On the motivic Galois theory of some families of Feynman integrals



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Abstract

Feynman integrals are the building blocks of scattering amplitudes in quantum field theory, and are therefore central to comparing theory with experiment in high-energy physics. Following the work of Brown [12], we can canonically associate a family of motivic periods to any convergent Feynman integral viewed as a function of the kinematic variables. Moreover, the action of the motivic Galois group is conjectured to be closed on the space of motivic periods arising from Feynman integrals, factoring through a subquotient referred to as the *cosmic Galois group*. Brown's "small graphs principle" implies that it is possible to deduce strong constraints on Feynman integrals to all loop orders from results on graphs with a small number of edges [12]. This serves as motivation for explicitly computing this group action, or, dually, the coaction of the Hopf algebra of functions on the motivic Galois group.

In this thesis, we find an explicit formula for the coaction on the 'motivic lifts' of Feynman integrals associated to one-loop Feynman graphs, as well as the two-loop sunrise graph, which is known to evaluate, in the equal-mass case, to a value of the elliptic dilogarithm [8]. We also study the coaction in some cases when the underlying geometry degenerates. It is conjectured in the physics literature [1] that dimensional regularization is compatible with the motivic Galois coaction. We show that computations in twisted cohomology reflect the expected effect of expanding the space of motivic Feynman integrals by introducing the 'motivic lifts' of dimensionally regularized Feynman integrals.

We find a new class of single-valued periods associated to the functions on the de Rham realization of the motivic Galois group arising in the coaction on the elliptic dilogarithm, which generalize Brown's non-holomorphic modular forms with two weights [11]. Finally, we apply the theory of the relative completion of the torsor of paths on a modular curve, due to Hain [31, 32] and Brown [13] to find a new proof that the sunrise Feyman integral can be expressed in terms of integrals of modular forms.

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Contents

1 Introduction **1**

1.1 Context 1

1.2 Contents 3

2 Brief overview of the technical background **9**

2.1 Category of realizations 9

2.2 Families of $\mathcal{H}(S)$-periods 10

2.3 Families of de Rham periods and coaction 12

2.4 Motivic periods 13

2.4.1 Mixed Tate Hodge structures 15

2.4.2 Examples of motivic and de Rham periods 16

2.5 Feynman integrals and their motivic lifts 19

2.5.1 Feynman integrals 19

2.5.2 Motivic Feynman amplitudes 21

3 Motivic Galois coaction and one-loop Feynman graphs **26**

3.1 Motives of one-loop graphs 26

3.1.1 Four and more internal edges 26

3.1.2 Semi-simplification of the motive 27

3.1.3 Reduction to four edges 29

3.1.4 Fewer than four edges 32

3.1.4.1 The triangle graph 32

3.1.4.2 The bubble graph 32

3.2 The 4-edge graph coaction 34

3.2.1 The motivic side of the coaction 34

3.2.2 de Rham side of the coaction 37

3.2.2.1 Connection and the de Rham periods 37

3.2.2.2 Residues and the de Rham projection 41

3.2.2.3 Comparing with an analytic expression for the Feynman integral . . . 42
3.3 The triangle graph: motives and coaction . . . 44
3.3.1 The triangle graph with non-vanishing masses . . . 44
3.3.1.1 Motive of the triangle graph with non-vanishing masses 44
3.3.1.2 Coaction on the triangle graph with non-vanishing masses and momenta . . . 47
3.3.2 Triangle graph with vanishing masses . . . 48
3.3.2.1 Motive . . . 48
3.3.2.2 Coaction on the triangle graph with two or more vanishing masses . . . 52
3.4 Twisted cohomology and dimensional regularization . . . 56
3.4.1 Twisted motives of Feynman graphs . . . 56
3.4.1.1 Massless triangle example . . . 60
3.4.1.2 Twisted cohomology groups associated to one-loop graphs in dimension $d-2\epsilon$ and reduction to $d+1$ edges . . . 62
3.4.1.3 Coaction . . . 65

4 The sunrise, motivic Galois coaction and integrals of modular forms 67
4.1 The motive $H^1(\mathcal{E}, \{P, Q\})$ . . . 67
4.1.1 de Rham realization, Betti realization and periods . . . 68
4.1.1.1 de Rham realization . . . 68
4.1.1.2 Betti realization . . . 70
4.1.1.3 Periods . . . 71
4.1.2 Motivic periods, de Rham periods, and the coaction . . . 71
4.1.3 Single-valued periods . . . 73
4.1.3.1 Defining single-valued periods over a general base . . . 73
4.1.3.2 Single-valued periods and elliptic curves . . . 75
4.2 The sunrise . . . 77
4.2.1 The motive . . . 78
4.2.2 Basis of de Rham cohomology . . . 80
4.2.2.1 The Feynman integrand and the holomorphic form on $\mathcal{E}$ 80
4.2.2.2 Weight 0 . . . 81
4.2.2.3 Subdivision of edges . . . 82
4.2.2.4 Completing the de Rham basis . . . 87
4.2.3 The motivic Galois coaction . . . 89

4.2.3.1 Arbitrary masses 89
4.2.3.2 Changing the de Rham basis 92
4.2.3.3 Equal masses 93
4.2.4 The sunrise integral and integrals of modular forms via the relative completion 94
4.2.4.1 Tannakian definition of the relative completion 94
4.2.4.2 Betti and de Rham relative completion of π_1 95
4.2.4.3 Motivic periods of the relative fundamental groupoid 97
4.2.4.4 The sunrise Feynman integral 102

A Residues of Feynman integrands of the sunrise with subdivided edges 106

Bibliography 108

List of Figures

1.1 The box graph. Internal edges which have non-vanishing mass are denoted by double lines. 4
1.2 The triangle graph. Internal edges denoted with a single line have vanishing mass. 5

2.1 The geometry underlying the motivic logarithm. 16

3.1 The tirangle graph with generic masses. 32
3.2 The bubble graph with generic masses. 33
3.3 The bubble graph with one vanishing mass. 34
3.4 The bubble graph obtained by contracting edges. 35
3.5 The geometry underlying the triangle graph with one vanishing mass. 48
3.6 The geometry underlying the triangle graph with one vanishing mass after blowing up. 50

4.1 The sunrise Feynman graph. 67
4.2 The geometry underlying the sunrise graph after blowing up. 78
4.3 Example of obtaining a graph from a given graph by subdividing an edge. 82
4.4 Graphs with subdivided edges corresponding to the elements of the chosen de Rham basis. 89

Chapter 1

Introduction

1.1 Context

Scattering amplitudes play a central role in comparing theory with experiment in high-energy physics. Their perturbative expansion can be organized using Feynman graphs according to loop-order. Each graph is associated to a Feynman integral, and therefore high-precision predictions for particle interactions depend on computing Feynman integrals of high loop order. Up to a factor, which is a special value of the gamma function, the following is the Feynman integral associated to a Feynman graph G, in parametric form:

$$\int_\sigma \omega_G(m,q), \tag{1.1}$$

where

$$\omega_G(m,q) = \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G}\right)^{N_G - h_G d/2} \Omega_G. \tag{1.2}$$

Here $\Psi_G, \Xi_G(m,q)$ are certain polynomials in the variables α_e indexed by the edges of G, d, N_G, h_G are fixed integers, and $\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_{N_G}$. The domain of integration is $\sigma = \{[\alpha_1 : \ldots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G - 1}(\mathbb{R})$. In physics, polynomials Ψ_G and $\Xi_G(m,q)$ are called *Symanzik polynomials*, and are often denoted by $\mathcal{U}$ and $\mathcal{F}$, respectively. They are defined in 2.5.2. The polynomial $\Xi_G(m,q)$ depends on parameters which represent masses and momenta of particles. When they converge Feynman integrals define families of periods, and for algebraic values of masses and momenta they are periods in the sense of Kontsevich-Zagier [35].

The fact that Feynman integrals can be viewed as families of periods enables a motivic approach to their study. In turn, this allows for a deeper understanding of how functions which traditionally arise in number theory such as multiple polylogarithms and elliptic polylogarithms arise in quantum field theory [4, 8, 5, 6].

When they are viewed as periods, a new structure satisfied by Feynman integrals arises, as all periods conjecturally carry an action of the motivic Galois group [3]. Brown defined a subquotient of the motivic Galois group called the *cosmic Galois group*, a phrase coined by P. Cartier, the action of which he conjectured [12, Conjecture 1] should be closed on the enlarged space of *regularized* Feynman integrals, and proved a weaker version of this statement [12, Theorem 8.11]. This is surprising since Feynman integrals are sparse in the space of all periods, so there is a priori no reason why the action of the motivic Galois group should preserve the space of Feynman integrals. An even stronger conjecture than Brown's, namely that Feynman integrals arising in specific theories are preserved by the cosmic Galois group, has been verified experimentally in many examples; Panzer and Schnetz considered integrals appearing in ϕ^4 theory [40], Caron-Huot, Dixon, Dulat, von Hippel and McLeod in $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory [20], Schnetz in the electron anomalous magnetic moment $g-2$ [46] and Schlotterer and Stieberger in string perturbation theory [45].

As suggested by Brown's conjecture, an important aspect of the theory of the cosmic Galois group is the notion of regularization of Feynman integrals. A common approach to regularizing Feynman integrals in physics, called *dimensional regularization*, involves setting the number of space-time dimensions to $d-2\epsilon$ and letting ϵ vary over the complex numbers. We may then consider the Laurent expansion around $\epsilon = 0$, the coefficients of which are periods. The prototype for the kind of integral we encounter in this way is the beta function, which we can expand around $(0,0)$ to obtain

$$\begin{aligned}\beta(s_0,s_1) &= \int_0^1 x^{s_0}(1-x)^{s_1}\frac{dx}{x(1-x)} \\ &= \left(\frac{1}{s_0}+\frac{1}{s_1}\right)\exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}\zeta(n)}{n}\left((s_0+s_1)^n - s_0^n - s_1^n\right)\right).\end{aligned} \tag{1.3}$$

In [1] Abreu, Britto, Duhr and Gardi considered certain dimensionally regularized one-loop Feynman integrals, and they experimentally observed that in each case the motivic Galois coaction, dual to the group action, applied to the coefficients of an expansion simultaneously, term by term in the regularization parameter ϵ, can be packaged into a concise formula in terms of other dimensionally regularized Feynman integrals related to subquotient graphs. In other words, the action of the motivic Galois group on the coefficients, which are periods of infinitely many different motives, can be written as a formula with finitely many terms. This was checked for the first few terms in the expansion in [1] and conjectured to hold to all orders in ϵ [1, §4.3]. More evidence in this direction is provided by [15], where Brown and Dupont studied

such expansions further in the case of Lauricella hypergeometric functions, comparing a 'global' coaction on the level of functions, with the 'local' coaction applied term by term in their Laurent expansions.

The motivation, from a physicist's perspective, to study the cosmic Galois group is that it is a new group of symmetries on Feynman integrals, which provides strong and concrete constraints that can help compute them. In particular, given a Feynman graph and the associated integral we expect its motivic Galois conjugates of low weight[1] to be periods of motives associated to subquotient graphs with the number of edges of those graphs controlled by the weight of the conjugates. This way, easy results for small graphs can provide very strong constraints on Feynman integrals to all loop orders. This is referred to as *the small graphs principle* [12, 8.4, 9.3].

The action of the motivic Galois group on periods is still very mysterious in general, which provides motivation for this work from a purely mathematical perspective. In this sense Feynman integrals provide a very rich testing ground, due to the additional structure they are endowed with, as well as the availability of computational data from physics.

1.2 Contents

After this introduction we provide a brief overview of the technical background and terminology. We will be working in a Tannakian category of realizations $\mathcal{H}(S)$ over a smooth base scheme S defined over $\mathbb{Q}$, following [24, §1.21] and [12, §7.2]. In order to study the motivic Galois theory of Feynman integrals we must first lift them to families of motivic periods, defined as functions on the scheme of isomorphisms between the de Rham and Betti fiber functors of $\mathcal{H}(S)$. In [12], Brown provides such a lifting, associating to each Feynman integral $I_G(m,q)$ a family of motivic periods called the *motivic Feynman amplitude* $I^{\mathfrak{m}}_G(m,q)$, with mild constraints on the masses and momenta of particles. We also recall the definition of the Tannaka group of $\mathcal{H}(S)$ with respect to the de Rham fiber functor. The results will be stated in terms of the coaction of the Hopf algebra of functions on the de Rham Tannaka group, the elements of which are called *de Rham periods*.

The first section of the third chapter contains general results on the (realizations of) motives of one-loop graphs in four dimensions of space-time, and can be regarded as restating the results of [7] in terms of motivic periods. In particular it contains

[1] The weight filtration comes from the mixed Hodge structure on the cohomology groups associated to Feynman graphs.

the reduction of the motivic Feynman amplitude of any graph with more than four edges to a linear combination, over the function field of the space of kinematics S, of motivic Feynman amplitudes of four-edge graphs, which amounts to lifting the result of Nickel[37] to motivic periods. In the following section we compute the coaction on the motivic Feynman amplitude on the four-edge one-loop graph with generic kinematics, which gives us the coaction for any one-loop graph with generic kinematics in four dimensions by the results of the previous section. Bearing in mind the definition of the motivic Feynman amplitude $I_G^{\mathfrak{m}}$[2] of a graph and its de Rham version $I_G^{\mathfrak{dr}}$ (2.5.9), the definition of the motivic Galois coaction (2.4), as well as the definitions of the de Rham logarithm $\log^{\mathfrak{dr}}(x)$ and the Lefschetz de Rham period $\mathbb{L}^{\mathfrak{dr}}$ in 2.4.2, the main result of the section is the following:

Theorem 3.2.5. *Let G be a one-loop Feynman graph with 4 edges (Figure 1.1), with generic non-vanishing masses and momenta. Let $I_G^{\mathfrak{m}}$ be its associated motivic Feynman amplitude in $d = 4$ dimensions. Then the motivic Galois coaction on $I_G^{\mathfrak{m}}$ is:*

$$\Delta I_G^{\mathfrak{m}} = I_G^{\mathfrak{m}} \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + \sum_{1\le j<k\le 4} I^{\mathfrak{m}}_{G/\{e_j,e_k\}}(\theta^1_{G/\{e_j,e_k\}}) \otimes P_{j,k} \log^{\mathfrak{dr}}(f_{j,k}) \mathbb{L}^{\mathfrak{dr}} + 1 \otimes I_G^{\mathfrak{dr}}, \tag{1.4}$$

where $I^{\mathfrak{m}}_{G/\{e_j,e_k\}}\left(\theta^1_{G/\{e_j,e_k\}}\right)$ is the motivic Feynman amplitude of the bubble graph obtained by contracting the edges e_j and e_k of G in $d = 2$ dimensions,

$$f_{j,k} = \frac{\sqrt{(U)^2_{j,k} - (U)_{j,j}(U)_{k,k}} - (U)_{j,k}}{\sqrt{(U)^2_{j,k} - (U)_{j,j}(U)_{k,k}} + (U)_{j,k}}, \tag{1.5}$$

where $C, D_{j,k}$ are the matrices of the quadratic forms Ξ_G, $\Xi_{G/\{e_j,e_k\}}$ (3.20) respectively, $U = C^{-1}$, and $P_{j,k} = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$.

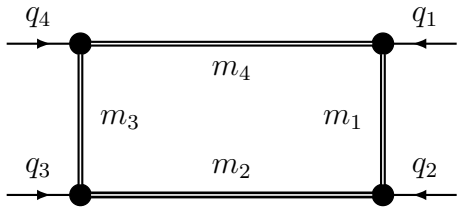


Figure 1.1: The box graph. Internal edges which have non-vanishing mass are denoted by double lines.

[2]We will often drop the notation (m, q) which denotes masses and momenta of particles for brevity. Dependence on masses and momenta is assumed unless otherwise specified.

In the fourth section of the third chapter we will concern ourselves with the three-edge one-loop graph. The result for generic kinematics is quite similar to the box graph case, but the vanishing of masses reveals subtleties. Namely, identifying the motivic periods of lower weight in the coaction with motivic periods of quotient graphs as in the previous theorem is not always possible without introducing regularization. The coaction on the motivic Feynman amplitude associated to the massless three-edge one-loop graph (Figure 1.2) is given by the following:

Theorem 3.3.6. *Let G be the three-edge one-loop graph, with all internal masses vanishing and non-trivial external momenta. Denote the three external momenta by q_1, q_2, q_3, and the associated motivic Feynman amplitude by $I_G^{\mathfrak{m}}$. Then*

$$\begin{aligned}\Delta I_G^{\mathfrak{m}} = {} & c_1 \log^{\mathfrak{m}}\left(\frac{q_2^2}{q_3^2}\right) \otimes \log^{\mathfrak{dr}}(f_1)\mathbb{L}^{\mathrm{dR}} + c_2 \log^{\mathfrak{m}}\left(\frac{q_1^2}{q_3^2}\right) \otimes \log^{\mathfrak{dr}}(f_1)\mathbb{L}^{\mathrm{dR}} + \\ & + c_3 \log^{\mathfrak{m}}\left(\frac{q_2^2}{q_3^2}\right) \otimes \log^{\mathfrak{dr}}(f_2)\mathbb{L}^{\mathrm{dR}} + c_4 \log^{\mathfrak{m}}\left(\frac{q_1^2}{q_3^2}\right) \otimes \log^{\mathfrak{dr}}(f_2)\mathbb{L}^{\mathrm{dR}} + \\ & + I_G^{\mathfrak{m}} \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + 1 \otimes I_G^{\mathfrak{dr}}\end{aligned}$$

where

$$f_1 = \frac{\left(q_1^2 + q_2^2 - q_3^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}\right)^2}{4q_1^2q_2^2}$$

and

$$f_2 = f_1 \frac{q_1^2 + q_3^2 - q_2^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}}{q_1^2 + q_3^2 - q_2^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}}$$

for some undetermined constants $c_i \in k_S$.

