MAGNETIC MONOPOLES AND HYPERBOLIC THREE-MANIFOLDS

by

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ABSTRACT

Let $M = H^3/\Gamma$ be a complete , non-compact , oriented , geometrically finite hyperbolic 3-manifold without cusps. By constructing a conformal compactification of $M \times S^1$ we functorially associate to M an oriented , conformally flat , compact 4-manifold X (without boundary) with an S^1-action. X determines M as a hyperbolic manifold.

Using our functor and the differential geometry of conformally flat 4-manifolds we prove that any Γ as above with a limit set of Hausdorff dimension ≤ 1 is Schottky , Fuchsian or extended Fuchsian. Furthermore , the Hodge theory for $H^2(X;\mathbb{R})$ carries over to $H^1(M,\delta M;\mathbb{R})$ and $H^2(M;\mathbb{R})$ which correspond to the spaces of harmonic L^2-forms of degree 1 and 2 on M. Comparison of lattices through the Hodge star gives an invariant $h(M) \in GL(H^2(M;\mathbb{R}))/GL(H^2(M;\mathbb{Z}))$ of the hyperbolic structure.

Secondly we pay attention to magnetic monopoles on M which correspond to S^1-invariant solutions of the anti-self-duality equations on X. The basic result is that we associate to M an infinite collection of moduli spaces of monopoles , labelled by boundary conditions. We prove that the moduli spaces are not empty (under reasonable conditions) , compute their dimension , prove orientability , the existence of a compactification and smoothness for generic S^1-invariant conformal structures on X. For these results one doesn't need a hyperbolic structure on M , the existence of a conformal compactification X suffices.

A twistor description for monopoles on a hyperbolic M can be given through the twistor space of X , and monopoles turn out to correspond to invariant holomorphic bundles on twistor space. We analyse these bundles. Explicit formulas for monopoles can be found on handlebodies M , and for M = surface $\times\ \mathbb{R}$ we describe the moduli spaces in some detail.

Acknowledgements

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CHAPTER I

INTRODUCTION

The basic objects of study in this thesis are complete , 3-dimensional Riemannian manifolds M. We shall assume that M is the interior of a compact manifold $\bar{M}$ with non-empty boundary. It is a corollary of Thurston's work that a large class of such 3-manifolds M actually can be given a complete hyperbolic metric.

The first topic of this thesis is to study the global differential geometry of such hyperbolic M. Secondly we shall study gauge fields on 'arbitrary' 3-manifolds M. More specifically our interest will lie in magnetic monopoles , which are the minima of the Yang-Mills-Higgs functional. If M is hyperbolic one may expect more detailed information about the gauge fields through the use of geometric methods , such as twistors etc. , and we pay attention to this as well.

In this introduction we will recall some background material , explain our results in a non-technical fashion and finally discuss the prospects of extending this work. The specific methods leading to the results have not been given here , they are exposed in the introductions to the various chapters.

§1 Global differential geometry of hyperbolic 3-manifolds

If M is a 3-manifold as above and M admits a hyperbolic structure then this means that M equals (as a Riemannian manifold) hyperbolic 3-space modulo a discrete group of isometries , i.e. $M = H^3/\Gamma$. If M has a boundary surface of genus ≥2 then the group Γ is not uniquely determined by M as a topological manifold , and continuous deformations of the Riemannian structure can be realised by deforming the homomorphism $\Gamma \to \mathrm{Isom}(H^3) \cong PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm 1\}$. Thus the situation is much the same as that for Riemann surfaces which admit families of hyperbolic structures (or equivalently conformal structures up to equivalence) too. This analogy goes further : the hyperbolic structure on M induces a conformal structure on the boundary surfaces S_j , $j = 1,\dots,N$, and Thurston's version of Mostow's theorem says that the hyperbolic structures on M are in one-one correspondence with all conformal structures on the S_j. This development of relating the hyperbolic geometry in dimension 3 to conformal geometry in dimension 2 started with Poincaré in 1883 , who was the first to realise that groups acting on the 2-sphere by conformal transformations also act on hyperbolic 3-space by isometries.

Here we shall pursue a similar line of thought. The group Γ which acts on H^3 , also acts by conformal transformations on the 4-sphere S^4. Just like the action of Γ on S^2 one first has to omit a set Λ from S^4 before taking the quotient. The resulting 4-manifold $X = (S^4-\Lambda)/\Gamma$ turns out to be a smooth , compact manifold without boundary , with an S^1-action (some assumptions on Γ are necessary). Alternatively one can view X as a conformal compactification of $M \times S^1$. This then provides a link between hyperbolic geometry in dimension 3 and S^1-equivariant , conformal geometry in dimension 4.

The compactness of X allows one to make quick progress with some problems which are hard on M. For example the study of elliptic p.d.e. on compact manifolds is standard , whereas on non-compact ones , such as M , it is not. Thus we develop Hodge theory for M through that of X. As a corollary we can define an invariant of the geometric structure of M.

Also the S^1-equivariant self-duality equations on X (non-linear elliptic p.d.e.) have a natural interpretation on M , namely they are the Bogomol'nyi equations for magnetic monopoles on M. Thus magnetic monopoles on M are the S^1-invariant instantons on X. More will be said about monopoles in §2.

Conformal geometry in dimension 4 has had much inspiration from the Penrose twistor program. This allows for descriptions of solutions of the self-duality equations in terms of holomorphic bundles on a twistor space Z_X which is a complex 3-dimensional manifold. The S^1-action on X induces a $\mathbb{C}^* = \mathbb{C}-\{0\}$-action on Z_X which is essentially the geodesic flow on M. The geodesic flow is exactly the tool used to prove Mostow's theorem mentioned above , and in due course one may hope to come up with a different approach towards this , using the twistor spaces. The magnetic monopoles correspond to certain $\mathbb{C}^*$-invariant holomorphic bundles on Z_X and we argue that these can be studied on the so called mini-twistor space $Z_X/\mathbb{C}^*$ as far as this makes sense. This would give the same description for monopoles on H^3/Γ which has previously been made for monopoles on $\mathbb{R}^3$ and H^3. However , we can neither prove that the bundles on the mini-twistor space determine the monopole nor can we decide exactly which conditions should be put on bundles on the mini-twistor space , to ensure that they arise from a monopole.

Just as Mostow's theorem assures that the geometric structure of M is encoded in the boundary , we have proved elsewhere , using twistors , that monopoles on H^3 are determined by their restriction to the boundary. Such a theorem should also be true for manifolds of the form H^3/Γ and maybe the Mostow theorem is in some sense a specialization of these twistor methods for monopoles : more precisely one may try to prove the Mostow theorem by treating the flat $PSL(2,\mathbb{C})$ bundle over M which encodes the monodromy of the hyperbolic structure , as a PSL(2,ℂ)-monopole bundle.

Schoen and Yau have related well known invariants of the group Γ to the existence of metrics on X of non-negative constant scalar curvature , in the given conformal class. Now conformally flat 4-manifolds with a metric of non-negative constant scalar curvature have been studied in detail by LeBrun. Combining the work of Schoen and Yau with that of LeBrun we end up with a classification of those groups Γ for which the so-called Hausdorff dimension of Λ is ≤ 1.

Finally we have made first steps in using the theory of automorphic forms for Γ to construct explicit solutions of the self-duality equations. It is conceivable that this can be developed further.

The situation of having distinguished geometrical structures on 3-manifolds lends itself to further comparison with algebraic geometry. All these problems have close relatives in algebraic geometry : Hodge theory is similar to sheaf cohomology , the space of hyperbolic structures on 3-manifolds compares well to moduli spaces of algebraic varieties , and , finally , the theory of monopoles is analogous to that of holomorphic vector bundles on algebraic varieties. This then concludes our introduction to the contents of chapter 2.

§2 Monopoles on 3-manifolds

As we explained in §1 magnetic monopoles on a suitable hyperbolic 3-manifold M can be viewed as S^1-invariant instantons , i.e. S^1-invariant solutions of the (anti-) self-duality equations on a compact 4-manifold X. Rather than just study the conformally flat manifolds X arising from the hyperbolic 3-manifolds we will consider (in chapter 3) any S^1-invariant conformal structure on such manifolds and study the

S^1-invariant instantons. Calling on the fundamental work of Donaldson and others concerning instantons we can obtain existence theorems , deformation theory and convergence properties for monopoles in a rather straightforward way. The only drawback of our method seems to be that the asymptotic value of the length of the Higgs field of the monopole , one of the asymptotic boundary conditions , is forced to be an integer. This contrasts with the 3-dimensional approach where this number can vary continuously.

The relation of our 4-dimensional approach to the conventional 3-dimensional approach can be seen most clearly by noting that the fixed surfaces of the S^1-action in X are in one-one correspondence with the boundary surfaces of M. For example S^1-equivariant index computations become computations on the fixed surfaces through the Lefschetz theorems , much in the same way as Callias developed index theorems on $\mathbb{R}^n$ playing back computations to the boundary sphere at infinity. Another example arises in the study of limits : any sequence of instantons on X has a subsequence which converges away from finitely many points in X. For S^1-invariant instantons these points obviously have to lie in the fixed point set of the S^1-action. The interpretation is easy : the only way in which a sequence of monopoles in M can fail to converge is because some lumps move off to the boundary surface at infinity.

Perhaps the most promising possibilities lie of extending our work lie in this chapter. Donaldson has used homological properties of the moduli spaces of instantons to define invariants for the smooth 4-manifold on which the instantons are defined. Similarly one can use the moduli spaces of monopoles to define topological invariants of the 3-manifold on which these are defined. A special case of this gives invariants for oriented links , some of which will depend on the orientation of the components of the link. The details and further properties of these will be worked out elsewhere.

§3 Monopoles on Surface × ℝ

One of the most remarkable facts concerning monopoles on H^3 and $\mathbb{R}^3$ is that the moduli spaces of monopoles have been determined completely. We found that the methods employed also apply to H^3/Fuchsian group (a Fuchsian group is an 'easy' discrete group of isometries) , and in chapter 4 we explain the structure of the moduli spaces. In this case our conformal 4-manifold X is $S \times S^2$, where S is a compact Riemann surface with a hyperbolic metric ; here the circle acts on S^2 by earth rotation. X being an algebraic surface , a fundamental theorem of Donaldson implies that monopoles are essentially the S^1-invariant stable holomorphic bundles on X.

The moduli spaces of such stable bundles can be understood in detail , and a picture emerges which shows that our monopole moduli spaces - in their full global glory - are quite complicated objects. Monopoles on H^3 are in one-one correspondence with rational functions on S^2 , which relate the fields at infinity to the interior. On $S \times \mathbb{R}$ we get rational functions twice , once for each end. In addition we see extra parameters arise from the non-trivial fundamental group. The rational functions are not unconstrained which they were in the H^3 case.

That constraints exist appears most drastically in some non-existence theorems for monopoles. The boundary conditions assign to each end of the 3-manifold a non-negative number called the mass of the monopole. If these masses are not equal , then it appears that the lower mass end exerts a gravitational attraction on monopoles at the high mass end. Monopoles exist only if the magnetic charge (another boundary condition) at the high mass end , is bigger than the mass difference. Physically one can learn from this that non-trivial topology in the universe can have quite complicated effects on existence of monopoles.

CHAPTER II

A KALUZA-KLEIN APPROACH TO HYPERBOLIC THREE-MANIFOLDS

§1 Introduction

In the recent past Thurston has caused a revolution in three-dimensional topology with the creed : 'Every 3-manifold is essentially geometric'. In particular a large class of 3-manifolds with boundary can be supplied with a hyperbolic structure. This situation is much the same as that for two-dimensional surfaces , which can also be given hyperbolic structures.

Another even more recent revolution in mathematics came about when mathematicians started paying close attention to methods employed in theoretical physics. In particular S.K. Donaldson found deep applications of Yang-Mills theory to four-dimensional topology.

On three-dimensional manifolds there exists a set of partial differential equations , the Bogomol'nyi equation , which describe magnetic monopoles in M. This equation is closely related to the Yang-Mills equation in dimension four , and can only be formulated in presence of a Riemannian metric and orientation on the 3-manifold. In the last three sections of this paper we shall study some aspects of this equation on hyperbolic 3-manifolds.

Kaluza-Klein theory , another favorite of theoretical physics , leads to a natural way to study these equations , thereby circumventing a large amount of analysis associated with more direct approaches. Basically Kaluza-Klein theory amounts to studying space through the geometry of a fibre bundle over space. In our case this fibre bundle over a hyperbolic

3-manifold is simply the product of the manifold with the circle. The analytical problems alluded to above are largely due to the fact that a 3-manifold with boundary , supplied with a hyperbolic metric , is very non-compact as a metric space. Although this is not changed by taking the product with a circle , it turns out that this 4-manifold has a natural conformal compactification (another favorite ingredient in physical theories).

The upshot is (§2) that we canonically associate a conformally flat , compact 4-manifold (without boundary) with a circle action , to a hyperbolic 3-manifold (provided some conditions hold see §2). This provides a link between conformal geometry in dimension 4 , and hyperbolic geometry in dimesion 3. This is very similar to Poincaré's observation in 1883 that hyperbolic geometry in dimension 3 is related to conformal geometry in dimension 2 , by considering the boundary surfaces of a hyperbolic 3-manifold.

In going over to the 4-manifold , no information is lost. This allows one to deduce precise facts concerning the 3-manifold from known facts about conformally flat 4-manifolds ; therefore , before we start studying the Bogomol'nyi equation , we study some global differential geometric questions about hyperbolic 3-manifolds in the light of the conformal compactifications.

In particular we can exploit recent work of Schoen and Yau to classify a family of hyperbolic 3-manifolds (§3). On the analytical side , knowledge about conformally invariant differential operators in dimension 4 can be exploited to get a Hodge theory for hyperbolic 3-manifolds (§4). Additionally this gives an invariant of the hyperbolic structure , of a type familiar from algebraic geometry.

After these digressions we start studying magnetic monopoles on the hyperbolic 3-manifolds by relating them to S^1-invariant instantons on the 4-manifolds. Relevant definitions and background can be found in §5.

The twistor spaces associated to the conformally flat 4-manifolds are studied in §6. Not only do these provide a way to study monopoles , they also encode a wealth of geometrical information belonging to the 3-manifold such as the entire geodesic flow. Finally in §7 , we use the twistor theory to construct some explicit formulas for monopoles on handlebodies. Here we naturally encounter the Eisenstein series associated to the hyperbolic 3-manifold.

We end this introduction by briefly indicating what kind of future developments can be expected. The compact 4-manifolds should allow for easy study of many natural differential operators on the 3-manifold ; in §4 it is indicated how. The twistors spaces may provide a natural environment to study theorems about the 3-manifold which rely on properties of the geodesic flow. In particular one may try to prove Mostow's theorem (and Thurston's generalisation of it) along the lines outlined in §6. Finally it is conceivable that the real line bundles , which are exploited in §7 , may give rise to a bigger class of explicit formulas for monopoles.

§2 Conformal compactifications and their topology

Let $\bar{M}$ be an oriented, irreducible, atoroidal, compact, three-dimensional manifold with non-empty boundary $\delta\bar{M}$. *Atoroidal* means that every map $T^2 \to \bar{M}$ has a kernel on the level of fundamental groups. For simplicity we shall avoid cusps and thus we assume that :

2.1 either no component of $\delta\bar{M}$ is of genus 1 or $\bar{M} = \bar{D}^2 \times S^1$.

Thurston's uniformization theorem (see Morgan [28]) asserts that there is a complete, geometrically finite, hyperbolic structure on $M = \bar{M}\setminus\delta\bar{M}$. This means that M can be realised as follows (see Bers [7], Maskit [26], Morgan [28], Beardon [6] for background).

Recall that $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm 1\}$ is the isometry group of hyperbolic 3-space H^3, and that the right action of an isometry on $H^3 = SU(2)\backslash SL(2,\mathbb{C})$ extends over the boundary $S^2 \cong \delta H^3$ as an action by a fractional linear transformation of S^2. A *Kleinian group* Γ without cusps is a discrete subgroup of $PSL(2,\mathbb{C})$ all elements of which are loxodromic (i.e. have exactly two fixed points in $\bar{H}^3 = H^3 \cup S^2$), and which acts freely and properly on a non-empty open set $\Omega \subset S^2$ (Felix Klein, the man of the dicrete groups, and Oscar Klein, of the Kaluza-Klein theories mentioned in the introduction, are not the same). Proper means that the map $\Omega \times \Gamma \to \Omega \times \Omega : (x,\gamma) \to (x\gamma, x)$ is proper. Proper actions are well behaved, and a proper free action has a smooth quotient, see Gleason [13].

There is a preferred region $\Omega(\Gamma)$, in which Γ acts properly. Define the *limit set* $\Lambda(\Gamma)$ of the group Γ to be the set of all $y \in S^2$

such that there is a sequence of different elements $\gamma_j \in \Gamma$ and an $x \in S^2$ with $\gamma_j \cdot x \to y$. The ***region of discontinuity*** $\Omega(\Gamma)$ is the complement $S^2 - \Lambda(\Gamma)$, and Γ acts properly on $\Omega(\Gamma)$. The limit set may be quite wild and has Hausdorff dimension $\dim_H \Lambda(\Gamma) \in [0,2]$. If no confusion is possible we shall denote $\Omega(\Gamma)$ by Ω and $\Lambda(\Gamma)$ by Λ.

The number of components of Ω is 1 , 2 or infinite , and Ω/Γ is a collection of N Riemann surfaces $S_1, \ldots, S_N$, where N is the number of Γ-orbits in the set of components of Ω (N can be infinite). It is well known that the Γ-action on H^3 is proper and that it extends to a proper action on $\bar{H}^3 - \Lambda$; therefore $(\bar{H}^3 - \Lambda)/\Gamma$ is a smooth manifold with boundary $\Omega/\Gamma = \cup_j S_j$.

In order to ensure that $(\bar{H}^3 - \Lambda)/\Gamma$ is compact we introduce another notion. The group Γ is said to be ***geometrically finite*** iff there is a finitely sided fundamental polyhedron (Maskit [26]) for the Γ-action on H^3. In this case the quotient $M = H^3/\Gamma$ is the interior of a compact , smooth manifold $\bar{M} = (\bar{H}^3 - \Lambda)/\Gamma$ which has boundary $\delta M = \Omega/\Gamma$, now equal to a finite collection of compact Riemann surfaces without boundary. In this case the hyperbolic structure on M is said to be geometrically finite. If $\Gamma = \{e\}$ we have N=1 , $S_1 = S^2$, and if Γ is cyclic then N=1 , $S_1 = T^2$; in both of these cases Ω is connected. In all other cases every S_j is a surface of genus ≥ 2.

The conjugacy class of Γ in $PSL(2,\mathbb{C})$ is not uniquely determined by M as a smooth manifold ; in fact continuous deformations of the complete hyperbolic structure on M can be realized by deforming the embedding $\Gamma \to PSL(2,\mathbb{C})$. Thus the situation is much the same as that for Riemann surfaces , which also admit families of hyperbolic structure (or equivalently complex structures).

As a metric space , M endowed with such a hyperbolic structure is highly non-compact , and the boundary surfaces lie at infinity , i.e. they are the celestial surfaces in M. Following the physical idea of a Kaluza-Klein theory we shall study the fibre bundle $M \times S^1$ over M instead of M itself. Another popular notion in physics is that of a *conformal compactification* : $M \times S^1$ has a natural conformal compactification X without boundary (or X_Γ if we want to indicate the dependence on Γ) , i.e. there is an injective conformal immersion $M \times S^1 \to X$ onto a dense subset. To get X we spin $\bar{M}$ around $\delta\bar{M}$, see figure 2.1 , i.e. X is $\bar{M} \times S^1$ with the circles over δM identified to a point. This gives a compact 4-manifold X with an S^1-action. The action is free away from the fixed point set , which is isomorphic to the boundary $\delta\bar{M} = \cup_{j=1,N} S_j$. The normal bundles of the S_j are trivial and of S^1-weight 1. For example take $M \cong S \times \mathbb{R}$ with S a surface. Then X is the compactification of $S \times \mathbb{R} \times S^1 \cong S \times \mathbb{C}^*$, that is $X \cong S \times S^2$, where S^1 acts on S^2 by earth rotation and has two fixed surfaces $S \times \{0,\infty\}$ in X.

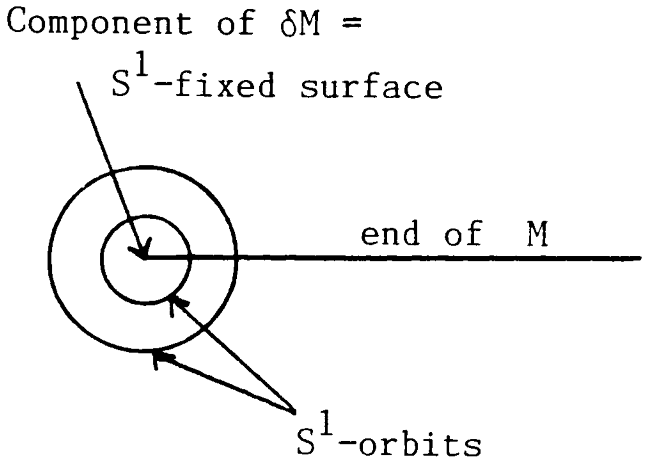

Figure 2.1

In order to relate the hyperbolic structure on M to a conformal structure on X we proceed more formally. Recall that $H^3 = \{(x,y,t) \in \mathbb{R}^3 \ ; \ t > 0\}$ with metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. It follows that :

$$2.2 \qquad i : H^3 \times S^1 \to (\mathbb{R}^2 \oplus \mathbb{R}^2) - (\mathbb{R}^2 \oplus 0) \cong \mathbb{R}^4 - \mathbb{R}^2 \cong S^4 - S^2 : \\ ((x,y,t),\vartheta) \to (x,y,t\cos\vartheta,t\sin\vartheta)$$

is an orientation preserving , conformal diffeomorphism. The map i intertwines the S^1-action on $H^3 \times S^1$ with rotations in the second summand of $\mathbb{R}^2 \oplus \mathbb{R}^2$. The S^1-action extends to S^4 with fixed point set $S^2 = (\mathbb{R}^2 \oplus 0) \cup \{\infty\} \subset S^4$. This fixed point set corresponds to $\delta H^3 \times S^1$ under :

2.3 $$i' : \bar{H}^3 \times S^1 \to S^4 ,$$

the continuous extension of i. To get further we shall show that the compactification S^4 of $H^3 \times S^1$ is natural enough to transfer group actions from H^3 to S^4. The maps i and i' are equivariant with respect to the group $S^1 \times PSL(2,\mathbb{C})$, which will act on the right on S^4 by *conformal transformations.* To see this , recall that the $PSL(2,\mathbb{C})$-action on S^4 , which is the quaternionic projective line $\mathbb{H}P^1 = \mathbb{H}^* \backslash (\mathbb{H}^2 - \{0\})$ (i.e. divide out the left action of multiplication by invertible quaternions) , is by fractional linear transformations :

2.4 $$([x,y] , \begin{bmatrix} a & c \\ b & d \end{bmatrix}) \to [xa + yb , xc + yd]$$

As a result a geometrically finite Kleinian group Γ acts on S^4. The limit set Λ' of the Γ-action on S^4 equals $i'(\Lambda \times S^1)$, so it is contained in the S^1-fixed point set $S^2 \subset S^4$. Clearly Λ' is isomorphic to Λ , and we shall simply identify Λ and Λ'. The restriction :

$$i' : (\bar{H}^3 - \Lambda) \times S^1 \to S^4 - \Lambda$$

is proper , equivariant and surjective. This implies immedeately that the Γ-action on $S^4 - \Lambda$ is proper. Since Γ is geometrically finite the quotient $X = (S^4 - \Lambda)/\Gamma$ is compact and without boundary. Finally , the fact that the Γ-action is free ensures that X is smooth and inherits a smooth S^1-action.

The S^1-action is free away from the fixed surfaces S_j, which correspond as conformal surfaces to $\Omega/\Gamma = (\delta H^3-\Lambda)/\Gamma \cong i'((\delta H^3-\Lambda) \times S^1)/\Gamma$. It is useful to realise that i and i' induce maps $i: M\times S^1 \to X$ and $i': \bar{M}\times S^1 \to X$. Summarizing we have proved:

Theorem 2.1 : Let $\bar{M}$ be an oriented, geometrically finite, complete hyperbolic 3-manifold with non-empty boundary $\delta\bar{M} = \cup S_j$ satisfying 2.1. Then $M \times S^1$ has an oriented, smooth conformal compactification X (without boundary) upon which S^1 acts. X is conformally flat, and the S^1-action is free away from its fixed surfaces S_j $(j=1,\dots,N)$ which correspond as conformal surfaces to the boundary surfaces of $\bar{M}$. The normal bundles N_j of S_j in X are topologically trivial and of S^1-weight 1. The hyperbolic structure on M can be reconstructed from X by giving $X-(\cup S_j)$ that metric in the conformal class for which the S^1-orbits have length 2π. Then M is the Riemannian quotient of $X-(\cup_j S_j)$ by S^1.∎

The existence of a conformal compactification is not automatic. It is easy to see that $\mathbb{R}^3 \times S^1$ cannot be compactified by adding an S^2 at infinity.

The topology of X is easily described :

Proposition 2.2 : a) $\pi_1(X,m) \cong \pi_1(\bar{M},m)$ for $m \in S_1$ (S_1 a fixed or boundary surface).

b) There are natural isomorphisms $H_2(\bar{M},\delta\bar{M};\mathbb{Z}) \to H_3(X;\mathbb{Z})$ and

$H_2(\bar{M};\mathbb{Z}) \oplus H_1(\bar{M},\delta\bar{M};\mathbb{Z}) \to H_2(X;\mathbb{Z})$

The two summands of $H_2(X;\mathbb{Z})$ (modulo torsion) are isotropic and dual to each other under the intersection form Q on $H_2(X;\mathbb{Z})$; consequently the signature $\sigma(X) = 0$, and $Q = n$ times $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with $n = \mathrm{rk}\, H_2(\bar{M};\mathbb{Z})$.

c) $\chi(X) = \Sigma_j\, \chi(S_j)$ with χ denoting the Euler characteristic.

d) Spin structures on X exist and the double cover of S^1 acts naturally and effectively on any spin structure.

Proof: a) Of course this is what one expects to be true : $\pi_1(\bar{M} \times S^1, m) \cong \pi_1(\bar{M}, m) \times \mathbb{Z}$, but the $\mathbb{Z}$ factor is killed by shrinking the circles to a point. Formally , remark that a tubular neighbourhood of $\cup S_j$ looks like $(\cup S_j) \times D^2$, and apply the Seifert-van Kampen theorem.

b) Define $j:\bar{M} \to X$ by $j(m) = i'(m,1)$. Up to S^1-rotation j is defined uniquely by the conformal structure of X. This induces a homomorphism $j_*:H_2(\bar{M};\mathbb{Z}) \to H_2(X;\mathbb{Z})$. Next remark that if c is a singular chain in $C_j(\bar{M},\delta\bar{M};\mathbb{Z})$ then $i'_*(c \times S^1)$ is a chain in $C_{j+1}(X;\mathbb{Z})$, because the circle shrinking to a point enforces $\delta i'_*(c \times S^1) = 0$. Taking a careful look at the Mayer-Vietoris sequence applied to $(\cup S_j) \times D^2$ and $M \times S^1$ shows that this gives natural isomorphisms as indicated in the proposition. The properties of the intersection form Q follow from the the intersection pairing : $H_2(\bar{M};\mathbb{Z}) \times H_1(\bar{M},\delta\bar{M};\mathbb{Z}) \to \mathbb{Z}$.

c) This is easy , using either a and b , or equivariant Lefschetz formulas.

d) Every orientable 3-manifold admits a spin structure , see Stiefel [31]. Give S^1 the spin structure corresponding to the connected double cover , which extends to the disc in $\mathbb{R}^2$; therefore a product spin structure on $M \times S^1$ extends to X. Clearly every spin structure on X arises in this way. The double cover of S^1 is needed to define an action on the spin structure of the orbits in X. ∎

The spin bundle of H^3 is the $\mathrm{Spin}(3) \cong SU(2)$ bundle $SL(2,\mathbb{C}) \to H^3 = SU(2)\backslash SL(2,\mathbb{C})$; thus a spin structure on X is in fact nothing else but a lift of the homomorphism $r:\Gamma \to PSL(2,\mathbb{C})$ which defines M , to a homomorphism $r': \Gamma \to SL(2,\mathbb{C})$.

If N denotes the number of boundary components of $\bar{M}$ (as before) then it follows from the exact sequence of the pair $(M,\delta M)$ that:

2.5 $$\mathrm{rk}\ H_2(X;\mathbb{Z}) = 2\cdot \mathrm{rk}\ H_1(\bar{M},\delta\bar{M};\mathbb{Z}) \geq 2\cdot(N - 1)$$

Another useful fact to keep in in mind is:

2.6 $$\mathrm{rk}\{\ \ker(H_1(\delta\bar{M};\mathbb{Z}) \to H_1(\bar{M};\mathbb{Z}))\ \} = \tfrac{1}{2}\cdot \mathrm{rk}\ H_1(\delta\bar{M};\mathbb{Z})\ ,$$

which can easily be deduced from Alexander duality and the exact sequence of the pair $(\bar{M},\delta\bar{M})$.

Examples 2.3 : 1) If Γ is the cyclic group generated by $\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$ with $\lambda \in \mathbb{C}^*$ then the limit set equals $\{0,\infty\}$ in the coordinates on δH^3 supplied by the upper half space model. It is easy to see that $M = H^3/\Gamma = D^2 \times S^1$. To find X , it is easiest to divide out the Γ-action on $S^4-\Lambda = \mathbb{C}^2-\{0\}$ which is given by $(z_0,z_1) \to (\lambda^2 z_0, |\lambda|^2 z_1)$. As a result X is a Hopf surface diffeomorphic to $S^3 \times S^1$. The S^1-action is given by $(z_0,z_1) \to (z_0,e^{i\vartheta} z_1)$, so the fixed surface is the two-torus $\mathbb{C}^*/\langle\lambda^{2k}\rangle$.

2) If Γ is Fuchsian , i.e. $\Gamma \subset PSL(2,\mathbb{R})$, then H^2/Γ is a compact Riemann surface without boundary S of genus ≥ 2 with metric ds^2. The 3-manifold M is diffeomorphic to $\mathbb{R} \times S$ with metric given by $dl^2 + \cosh^2 l\cdot ds^2$. Clearly it follows that X must be diffeomorphic to $S^2 \times$

S. A little computation shows that X is even conformally equivalent to $S^2 \times S$. Thus X is conformally equivalent to the Kähler surface $\mathbb{C}P^1 \times S$.

From the point of view of Kleinian groups , we remark that Ω is the union of two round discs which are both invariant under Γ. The limit set is a smooth circle.

3) A Kleinian group Γ which is not itself Fuchsian , but which contains a Fuchsian subgoup Γ_0 of index two is said to be an extended Fuchsian group. For details see Maskit [27]. The limit sets $\Lambda(\Gamma_0)$ and $\Lambda(\Gamma)$ are equal , and any $\gamma \in \Gamma-\Gamma_0$ swaps the two components of Ω. Such an element γ also gives rise to a fixed-point-free , orientation reversing involution σ of S (compare 2) , and one deduces from this that $M \cong H^3/\Gamma$ is a nontrivial $\mathbb{R}$-bundle over S/σ. Remark that $\delta\bar{M} \cong S$.

A standard way to get more interesting 3-manifolds is through the Klein-Maskit combination theorems (Maskit [26] , Morgan [28]). We shall explain how the simplest of these relates to the 4-manifolds involved. Let Γ_0 and Γ_1 be geometrically finite groups without cusps and $M_j = H^3/\Gamma_j$. Every pair of points $x_j \in \delta\bar{M}_j$ has neighbourhoods K_j in M_j isometric to a hyperbolic half space i.e. to a component of H^3-H^2. The $H^2_j = \delta K_j-\delta\bar{M}_j$ are embedded in M_j and $\delta H^2_j \cap \bar{M}_j = H^2_j \cap \delta\bar{M}_j$ are circles which bound discs in $\delta\bar{M}_j$. Define $M = M_0 \# M_1$ to be $M_0\backslash K_0 \cup_\rho M_1\backslash K_1$, where ρ is an isometry $\delta K_0 \to \delta K_1$. The metric structure of $M = M_0 \# M_1$ depends on ρ , the choice of x_j and the choice of the half spaces K_j. M is called a ***boundary connected sum*** of M_0 and M_1.

