Equivariant localization in supergravity and holography



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Abstract

In this thesis we use *equivariant localization* to compute observables in supergravity and holography. Equivariant localization is a powerful tool which allows to compute integrals without the need for an explicit metric, relying instead uniquely on global data, as we will review. Applications of this method to supergravity solutions of the form $M \times X$ where M is asymptotically locally AdS, are generally split into two categories: either localization on the spacetime M or on the internal space X. This thesis gives an example in each category. First, we obtain a universal formula for the on-shell action of Euclidean 4d $\mathcal{N} = 2$ gauged supergravity coupled to arbitrary vector matter. Second, we compute the free energy of various classes of 3d SCFTs by localizing on the internal space of $\mathrm{AdS}_4 \times X_6$ $\mathcal{N} = 2$ solutions in massive type IIA. This second application further offers a geometric interpretation of F-maximization in field theory, and the first example of boundary contributions to the localization formula. Finally, we move past the distinction stated above by using equivariant localization directly in ten dimensions: on both the spacetime and internal space simultaneously. This last application recovers some results from the previous chapters and opens the door to many fascinating extensions.

Statement of Originality

This thesis is based on results from the following publications, to which the author contributed substantially:

[1] Pietro Benetti Genolini, Jerome P. Gauntlett, Yusheng Jiao, Alice Lüscher, and James Sparks *Equivariant localization for $D = 4$ gauged supergravity*, JHEP 08 (2025) 211, arxiv:2412.07828

[2] Christopher Couzens and Alice Lüscher *A geometric dual of F-maximization in massive type IIA*, JHEP 08 (2024) 218, arxiv:2406.15547

[3] Christopher Couzens, Florian Gaar, and Alice Lüscher *Fun in massive IIA: localizing punctures and boundaries*, 2026

[4] Christopher Couzens, Alice Lüscher, and James Sparks *Ten-dimensional localization*, arxiv:2607.05529

The author also contributed to [5–11] during her DPhil, which will not be the subject of this thesis but only referred to. Some results in [3] appeared previously in the Master's thesis of co-author Florian Gaar.

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Contents

1 Introduction **2**

1.1 Basics of holography 2

1.1.1 Holographic observables 3

1.2 Equivariant localization 6

1.2.1 BVAB theorem 7

1.2.2 Boundary contributions 9

1.2.3 Application in supergravity 11

1.3 Outline of the thesis 12

2 Localization in four-dimensional gauged supergravity **15**

2.1 $\mathcal{N} = 2$ gauged supergravity in $D = 4$ 16

2.1.1 Lorentzian theory 17

2.1.2 Euclidean theory 20

2.1.3 Bilinears 22

2.1.4 Equivariantly closed forms 23

2.2 Evaluation of the action 26

2.2.1 Reality conditions 26

2.2.2 Using the BVAB theorem 27

2.2.3 Preliminaries 29

2.2.4 Action contributions from nuts 32

2.2.5 Action contributions from bolts 35

2.2.6 Gravitational free energy 37

2.3 Flux integrals, UV-IR relations, and holography 38

2.3.1 Flux integrals 38

2.3.2 UV-IR relations 39

2.3.3 Holography 42

2.4 Real solutions 44

2.4.1 The STU model . . . 45
2.4.2 Deformations of Euclidean AdS_4, $A^I = 0$ . . . 46
2.4.3 Deformations of Euclidean AdS_4, $A^I \neq 0$ . . . 50
2.4.4 Taub-bolt saddle solutions . . . 52
2.5 Complex solutions and black holes . . . 57
2.5.1 Black holes and indices . . . 58
2.5.2 Topologically twisted black holes . . . 59
2.5.3 Rotating black holes . . . 61
2.6 Orbifold solutions . . . 65
2.6.1 Spindle solutions . . . 66
2.6.2 Seifert three-manifolds . . . 69
2.7 Discussion . . . 71

3 Internal space localization in massive type IIA **73**
3.1 General AdS_4 solutions of massive type IIA . . . 74
3.1.1 Setup . . . 75
3.1.2 Degenerations and global regularity . . . 77
3.1.3 Polyforms . . . 81
3.1.4 Localization . . . 84
3.2 Examples . . . 86
3.2.1 D4-branes wrapped on a Riemann surface . . . 86
3.2.2 D4-branes wrapped on a spindle . . . 90
3.2.3 HS^2 bundle over B_4 . . . 94
3.2.4 Cylinder bundle with O4 and O8 . . . 98
3.2.5 S^2 bundle over B_4 . . . 100
3.2.6 Suspension of a Sasaki–Einstein manifold . . . 105
3.3 Discussion . . . 109

4 Ten-dimensional localization **112**
4.1 Ten-dimensional polyforms . . . 113
4.1.1 Preserving supersymmetry in ten dimensions . . . 114

4.1.2 Constructing the polyforms . . . 116
4.1.3 Partially on-shell action . . . 118
4.2 10 = 6+4: Romans supergravity coupled to a vector . . . 121
4.2.1 Isolated fixed points . . . 124
4.2.2 Fixed surfaces . . . 126
4.2.3 Fixed B_4 . . . 128
4.3 10 = 4 + 6: D2-D8 Chern-Simons theories . . . 131
4.3.1 Isolated fixed points . . . 133
4.3.2 Fixed surfaces . . . 135
4.4 Discussion . . . 139

5 Outlook **141**

A Bilinear equations in 4d gauged supergravity **143**
A.1 Killing spinor equations . . . 143
A.2 Bilinear relations . . . 143
A.3 Some useful supergravity relations . . . 145
A.4 Equivariantly closed forms . . . 146

B Minimal 4d gauged supergravity **149**

C Some cohomology and homology **151**
C.1 Homology relations used in chapter 3 . . . 151
C.2 Homology relations used in chapter 4 . . . 154

D Details of the 10d derivations **156**
D.1 Obtaining the polyforms . . . 156
D.2 Putting the action partially on-shell . . . 158

Bibliography **160**

Conventions

Throughout the thesis we will use a set of symbols whose meaning is implied. Here we present a list of these.

- vol_Y denotes the volume form of a space Y, Vol_Y its total volume, and $\text{Vol}_Y(b_i)$ its Sasakian volume.
- S^d denotes the d-dimensional sphere.
- We use D for the dimension of AdS and $d = D - 1$ for the dimension of the CFT
- $\mathcal{F}$ denotes a prepotential function and $\mathcal{F}^I$ its derivative.
- $F^I = \mathrm{d}A^I$ denote the field strengths in 4d.
- The Maxwell RR fluxes are denoted by F^{RR}.
- The Page p-fluxes are denoted by F_p and the polyform containing them all by F.
- F also denotes a CFT free energy (where the CFT is specified contextually). It should be clear from the context that it is not the F from the previous point
- F_{S^d} denotes the free energy of the CFT dual to $\text{EAdS}_D \times S^{10-D}$ (i.e. placed on the conformal boundary S^d, $d = D - 1$).
- Euclidean actions are denoted with the letter I and Lorentzian with S.
- We will use M_D to denote the D-dimensional spacetime manifold and X_{10-D} for the internal space manifold.
- G_D denotes the effective Newton constant in D dimensions.
- We set $L_{\text{AdS}} = 1$ and $g_s = 1$.

Clashes of nomenclature will be avoided as much as possible but there are a few instances where the same symbol is used for different objects. If unclear from the context, it will be made explicit on a case by case basis which object is intended. In particular, we apologize for the plethora of F's and hope the above makes their respective use clearer.

Chapter 1

Introduction

1.1 Basics of holography

One of the most fascinating conceptual developments in recent years is *holography*, which conjectures the equivalence between a theory of gravity and a quantum theory in one less dimension and offers the perfect playground to study quantum gravity [12, 13]. A realization of the holographic principle is the AdS/CFT correspondence, where the gravity theory is string theory on anti de Sitter (AdS) space and the corresponding quantum theory is a supersymmetric conformally invariant field theory (SCFT). Such an equivalence between two theories is also known as a *duality*. In practice, for the calculations to be more explicit and tractable, one typically considers the classical limit of string theory, known as *supergravity*, for the gravity theory. Holography then implies that results in supergravity translate into predictions for the dual quantum theories at strong coupling, and vice-versa [14].

The canonical example of holography is the $\mathrm{AdS}_5 \times S^5$ solution in type IIB being dual to 4d $\mathcal{N} = 4$ super-Yang–Mills proposed in [15] and refined in [16, 17]. More precisely, this duality arises by considering a stack of N coincident D3-branes. The gravity theory is defined on the backreacted geometry, while the field theory lives on the worldvolume of the branes and admits a $\mathrm{SU}(N)$ gauge symmetry. In particular, the near-horizon geometry admits an AdS factor and the dual field theory is conformal. Note that the symmetry group of AdS_5 matches the conformal group in 4 dimension $\mathrm{SO}(2, 4)$, while the symmetry of the sphere $\mathrm{SO}(6)$ corresponds to the R-symmetry group of $\mathcal{N} = 4$ SYM. Finally, when N is "large", or rather $N \to \infty$, referred to as large-N limit, stringy effects are suppressed, such that the supergravity approximation of string theory becomes trustworthy. On the supergravity side, N corresponds to the units of F_5 flux through the S^5 and counts the number of branes making up the background. In the dual field theory, strings stretching between these N branes lead to an $\mathrm{SU}(N)$ gauge symmetry.

The AdS/CFT correspondence actually extends beyond this example. For instance, simply replacing the S^5 by a generic Sasaki–Einstein five-manifold gives rise to a rich class of four-dimensional dual SCFTs [18, 19]. Another possibility is to consider a stack of M2- or M5-branes in M-theory, leading respectively to $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ supergravity backgrounds dual to the 3d ABJM theory [20] and the 6d $\mathcal{N} = (2, 0)$ non-Lagrangian theory [21,22]. More generally, one can consider supergravity solutions of the form $M_D \times X_{10-D}$.[1] On one hand, M_D is referred to as space-time, it is asymptotically locally AdS_D, and the $(D-1)$-dimensional dual CFT lives on its boundary ∂M_D. Therefore different choices of topology for M_D places the CFT on various non-trivial backgrounds. On the other hand, X_{10-D} is referred to as the internal space, and its topology encodes important properties of the dual CFT. This correspondence between internal geometry and field-theoretic data is a central theme of holography.

1.1.1 Holographic observables

Once a holographic pair is identified, the next step is to obtain *observables*. In our context, observables are physically meaningful quantities that can be computed from supergravity solutions and compared with corresponding quantities in the dual field theory. Examples include on-shell actions, conserved charges, and black hole entropies.

More precisely, an observable we are going to compute repeatedly in the following is the dual CFT *free energy*, which is defined as the log of its partition function, and is captured holographically by the Euclidean gravitational on-shell action

$$F = -\log Z = I_{\mathrm{Grav}} \,. \tag{1.1.1}$$

This provides a measure of the number of degrees of freedom of the theory. The gravity effective action takes the form

$$I_{\mathrm{Grav}} = -\frac{1}{16\pi G_D} \int_{M_D} \mathrm{vol}_{M_D} (R + \dots) \,, \tag{1.1.2}$$

[1] The total dimension is 10 for string theory and 11 for M-theory such that we would consider X_{11-D} for the latter. However, this thesis focuses on examples in string theory.

where the dots encode the matter content of the theory under consideration, and G_D is the effective Newton constant in D dimensions

$$\frac{1}{G_D} = \frac{1}{G_{10}} \int_X \widetilde{\text{vol}}_X \,, \quad G_{10} = 8\pi^6 \ell_s^8 \,, \tag{1.1.3}$$

where the tilde on the volume denotes the possibility of it being a warped volume (so including powers of the warp factor), and the ten-dimensional Newton constant can be expressed in terms of the string length ℓ_s.[2]

There are then two classes of examples one can consider in holography: one typically either focuses on fluctuations about the AdS factor for a fixed internal space, or keeps a fixed AdS factor and investigates properties related to the internal space topology (by "fixed" here we mean nonfluctuating, decided from the start, and ignored in the computations). Let us discuss them separately.

Firstly, let us specify the internal space topology and look at fluctuations of the AdS factor. A common approach is to perform a *consistent truncation* on the internal space, yielding a lower-dimensional effective supergravity theory [23]. This procedure gives rise to the effective Newton constant in (1.1.3) and dictates the matter content hidden in the dots in (1.1.2). The aim then is to perform the integral over M_D in (1.1.2) to obtain the on-shell action. Recall that M_D needs to be asymptotically locally AdS_D, and that the dual CFT lives on its boundary ∂M_D. The simplest instance of $M_D \equiv \text{AdS}_D$ simply gives rise to the free enery of the CFT on S^d. In the following, we will not perform any truncation, but rather start directly from the effective theory. In particular, this is the class of examples we consider in chapter 2.

More interestingly one can consider AdS black holes in these effective supergravity theories (i.e. considering M_D to have a black hole topology). The Euclidean on-shell action computes the thermodynamic free energy of the black hole, from which the *entropy* is obtained by a Legendre transform. Holographically, this computes the free energy of the dual field theory on the non-trivial background ∂M_D, which is typically referred to as an *index*. Note that by definition the entropy counts the number of quantum states that compose a system. From

[2]Note that we set the AdS radius L_{AdS} and the string coupling $g_s = 1$.

the black hole perspective, its entropy is proportional to the area of the horizon [24,25], and it is unclear what states it may be counting. On the other hand, the index provides a counting and interpretation of the quantum states from the field theory. This microstate counting is one of the great success of holography, see [26] for a review.

Secondly, one can study a wide variety of CFTs by considering various topologies for the internal space while keeping a fixed AdS factor. In this setup, the sphere partition function of the CFT is computed by the (warped) volume of the internal space.[3] Indeed, combining the relations (1.1.1)-(1.1.3) gives

$$F = \frac{2\pi}{(2\pi\ell_s)^8} I_{\text{AdS}}^{\text{ren}} \int_X \widetilde{\text{vol}}_X \,. \tag{1.1.4}$$

More precisely, since the AdS factor is held fixed, the renormalized Euclidean on-shell action of the AdS vacuum $I_{\text{AdS}}^{\text{ren}}$ is a universal constant. The dependence on the compactification manifold enters solely through the effective Newton constant obtained after integrating over the internal space. Consequently, at two-derivative level the free energy is proportional to the warped volume of the internal manifold. In particular, this is the class of examples we consider in chapter 3.

Further observables can be extracted from the internal space topology. For instance, the fluxes are quantized through cycles of the internal space, say a p-flux through a p-cycle Γ_p^a

$$\frac{1}{(2\pi\ell_s)^{p-1}} \int_{\Gamma_p^a} F_p \equiv N_a \in \mathbb{Z} \,. \tag{1.1.5}$$

These integers typically count the number of branes sourcing such flux i.e. D_{p-2} branes in the supergravity picture, while it therefore corresponds to some gauge or flavor symmetry group in the field theory. Another example is the conformal dimension of certain BPS operators. Some CFT operators are dual to branes wrapped on calibrated cycles in the internal space. Their conformal dimension is then given by the (warped) volume of these cycles [27, 28]

$$\Delta(\Sigma_p) = \frac{1}{(2\pi\ell_s)^{p+1}} \int_{\Sigma_p} \widetilde{\text{vol}}_{\Sigma_p} \,. \tag{1.1.6}$$

[3] The central charge of the CFT at large N can be computed similarly.

Finally, one could imagine working directly with the ten-dimensional solution rather than in one of the two splits between AdS or internal space we just explained. We explore this in chapter 4.

1.2 Equivariant localization

An issue that one quickly runs into when doing computations in supergravity and holography is that the supergravity equations of motion are notoriously difficult to solve in closed form (second order non-linear PDE). Finding explicit, analytic solutions is an art and often requires one to impose large amounts of symmetry, as such the most general solutions are often beyond current techniques. This is problematic since the observables defined above are given by integrals, which seemingly require the knowledge of an explicit solution (and explicit volume form) to perform. Therefore, in order to obtain results, new computational techniques are needed.

Recently, *equivariant localization* has been proposed as a powerful method to compute observables in supergravity [29]. This method relies on a deeper understanding of the supergravity equations combined with a mathematical theorem, known as BVAB theorem [30, 31], to circumvent the difficulties involved in solving the supergravity equations of motion by completely bypassing the need to solve them altogether. Indeed, the theorem allows one to perform integrals over spacetime, knowing only the topological data (global shape of the space) rather than an explicit metric (local details of the space).

Schematically, the BVAB theorem recasts n-dimensional integrals of n-forms into lower dimensional contributions (such as scalar quantities Φ_0, but more generally lower dimensional integrals) at some "special points" of the manifold:

$$\text{Observable} = \int_{\mathcal{M}_n} \Phi_n = \text{“} \sum_{\substack{\text{special} \\ \text{points}}} \Phi_0 \text{”} , \tag{1.2.1}$$

where the quotation marks express that it is merely a schematic statement. Ultimately this method allows us to compute observables without the knowledge of an explicit solution, only assuming it exists. In the next subsection, we give the precise statement of the theorem

making clear how these lower dimensional contributions are related to Φ_n and what makes the subspaces contributing "special".

1.2.1 BVAB theorem

Before we start to apply localization to our setup it is instructive to first review the formalism. One additional point that we wish to emphasize is the modification of the usual localization formulae in the presence of a boundary. This section closely follows [29,32], which initiated the idea of applying localization to supergravity, adding in a discussion on boundary contributions [2, 33]. See also [34, 35] for an introduction to equivariant localization.

Consider a d-dimensional manifold $\mathcal{M}$ with d even, equipped with a U(1) Killing vector ξ.[4] We define the *equivariant exterior derivative*

$$\mathrm{d}_\xi \equiv \mathrm{d} - \xi \lrcorner \,. \tag{1.2.2}$$

As the exterior derivative d raises the degree of a form by one, while the contraction with a vector field lowers the degree of a form by one, d_ξ naturally acts on polyforms, a formal object composed of differential forms of different degrees. Moreover, it has the property that $\mathrm{d}_\xi^2 = -\mathcal{L}_\xi$ (the Lie derivative), such that it defines an equivariant cohomology on the space of invariant polyforms Φ. A polyform is said to be *equivariantly closed* if

$$\mathrm{d}_\xi \Phi = 0 \,. \tag{1.2.3}$$

The integral over an invariant even dimensional submanifold $\Gamma \subset \mathcal{M}$ of an equivariantly closed polyform Φ can be evaluated using the BVAB theorem [30, 31], which states that such an integral localizes to the fixed point set of the group action. Explicitly let us denote by $\Sigma \subset \Gamma$ a fixed submanifold of ξ (i.e. $\xi = 0$ on Σ) of codimension $2k$ and $f : \Sigma \hookrightarrow \Gamma$ the embedding of the fixed point locus, then the BVAB theorem gives

$$\int_\Gamma \Phi = \sum_\Sigma \frac{1}{d_{2k}} \int_\Sigma \frac{f^*\Phi}{e_\xi(\mathcal{N})} \,, \tag{1.2.4}$$

[4] For d odd, the statement of the localization theorem changes substantially [36]. One could replace the U(1) action with any compact Lie group action with minimal modifications.

where $e_\xi(\mathcal{N})$ is the equivariant Euler form of the normal bundle and d_{2k} is the order of the orbifold structure group of Σ, meaning that for $N_\Sigma = \mathbb{R}^{2k}/G$ locally, $d_{2k} \in \mathbb{N}$ is the order of the finite group G.

We may simplify the BVAB formula under the assumption that the normal bundle N_Σ of Σ in Γ decomposes as a sum of line bundles $N_\Sigma = \oplus_{i=1}^k L_i$.[5] Then the BVAB formula, with this splitting assumption reduces to

$$\int_\Gamma \Phi = \sum_\Sigma \frac{1}{d_{2k}} \frac{(2\pi)^k}{\prod_{i=1}^k \epsilon_i} \int_\Sigma \frac{f^*\Phi}{[1 + \frac{2\pi}{\epsilon_i} c_1(L_i)]}\,, \tag{1.2.5}$$

where ϵ_i are the weights of ξ on N_Σ, i.e. in local coordinates

$$\xi = \sum_{i=1}^k \epsilon_i \partial_{\varphi_i} \tag{1.2.6}$$

with ∂_{φ_i} rotating L_i, and $c_1(L_i)$ are the first Chern classes. In practice the denominator is to be expanded in a power series in the Chern class, as illustrated below, which truncates at order $\dim(\Sigma)$ at most. In summary, the game to be played is to build equivariantly closed polyforms whose top form is a physical quantity that we want to compute. The physical observable, obtained by integrating the top form over some (sub-)manifold of our internal space, is then evaluated using the BVAB formula. This gives the observable as a sum of contributions of the lower forms evaluated on fixed submanifolds.

To make the BVAB formula a little more explicit let us see how the general fixed point formula (1.2.5) reads for the cases of interest in this thesis. We will consider 2-, 4-, and

[5] This assumption is satisfied by almost all of the examples considered in this thesis. It is used to write the Euler class of N_Σ in terms of first Chern class in the formula (1.2.5). This more general formula will be important in section 3.2.6 where such a decomposition in terms of line bundles is not possible.

6-dimensional manifolds or cycles thereof. The BVAB formula then gives:

$$\begin{aligned}
\int_{\mathcal{M}_2}\Phi_2 &= \sum_{\Sigma_0}\frac{2\pi}{d_0}\frac{\Phi_0}{\epsilon_1}\Big|_{\Sigma_0}\,, \\
\int_{\mathcal{M}_4}\Phi_4 &= \sum_{\Sigma_0}\frac{(2\pi)^2}{d_0}\frac{\Phi_0}{\epsilon_1\epsilon_2}\Big|_{\Sigma_0} + \sum_{\Sigma_2}\frac{2\pi}{d_2}\int_{\Sigma_2}\left[\frac{\Phi_2}{\epsilon_1} - \frac{2\pi\Phi_0}{\epsilon_1^2}c_1(L)\right]\,, \\
\int_{\mathcal{M}_6}\Phi_6 &= \sum_{\Sigma_0}\frac{(2\pi)^3}{d_0}\frac{\Phi_0}{\epsilon_1\epsilon_2\epsilon_3}\Big|_{\Sigma_0} + \sum_{\Sigma_2}\frac{(2\pi)^2}{d_2}\frac{1}{\epsilon_1\epsilon_2}\int_{\Sigma_2}\left[\Phi_2 - 2\pi\Phi_0\Big(\frac{c_1(L_1)}{\epsilon_1}+\frac{c_1(L_2)}{\epsilon_2}\Big)\right] \\
&\quad + \sum_{\Sigma_4}\frac{2\pi}{d_4}\int_{\Sigma_4}\left[\frac{\Phi_4}{\epsilon_1} - \frac{2\pi\Phi_2}{\epsilon_1^2}\wedge c_1(L) + \frac{(2\pi)^2\Phi_0}{\epsilon_1^3}c_1(L)\wedge c_1(L)\right]\,.
\end{aligned} \tag{1.2.7}$$

1.2.2 Boundary contributions

The manifold we want to localize on may also admit boundaries and this requires an extension of the BVAB formula (1.2.5). We make the restriction that the boundary of the manifold does not have any fixed points and let $\imath : \partial\mathcal{M} \hookrightarrow \mathcal{M}$ be the embedding of the boundary into $\mathcal{M}$. Then to the BVAB formula in (1.2.5) we add in an additional boundary contribution [33]:

$$\int_{\mathcal{M}}\Phi = \text{BVAB}(1.2.5) + \int_{\partial\mathcal{M}}\imath^*\frac{\eta\wedge\Phi}{\mathrm{d}_\xi\eta}\,, \tag{1.2.8}$$

where η is the one-form dual to ξ, such that $\xi\lrcorner\eta = |\xi|^2$. We understand the inverse of the polyform $\mathrm{d}_\xi\eta$ via a formal geometric series,

$$\frac{1}{\mathrm{d}_\xi\eta} = -\frac{1}{|\xi|^2(1-|\xi|^{-2}\mathrm{d}\eta)} = -\frac{1}{|\xi|^2}\left[1+\sum_{a=1}^{k-1}\left(\frac{\mathrm{d}\eta}{|\xi|^2}\right)^a\right]. \tag{1.2.9}$$

It is obvious from the formula that the boundary must not have fixed points of the U(1) given the factors of $|\xi|^2$ appearing in the denominator. This was a choice that we made and there exists a more general formula when this is not true, [33].

To better explain the contributions from the boundary let us study an example. Consider a four-sphere and let us put in an arbitrary boundary which preserves a choice of U(1) action

that we will specify shortly. Let us take the following metric on the four-sphere[6]

$$\mathrm{d}s^2 = \mathrm{d}\alpha^2 + \sin^2\alpha\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi_1^2 + \cos^2\theta\mathrm{d}\phi_2^2\big)\,. \tag{1.2.10}$$

For the round four-sphere the ranges of the coordinates are: $\alpha \in [0,\pi]$, $\theta \in [0,\frac{\pi}{2}]$, and $\phi_1,\phi_2 \in [0,2\pi]$. We will localize using the Killing vector

$$\xi = b_1\partial_{\phi_1} + b_2\partial_{\phi_2}\,, \tag{1.2.11}$$

with both b's non-zero. The dual 1-form is given by

$$\eta = \sin^2\alpha\big(b_1\sin^2\theta\mathrm{d}\phi_1 + b_2\cos^2\theta\mathrm{d}\phi_2\big)\,, \tag{1.2.12}$$

such that the norm of this Killing vector field is

$$|\xi|^2 = \sin^2\alpha(b_1^2\sin^2\theta + b_2^2\cos^2\theta)\,, \tag{1.2.13}$$

which clearly vanishes at the poles of the S^4 at $\alpha = 0,\pi$. Rather than taking the round four-sphere we will restrict α to $\alpha \in [0,\alpha_0]$ with $0 < \alpha_0 < \pi$ and call the space X_{α_0}. For $\alpha_0 = \frac{\pi}{2}$ this is the usual four-dimensional hemi-sphere. It is simple to explicitly compute the volume for this metric and we find

$$\mathrm{Vol}(X_{\alpha_0}) = \frac{4\pi^2}{3} + \frac{2\pi^2\cos\alpha_0(\cos\alpha_0^2 - 3)}{3}\,. \tag{1.2.14}$$

We now want to reproduce this using our localization formulae. We have a fixed point at the pole and a boundary term at $\alpha = \alpha_0$. Using the results in [32] the equivariantly closed

[6] We use the same conventions as in [32] in order to use their polyforms.

polyform for the (weighted) volume form is given by $\Phi = \Phi_4 + \Phi_2 + \Phi_0$ with:

$$\begin{aligned}
\Phi_4 &= \frac{1}{b_1 b_2}\text{vol} = \frac{1}{b_1 b_2}\sin^3\alpha \sin\theta\cos\theta \mathrm{d}\alpha\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\,,\\
\Phi_2 &= -\frac{1}{2}\sin^3\alpha\left(\frac{1}{b_1}\sin^2\theta\mathrm{d}\phi_1 + \frac{1}{b_2}\cos^2\theta\mathrm{d}\phi_2\right)\wedge\mathrm{d}\alpha\,,\\
\Phi_0 &= \frac{1}{6}(3\cos\alpha - \cos^3\alpha)\,.
\end{aligned} \tag{1.2.15}$$

The weighted volume is then

$$\begin{aligned}
\int_{X_{\alpha_0}}\frac{1}{b_1 b_2}\text{vol}(X_{\alpha_0}) &= \frac{(2\pi)^2}{b_1 b_2}\Phi_0\Big|_{\alpha=0} + \int_{\partial X_{\alpha_0}} \imath^*\frac{1}{|\xi|^2}\left[\eta\wedge\Phi_2 + \frac{\Phi_0}{|\xi|^2}\eta\wedge\mathrm{d}\eta\right]\\
&= \frac{4\pi^2}{3b_1 b_2} + \int_{\partial X_{\alpha_0}}\frac{b_1 b_2\cos\alpha_0(3-\cos^2\alpha_0)\cos\theta\sin\theta}{3(b_1^2\sin^2\theta + b_2^2\cos^2\theta)^2}\mathrm{d}\theta\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2
\end{aligned} \tag{1.2.16}$$

Note that the contribution only arises from the second term involving Φ_0 since $f^*\Phi_2 = 0$. We now need to perform the integral over ∂X_{α_0}. Note that this is just an integral over a round three sphere and one could in principle use equivariant localization again to perform this integral. We will refrain from doing this but it is not difficult to see that the final result is

$$\int_{X_{\alpha_0}}\frac{1}{b_1 b_2}\text{vol}(X_{\alpha_0}) = \frac{2\pi^2(2-3\cos\alpha_0+\cos^3\alpha_0)}{3b_1 b_2}\,, \tag{1.2.17}$$

and therefore we find that we recover the correct result in (1.2.14). It is important to note that for generic $\alpha_0 \neq \frac{\pi}{2}$ there is a contribution from the boundary. One expects that it is possible to use equivariant localization to also evaluate the integrals for the boundary contributions and this would be an interesting problem to study in the future.

1.2.3 Application in supergravity

The program of applying the BVAB theorem in supergravity was initiated in [29, 32]. The remarkable aspect of this approach is that observables of highly non-trivial supergravity solutions can be determined solely from topological and fixed-point data, even when no explicit solution is known. The application of the method relies on one key feature of supergravity solutions, which is that they typically admit a U(1) symmetry. Namely the one associated to supersymmetry $\mathrm{U}(1)_R$, which can then be used to perform localization. Moreover, the

polyforms can be built out of Killing spinor bilinears, and their equivariant closure deduced from the Killing spinor equations.

More precisely, by definition a supersymmetric solution solves both the equations of motion and the Killing spinor equations. The R-symmetry Killing vector is constructed as a Killing spinor bilinear, while the equivariantly closed forms are constructed using various Killing spinor bilinears as well as the supergravity fields. Integrals of these equivariantly closed forms can then be evaluated using equivariant localization, which expresses the integrals in terms of the fixed point set of the R-symmetry Killing vector. The precise details of these constructions, as well as the type of physical observables one can compute, depend on which particular supergravity theory one is studying and which class of solutions one is considering.

A number of different examples have now been explored from this point of view and in each case the formalism allows one to profitably analyze general classes of solutions, most of which are unlikely to ever be found in explicit form. In the context of supersymmetric supergravity solutions with an asymptotically AdS factor and an internal space, applications have mostly focused on integrals over one of these factors separately, as in the computation of on-shell actions in supergravity [1, 6–8, 29, 37–49], or focusing on the internal space [2, 9, 10, 29, 32, 50]. Some complementary approaches of equivariant localization in supergravity have also been explored in [51–60].

1.3 Outline of the thesis

The thesis is organized as follows:

In chapter 2, we use equivariant localization in 4D $\mathcal{N} = 2$ gauged supergravity. In section 2.1, we introduce the Euclidean $D = 4$ $\mathcal{N} = 2$ gauged supergravity theory that we study and explain how the equivariantly closed forms can be constructed using the spinor bilinears. In section 2.2, we explain how the on-shell action can be expressed in terms of the fixed point data using the BVAB formula. In $D = 4$ the fixed point set consists of isolated points ("nuts") or two-dimensional submanifolds ("bolts") and including both cases leads to our final expression for the gravitational free energy given in (2.2.33). In section

2.3, we use equivariant localization to compute various flux integrals. By considering flux integrals associated with certain non-compact two-dimensional submanifolds which intersect the boundary, we can relate UV boundary data to IR fixed point data. We consider various different examples to illustrate the formalism in section 2.4 and 2.5. Some of these were briefly considered in [8], and, as well as expanding on some of the details, we also treat new examples. Spindle solutions, with orbifold singularities, are considered in section 2.6. We conclude the chapter with a discussion in section 2.7.

Then in chapter 3, we use equivariant localization on the internal space of $\mathcal{N} = 2$ $\mathrm{AdS}_4 \times X_6$ solutions in massive type IIA. The plan of this chapter is the following. In section 3.1 we review the solutions of [61], study the various options for constructing a well-defined background using O8 sources and then construct various equivariantly closed polyforms and present the general localized integrals for these polyforms. In section 3.2 we use our results to study distinct examples of possible geometries, each with different behaviours. In particular in section 3.2.4, we consider an example giving rise to non-vanishing boundary contributions to the localization formula. Finally, we comment on the results of this chapter in section 3.3.

Finally, chapter 4 uses equivariant localization directly in ten dimensions. In section 4.1, we first review the supergravity backgrounds of [62], and further exploit their structure to build ten-dimensional equivariantly closed polyforms for the fluxes and on-shell actions in type II. The next two sections are dedicated to utilizing these polyforms for two classes of solutions in massive type IIA. In section 4.2 we consider a hemisphere HS^4 fibration over a six-dimensional spacetime, giving rise to an extension of Romans supergravity, while section 4.3 focuses on X_6 fibrations over a four-dimensional spacetime where X_6 is the suspension of a five-dimensional Sasaki–Einstein manifold. In both sections, we derive a master formula computing the on-shell action for any topology, and further illustrate it for specific examples.

We conclude with some outlook in section 5. In particular, we mention some extensions which appear in further papers of the author, but are not reported on in this thesis, and some interesting directions to explore in future work.

Technical details are relegated to various appendices. In appendix A we have collected some technical details regarding the spinor bilinears in four dimensions. A brief discussion regarding how one can obtain results for minimal gauged supergravity is presented in appendix

B, which allows one to connect with the results of [51] which were obtained using different methods. Some material on cohomology relations is relegated to appendix C. Finally some technical derivations in ten dimensions are detailed in appendix D.

Chapter 2

Localization in four-dimensional gauged supergravity

This chapter is based on [1], which expands upon the results presented in [8]. The general setting is Euclidean $D = 4$, $\mathcal{N} = 2$ gauged supergravity coupled to an arbitrary number of vector multiplets. This includes the STU model, whose solutions uplift on a seven-sphere to give $D = 11$ solutions holographically dual to ABJM theory. By setting all the vector multiplets to zero, it also includes minimal $\mathcal{N} = 2$ gauged supergravity, whose solutions can be uplifted on a Sasaki–Einstein seven-manifold SE_7, as well as other spaces, which are dual to other $\mathcal{N} = 2$ SCFTs in $d = 3$. The equivariant formalism is also relevant in holography even when the $D = 4$ gauged supergravity does not arise from a consistent Kaluza–Klein truncation.

Associated with the most general supersymmetric solutions of $D = 4$, $\mathcal{N} = 2$ gauged supergravity, there are two types of equivariantly closed forms. The first and simplest is the equivariant completion of the two-form gauge field strengths, with the zero-form component canonically expressed in terms of the scalar Killing spinor bilinears along with the supergravity scalar fields. Using the BVAB theorem, these forms can be used to obtain expressions for the integrated fluxes in terms of the fixed point set, which is assumed to lie in the bulk of the solution (i.e. not on the boundary). In particular this allows one to obtain important UV-IR relations which relate IR fixed point data in the bulk of the solution to UV data associated with deformations of the boundary SCFT. The second is the equivariant extension of a four-form whose integral gives the bulk contribution to the Euclidean on-shell action of a given solution. Using the BVAB formula this gives a contribution at the fixed point set plus a boundary contribution. The full on-shell action, which we refer to as the gravitational free energy, is supplemented by additional, standard boundary terms: the Gibbons–Hawking–York term, counterterms and, in order to ensure supersymmetry, a Legendre transform associated

with implementing an alternative quantization for half of the scalar fields. A key result is that when evaluated on a supersymmetric solution, these boundary terms exactly cancel the boundary contribution coming from the bulk integral, leading to an expression for the gravitational free energy which is just expressed in terms of the fixed point set in the bulk. From a holographic perspective it is more natural to express the gravitational free energy in terms of UV data, and this can be achieved using the UV-IR relations.

We illustrate the formalism by discussing various examples. These include cases associated with known supergravity solutions as well as cases where the solutions are unlikely to be found in explicit form. Amongst the results we shall obtain are proofs of some conjectures for the free energy of mass-deformed three-dimensional holographic $\mathcal{N} = 2$ SCFTs in the large N limit as well as extensions thereof. Notably, we provide a general proof of a conjecture [63] for the free energy on S^3 (see (2.4.16), (2.4.17)), and extend a conjecture of [64] for the free energy on the direct product of a circle and a Riemann surface to a much broader family of Seifert manifolds (see (2.4.30)). In addition, we discuss solutions with a "cigar" $\mathbb{R}^2$ factor that appear as supersymmetric non-extremal deformations of Wick-rotated Lorentzian black holes and in each case we show that the on-shell action is largely independent of the deformation parameters, as expected from their holographic interpretation as gravity duals to supersymmetric indices. In the companion work [7] we further analyzed toric cases, which have two linearly independent Killing vectors, allowing us to develop the formalism further and to consider additional classes of examples.

2.1 $\mathcal{N} = 2$ gauged supergravity in $D = 4$

We will study a $D = 4$ Euclidean theory that is obtained by a Wick rotation of Lorentzian $\mathcal{N} = 2$ gauged supergravity. We show that supersymmetric solutions of this theory admit equivariantly closed forms constructed from spinor bilinears. Later, in section 2.2, we will impose additional reality constraints on the Euclidean theory to further utilise the formalism.

2.1.1 Lorentzian theory

In Lorentzian signature, the bosonic fields consist of the metric $g_{\mu\nu}$, Abelian gauge fields A^I with $I = 0, \ldots, n$, and complex scalars z^i with $i = 1, \ldots, n$. The scalars parametrize a special Kähler manifold with Kähler potential $\mathcal{K}$ and Hermitian metric $\mathcal{G}_{i\bar{j}} \equiv \partial_i \partial_{\bar{j}} \mathcal{K}$, where we denote $\partial_i \equiv \partial_{z^i}, \partial_{\bar{j}} \equiv \partial_{\bar{z}^{\bar{j}}}$. The special Kähler manifold is also the base of a symplectic bundle with covariantly holomorphic sections

$$\begin{pmatrix} L^I \\ M_I \end{pmatrix} \equiv \mathrm{e}^{\mathcal{K}/2} \begin{pmatrix} X^I \\ \mathcal{F}_I \end{pmatrix} , \tag{2.1.1}$$

where $X^I = X^I(z)$ are holomorphic. We also have the symplectic constraint

$$\mathrm{e}^{\mathcal{K}}(X^I \overline{\mathcal{F}}_I - \mathcal{F}_I \overline{X}^I) = \mathrm{i} \, . \tag{2.1.2}$$

We will assume there exists a prepotential $\mathcal{F} = \mathcal{F}(X^I)$, which is a holomorphic function of X^I, homogeneous of degree 2, such that

$$\mathcal{F}_I = \frac{\partial \mathcal{F}}{\partial X^I} \, . \tag{2.1.3}$$

We then define the kinetic matrix

$$\mathcal{N}_{IJ} = \overline{\mathcal{F}}_{IJ} + \mathrm{i} \frac{N_{IK} X^K N_{JL} X^L}{N_{NM} X^N X^M} \equiv \mathcal{R}_{IJ} + \mathrm{i} \mathcal{I}_{IJ} \, , \tag{2.1.4}$$

where $\mathcal{F}_{IJ} = \partial_I \partial_J \mathcal{F}$ and $N_{IJ} = 2 \mathrm{Im} \mathcal{F}_{IJ}$, and we have defined $\partial_I \equiv \partial_{X^I}$, $\mathcal{R}_{IJ} \equiv \mathrm{Re} \mathcal{N}_{IJ}$ and $\mathcal{I}_{IJ} \equiv \mathrm{Im} \mathcal{N}_{IJ}$. We also observe that

$$\mathcal{N}_{IJ} X^J = \mathcal{F}_I \, . \tag{2.1.5}$$

The action for the bosonic fields, in the conventions of [6], is given by

$$S = \frac{1}{16\pi G_4}\int \Big[R - 2\mathcal{G}_{i\bar{j}}\partial^\mu z^i \partial_\mu \bar{z}^{\bar{j}} - \mathcal{V}(z,\bar{z}) \\ + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{8}\mathcal{R}_{IJ}\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\Big]\,\mathrm{vol}_4\,. \tag{2.1.6}$$

Here G_4 is the Newton constant, and the scalar potential is

$$\mathcal{V} \equiv \zeta_I\zeta_J \mathrm{e}^{\mathcal{K}}\left(\mathcal{G}^{i\bar{j}}\nabla_i X^I \nabla_{\bar{j}}\overline{X}^J - 3X^I\overline{X}^J\right)\,, \tag{2.1.7}$$

where $\zeta_I \in \mathbb{R}$ are the Fayet–Iliopoulos gauging parameters and we have defined

$$\nabla_i X^I \equiv (\partial_i + \partial_i\mathcal{K})X^I\,. \tag{2.1.8}$$

Note that

$$\nabla_i L^I \equiv \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)L^I = \mathrm{e}^{\mathcal{K}/2}\nabla_i X^I\,, \\ \nabla_{\bar{i}} L^I \equiv \left(\partial_{\bar{i}} - \frac{1}{2}\partial_{\bar{i}}\mathcal{K}\right)L^I = 0\,. \tag{2.1.9}$$

Under a Kähler transformation $\mathcal{K} \to \mathcal{K} + h(z) + \bar{h}(\bar{z})$ we have $X^I \to \mathrm{e}^{-h}X^I$ and $L^I \to \mathrm{e}^{-\frac{h}{2}+\frac{\bar{h}}{2}}L^I$. We can also introduce the holomorphic and real superpotentials

$$W \equiv \zeta_I X^I\,, \qquad \mathcal{W} \equiv -\sqrt{2}\mathrm{e}^{\mathcal{K}/2}|W|\,, \tag{2.1.10}$$

in terms of which the scalar potential (2.1.7) takes the form

$$\mathcal{V} = \mathrm{e}^{\mathcal{K}}\left(\mathcal{G}^{i\bar{j}}\nabla_i W \nabla_{\bar{j}}\overline{W} - 3W\overline{W}\right) = 2\mathcal{G}^{i\bar{j}}\partial_i\mathcal{W}\partial_{\bar{j}}\mathcal{W} - \frac{3}{2}\mathcal{W}^2\,. \tag{2.1.11}$$

A solution is supersymmetric if there are solutions to the Killing spinor equations

$$
\begin{aligned}
0 &= \nabla_\mu \epsilon + \frac{\mathrm{i}}{2}\mathcal{A}_\mu \Gamma_5 \epsilon - \frac{\mathrm{i}}{2} A^R \epsilon + \frac{1}{2\sqrt{2}} \Gamma_\mu \mathrm{e}^{\mathcal{K}/2} \left(W \mathbb{P}_- + \overline{W} \mathbb{P}_+ \right) \epsilon \\
&\quad - \frac{\mathrm{i}}{4\sqrt{2}} \mathcal{I}_{IJ} F^J_{\nu\rho} \Gamma^{\nu\rho} \Gamma_\mu \left(L^I \mathbb{P}_- + \overline{L}^I \mathbb{P}_+ \right) \epsilon \,, \\
0 &= \frac{\mathrm{i}}{2\sqrt{2}} \mathcal{I}_{IJ} F^J_{\mu\nu} \Gamma^{\mu\nu} \left[\mathcal{G}^{\bar{i}j} \nabla_j L^I \mathbb{P}_- + \mathcal{G}^{i\bar{j}} \nabla_{\bar{j}} \overline{L}^I \mathbb{P}_+ \right] \epsilon + \Gamma^\mu \left(\partial_\mu z^i \mathbb{P}_- + \partial_\mu \overline{z}^{\bar{i}} \mathbb{P}_+ \right) \epsilon \\
&\quad - \frac{1}{\sqrt{2}} \left[\mathcal{G}^{\bar{i}j} \nabla_j W \mathbb{P}_- + \mathcal{G}^{i\bar{j}} \nabla_{\bar{j}} \overline{W}^I \mathbb{P}_+ \right] \epsilon \,,
\end{aligned}
\tag{2.1.12}
$$

where ϵ is a Dirac spinor, Γ_μ generate Cliff$(1,3)$, $\Gamma_5 \equiv \mathrm{i}\Gamma_{0123}$, $\mathbb{P}_\pm \equiv \frac{1}{2}(1 \pm \Gamma_5)$. In addition the R-symmetry gauge field is defined by the following linear combination of the gauge fields

$$
A^R \equiv \frac{1}{2} \zeta_I A^I_\mu \,, \tag{2.1.13}
$$

while the composite Kähler connection $\mathcal{A}$ is defined by

$$
\mathcal{A}_\mu = -\frac{\mathrm{i}}{2} \left(\partial_i \mathcal{K} \partial_\mu z^i - \partial_{\bar{i}} \mathcal{K} \partial_\mu \overline{z}^{\bar{i}} \right) . \tag{2.1.14}
$$

After specifying the prepotential $\mathcal{F}$ and the gaugings ζ_I, there are still two ambiguities that change the parametrization of the solutions. First, we have to fix the sections $X^I(z)$ (see e.g. [65] for various choices). Second, within the Kähler transformations mentioned earlier are those with constant pure imaginary parameter, $h = \mathrm{i}\alpha$, which act like an Abelian group, with $\mathcal{A}$ the corresponding connection:[1]

$$
\epsilon \to \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}\Gamma_5} \epsilon, \quad L^I \to \mathrm{e}^{-\mathrm{i}\alpha} L^I, \quad \mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \alpha \,. \tag{2.1.15}
$$

This allows us to rotate the phases of the sections in (2.1.12), as explained in e.g. [6]. In (2.1.12) we have made the same choice that was used in section 5.3 of [6], to facilitate later comparison with the STU model.

[1] In the superconformal construction of supergravity, this symmetry appears among the Kähler transformations after the gauge-fixing of the chiral $U(1)$ in the $SU(2) \times U(1)$ R-symmetry group of the four-dimensional superconformal group (see for instance [66] for a review).

2.1.2 Euclidean theory

We now define the Euclidean theory using the procedure explained in [67]. In general one should double all degrees of freedom and consider complex metric and gauge fields as well as take z^i and $\bar{z}^{\bar{i}}$ to be independent scalar fields. In order to highlight the latter, we write $\bar{z}^{\bar{i}} \to \tilde{z}^{\tilde{i}}$, as well as $\overline{X}^I \to \widetilde{X}^I$, $\overline{L}^I \to \widetilde{L}^I$ and $\overline{W} \to \widetilde{W}$. In particular we denote $\overline{\mathcal{F}(X^I)} \to \widetilde{\mathcal{F}}(\widetilde{X}^I)$, where for example the quantity N_{IJ} in (2.1.4) becomes $N_{IJ} = 2\mathrm{Im}\mathcal{F}_{IJ} \to -\mathrm{i}(\partial_{X^I}\partial_{X^J}\mathcal{F} - \partial_{\widetilde{X}^I}\partial_{\widetilde{X}^J}\widetilde{\mathcal{F}})$, with similar expressions for other quantities. In principle in Euclidean signature one might consider taking $\mathcal{F}$, $\widetilde{\mathcal{F}}$ to be independent functions, but we shall later impose $\widetilde{\mathcal{F}}(\widetilde{X}^I) = -\mathcal{F}(\widetilde{X}^I)$, which is satisfied by the STU model, for example, which has $\mathcal{F}(X^I) = -2\mathrm{i}\sqrt{X^0X^1X^2X^3}$. At a later point we will restrict to real metric and gauge fields, but we continue with the general case for now.