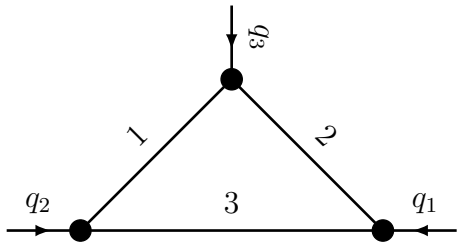


Figure 1.2: The triangle graph. Internal edges denoted with a single line have vanishing mass.

In the case of the previous theorem one cannot straightforwardly think of the motivic logarithms appearing in the coaction as motivic periods of the motives of quotient graphs. This will be clear to physicists, who know that the Feynman integrals associated to the one-loop graphs with two edges and both internal masses vanishing

are divergent. If we choose a basis of the de Rham realization of the motive of the triangle graph with all internal masses vanishing we can write down the coaction formula above with explicit constants c_i.

In the final section of the third chapter we turn towards dimensionally regularized Feynman integrals. Namely, we study Feynman integrals in $d - 2\epsilon$ dimensions of space-time for any even d and a fixed generic complex number ϵ. These are integrals of the form

$$\int_\sigma f^\epsilon \omega_G, \text{ where } f = \frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}}, \tag{1.6}$$

which we represent as matrix coefficients of twisted cohomology. We prove that we can reduce the integrals associated to Feynman graphs with more than $d+1$ edges in $d-2\epsilon$ dimensions to linear combinations of Feynman integrals associated to graphs with $d+1$ edges. We then compute the coaction of the Tannaka group of an appropriate Tannakian category with respect to the fiber functor given by twisted de Rham cohomology. Let $I^{\mathfrak{m}}_{G,\mathcal{T}}$ be the matrix coefficient corresponding to the Feynman integral (1.6) of a one-loop graph G with $d+1$ edges. The main result of the section is the following:

Theorem 3.4.7. *Let G be a graph as above. The coaction on $I^{\mathfrak{m}}_{G,\mathcal{T}}$, with respect to cohomology with twisted coefficients, is*

$$\Delta_{\mathcal{T}}(I^{\mathfrak{m}}_{G,\mathcal{T}}) = \sum_{I\in\mathbb{I}} I^{\mathfrak{m}}_{G/e_I,\mathcal{T}} \otimes f^{\mathfrak{dr}}_{G/e_I,G,\mathcal{T}}, \tag{1.7}$$

where $\mathbb{I}$ is a set of subsets I of $\{1,\dots,N_G\}$ such that $0 \le |I| < N_G$, excluding those I such that G/e_I is a massless tadpole, i.e. a massless self-edge graph.

This amounts to computing the 'global' coaction of [15] in the case of one-loop Feynman integrals. It is a step towards understanding the interplay between the motivic Galois coaction and dimensional regularization, and we show that it reflects the expected results of [1]. However, some caveats are in order. Namely, by working with twisted cohomology we are stepping outside the realm of motives. Additionally, we have fixed a generic value for the parameter ϵ in our set-up here, and we cannot therefore take the Laurent expansion at $\epsilon = 0$. In order to make a statement about the coaction on the level of the whole expansion, as predicted by [1], we would need to work in a category where ϵ is a formal parameter. This is the same issue as the one encountered in [15], where they do study certain formal power series whose coefficients are periods, although these are seen as periods of the motivic fundamental groupoid of the punctured sphere, and the resulting formula cannot be directly compared with

the formula arising from twisted cohomology, i.e. there is no functor between the two categories. However, these examples are evidence that there should exist a category of 'motives' with motivic periods corresponding to integrals of the form (1.6) which can be viewed as functions of ϵ. Such motives should admit expansions in classical motives on one hand, and specialization to generic ϵ on the other.

In the final chapter we turn our attention to a two-loop graph, namely the *sunrise graph* and its associated periods. The Feynman integral associated to the sunrise in $d = 2$ dimensions of space-time has been expressed in terms of elliptic dilogarithms and integrals of modular forms in two different ways: via the differential equation it satisfies and via motivic cohomology/K-theory [8, 6]. In the physics literature, this Feynman integral has been studied in dimensional regularization [2] and it was shown that it can be expressed in terms of iterated integrals of modular forms to all orders in the ϵ expansion. We compute the motivic Galois coaction on the motivic Feynman amplitude associated to the sunrise graph, which is the first such computation for a (motivic) elliptic polylogarithm. In the sunrise motivic Feynman amplitude we once again encounter an example for which we cannot express all of the conjugates under the coaction as motivic Feynman amplitudes of subquotient graphs. Unlike in the one-loop case, this is true for the sunrise even for generic kinematics. Moreover, while we do not study this here, we expect this to be true even after expanding the space of motivic Feynman integrals to include regularization. However, we can recover our ability to associate Feynman graphs to motivic Galois conjugates by including the motivic Feynman amplitudes of Feynman graphs obtained by subdividing edges of the original graph, i.e. replacing an edge with two edges of the same mass as the original edge meeting at a (new) vertex. A motivic Feynman amplitude of a graph obtained by subdividing an edge is a priori associated to a higher dimensional space than that associated to the original graph, so we must show that it is equivalent to a motivic period of the motive of the original graph. This construction is done in general, i.e. it applies to all Feynman graphs, not only the sunrise. Having done this we obtain the following result.

Theorem 4.2.7. *The motivic Galois coaction on the motivic Feynman amplitude associated to the sunrise Feynman graph with generic kinematics can be written as*

$$\begin{aligned}\Delta(I^{\mathfrak{m}}_G) = I^{\mathfrak{m}}_G \otimes K_1^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G_{s(e_1)}} \otimes K_2^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G\smallsetminus e_3} \otimes I^{\mathfrak{dr}}_{G,G\smallsetminus e_3} + \\ + I^{\mathfrak{m}}_{G_{s(e_1,e_2^2)}} \otimes F^{\mathfrak{dr}}_{\underline{b_1}} + I^{\mathfrak{m}}_{G_{s(e_1^2,e_3)}} \otimes F^{\mathfrak{dr}}_{\underline{b_2}} + I^{\mathfrak{m}}_{G_{s(e_2,e_3^2)}} \otimes F^{\mathfrak{dr}}_{\underline{b_3}}\,.\end{aligned} \tag{1.8}$$

In the above theorem we denote by $G_{s(e_i^{k_i})}$ the graph obtained from G by subdividing the ith edge k_i times, and by $G \smallsetminus e_3$ the graph obtained from G by deleting

the edge e_3. Furthermore, $K_1^{\mathfrak{dr}}$, $K_2^{\mathfrak{dr}}$, and $F_{\underline{b_1}}^{\mathfrak{dr}}$ are *de Rham elliptic integrals* which we introduce in the final chapter. To de Rham periods in general, i.e. functions on the de Rham Tannaka group of the category $\mathcal{H}(S)$, one can associate single-valued periods. This way one recovers, for example, the Bloch-Wigner dilogarithm, single-valued multiple zeta values [10] and Brown's non-holomorphic modular forms with two weights [11]. We observe that a new class of single-valued period arises from the families of de Rham periods which appear in the elliptic coaction which generalize Brown's non-holomorphic bi-modular forms. We briefly outline their properties.

Finally, we recall the theory of the relative completion of the torsor of paths between two base points on a modular curve, and its periods, developed by Hain [31, 32] and Brown [13]. We apply this to show that the sunrise Feynman integral can be expressed in terms of integrals of modular forms in the equal-mass case, reproving the result of [8].

Chapter 2

Brief overview of the technical background

We recall the definitions, constructions and main results on motivic periods and motivic Feynman amplitudes. This is a summary of the results of Brown [14], [12].

2.1 Category of realizations

Let S be a smooth geometrically connected scheme over $k \subset \mathbb{C}$. And let $\mathcal{H}(S)$ be the category of triples $(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c)$, where $\mathbb{V}_{\mathrm{B}}$ is a local system of finite dimensional k-vector spaces on $S(\mathbb{C})$, and $(\mathcal{V}_{\mathrm{dR}}, \nabla)$ is an algebraic vector bundle on S with a flat connection with regular singularities at infinity. Note that for the definition of regular singularities at infinity one needs a good compactification of S, but it does not depend on the choice of a good compactification [33]. Furthermore $\mathbb{V}_{\mathrm{B}}$ is equipped with an increasing *weight* filtration $W_\bullet \mathbb{V}_{\mathrm{B}}$ of local sub-systems, and $(\mathcal{V}_{\mathrm{dR}}, \nabla)$ is equipped by an increasing *weight* filtration by algebraic sub-vector bundles with a flat connection with regular singularities at infinity $W_\bullet \mathcal{V}_{\mathrm{dR}}$ and a decreasing *Hodge* filtration by algebraic sub-vector bundles $F^\bullet \mathcal{V}_{\mathrm{dR}}$ which satisfy Griffiths transversality

$$\nabla : F^p \mathcal{V}_{\mathrm{dR}} \to F^{p-1} \mathcal{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \Omega^1_{S/k}.$$

Finally, $c : (\mathcal{V}_{\mathrm{dR}}, \nabla)^{an} \xrightarrow{\sim} \mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}}$ is an isomorphism of analytic vector bundles with connection which respects the weight filtration, where the connection on the right hand side is one for which the sections of $\mathbb{V}_{\mathrm{B}}$ are flat. We further require that $\mathbb{V}_{\mathrm{B}}$, equipped with the filtrations W and cF, is a graded-polarizable variation of mixed Hodge structures. The morphisms in $\mathcal{H}(S)$ respect the above data. A special case is $S = \mathrm{Spec}(k)$, where k is often taken to be $\mathbb{Q}$ or a number field.

The category $\mathcal{H}(S)$ is Tannakian, see [25]. It is equipped with the following fiber functors. For any simply connected $Z \subset S(\mathbb{C})$, and any $z \in Z$ let:

$$\omega_B^z : \mathcal{H}(S) \to \mathrm{Vec}_k$$

be the fiber functor which send a triple $(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c)$ to the fiber of $\mathbb{V}_{\mathrm{B}}$ over z for z defined over k. For a simply connected $Z \subset S(\mathbb{C})$ and any two $z, z' \in Z$ the fiber functors ω_z and ω'_z are canonically isomorphic. We therefore define ω_{B}^Z to be the fiber functor at any $z \in Z$. We will also consider the fiber functors:

$$\omega_{\mathrm{dR}}^Y : \mathcal{H}(S) \to \mathrm{Proj}(\mathcal{O}_{S,Y}), \quad \omega_{\mathrm{dR}}^{gen} : \mathcal{H}(S) \to \mathrm{Vec}_{k_S}, \text{ and } \quad \omega_{\mathrm{dR}}^y : \mathcal{H}(S) \to \mathrm{Proj}(\mathcal{O}_y),$$

where $Y \subset S(\mathbb{C})$, and $\mathcal{O}_{S,Y} = \lim_{\to U} \mathcal{O}_U$ for U ranging over all affine opens in S such that $Y \subset U(\mathbb{C})$. The functors $\omega_{\mathrm{dR}}^{gen}$ and ω_{dR}^y are special cases of ω_{dR}^Y when $Y = \varnothing$, and $Y = y \in S(k) \subset S(\mathbb{C})$ respectively, and send the vector bundle $(\mathcal{V}_{\mathrm{dR}}, \nabla)$ to its stalk over the generic point of S and the stalk at y respectively. The ring $\mathcal{O}_y$ is the local ring at y, and k_S is the function field of S.

2.2 Families of $\mathcal{H}(S)$-periods

The *ring of* $\mathcal{H}(S)$*-periods* is defined as:

$$\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},Z,gen} = \mathcal{O}\left(\mathrm{Isom}_{\mathcal{H}(S)}^{\otimes}(\omega_{\mathrm{dR}}^{gen}, \omega_B^Z)\right).$$

It is a (k, k_S)-bimodule generated by equivalence classes of triples $(\mathcal{V}, \sigma, \omega)$, where $\mathcal{V} = (\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c) \in \mathrm{ob}(\mathcal{H}(S))$, $\sigma \in (\omega_B^Z \mathcal{V})^\vee$, and $\omega \in \omega_{\mathrm{dR}}^{gen} \mathcal{V}$, modulo the relations:

$$\begin{aligned} (\mathcal{V}, \lambda_1\sigma_1 + \lambda_2\sigma_2, \omega) &\sim \lambda_1(\mathcal{V}, \sigma_1, \omega) + \lambda_2(\mathcal{V}, \sigma_2, \omega) \\ (\mathcal{V}, \sigma, \lambda_1\omega_1 + \lambda_2\omega_2) &\sim \lambda_1(\mathcal{V}, \sigma, \omega_1) + \lambda_2(\mathcal{V}, \sigma, \omega_2), \end{aligned} \tag{2.1}$$

where $\lambda_1 \in \mathbb{Q}, \lambda_2 \in k_S$. Furthermore, for any morphism $\phi : \mathcal{V}_1 \to \mathcal{V}_2$ in the category $\mathcal{H}(S)$ we have:

$$(\mathcal{V}_1, (\omega_B^Z(\phi))^t(\sigma_2), \omega_1) \sim (\mathcal{V}_2, \sigma_2, \omega_{\mathrm{dR}}^{gen}(\phi)(\omega_1)),$$

where $(\omega_B^Z(\phi))^t$ denotes the dual morphism to $\omega_B^Z(\phi)$. We denote the equivalence classes of triples by $[(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), \sigma, \omega]^{\mathfrak{m}}$.

The ring $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},Z,gen}$ is equipped with a *period homomorphism*:

$$\mathrm{per} : \quad \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},Z,gen} \to M_Z(S(\mathbb{C})), \tag{2.2}$$

where $M_Z(S(\mathbb{C}))$ denotes the ring of multivalued meromorphic functions on $S(\mathbb{C})$ with a prescribed branch along Z. To define it let $\pi : \widetilde{S}(\mathbb{C})_Z \to S(\mathbb{C})$ be the universal

cover of $S(\mathbb{C})$ based at Z (recall that Z is simply connected). Since $\widetilde{S}(\mathbb{C})_Z$ is simply connected the local system $\pi^*(\mathbb{V}_{\mathrm{B}}^\vee)$ is trivial, and $\sigma \in \Gamma(Z, \mathbb{V}_{\mathrm{B}}^\vee)$ extends to a unique global section $\Gamma(\widetilde{S}(\mathbb{C})_Z, \pi^*(\mathbb{V}_{\mathrm{B}}^\vee))$. Let $x \in \widetilde{S}(\mathbb{C})_Z$, and $N(x)$ be a small enough neighbourhood of x so that the restriction of π is an isomorphism. Define a local section $\sigma_x = (\pi|_{N(x)}^{-1})^*\sigma \in \Gamma(\pi(N(x)), \mathbb{V}_{\mathrm{B}}^\vee)$.

Since $\omega \in \Gamma(\mathrm{Spec}(k_S), \mathcal{V}_{\mathrm{dR}})$, there exists an non-empty affine open $U \subset S$ such that $\omega \in \Gamma(U, \mathcal{V}_{\mathrm{dR}})$. Let $W \subset S$ be an affine open such that $\pi(x) \in W(\mathbb{C})$, and furthermore we can make $\pi(N(x)) \in W(\mathbb{C})$ by making $N(x)$ smaller if necessary. Since S is irreducible $U \cap W \neq \varnothing$. The section $\omega|_{U\cap W}$ can have poles along $W \smallsetminus U$, but we can 'clear denominators', i.e., there is $\alpha \in \mathcal{O}_W$ such that $\alpha\omega \in \Gamma(W, \mathcal{V}_{\mathrm{dR}})$. By restriction, and passing to the associated analytic vector bundle we can view $\alpha\omega$ as an element in $\Gamma(\pi(N(x)), \mathcal{V}_{\mathrm{dR}}^{an})$. Then the comparison isomorphism $c : \mathcal{V}_{\mathrm{dR}}^{an} \to \mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}}$ gives an element:

$$\mathrm{per}([(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), \sigma, \alpha\omega]^{\mathfrak{m}}) = \sigma_x(c(\alpha\omega)) \in \Gamma(\pi(N(x)), \mathcal{O}_{S^{an}}),$$

which can be viewed as a locally analytic function on $N(x)$.The period homomorphism is defined on $[(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), \sigma, \omega]^{\mathfrak{m}}$ by dividing by the function α. It locally has poles along the zeroes of α.