The ***first combination theorem*** expresses the fact that $M = H^3/\Gamma$ with Γ a Kleinian group which is isomorphic to the free product of Γ_0 and

Γ_1 . In PSL(2,$\mathbb{C}$) the group Γ is generated by Γ_0 and $g\Gamma_1 g^{-1}$ for a suitable $g \in$ PSL(2,$\mathbb{C}$). It is easy to verify this.

Reverting to the 4-manifolds , we see that we are identifying , by S^1-equivariant conformal maps , balls B_j around the points x_j in the fixed surfaces. Thus X_Γ equals X_{Γ_0} # X_{Γ_1} with # now denoting a conformal connected sum. Summarizing we get :

Proposition 2.4 : If Γ is the Kleinian group corresponding to a boundary connected sum of H^3/Γ_0 and H^3/Γ_1 then Γ is a Kleinian group such that X_Γ is the S^1-equivariant conformal connected sum of X_{Γ_0} and X_{Γ_1} at points in the fixed surfaces.

Example 2.5 : A classical Schottky group Γ of genus g is a free product of g cyclic groups (compare example 2.3 (1)) , formed as in the gluing construction described before proposition 2.4. The 3-manifold M_Γ is a handlebody of genus g , and by proposition 2.4 , X_Γ equals the connected sum $(S^3 \times S^1)^{\#g}$. In fact if Γ is any geometrically finite free Kleinian group without cusps , then H^3/Γ is a handlebody ; this follows from standard results in 3-manifold topology (see Hempel [16]). We shall refer to such free groups as Schottky groups.

§3 Applications of differential geometry

In the previous section we constructed a compact, oriented, conformally flat 4-manifold X starting from a suitable (see §2) hyperbolic 3-manifold. By Schoen's solution of the Yamabe problem [29] there is a metric in the conformal class of X, for which the scalar curvature is a constant. The sign of this constant −, 0 or + is called the *type* of X. A lot is known about X of non-negative type, and we shall classify 3-manifolds M which give rise to X of non-negative type.

In a different direction Schoen and Yau [30] proved that if X is the quotient of $S^n - \Lambda$ by a discrete group of conformal transformations, then X is of non-negative type if and only if the *Hausdorff dimension* of Λ satisfies $\dim_H \Lambda \leq (n-2)/2$. Hence our classification is that of M for which $\dim_H \Lambda \leq 1$.

Up to now, the only Kleinian groups to have been classified are the so called *function groups*, those Kleinian groups which leave a component of $\Omega = S^2 - \Lambda$ invariant. This has been done by Maskit. A special case of this, which we shall use repeatedly below, occurs when Ω is connected. In this case the Kleinian group is Schottky (see example 2.5).

Theorem 3.1 : a) If the type of X is + then M is a handlebody equal to H^3/Γ with Γ a Schottky group.

b) If the type of X is 0 then one of the following holds :

1) M equals $\mathbb{R} \times S = H^3/\Gamma$ with Γ Fuchsian and S a compact surface.

2) M equals H^3/Γ with Γ extended Fuchsian (see example 2.3 (3))

3) M is a handlebody as in a)

Proof : a) By theorem 4.6 of Schoen and Yau [30] , $\dim_H \Lambda \leq 1$ implies $\pi_2(X) = 0$. Going over to the simply connected cover $S^4-\Lambda$, the Hurewicz theorem implies $\pi_2(X) = \pi_2(S^4-\Lambda) = H_2(S^4-\Lambda;\mathbb{Z}) = 0$. We shall prove below that this implies that $S^2-\Lambda$ is connected. By Maskit's classification theorems (see Maskit [26]) it follows that Γ is Schottky.

Lemma : If $H_2(S^4-\Lambda;\mathbb{Z}) = 0$ then $S^2-\Lambda$ is connected.

Proof : Assume that $S^2-\Lambda$ is not connected. Removing a point from $S^2-\Lambda$ obviously doesn't affect the statement as $H_2(S^4-\Lambda;\mathbb{Z}) = H_2(\mathbb{R}^4-\Lambda;\mathbb{Z})$. We shall take two steps in an induction procedure. Write $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ and assume $\Lambda \subset \mathbb{R}^2 \oplus 0$. Cover $\mathbb{R}^3-\Lambda$ with $U_\pm = \mathbb{R}^3-[\Lambda \times \{x\in\mathbb{R} ; \pm x \geq 0\}]$. Clearly U_+ and U_- are contractible and their union equals $\mathbb{R}^3-\Lambda$. From the Mayer-Vietoris sequence it follows that $H_1(\mathbb{R}^3-\Lambda;\mathbb{Z}) \neq 0$. Next assume $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ with $\Lambda \subset \mathbb{R}^2 \oplus 0$. Cover $\mathbb{R}^4-\Lambda$ by $V_\pm = \mathbb{R}^4-[\Lambda \times \{0\} \times \{x\in\mathbb{R} ; \pm x \geq 0\}]$ and repeat the argument to show that $H_2(X;\mathbb{Z}) \neq 0$. This proves the lemma.

b) First assume $H^2(X,\mathbb{R}) \neq 0$ and give X a metric of zero scalar curvature in the conformal class. From proposition 2.2 we see that the intersection form is indefinite , so there is a self-dual harmonic 2-form ω on X. A Weizenbock formula asserts that on 2-forms $(d + d^*)^2 = \nabla^*\nabla$ with ∇ the total covariant derivative. It follows that ω is covariantly constant , and a multiple of ω serves as Kähler form for an integrable complex structure on X : compare LeBrun [23]. LeBrun proceeds to classify these as (1) a K3 surface , (2) a four dimensional torus modulo a finite group and (3) a flat $\mathbb{C}P^1$ bundle with the local product metric , over a Riemann surface S which carries a metric of curvature -1. From proposition 2.2 we see that only (3) is possible in our case because (1) has the wrong Euler characteristic , and (2) with Euler characteristic 0 should have had $H_2(X;\mathbb{R}) = 0$.

The Kähler form of X is the unique self-dual harmonic 2-form on X. This is preserved by the conformal S^1-action , thus the action is a holomorphic action on X. As a result the vector field v induced by the S^1-action on X is holomorphic. The fibration $\pi : X \to \mathbb{C}P^1$ helps us further : we get a map $\pi_*: TX \to \pi^*TS$ and π_*v is a section of π^*TS. Such a section is constant on fibres , so it is a pull-back of a section of TS. The only holomorphic section of TS is 0 ; so v is a vertical vector field.

From theorem 2.1 we see that zeroes of v must be simple , hence two per fibre. One of these is a sink , the other a source of $i\cdot v$ so we get two sections $S \to X$. This proves that X is the projectivization of a direct sum of holomorphic line bundles , say $X = P(L_0 \oplus L_1)$. The next step is to remember that the circle bundle $X-[P(L_0) \cup P(L_1)]$ over H^3/Γ may have no monodromy. Infinitesimally this implies that $L_0 \otimes L_1^*$ is a trivial line bundle. So $X = S \times \mathbb{C}P^1$ and consequently Γ must be Fuchsian.

Next we come to the case $H^2(X,\mathbb{R}) = 0$. If $S^2-\Lambda$ has only one component then we can apply Maskits classification theorem as in a) , and conclude that Γ is Schottky ; therefore we shall concentrate on the case that Ω has at least two components.

If Ω_0 is one of these components then the stabilizer $\Gamma_0 \subset \Gamma$ of Ω_0 is a geometrically finite Kleinian group , and has Ω_0 as a component , see Marden [25] corollary 6.5 (it should be remarked that subgroups are not automatically geometrically finite). As $S^2-\Omega_0$ is Γ-invariant and has non-empty interior , it follows that H^3/Γ_0 must have at least two ends. By formula 2.5 and the the fact that $\dim_H \Lambda(\Gamma_0) \leq \dim_H \Lambda(\Gamma) \leq 1$, the above implies that Γ_0 is Fuchsian. Thus every component of Ω is a round disc.

Before we proceed let us briefly recall what effect a conformal rescaling of the metric has on the scalar curvature. If on the 4-manifold X one has $g_1 = u^2 \cdot g_0$ then $\frac{1}{6} \cdot u^3 \cdot R(g_1) = (d^* du + \frac{1}{6} R(g_0) u)$, where d^* is taken with respect to g_0. Since here metrics of zero scalar curvature are involved , this equation looses its nonlinear character. An immedeate consequence is that metrics of zero scalar curvature are unique up to constants multiples and hence S^1-invariant.

We have the hyperbolic covering $H^3/\Gamma_0 \rightarrow H^3/\Gamma$, and on the 4-manifolds corresponding to each of these there exists an S^1-invariant metric of zero scalar curvature. Denote these by g_0 and g , and denote the hyperbolic metric on the 3-manifolds by g_h. Then we have positive functions $u_0 : H^3/\Gamma_0 \rightarrow \mathbb{R}_{>0}$ and $u : H^3/\Gamma \rightarrow \mathbb{R}_{>0}$ such that $g_0 = u_0^2 \cdot g_h$ and $g = u^2 \cdot g_h$. By the above u_0 and u are in the kernel of $(d^*d - 1)$ on H^3/Γ_0 and H^3/Γ respectively (here d^* is w.r.t. the hyperbolic metric).

Results of Sullivan [32] imply that positive solutions of d^*d-1 on H^3/Γ_0 are unique (up to positive scalar factors) becasue $\dim_H \Lambda(\Gamma_0) = 1$. Therefore the pullback of u equals u_0 , and hence the cover $X_{\Gamma_0}-(S_1 \cup S_2) \rightarrow X_{\Gamma}-S_1$ is an isometry (S_i are the fixed surfaces). The map can readily be extended to an isometry $X_{\Gamma_0}-S_2 \rightarrow X_{\Gamma}$ and then extends to a double cover $X_{\Gamma_0} \rightarrow X_{\Gamma}$. It follows that Γ is extended Fuchsian as claimed.∎

Corollary 3.2 : Let Γ be a geometrically finite Kleinian group without cusps. If $\dim_H \Lambda(\Gamma) < 1$, then Γ is Schottky. If $\dim_H \Lambda(\Gamma) = 1$ then Γ is Schottky , Fuchsian or extended Fuchsian.

Proof : The only new fact used is that a set of Hausdorff dimension < 1 cannot disconnect the plane.∎

Remark: 1) We shall see in section 7 that $\dim_H \Lambda(\Gamma) < 1$ implies that the type of X is +. Naturally one wonders if the equality $\dim_H \Lambda(\Gamma) = (n-2)/2$ corresponds exactly to type zero in Schoen-Yau [30] , theorem 4.7.

2) Existence of Schottky groups with limit set of any dimension smaller than 2 has been proved , Thurston [33].

3) In Gromov - Lawson [14] the conclusion is drawn that for so-called classical Schottky groups Γ the manifold X_Γ admits a metric of positive constant scalar curvature.

4) R. Bowen [9] has proved that any quasifuchsian group with $\dim_H \Lambda = 1$ is Fuchsian. Of course this is a special case of theorem 3.1.

It will be interesting to see if further developments in the theory of compact , 4-dimensional , conformally flat manifolds are going to have similar applications to Kleinian groups. On the other hand it seems likely that a purely 3-dimensional proof of theorem 3.1 could be found as well. The crucial element seems to be to exploit the existence of a harmonic two form , in the way LeBrun did. LeBrun arrives at his flat $\mathbb{C}P^1$ bundle through a foliation argument which presumably can be mimicked in the 3-manifold.

§4 Hodge theory for hyperbolic 3-manifolds

Apart from the topological and geometrical applications which we discussed in §3 , our Kaluza-Klein approach also has some more analytical applications.

Recall that the Hodge-star $* : \Omega^n(Y) \to \Omega^n(Y)$, on a 2n-dimensional oriented Riemannian manifold Y , depends only on the conformal structure underlying the metric. This has two consequences :

1) The L^2-norm $\|\omega\|^2 = \int \omega \wedge *\omega$, of $\omega \in \Omega^n(Y)$, is conformally invariant.

2) The harmonic n-forms , i.e. the $\omega \in \Omega^n(Y)$ s.t. $d\omega = d{*\omega} = 0$, depend only on the conformal structure of Y.

Of course conformal rescaling lies at the heart of our construction in §2 , and we shall now show how the above applies to this situation. Let X be the conformal compactification of $M \times S^1$ as in §2. Harmonic 2-forms on X are automatically S^1-invariant because they are in one-one correspondence with the elements of $H^2(X;\mathbb{R}) = H^2(M;\mathbb{R}) \oplus H^1(M,\delta M;\mathbb{R})$. By restriction to the open subset $M \times S^1 \subset X$ and a conformal rescaling of the metric on $M \times S^1$, 2) above implies that we get S^1-invariant harmonic 2-forms on $M \times S^1$ with respect to the product metric.

An S^1-invariant form can be written as $\omega = \rho^*\alpha + \rho^*\beta \wedge d\theta$, with $\alpha \in \Omega^2(M)$, $\beta \in \Omega^1(M)$ and $\rho : M \times S^1 \to M$ the projection. A short computation shows that such S^1-invariant forms ω are harmonic iff α and β are harmonic on M. If ω is a harmonic 2-form on $M \times S^1$ arising from a form on X then it follows from proposition 2.2 that $\alpha \in \Omega^2(M)$ and $\beta \in \Omega^1(M)$ are harmonic representatives for the class $\omega \in$

$H^2(M;\mathbb{R}) \oplus H^1(M,\delta M;\mathbb{R})$. The forms α and β have finite L^2-norm on M by 1) above.

Conversely any S^1-invariant , harmonic 2-form $\tilde{\omega}$ on $M \times S^1$ with finite L^2-norm arises in this way. By 1) above one can always consider $\tilde{\omega}$ to be an L^2-form ω on X because $\cup S_j = X \setminus M \times S^1$ has measure 0. Applying the first order elliptic operator $d \oplus d^*$ to ω gives a distributional form in $L^2_{-1}(\Lambda^*(X))$ of distributional order ≤ 1 , which has support in the codimension 2 manifold $\cup_j S_j \subset X$. The following lemma shows that this implies that $(d \oplus d^*)\omega = 0$, which proves that ω is a smooth harmonic form on X , as we claimed.

Lemma 4.1 : Let μ be a distribution of order ≤ 1 in $L^2_{-1}(\mathbb{R}^n)$. If supp μ is contained in $\mathbb{R}^{n-2}$ then $\mu = 0$.

Proof : Without loss of generality assume that μ is compactly supported. The structure theorem for distributions carried by submanifolds (see Hörmander [20] theorem 2.3.5) asserts that μ is a finite linear combination of distributions ν of the form $\langle \nu, f\rangle = \langle \eta , \delta_{\mathbb{R}^{n-2}} \cdot D^k_{\vec{n}} \cdot f\rangle$, where η is a compactly supported distribution on $\mathbb{R}^{n-2}$, $\delta_{\mathbb{R}^{n-2}}$ is restriction to $\mathbb{R}^{n-2}$ and $D^k_{\vec{n}}$ is a k-th derivative $(0 \leq k \leq 1)$ in a direction $\vec{n}$ normal to $\mathbb{R}^{n-2}$.

The Fourier transform $\hat{\mu}(u,x,y)$ is a smooth function on $\mathbb{R}^{n-2} \oplus \mathbb{R} \oplus \mathbb{R}$ of the form $f_0(u) + f_1(u) \cdot x + f_2(u) \cdot y$. It is easy to see from this that the L^2_{-1} -norm cannot be finite , unless $\mu = 0$.∎

Denote by $\mathcal{H}^i(M)$ the vectorspace of harmonic (i.e. closed and coclosed) i-forms on M with finite L^2-norm. Summarizing the above we have proved :

Theorem 4.2 : The natural maps $\mathcal{H}^1(M) \to H^1(M,\delta M;\mathbb{R})$ and $\mathcal{H}^2(M) \to H^2(M;\mathbb{R})$ are isomorphisms.∎

At this point it is worth making a short digression to give a geometrical application of the above. The Hodge star of the hyperbolic 3-manifold M gives an isomorphism $*_3 : \mathcal{H}^1(M) \to \mathcal{H}^2(M)$. Both $\mathcal{H}^1(M)$ and $\mathcal{H}^2(M)$ contain an integral lattice of maximal rank coming from integral cohomology. These lattices do not generally coincide under $*_3$; in fact their intersection is empty unless the 4-manifold carries a self-dual harmonic form which represents an integral cohomology class. The relative position of the two lattices in $H^2(M;\mathbb{R})$ is described by :

$$h(M) \in GL(H^2(M;\mathbb{R}))/GL(H^2(M;\mathbb{Z})\otimes\mathbb{Z}) \quad , \tag{4.1}$$

which is an invariant of the hyperbolic structure of M. Similar invariants are very popular in algebraic algebraic geometry. There discrete lattices in a complex vector space give rise to invariants associated to the complex structure of manifolds.

We proceed to sketch how the above theory relating solutions of elliptic p.d.e. on M to invariant solutions on X generalizes as follows. Suppose $D : \Gamma(E) \to \Gamma(F)$ is a conformally invariant first order (possibly over-determined) elliptic operator acting on sections of the vector bundle E over X. This class of operators was studied in detail by Hitchin [17] , and comprises , among others , Dirac and twistor operators on X and the operator $d + d^*$ on 2-forms which we studied above. Again restriction of S^1-invariant solutions on X to $M \times S^1$ gives solutions to a closely related geometric p.d.e. on M.

Conversely we can start with a solution on M and require that it has a finite L^2-norm on $X\setminus(\cup S_j)$. In general this is not the same as having a finite L^2-norm on M , but it is the same as having a finite weighted L^2-norm on M. The weighting function is a suitable power of the function on M which conformally rescales the hyperbolic metric on M to a metric on X. Such a function is determined up to multiplication by functions $\phi: M \to \mathbb{R}_{>0}$ which are bounded above and below. The exact value of the power needed is an inhomogeneous linear function of the conformal weight of E. The extension over the fixed surfaces S_j goes now as in lemma 4.1. We shall not make use of this in the sequel and therefore leave the details to the reader.

§5 Monopoles and Instantons

Our goal is now to exploit the compactification X of $M \times S^1$ (see §2) to get monopoles on M from S^1-invariant instantons on X. We shall also relate the instanton number on X to various topological invariants of the monopoles on M. General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [21]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let P be a principal SU(2)-bundle over X , with $c_2(P) = k \geq 0$. Recall that X comes naturally with a conformal structure. This enables us to talk about instantons or anti-self-dual connections A on P. These are defined to be the solutions of the anti-self-duality equation :

5.1 $$F^A = -*_4\, F^A \qquad (*_4 \text{ the Hodge star on } \Lambda^2(X))$$

Here F^A is the curvature of A , a section of $\Lambda^2(X) \otimes \mathfrak{g}_P$ with $\mathfrak{g}_P = P \times_{Ad} su(2)$. The instantons are the absolute minima of the Yang-Mills functional :

5.2 $$YM(A) = (16\pi^2)^{-1} \int_X \langle F^A \wedge *F^A \rangle$$

where $\langle \alpha, \beta \rangle = -2 \cdot tr(\alpha\beta)$ in an invariant inner product on su(2). For an instanton $YM(A) = k$.

Next assume that the double cover $\tilde{S}^1$ of S^1 acts on P by bundle automorphisms , covering the action on X ; the double cover will be needed in order to include the spin bundles of X. Our interest will now lie in $\tilde{S}^1$-invariant instantons on P. To relate these to objects on M introduce the map :

$$j : M \to X : m \to i'(m,1) \qquad \text{(compare 2.2)} ,$$

which is a diffeomorphism onto its image. Let v be the vectorfield on P induced by the $\tilde{S}^1$-action. If we interprete an $\tilde{S}^1$-invariant connection A as a 1-form on P , then define the Higgs-field Φ to be the su(2)-valued function $j^*A(\frac{1}{2}v)$ on j^*P. It is easy to see that Φ is a section of j^*g_P. Further $A_3 = j^*A$ defines a connection on the bundle j^*P over M. A little computation shows that the $\tilde{S}^1$-invariant connection A is anti-self-dual iff (A_3,Φ) satisfy the so called Bogomol'nyi equation on M:

5.3 $$d^{A_3}\Phi = -*_3F^{A_3} \quad .$$

As 5.3 is the standard equation describing magnetic monopoles on three dimensional manifolds , this leads to the definition:

Definition 5.1 : A monopole on P is an $\tilde{S}^1$-invariant instanton on P.

Normally one defines a monopole by imposing certain asymptotic conditions rather than requiring it to extend over a compact manifold. It seems however that this amounts to the same. We shall comment a bit more on this matter below.

If GA(P) denotes the group of $\tilde{S}^1$-invariant gauge transformations on P , then GA(P) leaves the set of monopoles invariant. Just as for instantons one can therefore define a monopole moduli space , equal to :

5.4 {solutions of 5.3}/GA(P)

In the next chapter it will be shown that under some assumptions these moduli spaces are non-empty finite dimensional manifolds.

We shall now return to our $\tilde{S}^1$-equivariant bundle P and relate topological invariants of the action to asymptotic invariants of (A_3, Φ) on M. Resctricted to one of the fixed surfaces S_j , $\tilde{S}^1$ acts by gauge transformations on P. The fibres of $E = P \times_{SU(2)} \mathbb{C}^2$ over S_j decompose into eigenspaces for the $\tilde{S}^1$ action. Denote by $m_j \in \mathbb{Z}_{\geq 0}$ the $\tilde{S}^1$-weight which is non-negative.

If $m_j > 0$ then :

5.5 $$E_{|S_j} \cong L_j \oplus L_j^*$$

where L_j is the complex line bundle in E of weight m_j and L_j^* that of weight $-m_j$; because $c_1(E_{|S_j}) = 0$, L_j^* is also the dual of L_j. In order to define the first Chern classes of L_j it is convenient to have an orientation of S_j. Recall that X is oriented and that a neighbourhood of S_j in X looks like $S_j \times \mathbb{R}^2$. The $\mathbb{R}^2$ is oriented by the S^1-action , and this induces an orientation of S_j. Now write $c_1(L_j) = -k_j \cdot x_j$ with $k_j \in \mathbb{Z}$ and x_j the positive generator of $H^2(S_j;\mathbb{Z})$. If $m_j = 0$ then $E_{|S_j}$ is trivial as an $\tilde{S}^1$-equivariant vector bundle. We shall leave k_j undefined in this case.

There is one important constraint on the m_j. This becomes clear by remarking that $-1 \in \tilde{S}^1$ acts as a gauge transformation on all of E , i.e. as +1 or as -1. This implies that either all m_j are even or they are all odd. In the next chapter it will be shown that any set of invariants (m_j,k_j) satisfying this constraint arises from a suitable $\tilde{S}^1$-equivariant bundle , and that the $\tilde{S}^1$-isomorphism class is determined by (m_j,k_j).

Definition 5.2 : The moduli space of monopoles on a principal SU(2)-bundle P with invariants (m_j,k_j) will be denoted by $\mathcal{M}(m_j,k_j)$.

Having defined the relevant invariants of P , the question now arises what they amount to in terms of asymptotic conditions for a pair (A_3,Φ) on M. The vector field v on P turns vertical over S_j. This shows that :

5.6 $$|\Phi(y)| \to m_j \quad \text{if} \quad y \to S_j \subset \delta M.$$

This is the Prasad-Sommerfeld boundary condition used in physics and the numbers m_j are called the *masses* of the monopole.

The solutions of the Bogomol'nyi equation 5.3 are minima of the energy functional :

5.7 $$E(A_3,\Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 \, dV_3 .$$

If the pair (A_3,Φ) arises from an invariant connection A on P then $E(A_3,\Phi) = YM(A)$. If we assume that (A_3,Φ) satisfies 5.4 , then :

$$|d_{A_3}\Phi|^2 \, dV_3 = |F^{A_3}|^2 \, dV_3 = \langle F^{A_3} \wedge d_{A_3}\Phi\rangle = d\langle F^{A_3}.\Phi\rangle \quad ,$$

by the Bianchi identity. It follows that :

$$E(A_3,\Phi) = -2\ \Sigma_j\ (8\pi)^{-1}\cdot\int_{S_j} \langle F^{A_3}\cdot\Phi\rangle \quad .$$

The minus sign appears because the boundary orientation of S_j does not agree with orientation we have given it above. A moments reflection shows that $2\cdot(8\pi)^{-1}\cdot\int_{S_j} \langle F^{A_3}\cdot\Phi\rangle = -m_j\cdot k_j$. Putting things together we get :

5.7 $$\Sigma\ m_j\cdot k_j = E(A_3,\Phi) = YM(A) = k.$$

This is essentially the localization formula in equivariant cohomology applied to the equivariant $c_2(P)$, see Atiyah [2].

Exactly what the physical symmetry breaking would lead one to expect does indeed happen : far away in M , that is near an S_j with $m_j \neq 0$, the connection almost becomes a U(1)-connection on L_j , the bundle of eigenvectors of Φ of eigenvalue $\frac{1}{2}\cdot m_j$. The *charges* k_j appear as first Chern classes of these line bundles on the boundary surfaces. This is of course nothing but the quantized charge of a U(1)-monopole , a so called Dirac monopole , on L_j. Dirac monopoles have singularities , but the genuine non-Abelian character of SU(2)-monopoles in the core of M allows for non-singular solutions.

From 5.7 we see that $\Sigma\ m_j\cdot k_j \geq 0$ is necessary for the existence of monopoles , however this is by no means sufficient as we shall see below and in the next chapter.

We shall end this section by giving some simple examples of monopoles.

Examples 5.3 : 1) Monopoles with all $m_j=0$.

For these monopoles $YM(A) = 0$, so we are dealing with flat connections. The Higgs field Φ vanishes, this follows from the Bogomol'nyi equation. It is not hard to see that the moduli space $\mathcal{M}(0,0)$ equals the space of all representations $\pi_1(X) \to SU(2)$ modulo conjugacy : one assign to a flat connection its holonomy representation. This space can be very non-trivial ; e.g. if $M = H^3/\text{Fuchsian group} \cong S \times \mathbb{R}$, with S a surface, then $\mathcal{M}(0,0)$ is the space of representations of $\pi_1(S) \to SU(2)$ modulo conjugacy. By the theorem of Narasimham-Seshadri this is the same as the moduli space of semi-stable $SL(2,\mathbb{C})$-bundles on S, for any complex structure on S. The topology of this $\mathcal{M}(0,0)$ was investigated by Atiyah-Bott [4].

2) Next keep $k_j=0$ but take at least one m_j to be nonzero. The connections are still flat so Φ is covariantly constant. This shows that $\mathcal{M}(m_j,0) = \emptyset$ unless all m_j are equal. Further $\mathcal{M}(m,o) \cong \text{Repr}(\pi_1(M),S^1) \cong \text{Repr}(H_1(M;\mathbb{Z}),S^1) \cong H_1(X;\mathbb{Z})_{tor} \times \{H_1(X;\mathbb{R})/H_1(X;\mathbb{Z})\}$.

3) For $M \cong H^3$ all monopoles were determined by Atiyah [2]. The moduli space $\mathcal{M}(m,k)$ equals $\{\phi:S^2 \to S^2;\ \phi \text{ rational},\ \text{degree } \phi = k,\ \phi(\infty)=0\}$, modulo multiplication by complex scalars of length 1. The monopole associated to the rational function $\sum_j \exp(i\alpha_j)\cdot\lambda_j/(z-a_j)$ with $\lambda_j \in \mathbb{R}_{>0}$, $a_j \in \mathbb{C}$, represents k lumps, centered at approximately $(a_j,\lambda_j) \in \mathbb{R}^3_+ \cong H^3$, with relative phase factors $\exp(i(\alpha_{j_1}-\alpha_{j_2}))$.

3) Monopoles arising from Riemannian curvature.

If X is a oriented Riemannian 4-manifold then one can write the curvature tensor $R : \Lambda^2 \to \Lambda^2$ as $\begin{bmatrix} W_+ + (R_{sc}/3) & B \\ B^* & W_- + (R_{sc}/3) \end{bmatrix}$ relative to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, in which B equals the Ricci curvature and $W_\pm$ the Weyl tensor. If X is a conformally flat spin manifold with a metric of zero scalar curvature then this curvature tensor equals

$\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. It follows that the connection on the spin bunlde S_+ is anti-self-dual. Recall (see §3) that for Γ Fuchsian , extended Fuchsian or a suitable Schottky group X_Γ admits such a metric. The connection on S_+ is a monopole because the metrics are S^1-invariant. The mass(es) is (are) 1 by proposition 2.2 , and the charges k_j equal $g-1$, where g is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in $\mathrm{Repr}(\pi_1(M), S^1)$, compare 2).

In section 7 we shall come to grips with more nontrivial monopoles on certain handlebodies. A general existence theory will be developed in the next chapter.

§6 Twistor spaces

To a conformally flat oriented 4-manifold X there are naturally associated two complex manifolds Z_+ and Z_- , the twistor spaces of X. Applying our construction of §2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold M , such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let X be the conformal compactification of $M \times S^1$, with M a hyperbolic 3-manifold H^3/Γ as in §2. We shall state those properties of $Z_\pm$ that we will need , and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details.

If S_+ (S_-) is the spin bundle of positive (negative) chirality on X , then Z_+ (Z_-) can be realised as the $\mathbb{C}P^1$-bundles over X :

$$P(S_+) \to X \qquad (P(S_-) \to X) ,$$

where P() denotes projectivization of vectorbundles. A remarkable fact is that Z_+ and Z_- are complex manifolds with a complex structure encoded in the conformal structure of X. However , the twistor spaces are only Kahler if $X \cong S^4$ or $X \cong \mathbb{C}P^2$, which in our case results in $\Gamma=\{e\}$ (see Hitchin [18]). There is an orientation reversing isometry of X arising from conjugation of the circles. This interchanges the two spin bundles and makes Z_+ holomorphically equivalent to Z_-. Henceforth we shall only consider Z_+ and denote it by Z.

Z carries an anti-holomorphic involution :

$$\sigma : Z \to Z \quad , \sigma^2 = 1 .$$

This involution is a bundle map , inducing the identity on the base X , and is equal to the antipodal map upon restriction to the fibres. The complex structure on Z is such that (orientation preserving) conformal transformations on X lift to holomorphic transformations of Z . So our S^1-action on X lifts to an action on Z by holomorphic transformations and complexifies to a holomorphic $\mathbb{C}^*$-action on Z. We shall show that this $\mathbb{C}^*$-action is essentially the geodesic flow in H^3/Γ (as one would expect from Hitchin [19]).

The naturality with respect to conformal transformations has one further important application.

Recall (see Atiyah [1]) that the twistor space of S^4 is $\mathbb{C}P^3$ with projection and real structure:

$$\pi : \mathbb{C}P^3 \to S^4 = \mathbb{H}P^1 : [z_0,z_1,z_2,z_3] \to [z_0 + z_1\cdot j \ , \ z_2 + z_3\cdot j]$$

$$\sigma : \mathbb{C}P^3 \to \mathbb{C}P^3 : [z_0,z_1,z_2,z_3] \to [-\bar{z}_1,\bar{z}_0,-\bar{z}_3,\bar{z}_2]$$

As $X = (S^4-\Lambda)/\Gamma$ it follows that the twistor space of X is the quotient :

$$Z = [\mathbb{C}P^3-\pi^{-1}(\Lambda)]/\Gamma .$$

To study Z it will be useful to know how $\mathbb{C}^*$ and PSL(2,$\mathbb{C}$) act on $\mathbb{C}P^3$. The $\mathbb{C}^*$ action is described by $[z_0,z_1,z_2,z_3] \to [z_0,\lambda\cdot z_1,z_2,\lambda\cdot z_3]$, and the right PSL(2,$\mathbb{C}$)-action by mapping $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to $\begin{bmatrix} a & 0 & c & 0 \\ 0 & \bar{a} & 0 & \bar{c} \\ b & 0 & d & 0 \\ 0 & \bar{b} & 0 & \bar{d} \end{bmatrix} \in$ PSL(4,$\mathbb{C}$) which acts naturally on $\mathbb{C}P^3$, compare 2.3. Clearly the S^1-action fixes precisely two lines in $\mathbb{C}P^3$ namely :

6.1 $$P_1^+ = \{[z_0,0,z_2,0] \in \mathbb{C}P^3\} \quad \text{and}$$
$$P_1^- = \{[0,z_1,0,z_3] \in \mathbb{C}P^3\}$$

These lines are also invariant under the hyperbolic isometries. The projections to the fixed point set $S^2 \subset S^4$ are the orientation preserving map $P_1^+ \to S^2 : [z_0,z_2] \to [z_0,z_2]$ and the orientation reversing map $P_1^- \to S^2 : [z_1,z_3] \to [\bar{z}_1,\bar{z}_3]$ respectively. Here we have used homogeneous quaternionic coordinates on $S^4 = \mathbb{H}P^1$. The real structure maps P_1^+ to P_1^- and vice versa.