Following [67] we obtain the Wick rotated action

$$I = -\frac{1}{16\pi G_4}\int\Big[\Big(R - 2\mathcal{G}_{i\tilde{j}}\partial^\mu z^i\partial_\mu\tilde{z}^{\tilde{j}} - \mathcal{V}(z,\tilde{z})\Big)\,\mathrm{vol}_4 + \frac{1}{2}\mathcal{I}_{IJ}F^I\wedge *F^J - \frac{\mathrm{i}}{2}\mathcal{R}_{IJ}F^I\wedge F^J\Big]\,, \tag{2.1.16}$$

where $\mathcal{G}_{i\tilde{j}} \equiv \partial_i\partial_{\tilde{j}}\mathcal{K}$ and

$$\mathcal{V} = \mathrm{e}^{\mathcal{K}}\left(\mathcal{G}^{i\tilde{j}}\nabla_i W\nabla_{\tilde{j}}\widetilde{W} - 3W\widetilde{W}\right) = 2\mathcal{G}^{i\tilde{j}}\partial_i\mathcal{W}\partial_{\tilde{j}}\mathcal{W} - \frac{3}{2}\mathcal{W}^2\,, \tag{2.1.17}$$

with

$$W \equiv \zeta_I X^I\,, \qquad \widetilde{W} \equiv \zeta_I\widetilde{X}^I\,, \qquad \mathcal{W} \equiv -\sqrt{2}\mathrm{e}^{\mathcal{K}/2}\sqrt{W\widetilde{W}}\,. \tag{2.1.18}$$

The associated equations of motion are given by

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{2}\mathcal{I}_{IJ}\Big(F^I_{\mu\rho}F^J{}_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}\Big) + 2\mathcal{G}_{i\tilde{j}}\partial_\mu z^i\partial_\nu\tilde{z}^{\tilde{j}} + \frac{1}{2}g_{\mu\nu}\mathcal{V}\,,\\ \nabla^\mu(\mathcal{G}_{i\tilde{j}}\nabla_\mu\tilde{z}^{\tilde{j}}) &= -\frac{1}{8}\partial_i\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{\mathrm{i}}{8}\partial_i\mathcal{R}_{IJ}F^I_{\mu\nu}(*F^J)^{\mu\nu} + \partial_i\mathcal{G}_{k\tilde{j}}\partial^\mu z^k\partial_\mu\tilde{z}^{\tilde{j}} + \frac{1}{2}\partial_i\mathcal{V}\,,\\ 0 &= \mathrm{d}\left(\mathcal{I}_{IJ}*F^J - \mathrm{i}\mathcal{R}_{IJ}F^J\right)\,. \end{aligned} \tag{2.1.19}$$

For later use we notice that if we take the trace of the Einstein equation we obtain an

expression for the on-shell action given by

$$I_{\rm OS} = \frac{\pi}{2G_4}\frac{1}{(2\pi)^2}\int \Phi_4\,, \tag{2.1.20}$$

where Φ_4 is the four-form given by

$$\Phi_4 \equiv -\frac{1}{2}\mathcal{V}\,\mathrm{vol}_4 - \frac{1}{4}\mathcal{I}_{IJ}F^I\wedge *F^J + \frac{\mathrm{i}}{4}\mathcal{R}_{IJ}F^I\wedge F^J\,. \tag{2.1.21}$$

The supersymmetry transformations are parametrized by two Dirac spinors ϵ and $\tilde\epsilon$ and the corresponding Killing spinor equations are given by

$$\begin{aligned}
0 &= \nabla_\mu\epsilon + \frac{\mathrm{i}}{2}\mathcal{A}_\mu\gamma_5\epsilon - \frac{\mathrm{i}}{2}A^R_\mu\epsilon + \frac{1}{2\sqrt{2}}\gamma_\mu \mathrm{e}^{\mathcal{K}/2}\left(W\mathbb{P}_- + \widetilde{W}\mathbb{P}_+\right)\epsilon \\
&\quad - \frac{\mathrm{i}}{4\sqrt{2}}\mathcal{I}_{IJ}F^J_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\left(L^I\mathbb{P}_- + \widetilde{L}^I\mathbb{P}_+\right)\epsilon\,, \\
0 &= \frac{\mathrm{i}}{2\sqrt{2}}\mathcal{I}_{IJ}F^J_{\nu\rho}\gamma^{\nu\rho}\left(\mathcal{G}^{\tilde{i}j}\nabla_j L^I\mathbb{P}_- + \mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I\mathbb{P}_+\right)\epsilon + \gamma^\mu\left(\partial_\mu z^i\mathbb{P}_- + \partial_\mu\tilde{z}^{\tilde{i}}\mathbb{P}_+\right)\epsilon \\
&\quad - \frac{1}{\sqrt{2}}\left(\mathcal{G}^{\tilde{i}j}\nabla_j W\mathbb{P}_- + \mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{W}\mathbb{P}_+\right)\epsilon\,,
\end{aligned} \tag{2.1.22}$$

and

$$\begin{aligned}
0 &= \nabla_\mu\tilde\epsilon + \frac{\mathrm{i}}{2}\mathcal{A}_\mu\gamma_5\tilde\epsilon + \frac{\mathrm{i}}{2}A^R_\mu\tilde\epsilon + \frac{1}{2\sqrt{2}}\gamma_\mu \mathrm{e}^{\mathcal{K}/2}\left(W\mathbb{P}_- + \widetilde{W}\mathbb{P}_+\right)\tilde\epsilon \\
&\quad + \frac{\mathrm{i}}{4\sqrt{2}}\mathcal{I}_{IJ}F^J_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\left(L^I\mathbb{P}_- + \widetilde{L}^I\mathbb{P}_+\right)\tilde\epsilon\,, \\
0 &= -\frac{\mathrm{i}}{2\sqrt{2}}\mathcal{I}_{IJ}F^J_{\nu\rho}\gamma^{\nu\rho}\left(\mathcal{G}^{\tilde{i}j}\nabla_j L^I\mathbb{P}_- + \mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I\mathbb{P}_+\right)\tilde\epsilon + \gamma^\mu\left(\partial_\mu z^i\mathbb{P}_- + \partial_\mu\tilde{z}^{\tilde{i}}\mathbb{P}_+\right)\tilde\epsilon \\
&\quad - \frac{1}{\sqrt{2}}\left(\mathcal{G}^{\tilde{i}j}\nabla_j W\mathbb{P}_- + \mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{W}\mathbb{P}_+\right)\tilde\epsilon\,.
\end{aligned} \tag{2.1.23}$$

In these expressions γ_μ, which are taken to be Hermitian, generate Cliff(4) with $\gamma_5 \equiv \gamma_{1234}$ and now $\mathbb{P}_\pm \equiv \frac{1}{2}(1\pm\gamma_5)$.

To recover the $\mathcal{N}=2$ supersymmetric theory in Lorentzian signature we demand that the metric and gauge fields are real and we also impose that the complex scalars are related by $\tilde{z}^{\tilde\alpha} = \overline{(z^\alpha)}$, which implies $\widetilde{X}^I = \overline{X}^I$, etc. We also demand in the Euclidean theory $\tilde\epsilon = \epsilon^c \equiv -\mathcal{C}^{-1}\epsilon^*$, where $\mathcal{C} = -\mathcal{C}^T = -\mathcal{C}^{-1}$ and $\gamma_\mu^T = \mathcal{C}\gamma_\mu\mathcal{C}^{-1}$, $\gamma_\mu^* = \mathcal{C}\gamma_\mu\mathcal{C}^{-1}$, then the Killing

spinor equations for $\tilde{\epsilon}$ are equivalent to those of ϵ, so after Wick rotation we recover the Lorentzian theory.

Later we will study a sub-class of solutions of the Euclidean theory where we impose that the metric and gauge fields are both real, and further just consider solutions which preserve supersymmetry with $\tilde{\epsilon} = \epsilon^c$. However, we continue with the general Euclidean theory for now.

2.1.3 Bilinears

For an arbitrary spinor λ of the Euclidean theory we define the Majorana conjugate via $\bar{\lambda} = \lambda^T \mathcal{C}$. We can then construct bilinears from the Killing spinors ϵ, $\tilde{\epsilon}$ via:

$$S \equiv \bar{\tilde{\epsilon}}\epsilon\,, \quad P \equiv \bar{\tilde{\epsilon}}\gamma_5\epsilon\,, \quad \xi^\flat \equiv -\mathrm{i}\bar{\tilde{\epsilon}}\gamma_{(1)}\gamma_5\epsilon\,, \quad K \equiv \bar{\tilde{\epsilon}}\gamma_{(1)}\epsilon\,, \quad U \equiv \mathrm{i}\bar{\tilde{\epsilon}}\gamma_{(2)}\epsilon\,, \tag{2.1.24}$$

where S, P are scalars, $\xi^\flat, K$ are one-forms and U is a two-form. In general these are all complex quantities. Notice that in the special case when $\tilde{\epsilon} = \epsilon^c$, which we later restrict to, we then have $\bar{\tilde{\epsilon}} = \epsilon^\dagger$ and all the bilinears defined in (2.1.24) are real.

The Killing spinor ϵ is necessarily non-vanishing [68], and is globally well-defined. It is charged with respect to the R-symmetry gauge field A^R given in (2.1.13).[2] Correspondingly, the bilinears in (2.1.24), which are uncharged under the R-symmetry, are all globally defined forms on M.

These bilinears satisfy various algebraic and differential conditions, some of which we discuss in appendix A. Of most significance is that the vector field ξ, dual to $\xi^\flat$, is a Killing vector. It is immediate from the equations in appendix A that $\mathcal{L}_\xi z^\alpha = 0 = \mathcal{L}_\xi \tilde{z}^{\tilde{\alpha}}$, while $\mathcal{L}_\xi F^I = 0$ is established in the next subsection. We also have the important equation

$$\mathrm{d}\xi^\flat = -\sqrt{2}\zeta_I\big(L^I U_{[+]} - \widetilde{L}^I U_{[-]}\big) - \sqrt{2}\mathcal{I}_{IJ}\big(C^I F^J_{[-]} - \widetilde{C}^I F^J_{[+]}\big)\,. \tag{2.1.25}$$

[2] It is also charged under the Kähler gauge connection $\mathcal{A}$ in (2.1.14), and when the scalars z^i, $\tilde{z}^{\tilde{i}}$ are global functions on M (which we assume), $\mathcal{A}$ is then a global one-form on M.

Here we have introduced the notation

$$U_{[\pm]} \equiv \frac{1}{2}(U \pm *U)\,, \tag{2.1.26}$$

for the self-dual/anti-self-dual parts of a two-form U. We have also defined the following combinations of scalar fields and scalar spinor bilinears,

$$C^I \equiv L^I(S-P)\,, \quad \widetilde{C}^I \equiv \widetilde{L}^I(S+P)\,. \tag{2.1.27}$$

It is illuminating to identify combinations of the spinor bilinears that transform linearly under the symmetry (2.1.15). If we make the charge assignments that L^I has charge -1 and $\widetilde{L}^I$ has charge $+1$ we find that bilinears with well-defined charges are given by:

$$\begin{aligned} \text{charge } 0\text{:} &\quad \xi^\flat\,,\ K\,,\\ \text{charge } -1\text{:} &\quad S+P\,,\ U_{[-]}\,,\\ \text{charge } +1\text{:} &\quad S-P\,,\ U_{[+]}\,. \end{aligned} \tag{2.1.28}$$

In particular we see that C^I and $\widetilde{C}^I$ are invariant (i.e. charge 0). It is useful to note that

$$\begin{aligned} \mathrm{d}C^I &= \mathrm{d}[L^I(S-P)] = \nabla_i L^I(S-P)\mathrm{d}z^i + L^I(\mathrm{d}-\mathrm{i}\mathcal{A})(S-P)\,,\\ \mathrm{d}\widetilde{C}^I &= \mathrm{d}[\tilde{L}^I(S+P)] = \nabla_{\tilde{i}}\widetilde{L}^I(S+P)\mathrm{d}\tilde{z}^{\tilde{i}} + \tilde{L}^I(\mathrm{d}+\mathrm{i}\mathcal{A})(S+P)\,. \end{aligned} \tag{2.1.29}$$

2.1.4 Equivariantly closed forms

Using the bilinears introduced in the previous subsection, we can construct some polyforms that are equivariantly closed for supersymmetric solutions with respect to the equivariant exterior derivative

$$\mathrm{d}_\xi \equiv \mathrm{d} - \xi \lrcorner\,. \tag{2.1.30}$$

This satisfies $\mathrm{d}_\xi^2 = -\mathcal{L}_\xi$, and is thus nilpotent when acting on polyforms that are invariant under $\mathcal{L}_\xi$, the Lie derivative with respect to ξ.

Associated with the Abelian field strengths we define

$$\Phi^I_{(F)} \equiv F^I + \Phi^I_0 \,, \tag{2.1.31}$$

where

$$\Phi^I_0 \equiv \sqrt{2}\big(C^I - \widetilde{C}^I\big) \,, \tag{2.1.32}$$

and recall that C^I, $\widetilde{C}^I$ were defined in (2.1.27). Using the equations in appendix A one can show that $\mathrm{d}_\xi \Phi^I_{(F)} = 0$. Generically $\Phi^I_{(F)}$ is a complex polyform, but note that with the reality conditions imposed in (2.2.1), below, it is real. Since F^I is closed by the Bianchi identity we have $\mathcal{L}_\xi F^I = \mathrm{d}(\xi \lrcorner F^I) = 0$, where the latter follows immediately from $\Phi^I_{(F)}$ being equivariantly closed. Notice that $\Phi^I_{(F)}$ in (2.1.31), (2.1.32) is gauge-invariant and globally defined.

If we choose a gauge with $\mathcal{L}_\xi A^I = 0$, notice that equivariant closure of (2.1.31) implies

$$\xi \lrcorner A^I = -\Phi^I_0 + c^I \,, \tag{2.1.33}$$

where c^I are *a priori* arbitrary integration constants. In particular via a further gauge transformation one can set $c^I = 0$ if desired. We can define the *supersymmetric gauge* to be the slightly weaker one for which $\zeta_I c^I = 0$. From (2.1.13) this is equivalent to the R-symmetry gauge field satisfying

$$\xi \lrcorner A^R_{\mathrm{SUSY}} = -\frac{1}{2}\zeta_I \Phi^I_0 = -\frac{1}{\sqrt{2}}\zeta_I \big(C^I - \widetilde{C}^I\big) \,. \tag{2.1.34}$$

At a given fixed point, where $\xi = 0$, a regular gauge field would satisfy $\xi \lrcorner A^I = 0$. So when $\zeta_I \Phi^I_0 \neq 0$ at a fixed point, which is true in all known examples, the R-symmetry gauge field A^R_{SUSY} is *not* in a regular gauge at that point.

There is also an equivariant completion of the four-form Φ_4 associated with the on-shell

action in (2.1.20). Specifically we find that

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0\,, \tag{2.1.35}$$

is equivariantly closed, $\mathrm{d}_\xi\Phi = 0$, where

$$\begin{aligned}
\Phi_4 &= -\frac{1}{2}\mathcal{V}\mathrm{vol}_4 - \frac{1}{4}\mathcal{I}_{IJ}F^I\wedge *F^J + \frac{\mathrm{i}}{4}\mathcal{R}_{IJ}F^I\wedge F^J\,,\\
\Phi_2 &= \frac{1}{\sqrt{2}}\zeta_I(L^I U_{[+]} + \widetilde{L}^I U_{[-]}) - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\Big(C^I F^J_{[+]} + \widetilde{C}^I F^J_{[-]}\Big) + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{R}_{IJ}F^J\big(C^I - \widetilde{C}^I\big)\,,\\
\Phi_0 &= \mathrm{i}\mathrm{e}^{\mathcal{K}}\Big[\mathcal{F}(X)(S-P)^2 + \widetilde{\mathcal{F}}(\widetilde{X})(S+P)^2\\
&\qquad - (\partial_I\mathcal{F}(X)\widetilde{X}^I + \partial_I\widetilde{\mathcal{F}}(\widetilde{X})X^I)(S^2-P^2)\Big]\,.
\end{aligned} \tag{2.1.36}$$

Since $\mathcal{F}$, $\widetilde{\mathcal{F}}$ are homogeneous of degree two, we can also write

$$\Phi_0 = \mathrm{i}\Big[\mathcal{F}(C) + \widetilde{\mathcal{F}}(\widetilde{C}) - \partial_I\mathcal{F}(C)\widetilde{C}^I - \partial_I\widetilde{\mathcal{F}}(\widetilde{C})C^I\Big]\,, \tag{2.1.37}$$

where the partial derivative notation $\partial_I\mathcal{F}$ means derivative with respect to the argument of $\mathcal{F}$ (which is X^I in (2.1.36), but C^I in (2.1.37)). It is also interesting to point out that using (2.1.25) and (2.1.32) in (2.1.36) we can write Φ_2 in the form

$$\Phi_2 = -\frac{1}{2}*\mathrm{d}\xi^\flat - \frac{1}{2}(\mathcal{I}_{IJ}*F^I - \mathrm{i}\mathcal{R}_{IJ}F^I)\Phi_0^J\,. \tag{2.1.38}$$

Notice that the polyform Φ is gauge-invariant and globally defined. Recalling (2.1.28) we also observe that the equivariantly closed forms $\Phi^I_{(F)}$ and Φ are all invariant under the Abelian transformation (2.1.15), as expected.

Finally, here we have shown that for supersymmetric solutions of the gauged supergravity theory we are considering, canonical expressions for Φ_0^I and Φ_2, Φ_0 can be obtained using spinor bilinears as given in (2.1.32) and (2.1.36), respectively. Notice that we could trivially modify the expressions for Φ_0 and Φ_0^I by adding constants $\Phi_0^I \to \Phi_0^I + c_0^I$, $\Phi_0 \to \Phi_0 + c_0$ while maintaining equivariant closure and similarly we could also modify $\Phi_2 \to \Phi_2 + \zeta_2$ where ζ_2 is a closed globally defined basic, two-form (recall a basic form α satisfies $\xi \lrcorner \alpha = 0$ and

$\mathcal{L}_\xi \alpha = 0$).[3] An interesting and important feature of the canonical expressions (2.1.32) and (2.1.36) is that they have the property that the total contribution to the gravitational free energy from the conformal boundary vanishes.[4]

2.2 Evaluation of the action

In this section we explain how we can use the BVAB theorem to evaluate the on-shell action via localization, as well as obtain important relations between UV and IR quantities.

2.2.1 Reality conditions

To do this we will now study a sub-class of solutions of the Euclidean theory where we impose that the metric and gauge fields are both real, and further just consider solutions which preserve supersymmetry with $\tilde{\epsilon} = \epsilon^c$: consistency of these assumptions implies additional restrictions which we summarise together as

$$\begin{aligned} &g_{\mu\nu} \in \mathbb{R}\,, && A^I_\mu \in \mathbb{R}\,, && z^i, \tilde{z^i} \in \mathbb{R}\,, \\ &\mathcal{A} \in \mathrm{i}\mathbb{R}\,, && L^I, \widetilde{L}^I \in \mathbb{R}\,, && \mathcal{G}^{\tilde{i}j}, \mathcal{G}^{i\tilde{j}} \in \mathbb{R}\,, \\ &\tilde{\epsilon} = \epsilon^c \equiv \mathcal{C}\epsilon^* && \Rightarrow \bar{\tilde{\epsilon}} = \epsilon^\dagger\,. \end{aligned} \tag{2.2.1}$$

The bilinears (2.1.24) are all real. Crucially, we will not impose $\tilde{z^i} = \overline{(z^i)}$, which would be $z^i = \tilde{z^i}$, given (2.2.1). Solutions that satisfy this additional condition will also be solutions of the Lorentzian theory, after Wick rotation.

We also notice that these reality conditions are only compatible with the symplectic constraint given (in the Lorentzian theory) in (2.1.2), provided that the prepotential is purely

[3] Ambiguities in the equivariant forms of the above form are well known and are associated with differing contributions from the fixed point sets arising from Stokes' theorem. For an example in the compact context, consider the equivariant integration giving the volume of $\mathbb{CP}^2$ written as an S^3 fibred over an interval. As described in detail in [32], the Reeb vector of S^3 has an isolated fixed point and a two-dimensional bolt, and the ambiguity in the volume polyform can be fixed so that the volume is given entirely by each of these or by a combination of the two.

[4] For minimal gauged supergravity, without using localization it was shown by a direct computation that the boundary contribution to the on-shell action vanishes if one works in a supersymmetric gauge [51] and further sets, in their notation, $c_\varphi = c_\sigma = 0$. In effect these constants are analogous to the constants c_0^0 and c_0. The freedom to shift by ζ_2 was not discussed in [51], which does lead to boundary contributions to the on-shell action.

imaginary, and so in the sequel we shall take

$$\mathcal{F} \in \mathrm{i}\mathbb{R} \qquad \Rightarrow \qquad \widetilde{\mathcal{F}}(\widetilde{X}) = -\mathcal{F}(\widetilde{X})\,. \tag{2.2.2}$$

2.2.2 Using the BVAB theorem

We assume our Euclidean theory admits a supersymmetric AdS_4 (i.e. H^4) vacuum with vanishing gauge fields and constant scalar fields satisfying $\tilde{z}^{\tilde{i}} = z^i$, and hence it is also a supersymmetric vacuum of the Lorentzian theory.[5] This solution defines what we refer to as the vacuum of the dual SCFT_3. We are interested in general Euclidean solutions on a four-dimensional manifold M which asymptotically approach this AdS_4 solution with conformal boundary ∂M. As usual, depending on the precise asymptotic behaviour near the boundary, such solutions will be dual to deformations of the SCFT, and allow for an arbitrary boundary metric, non-trivial gauge fields on the boundary as well as sources for the operators dual to the bulk scalar fields. To evaluate the gravitational free energy of such solutions we need to evaluate the bulk on-shell Euclidean action (2.1.20), and then supplement it with suitable boundary terms to implement holographic renormalization. We first focus on (2.1.20).

Consider, momentarily, equivariantly closed forms on a manifold M in general even dimension $2n$. Denote the fixed point set where $\|\xi\|^2 = 0$ by M_0, which we assume is in the interior of M, so $M_0 \cap \partial M = \emptyset$. Since $\mathrm{d}_\xi \xi^\flat = \mathrm{d}\xi^\flat - \|\xi\|^2$, on $M \setminus M_0$ we can define an inverse $(\mathrm{d}_\xi \xi^\flat)^{-1} = -\frac{1}{\|\xi\|^2}(1 - \frac{\mathrm{d}\xi^\flat}{\|\xi\|^2})^{-1} = -\frac{1}{\|\xi\|^2}\sum_{k=0}^{n}(\frac{\mathrm{d}\xi^\flat}{\|\xi\|^2})^k$. Using this we can define a polyform $\Phi_\xi = \xi^\flat \wedge (\mathrm{d}_\xi \xi^\flat)^{-1}$ satisfying $\mathrm{d}_\xi \Phi_\xi = 1$. Thus, for an arbitrary equivariantly closed form Φ on M, on $M \setminus M_0$ we can write $\Phi = \Phi\, \mathrm{d}_\xi \Phi_\xi = \mathrm{d}_\xi(\Phi\, \Phi_\xi)$ and hence we see it is equivariantly exact on $M \setminus M_0$. Since the top form of an equivariantly exact form is exact, this shows that the integral of Φ on M, if closed, will only receive contributions from the fixed points (as given by the BVAB formula [30, 31]). It also shows that if M has a boundary that does not include any fixed points we can obtain an explicit expression for an additional boundary contribution to the integral using Stokes' theorem and the expression for $(\mathrm{d}_\xi \xi^\flat)^{-1}$ [2].

We now return to our $D = 4$ theory. We want to evaluate the bulk on-shell action (2.1.20) on a manifold with boundary and we will assume that the fixed point set does not include the

[5] There could be more than one such AdS_4 solution, and one can treat each one similarly.

boundary. To simplify the exposition we shall assume that M is a manifold, and in particular that the fixed point set does not have orbifold singularities.[6] We write $I_{\rm OS} = I_{\rm OS}^{\rm FP} + I_{\rm OS}^{\partial M}$ where $I_{\rm OS}^{\rm FP}$ is the contribution from the fixed point set in the bulk given by the BVAB formula and $I_{\rm OS}^{\partial M}$ is the boundary contribution. Explicitly, on the one hand we have

$$I_{\rm OS}^{\rm FP} = \frac{\pi}{2G_4}\left\{ \sum_{\rm nuts} \frac{\Phi_0}{b_1 b_2} + \sum_{\rm bolts} \int_\Sigma \frac{\Phi_2}{2\pi b} - \frac{\Phi_0\, c_1(L)}{b^2} \right\}. \tag{2.2.3}$$

Here the sums in (2.2.3) are over connected components of the fixed point set. The b_i and b are weights of the linear action of ξ on the normal bundle to the (connected components of the) fixed point set, and $c_1(L)$ is the first Chern class of the normal bundle to the bolt surface $\Sigma \subset M$. The boundary contribution, on the other hand, is given by

$$\begin{aligned} I_{\rm OS}^{\partial M} &= -\frac{\pi}{2G_4}\frac{1}{(2\pi)^2}\int_{\partial M} \frac{1}{\|\xi\|^2}\xi^\flat \wedge \left(\Phi_2 + \Phi_0 \frac{\mathrm{d}\xi^\flat}{\|\xi\|^2}\right) \\ &= -\frac{\pi}{2G_4}\frac{1}{(2\pi)^2}\int_{\partial M} \eta \wedge (\Phi_2 + \Phi_0\, \mathrm{d}\eta)\,. \end{aligned} \tag{2.2.4}$$

where on $M \setminus M_0$ we have defined $\eta \equiv \frac{1}{\|\xi\|^2}\xi^\flat$, satisfying $\xi \lrcorner \eta = 1$.[7]

We next discuss the boundary terms associated with holographic renormalization. As usual these include a Gibbons–Hawking–York term, counterterms to cancel divergences, as well as additional finite boundary terms that implement a supersymmetric renormalization scheme. The supersymmetric AdS_4 vacuum solution with constant scalars and vanishing gauge fields is dual to a $d = 3$, $\mathcal{N} = 2$ SCFT. In this set-up each vector multiplet is dual to a conserved current multiplet; the massless gauge fields are dual to the currents with conformal scaling dimension $\Delta = 2$ while the scalars are necessarily dual to operators of dimensions $\Delta = 1, 2$ to fill out the multiplet. The latter corresponds to bulk scalar fields with $m^2 = -2/\ell^2$, where ℓ is the radius of the AdS_4 vacuum, and in order to obtain scalar operators with $\Delta = 1$ we need to implement alternate quantization on half of the scalar fields via boundary terms that implement a Legendre transform [69]. That this mass spectrum around a given supersymmetric vacuum is indeed realised was proven in appendix A of [63].

[6] We will discuss examples where this is relaxed in section 2.6.

[7] Note that η here is normalized differently from the introduction chapter where we used $\eta \equiv \xi^\flat$ instead.

Remarkably, it turns out that the boundary terms associated with holographic renormalization exactly cancel the boundary terms arising in (2.2.4), provided we use the canonical expressions (2.1.36) for the lower degree forms in the polyform. The computation, which extends that of [51], who obtained a similar result for minimal gauged supergravity, is lengthy and presented in the appendix of [1]. The total on-shell action is thus given by the fixed point contribution in the bulk, $I_{\rm OS}^{\rm FP}$, given by (2.2.3), which we now evaluate. We begin by establishing some preliminary results.

2.2.3 Preliminaries

2.2.3.1 Canonical frame

With $\tilde{\epsilon} = \epsilon^c$, the bilinears in (2.1.24) are real. As in [51] we can construct a canonical orthonormal frame E^a and write the bilinears as

$$P = S\cos\theta\,, \quad K = -S\sin\theta\,\mathrm{E}^3\,, \quad \xi^\flat = S\sin\theta\,\mathrm{E}^4\,, \quad U = -S(\mathrm{E}^{12} - \cos\theta\,\mathrm{E}^{34})\,, \tag{2.2.5}$$

where we have introduced the angle θ for convenience. The volume element is given by $\mathrm{vol}_4 = \mathrm{E}^{1234}$, so we have

$$U_{[\pm]} = -(1 \mp \cos\theta)\frac{S}{2}\left(\mathrm{E}^{12} \pm \mathrm{E}^{34}\right)\,. \tag{2.2.6}$$

As we discuss further below, it is important to note that at a fixed point where $\xi = 0$ this frame breaks down. Thus, strictly speaking it is a frame on $M \setminus M_0$, rather than on M.

2.2.3.2 Charge of the Killing spinor

We now compute the Lie derivative of the Killing spinor ϵ with respect to ξ, defined as

$$\mathcal{L}_\xi\epsilon \equiv \xi^\mu\nabla_\mu\epsilon + \frac{1}{8}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon\,. \tag{2.2.7}$$

To do so, we find it convenient to multiply this equation with $\bar{\tilde{\epsilon}}$ and consider $\bar{\tilde{\epsilon}}\mathcal{L}_\xi\epsilon = \xi^\mu\,\bar{\tilde{\epsilon}}\nabla_\mu\epsilon - \frac{\mathrm{i}}{8}\mathrm{d}\xi^\flat_{\mu\nu}U^{\mu\nu}$. To proceed, we substitute for $\nabla_\mu\epsilon$ using the Killing spinor equation (2.1.22) and

$\mathrm{d}\xi^\flat$ from (2.1.25). Given the reality conditions (2.2.1), we then utilise the identity structure (2.2.5) as well as $\xi \lrcorner \mathcal{A} = 0$, to find

$$\bar{\tilde{\epsilon}}\mathcal{L}_\xi \epsilon = \frac{\mathrm{i}}{4}\zeta_I \left(\xi \lrcorner A^I + \Phi_0^I\right) S\,, \tag{2.2.8}$$

where Φ_0^I is the lowest component of the equivariantly closed flux polyform (2.1.31), (2.1.32). We therefore deduce that

$$\mathcal{L}_\xi \epsilon = \mathrm{i}Q\epsilon\,, \qquad Q = \frac{1}{2}\xi \lrcorner A^R + \frac{1}{4}\zeta_I \Phi_0^I\,. \tag{2.2.9}$$

Clearly the charge Q of the Killing spinor depends on the choice of gauge for the R-symmetry gauge field $A^R \equiv \frac{1}{2}\zeta_I A^I$ in (2.1.13). If we work in a gauge with $\mathcal{L}_\xi A^I = 0$, from (2.1.33) we have $\xi \lrcorner A^I = -\Phi_0^I + c^I$, where c^I are constants, and hence

$$Q = \frac{1}{4}\zeta_I c^I\,. \tag{2.2.10}$$

Notice that in the supersymmetric gauge (2.1.34), which has $\zeta_I c^I = 0$, we have $Q = 0$ and the spinor is uncharged under the action of ξ, and so too is the frame E^a, $\mathcal{L}_\xi \mathrm{E}^a = 0$.

Having established that the Killing spinor has a definite charge Q under the action of ξ, we can now establish a useful result at a fixed point, where $\xi = 0$, by generalizing an argument from [32]. Start with $\xi^\mu \nabla_\mu \epsilon$ and substitute the expression for $\nabla_\mu \epsilon$ using the Killing spinor equation (2.1.22). Using $\xi \lrcorner \mathcal{A} = 0$, we then immediately find that at a fixed point $\xi^\mu \nabla_\mu \epsilon = \frac{\mathrm{i}}{2}\xi^\mu A^R_\mu$, which may be non-vanishing depending on the choice of gauge. Combining this result with (2.2.9) we deduce that on the fixed point set we have

$$\frac{1}{2}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon = \mathrm{i}\zeta_I \Phi_0^I \epsilon \qquad (\text{at } \xi = 0)\,, \tag{2.2.11}$$

and notice that this is a gauge-invariant result.

2.2.3.3 Chirality of spinors at fixed point set

Recall that the Killing spinor ϵ is globally well defined and non-vanishing in the bulk. Given the reality conditions (2.2.1) it is a globally well-defined section of $\mathcal{S}M \otimes \mathscr{L}^{1/2}$, where $\mathcal{S}M$ is the spin bundle of M, and $\mathscr{L}$ is the R-symmetry line bundle on which A^R is a connection. The bilinears in (2.1.24) are all globally defined and are uncharged under the R-symmetry. Given the reality conditions (2.2.1), we also have that S is the square of the norm of the Killing spinor, and hence S is also non-vanishing. On the other hand $\|\xi\|^2 = S^2 \sin^2\theta$ and so at a fixed point we necessarily have $\sin\theta = 0$ and hence $\cos\theta = \pm 1$. Correspondingly, at a fixed point we have $P = \pm S$ and the Killing spinor is necessarily chiral at the fixed point too:

$$\gamma_5 \epsilon = \pm \epsilon \qquad (\text{at } \xi = 0)\,. \tag{2.2.12}$$

It is convenient to append this chirality data to the fixed points and label them as nuts$_\pm$ and bolts$_\pm$. From (2.1.27) we then have, correlated with this chirality,

$$(C^I, \widetilde{C}^I)|_+ = (0, 2S\widetilde{L}^I)|_+\,, \qquad \text{or} \qquad (C^I, \widetilde{C}^I)|_- = (2SL^I, 0)|_-\,, \tag{2.2.13}$$

and with our reality conditions (2.2.1), these are real. It will be convenient to define, at the fixed points,

$$\begin{aligned} u^I_+ &\equiv \frac{\widetilde{X}^I}{\zeta_J \widetilde{X}^J}\Big|_+ = \frac{\widetilde{L}^I}{\zeta_J \widetilde{L}^J}\Big|_+ = \frac{\widetilde{C}^I}{\zeta_J \widetilde{C}^J}\Big|_+\,, \\ u^I_- &\equiv \frac{X^I}{\zeta_J X^J}\Big|_- = \frac{L^I}{\zeta_J L^J}\Big|_- = \frac{C^I}{\zeta_J C^J}\Big|_-\,, \end{aligned} \tag{2.2.14}$$

where we have assumed $\zeta_J X^J\big|_\pm \neq 0$. Clearly we have the constraints

$$\zeta_I u^I_+ = 1\,, \qquad \zeta_I u^I_- = 1\,. \tag{2.2.15}$$

From (2.2.6), at the fixed point we also have

$$\begin{aligned} &+ \text{ chirality}: \qquad U = U_{[-]}\,, \\ &- \text{ chirality}: \qquad U = U_{[+]}\,. \end{aligned} \tag{2.2.16}$$

As remarked earlier, at a fixed point the canonical frame E^a is no longer well-defined. Thus, an important point is that we cannot immediately use (2.1.25) and (2.2.6) to express $\mathrm{d}\xi^\flat$ in terms of this frame to obtain the weights of the Killing vector at the fixed point, and so we proceed differently below.

2.2.4 Action contributions from nuts

A nut is by definition an isolated fixed point. We remind the reader that to simplify the exposition we are assuming there are no orbifold singularities on M (examples with orbifolds will be considered in section 2.6). Thus, near a nut we can introduce local polar coordinates ρ_i, φ_i, with $\Delta\varphi_i = 2\pi$, that parametrize two copies of $\mathbb{R}^2 \cong \mathbb{C}$ with the nut located at $\rho_1 = 0 = \rho_2$. The supersymmetric Killing vector at the nut can be written

$$\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}\,, \tag{2.2.17}$$

where b_i are the weights of the action at the nut. Indeed, introducing a local orthonormal frame $\bar{e}^a$ with $\bar{e}^a = (\mathrm{d}\rho_1, \rho_1 \mathrm{d}\varphi_1, \mathrm{d}\rho_2, \rho_2 \mathrm{d}\varphi_2)$, at the nut we have $\mathrm{d}\xi^\flat = 2b_1 \bar{e}^{12} + 2b_2 \bar{e}^{34}$. If the orbits of ξ close, which does not have to be the case, then we can write $b_1/b_2 = p/q$ for coprime integers p, q.

Notice that in writing (2.2.17) we have made orientation choices for each $\mathbb{R}^2 \cong \mathbb{C}$ factor. However, for a given overall orientation (that we always fix by supersymmetry to be E^{1234}), we can reverse the orientations of both $\mathbb{R}^2$ factors (equivalently, a simultaneous complex conjugation of both $\mathbb{C}$ factors) or interchange the two $\mathbb{R}^2$ factors, leaving the overall orientation

on M fixed. These transformations act on the weights via

$$(b_1, b_2) \to (-b_1, -b_2)\,,$$
$$(b_1, b_2) \to (b_2, b_1)\,, \tag{2.2.18}$$

respectively, and our final answer should respect this. They also act on the local frame at the nut via $\bar{e}^{12}, \bar{e}^{34} \to -\bar{e}^{12}, -\bar{e}^{34}$ and $\bar{e}^{12} \leftrightarrow \bar{e}^{34}$.

We can also show that the chiral Killing spinor at the fixed point in this frame must satisfy

$$\gamma^{\bar{1}\bar{2}}\epsilon = -\mathrm{i}\kappa\epsilon\,, \qquad \gamma^{\bar{3}\bar{4}}\epsilon = \pm\mathrm{i}\kappa\epsilon\,, \tag{2.2.19}$$

where $\kappa = \pm 1$ and, moreover, provided that $\zeta_I \Phi_0^I \neq 0$ at the fixed point (as we are assuming[8]), we have

$$\begin{aligned} +\text{ chirality}: &\qquad b_1 - b_2 = -\frac{\kappa}{2}\zeta_I\Phi_0^I = \frac{\kappa}{\sqrt{2}}\zeta_I\tilde{C}^I\,,\\ -\text{ chirality}: &\qquad b_1 + b_2 = -\frac{\kappa}{2}\zeta_I\Phi_0^I = -\frac{\kappa}{\sqrt{2}}\zeta_I C^I\,. \end{aligned} \tag{2.2.20}$$

To show this we can use (2.2.11) to get, at the fixed point, $2(b_1\gamma^{\bar{1}\bar{2}} + b_2\gamma^{\bar{3}\bar{4}})\epsilon = \mathrm{i}\zeta_I\Phi_0^I\epsilon$. Multiplying this expression by $\gamma^{\bar{1}\bar{2}}$ and using the fact that the spinor has a definite chirality at the fixed point, we immediately deduce that the spinor is an eigenspinor of $\gamma^{\bar{1}\bar{2}}$, which has eigenvalues given by plus or minus i, and the result follows after using (2.1.32) and (2.2.13). An alternative derivation utilises (2.2.9): we first note that if we choose a gauge so that the R-symmetry gauge field is regular at the fixed point, $\xi \lrcorner A^R = 0$, we have $\mathcal{L}_\xi\epsilon = \mathrm{i}Q\epsilon$ with $Q = \frac{1}{4}\zeta_I\Phi_0^I$. On the other hand a spinor of definite chirality that is regular at the fixed point will necessarily[9] have the property that $\mathcal{L}_{\partial_{\varphi_1}}\epsilon = -\frac{\mathrm{i}}{2}\kappa\epsilon$ and $\mathcal{L}_{\partial_{\varphi_2}}\epsilon = \pm\frac{\mathrm{i}}{2}\kappa\epsilon$.

[8] This assumption eliminates having $b_1 = \pm b_2$ for $\pm$ chirality, respectively. It is worth highlighting, however, that there are certainly examples with $b_1 = \mp b_2$ for $\pm$ chirality.

[9] Consider flat $\mathbb{R}^2$ in a Cartesian frame $e^1 = \mathrm{d}x$, $e^2 = \mathrm{d}y$. In this frame a constant spinor is regular. However, since the Cartesian frame has charge 1 under rotations of ∂_φ the two different, two-dimensional spinor chiralities will have charges $\pm 1/2$ with respect to $\mathcal{L}_{\partial_\varphi}$, respectively. One can then take a tensor product of two copies of this to obtain an analogous result for $\mathbb{R}^4$. This can then be applied locally near a nut fixed point.

For a given fixed point, we can use the transformations (2.2.18) mentioned above to fix the sign of κ. However, it is important to emphasize that when there are multiple components of the fixed point set, each comes with its chirality and κ, and relative signs and orientation conventions at different fixed points need to be determined carefully. In some examples studied in this chapter, we will see how these signs can be constrained, and in [7] we show in the context of toric examples how these signs can be dealt with in a systematic way.

We can now obtain an expression for the contribution of an individual nut to the on-shell action in the BVAB formula (2.2.3). For an individual nut_+, from (2.2.13) we have $C^I = 0$ and since $\mathcal{F}$ is homogeneous of degree two, from (2.1.37) we have $\Phi_0 = \mathrm{i}\widetilde{\mathcal{F}}(\widetilde{C}) = -\mathrm{i}\mathcal{F}(\widetilde{C})$, where the last equality follows from (2.2.2). Thus, the expression (2.2.3) can now be written as

$$I_{\text{OS}}^{\text{nut}_+} = \frac{\pi}{2G_4}\frac{\Phi_0}{b_1 b_2} = -\frac{\pi}{2G_4}\frac{\mathrm{i}\mathcal{F}(\widetilde{C})}{b_1 b_2}\,. \tag{2.2.21}$$

We next use (2.2.20) to rewrite this as

$$\begin{aligned} I_{\text{OS}}^{\text{nut}_+} &= -\frac{\pi}{G_4}\frac{(b_1-b_2)^2}{b_1 b_2}\frac{\mathrm{i}\mathcal{F}(\widetilde{C})}{(\zeta_I \widetilde{C}^I)^2} \\ &= -\frac{\pi}{G_4}\frac{(b_1-b_2)^2}{b_1 b_2}\mathrm{i}\mathcal{F}(u_+)\,, \end{aligned} \tag{2.2.22}$$

where we used the definition of the u_+^I variables in (2.2.14) and that $\widetilde{\mathcal{F}}$ is homogeneous of degree two. A similar computation shows that the contribution for an individual nut_- is given by

$$I_{\text{OS}}^{\text{nut}_-} = \frac{\pi}{G_4}\frac{(b_1+b_2)^2}{b_1 b_2}\mathrm{i}\mathcal{F}(u_-)\,. \tag{2.2.23}$$

Notice that these expressions only depend on the ratio b_1/b_2 at the nut, and not the individual values of b_1, b_2, and are invariant under (2.2.18). We also notice that the value of $\kappa \in \{\pm 1\}$ does not appear in these expressions.

2.2.5 Action contributions from bolts

A bolt is by definition a connected two-dimensional fixed point set. Again, to simplify the exposition we assume there are no orbifold singularities. In this case we can introduce local coordinates $(x^1, x^2, \rho_2, \varphi_2)$, with $\Delta\varphi_2 = 2\pi$ and (x^1, x^2) parametrizing the bolt, located at $\rho_2 = 0$. The supersymmetric Killing vector is given by

$$\xi = b_2 \partial_{\varphi_2} \,. \tag{2.2.24}$$

We can introduce a frame $\bar{e}^a$ with $(\bar{e}^1, \bar{e}^2) = (\mathrm{d}x^1, \mathrm{d}x^2)$ tangent to the bolt and $(\bar{e}^3, \bar{e}^4) = (\mathrm{d}\rho_2, \rho_2 \mathrm{d}\varphi_2)$, so that b_2 is the weight of the action of ξ on the two-dimensional normal space to the bolt: $\mathrm{d}\xi^\flat = 2b_2 \bar{e}^{34}$. Notice that if we take $\varphi_2 \to -\varphi_2$, so $b_2 \to -b_2$, and also change the orientation of the bolt $\bar{e}^{12} \to -\bar{e}^{12}$ then we keep the overall orientation fixed.

As shown before, the spinor is necessarily chiral at the bolt. Using (2.2.11) we immediately deduce that $\zeta_I \Phi_0^I \neq 0$ at the bolt and furthermore, similar to case of nuts, we can take the chiral Killing spinor at the fixed bolt$_\pm$ in this frame to satisfy

$$\gamma^{\bar{1}\bar{2}}\epsilon = -\mathrm{i}\kappa\epsilon \,, \qquad \gamma^{\bar{3}\bar{4}}\epsilon = \pm\mathrm{i}\kappa\epsilon \,, \tag{2.2.25}$$

where $\kappa = \pm 1$ with

$$\begin{aligned} +\text{ chirality}: &\qquad b_2 = -\frac{\kappa}{\sqrt{2}}\zeta_I \widetilde{C}^I \,, \\ -\text{ chirality}: &\qquad b_2 = -\frac{\kappa}{\sqrt{2}}\zeta_I C^I \,, \end{aligned} \tag{2.2.26}$$

using (2.1.32) and (2.2.13). As with the case of nuts, we can also obtain these results using (2.2.9) and regularity of the spinor at the bolt. The sign of κ can be fixed at a given bolt by appropriate choice of conventions, but when there are multiple bolts and/or nuts, relative signs can be important.

Next, using (2.2.13) in (2.1.25) we can rewrite the expression for Φ_2 in (2.1.36) at a bolt

with positive and negative chirality as

$$\Phi_2|_+ = \frac{1}{2}\mathrm{d}\xi^\flat|_+ - \frac{\mathrm{i}}{\sqrt{2}}\widetilde{\mathcal{N}}_{IJ}\widetilde{C}^I F^J = \frac{1}{2}\mathrm{d}\xi^\flat|_+ + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{F}_I(\widetilde{C})F^I\,,$$
$$\Phi_2|_- = -\frac{1}{2}\mathrm{d}\xi^\flat|_- + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{N}_{IJ}C^I F^J = -\frac{1}{2}\mathrm{d}\xi^\flat|_- + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{F}_I(C)F^I\,, \tag{2.2.27}$$

respectively, where to get the first expressions we also used $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + \mathrm{i}\mathcal{I}_{IJ}$ and $\widetilde{\mathcal{N}}_{IJ} = \mathcal{R}_{IJ} - \mathrm{i}\mathcal{I}_{IJ}$, while to get the second we used (2.1.5) and that $\mathcal{F}$, $\widetilde{\mathcal{F}}$ are homogeneous of degree two, with $\mathcal{F}_I(C) \equiv \frac{\partial}{\partial C_I}\mathcal{F}(C)$ and $\mathcal{F}_I(\widetilde{C}) \equiv \frac{\partial}{\partial \widetilde{C}_I}\mathcal{F}(\widetilde{C})$. We also note that smoothness of the gauge field at the bolt implies that $\xi \lrcorner F^I = 0$ and hence from (2.1.31) we have Φ_0^I are constant on the bolt. Then from (2.1.32) and (2.2.13) we deduce $\widetilde{C}^I$ and C^I are constant on the bolt.[10] Thus, integrating Φ_2 over a bolt, from (2.2.27) we have

$$\frac{1}{2\pi}\int_{\mathrm{bolt}_+}\Phi_2 = \mathrm{i}\sqrt{2}\mathcal{F}_I(\widetilde{C})\,\mathfrak{p}^I\,,$$
$$\frac{1}{2\pi}\int_{\mathrm{bolt}_-}\Phi_2 = \mathrm{i}\sqrt{2}\mathcal{F}_I(C)\,\mathfrak{p}^I\,, \tag{2.2.28}$$

where we have defined the magnetic flux through each bolt via

$$\mathfrak{p}^I \equiv \frac{1}{4\pi}\int_{\mathrm{bolt}} F^I\,. \tag{2.2.29}$$

With these ingredients, we can evaluate the contribution to the on-shell action for a given bolt. For a positive chirality bolt, recalling the expression from (2.1.37) and using (2.2.13) we have $\Phi_0 = -\mathrm{i}\mathcal{F}(\widetilde{C})$. Then from (2.2.3) we find

$$\begin{aligned} I_{\mathrm{OS}}^{\mathrm{bolt}_+} &= \frac{\pi}{2G_4}\int_{\mathrm{bolt}_+}\left[\frac{\Phi_2}{2\pi b_2} - \frac{\Phi_0\, c_1(L)}{b_2^2}\right] \\ &= \frac{\pi}{2G_4}\left[\frac{\mathrm{i}\sqrt{2}}{b_2}\mathcal{F}_I(\widetilde{C})\,\mathfrak{p}^I + \frac{\mathrm{i}}{b_2^2}\mathcal{F}(\widetilde{C})\int_{\mathrm{bolt}_+} c_1(L)\right]. \end{aligned} \tag{2.2.30}$$

[10] It is not clear to us whether in addition X^I, $\widetilde{X}^I$ and S are also constant on a bolt.

We next use (2.2.26) and the definition of the u_+^I variables in (2.2.14) to rewrite this as

$$I_{\rm OS}^{{\rm bolt}_+} = \frac{\pi}{G_4}\left[-\kappa {\rm i}\mathcal{F}_I(u_+)\mathfrak{p}^I + {\rm i}\mathcal{F}(u_+)\int_{{\rm bolt}_+} c_1(L)\right], \tag{2.2.31}$$

while for a negative chirality bolt we similarly find

$$I_{\rm OS}^{{\rm bolt}_-} = \frac{\pi}{G_4}\left[-\kappa {\rm i}\mathcal{F}_I(u_-)\mathfrak{p}^I - {\rm i}\mathcal{F}(u_-)\int_{{\rm bolt}_-} c_1(L)\right], \tag{2.2.32}$$

where $\mathcal{F}_I(u_\pm) \equiv \frac{\partial}{\partial u_\pm^I}\mathcal{F}_I(u_\pm)$.

2.2.6 Gravitational free energy

By combining the results for the nuts and bolts we obtain the following general result for the gravitational free energy for solutions satisfying the reality condition (2.2.1):

$$\begin{aligned} F_{\rm grav} = \frac{\pi}{G_4}\Bigg[&\sum_{{\rm nuts}_\pm} \mp\frac{(b_1 \mp b_2)^2}{b_1 b_2}{\rm i}\mathcal{F}(u_\pm) \\ &+ \sum_{{\rm bolts}_\pm}\left(-\kappa {\rm i}\mathcal{F}_I(u_\pm)\mathfrak{p}^I \pm {\rm i}\mathcal{F}(u_\pm)\int_{{\rm bolt}_\pm} c_1(L)\right)\Bigg]. \end{aligned} \tag{2.2.33}$$

Some comments are in order. First, the result (2.2.33) is expressed in terms of quantities evaluated at the fixed point set. However, in the next section we explain how one can relate the fixed point IR data to UV data on the conformal boundary ∂M, and in subsequent sections we demonstrate how this works in more detail for various specific examples. We also notice the expression is written in terms of $u_\pm^I$ defined in (2.2.14). This is a very useful set of variables, as they are constant on fixed point sets, and moreover are invariant under $U(1)_R$ rotations (as they are ratios of terms that are linear in L^I). Therefore, the final result is independent of the phase chosen for the symplectic sections, as it should be. We also emphasize that the signs κ need to be specified for each individual fixed point component, i.e. one κ for each nut and for each bolt, and moreover these need to be determined in a globally consistent way; this is carried out for toric examples in [7].

