ISSN 1471-0498

DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

DEPENDENCE AND UNIQUENESS IN BAYESIAN GAMES

A.W. Beggs

Number 603
April 2012

Manor Road Building, Oxford OX1 3UQ

Dependence and Uniqueness in Bayesian Games

A. W. Beggs
Wadham College
Oxford
OX1 3PN
UK

alan.beggs@economics.ox.ac.uk

Current Version: April 2012

Abstract

This paper studies uniqueness of equilibrium in symmetric 2×2 bayesian games. It shows that if signals are highly but not perfectly dependent then players play their risk-dominant actions for all but a vanishing set of signal realizations. In contrast to the global games literature, noise is not assumed to be additive. Dependence is modeled using the theory of copulas.

Keywords: bayesian games, global games, uniqueness, copulas, risk dominance.

JEL Nos: C72, D82

1. Introduction

In an influential contribution, Carlsson and van Damme (1993) showed that in 2×2 games a small amount of noise in players' signals may lead to uniqueness of equilibrium even if there are multiple equilibria without noise. This has been widely used as an equilibrium-selection device in applications: see for example Morris and Shin (2003) for a survey.

In their analysis, Carlsson and van Damme (1993) assume that signals are subject to additive noise. As the noise becomes small the player's signals become almost perfectly correlated and indeed almost perfectly dependent. This paper examines uniqueness in the environment of 2×2 symmetric games. It replaces the assumption of additive errors by a symmetry assumption and natural monotonicity assumptions on beliefs. It shows that if signals are sufficiently, but not perfectly, dependent then Carlsson and van Damme (1993)'s result continues to hold.

The modeling of the relationship between signals draws on the theory of copulas. This has become influential as giving a flexible and general way of modeling dependence between random variables in statistics and also in economics. Nelsen (2006) provides a general introduction and Patton (2009) an introduction for finance. In particular copulas allow one to vary the degree of dependence between two variables while keeping their marginal distributions fixed. This makes them a natural tool for analyzing the effect of dependence on equilibrium.

In the case of the symmetric games, Carlsson and van Damme (1993) show that as signals become highly dependent each player becomes increasingly likely to play their risk dominant action given their signal. In essence this is because each believes there is roughly a 50:50 chance the other will play either of their actions. This paper shows that beliefs continue to have this property in the current framework and so players choose their risk-dominant actions for all but a shrinking set of signal realizations as signals become increasingly dependent.

The paper proceeds as follows. Section 2 provides an example of the framework considered. Section 3 provides an introduction to copulas. Section 4 lays out the general model and Section 5 states the main results. It is shown that as signals tend towards perfect dependence, equilibrium becomes unique. The initial presentation assumes the marginal distribution of signals is uniform and that player's values are private. The former assumption is simply a convenient normalization and Section 6 shows explicitly that this is so. Section 7 shows how to extend the analysis to common values and compares the assumptions made to those of Carlsson and van Damme (1993). Section 8 concludes.

2. Example

Consider the following game:

	0	1
0	$1, 1$	$1.1, t_2 + 1.1$
1	$t_1 + 0.1, 1.1$	$t_1 + 1, t_2 + 1$

Figure 1

where t_1 and t_2 are signals, observed by the row player (player 1) and column player (player 2) respectively.

Suppose for a moment that the signals are perfectly correlated and equal: $t_1 = t_2 = t$ where t is uniformly distributed on $[0, 1]$. The game then becomes

	0	1
0	$1, 1$	$1.1, t + 1.1$
1	$t + 0.1, 1.1$	$t + 1, t + 1$

Figure 2

There is a continuum of equilibria in this model. For example, for any k with $0.1 \leq k \leq 0.9$ it is an equilibrium for each player to use the following strategy: play

$$\begin{cases} \text{Play } 0 & t < k \\ \text{Play } 1 & t \geq k \end{cases}$$

.

Carlsson and van Damme (1993) show that if t_i are noisy signals of t with additive error:

$$t_i = t + \sigma\epsilon_i$$

where the ϵ_i are independent of t, then play in the game in Figure 1 converges to a unique equilibrium as σ becomes small.[1] As $\sigma \to 0$ the signals tend towards perfect correlation and, in the sense explained in the next section, towards perfect dependence.[2] They show that when σ is small,

$$\Pr(T_2 \leq t_2 | T_1 = t_1) + \Pr(T_1 \leq t_1 | T_2 = t_2) \approx 1 \tag{1}$$

if t_1 and t_2 are in the interior of $[0,1]$. In particular when $t_1 = t_2 = t$,

$$\Pr(T_2 \leq t | T_1 = t) \approx \frac{1}{2} \tag{2}$$

If both players switch between actions at the common level k, then it follows from (2) that each must believe there is a 50:50 chance that the other will play action 0 or action 1. Each must therefore be indifferent between his actions given this play by the other player. This is true for all switch points in the interior and it is easy to see that there is a unique k with this property.

The argument Carlsson and van Damme (1993) give for (1) and (2) depends heavily on the assumption of additive errors.[3] The aim in the current paper is see if this assumption can be relaxed. t_1 and t_2 will be assumed to have common distribution function $C^\theta(t_1, t_2)$ and to tend to perfect dependence as $\theta \to \infty$.

3. Copulas

This section provides a brief review of the properties of copulas. A standard reference is Nelsen (2006), on which this section draws heavily.

A two-dimensional **copula** $C(u, v)$ is a cumulative distribution function defined on $[0,1] \times [0,1]$ with uniform marginal distributions. Equivalently it is a function with domain $[0,1] \times [0,1]$ such that

1) C is grounded: $C(u, 0) = C(0, v) = 0$ for all u,v in $[0,1]$

[1] The body of their paper considers common values but, as they note, their arguments extend to the case of private values, as in this example.

[2] This is shown formally in Section 7.2.

[3] One can of course always write $t_i = t + v_i$, where $v_i = t - E(t|t_i)$ but in general the v_i will not be independent of t nor will σ enter multiplying v_i.

2) C is 2-increasing: for all u_1, u_2, v_1, v_2 in $[0,1]$ with $u_1 \leq u_2$, $v_1 \leq v_2$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$$

3) For every u, v in $[0,1]$, $C(u,1) = u$ and $C(1,v) = v$.

The condition that C is 2-increasing requires that any rectangle $[u_1, u_2] \times [v_1, v_2]$ is assigned positive volume (probability mass) by C. The third condition is the requirement of uniform marginals.

The importance of copulas derives from the following theorem. Let $\overline{\mathcal{R}} = [-\infty, \infty]$ be the extended real line and Ran denote range.

Sklar's Theorem *Let H be a joint distribution function H with marginals F and G. Then there exists a copula C such that for all x and y in $\overline{\mathcal{R}}$*

$$H(x,y) = C(F(x), G(y)) \tag{3}$$

If F and G are continuous then C is unique; otherwise C is uniquely determined on $\operatorname{Ran} F \times \operatorname{Ran} G$. Conversely if C is a copula and F and G are distribution functions, then the function H defined by (3) is a joint distribution function with margins F and G.