Non-trivial $\mathbb{C}^*$-orbits in $\mathbb{C}P^3$ are in one-one correspondence with a pair of begin- and end-points $(z,w) \in P_1^+ \times P_1^-$. Upon projecting the orbit $\mathcal{O}$ corresponding to (z,w) down to H^3 :

$$\mathcal{O} \subset \mathbb{C}P^3 \quad \to \quad \pi(\mathcal{O}) \subset S^4 = \overline{H^3\times S^1} \quad \to \quad g(\mathcal{O}) \subset H^3$$

one easily sees that $g(\mathcal{O})$ is an oriented geodesic in H^3 from $z \in S^2 = \delta H^3$ to $\bar{w} \in S^2$. The constant geodesics at infinity are included. Further for $p \in \mathcal{O} \subset \mathbb{C}P^3$ and $\lambda \in \mathbb{C}^*$ we have that the distance of $\pi(p)$ and $\pi(\lambda p)$ on $g(\mathcal{O})$ equals $\log|\lambda|$. As the $\mathbb{C}^*$-action commutes with the Γ-action , this shows that the $\mathbb{C}^*$-action is essentially geodesic flow in M.

It is now possible to describe Z in detail. First of all the fixed points of the $\mathbb{C}^*$-action on Z are surfaces S_j^+ , S_j^- , which project down to $S_j \subset X$. The surfaces S_j^+ , S_j^- equal the components of $[P_1^+-\Lambda]/\Gamma$ and $[P_1^--\Lambda]/\Gamma$ respectively. The real structure maps S_j^+ to S_j^- .

The nontrivial $\mathbb{C}^*$-orbits in Z come in three types. Good orbits emanate from a plus surface , say S_j^+ , and end on a minus surface , say S_k^- . The closure of one of these orbits in Z is a $\mathbb{C}P_1$. Note that these orbits are not determined by their two 'endpoints'. This corresponds precisely to the fact that two geodesics in M may have the same two endpoints , but in between one of them may run through different loops than the other. Denote by Ω_j^+ (Ω_j^-) the pre-image in P_1^+ (P_1^-) of S_j^+ (S_j^-) under the quotient map. From the above we get the following

Proposition 6.1 : The good orbits from S_j^+ to S_k^- are in one-one correspondence with oriented geodesics in $M \cong H^3/\Gamma$, which go from S_j to S_k . These have the complex analytic parameter space $[\Omega_j^+ \times \Omega_k^-]/\Gamma$, which is a holomorphic Ω_k^- bundle over S_j^+ or equivalently an Ω_j^+ bundle over S_k^-.

Considering all good orbits emanating from S_j^+ and ending on some S_k^- , one gets that these are holomorphically parametrized by a $\bigcup_k \Omega_k^- = P_1^- - \Lambda$ bundle over S_j^+ . Indeed , all orbits emanating from S_j^+ have a nice algebraic parameter space , which is equal to the projectivized holomorphic normal bundle $P(N_j^+)$ of S_j in Z. This is a $\mathbb{C}P_1$-bundle over S_j^+ . The bad orbits correspond to geodesics in M which , in the universal cover , start in Ω_j and end in Λ . Of course similar statements hold concerning arriving geodesics and the projectivized normal bundle of S_j^- . Concerning the normal bundles we have the following

Proposition 6.2 : There are injective , open , locally biholomorphic maps $\psi_j^\pm : N_j^\pm \to Z$, where $N_j^\pm$ is the holomorphic normal bundle of $S_j^\pm$ in Z. The $\mathbb{C}^*$-multiplication on the bundle N_j^+ is intertwined with the

$\mathbb{C}^*$-action on Z by ψ_j^+ , whereas ψ_j^- intertwines multiplication by the inverse with the $\mathbb{C}^*$-action on Z. The projectivized normal bundles $P(N_j^+)$ $(P(N_j^-))$ are an algebraic parameter space for all geodesics in M going out from (arriving at) S_j .

Proof : This is easy for the normal bundles of P_1^+ and P_1^- in $\mathbb{C}P^3$. Because the Γ action is linear and commutes with the $\mathbb{C}^*$-action the result also holds in Z. ∎

Remark 6.3 : 1) The relation of the normal bundles with Eichler's modules. If $\mathcal{H} \rightarrow \mathbb{C}P^1$ is the positive Hopf bundle , then $H^0(\mathbb{C}P^1,\mathcal{H}^n) = \Pi_n$ is an $SL(2,\mathbb{C})$-module , called an ***Eichler module*** , see Bers [7]. Hence after choice of a spin structure $\Gamma \rightarrow SL(2,\mathbb{C})$ a Γ-module (compare the discussion after proposition 2.2). A short computation shows that the normal bundle of S_j^+ in Z is isomorphic to :

$$N_j^+ = (\Omega_j^+ \times_\Gamma \bar{\Pi}_1) \otimes V_{+,j} ,$$

where $V_{+,j}$ is the positive spin bundle of S_j^+.

2) In general for complex submanifolds $V \subset W$ there are obstructions for locally embedding the normal bundle in a holomorphic way , see Kodaira [22] .

Finally there are some very bad orbits , corresponding to geodesics going from Λ to Λ in the universal cover. In M they keep spiralling around , and never find an endpoint in either direction. For example closed geodesics are among these. Points in non-trivial orbits have a non-trivial stabilizer iff the corresponding geodesic is closed. In figure 6.1 we have sketched the orbit situation.

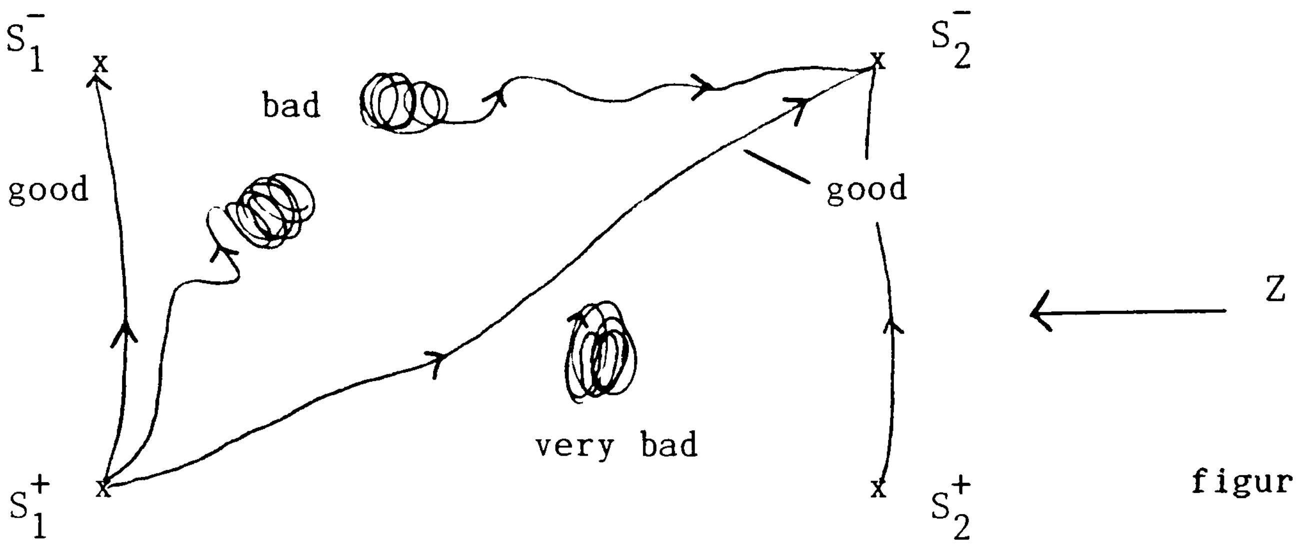

figure 6.1

The next objective of this section is to give a holomorphic description of monopoles. The relation between twistor spaces and anti-self-dual conections lies in the Atiyah-Ward correspondence (see Atiyah-Hitchin-Singer [5] , for the instanton case) :

Theorem 6.4 : Let $P \to X$ be an $\tilde{S}^1$-equivariant SU(2)-bundle , and A a monopole on P. Put $E = P \times_{SU(2)} \mathbb{C}^2$. Then π^*A induces a $\tilde{\mathbb{C}}^*$-invariant holomorphic structure on $F = \pi^*E$ such that :

1) F is trivial on the fibres of π.

2) The natural antiholomorphic antilinear bundle map $\sigma : F \to \bar{F}^*$, covering σ on Z , induces an S^1-invariant Hermitian metric on the vector spaces $H^0(\pi^{-1}(x),F)$.

3) $\Lambda^2 F$ is holomorphically trivial.

Conversely a $\mathbb{C}^*$-invariant holomorphic $\mathbb{C}^2$-bundle F over Z , with a real structure $\sigma : F \to \bar{F}^*$ satisfying 1,2 and 3 arises from a unique monopole on $P \to X$.∎

Real structures on indecomposable holomorphic bundles F over twistor space are unique. Hence all the information is encoded in the holomorphic structure. However , existence of real structures is not automatic. The gauge equivalence relation for monopoles on $P \to X$ is the same as holomorphic $\tilde{\mathbb{C}}^*$-equivariant equivalence , preserving real structures , for the holomorphic bundles F on Z.

Let A be a monopole on $P \to X$, with all $m_j \neq 0$ and even , for simplicity. In this case we need not consider double coverings of groups and we shall denote the weights of S^1 by $p_j = \frac{1}{2} \cdot m_j$. Denote by $F = \pi^*(P \times_{SU(2)} \mathbb{C}^2)$ the holomorphic bundle Z , with real structure σ. By theorem 6.4 the holomorphic stucture on F is $\mathbb{C}^*$-invariant. An important aspect of monopole geometry of $\mathbb{R}^3$ and H^3 is to consider the quotient bundle $\mathcal{F} = F/\mathbb{C}^*$ on $Z/\mathbb{C}^*$ as far as this makes sense. On $Z/\mathbb{C}^*$, $\mathcal{F}$ will be an extension of certain standard line bundles , and this has been put to constructive use in the $\mathbb{R}^3$ case , see Hitchin [19]. It will be shown that a more complicated but essentially similar picture persists in our more general case. As yet , the constructive power seems to be rather limited.

Restricting F to S_j^+ it splits holomorphically , since the $\mathbb{C}^*$ action is fibre-wise , with nonzero weights $\pm p_j$:

6.2
$$F_{|S_j^+} = L_j^+ \oplus (L_j^+)^*$$
$$F_{|S_j^-} = L_j^- \oplus (L_j^-)^*$$

Here L_j^+ has $\mathbb{C}^*$-weight p_j and $c_1(L_j^+) = -k_j$, as in §5. For L_j^- we have $\mathbb{C}^*$-weight $-p_j$ and $c_1(L_j^-) = -k_j$. The real structure gives an anti-linear isomorphism $L_j^+ \to L_j^-$.

Proposition 6.5 : On $N_j^+ \subset Z$ $(N_j^- \subset Z)$ there are line bundles K_j^+ (K_j^-) , extending the $L_j^\pm$ of 6.2 (which were defined on the zero sections $S_j^\pm$ of $N_j^\pm$) , such that on the $N_j^\pm$ the bundle F is an extension :

$$0 \to K_j^+ \to F_{|N_j^+} \to (K_j^+)^* \to 0$$

$$0 \to K_j^- \to F_{|N_j^-} \to (K_j^-)^* \to 0$$

The real structure interchanges these two extensions.

Proof : Recall that sections of $P(F)$ correspond to line sub-bundles of F. We shall look at the $\mathbb{C}^*$-action on $P(F)$ restricted to the fibres $(N_j^+)_z$ with $z \in S_j^+$. Over $(N_j^+)_z$ we have two fixed points in $P(F)$ namely $[(L_j^+)_z]$ and $[(L_j^+)_z^*]$, lying in the fibre above $0 \in (N_j^+)_z$. At $f = [(L_j^+)_z]$ the weights of the infinitesimal $\mathbb{C}^*$-action on $T_fP(F)$ are $(+1,+1,-p_j)$. This means that most of the $\mathbb{C}^*$-orbits will actually flow to $[(L_j^+)_z^*]$, compare figure 6.2.

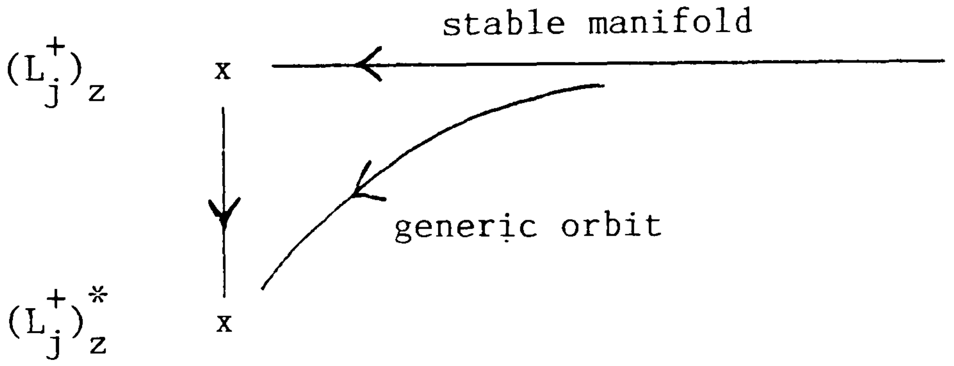

figure 6.2

By the stable manifold theorem with holomorphic parameter $z \in S_j^+$, we get a $\mathbb{C}^*$-invariant , $\mathrm{codim}_{\mathbb{C}}$ 1 , complex submanifold $[L_j^+]$ of $P(F)$, consisting of precisely those orbits that flow into L_j^+. For the stable manifold theorem see Hadamard [15] . On N_j^- the situation is of course similar.∎

In the case of monopoles on H^3 these extensions extend as bundle maps from $N_j^+ = \mathbb{C}P^3 - P_1^-$ to $\mathbb{C}P^3$ (also for N_j^-) but in our more general situations there can be obstructions to this , see Braam-Hurtubise [11].

The extensions of proposition 6.5 descend to the quotient $P(N_j^{\pm})$, and we proceed by identifying them there. Holomorphic line bundles on the ruled surfaces are of the form :

$$\rho^* L \otimes O(n)$$

where $\rho : P(N_j^{\pm}) \to S_j^{\pm}$ is the projection , L a line bundle on $S_j^{\pm}$, and $O(n)$ the n-th power of the positive Hopf bundle on $P(N_j^{\pm})$, which has fibre $(\mathbb{C}v)^*$ at the point $[v] \in P(N_j^{\pm})$. On the fibres of $N_j^{\pm}$ the structure of the bundle follows from :

Lemma 6.6 : Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ by scalar multiplication. A $\mathbb{C}^*$-equivariant $\mathbb{C}^2$-bundle $E \to \mathbb{C}^2$ is equivariantly isomorphic to $E_0 \times \mathbb{C}^2$ with E_0 the representation of $\mathbb{C}^*$ on the fibre over $0 \in \mathbb{C}^2$.

Proof : (see Atiyah [2]) On $\mathbb{C}^2\backslash\{0\}$ a $\mathbb{C}^*$-equivariant bundle is the same as a bundle on $\mathbb{C}P^1$, i.e. a sum of powers of the Hopf bundle. This establishes the given isomorphism on $\mathbb{C}^2\backslash\{0\}$. By Hartog's theorem it extends to $\mathbb{C}^2$.∎

The point of the lemma is that it identifies $K_j^{\pm}$ as the pull back of $L_j^{\pm}$ under the projection $N_j^{\pm} \to S_j^{\pm}$, with $\mathbb{C}^*$ acting on it by a character of weight $\pm p_j$. Now one concludes readily that the extension on $P(N_j^+)$ reads :

6.3 $$0 \to \mathcal{L}_j^+ \to \mathcal{F} \to (\mathcal{L}_j^+)^* \to 0 \text{ with}$$
$$\mathcal{L}_j^+ = \rho^* L_j^+ \otimes O(p_j) \text{ and } \mathcal{F} = [F_{|N_j^+\backslash\{0\}}]/\mathbb{C}^* .$$

Similarly on $P(N_j^-)$ we get :

6.4 $$0 \to \mathcal{L}_j^- \to \mathcal{F} \to (\mathcal{L}_j^-)^* \to 0 \text{ with}$$

$$\mathcal{L}_j^- = \rho^* L_j^- \otimes O(p_j) \quad \text{and} \quad \mathcal{F} = [F_{|N_j^- \setminus \{0\}}]/\mathbb{C}^* .$$

This results in :

Theorem 6.7 : The monopole A defines extensions of $\mathcal{F}$ on $P(N_j^+)$ and $P(N_j^-)$ for $j=1,\ldots,N$ as in 6.3 and 6.4. These extensions are interchanged by the real structure.∎

In the case of monopoles on H^3 these restrictions are essentially all the data one obtains about the quotient bundles and the monopole is determined by the extensions and the real structure : see Atiyah [2]. In our case the intersection of N_i^+ with N_j^- will generally have many components and we get extra data in the form of a set of invariant identifications:

$$6.5 \qquad g_{ij} : N_i^+ \cap N_j^- \to \mathrm{Hom}(F_{|N_i^+} , F_{|N_j^-}) .$$

Conjecture : Under general conditions on the hyperbolic structure on M bundles F arising from irreducible monopoles are determined by the extensions 6.3 , 6.4 and the real structure on these.∎

One can prove that if F_0 and F_1 are two holomorphic bundles on Z such that upon restriction to $\cup_i (N_i^+ \cup N_i^-)$ they become isomorphic , then they are isomorphic on Z. In order to prove the conjecture it remains to show that for irreducible monopoles no information is contained in the g_{ij}. Evidence for this conjecture comes from Thurston's version of Mostow's theorem (see Morgan [28]). This theorem implies that the flat PSL(2,$\mathbb{C}$)-bundles encoding the holonomy of the hyperbolic structure are determined by their restriction to the fixed surfaces , despite the fact that the fundamental group of Z is not necessarily generated by that of the fixed surfaces. In fact one may hope to reverse this procedure : a proof of the conjecture would be a good first step towards a proof of Mostow's theorem.

It might be a good point to stress that although Z is not Kahler , suddenly algebraic objects such as elements of Picard groups and ruled surfaces have appeared. This makes algebraic geometry enter the picture , perhaps somewhat unexpectedly.

Next we shall consider spectral curves , of which we shall obtain a whole bunch instead of just a single one , as obtained in the case of $\mathbb{R}^3$ and H^3 (see Hitchin [19] and Atiyah [2]). Just as in the $\mathbb{R}^3$ and H^3 case we should compare two extensions. On $P(N_j^+ \cap N_k^-)$ we have :

6.6
$$0 \to \mathcal{L}_j^+ \to \mathcal{F} \to (\mathcal{L}_j^+)^* \to 0 \quad \text{and}$$
$$0 \to \mathcal{L}_k^- \to \mathcal{F} \to (\mathcal{L}_k^-)^* \to 0 \quad .$$

Definition 6.8 : The spectral curve

$$C_{jk} \subset P(N_j^+ \cap N_k^-) = (\Omega_j^+ \times \Omega_k^-)/\Gamma \qquad j,k=1,\ldots\ldots,n$$

is the zero set of the canonical map :

$$\mathcal{L}_j^+ \to (\mathcal{L}_k^-)^* \quad ,$$

arising from 6.6. ■

Hence for a manifold with N ends , we get N^2 spectral curves. However , the real structure clearly interchanges C_{jk} with C_{kj} , so effectively we are left with $(N^2 + N)/2$ spectral curves , N of which , namely the C_{jj} , have to satisfy reality constraints. The curves can be interpreted geometrically as follows :

Proposition 6.9 : The following three are equivalent :

1) A $\mathbb{C}^*$ orbit $\mathcal{O} \in (\Omega_j^+ \times \Omega_k^-)/\Gamma$ lies in C_{jk} .

2) The bundle F restricted to $\bar{\mathcal{O}} \cong P_1 \subset Z$ is isomorphic to $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. (For other good orbits it will be isomorphic to $\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(-p_j+p_k)$.)

3) The Hitchin equation (compare Hitchin [19]):

$$\frac{\partial s}{\partial l} + A_l.s + i\Phi.s = 0 \quad , \quad s : g(\mathcal{O}) \to \mathbb{C}^2$$

on the corresponding geodesic $g(\mathcal{O}) \subset H^3/\Gamma$ has a bounded solution.

Proof : To see the equivalence of 1) and 2) we first digress on bundles on $\mathbb{C}P^1$. The result of lemma 6.6 also holds if one replaces $\mathbb{C}^2$ by $\mathbb{C}$; this follows by using an arbitrary projection $\mathbb{C}^2 \to \mathbb{C}$ and pulling back. Thus $E_{|\bar{\mathcal{O}}}$ trivializes in a $\mathbb{C}^*$-equivariant way as :

$$\begin{aligned} L_j^+ \oplus (L_j^+)^* &\quad \text{on } \bar{\mathcal{O}}-\{\infty\} \\ L_j^- \oplus (L_j^-)^* &\quad \text{on } \bar{\mathcal{O}}-\{0\}. \end{aligned}$$

The $\mathbb{C}^*$-equivariant automorphisms of $E_{|\bar{\mathcal{O}}-\{\infty\}}$ are easily seen to be of the form $\begin{bmatrix} a & b\cdot z^{2p_j} \\ 0 & c \end{bmatrix}$, and thus form a Borel subgroup of $GL(2,\mathbb{C})$. The situation is the same at infinity , and from this it follows that isomorphism classes of $\mathbb{C}^*$-equivariant holomorphic bundles on $\mathbb{C}P^1$ are given by the set of two elements $B\backslash GL(2,\mathbb{C})/B$. The exceptional case is that in which the transition function maps L_j^+ to L_j^- , i.e. $\mathcal{O} \in C_{jk}$. Then $F_{|\bar{\mathcal{O}}}$ equals $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$, otherwise it is isomorphic to

$\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(p_k-p_j)$.

To prove the equivalence of 2) and 3) , we first remark that $F_{|\bar{\sigma}}$ has a bounded $\mathbb{C}^*$-invariant holomorphic nonzero section , iff $F_{|\bar{\sigma}} \cong \mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. This follows from the standard description of sections of line bundles over $\mathbb{C}P^1$ as homogeneous polynomials and from the fact that the weights of the action are is p_j at 0 and $-p_k$ at ∞. The Hitchin equation is nothing but the Cauchy-Riemann equation for invariant sections , see Hitchin [19]. Therefore the proposition follows. ∎

Remark 6.10 : 1) One expects that the spectral curves will generally not be compact and more or less resemble a curve of infinite genus. This is because on the universal cover H^3 we are dealing with a monopole of infinite charge.

2) It should also be remarked that the complex manifolds $(\Omega_j^+ \times \Omega_k^-)/\Gamma$ in which the spectral curves lie are far from nice generally. In the case of cyclic groups they are a $\mathbb{C}^*$-bundle over a torus , see Braam-Hurtubise [11] and for quasi-Fuchsian groups they are disc bundles over a Riemann surface of genus ≥ 2. Generally they will be Ω_j^+ bundles over S_k^- and the fibre will have infinitely many components ; see §2 where we discussed Kleinian groups.

§7 Atiyah-Ward ansatzes , summing 't Hooft solutions and Eisenstein series

From the description of Z as $P(S_+)$, it follows that on Z there exists a line bundle L , which upon restriction to the fibre over $x \in X$, equals the negative Hopf bundle on $P(S_{+,x})$. It turns out that L is naturally holomorphic , and to tie in with the $(\mathbb{C}P^3, S^4)$ case we shall denote the $(-q)$-th power of L by $\mathcal{O}(q)$.

If $F \to \mathbb{C}P^3$ is an instanton bundle on the twistor space of S^4 then *Atiyah-Ward ansatzes* , that is an explicit formula for the instanton on S^4 , arise from a suitable description of F as holomorphic bundle. Let s be a section of $F \otimes \mathcal{O}(q) = F(q)$. Generically s will be nonzero away from a complex curve $C_s \subset Z$ and give rise to an extension class $e_s \in H^1(Z-C_s, \mathcal{O}(-2q))$. Elements of such sheaf cohomology groups correspond to solutions ϕ_s of linear p.d.e. on open sets of S^4 : this is the celebrated Penrose correspondence. Explicit formulas for the instanton , such as those of 't Hooft , can be constructed in terms of this ϕ_s. Finally every instanton on S^4 can theoretically be computed in this way. For background see Atiyah [1].

We shall see that on our manifolds $X = (S^4-\Lambda)/\Gamma$, for $\Gamma \neq \{e\}$, the situation is rather different , but that nevertheless in some cases explicit constructions can be made again. As before attention will only be paid to $\tilde{S}^1$-invariant instantons , i.e. monopoles. In those cases which we treat in detail , it will appear that we are essentially summing together a monopole , much in the same way as automorphic forms are constructed by summing kernels.

Recall from §2 and §3 , that X comes with a natural conformal structure , and that X can be given a metric in the conformal class with constant scalar curvature R_X. We proved that the majority of X's give rise to negative R_X. Assume a spin structure on X has been fixed , then the line bundle $\mathcal{O}(q)$ above is well defined.

Proposition 7.1 : If $R_X < 0$, then no monopole on X arises from an Atiyah-Ward construction , since $H^0(Z,F(q)) = 0$ for all $q \in \mathbb{Z}\setminus\{0\}$.

Proof : For $q < 0$ any section would vanish on the fibres $\pi^{-1}(x)$,and hence be zero ; this is independent of the sign of R_X. For $q \geq 0$, we know from Hitchin [17] , that elements of $H^0(Z,F(q))$ are in one-one correspondence with solutions of the twistor equation on X with coefficients in $E = P \times_{SU(2)} \mathbb{C}^2$:

$$\bar{D}_q\, s = 0$$

$$\bar{D}_q = \mathcal{P}\circ\nabla_A : \Gamma(S^q(S_+) \otimes E) \to \Gamma(S^{q+1}(S_+) \otimes S_- \otimes E) ,$$

with S^q the q-th symmetric product , $\mathcal{P} : \Lambda^1 \otimes S^q(S_+) \to S^{q+1}(S_+) \otimes S_-$ the projection , and A the anti-self-dual SU(2)-connection on $E \to X$. For these equations we have a vanishing theorem of Weizenbock type in the case of constant negative scalar curvature , see Besse [8].∎

Hence attention here needs only be paid to the $R_X \geq 0$ manifolds , which were classified in theorem 3.1. But even here there is a very fundamental difference between the case $X = S^4$, i.e. $\Gamma = \{e\}$, and the cases of non-trivial Γ.

On $X = S^4$, $Z = \mathbb{C}P^3$, the dimensions of $H^0(Z,\mathcal{O}(q))$ (and also of the invariant part $H^0(Z,\mathcal{O}(q))^{S^1}$) increase with q. Tracing through the (equivariant) Riemann-Roch formula (as in Hitchin [18]) , one learns that the increasing character is due to the fact that for the fixed point sets

$S^+ = P_1^+$, $S^- = P_1^- \subset Z = \mathbb{C}P^3$ we have $\chi(S^\pm) > 0$. For $\Gamma \neq \{e\}$ these Euler characteristics satisfy $\chi(S^\pm) \leq 0$. This leads one to suspect that it may not always be possible to find sections of $F(q)$, which would be needed to obtain Atiyah-Ward ansatzes in general.

After all these negative remarks , let us proceed to show that , at least in some cases , the construction works satisfactorily. To simplify things even further , we shall assume that X is a manifold with $R_X > 0$; by theorem 3.1 , X arises from a Schottky group. Consider on X the conformally invariant Laplacian D_0 acting on densities of conformal weight 1 , with values in densities of weight 3 , which equals :

$$D_0 = d^*d + \frac{1}{6}\cdot R_X \ .$$

Since $R_X > 0$, we get $\ker D_0 = 0$, and hence unique fundamental solutions ϕ_x exist satisfying :

$$D_0 \cdot \phi_x = \delta_x \qquad x \in X.$$

Through the twistor correspondence (see Atiyah [3] , [1] , and Hitchin [17]) ϕ_x corresponds to a class :

$$\varphi_x \in H^1(Z-\pi^{-1}(x), \mathcal{O}(-2)) \quad ,$$

and hence ϕ_x gives rise to a vector bundle F on $Z-\pi^{-1}(x)$, which is an extension :

$$0 \to \mathcal{O}(-1) \to F \to \mathcal{O}(1) \to 0.$$

In fact one can show (Atiyah [3]) that the bundle F extends to a bundle F on Z , such that F(1) has a holomorphic section vanishing precisely on $\pi^{-1}(x)$. The maximum principle applied to D_0 ensures that $\phi_x(y) > 0$, for all $y \in X$, and this implies that F is trivial on the real lines $\pi^{-1}(x)$. Since ϕ_x is real , F gets a real structure. Thus F is an instanton bundle.

All positive scalar multiples of ϕ_x give the same instanton. To get a monopole rather than just an instanton we have to assume $x \in S_1$, the fixed surface in X. The weight m_1 of a monopole constructed in this way equals 1 , because the Hopf bundle $\mathcal{O}(1)$ is of weight 1. The charge also equals 1.

Obviously the process can be generalized by using a positive linear combination of k fundamental solutions :

$$\varphi = \Sigma\, \lambda_j \varphi_{x_j} \qquad \lambda_j > 0 \quad , \; j=1,\ldots,k \quad ,$$

which is called an *'t Hooft potential.* If the x_j lie in $S_1 \subset X$, then the 't Hooft potential will be invariant , and it follows that we have created a monopole of mass 1 and charge k. Note that the number of parameters we get is only 3k-1 : we have 2 for every $x_j \in S_1$, and 1 for every λ_j . These solutions therefore don't give an open set in the $4k - \frac{1}{2}\cdot\chi(S)$ dimensional moduli space , unless $\chi = 2$.

We proceed to identify these potentials ϕ. In the course of this , explicit formulas for the connection A will also be given. Besides , a generalization of the Atiyah-Ward construction will emerge which points the way out of the negative picture we sketched above.

Pulling back ϕ_x to $S^4-\Lambda$, under the quotient map , one gets a generalized funtion $\tilde{\phi}_x$ on $S^4-\Lambda$ satisfying :

$$D_0 \tilde{\varphi}_x = \sum_{\gamma\in\Gamma} \delta_{\gamma y}$$

with $y \in S^4-\Lambda$ mapping to x. Of course the next step is to try to reverse this and to put :

7.1 $$\tilde{\varphi}_x = \sum_{\gamma\in\Gamma} \psi_{\gamma y}$$

where ψ_y is a fundamental solution on S^4 of D_0 at y. In the flat metric on $\mathbb{R}^4 \subset S^4$, fundamental solutions are equal to :

7.2 $$\psi_y(r) = (2\pi\|y-r\|)^{-2}.$$

Since the flat metric is not Γ-invariant , weight factors will occur in 7.1. It is easier to see what happens if one uses the Γ-invariant metric on $H^3 \times S^1$:

$$t^{-2}(dx_1^2 + dx_2^2 + dt^2) + d\theta^2 \qquad (x_1,x_2,t,\theta) \in H^3\times S^1$$

Under conformal rescaling , 7.2 transforms to the θ-independent *summation kernel* of the Eisenstein series on H^3 (compare Mandouvalos [24]):

$$E(y,h) = t/[(x_1-y_1)^2+(x_2-y_2)^2+t^2] \qquad y \in \mathbb{R}^2 \subset S^2 \quad , h=(x_1,x_2,t) \in H^3$$

Summing , we get for 7.1:

7.3 $$E_\Gamma(y,h) = \sum_{\gamma\in\Gamma} E(y,\gamma h) \quad ,$$

which is the ***Eisenstein series*** for Γ , see Mandouvalos [24]. As settled by Poincaré already , 7.3 is convergent if $\delta(\Gamma) < 1$, where $\delta(\Gamma)$ is the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ. The groups Γ for which this holds are the cyclic groups and classical Schottky groups (with their defining circles wide apart , compare Bers [7]). In passing by we note that $\delta(\Gamma) < 1$ implies that X is of positive type because we have a positive Green's function for $d^*d + \frac{1}{6}\cdot R_{sc}$.