Second, it is also worth highlighting an observation made in [51] (there in the context of minimal gauged supergravity). The original Killing spinor equation is linear in ϵ, so for

any particular solution we can multiply by an arbitrary non-zero complex number and get another solution. This procedure rescales the Killing vector ξ by an arbitrary positive number and notice that (2.2.33) is invariant under this rescaling. Furthermore, given the reality conditions (2.2.1), we notice that any solution with gauge field and Killing spinor (A, ϵ) can be transformed into another solution with $(-A, \epsilon^c)$, with ϵ^c having the same chirality as ϵ at fixed points, which transforms the Killing vector $\xi \to -\xi$. Correspondingly the final answer should be invariant under $\mathfrak{p}^I \to -\mathfrak{p}^I$ and $\xi \to -\xi$, which implies that *all* $\kappa \to -\kappa$, and indeed it is.

Third, if we set all of the vector multiplets to zero we obtain minimal gauged supergravity and, as described in appendix B, we recover the result obtained in [51].

Lastly, it is also possible to include orbifold singularities both in the bulk and on the boundary. This requires minor modifications which we discuss in section 2.6.

2.3 Flux integrals, UV-IR relations, and holography

In the previous section we showed how equivariant localization can be used to evaluate the on-shell action of a supersymmetric solution of the gauged supergravity theory, obtaining an expression in terms of quantities evaluated at the fixed point set of the R-symmetry. We now show that it is possible to apply equivariant localization to the polyforms of the gauge curvatures (2.1.31). In particular, this allows one to relate expressions evaluated at the fixed point sets to boundary data.

2.3.1 Flux integrals

We first consider integrals of the field strengths F^I over two-cycles in M. For $\Sigma \subset M$ a closed two-dimensional submanifold representing a cycle $[\Sigma] \in H_2(M, \mathbb{Z})$, we define the associated magnetic flux, $\mathfrak{p}^I = \mathfrak{p}^I_{[\Sigma]}$, as

$$\mathfrak{p}^I \equiv \frac{1}{2}\int_\Sigma c_1(F^I) = \frac{1}{4\pi}\int_\Sigma F^I \,. \tag{2.3.1}$$

For a general gauged supergravity model there is no *a priori* reason for these to be quantized. However, for gauged supergravity models that can be uplifted to $D = 10, 11$ supergravity on compact manifolds, the magnetic fluxes are quantized. In section 2.4 we discuss various classes of solutions of the STU model, which can be uplifted on S^7 (or a quotient thereof) to obtain solutions of $D = 11$ supergravity. We have chosen the normalization in (2.3.1) so that $\mathfrak{p}^I \in \mathbb{Z}$ for the STU model with solutions uplifted on S^7.

When Σ is a bolt, we have already seen how these fluxes directly enter into the localized version of the on-shell action (2.2.33) via (2.2.28). If instead ξ is tangent to Σ, hence rotating it, one can also use localization to relate these fluxes to the fixed point data at nuts, as illustrated in the examples later (see sections 2.5.3 and 2.6).

We also note that the global existence of the Killing spinor, as discussed in section 2.1.3, imposes constraints on the flux of the R-symmetry gauge field. As remarked in section 2.2.3.3, ϵ is charged under A^R and at the fixed point set becomes chiral and non-zero. Consider a bolt $\Sigma_\pm$: in this case, the existence of a non-zero section means that one of the line bundles in which $\mathcal{S}M|_{\Sigma_\pm} \otimes \mathscr{L}^{1/2}$ can be decomposed must be a trivial line bundle. It is then straightforward to extend the analysis of [51] to conclude that for a bolt of chirality ± 1 the R-symmetry flux is given by

$$\mathfrak{p}^R_\pm = \frac{1}{2}\zeta_I \mathfrak{p}^I_\pm = \frac{\kappa}{2}\int_{\Sigma_\pm} \left[\pm c_1(L) - c_1(T\Sigma_\pm)\right] . \tag{2.3.2}$$

2.3.2 UV-IR relations

We can also obtain important "UV-IR relations" by considering flux integrals over non-compact two-dimensional submanifolds, whose boundaries end on the asymptotic boundary $\partial M = M_3$. The submanifolds are assumed to have topology $\mathbb{R}^2$, with a UV boundary circle $S^1_{\rm UV} \equiv \partial\mathbb{R}^2 \subset M_3$, and be left invariant under the action of ξ. On such an $\mathbb{R}^2$ we may introduce a local coordinate φ, with $\Delta\varphi = 2\pi$, and write $\xi|_{\mathbb{R}^2} = b\partial_\varphi$. The origin of $\mathbb{R}^2$, which lies on the fixed point set, is referred to as the "IR". From (2.3.1) we can define the gauge-invariant flux threading the $\mathbb{R}^2$ via

$$\Delta^I \equiv \frac{1}{4\pi}\int_{\mathbb{R}^2} F^I . \tag{2.3.3}$$

Using Stokes' theorem we can then write $\Delta^I = \frac{1}{4\pi}(\int_{S^1_{\rm UV}} A^I - \int_{S^1_{\rm IR}} A^I)$, where $S^1_{\rm IR}$ is a small circle surrounding the origin of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is topologically trivial we can choose a regular gauge so that A^I is a global one-form on $\mathbb{R}^2$.[11] In this case the IR integral vanishes and Δ^I fixes the holonomy of the gauge field around $S^1_{\rm UV}$ in the boundary theory:

$$\Delta^I = \frac{1}{4\pi}\int_{S^1_{\rm UV}} A^I \,. \tag{2.3.4}$$

Note that Δ^I is not constant, in general; it will be independent of the coordinate on $S^1_{\rm UV}$, which is tangent to ξ, but it can depend on the remaining coordinates parametrizing the position of $S^1_{\rm UV}$.

Continuing, we can now consider the lowest components of the equivariantly closed forms for the fluxes in (2.1.31), (2.1.32) both in the UV and IR:

$$\sigma^I \equiv -\left.\frac{\mathrm{i}}{4\pi}\Phi_0^I\right|_{S^1_{\rm UV}} \,, \qquad y^I \equiv \left.\frac{1}{2b}\Phi_0^I\right|_{\rm IR} \,, \tag{2.3.5}$$

with the factors added for convenience. Notice that these quantities arise from evaluating the combinations of spinor bilinears and scalars appearing in (2.1.32) and are gauge-invariant. Also notice from the argument below (2.2.27) that y^I is constant on a connected component of the fixed point set, with its value dependent on which component of the fixed point set the $\mathbb{R}^2$ ends on in the bulk. By contrast, like Δ^I, σ^I is not constant in general, but it is independent of the coordinate on $S^1_{\rm UV}$: this follows from the fact that since $\mathrm{d}\Phi_0^I = \xi \lrcorner F^I$ we have $\mathcal{L}_\xi \Phi_0^I = 0$ and that ξ is tangent to $S^1_{\rm UV}$. Here σ^I depends on both the UV scalar mass deformations in the dual boundary theory as well as the boundary geometry, as we discuss further in the next subsection. From the equivariant closure of $\Phi^I_{(F)}$ one then immediately deduces the following relation:

$$\Delta^I + \mathrm{i}\beta\sigma^I = y^I \,, \tag{2.3.6}$$

where $\beta \equiv 2\pi/b$ so that defining $\psi \equiv \varphi/b$, we can also write $\xi = \partial_\psi$ with $\Delta\psi = \beta$. Notice that

[11] In the case that there are more than one such $\mathbb{R}^2$ and not homologous to each other, in general it will not be possible to choose a gauge that is regular on both of them.

with our Euclidean reality conditions (2.2.1) the σ^I are purely imaginary. We also notice that while Δ^I and σ^I are not constant, in general, the linear combination in (2.3.6) is constant (since y^I is constant).

We emphasize that equation (2.3.6) is a non-trivial UV-IR relation. On the right-hand side we have the IR quantities y^I which depend on the values of the scalars and bilinear fields at a specific part of the fixed point set. While on the left-hand side we have the UV boundary quantities σ^I, characterizing mass deformations, as well as the Δ^I which can be thought of as UV holonomies via (2.3.4). All of the y^I, σ^I and Δ^I depend on which $\mathbb{R}^2$ factor one uses to define them. Now the gravitational free energy (2.2.33) is expressed in terms of IR quantities evaluated at the fixed point set, but for applications to holography it is more natural to write it in terms of the boundary data, and this is what the UV-IR relation (2.3.6) allows one to do. A relation analogous to (2.3.6) was conjectured and verified in some examples for which $M \cong \mathbb{R}^2 \times \Sigma_g$ with Σ_g a Riemann surface, in [70].

Concretely, we now relate the IR fixed point quantities y^I to the fixed point data $u^I_\pm$ defined in (2.2.14) that enters the on-shell action in (2.2.33) as follows. We first assume that the origin of the $\mathbb{R}^2 \subset M$ lies on a bolt. Then, if we identify φ and the weight b with φ_2 and b_2, using the same local orthonormal frame which is associated with the projections (2.2.25), and in particular with the same specific κ for that bolt, we deduce that

$$y^I_\pm = \pm\kappa u^I_\pm \,, \tag{2.3.7}$$

for positive or negative chirality bolts, respectively.

For a nut$_\pm$ we similarly have

$$y^I_\pm = -\frac{\kappa}{b}(b_1 \mp b_2)u^I_\pm \,, \tag{2.3.8}$$

where κ is the same specific κ for the nut, as in (2.2.19). Notice here that with an appropriate choice of convention we could always take $b = b_2$, since we may swap $b_1 \leftrightarrow b_2$ using the transformations (2.2.18) (noting from (2.2.19) that for positive chirality this then also requires one to take $\kappa \to -\kappa$). However, we point out that for general examples of toric M,

studied in [7], it is convenient to choose a global ordering of b_1, b_2 at each fixed point, and correspondingly take either $b = b_1$ or $b = b_2$ in (2.3.8) according to this choice, for different copies of $\mathbb{R}^2 \subset M$.

2.3.3 Holography

We are assuming that the general gauged supergravity theory admits a supersymmetric AdS_4 vacuum with constant scalars and vanishing gauge fields. From the equations of motion (2.1.19) the radius squared of the AdS_4 is given by $\ell^2 = -6/\mathcal{V}_*$, where the subscript $*$ denotes evaluation at the vacuum solution. To simplify the subsequent formulae it will be convenient to choose units with $\ell = 1$. Supersymmetry requires $\nabla_i W_* = \nabla_{\tilde{i}} \widetilde{W}_* = 0$ and hence we deduce

$$\mathcal{V}_* = -3(\zeta_I L^I)_*(\zeta_I \widetilde{L}^I)_* = -6\,. \tag{2.3.9}$$

In our conventions, the free energy of the dual SCFT on a round S^3 in the large N limit, F_{S^3}, is related to the $D = 4$ Newton constant via

$$F_{S^3} = \frac{\pi}{2G_4} 4\mathrm{i}\mathcal{F}(L^I_*)\,. \tag{2.3.10}$$

Note that by a field redefinition we can always assume that the given AdS_4 vacuum lies at the origin of the scalar field space, i.e. $z^i_* = \tilde{z}^{\tilde{i}}_* = 0$, and we will now assume that this is the case.

A supersymmetric solution induces a structure on its conformal boundary ∂M, which can be found analysing the asymptotic expansion of the gravity fields near ∂M. In particular, ξ restricted to the boundary defines a Killing vector ξ with a transversely holomorphic foliation. Assuming[12] the orbits of ξ all close, we can write $S^1 \hookrightarrow \partial M \to \Sigma_2$, which is associated to a orbifold bundle $\mathcal{L}$, i.e. Σ_2 will in general be an orbifold (and note we are still assuming that ∂M is a manifold). The metric on ∂M can be written in terms ξ and a local transverse

[12]If ξ has a non-closed orbit a standard argument shows that ξ may be approximated by a sequence of vector fields with closed orbits. Note that when ξ has closed orbits the ratios of weights b_1/b_2 at the nuts are rational numbers.

complex coordinate u as

$$\mathrm{d}s^2_{\mathrm{bdry}} = \eta^2_{(0)} + 4\mathrm{e}^{V_{(0)}}\mathrm{d}u\mathrm{d}\overline{u}\,, \tag{2.3.11}$$

where $V_{(0)} = V_{(0)}(u,\overline{u})$ is a local function and $\eta_{(0)}$ is the global almost contact form on ∂M dual to ξ using this boundary metric and $\xi \lrcorner \eta_{(0)} = 1$ with $\|\xi\|^2 = 1$.[13] The two-dimensional metric on the leaf space is given by

$$\mathrm{d}s^2_2 = 4\mathrm{e}^{V_{(0)}}\mathrm{d}u\mathrm{d}\overline{u}\,. \tag{2.3.12}$$

The R-symmetry gauge field induced on the boundary is determined by the geometry, and its curvature is

$$F^R_{(0)} = \eta_{(0)} \wedge \mathrm{d} *_2 \mathrm{d}\eta_{(0)} + \frac{1}{4}\left[R_{(2d)} - 8\|\mathrm{d}\eta_{(0)}\|^2_2\right]\mathrm{vol}_2\,, \tag{2.3.13}$$

where $R_{(2d)} = -\mathrm{e}^{-V_{(0)}}\partial^2_{u\overline{u}}V_{(0)}$ is the Ricci scalar of (2.3.12), and vol_2 its volume form.

Deformations of the dual $d = 3$ SCFT are obtained by considering different backgrounds $M_3 = \partial M$ and also by turning on background gauge fields as well as allowing suitable boundary conditions for the scalar fields. Since we are studying supersymmetric solutions in the bulk, these deformations are necessarily associated with deformations of the SCFT that preserve supersymmetry. In particular, as remarked earlier, it is possible to turn on scalar mass deformations of the boundary SCFT, parametrized by σ^I, which are related to the lowest component Φ^I_0 of the equivariant form $\Phi^I_{(F)}$ by (2.3.5). The boundary behaviour of Φ^I_0, defined in (2.1.32), has the following general form

$$\Phi^I_0\Big|_{\mathrm{bdry}} = \sqrt{2}\left[L^I_*(*_2\mathrm{d}\eta_{(0)}) + \left(\nabla_i L^I\right)_*\left(z^i_{(1)} - \tilde{z}^{\tilde{i}}_{(1)}\right)\right]\,. \tag{2.3.14}$$

The combination $z^i_{(1)} - \tilde{z}^i_{(1)}$, appearing at the first subleading order in the expansion of the scalar fields near the conformal boundary, is proportional to the source for the boundary operator of dimension 2. From this expression, by restricting to the boundary S^1 of the

[13] Note that a conformal transformation may be required to put the boundary metric in this form.

specific $\mathbb{R}^2$ submanifold associated with σ^I, we immediately obtain

$$\mathrm{i}\sigma^I = \frac{\sqrt{2}}{4\pi}\left[L^I_*(*_2\mathrm{d}\eta_{(0)}) + \left(\nabla_i L^I\right)_*\left(z^i_{(1)} - \tilde{z}^i_{(1)}\right)\right]\Big|_{S^1_{\mathrm{UV}}}\,. \tag{2.3.15}$$

These general results, valid for arbitrary supersymmetric backgrounds on ∂M, substantially generalize a relation that was conjectured in the context of $\partial M \cong S^3$ with vanishing gauge fields, and verified in some examples, in [63].

Notice that there are n independent mass deformations, associated with $(z^i_{(1)} - \tilde{z}^i_{(1)})$, but there are $n+1$ components in σ^I. However, the combination $\zeta_I\sigma^I$ is constrained by the boundary geometry via

$$\mathrm{i}\zeta_I\sigma^I = \frac{1}{2\pi}*_2\,\mathrm{d}\eta_{(0)}\Big|_{S^1_{\mathrm{UV}}}\,. \tag{2.3.16}$$

In the special case that σ^I is constant (for any choice of S^1_{UV}), i.e. $\Phi^I_0\big|_{\mathrm{bdry}}$ in (2.3.14) is constant, we can integrate over the entire base, obtaining

$$\int_{\Sigma_2}\mathrm{i}\zeta_I\sigma^I\,\mathrm{vol}_2 = \frac{1}{2\pi}\int_{\Sigma_2}\mathrm{d}\eta_{(0)} = \frac{1}{b}\int_{\Sigma_2}c_1(\mathcal{L})\,, \tag{2.3.17}$$

where $c_1(\mathcal{L}) \in H^2(\Sigma_2,\mathbb{Q})$ is the first Chern class of the orbibundle $\mathcal{L}$.

The fact that the boundary R-symmetry gauge field is fixed by the geometry as in (2.3.13) allows us to immediately derive some necessary conditions for the possibility of constructing solutions where F^R vanishes. Namely, for (2.3.13) to vanish, we need $*_2\mathrm{d}\eta_{(0)}$ to be constant on Σ_2, and its square must be proportional to the Ricci scalar curvature (see [71] for a more detailed discussion). This will constrain the examples considered in the next section.

2.4 Real solutions

We now consider some specific examples of solutions on manifolds M, subject to the reality conditions (2.2.1) and recall that we also imposed the condition (2.2.2). Our formalism is applicable for general gauged supergravity theories with prepotential $\mathcal{F}$ but we will also focus on the specific case of the STU model, which includes minimal gauged supergravity as a special case.

2.4.1 The STU model

The STU model has three vector multiplets with prepotential given by

$$\mathcal{F}(X^I) = -2\mathrm{i}\sqrt{X^0X^1X^2X^3}\,, \tag{2.4.1}$$

and we take $\zeta_I = 1$ for all $I = 0, 1, 2, 3$. Solutions of the STU model uplift on S^7 to give solutions of $D = 11$ supergravity [72, 73]. For example, the AdS_4 vacuum with vanishing scalars and gauge fields uplifts to the $\mathrm{AdS}_4 \times S^7$ solution dual to the ABJM theory [20]. Minimal gauged supergravity can be recovered from the general set-up by setting all the vector multiplets to zero, as discussed in appendix B. Alternatively, it can be obtained from the STU model by setting $X^0 = X^1 = X^2 = X^3 = \frac{1}{4}$ and $A^0 = A^1 = A^2 = A^3$. Solutions of minimal gauged supergravity can also be uplifted to $D = 11$ in other ways [74], including on an arbitrary seven-dimensional Sasaki–Einstein space.

In Lorentzian signature, the complex scalars z^i of the three vector multiplets of the STU model parametrize three copies of the Poincaré disc with $|z^i| < 1$ and Kähler potential given by

$$\mathcal{K} = -\sum_{i=1}^{3} \log\left(1 - z^i\bar{z}^{\bar{i}}\right)\,. \tag{2.4.2}$$

The associated Hermitian metric and composite connection (2.1.14) take the form

$$\mathcal{G}_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{(1 - z^i\bar{z}^{\bar{j}})^2}\,, \qquad \mathcal{A} = -\frac{\mathrm{i}}{2}\sum_{i=1}^{3}\frac{\bar{z}^{\bar{i}}\mathrm{d}z^i - z^i\mathrm{d}\bar{z}^{\bar{i}}}{1 - z^i\bar{z}^{\bar{i}}}\,. \tag{2.4.3}$$

When needed, we will fix a specific parametrization of the symplectic sections X^I in terms of the physical scalars given by

$$\begin{aligned} X^0 &= \frac{1}{2\sqrt{2}}(1+z^1)(1+z^2)(1+z^3)\,, & X^1 &= \frac{1}{2\sqrt{2}}(1+z^1)(1-z^2)(1-z^3)\,,\\ X^2 &= \frac{1}{2\sqrt{2}}(1-z^1)(1+z^2)(1-z^3)\,, & X^3 &= \frac{1}{2\sqrt{2}}(1-z^1)(1-z^2)(1+z^3)\,. \end{aligned} \tag{2.4.4}$$

This "Freedman–Pufu" choice (also used in [70]) will be useful in comparing with the solutions in [67]; for a clear discussion of other choices of parametrizations and how they are related to

each other, see [65]. In particular, this is helpful if one wants to uplift the STU solutions to $D = 11$ using [72]. The holomorphic superpotential (2.1.10) and the scalar potential (2.1.11) take the form

$$W = \sqrt{2}(1 + z^1 z^2 z^3)\,, \qquad \mathcal{V} = 2\left(3 - \sum_{i=1}^{3} \frac{2}{1 - z^i \bar{z}^{\bar{i}}}\right)\,. \tag{2.4.5}$$

The AdS_4 vacuum with unit radius corresponds to $z^i = z^{\bar{i}} = 0$. The explicit expressions for $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ to give the Lorentzian action (2.1.6) or Euclidean action (2.1.16) for the STU model are rather long and so we don't give them here. For minimal gauged supergravity, the explicit form of the Euclidean action and the supersymmetry transformations are given in appendix B. Regarding the expression for σ^I in (2.3.15), for the AdS_4 vacuum in the parametrization of (2.4.4) we have

$$L^I_* = \frac{1}{2\sqrt{2}}\,, \tag{2.4.6}$$

and

$$\begin{aligned}
(\nabla_i L^0)_* &= \frac{1}{\sqrt{2}}(1,1,1)\,, & (\nabla_i L^1)_* &= \frac{1}{\sqrt{2}}(1,-1,-1)\,,\\
(\nabla_i L^2)_* &= \frac{1}{\sqrt{2}}(-1,1,-1)\,, & (\nabla_i L^3)_* &= \frac{1}{\sqrt{2}}(-1,-1,1)\,.
\end{aligned} \tag{2.4.7}$$

For the STU model, we can uplift on $S^7/\mathbb{Z}_k$ to get solutions dual to ABJM theory. In this case, from (2.3.10) the four-dimensional Newton constant is related to the free energy of ABJM theory on S^3 in the large N limit via

$$F_{S^3} = \frac{\pi}{2G_4} \overset{\text{ABJM}}{=} \frac{\sqrt{2k}\pi}{3} N^{3/2}\,. \tag{2.4.8}$$

2.4.2 Deformations of Euclidean AdS_4, $A^I = 0$

For our first class of examples we take M to be topologically $\mathbb{R}^4 = \mathbb{R}^2_1 \oplus \mathbb{R}^2_2$, with $U(1)^2$ isometry and conformal boundary $\partial M \cong S^3$. The boundary metric on S^3 has $U(1)^2$ isometry but we make no further assumption and in general it can be squashed. We don't make any

assumption on the gauge fields for the moment. We assume that $\xi = \sum_{i=1}^{2} b_i \partial_{\varphi_i}$ where ∂_{φ_i} rotate each $\mathbb{R}_i^2$ and we assume that $b_i \neq 0$ so that there is an isolated fixed point at the origin of $\mathbb{R}^4$. The chirality of the spinor at this nut can be ± 1 and correspondingly from (2.2.33) we have an expression for the free energy in terms of the fixed point data given for the two cases, respectively, by

$$F_{\text{grav}} = \mp \frac{(b_1 \mp b_2)^2}{b_1 b_2} \mathrm{i} \mathcal{F}(u_\pm) \frac{\pi}{G_4} \,. \tag{2.4.9}$$

Recall from (2.2.15) we have the constraints $\zeta_I u_\pm^I = 1$. For a holographic interpretation, we can also write this expression in terms of UV boundary data using the results of section 2.3. This example is particularly simple because there is a single nut in the bulk, and moreover, since the topology is $\mathbb{R}^4$, it is possible to choose a global gauge for the gauge fields when present. We now focus on various sub-cases and make contact with various known solutions.

We first consider solutions with vanishing gauge fields. This will allow us to make contact with the solutions explicitly constructed in [63, 67] associated with real mass deformations of the dual SCFTs on S^3.

With vanishing gauge fields, trivially we have $\Delta^I = 0$. Next, suppose we consider $\mathbb{R}_1^2$ in the decomposition $\mathbb{R}^4 = \mathbb{R}_1^2 \oplus \mathbb{R}_2^2$ so that on this submanifold we have $\xi|_{\mathbb{R}_1^2} = b_1 \partial_{\varphi_1}$. Then from (2.3.6) and (2.3.8) we deduce for the nut$_\pm$ cases we have:

$$y_\pm^I = -\frac{\kappa}{b_1}(b_1 \mp b_2) u_\pm^I = \frac{1}{b_1} 2\pi \mathrm{i} \sigma_{(1)}^I \,, \tag{2.4.10}$$

where the subscript in $\sigma_{(1)}^I$ indicates that we are considering $\mathbb{R}_1^2$, and hence

$$(b_1 \mp b_2) u_\pm^I = -\kappa 2\pi \mathrm{i} \sigma_{(1)}^I \,. \tag{2.4.11}$$

If instead we had chosen $\mathbb{R}_2^2$ we would obtain (2.4.11) but with $\sigma_{(2)}^I$ on the right-hand side. Since the $u_\pm^I$ on the left-hand side are necessary equal (evaluated at the origin of $\mathbb{R}^4$), we can therefore define the constant deformation parameters

$$\sigma^I \equiv \sigma_{(1)}^I = \sigma_{(2)}^I \,. \tag{2.4.12}$$

In fact, with vanishing gauge fields we can deduce a lot more about the UV deformations. Indeed the analysis of the bilinear equations shows that outside of the fixed point set, $\mathrm{d}z^i$ and $\mathrm{d}\tilde{z}^i$ are proportional to the bilinear K (see equation (A.2.4)). As a consequence, the scalars z^i, $\tilde{z}^i$ only depend on the asymptotic radial coordinate, and so the UV mass deformations $z^i_{(1)}$ and $\tilde{z}^{\tilde{i}}_{(1)}$ are necessarily constant everywhere on ∂M. Next, from the vanishing of (2.3.13) we must have $*_2\mathrm{d}\eta_{(0)}$ is also a constant on the boundary (and also $R_{(2d)} = 8\|\mathrm{d}\eta_{(0)}\|_2^2$). Thus, we see that $\Phi_0^I|_{\mathrm{bdry}}$, appearing in (2.3.14), is constant and thus the σ^I in (2.3.15), for any boundary S^1 are all the same constants. Then we can invoke (2.3.16), (2.3.17), or more immediately (2.4.11), to conclude that the deformation parameters σ^I are constrained to satisfy

$$2\pi\mathrm{i}\zeta_I\sigma^I = -\kappa(b_1 \mp b_2)\,. \tag{2.4.13}$$

Using the fact that the prepotential is homogeneous of degree two, we can then write (2.4.9) in terms of UV data as

$$F_{\mathrm{grav}} = \mp\frac{b_2}{b_1}\mathrm{i}\mathcal{F}\left(\frac{2\pi\mathrm{i}\sigma}{b_2}\right)\frac{\pi}{G_4}\,. \tag{2.4.14}$$

A particularly simple case is when $|b_1| = |b_2|$, which is consistent with a conformal boundary that is the round S^3 with $SO(4)$ isometry, which we also assume to be the case. It is not possible to have a nut$_\pm$ solution with $b_1 = \pm b_2$ since then $\sigma^I = 0$ (see footnote 8) and there would be no deformations. However, we can consider the special case that $b_1 = \mp b_2$ with a nut$_\pm$. Normalizing the Killing spinor to have $b_2 = 1$, the UV-IR relation simplifies to $u^I_\pm = \pm\kappa\pi\mathrm{i}\sigma^I$, with the mass parameters σ^I constrained by

$$2\pi\mathrm{i}\zeta_I\sigma^I = \pm 2\kappa\,, \tag{2.4.15}$$

and given in terms of boundary quantities by (2.3.15), which for $\partial M \cong S^3$ with the anti-Hopf fibration gives

$$\sqrt{2}\pi\mathrm{i}\sigma^I = L^I_* + \frac{1}{2}\left(\nabla_i L^I\right)_*\left(z^i_{(1)} - \tilde{z}^i_{(1)}\right)\,, \tag{2.4.16}$$

with all terms constant. Then $F_{\rm grav}$ is given by

$$F_{\rm grav} = 4\mathrm{i}\mathcal{F}(u_\pm)\frac{\pi}{G_4} = \mathrm{i}\mathcal{F}\left(\sqrt{2}\pi\mathrm{i}\sigma\right)\frac{2\pi}{G_4}\,. \tag{2.4.17}$$

In fact this expression[14] was conjectured to be the free energy of mass-deformed, three-dimensional $\mathcal{N}=2$ SCFTs on S^3 in the large N limit in [63] (building on [67]). Thus, our localization approach provides a general supergravity proof of this conjecture, assuming that the solutions actually exist.

For the specific case of the STU model, an explicit solution with $\mathbb{R}^4$ topology, vanishing gauge fields, and $b_1 = \mp b_2$ was constructed in [67]. Its conformal boundary is S^3 with round metric, and in fact the bulk solution also has $SO(4)$ isometry. The UV mass deformations are constant on the S^3. The free energy was also computed in [67], in agreement with (2.4.17) after restricting to the STU model using (2.4.1), and precisely matches the large N limit of the partition function of ABJM theory on S^3 with real mass deformations that was computed in [75].

More generally, we can also consider $|b_1| \neq |b_2|$, which is naturally associated with having a non-round metric on the boundary S^3. The relation between the chirality of the spinor at the nut and the sign of b_1/b_2 can only be determined after considering the global structure of the solutions. Such analysis is subtle and not available for a generic supergravity theory, though it has been done for self-dual metrics that are minimal supergravity solutions on $\mathbb{R}^4$ in [76]. To simplify the following discussion, with no essential loss of generality, we will restrict to the case

$$b_1/b_2 > 0\,, \qquad \hat{b}^2 = b_1/b_2\,, \tag{2.4.18}$$

and then the gravitational free energy can be written

$$F_{\rm grav} = \mp\left(\hat{b} \mp \frac{1}{\hat{b}}\right)^2 \mathrm{i}\mathcal{F}(u_\pm)\frac{\pi}{G_4}\,, \tag{2.4.19}$$

[14] One should identify $(z^i_{(1)} - \tilde{z}^i_{(1)})_{\rm here}$, which we argued above is constant, with $(\mathrm{i}e^\alpha_a \mathfrak{m}^a)_{\rm there}$, where e^α_a is a frame for the boundary Kähler metric on the space of scalar fields.

with $\zeta_I u_\pm^I = 1$. With vanishing gauge fields the UV-IR relation can be written

$$u_\pm^I = -\kappa \frac{2\pi \mathrm{i}}{b_2(\hat{b}^2 \mp 1)} \sigma^I \,, \tag{2.4.20}$$

with the constant σ^I given in (2.4.16).

A field theory conjecture for the large N limit of the free energy of ABJM theory with real and constant mass deformations on a squashed S^3, with squashing parameter $\hat{b}$, appears in [77] (see also the recent [78]). There, the three independent real mass deformations are parametrized in terms of $\hat{\sigma}_a$,[15] $a = 1, 2, 3, 4$, constrained by $\sum_a \hat{\sigma}_a = 2$. In this notation, the large N limit of the ABJM free energy has the form

$$F = \left(\hat{b} + \frac{1}{\hat{b}}\right)^2 \sqrt{\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 \hat{\sigma}_4}\, F_{S^3} \,. \tag{2.4.21}$$

Our result (2.4.19) specialized to the STU model clearly matches the field theory conjecture, provided that we restrict to the nut$_-$ solutions, and we identify

$$u_-^I = -\kappa \frac{2\pi \mathrm{i}}{b_2(\hat{b}^2 + 1)} \sigma^I \quad \leftrightarrow \quad \frac{1}{2} \hat{\sigma}_{I+1} \,. \tag{2.4.22}$$

It would be interesting to construct such nut$_-$ solutions with vanishing gauge fields (another possibility, however, is that such solutions require the existence of bulk gauge fields). It would also be interesting to know whether or not there are analogous nut$_+$ solutions which would correspond to saddle points in the dual SCFT in addition to those considered in [77].

2.4.3 Deformations of Euclidean AdS$_4$, $A^I \neq 0$

We now consider solutions with non-vanishing gauge fields. To simplify the discussion, we again take

$$b_1/b_2 > 0 \,, \qquad \hat{b}^2 = b_1/b_2 \,, \tag{2.4.23}$$

[15] These were labelled Δ_a in [77].

and correspondingly for nut$_\pm$ solutions we have the same fixed point result (2.4.19) as in the previous section, that is

$$F_{\rm grav} = \mp \left(\hat{b} \mp \frac{1}{\hat{b}}\right)^2 \mathrm{i}\mathcal{F}(u_\pm)\frac{\pi}{G_4}\,, \tag{2.4.24}$$

with $\zeta_I u^I_\pm = 1$.

The UV-IR relation, though, is now different. Considering $\mathbb{R}^2_1$ as before, from (2.3.6), (2.3.8) we now have

$$u^I_\pm = -\kappa \frac{\hat{b}^2}{\hat{b}^2 \mp 1}\left(\Delta^I_{(1)} + \frac{2\pi}{b_1}\mathrm{i}\sigma^I_{(1)}\right)\,. \tag{2.4.25}$$

The individual values of $\Delta^I_{(1)}$ and $\sigma^I_{(1)}$, associated with $\mathbb{R}^2_1$, might not be constant, but the particular combination on the right-hand side of (2.4.25) is constant (since the left-hand side is). Also, the value of $\sigma^I_{(1)}$ is related to boundary deformations via (2.3.15) and in particular $\zeta_I\sigma^I_{(1)}$ is constrained by the boundary geometry by (2.3.16). Considering instead $\mathbb{R}^2_2$, we similarly obtain

$$u^I_\pm = -\kappa \frac{1}{\hat{b}^2 \mp 1}\left(\Delta^I_{(2)} + \frac{2\pi}{b_2}\mathrm{i}\sigma^I_{(2)}\right)\,, \tag{2.4.26}$$

which is in fact equal to the right-hand side of (2.4.25). We also have $\zeta_I\sigma^I_{(2)}$ is constrained by (2.3.16) but in general $\zeta_I\sigma^I_{(1)} \neq \zeta_I\sigma^I_{(2)}$. Notice that the UV data is necessarily constrained via

$$b_1\Delta^I_{(1)} + 2\pi\mathrm{i}\sigma^I_{(1)} = b_2\Delta^I_{(2)} + 2\pi\mathrm{i}\sigma^I_{(2)}\,. \tag{2.4.27}$$

Since the prepotential is homogeneous of degree two, using the UV variables associated with $\mathbb{R}^2_2$, for example, we can write

$$F_{\rm grav} = \mp\frac{1}{\hat{b}^2}\mathrm{i}\mathcal{F}\left(\Delta_{(2)} + \frac{2\pi}{b_2}\mathrm{i}\sigma_{(2)}\right)\frac{\pi}{G_4}\,. \tag{2.4.28}$$

It is striking that $F_{\rm grav}$ depends on a specific combination of UV deformation data that is constant, even if the full UV deformation data is not constant; this applies more generally to subsequent examples we shall consider.

2.4.4 Taub-bolt saddle solutions

The second class of examples we consider consists of a spacetime M with topology

$$\mathbb{C} \hookrightarrow \mathcal{O}(-p) \to \Sigma_g \,, \tag{2.4.29}$$

where Σ_g is a Riemann surface of genus g and $-p$ is the degree of the complex line bundle constructed over it. We assume that $\xi = b\partial_\varphi$ rotates the $\mathbb{C}$ fibre, where $\varphi \sim \varphi + 2\pi$. If $g > 1$ this is necessarily the case, but if $g = 0$, 1 it is also possible that ξ rotates both fibre and base, in which case the symmetry is at least toric and is discussed in section 2.5.1 (and in more detail and generality in [7]). The conformal boundary ∂M is the three-manifold $\mathcal{M}_{g,p}$, which is by definition a circle bundle over Σ_g with first Chern number $-p$.

By construction, the R-symmetry Killing vector vanishes at the origin of the $\mathbb{C} \cong \mathbb{R}^2$ fibre, and thus fixes the base Σ_g, which is a bolt with normal bundle $\mathcal{O}(-p)$. The gravitational free energy can be immediately obtained from (2.2.33): there is only one bolt, so we can choose $\kappa = +1$, and the only remaining sign characterizing the solution is the chirality of the spinor at the bolt

$$F_{\text{grav}} = -\left[\pm p\, \mathrm{i}\mathcal{F}(u_\pm) + \mathrm{i}\mathcal{F}_I(u_\pm)\mathfrak{p}^I_\pm\right] \frac{\pi}{G_4} \,. \tag{2.4.30}$$

Here we recall that $\zeta_I u^I_\pm = 1$ and because of supersymmetry (see (2.3.2)), we have

$$\zeta_I \mathfrak{p}^I_\pm = \mp p - 2(1-g) \,. \tag{2.4.31}$$

For the purposes of holography, it is more convenient to express the result in terms of the "UV" data discussed in section 2.3. There is a natural family of $\mathbb{R}^2$ non-compact submanifolds, namely copies of the fibre $\mathbb{C} \cong \mathbb{R}^2$ of (2.4.29) at a chosen point on the Riemann surface base Σ_g. Equations (2.3.6) and (2.3.7) relate $u^I_\pm$, the constant values of u^I at the chosen point on the bolt Σ_g of chirality $\pm$, and the holonomy and flux of the gauge field at the boundary

$$u^I_\pm = \pm\left(\Delta^I + \mathrm{i}\beta\sigma^I\right) \,. \tag{2.4.32}$$

Localization in the bulk hence allows us to deduce that the right-hand side, which are *a priori*

functions on the conformal boundary $\partial M \cong \mathcal{M}_{g,p}$, are independent of the Riemann surface directions, and are therefore constant, although note that the individual elements Δ^I and σ^I may not be constant.

If we furthermore assume that the σ^I are constant, i.e. $\Phi_0^I\big|_{\text{bdry}}$ in (2.3.14) is constant, in this case the constraint (2.3.17) simplifies and we can write

$$2\pi \mathrm{i}\zeta_I \sigma^I = -\frac{p}{\mathrm{Vol}(\Sigma_g)}\frac{2\pi}{b}\,. \tag{2.4.33}$$

Together with the constraint on $u_\pm^I$, this also implies that the holonomy of the R-symmetry gauge field should satisfy

$$\zeta_I \Delta^I = \pm 1 + \frac{p}{\mathrm{Vol}(\Sigma_g)}\frac{2\pi}{b^2}\,. \tag{2.4.34}$$

For the STU model, we note that the result (2.4.30) can be written as

$$F_{\text{grav}} = -2\sqrt{u_\pm^0 u_\pm^1 u_\pm^2 u_\pm^3}\left[\pm 2p + \sum_{I=0}^{3}\frac{\mathfrak{p}_\pm^I}{u_\pm^I}\right]F_{S^3}\,. \tag{2.4.35}$$

Solutions with the topology (2.4.29) and $p \neq 0$ are not known in the STU model. Within the STU model, solutions with non-vanishing scalars are only known via analytical and numerical analysis when $p = 0$ [70]. The topology in this case is that of a direct product $\mathbb{C} \times \Sigma_g$. The resulting gravitational free energy obtained from (2.4.35) is

$$F_{\text{grav}} = -2\sqrt{u_\pm^0 u_\pm^1 u_\pm^2 u_\pm^3}\sum_{I=0}^{3}\frac{\mathfrak{p}_\pm^I}{u_\pm^I}F_{S^3}\,, \tag{2.4.36}$$

and it is a function of the fluxes $\mathfrak{p}_\pm^I$ and the gauge field variables Δ^I, σ^I that are constrained to satisfy (2.4.31) and, if we also assume that σ^I are constant, (2.4.33) and (2.4.34), namely

$$\sum_{I=0}^{3}\mathfrak{p}_\pm^I = 2(g-1)\,, \qquad \sum_{I=0}^{3}\Delta^I = \pm 1\,, \qquad \sum_{I=0}^{3}\sigma^I = 0\,. \tag{2.4.37}$$

Formula (2.4.36) and the constraints (2.4.37) match the expressions found by explicit solution of the equations of motion in [70].[16] The dual field theory is ABJM on $S^1 \times \Sigma_g$ deformed

[16]The map between [70] and (2.4.36) is given by $\left(\xi^2 g^2\right)_{\text{there}} = \frac{1}{2}$ and $\hat{u}^I_{\text{there}} = 2\pi u^I$.

by an R-symmetry flux through Σ_g (a "topological twist"), as required in order to preserve supersymmetry, and the free energy has a Hilbert space interpretation as a topologically twisted index [79–81]. As expected, (2.4.36) matches the large N limit of the topologically twisted index computed in [82], and the constraints (2.4.37) match those required to have the large N limit scaling like $N^{\frac{3}{2}}$ and thus a bulk gravity dual.

Explicit solutions with the topology (2.4.29) and $p \neq 0$ are known in minimal supergravity, both for $g = 0$ [83] and for $g > 0$ [84].[17] Applying the results of appendix B to (2.4.30), we find that the gravitational free energy of these solutions of minimal gauged supergravity is given by[18]

$$F_{\text{grav}} = \frac{\pi}{2G_4}\left(\pm\frac{p}{4} + 1 - g\right) , \tag{2.4.38}$$

where from (2.4.31) the flux through the bolt is given by

$$\mathfrak{p}^0_\pm = \mp\frac{p}{4} - \frac{1-g}{2} . \tag{2.4.39}$$

If we also assume that σ^0 is constant (which would follow from the spinor bilinear P being constant on the boundary or, equivalently, the right-hand side of (2.3.16) being constant on the boundary) then from (2.4.33), (2.4.34) we have the UV data

$$\mathfrak{p}^0_\pm = \mp\frac{p}{4} - \frac{1-g}{2} , \qquad 2\pi\mathrm{i}\sigma^0 = -\frac{p}{\text{Vol}(\Sigma_g)}\frac{2\pi}{4b} , \qquad \Delta^0 = \pm\frac{1}{4} + \frac{p}{\text{Vol}(\Sigma_g)}\frac{2\pi}{4b^2} . \tag{2.4.40}$$

It is immediate to check that F_{grav} matches the on-shell action computed for the explicit solutions (see (3.73), (3.74) of [84]), and that the flux through the bolt is also in agreement (see (3.48) of [84], accounting for the factor of 2 difference in the normalization used in (2.3.1)). Moreover, it is also straightforward to check that the expression for σ^0 given in (2.4.40) is in agreement with that constructed out of the Killing spinors of the solution (taking care that the choice of radial coordinate in [84] effectively fixes the otherwise arbitrary norm of the Killing spinor, and thus in turn the weights of the Killing vector and σ^0).

[17] In the case of [83], only the non-self-dual $\frac{1}{4}$-BPS solutions satisfy the assumptions in this section, as the R-symmetry Killing vector rotates only the fibre of the fibration. Whilst the $\frac{1}{2}$-BPS solutions have the same topology (2.4.29), the R-symmetry Killing vector rotates the sphere as well. This case is then captured more effectively using methods from toric geometry discussed in [7].

[18] When comparing with [51, (4.35)], recall that a $\pm$ solution there corresponds to a $\mp$ solution here.

More interestingly, the expressions obtained from application of the localization theorem provide predictions for the large N limit of the free energy on $\partial M \cong \mathcal{M}_{g,p}$ even in cases when the gravity solution is not known, and they match the field theory result, when this is known (see also [85]). For instance, (2.4.35) gives a prediction for the large N limit of the free energy of ABJM theory on $\mathcal{M}_{g,p}$ without the need of an explicit solution of the STU model on (2.4.29). Remarkably, the two possibilities corresponding to the two chiralities of the spinor at the bolt match the two solutions of the extremization problem in the large N limit of ABJM found in [84], which read

$$
\begin{aligned}
-\log Z^{\text{ABJM}}_{\mathcal{M}_{g,p}} &= -\frac{2\pi N^{3/2}}{3}\sqrt{2k[m_1][m_2][m_3][m_4]}\left(-2p+\sum_{i=1}^{4}\frac{\mathfrak{n}_i}{[m_i]}\right), \\
-\log Z^{\text{ABJM}}_{\mathcal{M}_{g,p}} &= -\frac{2\pi N^{3/2}}{3}\sqrt{2k[\hat{m}_1][\hat{m}_2][\hat{m}_3][\hat{m}_4]}\left(+2p+\sum_{i=1}^{4}\frac{\hat{\mathfrak{n}}_i}{[\hat{m}_i]}\right).
\end{aligned}
\tag{2.4.41}
$$

The variables $[m_i]$ and $[\hat{m}_i] \equiv 1-[m_i]$ are interpreted as the (fractional part of the) masses and $\mathfrak{n}_i$, $\hat{\mathfrak{n}}_i$ as the magnetic charges of the chiral fields, and they satisfy the constraints

$$
\sum_{i=1}^{4}[m_i] = \sum_{i=1}^{4}[\hat{m}_i] = 1\,, \quad \sum_{i=1}^{4}\mathfrak{n}_i = p-2(1-g)\,, \quad \sum_{i=1}^{4}\hat{\mathfrak{n}}_i = -p-2(1-g)\,. \tag{2.4.42}
$$

If we identify the sets of variables via

$$
\mathfrak{u}_+^I \leftrightarrow [\hat{m}_i]\,, \quad \mathfrak{u}_-^I \leftrightarrow [m_i]\,, \quad \mathfrak{p}_+^I \leftrightarrow \hat{\mathfrak{n}}_i\,, \quad \mathfrak{p}_-^I \leftrightarrow \mathfrak{n}_i\,, \tag{2.4.43}
$$

then after using (2.4.8) we immediately see that our expressions for the free energy and constraints (2.4.35) and (2.4.31) correctly reproduce (2.4.41) and (2.4.42). Note that on the supergravity side the UV/IR relation (2.4.32) allows for both σ^I (parametrizing mass deformations/boundary geometry) and Δ^I (parametrizing holonomies), which are not necessarily constant in general, and still giving the same free energy.

Even more generally, the authors of [84] computed the large N limit of the free energy on $\mathcal{M}_{g,p}$ for a large class of quiver gauge theories with $U(N)$ gauge factors, bifundamental and (anti-)fundamental chiral multiplets, and Chern–Simons levels with some additional conditions imposed which ensures there is a well-behaved M-theory description at large N,

with degrees of freedom scaling like $N^{3/2}$. The theory is assumed to have a polynomial superpotential

$$W = \sum_{\alpha} \prod_{i} \Phi_i^{q_i^{\alpha}} \,, \tag{2.4.44}$$

where Φ_i are the chiral multiplets, labelled by i, while α labels monomial terms in the superpotential. The $q_i^{\alpha} \in \mathbb{Z}_{\geq 0}$ hence specify W. The large N limit is computed by extremizing an effective twisted superpotential $\mathcal{W}$ (also referred to as Bethe potential) [82].[19] The authors of [84], building on the work of [64, 87], find two solutions to the extremization problem, namely[20]

$$\begin{aligned} -\log Z_{\mathcal{M}_{g,p}} &= p\, 2\pi \mathrm{i} \mathcal{W}([m_i]) - \sum_i \mathfrak{n}_i \partial_i \left[2\pi \mathrm{i} \mathcal{W}([m_i]) \right] \,, \\ -\log Z_{\mathcal{M}_{g,p}} &= -p\, 2\pi \mathrm{i} \mathcal{W}([\hat{m}_i]) - \sum_i \hat{\mathfrak{n}}_i \partial_i \left[2\pi \mathrm{i} \mathcal{W}([\hat{m}_i]) \right] \,, \end{aligned} \tag{2.4.45}$$

where the variables are constrained to satisfy

$$\begin{aligned} &\sum_i q_i^{\alpha} [m_i] = 1 = \sum_i q_i^{\alpha} [\hat{m}_i] \,, \\ &\sum_i q_i^{\alpha} \mathfrak{n}_i = p - 2(1-g) \,, \qquad \sum_i q_i^{\alpha} \hat{\mathfrak{n}}_i = -p - 2(1-g) \,. \end{aligned} \tag{2.4.46}$$

Note there is one constraint for each monomial term in the superpotential W. In this case we should not use the simple dictionary (2.4.43), but it is clear that if we write

$$\begin{aligned} &\frac{\pi}{G_4} \mathrm{i} \mathcal{F}(\mathfrak{u}_+) \leftrightarrow 2\pi \mathrm{i} \mathcal{W}_{\text{ext}}([\hat{m}_i]) \,, \quad \frac{\pi}{G_4} \mathrm{i} \mathcal{F}(\mathfrak{u}_-) \leftrightarrow 2\pi \mathrm{i} \mathcal{W}_{\text{ext}}([m_i]) \,, \\ \zeta_I \mathfrak{u}_+^I \leftrightarrow \sum_i q_i^{\alpha} [\hat{m}_i] \,, &\quad \zeta_I \mathfrak{u}_-^I \leftrightarrow \sum_i q_i^{\alpha} [m_i] \,, \quad \zeta_I \mathfrak{p}_+^I \leftrightarrow q_i^{\alpha} \hat{\mathfrak{n}}_i \,, \quad \zeta_I \mathfrak{p}_-^I \leftrightarrow q_i^{\alpha} \mathfrak{n}_i \,, \end{aligned} \tag{2.4.47}$$

then (2.4.30) and (2.4.31) match (2.4.45) and (2.4.46). We remark that the two sets of conditions (2.4.31) and (2.4.46) have been derived in different frameworks: the first set purely

[19] This is the superpotential of a $(2,2)$ two-dimensional theory obtained reducing the theory to two-dimensional $\mathbb{R}^2$. The fact that $\mathcal{W}$ is relevant also for three-dimensional quantum field theories on $\mathcal{M}_{g,p}$ comes from the realization that their partition function on $\mathcal{M}_{g,p}$ is in fact determined by the low-energy description on the Coulomb branch of the theory formulated on $\mathbb{R}^2 \times S^1$ [86].