A proof can be found in Nelsen (2006) Section 2.3. Sklar's Theorem has made copulas a popular approach to modelling dependence between random variables as one can vary the degree of dependence between them through the choice of C, whilst keeping their marginal distributions constant.

If F is a continuous univariate distribution one can define

$$F^{-1}(t) = \inf\{x | F(x) \geq t\} \tag{4}$$

and similarly for G. If F is strictly increasing F^{-1} is the ordinary inverse. One can then recover the copula C in Sklar's Theorem as

$$C(u,v) = H(F^{-1}(u), G^{-1}(v)) \tag{5}$$

It is well known that if a random variable X has continuous distribution function F then $U = F(X)$ is uniformly distributed on $(0,1)$.[4] Sklar's theorem

[4]See for example Billingsley (1995) Exercise 14.3.

can be thought of as a generalization of this result. By transforming random variables appropriately, one can always assume they have uniform marginal distributions.

In what follows U and V will denote two random variables uniformly distributed on $[0,1]$. If they have copula C then, provided C is appropriately differentiable, their joint density is

$$c(u,v) = \frac{\partial^2 C}{\partial u \partial v} \tag{6}$$

and

$$\Pr(V \leq v | U = u) = \frac{\partial C(u,v)}{\partial u} \tag{7}$$

The latter conditional distribution will be written as $C(v|u)$. A sufficient condition for the derivatives mentioned to exist everywhere is that the density c is continuous.

If U and V are independent then they have the product copula

$$\Pi(u,v) = uv \tag{8}$$

If U and V are uniform and perfectly positively dependent, that is $\Pr(U = V) = 1$, then they have copula

$$M(u,v) = \min\{u,v\} \tag{9}$$

If they are perfectly negatively dependent, that is $\Pr(U+V) = 1$, then they have copula

$$W(u,v) = \max\{u+v-1, 0\} \tag{10}$$

An arbitrary copula must lie between these two bounds:

$$W(u,v) \leq C(u,v) \leq M(u,v) \tag{11}$$

which is referred to as the Fréchet-Hoeffding bounds inequality and W as the Fréchet-Hoeffding lower bound and M the Fréchet-Hoeffding upper bound.[5]

One can define a partial ordering on copulas, the concordance ordering, by

$$C_1 \preceq C_2 \iff C_1(u,v) \leq C_2(u,v) \quad \forall u,v \tag{12}$$

[5] In higher dimensions the analogy of these bounds holds but the generalization of W need not be a copula.

Since for a copula C, $C(u,v) = \Pr(U \le u, V \le v)$, the concordance ordering can be interpreted as implying that $C_1 \preceq C_2$ means that the two random variables are more likely to take low (and high values) together if they have copula C_2 rather than C_1 and are in that sense more highly correlated or more dependent. One can show that U and V have higher Spearman's and Kendall rank correlation coefficients under C_2 than C_1.[6] One can also show that the concordance ordering is equivalent to the supermodular ordering: $C_1 \preceq C_2$ if and only if $\int u\, dC_1 \le \int u\, dC_2$ for all supermodular functions u.[7] M and W are the maximal and minimal elements respectively in the concordance order.

Some commonly used copulas are:

1) Gaussian Copulas: Let $\Phi_\Sigma(x,y)$ be the joint distribution function of a bivariate Normal distribution with covariance matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, that is corresponding to two random varibles with unit variance and correlation ρ. Let Φ denote the standard univariate Normal distribution. The Gaussian copula with correlation matrix Σ is

$$C_\Sigma(u,v) = \Phi_\Sigma(\Phi^{-1}(u), \Phi^{-1}(v)) \tag{13}$$

2) Archimedean Copulas: Let ϕ be a continuous, strictly decreasing, convex function from $[0,1]$ to $[0,\infty)$ such that $\phi(1) = 0$. Then

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))) \tag{14}$$

is a copula. Important examples include

a) (Clayton) $\phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ with $\theta > 0$ and
$C_\theta(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$.

b) (Gumbel) $\phi_\theta(t) = (-\ln t)^\theta$ with $\theta \ge 1$ and
$C_\theta(u,v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right)$.

c) (Frank) $\phi_\theta(t) = -\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$ with $\theta > 0$ and
$C_\theta(u,v) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right)$.

[6] See for example Joe (1997) exercise 2.10 on p. 54.

[7] See for example Müller and Stoyan (2002) Theorem 3.8.2. Note, however, that this is equivalence does not hold in more than two dimensions and that the definition of the concordance order needs to be adapted in higher dimensions — see Müller and Stoyan (2002) Chapter 3.

As $\rho \to 1$ the Gaussian copula tends to perfect positive correlation, which in the Gaussian context is the same as perfect positive dependence. The same holds as $\theta \to \infty$ in each of the Archimedean examples:

$$\lim_{\theta \to \infty} C_\theta(u, v) = M(u, v) \tag{15}$$

. Moreover these families are ordered by concordance by higher θ (or ρ): $\theta \geq \theta' \Rightarrow C^\theta \succeq C^{\theta'}$.[8]

4. The Model

There are two players $i = 1, 2$. Each player observes a real-valued signal t_i. After observing his signal, each player takes an action a_i, which can take on the values 0 or 1. Player i has utility function $U_i(a, t_i)$ where $a = (a_1, a_2)$, that is his payoff depends only on his own valuation. The case of common values is discussed in Section 7.

Assumption 1 *The game is symmetric:* $U_1(a_1, a_2, t_i) = U_2(a_2, a_1, t_i)$ *for all* a_1, a_2 *and* t_i. U_i *is continuous in* t_i *for each* a.

Let $\Delta U_i(a_j, t_i) = U_i(1, a_j, t_i) - U_i(0, a_j, t_i)$ be the incremental payoff in playing action 1 rather than 0 if the other player takes action a_j and the signal is t_i.

Assumption 2 *For each* i, $\Delta U_i(1, t_i) \geq \Delta U_i(0, t_i)$ *for all* t_i *and* $\Delta U_i(a_j, t_i)$ *is strictly increasing in* t_i *for each* a_j.

In other words payoff functions have increasing differences in (a_1, a_2) and (a_i, t_i), so the underlying game is a coordination game.

Assumption 3 *Each player's signal is uniformly distributed on* $[0, 1]$.

As can be seen from the previous section this assumption is a sense without loss of generality as one can transform the signals so it holds. The next section shows explicitly that the results hold if it is relaxed.

[8] See for example Joe (1997) pp. 140–142 for these facts. Note that he refers to the Clayton copula as the Kimeldorf- Sampson copula.

Assumption 4 *There exist $\underline{t} > 0$ and $\bar{t} < 1$ such that for each i, action 0 is strictly dominant for player i if and only if $t_i < \underline{t}$ and action 1 is strictly dominant if and only if $t_i > \bar{t}$.*

This is a standard assumption in the literature on global games and is often referred to as a limit dominance assumption.[9]

The joint cumulative distribution function of the signals $C^\theta(t_1, t_2)$ is parameterized by θ. θ belongs to some subset of the real line which is unbounded above. θ will sometimes be suppressed from the notation.