Remark 2.2.1. If we are given a region $Y \subset S(\mathbb{C})$ that we know is free of poles of the integrand, then we may replace the fiber functor $\omega_{\mathrm{dR}}^{gen}$ with ω_{dR}^{Y} in the previous construction, rather than work over the generic fiber.

The ring of $\mathcal{H}(S)$-periods comes equipped with a *connection*

$$\nabla : \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},Z,gen} \to \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},Z,gen} \otimes_{k_S} \Omega^1_{k_S/k}$$

which acts on families of motivic periods by:

$$\nabla[\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} = [\mathcal{V}, \sigma, \nabla\omega]^{\mathfrak{m}}.$$

For details see [14, §7.4].

Remark 2.2.2. Generally, for a Tannakian category $\mathcal{T}$ with two fiber functors ω_1, ω_2 defined over field k_1, k_2 respectively, we can define equivalence classes $[M, \sigma, \omega]^{\mathfrak{m}}$ where $M \in \mathrm{Ob}(\mathcal{T})$, $\sigma \in \omega_2(M)^\vee$, and $\omega \in \omega_1(M)$ analogously to the above. We refer to such equivalence classes as *matrix coefficients*. This is because for R an $k_2 \otimes_{\mathbb{Q}} k_1$-algebra an R-point of $\mathrm{Isom}_{\mathcal{T}}^{\otimes}(\omega_1, \omega_2)$ is a collection of isomorphisms of R-modules

$$\phi_M : \omega_1(M) \otimes_{k_1} R \xrightarrow{\sim} \omega_2(M) \otimes_{k_2} R$$

functorial in $M \in \mathrm{Ob}(\mathcal{T})$ [25, Proposition 6.6]. The corresponding homomorphism

$$\phi : \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{T}}(\omega_1, \omega_2)) \to R$$

is given by $\sigma(\phi_M(\omega))$ on matrix coefficients. In other words, choosing elements σ and ω corresponds to choosing a coefficient of the matrix representing the isomorphism ϕ_M.

2.3 Families of de Rham periods and coaction

We define the ring of *de Rham periods* as:

$$\mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)} = \mathcal{O}(\mathrm{Aut}^{\otimes}_{\mathcal{H}(S)}(\omega^{gen}_{\mathrm{dR}})).$$

It is spanned, over k_S, by equivalence classes of triples $[(\mathbb{V}_{\mathrm{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), v, \omega]^{\mathfrak{dr}}$, where $v \in (\omega^{gen}_{\mathrm{dR}}\mathcal{V})^{\vee}$, and $\omega \in \omega^{gen}_{\mathrm{dR}}\mathcal{V}$, defined analogously to $\mathcal{H}(S)$-periods. Furthermore, $\mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)}$ is a Hopf algebra, and the ring $\mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)}$ has a right *Galois coaction* by $\mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)}$:

$$\Delta : \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)} \to \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)} \otimes_{k_S} \mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)}, \tag{2.3}$$

given by the formula:

$$\Delta[\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} = \sum_{e_i} [\mathcal{V}, \sigma, e_i]^{\mathfrak{m}} \otimes [\mathcal{V}, e_i^{\vee}, \omega]^{\mathfrak{dr}} \tag{2.4}$$

where the $\{e_i\}$ is a basis of $\omega^{gen}_{\mathrm{dR}}\mathcal{V}$, and $e_i^{\vee}$ is the dual basis. Dual to the Galois coaction is the left *Galois action* of the affine group scheme given by $G^{\mathfrak{dr},gen}_{\mathcal{H}(S)} = \mathrm{Spec}\left(\mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)}\right)$:

$$G^{\mathfrak{dr},gen}_{\mathcal{H}(S)} \times \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)} \to \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)}$$

given by:

$$g[\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} = (1 \otimes g)\Delta[\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} = \sum_{e_i} [\mathcal{V}, \sigma, e_i]^{\mathfrak{m}} \otimes g\left([\mathcal{V}, e_i^{\vee}, \omega]^{\mathfrak{dr}}\right), \tag{2.5}$$

where $g \in G^{\mathfrak{dr},gen}_{\mathcal{H}(S)}(R)$, for R a k_S-algebra.

Remark 2.3.1. An element g of $G^{\mathfrak{dr},gen}_{\mathcal{H}(S)}(R)$ is a collection of isomorphisms

$$g_{\mathcal{V}} : \omega^{gen}_{\mathrm{dR}}(\mathcal{V}) \otimes R \to \omega^{gen}_{\mathrm{dR}}(\mathcal{V}) \otimes R$$

functorial in $\mathcal{V} \in \mathrm{Ob}(\mathcal{H}(S))$. The action of $G^{\mathfrak{dr},gen}_{\mathcal{H}(S)}$ on $\mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)}$ can therefore equivalently be defined by

$$g[\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} = [\mathcal{V}, \sigma, g_{\mathcal{V}}(\omega)]^{\mathfrak{m}}.$$

2.4 Motivic periods

We are particularly interested in objects of $\mathcal{H}(S)$ of a prescribed geometric origin. By this we mean the following. Recall that a simple normal crossing divisor is a normal crossing divisor such that each of its irreducible components is smooth. Assume, as before, that S is a smooth geometrically connected scheme over $\mathbb{Q}$, and let $D \subset X$ be a family of simple normal crossing divisors relative to a smooth morphism $\pi : X \to S$ which, on the underlying analytic varieties, we assume to be a locally trivial fibration of stratified analytic varieties, for the stratification on X induced by D – see [29] and [42, Ch. IV]. Let $j : X \smallsetminus D \hookrightarrow X$ be the inclusion, and denote by D_i the irreducible components of D, for $i \in I$. Denote by $D_J = \cup_{j\in J} D_j$, for $\emptyset \neq J \subset I$, and let $D_\emptyset = X$.

Because π is a locally trivial fibration, the sheaf

$$H^n_{\mathrm{B}}(X, D)_{/S} = R^n\pi_* j_! \mathbb{Q}, \tag{2.6}$$

where $\mathbb{Q}$ is the constant sheaf on $(X \smallsetminus D)(\mathbb{C})$, is a local system over $S(\mathbb{C})$ with its analytic topology. It is computed by the hypercohomology of the complex of sheaves:

$$\mathbb{Q}_{D_\bullet/S} : \mathbb{Q}_X \to \bigoplus_{|J|=1} \mathbb{Q}_{D_J} \to \bigoplus_{|J|=2} \mathbb{Q}_{D_J} \to \dots \tag{2.7}$$

where $\mathbb{Q}_{D_J}$ is the constant sheaf $\mathbb{Q}$ on $D_J(\mathbb{C})$ extended by 0 to $X(\mathbb{C})$.

Let $\Omega^\bullet_{D_J/S}$ denote the sheaf on X which is the direct image of the corresponding sheaves of Kähler differentials on D_J, and which vanishes outside of D_J. Consider the double complex of sheaves on X

$$\Omega^\bullet_{D_\bullet/S} : \quad \Omega^\bullet_{X/S} \to \bigoplus_{|J|=1} \Omega^\bullet_{D_J/S} \to \dots \to \bigoplus_{|J|=d} \Omega^\bullet_{D_J/S} \tag{2.8}$$

where d is the relative dimension of X over S, and horizontal maps are pullbacks along inclusions $D_J \hookrightarrow D_{J\smallsetminus i_j}$ multiplied by $(-1)^k$ if i_j is the kth element of J. We will denote such restrictions, including the signs, by $r^J_{i_j}$ and write simply r_{i_j} when $J = \{i_j\}$. Then define

$$H^n_{\mathrm{dR}}(X, D)_{/S} = \mathbb{R}^n\pi_*(\mathrm{Tot}^\bullet(\Omega^\bullet_{D_\bullet/S})), \tag{2.9}$$

where $\mathrm{Tot}^\bullet$ denotes the total complex. It should have a flat connection

$$\nabla : H^n_{\mathrm{dR}}(X, D)_{/S} \to H^n_{\mathrm{dR}}(X, D)_{/S} \otimes \Omega^1_{S/k}$$

by a relative version of [34].

By [22, Proposition 2.28], using the assumption that π is topologically trivial, we have an isomorphism:

$$c^{-1} : H^n_{\mathrm{B}}(X, D)_{/S} \otimes_{\mathbb{Q}} \mathcal{O}_S^{an} \xrightarrow{\sim} (H^n_{\mathrm{dR}}(X, D)_{/S})^{an}.$$

It follows from Saito's theory of mixed Hodge modules [43, 44] that $H^n_{\mathrm{B}}(X, D)_{/S}$ is a (admissible, graded-polarizable) variation of mixed Hodge structures. We provide a very brief sketch. Mixed Hodge modules generalize variations of mixed Hodge structures. There exists a functor

$$\mathrm{rat}_S : D^{\mathrm{b}}\mathrm{MHM}(S) \to D^{\mathrm{b}}_{\mathrm{cs}}(S; \mathbb{Q}),$$

where $D^{\mathrm{b}}\mathrm{MHM}(S)$ is the bounded derived category of mixed Hodge modules on S and $D^{\mathrm{b}}_{\mathrm{cs}}(S; \mathbb{Q})$ is the bounded derived category of cohomologically constructible complexes of sheaves of $\mathbb{Q}$-vector spaces on S, and the six-functor formalism of the latter lifts to the former. There exists a unique element we denote $\mathbb{Q}^{\mathrm{Hdg}} \in \mathrm{MHM}(\mathrm{pt})$ such that $\mathrm{rat}_{\mathrm{pt}}(\mathbb{Q}^{\mathrm{Hdg}}) = \mathbb{Q}(0)$. As before, we use the same notation for their pull-backs to X. It is enough to show that the cohomology sheaf we are interested in, $R^n\pi_* j_! j^* \mathbb{Q}(0)$, is in the image of rat_S. We have an element of $D^{\mathrm{b}}\mathrm{MHM}(S)$ given by $\pi_* j_! j^* \mathbb{Q}^{\mathrm{Hdg}}$ such that

$$\mathrm{rat}_S(\pi_* j_! j^* \mathbb{Q}^{\mathrm{Hdg}}) = \pi_* j_! j^* \mathbb{Q}(0).$$

Moreover, the functor rat_S is compatible with taking cohomology in the following way

$$\mathrm{rat}_S(\mathcal{H}^q M^\bullet) = {}^p\mathcal{H}^q(\mathrm{rat}_S(M^\bullet)),$$

for any $M^\bullet \in D^{\mathrm{b}}\mathrm{MHM}(S)$, where ${}^p\mathcal{H}^q$ denotes the perverse cohomology functor [41, Lemma 14.5]. In our situation, where the underlying sheaf $\pi_* j_! j^* \mathbb{Q}(0)$ is such that its cohomology sheaves are local systems (by our assumptions on π) perverse cohomology is just the usual cohomology with a shift in the degree, i.e.

$$H^n_{\mathrm{B}}(X, D)_{/S} = \mathcal{H}^n(\pi_* j_! j^* \mathbb{Q}(0)) = {}^p\mathcal{H}^{n-dim(S)}(\pi_* j_! j^* \mathbb{Q}(0))[-dim(S)].$$

Therefore we have that

$$H^n_{\mathrm{B}}(X, D)_{/S} = \mathrm{rat}_S(\mathcal{H}^{n-dim(S)}(\pi_* j_! j^* \mathbb{Q}^{\mathrm{Hdg}})[-dim(S)]).$$

Putting everything together we get an element

$$(H^n_{\mathrm{B}}(X, D)_{/S}, H^n_{\mathrm{dR}}(X, D)_{/S}, c) \tag{2.10}$$

of the category $\mathcal{H}(S)$, which we denote $H^n(X, D)_{/S}$.

Definition 2.4.1. We refer to the $\mathcal{H}(S)$-periods associated to objects $H^n(X,D)_{/S}$ of $\mathcal{H}(S)$ as *families of motivic periods*. Their images under the period homomorphism are families of periods which can be written as integrals where the integration variables are given by the coordinate functions on X, and S is the space of parameters – see 2.4.2 for examples.

For the purposes of this thesis we will define explicitly an open S as a complement of a certain closed subset of an irreducible affine algebraic variety over which $H^n_{\mathrm{B}}(X,D)_{/S}$ will be a local system, and we will work with a family of divisors $D \subset X$ which is simple normal crossing over the open subspace S.

Definition 2.4.2. Let $D^I = \cup_{j \notin I} D_j$, and $k = |I|$. *Face maps* are morphisms in the category $\mathcal{H}(S)$

$$H^{n-k}(D_I, D^I \cap D_I)_{/S} \to H^n(X,D)_{/S}, \tag{2.11}$$

defined by the inclusion of complexes $\Omega^{\bullet-k}_{D^I_\bullet/S} \to \Omega^\bullet_{D_\bullet/S}$, and $\mathbb{Q}_{D^I_\bullet/S}[-k] \to \mathbb{Q}_{D_\bullet/S}$, on the de Rham and Betti realizations respectively.

2.4.1 Mixed Tate Hodge structures

In this thesis we will focus on a special case of families of motivic periods coming from objects of $\mathcal{H}(S)$ associated to variations of mixed Tate Hodge structures. To make this precise we recall a few definitions. A *Tate Hodge structure* $\mathbb{Q}(-m)$ is the pure Hodge structure of weight $2m$ defined by

$$H_{\mathbb{Q}} = (2\pi i)^{-m}\mathbb{Q}, \quad H_{\mathbb{C}} = H^{m,m}(\mathbb{Q}(-m)_{\mathbb{C}}).$$

We can pull back $\mathbb{Q}(-m)$ to S via the structure map $S \to \mathrm{Spec}(\mathbb{Q})$ to obtain a constant *variation of Tate structure*, denoted $\mathbb{Q}(-m)_{/S}$.

A mixed Tate Hodge structures are mixed Hodge structures H such that the weight graded quotients are $gr^W_{2m}H = \oplus\mathbb{Q}(-m)$ and $gr^W_{2m+1}H = 0$. They are extensions of Tate Hodge structures. We will be working with objects of $\mathcal{H}(S)$ such that their fibers over $S(\mathbb{C})$ are mixed Tate Hodge structures. In this particular case, the choice of terminology where we refer to "motivic periods" is justified by the fact that the Hodge realization functor is fully faithful on mixed Tate motives over number fields [26].

2.4.2 Examples of motivic and de Rham periods

Let $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $X = S \times \mathbb{G}_m$, and $\pi : X \to S$ be the projection. Let x be the coordinate on S, y be the coordinate on $\mathbb{G}_m$, and define $D = \{y = 1\} \cup \{y = x\}$ (see Figure 2.1). We consider the object $\mathcal{V} = (H^1_B(X, D)_{/S}, H^1_{\mathrm{dR}}(X, D)_{/S}, c) \in \mathcal{H}(S)$.

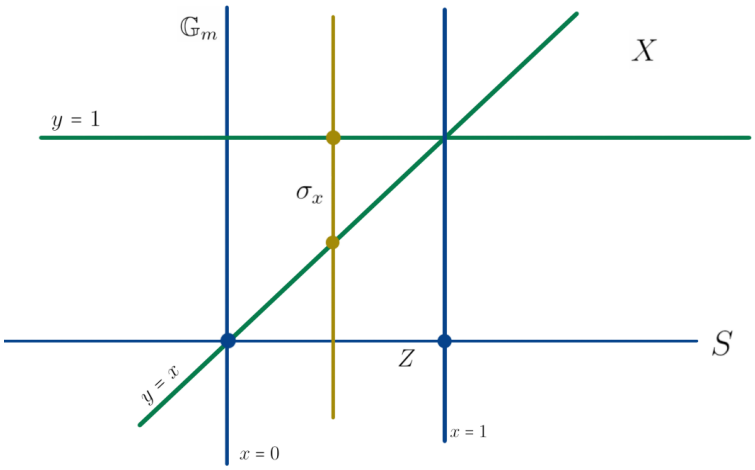


Figure 2.1: The geometry underlying the motivic logarithm.

We can choose Z to be the real interval $(0, 1)$ of $S(\mathbb{C})$, and define for all $x \in Z$ a cycle $\sigma_x \in \mathbb{G}_m(\mathbb{C})$ – a straight line path from 1 to x, which defines a class in $(\omega^Z_B(H^1(X, D)_{/S}))^\vee$. Note that in this case the differential form $\frac{dy}{y}$ defines a class in $\Gamma(S, (H^1_{\mathrm{dR}}(X, D)_{/S}))$, and we do not have to restrict to working over the generic point only. We define the motivic logarithm as a family of motivic periods

$$\log^{\mathfrak{m}}(x) = \left[\mathcal{V}, [\sigma_x], \left[\frac{dy}{y}\right]\right]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)}$$

Its period is the logarithm

$$\mathrm{per}(\log^{\mathfrak{m}}(x)) = \int_{\sigma_x} \frac{dy}{y} = \log(x)$$

for $x \in Z$. Note in Figure 2.1 that the if we did not exclude $x = 0, 1, \infty$ from the base S the geometry would degenerate for those values of the parameter x. Given two points $x_1, x_2 \subset S(\mathbb{C})$ the topological fundamental groupoid $\pi_1(S(\mathbb{C}), x_1, x_2)$ acts on $\log^{\mathfrak{m}}(x_1)$ by continuation along paths via its action on σ_{x_1}. If a path between x_1 and x_2 winds around one of the points $x = 0, 1, \infty$ we pick up monodromy. Later in this thesis we will consider logarithms over a higher dimensional, but still irreducible affine, base than $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, for which case we must extend the definition of the motivic logarithm given above. However note that we will always be given a subset Z of the

complex points of our base over which the branch of the logarithm is unambiguous, and the definition is thus extended in an obvious way.