[20] Here we have corrected the expression (5.16) in [84], taking the variable of $\mathcal{W}_{\text{ext}}$ to be $[\hat{m}_i] \equiv 1 - [m_i]$, and changing the sign of the first term.

by analysis of supersymmetric solutions in gauged supergravity, the second set by a large N limit of the free energy of quiver gauge theories. A crucial step in the match is the identification of the gauged supergravity prepotential evaluated at the fixed sets with the extremum of the effective twisted superpotential. This relation had first been observed for the topology $p = 0$ (that is, when the dual observable is the topologically twisted index) in [64]. Here we have shown that the conjectural holographic relation is valid for any p. In turn, the gauged supergravity prepotential is also related to the large N limit of the partition function on S^3 by the conjecture proved in section 2.4.2 [63, 64].

Building on these results, we can further conjecture that quite generally for this class of gauge theories, the on-shell action (2.2.33) on M gives the large N limit of a holographic theory on ∂M provided $\mathrm{i}\mathcal{F}$ is identified with the extremum of the effective twisted superpotential $2\pi\mathrm{i}\mathcal{W}_{\mathrm{ext}}$.

2.5 Complex solutions and black holes

The examples considered in the previous section are solutions satisfying the reality conditions (2.2.1). However, the equations of motion (2.1.19) and the Killing spinor equations (2.1.22) are analytic in the supergravity fields, so we can also consider analytically continued *complex* solutions. We shall now see that, despite the fact that the derivation in section 2.2 relies on the reality assumptions (2.2.1), the final result (2.2.33) still holds even for the complexified solutions, for some specific examples.

As a first example, we return to the solution of the STU model on $\mathbb{R}^4$ with vanishing gauge fields found in [67].In order to derive the gravitational free energy (2.4.17), we assumed that $z^i, \tilde{z}^i \in \mathbb{R}$. In particular, these scalar fields depend on three constants c_i, which we took to be real. However, in the construction of the explicit solutions in [67] one can analytically continue c_i and take them to be complex. On the other hand, the result (2.4.17), which here in particular takes exactly the form of the results in [67] even if $c_i \in \mathbb{C}$, which is suggestive that the localization approach can be extended when the reality constraints are relaxed.

2.5.1 Black holes and indices

Another interesting class of complex supersymmetric solutions appears when considering non-extremal deformations of Wick-rotated supersymmetric black holes [88]. By supersymmetric black holes, we mean (real) Lorentzian and supersymmetric black hole solutions. Typically the supersymmetric Lorentzian solutions have pathologies, such as naked singularities or closed time-like curves, unless one imposes that they are also extremal with vanishing temperature. In this case the metric includes a near-horizon AdS_2 factor with an infinite throat [89], which makes it challenging to compute the on-shell Euclidean action after Wick rotation. To resolve this issue, it has been proposed that one should consider a broader class of Euclidean solutions, with complex metrics in general, that are supersymmetric but non-extremal [88, 90, 91].[21] In this setting it is possible to compute the on-shell action to obtain the free energy and develop a thermodynamic picture, including a quantum statistical relation. Importantly, the on-shell action computed on the supersymmetric non-extremal solutions can be expressed in terms of "reduced" chemical potentials which remain finite (and complex) when taking the extremal limit. The reduced chemical potentials are the natural variables used to express the dual supersymmetric observables, and we shall find them appearing below when considering the UV-IR relations. Furthermore, by taking a Legendre transform of the on-shell action one can recover the entropy of the extremal supersymmetric Lorentzian solutions.

The field theory observable dual to the on-shell action can be identified as a supersymmetric index on $\Sigma \times S^1$. That is, it can be interpreted as a trace over the Hilbert space of the dual SCFT quantized on Σ, with supersymmetric boundary conditions that guarantee that the resulting observable is independent of the size of S^1 [93]. This explains why one can use supersymmetric non-extremal solutions to extract information about the extremal solutions.

We are therefore interested in complex supersymmetric and non-extremal solutions of the Euclidean gauged supergravity theory. Such solutions do not satisfy, in general, the reality conditions that we imposed in (2.2.1) in order to derive the localization formula. Nevertheless, proceeding formally we obtain formulae consistent with known results, which

[21] For a discussion of the relation with the extremization advocated in [82, 92] see [70].

strongly indicates that the localization results can be extended to this larger class of Euclidean solution.

The relevant supersymmetric, non-extremal solutions have M with topology $\Sigma \times \mathbb{R}^2$. Topologically the conformal boundary is $\partial M = \Sigma \times S^1$, but the metric may not in general be a direct product. Different choices of Σ and Killing vector ξ correspond to different black hole solutions after performing a Wick rotation and taking the extremal limit. Assuming that the solutions exist we can formally compute the gravitational free energy and compare with some known solutions.

2.5.2 Topologically twisted black holes

We first consider the case of a direct product $M = \Sigma_g \times \mathbb{R}^2$ where Σ_g is a compact Riemann surface of genus g and ξ rotates only the $\mathbb{R}^2$ factor. This case was also considered in section 2.4.4 as the subcase of a bolt with trivial normal bundle $\mathcal{O}(0)$: the resulting gravitational free energy for a generic prepotential is

$$F_{\text{grav}} = -\frac{\pi}{G_4} \mathrm{i} \mathcal{F}_I(u_\pm) \mathfrak{p}_\pm^I \,, \tag{2.5.1}$$

with the constraint on the flux given by

$$\zeta_I \mathfrak{p}_\pm^I = 2(g-1) \,, \tag{2.5.2}$$

and, assuming σ^I are constant, the UV constraints

$$\zeta_I \Delta^I = \pm 1 \,, \qquad \zeta_I \sigma^I = 0 \,. \tag{2.5.3}$$

Recall that the σ^I are given as in (2.3.15). If we further set the mass deformations to zero, we can take

$$\sigma^I = 0 \,, \tag{2.5.4}$$

(since the bolt has trivial normal bundle) and we then have

$$F_{\rm grav} = -\frac{\pi}{G_4}\mathrm{i}\mathcal{F}_I(\Delta)\mathfrak{p}^I_\pm \,. \tag{2.5.5}$$

As already mentioned, the dual observable is the topologically twisted index on $\Sigma_g \times S^1$ [79–81].

Solutions of this topology, with $\sigma^I = 0$, are supersymmetric deformations of static dyonic black holes with supersymmetry preserved via a topological twist on Σ_g (see e.g. [94–97] for some early references). Equation (2.5.5) with $\zeta_I\Delta^I = \pm 1$ provides an expression for the grand-canonical free energy of these solutions in terms of the magnetic charges and the reduced electrostatic potentials.[22] As suggested in [88], extending the canonical prescriptions of [98] to complex solutions, the entropy of the Lorentzian extremal supersymmetric static dyonic black hole can be obtained by taking the real part of the (constrained) Legendre transform of $F_{\rm grav}$ with respect to the Δ^I, obtaining a function of the conjugate variable (the electric charges). Concretely, one extremizes

$$S = -F_{\rm grav} - \mathrm{i}q_I\Delta^I \,, \tag{2.5.6}$$

with the constraint $\zeta_I\Delta^I = \pm 1$, and then imposes that one obtains a real positive solution $S_*(q,\mathfrak{p})$. The entropy of static extremal supersymmetric black hole solutions in a generic $\mathcal{N}=2$ gauged supergravity is given in terms of a (quartic) invariant of the special geometry describing the scalar manifold [99]. It would be interesting to show that this result can be recovered using the extremization of (2.5.6) described above.

Analytic solutions of the above type are known for the special case of the X^0X^1 model, which is a truncation of the STU model obtained by setting pairwise scalars and gauge fields to be equal [100]. For these solutions, the expression (2.5.5) matches the gravitational free

[22] The latter follows from the fact that the electrostatic potential is defined as the difference of $\xi \lrcorner A^I$ between the conformal boundary and the horizon, and this agrees with the definition of Δ^I, which are necessarily constant in this topology when $\sigma^I = 0$ (see below (2.3.3)).

energy. Specifically, for the X^0X^1 model we find

$$F_{\text{grav}} = \mp\frac{2\pi}{G_4}\left(\Delta^1_\pm \mathfrak{p}^0_\pm + \Delta^0_\pm \mathfrak{p}^1_\pm\right) , \qquad 2\Delta^0_\pm + 2\Delta^1_\pm = \pm 1 , \quad \mathfrak{p}^0_\pm + \mathfrak{p}^1_\pm = g-1 , \tag{2.5.7}$$

which matches eq. (4.15) of [100] upon identifying

$$\left(2\Delta^I\right)_{\text{here}} \leftrightarrow (\varphi_I)_{\text{there}} , \qquad \left(\mathfrak{p}^I_\pm\right)_{\text{here}} \leftrightarrow (2G_4 P_I)_{\text{there}} . \tag{2.5.8}$$

For this case, the constrained extremization of (2.5.6) gives $q_0 = q_1$ and $\mathfrak{p}^0_\pm = \mathfrak{p}^1_\pm = (g-1)/2$, from which $S = \frac{\pi}{G_4}\mathfrak{p}^0_\pm \mp \frac{\mathrm{i}}{2}q_0$. Demanding that this is real then imposes $q_0 = 0$ and the entropy is given by

$$S_* = \frac{\pi}{G_4}\mathfrak{p}^0_\pm , \tag{2.5.9}$$

with $\mathfrak{p}^0_\pm = (g-1)/2$.

2.5.3 Rotating black holes

Next we consider solutions with topology $S^2 \times \mathbb{R}^2$ with $\xi = \varepsilon\partial_{\varphi_1} + \partial_{\varphi_2}$, where $\varphi_{1,2} \sim \varphi_{1,2} + 2\pi$ are angular coordinates on S^2 and $\mathbb{R}^2$, respectively. For $\varepsilon \neq 0$, there are two isolated fixed points, corresponding to the origin of $\mathbb{R}^2$ at the North and South poles of S^2, each of which is labelled with a chirality $\chi_{N,S}$ and a sign $\kappa_{N,S}$ (as introduced in section 2.2.4); when $\varepsilon = 0$ there is a bolt fixed point set as just discussed in section 2.5.2. The weights of the $U(1)$ action at the poles (in the conventions of [7]) are given by

$$(b^1_N, b^2_N) = (-1, \varepsilon) , \qquad (b^1_S, b^2_S) = (-\varepsilon, -1) . \tag{2.5.10}$$

A first expression for the gravitational free energy, expressed in terms of fixed point data, can be obtained from (2.2.33):

$$F_{\text{grav}} = \frac{\pi}{G_4}\frac{1}{\varepsilon}\left[\chi_N\left(1+\chi_N\varepsilon\right)^2 \mathrm{i}\mathcal{F}(u_N) - \chi_S\left(1-\chi_S\varepsilon\right)^2 \mathrm{i}\mathcal{F}(u_S)\right] . \tag{2.5.11}$$

An expression more useful for applications to holography can be found following the discussion in section 2.3. We first consider the flux of the gauge fields through the S^2: applying the

BVAB theorem on $\Phi^I_{(F)}$ in (2.1.31) gives

$$\mathfrak{p}^I = \frac{1}{2\varepsilon}\left(\Phi^I_0|_N - \Phi^I_0|_S\right) . \tag{2.5.12}$$

Moreover, using (2.2.20) we find a condition for the R-symmetry gauge field flux

$$\zeta_I \mathfrak{p}^I = \frac{\kappa_N + \chi_S \kappa_S}{\varepsilon} + \chi_N \kappa_N - \kappa_S \, . \tag{2.5.13}$$

The fixed point set here contains disconnected components (namely two nuts), and thus we need to impose a global regularity condition in order to relate κ_N and κ_S, as mentioned in section 2.2.4. Namely, the spinor must be regular on the S^2 submanifold. This implies that it should have a definite fixed charge under the $U(1)$ generated by ∂_{φ_2}, which recall rotates the $\mathbb{R}^2$ direction and hence by construction fixes the S^2 submanifold. In the notation of the local frame at the nut introduced in section 2.2.4, we see from the weights in (2.5.10) that the normal directions to S^2 at the North pole are $\bar{e}^{12}$, and thus, from (2.2.19), we find that the charge of the Killing spinor at the North pole is $-\kappa_N/2$. On the other hand, again from the weights in (2.5.10), we see that the normal directions to S^2 at the South pole are $\bar{e}^{34}$, and from (2.2.19) we find that the charge of the Killing spinor at the South pole is $\chi_S\kappa_S/2$. Requiring that the spinor is regular everywhere on S^2 then implies that we should choose $\kappa_N = -\chi_S\kappa_S$ (see also [7]). The R-symmetry flux (2.5.13) hence simplifies to

$$\zeta_I \mathfrak{p}^I = -\kappa_S(1+\sigma) \, , \tag{2.5.14}$$

where $\sigma \equiv \chi_N\chi_S$.[23] The case $\sigma = +1$, when the chiralities are the same at each pole, corresponds to a topological twist with $\zeta_I\mathfrak{p}^I = -2\kappa_S$, which is the case that allows us to take the limit $\varepsilon \to 0$ recovering (2.5.2) (with $\kappa_S = +1$); while the case $\sigma = -1$, when the chiralities are opposite at each pole, corresponds to an anti-twist [68], has $\zeta_I\mathfrak{p}^I = 0$ (these two cases are referred to as the A-twist and the identity twist, respectively, in [53]).

For the fluxes through the non-compact $\mathbb{R}^2$ submanifolds with origin at the North and

[23]The sign σ should not be confused with the scalar variables σ^I. We also note that if we want to further compare with the notation of [7] we should also relate $\kappa_N\chi_N = (\sigma_1)_{\text{there}}$ and $\kappa_S = (-\sigma_0)_{\text{there}}$.

South poles, from the discussion in section 2.3.2 we introduce

$$y_N^I \equiv \frac{1}{2b_N^1}\Phi_0^I|_N = -\frac{1}{2}\Phi_0^I|_N\,, \qquad y_S^I \equiv \frac{1}{2b_S^2}\Phi_0^I|_S = -\frac{1}{2}\Phi_0^I|_S\,, \qquad y_N^I = y_S^I - \varepsilon\mathfrak{p}^I\,, \quad (2.5.15)$$

which are then related to $u_{N,S}^I$ by (2.3.8). This allows us to write F_{grav} as

$$F_{\text{grav}} = \frac{\pi}{G_4}\frac{1}{\varepsilon}\left[\chi_N \mathrm{i}\mathcal{F}(y_S - \varepsilon\mathfrak{p}) - \chi_S \mathrm{i}\mathcal{F}(y_S)\right]\,. \quad (2.5.16)$$

In terms of holographic boundary data we have

$$y_N^I = \Delta_N^I - 2\pi\mathrm{i}\sigma_N^I\,, \qquad y_S^I = \Delta_S^I - 2\pi\mathrm{i}\sigma_S^I\,, \quad (2.5.17)$$

where $\Delta_{N,S}^I$, $\sigma_{N,S}^I$ are constants (since they are independent of the boundary S^1 coordinate bounding the associated $\mathbb{R}^2$); in general they are not equal to each other. From (2.3.8) we have the constraints

$$\zeta_I y_N^I = \chi_S\kappa_S\left(1 + \sigma\chi_S\varepsilon\right)\,, \qquad \zeta_I y_S^I = \chi_S\kappa_S\left(1 - \chi_S\varepsilon\right)\,, \quad (2.5.18)$$

which gives rise to corresponding constraints on $\Delta_{N,S}^I$, $\sigma_{N,S}^I$ in addition to the constraint (2.3.16). While the holonomies and mass deformations on the boundary are not constant in general, it is only this constant UV data $\Delta_{N,S}^I$, $\sigma_{N,S}^I$ that appears in the free energy. Recall from (2.3.15) that this data includes both the source for boundary dimension-2 mass operator as well as $*_2\mathrm{d}\eta_{(0)} = 0$, evaluated along the S^1 at the N, S poles, respectively.

The solutions described in this section would be supersymmetric, non-extremal (complex) deformations of Wick-rotated supersymmetric rotating dyonic black holes. The expression (2.5.16) agrees with the known supergravity solutions, and also with the conjectures based on the study of the large N behavior of the dual field theory observable.

For example, if we further assume that we are preserving supersymmetry with the anti-

twist (i.e. $\sigma = -1$ or equivalently $\chi_N = -\chi_S$), and restrict to the STU model we find

$$F_{\rm grav} = \frac{2\pi}{G_4}\frac{1}{\varepsilon}\chi_N\left[\sqrt{(y^0-\varepsilon\mathfrak{p}^0)(y^1-\varepsilon\mathfrak{p}^1)(y^2-\varepsilon\mathfrak{p}^2)(y^3-\varepsilon\mathfrak{p}^3)}+\sqrt{y^0y^1y^2y^3}\right],$$
$$\sum_{I=0}^{3} y^I = -\kappa_S\left(\chi_N+\varepsilon\right), \qquad \sum_{I=0}^{3}\mathfrak{p}^I = 0\,. \tag{2.5.19}$$

There are no known supersymmetric non-extremal deformations of rotating black holes in the STU model of such a general form. However, we notice that when $\mathfrak{p}^I = 0$, (2.5.19) reproduces the large N limit of the leading singularity of the Cardy limit of the superconformal index of ABJM theory [101,102]. Also, supersymmetric non-extremal solutions obtained by deforming a rotating electrically charged black hole are known in the X^0X^1 model [90]. Restricting to the X^0X^1 model by setting $y^0 = y^1$, $y^2 = y^3$ and also setting $\mathfrak{p}^I = 0$, (2.5.19) matches the on-shell action computed using holographic renormalization in [90]. Interestingly, (2.5.19) also allows the possibility of adding magnetic charges through the S^2, provided they sum to zero: this is consistent with the known results about supersymmetric rotating black holes in the STU model [103]. We also note that the limit $\varepsilon \to 0$ is not defined for this expression.

Analogously, if we instead assume that $\sigma = +1$, so that we are preserving supersymmetry with the topological twist, and restrict to the STU model, then

$$F_{\rm grav} = \frac{2\pi}{G_4}\frac{1}{\varepsilon}\chi_N\left[\sqrt{(y^0-\varepsilon\mathfrak{p}^0)(y^1-\varepsilon\mathfrak{p}^1)(y^2-\varepsilon\mathfrak{p}^2)(y^3-\varepsilon\mathfrak{p}^3)}-\sqrt{y^0y^1y^2y^3}\right],$$
$$\sum_{I=0}^{3} y^I = \kappa_S\left(\chi_N-\varepsilon\right), \qquad \sum_{I=0}^{3}\mathfrak{p}^I = -2\kappa_S\,. \tag{2.5.20}$$

Supersymmetric non-extremal rotating solutions of the STU model with supersymmetry on S^2 preserved via a topological twist are not known, but if they exist, (2.5.20) would be their action. Moreover, (2.5.20) reproduces the large N limit of the refined topologically twisted index [79] as computed in [104]. We also note that the limit $\varepsilon \to 0$ is well defined for this expression, in contrast to the previous case.

As already discussed, one use of the supersymmetry-preserving non-extremal solutions discussed in this section is to obtain the entropy of the Lorentzian extremal supersymmetric black holes. To do so, one should take the real part of the Legendre transform of $F_{\rm grav}$ with

respect to Δ^I (in terms of which y^I are expressed) and ε, subject to the constraint in (2.5.20), obtaining the micro-canonical entropy as a function of the electric charges and the angular momentum. Indeed, starting from (2.5.19) and (2.5.20), one obtains the entropy of the known supersymmetric black holes in the STU model, respectively found in [103] and [105].

2.6 Orbifold solutions

Until now we have assumed that M is a manifold and that the fixed point sets do not have orbifold singularities. However, this is not necessary, and the results can be straightforwardly generalized to include the case that a fixed point set $F \subset M$ is an orbifold with orbifold structure group with order $d_F \in \mathbb{N}$, by extension of the argument in section 2.2.2. Under these assumptions, the contribution from the fixed points to the gravitational free energy written in (2.2.3) takes the form

$$I_{\rm OS}^{\rm FP} = \frac{\pi}{2G_4}\left\{\sum_{F_0}\frac{1}{d_{F_0}}\frac{\Phi_0}{b_1 b_2} + \sum_{F_2}\frac{1}{d_{F_2}}\int_{F_2}\frac{\Phi_2}{2\pi b} - \frac{\Phi_0 c_1(L)}{b^2}\right\}, \tag{2.6.1}$$

denoting by F_0 the nuts and F_2 the bolts. As explained in section 2.2.2, the boundary contribution consists of a term obtained from integration of the bulk on-shell action, namely (2.2.4), the Gibbons–Hawking–York term, holographic renormalization counterterms, and finally the finite terms implementing the Legendre transform needed to match the functional dependence of the boundary field theory free energy. Using the canonically defined expressions for the equivariant forms constructed from the bilinears, we have shown that the gravitational free energy of a supersymmetric solution on a manifold M does not receive any contribution from the boundary, and $F_{\rm grav} = I_{\rm OS}^{\rm FP}$. The computations of the contributions of nuts and bolts carry over without modification, and we find the final result extending (2.2.33) is

$$\begin{aligned} F_{\rm grav} = \frac{\pi}{G_4}\Bigg[&\sum_{F_{0,\pm}} \mp\frac{1}{d_{F_0}}\frac{(b_1 \mp b_2)^2}{b_1 b_2}\mathrm{i}\mathcal{F}(u_\pm) \\ &+ \sum_{F_{2,\pm}}\frac{1}{d_{F_2}}\left(-\kappa\mathrm{i}\mathcal{F}_I(u_\pm)\mathfrak{p}^I \pm \mathrm{i}\mathcal{F}(u_\pm)\int_{F_{2,\pm}} c_1(L)\right)\Bigg]. \end{aligned} \tag{2.6.2}$$

This formula is expressed in terms of the bulk fixed point values $u_\pm$, and the relation with the UV boundary data is discussed in section 2.3. In the presence of non-trivial orbifold structure, the relation (2.3.6) is modified to include the order of the orbifold group

$$d_F \Delta^I + \mathrm{i}\beta\sigma^I = y^I \,, \tag{2.6.3}$$

where d_F is the order of the orbifold group of the fixed set at the origin of the chosen $\mathbb{R}^2$ submanifold (whether a bolt or a nut).

2.6.1 Spindle solutions

As a concrete example, we can generalize the Euclidean supersymmetric black hole spacetime with topology $S^2 \times \mathbb{R}^2$ discussed in section 2.5.3, by replacing the S^2 horizon with a spindle $\Sigma = \mathbb{WCP}^1_{[n_N,n_S]}$. We consider $\xi = \varepsilon\partial_{\varphi_1} + \partial_{\varphi_2}$, with ∂_{φ_1} rotating Σ and ∂_{φ_2}, rotating $\mathbb{R}^2$, which has two isolated fixed points at the North and the South of the sphere with the order of the orbifold groups given by $d_{N/S} = n_{N/S}$. The weights of the $U(1)$ action at the poles generalize those in (2.5.10) (in conventions[24] taken from [7]) and are given by

$$(b^1_N, b^2_N) = (-1, \varepsilon/n_N) \,, \qquad (b^1_S, b^2_S) = (-\varepsilon/n_S, -1) \,, \tag{2.6.4}$$

with the associated gravitational free energy given by

$$F_{\text{grav}} = \frac{\pi}{G_4}\frac{1}{\varepsilon}\left[\chi_N \left(1 + \chi_N \frac{\varepsilon}{n_N}\right)^2 \mathrm{i}\mathcal{F}(u_N) - \chi_S \left(1 - \chi_S \frac{\varepsilon}{n_S}\right)^2 \mathrm{i}\mathcal{F}(u_S)\right] . \tag{2.6.5}$$

In order to relate this to boundary data and constrain the fluxes, we follow the same procedure as was done after (2.5.11). Again ensuring that the spinor is globally defined requires $\kappa_N = -\chi_S\kappa_S$, and we deduce

$$\zeta_I \mathfrak{p}^I = -\kappa_S \sigma \frac{n_S + \sigma n_N}{n_N n_S} \,, \tag{2.6.6}$$

where we have introduced $\sigma \equiv \chi_N\chi_S = \pm 1$ which parametrizes whether supersymmetry is

[24] Note that one should also relate $\kappa_N\chi_N = (\sigma_1)_{\text{there}}$ and $\kappa_S = (-\sigma_0)_{\text{there}}$.

being preserved by a twist or anti-twist, respectively [68]. We next introduce the variables y^I associated with the North and the South poles of the spindle, as in (2.3.5), finding the same relation between the two as in (2.5.15), namely

$$y_N^I = y_S^I - \varepsilon \mathfrak{p}^I \,. \tag{2.6.7}$$

Since the relation between y^I and u^I is again (2.3.8), the expression for the gravitational free energy on the spindle in terms of y^I and $\mathfrak{p}^I$ is also the same as that on the two-sphere (2.5.16), namely

$$F_{\rm grav} = \frac{\pi}{G_4}\frac{1}{\varepsilon}\left[\chi_N \mathrm{i}\mathcal{F}(y_S - \varepsilon\mathfrak{p}) - \chi_S \mathrm{i}\mathcal{F}(y_S)\right] \,. \tag{2.6.8}$$

However, the constraint on the gauge field fluxes (2.6.6) differs from the S^2 case, as do the relationships between y^I and the boundary holographic data which now read

$$y_N^I = n_N \Delta_N^I - 2\pi\mathrm{i}\sigma_N^I \,, \qquad y_S^I = n_S \Delta_S^I - 2\pi\mathrm{i}\sigma_S^I \,, \tag{2.6.9}$$

and again $\Delta_{N,S}^I$, $\sigma_{N,S}^I$ are constants. Furthermore, the constraints $\zeta_I u_\pm^I = 1$ with (2.3.8) imply

$$\zeta_I y_N^I = -\kappa_N \left(1 + \chi_N \frac{\varepsilon}{n_N}\right) \,, \qquad \zeta_I y_S^I = \kappa_S \left(\chi_S - \frac{\varepsilon}{n_S}\right) \,. \tag{2.6.10}$$

If we assume that $*_2 \mathrm{d}\eta_{(0)} = 0$, associated with a direct product $\Sigma \times \mathbb{R}^2$ geometry, then from (2.3.16) we have $\zeta_I \sigma_{N,S}^I = 0$, so then $\zeta_I y_{N,S}^I = \zeta_I \Delta_{N,S}^I$, and also from (2.3.15) the σ^I are directly fixed by the mass deformations.

Remarkably, these results for the gravitational free energy of Euclidean supersymmetric spindle solutions precisely agree with minus the off-shell entropy function introduced in [6]. The change of variable is simply $y_{N,S}^I = -x_\pm^I/2$, $\varepsilon = b_0$, and then we obtain agreement with (5.36)–(5.39) of [6] after making the choice[25] $\kappa_N = 1$, and $\kappa_S = -\chi_S$. In [6] we had considered the equivariant localization of a two-form obtained by reduction of the four-dimensional action on a geometry $\mathrm{AdS}_2 \times \Sigma$ and, generalizing Sen's approach [106], we argued that extremizing the resulting function over the unconstrained variables would give the entropy of the relevant black hole with near-horizon $\mathrm{AdS}_2 \times \Sigma$. This indeed reproduces the entropy of vari-

[25] To compare with [8] one should also set $\chi_N = -1$.

ous accelerating black hole solutions, with different subfamilies (and/or near-horizon limits) studied in [107–111]. Here we strengthen this interpretation of the result, as we are proving that, assuming the existence of a family of supersymmetric deformations of the extremal black holes, the on-shell action of the family agrees with the "entropy function" computed from the near-horizon geometry, and is thus independent of the deformation parameters . Indeed the on-shell action only depends on the constant UV data $\sigma^I_{N,S}$ and $\Delta^I_{N,S}$. This is not unexpected from the holographic interpretation, as the dual observable is a supersymmetric index and so will be independent of most UV deformations [93]. In fact, the class of spaces considered here is larger than that of [6], since it also includes the possibility of metrics that are not just the direct product $\mathbb{R}^2 \times \Sigma$, such as dyonic black holes. From the perspective in this chapter the extremization simply performs a Legendre transform, passing from canonical to micro-canonical ensemble, hence reproducing the entropy.

In [110] instead the on-shell action of a complex locus of supersymmetric but non-extremal accelerating black holes in minimal gauged supergravity was computed. These black holes are rotating, accelerating, and carry dyonic charge. Formally our gravitational free energy (2.6.8) (in the minimal gauged supergravity case) agrees with the on-shell action computed in [110], identifying $\varepsilon = \mp\omega/2\pi\mathrm{i}$ and setting $\sigma = -1$ (for the "anti-twist"), where ω is a complex chemical potential associated with rotation, and the sign here is associated to a choice of branch of the complex solutions. This agreement is only formal due to the reality conditions we have imposed in the present work, but as commented already we expect it to be possible to relax these conditions, allowing complex solutions. Importantly, the formula (2.6.8) for the gravitational free energy is also in precise agreement with the large N field theory result for the "spindle index" [112, 113], obtained recently in [114], providing a microscopic "counting" of the entropy of accelerating black holes.

Additional topologies including spindles with toric symmetry are further considered in the related work [7]. Moreover, infinite classes of explicit supersymmetric solutions of minimal gauged supergravity on manifolds that are total spaces of orbifold line bundles over spindles have been recently constructed in [38]. As shown there, the direct computation of the on-shell action using holographic renormalization agrees with the result of the application of the equivariant localization theorem (2.6.2).

2.6.2 Seifert three-manifolds

We conclude this section with one final class of examples, which generalizes the examples in section 2.4.4, by considering four-dimensional solutions which have a three-dimensional boundary that is a Seifert manifold. A Seifert manifold can be viewed as a circle orbibundle over a two-dimensional orbifold $\Sigma_{g,k}$ (see e.g. [71,115] for a review). We want to consider the four-dimensional solution[26] to have the same topology as the complex line orbibundle that is associated with the S^1 orbibundle:

$$\mathbb{C} \hookrightarrow L \to \Sigma_{g,k} \,. \tag{2.6.11}$$

The orbifold $\Sigma_{g,k}$ has genus g and k orbifold points, where in a given patch containing the ith orbifold singularity the coordinate system is modelled on $\mathbb{C}/\mathbb{Z}_{n_i}$ with n_i a positive integer. That is, if z_i is a local coordinate on $\mathbb{C}$ then we identify $z_i \sim \omega_i z_i$ with $\omega_i = \mathrm{e}^{2\pi\mathrm{i}/n_i}$. To specify the line orbibundle near that point, each patch is modelled on $(\mathbb{C}\times\mathbb{C})/\mathbb{Z}_{n_i}$ with the quotient specified by $(z_i, s_i) \sim (\omega_i z_i, \omega_i^{m_i} s_i)$ with "charges" $m_i \in \mathbb{Z}_{n_i}$. The integrated orbifold first Chern class is then given by

$$\int_{\Sigma_{g,k}} c_1(L) = \deg(L) + \sum_{i=1}^{k} \frac{m_i}{n_i} \,, \tag{2.6.12}$$

which in general is a rational number. Here[27] we have taken $0 \le m_i < n_i$, and this formula then defines the *degree* $\deg(L) \in \mathbb{Z}$. We will also need the rational-valued orbifold Euler number for $\Sigma_{g,k}$, which is given by

$$\int_{\Sigma_{g,k}} c_1(T\Sigma_{g,k}) = 2(1-g) + \left(\sum_{i=1}^{k} \frac{1}{n_i}\right) - k \,. \tag{2.6.13}$$

Now, by construction, the R-symmetry Killing vector vanishes at the origin of the $\mathbb{C} \cong \mathbb{R}^2$ fibre in (2.6.11), and hence fixes the base $\Sigma_{g,k}$, which is a bolt with normal bundle L. The

[26] There may be other four-dimensional manifolds with the same Seifert manifold boundary, but we do not consider them here.

[27] This can be compared with (2.17) of [68] with the sign difference in the m_i accounted for by the fact that a single azimuthal angular coordinate φ was used at both poles in [68].

gravitational free energy can be obtained from (2.6.2). There is only one bolt, so we choose $\kappa = +1$, and the only remaining sign characterizing the solution is the chirality of the spinor at the bolt

$$F_{\rm grav} = \left(-\mathrm{i}\mathcal{F}_I(u_\pm)\,\mathfrak{p}^I_\pm \pm \mathrm{i}\mathcal{F}(u_\pm) \int_{\Sigma_{g,k}} c_1(L) \right) \frac{\pi}{G_4}\,. \tag{2.6.14}$$

As usual we have $\zeta_I u^I_\pm = 1$ and because of supersymmetry (see (2.3.2)), we have

$$\mathfrak{p}^R_\pm = \frac{1}{2}\zeta_I \mathfrak{p}^I_\pm = \frac{1}{2}\int_{\Sigma_{g,k}} \left[\pm c_1(L) - c_1(T\Sigma_{g,k}) \right]\,. \tag{2.6.15}$$

where the integrated Chern classes are given in (2.6.12), (2.6.13).

These results generalize those of (2.4.30). We can relate the free energy to UV data following the discussion in section 2.4.4. Assuming that the corresponding supergravity solutions exist, this then provides a prediction for the large N limit of the partition function of $\mathcal{N} = 2$ SCFTs on Seifert manifolds obtained in [115].

It is also possible to make a connection with the toric results of [7] by considering the special case of a spindle with $g = 0$ and $k = 2$. With normal bundle $\mathcal{O}(-p) \to \mathbb{WCP}^1_{[n_1,n_2]}$, by definition we have $\int_{\Sigma_{g,k}} c_1(L) = -\frac{p}{n_1 n_2}$ with $p \in \mathbb{Z}$, and $\int c_1(T\Sigma_{g=0,k=2}) = \frac{1}{n_1} + \frac{1}{n_2}$ to get

$$F_{\rm grav} = \left(-\mathrm{i}\mathcal{F}_I(u_\pm)\,\mathfrak{p}^I \mp \mathrm{i}\mathcal{F}(u_\pm)\frac{p}{n_1 n_2} \right) \frac{\pi}{G_4}\,, \tag{2.6.16}$$

and (2.6.15) becomes

$$\zeta_I \mathfrak{p}^I_\pm = \mp\frac{p}{n_1 n_2} - \left(\frac{1}{n_1} + \frac{1}{n_2} \right)\,. \tag{2.6.17}$$

This result can now be compared with [7]. In the limit that the R-symmetry Killing vector just rotates the fibre, in [7] we need to set the signs $\sigma_0 = \sigma_2$. Also identifying $\sigma_1 y^I_{N,S}$ with $u^I_\pm$, we then see that setting $\sigma_0 = -1$ one has $\chi_S = \sigma_1 = \pm$ in the above notation, and the formulae (2.6.16), (2.6.17) precisely agree with (43), (44) in [7], respectively.

Furthermore, for the case of trivial fibration over a spindle, obtained by setting $p = 0$, we can also see these results are consistent with those of section 2.6 in the following sense. In section 2.6 we should assume that the R-symmetry Killing vector does not rotate the spindle

and set $\varepsilon = 0$. Further restricting to the twist class $\sigma = +1$, which can be achieved by taking $\chi_N = \chi_S = -1$, and also setting $\kappa_S = +1$, we find precise agreement with (2.6.16), (2.6.17).

2.7 Discussion

We have shown that supersymmetric solutions of $D = 4$, $\mathcal{N} = 2$ Euclidean gauged supergravity coupled to vector multiplets have a set of equivariantly closed forms. These forms, which are globally defined and gauge-invariant, are expressed in terms of spinor bilinears and the supergravity fields. We have explained how this structure allows one to compute various flux integrals and also the on-shell action of a supersymmetric solution using the BVAB fixed point formula. The on-shell action is naturally expressed in terms of bulk fixed point data but we also explained how it can be expressed in terms of holographic boundary data. We illustrated the formalism with various concrete examples.

In developing the formalism we focussed on supersymmetric solutions that satisfy a particular reality condition given in (2.2.1). In particular, we assumed that the metric, the gauge fields and the scalars z^i and $\tilde{z}^i$ are real, and also imposed a corresponding reality condition on the Killing spinors. While this still covers rich classes of solutions, there is good evidence that these assumptions can be relaxed. For example, explicit solutions were constructed in [67] that are dual to ABJM theory on S^3, with real metric and gauge fields but with complex $z^i, \tilde{z}^i$. For real $z^i, \tilde{z}^i$ we precisely recover the the on-shell action computed in [67] and we also do if we formally assume that $z^i, \tilde{z}^i$ are complex. Similarly, in section 2.5, again proceeding formally, we were also able to obtain the on-shell action and hence the entropy of classes of black hole solutions which have complex metrics. Thus, it is of interest to systematically analyze what happens when the reality conditions (2.2.1) are relaxed.

It is also of interest to generalize the supergravity models to include hypermultiplets in addition to vector multiplets. Based on the results of [6] we anticipate that there will be additional equivariantly closed forms which, moreover, will lead to the imposition of specific constraints on the variables appearing in the on-shell action. Another interesting generalization is to consider supergravity theories with higher derivatives, as this will lead to additional general predictions for the dual SCFTs beyond the leading large N contributions

that we have obtained here. It seems likely that the results for the supersymmetric index of supersymmetric black holes in ungauged $\mathcal{N} = 2$ supergravity, recently presented in [116], can be obtained in this way.

Some of the $\mathcal{N} = 2$ gauged supergravity theories that we have considered can be obtained from a consistent KK truncation of $D = 10$, 11 supergravity. For example, solutions of the STU model can be uplifted on S^7 [72] to obtain solutions dual to ABJM theory, while solutions of minimal gauged supergravity can also be uplifted on SE_7 (seven-dimensional Sasaki–Einstein manifolds) and other manifolds to $D = 11$ [74] giving solutions dual to other SCFTs. For such theories, our expressions for the $D = 4$ on-shell action immediately lead to expressions for the on-shell action of the corresponding supersymmetric solutions of $D = 10$, 11 supergravity. However, we believe that even when such a consistent KK truncation does not exist, the corresponding gauged supergravity sector that we study is sufficient to be able to evaluate the gravitational free energy of the associated $D = 10, 11$ solutions, at least in some cases. For example, in section 2.4.4 we obtained results for the on-shell action that are in agreement with field theory results, in the large N limit, for general classes of $\mathcal{N} = 2$ SCFTs [84] that have $D = 11$ holographic duals involving SE_7 but are not associated with consistent KK truncations (aside from the connection with minimal gauged supergravity). This comparison requires one to identify the prepotential of the gauged supergravity with the twisted superpotential or Bethe potential arising in the field theory computation (*cf.* [64]). This is partly explored from the $D = 10$ perspective in chapter 4, but it would be interesting to study this further.

Chapter 3

Internal space localization in massive type IIA

As mentioned in the introduction, finding metrics satisfying the non-linear Einstein equations is a notoriously difficult problem. Naively, this seems to pose a computational problem to computing observables holographically. However, recent advances in defining extremal problems in gravity allows one to extract out certain holographic observables. These geometric extremal problems have a corresponding field theoretic dual, whether it be: a-maximization [117]; F-extremization, [118]; c-extremization [119,120], or $\mathcal{I}$-extremization [82]. This program of studying geometric extremal problems began with the geometric dual of a-maximization/F-extremization in [28, 121] and more recently has been extended to $\mathcal{I}$/c-extremization [122] see also [123–128].

Recently in [29,32] equivariant localization was used to setup such extremal problems in a variety of different setups [6, 37, 50].[1] This allows one to compute physical observables of the dual SCFT without knowing the full supergravity solution. In this chapter, based on [2, 3], we apply these methods to (massive) type IIA on $\mathrm{AdS}_4 \times X_6$. Such solutions have been classified in [61], see also [129]. Concretely we use equivariant localization to perform flux quantization, compute the holographic free energy, and conformal dimensions of certain BPS operators. We use the formalism to derive certain gravitational block formulae conjectured in the literature [130], reproduce known results and make predictions for new solutions.

At first sight, given that our internal manifold X_6 shares many similarities with the internal manifold of AdS_5 solutions in M-theory [131] which were equivariantly localized in [32], this may seem a trivial extension of their work. We find that although there are similarities there is also a rich structure of possible internal geometries which are not possible in the AdS_5 M-theory solutions. In particular one novel aspect of our work is the inclusion

[1] See also [54,55] for a complementary but distinct approach and [51] for an earlier incarnation of localization.

of boundaries on the internal space on which we localize. These boundaries arise in our construction from the presence of brane sources which cap off the space. One finds that such singular geometries can also be localized and this opens up a wide avenue of solutions where these techniques can be applied.

AdS_4 solutions of (massive) type IIA supergravity preserving $\mathcal{N}=2$ supersymmetry with an SU(2) structure were classified in [61]. The R-symmetry of the putative dual field theories is (at least) U(1), ergo, via the usual AdS/CFT lore there is a U(1)-isometry of the metric and an associated Killing vector field. We will use this U(1) isometry to localize integrals in our setup. Given the geometric setup in [61], using brute force, we construct a set of equivariantly closed polyforms. To contruct these polyforms we use a subset of the torsion conditions. Since only a subset are imposed we are to some extent "off-shell" and one needs to extremize over the free parameters to find the on-shell results. This provides a geometric interpretation to F-maximization principles of the dual SCFTs [75, 118, 132]. A holographic approach to F-maximization was already discussed in [67, 133] for Sasaki–Einstein geometries and a class of massive type IIA solutions. In this work we extend this geometric extremal problem to all $\mathcal{N}=2$ AdS_4 solutions in (massive) type IIA with an SU(2) structure.

3.1 General AdS_4 solutions of massive type IIA

$\mathcal{N}=2$ preserving AdS_4 solutions of massive type IIA with an SU(2) structure were classified in [61]. It was shown that there are two distinct classes, named class K (Kähler) and class HK (hyper-Kähler). Both classes are topologically an S^2 bundle over a four-dimensional space M_4 which is Kähler or hyper-Kähler respectively. In both cases the 10d metric takes the form

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2A}\left[\mathrm{d}s_{\mathrm{AdS}_4}^2 + \mathrm{d}s_{X_6}^2\right] , \tag{3.1.1}$$

with X_6 dependent on the class. The class K solutions will be the main focus of this work and we will ignore the class HK solutions since the solutions in the HK class are essentially unique and therefore localization is not needed to compute observables in these theories. We work in conventions where the length scale on AdS_4 is set to 1. This may be reinstated with

a little dimensional analysis.

3.1.1 Setup

The internal metric for solutions of class K takes the form

$$\mathrm{d}s^2_{X_6} = \frac{1}{\mathrm{e}^{4A} - y^2}\mathrm{d}y^2 + \frac{1}{4}(1 - \mathrm{e}^{-4A}y^2)D\psi^2 + \frac{y}{F_0^2\mathrm{e}^{4A}y^2 + l^2}g^{(4)}_{ij}(y,x)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{3.1.2}$$

with $g^{(4)}$ a four-dimensional Kähler metric at fixed y coordinate. We also defined

$$D\psi = \mathrm{d}\psi + \rho\,, \tag{3.1.3}$$

where ρ is a one-form defined on the SU(2)-structure base but may depend on the coordinate y. The SU(2) structure forms on M_4 satisfy:[2]

$$\partial_\psi j = 0\,, \qquad \partial_y j = \frac{1}{2}(F_0^2y^2 + l^2y^{-2})\mathrm{d}_4\rho\,, \qquad \mathrm{d}_4 j = 0\,, \tag{3.1.4}$$

$$\partial_\psi \omega = 0\,, \qquad \partial_y \omega = -\frac{1}{2}(F_0^2y^2 + l^2y^{-2})T\omega\,, \qquad \mathrm{d}_4\omega = \mathrm{i}P\wedge\omega\,, \tag{3.1.5}$$

where

$$P \equiv -\rho + \mathrm{i}\frac{2\mathrm{e}^{4A}(F_0^2y^4 + l^2)}{(\mathrm{e}^{4A} - y^2)(F_0^2\mathrm{e}^{4A}y^2 + l^2)}\mathrm{d}_4A\,, \tag{3.1.6}$$

$$T \equiv \frac{\partial_y(\mathrm{e}^{4A}y^2)}{(\mathrm{e}^{4A} - y^2)(F_0^2\mathrm{e}^{4A}y^2 + l^2)}\,. \tag{3.1.7}$$

Moreover, by splitting the exterior derivative of $\mathrm{d}\rho$ into its self-dual and anti self-dual components one finds:

$$(\partial_4\rho)^+ = -Tj\,, \tag{3.1.8}$$

There are two constant parameters, F_0 the Romans mass and l. It is necessary that at least one of these parameters is non-zero. A non-zero value of F_0 signifies the presence of D8-branes in the setup. On the other hand the presence of a non-zero l signifies the presence of D2- and D6-branes as can be seen more clearly by considering the fluxes supporting the

[2] In the original paper [61] the j used here is denoted by $\hat{\jmath}$ and is normalized differently from j there.

solution. Note that if both are non-zero by a redefinition of the y coordinate and rescaling of the metric and fluxes l may be set without loss of generality to a non-zero value, such as $l = 1$.

The flux quantization of the Romans mass F_0 imposes

$$n_0 = 2\pi\ell_s F_0 \in \mathbb{Z}\,. \tag{3.1.9}$$

The magnetic parts of the RR-fluxes are:

$$\begin{aligned}
F_2^{\rm RR} &= l\left(\frac{1}{2y}\mathrm{d}\rho - \frac{1}{2}\mathrm{d}(\mathrm{e}^{-4A}yD\psi) + \frac{1}{F_0^2\mathrm{e}^{4A}y^2+l^2}j\right)\,,\\
F_4^{\rm RR} &= -\frac{F_0}{2}\Bigg[\frac{\mathrm{e}^{4A}y^2}{(F_0^2\mathrm{e}^{4A}y^2+l^2)^2}j\wedge j\\
&\quad -\left(\mathrm{d}(\mathrm{e}^{-4A}yD\psi) - y^{-1}\mathrm{d}\rho\right)\wedge\left(\frac{\mathrm{e}^{4A}y^2}{F_0^2\mathrm{e}^{4A}y^2+l^2}j + \frac{y^2}{2}\mathrm{d}y\wedge D\psi\right)\Bigg]\,,\\
F_6^{\rm RR} &= \frac{3l}{4}\frac{y^2}{(F_0^2\mathrm{e}^{4A}y^2+l^2)^2}\mathrm{d}y\wedge D\psi\wedge j\wedge j\,.
\end{aligned} \tag{3.1.10}$$

For the NS-NS sector it is useful to give two different expressions for the NS-NS two-form related by gauge transformations:

$$\begin{aligned}
B_1 &= -\frac{F_0}{l}\left(\frac{\mathrm{e}^{4A}y^2}{F_0^2\mathrm{e}^{4A}y^2+l^2}j + \frac{y^2}{2}\mathrm{d}y\wedge D\psi\right)\,,\\
B_2 &= \frac{l}{F_0}\left(\frac{1}{F_0^2\mathrm{e}^{4A}y^2+l^2}j + \frac{1}{2y^2}\mathrm{d}y\wedge D\psi\right)\,.
\end{aligned} \tag{3.1.11}$$

Note that the former is ill-defined in the $l\to 0$ limit while the latter is ill-defined in the $F_0\to 0$ limit so one should choose the correct gauge when specialising to either case. Finally the dilaton is given by:

$$\mathrm{e}^{2\phi} = \frac{\mathrm{e}^{6A}}{F_0^2\mathrm{e}^{4A}y^2+l^2}\,, \tag{3.1.12}$$

which makes manifest the necessity to have at least one of the parameters F_0 or l non-zero.

The above fluxes are the (magnetic) ones which appear in the equations of motion. They, however, are not the ones one would use to quantize the fluxes. These are the Page fluxes: $F = F^{\rm RR}\wedge \mathrm{e}^{-B}$ in polyform notation. Here $F^{\rm RR}$ is the polyform of magnetic fluxes, B is either of the representatives of B given in (3.1.11) and one extracts out the part of form

degree p. Recall that Page fluxes are closed, hence define conserved charges, but are not gauge invariant. The quantization condition reads

$$\frac{1}{(2\pi\ell_s)^{p-1}}\int_{\Sigma_p} F_p \in \mathbb{Z}\,. \tag{3.1.13}$$

In the general class ($l \neq 0$) the Page fluxes read:

$$\begin{aligned} F_2 &= \frac{l}{2}\mathrm{d}\Big[y^{-1}\big(1-\mathrm{e}^{-4A}y^2\big)D\psi\Big] - F_0(B_1-B_2)\,,\\ F_4 &= \frac{F_0}{2l^2}\Big[\frac{\mathrm{e}^{4A}y^2}{F_0^2\mathrm{e}^{4A}y^2+l^2}j\wedge j + y^2\mathrm{d}y\wedge D\psi\wedge j\Big]\,,\\ F_6 &= \frac{y^2}{4l^3}\frac{F_0^2\mathrm{e}^{4A}y^2+3l^2}{F_0^2\mathrm{e}^{4A}y^2+l^2}\mathrm{d}y\wedge D\psi\wedge j\wedge j\,. \end{aligned} \tag{3.1.14}$$

Observe that these fluxes do not admit a well-defined $l=0$ limit, though one can safely take the $F_0=0$ limit. If $l=0$ one notices that both F_2 and F_6 vanish identically and that one may pick a gauge in which $B=0$ too, i.e. pick $B=B_2$. It follows that the only non-trivial flux is F_4 and its charge defines a conserved quantity. We must therefore specify as initial data whether $l=0$ or not and we will explicitly state this when relevant. In section 3.1.3.1 we will construct equivariantly closed polyforms for $l\neq 0$ and in section 3.1.3.2 the analogous polyforms for the $l=0$ case.