Assumption 5 *For each θ,*
(i) $C^\theta(t_1, t_2)$ is continuous, has a joint density and is symmetric: $C^\theta(t_1, t_2) = C^\theta(t_2, t_1)$ for all t_1 and t_2.
(ii) The conditional distribution $C^\theta(v|u)$ is decreasing in u.
(iii) $C^\theta(u|u)$ is increasing in u.

Since $C^\theta(v|u) = \Pr^\theta(T_2 \leq v | T_1 = u)$, the second assumption is one of first-order stochastic dominance. Note that from (7) this is equivalent to concavity of C^θ in each variable holding the other fixed since $C^\theta(v|u) = \partial C^\theta / \partial u$. It can be checked that it holds for all the copulas mentioned in the previous section. The results of Müller and Scarsini (2005) show that it holds for any two-dimensional Archimedean copula where the derivative of minus the inverse generator, $-\phi^{-1}$, is log-convex (recall that ϕ and so ϕ^{-1} are decreasing).[10] It is used to ensure that optimal strategies are monotone.

Given the symmetry of C the third assumption is equivalent to the assertion that $C(u, u)$ is convex in u (since $C(v|u) = \partial C / \partial u$). This is certainly satisfied by the independence copula Π, since then $C(u, u) = u^2$, and the case of perfect positive dependence, since then $C(u, u) = u$. It is satisfied by a wide variety of copulas, including those mentioned in the previous section.[11]

To see that it is a natural assumption note that since $C(u, u) = \Pr(T_1 \leq, T_2 \leq u)$ (iii) is further equivalent to the assumption that $Z = \max\{T_1, T_2\}$ has an increasing density or, since T_1 and T_2 are uniform, that the likelihood

[9] See for example Morris and Shin (2003).

[10] See Müller and Scarsini (2005) Theorem 2.8(d) and note that they define an Archimedean copula to have form $\psi(\psi^{-1} + \psi^{-1}(y))$.

[11] Indeed as Nelsen et al. (2008) p. 480 note it holds for all but 1 copula (4.2.18) in the table of 22 Archimedean copulas on pp.116–118 of Nelsen (2006).

ratio of Z to T_1 (or T_2) is increasing. In other words, Z dominates T_1 (and T_2) in the likelihood ratio order. Z always dominates T_i in the sense of first-order dominance but is natural to assume that it also dominates in the stronger likelihood ratio order.[12] (iii) is needed for technical reasons in the proof, as will be explained later.

Assumption 6 $\lim_{\theta\to\infty} C^\theta(u,v) = \min\{u,v\} = M(u,v)$ *for all* u, v.

In other words the joint distribution tends to perfect positive dependence as θ becomes large. Signals are common knowledge in the limiting case of perfect dependence. It is not assumed that θ orders the copulas according to increasing concordance, although this is a natural assumption. If the latter holds, as it does in the examples of the previous section, then increasing θ corresponds to signals becoming increasingly dependent.

A **strategy** for player i is a measurable function $[0,1] \longrightarrow A_i$. Strategy σ_i is greater than strategy σ_i' if $\sigma_i(t_i) \geq \sigma_i'(t_i)$ for almost all t_i, with the obvious order on $\{0,1\}$. A **monotone strategy** is a strategy for which there exists a number k_i, or cutoff, such that player i plays action 0 if $t_i < k_i$ and action 1 if $t_i > k_i$. Since C has a joint density, the action chosen at k_i does not affect either player's expected payoffs. Note that if $k_i > k_i'$ then the strategy corresponding to cutoff k_i is smaller than the strategy corresponding to k_i'.

Player i's **interim** payoff conditional on receiving signal t_i and taking action a_i given j's strategy σ_j, $j \neq i$, is denoted $V_i(a_i, t_i; \sigma_j)$. If player j employs a monotone strategy with cutoff k_j then V_i can be written as

$$V_i(a_i, t_i; k_j) = C(k_j|t_i)U_i(a_i, 0, t_i) + (1 - C(k_j|t_i))U_i(a_i, 1, t_i) \tag{16}$$

Player i's **ex ante** payoff given the strategies of the two players, $W_i(\sigma_i, \sigma_j)$, is $E(V_i(\sigma_i(t_i), t_i; \sigma_j))$. (σ_1^*, σ_2^*) is an **equilibrium** if for each i, $j \neq i$, $W_i(\sigma_i^*, \sigma_j^*) \geq W_i(\sigma_i, \sigma_j^*)$ for each strategy σ_i of player i.

Attention will focus on monotone strategies. The results of Van Zandt and Vives (2007) show that in the current framework greatest and least equilibria exist and these employ monotone strategies. This will be used to bound behavior in all equilibria.

[12] See, for example, Müller and Stoyan (2002) p. 12, Theorem 1.4.4 for a proof that the likelihood ratio order is stronger than first-order stochastic dominance.

5. Main Results

This section presents the main result. It is shown that as the signals approach perfect positive dependence players will play the risk dominant strategies, for each signal, with arbitrarily high probability.

Since the game is symmetric an action is risk dominant, for a given signal, if a player prefers to play it if he believes his opponent is equally likely to play either of her actions. Under the assumptions of the previous section, there is a unique switching point at which action 1 becomes risk dominant:

Lemma 1 *Under Assumption 1, Assumption 2 and Assumption 4 there is a unique signal t^* such that if player i believes that there is a 50:50 chance that player j will play action 0 or action 1, he strictly prefers to play action 0 if t_i is below t^*, strictly prefers action 1 if it is above and is indifferent if $t_i = t^*$:*

$$\frac{1}{2}\Delta U_i(0, t^*) + \frac{1}{2}\Delta U_i(1, t^*) = 0 \tag{17}$$

This follows immediately from the limit dominance assumption and the fact that the ΔU_i are strictly increasing and continuous.

The result below shows that for large θ, players play the risk dominant action with arbitrarily high probability:

Theorem 1 *Under Assumptions 1–6, for any $\delta > 0$ there exists θ' such that for all $\theta \geq \theta'$, in any equilibrium, player i plays action 0 if $t_i \leq t^* - \delta$ and action 1 if $t_i \geq t^* + \delta$.*

The proof of the theorem follows from the following generalisation of (2) proven in the Appendix:

Lemma 2 *For any interval $[u, v]$ in the interior of $[0, 1]$ and $\delta > 0$ there exists θ' such that for all $\theta \geq \theta'$, for all $t \in [u, v]$*

$$\left|\Pr^\theta(T_2 \leq t | T_1 = t) - \frac{1}{2}\right| < \delta \tag{18}$$

$\Pr^\theta$ denotes probability under C^θ. The intuition for the result can be seen in Figure 3. $\Pr^\theta(T_2 \leq t | T_t = t) = C^\theta(t|t)$. Since T_1 has a uniform marginal, integrating $C^\theta(t|t)$ between $u-\epsilon$ and $u+\epsilon$ gives the probability mass of the shaded area A below the 45° shown in the figure:

$$\Pr^\theta(u-\epsilon \le T_1 \le u+\epsilon, T_2 \le T_1) = \int_{u-\epsilon}^{u+\epsilon} C^\theta(t|t)\, dt \tag{19}$$

