

A Harary-Sachs Theorem for Hypergraphs

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Abstract

We generalize the Harary-Sachs theorem to k -uniform hypergraphs: the codegree- d coefficient of the characteristic polynomial of a uniform hypergraph \mathcal{H} can be expressed as a weighted sum of subgraph counts over certain multi-hypergraphs with d edges. We include a detailed description of the aforementioned multi-hypergraphs and a formula for their corresponding weights.

Keywords: Characteristic Polynomial, Hypergraph, Subgraph Statistics, Coefficient

1. Introduction

An early, seminal result in spectral graph theory of Harary [5] (and later, more explicitly, Sachs [8]) showed how to express the coefficients of a graph's characteristic polynomial as a certain weighted sum of the counts of various subgraphs of G (a thorough treatment of the subject is given in [1], Chapter 7).

Theorem 1 ([5],[8]) *Let G be a labeled simple graph on n vertices. If H_i denotes the collection of i -vertex graphs whose components are edges or cycles, and c_i denotes the coefficient of λ^{n-i} in the characteristic polynomial of G , then*

$$c_i = \sum_{H \in H_i} (-1)^{c(H)} 2^{z(H)} [\#H \subseteq G]$$

where $c(H)$ is the number of components of H , $z(H)$ is the number of components which are cycles, and $[\#H \subseteq G]$ denotes the number of (labeled) subgraphs of G which are isomorphic to H .

The goal of the present paper is to provide an analogous result for the characteristic polynomial of a hypergraph. The full result is given in Theorem 14, but to state here simply: fix $k \geq 2$ and let H_d denote the set of k -valent (i.e., k divides the degree of each vertex) k -uniform multi-hypergraphs on d edges. For a k -uniform hypergraph \mathcal{H} the

codegree- d coefficient (i.e., the coefficient of $x^{\deg-d}$) of the characteristic polynomial of the n -vertex hypergraph \mathcal{H} can be written

$$c_d = \sum_{H \in H_d} (-(k-1)^n)^{c(H)} C_H(\#H \subseteq \mathcal{H})$$

where $c(H)$ is the number of components of H , C_H is a constant depending only on H , and $(\#H \subseteq \mathcal{H})$ is the number of times H occurs (in a certain sense that is a minor generalization of the subgraph relation) in \mathcal{H} .

The quantity $(\#H \subseteq \mathcal{H})$ is straightforward to compute. However, computing C_H is more complicated. This notion of an *associated coefficient of a hypergraph* first appeared in [2], where the authors provide a combinatorial description of the codegree k and codegree $k+1$ coefficient, denoted c_k and c_{k+1} respectively, for the normalized adjacency characteristic polynomial of a k -uniform hypergraph.

Theorem 2 [2] *Let \mathcal{H} be a k -uniform hypergraph. Then*

$$c_k = -k^{k-2}(k-1)^{n-k}|E(\mathcal{H})|$$

and

$$c_{k+1} = -C_k(k-1)^{n-k}(\# \text{ of simplices in } \mathcal{H}),$$

where C_k is some constant depending on k .

This idea was further studied by Shao, Qi, and Hu where the authors prove (restating Theorem 4.1 of [9]),

$$c_d = (k-1)^{n-1} \sum_{D \in \mathbf{D}} f_D |\mathfrak{E}(D)|$$

where \mathbf{D} is a certain large family of digraphs, f_D is a function of D and $\mathfrak{E}(D)$ is the set of Euler circuits in D . The authors then use their formula to provide a general description of $\text{Tr}_2(T)$ and $\text{Tr}_3(T)$ for a general tensor T . Our first few results are similar to that of [9] (as described in more detail below), and we use them to provide an explicit combinatorial description of H_D and the resulting C_H .

Here we present some requisite background maintaining the notation of [2]. A (*cubic*) *hypermatrix* \mathcal{A} over a set \mathbb{S} of *dimension* n and *order* k is a collection of n^k elements $a_{i_1 i_2 \dots i_k} \in \mathbb{S}$ where $i_j \in [n]$. A hypermatrix is said to be *symmetric* if entries with identical multisets of indices are the same. That is, \mathcal{A} is symmetric if $a_{i_1 i_2 \dots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(k)}}$ for all permutations σ of $[k]$. An order k dimension n symmetric hypermatrix \mathcal{A} uniquely defines a homogeneous degree k polynomial in n variables (a.k.a. a “ k -form”) by

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1, i_2, \dots, i_k=1}^n a_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