3.1.2 Degenerations and global regularity

Before we set up the localization problem we need to first discuss the possible global completions of the metric. It turns out that there is a very rich structure for the class K solutions. In order to have well defined, globally complete (and compact)[3] solutions we need to fix the length of the line interval with coordinate y. The usual way, amenable to localization, is for the localizing circle to shrink at some value of y. Depending on the shrinking of the remaining part of the internal metric one obtains either fixed points or fixed surfaces of co-dimension 2 or 4.

There is however another way in which to globally complete the metric: the presence

[3]One can easily release the compact condition by studying solutions corresponding to 3d conformal defects in a higher dimensional parent theory, see for example [134–136] for solutions of this form in M-theory. We will not pursue this here and thank Pieter Bomans for discussions on this topic.

of brane singularities. Chosen correctly these cap off the space, giving a global completion, despite the localizing circle remaining of non-zero size. From the localization viewpoint these lead to boundaries in the internal space that we must localize over. We will consider geometries admitting brane singularities. These introduce boundaries on the internal space which give additional options for capping off the spacetime. To introduce these boundaries we need to study, locally near the singularities, the behaviour of the solution and in particular the consequences for the G-structure forms and warp factor.

O8

We will be interested in capping off the spacetime with an O8-plane singularity with divergent dilaton. The metric and dilaton behaviour of such an O8 singularity is

$$\begin{aligned} \mathrm{d}s^2 &\sim \frac{1}{\sqrt{r}}\mathrm{d}s_9^2 + \sqrt{r}\mathrm{d}r^2\,, \\ \mathrm{e}^{2\phi} &\sim r^{-5/2}\,, \end{aligned} \tag{3.1.15}$$

where we have kept only the leading order pieces and the plane is located at $r = 0$. Note that there are other choices of O8-plane which would not have the above divergence structure, this choice is taken to match with the O8-plane in the Brandhuber–Oz solution [137] that will be relevant later.

We can now study the existence of such a degeneration for the metric in (3.1.1) and dilaton in (3.1.12). It is not difficult to show that this is only possible for $l = 0$ and $y = 0$. The argument is simple, one immediately sees that the warp factor must diverge as $\mathrm{e}^{2A} \sim r^{-1/2}$. Requiring that the dilaton has the correct divergence fixes $l = 0$. Finally requiring that the metric in (3.1.2) degenerates correctly fixes $y = 0$ to be the location of the branes, that is it identifies y with r in (3.1.15).

In conclusion to have an O8-plane it follows that $l = 0$ and as $y \to 0$ the warp factor must diverge as

$$\mathrm{e}^{2A} \sim y^{-1/2}\,. \tag{3.1.16}$$

Around $y = 0$ the metric and dilaton have the following behaviour

$$\begin{aligned} \mathrm{d}s^2 &\sim \frac{1}{\sqrt{y}}\left(\mathrm{d}s^2_{\mathrm{AdS}_4} + \frac{1}{4}D\psi^2 + \frac{1}{F_0^2}g^{(4)}_{ij}(0,x)\mathrm{d}x^i\mathrm{d}x^j\right) + \sqrt{y}\mathrm{d}y^2\,, \\ \mathrm{e}^{2\phi} &\sim y^{-5/2}\,. \end{aligned} \tag{3.1.17}$$

One needs to further refine the warp factor divergence to the next order. In general expanding the warp factor around $y = 0$ we have

$$\mathrm{e}^{2A} \sim \frac{a_1}{\sqrt{y}} + a_2\sqrt{y}\,. \tag{3.1.18}$$

For $a_2 = 0$ it follows that F_4 as given in (3.1.36) vanishes at $y = 0$. For $a_2 \neq 0$ we find that there is a term proportional to the volume form on M_4. Comparing to the explicit solutions in [61] a_2 plays the role of the parameter σ in section 4.1.4 there. We see that the value of a_2 changes the homology relation we will impose later since the flux does not vanish on the boundary of M_6.

O4

Consider now ending the space with an orientifold-4 plane. The metric of a fully localized O4-plane at the tip of a cone $C(Y_4)$ is given by:[4]

$$\mathrm{d}s^2 = H_{\mathrm{O4}}^{-1/2}\mathrm{d}s^2(W_5) + H_{\mathrm{O4}}^{1/2}\big(\mathrm{d}r^2 + r^2\mathrm{d}s^2(Y_4)\big)\,, \qquad H_{\mathrm{O4}} = 1 - \left(\frac{r_0}{r}\right)^3\,. \tag{3.1.19}$$

The metric breaks down as $r \to r_0$ in which case the solution locally takes the form

$$\mathrm{d}s^2 \sim \frac{1}{\sqrt{r-r_0}}\mathrm{d}s^2(W_5) + \sqrt{r-r_0}\big(\mathrm{d}r^2 + \mathrm{d}s^2(Y_4)\big)\,, \quad \mathrm{e}^{2\phi} \sim \frac{1}{\sqrt{r-r_0}}\,. \tag{3.1.20}$$

We now want to study the form of the metric in the classification form consistent with this boundary. As a first observation we must require that the brane wraps the R-symmetry direction or at a locus where the R-symmetry vector shrinks. From the form of the metric we see that this is not possible and therefore we conclude that the R-symmetry direction is

[4]For discussion on orientifold planes and further references see [138].

part of the world-volume of the brane.

With a little hindsight we can identify the radial coordinate r in the expansion above with y and fix the singularity to be at $y = y_b$. In order for the metric to have the correct O4-plane metric it follows that the warp factor has the expansion:

$$\mathrm{e}^{2A} \sim \frac{a_1}{\sqrt{y - b_0}} + a_2\sqrt{y - y_b} + ... \tag{3.1.21}$$

where the a's are allowed to depend on the coordinates on the four-dimensional base. The metric becomes:

$$\mathrm{d}s^2 \sim \frac{a_1}{\sqrt{y - y_b}}\left[\mathrm{d}s^2(\mathrm{AdS}_4) + D\psi^2\right] + \frac{(y - y_b)}{a_1^2}\left[\mathrm{d}y^2 + \frac{1}{F_0^2}\mathrm{d}s_4^2\Big|_{y=y_b}\right], \tag{3.1.22}$$

where we note that the singularity is only consistent with a non-zero Romans mass and that the metric on $\mathrm{d}s_4^2|_{y=y_b}$, is smooth in the $y \to y_b$ limit.

Now consider the torsion conditions in this limit. First observe that the torsion conditions on B_4 reduce to

$$\mathrm{d}_4 j = 0\,, \qquad \mathrm{d}_4\omega = -\mathrm{i}\rho \wedge \omega\,, \tag{3.1.23}$$

and therefore it follows that B_4 is complex and in fact Kähler. Moreover the U(1) is fibred over the complex base with the anti-canonical class of the base.

It is now instructive to use the identity (3.1.8) to show that the Ricci scalar is fixed to be for the O4-plane

$$R\big|_{y=y_b} = -4T(y = y_b)\,. \tag{3.1.24}$$

From our expansion we find that the function T and dilation are given by

$$T(y_b) = -\frac{1}{F_0^2 a_1^2}\,, \quad \mathrm{e}^{2\phi} = \frac{a_1}{F_0^2 y_b^2\sqrt{y - y_b}}\,, \tag{3.1.25}$$

and therefore the expansion parameter is fixed in terms of the Ricci scalar of the base at $y = y_b$. In particular one observes that the base should have positive curvature and that the function a_1 is necessarily positive.

We conclude that the boundary five-manifold is a U(1) fibration over a Kähler space. The

U(1) is fibred over the base with respect to the inverse canonical class of the base.

O6

For an O6-plane, the metric and dilaton behave as

$$\partial s^2 \sim \frac{1}{\sqrt{r}}\partial s(W_7)^2 + \sqrt{r}(\mathrm{d}r^2 + \mathrm{d}s^2(Y_2))\,, \qquad e^{2\phi} \sim r^{-3/2}\,, \tag{3.1.26}$$

near $r = 0$, which is where the orientifold plane is located. We want the O6 to extend along AdS_4 and three additional directions in X_6, while being smeared over two directions in X_6. This implies that, near the O6-plane, the warp factor must behave as

$$e^{2A} \sim r^{-1/2}\,. \tag{3.1.27}$$

Using the behaviour of the warp factor, it follows that near the O6-plane, the dilaton takes the form

$$e^{2\phi} = \frac{e^{6A}}{F_0^2 y^2 e^{4A} + l^2} \sim \frac{r^{-3/2}}{F_0 \frac{y^2}{r} + l^2} \overset{!}{\sim} r^{-3/2}\,. \tag{3.1.28}$$

For the correct behaviour, we need $l \neq 0$ and y^2/r to be finite as $r \to 0$. Comparing the scaling of the one-form η we find that $y = r$ and therefore the O6-plane is necessarily located at $y = 0$ and only occurs for $l \neq 0$. Moreover the O6-plane wraps the R-symmetry circle and two directions within M_4.

This concludes the different options for capping off the spaces that we will consider. In each case these brane singularities introduce a boundary term in the geometry.

3.1.3 Polyforms

In order to perform equivariant localization we need to first pick an action on which we wish to localize. There is an obvious U(1)-action given by the R-symmetry Killing vector $\xi = \partial_\psi$.[5] The Killing vector $\xi = \partial_\psi$ satisfies $\xi \lrcorner D\psi = 1$, and contractions into the SU(2) structure forms vanish: $\xi \lrcorner j = \xi \lrcorner \omega = 0$. As reviewed in section 1.2, we want to construct various equivariantly closed polyforms i.e. polyforms Φ which satisfy $\mathrm{d}_\xi \Phi \equiv (\mathrm{d} - \xi \lrcorner)\Phi = 0$. This

[5]Observe that we have taken a different normalization for ξ to that in [61] for which $\xi_{\text{there}} = 4\xi_{\text{here}}$.

allows us to localize integrals of the top-form using the BVAB fixed point formula. In this section we will construct these polyforms and give generic formulae for the BVAB theorem applied to these polyforms in section 3.1.4 before using them later in section 3.2 to study various choices for X_6. In constructing it is necessary to separate the two cases of $l \neq 0$ and $l = 0$, though one can safely set $F_0 = 0$ in the former case.

There are a number of different equivariantly closed polyforms that we want to construct. Firstly, we need to perform flux quantization and therefore we need to equivariantly complete the Page fluxes, secondly, we want to compute the free-energy of the various solutions, this is computed via:

$$F = \frac{16\pi^3}{(2\pi\ell_s)^8} \int_{X_6} \mathrm{e}^{8A-2\phi} \mathrm{vol}_{X_6} \,. \tag{3.1.29}$$

We therefore need to find a polyform with top-component $\mathrm{e}^{8A-2\phi}\mathrm{vol}_{X_6}$. Finally, there are various probe branes which we can consider. These wrap calibrated cycles in the internal space and depending on the type of cycle wrapped give rise to observables in the three-dimensional dual SCFT such as conformal dimensions of BPS particles. We will focus on D2-branes wrapping calibrated two-cycles [139] though it would be interesting to consider other probe branes in the geometry.

3.1.3.1 The $l \neq 0$ case

After a slightly tedious, but straightforward computation, one can compute all the various polyforms outlined above. We suppress the details of the computations, presenting just the final polyforms. The polyform for the free energy is

$$\begin{aligned} \Phi = \mathrm{e}^{8A-2\phi} \star_1 - \frac{1}{12}\left(\frac{y^3}{F_0^2\mathrm{e}^{4A}y^2+l^2} j\wedge j + y\mathrm{d}y\wedge D\psi\wedge j\right) \\ + \frac{1}{24}\left(y^2 j + \frac{1}{2}(F_0^2 y^4 + l^2)\mathrm{d}y\wedge D\psi\right) - \frac{y}{48}\left(\frac{F_0^2 y^4}{5} + l^2\right) . \end{aligned} \tag{3.1.30}$$

For the Page fluxes we find:

$$\begin{aligned}
\Phi^{F_2} &= F_2 - \frac{y}{6l}(F_0^2 y^2 - 3l^2 \mathrm{e}^{-4A})\,, \\
\Phi^{F_4} &= F_4 - \frac{F_0}{6l^2}\left(y^3 j + \frac{F_0^2 y^5 + l^2 y}{2}\mathrm{d}y \wedge D\psi\right) + \frac{F_0 y^2}{72l^2}(F_0^2 y^4 + 3l^2)\,, \\
\Phi^{F_6} &= F_6 - \frac{1}{12l^3}\left(y^3 \frac{F_0^2 \mathrm{e}^{4A} y^2 + 3l^2}{F_0^2 \mathrm{e}^{4A} y^2 + l^2} j \wedge j + (F_0^2 y^5 + 3l^2 y)\mathrm{d}y \wedge D\psi \wedge j\right) \\
&\quad + \frac{1}{144l^3}\Big(2y^2(F_0^2 y^4 + 9l^2)j + (F_0^2 y^4 + 9l^2)(F_0^2 y^4 + l^2)\mathrm{d}y \wedge D\psi\Big) \\
&\quad - \frac{y}{1296l^3}(F_0^2 y^4 + 9l^2)^2\,.
\end{aligned} \tag{3.1.31}$$

These polyforms will allow us to perform flux quantization and to compute the free energy.

There are additional polyforms that one can construct. For example we have that there is the closed global two-form:

$$Y = j + \frac{F_0^2 y^2 + l^2 y^{-2}}{2}\mathrm{d}y \wedge D\psi\,, \tag{3.1.32}$$

with equivariantly closed the polyform

$$\Phi^Y = Y - \frac{1}{6y}\left(F_0^2 y^4 - 3l^2\right)\,. \tag{3.1.33}$$

Note that $\Phi_2^{F_4} \sim y^3 Y$. There is an analogous closed global four-form that we can construct, given by

$$Z = j \wedge j + (F_0^2 y^2 + l^2 y^{-2})\mathrm{d}y \wedge D\psi \wedge j\,, \tag{3.1.34}$$

with associated polyform

$$\Phi^Z = Z - \frac{1}{6}\left(F_0^2 y^3 - \frac{3l^2}{y}\right)\Big(2j + (F_0^2 y^2 + l^2 y^{-2})\mathrm{d}y \wedge D\psi\Big) + \frac{(F_0^2 y^4 - 3l^2)^2}{36y^2}\,. \tag{3.1.35}$$

These will turn out to be useful when the geometry contains homologically trivial two-cycles, see section 3.2.6.

3.1.3.2 The $l = 0$ case

Not all the polyforms in section 3.1.3.1 are well-defined for $l = 0$, in particular the Page flux polyforms. When $l = 0$ it is easy to see that the B-field vanishes (i.e. becomes pure gauge) and therefore the Page fluxes and Maxwell fluxes become equivalent. Moreover, it follows that both the magnetic F_2 and F_6 fluxes become trivial, leaving only F_4 and the Romans mass as the non-trivial fluxes. The non-trivial four-form flux (which is closed and therefore defines a conserved charge) is

$$F_4 = -\frac{F_0}{2}\left[\frac{1}{F_0^4 \mathrm{e}^{4A} y^2} j \wedge j - \left(\mathrm{d}(\mathrm{e}^{-4A} y D\psi) - y^{-1}\mathrm{d}\rho\right) \wedge \left(\frac{1}{F_0^2} j + \frac{y^2}{2}\mathrm{d}y \wedge D\psi\right)\right]. \quad (3.1.36)$$

The polyform for this flux is

$$\Phi^{F_4} = F_4 - \frac{F_0}{2}\left[\frac{y}{F_0^2 \mathrm{e}^{4A}} j + \frac{y}{2}\mathrm{d}y \wedge D\psi\right] + \frac{F_0 y^2}{8}\,. \quad (3.1.37)$$

The polyforms for the free energy, Y, and Z, are obtained by taking the smooth $l = 0$ limit of the expressions above.

3.1.4 Localization

Physical observables such as the free energy, flux quantization and conformal dimensions of dual operators are computed by integrating the closed polyforms from the previous subsection. Using the fixed point formulae (1.2.7), these integrals are particularly easy to perform. From our earlier discussion 3.1.2 there are two choices of boundary conditions we will consider. Either the R-symmetry circle shrinks at $\mathrm{e}^{4A} = y^2 \neq 0$ or the boundary condition for an O8-plane ($l = 0$ only) where $\mathrm{e}^{2A} = y = 0$. In each case, y is fixed, and we use this in the localization formulae. Notice that in the $l = 0$ case all the lower dimensional parts of the polyforms vanish upon substituting $y = 0$. Given the form of the contributions from the boundary in (1.2.8) it is clear that for the O8-plane there are no contributions from the boundary! We emphasize that this is not generic, other boundaries would contribute, the O8-plane is special.

Therefore when using the localization formulae we need only sum over the contributions

from the shrinking R-symmetry circle. In the $l = 0$ case we present, for exposition of the formulae, the general BVAB results. The on-shell action reads

$$F = \frac{1}{24\pi^2\ell_s^8}\Bigg\{ -\sum_{\Sigma_0}\frac{1}{d_0}\frac{F_0^2}{20}\frac{y^5}{\epsilon_1\epsilon_2\epsilon_3}\Big|_{\Sigma_0} + \sum_{\Sigma_2}\frac{1}{d_2}\frac{1}{\epsilon_1\epsilon_2}\int_{\Sigma_2}\left[\frac{y^2}{4\pi}j + \frac{F_0^2y^5}{20}\Big(\frac{c_1(L_1)}{\epsilon_1}+\frac{c_1(L_2)}{\epsilon_2}\Big)\right] \\ -\sum_{\Sigma_4}\frac{1}{d_4}\int_{\Sigma_4}\left[\frac{1}{F_0^2}\frac{1}{(2\pi)^2\epsilon_1}\frac{1}{y}j\wedge j + \frac{y^2}{4\pi\epsilon_1^2}j\wedge c_1(L) + \frac{F_0^2y^5}{20\epsilon_1^3}c_1(L)\wedge c_1(L)\right]\Bigg\}, \tag{3.1.38}$$

For $l = 0$ there are only the 4-form fluxes, which read

$$N^I = \frac{1}{(2\pi\ell_s)^3}\int_{\Gamma_4^I}F_4 = \frac{F_0}{4\pi\ell_s^3}\left[\sum_{\Sigma_0}\frac{1}{d_0}\frac{y^2}{4\epsilon_1\epsilon_2}\Big|_{\Sigma_0} - \sum_{\Sigma_2}\frac{1}{d_2}\int_{\Sigma_2}\left[\frac{1}{2\pi\epsilon_1}\frac{1}{F_0^2y}j + \frac{y^2}{4\epsilon_1^2}c_1(L)\right]\right], \tag{3.1.39}$$

where recall that $\Sigma_{0,2}$ denote fixed submanifolds inside Γ_4^I. The four-cycles Γ_4^I are four-cycles in X_6, which are not entirely fixed by the action of ξ. On the other hand when they are entirely fixed one has

$$N = \frac{1}{(2\pi\ell_s)^3}\int_{\Sigma_4}F_4 = -\frac{1}{(2\pi\ell_s)^3}\int_{\Sigma_4}\frac{1}{2F_0^3y^4}\,j\wedge j\,. \tag{3.1.40}$$

There is another observable that we will compute, namely the conformal dimensions of certain BPS operators in the dual conformal field theory. These are given by the action of D2-branes wrapped on calibrated two-cycles Σ_2 in X_6 [139]

$$\Delta(\Sigma_2) = \frac{1}{(2\pi)^2\ell_s^3}\int_{\Sigma_2}F_0\mathrm{e}^{2A}y\,\mathrm{vol}_{\Sigma_2} = \frac{1}{(2\pi)^2\ell_s^3}\int_{\Sigma_2}Y'\,, \tag{3.1.41}$$

where

$$Y' = F_0\Big[\frac{y}{F_0^2\mathrm{e}^{4A}}j + \frac{y}{2}\mathrm{d}y\wedge D\psi\Big] = -2\Phi_2^{F_4}\,, \tag{3.1.42}$$

with $\Phi_2^{F_4}$ the two-form part of the polyform for F_4 in equation (3.1.37). The BVAB formula then gives

$$\Delta(\Sigma_2) = -\frac{F_0}{2\pi\ell_s^3}\sum_{\Sigma_0}\frac{1}{d_0}\frac{y^2}{4\epsilon_1}\,, \qquad \Delta(\Sigma_2) = \frac{1}{(2\pi)^2\ell_s^3}\int_{\Sigma_2}\frac{1}{F_0y}j\,, \tag{3.1.43}$$

where the first expression holds when Σ_2 is not entirely fixed by ξ, and the second when it

is. The localization of the $l \neq 0$ polyforms from 3.1.3.1 follows in a similar way. Since the expressions are not particularly insightful, we do not write them out here but rather use them directly in the examples.

3.2 Examples

In the following we specify the general formulae form the previous section on explicit examples for the topology of X_6. We will consider the following examples:

3.2.1 5d SCFTs compactified on a Riemann surface,

3.2.2 5d SCFTs compactified on a spindle,

3.2.3 3d SCFTs from massive deformed 3d quivers,

3.2.4 3d SCFTs from deformed 3d quivers reduced from M-theory,

3.2.5 3d Chern–Simons theories from M-theory on a Sasaki–Einstein manifold,

3.2.6 3d Chern–Simons theories from suspension of a Sasaki–Einstein manifold.

3.2.1 D4-branes wrapped on a Riemann surface

As a first application of our proposal we study the near-horizon of the 5d SCFTs arising from N D4-branes probing a type IIA background with an O8-plane and N_f D8-branes compactified on a Riemann surface. The AdS_4 supergravity solutions were studied in [139]. Recall that the 5d SCFTs are dual to the Brandhuber–Oz AdS_6 solutions [137] which have internal space a topological four-dimensional hemi-sphere HS^4.

Consider X_6 to be a four-dimensional hemi-sphere bundle over a Riemann surface with projection map $\pi : X_6 \to \Sigma_g$. We view the hemi-sphere bundle as being embedded in the $\mathbb{R}^5_+$ bundle $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathbb{R}_+$, with the $\mathcal{L}_i$ two complex line bundles. We can take coordinates $\{z_1, z_2, x\}$ on $\mathbb{R}^5_+$ with $x \geq 0$ and the hemi-sphere is embedded as $|z_1|^2 + |z_2|^2 + x^2 = 1$. The north pole is then located at the point $\{z_1 = z_2 = 0, x = 1\}$ and the boundary, which is a three-sphere, is located at $x = 0$.

We take the total space to be a Calabi–Yau threefold with bundle:[6]

$$\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2) \to \Sigma_g \,, \tag{3.2.1}$$

where we identify $\mathcal{L}_i = \mathcal{O}(-p_i)$. In order for this to be Calabi–Yau the degrees of the line bundles must satisfy $p_1 + p_2 = \chi(\Sigma_g) = 2(1-g)$. We may then write the R-symmetry vector as

$$\xi = \sum_{i=1}^{2} b_i \partial_{\varphi_i} \,, \tag{3.2.2}$$

with each ∂_{φ_i} rotating the line bundle $\mathcal{L}_i$ defined above. The b_i are then directly the weights ϵ_i in the localization formulae (by definition). The first Chern classes of the line bundles $\mathcal{L}_i$ in terms of p_i are

$$\int_{\Sigma_g} c_1(\mathcal{L}_i) = -p_i \,. \tag{3.2.3}$$

We now need to consider the fixed point set of ξ. This is simply a copy of Σ_g at the pole of HS^4, and we denote this as Σ_g^p. There is also a boundary (from the perspective of the 6d internal metric) at $y = 0$ which is the boundary of the hemi-sphere and where all contributions to the localization formulae vanish.

There are three four-cycles that we must quantize the flux over. The first one is the full four-dimensional hemi-sphere HS^4 itself. Performing the integration we have

$$N = \frac{1}{(2\pi\ell_s)^3} \int_{HS^4} F_4 = \frac{1}{2\pi\ell_s^3} \frac{1}{b_1 b_2} \frac{F_0 y_p^2}{8} \,, \tag{3.2.4}$$

from which we deduce[7]

$$y_p = 4\sqrt{\frac{\pi l_s^3}{F_0} b_1 b_2 N} \,. \tag{3.2.5}$$

The other four-cycles, which we denote by C_4^i, are two-dimensional hemi-sphere bundles over the Riemann surface. The two-dimensional hemispheres are given by the embedding $HS^{2,i} \subset \mathcal{L}_i \oplus \mathbb{R}_+$ and the four-cycles are constructed by fibering these over the Riemann

[6] Observe that in comparing with [139] we take $p_1 = -p_{\text{there}}$ and $p_2 = -q_{\text{there}}$.

[7] Since we are solving for a quadratic there is of course the negative root too. The two choices give the same final results, up to a choice of orientation. It is not hard to see that they are identical after the redefinition $y \to -y$ which just interchanges the choice of pole of the hemi-sphere we are working with.

surface. One finds

$$N_i = \frac{1}{(2\pi\ell_s)^3}\int_{C_4^i} F_4 = \frac{1}{(2\pi\ell_s)^3}\left(\int_{\Sigma_g^p} -\frac{2\pi}{b_i}\frac{1}{2F_0 y_p} j + \left(\frac{2\pi}{b_i}\right)^2 \frac{F_0 y_p^2}{8} p_i\right), \tag{3.2.6}$$

where the second term has been integrated using (3.2.3). Moreover we find that $N_1 = -p_2 N$ and $N_2 = -p_1 N$. This is a result of cohomological considerations, see appendix C. Then we may use this to find an expression for the integral of j over the fixed Riemann surface as

$$\int_{\Sigma_g^p} j = 32\sqrt{\pi^5 \ell_s^9 F_0 b_1 b_2 N^3}(b_1 p_2 + b_2 p_1)\,. \tag{3.2.7}$$

Finally the free energy is given by

$$\begin{aligned} F &= -\frac{16\pi^3}{(2\pi\ell_s)^8}\frac{(2\pi)^2}{b_1 b_2}\left(\frac{y_p^2}{24}\int_{\Sigma_g^p} j - \frac{2\pi}{b_1 b_2}\frac{F_0^2 y_p^5}{240}\left(\frac{p_1}{b_1}+\frac{p_2}{b_2}\right)\right) \\ &= -\frac{16\sqrt{2}\pi}{5 n_0^{1/2}}\sqrt{b_1 b_2}(b_1 p_2 + b_2 p_1) N^{5/2}\,. \end{aligned} \tag{3.2.8}$$

Given that the Killing spinors have weight $\frac{1}{2}$ under ξ (in our conventions) the coefficients b_i satisfy the constraint $b_1 + b_2 = 1$, see [32] for the analogous statement for S^4 bundles. We may therefore write them in terms of a single parameter ε

$$b_1 = \frac{1}{2}(1+\varepsilon)\,, \quad b_2 = \frac{1}{2}(1-\varepsilon)\,. \tag{3.2.9}$$

Similarly, since $p_1 + p_2 = 2(1-g)$ we may parameterize them in terms of a new variable z (valid for $g \neq 1$)

$$p_1 = (1-g)(1+\kappa^{-1}z)\,, \quad p_2 = (1-g)(1-\kappa^{-1}z)\,, \tag{3.2.10}$$

with κ the extrinsic curvature $\kappa = 1$ for $g = 0$, $\kappa = 0$ for $g = 1$ and $\kappa = -1$ for $g > 1$. In terms of these variables the free energy reads:

$$F = -\frac{8\sqrt{2}\pi}{5\kappa n_0^{1/2}}\sqrt{(1-\varepsilon^2)}(g-1)(\varepsilon z - \kappa) N^{5/2}\,. \tag{3.2.11}$$

This is an *off-shell* result and should be extremized for the parameter ε, finding

$$\varepsilon_\pm = \frac{\kappa \pm \sqrt{\kappa^2 + 8z^2}}{4z}\,. \tag{3.2.12}$$

We need to keep the ε_+ solution only since the ε_- gives a negative free energy.[8] After extremization one finds the beautiful on-shell free energy

$$F = \frac{2\pi(1-g)N^{5/2}}{5\kappa n_0^{1/2}}(\sqrt{\kappa^2+8z^2}-3\kappa)\sqrt{1-\frac{\kappa(\sqrt{\kappa^2+8z^2}+\kappa)}{4z^2}}\,. \tag{3.2.13}$$

This matches with the field theory result obtained in [139, 140] after a little rewriting and gives a derivation of the off-shell gravitational block formula of [140] for the Riemann surface directly in massive type IIA.

We can also use our localization formulae to compute the conformal dimensions of certain BPS operators which correspond to D2-branes wrapping two-cycles. For the three obvious two-cycles in the geometry we find:

$$\begin{aligned}
\Delta(HS^{2,1}) &= -\frac{F_0}{2\pi\ell_s^3}\frac{y_p^2}{4b_1} = 2b_2N = (1-\varepsilon)N\,,\\
\Delta(HS^{2,2}) &= -\frac{F_0}{2\pi\ell_s^3}\frac{y_p^2}{4b_2} = 2b_1N = (1+\varepsilon)N\,,\\
\Delta(\Sigma_g^p) &= -\frac{1}{(2\pi)^2\ell_s^3}\int_{\Sigma_g^p}\frac{1}{F_0 y}j = -2(b_1p_2+b_2p_1)N = \frac{2(1-g)}{\kappa}(z\varepsilon-\kappa)N\,.
\end{aligned} \tag{3.2.14}$$

These are the general off-shell results plugging in the on-shell value of $\varepsilon = \varepsilon_+$ gives

$$\begin{aligned}
\Delta(HS^{2,i}) &= \left(1 \mp \frac{\sqrt{\kappa^2+8z^2}+\kappa}{4z}\right)N\,,\\
\Delta(\Sigma_g^p) &= \frac{(1-g)}{2\kappa}\left(\sqrt{\kappa^2+8z^2}-3\kappa\right)N\,,
\end{aligned} \tag{3.2.15}$$

with the $\mp$ signs corresponding to $i = 1, 2$ respectively. The on-shell expressions match [139]. Note that requiring these to be positive fixes $|z| > 1$ for $\kappa = 1$ and $|z| > 0$ for $\kappa = -1$. The

[8] One can raise the point of whether there is a metric for both of these choices of ϵ. The logical claim would be that there is an obstruction to finding a metric with the ϵ_- solution. A putative way of discerning this is by studying the conformal dimensions of BPS particles obtained by wrapping D2 branes on two-cycles. From the results below (see (3.2.14)) we find that the ϵ_- solution gives two positive conformal dimensions and one negative. The latter is then signalling an obstruction to finding explicit metrics.

torus case can be obtained by setting $\kappa = 0$ and $g = 1$ in both expressions with $\frac{1-g}{\kappa} \to 1$. This is indeed the correct field theory constraints and it is satisfying that it pops out from gravity.

Before we wrap up this section it is useful to write the analogous results of the conformal dimensions if we took the $\varepsilon = \varepsilon_-$ solution to the extremal problem. We would have found:

$$\begin{aligned} \Delta(HS^{2,i}) &= \left(1 \mp \frac{\kappa - \sqrt{\kappa^2 + 8z^2}}{4z}\right) N\,, \\ \Delta(\Sigma_g^p) &= \left(\frac{(g-1)(3\kappa + \sqrt{\kappa^2 + 8z^2})}{2\kappa}\right) N\,. \end{aligned} \tag{3.2.16}$$

One can see that the sign of $\Delta(\Sigma_g^p)$ and $\Delta(HS^2, i)$ are different despite all being required to be of the same (positive) sign. This then gives an obstruction to the existence of metrics with $\varepsilon = \varepsilon_-$. Recall that a similar obstruction has been observed in the geometric dual of $\mathcal{I}$/c-extremization [122, 141], it would be interesting to further refine these statements on obstructions to the existence of metrics.

3.2.2 D4-branes wrapped on a spindle

Having considered D4-branes wrapping a constant curvature Riemann surface we now turn our attention to D4-branes wrapping a spindle.[9] A spindle is the two-dimensional weighted projective space $\Sigma = \mathbb{WCP}^1_{[n_+,n_-]}$. It is topologically a two-sphere but with conical deficit angles $2\pi(1 - n_\pm)$ at the poles and first appeared in supergravity theories in [107, 143] with many other solutions in various theories appearing since. The explicit supergravity solution corresponding to D4-branes wrapped on a spindle was constructed in [140]. As shown in [68] there are different types of spindle solutions distinguished by the mechanism to preserve supersymmetry: known as the *twist* and *anti-twist*. In the twist case setting the two orbifold parameters to be trivial, $n_+ = n_- = 1$, we recover a two-sphere and indeed this reduces to the results of the previous section. In [140] a gravitational block formula for the free energy of the dual SCFT was conjectured which we will derive from our results. Recently this was

[9] One can also replace the spindle with a topological disc [142] which are different global completions of the same local solution. We will not concern ourselves with discs in this work though the extension is straight-forward once one accepts introducing an additional boundary contribution.

recovered in [37] by using equivariant localization in 6d $\mathrm{U}(1)^2$ gauged supergravity with an $\mathrm{AdS}_4 \times \Sigma$ ansatz.

Similarly to the previous section, we consider a four-dimensional hemi-sphere orbibundle over a two-dimensional space, in this instance a spindle. As before we view the hemi-sphere via the embedding $HS^4 \subset \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathbb{R}_+ \subset \mathbb{R}^5$. In the twist class the fibration of the two line orbibundles over the spindle $\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2) \to \Sigma$ is Calabi-Yau, which enforces the condition $p_1 + p_2 = n_+ + n_-$. To impose the anti-twist one should instead take $p_1 + p_2 = n_+ - n_-$. This may be unified by introducing the sign $\sigma = \pm 1$ such that $p_1 + p_2 = n_+ + \sigma n_-$ with the twist given by $\sigma = 1$ and the anti-twist given by $\sigma = -1$.

The main difference in the localization in comparison with the Riemann surface analysis resides in the fact that the R-symmetry vector can now also rotate the spindle, and therefore the R-symmetry vector takes the form

$$\xi = \sum_{i=1}^{2} b_i \partial_{\varphi_i} + \varepsilon \partial_{\varphi_0} \,, \tag{3.2.17}$$

where ∂_{φ_i} rotate the $\mathcal{L}_i$ and ∂_{φ_0} the spindle. Therefore the weights at the fixed points are identified with $\epsilon_i = b_i$, and $\epsilon_3 = \mp\varepsilon/n_\pm$. The fixed point set is qualitatively very different from before as we now have isolated fixed points (located at the poles of the spindle and hemi-sphere) rather than a fixed two-dimensional surface. Consequently, even though the setups seem closely related, the localization analysis is fairly different as we shall see now.

Let us first quantize the flux. We get a copy of $HS^4/\mathbb{Z}_{n_\pm}$ at each pole of the spindle through which we can compute the fluxes[10]

$$N_\pm = \frac{1}{(2\pi\ell_s)^3} \int_{HS^4/\mathbb{Z}_{n_\pm}} F_4 = \frac{1}{2\pi\ell_s^3} \frac{1}{n_\pm} \frac{1}{b_1^\pm b_2^\pm} \frac{F_0 y_\pm^2}{8} \,. \tag{3.2.18}$$

Moreover the homologies of these cycles are related, see appendix C and [144]. We have

$$n_+ N_+ = n_- N_- \equiv N \,, \tag{3.2.19}$$

[10]This is the same flux equation as (3.2.4) but doubled and with an orbifold factor $d = n_\pm$. Note that the subscript p there denoted the pole of the hemi-sphere, while now the $\pm$ denote the poles of the spindle. These fixed points are still at the pole of the hemisphere. To be fully precise we should use the subscript $p, \pm$, but we drop the p for ease of notation.

such that[11]

$$y_+ = 4\sqrt{\frac{\pi l_s^3}{F_0} b_1^+ b_2^+ N}\,, \quad y_- = 4\sigma\sqrt{\frac{\pi l_s^3}{F_0} b_1^- b_2^- N}\,. \tag{3.2.20}$$

The flux quantization through other cycles is not needed to obtain the free energy (this is to be contrasted with the Riemann surface computation), and so we postpone performing this for the moment.

We can directly plug the expressions for $y_\pm$ back into the free energy. This equation is quite different from the section before since we are now localizing on isolated fixed points rather than fixed surfaces,

$$\begin{aligned} F &= -\frac{1}{24\pi^2\ell_s^8}\frac{F_0^2}{20}\left(\frac{1}{n_+}\frac{y_+^5}{b_1^+ b_2^+(-\varepsilon/n_+)} + \frac{1}{n_-}\frac{y_-^5}{b_1^- b_2^-(\varepsilon/n_-)}\right) \\ &= \frac{32\sqrt{2}\pi}{15 n_0^{1/2}}\frac{(b_1^+ b_2^+)^{3/2} - \sigma(b_1^- b_2^-)^{3/2}}{\varepsilon} N^{5/2}\,. \end{aligned} \tag{3.2.21}$$

This derives the conjectured gravitational block formula in [140] directly in massive type IIA. The ambiguity in which gluing to pick is then related to the sign of the fixed points $y_\pm$, this is similar to what happens in the explicit solutions in the 5d and 4d cases [68, 109]. It would be interesting to understand any obstructions to finding explicit metrics for these different choices.

To proceed with the extremization it is useful to identify[12]

$$b_i^\pm = b_i + \frac{\varepsilon}{2}\left(\pm\frac{p_i}{n_+ n_-} + \frac{n_+ - \sigma n_-}{2 n_+ n_-}\right)\,, \quad b_1 + b_2 = 1\,, \tag{3.2.22}$$

which can be read off from the results in [144] or [140]. More concretely one can realize this by noting that this solves the conditions:

$$b_1^\pm + b_2^\pm = 1 \pm \frac{\varepsilon}{n_\pm}\,, \qquad b_i^+ - b_i^- = \frac{p_i}{n_+ n_-}\varepsilon\,. \tag{3.2.23}$$

The former conditions set the overall charge of the holomorphic volume form to be 1 as

[11] We see that there is a sign ambiguity with the roots of $y_\pm$. Without loss of generality we can take y_+ to be positive and then we introduce a relative sign between the two roots. With hindsight comparing with the literature [140] we identify this sign with σ, however we emphasize that this is not dictated by the above.

[12] Notice that we have $2b_i^\pm = \Delta_i^\pm$ when comparing with the notation of [140].

required and the latter are equivalent to (3.24) of [144] which define the gluing of the different patches used to build X_6.

One should now extremize this off-shell free energy over ε and b_i subject to the constraint (3.2.22). Since this has been done already in [140] (see also [37]) we will not give the extremization explicitly and refer the interested reader to those works. Upon extremization this gives the free energy for the compactifcation of the 5d $\mathcal{N}=1$ USp($2N$) gauge theory with $N_f = 8-n_0$ massless hypermultiplets in the fundamental representation and one hypermultiplet in the antisymmetric representation of USp($2N$) on a spindle.

Having recovered the off-shell gravitational block formula for the free energy we return to the quantization of the fluxes. There are other four cycles through which we can quantize the flux, and while they were not needed for the free energy computation, they are interesting quantities in their own right. The cycles we want to consider are the total space of the $HS^{2,i} \subset \mathcal{L}_i \oplus \mathbb{R}_+$ bundle over the spindle and denoted by C_4^i. The localization formula gives

$$N_i = \frac{1}{(2\pi\ell_s)^3}\int_{C_4^i} F_4 = \frac{1}{2\pi\ell_s^3}\frac{F_0}{8}\left(\frac{1}{n_+}\frac{y_+^2}{b_i^+(-\varepsilon/n_+)} + \frac{1}{n_-}\frac{y_-^2}{b_i^-(\varepsilon/n_-)}\right), \tag{3.2.24}$$

and after simplifying we find

$$N_1 = \frac{b_2^- - b_2^+}{\varepsilon}N = \frac{p_2}{n_+n_-}N\,, \quad N_2 = \frac{b_1^- - b_1^+}{\varepsilon}N = \frac{p_1}{n_+n_-}N\,. \tag{3.2.25}$$

This is the same result as in the M5-brane case [32] and can be explained using cohomological arguments similar to those in appendix C.

Before finishing the section we can study certain BPS operators in the dual field theory which correspond to D2-branes wrapping calibrated cycles. There are two types of calibrated cycle, one is the two cycle consisting of the spindle at the pole of the hemi-sphere and the second type are the two-dimensional hemispheres $HS^{2,i}$ discussed above at a pole of the spindle. This is similar to the M5-brane case as explained in [145]. For the D2-branes wrapping a copy of the spindle at the pole of the HS^4, the off-shell conformal dimension is

$$\Delta(\Sigma^p) = -\frac{F_0}{2\pi\ell_s^3}\left(\frac{1}{n_+}\frac{y_+^2}{4(-\varepsilon/n_+)} + \frac{1}{n_-}\frac{y_-^2}{4(\varepsilon/n_-)}\right) = -2\frac{b_1^-b_2^- - b_1^+b_2^+}{\varepsilon}N\,. \tag{3.2.26}$$

For the second type of BPS operator we have

$$\Delta(HS^{2,i}_{\pm}) = \frac{F_0}{2\pi\ell_s^3}\frac{y_\pm^2}{4b_i^\pm} = 2\frac{b_1^\pm b_2^\pm}{b_i^\pm}N\,. \tag{3.2.27}$$

An obvious obstruction to finding explicit metrics is to require these conformal dimensions to be positive, however it is clear that since the $y_\pm$ always appear squared this will not fix the sign ambiguity noted in footnote 11. We leave understanding these points to future work.

3.2.3 HS^2 bundle over B_4

We now turn our attention to hemi-sphere bundles over a four-dimensional base with the requirement that X_6 is complex, see section 4 of [61]. We will impose that there is an O8-plane which implies that we consider X_6 as the total space of an HS^2 bundle over a 4-dimensional base B_4, with HS^2 the two-dimensional hemi-sphere[13]

$$HS^2 \hookrightarrow X_6 \to B_4\,. \tag{3.2.28}$$

Recall from section 3.1.2 that this implies that we necessarily need to take $l = 0$. While B_4 could in principle be any compact manifold, for X_6 to be a complex manifold B_4 is either Kähler-Einstein or the product of two Riemann surfaces which can be seen by using the arguments of [131, 146]. Explicit solutions of this type have been constructed in [61] and further studied in [55]. This section recovers and generalizes their results.

We assume that the vector field ξ rotates just the HS^2 fibre and leaves B_4 fixed. The known solutions are constructed by fibring the hemi-sphere bundle using the anti-canonical line bundle $\mathcal{L}$ over B_4. Note that since we are considering the hemi-sphere in our localization computations, we have a spacetime with a boundary. As explained earlier, the general formulae as written in section 3.1.4 do not contain such boundary contributions, however since the boundary is for an O8 plane located at $y = 0$ and the U(1) action acts freely on the boundary, the boundary terms do not contribute and we may use those results. Recall that this requires the warp factor e^{2A} to degenerate in a certain way, and is what fixes the

[13] We refer interchangeably to the hemi-sphere as a disc (compact with boundary) since they are topologically identical. The hemi-sphere notation makes explicit the choice of boundary condition.

boundary conditions. Our localization locus consists of a single fixed point at the pole of the hemi-sphere, denoted y_p and the vanishing boundary contribution at $y = 0$. The fixed point at the pole is actually a fixed copy of B_4 with normal bundle $\mathcal{L}$.

Following [32] we define

$$c_\alpha = \int_{\Gamma_\alpha} j\,, \quad n_\alpha = \int_{\Gamma_\alpha} c_1(\mathcal{L})\,, \tag{3.2.29}$$

where Γ_α form a basis for the 2-cycles of the copy of B_4 at the pole, and $\mathcal{L}$ is its the normal line bundle. Moreover the weight at the pole is $\epsilon = 1$ the sign fixed by our choice of $c_1(\mathcal{L})$. Using this we can quantize the flux. The first possibility is to consider four-cycles C_4^α which are the total space of the HS^2 bundle over Γ_α, giving N_α

$$N_\alpha = \frac{1}{(2\pi\ell_s)^3}\int_{C_4^\alpha} F_4 = \frac{F_0}{4\pi\ell_s^3}\left[\frac{1}{2\pi}\frac{1}{F_0^2 y_p}c_\alpha - \frac{y_p^2}{4}n_\alpha\right]\,, \tag{3.2.30}$$

from which we can deduce an expression for c_α in terms of N_α and n_α

$$c_\alpha = \frac{\pi F_0 y_p}{2}(F_0 y_p^2 n_\alpha - 16\pi\ell_s^3 N_\alpha)\,. \tag{3.2.31}$$

Then we can also quantize the flux through the full fixed B_4 giving N_p

$$\begin{aligned} N_p &= \frac{1}{(2\pi\ell_s)^3}\int_{B_p^4} F_4 = -\frac{1}{(2\pi\ell_s)^3}\frac{1}{2F_0^3 y_p^4}\langle c, c\rangle \\ &= \frac{1}{2}\langle N, n\rangle - \frac{F_0 y_p^2}{64\pi\ell_s^3}\langle n, n\rangle - \frac{4\pi\ell_s^3}{F_0 y_p^2}\langle N, N\rangle\,, \end{aligned} \tag{3.2.32}$$

where we used (3.2.31) in the second line. To keep the equations succinct it is useful to define the bracket $\langle c, c\rangle \equiv I_{\alpha\beta}c_\alpha c_\beta$ and similarly for other quantities in the following. Here $(I_{\alpha\beta})$ is the inverse of the intersection form for B_4, see appendix C. Finally as shown in appendix C, the cycles are not independent such that we get some topological relation between the various fluxes which reads[14]

$$N_p = -\langle N, n\rangle + M\,. \tag{3.2.33}$$

[14]When comparing with [55, 61] the parameter M is related to having a non-zero value of σ in the notation of [61].

Combining the two previous relations for N_p, we can solve for y_p, giving the solutions

$$y_p = 4\sqrt{\frac{\pi\ell_s^3}{F_0}\frac{3\langle N,n\rangle - 2M \pm \sqrt{(3\langle N,n\rangle - 2M)^2 - \langle N,N\rangle\langle n,n\rangle}}{\langle n,n\rangle}}\,. \tag{3.2.34}$$

Inserting these results for the y_p into the free-energy localized form we obtain two positive solutions

$$\begin{aligned} F &= -\frac{1}{24\pi^2\ell_s^8}\left[\frac{1}{(2\pi F_0)^2 y_p}\langle c,c\rangle - \frac{y_p^2}{4\pi}\langle c,n\rangle + \frac{F_0^2 y_p^5}{20}\langle n,n\rangle\right] \\ &= \frac{2y_p}{3\ell_s^2}\langle N,N\rangle - \frac{F_0^2 y_p^5}{1920\pi^2\ell_s^8}\langle n,n\rangle \\ &= \frac{16\sqrt{2}\pi}{15 n_0^{1/2}}\frac{1}{\langle n,n\rangle^{3/2}}\sqrt{3\langle N,n\rangle - 2M \pm \sqrt{(3\langle N,n\rangle - 2M)^2 - \langle N,N\rangle\langle n,n\rangle}} \\ &\times\left(3\langle N,N\rangle\langle n,n\rangle + (3\langle N,n\rangle - 2M)\left(3\langle N,n\rangle \pm \sqrt{(3\langle N,n\rangle - 2M)^2 - \langle N,N\rangle\langle n,n\rangle}\right)\right)\,, \end{aligned} \tag{3.2.35}$$

where the first line is just the localization (3.1.38), then we replaced c_α in the second line, and y_p in the last line, giving our final result. In summary this is the result for a general topology on B_4. Picking a topology amounts to specifying the N,n-brackets and we will consider different examples shortly in the following and compare with known solutions. Note that the parameter M generalizes the results in [55]. We also note that there are two different choices, both giving a well defined free energy for suitable constraints on the bracket. The comments about obstructions may, once again be repeated virtually verbatim.