In order to lift $2\pi i$ to its motivic version it is enough to define it over $S = \mathrm{Spec}(\mathbb{Q})$, and it can then be pulled back to a constant family over $S' \to S$. Consider the object $H = (H^1_{\mathrm{B}}(\mathbb{G}_m), H^1_{\mathrm{dR}}(\mathbb{G}_m), c) \in \mathrm{ob}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\mathrm{Spec}(\mathbb{Q}))$, and let γ_0 be a positively oriented circle around 0. We have that $[\gamma_0] \in H_B^\vee$, and

$$\mathrm{per}\left(\left[H, [\gamma_0], \left[\frac{dy}{y}\right]\right]^{\mathfrak{m}}\right) = \int_{\gamma_0} \frac{dy}{y} = 2\pi i$$

We denote

$$\mathbb{L}^{\mathfrak{m}} = \left[H, [\gamma_0], \left[\frac{dy}{y}\right]\right]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}},$$

where we have dropped Z, gen from the notation since we are working over a point, and we refer to it as the *Lefschetz motivic period*. We denote the constant family of motivic periods obtained by pulling back $\mathbb{L}^{\mathfrak{m}}$ to S via its structure morphism by the same symbol.

We will also consider de Rham verisons of these motivic periods. We choose a basis of the de Rham realization $\left\{\left[\frac{dy}{y}\right], \left[\frac{dy}{x-1}\right]\right\}$, and denote its dual basis $\left\{\left[\frac{dy}{y}\right]^\vee, \left[\frac{dy}{x-1}\right]^\vee\right\}$. For the de Rham logarithm define

$$\log^{\mathfrak{dr}}(x) = \left[\mathcal{V}, \left[\frac{dy}{x-1}\right]^\vee, \left[\frac{dy}{y}\right]\right]^{\mathfrak{dr}}.$$

Finally, for the Lefschetz de Rham period define:

$$\mathbb{L}^{\mathfrak{dr}} = \left[H, \left[\frac{dy}{y}\right]^\vee, \left[\frac{dy}{y}\right]\right]^{\mathfrak{dr}} \tag{2.12}$$

where H is as above. This can also be pulled back to a constant family of de Rham periods on S, which we denote by $\mathbb{L}^{\mathfrak{dr}}$ as well.

Remark 2.4.3. This definition of the de Rham logarithm is motivated by it being the image of the motivic logarithm, defined earlier, under the *de Rham projection*. This is a homomorphism from the ring of motivic periods to the ring of de Rham periods studied in detail in [16], and defined as follows.

An object in $\mathcal{H}(S)$ is said to be *separated* if its Hodge numbers $h^{p,q}$ vanish unless $p = q = 0$ of $p, q > 0$. In particular objects of mixed Tate type are separated. For a separated object $H^n(X \setminus A, B)$, where X is a smooth algebraic variety of dimension n defined over a field k with two divisors A, B which are simple normal crossing, there exists a morphism [16, Theorem 1.7]

$$c_0^\vee : H^{\mathrm{B}}_n(X \setminus A, B) \to H^n_{\mathrm{dR}}(X \setminus B, A) \otimes_k \overline{k}.$$

The image $c_0^\vee([\sigma])$ of a class represented by a path σ can be represented by a global algebraic differential form ν_σ on X with logarithmic singularities along B. Brown and Dupont give us a recipe for computing it [16, Theorem 4.16] as the unique form such that

$$res_{B_{i_n}} \cdots res_{B_{i_n}}((2\pi i)^{-n}\nu_\sigma) = (-1)^{\frac{n(n-1)}{2}} \partial_{B_{i_n}} \cdots \partial_{B_{i_n}}(\sigma),$$

where on the left hand side we take iterated residues along the irreducible components of B, denoted B_{i_k}, and on the right hand side we take iterated boundaries along B_{i_k}. As a special case of Verdier duality we get the perfect pairing compatible with mixed Hodge structures [16, §3]

$$H^k(X \setminus B, A) \otimes H^{2n-k}(X \setminus B, A) \to \mathbb{Q}(-n). \tag{2.13}$$

This can be interpreted in terms of differential forms, so we get a morphism:

$$\begin{aligned} \pi^{\mathfrak{dr}} : \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(S)} &\to \mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)} \\ [\mathcal{V}, \sigma, \omega]^{\mathfrak{m}} &\mapsto [\mathcal{V}, c_0^\vee(\sigma), \omega]^{\mathfrak{dr}}. \end{aligned} \tag{2.14}$$

We now come back to our example. Our claim is that with our definition of the de Rham logarithm above we have

$$\pi^{\mathfrak{dr}}(\log^{\mathfrak{m}}(x)) = \log^{\mathfrak{dr}}(x).$$

First note that from the description in terms of residues and boundaries above we have $c_0^\vee(\sigma_x) = \frac{1}{2\pi i} d\log(\frac{y-x}{y-1})$. It only remains to show that the form $\frac{1}{2\pi i} d\log(\frac{y-x}{y-1})$ corresponds to the dual form $[\frac{dy}{x-1}]^\vee$ via the pairing (2.13) in de Rham. To compute it we one first finds a different representative of the class $[\frac{dy}{x-1}]$ in $H^1_{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\})$. Take any smooth function g with compact support such that $g(x) - g(1) = 1$, then $[\frac{dy}{x-1}] = [dg]$. The de Rham pairing is computed by

$$\int_{\mathbb{P}^1(\mathbb{C})} d\log\left(\frac{y-x}{y-1}\right) \wedge dg = \int_{\mathbb{P}^1(\mathbb{C})} d(g d\log(\frac{y-x}{y-1})),$$

which can be shown to be equal to $2\pi i(g(t) - g(1)) = 2\pi i$ by integrating on the the complement of small disks of radius ϵ around each point $\{0, 1, t, \infty\}$. The result then follows from Stokes theorem and taking $\epsilon \to 0$. Finally, one can see that the same pairing of the forms $d\log(\frac{y-x}{y-1})$ and $\frac{dy}{y}$ is 0.

2.5 Feynman integrals and their motivic lifts

2.5.1 Feynman integrals

Definition 2.5.1. A Feynman graph is a multigraph G, defined by a triple:

$$(V_G, E_G, E_G^{ext})$$

where V_G are the vertices, E_G is the set of internal edges of the graph which are not oriented, with the endpoints of each element of E_G encoded by a map $\partial : E_G \to \mathrm{Sym}^2 V_G$, and E_G^{ext} is the set of external half-edges, with the endpoint of each element of E_G^{ext} encoded by a map $\partial : E_G^{ext} \to V_G$. We will only consider connected graphs.

To each internal edge $e \in E_G$ we assign its *particle mass* $m_e \in \mathbb{R}$. To each external edge $i \in E_G^{ext}$ we assign a *momentum*, which is a vector $q_i \in \mathbb{R}^d$ where $d \geq 0$ is the dimension of space-time. A condition on momenta $\sum_{i \in E_G^{ext}} q_i = 0$, called *momentum conservation*, is assumed. We write G/e_i for the graph with the edge e_i contracted, and G/e_I for the graph with the edges $\{e_i\}_{i \in I}$ contracted. If a Feynman graph has M non-zero masses and F non-zero external momenta we say that it is a graph of 'type (F, M)'.

Definition 2.5.2. Let G be a Feynman graph. Associate to each internal edge $e \in E_G$ a variable α_e. Then the *first Symanzik polynomial* is a homogeneous polynomial defined to be

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$$

where the sum is over all spanning trees T of the graph G. We also define the following homogeneous polynomial

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e$$

where the sum ranges over all spanning 2-trees $T = T_1 \cup T_2$ of G. A spanning 2-tree of a graph G is a subgraph with 2 connected components, each of which is a tree, with the minimal number of edges such that its vertices include all of the vertices of the original graph G. We define $q^{T_1} = \sum_{i \in E_G^{ext}} q_i$ as the sum of all incoming momenta entering T_1. By momentum conservation $q^{T_1} = -q^{T_2}$. We denote the Euclidean scalar product of the vector $q \in \mathbb{R}^d$ with itself by q^2. The *second Symanzik polynomial*, also homogeneous, is then defined to be:

$$\Xi_G(m, q) = \Phi_G(q) + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G. \tag{2.15}$$

To abbreviate the dependencies of Ξ_G on momenta and masses in the above definitions we write $q = \{q_1, ..., q_F\}$ and $m = \{m_1, ..., m_M\}$.

Let G be a Feynman graph with N_G edges, h_G loops. In parametric form, the Feynman integral which is of interest in physics is the following projective integral:

$$\Gamma\left(N_G - h_G\frac{d}{2}\right)\int_\sigma \omega_G(m,q) \tag{2.16}$$

where

$$\omega_G(m,q) = \frac{1}{\Psi_G^{d/2}}\left(\frac{\Psi_G}{\Xi_G}\right)^{N_G - h_G d/2}\Omega_G \tag{2.17}$$

and Ψ_G, and $\Xi_G(m,q)$ are the first and second Symanzik polynomials of G, respectively. The domain of integration is:

$$\sigma = \{[\alpha_1 : \ldots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G-1}(\mathbb{R})$$

and

$$\Omega_G = \sum_{i=1}^{N_G}(-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_{N_G} \tag{2.18}$$

where $\widehat{d\alpha_i}$ means that we omit $d\alpha_i$. The derivation of this form of the Feynman integral from its momentum space representation using the Schwinger trick is nicely explained in [39]. Also see [4].

Remark 2.5.3. It is important to note that the integral (2.16) can diverge, for example when when $N_G = h_G\frac{d}{2}$, in which case the Γ prefactor will have a pole. It is crucial to make sense of this situation in physics, which is where the theory of renormalization comes in. For a detailed treatment of this, in the context of the parametric representation of Feynman integrals, which is suitable for an algebro-geometric approach see [19]. In this thesis we will only consider the cases when (2.16) converges. We note, however, that in [12], it was already speculated that the theory of the cosmic Galois group should extend to the setting of [19].

Definition 2.5.4. We will consider the following projective integral:

$$I_G(m,q) = \int_\sigma \omega_G(m,q). \tag{2.19}$$

Therefore, since we are considering cases in which the (2.16) converges, we have simply dropped the prefactor which is a value of the Gamma function.

2.5.2 Motivic Feynman amplitudes

In order to study their Galois theory we need to lift the functions $I_G(m,q)$ to families of motivic periods. Concretely, for each Feynman graph G of type (F,M) we need an element $mot_G \in \mathrm{Ob}(\mathcal{H}(S))$, and a family of motivic periods

$$[mot_G,[\sigma],[\omega]]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m},Z,gen}_{\mathcal{H}(\mathcal{S})},$$

for a certain space of kinematic parameters S, and $Z \subset S(\mathbb{C})$ simply connected region as in 2.1 and 2.2, such that

$$\mathrm{per}([mot_G,[\sigma],[\omega]]^{\mathfrak{m}}) = I_G(m,q)\,.$$

It defines a multi-valued meromorphic function on $S(\mathbb{C})$. We will define S and Z explicitly below.

The graph polynomial (2.15), and therefore the Feynman integral, are invariant under the action of the orthogonal group in d dimensions. Therefore they only depend on $s_{i,j} = s_{j,i} = q_i \cdot q_j$, the Euclidean product of q_i, q_j, $1 \leq i \leq j \leq F$. Recall that they also satisfy momentum conservation by assumption. Furthermore, if a graph G has a vertex $v \in V$ such that the incoming momentum is non-trivial, i.e., $q^{\{v\}} \neq 0$, then the polynomial $\Phi_G(q) \not\equiv 0$ if $s_I = \sum_{i,j\in I} s_{i,j} \not\equiv 0$ for all $I \subsetneq \{1,\dots,F\}$ [12, Lemma 1.12]. If the previous condition holds along with $s_I + m_j^2 \not\equiv 0$ for all $I \subsetneq \{1,\dots,F\}$ and $j \in \{1,...,|E_G|\}$ then the polynomial $\Xi_G(m,q) = 0$ if and only if all internal masses and all external momenta are trivial [12, Lemma 1.13]. This leads to the following definition.

Definition 2.5.5. Define $K_{F,M} = \mathbb{A}^{\binom{F}{2}} \times \mathbb{G}_m^M$ to be the *space of kinematics*. Let $K^{gen}_{F,M} \subset K_{F,M}$ be the open complement of the union of spaces $s_I + m_j^2 = 0$, where $s_I = \sum_{i,j\in I} s_{i,j}$, for $I \subsetneq \{1,\dots,F\}$, and $j \in \{0,1,\dots M\}$, and we set $m_0 = 0$. We also define $U^{gen}_{F,M} \subset K^{gen}_{F,M}(\mathbb{C})$ to be the region in $K^{gen}_{F,M}(\mathbb{C})$ where $\mathrm{Re}(s_I) > 0$, and $\mathrm{Re}(m_j^2) > 0$, called the *extended Euclidean sheet*. Denote by $k_{F,M} = \mathrm{Frac}(\mathcal{O}(K_{F,M}))$ the field of fractions of $\mathcal{O}(K_{F,M})$. Note that $k_{F,M} \cong \mathbb{Q}((s_{i,j})_{1\leq i\leq j\leq F}, (m_k)_{1\leq k\leq M})$.

The lifting of Feynman integrals to families of motivic periods is provided by Brown in [12] for Feynman graphs of any loop order. Key results for this lifting are certain factorization properties of Symanzik polynomials, which lead to the concept of *motic subgraphs*. We recall the definition.

Definition 2.5.6. A subgraph γ is called *mass-momentum spanning* if it contains all internal edges $e \in E_G$ which have non-vanishing mass m_e, $\partial E_G^{ext} \subset V_\gamma$, and all the vertices ∂E_G^{ext} all lie in a single connected component of γ.

A subgraph spanned by the edges $\gamma \subset E_G$, which we also denote by γ, is called *motic* if, for every subgraph spanned by the edges $\gamma' \subsetneq E_\gamma$, which is mass-momentum spanning in γ, one has $h_{\gamma'} < h_\gamma$, where we denote by h_γ the loop number of γ. In other words, a subgraph $\gamma \subset G$ is motic if and only if removing any edge of γ makes it non-mass-momentum spanning or reduces its loop number h_γ.

Recall that α_e are projective coordinates. Let $\Delta_G = \bigcup_{e \in E_G} \Delta_e \subset \mathbb{P}^{N_G-1}$, where $V(f)$ is defined by the vanishing locus of f, and note that $\partial\sigma \subset \Delta_G$. Denote $\Delta_\gamma = \bigcup_{e \in E_\gamma} \Delta_e \subset \mathbb{P}^{N_G-1}$, for a subgraph $\gamma \subset G$, and let $\Delta_\varnothing = \mathbb{P}^{N_G-1}$. These schemes are defined a priori over $\mathrm{Spec}(\mathbb{Q})$, and we write, by abuse of notation, $\mathbb{P}^{N_G-1}$, Δ_G, and Δ_γ for their base change to $K_{F,M}^{gen}$ as well.

By [12, Proposition 6.2] a linear subspace Δ_γ corresponding to a motic subgraph γ is contained in $V(\Xi_G(m,q))$, and if γ is not mass-momentum spanning then Δ_γ is also contained in $V(\Psi_G)$, over each fiber over $K_{F,M}^{gen}$. Therefore, over each fiber, the boundary of the domain of integration meets the singularities of the integrand ω_G, causing potential divergences. We blow up $\mathbb{P}^{N_G-1}$ along the linear subspaces corresponding to motic subgraphs, at first blowing up those linear subspaces of dimension 0, then the strict transforms of linear subspaces of dimension one etc. Denote the blow-up by $\pi_G : P^G \to \mathbb{P}^{N_G-1}$. Let $X_G = V(\Xi_G(m,q)) \bigcup V(\Psi_G), X'_G = V(\Xi_G(m,q))$, and Y_G, Y'_G be their strict transforms. Let $D = \pi_G^{-1}(\Delta_G)$.

We recall the definition of a Landau variety following [42, Ch. 4], based on stratified Morse theory [29]. Consider the underlying analytic varieties of the pair $(P^G, D \bigcup Y_G)$. Then the divisor $Y = D \bigcup Y_G$ gives rise to a stratification of P^G

$$P^G \supset Y \supset Y^{(1)} \supset Y^{(2)} \supset \dots$$

where $Y^{(k)}$ is the skeleton of D of codimension k. It is a stratification [42, Ch. 3] since the open strata $Y^{(k)} \setminus Y^{(k+1)}$ are smooth, and the boundary of each irreducible component of $Y^{(k)} \setminus Y^{(k+1)}$, denoted A_k, has the property that the boundary $\bar{A}_k \setminus A_k$ is a union of strata of lower dimension. Moreover they satisfy Whitney's conditions A and B[1]. Define the critical set of A_k to be the set where π fails to be submersive:

$$cA_k = \{x \in A_k | \operatorname{rk} T_x\pi < \dim K_{F,M}^{gen}\}.$$

[1]Whitney's conditions A and B are local conditions defined for every two strata A_1 and A_2 such that $A_1 \subset \partial A_2$. Condition A asks that for every point a of A_1 and b of A_2 in a neighbourhood of a the angle of $T_b(A_2)$ and $T_a(A_1)$ tends to 0 with $|b-a|$. Condition B asks that for $r : A_2 \to A_1$ a local retraction the angle between $T_b(A_2)$ and the vector $\overrightarrow{b, r(b)}$ tends to zero with $|b-a|$.