If we write $\mathbf{x}^{\otimes r}$ for the order r dimension n hypermatrix with i_1, i_2, \dots, i_k entry $x_{i_1} x_{i_2} \dots x_{i_r}$ and x^r for the vector with i -th entry x_i^r then the above expression can be written as

$$\mathcal{A} \mathbf{x}^{\otimes k-1} = \lambda \mathbf{x}^{k-1}$$

where the multiplication denoted by concatenation is tensor contraction. Call $\lambda \in \mathbb{C}$ an *eigenvalue* of \mathcal{A} if there is a non-zero vector $\mathbf{x} \in \mathbb{C}^n$, which we call an *eigenvector*, satisfying

$$\sum_{i_2, i_3, \dots, i_k=1}^n a_{ji_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k} = \lambda x_j^{k-1}.$$

Next we offer an important result from commutative algebra to proceed the definition of the adjacency characteristic polynomial of a hypergraph.

Theorem 3 (*The Resultant, [4]*) Fix degrees d_1, d_2, \dots, d_n . For $i \in [n]$, consider all monomials \mathbf{x}^α (where α is itself a vector) of total degree d_i in x_1, \dots, x_n . For each such monomial, define a variable $u_{i,\alpha}$. Then there is a unique polynomial $\text{RES} \in \mathbb{Z}[\{u_{i,\alpha}\}]$ with the following three properties:

1. If $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ are homogeneous polynomials of degrees d_1, \dots, d_n respectively, then the polynomials have a non-trivial common root in \mathbb{C}^n exactly when $\text{RES}(F_1, \dots, F_n) = 0$. Here, $\text{RES}(F_1, \dots, F_n)$ is interpreted to mean substituting the coefficient of \mathbf{x}^α in F_i for the variable $u_{i,\alpha}$ in RES .
2. $\text{RES}(x_1^{d_1}, \dots, x_n^{d_n}) = 1$.
3. RES is irreducible, even in $\mathbb{C}[\{u_{i,\alpha}\}]$.

Moreover, for $i \in [n]$, RES is homogeneous in the variable $\{u_{i,\alpha}\}$ with degree $\prod_{j \in [n], j \neq i} d_j$.

Definition 1 ([7]) The symmetric hyperdeterminant of \mathcal{A} , denoted $\det(\mathcal{A})$ is the resultant of the polynomials which comprise the coordinates of $\mathcal{A}x^{\otimes k-1}$. Let λ be an indeterminate. The characteristic polynomial $\phi_{\mathcal{A}}(\lambda)$ of a hypermatrix \mathcal{A} is $\phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I} - \mathcal{A})$.

We consider the normalized adjacency matrix of a k -uniform hypergraph, $\mathcal{H} = (V, E)$. We refer to such hypergraphs as k -graphs and we reserve the language of *graph* for the case of $k = 2$. For a k -graph $\mathcal{H} = ([n], E)$ we denote the (normalized) adjacency hypermatrix $\mathcal{A}_{\mathcal{H}}$ to be the order k dimension n hypermatrix with entries

$$a_{i_1, i_2, \dots, i_k} = \frac{1}{(k-1)!} \begin{cases} 1 & : \{i_1, i_2, \dots, i_k\} \in E(H) \\ 0 & : \text{otherwise.} \end{cases}$$

For simplicity, we denote $\phi(\mathcal{H}) = \phi_{\mathcal{A}_{\mathcal{H}}}(\lambda)$ and write

$$\phi(\mathcal{H}) = \sum_{i=0}^t c_i \lambda^{t-i}$$

where $t = n(k-1)^{n-1}$ by Theorem 3. Throughout, we make use of the notation $\phi_d(\mathcal{H}) = c_d$ for the codegree- d coefficient of $\phi(\mathcal{H})$.

Our approach relies on the following trace formula for the hyperdeterminant of a tensor. In [6], Morozov and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using Schur polynomials in the generalized traces of the order k , dimension n hypermatrix \mathcal{A} . Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear map and let I be the unity map,

$I = (x_1, x_2, \dots, x_n)^T \rightarrow (x_1, x_2, \dots, x_n)^T$. Famously,

$$\log \det(I - f) = \text{tr} \log(I - f) = - \sum_{k=1}^{\infty} \frac{\text{tr}(f^k)}{k}.$$

The characteristic polynomial is defined as the resultant of a certain system of equations, so calculating the characteristic polynomial requires computation of the resultant. Moroz and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using *Schur polynomials* in the generalized traces of the order k , dimension n hypermatrix \mathcal{A} .