Finally we can also compute the conformal dimensions using (3.1.43). We can either consider a D2 brane wrapping the fibre HS^2 itself giving

$$\Delta(HS^2) = \frac{F_0}{2\pi\ell_s^3}\frac{y_p^2}{4} = \frac{2}{\langle n,n\rangle}\left(3\langle N,n\rangle - 2M \pm \sqrt{(3\langle N,n\rangle - 2M)^2 - \langle N,N\rangle\langle n,n\rangle}\right)\,, \tag{3.2.36}$$

or alternatively wrapping Γ_α which results in

$$\begin{aligned} \Delta(\Gamma_\alpha) &= \frac{1}{(2\pi)^2\ell_s^3}\frac{c_\alpha}{F_0 y_p} \\ &= \frac{2}{\langle n,n\rangle}\left((3\langle N,n\rangle - 2M)n_\alpha + (\langle n,n\rangle \pm \sqrt{(3\langle N,n\rangle - 2M)^2 - \langle N,N\rangle\langle n,n\rangle})N_\alpha\right)\,. \end{aligned} \tag{3.2.37}$$

3.2.3.1 Kähler–Einstein base

For our first example we consider $B_4 = KE_4^+$ to be a positively curved Kähler–Einstein four-manifold. Since we have taken a Kähler–Einstein base the fluxes are proportional to the Chern numbers $N_\alpha = kn_\alpha$. Then we have

$$\langle n, n\rangle = \bar{M}\,, \quad \langle N, n\rangle = k\bar{M}\,, \quad \langle N, N\rangle = k^2\bar{M}\,, \tag{3.2.38}$$

where $\bar{M}$ is a topological invariant (the integral of the Ricci form squared), and we further define

$$k = \frac{N}{h}\,. \tag{3.2.39}$$

Plugging these in the general expression for the free energy and setting the flux number $M = 0$ straightforwardly gives

$$F = \frac{32\sqrt{2}\pi}{5n_0^{1/2}}(3 \pm 2\sqrt{2})\bar{M}\left(\frac{N}{h}\right)^{5/2}\,. \tag{3.2.40}$$

This matches exactly the results obtained in [55]. Additionally the conformal dimensions read

$$\Delta(HS^2) = \frac{2(3 \pm 2\sqrt{2})}{h}N\,, \qquad \Delta(\Gamma_\alpha) = \frac{4(1 \pm \sqrt{2})}{h}Nn_\alpha\,, \tag{3.2.41}$$

which have not appeared in the literature previously. Note that for the $-$ sign that $\Delta(\Gamma_\alpha) < 0$, one suspects that this then indicates an obstruction to finding a metric in this case, whilst the other case seems pathology free.

3.2.3.2 Product base

The second option is to take B_4 to be the product of two Riemann surfaces: $B_4 = \Sigma_{g_1} \times \Sigma_{g_2}$. Then the brackets are

$$\langle n, n\rangle = 2\chi_1\chi_2\,, \quad \langle N, n\rangle = \chi_1 N_2 + \chi_2 N_1\,, \quad \langle N, N\rangle = 2N_1N_2\,. \tag{3.2.42}$$

It is then straightforward to insert these in the general formula (3.2.35) to obtain the free energy. We secretly assumed that our base had a positive overall curvature which implies that at least one of the Riemann surfaces should be a sphere for B_4 to be positively curved and our computations to apply.

In particular considering the product of two spheres, $\chi_1 = \chi_2 = 2$ and setting $M = 0$, we can parametrize the fluxes in terms of a single variable $N_1 = (1+z)N$, $N_2 = (1-z)N$, with $|z| < 1$ and obtain

$$F = \frac{32\pi}{5n_0^{1/2}}\sqrt{3 \pm \sqrt{8+z^2}}(\sqrt{8+z^2} \pm (2+z^2))N^{5/2}\,. \tag{3.2.43}$$

Again this result matches with the one of [55]. We also have

$$\begin{aligned} &\Delta(HS^2) = \left(3 \pm \sqrt{8+z^2}\right)N\,, \\ &\Delta(S_1^2) = 2\left(2 - z \pm \sqrt{8+z^2}\right)N\,, \quad \Delta(S_2^2) = 2\left(2 + z \pm \sqrt{8+z^2}\right)N\,, \end{aligned} \tag{3.2.44}$$

where the copies of the S^2 from the base are taken at the pole of the HS^2 fibre. Similarly to the Kähler–Einstein case, with the minus sign, the $\Delta(S_i^2)$ become negative and one would suspect that this indicates an obstruction to finding explicit metrics for this solution.

3.2.4 Cylinder bundle with O4 and O8

In this example we consider a cylinder bundle over a Kähler–Einstein base. The data of such a base has been defined in 3.2.3.1. The cylinder is capped off by the presence of both O4 and O8 orientifold planes, one at each end. From our analysis in section 3.1.2 it follows that the O8 plane is situated at $y = 0$, and we fix the O4 plane to be at $y_b \neq 0$. As before, the presence of the O8 plane imposes $l = 0$. This example is particularly interesting because the fixed point set is empty and the localized integrals only pick up contributions from the boundary terms.

For simplicity we will restrict to regular fibrations, which covers a large class of geometries. For the general case of irregular fibrations, an interesting way to perform the integrals would be to build equivariantly closed forms for the boundary contributions and further use odd-

dimensional localization. We leave this approach for future work.

The first four-cycle we consider is the copy of the base at the non-zero boundary. It is well-defined for the fibrations under consideration, and its flux counts the D4's on top of the O4-plane

$$N_b = \frac{1}{(2\pi\ell_s)^3}\int_{B_4^b} F_4 = \frac{1}{8\pi^2\ell_s^3}\frac{1}{F_0 y_b}\langle c^b, n\rangle\,, \quad \lambda_\alpha^b = \int_{\Gamma_\alpha^b} j\,, \tag{3.2.45}$$

where the latter integral should be understood as j restricted to the boundary integrated over the two-cycle Γ_α in the base. The second type of four-cycles arising in this geometry are the cylinder bundles over the two-cycles in the base. The flux integrals pick up contributions at the two S^1 boundaries of the cylinder. The contribution from $y = 0$ vanishes, and thus we find

$$N_\alpha = -\frac{1}{(2\pi\ell_s)^3}\int_{S^1\ltimes\Gamma_\alpha^b} \imath^*\frac{\eta\wedge\Phi^{F_4}}{\mathrm{d}_\xi\eta} = -\frac{1}{(2\pi\ell_s)^3}\frac{F_0 y_b^2}{8}\int_{S^1\ltimes\Gamma_\alpha^b} D\psi\wedge\mathrm{d}_4\rho\,. \tag{3.2.46}$$

In general, this integral is non trivial to perform because of the fibration of the boundary over the base however the quasi-regular assumption allows us to make progress. First note that $D\psi$ is a globally well-defined one-form on the total space ∂X_6, and may be normalized by $\int_{S^1} D\psi = 2\pi$. Then since both j and $\mathrm{d}_4\rho$ are basic one can compute the integral by fibre integration. In particular for the three-cycles $S^1 \ltimes \Gamma_\alpha^b$ one has

$$\int_{S^1\ltimes\Gamma^\alpha} D\psi\wedge\mathrm{d}_4\rho = \int_{\Gamma_\alpha}\pi_*(D\psi)\,\mathrm{d}_4\rho\,. \tag{3.2.47}$$

This requires that the fibration is regular or quasi-regular with the latter understood to include appropriate orbifold factors. For an irregular fibration one is not able to perform this splitting as above. Finally note that $\mathrm{d}_4\rho = -c_1(\mathcal{L})$ and therefore we have

$$N_\alpha = \frac{F_0 y_b^2}{16\pi\ell_s^3} n_\alpha\,. \tag{3.2.48}$$

This allows us to read off the proportionality constant k and we find an expression for y_b

$$y_b = \sqrt{\frac{16\pi\ell_s^3}{F_0}k}\,. \tag{3.2.49}$$

Similarly, the free energy picks up a single contribution coming from the boundary

$$\begin{aligned} F &= \frac{16\pi^3}{(2\pi\ell_s)^8}\int_{\partial X_6} \imath^* \frac{\eta\wedge\Phi^{\mathcal{F}}}{\mathrm{d}_\xi\eta} \\ &= \frac{16\pi^3}{(2\pi\ell_s)^8}\int_{S^1\ltimes B_b^4} D\psi\wedge\left[\frac{y_b^2}{24}j(y_b)\wedge\mathrm{d}_4\rho+\frac{F_0^2y_b^5}{240}\mathrm{d}_4\rho\wedge\mathrm{d}_4\rho\right]. \end{aligned} \tag{3.2.50}$$

As above, to perform this integral we use the fibre-wise integration under the assumption of a (quasi-)regular fibration. Then

$$F = \frac{16}{15}\frac{\sqrt{2}\pi}{n_0^{1/2}}(2kM+5N_b)k^{3/2}\,. \tag{3.2.51}$$

Note that the field theories dual to this geometry are unknown, and the expression above gives a prediction for their free energy. We expect these theories to be 3d quiver gauge theories, arising as deformations of quivers dual to dimensional reductions of AdS_4 solutions of M-theory. Similar solutions have been described in [147].

Finally we can also compute the conformal dimension of BPS operators dual to D2-branes wrapped on calibrated two-cycles. The only two-cycle in the present geometry is the cylinder itself and we find

$$\Delta = -\frac{1}{2\pi^2\ell_s^3}\int_{S_b^1}\frac{\eta\wedge\Phi^{F_4}}{\partial_\xi\eta} = \frac{F_0y_b^2}{8\pi\ell_s^3} = 2k. \tag{3.2.52}$$

3.2.5 S^2 bundle over B_4

Next let us consider an S^2 bundle over a four-dimensional base B_4. In this case we will no longer force the presence of an O8-plane which accounts for considering an S^2 rather than HS^2 and allows us to consider the $l\neq 0$ class. Moreover we are no longer able to gauge away l, as one can do when both F_0 and l are non-zero, and we will see that we in fact need to extremize over it.

To make things more tractable we will restrict to the massless case and make some comments about the massive case at the end of this section. Recall that solutions of this form can be constructed by reducing certain Sasaki–Einstein manifolds from M-theory. Additionally one can consider the massive deformation of these theories by turning on a non-trivial Romans mass, see [129, 147–149].

Since we are considering an S^2 bundle over a four-dimensional base the fixed point locus corresponds to the base B_4 at the two poles of the sphere. We use the anti-canonical line bundle $\mathcal{L}$ of B_4 to fibre the S^2 over B_4. We assume that the R-symmetry vector ξ acts only on the S^2 such that the fixed point set is precisely copies of B_4 at the north and south pole of the sphere, which we call $B_4^{N,S}$. The normal bundles are then

$$L^N = \mathcal{L}^{-1}\,, \qquad L^S = \mathcal{L}\,. \tag{3.2.53}$$

As explained in the previous section, albeit for the hemi-sphere, there are two types of two-cycles to consider. One is the S^2 fibre and the other is a two-cycle in B_4 at either of the poles. We denote by b_2 the 2nd Betti number on B_4 and define $\Gamma_\alpha^{N,S}$ to be the two-cycles on B_4 at the north and south pole of the sphere respectively. These cycles are not independent, following [32] (see also appendix C), we have

$$[\Gamma_\alpha^S] - [\Gamma_\alpha^N] = n_\alpha [S^2_{\text{fibre}}] \in H^2(X_6, \mathbb{Z}) \tag{3.2.54}$$

where

$$n_\alpha \equiv \int_{\Gamma_\alpha} c_1(\mathcal{L}) \in \mathbb{Z}\,. \tag{3.2.55}$$

The four-cycles have a similar structure, they are the copies of B_4 at the two poles, $B_4^{N,S}$ and four-cycles constructed by taking the sphere bundle over one of the cycles Γ_α which we denote by C_α. Again these cycles are not independent and we have the relation

$$[B_4^S] - [B_4^N] = \sum_{\alpha=1}^{b_2} n_\alpha [C_\alpha]\,. \tag{3.2.56}$$

Analogously to the definition of c_α in (3.2.29), we define

$$c_\alpha^{N,S} \equiv \int_{\Gamma_\alpha^{N,S}} j\,. \tag{3.2.57}$$

Using the homological relation (3.2.54) with the equivariantly closed form Φ^Y, (3.1.33) we

find the constraint

$$c_\alpha^S - c_\alpha^N = \frac{l^2\pi}{\epsilon}\left(\frac{1}{y^N} - \frac{1}{y^S}\right) n_\alpha \,, \tag{3.2.58}$$

where $\pm\epsilon$ is the weight of ξ at the north and south pole respectively, and the charge of the Killing spinor fixes $\epsilon = 1$. This is analogous to the constraint (3.2.29) in the previous section, though different due to the bundle structure. One could also integrate the equivariantly closed four-form Φ^Z over the four-cycles and use the homology relation (3.2.56) and would reassuringly find that this is satisfied given (3.2.58).

Next let us consider the quantization of the fluxes. In the massless case it is not difficult to see that the four-form flux vanishes and we can choose the gauge in which the B field vanishes too. We therefore have two types of fluxes to quantize, the two-form F_2 and the six form F_6.[15] Let us first quantize the two-form flux. To avoid a proliferation of flux integers called n we denote these quantized fluxes associated to the two-form F_2, by k. In the field theory there is a relation between these and the Chern–Simons levels of the dual field theory, see for example [147, 150]. We have

$$\begin{aligned} k_2 &= \frac{1}{2\pi\ell_s}\int_{S^2_{\text{fibre}}} F_2 = \frac{l}{2\epsilon\ell_s}\left[\frac{1}{y_N} - \frac{1}{y_S}\right] , \\ k_\alpha^{N,S} &= \frac{1}{2\pi\ell_s}\int_{\Gamma_\alpha^{N,S}} F_2 = \frac{c_\alpha^{N,S}}{2l\ell_s\pi} \,. \end{aligned} \tag{3.2.59}$$

We see that we may rewrite the constraint (3.2.58) as

$$c_\alpha^S - c_\alpha^N = 2l\ell_s\pi k_2 n_\alpha \quad \Leftrightarrow \quad k_\alpha^S - k_\alpha^N = k_2 n_\alpha \,. \tag{3.2.60}$$

The quantization conditions fixes the $c_\alpha^{N,S}$ in terms of the flux parameters $k_\alpha^{N,S}$ (associated to D6-branes wrapping four-cycles or KK-monopoles in the M-theory setup). Note that using the constraint (3.2.60) one sees that for integer n_α, k_2, and k_α^N necessarily $k_\alpha^S \in \mathbb{Z}$ as required.

[15] There is a subtlety here that we are sweeping under the carpet slightly. Since we may fix a gauge where $B = 0$ one can use the Maxwell fluxes to quantize the fluxes. Rather than computing a whole new set of polyforms we will set $F_0 = 0$ in our Page flux polyforms and since the polyforms use the gauge in which $B = 0$ if we set $F_0 = 0$ these will be equivalent. This still leaves possible the option of turning on a closed term in B and computing the Page flux with this choice. Notice that this choice is made in [150] for example. This ambiguity leaves a gauge freedom which we will encounter later and is related to the parameter l not being fixed by anything.

We use the definition of k_2 to fix one of the roots, without loss of generality we solve for y_N:

$$y_N = \frac{l y_S}{l + 2\ell_s k_2 y_S \epsilon}\,. \tag{3.2.61}$$

Next let us compute the quantization of F_6. We have

$$\begin{aligned} N &= -\frac{1}{(2\pi\ell_s)^5}\int_{X_6} F_6 \\ &= \frac{l^4 k_2^2 \pi^2 (y_S - y_N) + l^2 k_2 \epsilon \pi (c_N y_N^2 - c_S y_S^2) + \epsilon^2 (c_S^2 y_S^3 - c_N^2 y_N^3)}{64 l^3 \epsilon^3 \pi^4 \ell_s^5}\,. \end{aligned} \tag{3.2.62}$$

We need to solve this for y_S however we should be smart about this. We redefine the parameter l as

$$l = -L y_S k_2 \ell_s\,, \tag{3.2.63}$$

with L the new free parameter. We find

$$N = \frac{y_S^2(\langle k^N, k^N\rangle L^2 + \langle k^S, k^S\rangle (L-2)^2 + \langle k^N, k^S\rangle L(L-2))}{8L(L-2)^3 \ell_s^4 k_2 \pi^2}\,, \tag{3.2.64}$$

where for ease of presentation we have written N in terms of k_α^S rather than n_α using (3.2.60) and set $\epsilon = 1$ as required by supersymmetry. We may finally compute the free energy, to save presenting the messy intermediate results we just give the result

$$F = \frac{4\pi}{3}\sqrt{\frac{2k_2^3 L^3 (L-2)^3}{\langle k^N, k^N\rangle L^2 + \langle k^S, k^S\rangle (L-2)^2 + \langle k^N, k^S\rangle L(L-2)}}\, N^{3/2}\,. \tag{3.2.65}$$

We now need to extremize over the remaining free parameter L. It is not hard to see that the condition to solve is the cubic[16]

$$\langle k^N, k^N\rangle L^2(2L-1) + \langle k^S, k^S\rangle (L-L)^2(2L-3) + 2\langle k^N, k^S\rangle L(2 - 3L + L^2) = 0\,. \tag{3.2.66}$$

One can solve this simply using Mathematica, obtaining a result in terms of cubic roots of the parameters k_2, k_α^N, and n_α. It is not very instructive to present the roots, though one

[16] Another extremal value is $L = 0$ or $L = 2$ however these will lead to a vanishing free energy and so we can discount them.

can show that a single root exists since the discriminant of (3.2.66) is negative.

Having computed the general on-shell free energy we can compare to the reduction of $Y^{p,k}(\mathrm{KE}_4)$ to massless type IIA. We will follow the conventions of [151] for $Y^{p,k}(\mathrm{KE}_4)$. The free energy is

$$F = \sqrt{\frac{2\pi^6}{27\mathrm{Vol}(Y^{p,k}(\mathrm{KE}_4))}}N^{3/2}\,. \tag{3.2.67}$$

We can extract out the volume from [151],

$$\mathrm{Vol}\big(Y^{p,k}(\mathrm{KE}_4)\big) = \mathrm{Vol}(\mathrm{KE}_4)\frac{2\pi^2}{3\cdot 4^4}\frac{x_2-x_1}{p(x_2-1)(1-x_1)}(x_2^3-x_1^3)\,, \tag{3.2.68}$$

where the x's are solutions of

$$\begin{aligned} 0 &= 3p^3x_1^3 + 2p^2(6b-5p)x_1^2 + p(18b^2-28bp+11p^2)x_1 + 4(3b^3+4bp^2-6b^2p-p^2)\,,\\ 0 &= 3p^3x_2^3 + 2p^2(p-6b)x_2^2 + p(18b^2-8bp+p^2)x_2 + 4b(3bp-3b^2-p^2)\,. \end{aligned} \tag{3.2.69}$$

The dictionary between our results and those in [151] are[17],

$$k_2 \equiv p, \quad k_\alpha^S = \frac{k}{h}n_\alpha \equiv bn_\alpha\,, \quad k_\alpha^N = (b-p)n_\alpha\,, \quad \mathrm{Vol}(\mathrm{KE}_4) = \frac{\pi^2}{2}\langle n,n\rangle\,, \tag{3.2.70}$$

where $h = \gcd(n_\alpha)$. We find a perfect match.

Observe that we have found an extremal problem for a class of Sasaki–Einstein manifolds using equivariant localization. This is of the same spirit as the use of equivariant localization for odd dimensional manifolds in [36]. One uses a second U(1) to reduce along, then localizes with the remaining U(1). It would be interesting to consider more general reductions, for example one could write a seven-dimensional Sasaki–Einstein manifold as a fibration of $L^{a,b,c}$ over a round two-sphere and then reduce along one of the non R-symmetry directions. One would then not obtain a Kähler–Einstein base as we have assumed above. One can also consider turning on a non-trivial Romans mass. Our localized integrals can be used to study this case however one finds the need to solve difficult algebraic equations which we were unable to solve without resorting to numerics. This is certainly an interesting class of solutions and

[17] Note that the north pole here is the south pole in [151] and vice versa.

it would be interesting to see if there is a change of parameters which makes the problem tractable.

3.2.6 Suspension of a Sasaki–Einstein manifold

An interesting class of 3d $\mathcal{N}=2$ SCFTs can be constructed by considering D2-branes probing the suspension of a Sasaki–Einstein manifold in the presence of D8-branes. The dual field theories are 3d Chern–Simons theories with the level specified by the Romans mass, see for example [133, 147]. We will first consider the case where the Sasaki–Einstein manifold is the round five-sphere S^5 before considering a more general Sasaki–Einstein manifold. Note that this class keeps both l and F_0 non-zero.

3.2.6.1 Round five-sphere

Consider first the suspension of the round five-sphere, $\mathcal{S}(S^5)$. There are two fixed points at the poles of the suspension, which we call the North and South pole respectively. We may write the R-symmetry vector as

$$\xi = \sum_{i=1}^{3} \epsilon_i \partial_{\varphi_i} \,, \tag{3.2.71}$$

where the ∂_{φ_i} generate the $\mathrm{U}(1)^3$ toric action of the S^5. Consider the three linearly embedded four-spheres $S_i^4 \subset \mathcal{S}$ which are invariant under the action of ξ. These are all trivial in homology such that we necessarily have

$$0 = \int_{S_i^4} \Phi^Z = \frac{(2\pi)^2 \epsilon_i}{\epsilon_1 \epsilon_2 \epsilon_3} \left[\frac{(F_0^2 y_N^4 - 3l^2)^2}{36 y_N^2} + \frac{(F_0^2 y_S^4 - 3l^2)^2}{36 y_S^2} \right] . \tag{3.2.72}$$

Since the two terms are positive definite each must vanish individually and therefore

$$|y_N| = |y_S| = \left(\frac{3l^2}{F_0^2} \right)^{1/4} . \tag{3.2.73}$$

Note that we can consider three linearly embedded two spheres also: $S_i^2 \subset \mathcal{S}(S^5)$ which are invariant under ξ and are again trivial in homology. Integrating the closed two-form Y over

these cycles gives

$$0 = \int_{S_i^2} \Phi^Y = -\frac{\pi}{3\epsilon_i}\left[\frac{1}{y_N}(F_0^2 y_N^4 - 3l^2) - \frac{1}{y_S}(F_0^2 y_S^4 - 3l^2)\right], \tag{3.2.74}$$

which is true after application of (3.2.73).

Next consider the quantization of the fluxes. We only have a six-cycle on which we may integrate our fluxes and therefore we introduce the quantum number N counting the number of D2-branes as

$$N = \frac{1}{(2\pi\ell_s)^5}\int_{X_6} F_6 = \frac{(2\pi)^3}{1296 l^3 (2\pi\ell_s)^5 \epsilon_1\epsilon_2\epsilon_3}\left[y_N(F_0^2 y_N^4 + 9l^2)^2 - y_S(F_0^2 y_S^4 + 9l^2)^2\right]. \tag{3.2.75}$$

Restricting to a positive N we fix $y_N = -y_S > 0$. The quantization of N and the Romans mass imposes

$$\ell_s^{-1} = \frac{2^{1/9}3^{7/18}n_0^{1/9}N^{2/9}\pi^{1/3}(\epsilon_1\epsilon_2\epsilon_3)^{2/9}}{l^{1/3}}, \qquad F_0 = \frac{n_0}{2\pi\ell_s}. \tag{3.2.76}$$

It is now trivial to use our results to compute the free energy, finding

$$\begin{aligned} F &= \frac{(2\pi)^3}{240\epsilon_1\epsilon_2\epsilon_3}\frac{16\pi^3}{(2\pi\ell_s)^8}\left[y_N(F_0^2 y_N^4 + 5l^2) - y_S(F_0^2 y_S + 5l^2)\right] \\ &= \frac{9}{5}2^{1/3}3^{1/6}n_0^{1/3}N^{5/3}\pi(\epsilon_1\epsilon_2\epsilon_3)^{2/3}. \end{aligned} \tag{3.2.77}$$

We can now extremize the above functional over the ϵ_i subject to the constraint $\sum_{i=1}^3 \epsilon_i = 1$, one finds the symmetric solution

$$\epsilon_i = \frac{1}{3}. \tag{3.2.78}$$

Inserting this into the off-shell free energy we obtain:

$$F = \frac{2^{1/3}3^{1/6}n_0^{1/3}\pi^3}{5\mathrm{Vol}(S^5)^{2/3}}N^{5/3}, \tag{3.2.79}$$

where $\mathrm{Vol}(S^5) = \pi^3$. This is indeed the correct result for the free energy of D2-branes probing the suspension of the round five-sphere.

Notice that the extremization that we are performing is nothing other than extremizing

the volume of the five-sphere. In order to rewrite this in a more amenable form we note that the normalization of the Reeb vector field of the S^5 in comparison to our localizing Killing vector is such that the spinor has charge 3 under the Reeb and 1 under ξ. Therefore in order to compare with the canonically normalized Reeb vector we should identify

$$b_i = 3\epsilon_i \,, \tag{3.2.80}$$

where b_i are the mixing parameters of the Reeb vector field. With these conventions the off-shell volume of the S^5 is

$$\mathrm{Vol}(S^5)[b_i] = \frac{\pi^3}{b_1 b_2 b_3} \,, \tag{3.2.81}$$

subject to the constraint $\sum_{i=1}^{3} b_i = 3$. It therefore follows, in the S^5 case, that we have recovered that a-maximization = F-maximization [133] since the off-shell free energy can be written as

$$F = \frac{2^{1/3} 3^{1/6} n_0^{1/3} \pi^3}{5(\mathrm{Vol}(S^5)[b_i])^{2/3}} N^{5/3} \,, \tag{3.2.82}$$

and extremizing over b is equivalent to maximizing the 'a'-central charge of the 4d parent theory.

3.2.6.2 Quasi-regular Sasaki–Einstein

Having recovered the $\mathcal{S}(S^5)$ case we turn our attention to studying the more general $\mathcal{S}(\mathrm{SE}_5)$ case. We restrict to the quasi-regular Sasaki–Einstein case in the following. The Reeb vector field is once again aligned with the $\mathrm{U}(1)_R$ vector field ξ which we use to localize, with the factor of proportionality once again 3. Recall a basic fact of Sasaki–Einstein geometry that the Reeb vector field has constant square norm. This implies that the action generated by the Reeb vector field is locally free on the suspension away from the poles. Hence, the fixed point contribution is entirely from the singular points at the two poles of the suspension. Since, for a generic base, the poles of the suspension are badly singular one cannot use the previous localization formulae with the line bundle decomposition and one needs to use the more general result (1.2.4).

Consider now two-cycles on the base, in $\mathcal{S}(\mathrm{SE}_5)$ these are trivial in homology and therefore

we find, analogously to the S^5 case that

$$|y_N| = |y_S| = \left(\frac{3l^2}{F_0^2}\right)^{1/4} . \tag{3.2.83}$$

As in the previous case there are no two-cycles nor four-cycles over which we need to integrate the fluxes and we are left with a single flux integral to perform. We have

$$\begin{aligned} N &= \frac{1}{(2\pi\ell_s)^2}\int_{\mathcal{S}(\mathrm{SE}_5)} F_6 \\ &= \frac{1}{1296(2\pi\ell_s)^5}\left[\frac{y_N(F_0^2 y_N^4 + 9l^2)^2}{e(\mathcal{N}_N)} + \frac{y_S(F_0^2 y_S^4 + 9l^2)^2}{e(\mathcal{N}_S)}\right] \\ &= \frac{1}{9(2\pi\ell_s)^5}\left[\frac{y_N}{e(\mathcal{N}_N)} + \frac{y_S}{e(\mathcal{N}_S)}\right] . \end{aligned} \tag{3.2.84}$$

We have kept the normal bundles at the two poles arbitrary for the moment. The quantization conditions become

$$\ell_s^{-1} = \frac{2\cdot 3^{7/18}\pi n_0^{1/9} N^{2/9}}{\mathcal{E}} , \quad F_0 = \frac{n_0}{2\pi\ell_s} , \tag{3.2.85}$$

where

$$\mathcal{E} = \frac{1}{e(\mathcal{N}_N)} - \frac{1}{e(\mathcal{N}_S)} . \tag{3.2.86}$$

We may similarly compute the free energy which gives

$$\begin{aligned} F &= \frac{\pi^3}{15(2\pi\ell_s)^8}\left[\frac{y_N(F_0^2 y_N^4 + 5l^2)}{e(\mathcal{N}_N)} + \frac{y_S(F_0^2 y_S^4 + 5l^2)}{e(\mathcal{N}_S)}\right] \\ &= \frac{8\pi^3 y_N}{15(2\pi\ell_s)^8}\left[\frac{1}{e(\mathcal{N}_N)} - \frac{1}{e(\mathcal{N}_S)}\right] , \end{aligned} \tag{3.2.87}$$

and inputting the quantization conditions we have

$$F = \frac{72\times 3^{1/6} n_0^{1/3}\pi^3}{5\mathcal{E}^{2/3}} N^{5/3} . \tag{3.2.88}$$

We now want to understand what $\mathcal{E}$ is computing. Let us take $\mathcal{N}_N = -\mathcal{N}_S$ for simplicity.

Then for a Calabi–Yau cone we have that[18]

$$\mathcal{E} = 2^4 3^3 \mathrm{Vol}(\mathrm{SE}_5)\,, \tag{3.2.89}$$

and we obtain the final result

$$F = \frac{2^{1/3} 3^{1/6} \pi^3}{5 \mathrm{Vol}(\mathrm{SE}_5)^{2/3}} n_0^{1/3} N^{5/3}\,. \tag{3.2.90}$$

Strictly this is an off-shell result and one needs to vary the volume $\mathrm{Vol}(\mathrm{SE}_5)$ with respect to the Reeb vector field. We have therefore recovered in the general quasi–Regular case that a-maximization = F-maximization [133].

We note that there seems to be an interesting option to have a suspension where the poles of the cone have different singularities. One can view this as having a flop between the two poles. It is not clear that such a metric exists but it would be interesting to study whether this is obstructed.

3.3 Discussion

Using equivariant localization we have constructed the geometric dual of F-maximization in massive type IIA. There is a rich structure of topologies for the internal manifolds X_6, beyond those for the closely related AdS_5 solutions in M-theory [131] recently studied using localization in [29, 32], that we have discussed. One of the key differences of our work with theirs is the presence of boundary terms in the internal space. For a subset of examples we have introduced boundary conditions associated to the presence of an O8-plane. Rather than the localizing circle shrinking at this locus it remains of finite size (in the internal manifold) and therefore leads to a boundary contribution. For the O8-plane we saw that there is no contribution to the free energy, flux quantization and conformal dimensions of a class of BPS operators. However this is not generic and other choices of defect branes, which are used to cap off the space, would give contributions. On the other hand, we did also consider an

[18] One can make this more precise by using the results of [36, 121, 152]. One can understand the factors of 3 due to the same normalization of the Reeb vector field in the previous S^5 example, 3 of the 2's due to the formula for the volume and one factor of the 2's since $\mathcal{E}$ double counts the volume when $\mathcal{N}_N = \mathcal{N}_S$.

example including O4-planes and leading to non-vanishing boundary contributions.

It would be interesting to extend our analysis to further examples too. Moreover, this could open up avenues for studying more general irregular punctures, those of type 3, in the holographic duals of Argyres–Dougles theories recently studied in [153–157]. Finding explicit supergravity solutions seems exceedingly difficult, yet if one can understand concretely the degeneration structure at such a singularity one can utilize a similar logic to this paper.

As we have emphasized repeatedly throughout, though the equivariant localization may allow for a particular choice of topology it is not immediate that explicit metrics can be found realizing these topologies. It is therefore interesting to consider obstructions to such metrics in an analogous way to how the Futaki invariant gives an obstruction to the existence of Sasaki–Einstein metrics, [158]. It is natural to conjecture that requiring that the conformal dimensions of wrapped D2-branes are all positive gives rise to such an obstruction, however it is unclear whether this is sufficient or necessary. One could therefore consider studying more refined observables for example the eigenvalue density of the matrix model associated to the dual field theory, see for example [5, 159], and flavour central charges associated to the defect branes [139, 160, 161].

Observe that as a by-product of our construction we have constructed the consistent truncation of the general AdS_4 solutions to four-dimensional Einstein–Maxwell supergravity.[19] To see this clearly one should gauge the R-symmetry vector $D\psi$ with the 4d graviphoton, and in the polyforms for the fluxes wedge with the graviphoton field strength so that the form degree of the total polyform is simply the form degree of the top component. One could then imagine using the polyforms constructed here and the polyforms for 4d Einstein–Maxwell [29] to construct polyforms for the full 10d solution. In this way one could study black holes in these massive type IIA backgrounds directly and make contact with the field theory results in [162].

Note that though our supergravity solutions preserve supersymmetry there is no requirement to actually have any supersymmetry to use equivariant localization, rather one just needs some symmetry. There are often non-supersymmetric sister AdS solutions of certain classes of supersymmetric AdS solutions. For AdS_7 in massive type IIA there are the

[19] One should check that this preserves supersymmetry but this should be automatic by construction.

non-supersymmetric sister solutions of [163] for example, and for the AdS_4 Sasaki–Einstein solutions in M-theory there are various different non-susy families, for example the Englert solutions [164] and the Pope–Warner solutions [165,166]. There are in fact non-supersymmetric solutions which are sister solutions of the supersymmetric solutions studied here [167]. Given that the solutions have a very similar structure, in particular a U(1) symmetry, one could apply equivariant localization techniques to these non-supersymmetric cases too.

Chapter 4

Ten-dimensional localization

As explained in the introduction, there are two main classes of solutions of interest when considering supersymmetric backgrounds relevant for holography and AdS/CFT. The first class studies supersymmetric AdS solutions embedded in ten- and eleven-dimensional supergravity, i.e. backgrounds of the form $\mathrm{AdS}_D \times X_{10-D}$. Via the AdS/CFT correspondence these are dual to $(D-1)$-dimensional CFTs on a Minkowski background. The details of the CFT, for example the symmetries and field content, are encoded in the choice of compact internal space X_{10-D} and one can learn about the supersymmetric protected observables such as anomalies, free energy and conformal dimensions of BPS operators by studying this geometry. There is a rich history of classifying the most general supersymmetry-preserving AdS solutions using G-structure techniques [168–170], and more recently generalized complex geometry [62, 171].

The second class of solutions involves replacing AdS with a space M_D which is asymptotically locally AdS. One now fibres X_{10-D} over M_D so that the spacetime takes the form $M_D \ltimes X_{10-D}$, where X_{10-D} may also be warped. This is then interpreted as placing the CFT on the non-trivial background ∂M_D and tells us about deformations of the CFT. This allows us, for example, to probe how the observables of the CFT depend on the background, and we can compute protected observables such as indices and twisted partition functions. Typically one works in a D-dimensional gauged supergravity theory which is a consistent truncation of the ten-dimensional theory.

While consistent truncations provide an invaluable bridge between higher-dimensional string theory and lower-dimensional gauged supergravity, they are only available for special internal manifolds and restricted field content. Indeed, for some of the examples we will present, such consistent truncations are not currently known. Furthermore, many questions can only be answered when studying the higher-dimensional parent theories. For example, studying probe branes in a black hole background requires analysing the embedding of the branes in the full parent theory, and not merely within the consistent truncation of which

the black hole is a solution [11, 172–175]. Moreover, questions such as studying resolutions of punctured Riemann surfaces require a higher-dimensional embedding (or coupling the gauged supergravity to non-Abelian vector multiplets with no known uplifts [176, 177]). For these, and many other reasons, a higher-dimensional description, involving both the external and internal spacetime, is thus desirable. This, however, is no easy task, as one needs to solve Einstein's equations in ten and eleven dimensions. Even with supersymmetry one still typically has non-linear partial differential conditions to solve.

In this chapter, based on [4], we apply equivariant localization directly to (massive) type IIA and type IIB supergravity. This allows us to simultaneously address both cases outlined above. For supersymmetric solutions preserving a minimal amount of supersymmetry, the ten-dimensional manifold generically has a Killing vector [62]. We use a subset of the necessary and sufficient conditions for such a supersymmetric solution to exist [62,178] to compute equivariantly closed polyforms for the Page fluxes and the on-shell action. This allows us to appropriately compute the quantization of fluxes for any supersymmetric supergravity background and to then compute the on-shell action. We find that there is a very universal structure for these polyforms. Our results are applicable to whatever topologies and fluxes one wishes to consider. We will apply this formalism to geometries with asymptotically AdS spaces; however, it is applicable more broadly, including to asymptotically flat black holes and even non-conformal branes.

4.1 Ten-dimensional polyforms

We are interested in computing the (partially) on-shell action of supersymmetric solutions of type IIA and type IIB supergravity. In order for the solutions to be well-defined string backgrounds we must also quantize the fluxes. Our goal in this section is to construct equivariantly closed polyforms for the Page fluxes before using these to construct a set of equivariantly closed polyforms for the partially on-shell action. A key tool in constructing these polyforms is the classification of supersymmetric solutions using generalized geometry [62, 178]. To keep the section self-contained we will first review the necessary conditions for a solution to preserve supersymmetry.

4.1.1 Preserving supersymmetry in ten dimensions

In the following we will consider the two type II theories concurrently, and in order to do so we must introduce a little notation. Let g be the metric, ϕ the dilaton, and $H = \mathrm{d}B$ the NS-NS three-form. In addition, using the democratic approach, we consider the polyform of all RR fluxes given by

$$F^{\mathrm{RR}} = \sum_k F_k^{\mathrm{RR}}\,, \tag{4.1.1}$$

where k runs over all even numbers between 0 to 10 for type IIA and over all odd numbers in this range for type IIB. This democratic formulation doubles up the fluxes, where one must then impose the self-duality constraint

$$F^{\mathrm{RR}} = \star\lambda(F^{\mathrm{RR}})\,, \tag{4.1.2}$$

where λ acts on a k-form as $\lambda(\omega_k) \equiv (-1)^{\left\lfloor \frac{k}{2} \right\rfloor}\omega_k$. Furthermore, we denote by d_H the twisted differential operator defined by

$$\mathrm{d}_H = \mathrm{d} - H\wedge\,. \tag{4.1.3}$$

In the democratic formulation the Maxwell equations for the RR fluxes simply become

$$\mathrm{d}_H F^{\mathrm{RR}} = 0\,. \tag{4.1.4}$$

Supersymmetric solutions of type IIA and type IIB supergravity admit two Majorana–Weyl Killing spinors ϵ_1 and ϵ_2, which in our conventions satisfy:

$$\begin{cases} \Gamma_{11}\epsilon_1 = \epsilon_1\,, \quad \Gamma_{11}\epsilon_2 = -\epsilon_2 & \text{type IIA}\,, \\ \Gamma_{11}\epsilon_1 = \epsilon_1\,, \quad \Gamma_{11}\epsilon_2 = \epsilon_2 & \text{type IIB}\,. \end{cases} \tag{4.1.5}$$

Following [62, 178, 179] one constructs the vector bilinears[1]

$$K_\alpha = \overline{\epsilon_\alpha}\Gamma^M\epsilon_\alpha\partial_M\,, \tag{4.1.6}$$

[1] In comparison with [62, 178] we have rescaled the bilinear forms by a factor of 32, more in line with the conventions in [179].

which are null as follows by the ϵ_α being Majorana–Weyl. It follows that the Killing spinors satisfy the projection conditions

$$K_\alpha \cdot \epsilon_\alpha = 0\,, \tag{4.1.7}$$

with the dot denoting Clifford multiplication. One then constructs the vector field K and one-form $\widetilde{K}$ defined as

$$K = \frac{1}{2}(K_1 + K_2)\,, \qquad \widetilde{K} = \frac{1}{2}(K_1 - K_2)\,. \tag{4.1.8}$$

Since the K_i are null, it follows that K is either null (if the two K_i are parallel) or otherwise time-like: $K^2 < 0$. One can show that supersymmetry implies

$$\mathrm{d}\widetilde{K} = K \lrcorner H\,. \tag{4.1.9}$$

Consider now the mixed bispinor

$$\widehat{\Phi} = 32\,\epsilon_1 \otimes \overline{\epsilon_2}\,, \tag{4.1.10}$$

which satisfies $\widehat{\Phi} = \star\lambda(\widehat{\Phi})$. One should understand this as a polyform upon application of the Clifford map. If the two Killing spinors have the same chirality then $\widehat{\Phi}$ is a sum of odd forms only, whilst if the spinors have opposite chirality then it is a sum of even forms only; therefore

$$\widehat{\Phi} = \begin{cases} \widehat{\Phi}_0 + \widehat{\Phi}_2 + \widehat{\Phi}_4 + \widehat{\Phi}_6 + \widehat{\Phi}_8 + \widehat{\Phi}_{10} & \text{type IIA}\,, \\ \widehat{\Phi}_1 + \widehat{\Phi}_3 + \widehat{\Phi}_5 + \widehat{\Phi}_7 + \widehat{\Phi}_9 & \text{type IIB}\,. \end{cases} \tag{4.1.11}$$

It follows that $\widehat{\Phi}$ satisfies

$$(\widetilde{K} \wedge + K \lrcorner)\,\widehat{\Phi} = 0\,. \tag{4.1.12}$$

One can derive a set of necessary and sufficient conditions for the preservation of supersymmetry. In particular, one finds that K is a Killing vector for the full solution, the Killing

spinors are uncharged under it, $\mathcal{L}_K \epsilon_i = 0$, and that one necessarily has

$$\mathrm{d}_H\big(\mathrm{e}^{-\phi}\,\widehat{\Phi}\big) = -(\widetilde{K} + K\lrcorner)\,F^{\mathrm{RR}}\,. \tag{4.1.13}$$

There are further conditions that need to be satisfied, known as pairing constraints, and we refer the reader to [62, 178] for the full set of necessary and sufficient conditions; however, it turns out that equations (4.1.9), (4.1.12) and (4.1.13) are sufficient for us to construct the equivariantly closed polyforms that we will need.

4.1.2 Constructing the polyforms

Having reviewed the necessary background material we will turn to constructing equivariantly closed polyforms, with respect to the Killing vector K, for both type II theories. The key observation is that the supersymmetry conditions outlined above are precisely sufficient to find a completion of all the Page fluxes and the on-shell action into equivariantly closed polyforms. This then allows us to appropriately quantize the fluxes and then compute the on-shell action. The rest of this section outlines the computation of these polyforms, with a more detailed exposition presented in appendix D.1.

First recall that in the presence of non-trivial NS-NS flux the Maxwell charges are not conserved, as is evident from (4.1.4). Instead, the conserved charge is the Page charge. As such we are interested in constructing equivariantly closed polyforms for the Page fluxes. Using the democratic formulation, the Page flux polyform F takes the simple form

$$F = \mathrm{e}^{-B} \wedge F^{\mathrm{RR}}\,, \tag{4.1.14}$$

where B is the potential for the NS-NS three-form. From (4.1.4) it follows that F is closed and therefore each of the terms of the same form degree may serve as the top component of an equivariantly completed polyform.

Since K preserves all of the bosonic fields we have that $\mathcal{L}_K H = 0$. Furthermore, by a suitable choice of gauge for B we can impose that in each patch the Lie derivative of B along

the flow generated by K also vanishes $\mathcal{L}_K B = 0$. In this gauge, and using (4.1.9), one finds

$$0 = \mathcal{L}_K B = \mathrm{d}(\widetilde{K} + K \lrcorner B)\,, \tag{4.1.15}$$

and we may therefore locally introduce a function Υ such that

$$\widetilde{K} + K \lrcorner B = \mathrm{d}\Upsilon\,. \tag{4.1.16}$$

Using (4.1.13) and (4.1.16) we find that K contracted into the Page fluxes is

$$K \lrcorner F = -\mathrm{d}\Big[\Upsilon F + \Phi\Big]\,, \tag{4.1.17}$$

where we defined the polyform

$$\Phi \equiv \mathrm{e}^{-\phi}\mathrm{e}^{-B} \wedge \widehat{\Phi}\,. \tag{4.1.18}$$

The term in the brackets is the first step in constructing the equivariantly closed polyforms for the Page fluxes. We can then compute K contracted into this polyform and proceed further along the descent – the reader interested in these details should consult appendix D.1. The final result is a polyform of polyforms for the Page fluxes:

$$\Psi^F = F + \sum_m \Psi_m\,, \tag{4.1.19}$$

where m runs from 1 to 5 in IIA and 1 to 4 in IIB, while

$$\Psi_m = \frac{(-1)^m}{m!}\Big(\Upsilon^m F + m\Upsilon^{m-1}\Phi\Big)\,. \tag{4.1.20}$$

This is the central result of this chapter and applies to the Page fluxes of both (massive) type IIA and type IIB supergravity under the single assumption of minimal supersymmetry.

The equivariantly closed polyform for the Page flux F_k is extracted by taking the sum of the k-form of F, i.e. F_k, the $(k-2)$-form of Ψ_1, the $(k-4)$-form of Ψ_2 and so forth until one ends up with either a zero-form for type IIA or a one-form for type IIB. Explicitly, the

polyforms for the first three Page fluxes in massive type IIA are

$$\begin{aligned}\Psi^{F_0} &= F_0\,,\\ \Psi^{F_2} &= F_2 - (\Upsilon F_0 + \Phi_0)\,,\\ \Psi^{F_4} &= F_4 - (\Upsilon F_2 + \Phi_2) + \frac{1}{2}(\Upsilon^2 F_0 + 2\Upsilon\Phi_0)\,,\end{aligned} \tag{4.1.21}$$

where we have used the short-hand $\Phi_k = \Phi\,|_{k\text{-form}}$. The polyforms for the higher-form fluxes, which we also use later, and the ones for type IIB have the same structure and are spelled out explicitly in appendix D.1. One should then quantize the k-form Page flux as

$$\frac{1}{(2\pi\ell_s)^{k-1}}\int_{\Sigma_k} F_k \in \mathbb{Z}\,. \tag{4.1.22}$$

In particular, one can quantize the Romans mass and define

$$n_0 = 2\pi\ell_s F_0\,. \tag{4.1.23}$$

We emphasize that these polyforms are valid for any minimally supersymmetric massive type IIA background, and those in appendix D.1 hold for any minimally supersymmetric type IIB background. Not only will these polyforms be instrumental in quantizing the Page fluxes, they are also sufficient for computing the on-shell action to which we now turn.

4.1.3 Partially on-shell action

The key observable that we wish to compute in this chapter is the (partially) on-shell string frame action of type II supergravity. For both type II theories, using the democratic formulation one can write a pseudo action which takes the form

$$S = \frac{1}{(2\pi)^7\ell_s^8}\int \mathrm{d}^{10}x\sqrt{-g}\left[\mathrm{e}^{-2\phi}\Big(R + 4|\mathrm{d}\phi|^2 - \frac{1}{2}|H|^2\Big) - \frac{1}{4}\sum_k |F_k^{\mathrm{RR}}|^2\right], \tag{4.1.24}$$

where for a (real) p-form α we define $|\alpha|^2 \equiv \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\alpha^{\mu_1\cdots\mu_p}$. By doubling the number of fields we remove the Chern–Simons terms from the action, which are effectively hidden in the last term [180]. Notice that it is the Maxwell fluxes that appear in (4.1.24), rather than

the Page fluxes we have just computed equivariantly closed polyforms for.

The action (4.1.24) is a pseudo action in the sense that we are summing over all the RR fluxes but have not imposed the self-duality constraint in (4.1.2). Imposing this at the level of the action implies that the RR term *vanishes*, much like the usual story in type IIB supergravity for the self-dual five-form F_5. On the other hand, the latter has been addressed recently in the holographic context in [181, 182], where it is shown that the correct on-shell action for $\mathrm{AdS}_5 \times M_5$ solutions in type IIB supergravity, supported by self-dual five-form flux, is given by splitting the flux into electric and magnetic components. Their result for the on-shell action is derived from the PST formalism [183], which requires an additional boundary term. Rather than attempting to carry out a similar analysis for type II (using for example[2] [184]), we instead propose that the analogous result holds generally: namely one uses the result (4.1.24), but separates the RR contributions into electric and magnetic, only including half the terms in the sum. Effectively, the RR terms in (4.1.24) cancel pairwise on-shell, and this prescription means one is choosing to include only half the terms. The inherent ambiguity in the signs in doing this has a physical interpretation: the different choices are related by a Legendre transform that interchanges between electric and magnetic ensembles [185]. For the purposes of this work we take this as a conjectured prescription for the on-shell action, and show this agrees both with lower-dimensional supergravity results obtained with holographic renormalization (when there is a consistent truncation), and also with field theory results that otherwise have no known gravity dual. We find perfect agreement with all classes of known supergravity solutions and large N field theory results.

Inserting the dilaton equation of motion and dropping a boundary term (see appendix D.2), the above prescription for the action for type IIA takes the form

$$S^{\mathrm{IIA}} = \frac{1}{2(2\pi)^7 \ell_s^8} \int_{M_{10}} \Big[s_1 F_0 F_{10} + s_2 F_2 \wedge F_8 + s_3 F_4 \wedge F_6 \Big] \,, \tag{4.1.25}$$

and for IIB

$$S^{\mathrm{IIB}} = \frac{1}{2(2\pi)^7 \ell_s^8} \int_{M_{10}} \Big[s_1 F_1 \wedge F_9 + s_2 F_3 \wedge F_7 + s_3 F_5 \wedge F_5 \Big] \,, \tag{4.1.26}$$

[2]This can also be made supersymmetric, and we expect the polyforms we have constructed to generalize appropriately. We thank Dmitri Sorokin for discussions on this.

where the s_i's are the signs depending on the ensemble that we are working in, i.e. which fluxes we held fixed. For all s_i being $+1$ we are in the ensemble where F_0, F_2 and F_4 define conserved magnetic charges which we hold fixed in IIA (or F_1, F_3, F_5 in IIB).[3] Other ensembles are then related by the change of sign of the relevant term.