Definition 2.5.7. Define the Landau variety L_G to be the codimension 1 part of $\pi(\cup_i cA_i)$, where the union is over all the strata of Y.

Note that we have been working with underlying analytic varieties of P^G, Y, and by [42, Lemma 5.2, Corollary 5.3] each $\pi(\cup_i cA_i)$ is a complex analytic set. In fact, the same proof as in (*loc. cit.*) tells us that it is in fact an algebraic variety, since it is expressed therein in terms of minors of a matrix of partial derivatives of local equations for X_G and D, which are algebraic. Then by Thom's Isotopy theorem π is a locally trivial fibration of stratified analytic varieties on the complement of L_G.

Definition 2.5.8. Let $S = K^{gen}_{F,M} \setminus L_G$. The *graph motive* of a Feynman graph G is an object of $\mathcal{H}(S)$ defined as

$$mot_G = \left(H_B^{N_G-1}\left(P^G \setminus Y_G, D \setminus Y_G \cap D\right)_{/S}, H_{\mathrm{dR}}^{N_G-1}\left(P^G \setminus Y_G, D \setminus Y_G \cap D\right)_{/S}, c\right).$$

We also define

$$mot'_G = \left(H_B^{N_G-1}\left(P^G \setminus Y'_G, D \setminus Y'_G \cap D\right)_{/S}, H_{\mathrm{dR}}^{N_G-1}\left(P^G \setminus Y'_G, D \setminus Y'_G \cap D\right)_{/S}, c\right).$$

Next we write down a canonical Betti class over $Z = U^{gen}_{F,M} \cap S(\mathbb{C}) \subset S(\mathbb{C})$. Let $\pi_G : P^G \to \mathbb{P}^{N_G-1}$ be the blow up along Δ_γ where these are viewed as schemes over $\mathrm{Spec}(\mathbb{Q})$, and let $\widetilde{\sigma}$ be the closure in the analytic topology of $\pi_G^{-1}(\mathring{\sigma})$, where $\mathring{\sigma}$ is the real open simplex given by $\alpha_e > 0$. We define a constant family of manifolds with corners over $U^{gen}_{F,M}$:

$$\sigma_G = \widetilde{\sigma} \times U^{gen}_{F,M}.$$

Then [12, Theorem 6.7] tells us that

$$\sigma_G \cap Y_G(\mathbb{C}) = \varnothing.$$

It uses the fact that the coefficients of Ψ_G are all positive and that $\mathrm{Re}(\Xi_G(m,q)) > 0$ when $\alpha_i > 0$ and $(m,q) \in U^{gen}_{F,M}$, i.e., parameters have positive real parts. Then we have

$$[\sigma_G] \in \Gamma(U^{gen}_{F,M}, (mot_G)^\vee_B)$$

as desired.

Finally, for a general definition of a motivic Feynman amplitude, one needs to prove that the pull-back of the differential form $\pi_G^*(\omega_G(m,q))$ doesn't acquire any new poles along the exceptional divisors after blowing up. For a general criterion in terms of the degrees of divergence of motic subgraphs see [12, §6.6]. We do not need this result in full generality, and will check for poles directly in section 3.3, when

we first encounter motic subgraphs and having to blow-up. When this is satisfied $\pi_G^*(\omega_G(m,q))$ is a global section of $\Omega^{N_G-1}_{P^G\smallsetminus Y_G/k_S}$ and defines a class

$$[\pi_G^*(\omega_G(m,q))] \in \Gamma(\mathrm{Spec}(k_S),(mot_G)_{\mathrm{dR}}),$$

where $\mathrm{Spec}(k_S)$ is the generic point of S.

Definition 2.5.9. The *motivic Feynman amplitude* is the family of motivic periods:

$$I_G^{\mathfrak{m}}(m,q) = [mot_G,[\sigma_G],[\pi_G^*(\omega_G(m,q))]]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m},U^{gen}_{F,M},gen}_{\mathcal{H}(S)}$$

When we have $[\pi_G^*(\omega_G(m,q))] \in \Gamma(\mathrm{Spec}(k_S),(mot'_G)_{\mathrm{dR}})$, e.g. when $N_G \geq (h_G+1)\frac{d}{2}$, the motivic Feynman amplitude can be viewed as a motivic period of the motive mot'_G. In this case we denote it by the same symbol:

$$I_G^{\mathfrak{m}}(m,q) = [mot'_G,[\sigma_G],[\pi_G^*(\omega_G(m,q))]]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m},U^{gen}_{F,M},gen}_{\mathcal{H}(S)}.$$

In what follows we will make it clear if we are working with mot_G or mot'_G.

Since the coaction of the de Rham Tannaka group will generate new integrands associated to the same graph motive we make the following definition.

Definition 2.5.10. A *motivic Feynman period* of type (F,M) is

$$I_G^{\mathfrak{m}}(\omega) = [mot'_G,[\sigma_G],[\omega]]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m},U^{gen}_{F,M},gen}_{\mathcal{H}(S)},$$

where G is a Feynman graph of type (F,M) and $\omega \in \Gamma(\mathrm{Spec}(k_S),(mot_G)_{\mathrm{dR}})$ is any relative de Rham cohomology class at the generic point. Note that all motivic Feynman amplitudes are motivic Feynman periods but the reverse does not hold.

Let us now consider the subspace

$$\mathcal{FP}^{\mathfrak{m}}_{F,M} \subset \mathcal{P}^{\mathfrak{m},U^{gen}_{F,M},gen}_{\mathcal{H}(S)}$$

spanned by motivic Feynman periods of type (F,M). Applying the general coaction formula to an element $I_G^{\mathfrak{m}}(\omega) \in \mathcal{FP}^{\mathfrak{m}}_{F,M}$ we get

$$\Delta(I_G^{\mathfrak{m}}(\omega)) = \sum_{e_i}[mot_G,\sigma_G,e_i]\otimes[mot_G,e_i^\vee,\omega],$$

where $e_i \in \Gamma(\mathrm{Spec}(k_S),(mot_G)_{\mathrm{dR}})$. We see that $\mathcal{FP}^{\mathfrak{m}}_{F,M}$ is a right comodule over

$$\mathcal{FP}^{\mathfrak{dr}}_{F,M} \subset \mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)},$$

where $\mathcal{FP}^{\mathfrak{dr}}_{F,M}$ is the space of *de Rham Feynman periods*, i.e. the subspace of the space of de Rham periods spanned by

$$[mot_G,f,v],\ \text{for } f \in \Gamma(\mathrm{Spec}(k_S),(mot_G)_{\mathrm{dR}})^\vee,\ \text{and } v \in \Gamma(\mathrm{Spec}(k_S),(mot_G)_{\mathrm{dR}}).$$

Definition 2.5.11. We define the *cosmic Galois group* for *convergent* Feynman integrals to be

$$C_{F,M} = \mathrm{Spec}(\mathcal{FP}^{\mathfrak{dr}}_{F,M}).$$

It is an affine group scheme over $k_{F,M}$.

Remark 2.5.12. When the motive of a Feynman graph is mixed Tate (see 2.4.1), as will be the case for one-loop graphs which we study in the next chapter, we define the *de Rham Feynman amplitude* as:

$$I^{\mathfrak{dr}}_G(m,q) = [mot_G, \epsilon, [\pi^*_G(\omega_G(m,q))]]^{\mathfrak{dr}} \in \mathcal{P}^{\mathfrak{dr},gen}_{\mathcal{H}(S)}, \tag{2.20}$$

where $\epsilon : \omega^{gen}_{\mathrm{dR}}(mot_G)_{\mathrm{dR}} \to \omega^{gen}_{\mathrm{dR}} W_0(mot_G)_{\mathrm{dR}}$. This map is simply the projection to the weight 0 part. Note that the weight filtration on the de Rham realization is split since the motive is Mixed Tate.

Chapter 3

Motivic Galois coaction and one-loop Feynman graphs

3.1 Motives of one-loop graphs

We specialize the previous discussion to the case of one-loop cycle graphs with N_G internal edges, and one external edge attached to each vertex, in $d = 4$ dimensions. In this case, the second Symanzik polynomial Ξ_G is homogeneous quadratic in the α_i, and the first Symanzik polynomial Ψ_G is linear in the α_i. The integral of interest is:

$$I_G(m,q) = \int_\sigma \frac{\Psi_G^{N_G-4}}{\Xi_G^{N_G-2}} \Omega_G. \tag{3.1}$$

3.1.1 Four and more internal edges

Let $N_G \geq 4$, and all masses and momenta non-vanishing, i.e., $F = M = N_G$. Note that in this case the polar locus of the integrand does not include the first Symanzik polynomial, therefore we can work with the restricted motive mot'_G.

In the case of non-vanishing masses and momenta, there are no motic subgraphs (see definition 2.5.6) of a one-loop graph. In particular there is no need to blow up, and denoting by Q the vanishing locus of Ξ_G, and following the last section we have:

$$mot'_G = (H_{\mathrm{B}}^{N_G-1}(\mathbb{P}^{N_G-1} \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}, H_{\mathrm{dR}}^{N_G-1}(\mathbb{P}^{N_G-1} \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}, c),$$

and

$$[\omega_G] \in \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}}), \quad [\sigma_G] \in \Gamma(U_{F,M}^{gen}, ((mot'_G)_B)^\vee).$$

The associated motivic Feynman amplitude is

$$I_G^{\mathfrak{m}}(m,q) = [mot'_G, [\sigma_G], [\omega_G]]^{\mathfrak{m}} \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, U_{F,M}^{gen}, gen}.$$

3.1.2 Semi-simplification of the motive

Lemma 3.1.1. *Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Then*

$$H^q(\mathbb{P}^n \setminus Q, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-m), & \text{if } q = n = 2m-1 \\ \mathbb{Q}(0), & \text{if } q = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

Proof. Recall that for a quadric Q of dimension d we have $H^{2q}(Q, \mathbb{Q}) \cong \mathbb{Q}(-m)$ for $1 \leq q \leq d$ if d odd. If $d = 2l$ is even then $H^d(Q, \mathbb{Q}) \cong \mathbb{Q}(-l) \oplus \mathbb{Q}(-l)$. Let $n = 2m-1$ and consider the Gysin long exact sequence:

$$\dots H^n(\mathbb{P}^n) \to H^n(\mathbb{P}^n \setminus Q) \xrightarrow{res} H^{n-1}(Q)(-1) \xrightarrow{Gysin} H^{n+1}(\mathbb{P}^n) \to H^{n+1}(\mathbb{P}^n \setminus Q) \to \dots$$

Note that $H^{n-1}(Q)(-1) \cong \mathbb{Q}(-m) \oplus \mathbb{Q}(-m)$ and that the Gysin morphism is surjective since $\mathbb{P}^n \setminus Q$ is a closed subset of dimension n of the affine space $\mathbb{P}^N \setminus H$, where H is a hyperplane, and the closed embedding is given by the Veronese embedding $i : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+2}{n} - 1$. Since $H^n(\mathbb{P}^n) = 0$ we have that $H^n(\mathbb{P}^n \setminus Q) \cong \mathbb{Q}(-m)$. From the same long exact sequence we can see that $H^{n+k}(\mathbb{P}^n \setminus Q) = 0$ when $0 \leq k \leq n$ because in that case $H^{n+k}(Q)(-1) \cong H^{n+k}(\mathbb{P}^n)$, and $H^{n+k}(\mathbb{P}^n) = 0$ if k even and $H^{n+k-1}(Q) = 0$ if k is odd. By Poincaré duality we get the remaining cohomology. The result is proved analogously for $n = 2m$ even. □

Recall that the rank of a quadric $Q = V(f) \subset \mathbb{P}^n$, where f is a homogeneus polynomial of degree 2, is the rank of the matrix C where $f = \vec{\alpha} C \vec{\alpha}^t$, and α is the row vector $(\alpha_1, \dots, \alpha_{n+1})$. Let $\Delta_i = V(\alpha_i)$, $\Delta_I = \cap_{i \in I} \Delta_i$, and $\Delta_\varnothing = \mathbb{P}^n$. The rank of $Q = V(\Xi_G(m, q)) \subset \mathbb{P}^{N_G - 1}$, for $N_G \geq 6$, in $d = 4$ space-time dimensions is at most 6, and it is exactly 6 for generic values of masses and momenta [7, Lemma 6.3], i.e., the locus where the rank is strictly less than 6 is in the Landau variety L_G. When $N_G \leq 5$ the quadric Q is generically smooth. In order to determine the weight graded pieces of mot'_G, we first prove the following result, which applies to each fiber of mot'_G over $S(\mathbb{C})$.

Proposition 3.1.2. *Let $n \geq 3$, and $\Delta_G \subset \mathbb{P}^n$ be the standard simplex. Let $Q \subset \mathbb{P}^n$ be a quadric of rank $min(6, n+1)$ with Q and Δ_G in general position. Then the weight graded pieces of the motive $H = H^n(\mathbb{P}^n \setminus Q, \Delta_G \setminus Q \cap \Delta_G)$ are:*

$$gr_0^W H = \mathbb{Q}(0), \quad gr_2^W H = \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1), \quad gr_4^W H = \bigoplus_{\binom{n+1}{n-3}} \mathbb{Q}(-2). \tag{3.3}$$

When $n \leq 4$ all the other weight-graded pieces, except for these three, vanish. When $n \geq 5$ we also have:

$$gr_6^W H = \bigoplus_{\binom{n+1}{n-5}-\binom{n+1}{n-6}} \mathbb{Q}(-3), \tag{3.4}$$

and all the others vanish.

Proof. We apply the relative cohomology spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\Delta_I \setminus (Q \cap \Delta_I)) \Rightarrow H^{p+q}(\mathbb{P}^n \setminus Q, \Delta_G \setminus Q \cap \Delta_G).$$

To compute $W_4 H$ we note that our assumption on the rank of Q implies that $Q \cap \Delta_I$ is smooth when Δ_I is a face of Δ_G of dimension ≤ 5. Therefore, by the previous lemma, the first row $E_1^{\bullet,1}$ is zero except for the entry

$$E_1^{n-1,1} \cong \bigoplus_{|I|=n-1} H^1(\Delta_I \setminus Q \cap \Delta_I) \cong \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1).$$

Hence $gr_2^W H \cong \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1)$ as required. Again, by the previous lemma, row $E_1^{\bullet,2}$ is all zero, while the row $E_1^{\bullet,3}$ is zero except for

$$E_1^{n-3,3} \cong \bigoplus_{|I|=n-3} H^3(\Delta_I \setminus Q \cap \Delta_I) \cong \bigoplus_{\binom{n+1}{n-3}} \mathbb{Q}(-2).$$

It remains to show the claim for the graded weight 6 piece of H, as well as that there are no higher weight graded pieces. By the assumption on the rank of Q, if $n \geq 6$ then $Q \subset \mathbb{P}^n$ is a generalized cone over a smooth quadric $Q_0 \subset \mathbb{P}^5$. Hence, $\mathbb{P}^n \setminus Q$ is a fiber bundle with fibers $\mathbb{A}^{n-5}$ over $\mathbb{P}^5 \setminus Q_0$. By the Leray spectral sequence we get

$$H^i(\mathbb{P}^n \setminus Q, \mathbb{Q}) \cong H^i(\mathbb{P}^5 \setminus Q_0, \mathbb{Q}) \text{ for all } i \geq 0 \tag{3.5}$$

Therefore we can apply the previous lemma again to conclude that for all $q \geq 6$ and $p \geq 0$ the $E_1^{p,q}$ vanish. Hence there are no weight graded pieces higher than 6. We also get $E_1^{n-4,5} = 0$ and

$$E_2^{n-5,5} \cong \mathrm{gr}_6^W H \cong \mathrm{coker}\left(E_1^{n-6,5} \to E_1^{n-5,5}\right) \cong \bigoplus_{\binom{n+1}{n-5}-\binom{n+1}{n-6}} \mathbb{Q}(-3).$$

Finally, the bottom row $E_1^{\bullet,0}$ computes the cohomology of the simplicial complex Δ_G which is homologicaly equivalent to a sphere, giving us the graded weight 0 part of the proposition. □

We must be careful when pulling back this result to S. It is not the case that, for S defined as above, mot'_G is isomorphic to $\mathbb{Q}(0)_{/S} \oplus \mathbb{Q}(-1)^{m_1}_{/S} \oplus \mathbb{Q}(-2)^{m_2}_{/S} \oplus \mathbb{Q}(-3)^{m_3}_{/S}$, for m_1, m_2, m_3 defined in the previous Proposition, and $\mathbb{Q}(-k)_{/S}$, constant variation of Tate Hodge structure defined in 2.4.1. The reason is that the cohomology of a smooth quadric in even dimensions has rank 2 in middle degree, and the monodromy might interchange the two classes. However, if we pass to the covering of S which includes the square roots of the determinant of the matrix associated to the quadric Q associated to the graph G, as well as the square roots of the determinants of all the quadrics $Q|_{\Delta_I}$, where Δ_I is a face of Δ_G as before, we do indeed get that mot'_G is a constant variation. This explains the prefactors $P_{j,k}$ in Theorem 3.2.5. We assume that this has been done and denote the new base by S as well.