For brevity area will be used as shorthand for probability mass of the region referred to. Symmetry of C^θ implies that the areas A and B are equal — note that for finite θ the diagonal has zero mass. The total shaded area $A+B$ equals $C^\theta(u+\epsilon, u+\epsilon) - C^\theta(u-\epsilon, u-\epsilon)$, which converges to $u+\epsilon-(u-\epsilon) = 2\epsilon$ since $C^\theta(s,t)$ converges to $M(s,t) = \min\{s,t\}$ for all s and t. This implies that $C^\theta(t|t)$ must be on average approximately 1/2 between $u-\epsilon$ and $u+\epsilon$ for large θ. This argument can be applied to any interval in the interior. The assumption that $C^\theta(u|u)$ is increasing is used to show that $C^\theta(u|u)$ must converge to 1/2 uniformly at every point. It may not necessary.[13]

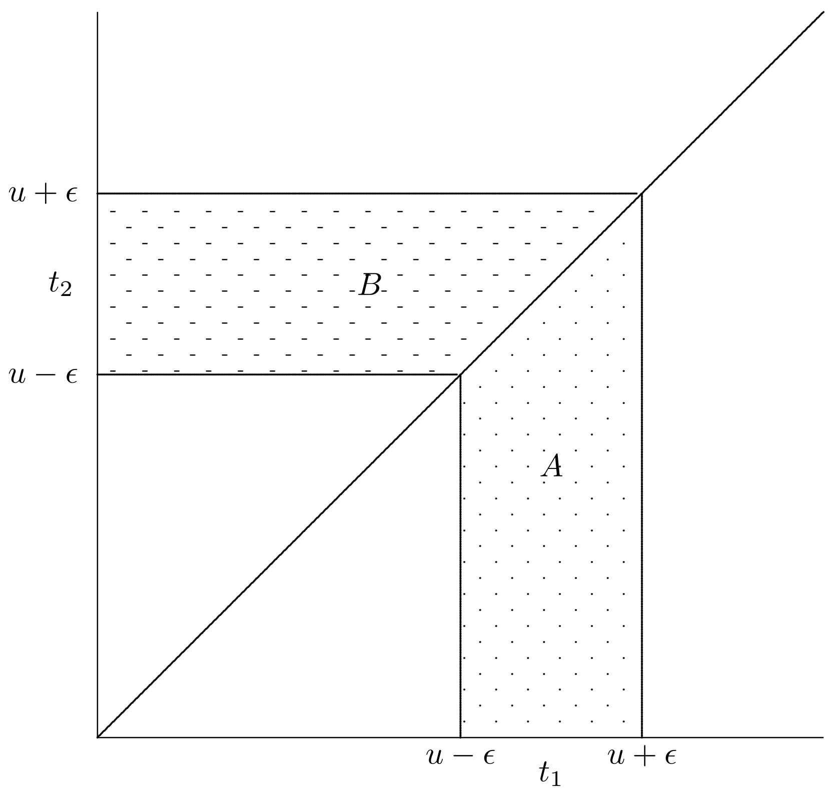

Figure 3

[13] In the language of functional analysis without it the argument shows that $C^\theta(u|u)$ converges weakly to the constant function 1/2, that is $\int_S C^\theta(u|u)\, du \to \int_S 1/2\, du$ for every interval, and by an approximation argument any set S, in the interior of $[0,1]$. In general weak convergence does not imply pointwise convergence.

The result of the theorem follows immediately from this in the case of monotone equilibria. The proof is completed by using the results of Van Zandt and Vives (2007) to show that one can bound play in all equilibria by that in symmetric monotone equilibria.

The lemma is illustrated for the Clayton copula $C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ in the figure below. As $\theta \to 0$ signals become independent and as $\theta \to \infty$, perfect dependence is approached. Note that $C^{\theta}(1|1) = 1$ for all θ, so convergence does not place at the endpoints, which is the reason for the restriction to compact subsets of the interior:

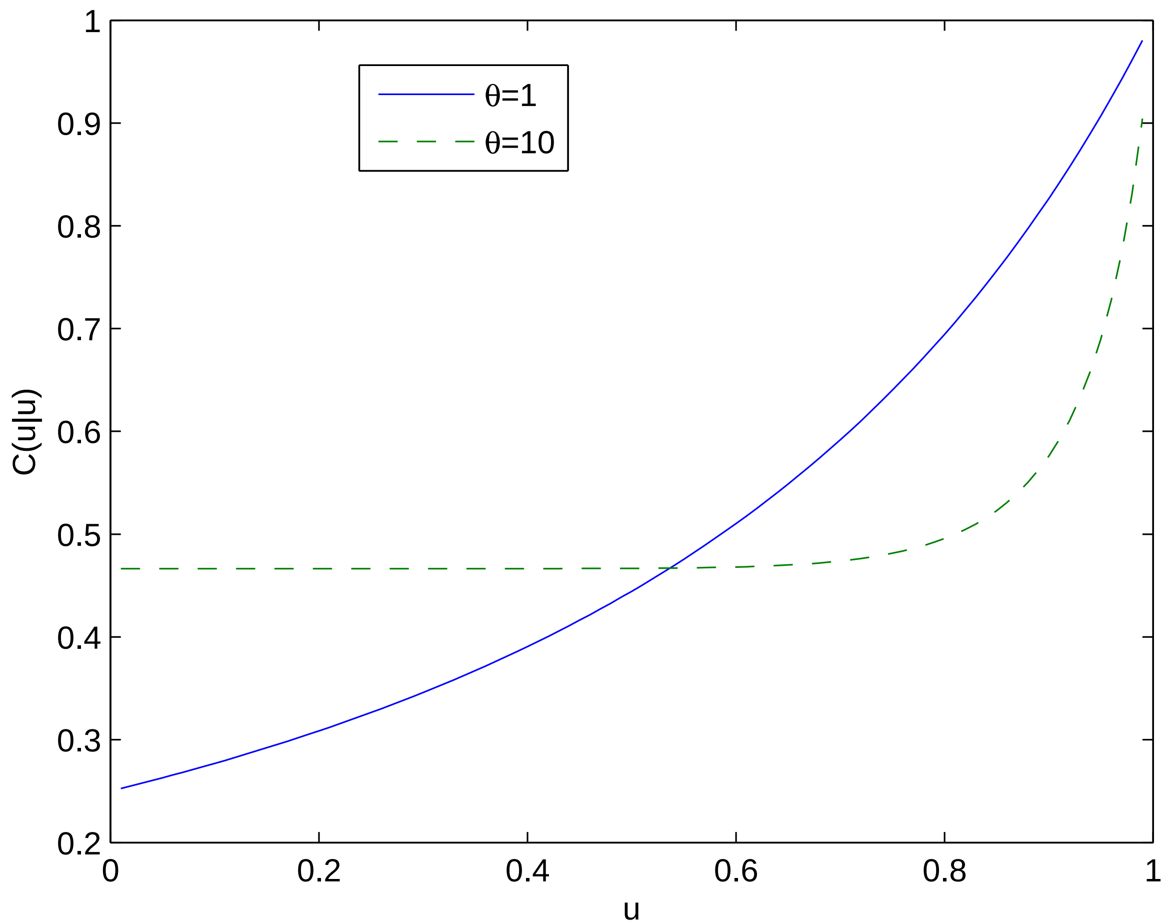

Figure 4

One can also allow payoffs to depend on θ:

Assumption 7 *Players have utility functions $U_i^{\theta}(a_1, a_2, t_i)$, $i = 1, 2$, which satisfy Assumption 1 and Assumption 2 for each θ. They converge uniformly* *as $\theta \to \infty$ to functions U_i satisfying Assumption 1, Assumption 2 and Assumption 4.*

Theorem 2 *Theorem 1 holds under Assumption 7.*

The proof in the Appendix is an immediate generalisation of that for Theorem 1 since payoffs under U^θ are arbitrarily close to those under U_i for large θ.

