \mathcal{F}$.

Now we begin to prove the main theorem of this chapter.

Proof of Theorem 9.1.7. Fix positive contractions $e, f \in B_\omega \cap \phi(A)'$ such that $e \in J_{B_\omega}$ and f has the property that for any $a \in A_+ \setminus \{0\}$, there exists $\gamma_a > 0$ with

$$\tau(\phi(a) f^n) > \gamma_a, \quad \tau \in T_\omega(B_\omega),\ n \in \mathbb{N}. \tag{9.2.1}$$

We will show that there exists $s \in B_\omega$ such that

$$fs = s \text{ and } s^* \phi(a) s = \phi(a) e, \quad a \in A. \tag{9.2.2}$$

By the Kirchberg ϵ-test (Lemma 2.2.1), it suffices to prove the following: For a finite subset $\mathcal{F}$ of contractions in A and $\epsilon > 0$, there exists a contraction $s \in B_\omega$ such that

$$fs = s \text{ and } s^* \phi(x) s \approx_\epsilon \phi(x) e, \quad x \in \mathcal{F}. \tag{9.2.3}$$

By Theorem 8.3.9, there exists $L, N \in \mathbb{N}$, pairwise inequivalent pure states $\rho_1, \cdots, \rho_L$ on A and elements $c_i, d_{i,l} \in A$ such that

$$x \approx_{\epsilon/3} \sum_{i,j=1}^{N} \sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* x d_{j,\ell}) c_i^* c_j, \quad x \in \mathcal{F}. \tag{9.2.4}$$

We apply the excision lemma (Lemma 9.2.1) to the following finite subset in A,

$$\{d_{i,\ell}^* x d_{j,\ell'} : x \in \mathcal{F}, 1 \leq i, j \leq N, 1 \leq \ell, \ell' \leq L\}, \tag{9.2.5}$$

and $\epsilon_1 = \epsilon/(3N^2 L \max_k \|c_k\|^2)$. There exists positive contractions $a_1, \cdots, a_L \in A$ such that for $\ell = 1, \cdots, L$, we have $\rho_\ell(a_\ell) = 1$ and

$$a_\ell d_{i,\ell}^* x d_{j,\ell} a_\ell \approx_{\epsilon_1} \rho_\ell(d_{i,\ell}^* x d_{j,\ell}) a_\ell^2, \quad 1 \leq i, j \leq N,\ x \in \mathcal{F}, \tag{9.2.6}$$

and for $\ell \neq \ell'$,

$$a_\ell d_{i,\ell}^* x d_{j,\ell'} a_{\ell'} \approx_{\epsilon_1} 0, \quad 1 \leq i,j \leq N,\ x \in \mathcal{F}. \tag{9.2.7}$$

For each $\ell = 1, \cdots, L$, let $S_\ell \subseteq A_+$ be the countable set provided by [9, Lemma 4.9] with ϕ in place of π and a_ℓ in place of a. Applying [9, Lemma 1.18] with $x = 0$, $T = \phi(A)$ and $S_0 = \phi(S_1 \cup \cdots \cup S_L)$, there exists $f' \in B_\omega \cap \phi(A)'$ such $ff' = f'$ and for any $b \in S_1 \cup \cdots \cup S_L$, we have

$$\tau(\phi(b)(f')^n) > \gamma_b, \quad \tau \in T_\omega(B_\omega),\ n \in \mathbb{N},\ b \in S_0. \tag{9.2.8}$$

In particular, this holds for $n = 1$ and thus for each $\ell = 1, \cdots, L$, an application of [9, Lemma 4.9] shows that there exists $r_\ell \in B_\omega$ such that for $\ell = 1, \cdots, L$,

$$\phi(a_\ell) r_\ell = f r_\ell = r_\ell \text{ and } r_\ell^* r_\ell = e. \tag{9.2.9}$$

It follows that

$$r_\ell^* \phi(a_\ell^2) r_\ell = r_\ell^* r_\ell = e. \tag{9.2.10}$$

Now take

$$s = \sum_{i=1}^{N} \sum_{\ell=1}^{L} \phi(d_{i,\ell}) \phi(a_\ell) r_\ell \phi(c_i). \tag{9.2.11}$$

Since $f r_\ell = r_\ell$ and f commutes with $\phi(A)$, we have $fs = s$. Moreover, for any $x \in \mathcal{F}$,

$$\begin{aligned}
s^* \phi(x) s &= \sum_{i,j=1}^{N} \sum_{\ell,\ell'=1}^{L} \phi(c_i)^* r_\ell^* \phi(a_\ell d_{i,\ell}^* x d_{j,\ell'} a_{\ell'}) r_{\ell'} \phi(c_j) && (9.2.12)\\
&\overset{(9.2.7)}{\approx_{\epsilon/3}} \sum_{i,j=1}^{N} \sum_{\ell=1}^{L} \phi(c_i)^* r_\ell^* \phi(a_\ell d_{i,\ell}^* x d_{j,\ell} a_\ell) r_\ell \phi(c_j) && (9.2.13)\\
&\overset{(9.2.6)}{\approx_{\epsilon/3}} \sum_{i,j=1}^{N} \sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* x d_{j,\ell}) \phi(c_i)^* r_\ell^* r_\ell \phi(c_j) && (9.2.14)\\
&= \sum_{i,j=1}^{N} \sum_{\ell=1}^{L} \rho_\ell(d_{i,\ell}^* x d_{j,\ell}) \phi(c_i)^* e \phi(c_j) && (9.2.15)\\
&\overset{(9.2.4)}{\approx_{\epsilon/3}} \phi(x) e. && (9.2.16)
\end{aligned}$$

This concludes the proof. □

Now we give the proof for Theorem I, through "relative central surjectivity".