Definition 2 Define the d -th Schur polynomial $P_d \in \mathbb{Z}[t_1, \dots, t_d]$ by $P_0 = 1$ and, for $d > 0$,

$$P_d(t_1, \dots, t_d) = \sum_{m=1}^d \sum_{d_1 + \dots + d_m = d} \frac{t_{d_1} \cdots t_{d_m}}{m!}.$$

More compactly, one may define P_d by

$$\exp \left(\sum_{d=1}^{\infty} t_d z^d \right) = \sum_{d=1}^{\infty} P_d(t_1, \dots, t_d) z^d.$$

Let f_i denote the i th coordinate of $\mathcal{A}\mathbf{x}^{\otimes k-1}$. Define A to be an auxiliary $n \times n$ matrix with distinct variables A_{ij} as entries. For each I , we define the differential operator

$$\hat{f}_i = f_i \left(\frac{\partial}{\partial A_{i1}}, \frac{\partial}{\partial A_{i2}}, \dots, \frac{\partial}{\partial A_{in}} \right)$$

in the natural way. In [2], Cooper and Dutle use the aforementioned Morozov-Shakirov formula to show that the d -th trace of $\mathcal{A}_{\mathcal{H}}$,

$$\text{Tr}_d(\mathcal{A}_{\mathcal{H}}) = (k-1)^{n-1} \sum_{d_1 + \dots + d_n = d} \left(\prod_{i=1}^n \frac{\hat{f}_i^{d_i}}{(d_i(k-1))!} \text{tr}(A^{d(k-1)}) \right) \quad (1)$$

where $\text{tr}(A^{d(k-1)})$ is the standard matrix trace (for a more detailed explanation, see [2]). We prove Theorem 14 with the aid of the following reformulation of Equation 1:

$$\text{Tr}_d(\mathcal{A}_{\mathcal{H}}) = (k-1)^n \sum_{H \in H_d} C_H(\#H \subseteq \mathcal{H}).$$

The paper is arranged as follows. In the following section we define the associated digraph of an operator in $\text{Tr}_d(\mathcal{H})$, as in Equation 1. In particular, we provide a formula for a summand of $\text{Tr}_d(\mathcal{H})$ in terms of associated digraphs. In Section 3 we give a combinatorial description of the differential operators which have non-zero contribution to $\text{Tr}_d(\mathcal{H})$ and characterize these operators in terms of associated digraphs. In Section 4 we show that associated digraphs correspond to a particular type of hypergraph, termed Veblen hypergraphs. We conclude with a proof of our main result.

2. The associated digraph of an operator

Recall the d -th trace of $\mathcal{A}_{\mathcal{H}}$ from Equation 1,

$$\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}}) = (k-1)^{n-1} \sum_{d_1+\dots+d_n=d} \left(\prod_{i=1}^n \frac{\hat{f}_i^{d_i}}{(d_i(k-1))!} \mathrm{tr}(A^{d(k-1)}) \right)$$

where $\mathrm{tr}(A^{d(k-1)})$ is the standard matrix operation. Let $\hat{f}_{d_1, d_2, \dots, d_n}$ be an addend of $\prod_{i=1}^n \hat{f}_i^{d_i}$ in $\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}})$. When the context is clear we suppress the subscript and simply write \hat{f} . Given $\alpha = (i_1, i_2, \dots, i_{d(k-1)})$ let

$$A_{\alpha} := A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{d(k-1)-1}, i_{d(k-1)}} A_{i_{d(k-1)}, i_1} \quad (2)$$

and recall that

$$\mathrm{tr}(A^{d(k-1)}) = \sum_{\alpha} A_{\alpha}$$

where the factors of A_{α} are commutative. Adhering to the terminology of [2] we say A_{α} is k -valent if k divides the number of times i occurs in a subscript of A_{α} . We utilize divisibility notation for monomials in $\mathrm{tr}(A^{d(k-1)})$, e.g., using $g|h$ to denote that g occurs as a factor of the formal product h . We say that A_{α} *survives* \hat{f} if $\hat{f}A_{\alpha} \neq 0$.

Definition 3 For a differential operator $\hat{f}_{d_1, d_2, \dots, d_n}$ the associated digraph of \hat{f} , denoted $D_{\hat{f}}$, is the directed multigraph where there are d_i distinguishable edges directed from i to j given $\left(\frac{\partial}{\partial A_{i,j}}\right)^{d_i} | \hat{f}$ and isolated vertices are ignored.

We suppress the subscript and write D when \hat{f} is understood. We recall the following graph theoretic definitions according to [3].