To compute the gauge potentials , it is easiest to go back to the flat metric on $\mathbb{R}^4$. The more general potentials there look like :

$$7.4 \qquad \phi(h,\theta) = \sum_{i=1}^{k} \lambda_i \cdot t^{-1} \cdot E_\Gamma(x_i,h) \quad ,$$

and the formulas of 't Hooft give for the connection (see Atiyah-Hitchin-Singer [5]) :

$$A = \sum_i P_+(-1/2 \; d\log\varphi \wedge e_i) \otimes e_i \quad \in \Gamma(\mathbb{R}^4, \Lambda^2_+ \otimes \Lambda^1) \; ,$$

with e_i an orthonormal , covariantly constant framing of $T^*\mathbb{R}^4$ and Λ^2_+ identified with su(2). To see what this looks like , assume that Γ is cyclic , generated by $\begin{bmatrix} \lambda^{1/2} & \\ & \lambda^{-1/2} \end{bmatrix}$, $\lambda \in \mathbb{R}_{>0}$. Then :

$$7.5 \qquad \varphi(r) = \sum_{n=1}^{\infty} \Big[\sum_i \Big\{ \frac{\lambda^n \lambda_i}{\|\lambda^n r - y_i\|^2} + \frac{\lambda^{-n} \lambda_i}{\|r - \lambda^{-n} y_i\|^2} \Big\} \Big] + \sum_i \frac{\lambda_i}{\|r - y_i\|^2}$$

$$\text{with } y_i \in \mathbb{R}^2 \subset S^2 \text{ and } r \in S^4 \setminus \Lambda = \mathbb{R}^4 \setminus \{0\}.$$

So we see that for $\lambda >> \lambda_i$ and $1 \leq \|r\|$, $\|y\| \leq \lambda$, the second term dominates strongly and the monopole will look much like a 'grafted S^4-monopole'. On making λ smaller , nearby nonlinear interaction makes the monopole look more complicated.

Finally we discuss a modification of this construction which supplies a few more solutions. Suppose we put $k=1$ and consider the harmonic function:

$$\phi_\alpha(r) = \sum_{n\in\mathbb{Z}} \lambda^{(1+\alpha)\cdot n} \cdot \|\lambda^n r - y\|^{-2} ,$$

which converges for $-1 < \alpha < 1$. Then $\phi_\alpha(\lambda r) = \lambda^{-\alpha-1}\cdot\phi_\alpha(r)$, so the instanton is invariant. This results in a 3-parameter family of monopoles.

Now ϕ_α describes a fundamental solution of the Laplacian acting on sections of a flat real line bundle with monodromy λ^α along the non-trivial loop in H^3/Γ , so we have constructed a bundle F on twistor space , which is an extension of $L(1)$ by $L^*(-1)$, where L is a real flat line bundle in the Picard group of Z with monodromy $\lambda^{\frac{1}{2}\cdot\alpha}$.

Certainly the same procedure can be used for Schottky groups Γ of genus g , by twisting the sum with a character $\Gamma \rightarrow \mathbb{R}_{>0}$ close to 1. This gives a $g+2$ parameter family of monopoles. It is currently under investigation if this method gives results for other groups as well.

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CHAPTER III

MAGNETIC MONOPOLES ON THREE-MANIFOLDS

§0 Introduction

In this chapter we will investigate magnetic monopoles on an oriented , complete Riemannian 3-manifold M. Basically the result is that we associate to M a collection of moduli spaces of solutions of the (magnetic) monopole equations , provided that M is not compact and that the Riemannian metric on M is 'good near infinity'. This situation is much the same as that for an oriented , Riemannian 4-manifold , which has a collection of instanton moduli spaces associated to it. More specifically , we shall prove that monopoles exist under reasonable conditions , we compute the dimensions of the moduli spaces and study smoothness , orientability and asymptotic behaviour. Having obtained these moduli spaces together with their basic properties , the next step would be to exploit the topology of the moduli spaces to define topological invariants for 3-manifolds , just as instanton moduli spaces give invariants for smooth 4-manifolds. This will be discussed in a forthcoming paper.

To carry non-trivial monopoles M should not be compact. This gives rise to hard analytical problems on the 3-manifold , such as those considered in the work of Taubes and Floer. To avoid this we shall exploit the fact that a monopole is a 'time'-invariant instanton on $M \times S^1$. Using the conformal invariance of the instanton equations , we find ourselves working on a conformal compactification X of $M \times S^1$, and studying S^1-invariant instantons on X. The fixed point set of the S^1-action on X now plays the role which the boundary of M played in a direct

3-dimensional approach. In order to exploit as much as possible the available knowledge about instantons we carefully compare various definitions of S^1-invariance. This enables us to realise the monopole moduli spaces as submanifolds of instanton moduli spaces. This will be carried out in §1.

To compute dimensions of the monopole moduli spaces we have to compute the index of a certain Dirac operator. In order to do so , one has to come to grips with topological invariants of S^1-equivariant bundles on X. After this we can apply the Atiyah-Singer index formula as for instanton moduli spaces , only we now use the equivariant index formula. This is the subject matter of §2.

In §3 we describe the configuration spaces for the monopole problem and carry out deformation theory. This carries over almost word for word from the instanton case , inserting 'S^1-equivariant' at appropiate places.

Instanton moduli spaces are smooth for a generic metric , are orientable and have a compactification. All this carries over to monopole moduli spaces and this is the contents of §4. It is here that we start profiting from our careful preparations in §1.

In §5 we finally tackle the existence problem for monopoles by adapting Donaldson's alternating procedure. This gives the asymptotic structure of the moduli spaces and also provides information about monopoles on connected sums of 3-manifolds. We have included some examples of these asymptotic models. Also we have made some further comparison with the direct approaches of Taubes and Floer.

As the reader presumably realizes , all our main results rely entirely on fundamental work concerning instantons , which has been carried out by S.K. Donaldson , C.H. Taubes , K.K. Uhlenbeck and others.

The relation of our 4-dimensional approach with the conventional

3-dimensional approach can be seen most clearly by noting that the fixed surfaces of the S^1-action on the 4-manifold are in one-one correspondence with the boundary surfaces of the 3-manifold. For example , S^1-equivariant index computations become computations on the fixed surfaces through the Lefschetz theorems , much in the same way as Callias [11] developed index theorems on $\mathbb{R}^n$, playing back computations to the boundary sphere at infinity. Another example arises in the study of limits : any sequence of instantons on a compact 4-manifold has a subsequence which converges away from a finite set of points. For S^1-invariant instantons these points obviously have to lie in thhe fixed point set of the circle action. The interpretation is easy : a sequence of monopoles on the 3-manifold can only fail to converge because some lumps move off to the boundary surfaces at infinity.

§1 Compactifications , monopoles and instantons

Let $\bar{M}$ be a compact , oriented 3-manifold with boundary $\delta M = \cup_{j=1,\dots,N} S_j$, each S_j a compact Riemann surface without boundary. We shall assume that the interior $M = \bar{M}\setminus\delta M$ of $\bar{M}$ carries a complete Riemannian metric. The boundary surfaces S_j then lie at infinity. General references for the discussion of the Yang-Mills equations which follows are Freed-Uhlenbeck [20] and Jaffe-Taubes [22]. More specifically our approach is a generalization of that in Atiyah [2].

Let $Q \to M$ be a pricipal fibre bundle with fibre a compact Lie group G. In the sequel we shall take G to be SU(2) or SO(3) for simplicity. Let A_3 be a connection on Q , and Φ a *Higgs field* , i.e. a section of $g_Q = Q \times_{Ad} g$, with g denoting the Lie algebra of G. The *Bogomol'nyi equations* for (A_3,Φ) , the solutions of which are the *(magnetic) monopoles* on M , read :

1.1 $$d_{A_3}\Phi = -*_3 F^{A_3} \qquad (*_3 \text{ the Hodge star on } M)$$

with d_{A_3} denoting covariant derivative and F^{A_3} the curvature of A_3. The standard boundary conditions for 1.1 are :

1.2 $$E(A_3,\Phi) = (8\pi)^{-1}\int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 \, dV_M < \infty \text{ and}$$
$$|\Phi(x)| \to m_j \text{ if } x \to S_j \quad (m_j \in \mathbb{R}_{\geq 0} , j=1,\dots,N).$$

The inner product on the Lie algebra su(2) has been chosen as $\langle\alpha,\beta\rangle = -2\cdot \mathrm{tr}(\alpha\cdot\beta)$ and so(3) is isometric to su(2) through the adjoint representation. The m_j are called the *masses* of the monopole. As we indicated in the introduction, we shall not analyse these equations directly but revert to 4-dimensional geometry to study them.

Let $P' = Q \times S^1 \rightarrow M \times S^1$ be the pullback of Q under $\pi : M \times S^1 \rightarrow M$. Note that P' automatically has an S^1-action. Denote by $\frac{\partial}{\partial\theta}$ the vector field on P' induced by the action, and by $d\theta$ the dual one-form. Then $A = \pi^* A_3 + (\pi^*\Phi)d\theta$ is an S^1-invariant connection on P', and every S^1-invariant connection on P' can uniquely be written in this way. It is easy to check that (A_3, Φ) satisfies 1.1 iff A satisfies the *anti-self-duality equation* :

$$F^A = -*_4 F^A \qquad 1.3$$

Here $*_4$ is the Hodge star on $M \times S^1$ considered as an oriented Riemannian product. Solutions of the anti-self-duality equation are called *instantons*.

The next step is to exploit the conformal invariance of 1.3. Suppose that $M \times S^1$ has an oriented, S^1-equivariant conformal compactification X with $X \backslash (M \times S^1) = \cup_j S_j$. Topologically we assume that X is M spun around its boudary surfaces, compare figure 1.1. The manifold X has an S^1-action which is free away from the fixed surfaces S_j, which correspond exactly to the components of the boundary of $\bar{M}$. The normal bundles of the S_j are topologically trivial and of S^1-weight 1. Geometrically we assume that X carries a conformal structure which coincides with the conformal structure underlying the product metric on

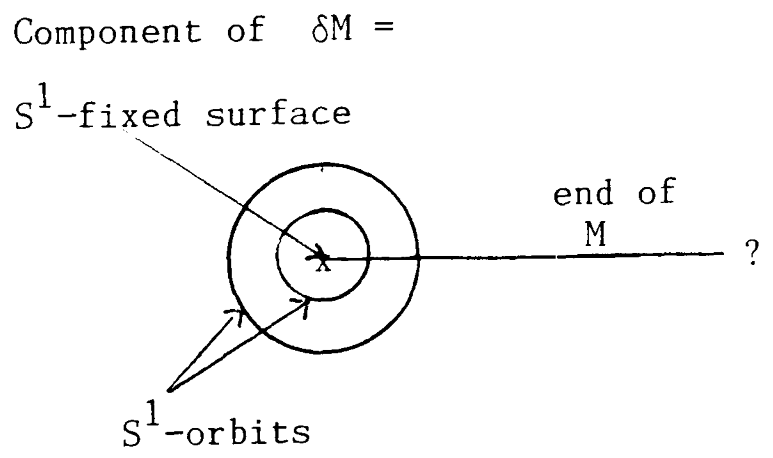

figure 1.1

$M \times S^1$. In chapter 2 we showed that for a large class of hyperbolic 3-manifolds M such a compactification exists naturally. It is always possible to give M a metric which is hyperbolic near the boundary and such that the compactification exists. Hyperbolicity is not necessary for the existence of the compactification : by deforming the conformal structure on X we see that there is an infinite dimensional space of metrics on M for which a conformal compactification of $M \times S^1$ exists. Nevertheless , for 'most' metrics on M the compactification seems not to exist.

Given X with the S^1-action consider an S^1-equivariant principal $SO(3)$-bundle P , i.e. a principal $SO(3)$-bundle with an S^1-action by bundle automorphisms covering the S^1-action on X. We shall be interested in S^1-invariant instantons modulo the group of S^1-invariant *gauge transformations* $GA(P) \cong \mathcal{G}(P)^{S^1}$, where $\mathcal{G}(P) = \Gamma(P \times_{Ad} G)$ is the group of all gauge transformations , i.e. bundle automorphisms fixing the base.

Definition 1.1 : An $SO(3)$-*monopole* on P is an S^1-invariant instanton on P. The *monopole moduli space* equals $\{S^1\text{-invariant instantons}\}/GA(P)$ and will be denoted by $\mathcal{M}(P)$.

Upon restriction to $M \times S^1 \subset X$ our monopoles certainly give rise to 'ordinary' monopoles on M. To check the boundary conditions 1.2 , first remark that $E(A_3,\Phi)$ is equal to the *Yang-Mills functional* of A :

$$YM(A) = (16\pi^2)^{-1} \int_X |F^A|^2 \, dV_X \; , \tag{1.4}$$

so this is certainly finite. The limiting behaviuor of the Higgs field can be seen as follows. The S^1-action on $P_{|S_j}$ is by gauge transformations.

This gives rise to a homomorphism $S^1 \to \mathcal{G}(P_{|S_j}) \cong \Gamma(S_j, P_{|S_j} \times_{Ad} SO(3))$, which is essentially a family of homomorphisms $S^1 \to SO(3)$. Such a family is constant up to conjugation , and it follows that the vectorfield $\frac{\partial}{\partial\theta}$ on P , which is induced by the action , is vertical and of constant integral length on $P_{|S_j}$. Thus $|A(\frac{\partial}{\partial\theta})|(x) = m_j \in \mathbb{Z}_{\geq 0}$ if $x \in S_j$ and therefore the boundary conditions 1.2 are satisfied.

Remark 1.2 1) Atiyah conjectured in [2] that any solution of the Bogomol'nyi equation 1.1 on hyperbolic 3-space , which satisfies 1.2 with integral m_j arises from a monopole on some S^1-equivariant bundle over S^4. The integrality condition is necessary as there are solutions of 1.1 satisfying 1.2 for non-integral m_j. One may conjecture the same in our more general situation.

2) The integrality of the m_j's is forced upon the monopole by our method and this seems to be a drawback of our method. A. Floer has exploited the fact that in a direct 3-dimensional approach the ***masses*** m_j can be varied continuously , see [17],[18].

In our setup the definition of an SU(2)-monopole is slightly more complicated. To explain where the complications come from we make a short digression to discuss various definitions of invariant connections. Let X be a manifold with an S^1-action and suppose P is any principal SU(2)-bundle over X. Let $\mathcal{A}^*(P)$ be the space of irreducible connections on P and $\mathcal{G}(P)$ the group of gauge transformations. It is always possible the find a bundle automorphism of P covering the diffeomorphism of X induced by the action of an element of S^1 on X. Moreover such bundle automorphisms are unique up to gauge transformations , so S^1 acts on $\mathcal{B}^* = \mathcal{A}^*(P)/\mathcal{G}(P)$ in a natural way. If $[A] \in \mathcal{B}^*$ is a fixed point and if S^1 also acts on P (and therefore on $\mathcal{A}^*(P)$), then A is invariant up to gauge transformations. We shall relate these fixed points to invariant connections under some action of S^1 on P.

Let $\mathcal{H}$ be the group of all bundle automorphisms of P which cover the action of S^1 on X. Then an S^1-action on P is a homomorphism $S^1 \to \mathcal{H}$ such that under composition with the natural map $\pi : \mathcal{H} \to \mathrm{Diff}(X)$ we end up with the S^1-action on X. Remark that there is an exact sequence :

$$\{e\} \to \mathcal{G}(P) \to \mathcal{H} \to S^1 \to \{e\}.$$

A class of connections $[A] \in \mathcal{B}^*$ is a fixed point iff the stabilizers $\mathcal{G}_A \subset \mathcal{G}(P)$ and $\mathcal{H}_A \subset \mathcal{H}$ satisfy that :

1.5 $$\{e\} \to \mathcal{G}_A \to \mathcal{H}_A \to S^1 \to \{e\}$$

is exact. Note that for irreducible $SU(2)$-connections A the stabilizer $\mathcal{G}_A = \{\pm 1\}$. (For irreducible $SO(3)$-connections the stabilizer is trivial.) Thus the only two possibilities for $\mathcal{H}_A$ in 1.5 are $\mathcal{H}_A \cong \tilde{S}^1$, the double cover of S^1, and $\mathcal{H}_A \cong (\mathbb{Z}/2\mathbb{Z}) \times S^1$. It follows that there exists a unique $\tilde{S}^1$-action $\phi : \tilde{S}^1 \to \mathcal{H}$ on P stabilizing A if $[A]$ is a fixed point in $\mathcal{B}^*$. A gauge transform $g \cdot A$ of A is stabilized by the action $g \circ \phi \circ g^{-1}$. (For irreducible $SO(3)$-connections the situation is simpler : $\mathcal{H}_A = S^1$ and there is a unique S^1-action on P stabilizing A.)

In the next section we shall see that, up to conjugation by gauge transformations, the $\tilde{S}^1$-actions can easily be classified. Denote these actions by $\phi_j : \tilde{S}^1 \to \mathcal{H}$, with j in some indexing set J, and let $\mathcal{A}^*_j \subset \mathcal{A}^*$ be the set of irreducible connections which are invariant under ϕ_j.

We have just seen that $\cup_{j\in J} \mathcal{A}^*_j \to (\mathcal{B}^*)^{S^1}$ is surjective. Denoting by $\mathcal{G}_j$ the ϕ_j-invariant gauge transformations we see that also :

1.6
$$\cup_{j\in J} (\mathcal{A}^*_j/\mathcal{G}_j) \to (\mathcal{B}^*)^{S^1}$$

is surjective. The final claim is :

Theorem 1.3 : The map in 1.6 is a bijection.

Proof : We need only show injectivity in view of the above. Now an irreducible connection is stabilized by a unique $\tilde{S}^1$-action. Suppose that $A_0 = g\cdot A_1$ with $[A_i] \in (\mathcal{B}^*)^{S^1}$, $g \in \mathcal{G}(P)$ and $A_i \in \mathcal{A}_{j_i}$, $i=0,1$, for some $j_i \in J$. We have to prove $g \in \mathcal{G}_{j_0}$. From the uniqueness we get that $\phi_{j_1} = g\circ\phi_{j_0}\circ g^{-1}$. But then $j_0=j_1$ and $g \in \mathcal{G}_{j_0}$, hence 1.6 is injective. ∎

Remark : Obviously the above discussion generalizes to connections invariant under the action of some connected Lie group K which acts on X. Forgacs and Manton [19] gave a proof of theorem 1.3 for these invariant connections which superficially looks very different. However , solutions of their crucial partial differential equation are sections of the exact sequence 1.5 , so essentially the two approaches agree.

It is now easy to see what the definition of an SU(2)-monopole should be. Let X be as above , and let P be a principal SU(2)-bundle on which the double cover $\tilde{S}^1$ of S^1 acts by bundle automorphisms covering the action of S^1 on X.

Definition 1.4 : An SU(2)-*monopole* on P is an $\tilde{S}^1$-invariant instanton on P. The *moduli space of monopoles* is the space $\mathcal{M}(P)$ = {$\tilde{S}^1$-invariant instantons on P}/{$\tilde{S}^1$-inavariant gauge transformations}.∎

Example 1.5 1) The four sphere S^4 is the S^1-equivariant conformal compactification of $H^3 \times S^1$, with H^3 hyperbolic 3-space. The 't Hooft instanton is an instanton on the spin bundle S_- of S^4 which is an SU(2)-bundle. This instanton is S^1-invariant up to gauge tranformations if the centre lies in the fixed point set $S^2 \subset S^4$, but there is no S^1-action on S_-, which stabilizes this instanton. There is however an $\tilde{S}^1$-action on S_- stabilizing the 't Hooft instanton ; thus the basic instanton is also a hyperbolic monopole with mass equal to 1.

2) Next we discuss in some detail the hyperbolic version of the 't Hooft-Polyakov monopole of arbitrary mass m on H^3. In geodesic normal coordinates the metric of H^3 reads :

$$ds^2 = dl^2 + \sinh^2 l\cdot(d\psi^2 + \sin^2\psi\cdot d\phi^2) \; ,$$

with $l \in \mathbb{R}_{\geq 0}$ and (ψ,ϕ) the standard coordinates on S^2. If $\tau_j \in su(2)$ is a basis satisfying $[\tau_i,\tau_j] = -\epsilon_{ijk}\cdot\tau_k$ then the connection and Higgs field for the 1-monopole of mass $m = \alpha - 1$, with centre at $l = 0$ are equal to :

$$A_3 = \left[\frac{\alpha\cdot\sinh l}{\sinh(\alpha l)}\right]\cdot\tau_2\cdot d\psi + \left[\frac{\alpha\cdot\sinh l}{\sinh(\alpha l)}\cdot\sin(\psi)\cdot\tau_1 + \cos(\psi)\cdot\tau_3\right]\cdot d\phi$$
$$\Phi = [\coth l - \alpha\cdot\coth(\alpha l)]\cdot\tau_3 \; ,$$

see Chakrabarti [12]. We see that Φ vanishes at the point $l=0$ and this defines the *centre of the monopole* to be the point $l=0$. All other monopoles of charge one and mass m can be obtained from the given one by applying an isometry of H^3. Having written down the formulas it is worth making some further remarks.

First of all note that on the two-sphere at ∞ the curvature F^{A_3} equals :

$$F^{A_3}\big|_{S^2_\infty} = \sin(\psi)\cdot\tau_3\cdot d\phi \wedge d\psi$$

which is the ordinary volume element. We see that near infinity the connection approximates a $U(1)$-connection on the two-sphere and Φ becomes the unique covariantly constant section of $P \times_{\mathrm{Ad}} su(2)$ of length m for this connection. Also we see that the bigger m is the more localized the non-Abelian part of the connection is.

If the mass m is an integer then the monopole can also be interpreted as an S^1-invariant instanton on S^4 , but for non-integral mass there is monodromy around $S^2 \subset S^4$.

More examples can be found in §5 of chapter 2 and in §6 below. We proceed to study the topology of the equivariant bundles occurring in the definitions above.

§2 Topology of bundles and index computations

Our aim here is twofold. We start by discussing the topological invariants of the equivariant bundles occurring in the definitions of monopoles, and, secondly, we compute the equivariant index of the omnipresent Dirac operator, which governs the deformation theory of monopoles.

We shall start discussing $\tilde{S}^1$-equivariant principal SU(2)-bundles. Afterwards we will indicate the changes which have to be made for SO(3)-bundles. As we explained in §1, for every fixed surface $S_j \subset X$ we get a ***mass*** or ***weight*** $m_j \in \mathbb{Z}_{\geq 0}$. Let $E = P \times_{SU(2)} \mathbb{C}^2$. If $m_j \neq 0$ then $E_{|S_j}$ splits as :

2.1 $$L_j \oplus L_j^* ,$$

where $\tilde{S}^1$ acts on L_j (L_j^*) by scalar multiplication with weight m_j $(-m_j)$. Now a further invariant is the *charge* k_j, which is defined by :

2.2 $$k_j \cdot x_j = c_1(L_j^*) \in H^2(S_j;\mathbb{Z}) ,$$

with $k_j \in \mathbb{Z}$ and x_j a positive generator of $H^2(S_j;\mathbb{Z})$; this requires an orientation of S_j : X gets an orientation from the orientation of the 3-manifold, the normal bundle N_j of S_j in X is oriented by the S^1-action, thus S_j inherits an orientation. If $m_j = 0$ then $E_{|S_j}$ is trivial as an S^1-equivariant bundle, and we shall leave k_j undefined.

There is one important constraint on the m_j. Recall that $-1 \in \tilde{S}^1$ acts trivially on X, so it acts on E by a gauge transformation of order two , that is , it acts as -1 or $+1$ on E. It follows that all m_j are either even or odd. That this is the only constraint follows from :

Proposition 2.1 : Isomorphism classes of $\tilde{S}^1$-equivariant SU(2)-bundles are in one-one correspondence with tuples of integers $(m_j,k_j)_{j=1,\dots,N}$ where $k_j \in \mathbb{Z}$ is undefined for $m_j=0$, and the $m_j \in \mathbb{Z}_{\geq 0}$ are all either even or all odd.

Proof: Clearly the m_j and k_j depend only on the isomorphism class.

To understand the structure of these $\tilde{S}^1$-equivariant SU(2)-bundles first consider the restriction to the open set $M \times S^1 \subset X$. Use the double cover $p : M \times \tilde{S}^1 \to M \times S^1$ to form the pullback $P' = p^*P_{|M\times S^1}$, which carries a free $\tilde{S}^1$-action. Because the action is free , P' is a pullback of an SU(2)-bundle P'' on M. As all SU(2)-bundles on M are trivial , it follows that the $\tilde{S}^1$-equivariant isomorphism class of $P_{|M\times S^1}$ is determined by the sign of the action of $-1 \in \tilde{S}^1$.

Next we concentrate on a neighbourhood of the $S_j \subset X$; such a neighbourhood is always S^1-equivariantly diffeomorphic to $S_j \times \mathbb{C}$, because the normal bundles N_j of S_j are trivial. It is easy to see that $E_{|S_j\times\mathbb{C}}$ is $\tilde{S}^1$-equivariantly isomorphic to $\pi^*E_{|S_j}$ with $\pi : S_j \times \mathbb{C} \to S_j$ the projection. Clearly $\tilde{S}^1$-equivariant isomorphism classes of $\tilde{S}^1$-equivariant SU(2)-bundles on S_j are determined by a pair (m_j,k_j).

Finally we need an $\tilde{S}^1$-equivariant transition function $(S_j\times\mathbb{C}) \cap (M\times S^1) \cong S_j\times(\mathbb{C}\setminus\{0\}) \to SU(2)$. This is the same as a $\mathbb{Z}/2\mathbb{Z}$ - equivariant transition function on a slice $S_j\times\mathbb{R}_{>0}$. Such a function exists if and only if the parity of m_j agrees with the sign of the action on $P_{|M\times S^1}$. The transition functions in question are trivial up to equivalence , as $\mathrm{Maps}(S_j,SU(2))$ is connected.

This proves the proposition. ∎

For an S^1-equivariant SO(3)-bundle Q let $\mathfrak{g}_Q = Q \times_{Ad} so(3)$ be the bundle of Lie algebras. If the action on the restriction of Q to S_j is non-trivial, then the S^1-action splits $\mathfrak{g}_{Q|S_j}$ as :

$$\mathfrak{g}_{Q|S_j} = K_j \oplus \mathbb{R} , \tag{2.3}$$

where S^1 acts on K_j by scalar multiplication of weight $m_j \in \mathbb{Z}_{>0}$ thereby turning K_j into a complex line bundle with :

$$c_1(K_j) = -q_j \cdot x_j \quad (q_j \in \mathbb{Z}).$$

If the action on $Q_{|S_j}$ is trivial, we shall put $m_j=0$ and leave q_j undefined. If Q is associated to an $\tilde{S}^1$-equivariant SU(2)-bundle via the adjoint representation, then the two definitions of m_j agree, and $q_j = 2 \cdot k_j$, because $K_j = L_j \otimes L_j$. Using the map $j: M \to X : m \to (m,1)$ another invariant of SO(3)-bundles is the second Stiefel-Whitney class $w_2(j^*Q) \in H^2(M;\mathbb{Z}/2\mathbb{Z})$. The following proposition shows that these invariants determine the bundle, and tells which relations exist among the invariants.

Proposition 2.2 : Isomorphism classes of S^1-equivariant SO(3)-bundles on X are in one-one correspondence with sets of invariants $((m_j,q_j)_{j=1,N}, w_2)$ with $m_j \in \mathbb{Z}_{\geq 0}$, $q_j \in \mathbb{Z}$, and $w_2 \in H^2(M;\mathbb{Z}/2\mathbb{Z})$. The only constraints on these variables are that $w_2(Q)_{|S_j} = q_j \cdot x_j \pmod 2$ for all j for which q_j is defined.

Proof : The proof is very similar to that of proposition 2.1. Now $Q_{|M \times S^1}$

is the pullback of an SO(3)-bundle Q' on M. Such Q' are determined by $w_2 = w_2(Q') \in H^2(M;\mathbb{Z}/2\mathbb{Z})$, see Freed-Uhlenbeck [20] , appendix E. Any $w_2 \in H^2(M;\mathbb{Z}/2\mathbb{Z})$ occurs as the class of a bundle. To see this , choose a map $M \to K(\mathbb{Z}/2\mathbb{Z},2)$; to get a lift $M \to BSO(3)$, apply obstruction theory to the pullback of the fibration $BSU(2) \to BSO(3) \to K(\mathbb{Z}/2\mathbb{Z},2)$ under the given map (for the fibration see Freed a.o. , loc. cit.). Existence of the transition fuctions requires that $Q_{|S_j}$ is isomorphic to Q restricted to the end of M going out towards S_j. This implies the relation between q_j and w_2 , because $w_2(Q_{|S_j}) = q_j \cdot x_j \pmod 2$ if q_j is defined.∎

There is difference in the nature of the constraints occurring in the two propositions above. The equality of the parity in proposition 2.1 is an artifact of our construction , whereas the constraint on the w_2 would also appear in a direct approach on M. Let us finally remark that if P and Q are as above , then there are various ways of proving that :

2.4
$$c_2(P) = \Sigma_j\ m_j \cdot k_j \quad ,$$
$$p_1(Q) = \Sigma_j\ 2 \cdot m_j \cdot q_j \quad ,$$

see chapter 2 ,§5 and Atiyah [2].

Next we turn to the index calculation. We treat only the case of an $\tilde{S}^1$-equivariant SU(2)-bundle P. Let $\mathfrak{g}_P = P \times_{Ad} su(2)$ be the bundle of Lie algebras associated to P. Denote by Λ^j the vector bundle of j-forms on X , by Λ^2_+ the bundle of self-dual 2-forms on X and by P_+ the projection $\Lambda^2 \to \Lambda^2_+$. The basic elliptic complex {P} occuring in the analysis of instantons reads ($A \in \mathfrak{A}(P)^{\tilde{S}^1}$ a monopole on X) :

2.5 $$\Gamma(\mathfrak{g}_P) \xrightarrow{d_A} \Gamma(\Lambda^1 \otimes \mathfrak{g}_P) \xrightarrow{P_+ d_A} \Gamma(\Lambda^2_+ \otimes \mathfrak{g}_P) \ .$$

Here the complex , and its cohomology vector spaces H^j are acted upon by $\tilde{S}^1$. In what follows we shall consider only this circle action and therefore denote it by S^1 ; thus S^1-equivariant , S^1-invariant etc. is to be understood with respect to the $\tilde{S}^1$-action. For our study of monopoles we shall be interested in the weight zero subspace H^1_0 of H^1 : under the assumption $H^0_0 = H^2_0 = 0$, H^1_0 is the tangent space at A to the moduli space of monopoles on X (see §3).

The Atiyah-Segal-Singer Lefschetz-formula expresses the S^1-character

2.6 $$\mathrm{ind}_{S^1}(\{P\}\otimes\mathbb{C}) \ \in \ R(S^1)$$

of the virtual representation $HT = (H^0 - H^1 + H^2) \otimes \mathbb{C}$, in terms of topological data. Assuming $H^0 = H^2 = 0$, we find

2.7 $$\begin{aligned} \dim_{\mathbb{C}} (H^1_0 \otimes \mathbb{C}) &= \dim_{\mathbb{R}} H^1_0 \\ &= - \int_{S^1} \mathrm{ind}(\{P\}\otimes\mathbb{C})(t)\ dt \\ &= - \text{ constant term in } \mathrm{ind}(\{P\}\otimes\mathbb{C})(t) \in \mathbb{Z}[t,t^{-1}] \ . \end{aligned}$$

To compute this index first recall from Atiyah-Hitchin-Singer [6] that the operator

$$d_A \oplus (P_+ d_A)^* \ : \Gamma((\mathfrak{g}_P \oplus (\mathfrak{g}_P \otimes \Lambda^2_+)) \otimes \mathbb{C}) \ \rightarrow \ \Gamma(\mathfrak{g}_P \otimes \Lambda^1 \otimes \mathbb{C})$$

is nothing but the Dirac operator on $S_+ \otimes \mathfrak{g}_P$:

2.8 $$D_A : \Gamma(S_+ \otimes S_+ \otimes \mathfrak{g}_P) \to \Gamma(S_- \otimes S_+ \otimes \mathfrak{g}_P) ,$$

where S_+ , S_- are spin bundles of positive and negative chirality on X . Denote the complex 2.8 by $\{P\}\otimes\mathbb{C}$ too.