The solution concept used is that of ex ante Bayesian equilibrium. One would obtain the same result if one looked at the set of rationalizable strategies or the set of strategies not eliminated by iterated deleted of strictly dominated strategies. Milgrom and Roberts (1990) show that the least and greatest equilibria in a supermodular game, such as the current game, bound the set of serially undominated strategies and a fortiori the set of rationalizable strategies.

Corollary 1 *Theorem 1 and Theorem 2 remain true if 'in any equilibrium' is replaced by 'in any strategy surviving iterated deletion of strictly dominated strategies' or 'in any rationalizable strategy'.*

6. Non-Uniform Marginals

As noted earlier, the assumption that marginal distributions are uniform is in a sense without loss of generality. For completeness, this section shows explicitly how to transform the case with non-uniform marginals to one with uniform marginals, so the results of the previous section can be applied. It also shows that the marginal distributions may be allowed to vary.

Assumption 8 *Each player's signal has marginal distribution function G, which is strictly increasing and continuous on its support. If the support is unbounded then players payoff functions are bounded.*

The requirement that G is strictly increasing on its support and continuous rules out atoms and implies that the support is connected. Since G is strictly increasing, the game where players observe signals S_i with marginal distribution G is equivalent to one where they observe signals $T_i = G(S_i)$ with uniform marginals and appropriately transformed joint distribution — this is in essence the content of Sklar's theorem.

In this transformed game player i has utility function

$$\tilde{U}_i(a, t_i) = U_i(a, G^{-1}(t_i)) \tag{20}$$

If the support of G is unbounded above then $G^{-1}(1) = \infty$, so $\tilde{U}_i(a, 1)$ is not defined and similarly for for $\tilde{U}_i(a, 0)$ if it is unbounded below. One can, however, assign arbitrary values in these cases. In the unbounded case some additional assumption is required to ensure expectations are finite, for which payoff functions being bounded is sufficient.

If U_i satisfies the assumptions in Section 2 , then so will $\tilde{U}_i$ (with different cutoffs in Assumption 4) if the support is compact. The same is true in the non-compact case except that $\tilde{U}_i$ will in general only be continuous in the interior of $[0, 1]$ but this is all that is required since limit dominance guarantees that any equilibria must be strictly in the interior.

One can therefore one can apply Theorem 1 above to $\tilde{U}_i$ and deduce:

Corollary 2 *If Assumption 3 is replaced by Assumption 8 then Theorem 1 holds with the value of t^* satisfying the corresponding version of Lemma 1.*

In terms of checking assumptions note that from (3) that if the joint distribution of players' signals is H^θ then the corresponding copula is defined by

$$H^\theta(x, y) = C^\theta(G(x), G(y)) \tag{21}$$

One can, however, check some of the assumptions on C^θ by looking directly H^θ. For example since G is strictly increasing $\Pr^\theta(T_2 \leq v | T_1 = u)$ equals $\Pr^\theta(S_2 \leq G^{-1}(v) | S_1 = G^{-1}(u))$, so C^θ satisfies first-order stochastic dominance if and only if H^θ does. Similarly $\Pr^\theta(T_2 \leq u | T_1 = u)$ is increasing in u if and only if $\Pr^\theta(S_2 \leq s | S_1 = s)$ is increasing in s under H^θ and the characterisation that this is equivalent to the distribution of $\max\{S_1, S_2\}$ dominating that of the marginal distributions of S_1 and S_2 in the likelihood ratio order remains valid.[14]

A one-one transform of signals does not alter the information received by the players. In this sense the restriction to uniform marginals is simply a convenient normalization. Furthermore applying different transformations to different

[14] Convexity is not however invariant to monotone transformations of the dependent variables, so $H^\theta(s, s)$ need not be convex in s even if $C^\theta(u, u)$ is convex in u.

players also preserves the information structure. It follows that one can allow different marginal distributions, G_i, for different players provided the transformed utility functions

$$\tilde{U}_i(a, t_i) = U_i(a, G_i^{-1}(t_i)) \tag{22}$$

obey the assumptions above and in particular are symmetric. Symmetry is therefore best understood as a requirement when the marginals have been transformed to common form.

In the case of additive noise, discussed in the example, the support of signals and the marginal distribution of signals may depend on θ: G^θ. One can allow for the marginal distributions changing:

Assumption 9 *For each θ, each player's signal has marginal distribution function G^θ, which is strictly increasing and continuous on its support, and G^θ converges pointwise to G, which is also strictly increasing and continuous on its support. If the support of any of the G^θ is unbounded, payoff functions are bounded.*

Note that if a sequence of distribution functions converges pointwise to a continuous distribution function then the convergence is in fact uniform.[15] By using the device of (20), one can therefore deduce the result under Assumption 9 from Theorem 2:

Corollary 3 *Theorem 1 holds with Assumption 9 replacing Assumption 3.*

7. Generalizations and Discussion

The first sub-section shows how the results can be extended to the case of common values. The second compares the assumptions made with those of Carlsson and van Damme (1993). The final subsection discusses other possible extensions.

7.1 Common Values

The analysis so far has assumed that valuations are private, in other words that players know their own valuation. Much of the literature on global games

[15] See for example Billingsley (1995) Exercise 14.8.

has concentrated on the case of common values, though Morris and Shin (2006) is a notable exception. The proof technique used, however, is to argue that when noise is small, players almost know their valuation, so a common values game is approximately a private values game — see for example Frankel et al. (2003). One can perform a similar exercise here.

Suppose that players payoffs depend on an unobserved parameter, τ, so that their payoff functions are $\mathcal{U}_i(a,\tau)$ and $\Delta\mathcal{U}_i(a_j,\tau)$ denotes the incremental payoff to playing action 1 rather 0 given the other player's action. Players do not observe τ directly but receive signals t_i as before. The joint distribution of (t_1,t_2) is assumed to satisfy Assumption 3, Assumption 5 and Assumption 6.