Definition 4 A walk in a graph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. A walk is closed if $v_0 = v_k$. A closed walk in a graph is an Euler tour if it traverses every edge of the graph exactly once. An Euler circuit is an Euler tour up to cyclic permutation of its edges, i.e., an Euler tour with no distinguished beginning.

We denote the set of Euler tours of a graph G which begin at the edge $e \in E(G)$ by $\mathfrak{E}_e(G)$ and we denote the set of Euler circuits of G by $\mathfrak{E}(G)$. Recall that a digraph D has an Euler circuit if and only if $\deg^+(v) = \deg^-(v)$ for all $v \in V(D)$ and D is weakly connected.

Definition 5 Let $w = (v_i)_{i=0}^m$ be a sequence of (not necessarily distinct) vertices of D . We say that w describes an Euler tour in D if there exist distinct edges e_0, \dots, e_{m-1} such that $v_0 e_0 v_1 e_1 \dots e_{m-1} v_m$ is an Euler tour in D . Moreover we say that such Euler tours are described by w .

Note that the use of Euler tour in the previous definition is well-founded as e_0 is distinguished as the first edge.

Lemma 4 Consider $\mathrm{Tr}_d(\mathcal{H})$, $\hat{f} \mathrm{tr}(A^{d(k-1)}) \neq 0$ if and only if D is Eulerian. In this case

$$\hat{f} \mathrm{tr}(A^{d(k-1)}) = |E(D)| |\mathfrak{E}(D)|. \quad (3)$$

Proof: Consider $\text{Tr}_d(\mathcal{H})$. Fix a term A_α of $\text{tr}(A^{d(k-1)})$ and a differential operator \hat{f} of $\prod_{i=1}^n \hat{f}_i^{d_i}$. Suppose $\hat{f}A_\alpha \neq 0$. Whence $\hat{f}A_\alpha \neq 0$ the factors of \hat{f} are in one-to-one correspondence with the factors of A_α . It follows that the edges of $D_{\hat{f}}$ are in one-to-one correspondence with the factors of A_α . Notice that for $i \in V(D)$, $\deg^+(i) = \deg^-(i)$ by Equation 2. Further, D is strongly connected as the sequence of indices from i to j (cyclically, if necessary) is a walk from vertex i to vertex j . Therefore, D is Eulerian. Suppose now that D is Eulerian and let $\alpha = (v_i)_{i=0}$ describe an Euler tour in D . We claim that $\hat{f}A_\alpha$ is equal to the number of Euler tours in $D_{\hat{f}}$ described by α . Let $m(i, j)$ denote the number of edges from i to j in D . Notice that

$$A_\alpha = \prod_{i,j \in V(D)} A_{i,j}^{m(i,j)} \text{ and } \hat{f} = \prod_{i,j \in V(D)} \frac{\partial^{m(i,j)}}{\partial A_{i,j}^{m(i,j)}}.$$

Clearly

$$\hat{f}A_\alpha = \prod_{i,j \in V(D)} m(i, j)!$$

Moreover, α describes $\prod_{i,j \in V(D)} m(i, j)!$ Euler tours in D by straightforward enumeration. Observe that there are $|E(D)||\mathfrak{E}(D)|$ Euler tours in D as we may distinguish any edge from D as the first edge of an Euler circuit. Thus,

$$\hat{f} \text{tr}(A^{d(k-1)}) = \sum_{\alpha} \hat{f}A_\alpha = |E(D)||\mathfrak{E}(D)|$$

as every Euler tour is described by exactly one α . ■

We conclude this section with a remark about the evaluation of Equation 3. Conveniently, $|\mathfrak{E}(D)|$ can be computed using the BEST theorem, originally appearing in [11] as a variation of a result of [10].

Theorem 5 (*BEST Theorem*) *The number of Euler circuits in a connected Eulerian graph G is*

$$|\mathfrak{E}(G)| = \tau(G) \prod_{v \in V} (\deg(v) - 1)!$$

where $\tau(G)$ is the number of arborescences (i.e., the number of rooted subtrees of G with a specified root).

For simplicity we abbreviate $\tau(f) = \tau(D_{\hat{f}})$. Combining the BEST theorem and the observation that $|E(D)| = d(k-1)$ yields the following.

Corollary 6

$$\hat{f} \text{tr}(A^{d(k-1)}) = d(k-1)\tau(f) \prod_{v \in V(D)} (\deg^-(v) - 1)!$$

As a final note, recall that $\tau(G)$ can be computed using the Matrix Tree Theorem, which makes the computation of the right-hand side of the equality in Corollary 6 efficient.