A computationally pleasant way to find the index is to invoke equivariant cohomology , see Atiyah-Bott [3] for background. Let ES^1 and BS^1 be the universal bundle and classifying space for S^1. For any S^1-manifold Y , define the homotopy quotient Y_{S^1} to be the associated bundle $Y_{S^1} = ES^1 \times_{S^1} Y$ over BS^1. Then the equivariant cohomology $H^*_{S^1}(Y;\mathbb{Z})$ is by definition $H^*(Y_{S^1};\mathbb{Z})$, which is a module over $H^*(BS^1;\mathbb{Z})$. For an S^1-equivariant vector bundle $V \to Y$ define $ch_{S^1}(V)$ to be $ch(V_{S^1}) \in H^*_{S^1}(Y;\mathbb{Z})$ and similarly for other characteristic classes. For S^1-representations V , the map :

$$\begin{aligned} R(S^1) = \mathbb{Z}[t,t^{-1}] \quad &\to \quad H^{**}(BS^1) = \text{completion of } H^*_{S^1}(\text{point}) = \mathbb{Z}[[u]] \\ V \quad &\to \quad ch_{S^1}(V) = ch(ES^1 \times_{S^1} V) \end{aligned}$$

is given by $\Sigma\, a_j t^j \to \Sigma\, a_j (e^u)^j$, see Atiyah-Hirzebruch [4] 4.3. Therefore 2.7 equals minus the constant term in an expansion of $ch(HT_{S^1})$ in e^u .

Now the index formula (Atiyah-Singer [7]) gives :

$$ch_{S^1}(HT) = \pi_*^{S^1}\{ch_{S^1}(\mathfrak{g}_P \otimes S_+)\cdot\hat{A}_{S^1}(X)\} ,$$

with $\pi_*^{S^1}$ the push-forward to a point in equivariant cohomology. By the localization formula in equivariant cohomology (Atiyah - Bott [3]) this equals :

$$\sum_j \int_{S_j} e_{S^1}(N_j)^{-1}\cdot ch_{S^1}(\mathfrak{g}_P \otimes S_+)_{|S_j} \cdot \hat{A}_{S^1}(X)_{|S_j} ,$$

with $e_{S^1}(N_j)$ the equivariant Euler class of the normal bundle N_j of S_j. We proceed to compute the relevant characteristic classes restricted to S_j.

The equivariant Euler class of N_j is equal to

$$2\cdot u \in H^*_{S^1}(S_j) \cong H^*(S_j) \otimes H^*(BS^1) \cong H^*(S_j) \otimes \mathbb{Z}[u] .$$

Using the decomposition (2.1) we get (x_j a positive generator of $H^2(S_j)$):

$$\begin{aligned} ch_{S^1}(\mathfrak{g}_P)_{|S_j} &= ch_{S^1}(L_j^2) + ch_{S^1}(L_j^{-2}) + ch_{S^1}(\mathbb{C}) \\ &= e^{2(-k_j x_j + m_j u)} + e^{2(k_j x_j - m_j u)} + 1 \\ &= (1 + 2\cdot\cosh(2\cdot m_j\cdot u)) - 4\cdot k_j\cdot x_j\cdot\sinh(2\cdot m_j\cdot u) . \end{aligned}$$

Furthermore , $S_+ \otimes S_+ = (\Lambda^2_+ \otimes \mathbb{C}) \oplus \mathbb{C}$, and $\Lambda^2_+ \otimes \mathbb{C}_{|S_j} \cong \mathbb{C} \oplus (\Lambda^{1,0}(S_j) \otimes_{\mathbb{C}} N_j^*) \oplus (\Lambda^{0,1}(S_j) \otimes_{\mathbb{C}} N_j)$, where N_j has been given a $\mathbb{C}$-structure , using the S^1-action. It follows that :

$$\begin{aligned} ch_{S^1}(S_+) &= e^{(-u + \frac{1}{2}\cdot c_j\cdot x_j)} + e^{(u - \frac{1}{2}\cdot c_j\cdot x_j)} \\ &= 2\cdot\cosh(u) - c_j\cdot x_j\cdot\sinh(u) \end{aligned}$$

Where $c_j\cdot x_j = c_1(\Lambda^{1,0}S_j) = 2\cdot(g_j - 1)\cdot x_j$, with g_j is the genus of S_j . Finally we need the equivariant $\hat{A}$ genus of X restricted to S_j . It follows from $T^*X_{|S_j} \cong T^*S_j \oplus N_j^*$, that $(p_1)_{S^1}(T^*X)_{|S_j} = (c_1)^2_{S^1} - 2(c_2)_{S^1}$ $= (c_j\cdot x_j - 2\cdot u)^2 + 4\cdot u\cdot c_j\cdot x_j = (2u)^2$, so

$$\hat{A}_{S^1|S_j} = \frac{2\cdot u}{e^{u} - e^{-u}}$$
$$= u\cdot \sinh^{-1}(u).$$

Combining the formulas above we obtain :

$$\begin{aligned}
ch_{S^1}(HT) &= \pi_*^{S^1}\{ch_{S^1}(g_P \otimes S_+)\cdot \hat{A}_{S^1}(X)\} \\
&= \sum_j \left[\int_{S_j} e_{S^1}(N_j)^{-1}\cdot ch_{S^1}(g_P \otimes S_+)_{|S_j}\cdot \hat{A}_{S^1}(X)_{|S_j} \right] \\
&= \sum_j \left[\int_{S_j} \tfrac{1}{2}\cdot \sinh^{-1}(u)\cdot \Big(-2\cdot\cosh(u)\cdot 4\cdot k_j\cdot x_j\cdot \sinh(2\cdot m_j\cdot u) \right. \\
&\qquad\qquad \left. c_j\cdot x_j\cdot (1 + 2\cdot\cosh(2\cdot m_j\cdot u)\cdot \sinh(u)) \Big) \right]
\end{aligned}$$

$$= \sum_{m_j\neq 0} (-4k_j - \tfrac{1}{2}\cdot c_j) - \sum_{m_j=0} \frac{3}{2}\cdot c_j + \sum_{n\neq 0} (\cdot\cdot\cdot\cdot)e^{nu} .$$

This results in :

Theorem 2.3 : The constant term in $-\mathrm{ind}(\{P\}\otimes\mathbb{C})(t)$ is equal to :

2.9 $$I(m_j,k_j) = \sum_{m_j\neq 0} (4\cdot k_j + (g_j - 1)) + \sum_{m_j=0} 3\cdot(g_j - 1) .$$

Moreover if all m_j are $\neq 0$, then this simplifies to :

2.10 $$I(m_j,k_j) = \sum_j 4\cdot k_j - (1 - b^1(M) + b^2(M)) ,$$

and

2.11 $$I(0,0) = 3\cdot(b^1(M) - b^2(M) - 1).$$

where $b^i(M) = \dim H^i(M;\mathbb{R})$.

Proof : Only 2.10 and 2.11 remain to be proved and follow directly from chapter 2 , proposition 2.2.∎

Remarks 2.4 : 1) For S^1-equivariant SO(3)-bundles Q with invariants $((m_j,q_j),w_2)$ the negative of the constant term in the S^1-equivariant index of the Dirac operator on $\mathfrak{g}_Q \otimes S_+$ equals:

2.12 $$I((m_j,q_j),w_2) = \sum_{m_j \neq 0} (2 \cdot q_j + (g_j - 1)) + \sum_{m_j = 0} 3 \cdot (g_j - 1) .$$

The proof is the same as that of theorem 2.3.

2) If X is a Kahler manifold we could have used the equivariant Riemann-Roch formula for the Dolbeault complex on $\mathfrak{g}_P \otimes \mathbb{C}$. One sees easily that $a \to a^{0,1}$ induces an isomorphism $H^1 \to H^1(\mathfrak{g}_P \otimes \mathbb{C})$.

The restrictions to S_j of the characteristic classes occurring in this procedure , show u^{-1} terms. These terms cancel upon summation . Conversely , one deduces in this way , that many of the X built from 3-manifolds are not complex manifolds.

§3 Deformation theory of monopoles

Let $A_0 \in \mathcal{A}(P)^{S^1}$ be a monopole , i.e. :

$$P_+ F^{A_0} = \tfrac{1}{2}(F^{A_0} - *_4 F^{A_0}) = 0$$

We shall look for nearby solutions $A = A_0 + p$ of these equations which are not gauge-equivalent to A_0. This deformation theory will turn out to be very similar to the deformation theory for instantons , see Donaldson [14] §4 , for the most complete treatment thus far. Before we start , it is necessary to establish some basic facts concerning the configuration space.

Choose a metric representing the conformal structure on X , and let P be an $\tilde{S}^1$-equivariant SU(2)-bundle ; we leave the SO(3) case to the reader. Recall that $\mathcal{A}(P)$, the space of connections on P , has completions

$$\mathcal{A}_{p,k}(P) = A + L^p_k[\Lambda^1(\mathfrak{g}_P)] \qquad (A \in \mathcal{A}(P)\ ,\ p > 1\ ,\ k \geq 0)\ ,$$

turning it into an affine Banach manifold ; here $L^p_k(E)$ means sections of the Hermitian vector bundle E whose derivatives up to order k are in L^p. $\tilde{S}^1$ acts continuously by affine transformations , but not smoothly on $\mathcal{A}_{p,k}(P)$. Define the configuration space $C_{p,k}(P)$ of the monopole problem to be :

3.1 $$C_{p,k}(P) = \mathcal{A}_{p,k}(P)^{\tilde{S}^1}.$$

Then $C_{p,k}(P)$ is a closed affine submanifold of $\mathcal{A}_{p,k}(P)$, equal to the

L^p_k-closure of $\mathfrak{a}(P)^{\tilde{S}^1}$ in $\mathfrak{a}(P)$. The underlying vector space is $\mathfrak{W}^1_{p,k}$ where $\mathfrak{W}^i_{p,k}$ and $\mathfrak{W}^i$ are defined as :

3.2 $$\mathfrak{W}^i_{p,k} = L^p_k(\Lambda^i(g_P))^{S^1} \quad , \quad \mathfrak{W}^i = \Gamma\Lambda^i(g_P)^{S^1}$$

In the same way :

3.3 $$GA_{p,k+1}(P) = \mathcal{G}_{p,k+1}(P)^{S^1}$$

is a closed Banach Lie subgroup of the group of gauge transformations $\mathcal{G}_{p,k+1}(P)$, provided $k+1-p/4 > 0$ (see Freed-Uhlenbeck [20]). The Lie algebra of $GA_{p,k+1}$ equals $\mathfrak{W}^0_{p,k+1}$. The action of $GA_{p,k+1}(P)$ on $C_{p,k}(P)$ is smooth , being the restriction of the $\mathcal{G}_{p,k+1}(P)$ action on $\mathfrak{a}_{p,k}(P)$. We shall assume that $k \geq 1$,$p \geq 2$ and $k-p/4 > 0$, then we are in the stable range for multiplication with L^p_k to be continuous and this will ensure continuity of the operators which we use below ; the condition $p \geq 2$ gives our L^p_k-spaces a positive definite inner product.

Remark : The various Sobolev spaces can be chosen in essentially different ways if one works on the 3-dimensional manifold itself. However , a direct approach on the 3-manifold seems hard at the moment because the Fredholm theory for the elliptic operators involved has yet to be developed. A. Floer has developed such a theory for *asymptotically Euclidean* 3-manifolds , see [16].

We proceed to construct a slice at $A \in C_{p,k}(P)$ for the $GA_{p,k+1}(P)$ action on $C_{p,k}(P)$. Identifying the tangent space $T_A C_{p,k}(P)$ with the space of S^1-invariant forms $\mathfrak{W}^1_{p,k}$ we see that :

3.4 $$T_A(GA_{p,k+1}(P)\cdot A) = d_A(\mathbb{W}^0_{p,k+1}) \subset T_A C_{p,k}(P)$$

Next remark that the Green operator $G_A : \mathbb{W}^0_{p,k+1} \to \mathbb{W}^0_{p,k-1}$ of $d^*_A \circ d_A$, which we define to be equal to $(d^*_A \circ d_A)^{-1}$ on $(\ker d_A)^{\perp_{L^2}}$ and zero on $\ker d_A$, is automatically $\tilde{S}^1$-invariant. Writing $p \in T_A C_{p,k}(P)$ as

3.5 $$p = d_A G_A d^*_A \cdot p + (1 - d_A G_A d^*_A)p \quad ,$$

we see that $(T_A(GA_{p,k+1}(P)\cdot A))^{\perp_{L^2}} \subset T_A C_{p,k}(P)$ equals :

3.6 $$H_A = \mathbb{W}^1_{p,k} \cap \ker d^*_A$$

and that $H_A \oplus T_A(GA_{p,k+1}(P)\cdot A) = T_A C_{p,k}(P)$. This is exactly the same as in the instanton case, apart from the fact that we only consider $\tilde{S}^1$-invariant objects here.

Precisely as in Atiyah-Hitchin-Singer [6], Donaldson [13], or Freed-Uhlenbeck [20], it follows that $S_A = A + W \subset C_{p,k}(P)$ is a slice for the $GA_{p,k+1}(P)$-action, with $W \subset H_A$ a small neighbourhood of 0. The stabilizer $\Gamma_A \subset GA_{p,k+1}(P)$ acts on H_A, and W can always be chosen in such a way that S_A is Γ_A-invariant. This stabilizer Γ_A consists of S^1-invariant, covariantly constant sections of $\mathfrak{g}(P) \cong P \times_{Ad} SU(2)$, and its Lie algebra equals $\ker (d_A) \cap \mathbb{W}^0$. If one of the integral invariants m_j of P is nonzero, then Γ_A is isomorphic to $S^1 \subset SU(2)$ for reducible A and to $\{\pm 1\}$ for irreducible A. If all $m_j = 0$ then $\Gamma_A \cong SU(2)$ if A is a connection with holonomy contained in $\{\pm 1\}$.

Just as for instantons $B_{p,k}(P) = C_{p,k}(P)/GA_{p,k+1}(P)$ is a smooth

Banach manifold away from the reducible connections and there are singularities at the reducible connections (cones on $\mathbb{C}P^{\infty}$) , which are caused by a jump of stabilizers. In any case $B_{p,k}(P)$ is a Hausdorff topological space , and this property is inherited by the moduli spaces of monopoles. Putting the results together we have proved :

Theorem 3.1 : For $k - 4/p > 0$, $p \geq 2$ the configuration space $C_{p,k}(P)$ (3.1) is a smooth affine Banach manifold with a smooth action of the Banach Lie group $GA_{p,k+1}(P)$ (3.3) on it. Slices for the action at any $A \in C_{p,k}(P)$ exist , and are equal to $A + W$, where W is a small neighbourhood of 0 in H_A (3.6). The quotient space $B_{p,k}(P) = C_{p,k}(P)/GA_{p,k+1}(P)$ is a Hausdorff topological space , and a smooth Banach manifold away from the reducible connections. ∎

We now proceed with a standard Lyapunov-Schmidt procedure to treat the deformation theory of a monopole , we stick to the assumptions made about p , k in theorem 3.1. Suppose that $A_0 \in C_{p,k}(P)$, is a monopole. Of interest are the zeroes of :

$$3.7 \quad K : H_{A_0} \rightarrow \mathfrak{W}^2_{+,p,k-1} \quad : \quad p \rightarrow P_+F^{A_0+p} = P_+(d_{A_0}p + \tfrac{1}{2}\cdot[p,p]) \quad ,$$

where $\mathfrak{W}^2_+$ is the space of $\tilde{S}^1$-invariant , self-dual 2-forms on X. Recall that to A_0 there is associated the elliptic complex 2.5 . Just as in 3.5 we can , using the Greens function for this complex , write direct sums :

$$3.8 \qquad H_{A_0} = H^1_0 \oplus (\mathrm{im}(P_+d_{A_0})^* \cap \mathfrak{W}^1_{p,k}) = H^1_0 \oplus C$$

$$\mathfrak{W}^2_{+,p,k-1} = V_{A_0} \oplus (\mathrm{im}(P_+ d_{A_0}) \cap \mathfrak{W}^2_{+,p,k-1}) = V_{A_0} \oplus I$$

with V_{A_0} a finite dimensional subspace of $\mathfrak{W}^2_+$, such that the L^2-projection to the $\tilde{S}^1$-invariant part H^2_0 of the 2-nd cohomology of 2.5 is an isomorphism. By Aronzajn's theorem [1] the forms in V_{A_0} can be assumed to have support in the complement of small open sets in X. The spaces C and I are defined to be the second summands of the sums occurring in the middle in 3.8.

By definition the derivative :

$$DK_{A_0} = P_+ d_{A_0} \quad : \quad C \rightarrow I$$

is an isomorphism. The implicit function theorem therefore supplies us with smooth maps , defined in a neighbourhood of $0 \in H^1_0$:

3.9
$$p \rightarrow \tilde{p} \quad : \quad H^1_0 \rightarrow H_{A_0}$$
$$p \rightarrow \phi(p) \quad : \quad H^1_0 \rightarrow V_{A_0}$$

which are Γ_{A_0}-equivariant and solve the eqation :

3.10
$$K(p) = P_+ F^{A_0+p} = \phi(p) \quad .$$

Conversily every small solution $q \in \mathfrak{W}^1$ of

$$d^*_{A_0} q = 0 \quad , \quad K(q) \in V_{A_0}$$

is of the form $q = \tilde{p}$. In the sequel we shall need the following estimates :

3.11 $$\|p - \tilde{p}\| = O(p^2) \quad , \quad \|\frac{d\tilde{p}}{dp} - \mathrm{id}\| = O(p) \quad .$$

Both of these readily follow from the implicit function theorem.

From the above we extract a local model of the moduli space as the quotient by Γ_{A_0} of the zeroes of the Γ_{A_0}-equivariant map $\phi_{|W} : W \to V_{A_0}$, with W a neighbourhood of $0 \in H^1_0$. Under the assumptions $p \geq 2$, $k-4/p>0$ all L^p_k-solutions of the Bogomol'nyi equation are automatically smooth , and the topology induced on the moduli spaces is independent of the precise choice of p,k.

The leading term of ϕ is generically quadratic and can be expressed very explicitly. The equation $\phi(p) = 0$ is equivalent with $< \omega , \phi(p) >_{L^2} = 0$, for all harmonic forms $\omega \in \ker (P_+ d_{A_0})^* \cong H^2_0$.

Now :

3.12 $$\begin{aligned} < \omega , \phi(p) >_{L^2} &= < \omega , P_+(d_{A_0}\tilde{p} + \tfrac{1}{2}[\tilde{p},\tilde{p}]) >_{L^2} \\ &= < \omega , \tfrac{1}{2}[\tilde{p},\tilde{p}] >_{L^2} \\ &= < \omega , \tfrac{1}{2}[p,p] >_{L^2} + O(p^3) \quad . \end{aligned}$$

Hence ϕ is a small C^1-perturbation of the quadratic form :

3.13 $$q : H^1_0 \to H^{2*}_0 \cong (\ker(P_+ d^*_{A_0}))^* \quad : \quad p \to \tfrac{1}{2}< ? , P_+[p,p]>_{L^2}$$

The stability theorem for submersions learns that if 0 is a regular value of the map :

$$q \ : \ H^1_0\setminus\{0\} \ \to \ (H^2_0)^*$$

then there exists a Γ_{A_0}-equivariant local homeomorphism around $0 \in U_{A_0}$, carrying the zero set of ϕ into the null-cone of q. This local homeomorphism is smooth away from the origin. Summarizing one has :

Theorem 3.2 : Let A_0 be a monopole on P, Γ_{A_0} its stabilizer in $GA(P)$, and H^1_0, H^2_0 the S^1-invariant parts of the cohomology groups of (2.5). There exists a Γ_{A_0}-invariant neighbourhood W of $0 \in H^1_0$, an embedding $\tilde{\ } : W \to H_{A_0}$, with derivative 1 at $0 \in W \subset H^1_0$ and a smooth map $\phi : W \to H^2_0$ (as in 3.9) such that:

1) $\tilde{\ }$ induces a Γ_{A_0}-equivariant homeomorphism of the zero set of ϕ in W to all S^1-invariant solutions of the anti-self-duality equation near A_0 in a slice.

2) the moduli space of monopoles is in a neighbourhood of A_0 the quotient of the zero set of the smooth function ϕ in W by Γ_{A_0}.

If $\Gamma_{A_0} = \{\pm 1\}$ and $H^2_0 = 0$, then the moduli space near A_0 is a smooth manifold of dimension given by theorem 2.3.∎

A very remarkable property of deformations of monopoles on $\mathbb{R}^3$ is that deformations $(a_3,\phi) \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^0(\mathfrak{g}_P)$ can after gauge transformation be assumed to have a finite L^2-norm ; this is rather non-trivial , see Atiyah-Hitchin [5]. A consequence of this fact is that monopoles on $\mathbb{R}^3$ can be moved slowly with finite energy. In hyperbolic spaces , which are a special case of our situation , this is very different. For example moving a $k=1$ monopole on H^3 by an isometry , induces a transformation on the curvature on $\delta H^3 = S^2$ which is the pullback under a fractional linear transformation (unless the isometry leaves the monopole invariant). A moments reflection shows that such a change cannot be removed by a gauge transformation and , indeed , the resulting infinitesimal deformations are not L^2. It will be interesting to see if motion of hyperbolic monopoles can be defined in a way which reflects geometrical properties of the moduli spaces as for $\mathbb{R}^3$ monopoles.

Related to this is the following. If (A_3, Φ) is any monopole on $\mathbb{R}^3$ then A restricted to the 2-sphere S^2_∞ exists , and is always the unique (up to gauge transformations) homogenous connection on the k-th power of the Hopf bundle on S^2. In contrast with this the restriction of A to S^2_∞ for hyperbolic monopoles depends on the monopole , and one can in fact show that it determines the monopole , see Braam [9].

§4 Smoothness , orientability and compactifications

In Freed-Uhlenbeck [20] it is proved that for a generic metric on X , the moduli spaces of irreducible instantons are smooth manifolds. Mimicking their approach we shall prove that for a generic S^1-invariant metric on X , the moduli spaces of irreducible monopoles are smooth. After this we show that it follows from Donaldson's orientability theorem [15] , that the monopole moduli spaces can be oriented in a canonical way , starting from an orientation of the real homology of X. Another important fact concerning instanton moduli spaces is that they have a compactification. Again , monopole moduli spaces can be compactified similarly.

Let P be an $\tilde{S}^1$-equivariant $SU(2)$-bundle with invariants (m_j,k_j).

Proposition 4.1 : For an open dense set of S^1-invariant metrics on X the space $\mathcal{M}^*(m_j,k_j)$ of irreducible monopoles in $\mathcal{M}(m_j,k_j)$ is a smooth (possibly empty) manifold of dimension $I(m_j,k_j)$ given by 2.9 , provided $\Sigma\, m_j \cdot k_j \neq 0$.

Proof : We shall outline the approach of Freed-Uhlenbeck [20] p.60 - 73 and indicate what changes one has to make to obtain our result. Fix an S^1-invariant C^q-metric g on X. The anti-self-duality equations read $P_+F^A = 0$, where P_+ is projection onto self-dual forms. Let $\mathfrak{D} = C^q(GL(TM))^{S^1}$ be the Banach manifold of S^1-invariant C^q-automorphisms of TX. Note that $\phi \in \mathfrak{D}$ acts on $\mathfrak{W}^2 = \Omega^2(g_P)$ and that the anti-self-duality equations for $A \in C(P)$ with respect to the metric $\phi^* g$ read $P_+(\phi^* F^A) = 0$.

Following Freed-Uhlenbeck study the map :

$$\mathfrak{P} : C^*(P) \times \mathfrak{D} \to \mathcal{W}^2_+ : (A,\phi) \to P_+(\phi^* F^A) \quad ,$$

with $C^*(P)$ denoting the space of irreducible invariant connections.

Lemma 4.2 : $\mathfrak{P}$ is smooth and has 0 as a regular value.

Proof : (sketch) As $\mathfrak{D}$ acts transitively on S^1-invariant C^k-metrics we should prove that if $\mathfrak{P}(A,\mathrm{id}) = 0$ then $d\mathfrak{P}_{(A,\mathrm{id})}$ is surjective. Assuming that this is not the case , there is an element $\omega \in \mathcal{W}^2_+$ in the cokernel of $d\mathfrak{P}$.

By a pointwise analysis we deduce just as in Freed-Uhlenbeck that on $X\setminus(\cup_j S_j)$ the rank of $F^A(x) : \Lambda^2_x(X) \to (g_P)_x$ is ≤ 1 ; it is here that the argument depends on $\Sigma\, m_j k_j \neq 0$. The S^1-invariance of the objects under consideration doesn't affect Freed and Uhlenbeck's proof , because the action on $X\setminus(\cup_j S_j)$ is free. The function $x \to \mathrm{rk}\, F^A(x)$ being lower semicontinuous it follows that $\mathrm{rk}\, F^A(x) \leq 1$ on X. But then Freed-Uhlenbeck conclude that A is reducible , contradicting $A \in C^*(P)$; thus $d\mathfrak{P}_{(A,\mathrm{id})}$ is surjective. This concludes the proof of lemma 4.2. ∎

From the lemma it follows that $\mathfrak{P}^{-1}(0) \subset C^*(P) \times \mathfrak{D}$ is a manifold. One finishes the proof by showing that $\mathfrak{P}^{-1}(0)/GA(P) \to \mathfrak{D}$ is a Fredholm map of index $I(m_j,k_j)$; the infinite dimensional Sard theorem gives the result. ∎

Proposition 4.2 : If $b^2(M) > b^1(M)$ then for an open dense set of C^k-metrics there are no reducible monopoles in $\mathcal{M}(m_j,k_j)$ if $\Sigma\, m_j k_j \neq 0$, i.e. $\mathcal{M}^*(m_j,k_j) = \mathcal{M}(m_j,k_j)$.

Proof : Proceeding exactly as in Freed-Uhlenbeck corollary 3.21 the result follows.∎

An open dense set of C^k-metrics contains smooth metrics. Combining propositions 4.1 and 4.2 we get:

Theorem 4.3 : For an open dense set of smooth S^1-invariant metrics the moduli space $\mathcal{M}^*(m_j,k_j)$ is a smooth manifold of dimension $I(m_j,k_j)$ if $\Sigma\, m_j \cdot k_j \neq 0$. If additionally $b^2(M) > b^1(M)$ holds , then one may assume that for a generic metric also $\mathcal{M}(m_j,k_j) = \mathcal{M}^*(M_j,k_j)$.∎

Another condition which ensures that $\mathcal{M}^*(m_j,k_j)$ is smooth is that on X there is a metric in the given conformal class of positive scalar curvature , see Atiyah-Hitchin-Singer [6]. In chapter 1 we analysed which X's arising from hyperbolic manifolds satisfy this condition.

Remark 4.4 : 1) If $Q \to X$ is an S^1-equivariant $SO(3)$-bundle with $w_2(Q) \neq 0$ then $\mathcal{M}(m_j,q_j,w_2) = \mathcal{M}^*(m_j,q_j,w_2)$. If additionally $\Sigma\, m_j \cdot q_j \neq 0$ then one can prove as above that $\mathcal{M}(m_j,q_j,w_2)$ is smooth for a generic smooth S^1-invariant metric.

2) A. Floer investigated monopole moduli spaces on asymptotically Euclidean 3-manifolds. One of his results is that these are smooth if the 3-manifold has positive definite Ricci curvature.

Next we shall show that the monopole moduli spaces are orientable :

Theorem 4.5 : Assume that $\mathcal{M}^*(m_j,k_j)$ is smooth and cut out transversely. An orientation of $H^1(M;\mathbb{R}) \oplus H^2(M;\mathbb{R})$ gives a canonical orientation of

$\mathcal{M}^*(m_j,k_j)$.

Proof : The operators (compare 2.8)

$$D_A : (\Lambda^2_+ \oplus \Lambda^0) \otimes g_P \to \Lambda^1 \otimes g_P$$

form a family of Dirac oprators on $\mathfrak{B}^* \times X \to \mathfrak{B}^*$, with $\mathfrak{B}^* = \mathfrak{A}^*(P)/\mathfrak{G}(P)$. Associated to this is a real determinant line bundle $\lambda \to \mathfrak{B}^*$ with fibre at [A] equal to :

$$\lambda_A = \Lambda^{\max}(\ker D_A)^* \otimes \Lambda^{\max}(\operatorname{coker} D_A) ,$$

where $\Lambda^{\max} V = \Lambda^{\dim V} V$ for any vector space V.

From Donaldson [15] we know that an orientation of $H^2_+(X;\mathbb{R}) \oplus H^1(X;\mathbb{R})$ canonically gives rise to a non-vanishing section of λ (up to positive pointwise multiplication by a positive function). Now $H^2_+(X;\mathbb{R}) \cong H^2(M;\mathbb{R})$ using the Hodge star (compare chapter 2 , propositîton 2.2). It follows that $H^2_+(X;\mathbb{R})$ gets a standard orientation from that of $H^2(M;\mathbb{R})$. As $H^1(X;\mathbb{R}) \cong H^1(M;\mathbb{R})$ our data also give a non-vanishing section of λ.

We may assume that $\mathcal{M}^*(m_j,k_j) \subset \mathfrak{B}^*$, by theorem 1.3. If $A \in \mathcal{M}^*(m_j,k_j)$ then $\ker D_A \cong T_A\mathcal{M}(m_j,k_j) \oplus N_A$ where N_A is an S^1-representation , in which weight 0 does not occur. This also holds for $\operatorname{coker} D_A$, because we assumed $\mathcal{M}(m_j,k_j)$ to be cut out transversely. Hence N_A and $\operatorname{coker} D_A$ are canonically oriented. Now :

$$\lambda_A \cong \Lambda^{\max}(T_A\mathcal{M}^*(m_j,k_j))^* \otimes \Lambda^{\max}(N_A)^* \otimes \Lambda^{\max}(\operatorname{coker} D_A) ,$$

so our data give a canonical non-vanishing section of $\Lambda^{\max}(T\mathcal{M}^*(m_j,k_j))$, i.e. an orientation of $\mathcal{M}^*(m_j,k_j)$. ∎

Remark 4.6 : For monopole moduli spaces $\mathcal{M}(Q)$ with Q an S^1-equivariant $SO(3)$-bundle , we get a canonical orientation from an orientation of $H^1(M;\mathbb{R}) \oplus H^2(M;\mathbb{R})$ if $w_2(Q) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ lies in the image of the natural map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}/2\mathbb{Z})$. This follows from Donaldson [15] , like theorem 4.5 , where we additionally use that X is always a Spin manifold , see chapter 2 , proposition 2.2.

The final topic to be discussed in this section is that of compactifying $\mathcal{M}(m_j,k_j)$. We fist recall briefly the situation for instantons. Let $[A_k] \in \mathcal{I}_p$ be a sequence of instantons on a bundle P with $c_2(P) = p$, where $\mathcal{I}_p$ denotes the *instanton moduli space*. From Uhlenbeck's weak compactness theorem it follows that , after going over to a subsequence $[A_k]$, there is a sequence of gauge transformations $g_k \in \mathcal{G}(P)$ and a finite set of points $\{x_1,\ldots\ldots,x_n\} \in X$, counted with multiplicities , such that :

4.1 1) On $X\backslash\{x_1,\ldots\ldots,x_n\}$, $g_k \cdot A_k \to A$ in the C^∞-topology , with A_∞ an instanton on a bundle $P' \to X$ with $c_2(P') = c_2(P) - n$.

2) The functions $|F^{A_k}|^2$ converge to $8\pi^2\delta_{x_j} + |F^{A_\infty}|^2$ in the sense of measures.

Using this control , it is possible to define a compactification $\bar{\mathcal{I}}_p$ of $\mathcal{I}_p$, equal to the closure of $\mathcal{I}_p$ in the space :

$$\mathcal{I} = \mathcal{I}_p \cup (\mathcal{I}_{p-1} \times X) \cup (\mathcal{I}_{p-2} \times \mathcal{S}^2(X)) \cup \ldots \cup (\mathcal{I}_0 \times \mathcal{S}^p(X))$$

which one gives a topology using 4.1 ; here $\mathcal{S}^j(X)$ is the space of unordered j-tuples of point in X. Observe that $\mathcal{G}$ is compact , so $\mathcal{G}_p$ is compact. We shall show here that a similar compactification of the monopole moduli spaces exists , adding lower strata which consist of pairs of a set of points in boundary surfaces of M and a monopole of lower charge.

Let $[A_i]$ be a sequence in $\mathcal{M}(m_j,k_j)$, and again consider the convergence 4.1. Clearly by the S^1-invariance the x_i lie in some S_j , so we change the indices to j_i meaning x_{j_i} is the i-th point in S_j. Denote by μ_{j_i} its multiplicity.

Proposition 4.7 : 1) $\mu_{j_i} = \lambda_{j_i} \cdot m_j$ for some $\lambda_{j_i} \in \mathbb{Z}_{>0}$.

2) $[A_\infty] \in \mathcal{M}(m_j, l_j)$ with $l_j = k_j - \Sigma_i \lambda_{j_i}$ and $\Sigma_j m_j \cdot l_j \geq 0$.