Assumption 10 *The game is symmetric:* $\mathcal{U}_1(a_1,a_2,\tau) = \mathcal{U}_2(a_2,a_1,\tau)$ *for all* a_1, a_2 *and* τ. $\mathcal{U}_i$ *is continuous in* τ *for each* a. $\Delta\mathcal{U}_i(1,\tau) \geq \Delta\mathcal{U}_i(0,\tau)$ *for each* τ *and* $\Delta\mathcal{U}_i(a_j,\tau)$ *is strictly increasing in* τ *for each* a_j .

Assumption 11 *There exist* $\underline{\tau}$ *and* $\overline{\tau}$ *such that action 0 is dominant for each player if* $\tau < \underline{\tau}$ *and action 1 is dominant if* $\tau > \overline{\tau}$.

The obvious version of Lemma 1 holds:

Lemma 3 *Under Assumption 10 and Assumption 11 there is a unique value* τ^* *such that if player* i *observes* τ *and believes that there is a 50:50 chance that player* j *will play action 0 or action 1, he strictly prefers to play action 0 if* τ *is below* τ^**, strictly prefers action 1 if it is above and is indifferent if* $\tau = \tau^*$:

$$\frac{1}{2}\Delta\mathcal{U}_i(0,\tau^*) + \frac{1}{2}\Delta\mathcal{U}_i(1,\tau^*) = 0 \tag{23}$$

Let $\mathcal{W}_i^\theta(a,(t_1,t_2)) = E(\mathcal{U}_i(a,\tau)|(t_1,t_2))$ be the expected value of $\mathcal{U}_i$ conditional on players' signals. $\Delta\mathcal{W}_i^\theta$ denotes the incremental payoff to player i from playing action 1 rather than 0 given the other player's action. To ensure the game is supermodular assume that:

Assumption 12 *For each* i, $\Delta\mathcal{W}_i^\theta(1,\mathbf{t}) \geq \Delta\mathcal{W}_i^\theta(0,\mathbf{t})$ *for all* $\mathbf{t} = (t_1,t_2)$ *and* $\Delta\mathcal{W}_i^\theta(a_j,\mathbf{t})$ *is increasing in* t_1 *and* t_2 *for each* a_j.

Given Assumption 10 a sufficient primitive condition for this assumption is that for each θ, τ, t_1 and t_2 are affiliated.[16] Note that affiliation is preserved

[16] See for example Appendix D of Krishna (2002).

if random variables are subject to strictly increasing transformations, so in particular is preserved if T_1 and T_2 are transformed to have uniform marginals. Affiliation also implies that $\Pr(T_2 \leq v|T_1 = u)$ is decreasing in u, that is $C(v|u)$ is decreasing in u.

Under these assumptions, for each θ the game is a symmetric supermodular game, so one can apply the methods of the previous section. If each player adopts a cutoff of k between their actions then the expected difference in payoffs between the actions of player i if he receives signal t_1 is

$$\int_0^k \Delta\mathcal{W}_i^\theta(0,(t_1,t_2))\, c^\theta(t_2|t_1)dt_2 + \int_k^1 \Delta\mathcal{W}_i^\theta(1,(t_1,t_2))\, c^\theta(t_2|t_1)\, dt_2 \tag{24}$$

If τ, t_1 and t_2 become almost perfectly dependent as $\tau \to \infty$ it is plausible that for large θ, this is approximately

$$\Delta\mathcal{U}_i(0,t_i)C^\theta(k|t_i) + \Delta\mathcal{U}_i(1,t_i)(1 - C^\theta(k|t_i)) \tag{25}$$

as each player can almost act as if $t_i = \tau$.

Assumption 13 *(24) converges uniformly in k and t_i to (25) for each i as $\theta \to \infty$.*

Since $\mathcal{U}_i$ is continuous a sufficient but not necessary condition for this is that for each $\delta > 0$ there is θ' such that for all $\theta \geq \theta'$, $(\tau - t_1)^2 + (\tau - t_2)^2 < \delta$. For example in the literature on global games it is usually assumed (see for example Carlsson and van Damme (1993)) that

$$t_i = \tau + \sigma u_i \tag{26}$$

where u_i are independent of τ and have bounded support, which implies[17] the condition (taking $\theta = 1/\sigma$). One therefore obtains (see Appendix):

Theorem 3 *Under Assumptions 3, 5, 6, 10, 11, 12 and 13 for any $\delta > 0$ there exists θ' such that for all $\theta \geq \theta'$, in any equilibrium, player i plays action 0 if $t_i \leq \tau^* - \delta$ and action 1 if $t_i \geq \tau^* + \delta$.*

[17]This is true even if the signals are transformed to have uniform marginals.

7.2 Comparison with Carlsson and van Damme (1993)

As noted in Section 2, Carlsson and van Damme (1993) assume an information structure of the form

$$t_i = t + \sigma\epsilon_i \tag{27}$$

where the ϵ_i have a joint density which is independent of the density of t. The joint marginal joint density of t_1 and t_2 is

$$h^\sigma(t_1, t_2) = \int_{-\infty}^{\infty} g(t)\frac{1}{\sigma^2} f\left(\frac{t_1 - t}{\sigma}, \frac{t_2 - t}{\sigma}\right) dt \tag{28}$$

where g is the density of t, strictly positive on its support, and f the joint density of the ϵ_i, and the cumulative distribution function of t_1 and t_2 is

$$H^\sigma(t_1, t_2) = \int_{\infty}^{\infty} g(t)F\left(\frac{t_1 - t}{\sigma}, \frac{t_2 - t}{\sigma}\right) dt \tag{29}$$

where F is the cumulative distribution function of the ϵ_i. The marginal distribution of t_i is

$$K^\sigma(t_i) = \int_{-\infty}^{\infty} g(t)F_i\left(\frac{t_1 - t}{\sigma}\right) dt \tag{30}$$

where F_i is the marginal distribution of ϵ_i. The copula corresponding to H^σ can be found using (5).

As $\sigma \to 0$, the signals become almost perfectly dependent (and equal to t) and, as shown in the Appendix, it is straightforward to check that Assumption 6 is satisfied by this structure:

Lemma 4 *The copula corresponding to H^σ satisfies Assumption 6 with $\theta = 1/\sigma$.*

If F is symmetric then so is H^σ and the corresponding copula. Parts (ii) and (iii) of Assumption 5 need not be satisfied by an additive error structure. If the density f is affiliated, in particular if the errors are independent, and log-concave in each argument then (t, t_1, t_2) is affiliated, so (t_1, t_2) is affiliated[18] and hence the conditional distribution of T_2 given T_1 is stochastically increasing, so part (ii) is satisfied. Part (iii) is satisfied if the errors ϵ_i are independent and identically distributed and g is log-concave (proof in Appendix). To summarise:

[18]See Milgrom and Weber (1982) Theorem 3.

Lemma 5 *(a) Part (i) of Assumption 5 is satisfied if F is symmetric. (b) Part (ii) is satisfied if f is affiliated and log-concave in each argument. (c) Part (iii) is satisfied if ϵ_1 and ϵ_2 are independent and identically distributed and g has a log-concave density.*

Note that these are sufficient not necessary conditions. They are fairly mild but the additive case is not fully covered by the results in the paper. On the other hand, the results apply to cases where the signal structure is not additive.