Theorem 7 (*Matrix Tree Theorem/Kirchhoff's Theorem*) For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ be the non-zero eigenvalues of $\mathcal{L}(G) = D(G) - A(G)$. Then

$$\tau(G) = \frac{\lambda_1 \lambda_2 \dots \lambda_{n-1}}{n}.$$

3. Euler operators and Veblen hypergraphs

In Lemma 4 we showed that the only differential operators \hat{f} for which $\hat{f} \operatorname{tr}(A^{d(k-1)}) \neq 0$ are the operators whose associated digraphs are Eulerian. The question remains: which \hat{f} have an Eulerian associated digraph? We answer this question with the following graph decoration.

Definition 6 We define the u -rooted directed star of a k -uniform edge e to be

$$S_e(u) = (e, \{uv : v \in e, u \neq v\}).$$

A rooting of a k -graph \mathcal{H} is an ordering $R = (S_{e_1}(v_1), S_{e_2}(v_2), \dots, S_{e_m}(v_m))$ such that $E(\mathcal{H}) = \{e_1, \dots, e_m\}$ and $v_i \leq v_{i+1}$. Given a rooting of \mathcal{H} we define the rooted multi-digraph of R to be

$$D_R = \bigcup_{i=1}^m S_{e_i}(v_i)$$

where the union sums edge multiplicities. We say that a rooting R is an Euler rooting if D_R is Eulerian. We denote the multi-set of rooted digraphs of \mathcal{H} as $S(\mathcal{H})$.

Note that two distinct rootings can yield the same rooted digraph. We suppress the subscript D_R and write D when the context is clear. We further refer to D as a rooted digraph of \mathcal{H} for convenience.

Definition 7 Given a rooted digraph $D \in S(\mathcal{H})$, we define the rooted operator of D to be

$$\hat{f}_D = \prod_{uv \in E(D)} \frac{\partial}{\partial A_{u,v}}.$$

Moreover, we denote

$$\hat{S}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

In the case when D is Eulerian we refer to \hat{f}_D as an Euler operator.

The notation of \hat{f}_D is consistent with our usage of \hat{f} whence

$$\hat{f}_D \mid \prod_{i=1}^n \hat{f}_i^{d_i}$$

where d_i is equal to the number of times vertex i appears as a root of D . If \hat{f} is a rooted operator then it is understood that there exists a (not necessarily unique) rooting R such that, with a slight abuse of notation, $\hat{f} = \hat{f}_{D_R}$. We call such a rooting an *underlying rooting* of a differential operator.

Lemma 8 *The associated digraph of an operator \hat{f} is Eulerian if and only if \hat{f} is an Euler operator.*

Remark 1 *By Lemma 8 the only operators which have non-zero contribution to $\text{Tr}_d(\mathcal{H})$ are Euler operators. We denote $R(\mathcal{H}) \subseteq S(\mathcal{H})$ to be the multi-set of Euler rooted digraphs of \mathcal{H} . We further denote*

$$\hat{R}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

Remark 2 *One can deduce Theorem 4.1 of [9] from Lemma 8 by a change of notation: our \hat{f}_D is their F , our set of Eulerian associated digraphs arising from \hat{f}_R is their $\mathbf{E}_{d,m-1}(n)$, and our $\mathfrak{E}(D)$ is their $\mathbf{W}(E)$.*

We now show that an Euler rooting is a rooting of a special type of hypergraph.

Definition 8 *A Veblen hypergraph¹ is a k -uniform, k -valent multi-hypergraph.*

Lemma 9 *An Euler rooting R is a rooting of precisely one labeled Veblen hypergraph.*

Proof: Suppose $S = (S_i)_{i=1}^m$ is a rooting of a connected k -graph \mathcal{H} . Since D_S is Eulerian we have for all $j \in V(\mathcal{H})$

$$\deg^+(j) = (k-1)|\{i : v_i = j\}| = |\{i : v_i \neq j, j \in e_i\}| = \deg^-(j).$$

Fix a vertex $v \in V(\mathcal{H})$. We compute

$$\deg_{\mathcal{H}}(v) = \deg_D^+(v) + \deg_D^-(v) = k|\{i : v_i = v\}|.$$

Observe that $k \mid \deg_{\mathcal{H}}(v)$; it follows that \mathcal{H} is Veblen by definition. Now suppose that \mathcal{H}_0 is a connected Veblen graph such that S is an Euler rooting of \mathcal{H}_0 . As S is a rooting of H_0 , $E(\mathcal{H}_0) = E(\mathcal{H})$ and since both hypergraphs are connected $V(\mathcal{H}_0) = V(\mathcal{H})$. It follows that \mathcal{H} is unique. \blacksquare

We combine Lemmas 8 and 9 into the following Lemma.