Proof : There is sequence of gauge transformations $g_i \in \mathcal{G}(P)$ (so they don't necessarily commute with the $\tilde{S}^1$-action) such that $g_i \cdot A_i \to A_\infty$, in the C^∞-topology on X with the points removed. Denote our original $\tilde{S}^1$-action on P by a homomorphism $\theta : \tilde{S}^1 \to \mathrm{Aut}(P)$.

Lemma : Suppose A_i , A_∞ , $B(t)$ are connections on some principal SU(2)-bundle over a smooth manifold with $B(t)$ depending smoothly on t. Let $h_i(t)$ be gauge transformations also depending smoothly on t. If $A_i \to A_\infty$ and $h_i(t) \cdot A_i \to B(t)$ both converge in the C^∞-topology and uniformly in t , then a subsequence $h_j(t)$ converges in the C^∞-topology to a family of gauge transformations $h(t)$, uniformly in t , where $h(t)$ has the property that $h(t) \cdot A_\infty = B(t)$.

Proof : The proof is standard and can be modelled on that of proposition A.5 in Freed-Uhlenbeck.∎

Now apply this to :

$$g_i \cdot A_i \rightarrow A_\infty$$

$$h_i(t) \circ g_i \cdot A_i \rightarrow \theta(t) \cdot A_\infty \ , \quad h_i(t) = \theta(t) \circ g_i \circ \theta(-t) \circ g_i^{-1}$$

then we may assume $h_i(t) \rightarrow h(t)$ and $t \rightarrow h(t)^{-1} \circ \theta(t)$ defines an action on $P_{|X \setminus \{x_{j_i}\}}$, stabilizing A_∞. This action is the limit of the actions $t \rightarrow g_i \circ \theta(t) \circ g_i^{-1}$, so its weights are equal to those of θ.

Considering A_∞ as a connection on some SU(2)-bundle P' has two disadvantages : A_∞ is $\tilde{S}^1$-invariant on $P'_{|X \setminus \{x_{j_i}\}}$, but not yet invariant on P' and , furthermore , it is not clear that the gauge transformation removing the singularity respects the $\tilde{S}^1$-action. The latter is needed to ensure that the multiplicities are multiples of m_j.

For this reason consider A_∞ as a singular connection on P. We shall study the $\tilde{S}^1$-equivariance properties of the gauge transformation which removes the singularities. Start by taking a small S^1-invariant ball B centered at one of the singular points x_{j_i} and a section $s : B \rightarrow P$ such that the action of $\tilde{S}^1$ on P is described by :

$$u \cdot s(b) = s(u \cdot b) \cdot \lambda(u) \qquad (u \in S^1 \ , \ b \in B)$$

for a homomorphism $\lambda : \tilde{S}^1 \rightarrow SU(2)$. Identifying $s^* A_\infty$ with A_∞ , the $\tilde{S}^1$-invariance of A_∞ reads :

4.2 $$u^* A_\infty = \lambda(u) \cdot A_\infty .$$

Here the action of a gauge transformation $g : B \rightarrow SU(2)$ on A_∞ is given by : $g \cdot A_\infty = gdg^{-1} + g \cdot A_\infty \cdot g^{-1}$.

The removability of singularities theorem (see Freed-Uhlenbeck [20]) , which depends on $\int_B |F^{A_\infty}|^2 \, dV < \infty$, asserts that (possibly after shrinking B) there exists an L_2^2-gauge transformation g such that :

1) $d^*(g \cdot A_\infty) = 0$
2) $g \cdot A_\infty(\frac{\partial}{\partial r}) = 0$ on δB
3) $\|g \cdot A_\infty\|_{L_1^2(B)} \leq cst \cdot \|F^{A_\infty}\|_{L^2(B)}$
4) g is smooth on $B \setminus \{0\}$.

With such a g , elliptic regularity implies that $g \cdot A_\infty$ is smooth. Furthermore , there are no infinitesimal L_2^2-deformations of g preserving the properties 1)...4) , apart from those arising from composition on the left with constant gauge transformations.

Next remark that $d^*(u^* g \cdot A_\infty) = 0$ and that

$$u^* g \cdot A_\infty = h(u) \cdot g \cdot A_\infty \tag{4.3}$$

for the gauge transformation $h(u) = (u^* g) \circ \lambda(u) \circ g^{-1}$ which is possibly singular at 0. It remains to show that $\frac{\partial h}{\partial u}(1) = w$ is in L_2^2 , because then it is constant and equal to $\frac{\partial \lambda}{\partial u}(1)$, up to conjugation. This implies that $g \cdot A_\infty$ is invariant as in 4.2 , and g is an $\tilde{S}^1$-equivariant gauge transformation which can only change $c_2(P)$ by multiples of m_j.

Now 4.3 gives that the Lie algebra valued function w satisfies the ordinary differential equation

$$dw + [g \cdot A_\infty, w] = L_{\frac{\partial}{\partial \theta}} (g \cdot A_\infty) \quad ,$$

so w is certainly L_2^2.

This finishes the proof of proposition 4.7.∎

It follows that we can define a compactification $\overline{\mathcal{M}}(m_j,k_j)$ of $\mathcal{M}(m_j,k_j)$ which is the closure of $\mathcal{M}(m_j,k_j)$ in the compact space of ideal monopoles $\mathcal{I}$:

$$\mathcal{I} = \bigcup_{\substack{\{(l_j)\ ;\ j \text{ is s.t. } m_j\neq 0 \\ 0\leq l_j\ ,\ \Sigma m_j l_j \geq 0\}}} \left[\mathcal{M}(m_j,k_j-l_j) \times \prod_{\substack{j \text{ s.t.} \\ m_j\neq 0}} \mathcal{S}^{l_j}(S_j)\right] ,$$

where $\mathcal{S}^{l}(S_j)$ is the l-th symmetric product of S_j , normalized such that $\mathcal{S}^{0}(S_j)$ is a set with one element. The topology on $\mathcal{I}$ is defined using 4.1 ; thus a point $(A_\infty,\{x_{j_i}\}) \in \mathcal{I}$ lies in $\overline{\mathcal{M}}(m_j,k_j)$ precisely if there is a sequence $A_i \in \mathcal{M}(m_j,k_j)$ such that 4.1 holds. This expresses the fact that a monopole can loose l_j lumps which move to definite points in surfaces S_j for which $m_j \neq 0$. It should be noted that l_j could become $> k_j$, although no example of this is known.

In §1 we indicated that $\mathcal{M}(m_j,k_j)$ could be considered as a subset of an instanton moduli space. The same now holds for the compactified moduli spaces , but attention should be drawn to the fact that the multiplicities of the tuples of points in $\mathcal{M}(m_j,k_j)$ differ by factors m_j from the multiplicities in the compactifications of instanton moduli spaces.

Taubes [23] has studied the convergence properties of sequences of monopoles on $\mathbb{R}^3$, and he shows that essentially the same compactification exists for monopoles on $\mathbb{R}^3$, as the one we constructed for monopoles on H^3.

§5 Existence theorems

For the construction of solutions of the Bogomol'nyi equation near the lower strata in the compactification we shall use Donaldson's [14] alternating procedure , which describes instantons on connected sums of two 4-manifolds. As we shall see , this needs only minor modifications in order to deal with S^1-invariant instantons. Let X_0 , X_1 be two Riemannian 4-manifolds with S^1-actions which are constructed from two 3-manifolds M_0 and M_1 as in §1. Let $x_0 \in X_0$ and $x_1 \in X_1$ be points lying in fixed surfaces , and first assume that the Riemannian metrics on X_0 and X_1 are conformally flat in neighbourhoods of x_0 and x_1. This holds automatically for those X's arising from Kleinian groups , see chapter 2. In this situation we can form an S^1-equivariant conformal connected sum $X_0 \# X_1$. Choose coordinates ξ , η on X_0 , X_1 , such that the metric in these coordinates is Euclidean and such that $\xi(x_0) = 0$, $\eta(x_1) = 0$. Now define the Riemannian manifold $X = X_0 \# X_1$ by the identification :

$$\eta = \frac{\lambda \bar{\xi}}{|\xi|^2}$$

where $\lambda > 0$ is sufficiently small , and $\xi \to \bar{\xi}$ is an orientation reversing S^1-equivariant isometry $T_{x_0}X_0 \to T_{x_1}X_1$. In chapter 2 , §2 , it has been explained that X is the 4-manifold which arises from the 3-manifold $M = M_0 \#_b M_1$, where $\#_b$ denotes a boundary connected sum. If we take $X_1 = S^4$, then $X_0 \# X_1 = X_0$ as conformal manifolds , and in this case we shall see that the construction below gives monopoles on X_0 'supported' near a boundary surface of X_0 , by 'gluing in' monopoles from

H^3.

For the following discussion compare figure 5.1 , in which arrows indicate the boundaries of the open sets which will be defined. Define spheres in X_0 of radii $N^{-1}\sqrt{\lambda}$ and $N\sqrt{\lambda}$, and shells :

$$R_{-1} = \{x \in X_0 \ ; \ |x-x_0| \in [kN^{-1}\sqrt{\lambda} \ , \ N^{-1}\sqrt{\lambda}]\}$$

$$R_1 = \{x \in X_0 \ ; \ |x-x_0| \in [N\sqrt{\lambda} \ , \ k^{-1}N\sqrt{\lambda}]\} \ .$$

Like Donaldson , we shall fix k , say at 0.9 , while N will have to be chosen large enough for the proof to work. More precisely , one needs:

$$(k^{-4}-1)\cdot(1-k)\cdot N\cdot(N-N^{-1})^{-3} < 8 \ ,$$

which is a relation independent of λ , but for our spheres and shells to be well defined , λ should be sufficiently small.

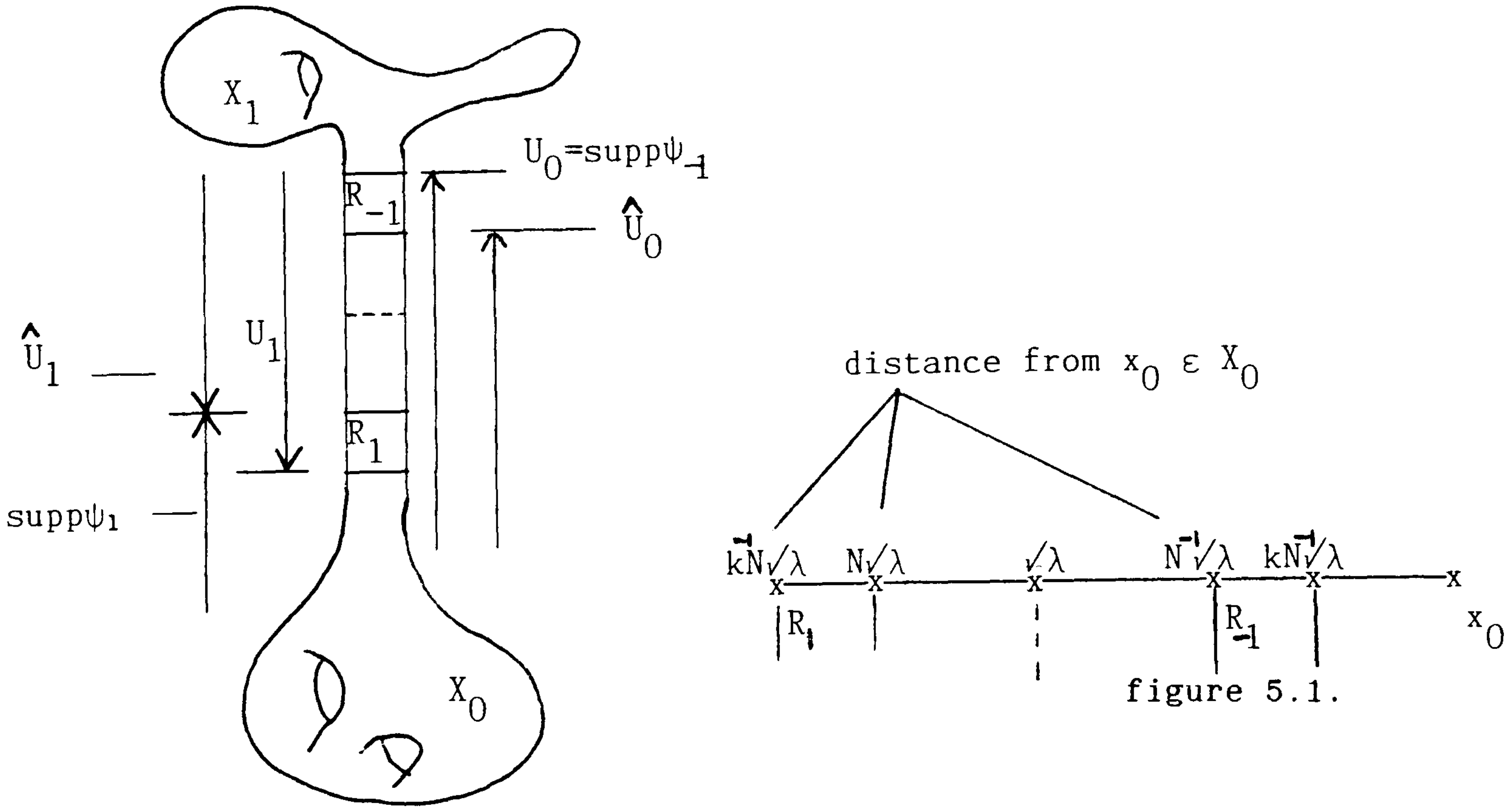

figure 5.1.

The identification maps R_{-1} and R_1 to shells in X_1 whose sizes are reversed. The image of R_j in X_1 will be denoted by R_j too. Let $U_0 \subset X_0$ be the complement of the ball $|\xi| \leq kN^{-1}\sqrt{\lambda}$, $\hat{U}_0 \subset X_0$ the complement of the ball $|\xi| \leq N^{-1}\surd\lambda$. Define U_1 , $\hat{U}_1 \subset X_1$ symmetrically , so that $X = X_0 \# X_1$ is covered by the open sets $\hat{U}_0$, $\hat{U}_1$, which intersect in an annulus bounded by the inner boundary spheres of the shells $R_1 \subset X_0$ and $R_{-1} \subset X_1$.

Suppose that A_0 , A_1 are monopoles on $\tilde{S}^1$-equivariant SU(2)-bundles $P_j \to X_j$, satisfying the following acyclicity condition on the $\tilde{S}^1$-invariant part of the cohomology groups of 2.5:

5.1 $$H^0_{0,A_j} = H^1_{0,A_j} = H^2_{0,A_j} = 0 \qquad j=1,2 .$$

Consider for $\eta_0 > 0$ the set of $\tilde{S}^1$-invariant connections on $X = X_0 \# X_1$ which can be represented in the form :

5.2 $$A = (A_0 + a \ , \ A_1 + a' \ , \ \rho)$$

$$a \text{ defined on } \hat{U}_0 \ , \ d^*_{A_0} a = 0 \ , \ \|a\|_{L^{2p}(\hat{U}_0)} < \eta_0$$

$$a' \text{ defined on } \hat{U}_1 \ , \ d^*_{A_1} a' = 0 \ , \ \|a'\|_{L^{2p}(\hat{U}_1)} < \eta_0$$

$$\rho \in C^\infty(\hat{U}_0 \cap \hat{U}_1 \ , \ \mathrm{Hom}^{\tilde{S}^1}(P_{0,x_0}, P_{1,x_1})) \quad \text{s.t.} \quad A_1 + a' = \rho\cdot(A_0 + a)$$

Following Donaldson , we take $p > 6$ to ensure that all non-linear maps involved will be smooth on the relevant Banach spaces. The last equality should be satisfied in $\hat{U}_0 \cap \hat{U}_1$, in exponential trivialisations emanating from x_0 and x_1. Recall that the points x_j lie in fixed surfaces in X_j , so the $\tilde{S}^1$-actions on P_j assign a mass m_j to x_j. For an identification ρ to exist , the masses m_j must be equal.

Theorem 5.1 : Under the conditions 5.1 and for sufficiently small λ , η_0

, the monopoles A on $X_0 \# X_1$, which can be represented in the form 5.2 are smoothly parametrized by $\mathrm{Hom}^{\tilde{S}^1}(P_{0,x_0}, P_{1,x_1})/\{\pm 1\}$.

Proof: This is an $\tilde{S}^1$-equivariant version of theorem 4.17 in Donaldson [14]. His proof is natural with respect to $\tilde{S}^1$-actions and carries over to our situation.∎

Notice that the set of gluing data :

$$I = \mathrm{Hom}^{\tilde{S}^1}(P_{0,x_0} , P_{1,x_1}) ,$$

is diffeomorphic to $SU(2)$ if the masses m_j at x_j vanish , and diffeomorphic to S^1 if $m_0 = m_1 \neq 0$.

Before proceeding , let us recall briefly how theorem 5.1 constructs monopoles on $X_0 \# X_1$. In trying to understand the construction , it may be helpful to compare it with the Mayer-Vietoris argument in de Rham cohomology. In fact , Donaldson's alternating procedure can be interpreted as a non-linear version of a proof of the statement $H^1(X;\mathbb{R}) \cong H^1(X_0;\mathbb{R}) \oplus H^1(X_1;\mathbb{R})$ on the level of harmonic differential 1-forms.

First , following Donaldson , glue A_0 to A_1 by using an exponential gauge around x_0 , x_1 : let ψ_1 , ψ_{-1} be S^1-invariant cutoff functions $X_0 \# X_1 \to \mathbb{R}$ such that :

$$\psi_1|_{X \setminus \hat{U}_1} = 1 \quad , \quad \mathrm{supp}\, \psi_1 \subset X \setminus U_1 \quad , \quad \mathrm{supp}\, d\psi_1 \subset R_1 \quad \text{and}$$
$$\psi_{-1}|_{\hat{U}_0} = 1 \quad , \quad \mathrm{supp}\, \psi_{-1} \subset U_0 \quad , \quad \mathrm{supp}\, d\psi_{-1} \subset R_{-1} \quad .$$

For $\rho \in I = \mathrm{Hom}^{\tilde{S}^1}(P_{0,x_0}, P_{1,x_1})$ define the glued connection to be :

5.3 $$A^0(\rho) = (A_0 + a_0\ ,\ A_1 + a_0'\ ,\ \rho) \text{ with}$$

$$a_0 = (\psi_1 - 1)\cdot A_0 + (1 - \psi_1)\cdot\rho^{-1}A_1\rho \text{ on } U_0\ , \text{ extended to } X_0 \text{ by } 0$$

$$a_0' = -\psi_1\cdot A_1 + \psi_1\cdot\rho A_0\rho^{-1} \text{ on } U_1\ , \text{ extended to } X_1 \text{ by } 0.$$

Clearly supp $P_+F^{A^0(\rho)} \subset R_1 \subset X$.

Next suppose (induction) that a_j , a_j' , with j even , are given , and assume that $\sigma = P_+F^{A_0+a_j}{}_{|U_0}$ satisfies supp $\sigma \subset R_1$. We shall solve an equation on X_0. Extend a_j over $X_0\setminus U_1$ by 0 and solve for the following equation for $b \in \mathcal{W}^1_{p,1}(X_0)$:

5.4 $$P_+(d_{A_0}b + [a_j\ ,\ b] + \tfrac{1}{2}\cdot[b\ ,\ b]) = -\sigma$$

$$d^*_{A_0}b = 0\ .$$

To see that there are solutions and that they are well behaved , interpret the left hand side of 5.4 as an operator :

$$Q(\cdot,\cdot)\ :\ \mathcal{W}^1_{p,1}(X_0) \times \mathcal{W}^1_{2p,0}(X_0) \to \mathcal{W}^2_{+,p,0}(X_0) \times \mathcal{W}^0_{p,0}(X_0)$$

$$(b,a) \to P_+\{d_{A_0}b + [a,b] + \tfrac{1}{2}\cdot[b,b]\}.$$

The derivative with respect to the first variable is precisely the deformation operator studied in §3 and is assumed to be invertible. Therefore the implicit function theorem supplies solutions of 5.4 for small a_j and σ. Next define :

$$a_{j+1} = a_j + \psi_{-1}\cdot b$$

$$a_{j+1}' = a_j' + \psi_{-1}\cdot\rho b\rho^{-1}\ .$$

The effect of this is that the self-dual part of $F^{(A_0+a_{j+1},A_1+a'_{j+1},\rho)}$ now has its support in $R_{-1} \subset X$. Now we 'alternate' , i.e. we solve a similar equation on X_1 by reversing the roles of X_0 and X_1. This completes one cycle of the alternating procedure.

Using estimates on the linearisation of 5.4 and the initial data , Donaldson proves that the a_j ,a'_j converge to some a_∞ , a'_∞ , and that $A^\infty(\rho) = (A_0+a_\infty , A_1+a'_\infty , \rho)$ has anti-self-dual curvature. Of course it is here that conditions on λ and η_0 appear. Next , a lemma gives that for $\rho \neq \rho'$ the connections $A^\infty(\rho)$ and $A^\infty(\rho')$ are not gauge equivalent. The construction is completed by putting the a_∞ , a'_∞ in the right gauge giving new $a_\infty(\rho)$, $a'_\infty(\rho)$ satisfying $d^*_{A_0} a_\infty(\rho) = d^*_{A_1}(\rho) = 0$.

We now have the right preliminaries to explain what must be modified if we relax the $\tilde{S}^1$-acyclicity condition on the deformation complexes of A_0 , A_1 and the condition that the metrics be conformally flat. Again , this is Donaldson's original result in an equivariant setup. First choose liftings V_{A_0} , V_{A_1} of H^2_{0,A_0} , H^2_{0,A_1} , given by forms which are supported away from the identification area , as in §3. Let :

$$p_0 \to \tilde{p}_0 \quad : \quad H^1_{0,A_0} \to \mathfrak{W}^1(X_0)$$

$$p_1 \to \tilde{p}_1 \quad : \quad H^1_{0,A_1} \to \mathfrak{W}^1(X_1)$$

be the deformations described in formula 3.9. Equation 5.4 is now replaced by :

5.5 $$P_+(d_{A_0} b + [a_j + \tilde{p}_0 , b] + \tfrac{1}{2}\cdot[b , b]) - \phi_0 = -\tilde{\sigma}$$

$$d^*_{A_0} b = 0 \quad ,$$

with $\phi_0 \in V_{A_0}$, and $\tilde{\sigma} = P_+F^{A_0+\tilde{p}_0+a_j}$. This equation has a unique solution $b \in (H^1_{0,A_0})^{\perp_{L^2}} \subset W^1_{p,1}(X_0)$, and solves the infinite dimensional part of the equations , just as the implicit function theorem took care of the infinite dimensional part in formulas 3.7 - 3.10. As above , we start the iteration by cutting off the deformed connections $A_0 + \tilde{p}_0$, $A_1 + \tilde{p}_1$ and solving for a_i , a_i'. For small p_i we get a limiting connection

$$A^{\infty}(\rho,p_0,p_1) = (A_0 + \tilde{p}_0 + a_{\infty}(\rho) \ , \ A_1 + \tilde{p}_1 + a_{\infty}'(\rho) \ , \ \rho)$$

satisfying :

$$P_+F(A^{\infty}(\rho,p_0,p_1)) = \phi_0(\rho,p_0,p_1) + \phi_1(\rho,p_0,p_1)$$

for smooth functions :

5.6 $$\phi_i : I \times H^1_{0,A_0} \times H^1_{0,A_1} \rightarrow V_{A_i} .$$

Therefore the $\tilde{S}^1$-equivariant version of Donaldson's theorem 4.53 reads :

Theorem 5.2 : Let g_0 , g_1 be S^1-invariant metrics on X_0 , X_1 , which are conformally flat near x_0 , x_1 , and A_0 , A_1 monopoles with respect to these metrics. If λ and η_0 are sufficiently small and if g is the S^1-invariant metric on $X = X_0 \# X_1$, then :

1) There is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-invariant open neighbourhood N of $I \times \{0\} \times \{0\}$ in $I \times H^1_{0,A_0} \times H^1_{0,A_1}$ and

2) There is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-equivariant map :

$$\Phi = (\varphi_0 \ , \ \varphi_1) : N \rightarrow H^2_{0,A_0} \times H^2_{0,A_1}$$

such that the monopoles with respect to the metric g on $X_0 \# X_1$ representable in the standard form 5.1 are parametrized (up to gauge equivalence) by $\Phi^{-1}(0)/(\Gamma_{A_0} \times \Gamma_{A_1})$. ∎

Theorem 5.2 is quite powerful. First of all it allows us to construct a $4k-1$ parameter family of monopoles on H^3 (recall that the corresponding X is S^4) of any mass $m \in \mathbb{Z}_{>0}$ and charge $k \in \mathbb{Z}_{>0}$: using induction , start with an $(m,k-1)$-monopole A_0 on H^3 , and let A_1 be the $(m,1)$-monopole on H^3. Now apply theorem 5.2. The H^2_0's vanish by the remark after proposition 4.2 , $I \cong S^1$ and $\Gamma_{A_1} = \{\pm 1\}$, so by induction it follows that we construct a $4k-1$ dimensional family of monopoles. Starting from the fact that the 1-monopole has 3 degrees of freedom , we do not even need the index formulas of §2 for this ; in fact theorem 5.2 can be used to give an alternative computation of the indices.

It is worth discussing another direct corollary of theorem 5.2. Recall (see chapter 2 , 2.5) that a 3-manifold M with $H^2(M;\mathbb{R}) = 0$ can have at most 1 boundary surface. Thus for monopoles on such an M there is just one charge and one mass.

Corollary 5.3 : Let M be a 3-manifold with $H^2(M;\mathbb{R}) = 0$, and such that a compactification X_0 of $M\times S^1$ as in §1 can be found , and such that X_0 is conformally flat near the fixed surface. For any mass $m \in \mathbb{Z}_{>0}$ a $4k-1+b^1(M)$ parameter family of monopoles of charge k exists.

Proof : Start with a flat monopole A_0 (i.e. $F^{A_0}=0$) of mass $m \in \mathbb{Z}_{>0}$ on M. These are parametrized up to gauge transformations by $(H_1(M;\mathbb{R})/H_1(M;\mathbb{Z})) \times H_1(M;\mathbb{Z})_{tor}$, see chapter 2 , example 5.3.2. The stabilizer of such a monopole is $\Gamma_{A_0} = S^1$ and $H^2_{0,A_0} \cong H^2_+(X;\mathbb{R}) \cong H^2(M;\mathbb{R}) = 0$. Now apply theorem 5.2 with A_1 an (m,k)-monopole on H^3. ∎

In particular this applies to hyperbolic handlebodies $M \cong H^3/\text{Schottky}$ group , and to complements of S^1's , i.e. knots , in a homology 3-sphere , provided the metric near the knot is chosen such that the compactification X exists.

Example 5.4 : Let M be a manifold as in theorem 5.3. Choose a mass $m \in \mathbb{Z}_{>0}$ and consider monopoles of charge 1. Using theorem 5.3 one constructs a map:

$$S \times (0,\epsilon) \times \{(H_1(M;\mathbb{R})/H_1(M;\mathbb{Z})) \times H_1(M;\mathbb{Z})_{tor}\} \rightarrow \mathcal{M}(m,1) ,$$

where the parameter in S is the attaching point of $\infty \in S^4$ to X , and the second one is the scale λ of the identification. Conjecturally this map extends to a diffeomorphism $M \times \{(H_1(M;\mathbb{R})/H_1(M;\mathbb{Z})) \times H_1(M;\mathbb{Z})_{tor}\} \rightarrow \mathcal{M}(m,1)$. The exact identification of points in M with monopoles should go through the zero of the Higgs field. Floer proved a theorem of this kind for asymptotically Euclidean manifolds with $H_1(M;\mathbb{Z}) = 0$.

We shall now relax the condition that the metrics on X_0 , X_1 be flat in the identifying region. If g_0 , g_1 are any S^1-invariant metrics on X_0 , X_1 then the connected sum , as a manifold , can be defined as before , using a geodesic coordinate system centered on x_0 , x_1. The gluing data ($\lambda>0$, $\sigma : T_{x_0}X_0 \rightarrow T_{x_1}X_1$) remain the same , just as the shells R_i and open sets U_i. We shall use Donaldson's notion of conformal structures being close. Let g be a metric on X. We shall say that g is conformally ϵ-close to g_0 , g_1 if there are functions f_i on U_i such that :

$$\|(g_i - f_i \cdot g)_{|U_i}\|_{L^\infty(U_i)} < \epsilon \ .$$

A short computation shows that metrics g on X exist , which are $C \cdot \lambda$ conformally close to g_0 , g_1 , with C a constant depending on the Riemannian curvature of g_0 and g_1. As in Donaldson [14] , 4.6 one concludes :

Theorem 5.5 : Let g_i be metrics on X_i and let A_i be monopoles with respect to these metrics. Choose a constant $K>0$. There are $\tilde{\eta}_0 > 0$ and $\tilde{\lambda} > 0$ such that for any $\lambda \leq \tilde{\lambda}$ and $\eta_1 \leq \tilde{\eta}_1$ and any metric g on X , which is $K \cdot \lambda^{\frac{1}{2}}$ close to g_0 , g_1 statement 1) and 2) of theorem 5.2 hold.∎

The formula of Donaldson [14] 4.57 , giving a highest order approximation of $<\varphi_0 \ , \ \omega>_{L^2}$ $(\omega \in H^2_{0,A_0})$ in the case of metrics conformally flat near x_0 , x_1 , remains unaltered ; only ω is an $\tilde{S}^1$-invariant form here. The formula reads :

$$<\varphi_0(\rho, p_0, p_1) \ , \ \omega>_{L^2} = q_0^{\omega}(\rho, p_0, p_1) + O(\lambda^3 + |p_0|^3 + (|p_0| + |p_1|)\lambda^2) \ ,$$

with q_0^{ω} the quadratic form we encountered in the deformation theory for monopoles , modified with a term involving the bundle clutching parameters ρ:

$$5.8 \qquad q_0^{\omega}(\rho, p_0 \cdot p_1) = <p_0 \wedge p_0 \ , \ \omega>_{L^2} - \tfrac{1}{2} \cdot v_3 \cdot \lambda^2 \cdot \mathrm{Tr}(F^{(A_1+p_1)}(x_1) \cdot \rho^{-1} \omega^{\sigma} \rho).$$

(In Donaldsons formula , 4 should be 4^{-1} and our inner product is two times his inner product.) Here v_3 is the 3-volume of the standard 3-sphere , σ is the orientation reversing isometry $T_{x_0}X_0 \to T_{x_1}X_1$, and $\omega^\sigma = (\sigma^{-1*}\omega)(x_1)$. Observe that q_0^ω is linear in ω , therefore $\dot{q}_0$ and $\dot{q}_1$ define a map :

$$q = \dot{q}_0 \oplus \dot{q}_1 : I \times H^1_{0,A_0} \times H^1_{0,A_1} \to H^2_{0,A_0} \oplus H^2_{0,A_1}$$

If q is C^1-close to Φ , then stability theorems imply that the zero set of Φ is modelled on that of q , like in §3.

The procedure above extends in the obvious way to construct S^1-invariant connections over $X_0 \# X_1 \# \ldots\ldots\ldots \# X_l$, where the X_i , $i = 1,\ldots,l$ are attached to X_0 at different points. Specifically we shall take X_0 to be our original manifold X , all X_i to be S^4 , and attach $\infty \in S^4$ to different points $x_{j_i} \in S_j \subset X$, where $m_j \neq 0$. The index j_i indicates the i-th point in S_j , as in §4. Let $\bar{\lambda} = \max \lambda_{j_i}$, where λ_{j_i} are the scales in the identifications. The metric on X can be seen to be conformally $(\mathrm{const}\cdot\bar{\lambda})$-close to the metrics on X and the S^4's.

To start the alternating procedure take for A_0 a monopole in $\mathcal{M}(m_j,k_j-l_j)$, where $-l_j$ is the number of j_i's ; so $l = \Sigma_j -l_j$. For A_{j_i} we take the standard one-monopoles I_{m_j} on S^4 with mass m_j , i.e. $I_{m_j} \in \mathcal{M}(m_j,1)$, which we discussed in §1.