3.1.3 Reduction to four edges

The main tool we use to reduce to the one-loop graph with 4 internal edges (Figure 1.1) is the following:

Lemma 3.1.3. *Let $n = N_G - 1 \geq 4$. Let $\Delta_i \subset \mathbb{P}^n$ be the face of Δ_G where $\alpha_i = 0$. Then the Feynman differential form $\omega_G(m, q)$ is exact, and we can find an $(n-1)$-form ω_{n-1} which is a global section of the sheaf $\Omega^{n-1}_{\mathbb{P}^n \setminus Q/k_S}$, along with constants $a_j \in k_S$, such that $d\omega_{n-1} = \omega_G$ and $\omega_{n-1}|_{\Delta_j} = a_j \omega_{G/e_j}$.*

Proof. See [7, Lemma 8.1] □

Proposition 3.1.4. *For generic values of masses and momenta and in $d = 4$ dimensions, the motivic Feynman amplitude of any one-loop graph G with $N_G \geq 5$ edges is a k_S-linear combination of motivic periods of graph motives of 4-edge quotient graphs of G.*

Proof. We start with the motivic period $[mot'_G, [\sigma_G], [\omega_G]]^{\mathfrak{m}}$, and we would like to show that there exist constants $a_I \in k_S$, for $I \subset \{1, \ldots, N_G\}$, such that we can write

$$[mot'_G, [\sigma_G], [\omega_G]]^{\mathfrak{m}} = \sum_{|I| = N_G - 4} a_I \cdot \left[mot'_{G/e_I}, \left[\sigma_{G/e_I}\right], \left[\omega_{G/e_I}\right] \right]^{\mathfrak{m}} \tag{3.6}$$

or equivalently:

$$I^{\mathfrak{m}}_G = \sum_{|I| = N_G - 4} a_I \cdot I^{\mathfrak{m}}_{G/e_I}. \tag{3.7}$$

Restricting the family $\mathbb{P}^{N_G - 1} \setminus Q$, and $\Delta_G \setminus Q \cap \Delta_G$ to the fiber at the generic point, we apply the previous lemma as follows. Because $\mathbb{P}^{N_G - 1} \setminus Q$ is affine and $\Delta_G \setminus Q \cap \Delta_G$

is a simple normal crossing divisor therein, $(mot'_G)_{\mathrm{dR}}$ is computed by the cohomology of the total complex of the double complex

$$\Gamma(\mathbb{P}^n \setminus Q, \Omega^\bullet_{\mathbb{P}^n \setminus Q}) \to \bigoplus_{|I|=1} \Gamma(\Delta'_I, \Omega^\bullet_{\Delta'_I}) \to \ldots \to \bigoplus_{|I|=n} \Gamma(\Delta'_I, \Omega^\bullet_{\Delta'_I}) \tag{3.8}$$

where $\Delta'_I = \Delta_I \setminus \Delta_I \cap Q$, and $\Omega^\bullet_{\Delta'_I}$ is the direct image under $\Delta_I \hookrightarrow \mathbb{P}^n \setminus Q$ of the sheaf of algebraic differential forms on Δ_I and vanishes outside of Δ_I. The horizontal morphisms are restrictions, i.e. pullback along the inclusion of faces $\Delta_I \hookrightarrow \Delta_{I \setminus i_j}$ with the sign $(-1)^k$ where i_j is the kth element of I. The differential of the total complex of (3.8) is defined by:

$$d^n_{Tot} = \sum_{n=p+q} d_0^{p,q} + (-1)^p d_1^{p,q} \tag{3.9}$$

where d_0 is the exterior derivative and d_1 is the restriction. Recall that we have specialized to the generic point, which we drop from the notation. Therefore the coefficients of the algebraic differential forms above lie in the field k_S. This double complex is obtained by specializing (2.8) to a point and using the fact that we are working with affine schemes.

Therefore, the class of the Feynman form $[\omega_G] \in (mot'_G)_{\mathrm{dR}}$ can be represented by

$$(\omega_G, 0 \ldots, 0) \in \bigoplus_{N_G - 1 = p+q} C^{p,q}, \text{ where } C^{p,q} = \bigoplus_{|I|=p} \Gamma\left(\Delta'_I, \Omega^q_{\Delta'_I}\right).$$

By Lemma 3.1.3 we can construct an $(N_G - 2)$-form

$$\omega' = (\omega, 0, \ldots, 0) \in \bigoplus_{N_G - 2 = p+q} C^{p,q}, \quad \text{where } \omega \in \Gamma\left(\mathbb{P}^{N_G - 1} \setminus Q, \Omega^{N_G - 2}_{\mathbb{P}^{N_G-1} \setminus Q}\right),$$

and an $(N_G - 1)$-form

$$\begin{aligned} \omega'' &= \left(0, \omega|_{\Delta_1}, \omega|_{\Delta_2}, \ldots, \omega|_{\Delta_{N_G}}, 0, \ldots, 0\right) \\ &= \left(0, a_1 \cdot \omega_{G/e_1}, a_2 \cdot \omega_{G/e_2}, \ldots, a_{N_G} \cdot \omega_{G/e_{N_G}}, 0, \ldots, 0\right) \in \bigoplus_{N_G - 1 = p+q} C^{p,q} \end{aligned} \tag{3.10}$$

such that $d_{Tot}(\omega') = \omega + \omega''$. Now, obviously we can write $[\omega'']$ as a linear combination, over k_S, of classes of $(N_G - 2)$-forms in $(mot_{G/e_j})_{\mathrm{dR}}$ for $1 \leq j \leq N_G$.

The de Rham component of a face map (Definition 2.4.2):

$$\Phi_{i,dR} : \Gamma(\mathrm{Spec}(k_S), (mot'_{G/e_i})_{\mathrm{dR}}) \to \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}}), \tag{3.11}$$

for all $1 \leq i \leq N_G$ sends

$$\left[\left(a_i \cdot \omega_{G/e_i}, 0, \ldots, 0\right)\right] \mapsto \left[\left(0, \ldots, 0, a_i \cdot \omega_{G/e_i}, 0, \ldots, 0\right)\right]$$

Summing over all the faces we get a morphism

$$\begin{aligned} \Phi_{\mathrm{dR}} : \bigoplus_i \Gamma(\mathrm{Spec}(k_S), (mot'_{G/e_i})_{\mathrm{dR}}) &\to \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}}) \\ \left(a_1 \cdot \left[\omega_{G/e_1}\right], \ldots, a_{N_G} \cdot \left[\omega_{G/e_{N_G}}\right]\right) &\mapsto [\omega_G] \end{aligned} \tag{3.12}$$

where we write $[\omega_{G/e_i}]$ for the class of the element $(\omega_{G/e_i}, 0, \ldots, 0)$.

The Betti components of face maps (2.4.2) are given by restriction to the faces of $\Delta_G \setminus Q \cap \Delta_G$, and again we sum over all the faces to get:

$$\begin{aligned} \Phi_B^\vee : \Gamma(U_{F,M}^{gen}, (mot'_G)_{\mathrm{B}}^\vee) &\to \bigoplus_i \Gamma(U_{F,M}^{gen}, (mot'_{G/e_i})_{\mathrm{B}}^\vee) \\ [\sigma_G] &\mapsto \left(\left[\sigma_{G/e_1}\right], \ldots, \left[\sigma_{G/e_{N_G}}\right]\right) \end{aligned} \tag{3.13}$$

which shows the equivalence of motivic periods:

$$[mot_G, [\sigma_G], [\omega_G]]^{\mathfrak{m}} = \sum_{1 \le i \le N_G} a_i \cdot \left[mot_{G/e_i}, \left[\sigma_{G/e_i}\right], \left[\omega_{G/e_i}\right]\right]^{\mathfrak{m}}$$

Iterating this same process we can then write each of the $(N_G - 2)$-forms on the right hand side above as a k_S-linear combination of $(N_G - 3)$-forms of graphs obtained by contracting two edges of G, up to a form that is exact in relative de Rham cohomology. This can be repeated until we get to a k_S-linear combination of 3-forms which belong to the de Rham realizations of motives of graphs with 4 edges obtained by contracting the $N_G - 4$ edges of the original graph. Gathering all the coefficients a_i at each stage into a_I for each set of $N_G - 4$ contracted edges I, we obtain the proof of the lemma. □

Applying the period map to (3.7) we recover an old result of Nickel [37]:

Corollary 3.1.5. *For generic values of masses and momenta the Feynman integral of any one-loop graph G in $d = 4$ dimensions is a k_S-linear combination of periods of graph motives of 4-edge quotient graphs of G.*

Remark 3.1.1. Note that using face maps to show equivalences of families of motivic periods in the previous proposition is the "motivic lift" of repeatedly applying Stokes' theorem on the corresponding integrals under the period map.

Remark 3.1.2. Since we have reduced to motives of graphs with 4 edges in the previous proposition, proposition 3.1.2 implies that all motivic Feynman amplitudes of one-loop graphs with generic masses and momenta are k_S-linear combinations of families of periods of motives that have the following weight graded pieces

$$\mathrm{gr}^W(mot_G) = \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 6} \oplus \mathbb{Q}(0)_{/S}$$

Griffiths transversality is used in [7, §9] to show that, up to a constant of integration, periods of heighest weight of such motives are always k_S-linear combinations of dilogarithms.

3.1.4 Fewer than four edges

3.1.4.1 The triangle graph

We will also consider the graph with $N_G = 3$ edges, henceforth *the triangle graph*. In the next figure we have the triangle graph when $F = M = N_G$, i.e., all masses and momenta are non-zero. Recall that denote the internal edges with non-vanishing masses with double lines.

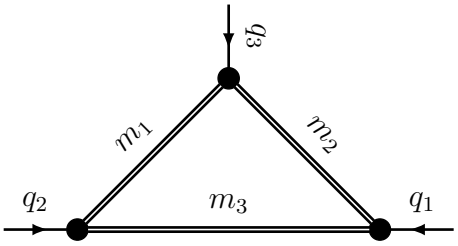


Figure 3.1: The tirangle graph with generic masses.

The first and second Symanzik polynomials of this graph are:

$$\begin{aligned} &\Psi_G = \alpha_1 + \alpha_2 + \alpha_3, \text{ and} \\ &\Xi_G(m,q) = q_1^2\alpha_2\alpha_3 + q_2^2\alpha_1\alpha_3 + q_3^2\alpha_1\alpha_2 + (m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3)\Psi_G. \end{aligned} \tag{3.14}$$

Note that the differential form corresponding to this graph ω_G (3.1) has the first Symanzik polynomial in the denominator. Because of this we must consider the full graph motive. Let $L = V(\Psi_G)$, and $Q = V(\Xi_G(m,q))$. Then, the motivic Feynman amplitude is a family of periods associated to the object:

$$mot_G = H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta_G \setminus (Q \cup L) \cap \Delta_G)_{/S}$$

which we refer to as *the motive of the triangle graph*. We will study the triangle graph and the Galois coaction on the associated motivic Feynman amplitude in section 3.3. We will also look at cases of the triangle graph where internal masses vanish.

3.1.4.2 The bubble graph

Finally, we will be considering the one loop graph with $N_G = 2$ edges, henceforth *the bubble graph*. The first case we will look at is $F = M = N_G$, when the masses and the momentum are non-vanishing.

The first and second Symanzik polynomials of this graph are:

$$\begin{aligned} \Psi_G &= \alpha_1 + \alpha_2, \text{ and} \\ \Xi_G(m,q) &= q_1^2\alpha_1\alpha_2 + (m_1^2\alpha_1 + m_2^2\alpha_2)\Psi_G. \end{aligned} \tag{3.15}$$

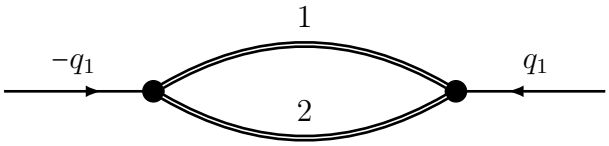


Figure 3.2: The bubble graph with generic masses.

The motive of the bubble graph is

$$mot_G = H^1(\mathbb{P}^1 \setminus (Q \cup L), \Delta_G \setminus (Q \cup L) \cap \Delta_G)_{/S}, \tag{3.16}$$

where $L = V(\Psi_G)$, and $Q = V(\Xi_G(m,q))$. It follows from the relative cohomology long exact sequence

$$\ldots \to H^0(\Delta'_G)_{/S} \to H^1(\mathbb{P}^1 \setminus (Q \cup L), \Delta'_G)_{/S} \to H^1(\mathbb{P}^1 \setminus (Q \cup L))_{/S} \to \ldots,$$

where $\Delta'_G = \Delta_G \setminus (Q \cup L)$, that it has rank 3.

In order to relate the bubble graph with the motivic coaction on one loop graphs with more than 2 edges, we are going to consider the motivic Feynman amplitude of this graph in $d = 2$ dimensions, in which case we have the integrand

$$\theta^1_G = \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{\Xi_G}.$$

It defines a class over the generic point of the de Rham realization of the restricted motive of the bubble graph:

$$mot'_G = H^1(\mathbb{P}^1 \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}$$

Denote by

$$[p_0 p_1 | p_2 p_3] = \frac{(p_2 - p_0)(p_3 - p_1)}{(p_2 - p_1)(p_3 - p_0)}$$

the cross-ratio of 4 points on $\mathbb{P}^1$. Then the period corresponding to the motivic Feynman amplitude of the bubble graph in $d = 2$ dimensions is:

$$\begin{aligned} \mathrm{per}\left(\left[mot_G, [\sigma_G], \left[\theta^1_G\right]\right]^{\mathfrak{m}}\right) &= \frac{1}{\sqrt{4|\det C|}} \log([p_0 p_1 | u_0 u_1]) \\ &= \frac{1}{x-y} \log\left(\frac{y}{x}\right), \end{aligned} \tag{3.17}$$

where $\{u_0, u_1\} = V(\Xi_G) \subset \mathbb{P}^1$, C is the matrix of the quadratic form Ξ_G, x, y are the coordinates of u_0, u_1 respectively in the chart where $\alpha_2 \neq 0$, and $p_0 = [0:1], p_1 = [1:0]$. We know from cohomology computations that $[mot_G, [\sigma_G], [\theta^1_G]]^{\mathfrak{m}}$ is a motivic logarithm (up to an explicit prefactor), and since these are determined by their periods we

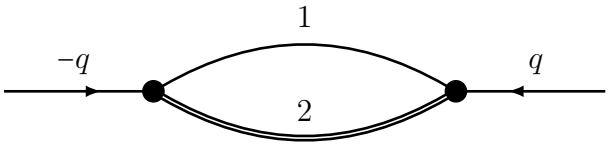


Figure 3.3: The bubble graph with one vanishing mass.

have that $[mot_G, [\sigma_G], [\theta_G^1]]^{\mathfrak{m}} = \frac{1}{\sqrt{4|\det C|}} \log^{\mathfrak{m}}([p_0 p_1 | u_0 u_1])$. In the case of the bubble graph with one vanishing mass: one of the two points of $Q = \{u_0, u_1\} \subset \mathbb{P}^1$ coincides with one of the points of $\Delta_G = \{[1:0], [0:1]\}$. Without loss of generality let $u_0 \in \Delta_G$. In that case the relevant motive will be:

$$mot''_G = H^1(\mathbb{P}^1 \setminus (u_1 \cup L), \Delta_G \setminus (u_1 \cup L) \cap \Delta_G)_{/S} \tag{3.18}$$

It has rank 2. The motivic period we will be interested, which is associated to this motive, is obtained by pairing the interval $(0, \infty)$ with a generator of $(mot''_G)_{\mathrm{dR}}$ represented by the form

$$\theta_G^2 = \frac{(x-y)(\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2)}{(\alpha_2 - x\alpha_1)(\alpha_2 - y\alpha_1)} \tag{3.19}$$

where x, y are coordinates of u_1 and L respectively in the coordinate chart of $\mathbb{P}^1$ where $\alpha_2 \neq 0$. The associated period is:

$$\mathrm{per}\left(\left[mot''_G, [\sigma_G], \left[\theta_G^2\right]\right]^{\mathfrak{m}}\right) = \frac{1}{x-y} \log\left(\frac{y}{x}\right).$$

Similarly to the previous case we see that $[mot''_G, [\sigma_G], [\theta_G^2]]^{\mathfrak{m}}$ is a motivic logarithm.