Since the partially on-shell action is given by a sum over the wedge products of equivariantly closed forms, we can simply compute the equivariantly closed polyform for the action. It is given by taking the wedge product of the equivariantly closed Page flux polyforms. For massive type IIA we find

$$\Psi = \left[s_1 F_0 \Psi^{F_{10}} + s_2 \Psi^{F_2} \wedge \Psi^{F_8} + s_3 \Psi^{F_4} \wedge \Psi^{F_6} \right] . \tag{4.1.27}$$

We thus obtain a method for both quantizing the fluxes and computing the partially on-shell action. In the remainder of this chapter we will exhibit the power of this formalism by computing the on-shell action of a variety of different classes of solution in type IIA. The IIB action can in principle be evaluated in a similar fashion. However, we postpone this to future work and just discuss the (massive) type IIA theory here.

In particular, we will focus on two classes of solutions: hemisphere HS^4 fibrations over M_6 and S^6 fibrations over M_4. For these two classes the signs in the action are respectively $\{s_1 = 1, s_2 = 1, s_3 = +1\}$ and $\{s_1 = 1, s_2 = 1, s_3 = -1\}$. The physical difference between the two classes is clear: in the first class F_4 is a magnetic flux which we hold fixed, whilst in the second F_6 defines a magnetic charge that we hold fixed. The change in ensemble precisely flips the sign s_3. For concreteness let us write the zero-form part of the action polyform for each of these

$$\Psi_0^{\text{elec}} = -\frac{2}{15}\Upsilon^3(F_0^2\Upsilon^2 + 5F_0\Upsilon\Phi_0 + 5\Phi_0^2) , \quad \Psi_0^{\text{mag}} = \frac{1}{30}\Upsilon^3(F_0^2\Upsilon^2 + 5F_0\Upsilon\Phi_0 + 10\Phi_0^2) . \tag{4.1.28}$$

Similar expressions hold for the higher-form parts. Moreover, turning off the Romans mass these expressions simply reduce to

$$\Psi_0^{\text{elec}} = -\frac{2}{3}\Upsilon^3\Phi_0^2 , \quad \Psi_0^{\text{mag}} = \frac{1}{3}\Upsilon^3\Phi_0^2 , \tag{4.1.29}$$

[3] In particular we always have $s_1 = +1$ for IIA and $s_3 = +1$ for IIB.

which are then "obviously" related by electro-magnetic duality. Using the embedding of the massless theory into 11d supergravity as spelt out in [178] these results should allow for the computation of both the on-shell action and flux quantization in M-theory.

As a final comment, in order to compare with field theory results we are really interested in computing the Euclidean on-shell action rather than the Lorentzian one. As such we need to perform a Wick rotation of the theory. This carries some subtleties in the properties of the spinors used to define the bilinears, where in particular there is no Majorana condition in ten-dimensional Euclidean space. One can follow the logic in [186,187] and complexify the theory working on a "holomorphic" slice and then restricting the theory to Euclidean signature. The Killing vector K is then taken to be complex in the Euclidean theory. Furthermore, we will use that this prescription multiplies the Lorentzian on-shell action by an overall $-\mathrm{i}$ and to distinguish the two we will denote the Euclidean (partially) on-shell action as $I = -\mathrm{i}S$. In summary, we will use localization to evaluate

$$I = \frac{-\mathrm{i}}{2(2\pi)^7 \ell_s^8} \int_{M_{10}} \Psi \,. \tag{4.1.30}$$

Before we move on to using our results, let us quickly summarize this section. Given any minimally supersymmetric type II background, the Page fluxes admit the universal equivariantly closed completion given in (4.1.19) and (4.1.20). Using these one should quantize the fluxes over all compact cycles and use them to impose the homology relations of the space-time under consideration.[4] One can then use this to compute the partially on-shell action (4.1.30) with the equivariantly closed polyform given in (4.1.27). The action depends on a set of signs which are related to the ensemble in which one is computing.

4.2 10 = 6+4: Romans supergravity coupled to a vector

As a first application of our polyforms we will consider solutions of Euclidean massive type IIA supergravity which are a four-dimensional hemisphere bundle[5] over a six-dimensional base

[4] This step of imposing all of the homology relations is not strictly necessary and typically follows from extremization of the action subject to the flux quantization conditions (this was first pointed out in [9]). The benefit of imposing all homology relations is that it typically simplifies the final extremization problem.

[5] This is the only solution which admits an AdS_6 vacuum solution in (massive) type IIA [188].

M_6. When the bundle is taken to be trivial and M_6 is Euclidean AdS_6 this is the topology of the famous Brandhüber–Oz solution [137]. In this section we will consider M_6 to have a more general topology than just AdS_6 and allow for a general HS^4 fibration over M_6. We derive a master formula (4.2.4), depending on the topology and global data of a putative solution, which can be written in terms of a function which we interpret as the prepotential of 6d $\mathrm{U}(1)^2$ gauged supergravity. This generalizes the results of [40], which considered the Abelian Romans theory. We also expect to be able to derive our results directly from a six-dimensional $U(1)^2$ gauged supergravity theory, which couples the minimal Romans theory studied in [40] to an additional vector multiplet. This theory should be a consistent truncation of massive type IIA. However, note that we have no need for such a consistent truncation here – we derive formulas by directly localizing in ten dimensions.

Consider the following embedding of the hemisphere in five dimensions

$$HS^4 \subset \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{R}_{\geq 0}\,, \tag{4.2.1}$$

with $|z_1|^2 + |z_2|^2 + x^2 = 1$, where $z_i \in \mathbb{C}_i$ are complex coordinates and $x \in \mathbb{R}_{\geq 0}$ is a polar coordinate on the hemisphere HS^4. The latter has boundary $\partial HS^4 = \{x = 0\} \cong S^3$. One can define two two-dimensional hemispheres $HS^2_i = \{z_i = 0\}$ within HS^4 which are trivial in relative homology $H_2(HS^4, S^3)$. We fibre HS^4 over M_6 using the $U(1)^2$ action on $\mathbb{C}_1 \oplus \mathbb{C}_2$ via two complex line bundles $\mathcal{L}_i$ over M_6 with Chern classes $c_1(\mathcal{L}_i) \in H^2(M_6, \mathbb{Z})$ for $i = 1, 2$.

We define the projection $\pi : M_{10} \to M_6$ from the total space of the bundle to the base. One can then project the Killing vector K down to the base M_6 so that $\xi \equiv \pi_* K$ is a vector field on M_6. Over a fixed point of ξ on M_6, we decompose the action of K on the normal space in M_{10} as

$$K = \xi + \sum_{i=1}^{2} b_i \partial_{\varphi_i}\,, \quad \xi = \sum_a \varepsilon_a \partial_{\psi_a}\,, \tag{4.2.2}$$

where ∂_{φ_i} rotates $\mathcal{L}_i$ and we assume that $b_i \neq 0$ so that the fixed points of K are then the fixed points of ξ on M_6 at the pole of HS^4. The fixed point in M_6 can be part of a fixed submanifold of dimension 0, 2 or 4, and the index a correspondingly takes values in $\{1, 2, 3\}$, $\{1, 2\}$ or $\{1\}$ respectively, such that ∂_{ψ_a} rotates the line bundle L_a in the normal space in

M_6 over the fixed point set. The weights ε_a, b_i at different fixed points are determined from the equivariant action of K on M_{10}. In particular, they depend on the fixed point p and to be precise we should write b_i^p but will often suppress the index so as to not clutter formulae. Since the Killing spinors are not charged under K it follows that the weights at a fixed point must satisfy the constraint

$$\sum_a \sigma_a \varepsilon_a + \sum_{i=1}^{2} b_i = 0\,, \tag{4.2.3}$$

where σ_a are signs depending on the properties of the spinor at the fixed point.

One can now analyse the contribution of the different fixed point sets to the on-shell action. The result of this analysis is the general master formula

$$\begin{aligned} I = \frac{8}{27} F_{S^5} \Bigg\{ & \sum_{\mathrm{p}} \mp \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \mathcal{F} + \sum_{\Sigma} \frac{1}{\varepsilon_1 \varepsilon_2} \left[\sum_{i=1}^{2} p_i \partial_{b_i} \mathcal{F} - \sum_{a=1}^{2} \left(\frac{m_a}{\varepsilon_a} \right) \mathcal{F} \right] \\ & + \sum_{B_4} \mp \frac{1}{\varepsilon} \left[\frac{1}{2} \sum_{i,j=1}^{2} \langle p_i, p_j \rangle \partial^2_{b_i b_j} \mathcal{F} - \frac{1}{\varepsilon} \sum_{i=1}^{2} \langle p_i, m \rangle \partial_{b_i} \mathcal{F} + \frac{1}{\varepsilon^2} \langle m, m \rangle \mathcal{F} \right] \Bigg\}\,, \end{aligned} \tag{4.2.4}$$

where

$$\mathcal{F} \equiv \mathcal{F}(b_1, b_2) = (b_1 b_2)^{3/2} \tag{4.2.5}$$

is the prepotential on HS^4 and F_{S^5} is the S^5 free energy

$$F_{S^5} = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{n_0}}\,. \tag{4.2.6}$$

The parameters appearing are just the weights, ε_a, b_i (which we emphasize depend on the fixed point and have a suppressed superscript) for the action of K and the (associated) Chern classes m_a, p_i for normal bundles inside M_6, or the HS^4 fibres, respectively. The $\langle\ ,\ \rangle$ in the final line denotes the intersection pairing $H^2(B_4, \mathbb{Z}) \times H^2(B_4, \mathbb{Z}) \to \mathbb{Z}$. In the following, we will derive this result by looking in turn at isolated fixed points p of ξ on M_6, fixed surfaces $\Sigma = \Sigma_g$, and fixed four-manifolds B_4 giving rise to the various contributions in (4.2.4). By way of exposition we present explicit applications of this formula for different topologies realizing each of the terms above. We take the Page fluxes (and their equivariant extensions) to represent relative classes in $H^*(M_{10}, \partial)$ in what follows, where their restriction being zero

on the D8/O8 boundary ∂ (an $S^3 \subset HS^4$ bundle over M_6) is a boundary condition. We note this is satisfied for uplifted solutions to the Romans theory.

4.2.1 Isolated fixed points

Consider a fixed point $\mathrm{p} \in M_6$ of ξ and consider the copy of HS^2_i lying above it in the fibre. On one hand the flux of F_2 through such a relative cycle is trivial, since $[F_2] \in H^2(M_{10}, \partial)$, and on the other hand, it can be evaluated using equivariant localization. Combining these observations, and taking Φ_0 to vanish on the boundary, immediately gives

$$0 = \int_{HS^2_i} F_2 = -\frac{2\pi}{b_i} \Psi_0^{F_2}|_{\mathrm{p}} = -\frac{2\pi}{b_i}(F_0 \Upsilon + \Phi_0)|_{\mathrm{p}} \quad \Rightarrow \quad F_0 \Upsilon|_{\mathrm{p}} = -\Phi_0|_{\mathrm{p}} \,. \tag{4.2.7}$$

Next we consider the flux of F_4 through the hemisphere, again at the fixed point[6] p

$$N \equiv \frac{1}{(2\pi\ell_s)^3} \int_{HS^4} F_4 = \frac{1}{(2\pi\ell_s)^3} \frac{(2\pi)^2}{b_1 b_2} (F_0 \tfrac{1}{2} \Upsilon^2 + \Phi_0 \Upsilon) \,|_{\mathrm{p}} \,, \tag{4.2.8}$$

and solving using (4.2.7) gives

$$\Upsilon|_{\mathrm{p}} = \pm \frac{2\sqrt{2}\pi \mathrm{i} \ell_s^2}{\sqrt{n_0}} \sqrt{b_1 b_2}\, N^{1/2} \,. \tag{4.2.9}$$

This is enough to turn to the evaluation of the action. If $\mathrm{p} \in M_6$ is an isolated fixed point, with weights ε_a, $a = 1, 2, 3$, for K on the normal space, then we can immediately localize

$$I_{\mathrm{p}} = \frac{\mathrm{i}}{2(2\pi)^7 \ell_s^8} \frac{(2\pi)^5}{\varepsilon_1 \varepsilon_2 \varepsilon_3 b_1 b_2} \Psi_0^{\mathrm{elec}}|_{\mathrm{p}} = \mp \frac{8\sqrt{2}\pi}{15\sqrt{n_0}} \frac{(b_1 b_2)^{3/2}}{\varepsilon_1 \varepsilon_2 \varepsilon_3} N^{5/2} \,. \tag{4.2.10}$$

Here the sign is that in (4.2.9). Note that scaling all five weights by the same constant leaves both (4.2.3) and the action contribution (4.2.10) invariant, as it should, so this choice of normalization is irrelevant.

To obtain the on-shell free energy for EAdS_6, with boundary five-sphere, one should extremize the above subject to the sum of the weights vanishing. Doing this one obtains (4.2.6). This is the free energy of the dual five-dimensional SCFT on S^5. We may finally

[6] Note that these hemispheres are homologous as classes in $H_4(M_{10}, \partial)$ (taking M_6 to be connected) so that N is independent of p.

rewrite the off-shell action result for an arbitrary isolated fixed point in the form

$$I_{\rm p} = \mp\frac{8}{27}F_{S^5}\frac{\mathcal{F}(b_1^{\rm p},b_2^{\rm p})}{\varepsilon_1^{\rm p}\varepsilon_2^{\rm p}\varepsilon_3^{\rm p}}\,, \tag{4.2.11}$$

where we re-establish the superscript p as a reminder that the weights in general depend on the fixed point.

Spindle As an example, we can consider the case $M_6 = \mathbb{R}_4 \times \Sigma$, where HS^4 is fibred over the spindle $\Sigma \equiv \mathbb{WCP}^1_{[n_+,n_-]}$. The hemisphere directions $\mathbb{R}^2_1 \oplus \mathbb{R}^2_2 = \mathbb{C}_1 \oplus \mathbb{C}_2$ are fibred as $\mathcal{O}(-p_1)\oplus\mathcal{O}(-p_2) \to \Sigma$, where the twist and anti-twist are $p_1 + p_2 = n_+ + \sigma n_-$ with $\sigma = \pm 1$. Here the poles of Σ are denoted with $\pm$ (not to be confused with the above sign) and are fixed under the action of ξ. The weights at the two fixed points are then $(\varepsilon_1, \varepsilon_2, \mp\varepsilon/n_\pm, b_1^\pm, b_2^\pm)$, with ε the weight of ξ in the spindle direction. The twisting precisely means that

$$b_i^+ - b_i^- = \frac{p_i}{n_+ n_-}\varepsilon\,, \tag{4.2.12}$$

while taking $\sigma_a = 1$, the charge condition (4.2.3) gives

$$\varepsilon_1 + \varepsilon_2 + b_1^+ + b_2^+ - \frac{\varepsilon}{n_+} = 0\,, \quad \varepsilon_1 + \varepsilon_2 + b_1^- + b_2^- + \sigma\frac{\varepsilon}{n_-} = 0\,. \tag{4.2.13}$$

Introducing appropriate orbifold factors in (4.2.11) and recalling $\varepsilon_3 = \mp\varepsilon/n_\pm$, the total action then reads

$$I = -\frac{8}{27}F_{S^5}\frac{(b_1^+ b_2^+)^{3/2} - \sigma(b_1^- b_2^-)^{3/2}}{\varepsilon_1\varepsilon_2\varepsilon}\,. \tag{4.2.14}$$

In particular, the EAdS_4 case has $\varepsilon_1 = \varepsilon_2 = -\frac{1}{2}$ and the charge condition and the action match the spindle solution in the previous chapter 3.2.2. This can be viewed as an $\mathrm{EAdS}_4 \times X_6$ solution with the internal space $X_6 \equiv HS^4 \ltimes \Sigma$, or as a spindle solution to Romans theory coupled to an additional flavour vector multiplet (which allows $p_1 \neq p_2$).

Squashed sphere The above analysis allows us to similarly recover the squashed sphere $S^3_b \times S^2_\varepsilon$ field theory result of [189], with the squashing parameter

$$b \equiv \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\,, \quad Q^2 \equiv \frac{(\varepsilon_1+\varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}\,. \tag{4.2.15}$$

We set $n_+ = 1 = n_-$ so that we have a sphere and look at the twist case $\sigma = +1$. The variables are related in the following way

$$\mathfrak{t}_i = p_i\,, \quad \Delta_i^\pm = -\frac{2(b_i^\pm \mp \frac{\tilde{\varepsilon}}{2}p_i)}{\varepsilon_1+\varepsilon_2}\,, \quad \tilde{\varepsilon} = \frac{2\varepsilon}{\varepsilon_1+\varepsilon_2}\,. \tag{4.2.16}$$

Then $\mathfrak{t}_1+\mathfrak{t}_2 = 2$, $\Delta_1^\pm + \Delta_2^\pm = 2$ and $\Delta_i^+ - \Delta_i^- = 0$, which matches the field theory variables. We can drop the $\pm$ superscripts, and write $\Delta_i^\pm \equiv \Delta_i$. Note the rescaling of ε is required to match the correct twisting condition, where $\Delta_i \pm \frac{\tilde{\varepsilon}}{2}p_i$ appear in the two blocks.

4.2.2 Fixed surfaces

Now let $\Sigma = \Sigma_g \subset M_6$ be a connected, compact surface that is fixed under ξ. This has normal bundle $N_{\Sigma\hookrightarrow M_6} = L_1 \oplus L_2$ in M_6, where we may write $L_a = \mathcal{O}(-m_a)$, $a = 1,2$ for $m_a \in \mathbb{Z}$. Similarly, the hemisphere line bundles restrict to $\mathcal{L}_i|_\Sigma = \mathcal{O}(-p_i)$, $i = 1,2$, for $p_i \in \mathbb{Z}$. The total normal bundle of $\Sigma \subset M_{10}$ is then $N_{\Sigma\hookrightarrow M_{10}} = L_1 \oplus L_2 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$. Equivalently,

$$m_a \equiv -\int_\Sigma c_1(L_a)\,, \quad p_i \equiv -\int_\Sigma c_1(\mathcal{L}_i)\,. \tag{4.2.17}$$

We denote the weights on L_a by ε_a, $a = 1,2$, and the weights on the disc directions in the hemisphere by b_i, $i = 1,2$, as before. Note these depend on the particular connected two-dimensional fixed point set Σ, but we suppress this in the notation. The weights are related through the charge constraint (4.2.3). The latter extends to surfaces to further impose that the sum of the first Chern classes of the relevant line bundles (including the one on Σ, which is equal to its Euler characteristic χ_Σ) vanishes, such that

$$m_1 + m_2 + p_1 + p_2 = \chi_\Sigma\,. \tag{4.2.18}$$

We start by quantizing the fluxes through the various cycles. Integrating F_2 through the HS_i^2 fibres over a point on Σ gives the same condition as for an isolated fixed point in (4.2.7). Likewise integrating F_4 through the HS^4 over a point on Σ gives again (4.2.9). However, we now have additional four-cycles to consider, which are HS_i^2 fibred over Σ, and we denote these by C_i. On one hand, the flux through these cycles is quantized and can be computed using localization

$$N_i \equiv \frac{1}{(2\pi\ell_s)^3}\int_{C_i} F_4 = \frac{1}{(2\pi\ell_s)^3}\int_{\Sigma^*}\left[\frac{2\pi}{b_i}\Psi_2^{F_4} + \frac{(2\pi)^2}{b_i^2}\Psi_0^{F_4}p_i\right], \tag{4.2.19}$$

where Σ^* is the fixed copy of Σ at the pole $*$ of the HS_i^2. Note that the flux of F_2 through Σ enters the first term, as well as Φ_2, using (4.1.21). On the other hand, we have the homology relations $N_1 = -p_2 N$, $N_2 = -p_1 N$, as reviewed in appendix C.2. Combining these observations one can isolate

$$\int_{\Sigma^*}\Phi_2 = \mp\frac{2\sqrt{2}\mathrm{i}\sqrt{b_1 b_2}\ell_s^2\pi}{\sqrt{n_0}}N^{1/2}\int_{\Sigma^*}F_2 + 4\pi^2\ell_s^3(b_1 p_2 + b_2 p_1)N\,. \tag{4.2.20}$$

The localized action is

$$I_\Sigma = \frac{\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^4}{\varepsilon_1\varepsilon_2 b_1 b_2}\int_{\Sigma^*}\left[\Psi_2 + 2\pi\Psi_0\left(\sum_{i=1}^2\frac{p_i}{b_i} + \sum_{a=1}^2\frac{m_a}{\varepsilon_a}\right)\right]. \tag{4.2.21}$$

Inserting the previous flux relations gives

$$I_\Sigma = \frac{8}{27}F_{S^5}\frac{1}{\varepsilon_1\varepsilon_2}\left[\sum_{i=1}^2 p_i\partial_{b_i}\mathcal{F}(b_1,b_2) - \mathcal{F}(b_1,b_2)\sum_{a=1}^2\frac{m_a}{\varepsilon_a}\right], \tag{4.2.22}$$

where F_{S^5}, given in (4.2.6), is used to eliminate N and recall that $\mathcal{F} = (b_1 b_2)^{3/2}$. Note that, unlike in the fixed point case, the dependence on the choice of sign in (4.2.9) drops out of the surface contribution.

Black brane Consider $M_6 = \mathbb{R}_4 \ltimes \Sigma_g$, with the HS^4 also fibred over Σ_g. This corresponds to an asymptotically AdS_6 black brane, with horizon the Riemann surface Σ_g, magnetic charges p_1 and p_2 and the normal bundle twisting parameters m_1, m_2. Such a setup was

previously studied in [40] for $b_1 = b_2$, i.e. the Romans theory. On the other hand, turning off the m's and setting $\varepsilon_1 = \varepsilon_2 = -\frac{1}{2}$, this is the $\mathrm{EAdS}_4 \times X_6$ solution with $X_6 = HS^4 \ltimes \Sigma_g$ studied in section 3.2.1. The charge constraints (4.2.3) and (4.2.18) (with $\sigma_i = 1$) read

$$\varepsilon_1 + \varepsilon_2 + b_1 + b_2 = 0\,, \quad m_1 + m_2 + p_1 + p_2 = \chi(\Sigma_g)\,. \tag{4.2.23}$$

Note that for $m_1 = m_2 = 0$, the latter is simply the Calabi–Yau condition for X_6.

Taking the two weights b_i equal and redefining the weights ϵ_i as

$$\epsilon_i \equiv -2w_i \frac{b_1}{w_1 + w_2} \tag{4.2.24}$$

trivializes the weight constraint and the on-shell action becomes

$$I = \frac{F_{S^5}}{27 w_1^2 w_2^2}(w_1 + w_2)^2 \Big[3w_1 w_2 \chi(\Sigma_g) + m_1 w_2(w_2 - 2w_1) + m_2 w_1(w_1 - 2w_2)\Big]\,, \tag{4.2.25}$$

which agrees with the results in [40] – see section (5.1) thereof.

4.2.3 Fixed B_4

Finally, let $B_4 \subset M_6$ be a fixed connected, compact four-manifold under the action of ξ. This has normal bundle $N_{B_4 \hookrightarrow M_6} = L$ a complex line bundle (on fixing an orientation), with Chern class $c_1(L) \in H^2(B_4, \mathbb{Z})$. We pick a basis of two-cycles $\Gamma_\alpha \subset B_4$ which form a basis for the free part of $H_2(B_4, \mathbb{Z})$, with Poincaré duals $\Psi_\alpha = \mathrm{PD}[\Gamma_\alpha] \in H^2(B_4, \mathbb{R})$. Given a closed[7] two-form such as $\Phi_2|_{B_4}$, we may write

$$\phi^\alpha \equiv \int_{\Gamma_\alpha} \Phi_2 \quad \Rightarrow \quad [\Phi_2] = \sum_{\alpha=1}^{\mathrm{b}_2} \hat{\phi}^\alpha \Psi_\alpha \in H^2(B_4, \mathbb{R})\,, \tag{4.2.26}$$

[7]The zero-form parts of all RR equivariant forms are constant on a given connected component of the fixed point set, which implies that Υ, Φ_0 are constant. Then similarly one can deduce that Φ_2 is closed from equivariant closure and K being zero on a fixed point set.

where $\mathrm{b}_2 = \dim H_2(B_4, \mathbb{R})$, and

$$\phi^\alpha = \int_{B_4} \Phi_2 \wedge \Psi_\alpha = \sum_{\beta=1}^{\mathrm{b}_2} Q_{\alpha\beta} \hat{\phi}^\beta \,, \tag{4.2.27}$$

where $Q_{\alpha\beta} \equiv \int_{B_4} \Psi_\alpha \wedge \Psi_\beta$ is the intersection matrix. This is an integer-valued unimodular symmetric matrix, with inverse $I = Q^{-1}$. Then, for example, one has

$$\int_{B_4} \Phi_2 \wedge \Phi_2 = \sum_{\alpha,\beta=1}^{\mathrm{b}_2} I_{\alpha\beta} \phi^\alpha \phi^\beta \equiv \langle \phi, \phi \rangle \,. \tag{4.2.28}$$

We similarly introduce

$$k^\alpha \equiv \frac{1}{2\pi\ell_s} \int_{\Gamma_\alpha} F_2 \,, \quad m^\alpha \equiv -\int_{\Gamma_\alpha} c_1(L) \,, \quad p_i^\alpha \equiv -\int_{\Gamma_\alpha} c_1(\mathcal{L}_i) \,. \tag{4.2.29}$$

Using our two-cycle basis Γ_α on B_4 we can define four-cycles C_i^α which are HS_i^2 fibred over Γ_α. Note that these cycles satisfy the homology relations $N_1^\alpha = -p_2^\alpha N$, $N_2^\alpha = -p_1^\alpha N$. The argument is the same as for fixed Σ, where now we simply apply this to $\Sigma = \Gamma_\alpha$. Then as before we can solve the analogue of (4.2.26) to give

$$\phi^\alpha = \mp \frac{4\sqrt{2}\mathrm{i}\sqrt{b_1 b_2}\ell_s^3\pi^2}{\sqrt{n_0}} k^\alpha N^{1/2} + 4\pi^2 \ell_s^3 (b_1 p_2^\alpha + b_2 p_1^\alpha) N \,. \tag{4.2.30}$$

There is the copy of B_4 at the pole of HS^4, which we denote B_4^*. We then have the homology relation

$$N^* \equiv \frac{1}{(2\pi\ell_s)^3} \int_{B_4^*} F_4 = \langle p_1, p_2 \rangle N \,, \tag{4.2.31}$$

which is proven similarly to the S^4 over B_4 case – see appendix C.2.

The localized action is then

$$\begin{aligned} I_{B_4} = \frac{\mathrm{i}}{2(2\pi)^7 \ell_s^8} \frac{(2\pi)^3}{\varepsilon b_1 b_2} \Bigg\{ \int_{B_4^*} \Bigg[\Psi_4 - 2\pi \Psi_2 \wedge \left(\frac{c_1(L)}{\varepsilon} + \sum_{i=1}^{2} \frac{c_1(\mathcal{L}_i)}{b_i} \right) \Bigg] \\ + (2\pi)^2 \Psi_0 \left(\frac{\langle p_1, p_1 \rangle}{b_1^2} + \frac{\langle p_1, p_2 \rangle}{b_1 b_2} + \frac{\langle p_2, p_2 \rangle}{b_2^2} + \frac{\langle m, m \rangle}{\varepsilon^2} + \sum_{i=1}^{2} \frac{\langle m, p_i \rangle}{\varepsilon b_i} \right) \Bigg\} . \end{aligned} \tag{4.2.32}$$

After substituting in all the forms and the above formulae, we obtain the simple result

$$I_{B_4} = \mp\frac{1}{\varepsilon}\Bigg[\frac{1}{2}\sum_{i,j=1}^{2}\langle p_i, p_j\rangle\partial^2_{b_i b_j}\mathcal{F}(b_1,b_2) - \frac{1}{\varepsilon}\sum_{i=1}^{2}\langle p_i, m\rangle\partial_{b_i}\mathcal{F}(b_1,b_2) + \frac{1}{\varepsilon^2}\langle m,m\rangle\mathcal{F}(b_1,b_2)\Bigg]\frac{8}{27}F_{S^5} \pm \frac{2\sqrt{2}\pi}{3\varepsilon}\sqrt{b_1 b_2}\langle k,k\rangle\frac{N^{3/2}}{n_0^{3/2}}\,. \tag{4.2.33}$$

The first term recovers the result in [190]. Note that the F_2 flux term k^α appears as an $N^{3/2}$ correction, and so is not seen at leading order in the large N limit (assuming that the F_2 flux is order 1 at large N). It would be interesting to investigate such terms in more detail; it is not obvious *a priori* that the formalism should compute such terms.

Product of Riemann surfaces As an example of the above formula consider B_4 to be the product of two Riemann surfaces: $B_4 = \Sigma_{g_1}\times\Sigma_{g_2}$. For such a B_4 there are two two-cycles, represented by copies of the Riemann surface at a point on the other one. As such the intersection matrix takes the simple form

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}\,. \tag{4.2.34}$$

Then the bracket gives

$$\langle p_i, p_j\rangle = p_i^{(1)}p_j^{(2)} + p_i^{(2)}p_j^{(1)}\,. \tag{4.2.35}$$

The condition for the spinors to be well-defined, implies that the line bundles over the Riemann surfaces are constrained to satisfy

$$\pm\chi(\Sigma_{g_\alpha}) = \sigma_1 p_1^\alpha + \sigma_2 p_2^\alpha + \sigma_3 m^\alpha\,. \tag{4.2.36}$$

One can now use this to expand out each of the terms in (4.2.33) and use (4.2.36) to eliminate one of the first Chern classes in favour of the Euler characteristic to get a more explicit, and substantially longer, result.

Setting $m = 0$ so that $M_6 = \Sigma_{g_1}\times\Sigma_{g_2}\times\mathbb{R}^2$ is simply a product, using (4.2.33) and taking

the upper sign with $\sigma_1 = \sigma_2 = +1$ in (4.2.36) we find

$$I = \frac{2\sqrt{2}\pi}{5\sqrt{n_0}\,\varepsilon} \frac{\left[b_2(b_2 p_1^{(2)} + 3b_1 p_2^{(2)})p_1^{(1)} + b_1(b_1 p_2^{(2)} + 3b_2 p_1^{(2)})p_2^{(1)}\right]}{(b_1 b_2)^{1/2}} N^{5/2}\,. \tag{4.2.37}$$

We may compare this to the large N field theory result (3.105) for $\log Z$ derived in [190–193] for the dual USp($2N$) Seiberg theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1 = \partial M_6$. Identifying our variables with theirs via $p_i^{(1)} = \mathfrak{s}_i$, $p_i^{(2)} = \mathfrak{t}_i$ and $b_i = \Delta_i$, our R-charge constraint $b_1 + b_2 = -\varepsilon$ matches their $\Delta_1 + \Delta_2 = 2\pi$ provided we normalize the weight $\varepsilon = -2\pi$ on the $\mathbb{R}^2$ factor (with the Chern numbers satisfying (4.2.36)). The result (4.2.37) then perfectly matches the field theory result for $-\log Z$.

4.3 10 = 4 + 6: D2-D8 Chern-Simons theories

We now turn to a second application of our results, where the geometry is a fibration of the suspension $X_6 = \mathcal{S}\mathrm{SE}_5$ over a four-dimensional base M_4 with SE_5 a five-dimensional Sasaki–Einstein manifold. When M_4 is Euclidean AdS_4 this describes 3d $\mathcal{N} = 2$ SCFTs arising from D2-branes probing the suspension of the Sasaki–Einstein manifold in the presence of D8-branes [133, 147]. From the field theory perspective, the choice of SE_5 determines the matter content of the theory, while the choice of the four-dimensional spacetime means placing the dual theory on the background ∂M_4.

For brevity, in this section we will focus on the S^5 case. However, it is straightforward to extend this analysis to other Sasaki–Einstein manifolds, such that we give the final result for these too. More details can be found in the appendix of [4]. The resultant master formula for each Sasaki–Einstein space depends only on the definition of a prepotential given by the off-shell Sasakian volume to the power $-2/3$. For the moment we will keep the discussion general before specializing to the S^5 case.

Let us first restrict to the toric Sasaki–Einstein case which allows us to introduce a basis for the $\mathrm{U}(1)^3$ action given by ∂_{φ_i}, $i = 1, 2, 3$. For each of these directions we may introduce a complex line bundle $\mathcal{L}_i$ with which we can fibre over the base M_4. Let us introduce the projection $\pi : M_{10} \to M_4$ from the bundle to the base. Projecting the vector field K to

the base we define $\xi \equiv \pi_* K$ which is a vector field on M_4. We may then decompose the ten-dimensional Killing vector as:

$$K = \xi + \sum_{i=1}^{3} b_i \partial_{\varphi_i} \,, \quad \xi = \sum_a \varepsilon_a \partial_{\psi_a} \,, \tag{4.3.1}$$

where a takes values $\{1, 2\}$ for isolated fixed points in M_4 and $\{1\}$ for bolts in M_4. As in the previous section it follows that the fixed points of K are the fixed points of ξ in M_4 at the two poles of the suspension.

The fact that the Killing spinor is uncharged under K fixes the weights at the fixed point to satisfy[8]

$$\sum_a \sigma_a \varepsilon_a + \sum_{i=1}^{3} b_i = 0 \,, \tag{4.3.2}$$

where σ_a are signs which are fixed at each fixed point. Furthermore, when considering the weights of the suspension the product of the weights (independent of the number of weights used to compute a cycle) picks up an overall sign between the north and south pole. We will take the sign to be positive at the north pole (when computing using toric geometry) and negative at the south pole. Note that the weights in the external spacetime are the same at the north and south pole.

For this class of theory it follows that the partially on-shell action is given by the following master formula:

$$I = \frac{9}{4} F_{S^3} \left\{ \sum_{\mathrm{p}} \frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F} + \sum_{\Sigma} \frac{1}{\varepsilon} \left[\sum_{i=1}^{3} p_i \partial_{b_i} \mathcal{F} - \left(\frac{m}{\varepsilon} \right) \mathcal{F} \right] \right\} , \tag{4.3.3}$$

where the prepotential is a homogeneous degree two function of the weights b_i, $i = 1, 2, 3$ determined by the choice of SE_5 in terms of the Sasakian volume. For the five-sphere the prepotential is

$$\mathcal{F}_{S^5} = (b_1 b_2 b_3)^{2/3} \,, \tag{4.3.4}$$

[8]We fix all of the signs for the weights on the S^5 to be 1 at each fixed point of M_4.

and F_{S^3} is the free energy of the CFT dual to the S^5 suspension placed on S^3

$$F_{S^3} = -\frac{2^{1/3}\pi}{5\times 3^{5/6}}(3\mathrm{i}+\sqrt{3})n_0^{1/3}N^{5/3}\,. \tag{4.3.5}$$

Note in particular that F_{S^3} is complex and it is a non-trivial result that our formalism recovers both the real and imaginary part.

In the remainder of this section we prove the master formula (4.3.3) for $X_6 = \mathcal{S}S^5 = S^6$. The logic is similar to the previous section. We first quantize the fluxes and impose the homology relations before substituting this into the action and extremizing over the free parameters. As in the previous section we first recover the contribution of fixed points before turning to the fixed surface contributions. We provide examples to further illustrate both types of contributions, which recover and extend known results in the literature.

4.3.1 Isolated fixed points

Let us first quantize the various fluxes through the various cycles. There are three $S_i^2 \subset S^6$ and similarly three $S_i^4 \subset S^6$. These are all invariant under K and trivial in homology, such that using localization

$$\begin{aligned} 0 &= \int_{S_i^2} F^2 = \frac{2\pi}{b_i}\left[(F_0\Upsilon_N + \Phi_0^N) - (F_0\Upsilon_S + \Phi_0^S)\right]\,, \\ 0 &= \int_{S_i^4} F^4 = \frac{(2\pi)^2}{b_j b_k}\frac{1}{2}\left[(F_0\Upsilon_N^2 + 2\Upsilon_N\Phi_0^N) - (F_0\Upsilon_S^2 + 2\Upsilon_S\Phi_0^S)\right]\,, \end{aligned} \tag{4.3.6}$$

from which we obtain

$$\Phi_0^N = -\Phi_0^S = -\frac{1}{2}F_0(\Upsilon_N - \Upsilon_S)\,. \tag{4.3.7}$$

We now turn to the quantization of F_6 through X_6, which counts the number of D2-branes

$$\begin{aligned} N &= \frac{1}{(2\pi\ell_s)^5}\frac{(2\pi)^3}{b_1b_2b_3}\frac{1}{3!}\left[(F_0\Upsilon_N^3 + 3\Upsilon_N^2\Phi_0^N) - (F_0\Upsilon_S^3 + 3\Upsilon_S^2\Phi_0^S)\right] \\ &= \frac{1}{b_1b_2b_3}\frac{n_0}{96\ell_s^6\pi^3}(\Upsilon_N - \Upsilon_S)^3\,. \end{aligned} \tag{4.3.8}$$

Inserting our previous result for $\Phi_0^{N,S}$, we can isolate an expression for Υ_N. The flux being cubic in Υ_N, it admits three roots

$$\Upsilon_N = -2^{5/3}\mathrm{e}^{2\pi \mathrm{i} n_N/3}\pi\ell_s^2(3b_1b_2b_3n_0^{-1}N)^{1/3} + \Upsilon_S\,, \tag{4.3.9}$$

where $n_N = \{0,1,2\}$ and labels the three distinct roots.[9]

We have now all the ingredients to evaluate the action. Localization gives

$$I_{\mathrm{p}} = \frac{\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^5}{\varepsilon_1\varepsilon_2 b_1b_2b_3}(\Psi_0^{\mathrm{mag}}|_N - \Psi_0^{\mathrm{mag}}|_S)\,, \tag{4.3.10}$$

where we insert the expressions for $\Phi_0^{N,S}$ and Υ_N obtained above. The explicit expression is not particularly enlightening as it stands, however extremizing over the remaining free parameter Υ_S we find that the critical point fixes

$$\Upsilon_S = -\Upsilon_N\,. \tag{4.3.11}$$

Inserting this back into the action gives the extremized result

$$I_{\mathrm{p}} = (-1)^{n_N}\mathrm{i}\mathrm{e}^{\mathrm{i}\pi n_N/3}\frac{3^{5/3}\pi}{2^{2/3}5}n_0^{1/3}\frac{(b_1b_2b_3)^{2/3}}{\varepsilon_1\varepsilon_2}N^{5/3}\,. \tag{4.3.12}$$

Note that this is an off-shell result and there is still an extremization over the weights to perform. In order to obtain a result with positive real part one should take $n_N = 1$ and the result is

$$I_{\mathrm{p}} = \frac{3^{7/6}(3-\mathrm{i}\sqrt{3})\pi}{2^{5/3}\times 5}\frac{(b_1b_2b_3)^{2/3}}{\epsilon_1\epsilon_2}n_0^{1/3}N^{5/3}\,. \tag{4.3.13}$$

On the other hand the choice $n_N = 2$ switches the sign of the real part, playing the same role as the $\pm$ sign in the fixed point contribution (4.2.11) we obtained in the previous section.

In the case of EAdS_4, there is a single fixed point and the extremization over the weights sets them to all be equal, such that (4.3.12) recovers the sphere free energy (4.3.5). In terms of this quantity and of the prepotential (4.3.4), the fixed point contribution to the action can

[9]The root with $n_N = 0$ leads to a purely imaginary action and should be ignored.

be rewritten as

$$I_{\rm p} = \frac{9}{4} F_{S^3} \frac{\mathcal{F}(b_i^{\rm p})}{\varepsilon_1^{\rm p} \varepsilon_2^{\rm p}} \,. \tag{4.3.14}$$

This takes the gravitational block form in [104].

Spindle In particular we can consider an accelerating black hole, whose horizon is a spindle $\mathbb{WCP}^1_{[n_+,n_-]}$. The spacetime has topology $\mathrm{AdS}_2 \times \Sigma$ and there are two pairs of fixed points: the 2 poles of the suspensions (N, S) at the 2 poles of the spindle $(\pm)$. Using that the on-shell action in (4.3.13) is homogeneous of degree zero under rescaling all of the weights we may choose to set the weight on AdS_2 to be $\varepsilon_1 = -1$. After the rescaling the weights on the spindle poles are $\varepsilon_2^{\pm} \equiv \mp \varepsilon / n_{\pm}$ and the charge condition (4.3.2) then reads

$$-\frac{\varepsilon}{n_+} + \sum_{i=1}^{3} b_i^+ = 1 \,, \qquad \sigma \frac{\varepsilon}{n_-} + \sum_{i=1}^{3} b_i^- = 1 \,, \tag{4.3.15}$$

where $\sigma = \pm 1$ for the twist/anti-twist. We can directly use (4.3.14) at the two poles of the spindle (and introducing the appropriate orbifold factor) to obtain

$$I = \frac{9}{4} F_{S^3} \frac{\mathcal{F}(b_i^+) - \sigma \mathcal{F}(b_i^-)}{\varepsilon} \,. \tag{4.3.16}$$

Note that upon extremization this generalizes the result of the universal spindle black hole in [194] which uplifts the spindle black hole solution of four-dimensional minimal gauged supergravity on the suspension. This is an example of a solution which would not currently be possible to study without our ten-dimensional method.

4.3.2 Fixed surfaces

Now we consider the contribution coming from a fixed surface Σ of ξ. The fixed point set then consists of two bolts: the copies of the fixed surface at the two poles of the suspension. The bolt has a normal line bundle $L = \mathcal{O}(m)$ in M_4, while the fibration of the internal space over the fixed surface is characterized by the integers p_i, $i = 1, 2, 3$ defined as:

$$\int_{\Sigma^N} c_1(L) = \int_{\Sigma^S} c_1(L) \equiv m \qquad \int_{\Sigma^N} c_1(\mathcal{L}_i) = -\int_{\Sigma^S} c_1(\mathcal{L}_i) \equiv p_i \,. \tag{4.3.17}$$

Recall that to each fixed point we have a set of signs σ depending on the eigenvalues of the spin generators of the spinors at the fixed point. Let σ_4 be the eigenvalue for the copy of $\mathbb{C}$ inside M_4 at the bolt. This is fixed for both poles. On the other hand the signs for the suspension should flip sign going from the north pole to the south. We may fix the signs at the north pole to all be $+1$ without loss of generality. Then the condition for the spinor to be well-defined imposes the condition:

$$\sigma_4\Big[m+\chi(\Sigma)\Big]=p_1+p_2+p_3\,. \tag{4.3.18}$$

In addition we have that the weights satisfy

$$\sigma_4\varepsilon+\sum_{i=1}^{3}b_i=0\,. \tag{4.3.19}$$

Having fixed the geometry let us quantize the fluxes. The cycles discussed in the previous section are still present, but there are now additional ones involving the fixed surface. As in the previous section the two-form and four-form fluxes through the compact cycles in the S^6 are again trivial

$$\int_{S_i^2}F^2=0\,,\quad \int_{S_i^4}F^4=0\quad\Longrightarrow\quad \Phi_0^N=-\Phi_0^S=-\frac{1}{2}F_0(\Upsilon_N-\Upsilon_S)\,, \tag{4.3.20}$$

while the F_6 flux through the S^6 remains unchanged

$$N=\frac{1}{(2\pi\ell_s)^5}\int_{X_6}F_6=\frac{1}{b_1b_2b_3}\frac{n_0}{96\ell_s^6\pi^3}(\Upsilon_N-\Upsilon_S)^3\,. \tag{4.3.21}$$

Once again we obtain an expression for Υ_N as one of the cubic roots of this flux, giving (4.3.9).

We now turn to the "new" cycles involving the bolt. We have the two-cycle which is the bolt at the north and south pole of S^6 denoted by $[\Sigma^N]$ and $[\Sigma^S]$ respectively. We also have four-cycles which are given by the linearly embedded spheres inside S^6 fibred over the Riemann surface. These cycles are trivial in homology and the flux through them should vanish. Finally we have new six-cycles given by the linearly embedded four-spheres fibred

over the Riemann surface. For the S_i^4 defined by $z_i = 0$ inside S^6, the six-cycle satisfies $[S_i^4 \ltimes \Sigma] = p_i[S^6]$ and we have a total of three such six-cycles.

The homology relations of the four-cycles $[S_i^2 \ltimes \Sigma]$ impose

$$0 = \int_{S_i^2 \ltimes \Sigma} F_4 = \frac{2\pi}{b_i}\left[\int_{\Sigma^N}\left(\Psi_2^{F_4} - \frac{2\pi}{b_i}\Psi_0^{F_4} c_1(\mathcal{L}_i)\right) - \int_{\Sigma^S}\left(\Psi_2^{F_4} + \frac{2\pi}{b_i}\Psi_0^{F_4} c_1(\mathcal{L}_i)\right)\right], \quad (4.3.22)$$

which after some simplification and using (4.3.20) gives the simple condition

$$\int_{\Sigma^N}(\Upsilon_N F_2 + \Phi_2) = \int_{\Sigma^S}(\Upsilon_S F_2 + \Phi_2)\,. \quad (4.3.23)$$

Since the two copies of Σ are homologous, $[\Sigma^N] = [\Sigma^S]$, and F_2 is closed we have

$$\frac{1}{2\pi\ell_s}\int_{\Sigma^N} F_2 = \frac{1}{2\pi\ell_s}\int_{\Sigma^S} F_2 \equiv M\,. \quad (4.3.24)$$

Then we may use this to eliminate the integral of Φ_2 over Σ^N as[10]

$$\int_{\Sigma^N}\Phi_2 = \int_{\Sigma^S}\Phi_2 + (\Upsilon_S - \Upsilon_N)M\,. \quad (4.3.25)$$

Finally the six-cycle homology relations impose

$$p_i N = \frac{1}{(2\pi\ell_s)^5}\int_{S_i^4 \ltimes \Sigma} F_6 \quad (4.3.26)$$

$$= \frac{1}{(2\pi\ell_s)^5}\frac{2\pi}{b_j b_k}\left[\int_{\Sigma^N}\Psi_2^{F_6} - \int_{\Sigma^S}\Psi_2^{F_6} - \left(2\pi\Psi_{0,N}^{F_6} - 2\pi\Psi_{0,S}^{F_6}\right)\left(\frac{p_j}{b_j} + \frac{p_k}{b_k}\right)\right], \quad (4.3.27)$$

for $i = 1, 2, 3$ with i, j, k all distinct in the above formula. We may solve these as

$$\int_{\Sigma^S}\Phi_2 = \frac{\pi(\Upsilon_N - \Upsilon_S)}{6}\left[6\ell_s M + F_0(\Upsilon_N - \Upsilon_S)\sum_{i=1}^{3}\frac{p_i}{b_i}\right]. \quad (4.3.28)$$

This exhausts the homology relations and we can now compute the action:

$$I_\Sigma = \frac{-\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^4}{\varepsilon b_1 b_2 b_3}\left[\int_{\Sigma^N}\Psi_2 - \int_{\Sigma^S}\Psi_2 - (2\pi\Psi_{0,N} - 2\pi\Psi_{0,S})\left(\frac{m}{\varepsilon} + \sum_{i=1}^{3}\frac{p_i}{b_i}\right)\right]. \quad (4.3.29)$$

[10]Since Φ_2 is not closed generically these need not be equal and the difference is parametrized by M.

In addition to the weights, the action still depends on Υ_S which should be extremized over. Performing the extremization for Υ_S we have two possible options, either $m = 0$ (which forces $M = 0$ and the dependence in Υ_S drops out) or for $m \neq 0$ Υ_S satisfies

$$\Upsilon_S = -\Upsilon_N - \frac{4\pi\ell_s^2 M\varepsilon}{mn_0}\,. \tag{4.3.30}$$

For this second, more general solution, we find

$$I_\Sigma = \frac{-\mathrm{i}e^{10\pi\mathrm{i}n_N/3}3^{5/3}\pi}{2^{2/3}\times 5}\frac{1}{\varepsilon}\Big[(b_1b_2b_3)^{2/3}\frac{m}{\varepsilon} - \sum_{i=1}^{3} p_i\partial_{b_i}(b_1b_2b_3)^{2/3}\Big]n_0^{1/3}N^{5/3} + \frac{\mathrm{i}\pi M^2 N}{mn_0}\,. \tag{4.3.31}$$

Note that setting $m = M = 0$ this recovers the $m = 0$ branch above which just drops the final term. In terms of the S^3 free energy (for $m \neq 0$) we find

$$I_\Sigma = \frac{9}{4}F_{S^3}\frac{1}{\varepsilon}\Big[\sum_{i=1}^{3} p_i\partial_{b_i}\mathcal{F}(b_i) - \mathcal{F}(b_i)\frac{m}{\varepsilon}\Big] + \frac{\mathrm{i}\pi M^2 N}{mn_0}\,, \tag{4.3.32}$$

where $\mathcal{F}(b_i) = (b_1b_2b_3)^{2/3}$ as in equation (4.3.4). Note that the term involving M is purely imaginary and assuming that M does not scale with N is subleading in the large N limit.