Lemma 10 *We have*

$$\hat{f} \text{tr}(A^{d(k-1)}) \neq 0$$

if and only if $\hat{f} = \hat{f}_D$ is a rooted operator. Moreover, the underlying rooting of \hat{f} is necessarily an Euler rooting of precisely one connected, labeled Veblen hypergraph.

In the following section we use Lemma 10 to express the codegree- d coefficient of a k -graph as a function of Veblen hypergraphs. Here we conclude with a note about Veblen's theorem.

Theorem 11 *(Veblen's theorem [12]) The set of edges of a finite graph can be written as a union of disjoint simple cycles if and only if every vertex has even degree.*

¹The nomenclature is a reference to Oswald Veblen (1880-1960) who proved an extension of Euler's theorem in 1912. We present a brief note about Veblen's namesake theorem at the conclusion of this section.

Unfortunately, Veblen's theorem does not extend to higher uniformity: the set of edges of a finite k -graph \mathcal{H} can not always be written as a union of disjoint simple k -regular k -graphs if and only if \mathcal{H} is k -valent. Consider the Veblen 3-graph \mathcal{T} which consists of three bottomless tetrahedrons each sharing a common base. To be precise,

$$\mathcal{T} = \left(\{a, b, c, 1, 2, 3\}, \bigcup_{i=1}^3 \left(\binom{\{a, b, c, i\}}{3} \setminus \{a, b, c\} \right) \right).$$

A drawing of \mathcal{T} is given in Figure 1. Since there are only three edges containing

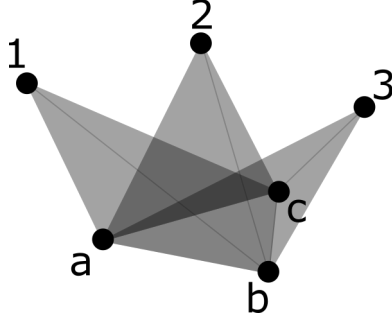


Figure 1: The Veblen 3-graph \mathcal{T} where edges are drawn as triangular faces.

$i \in [3]$ it must be the case that any partition into Veblen graphs places each edge containing i into the same class. Observe that for each i the vertices a , b , and c each have degree 2. Therefore, the only 3-valent edge partition is the trivial one.

4. The associated coefficient of a Veblen hypergraph

We now turn our attention to computing the codegree- d coefficient of a k -graph \mathcal{H} via Equation 1. From Lemma 10 we know that the only operators which satisfy $\hat{f} \operatorname{tr}(A^{d(k-1)}) \neq 0$ are rooted operators. Furthermore, as a differential operator of $\operatorname{Tr}_d(\mathcal{A}_{\mathcal{H}})$ is of degree d , the underlying Euler rooting of \hat{f} is a rooting of precisely one connected, labeled Veblen hypergraph with d edges. We equate $\operatorname{Tr}_d(\mathcal{H})$ to a weighted sum over Euler rootings of connected Veblen graphs with d edges which “appear” in \mathcal{H} . Consider the following generalization of the notion of subgraph.

Definition 9 For a labeled multi-hypergraph H , we call the simple k -graph formed by removing duplicate edges of H the flattening of H and denote it \underline{H} . We say that H is an infragraph of \mathcal{H} if $\underline{H} \subseteq \mathcal{H}$. Let $\mathcal{V}_d(\mathcal{H})$ denote the set of isomorphism classes of connected, labeled Veblen infragraphs with d edges of \mathcal{H} .

Definition 10 The associated coefficient of a connected Veblen hypergraph H is

$$C_H = \sum_{D \in R(H)} \left(\frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right).$$

The associated coefficient of a (possibly disconnected) Veblen hypergraph $H = \bigcup_{i=1}^m G_i$ is

$$C_H = \prod_{i=1}^m C_{G_i}.$$

Definition 11 For a k -graph \mathcal{H} and a Veblen k -graph $H = \bigcup_{i=1}^m G_i$ we define

$$(\#H \subseteq \mathcal{H}) = \frac{1}{|\text{Aut}(H)|} \prod_{i=1}^m |\text{Aut}(G_i)| |\{S \subseteq \mathcal{H} : S \cong \underline{G_i}\}|.$$