We shall now work out formula 5.8 for this case. There are no obstructions coming from the 4-spheres. Take $\omega \in H^2_{0,A_0} \subset \mathcal{W}^2_+(X)$, then our interest lies in the second term of 5.8. Let $E = P \times_{SU(2)} \mathbb{C}^2$, and recall that $E_{|S_j} = L_j \oplus L_j^*$, where $\tilde{S}^1$ acts on L_j with weight $+m_j$. If $x \in S_j$

and $Z_x \in \Lambda^2 T_x X$ is the two-vector dual to $dV_{S_j} + *_X\, dV_{S_j} \in \Lambda^2 T_x^* X$ (with dV_S denoting the volume 2-form using the orientation of S), then $\omega(Z_x)$ is a skew adjoint, S^1-invariant endomorphism of E_x. Define :

$$e(\omega(x)) = -i \cdot \omega(Z_x)_{|L_{j,x}} \in \mathbb{R}.$$

Proposition 5.6 : If we glue l standard one-monopoles A_{j_i} of masses m_j to A_0 then :

$$5.9 \quad \langle q_0(p_0, 0, \rho_{j_i}),\ \omega\rangle = \langle p_0 \Lambda p_0,\ \omega\rangle_{L^2(X_0)} - \sum_{\text{all } j_i} v_3 \cdot \lambda_{j_i}^2 \cdot e(\omega(x_{j_i}))$$

Proof : Denote by P_I, E_I etc. the bundles on S^4 associated to the standard one-monopoles. Then, as a 2-form with values in endomorphisms of $L_I \oplus L_I^*$, the curvature of A_I restricted to $S^2 \subset S^4$ equals :

$$F_I = \tfrac{1}{2} \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot (dV_{S^2} - *_4\, dV_{S^2}).$$

(Observe that $c_1(L_I) = -(2\pi i)^{-1} \cdot \int \tfrac{1}{2} \cdot i \cdot dV_{S^2} = -1$, as it should be.) So F_I takes values in the trivial summand $\mathbb{R}$ of the endomorphism bundle $\mathfrak{g}_I$. The S^1-equivariant clutching parameter $\rho : \mathfrak{g}_{P_0} \to \mathfrak{g}_I$ restricts to the identity map $\mathbb{R} \to \mathbb{R}$. Furthermore, the orientation reversing isometry $\sigma : T_{x_{j_i}} X \to T_\infty S^4$ maps dV_{S_j} to $-dV_{S^2}$, also because it is S^1-equivariant.

Now put $\omega(x_{j_i}) = \begin{bmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{bmatrix} \cdot (dV_{S_j} + *_4\, dV_{S_j}) + R$, with R orthogonal in $(\Lambda_+^2 X \otimes \mathfrak{g}_P)_{x_{j_i}}$ to the first term ; then $e(\omega(x_{j_i})) = \alpha$.

But:

$$
\begin{aligned}
&-\tfrac{1}{2}\cdot v_3\cdot\lambda^2_{j_i}\cdot \mathrm{tr}(F_I(\infty)\cdot\rho^{-1}\omega^{\sigma}\rho) = \\
&= v_3\cdot\lambda^2_{j_i}\cdot \mathrm{tr}\left(\tfrac{1}{2}\cdot\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\cdot\begin{bmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{bmatrix}\right) = \\
&= -v_3\cdot\alpha\cdot\lambda^2_{j_i} = -v_3\cdot\lambda^2_{j_i}\cdot e(\omega(x_{j_i}))\ ,
\end{aligned}
$$

so the proposition follows. ▬

Remark 5.7 : Floer [18] expressed the 'constraint formula' 5.9 a bit differently. By expanding ω in normal coordinates around S_j one recovers Floer's formula.

There is an important difference between formula 5.8 for instantons and 5.9 for monopoles. In 5.8 the sign of the second term can be changed by a different choice of ρ , provided ω and F^{A_1} do not vanish at the point in question. To get an existence theorem one has to take care of more than one ω , and therefore the condition for existence is , roughly , that one attaches sufficiently many 4-spheres at points which are in general position with respect to the ω's. For monopoles the signs of these terms are determined by the value distribution values of the $e(\omega(x))$, where x ranges through the fixed surfaces in X , and ω through H^2_0. This is a more global affair : indeed , examples exist , see example 5.8 and chapter 4 , which show that for existence it may be necessary to have k_j non-zero for more than just one of the j. Also this property of 5.9 makes it rather more cumbersome to formulate a general existence theorem than is the case for instantons , because one would first have to create an overview of the value distribution of the $e(\omega(x))$. This can certainly be carried out if all m_j are equal , because then we can start with the trivial monopole and H^2_0 is just the de Rham cohomology of M. If there are no obvious , e.g. reducible , solutions with unequal masses , then it seems very hard to prove a general existence theorem for such monopoles , because it is not clear how to 'start' the gluing procedures.

Next we continue the general discussion. The parameters in H^1_0 for the standard one-monopoles can , after grafting , be considered as small variations in the attaching points and scales of the identification. Notice that in coordinates in M near $x_{j_i} \in \delta M$, the zero of the Higgs field of a grafted monopole is approximately the point $(x_{j_i}, \lambda^{1/2})$, with the second coordinate a normal coordinate to δM. Our construction gives a family of connections parametrized by :

$$N_r =$$

$$(B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} \left[(B_r(x_{j_i}) \subset S_j) \times S^1 \times ((1-r)\lambda_{j_i}, (1+r)\lambda_{j_i})\right] ,$$

with $r>0$ sufficiently small , and $B_\epsilon(y)$ denoting a ball of radius ϵ around y. The monopoles in N_r are the zero set of $\Phi : N_r \to H^2_{0,A_0}$. There are no problems in defining the bigger family of connections described by :

$$N = (B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} \left[B_r(x_{j_i}) \times S^1 \times (0,\epsilon)\right]$$

(for some $\epsilon > 0$) such that the monopoles are still zeroes of a $\Gamma_{A_0} \times \{\pm 1\}^1$-equivariant smooth map $\Phi : N \to H^2_{0,A_0}$. To prove this it is necessary to check that letting $\lambda_{j_i} \to 0$ and varying the attaching points

x_{j_i} does not force one to shrink the ball in H^1_{0,A_0} drastically. For the scales this follows from the fact that all estimates in the alternating procedure improve upon letting $\lambda \to 0$, see Donaldson [14] 4.3 - 4.6. The convergence of the alternating procedure is also easily seen to be locally uniform under small variations in the attaching points.

We now insert a long example to illustrate the material discussed so far.

Example 5.8 : Monopoles on $S^2 \times \mathbb{R}$.

Let $M = S^2 \times \mathbb{R}$, with metric $ds^2 = \cosh^2 l \cdot ds^2_{S^2} + dl^2$. An easy computation shows that the conformal compactification X of $M \times S^1$ equals $S^2 \times S^2$, where the S^1-action is earth rotation in the second S^2. Recall that any complex line bundle on S^2 is isomorphic to $O(n)$, the n-th power of the positive Hopf bundle over S^2. The bundle $O(n)$ admits an $\tilde{S}^1$-action for any weights m_0 , m_∞ satisfying $2n = m_0 - m_\infty$. Let $\pi_j : X \to S^2$ be the projection on the j-th factor , and denote by $O(n,m)$ the bundle $\pi_1{}^*O(n) \otimes \pi_2{}^*O(m)$ over X.

The line bundles $O(k,-k)$ carry a unique (up to gauge transformations) anti-self-dual connection. For any weights m_0 , m_∞ satisfying $2k = m_0 - m_\infty$, there is an $\tilde{S}^1$-action with these weights , which leaves this reducible connection invariant. Hence one finds reducible monopoles on the vector bundle $E = O(k,-k) \oplus O(-k,k)$. We shall consider two cases :

1) Take $E = O(k,-k) \oplus O(-k,k)$ with the reducible A_0 connection as a monopole with $m_0=2k$, $k_0=k$, $m_\infty=0$. The cohomology of the Atiyah-Hitchin-Singer complex for A_0 is given by :

$$H^0_0 = \mathbb{R} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$H^2_0 = \mathbb{R}\cdot\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\cdot\omega$ with ω the Kahler form of $S^2\times S^2$ and,

$$\begin{aligned} H^1_0 &\cong H^1_{\text{sheaf}}(X\ ,\ \mathbb{C}\oplus O(2k,-2k)\oplus O(-2k,2k))^{\tilde{S}^1} \\ &\cong H^1_{\text{sheaf}}(S^2\ ,\ O(-2k))\otimes H^0_{\text{sheaf}}(S^2\ ,\ O(2k))^{\tilde{S}^1} \\ &\cong \mathbb{C}^{2k-1}\otimes\mathbb{C}\ . \end{aligned}$$

This can be proved using well known relations between gauge theory and algebraic geometry on surfaces , see chapter 4 , §2,3 and Donaldson [15]. The quadratic form 3.13 is given by :

$$H^1_0 \to \mathbb{R} : p_0 \to \tfrac{1}{2}\langle\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\cdot\omega\ ,\ P_+[p_0,p_0]\rangle_{L^2} = \tfrac{1}{2}\|p_0\|^2_{L^2}\ ,$$

see chapter 4 , §3 for a proof. This is positive definite , so the given reducible solution is isolated. In fact , using algebraic geometry , one can prove along the lines of chapter 4 , §5 that $\mathcal{M}((2k,0),(k,0))$ is a point. This exemplifies the statement that for existence it may be necessary to have more than one charge non-zero.

Next we try to glue in a monopole of mass $2k$ at a point in $S^2\times\{0\}$, in order to obtain elements of $\mathcal{M}((2k,0),(k+1,0))$. For the function q we find (compare 5.9):

$$\begin{aligned} q : N \cong \mathbb{C}^{2k-1} \times S^2 \times S^1 \times (0,\epsilon) &\to \mathbb{R} \\ (p_0\ ,\ x\ ,\ \rho\ ,\ \lambda) &\to |p_0|^2 - v_3\cdot\lambda^2 \end{aligned}$$

This is a family of non-degenerate quadratic forms parametrized by (x,ρ) , so one expects the moduli space of $((2k,0),(k+1,0))$-monopoles to look asymptotically like $\left[S^2\times S^1\times\{(p,\lambda);|p|^2-\lambda^2=0,\lambda>0\}\right]/S^1$, a bundle of cones over S^2. (Observe that this has real dimension $4(k+1)-4$, as predicted by the index theorem , whereas the dimension of $\mathcal{M}((2k,0),(k,0))$

is 'wrong'.) To make this rigorous, we need to know how Φ is approximated by q for λ and p small, and we shall make some remarks concerning this below.

It can also be proved that $\mathcal{M}((2k,0),(1,0))$ is empty for $1<k$, see chapter 4, §5. Therefore the compactification of this moduli space is much smaller then one might naively be led to expect, because many of the possible lower strata do not occur.

2) Consider the trivial bundle E with the trivial connection as an element of $\mathcal{M}((2m,2m),(0,0))$, $m \in \mathbb{Z}_{>0}$. For the trivial connection one has $H^1_0 = 0$, $H^0_0 = \mathbb{R}$, $H^2_0 = \mathbb{R}$, where the latter is generated by what is essentially the Kahler form of X, as in 1).

It can be proved that $\mathcal{M}((2m,2m),(k,0))$ is empty for any $k>0$, see chapter 4. This does not come totally unexpectedly, for the constraint formula 5.9 is easily seen to be definite if one applies it to attaching one-monopoles to one fixed surface. However, monopoles in $\mathcal{M}((2m,2m),(k,1))$ exist, the signs of $e(\omega(x))$ are different for the two fixed S^2's ($\omega \in H^2_0$). We now take up the general discussion again.

If we find a sequence of monopoles $[A_n]$ in N such that the coordinates $(p_0,x_{j_i},\rho_{j_i},\lambda_{j_i})_n$ converge to $(0,x_{j_i},\rho_{j_i},0)$ then the sequence $[A_n]$ converges to $([A_0],(x_{j_i}))$ in the compactified moduli space; this is easy to check.

Our final aim is to show that one can use the methods of this section to describe a complete neighbourhood of $([A_0],(x_{j_i}))$ in the compactified moduli space. Interpret $S^1 \times (0,\epsilon)$ as the punctured disc $B_\epsilon(0)\setminus\{0\} \subset \mathbb{C}$ and define :

$$\bar{N} = (B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} \{B_r(x_{j_i}) \times (B_\epsilon(0) \subset \mathbb{C})\}.$$

Putting one of the scales , say $\lambda_{j_0 i_0}$, equal to zero amounts to looking for a monopole in the lower stratum $S_{j_0} \times \mathcal{M}(m_j,k_j')$ of the compactified moduli space $\bar{\mathcal{M}}(m_j,k_j)$, with $k_j'=k_j$ for $j \neq j_0$ and $k_{j_0}' = k_{j_0} - 1$. The function q (5.9) extends naturally to a map $\bar{N} \to H^2_{0,A_0}$, and from the formula preceding 5.8 it follows that also Φ extends continuously to $\bar{N}$. Appealing to Donaldson [14] 5.4 , we can obtain more :

Proposition 5.9 : The function $\Phi : \bar{N} \to H^2_{0,A_0}$ is C^1 , and is C^1-approximated by q for small λ_{j_i} and p_0. Furthermore , $\Phi^{-1}\{0\}/(\Gamma_{A_0} \times \{\pm 1\}^l)$ is a family of gauge inequivalent ideal monopoles.∎

Now assume that $m_j=1$ for all j. Our one-monopoles are the basic one-instantons , with scale λ and centres in $S^2 \subset S^4$. A standard theorem concerning instantons now asserts that any monopole $[A] \in \mathcal{M}(m_j,k_j)$ sufficiently close to $([A_0] , (x_{j_i})) \in \bar{\mathcal{M}}(m_j,k_j)$ can be put in the standard form 5.3 ; see e.g. Donaldson [14] , proposition 4.11. It follows that $\Phi^{-1}\{0\}/(\Gamma_{A_0} \times \{\pm 1\}^l)$ is a full neighbourhood of $([A_0] , (x_{j_i}))$ in $\bar{\mathcal{M}}(m_j,k_j)$. Summarizing we obtain :

Theorem 5.10 : A neighbourhood of $([A_0] , (x_{j_i}))$ in $\bar{\mathcal{M}}(1,k_j)$ is modelled on the quotient of the zero set of :

$$\Phi : \bar{N} \to H^2_{0,A_0}$$

by $\Gamma_{A_0} \times \{\pm 1\}^{l}$. The stratification is induced by taking intersections of $\Phi^{-1}\{0\}$ with hyperplanes $\lambda_{j_i}=0$. The map Φ is a C^1-perturbation of the function $q : \bar{N} \to H^2_{0,A_0}$ (5.9), for small p_0 and λ_{j_i}. ■

Of course one expects a result of this kind for any choice of masses.

§6 References

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CHAPTER IV

MONOPOLES ON SURFACE × $\mathbb{R}$

§1 Introduction

If $\Gamma \subset PSL(2,\mathbb{R})$ is a Fuchsian group without cusps , then $M = H^3/\Gamma$ is diffeomorphic to $S \times \mathbb{R}$ with S a compact Riemann surface of genus $g \geq 2$ without boundary. Here we shall make a relatively detailed study of the moduli spaces of magnetic monopoles on M. This is an example of the material treated in chapters 2 and 3.

In §2 we recall that we viewed monopoles as S^1-invariant instantons on a conformal compactification X of $M \times S^1$. In this case X turns out to be the algebraic surface $S \times \mathbb{C}P^1$ with the product Kahler structure. A fundamental theorem of Donaldson identifies stable holomorphic bundles on an algebraic surface with the instantons , and so we start studying S^1-invariant stable bundles.

We consider first the relatively simple cases of flat monopoles and those with Abelian holonomy. Usually one expects that reducible connections form a singular part of the moduli space , but here we find examples where they are isolated.

In §4 and §5 we treat some technical material concerning holomorphic bundles on X and their stability. This is all quite straightforward , but we spell out most details. This culminates in precise criteria for an S^1-invariant bundle to be stable , i.e. criteria for the existence of monopoles. Also criteria for instability deserve special attention , because they show that under certian circumstances monopoles do not exist. More precisely , boundary conditions assign a mass and a charge to the monopole for each of the two ends of the universe $S \times \mathbb{R}$. The instability criteria prove that a difference in the two masses assigned to a monopole in M should be balanced by a sufficiently high magnetic charge. We engage in some physical speculation by elaborating this idea.

Finally , in §6 , we give a description of the moduli spaces , which is complete in its details up to problems concerning bundles and Picard groups on S. Also we describe scattering data for the monopole , which determine the monopole. All of this is suggested by the results found by Hitchin , Atiyah and Donaldson for monopoles on $\mathbb{R}^3$ and H^3.

§2 Fuchsian groups and generalities on holomorphic bundles

Let $\Gamma \subset SL(2,\mathbb{R})$ be a geometrically finite Kleinian group (compare chapter 2 , §2) without cusps ; such a group will be called ***Fuchsian***. The quotient of the upper-half-plane by the Γ action is a smooth compact Riemann surface S without boundary :

$$S = PSO(2)\backslash PSL(2,\mathbb{R})/\Gamma .$$

The genus g of S is > 1 , and S carries a hyperbolic metric. The $PSL(2,\mathbb{R})$ action on $H^3 = \{(x,y,t) \; ; \; x,y \in \mathbb{R} \; , \; t \in \mathbb{R}_{>0}\}$, preserves the hyperbolic planes H_η $(0 < \eta < \pi)$:

$$H_\eta = \{(x, r\cdot\cos \eta, r\cdot\sin \eta) \in \mathbb{R}^3_+ \; ; \; r > 0 \; , \; x \in \mathbb{R}\}$$

It follows that $M = H^3/\Gamma \cong S \times \langle 0,\pi\rangle$. Monopoles are S^1-invariant instantons on $M \times S^1$, therefore the conformal structure of $M \times S^1$ is of interest. The product metric is given by :

$$r^{-2}\sin^{-2}\eta[dx^2 + dr^2] + \sin^{-2}\eta \; d\eta^2 + d\theta^2 .$$

This is clearly conformally equivalent to the product metric on the compactification $X = S \times \mathbb{C}P_1$, which is given by :

$$ds^2 = r^{-2}[dx^2 + dr^2] + d\eta^2 + \sin^2\eta \; d\theta^2$$

The anti-self-duality equation for instantons being conformally invariant , we shall study S^1-invariant instantons on S^1-equivariant pricipal

SU(2)-bundles $P \to X$ (compare chapter 2 , §5 and chapter 3 , §2 for more details). Let $E = P \times_{SU(2)} \mathbb{C}^2$; then S^1 acts on $E_0 = E_{|S\times\{0\}}$ and $E_\infty = E_{|S\times\{\infty\}}$ by gauge transformations , and the essential invariants are the non-negative ***weights or masses*** p_0 , p_∞ of the circle action in the fibres. If $p_i \neq 0$ then the action induces a splitting into eigen-bundles : $E_i = L_i \oplus L_i^*$ on the surface S , where S^1 acts on L_i (L_i^*) by scalar multiplication of weight p_i $(-p_i)$. Giving X and S the complex orientation we can define the charges :

$$k_0 = c_1(L_0^*) \in \mathbb{Z} \cong H^2(S;\mathbb{Z}) \qquad (\text{if } p_0\neq 0)$$

$$k_\infty = c_1(L_\infty) \in \mathbb{Z} \cong H^2(S;\mathbb{Z}) \qquad (\text{if } p_\infty\neq 0).$$

This agrees with the definitions in chapters 2 and 3.

The Riemannian manifold X is conformally flat and has scalar curvature zero. Reasoning as in Atiyah-Hitchin-Singer [3] section 6 , we obtain

Proposition 2.1 : The moduli spaces $\mathcal{M}^*(p_i,k_i)$ (in the notation of chapter 3 , this would be $\mathcal{M}(2\cdot p_i,k_i)$) of irreducible monopoles on X are smooth (possibly empty) manifolds. Their real dimensions are (compare chapter 3):

$$4k_0 + 4k_\infty + 2g - 2 \quad \text{if} \quad p_0\neq 0 \quad \text{and} \quad p_\infty\neq 0$$

$$4k_0 + 4g - 4 \quad \text{if} \quad p_0\neq 0 \quad \text{and} \quad p_\infty=0$$

$$6g - 6 \quad \text{if} \quad p_0 = p_\infty = 0. \;\blacksquare$$

The first step in understanding the monopoles on X is to exploit the fact that X is a projective algebraic variety. In fact the Kahler the fact that X is a projective algebraic variety. In fact the Kahler form $\omega = (4\pi)^{-1}(dV_S + dV_{\mathbb{C}P^1})$ is a Hodge form ($dV_\bullet$ denote the volume forms). A multiple of it provides an embedding $X \to \mathbb{C}P^N$, for some N. The associated hyperplane section H is linearly equivalent to a multiple of $(g-1)\cdot(S \times \{p\}) + (\{s\} \times \mathbb{C}P^1)$, $s \in S$, $p \in \mathbb{C}P^1$.

As we shall proceed to explain , there has been established a precise correspondence between the bundles on an algebraic surface carrying an instanton , and the so called stable holomorphic bundles. The concept of stability depends on the divisor H.

First recall theorem 5-1 from Atiyah-Hitchin-Singer [3]:

Theorem 2.2 : Let P be a principal G bundle over a Kahler manifold , where G is a compact Lie group , with complexification $G_{\mathbb{C}}$. Suppose A is an anti-self-dual connection on P. Then A induces a holomorphic structure on $P_{\mathbb{C}} = P \times_G G_{\mathbb{C}}$, such that the extension of A to $P_{\mathbb{C}}$ is the unique G-connection the horizontal subspaces of which are invariant under the complex structure of $P_{\mathbb{C}}$.∎

So the bundles carrying an anti-self-dual connection are holomorphic. Donaldson proved a converse to this. Recall that a holomorphic $\mathbb{C}^N$ bundle E on X is (semi) stable with respect to H if and only if for all coherent sub-sheaves $J \subset \mathcal{O}(E)$:

$$2.1 \qquad c_1(J)\cdot H/rk(J) \quad (\leq) < \quad c_1(E)\cdot H/rk(E)$$

The converse to 2.2 can now be stated (see Donaldson [7] , [10])

Theorem 2.3 : A holomorphic Hermitian vector bundle E over X with trivial determinant $\Lambda^2 E$ is stable with respect to H if and only if E carries an irreducible anti-self-dual SU(2)-connection. This connection is then uniquely determined , up to unitary gauge transformations , by the holomorphic structure of E. (Notice that the concept of anti-self-duality depends on the metric on X hence on the Kahler form of X , and in this way is related to the divisor H.)

§3 Special cases of monopole moduli spaces

The first case to be considered is that of flat SU(2)-monopoles, i.e. flat S^1-invariant connections on an equivariant SU(2)-bundle P. These are sometimes called ***vacuum configurations***. Flat SU(2)-connections arise from representations of the fundamental group. It follows that P restricted to the lines $\{s\}\times\mathbb{C}P^1$ $(s\in S)$ is holomorphically trivial. Using Grauerts theorem (see e.g. Hartshorne [12] Corollary III.12.9), we see that $E = \pi^* F$, with $\pi : X \to S$ the projection, and F a holomorphic bundle on S. The connection on E is the pull-back of a flat connection on F. The action of S^1 is holomorphic on X, preserves the connection on E and is therefore holomorphic on E. This implies that the action of $z \in S^1$, on the trivial bundle $E_{|\{s\}} \times \mathbb{C}P^1$ is multiplication by the SU(2) matrix $\begin{bmatrix} z^p & 0 \\ 0 & z^{-p} \end{bmatrix}$, with $p = p_0 = p_\infty$. The action on $E_{|S \times \{0\}}$ is trivial if $p = 0$, and then F may well be indecomposable as long as it admits a flat connection. If $p \neq 0$ however, then $F = L \oplus L^*$, with L a line bundle of degree zero on S. Of course L is the eigen-bundle for the S^1-action at $S \times \{0\}$. This results in :

Theorem 3.1 : The moduli space $\mathcal{M}(p_i,k_i)$ of flat monopoles on X is nonempty if and only if $p_0 = p_\infty$ and $k_0=k_\infty=0$. If $p_0 = p_\infty = 0$, it is the compactification of the moduli space of stable holomorphic $\mathbb{C}^2$-bundles F on S with trivial determinant, which equals the union of this space with the Picard group $\mathrm{Pic}^0(S)$. If $p_0 = p_\infty \neq 0$, then it is isomorphic to the Picard group of S.

Proof. A holomorphic vector bundle F over S admits a unitary flat the theorem of Narasimhan and Seshadri ; see Donaldson [6]. This connection will be an SU(2)-connection iff $\Lambda^2 F = \mathbb{C}$ as a holomorphic bundle. In the reducible case $F = L \oplus L^*$, with $L \in \mathrm{Pic}^0 S$.

Next we attack the reducible monopoles. If an SU(2)-connection is reducible , i.e. the centralizer of its holonomy is non-zero , then the holonomy is actually contained in a U(1) , embedded in SU(2) as a maximal torus. In view of the fact that the flat case has already been treated , we may as well assume that that the holonomy is equal to U(1). Then the curvature doesn't vanish , so one of the charges must be non-zero.

Lemma 3.2 : The holonomy bundle through $p \in P$ is invariant under the S^1 action.

Proof : If not , the S^1 translates of the holonomy bundle would give a one parameter family of U(1) subbundles of P , in such a way that the connection on P is tangent to every member of this family. This would infinitesimally give rise to a covariantly constant non-zero section of $P \times_{SU(2)} (su(2)/u(1))$, see proposition 2.7.4 in Kobayashi-Nomizu [15]. But there are no such sections if the holonomy is equal to U(1) . ∎

The lemma reduces the problem of finding reducible monopoles to that of classifying equivariant complex line bundles with anti-self-dual connections. The integral , anti-self-dual forms on X are the multiples of $(4\pi)^{-1}(dV_S - dV_{\mathbb{C}P^1})$. Thus we are looking for holomorphic equivariant line bundles with Chern class $((1-g)\cdot m, m) \in H^2(X;\mathbb{Z}) = H^2(S;\mathbb{Z}) \oplus H^2(\mathbb{C}P^1;\mathbb{Z})$ $(m \in \mathbb{Z})$. From $\mathrm{Pic}(X) = \mathrm{Pic}(S) \times \mathrm{Pic}(\mathbb{C}P^1)$, it follows that these are of the form $\pi^* L \otimes \rho^* \mathcal{O}(m)$, with $L \in \mathrm{Pic}^{(1-g)\cdot m}(S)$, $\mathcal{O}(m) \in \mathrm{Pic}(\mathbb{C}P^1)$ the m-th power of the positive Hopf bundle on $\mathbb{C}P^1$, and

$$\pi : X \to S \quad \text{and} \quad \rho : X \to \mathbb{C}P^1 \,, \tag{3.1}$$

the projections. Now $\mathcal{O}(m)$ admits an S^1-action , preserving the harmonic connection for any weights p_0 , p_∞ satisfying $m = p_0 - p_\infty$. Together this results in :

Proposition 3.3 : Consider the subset R of $\mathcal{M}(p_i, k_i)$, consisting of reducible monopoles with holonomy $U(1)$. Then R is non-empty if :

1) $k_0 \neq 0$ and $k_0 = k_\infty = (g-1)\cdot(p_0 + p_\infty)$ or

2) $k_0 \neq 0$ and $k_0 = -k_\infty = (g-1)\cdot(p_0 - p_\infty)$.

If R is non-empty it is isomorphic to $\mathrm{Pic}^{-k_0}(S)$.

Proof : For the first alternative it follows that $(L \in \mathrm{Pic}^{-k_0}(S))$:

$$E = [\pi^*L \otimes \rho^*\mathcal{O}(p_0 + p_\infty)] \oplus [\pi^*L \otimes \rho^*\mathcal{O}(p_0 + p_\infty)]^* \,,$$

whereas for the second we have:

$$E = [\pi^*L \otimes \rho^*\mathcal{O}(p_0 - p_\infty)] \oplus [\pi^*L \otimes \rho^*\mathcal{O}(p_0 - p_\infty)]^* \,. \quad \blacksquare$$

It is instructive to pay attention to the deformations of elements in R within $\mathcal{M}(p_i, k_i)$.

Proposition 3.4 : Let $p_0 \geq p_\infty$ and $k_0 \geq g$. Then the set R of proposition 3.3 is isolated exactly if R resorts under case 2) or under case 1) with $p_\infty = 0$.

Proof : We shall apply deformation theory similar to that of chapter 3,

§3.

Let the monopole $[A] \in \mathcal{M}(p_i,k_i)$ be represented by the holomorpic bundle $K \oplus K^*$. where $K = \pi^*L \otimes \rho^*\mathcal{O}(m)$, for $L \in \mathrm{Pic}^{-k_0}(S)$, as in proposition 3.3. Clearly the S^1-invariant 0-th cohomology of the Atiyah-Hitchin-Singer complex is given by:

$$H^0_0 = \mathbb{R} = \mathbb{R}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

The elements of H^2 are covariantly constant (as $R_{sc}=0$) , and the condition $k_0 \geq g$ is easily seen to imply that :

$$H^2_0 = \mathbb{R}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}(dV_S + dV_{\mathbb{CP}^1}) \ .$$

To get H^1_0 , the S^1-invariant part of H^1 , we first remark that:

$$g_P \otimes \mathbb{C} = \mathcal{O}_X \oplus K^2 \oplus K^{-2} \quad .$$

with $g_P = P \times_{Ad} su(2)$. There is a natural isomorphism $H^1 \cong H^1(X, g_P \otimes \mathbb{C})$, where the r.h.s. is sheaf cohomology , see Donaldson [9] ; the isomorphism is given by $a \rightarrow a^{0,1}$. Combining this with the holomorphic Kunneth formula we obtain for H^1_0 :

3.2

$$H^1 = (H^0(\mathbb{CP}^1,\mathcal{O}(2k)) \otimes H^1(S,L^2)) \quad \oplus \quad (H^1(\mathbb{CP}^1,\mathcal{O}(-2k)) \otimes H^0(S,L^{-2})) \quad \oplus$$
$$H^1(X,\mathcal{O}_X) .$$

The standard description of cohomology of line bundles on $\mathbb{CP}^1$ tells us:

$$H^0(\mathbb{C}P^1,\mathcal{O}(2(p_0+p_\infty)))^{S^1} \cong \mathbb{C}$$

$$H^1(\mathbb{C}P^1,\mathcal{O}(-2(p_0+p_\infty)))^{S^1} \cong \begin{cases} \mathbb{C} & p_\infty>0 \\ 0 & p_\infty=0 \end{cases}$$

and

$$H^0(\mathbb{C}P^1,\mathcal{O}(2(p_0-p_\infty)))^{S^1} = \begin{cases} \mathbb{C} & p_\infty=0 \\ 0 & p_\infty>0 \end{cases}$$

$$H^1(\mathbb{C}P^1,\mathcal{O}(2(p_0-p_\infty)))^{S^1} = 0$$

It follows that for case 1):

3.3 $$H^1_o \cong \begin{cases} H^1(S,L^2) \oplus \qquad\qquad \oplus H^0(X,\mathcal{O}_X) & \text{if } p_\infty=0 \\ H^1(S,L^2) \oplus H^0(S,L^{-2}) \oplus H^0(X,\mathcal{O}_X) & \text{if } p_\infty>0 \end{cases}$$

and for case 2):

3.4 $$H^1_0 = \begin{cases} H^1(S,L^2) \oplus H^0(X,\mathcal{O}_X) & p_\infty=0 \\ H^0(X,\mathcal{O}_X) & p_\infty>0 \end{cases}$$

To get the actual deformations , we need to study a Lyapunov-Schmidt reduction as in chapter 3 , 3.12. First compute the quadratic form of chapter 3 , 3.13:

$$Q: H^1_0 \to \mathbb{R} \ : a \to \langle a \wedge a \ , \omega \rangle \quad ,$$

with ω a harmonic generator of H^2_0. Introduce the following notation :

$$a^{0,1} = \begin{bmatrix} \delta & \alpha \\ \beta & -\delta \end{bmatrix} \quad \text{then} \quad a = \begin{bmatrix} \delta + \delta^* & \alpha - \beta^* \\ \beta - \alpha^* & -\delta - \delta^* \end{bmatrix} \quad , \text{ with}$$

$$\alpha \oplus \beta \oplus \delta \in H^1_0 \quad \text{as in } 3.2.$$

A little computation shows :

$$3.5 \qquad \langle a \wedge a \, , \, \omega \rangle = \int_X |\alpha|^2 - |\beta|^2 \quad .$$

This form is clearly degenerate , but there is an easy way around this. Over the R we can define two real vector bundles $\mathcal{H}^1$ and $\mathcal{H}^2$. The bundle $\mathcal{H}^2$ is the bundle of second cohomology groups of the deformation complex belonging to the connections in the base , and the bundle $\mathcal{H}^1$ has as fibre over $A \in R$ the orhogonal complement of $H^1(X,\mathcal{O}_X)$ in $H^1_{0,A}$. The map $\phi : H^1_0 \to C \cong H^2_0$ (chapter 3 , 3.10) now gets replaced by a bundle map $\mathcal{H}^1 \to \mathcal{H}^2$, and the moduli space near R looks like the zero set of ϕ modulo S^1. Fibrewise the map ϕ is approximated in C^1-sense by a quadratic form q_A , which equals the restriction of 3.5 to the fibre of $\mathcal{H}^1$. This form is non-degenerate because the condition $k_0 \geq g$ ensures that the cohomology groups of L^2 and L^{-2} have positive dimension. From the compactness of R one obtains isolation in the definite case and nearby irreducible solutions in the non-degenerate case. This proves the proposition.∎

In the case that e.g. $p_\infty = 0$, $p_0 > 0$, notice that the number of linearly independent *non-integrable* infinitesimal deformations is equal to $4k_0 - 1 + g$, provided $k_0 \geq g$. The existence of these is quite remarkable , because one would not expect real , positive definite quadratic forms to enter the picture in an algebraic geometric problem (that of classifying stable bundles).