Black bolt To exemplify our results consider taking M_4 to be a black saddle with Riemann surface horizon and let m define the twisting of the thermal circle over the Riemann surface making it into a "black bolt", [162, 195, 196]. From the perspective of the four-dimensional black hole the p_i are the magnetic charges of three U(1) gauge fields. We fix $\sigma_4 = -1$ in the following so that (4.3.18) and (4.3.19) read

$$\chi(\Sigma_g) + m + \sum_{i=1}^{3} p_i = 0\,, \qquad -\varepsilon + \sum_{i=1}^{3} b_i = 0\,. \tag{4.3.33}$$

It is useful to make the redefinition

$$b_i = \varepsilon\Delta_i\,, \tag{4.3.34}$$

so that the action reads:[11]

$$I_{\text{black saddle}} = \frac{9}{4} F_{S^3} \Big[\sum_{i=1}^{3} p_i \partial_{\Delta_i} \mathcal{F}(\Delta) - m\mathcal{F}(\Delta) \Big] , \tag{4.3.35}$$

with $\mathcal{F}(\Delta) = (\Delta_1 \Delta_2 \Delta_3)^{2/3}$.

4.4 Discussion

In this chapter we have constructed equivariantly closed polyforms for the Page fluxes and the on-shell action of both type II supergravity theories directly in ten dimensions. Our construction relies only on the universal supersymmetry conditions of generalized geometry [62] and therefore applies to arbitrary minimally supersymmetric type IIA and type IIB backgrounds admitting a pair of Killing spinors. This provides a unified localization framework in which both flux quantization and the on-shell action can be computed for arbitrary backgrounds.

We illustrated the formalism for two large classes of massive type IIA solutions. The first was a four-dimensional hemisphere bundle over a six-dimensional external spacetime which generalizes the localization of Romans supergravity in [40] to the $\mathrm{U}(1)^2$ theory. We derived the master formula (4.2.4) for the on-shell action for a general topology with arbitrary number of isolated fixed points, fixed surfaces and four-dimensional fixed submanifolds. Similarly, we analysed the suspension of a toric Sasaki–Einstein five-manifold fibred over a four-dimensional spacetime, again obtaining a master formula for the on-shell action in (4.3.3). The master formula depends on a prepotential which encodes the dependence on the fibres. In particular the prepotential is a power of the Sasakian volume of the Sasaki–Einstein manifold. In each case the localized action is expressed entirely in terms of the equivariant weights and the topology of the fixed point sets and makes no assumptions about a consistent truncation on the fibre.

One of the main advantages of the ten-dimensional perspective is that the localization computation completely bypasses the need for a consistent truncation. This represents a substantial simplification of the problem. Indeed, for a general Sasaki–Einstein suspension

[11] We have set the two-form flux term to vanish.

there is no known consistent truncation containing three vector fields and so obtaining these results otherwise seems prohibitively difficult. Furthermore, for some questions, one needs to work in ten dimensions and a gauged supergravity description is not sufficient. For example, to compute the on-shell action of probe branes one needs to work directly in ten dimensions to understand the embedding of the brane in the geometry [11]. Moreover, one could consider solutions where the external spacetime has punctures, i.e. a punctured Riemann surface. One can resolve these singularities, but this however requires the ten-dimensional embedding and the gauged supergravity description is insufficient; see, for example, [9, 10].

There are several natural directions for future work. First, while we have focused on massive type IIA, the construction of the equivariant polyforms is equally valid in type IIB, where it would be interesting to derive analogous master localization formulae. Second, in the massless limit our expressions should admit a direct interpretation in eleven-dimensional supergravity, providing a route towards localization in M-theory. Third, though we have considered geometries which have an asymptotically AdS factor the localization works equally well for other geometries. One could, for example, consider the on-shell action of non-conformal branes or asymptotically flat black holes. In these cases one needs to be more careful with boundary terms, which for the asymptotically AdS spacetimes we have considered (conjecturally) cancel. Finally, our analysis suggests that subleading contributions, arising from fluxes through resolution cycles, can also be incorporated systematically – see [197] for an example where such fluxes are incorporated in field theory. Understanding the physical interpretation of these corrections, as well as extending the formalism to more general geometries and flux backgrounds, would further clarify the scope of ten-dimensional localization.

Chapter 5

Outlook

A recurring theme throughout this thesis has been that equivariant localization allows physical observables to be computed from global geometric data without requiring explicit knowledge of the underlying supergravity solution. The examples considered here demonstrate this principle in four-dimensional gauged supergravity, on six-dimensional internal manifolds, and directly in ten dimensions. However, there are many further setups in which localization has already proved useful.

Odd-dimensional localization In this work all the solutions had an even-dimensional localizing space. However there are various tools to also use equivariant localization in odd-dimensional spaces. First in case there is a trivial circle direction in the geometry, one can first reduce on this circle and produce an even-dimensional space over which one can now localize similarly to what appears in this thesis.[1] This has been used for instance in [10, 39, 41, 42]. Beyond this reduction, one may use some further theorems to localize over odd-dimensional spaces [36, 152]. These have been applied to five-dimensional supegravity in [44, 45, 49], and further to compute probe brane actions in [11].

Higher-derivative corrections Next, similar master formulae to the one we obtained for $D = 4$ supergravity were obtained for $D = 5$ and 6 in [41] and [40] respectively. Moreover [46–48] extended the $D = 4$ and 5 results to include higher derivative corrections. Extending localization formulae beyond the two-derivative approximation provides access to subleading $1/N$ corrections and offer a geometric framework for studying finite-N observables holographically.

[1] Notice that in section 3.2.5 we have actually localized over an odd-dimensional space, using this circle reduction to perform the localization.

Puncture contributions One significant additional advancement, developed in [9] and further applied in [3,10], is the use of localization to study specific contributions coming from introducing singular points, known as punctures, in the geometry. These are of particular interest from a field theory perspective where punctures play an important role and are actively studied as they give rise to a rich family of theories with an intricate symmetry structure. Through holography, our approach offers new results and a new perspective on punctures in the dual field theory; we easily recover field theory results but also generalize them to setups where punctures are technically difficult to study and very little is known. The technique we developed for these types of geometries can also be applied far more broadly and opens the door to many exciting directions we hope to explore in the future.

Probe branes Localization has also been used to compute the on-shell action of some probe branes in [11]. These branes probe aspects of the microscopic structure. Depending on the geometry they wrap, they represent different physical objects, including non-perturbative contributions to the black hole index and holographic duals to supersymmetric defects in the dual field theory. In all such cases, their contribution is governed by an integral over a calibrated cycle in the ten-dimensional geometry.

Beyond supersymmetry Finally, it would be interesting to develop equivariant localization beyond supersymmetry. Some insights have been explored in the appendix of [1], but to date there are no actual applications to non-supersymmetric systems in gravity. Ultimately, one would like to understand whether localization reflects a deeper geometric structure that survives beyond supersymmetric theories. While current applications rely crucially on Killing spinors and supersymmetry, the existence of localization formulae in purely geometric settings suggests that some aspects of the formalism may admit a broader interpretation.

Appendix A

Bilinear equations in 4d gauged supergravity

A.1 Killing spinor equations

Our starting point is the Killing spinor equations of the Euclidean theory for ϵ and $\tilde{\epsilon}$ given in (2.1.22) and (2.1.23). From these we can obtain the following useful differential and algebraic relations that hold for an arbitrary element of the Clifford algebra $\mathbb{A}$:

$$\begin{aligned}\nabla_\mu\left(\bar{\tilde{\epsilon}}\mathbb{A}\epsilon\right) = &-\frac{\mathrm{i}}{2}\mathcal{A}_\mu\,\bar{\tilde{\epsilon}}\{\gamma_5,\mathbb{A}\}\epsilon - \frac{1}{4\sqrt{2}}\zeta_I\bar{\tilde{\epsilon}}\left[(L^I+\widetilde{L}^I)\{\gamma_\mu,\mathbb{A}\} - (L^I-\widetilde{L}^I)[\gamma_5\gamma_\mu,\mathbb{A}]\right]\epsilon \\ &+\frac{\mathrm{i}}{8\sqrt{2}}\mathcal{I}_{IJ}F^J_{ab}\bar{\tilde{\epsilon}}\left[(L^I+\widetilde{L}^I)\left(\gamma_\mu\gamma^{ab}\mathbb{A}+\mathbb{A}\gamma^{ab}\gamma_\mu\right)\right. \\ &\left.-(L^I-\widetilde{L}^I)\left(\gamma_5\gamma_\mu\gamma^{ab}\mathbb{A}+\mathbb{A}\gamma^{ab}\gamma_\mu\gamma_5\right)\right]\epsilon\,, \end{aligned} \tag{A.1.1}$$

and

$$\begin{aligned}0 = &\bar{\tilde{\epsilon}}\left[\frac{1}{2}\partial_a(z^i+\tilde{z}^{\tilde{\imath}})[\mathbb{A},\gamma^a]_\pm + \frac{1}{2}\partial_a(z^i-\tilde{z}^{\tilde{\imath}})[\mathbb{A},\gamma_5\gamma^a]_\mp\right]\epsilon \\ &+\frac{\mathrm{i}}{4\sqrt{2}}\mathcal{I}_{IJ}F^J_{ab}\bar{\tilde{\epsilon}}\left[\left(\mathcal{G}^{\tilde{\imath}j}\nabla_j L^I+\mathcal{G}^{i\tilde{\jmath}}\nabla_{\tilde{\jmath}}\widetilde{L}^I\right)[\mathbb{A},\gamma^{ab}]_\pm - \left(\mathcal{G}^{\tilde{\imath}j}\nabla_j L^I-\mathcal{G}^{i\tilde{\jmath}}\nabla_{\tilde{\jmath}}\widetilde{L}^I\right)[\mathbb{A},\gamma_5\gamma^{ab}]_\pm\right]\epsilon \\ &-\frac{1}{2\sqrt{2}}\zeta_I\bar{\tilde{\epsilon}}\left[\left(\mathcal{G}^{\tilde{\imath}j}\nabla_j L^I+\mathcal{G}^{i\tilde{\jmath}}\nabla_j\widetilde{L}^I\right)\mathbb{A}(1\pm 1) - \left(\mathcal{G}^{\tilde{\imath}j}\nabla_j L^I-\mathcal{G}^{i\tilde{\jmath}}\nabla_{\tilde{\jmath}}\widetilde{L}^I\right)[\mathbb{A},\gamma_5]_\pm\right]\epsilon\,, \end{aligned} \tag{A.1.2}$$

where $[\cdot\,,\cdot]_\pm$ denote anti-commutator and commutator, respectively.

A.2 Bilinear relations

Recall that the spinor bilinears were defined in (2.1.24). Using appropriate choices of $\mathbb{A}$ in (A.1.1) and (A.1.2) allows us to obtain various relations on the bilinears. In particular, from

(A.1.1) we obtain the following differential constraints:

$$
\begin{aligned}
\mathrm{d}K &= 0\,,\\
(\mathrm{d}+\mathrm{i}\mathcal{A})(S+P) &= -\frac{\zeta_I}{\sqrt{2}}L^I\,K + \sqrt{2}\mathcal{I}_{IJ}L^I\,\xi\lrcorner F^J_{[-]}\,,\\
(\mathrm{d}-\mathrm{i}\mathcal{A})(S-P) &= -\frac{\zeta_I}{\sqrt{2}}\widetilde{L}^I\,K - \sqrt{2}\mathcal{I}_{IJ}\widetilde{L}^I\,\xi\lrcorner F^J_{[+]}\,,\\
(\mathrm{d}+\mathrm{i}\mathcal{A})U_{[-]} &= \frac{3}{2\sqrt{2}}\zeta_I L^I(\xi\lrcorner\mathrm{vol}_4) - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}L^I\,K\wedge F^J_{[+]}\,,\\
(\mathrm{d}-\mathrm{i}\mathcal{A})U_{[+]} &= \frac{3}{2\sqrt{2}}\zeta_I \widetilde{L}^I(\xi\lrcorner\mathrm{vol}_4) - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\widetilde{L}^I\,K\wedge F^J_{[-]}\,,
\end{aligned}
\tag{A.2.1}
$$

together with

$$
\mathrm{d}\xi^\flat = -\sqrt{2}\zeta_I\left[L^I U_{[+]} - \widetilde{L}^I U_{[-]}\right] - \sqrt{2}\mathcal{I}_{IJ}\left[C^I F^J_{[-]} - \widetilde{C}^I F^J_{[+]}\right]\,,
\tag{A.2.2}
$$

and hence

$$
*\mathrm{d}\xi^\flat = -\sqrt{2}\zeta_I\left[L^I U_{[+]} + \widetilde{L}^I U_{[-]}\right] + \sqrt{2}\mathcal{I}_{IJ}\left[C^I F^J_{[-]} + \widetilde{C}^I F^J_{[+]}\right]\,.
\tag{A.2.3}
$$

On the other hand from (A.1.2) we obtain the following set of useful algebraic relations for the bilinears:

$$
\begin{aligned}
0 &= \xi\lrcorner\mathrm{d}z^i = \xi\lrcorner\mathrm{d}\tilde{z}^{\tilde{i}}\,,\\
0 &= \mathrm{d}z^i(S-P) + \sqrt{2}\mathcal{I}_{IJ}\mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I\,\xi\lrcorner F^J_{[-]} - \frac{\zeta_I}{\sqrt{2}}\mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I K\,,\\
0 &= \mathrm{d}\tilde{z}^{\tilde{i}}(S+P) - \sqrt{2}\mathcal{I}_{IJ}\mathcal{G}^{\tilde{i}j}\nabla_j L^I\,\xi\lrcorner F^J_{[+]} - \frac{\zeta_I}{\sqrt{2}}\mathcal{G}^{\tilde{i}j}\nabla_j L^I K\,,\\
0 &= \mathrm{d}z^i\wedge U_{[+]} + \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I K\wedge F^J_{[+]} + \frac{\zeta_I}{2\sqrt{2}}\mathcal{G}^{i\tilde{j}}\nabla_{\tilde{j}}\widetilde{L}^I(\xi\lrcorner\mathrm{vol}_4)\,,\\
0 &= \mathrm{d}\tilde{z}^{\tilde{i}}\wedge U_{[-]} + \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\mathcal{G}^{\tilde{i}j}\nabla_j L^I K\wedge F^J_{[-]} + \frac{\zeta_I}{2\sqrt{2}}\mathcal{G}^{\tilde{i}j}\nabla_j L^I(\xi\lrcorner\mathrm{vol}_4)\,.
\end{aligned}
\tag{A.2.4}
$$

From (A.2.1) and (A.2.4) we can then derive

$$
\mathrm{d}\left[(\zeta_I L^I)U^+ + (\zeta_I\widetilde{L}^I)U^-\right] = -\frac{1}{\sqrt{2}}\mathcal{V}\,\xi\lrcorner\mathrm{vol}_4 + \frac{1}{2\sqrt{2}}\zeta_I F^I\wedge K\,,
\tag{A.2.5}
$$

and notice that on the right-hand side we have the R-symmetry field strength $F^R \equiv \mathrm{d}(\frac{1}{2}\zeta_I A^I_\mu)$ (see (2.1.13)).

Finally, we also record the result

$$\xi \lrcorner U_{[\pm]} = \mp\frac{1}{2}(S \mp P)K\,, \tag{A.2.6}$$

which follows from Fierz identities.

A.3 Some useful supergravity relations

We now record some useful relations for the Euclidean supergravity theory we are considering (obtained from the Lorentzian theory; see, for example, [66, 70, 198–200]).

One can show that

$$\mathrm{e}^{-\mathcal{K}} = -X^I N_{IJ} \widetilde{X}^J\,. \tag{A.3.1}$$

We also have

$$\mathcal{N}_{IJ} X^J = \mathcal{F}_I\,, \tag{A.3.2}$$

and hence

$$\mathcal{N}_{IJ} X^I X^J = 2\mathcal{F}\,, \qquad \mathcal{R}_{IJ} X^I \widetilde{X}^J = \frac{1}{2}\left(\mathcal{F}_I \widetilde{X}^I + \widetilde{\mathcal{F}}_I X^I\right)\,, \tag{A.3.3}$$

where we used that $\mathcal{F}$ is homogeneous of degree two to get the first equation.

From (A.3.1) we can deduce that

$$\nabla_i X^I N_{IJ} \widetilde{X}^J = 0\,, \tag{A.3.4}$$

and then using (A.3.2) we have

$$\nabla_i \mathcal{F}_I = \widetilde{\mathcal{N}}_{IJ} \nabla_i X^J\,. \tag{A.3.5}$$

Expressions for $\nabla_i X^I$ and $\nabla_i L^I$ are given in (2.1.8), (2.1.9) and we recall $\nabla_{\bar{i}} L^I = 0$. We then compute

$$\mathrm{d}L^I = \nabla_i L^I \mathrm{d}z^i - \mathrm{i}\mathcal{A}L^I \,. \tag{A.3.6}$$

A standard relation (see above references) which is important for our purposes is

$$\mathcal{G}^{i\tilde{j}} \nabla_i L^I \nabla_{\tilde{j}} \widetilde{L}^J + L^I \widetilde{L}^J = -\frac{1}{2} (\mathcal{I}^{-1})^{IJ} \,. \tag{A.3.7}$$

By taking the holomorphic covariant derivative of (A.3.2) and using (A.3.5) we deduce

$$\nabla_i (\mathcal{N}_{IJ} L^J) = \widetilde{\mathcal{N}}_{IJ} \nabla_i L^J \,. \tag{A.3.8}$$

Since $\mathcal{N}_{IJ}$ does not transform under Kähler transformations we deduce $(\partial_i \mathcal{N}_{IJ}) L^J = -2\mathrm{i}\mathcal{I}_{IJ} \nabla_i L^J$. From (A.3.2) we have $0 = \partial_{\bar{j}} \mathcal{F}_I = (\partial_{\bar{j}} \mathcal{N}_{IJ}) X^J$. After multiplying the latter by $\mathrm{e}^{\mathcal{K}/2}$ we obtain

$$(\mathrm{d}\mathcal{N}_{IJ}) L^J = -2\mathrm{i}\mathcal{I}_{IJ} \nabla_i L^J \mathrm{d}z^i \,, \quad (\mathrm{d}\widetilde{\mathcal{N}}_{IJ}) \widetilde{L}^J = 2\mathrm{i}\mathcal{I}_{IJ} \nabla_{\tilde{i}} \widetilde{L}^J \mathrm{d}\tilde{z}^{\tilde{i}} \,, \tag{A.3.9}$$

from which we derive the further relations

$$\begin{aligned} \mathrm{d}(\mathcal{N}_{IJ} L^I L^J) &= 2\mathcal{R}_{IJ} L^I \nabla_i L^J \mathrm{d}z^i - 2\mathrm{i}\mathcal{A}(\mathcal{N}_{IJ} L^I L^J) \,, \\ \mathrm{d}(\mathcal{R}_{IJ} L^I \widetilde{L}^J) &= \widetilde{\mathcal{N}}_{IJ} \widetilde{L}^J \nabla_i L^J \mathrm{d}z^i + \mathcal{N}_{IJ} L^J \nabla_{\tilde{i}} \widetilde{L}^I \mathrm{d}\tilde{z}^{\tilde{i}} \,. \end{aligned} \tag{A.3.10}$$

A.4 Equivariantly closed forms

We now obtain expressions for equivariantly closed forms associated with the fluxes and the on-shell action. Here we show that for supersymmetric configurations they can be canonically expressed in terms of supergravity fields and the bilinears.

We first consider the equivariantly closed forms for the fluxes. We first recall (2.1.29):

$$\begin{aligned}
\mathrm{d}C^I &\equiv \mathrm{d}[L^I(S-P)] = \nabla_i L^I \mathrm{d}z^i (S-P) + L^I(\mathrm{d}-\mathrm{i}\mathcal{A})(S-P)\,,\\
\mathrm{d}\widetilde{C}^I &\equiv \mathrm{d}[\widetilde{L}^I(S+P)] = \nabla_{\tilde{i}} \widetilde{L}^I \mathrm{d}z^{\tilde{i}} (S+P) + \widetilde{L}^I(\mathrm{d}+\mathrm{i}\mathcal{A})(S+P)\,.
\end{aligned} \tag{A.4.1}$$

Then using (A.2.1), (A.2.4) and (A.3.7) we find

$$\mathrm{d}\left[\sqrt{2}\left(C^I - \widetilde{C}^I\right)\right] = \xi \lrcorner F^I\,. \tag{A.4.2}$$

Thus, we immediately deduce that $\Phi^I_{(F)}$, given by

$$\Phi^I_{(F)} = F^I + \sqrt{2}\left(C^I - \widetilde{C}^I\right)\,, \tag{A.4.3}$$

is equivariantly closed.

We now turn to the on-shell action. We want to show that following polyform is equivariantly closed

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0\,, \tag{A.4.4}$$

where

$$\begin{aligned}
\Phi_4 &= -\frac{1}{2}\mathcal{V}\,\mathrm{vol}_4 - \frac{1}{4}\mathcal{I}_{IJ}F^I\wedge *F^J + \frac{\mathrm{i}}{4}\mathcal{R}_{IJ}F^I\wedge F^J\,, \\
\Phi_2 &= \frac{1}{\sqrt{2}}\zeta_I(L^I U_{[+]} + \widetilde{L}^I U_{[-]}) - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\big(C^I F^J_{[+]} + \widetilde{C}^I F^J_{[-]}\big) + \frac{\mathrm{i}}{\sqrt{2}}\mathcal{R}_{IJ}F^J\big(C^I - \widetilde{C}^I\big)\,, \\
\Phi_0 &= \mathrm{i}\mathrm{e}^{\mathcal{K}}\Big[\mathcal{F}(X)(S-P)^2 + \widetilde{\mathcal{F}}(\widetilde{X})(S+P)^2 - (\partial_I\mathcal{F}(X)\widetilde{X}^I + \partial_I\widetilde{\mathcal{F}}(\widetilde{X})X^I)(S^2-P^2)\Big]\,.
\end{aligned} \tag{A.4.5}$$

The expression for Φ_4 is associated with the on-shell action.Specifically, using (A.2.3) and (A.4.3) in (A.4.5) we have

$$\Phi_2 = -\frac{1}{2}*\mathrm{d}\xi^\flat - \frac{1}{2}(\mathcal{I}_{IJ}*F^I - \mathrm{i}\mathcal{R}_{IJ}F^I)\Phi^J_0\,. \tag{A.4.6}$$

Then $\mathrm{d}\Phi_2 = \xi\lrcorner\Phi_4$ can also be shown directly using the bilinear relations above, together with the Bianchi identity, $\mathrm{d}F^I = 0$, and (A.3.9).

We now show that the expression for Φ_0 given in (A.4.5) does indeed satisfy $\mathrm{d}\Phi_0 = \xi \lrcorner \Phi_2$. On the one hand, using Fierz identities we have

$$\begin{aligned}\xi \lrcorner \Phi_2 = &-\frac{1}{2\sqrt{2}}\zeta_I(C^I - \widetilde{C}^I)K - \frac{1}{\sqrt{2}}\mathcal{I}_{IJ}\big[(\xi \lrcorner F^J_{[+]})C^I + (\xi \lrcorner F^J_{[-]})\widetilde{C}^I\big] \\ &+ \frac{\mathrm{i}}{\sqrt{2}}\mathcal{R}_{IJ}(\xi \lrcorner F^J)(C^I - \widetilde{C}^I)\,. \end{aligned} \tag{A.4.7}$$

On the other hand, using (A.3.3) we see that Φ_0 can be written as

$$\Phi_0 = \mathrm{i}\Big[\frac{1}{2}\mathcal{N}_{IJ}L^I L^J(S-P)^2 + \frac{1}{2}\widetilde{\mathcal{N}}_{IJ}\widetilde{L}^I\widetilde{L}^J(S+P)^2 - \mathcal{R}_{IJ}L^I\widetilde{L}^J(S+P)(S-P)\Big]\,. \tag{A.4.8}$$

Taking the exterior derivative and using (A.3.10) we have

$$\begin{aligned}\mathrm{d}\Phi_0 = \mathrm{i}\Big[&\mathcal{R}_{IJ}L^I\nabla_i L^J \mathrm{d}z^i (S-P)^2 + \mathcal{R}_{IJ}\widetilde{L}^I\nabla_{\tilde{i}}\widetilde{L}^J\mathrm{d}\tilde{z}^{\tilde{i}}(S+P)^2 \\ &- (\widetilde{\mathcal{N}}_{IJ}\widetilde{L}^J\nabla_i L^I\mathrm{d}z^i + \mathcal{N}_{IJ}L^J\nabla_{\tilde{i}}\widetilde{L}^I\mathrm{d}\tilde{z}^{\tilde{i}})(S+P)(S-P)\Big] \\ +\mathrm{i}\Big[&\mathcal{N}_{IJ}L^I L^J(S-P)(\mathrm{d}-\mathrm{i}\mathcal{A})(S-P) + \widetilde{\mathcal{N}}_{IJ}\widetilde{L}^I\widetilde{L}^J(S+P)(\mathrm{d}+\mathrm{i}\mathcal{A})(S+P) \\ &- \mathcal{R}_{IJ}L^I\widetilde{L}^J((S+P)(\mathrm{d}-\mathrm{i}\mathcal{A})(S-P) + (S-P)(\mathrm{d}+\mathrm{i}\mathcal{A})(S+P))\Big]\,. \end{aligned} \tag{A.4.9}$$

The differential terms can be replaced using (A.3.10), (A.2.1), and subsequently (A.2.4). Further using (A.3.7), this reduces to

$$\begin{aligned}\mathrm{d}\Phi_0 = &\frac{\mathrm{i}}{2\sqrt{2}}(\mathcal{I}^{JK})^{-1}\Big[(\mathcal{N}_{IJ} - \mathcal{R}_{IJ})L^I(S-P) + (\widetilde{\mathcal{N}}_{IJ} - \mathcal{R}_{IJ})\widetilde{L}^I(S+P)\Big]\zeta_K K \\ &+ \frac{\mathrm{i}}{\sqrt{2}}\Big[\big(\mathcal{N}_{IJ}(\xi \lrcorner F^J_{[+]}) + \mathcal{R}_{IJ}(\xi \lrcorner F^J_{[-]})\big)L^I(S-P) \\ &\quad - \big(\widetilde{\mathcal{N}}_{IJ}(\xi \lrcorner F^J_{[-]}) + \mathcal{R}_{IJ}(\xi \lrcorner F^J_{[+]})\big)\widetilde{L}^I(S+P)\Big]\,, \end{aligned} \tag{A.4.10}$$

which recalling (2.1.27) is equal to (A.4.7). This completes the proof that Φ given in (A.4.4) and (A.4.5), canonically expressed in terms of the supergravity fields and bilinears, is equivariantly closed for supersymmetric configurations.

Appendix B

Minimal 4d gauged supergravity

We can obtain minimal $D = 4$, $\mathcal{N} = 2$ Euclidean gauged supergravity by setting all the vector multiplets to zero. This can be achieved, consistently with our conventions, by setting

$$X^0 = \widetilde{X}^0 = \frac{1}{2^{3/2}}\,, \quad \zeta_0 = 4\,, \quad \mathcal{F}(X^0) = -2\mathrm{i}(X^0)^2 = -\frac{\mathrm{i}}{4}\,, \tag{B.0.1}$$

so that $\mathcal{K} = 0$, $\mathcal{V} = -6$, $\mathcal{I}_{00} = -4$ and $\mathcal{R}_{00} = 0$. We also note that $u = \frac{1}{4}$, so

$$\mathrm{i}\mathcal{F}(u) = \frac{1}{8}\,, \qquad \mathrm{i}\frac{\partial\mathcal{F}(u)}{\partial u} = 1\,. \tag{B.0.2}$$

Writing the gauge field as $A \equiv A^0$, we then obtain from (2.1.16) the action

$$I = -\frac{1}{16\pi G_4}\int \left(R + 6 - F^2\right)\mathrm{vol}_4\,, \tag{B.0.3}$$

with $F^2 = F_{\mu\nu}F^{\mu\nu}$ and from (2.1.22), (2.1.23) the supersymmetry transformations

$$\begin{aligned} 0 &= \nabla_\mu\epsilon - \mathrm{i}A_\mu\epsilon + \frac{1}{2}\gamma_\mu\epsilon + \frac{\mathrm{i}}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon\,, \\ 0 &= \nabla_\mu\tilde{\epsilon} + \mathrm{i}A_\mu\tilde{\epsilon} + \frac{1}{2}\gamma_\mu\tilde{\epsilon} - \frac{\mathrm{i}}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\tilde{\epsilon}\,. \end{aligned} \tag{B.0.4}$$

We also note that the R-symmetry gauge field is defined as $A^R = \frac{1}{2}\zeta_I A^I = 2A$ and that the AdS_4 vacuum solution has unit radius, $\ell = 1$.

From (2.1.31) we have the equivariantly closed form

$$\Phi_{(F)} \equiv F - P\,, \tag{B.0.5}$$

and from (2.1.35), (2.1.36)

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0\,, \tag{B.0.6}$$

with

$$\begin{aligned}\Phi_4 &= 3\,\mathrm{vol}_4 + F \wedge *F\,,\\ \Phi_2 &= U + SF - P * F\\ &= -\frac{1}{2} * \mathrm{d}\xi^\flat - 2 * FP\,,\\ \Phi_0 &= -SP\,.\end{aligned} \tag{B.0.7}$$

For the case of real metric and gauge field and $\tilde{\epsilon} = \epsilon^c$ (as in section 2.2.1) the expression for Φ is in precise agreement with [29]; here we have shown that the equivariantly closed form still exists when these conditions are relaxed. The equivariant closure for $\Phi_{(F)}$ is equivalent to $\xi \lrcorner F = -\mathrm{d}P$, a condition which was noted in (3.7) of [51] in the real setting.

The result for the free energy (2.2.33) now reads

$$I^{\mathrm{FP}}_{\mathrm{OS}} = \frac{\pi}{2G_4}\left[\mp \sum_{\mathrm{nuts}_\pm} \frac{(b_1 \mp b_2)^2}{4b_1 b_2} + \sum_{\mathrm{bolts}_\pm} \left(-2\kappa \mathfrak{p}_\pm \pm \frac{1}{4}\int_{\mathrm{bolt}_\pm} c_1(L) \right) \right]. \tag{B.0.8}$$

The R-symmetry gauge field $A^R = 2A$ has flux through a bolt constrained by (2.3.2), which reads

$$\begin{aligned}\mathfrak{p}^R_\pm = 2\mathfrak{p}_\pm &\equiv 2\frac{1}{4\pi}\int_{\Sigma_\pm} F\\ &= \frac{\kappa}{2}\int_{\Sigma_\pm} \left[\pm c_1(L) - c_1(T\Sigma_\pm)\right]\,,\end{aligned} \tag{B.0.9}$$

(and matches (3.74) of [51] with $\kappa = +1$). We also note that eliminating $\mathfrak{p}_\pm$ from (B.0.8) is in agreement with (11) of [29].

Finally, for minimal gauged supergravity from (2.3.15) we obtain a relation between the boundary geometry and the bilinear P:

$$\mathrm{i}\sigma^0 = \frac{\mathrm{i}}{4\pi} P|_{S^1_{\mathrm{UV}}} = \frac{1}{8\pi} *_2 \mathrm{d}\eta_{(0)}|_{S^1_{\mathrm{UV}}}\,. \tag{B.0.10}$$

Appendix C

Some cohomology and homology

C.1 Homology relations used in chapter 3

In the main text we have considered a number of different fiber bundles with 2d hemi-sphere and 4d hemi-sphere fibers. In this section we will describe some of the various homology relations that we have used. Many of the results can be extracted from [201] and slight modifications, but for clarity we discuss these results and their applications to our setup. For a clear exposition of S^2 and S^4 bundles see appendix C of [32].

Consider a six-manifold M_6 which is the total space of a disc bundle over a four-manifold B_4 with projective map $\pi : M_6 \to B_4$. We consider the disc bundle inside the $\mathbb{R}^3_+$ bundle $\mathcal{L} \oplus \mathbb{R}_+ \to B_4$ where $\mathcal{L}$ is a complex line bundle. We may introduce coordinates (z, x) on $\mathbb{R}^3_+$, with $x > 0$ and the disc defined by $|z|^2 + x^2 = 1$.

We may define the Thom class for the bundle. Let $\rho(r)$ be a radial function which is 1 at the centre of the disc and 0 on the boundary. This then makes $\mathrm{d}\rho = \rho'(r)\mathrm{d}r$ a bump form on $\mathbb{R}$ with total integral 1. Further, define the global angular form on the disc to be η. Then the Thom class may be written as[1]

$$\Phi^{\mathrm{Thom}} = \frac{1}{2\pi}\mathrm{d}\left(\rho(r)\eta\right) . \tag{C.1.1}$$

This is compactly supported in the vertical direction, is closed and integrates to 1 when restricted to the fiber. Let $s : B_4 \to M_6$ be the zero section, then we have

$$s^*\Phi^{\mathrm{Thom}} = -e\,, \tag{C.1.2}$$

[1] We thank *Topology Boy* for clarifying this and other points in this appendix with us.

with e the Euler class of the bundle [201]. It then follows that the Poincaré dual of B_4 is

$$\Psi_p \equiv \mathrm{PD}[B_4] = \Phi^{\mathrm{Thom}}\,. \tag{C.1.3}$$

Let Γ_α be a basis for the free part of $H_2(B_4,\mathbb{Z})$ and let $\{\Psi_\alpha\}$ be the Poincaré duals of this basis: $\Psi_\alpha = \mathrm{PD}[\Gamma_\alpha] \in H^2(B_4)$. Further let the dimension of $H_2(B_4,\mathbb{R})$ be $b_2 \equiv \dim(H_2(B_4,\mathbb{R}))$. Given any closed two-form, τ on B_4 we may expand it in terms of this basis via

$$\tau = \sum_{\alpha=1}^{b_2} \hat{C}_\alpha \Psi_\alpha\,. \tag{C.1.4}$$

Alternatively we may expand by defining the constants C_α where

$$C_\alpha \equiv \int_{\Gamma_\alpha} \tau\,. \tag{C.1.5}$$

Using the definition of the Poincaré dual we may rewrite this as

$$C_\alpha = \int_{B_4} \tau \wedge \Psi_\alpha = \sum_{\beta=1}^{b_2} Q_{\alpha\beta}\hat{C}\,, \tag{C.1.6}$$

where Q is the intersection form

$$Q_{\alpha\beta} \equiv \int_{B_4} \Psi_\alpha \wedge \Psi_\beta = [\Gamma_\alpha] \cap [\Gamma_\beta]\,. \tag{C.1.7}$$

Recall that this is an integer-valued unimodular symmetric matrix. It follows that we may write

$$\int_{B_4} \tau \wedge \tau = \sum_{\alpha,\beta=1}^{b_2} Q_{\alpha\beta}\hat{C}_\alpha\hat{C}_\beta = \sum_{\alpha,\beta=1}^{b_2} I_{\alpha\beta}C_\alpha C_\beta\,, \tag{C.1.8}$$

where $I = Q^{-1}$. This allows us to expand the first Chern class of a line bundle over B_4 as

$$c_1(\mathcal{L}) = \sum_{\alpha=1}^{b_2} \hat{n}_\alpha \Psi_\alpha\,, \tag{C.1.9}$$

with $\hat{n}_\alpha \in \mathbb{Z}$ which is used in section 3.2.3. It is also useful to define the pairing $\langle\cdot\,,\cdot\rangle$ for

two-forms. Let τ and ω be two two-forms with expansions:

$$\tau = \sum_{\alpha=1}^{b_2} \hat{C}_\alpha \Psi_\alpha \,, \quad \omega = \sum_{\alpha=1}^{b_2} \hat{N}_\alpha \Psi_\alpha \,, \tag{C.1.10}$$

then

$$\int_{B_4} \tau \wedge \omega = \sum_{\alpha,\beta=1}^{b_2} Q_{\alpha\beta} \hat{C}_\alpha \hat{N}_\beta = \sum_{\alpha,\beta=1}^{b_2} I_{\alpha\beta} C_\alpha N_\beta \equiv \langle C, N \rangle \,. \tag{C.1.11}$$

The pairing is used in section 3.2.3 to simplify expressions.

Consider now a closed form-form on M_6. Using the Thom class we can expand the four-form F_4 as

$$F_4 = \Phi^{\text{Thom}} \wedge \sum_{\alpha=1}^{b_2} \hat{N}_\alpha \Psi_\alpha + M \widetilde{\text{vol}}(B_4) \,, \tag{C.1.12}$$

where $\widetilde{\text{vol}}(B_4)$ is normalized to integrate to 1 over B_4. Note that whereas the contribution from the Thom class vanishes on the boundary of the disc, the second term does not and signifies that additional branes are present along the boundary. We therefore find that

$$\begin{aligned} (2\pi\ell_s)^3 N &= \int_{B_4} F_4 \\ &= M + \hat{N}_\alpha \int_{B_4} s^*(\Phi^{\text{Thom}}) \wedge \Psi_\alpha \\ &= M - \hat{N}_\alpha \int_{B_4} e \wedge \Psi_\alpha \\ &= M - \langle N, n \rangle \,. \end{aligned} \tag{C.1.13}$$

Next consider two-cycles in M_6 defined by taking $\Gamma_\alpha \subset B_4$ evaluated at the centre of the disc. By the Thom isomorphism these have Poincaré duals

$$\text{PD}[\Gamma_\alpha] = \pi^* \Psi_\alpha \wedge \Phi^{\text{Thom}} \in H^4(M_6) \,. \tag{C.1.14}$$

Integrating this over B_4 at the pole we have

$$\int_{B_4} \pi^* \Psi_\alpha \wedge \Phi^{\text{Thom}} = n_\alpha \,, \tag{C.1.15}$$

and therefore we find the homology relation

$$[\Gamma_\alpha] = n_\alpha [HS^2_{\text{fibre}}]\,. \tag{C.1.16}$$

Note that this holds on closed forms which vanish on the boundary of the disc.

C.2 Homology relations used in chapter 4

In this appendix we prove the homology relations used in section 4.2. The extension of these arguments to the relations in section 4.3 is straightforward.

The manifold M_{10} has a "boundary" which is $\partial HS^4 \to M_6$. We denote this by $\partial \subset M_{10}$, noting that M_6 itself will have a (conformal) boundary which is different.[2] Then in what follows we will take the RR Page forms

$$F_{2n} \in H^{2n}(M_{10}, \partial)\,, \tag{C.2.1}$$

in particular meaning they have zero restriction to the D8/O8 boundary ∂, so $F_{2n}|_\partial = 0$. One can regard this as a choice of boundary conditions, where we note this is satisfied for uplifts of the minimal Romans theory (the RR forms in (2.1) of [40] are all uplifts of Romans fields that are multiplied by functions that go to zero on ∂HS^4). We will need also the stronger condition that also the lower degree parts of equivariantly closed forms for these fluxes are zero – concretely, that Υ and the bilinear form Φ are zero when restricted to ∂. To see this one should solve the ten-dimensional killing spinor equations near a supersymmetric D8/O8 boundary. Using the results in [188] we can see that indeed the spinors vanish on the D8/O8 locus and therefore both Υ and Φ do too. We will assume that

$$\Psi^{F_{2n}} \in H_K(M_{10}, \partial)\,, \tag{C.2.2}$$

with (C.2.1) then a corollary of this, where H_K denotes equivariant cohomology for d_K.

As a first consequence, it is immediate from (C.2.2) that there is no flux of the Page RR

[2] Thus ∂ itself has a boundary, namely $S^3 \to \partial M_6$.

two-form F_2 through the copies of HS^2_i at the fixed points. Indeed, since $H^2(HS^4, S^3) = 0$, necessarily $F_2 = \mathrm{d}\omega$ is exact restricted to a fibre HS^4 where $\omega|_{S^3} = 0$, but then $\int_{HS^2_i} F_2 = \int_{S^1_i} \omega = 0$ where $S^1_i \subset S^3$ lies on the boundary.

Next, consider the 6-manifold B_6 which is HS^4 fibred over a two-dimensional manifold Σ, with four-cycles C_i which are HS^2_i fibred over Σ. These and the fibre HS^4 are classes in $H_4(B_6, \partial, \mathbb{Z})$. Consider a closed four-form $F_4 \in H^4(B_6, \partial)$. Fix, say, C_1, and let the Poincaré dual be Ψ_1, which has support along the fibre of $\mathcal{L}_2$. This satisfies $\Psi_1 = \pi^* c_1(\mathcal{L}_2) \in H^2(B_6)$ by explicit construction of this Poincaré dual, where $\pi : HS^4 \to \Sigma$ is the projection. Then

$$N_1 \equiv \int_{C_1} F_4 = \int_{B_6} F_4 \wedge \Psi_1 = \int_{B_6} F_4 \wedge \pi^* c_1(\mathcal{L}_2) = N \int_{\Sigma} c_1(\mathcal{L}_2)\,, \tag{C.2.3}$$

where $N \equiv \int_{HS^4} F_4$ is the integral of F_4 over the HS^4 fibre.

Finally, consider HS^4 fibred over B_4, and let B_4^* be the copy of B_4 at the pole of HS^4. The Poincaré dual of B_4^* inside the total space B_8 is $\Psi_1 \wedge \Psi_2 = \pi^*[c_1(\mathcal{L}_1) \wedge c_1(\mathcal{L}_2)] \in H^4(B_8)$. Then

$$\int_{B_4^*} F_4 = \int_{B_8} F_4 \wedge \Psi_1 \wedge \Psi_2 = \int_{B_4} c_1(\mathcal{L}_1) \wedge c_1(\mathcal{L}_2) \int_{HS^4_{\text{fibre}}} F_4 = \langle p_1, p_2 \rangle N\,, \tag{C.2.4}$$

with notation as in the main text.

Appendix D

Details of the 10d derivations

In this appendix we present a more explicit derivation of the polyforms for the Page fluxes used in the main text, before giving the explicit polyforms in type IIB.

D.1 Obtaining the polyforms

First consider the Maxwell equations in the democratic formalism (4.1.4) which we may rewrite in the form

$$\mathrm{d}_H F^{\mathrm{RR}} = 0 \quad \Leftrightarrow \quad \mathrm{d}(\mathrm{e}^{-B} \wedge F^{\mathrm{RR}}) = 0 \quad \Leftrightarrow \quad \mathrm{d}F = 0\,, \tag{D.1.1}$$

where the Page flux F is defined in (4.1.14). Considering the contraction of K into the Page flux, one finds

$$K \lrcorner F = -(K \lrcorner B + \widetilde{K}) \wedge F - \mathrm{d}(\mathrm{e}^{-\phi}\mathrm{e}^{-B}\,\widehat{\Phi})\,. \tag{D.1.2}$$

Using the definition of Υ in equation (4.1.16) and the definition of Φ in (4.1.18) we find

$$K \lrcorner F = \mathrm{d}[-(\Upsilon F + \Phi)] \equiv \mathrm{d}\Psi_1\,. \tag{D.1.3}$$

This is the first step in constructing equivariantly closed forms with respect to K. Considering the contraction of K into Φ, we have

$$K \lrcorner \Phi = -(K \lrcorner B) \wedge \Phi - \tilde{K} \wedge \Phi = -\mathrm{d}\Upsilon \wedge \Phi\,, \tag{D.1.4}$$

and therefore

$$K \lrcorner \Psi_1 = \Upsilon \mathrm{d}(\Upsilon F + \Phi) + \mathrm{d}\Upsilon \wedge \Phi = \frac{1}{2}\mathrm{d}(\Upsilon^2 F + 2\Upsilon\,\Phi) \equiv \mathrm{d}\Psi_2\,. \tag{D.1.5}$$

Continuing in this manner we find that the full tower of equivariantly closed polyforms for the Page fluxes is given by

$$\Psi^F = \sum_{m=0} \Psi_m \,, \tag{D.1.6}$$

where

$$\Psi_m = \frac{(-1)^m}{m!}\Big(\Upsilon^m F + m\Upsilon^{m-1}\Phi\Big) \,. \tag{D.1.7}$$

We have therefore constructed a polyform of equivariantly closed polyforms for all of the Page fluxes in both type II theories. For massive type IIA these read

$$\begin{aligned}
\Psi^{F_0} &= F_0 \,, \\
\Psi^{F_2} &= F_2 - (\Upsilon F_0 + \Phi_0) \,, \\
\Psi^{F_4} &= F_4 - (\Upsilon F_2 + \Phi_2) + \frac{1}{2}(\Upsilon^2 F_0 + 2\Upsilon\Phi_0) \,, \\
\Psi^{F_6} &= F_6 - (\Upsilon F_4 + \Phi_4) + \frac{1}{2}(\Upsilon^2 F_2 + 2\Upsilon\Phi_2) - \frac{1}{3!}(\Upsilon^3 F_0 + 3\Upsilon^2\Phi_0) \,, \\
\Psi^{F_8} &= F_8 - (\Upsilon F_6 + \Phi_6) + \frac{1}{2}(\Upsilon^2 F_4 + 2\Upsilon\Phi_4) - \frac{1}{3!}(\Upsilon^3 F_2 + 3\Upsilon^2\Phi_2) + \frac{1}{4!}(\Upsilon^4 F_0 + 4\Upsilon^3\Phi_0) \,, \\
\Psi^{F_{10}} &= F_{10} - (\Upsilon F_8 + \Phi_8) + \frac{1}{2}(\Upsilon^2 F_6 + 2\Upsilon\Phi_6) - \frac{1}{3!}(\Upsilon^3 F_4 + 3\Upsilon^2\Phi_4) + \frac{1}{4!}(\Upsilon^4 F_2 + 4\Upsilon^3\Phi_2) \\
&\qquad - \frac{1}{5!}(\Upsilon^5 F_0 + 5\Upsilon^4\Phi_0) \,.
\end{aligned} \tag{D.1.8}$$

For type IIB one finds that the equivariantly closed polyforms for each of the fluxes are

$$\begin{aligned}
\Psi^{F_1} &= F_1 \,, \\
\Psi^{F_3} &= F_3 - (\Upsilon F_1 + \Phi_1) \,, \\
\Psi^{F_5} &= F_5 - (\Upsilon F_3 + \Phi_3) + \frac{1}{2}(\Upsilon^2 F_1 + 2\Upsilon\Phi_1) \,, \\
\Psi^{F_7} &= F_7 - (\Upsilon F_5 + \Phi_5) + \frac{1}{2}(\Upsilon^2 F_3 + 2\Upsilon\Phi_3) - \frac{1}{3!}(\Upsilon^3 F_1 + 3\Upsilon^2\Phi_1) \,, \\
\Psi^{F_9} &= F_9 - (\Upsilon F_7 + \Phi_7) + \frac{1}{2}(\Upsilon^2 F_5 + 2\Upsilon\Phi_5) - \frac{1}{3!}(\Upsilon^3 F_3 + 3\Upsilon^2\Phi_3) + \frac{1}{4!}(\Upsilon^4 F_1 + 4\Upsilon^3\Phi_1) \,.
\end{aligned} \tag{D.1.9}$$

D.2 Putting the action partially on-shell

Starting from the pseudo action (4.1.24), the equation of motion for the dilaton is

$$R - \frac{1}{2}|H|^2 = 4|\mathrm{d}\phi|^2 - 4\nabla^2\phi\,, \tag{D.2.1}$$

and we may go partially on-shell by eliminating the Ricci scalar. The partially on-shell action becomes

$$\begin{aligned} S &= -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\sum_k F_k^{\mathrm{RR}}\wedge\star F_k^{\mathrm{RR}} + \frac{1}{(2\pi)^7\ell_s^8}\int_{M_{10}}\mathrm{d}^{10}x\sqrt{-g}\nabla^2(\mathrm{e}^{-2\phi}) \\ &= -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\mp\langle F^{\mathrm{RR}}, F^{\mathrm{RR}}\rangle + \frac{1}{(2\pi)^7\ell_s^8}\int_{\partial M_{10}}\mathrm{d}^9x\sqrt{-h}\,n^\mu\nabla_\mu(\mathrm{e}^{-2\phi})\,, \end{aligned} \tag{D.2.2}$$

where n^μ is the normal to the boundary, h the induced metric on the boundary and $\langle\cdot\,,\cdot\rangle$ denotes the Mukai pairing. To write the Mukai pairing we have used the self-duality condition (4.1.2) with $\star\lambda(F) = \mp\lambda(\star F)$ in IIA/IIB respectively. This should then be interpreted as a formal rewriting; we will come back to this momentarily. Observe that the full NS-NS sector is a total derivative term for the partially on-shell action and in the following we will neglect such boundary terms.[1]

We still have a little work to relate this to our polyforms in the previous section. Fortunately for any two-form ω the Mukai pairing satisfies

$$\langle \mathrm{e}^\omega G, \mathrm{e}^\omega\widetilde{G}\rangle = \langle G, \widetilde{G}\rangle\,, \tag{D.2.3}$$

and therefore we may replace the Maxwell fluxes F^{RR} with the Page fluxes F in the above. Our final result, ignoring boundary contributions, is then

$$S = -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\mp\langle F, F\rangle\,, \tag{D.2.4}$$

where the $-$ sign is for type IIA and the $+$ sign for type IIB. In this form the partially on-

[1] These boundary terms may be important in other setups. Their analysis is beyond the scope of this paper, but we intend to come back to these contributions in future work.

shell action can be equivariantly completed using our previous results, provided we interpret (D.2.4) suitably using a choice of electric/magnetic polarization so as to obtain a non-zero result. This is discussed in the main text.

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