7.3 Other Extensions

Carlsson and van Damme (1993) also consider the case of asymmetric games. The notion of risk-dominance is arguably less compelling for such games. To extend the analysis to such games one would need to generalize their full result in equation (1). It is easy to see that if one drops symmetry, the argument in Section 4 implies the result on the 45° line. If one assumes that the density goes to zero at at uniform rate outside neighborhoods of the diagonal, then one will obtain (1) off the diagonal — in essence if one player receives a signal she know that it is almost impossible that the other player has received a higher signal. One would need to add a condition to link the two cases together so that convergence is uniform over compact subsets of the interior of the unit square. Given this the analysis could be extended to asymmetric games.

The results assume that payoff functions are supermodular. Some relaxation of this assumption is possible if one restricts attention to monotone equilibria. As noted by Athey (2001) a number of assumptions will lead to the existence of monotone equilibria. For example one could assume that payoff functions are log-supermodular in types and actions and strengthen Assumption 5(ii) by assuming signals are affiliated. The argument for Theorem 1, restricted to monotone equilibria, then would go through as before. One would, however, need some additional argument to rule out non-monotone equilibria or to extend the results to include these.

8. Conclusion

The paper has extended the analysis of Carlsson and van Damme (1993) by relaxing their assumption of additive noise. It has done so in the simplest context — two player, binary action, symmetric games — but this is the setting in which risk-dominance has greatest appeal and has been used most in applications.

Appendix

Proof of Theorem 1

The game satisfies the assumptions of Van Zandt and Vives (2007). It follows that there are least and greatest equilibria and these are monotone. Since the game is symmetric these must be symmetric. For if, say (k_1^*, k_2^*) is the least equilibrium in cutoff strategies, then (k_2^*, k_1^*) must also be the least equilibrium by symmetry. Hence $k_1^* = k_2^*$. To prove the result it is therefore sufficient to show that any symmetric equilibrium in cutoff strategies has the desired properties.

The interim payoff to player i at cutoff k from choosing action 1 rather than 0 is

$$C^\theta(k|k)\Delta U_i(0,k) + (1 - C^\theta(k|k))\Delta U_i(1,k) \tag{31}$$

On the other hand, since $\Delta U_i(0,k)$ and $\Delta U_i(1,k)$ are strictly increasing and continuous, so uniformly continuous on $[0,1]$, for any $\delta > 0$, one can find $\epsilon > 0$ so that

$$\begin{cases} \frac{1}{2}\Delta U_i(0,k) + \frac{1}{2}\Delta U_i(1,k) < -\epsilon & k < t^* - \delta \\ \frac{1}{2}\Delta U_i(0,k) + \frac{1}{2}\Delta U_i(1,k) > \quad \epsilon & k > t^* + \delta \end{cases} \tag{32}$$

From Assumption 4, an equilibrium cutoff k must belong to the compact interval $T = [\underline{t}, \overline{t}]$ in the interior of $[0,1]$. Applying Lemma 2, proved below, $C^\theta(k,k)$ is uniformly close to $\frac{1}{2}$ for all $k \in T$ for large enough θ, so applying this to (32), one obtains that for all θ greater than some θ' for $k \in T$

$$\begin{cases} C^\theta(k|k)\Delta U_i(0,k) + (1 - C^\theta(k,k))\Delta U_i(1,k) < 0 & k < t^* - \delta \\ C^\theta(k|k)\Delta U_i(0,k) + (1 - C^\theta(k|k))\Delta U_i(1,k) > 0 & k > t^* + \delta \end{cases} \tag{33}$$

Hence for $\theta \geq \theta'$ any symmetric equilibrium cutoff must belong to $(t^* - \delta, t^* + \delta)$, which proves the result.

Proof Lemma 2

The proof follows the lines indicated in the text. By letting u be its mid-point, any open interval in the interior of $[0,1]$ can be written as $(u - \epsilon, u + \epsilon)$. It will be shown that for any such interval

$$\int_{u-\epsilon}^{u+\epsilon} C^\theta(t|t)\, dt \to \epsilon \tag{34}$$

Since $C^\theta(u,v)$ converges to $M(u,v) = \min\{u,v\}$ the associated measures converge in distribution, so the measure under C^θ of any set whose boundary has measure zero under M converges to that under M (see Billingsley (1995) Theorem 29.1). In particular the areas of the rectangles $C = [0, u-\epsilon] \times [0, u-\epsilon]$ and $D = [0, u+\epsilon] \times [0, u+\epsilon]$ converge to their areas under M, since M is concentrated on the diagonal and is atomless. On the other hand, $A+B = D-C$, so as argued in the text $A+B$ converges to 2ϵ, which, together with symmetry, establishes (34).

Suppose now that $C^\theta(u|u)$ does not converge pointwise to 1/2 for some u in the interior of (0,1). If, say, $\liminf_\theta C^\theta(u|u) = \alpha < 1/2$, then one can find a subsequence θ_n, with $\theta_n \to \infty$ with $\lim_n C^{\theta_n}(u|u) = \alpha$. In particular, given $\delta > 0$ with $\alpha + \delta < 1/2$, $C^{\theta_n}(u|u) \leq \alpha + \delta$ for all n greater than some N. Since $C^\theta(t|t)$ is increasing $C^{\theta_n}(t|t) \leq \alpha + \delta < 1/2$ for all $t < u$ if $n \geq N$. This, however, contradicts (34): consider an interval $(v-\epsilon, v+\epsilon)$, $v + \epsilon < u$ — from what has been shown the left hand-side of (34) is bounded above by $(\alpha+\delta)2\epsilon < \epsilon$ for all $n \geq N$. Similarly if $\liminf_\theta C^\theta(u|u) = \beta > 1/2$, then $C^\theta(t|t) \geq \beta - \delta > 1/2$ for all $t \geq u$ for all sufficiently large θ, which leads to a contradiction of (34). It follows that $\liminf_\theta C^\theta(u,u) = 1/2$. A similar argument shows that $\limsup_\theta C^\theta(u,u) = 1/2$, so $C^\theta(u|u)$ indeed converges to 1/2.

To prove uniform convergence, let $[u,v]$ be a closed interval in the interior of $[0,1]$. Since $C^\theta(t|t)$ converges to 1/2 at u, given $\delta > 0$ one can find θ' such that $C^\theta(u|u) \geq 1/2 - \delta$ for all $\theta \geq \theta'$. Similarly given δ', one can find θ'' such that $C^\theta(v|v) \leq 1/2 + \delta$ for all $\theta \geq \theta''$. Let $\theta^* = \max\{\theta', \theta''\}$. Since $C^\theta(t|t)$ is increasing, $1/2 - \delta \leq C^\theta(t|t) \leq 1/2 + \delta$ for all $t \in [u,v]$ if $\theta \geq \theta^*$. This implies the stated uniform convergence.[19]

Proof of Theorem 2

The proof is almost identical to that of Theorem 1 with ΔU_i^θ replacing ΔU_i. Since U_i^θ converges uniformly to U_i, given $\delta > 0$ one can find ϵ such (32) holds for ΔU_i^θ for all large enough θ. Similarly since Assumption 4 holds for U_i, one can find a compact interval T in the interior of $[0,1]$ such that for all large enough

[19]The result also follows from the standard result that if a sequence of monotonic functions converges uniformly to a monotonic function F at each point of continuity of F then convergence is uniform on each compact interval of continuity of F (see for example Doob (1994) p. 166) but the direct proof is easy, so is given.