Theorem 9.2.2. *Let B_n be a sequence of unital, monotracial and non-elementary C^*-algebras with strict comparison and $QT(B_n) = T(B_n)$ for all $n \in \mathbb{N}$. We write B_ω for $\prod_\omega B_n$. Let A be a separable, unital and nuclear C^*-algebra and $\phi : A \to B_\omega$ a unital and full *-homomorphism. Then there exists a unital embedding $\mathcal{Z} \to B_\omega \cap \phi(A)'$.*

Proof. For each n, we denote the unique trace of B_n by τ_n and take $\mathcal{M}_n = \pi_{\tau_n}(B_n)''$. We write $B_\omega := \prod_\omega B_n$ to be the norm ultrapower, which has a unique trace τ_{B_ω} by Theorem 2.8.20. We write $B^\omega := \prod^\omega B_n$ and $\mathcal{M}^\omega := \prod^\omega \mathcal{M}_n$ for tracial ultrapowers. By [9, Lemma 3.10], we have

$$B_\omega/J_B = B^\omega \cong \mathcal{M}^\omega, \tag{9.2.17}$$

and $\mathcal{M}^\omega$ is a II_1 factor with a unique trace $\tau_{\mathcal{M}^\omega}$ by Proposition 7.2.2. Moreover, [9, Lemma 3.10] gives the short exact sequence at the level of central sequence algebras,

$$0 \longrightarrow J_B \cap \phi(A)' \xrightarrow{j_B} B_\omega \cap \phi(A)' \xrightarrow{q_B} \mathcal{M}^\omega \cap \bar{\phi}(A)' \longrightarrow 0, \tag{9.2.18}$$

where $\bar{\phi} = q_B \circ \phi$ and $q_B : B_\omega \to B_\omega/J_B$ is the quotient map. Since A is nuclear, it follows that $\bar{\phi}(A)'$ is hyperfinite by Connes' theorem. Thus, there exists a unital *-homomorphism $\Phi : M_2 \to \mathcal{M}^\omega \cap \bar{\phi}(A)'$. By [71], we can lift the unital embedding Φ to a completely positive order zero map $\varphi : M_2 \to B_\omega \cap \phi(A)'$.

Take $s_1 = \varphi(e_{1,1})$ and $s_2 = \varphi(e_{1,2})$. By the structure theorem for c.p. order zero maps (Theorem 2.6.10), we have $s_1^*s_1 = s_2s_2^*$, $s_1^*s_1 \perp s_2^*s_2$ and $\varphi(1)^2 = s_1^*s_1 + s_2^*s_2$. Take $e = 1 - (s_1^*s_1 + s_2^*s_2)$, which is an element in $B_\omega \cap \phi(A)'$. Since $\tau_{\mathcal{M}^\omega} \circ q_B = \tau_{B_\omega}$ and $q_B \circ \varphi = \Phi$ is a unital map, we have

$$\tau_{B_\omega}(e) = \tau_{B_\omega}(1 - \varphi(1)^2) = (\tau_{\mathcal{M}^\omega} \circ q_B)(1 - \varphi(1)^2) = 0. \tag{9.2.19}$$

This implies that $e \in J_B \cap \phi(A)'$. On the other hand, since $\Phi(e_{1,1})$ and $\Phi(e_{2,2})$ are Murray–von Neumann equivalent projections in $\mathcal{M}^\omega \cap \bar{\phi}(A)'$, it follows that

$$\tau_{\mathcal{M}^\omega}\big(\bar{\phi}(a)\Phi(e_{1,1})\big) = \tau_{\mathcal{M}^\omega}\big(\bar{\phi}(a)\Phi(e_{2,2})\big), \quad \text{for any } a \in A. \tag{9.2.20}$$

For any nonzero $a \in A_+$, we have that $\phi(a)$ is full in B_ω by fullness of ϕ and thus not contained in J_B. This implies that $\tau_{B_\omega}(\phi(a)) > 0$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned} 0 < \tau_{B_\omega}(\phi(a)) &= \tau_{\mathcal{M}^\omega}\big(\bar{\phi}(a)\Phi(1)\big) && (9.2.21)\\ &= 2\tau_{\mathcal{M}^\omega}\big(\bar{\phi}(a)\Phi(e_{1,1})\big) && (9.2.22)\\ &= 2\tau_{\mathcal{M}^\omega}\big(\bar{\phi}(a)\Phi(e_{1,1})^n\big) && (9.2.23)\\ &= 2\tau_{B_\omega}\big(\phi(a)\varphi(e_{1,1})^n\big) && (9.2.24)\\ &= 2\tau_{B_\omega}\big(\phi(a)s_1^n\big). && (9.2.25) \end{aligned}$$

By Theorem 9.1.7, the map ϕ has property (SI) and thus, there exists $s \in B_\omega \cap \phi(A)'$ such that $s^*s = e$ and $s_1 s = s$. It follows that

$$s^*s = 1 - (s_1^*s_1 + s_1^*s_1) \ \text{ and } \ ss^*s_1^*s_1 = ss^*. \tag{9.2.26}$$

We have shown that s_1, s_2, s are elements in $B_\omega \cap \phi(A)'$ satisfying the conditions in Lemma 2.7.3, which implies that there exists a unital *-homomorphism from $\mathcal{Z}_{2,3}$ to $B_\omega \cap \phi(A)'$. By Theorem 2.7.2, we get a unital embedding $\mathcal{Z} \hookrightarrow B_\omega \cap \phi(A)'$.

□

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