3.2 The 4-edge graph coaction

3.2.1 The motivic side of the coaction

The first and second Symanzik polynomials of the one-loop graph with 4 edges G are:

$$\begin{aligned} \Psi_G &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \Xi_G &= (q_2+q_3)^2 \alpha_1\alpha_3 + (q_1+q_2)^2 \alpha_2\alpha_4 + q_1^2 \alpha_1\alpha_4 + q_2^2 \alpha_1\alpha_2 + q_3^2 \alpha_2\alpha_3 + q_4^2 \alpha_3\alpha_4 \\ &\quad + (m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3 + m_4^2\alpha_4)\Psi_G. \end{aligned} \tag{3.20}$$

Recall that the general formula for the coaction is:

$$\Delta\,[mot'_G, [\sigma_G], [\omega_G]]^{\mathfrak{m}} = \sum_j [mot'_G, [\sigma_G], [\omega_j]]^{\mathfrak{m}} \otimes [mot'_G, [\omega_j]^\vee, [\omega_G]]^{\mathfrak{dr}}$$

where $\{[\omega_j]\}$ is a basis of $(mot'_G)_{\mathrm{dR}}$, and $\{[\omega_j]^\vee\}$ is the dual basis. We can use the relative cohomology spectral sequence to show that there is a surjective morphism from the direct sum of the motives of bubble graphs obtained by contracting any two edges of G to $gr_2^W mot'_G$. The first sheet of the spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\Delta_I \setminus (Q \cap \Delta_I)) \Rightarrow H^{p+q}(\mathbb{P}^n \setminus Q, \Delta_G \setminus Q \cap \Delta_G)$$

applied to any fiber over $t \in S(\mathbb{C})$ of $(mot'_G)_B$, by lemma 3.1.1, reads:

$$\begin{array}{ccccccc}
E^{0,3} \cong \mathbb{Q}(-2) & \longrightarrow & E^{1,3} \cong 0 & \longrightarrow & E^{2,3} \cong 0 & \longrightarrow & E^{3,3} \cong 0 \\
E^{0,2} \cong 0 & \longrightarrow & E^{1,2} \cong 0 & \longrightarrow & E^{2,2} \cong 0 & \longrightarrow & E^{3,2} \cong 0 \\
E^{0,1} \cong 0 & \longrightarrow & E^{1,1} \cong 0 & \longrightarrow & E^{2,1} \cong \mathbb{Q}(-1)^{\oplus 6} & \longrightarrow & E^{3,1} \cong 0 \\
E^{0,0} \cong \mathbb{Q}(0) & \longrightarrow & E^{1,0} \cong \mathbb{Q}(0)^{\oplus 4} & \longrightarrow & E^{2,0} \cong \mathbb{Q}(0)^{\oplus 6} & \longrightarrow & E^{3,0} \cong \mathbb{Q}(0)^{\oplus 4}.
\end{array} \tag{3.21}$$

We can see that the morphism

$$\bigoplus_{|I|=2} H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S} \to W_2 H^3(\mathbb{P}^3 \setminus Q, \Delta_G \setminus Q \cap \Delta_G), \tag{3.22}$$

given by the sum of face maps, is surjective by computing the relative cohomology spectral sequence for a fiber of $H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S}$, where $|I| = 2$, and noticing that the morphism of spectral sequences induced by the sum of face maps is surjective. Now we look at the de Rham and Betti realization of this sum of face maps.

Consider the bubble graph obtained by contracting edges e_i and e_j. In the next figure we have chosen to contract e_2 and e_3: The restricted motive of the bubble

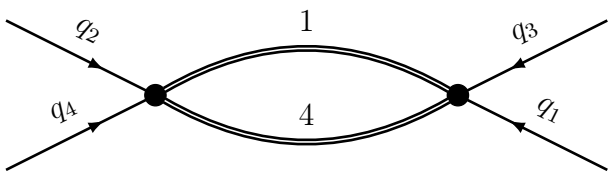


Figure 3.4: The bubble graph obtained by contracting edges.

graph (3.16) obtained by contracting edges e_i and e_j is:

$$mot'_{G/\{e_i,e_j\}} = H^1\left(\mathbb{P}^1 \setminus Q_{G/\{e_i,e_j\}}, \Delta_G \setminus Q_{G/\{e_i,e_j\}} \cap \Delta_G\right)_{/S}$$

where $Q_{G/\{e_i,e_j\}}$ is the vanishing locus of the second Symanzik polynomial of the bubble graph. Observe that $\Xi_{G/\{e_i,e_j\}} = \Xi_G|_{\{\alpha_i=0,\alpha_j=0\}}$, hence:

$$H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S} \cong mot'_{G/\{e_i,e_j\}},$$

where $I = \{i,j\}$. We can then choose a basis of the fiber over the generic point of $(mot_G)_{\mathrm{dR}}$ such that it contains the class of the Feynman form $[\omega_G]$ of G , which is of weight 4, and the 6 weight 2 classes denoted by $\left[\omega_G^{ij}\right]$, which we define to be the images of $\left[\theta^1_{G/\{e_i,e_j\}}\right]$ under the de Rham component of the face map (see Definition 2.4.2):

$$\begin{aligned} \Phi_{ij,dR} : &\Gamma(\mathrm{Spec}(k_S), (mot'_{G/\{e_i,e_j\}})_{\mathrm{dR}}) \to \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}}) \\ &\left[\theta^1_{G/\{e_i,e_j\}}\right] \mapsto \left[\omega_G^{ij}\right] \end{aligned} \tag{3.23}$$

for all $1 \leq i < j \leq 4$. The restriction to the faces of the domain of integration

$$\Phi_B^\vee : [\sigma_G] \mapsto [\sigma_{G/\{e_i,e_j\}}]$$

is the Betti component of the face map, which gives an equivalence of motivic periods:

$$\left[mot'_G, [\sigma_G], \left[\omega_G^{ij}\right]\right]^{\mathfrak{m}} = \left[mot'_{G/\{e_i,e_j\}}, [\sigma_{G/\{e_i,e_j\}}], \left[\theta^1_{G/\{e_i,e_j\}}\right]\right]^{\mathfrak{m}} = I^{\mathfrak{m}}_{G/\{e_i,e_j\}}\left(\theta^1_{G/\{e_i,e_j\}}\right),$$

where $I^{\mathfrak{m}}_{G/\{e_i,e_j\}}\left(\theta^1_{G/\{e_i,e_j\}}\right)$ is the motivic Feynman amplitude of the bubble graph in $d = 2$ dimensions. We have proved:

Proposition 3.2.1. *Let G be a one loop graph with $N_G = 4$. Then the motivic Galois coaction on its motivic Feynman amplitude in $d = 4$ dimensions with non-vanishing masses and momenta is:*

$$\Delta I^{\mathfrak{m}}_G = I^{\mathfrak{m}}_G \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + \sum_{1\leq i<j\leq 4} I^{\mathfrak{m}}_{G/\{e_i,e_j\}}\left(\theta^1_{G/\{e_i,e_j\}}\right) \otimes \left[mot'_G, \left[\omega_G^{ij}\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} + 1 \otimes I^{\mathfrak{dr}}_G. \tag{3.24}$$

Corollary 3.2.2. *We can rewrite the coaction in terms of motivic logarithms as*

$$\begin{aligned} \Delta I^{\mathfrak{m}}_G = &I^{\mathfrak{m}}_G \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + 1 \otimes I^{\mathfrak{dr}}_G \\ &+ \sum_{1\leq i<j\leq 4} P_{i,j} \log^{\mathfrak{m}}\left(\left[p_0 p_1 | u^0_{\{i,j\}} u^1_{\{i,j\}}\right]\right) \otimes \left[mot'_G, \left[\omega_G^{ij}\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}}, \end{aligned}$$

where $P_{i,j} = \frac{1}{\sqrt{4|\det D_{i,j}|}}$, and $D_{i,j}$ is the matrix of the quadratic form $\Xi_{G/\{e_i,e_j\}}$.

Proof. We have

$$mot'_{G/\{e_i,e_j\}} \cong H^1(\mathbb{P}^1 \setminus \{u^0_{\{i,j\}}, u^1_{\{i,j\}}, \{p_0, p_1]\})_{/S}$$

where

$$\begin{aligned} \{u^0_{\{i,j\}}, u^1_{\{i,j\}}\} &= Q_{G/\{e_i,e_j\}} \cap \Delta_{\{i,j\}} \\ \{p_0, p_1\} &= \{[1:0], [0:1]\} \end{aligned} \tag{3.25}$$

and the form $\theta^1_{G/\{e_i,e_j\}}$ has simple poles at $\{u^0_{\{i,j\}}, u^1_{\{i,j\}}\}$. The result follows. □

Remark 3.2.1. One can combine the previous proposition with Proposition 3.1.4 to obtain the coaction on all one-loop graphs with $N_G \geq 4$ edges in $d = 4$ dimensions with non-vanishing masses and momenta.

3.2.2 de Rham side of the coaction

Next, we need to determine the de Rham side of the coaction, i.e. to interpret the objects $\left[mot'_G, \left[\omega^{ij}_G\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}}$.

3.2.2.1 Connection and the de Rham periods

The connection allows us to differentiate a section $[\omega]$ of $H^n_{\mathrm{dR}}(X, D)_{/S}$ with respect to parameters, which in the case of Feynman amplitudes are the masses and momenta. Consider a section $[\omega]$ of $\mathcal{V}_{\mathrm{dR}}$ on an open affine $U \subset S$ with coordinate q. We can compose the connection with the contraction by a vector field $\frac{\partial}{\partial q}$ to obtain a map:

$$\mathcal{V}_{\mathrm{dR}} \xrightarrow{\nabla} \mathcal{V}_{\mathrm{dR}} \otimes \Omega^1_{S/k} \xrightarrow{\frac{\partial}{\partial q}} \mathcal{V}_{\mathrm{dR}}$$

which sends $[\omega]$ to its first derivative with respect to q, denoted $\nabla_q(\omega)$. Since $\mathcal{V}_{\mathrm{dR}}$ has finite rank we will get a relation between

$$[\omega], \nabla_q([\omega]), \nabla^2_q([\omega]), ..., \nabla^k_q([\omega])$$

for some finite k. This is the Picard-Fuchs equation satisfied by ω and therefore by $\mathrm{per}([\mathcal{V}, [\sigma], [\omega]]^{\mathfrak{m}})$, for some cycle of integration $\sigma \in (\omega^X_B(\mathcal{V}))^\vee$, defined over some simply connected $X \subset S(\mathbb{C})$.

We use the Gauss–Manin connection on the vector bundle

$$(mot'_G)_{\mathrm{dR}} = H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}$$

This vector bundle sits in a long exact sequence of vector bundles:

$$\ldots \to H^2_{\mathrm{dR}}(\Delta_G \setminus Q \cap \Delta_G)_{/S} \to H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S} \to H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q)_{/S} \to \ldots,$$

where $H^k_{\mathrm{dR}}(\Delta_G \setminus Q \cap \Delta_G)_{/S}$ are objects of $\mathcal{H}(S)$ which are obtained by truncating the complexes (2.7) and (2.8) on the left so that the non-zero components are in $|J| \geq n-1$,and we denote the section of $H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ over the generic point which is the image of $[\omega_G]$ by the same symbol. Since $H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ is a vector bundle with connection of rank 1 we know that the Feynman integrand satisfies a relation $\nabla_{q_1}([\omega_G]) + B(m,q)[\omega_G] = 0$, where ∇_{q_1} is the Gauss–Manin connection on $H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ composed with contraction by the vector field $\partial/\partial q_1$, and $B(m,q) \in k_S$. This relation lifts to a relation of sections of $H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}$. We compute it explicitly in the following lemma:

Lemma 3.2.3. *Let q_1 be one of the momentum parameters of a one-loop four-edge integral $I_G(m,q)$ with non-vanishing masses and momenta. Then*

$$[\omega_G] \in \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}})$$

satisfies a relation of Picard-Fuchs type

$$\nabla_{q_1}([\omega_G]) + B(m,q)[\omega_G] = [d\beta] \tag{3.26}$$

where β is a section of $\Omega^2_{\mathbb{P}^3 \setminus Q/\mathrm{Spec}(k_S)}$, and $B(m,q) \in k_S$. Both β and $B(m,q)$ can be computed explicitly.

Proof. We use the general description of rational forms on $\mathbb{P}^n$ with poles along a hypersurface [30]. A 2-form on $\mathbb{P}^3$ with a pole along Ξ_G of order 2 is of the form:

$$\beta = \frac{\sum_{i<j} (-1)^{i+j} \alpha_j A_i - \alpha_i A_j}{\Xi_G^2} d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge \widehat{d\alpha_j} \wedge \ldots \wedge d\alpha_4 \tag{3.27}$$

for some linear polynomials A_i in the variables α_i. One then computes the exterior derivative

$$d\beta = -2\frac{\sum_i A_i \frac{\partial \Xi_G}{\partial \alpha_i}}{\Xi^3}\Omega_G + \frac{\sum_i \frac{\partial A_i}{\partial \alpha_i}}{\Xi^2}\Omega_G$$

In [30, Proposition 4.6] Griffiths makes a quite general observation that whenever we have a rational form $\frac{A}{F^k}\Omega$ such that $A \in J(F)$, where $J(F)$ is the Jacobian ideal generated by partial derivatives of F, we can reduce the order of the pole up to an exact form. To compute $\nabla_{q_1}([\omega_G])$ one differentiates ω_G with respect to q_1 and writes the resulting form in terms of a basis of sections of $H^3(\mathbb{P}^3 \setminus Q)_{/S}$, which we have chosen

to be $[\omega_G]$. We can use the Groebner basis of $J(\Xi_G)$ to find the A_i's such that we can reduce the pole of $\frac{\partial}{\partial q_1}(\omega_G)$:

$$\frac{\partial}{\partial q_1}(\omega_G) = \frac{-2(2(q_1+q_2)\alpha_2\alpha_4 + 2q_1\alpha_1\alpha_4)}{\Xi_G^3}\Omega_G = \frac{\sum_i A_i \frac{\partial \Xi_G}{\partial \alpha_i}}{\Xi_G^3}\Omega_G = \frac{1}{2}\frac{\sum_i \frac{\partial A_i}{\partial \alpha_i}}{\Xi_G^2}\Omega_G - \frac{1}{2}d\beta \tag{3.28}$$

Defining

$$B(m,q) = \frac{1}{2}\sum_i \frac{\partial A_i}{\partial \alpha_i} \tag{3.29}$$

which indeed depends only on masses and momenta since A_i are necessarily linear, gives us the stated relation. □

Proposition 3.2.4. *Let $C, D_{j,k}$ are the matrices associated to the quadratic forms Ξ_G, $\Xi_{G/\{e_j,e_k\}}$ respectively, and $U = C^{-1}$. Let*

$$f_{ij}(m,q) = \frac{\sqrt{(U)_{i,j}^2 - (U)_{i,i}(U)_{j,j}} - (U)_{i,j}}{\sqrt{(U)_{i,j}^2 - (U)_{i,i}(U)_{j,j}} + (U)_{i,j}},$$

and $P = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$. Then

$$\left[mot'_G, \left[\omega_G^{ij}\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} = P\log^{\mathfrak{dr}}(f_{ij})\mathbb{L}^{\mathfrak{dr}}. \tag{3.30}$$

Proof. Let G and β be as in the previous lemma. We first show that $[d\beta]$, viewed as a section of $H^3_{\mathrm{dR}}(\mathbb{P}^3 \setminus Q, \Delta_G \setminus Q \cap \Delta_G)_{/S}$ over the generic point, can be written as a sum of images under the face maps (2.4.2)

$$\Phi_{ij,dR} : \Gamma(\mathrm{Spec}(k_S), (mot'_{G/\{e_i,e_j\}})_{\mathrm{dR}}) \to \Gamma(\mathrm{Spec}(k_S), (mot'_G)_{\mathrm{dR}}).$$

We will always work over the generic point of the space of kinematics throughout this proof, and we drop the explicit reference to $\mathrm{Spec}(k_S)$ to ease the notation.