θ equilibrium cutoffs must belong to T. The remainder of the proof is as before with ΔU_i^θ replacing ΔU_i.

Proof of Theorem 3

By Assumption 13 interim payoffs can be uniformly approximated by (31), replacing ΔU_i by $\Delta \mathcal{U}_i$. The proof then proceeds as in the proof of Theorem 1 replacing ΔU_i by $\Delta \mathcal{U}_i$.

Proof of Lemma 4

$$H^\sigma(t_1, t_2) = \int_\infty^\infty g(t) F\left(\frac{t_1 - t}{\sigma}, \frac{t_2 - t}{\sigma}\right) dt \tag{35}$$

Now

$$F\left(\frac{t_1 - t}{\sigma}, \frac{t_2 - t}{\sigma}\right) \to \begin{cases} 1 & t < \min\{t_1, t_2\} \\ 0 & t > \min\{t_1, t_2\} \end{cases} \tag{36}$$

as $\sigma \to 0$ since $\lim_{x \to -\infty} F(x, y) = \lim_{y \to -\infty} F(x, y) = 0$ for any x,y and $\lim_{x \to \infty, y \to \infty} F(x, y) = 1$ as F is a joint distribution function.[20] In the case $t_2 < t < t_1$ (or mutatis mutandis with inequalities reversed) note that $F(x, y) \leq F(x, \infty)$ and $\lim_{x \to -\infty} F(x, \infty) = 0$ as $F(x, \infty)$ is the marginal distribution of the first component.

So by the dominated convergence theorem

$$\lim_{\sigma \to 0} H^\sigma(t_1, t_2) = \int_{-\infty}^{\min\{t_1, t_2\}} g(t)\, dt = G(\min\{t_1, t_2\}) = \min\{G(t_1), G(t_2)\} \tag{37}$$

The last equality holds since G is increasing.

Now

$$K_i^\sigma(t_i) = \int_{-\infty}^\infty g(t) F_i\left(\frac{t_i - t}{\sigma}\right) dt \tag{38}$$

So by a similar argument to the above

$$\lim_{\sigma \to 0} K^\sigma(t_i) = G(t_i) \tag{39}$$

Now G is strictly increasing on its support, as g is strictly positive, and so it follows that G^{-1} is continuous. Hence it follows from van der Vaart (1998) p. 305 Lemma 21.2 that

$$\lim_{\sigma \to 0} (K^\sigma)^{-1}(t_i) = G^{-1}(t_i) \tag{40}$$

[20] See for example Billingsley (1995) p. 260.

where $(K^\sigma)^{-1}$ is defined as in (4). Since from (5)

$$C^\sigma(t_1, t_2) = H^\sigma((K^\sigma)^{-1}(t_1), (K^\sigma)^{-1}(t_2)) \tag{41}$$

(37) and (40) imply that

$$\lim_{\sigma \to 0} C^\sigma(t_1, t_2) = \min\{t_1, t_2\} \tag{42}$$

as was to be shown.

Proof of Lemma 5

Parts (a) and (b) were proven in the text. For part (c), note from the discussion in Sections 4 and 5 that part (iii) of Assumption 5 is equivalent to the assertion that the distribution of $S = \max\{T_1, T_2\}$ dominates that of T_1 and T_2 in the likelihood ratio order.

Now

$$s = \max\{t_1, t_2\} = \max\{\epsilon_1, \epsilon_2\} + t \tag{43}$$

under (27).

The density function of $\eta = \max\{\epsilon_1, \epsilon_2\}$ is

$$2h(\eta)H(\eta) \tag{44}$$

where h and H are the density and distribution functions of the ϵ_i (assumed identically distributed). η clearly dominates each ϵ_i in the likelihood ratio order.

According to Theorem 2.1(d) of Keilson and Sumita (1982), if X_1 and X_2 are two random variables such that X_1 dominates X_2 in the likelihood ratio order and ξ has a log-concave density function and is independent of X_1 and X_2 then $X_1 + \xi$ dominates $X_2 + \xi$ in the likelihood ratio order. Taking $X_1 = \eta$ and $X_2 = \epsilon_1$ and $\xi = t$ proves the result since $t_1 = \epsilon_1 + t$ and from (43), $s = \eta + t$.

References

Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69:861–889.

Billingsley, P. (1995). *Probability and Measure*. John Wiley and Sons, New York, third edition.

Carlsson, H. and van Damme, E. (1993). Global games and equilibrium selection. *Econometrica*, 61:989–1018.

Doob, J. (1994). *Measure Theory*. Springer-Verlag, New York.

Frankel, D., Morris, S., and Pauzner, A. (2003). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, 108:1–44.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. Chapman and Hall, London.

Keilson, J. and Sumita, U. (1982). Uniform stochastic ordering and related inequalities. *The Canadian Journal of Statistics*, 10:181–198.

Krishna, V. (2002). *Auction Theory*. Academic Press, San Diego.

Milgrom, P. and Roberts, J. (1990). Rationalizabilty, learning and equilibrium in games with strategic complementarities. *Econometrica*, 58:1255–1277.

Milgrom, P. and Weber, R. (1982). A theory of auctions and competitive bidding. *Econometrica*, 50:1089–1122.

Morris, S. and Shin, H. (2003). Global games: Theory and applications. In Dewatripont, M., Hansen, L., and Turnovsky, S., editors, *Advances in Economics and Econometrics: Theory and Applications, Proceedingsof the Eighth World Congress of the Econometric Society*, pages 56–114, Cambridge. Cambridge University Press.

Morris, S. and Shin, H. (2006). Heterogeneity and uniqueness in interaction games. In Blume, L. and Durlauf, S., editors, *The Economy as a Complex Evolving System, III*, pages 207–242, Oxford. Oxford University Press.

Müller, A. and Scarsini, M. (2005). Archimedean copulae and positive dependence. *Journal of Multivariate Analysis*, 93:434–445.

Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. John Wiley, Chichester.

Nelsen, R. (2006). *An Introduction to Copulas*. Springer, New York, 2nd edition.

Nelsen, R., Quesada-Molina, J., Rodríguez-Lallena, J., and Úbeda-Flores, M. (2008). On the construction of copulas and quasi-copulas with given diagonal sections. *Insurance: Mathematics and Economics*, 42:473–478.

Patton, A. (2009). Copula models for financial time series. In Andersen, T., Davis, R., Kreiss, J.-P., and Mikosch, T., editors, *Handbook of Financial Times Series*, pages 767–786, Berlin. Springer Verlag.

van der Vaart, A. (1998). *Asymptotic Statistics*. Cambridge University Press, Cambridge.

Van Zandt, T. and Vives, X. (2007). Monotone equilibria in bayesian games of strategic complementarities. *Journal of Economic Theory*, 134:339–360.