In the case when H is connected this simplifies to

$$(\#H \subseteq \mathcal{H}) = \frac{|\text{Aut}(\underline{H})|}{|\text{Aut}(H)|} |\{S \subseteq \mathcal{H} : S \cong \underline{H}\}| = |\text{Aut}(\underline{H})/\text{Aut}(H)| \cdot |\{S \subseteq \mathcal{H} : S \cong \underline{H}\}|.$$

Note that for $H = \bigcup_{i=1}^m G_i$, $(H \subseteq \mathcal{H})$ is not multiplicative over the components of H as

$$\prod_{i=1}^m (\#G_i \subseteq \mathcal{H}) = \frac{(\#H \subseteq \mathcal{H}) |\text{Aut}(H)|}{\prod_{i=1}^m |\text{Aut}(G_i)|}.$$

However, we have the following identity.

Lemma 12 Let $H = \bigcup_{i=1}^m G_i$ be a Veblen k -graph. If μ_H denotes the number of linear orderings of the components of H (where two components are indistinguishable if they are isomorphic) then

$$(\#H \subseteq \mathcal{H}) = \frac{\mu_H}{m!} \prod_{i=1}^m (\#G_i \subseteq \mathcal{H}).$$

Proof: Suppose there are t isomorphism classes of components of H with representatives H_1, H_2, \dots, H_t . Denote the number of components of H which are isomorphic to H_i as μ_i . Fix an ordering of the components which are isomorphic to H_i , $\{G_{1,i}, G_{2,i}, \dots, G_{\mu_i,i}\}$. The number of distinct linear orderings of the components of H where G_i and G_j are indistinguishable when $G_{r,i} \cong G_{s,i}$ is

$$\mu_H = \binom{m}{\mu_1, \mu_2, \dots, \mu_t}$$

so that

$$\frac{m!}{\mu_H} = \prod_{i=1}^t \mu_i!.$$

Note that for $a \in \text{Aut}(H)$, there exists $\sigma \in \mathfrak{S}_{\mu_i}$ such that $a(G_{j,i}) = G_{\sigma(j),i}$. In this way, $\text{Aut}(H)$ induces a permutation on the isomorphism classes of the components of H (note that this map is well-defined since the components are labeled). Let $\psi : \text{Aut}(H) \rightarrow \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \dots \times \mathfrak{S}_{\mu_t}$ be such a map. Notice $\ker(\psi)$ is the group of

automorphisms of H which maps each component to itself. Appealing to the First Isomorphism Theorem we have

$$\frac{|\text{Aut}(H)|}{\prod_{i=1}^m |\text{Aut}(G_i)|} = \prod_{i=1}^t \mu_i!.$$

The desired equality follows by substitution. \blacksquare

Remark 3 *The equation in Lemma 12 implies that $(\#H \subseteq \mathcal{H})$ is multiplicative over its components if and only if the components of H are pairwise non-isomorphic.*

Let

$$A(d, n) = \left\{ (d_1, \dots, d_n) : \sum d_i = d, d_i \geq 0 \right\}$$

be the set of *arrangements of d into n non-negative parts* and further let $A^+(d, n) \subset A(d, n)$ be the set of *arrangements of d into n positive parts*. For a k -graph \mathcal{H} and $a \in A^+(d, |E|)$ let $R^a(\mathcal{H})$ be the set of Euler rootings of all labeled, connected Veblen infragraphs of \mathcal{H} which have the property that vertex v_i is the root of exactly d_i edges. (N.B. We take $a \in A^+(d, |E|)$ as it is necessary that $d_i > 0$ for $D \in R^a(\mathcal{H})$ to be Eulerian.)

Remark 4 *Let $\mathcal{V}_d^*(\mathcal{H})$ denote the set of (possibly disconnected) Veblen infragraphs of \mathcal{H} with d edges up to isomorphism. Further let $\mathcal{V}_d(\mathcal{H}) \subseteq \mathcal{V}_d^*(\mathcal{H})$ denote the set of connected Veblen infragraphs of \mathcal{H} with d edges up to isomorphism.*

We now present a formula for $\text{Tr}_d(\mathcal{H})$ as a weighted sum over its Veblen infragraphs.