Monopoles with $k_0>0$ and $k_\infty<0$ are more like a pair of a monopole and an anti-monopole than like a monopole. Physically this situation seems unstable unless very special balancing occurs between the various forces which monopoles exert on each other (weak , electro-magnetic , gravitational).

§4 Families of $\mathbb{C}^*$-equivariant bundles on $\mathbb{CP}^1$.

The next step is of course to attack the general SU(2)-monopole on X. Our approach will be to split the problem into two parts. First we shall investigate the structure of a general S^1-equivariant holomorphic SU(2)-bundle. After that we turn to the study of stability of these bundles. In view of theorem 2.3 , this will give information about the structure of the moduli space of monopoles on X.

We start of by considering (families of) S^1 equivariant holomorphic $\mathbb{C}^2$-bundles on $\mathbb{CP}^1$, with structure group $SL(2,\mathbb{C})$. We follow the lines of thought of Atiyah [1] , section 6.

Lemma 4.1 : A holomorphic $\mathbb{C}^*$-equivariant vector bundle E over $\mathbb{C}^2$ is holomorphically $\mathbb{C}^*$-isomorphic to $\mathbb{C}^2 \times E_0$, where E_0 is the fibre over $0 \in \mathbb{C}^2$, with the $\mathbb{C}^*$-representation given by the action.

Proof : (Atiyah [1] lemma 6.1) On $\mathbb{C}^2\setminus\{0\}$, $\mathbb{C}^*$-bundles are the same as bundles on $\mathbb{CP}^1$. Now these are of the form $\bigoplus_i \mathcal{O}(n_i)$. This gives a $\mathbb{C}^*$-isomorphism $E_{|\mathbb{C}^2\setminus\{0\}} \cong V \times \mathbb{C}^2\setminus\{0\}$, for a representation V . This isomorphism necessarily extends to all of $\mathbb{C}^2$, as $\{0\}$ is of codimension two . This then also gives $V \cong E_0$.∎

Corollary 4.2 : Lemma 4.1 also holds with $\mathbb{C}^2$ replaced by $\mathbb{C}$. Any automorphism of an equivariant $\mathbb{C}^2$-bundle on $\mathbb{C}$ with weights p , $-p \neq 0$, is of the form

$$(z,(v_1,v_2)) \rightarrow (z,(av_1 + bz^{2p}v_2 , cv_2))$$

with $a, c \in \mathbb{C}^*$, and $b \in \mathbb{C}$. For $p = 0$, automorphisms are given by constant $GL(2,\mathbb{C})$ matrices.

Proof: Pull back from $\mathbb{C}$ to $\mathbb{C}^2$ to get the first statement. The statement concerning the automorphisms is obvious.∎

We are now ready to treat bundles on $\mathbb{C}P^1$ in a similar way. Fix a point $1 \in \mathbb{C}P^1$, such that $1 \neq 0$, $1 \neq \infty$.

Proposition 4.3 : Let E be a $\mathbb{C}^*$-equivariant $\mathbb{C}^2$-bundle on P_1 with structure group $SL(2,\mathbb{C})$ and weights p_0 , $p_\infty \geq 0$, and given trivializations :

$$E_0^+ \cong \mathbb{C} \quad , \quad E_\infty^- \cong \mathbb{C} \quad , \quad E_1 \cong \mathbb{C}^2 \quad ,$$

where E_0^+ is the p_0 eigenspace in E_0 for the $\mathbb{C}^*$-action , and E_∞^- the $-p_\infty$ eigenspace at ∞. The $\mathbb{C}^*$-isomorphism class of E is canonically given by two non-zero vectors $v_1, v_2 \in \mathbb{C}^2$, which represent the image of $1 \in E_0^+$ and $1 \in E_\infty^-$ in E_1 under any equivariant transition functions respecting the trivializations.

If $p_\infty = 0$, then $\mathbb{C}^*$ isomorphism classes of $\mathbb{C}^2$-bundles with weights and trivializations as above (apart from the ones at infinity) are given by a non-zero vector $v_1 \in \mathbb{C}^2$.

The bundle is holomorphically $\mathbb{C}^*$-isomorphic to

$$\mathcal{O}(p_0 - p_\infty) \oplus \mathcal{O}(-p_0 + p_\infty) \quad \text{if} \quad \dim \operatorname{span}(v_1, v_2) = 2 \quad , \text{ and to}$$

$$\mathcal{O}(p_0 + p_\infty) \oplus \mathcal{O}(-p_0 - p_\infty) \quad \text{if} \quad \dim \operatorname{span}(v_1, v_2) = 1 \; .$$

Proof : The proof is almost trivial using our lemmas : We shall assume $p_\infty \neq 0$, and leave the case $p_\infty = 0$ to the reader. Using trivialisations on $\mathbb{C}P^1 \setminus \{0\}$ and $\mathbb{C}P^1 \setminus \{\infty\}$ provided by corollary 4.2 , we see that transition functions

$$h_0 : E_0 \to E_1 \quad \text{at} \quad 1 \in \mathbb{C}P^1$$
$$h_\infty : E_0 \to E_1 \quad \text{at} \quad 1 \in \mathbb{C}P^1$$

give rise to $\mathbb{C}^*$-invariant transition functions on $\mathbb{C}^* \subset \mathbb{C}P^1$.

The only automorphisms of E which respect the action and trivializations in $\mathbb{C}P^1 \setminus \{\infty\}$ act on the fibre at $1 \in \mathbb{C}P^1$ by matrices:

$$\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} , \quad b \neq 0$$

Hence h_0 is equivalent to $h_0 \cdot \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$. Clearly the only invariant associated to this is the image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in E_0^+$, in E_1 .

At $\infty \in \mathbb{C}P^1$ the situation is similar to that at $0 \in \mathbb{C}P^1$. This can be seen by using the map $z \to z^{-1}$.

For the final statement one considers

$$A_{\infty 0} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot h_\infty^{-1} \cdot h_0 \cdot \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} ,$$

and the reader will easily convince himself that this can be made equal to the desired transition functions. ∎

Remark 4.5 : 1) Using the proof of 5.13 we see that a $\mathbb{C}^*$-equivariant $\mathbb{C}^N$-bundle on $\mathbb{C}P^1$, with weights p_0^i , p_∞^j , at 0 and $\infty \in \mathbb{C}P^1$ (i,j =

1,...,N) , is $\mathbb{C}^*$-isomorphic to a sum of N equivariant line bundles of the form

$$\mathcal{O}(p_0^i - p_\infty^j).$$

In fact , the equivalence relation on $\mathbb{C}^*$-transition functions at $1 \in \mathbb{CP}^1$ is given by $h \simeq a \cdot h \cdot b$, with $h \in GL(N,\mathbb{C})$, and a,b in parabolic subgroups stabilizing a flag of increasing weight spaces . These parabolic subgroups contain the Borel subgroup B of upper triangular matrices in $GL(N,\mathbb{C})$. Now $B \backslash GL(N,\mathbb{C})/B = S_N$, the group of permutation matrices , and this implies our statement .

2) In Atiyah [2] a similar , but slightly more abstract approach is used to study equivariant $G_{\mathbb{C}}$-bundles. We have preferred to take the above explicit approach.

Now suppose that U is connected and open in $\mathbb{C}^n$, and that E is a holomorphic $\mathbb{C}^2$-bundle on $U \times \mathbb{CP}^1$ with trivial determinant and weights p_0 , $p_\infty \geq 0$, which are naturally independent of $u \in U$.

Proposition 4.6 : If $p_\infty \neq 0$ and $p_0 \neq 0$, let I denote the set of $\mathbb{C}^*$-isomophism classes of such bundles with given preferred isomorphisms

$$E_0^+ \cong L_0 \quad , \quad E_\infty^- \cong L_\infty \quad , \quad E_{|U \times \{1\}} \cong E_1 \quad ,$$

for line bundles L_0 , L_∞ and a vector bundle E_1 on U. Then I equals :

$$\Gamma(U,(L_0^* \otimes E_1)\backslash\{0\}) \times \Gamma(U,(L_\infty^* \otimes E_1)\backslash\{0\}) \quad ,$$

where $\{0\}$ is the zero section. If $p_0 \neq 0$ but $p_\infty = 0$ the corresponding I equals :

$$\Gamma(U, (L_0^* \otimes E_\infty) \setminus \{0\}) ,$$

with $E_\infty = E_{|S \times \{\infty\}}$.

Proof : The proof follows as the lemmas and proposition above using the following standard lemma to adapt the proof of lemma 4.1 to families:

Lemma Let F be a holomorphic vector bundle on $U \times \mathbb{C}P^1$ such that $F_{|u \times \mathbb{C}P^1} \cong \bigoplus_i \mathcal{O}(n_i)$, for all $u \in U$, with n_i independent of u . Then for all $u \in U$ there is an open $V \ni u$ such that $F_{|V \times \mathbb{C}P^1} \cong \pi^* (\bigoplus_i \mathcal{O}(n_i))$, where $\pi : U \times \mathbb{C}P^1 \to U$ is the projection. ▬

Let $Y(p_i, k_i)$ be the set of isomorphism classes of $\mathbb{C}^*$-equivariant holomorphic $\mathbb{C}^2$-bundles on $X = S \times \mathbb{C}P^1$, with trivial determinant. Fix a $1 \in \mathbb{C}P^1$ as for proposition 4.1. This 1 can be choosen in a canonical way by observing that precisely one surface S in $H^3/\Gamma \cong S \times \mathbb{R}$ is totally geodesic. Also observe that the stable bundles in $Y(p_i, k_i)$ will correspond to our monopoles. Using the obvious restrictions we get a map :

4.1 $$y : Y(p_i, k_i) \to \operatorname{Pic}^{-k_0}(S) \times \operatorname{Pic}^{-k_\infty}(S) \times \begin{bmatrix} \text{isomorphism classes of } \mathbb{C}^2\text{-} \\ \text{bundles on } S \text{ with } c_1 = 0 \text{ and} \\ \text{trivial determinant} \end{bmatrix}$$

$$E \to (E_0^+ , E_\infty^- , E_{|S \times \{1\}})$$

The fibre $y^{-1}(E_0^+, E_\infty^-, E_1)$ can be described by choosing representatives

$$\begin{array}{rl} L_0 \text{ for } & E_0^+ \in \mathrm{Pic}^{-k_0}(S) \\ L_\infty \text{ for } & E_\infty^- \in \mathrm{Pic}^{-k_\infty}(S) \\ F \text{ for } E_1 \in & \begin{bmatrix}\text{isomorphism classes of vector} \\ \text{bundles on } S \text{ as above}\end{bmatrix} \end{array}$$

Proposition 4.6 tells us that $y^{-1}(L_0\ ,\ L_\infty\ ,\ F)$ is equal to :

4.2 $$[\Gamma(S,(L_0^* \otimes F)\setminus\{0\}) \times \Gamma(S,(L_\infty^* \otimes F)\setminus\{0\})]/G$$

where G is the group of automorphisms :

4.3 $$\mathrm{Aut}(L_0^*) \times \mathrm{Aut}(L_\infty^*) \times \mathrm{Aut}(F) \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathrm{Aut}(F)$$

acting in the natural way. Again a corresponding statement holds if $p_\infty = 0$. Notice that $y^{-1}(L_\infty\ ,\ L_0\ ,\ F)$ may well be empty.

The sections in 4.2 are essentially the rational functions occurring in Atiyah [1] and Donaldson [8]. If one assumes that F is indecomposable , then $G = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, and $y^{-1}(\mathrm{Pic}^{-k_0}, \mathrm{Pic}^{-k_\infty}, F)$ is equal to the subset of

$$\Gamma(S,P(F)) \times \Gamma(S,P(F))$$

consisting of the sections of the ruled surface $P(F) \to S$ of degrees k_0 , k_∞ . In the case considered by Atiyah $P(F) = \mathbb{C}P^1 \times \mathbb{C}P^1$, and we end up with rational functions. In that case the group G (4.3) is reduced to $\mathbb{C}^*$ by insisting on a fixed trivialisation at infinity.

Proposition 4.7 : Let $E \in Y(p_i,k_i)$, $p_0 \neq 0$. On $X \setminus S \times \{\infty\}$ E is an extension

$$0 \to \pi^*(E_0^+) \to E \to \pi^*(E_0^+)^* \to 0 .$$

If $p_\infty = 0$ then the extension extends to one on X :

$$0 \to \pi^*(E_0^+) \otimes \rho^*\mathcal{O}(p_0) \to E \to \{\pi^*(E_0^+) \otimes \rho^*\mathcal{O}(p_0)\}^* \to 0$$

Also in this case $E_{|S\times\mathbb{C}P^1\setminus\{0\}} = \pi^* E_\infty$.

Proof : In proposition 4.3 we saw that a vector $v \in E_0^+$ gives rise to a map $\rho^* E_0^+ \to E$ on $\{u\} \times \mathbb{C}P^1 \setminus \{\infty\}$, by using the trivialization $E_{|\mathbb{C}P^1\setminus\{\infty\}} = \mathbb{C} \times E_0$. This holds for families as well , and this proves the first part of the lemma.

On a $\{u\} \times \mathbb{C}P^1 \setminus \{0\}$ the bundle E is isomorphic to $\mathbb{C} \times E_\infty$, and the image of E_0^+ is a fixed vector subspace of E_∞ in this trivialization. In a canonical way this gives E a sub-bundle isomorphic to $\mathcal{O}(p_0)$ on $\{u\} \times \mathbb{C}P^1$. This is stable under variation of $u \in \mathbb{C}P^1$ and supplies the sub-bundle

$$\pi^* E_0^+ \otimes \rho^*\mathcal{O}(p_0) \to E$$

The last bit is proved similarly , using trivializations coming from the family version of corollary 4.2 .∎

Notice that the existence of the extension could have been obtained by appealing to chapter 2 , proposition 6.5.

§5 The stability of $\mathbb{C}^*$-equivariant bundles on X

So far for the equivariant bundles on X. The next question we shall address is that of the stability of elements in $Y(p_i,k_i)$. First of all it follows from Okonek-Spindler-Schneider [16], 1.2.2, that in 5.3 we may restrict J to sheaves for which $\mathcal{O}(E)/J$ is torsion free.
From Okonek a.o., loc. cit., 1.1.2., 1.1.15 and 1.1.16 it follows that J must be a line bundle. Actually, we can do a bit better :

Lemma 5.1 : Let E be a holomorphic $\mathbb{C}^2$ bundle on X which is not semi-stable. There is a unique torsion free quotient sheaf $\mathcal{O}(E)/J$ of rank 1, such that $c_1(J)\cdot H$ is maximal.
Proof : Existence of a quotient maximizing the first Chern class follows from the Kodaira vanishing theorem.

To prove uniqueness assume that there are J and J', satisfying the maximality condition. Use the extensions $J \to E \to E/J$ and its primed equivalent, to obtain the composition :

$$J \to E \to E/J' .$$

If this is nonzero, then one gets $c_1(J)\cdot H \leq c_1(E)\cdot H - c_1(J')\cdot H$, which shows that E would be semi-stable, contradictory to our assumption.∎

If $J \xrightarrow{i} E$ is such a destabilizing line bundle then clearly i must be equivariant for some action on J. Because $J \cong \rho^*\mathcal{O}(q) \otimes \pi^*N$, for some $N \in \mathrm{Pic}(S)$, it follows that any action on J arises from considering $\rho^*\mathcal{O}(q)$ as an equivariant pull back and giving π^*N a trivial

$\mathbb{C}^*$-action. Further $\mathcal{O}(q)$ is a subbundle of $E_{|\{s\}\times\mathbb{C}P^1}$ for generic $s \in S$. Thus the possible weights of J satisfy $w_0 = \pm p_0$, $w_\infty = \pm p_\infty$, and $q = w_0 - w_\infty$.

If E is semi-stable but not stable we can draw the same conclusion:

Lemma 5.2 : If E is semi-stable but not stable there is exists a equivariant destablizing line bundle J with $\mathbb{C}^*$-weights $\pm m_0$, $\pm m_\infty$.

Proof. Let $J \to E$ be any destabilizing, invertible subsheaf with $c_1(J).H = 0$. Give J any $\mathbb{C}^*$-action then $H^0(X,J^*\otimes E)$ is a $\mathbb{C}^*$-module. Let s be a weight vector. After modifying the action on J by a character we get an equivariant non-zero morphism

$$J \xrightarrow{s} E ,$$

making J an equivariant, destabilizing line bundle. Now s restricted to $S \times \{0\}$ cannot be zero, because then $J \otimes \mathcal{O}(S \times \{0\})$ would make E not to be semistable. Therefore J inherits the weights $\pm p_0$. At ∞ the situation is of course the same. ▬

Lemma 5.3 : Let $E \in Y(p_i,k_i)$ with $p_0 \neq 0$ and suppose $N \in \mathrm{Pic}(S)$. Giving π^*N a trivial $\mathbb{C}^*$-action consider the $\mathbb{C}^*$-equivariant coherent sheaf $F = \rho_*(\pi^*N \otimes E)$. F is a locally free sheaf on $\mathbb{C}P^1$, and the possible weights of F at $0 \in \mathbb{C}P^1$ are $\geq -p_0$.

Proof. Torsion of F would occur at discrete points $y \in \mathbb{C}P^1$, and give rise to holomorphic sections of $\pi^*N \otimes E$ supported on $S \times \{y\}$. This is clearly impossible, therefore F is torsion free, but on an algebraic curve this implies that F is locally free.

On $S \times \mathbb{C}$, with $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$, $\pi^*N \otimes E$ is an equivariant extension :

5.1 $$\pi^*(N \otimes E_0^+) \to \pi^*N \otimes E \to \pi^*(N \otimes E_0^{+*}) .$$

This gives rise to the complex of free $\mathbb{C}^*$-equivariant $\mathbb{C}[t]$ - modules

$$\mathbb{C}[t] \otimes H^0(S,N \otimes E_0^+) \xrightarrow{i} H^0(S \times \mathbb{C},\pi^*N \otimes E) \xrightarrow{p} \mathbb{C}[t] \otimes H^0(S,N \otimes E_0^{+*})$$

Here $H^0(S,N \otimes E_0^+)$ ($H^0(S,N \otimes E_0^{+*})$) has $\mathbb{C}^*$-weight p_0 $(-p_0)$, and $\mathbb{C}^*$ acts on $\mathbb{C}[t]$ by sending $t \to zt$. This sequence is exact at $F[\mathbb{C}] = H^0(S \times \mathbb{C},\pi^*N \otimes E)$ due to the cohomology sequence associated to the extension 5.1.

Let $s \in F[\mathbb{C}]$ be a weight vector of some weight m . These exist by corollary 4.2 . If $p(s) \neq 0$ then $p(s) = \Sigma\, q_n(t)h_n$, with $h_n \in H^0(S,N \otimes E_0^{+*})$, and q_n homogeneous of degree $m+p_0 \geq 0$. Hence $m \geq -p_0$. Should $p(s)=0$ then one proves in the same way that $m \geq p_0$. ∎

We have now collected all the tools to state

Theorem 5.4 : Let $E \in Y(p_i,k_i)$, $p_i \geq 0$, $k_i \geq 0$. Then E is not stable if and only if one of the following holds :

1) $p_\infty = 0$ and $(g-1)\cdot k_0 \leq p_0 \neq 0$ or

$p_0 = 0$ and $(g-1)\cdot k_\infty \leq p_\infty \neq 0$ or

$p_0 \neq 0$, $p_\infty \neq 0$ and all lines $\{s\} \times \mathbb{C}P^1$ jumping and $(g-1)\cdot k_0 = (g-1)\cdot k_\infty \leq p_0 + p_\infty$

2) E_1 has a line sub-bundle of degree n s.t. $(g-1)\cdot n \geq p_0 + p_\infty$

3) $0 \neq p_0 \geq p_\infty$ and $(g-1)\cdot k_0 \leq p_0 - p_\infty$ or

$0 \neq p_\infty \geq p_0$ and $(g-1)\cdot k_\infty \leq p_\infty - p_0$

4) $p_0 = p_\infty \neq 0$ and $k_0 = 0$, $k_\infty \neq 0$ or

$p_\infty = p_0 \neq 0$ and $k_\infty = 0$, $k_0 \neq 0$.

Proof. Suppose E is not stable. Let $J = \pi^*N \otimes \rho^*\mathcal{O}(q)$ be an equivariant destabilizing bundle as discussed above , with weights w_0 , w_∞ , and $N \in \mathrm{Pic}^n S$. Then $c_1(L)\cdot H = (g-1)\cdot n + q$. We shall prove that the conditions given in 1)....4) are necessary and sufficient for the following cases to occur:

1) $w_0 = p_0$, $w_\infty = -p_\infty$. Then $q = p_0 + p_\infty$. As $\mathcal{O}(q)$ is a subbundle of $E_{|\{s\}\times\mathbb{C}P^1}$ for generic $s \in \mathbb{C}P^1$ we see from 4.3 that then either all these lines have to be jumping or $p_\infty = 0$ or $p_0 = 0$. Treating the case $p_0 \neq 0$, observe that on $S \times \{0\}$ the line bundle N maps to E_0^+ , hence $\deg N \leq -k_0$. As J destabilizes it follows that $(g-1)\cdot k_0 \leq p_0 + p_\infty$.

Conversely , using the assumptions we see as in 4.7 that $\pi^*E_0^+ \otimes \rho^*\mathcal{O}(p_0+p_\infty)$ is a subbundle of E , and one uses 4.7 to destabilize E with this sub-bundle if the given conditions hold.

2) $w_0 = -p_0$, $w_\infty = p_\infty$. Now $q = -p_0 - p_\infty$, hence $(g-1)\cdot n \geq p_0 + p_\infty$.

Conversely , consider $F = \rho_*(\pi^*N \otimes E)$ as a $\mathbb{C}^*$-equivariant vector bundle on $\mathbb{C}P^1$. By lemma 5.3 , the weights of F are $\geq -p_0$ at $0 \in \mathbb{C}P^1$ and $\leq p_\infty$ at $\infty \in \mathbb{C}P^1$. From remark 4.5 it follows that $\rho^*\mathcal{O}(q) \otimes \pi^*N$ destabilizes E for some $q \geq p_0 + p_\infty$.

Now suppose $0 \neq p_0 \geq p_\infty$. The other half follows by symmetry , and 2).

3) $w_0 = p_0$, $w_\infty = p_\infty$, so $q = p_0 - p_\infty$. At $0 \in \mathbb{C}P^1$ we have $N \to E_0^+$. Hence $\deg N \leq -k_0$, so $(g-1)\cdot k_0 \leq p_0 - p_\infty$.

Conversely , this being the case , consider $F = \rho_*(\pi^*E_0^+ \otimes E)$.

This has p_0 among its weights at $0 \in \mathbb{C}P^1$ and we destabilize as in 2). Finally

4) $w_0 = -p_0$, $w_\infty = -p_\infty$, so $q = -p_0 + p_\infty$. Hence $(g-1)\cdot n \geq -q \geq 0$. But at $\infty \in \mathbb{C}P^1$, N maps into E^-_∞ , if $p_\infty \neq 0$,hence , $\deg N \leq -k_\infty \leq 0$. Thus $k_\infty = 0$, $p_0 = p_\infty$. The case $p_\infty = 0$ is dealt with in 1) , 2) .

This time we consider $F = \rho_*(\pi^* E^-_\infty \otimes E)$, which has $-p_\infty$ among its weights at $\infty \in \mathbb{C}P^1$. The weights at $0 \in \mathbb{C}P^1$ are $\geq -p_0$, and therefore the resulting map

$$\pi^* E^-_\infty \otimes \rho^* \mathcal{O}(q) \to E$$

destabilizes E for some $q \geq -p_0 + p_\infty$.∎

Holomorphic bundles with $p_0 > p_\infty$ and $(g-1)\cdot k_0 < (p_0 - p_\infty)$ cannot be stable. We shall give some speculative physical arguments to explain this. Suppose we had a monopole with such charges and masses. We could think of this configuration as having a monopole of mass p_0 and charge k_0 near 0. If this monopole were free to move it could go to the lower mass end ∞ in the universe $S \times \mathbb{R}$: the energy gain would be $p_0 - p_\infty$ ($E = mc^2$). But it may loose its charge in the process : 'energy' (?) loss $(g-1)\cdot k_0$. This is exactly the criterion for stability which we derived above : monopoles exist if converting their charge into mass is not 'thermo-dynamically' favorable.

To make this more precise one would have to construct an approximate monopole configuration with these masses and charges , and consider this as an initial value for the time-dependent Yang-Mills-Higgs equation. If the time dependent solution would show movement towards ∞ , then this would give strong evidence for our conjecture to be correct.

Even more wildly , one could argue as follows. Black holes in the universe could have a considerable influence on the topology of a space-like slice. The topology would almost certainly put constraints on possible monopole configurations , and this in turn could affect observability....

§6 General moduli spaces and scattering data

In this section we will combine the material of §4 and §5 to describe the moduli spaces of monopoles on X. After this we shall identify part of the holomorphic data needed to describe the monopole as scattering data for the monopole. For simplicity we shall take $p_\infty=o$ and $g=2$ throughout this section.

Our interest will lie in irreducible monopoles ; for that reason the invariants k_0 , p_0 satisfy $p_0 \neq k_0$. Recall that we have a projection:

$$\begin{array}{ccc} Y(p_0,k_0) & \rightarrow & \mathrm{Pic}^{-k_0} \times \begin{bmatrix} \text{isom. classes of } \mathbb{C}^2\text{-bundles on} \\ S \text{ , as in formula 4.1} \end{bmatrix} \\ E & \rightarrow & (E_0^+, E_\infty) \end{array}$$

The monopole moduli space is the subset of $Y(p_0,k_0)$ consisting of stable bundles. From theorem 5.4 we learn that $E \in Y(p_0,k_0)$ is stable if and only if :

$k_0 > p_0$

E_∞ admits no line sub-bundles of degree $\geq p_0$.

Let J be the set of isomorphism classes of holomorphic $\mathbb{C}^2$ bundles on S with trivial determinant. The set J has a filtration :

$$J_{-1} \subset J_0 \subset J_1 \subset \dots . \subset J$$

where J_q is the set of isomorphism classes of those holomorphic bundles which admit line sub-bundles of degree at most q. Thus J_{-1} is the same as the moduli space of stable bundles on S , however , J_0 is not the moduli space of semi-stable bundles , because two semi-stable bundles which give the same point in the moduli space need not be isomorphic. This filtration arises from a stratification of the space of holomorphic structures on an $\mathbb{C}^2$-bundle over S, see Atiyah-Bott [4].

The conclusion is :

$$\mathcal{M}(p_0,k_0) = y^{-1}(\mathrm{Pic}^{-k_0}(S) \times J_{p_0}).$$

We shall comment a bit further on this. Above all it is necessary to stress that y is not a continuous map ; in fact, we have not even defined a topology on the image space, and it is known that spaces like J have no good topology. The fibres of y are (4.2):

$$H^0(S,(E_0^{+*} \otimes E_\infty)\setminus\{0\})/\mathbb{C}^*\times\mathrm{Aut}(E_\infty) .$$

If k_0 is small then $\dim H^0(S,E_0^{+*} \otimes E_\infty)$ wil not be constant for varying E_0^+ and E_∞, and then these cohomology spaces may well be zero. A detailed study of this would be something like a theory of 'special divisors' for vector bundles. If k_0 is sufficiently big, then this pathology disappears. To see this, notice that for a vector bundle $E_\infty \in J_{p_0}$ one has $H^0(S,L^* \otimes E_\infty) = 0$ for any line budle L satisfying $\deg L > p_0$. By Serre-duality, $H^1(S,L^* \otimes E_\infty) \cong H^0(S,K \otimes L \otimes E_\infty)$, where K is the canonical bundle ; so $H^1(S,E_0^{+*} \otimes E_\infty) = 0$ for $k_0 > p_0 + 2\cdot g - 2$, and this being the case, the dimension of $H^0(S,E_0^{+*} \otimes E_\infty)$ is constant as a function of E_0^+ and E_∞. However it seems unavoidable that $\mathrm{Aut}(E_\infty)$ keeps jumping, this happens for example at $E_\infty \cong \mathbb{C}$.

There is a further problem of a more global nature. One may hope that there is a universal bundle :

$$E \to \mathrm{Pic}^{-k_0}S \times J_{p_0} \times S ,$$

such that the collection of cohomology groups is the direct image of this universal bundle under the projection $\mathrm{Pic}^{-k_0}S \times J_{p_0} \times S \to \mathrm{Pic}^{-k_0}S \times J_{p_0}$. However , there are topological obstructions to the existence of this universal bundle , see Atiyah-Bott [4]. The conclusion one may draw from this discussion is that it is relatively simple to say which holomorphic bundles carry a monopole , but the global structure of the moduli space is not so easy to investigate.

Finally we shall show that the elements in $y^{-1}(E_0^{+*} \otimes E_\infty)$ are scattering data for the monopole. Recall from proposition 4.7 , that a monopole gives rise to an S^1-invariant , holomorphic , nowhere vanishing section ϕ of:

$$E_0^{+*} \otimes \mathcal{O}(-p_0) \otimes E$$

on X , where $E_0^{+*} \otimes \mathcal{O}(p_0)$ has an S^1-action with weight p_0 at 0 and weight 0 at infinity. The element in $r \in y^{-1}(E_0^{+*} \otimes E_\infty)$ belonging to the monopole is ϕ restricted to $S \times \{\infty\}$.

Upon restricting to a line $\{s\} \times \mathbb{C}P^1$ there is an S^1-invariant section of $\mathcal{O}(p_0)$ which has a zero of order p_0 at 0 and which doesn't vanish at infinity. Thus also E restricted to this line has an S^1-invariant holomorphic section decaying near 0 and equal to multiple of r(s) at infinity.

Let (A_3,Φ) the pair of connection and Higgs field on $M = S \times \mathbb{R}$ describing the monopole (see chapter 2 , §5). The S^1-invariant holomorphic sections of $E_{|\{s\} \times (\mathbb{C}P^1-\{0,\infty\})}$ correspond to solutions of the Hitchin equation on $\{s\} \times \mathbb{R}$ in M (see Hitchin [13]):

$$\frac{\partial}{\partial l}\phi + A_3(\frac{\partial}{\partial l})\phi + i\cdot\Phi\cdot\phi = 0$$

for a $\mathbb{C}^2$-valued function ϕ. The solution space of this ordinary differential equation is two-dimensional , and can be identified with $(E_\infty)_S$: see corollary 4.2. Because Φ attains norm $2p_0$ near $-\infty$ on the line $\{s\} \times \mathbb{R}$, the generic solution of this equation will be growing exponentially for $l \to -\infty$. However there is solution , unique up to scalar factors , which decays for $l \to -\infty$, and , varying $s \in S$, this defines a line sub-bundle of E_∞ which is obviously equal to $r(E_0^+)$. This sub-bundle is called the *scattering data* for the monopole. We have proved :

Proposition 6.1 : A magnetic monopole on X with invariants p_0 , k_0 , $p_\infty=0$ is determined by the following two data :

1) an element $E_\infty \in J_{p_0}$

2) Scattering data , i.e. a section of $P(E_\infty)$ of degree k_0.

Furthermore , if $k_0>p_0$ then there exists exactly one monopole for any pair $(E_\infty$,scattering data) if E_∞ is indecomposable.

Monopoles on $\mathbb{R}^3$ and H^3 are determined by their scattering data. There are no further invariants in the form of non-trivial bundles , and the scattering data can be identified with rational functions , see Hitchin [13] , Donaldson [8] , Hurtubise [14] for $\mathbb{R}^3$ and Atiyah [1] for H^3. The main difference is therefore that the scattering data now live in some background provided by E_∞.

§7 References

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