Lemma 13 *For a k -graph \mathcal{H}*

$$\text{Tr}_d(\mathcal{H}) = d(k-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}).$$

Proof: For convenience let $|E| = |E(\mathcal{H})|$. We equate

$$\begin{aligned} \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}) &= \sum_{H \in \mathcal{V}_d(\mathcal{H})} \left(\left(\sum_{D \in R(H)} \frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right) (\#H \subseteq \mathcal{H}) \right) \\ &= \sum_{a \in A^+(d, |E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right). \end{aligned}$$

Recall

$$\text{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{d_1 + \dots + d_n = d} \left(\prod_{i=1}^n \frac{\hat{f}_i^{d_i}}{(d_i(k-1))!} \text{tr}(A^{d(k-1)}) \right).$$

Applying Lemma 10 we have

$$\text{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{a \in A^+(d, |E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\hat{f}_D \text{tr}(A^{d(k-1)})}{\prod_{i=1}^n (d_i(k-1))!} \right).$$

By Corollary 6 ,

$$\hat{f}_D \operatorname{tr}(A^{d(k-1)}) = d(k-1)\tau_D \prod_{v \in V(D_R)} (\deg^-(v) - 1)!$$

When $D \in R^a(\mathcal{H})$ with $a = (d_1, \dots, d_n)$, we have $\deg^-(v_i) = d_i(k-1)$. By substitution we have

$$\begin{aligned} \operatorname{Tr}_d(\mathcal{H}) &= d(k-1)^n \sum_{a \in A^+(d, |E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right) \\ &= d(k-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}). \end{aligned}$$

■

We are now prove a generalization of the Harary-Sachs formula for k -graphs.

Theorem 14 *For a simple k -graph \mathcal{H} ,*

$$\phi_d(\mathcal{H}) = \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H(\#H \subseteq \mathcal{H}).$$

Proof: Fix a k -graph \mathcal{H} and $d \geq 1$. From [2] we have by Equation 1

$$\phi_d(\mathcal{H}) = P_d \left(\frac{-\operatorname{Tr}_1(\mathcal{H})}{1}, \frac{-\operatorname{Tr}_2(\mathcal{H})}{2}, \dots, \frac{-\operatorname{Tr}_d(\mathcal{H})}{d} \right)$$

where

$$P_d(t_1, t_2, \dots, t_d) = \sum_{m=1}^d \sum_{d_1 + \dots + d_m = d} \frac{t_{d_1} t_{d_2} \dots t_{d_m}}{m!}.$$

Fix an $1 \leq m \leq d$ and an arrangement $a = (d_1, d_2, \dots, d_m) \in A^+(d, m)$. By Lemma 13,

$$\frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} = -(k-1)^n \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}).$$

Let $\mathcal{V}^a(\mathcal{H})$ denote the set of m -tuples of connected unlabeled Veblen infragraphs of \mathcal{H} whose i -th coordinate has d_i edges for $i \in [m]$, such that $\sum_i d_i = d$. We have

$$\begin{aligned} \prod_{i=1}^m \frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} &= (-1)^m (k-1)^{mn} \prod_{i=1}^m \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}) \\ &= (-(k-1)^n)^m \sum_{\substack{H = G_1 \cup \dots \cup G_m \\ (G_1, G_2, \dots, G_m) \in \mathcal{V}^a(\mathcal{H})}} C_H \prod_{i=1}^m (\#G_i \subseteq \mathcal{H}) \end{aligned}$$

For $m \in \mathbb{N}$, let $\mathcal{V}_d^m(\mathcal{H})$ be the set of unlabeled Veblen infragraphs of \mathcal{H} with d edges and m components. Appealing to Lemma 12 we may write

$$\begin{aligned}
\phi_d &= P_d \left(-\frac{\text{Tr}_1(\mathcal{H})}{1}, -\frac{\text{Tr}_2(\mathcal{H})}{2}, \dots, -\frac{\text{Tr}_d(\mathcal{H})}{d} \right) \\
&= \sum_{m=1}^d \sum_{d_1+\dots+d_m=d} \frac{1}{m!} \prod_{i=1}^m \frac{-\text{Tr}_{d_i}(\mathcal{H})}{d_i} \\
&= \sum_{m=1}^d \left(\sum_{a \in A^+(m,d)} (-(k-1)^n)^m \sum_{H \in \mathcal{V}^a(\mathcal{H})} C_H \frac{(\prod_{i=1}^m (\#G_i \subseteq \mathcal{H}))}{m!} \right) \\
&= \sum_{m=1}^d (-(k-1)^n)^m \sum_{H \in \mathcal{V}_d^m(\mathcal{H})} C_H \left(\frac{\mu_H}{m!} \prod_{i=1}^m (\#G_i \subseteq \mathcal{H}) \right) \\
&= \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H (\#H \subseteq \mathcal{H}).
\end{aligned}$$

■

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