Because $\mathbb{P}^3 \setminus Q$ is affine $[d\beta]$ is represented by the cocycle of the total complex of the double complex:

$$(d\beta, 0, \ldots, 0) \in \bigoplus_{3=p+q} \bigoplus_{|I|=p} \Gamma(\Delta_I \setminus Q \cap \Delta_I, \Omega^q_{\Delta_I \setminus Q \cap \Delta_I})$$

Then

$$d^2_{Tot}(\beta, 0, \ldots, 0) = (d\beta, 0, \ldots, 0) + (0, \beta|_{\Delta_1}, \beta|_{\Delta_2}, \beta|_{\Delta_3}, \beta|_{\Delta_4}, 0, \ldots, 0) \tag{3.31}$$

where d^2_{Tot} is the differential of the total complex defined in (3.9). As sections of $(mot'_G)_{\mathrm{dR}}$ over the generic point, we have

$$[d\beta] = -[(0, \beta|_{\Delta_1}, \beta|_{\Delta_2}, \beta|_{\Delta_3}, \beta|_{\Delta_4}, 0, \dots, 0)] \tag{3.32}$$

Note that each $\beta|_{\Delta_i} \in \Gamma(\Delta_i \smallsetminus Q \cap \Delta_i, \Omega^2_{\Delta_i \smallsetminus Q \cap \Delta_i})$, but since $H^2(\Delta_i \smallsetminus Q) \cong H^2(\mathbb{P}^2 \smallsetminus Q) \cong 0$ by (3.1.1), there must exist forms $\beta_i \in \Gamma(\Delta_i \smallsetminus Q \cap \Delta_i, \Omega^1_{\Delta_i \smallsetminus Q \cap \Delta_i})$ such that $d\beta_i = \beta|_{\Delta_i}$. Using this fact we see that we can write for each β_i:

$$d^1_{Tot}(0, ..., 0, \beta_i, ..., 0) = (0, \dots, \beta|_{\Delta_i}, \dots, 0) - (0, ..., 0, r^i_j(\beta_i), r^i_k(\beta_i), r^i_l(\beta_i), 0, ..., 0) \tag{3.33}$$

where r^i_j is the restriction of differential forms on Δ_i to forms on Δ_j with appropriate signs. Combining (3.32) and (3.33) for each i we get:

$$[d\beta] = -\left[\sum_{0 \le i < j \le 3} (0, ..., r^i_j(\beta_i) + r^j_i(\beta_j), 0, ..., 0)\right] \tag{3.34}$$

Note that in the previous equation we get two contributions for each 1-face $\Delta_{\{i,j\}}$ — one from first restricting β to the 2-face Δ_i, taking the primitive of the restriction, then in turn restricting that primitive to the 1-face $\Delta_{\{i,j\}}$, and the other by the same procedure, but starting by restricting to Δ_j. If we write

$$\beta_{ij} = (r^i_j(\beta_i) + r^j_i(\beta_j), 0, 0) \in \bigoplus_{3=p+q} \bigoplus_{\substack{|I|=p \\ i,j \subset I}} \Gamma(\Delta_I \smallsetminus Q \cap \Delta_I, \Omega^q_{\Delta_I \smallsetminus Q \cap \Delta_I}) \tag{3.35}$$

where the latter is the double complex which computes

$$H^1(\Delta_{\{i,j\}} \smallsetminus Q \cap \Delta_{\{i,j\}}, \Delta_{\{k,l\}} \cap \Delta_{\{i,j\}}),$$

for i, j, k, l pairwise distinct. Therefore we proved: $[d\beta] = -\sum_{1 \le i < j \le 4} \Phi_{ij}([\beta_{ij}])$.

Since $[\beta_{ij}]$ is a section of $(mot'_{G/\{e_i,e_j\}})_{\mathrm{dR}})$ over the generic point, we can write

$$[\beta_{ij}] = a_{ij}[\theta^1_{G/\{e_i,e_j\}}],$$

where $a_{ij} \in k_S$. For each $1 \le i < j \le 4$ we have equalities of motivic families of motivic periods:

$$\begin{aligned}
&\left[mot'_G, \left[\omega^{ij}_G\right]^\vee, \nabla_{q_1}([\omega_G])\right]^{\mathfrak{dr}} \left[mot'_G, \left[\omega^{ij}_G\right]^\vee, [B(m,q)\omega_G - \frac{1}{2}d\beta]\right]^{\mathfrak{dr}} \\
&= \left[mot'_G, \left[\omega^{ij}_G\right]^\vee, \left([B(m,q)\omega_G] + \frac{1}{2}\sum_{1 \le i < j \le 4} \Phi_{ij,dR}([\beta_{ij}])\right)\right]^{\mathfrak{dr}} \\
&= B(m,q)\left[mot'_G, \left[\omega^{ij}_G\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} + \frac{a_{ij}}{2}\left[mot'_{G/\{e_i,e_j\}}, \left[\theta^1_{G/\{e_i,e_j\}}\right]^\vee, \left[\theta^1_{G/\{e_i,e_j\}}\right]\right]^{\mathfrak{dr}} \\
&= B(m,q)\left[mot'_{G/\{e_i,e_j\}}, \left[\omega^{ij}_G\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} + \frac{a_{ij}}{2}\mathbb{L}^{\mathfrak{dr}}
\end{aligned} \tag{3.36}$$

where the second to last equality holds because $\Phi_{ij,dR}\left(\left[\theta^1_{G/\{e_i,e_j\}}\right]\right) = \left[\omega^{ij}_G\right]$.

On the other hand one can check that

$$\nabla_x(P\log^{\mathfrak{dr}}(x)\mathbb{L}^{\mathfrak{dr}}) = \frac{\partial}{\partial x}(P)\log^{\mathfrak{dr}}(x)\mathbb{L}^{\mathfrak{dr}} + P\frac{1}{x}\mathbb{L}^{\mathfrak{dr}}.$$

Comparing this expression with (3.36), we see that $a_{ij} = 2P\frac{1}{f_{ij}}\frac{\partial f_{ij}}{\partial q_1}$, and we obtain the result up to a constant:

$$\left[mot'_G, \left[\omega^{ij}_G\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} = P\log^{\mathfrak{dr}}(cf_{ij})$$

The constant c can be determined to be 1 by specializing to a convenient point in the space of kinematics. □

Remark 3.2.2. Note that an equivalence of de Rham periods induced by a face map $\Phi_{ij,dR}$ in the previous proposition is a version of Stokes' theorem for de Rham periods.

Putting Proposition 3.2.1 and Proposition 3.2.4 together we get

Theorem 3.2.5. *Let G be the 4-edge one-loop Feynman graph with generic non-vanishing masses and momenta. Let $I^{\mathfrak{m}}_G$ be its associated motivic Feynman amplitude in $d = 4$ dimensions. Then the motivic Galois coaction on $I^{\mathfrak{m}}_G$ is:*

$$\Delta I^{\mathfrak{m}}_G = I^{\mathfrak{m}}_G \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + \sum_{1\le j<k\le 4} I^{\mathfrak{m}}_{G/\{e_j,e_k\}}(\theta^1_{G/\{e_j,e_k\}}) \otimes P_{j,k}\log^{\mathfrak{dr}}(f_{j,k})\mathbb{L}^{\mathfrak{dr}} + 1 \otimes I^{\mathfrak{dr}}_G, \tag{3.37}$$

where $I^{\mathfrak{m}}_{G/\{e_j,e_k\}}\left(\theta^1_{G/\{e_j,e_k\}}\right)$ is the motivic Feynman amplitude of the bubble graph obtained by contracting the edges e_j and e_k of G in $d = 2$ dimensions,

$$f_{j,k} = \frac{\sqrt{(U)^2_{j,k} - (U)_{j,j}(U)_{k,k}} - (U)_{j,k}}{\sqrt{(U)^2_{j,k} - (U)_{j,j}(U)_{k,k}} + (U)_{j,k}}, \tag{3.38}$$

where $C, D_{j,k}$ are the matrices of the quadratic forms Ξ_G, $\Xi_{G/\{e_j,e_k\}}$ (3.20) respectively, $U = C^{-1}$, and $P_{j,k} = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$.

3.2.2.2 Residues and the de Rham projection

Since $\left[\omega^{ij}_G\right]$ and $[\omega_G]$ are sections over the generic point of

$$H^3_{\mathrm{dR}}(\mathbb{P}^3 \smallsetminus Q, (\Delta_i \cup \Delta_j) \smallsetminus Q \cap (\Delta_i \cup \Delta_j))_{/S},$$

and we are interested in the de Rham period $\left[mot'_G, \left[\omega^{ij}_G\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}}$, which only depends on $\Delta_i \cup \Delta_j \subset \Delta_G$, we can restrict our attention to the fiber over the generic point of

$$H^3(\mathbb{P}^3 \smallsetminus Q, (\Delta_i \cup \Delta_j) \smallsetminus Q \cap (\Delta_i \cup \Delta_j))_{/S}.$$

Consider the residue map:

$$H^3(\mathbb{P}^3 \setminus Q, (\Delta_i \cup \Delta_j) \setminus Q \cap (\Delta_i \cup \Delta_j)) \xrightarrow{res} H^2(Q, (\Delta_i \cup \Delta_j) \cap Q)(-1),$$

where we have restricted to the fiber over the generic point. Denote the former by H'_G, and the latter by H_Q. Consider the motivic period:

$$[H'_G, [\mathrm{Tube}(\sigma_{Lune})], [\omega_G]]^{\mathfrak{m}} = [H_Q, [\sigma_{Lune}], [res(\omega_G)(-1)]]^{\mathfrak{m}}$$

where σ_{Lune} is the spherical lune cut out by the two hyperplanes Δ_i and Δ_j on $Q(\mathbb{C})$, the '(-1)' notation in $[res(\omega_G)(-1)]$ refers to the Tate twist implying that we must multiply by $2\pi i$ when taking the period, and Tube denotes the tubular neighbourhood homomorphism. The image of $[H_Q, [\sigma_{Lune}], [res(\omega_G)(-1)]]^{\mathfrak{m}}$ under the period homomorphism can be computed in spherical coordinates (see [9, 5.3.3]). Since we are working with a smooth quadric we can reduce to the case of a sphere $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \alpha_4^2$, where one checks that the corresponding period is $\frac{1}{2i}\theta_{ij}$, where θ_{ij} is the angle between Δ_i and Δ_j.

To show the relation of this computation with the de Rham period we are interested in, one should show that $res^\vee\left(\left[\omega_Q^{ij}\right]^\vee\right) = \left[\omega_G^{ij}\right]^\vee$, where $res^\vee$ is the dual of the residue, and $c_0^\vee([\sigma_{Lune}]) = [\omega_Q^{ij}]$ (see 2.4.3 for the definition of the morphism $c_0^\vee$). Therefore, $\left[H_Q, \left[\omega_Q^{ij}\right]^\vee, [res(\omega_G)(-1)]\right]^{\mathfrak{dr}}$ is the image under the de Rham projection of $[H_Q, [\sigma_{Lune}], [res(\omega_G)(-1)]]^{\mathfrak{m}}$. If the above statement about $\left[\omega_G^{ij}\right]^\vee$ and $\left[\omega_Q^{ij}\right]^\vee$ is true, we have an equivalence of de Rham periods $\left[H_Q, \left[\omega_Q^{ij}\right]^\vee, [res(\omega_G)(-1)]\right]^{\mathfrak{dr}}$ and $\left[H'_G, \left[\omega_G^{ij}\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}}$, and have therefore associated a motivic period to the de Rham period we are interested in.

3.2.2.3 Comparing with an analytic expression for the Feynman integral

We are going to apply the motivic coaction to an expression for the Feynman integral of the 1-loop 4-edge graph in 4 dimensions as it appears in the physics literature [38, Proposition 8.], and check that it matches the coaction computed in the previous proposition. As a consequence, we will see that the coaction is both shorter and more symmetrical than the full expression of the Feynman integral in terms of dilogarithms as found in [38]. For a classical reference for a geometric approach to one-loop integrals and dilogarithms see also [21].

The expression given in [38] for I_G is as follows: let C be the 4×4 matrix associated to the quadratic form Ξ_G, and $U = C^{-1}$. Let

$$\begin{aligned}
\nu^0(r,s,t) &= \arctan\left(\frac{C_{t,4}U_{r,4}\sqrt{|\det U|}}{U_{r,s}U_{r,4}-U_{r,r}U_{s,4}}\right)\\
\nu^1(r,s,t) &= \arctan\left(\frac{C_{t,4}U_{r,4}\sqrt{|\det U|}}{(U_{r,s}U_{r,4}-U_{r,r}U_{s,4})\sqrt{1-C_{4,4}U_{4,4}}}\right)\\
\nu^2(r,s,t) &= \arctan\left(\frac{U_{r,4}\sqrt{U_{r,r}U_{s,s}-U_{r,s}^2}}{U_{r,s}U_{r,4}-U_{r,r}U_{s,4}}\right)\\
\nu^3(r,s,t) &= \arctan\left(\frac{U_{r,4}}{\sqrt{U_{r,r}U_{4,4}-U_{r,4}^2}}\right)
\end{aligned} \tag{3.39}$$

then the family of periods, depending on masses and momenta, of interest is:

$$\begin{aligned}
I_G = \frac{1}{16\sqrt{|\det C|}}\sum_{\{r,s,t\}\in S_3}&\big(2\,\mathrm{Im}\,\mathrm{Li}_2(\exp(2\nu^0(r,s,t)))+\\
&+\sum_{l=1}^{3}(-1)^l[\mathrm{Im}\,\mathrm{Li}_2(\exp(2\nu^0(r,s,t)+2\nu^l(r,s,t)))\\
&+\mathrm{Im}\,\mathrm{Li}_2(\exp(2\nu^0(r,s,t)-2\nu^l(r,s,t)))]\big).
\end{aligned} \tag{3.40}$$

Note that the above expression consists of a linear combination of 42 dilogarithms. We replace the dilogarithms with their motivic versions, and apply the coaction to each motivic dilogarithm, which reads:

$$\Delta\mathrm{Li}_2^{\mathfrak{m}}(x) = \mathrm{Li}_2^{\mathfrak{m}}(x)\otimes\mathbb{L}^{\mathfrak{dr}} + \mathrm{Li}_1^{\mathfrak{m}}(x)\otimes\log^{\mathfrak{dr}}(x)\mathbb{L}^{\mathfrak{dr}} + 1\otimes\mathrm{Li}_2^{\mathfrak{dr}}(x).$$

Taking care to note that the arguments are all on the unit circle, we get a linear combination of 42 terms of the form

$$\begin{aligned}
&\Delta\,\mathrm{Im}(\mathrm{Li}_2^{\mathfrak{m}}(z)) = \Delta\frac{1}{2i}\left(\mathrm{Li}_2^{\mathfrak{m}}(z)-\mathrm{Li}_2^{\mathfrak{m}}(\bar z)\right)\\
&=\frac{1}{2i}((\mathrm{Li}_2^{\mathfrak{m}}(z)-\mathrm{Li}_2^{\mathfrak{m}}(\bar z))\otimes(\mathbb{L}^{\mathfrak{dr}})^2+\mathrm{Li}_1^{\mathfrak{m}}(z)\otimes\log^{\mathfrak{dr}}(z)\mathbb{L}^{\mathrm{dR}}-\\
&-\mathrm{Li}_1^{\mathfrak{m}}\left(\frac{1}{z}\right)\otimes\log^{\mathfrak{dr}}\left(\frac{1}{z}\right)\mathbb{L}^{\mathrm{dR}}+1\otimes\left(\mathrm{Li}_2^{\mathfrak{dr}}(z)-\mathrm{Li}_2^{\mathfrak{dr}}(\bar z)\right))\\
&=\frac{1}{2i}((\mathrm{Li}_2^{\mathfrak{m}}(z)-\mathrm{Li}_2^{\mathfrak{m}}(\bar z))\otimes(\mathbb{L}^{\mathfrak{dr}})^2+\log^{\mathfrak{m}}\left(\frac{(1-z)^2}{z}\right)\otimes\log^{\mathfrak{dr}}(z)\mathbb{L}^{\mathrm{dR}}+\\
&+1\otimes\left(\mathrm{Li}_2^{\mathfrak{dr}}(z)-\mathrm{Li}_2^{\mathfrak{dr}}(\bar z)\right))
\end{aligned} \tag{3.41}$$

Notice that in Theorem 3.2.5 we have 6 terms of the form $\log^{\mathfrak{m}}(f_i)\otimes\log^{\mathfrak{dr}}(g_i)$ (up to prefactors), while in the coaction (3.41) on (3.40) we have 42 such terms.

All the computations below were done in Maple. In order to check that the two expressions are equivalent, we take the 6 motivic logarithms from Theorem 3.2.5 and the 42 motivic logarithms from the coaction on (3.40), and we find a basis for these 48 functions which includes the 6 motivic logarithms in Theorem 3.2.5. One can do this by applying the LLL algorithm to a matrix whose entries are the evaluations of the (q_1-derivative) of the 48 motivic logarithms at sufficiently many points in the space of generic kinematics. The basis found contains 27 motivic logarithms. One then checks that the relations found indeed hold on the level of the logarithms themselves and expresses the non-basis elements in terms of the basis. Plugging this back into the original expression for the coaction on $I^{\mathfrak{m}}_G$ in (3.40) and collecting the de Rham logarithms with each of the 27 basis motivic logarithms, one then repeats the procedure of finding a basis for the de Rham logarithms one is left with on the right hand side of the tensor product. The basis on the de Rham side contains 20 logarithms. Expressing the non-basis de Rham logarithms in terms of these 20 and plugging this back into the previous expression with the motivic side reduced to 27 terms one observes that everything cancels out but 6 terms:

$$\sum_{1\leq j<k\leq 4} 2\log^{\mathfrak{m}}\left(\left[p_0p_1|u^0_{\{j,k\}}u^1_{\{j,k\}}\right]\right)\otimes\log^{\mathfrak{dr}}\left(\frac{\sqrt{(U)^2_{j,k}-(U)_{j,j}(U)_{k,k}}-(U)_{j,k}}{\sqrt{(U)^2_{j,k}-(U)_{j,j}(U)_{k,k}}+(U)_{j,k}}\right)\mathbb{L}^{\mathfrak{dr}} \quad (3.42)$$

If $a_{jk}(m,q)$ are computed as in the previous proposition, and $D_{j,k}$ is the matrix associated to the quadratic form $\Xi_{G/\{e_j,e_k\}}$, we can check that

$$a_{jk}(m,q)=\frac{\sqrt{|\det D_{j,k}|}}{4\sqrt{|\det C|}}\frac{1}{f_{j,k}}\frac{\partial f_{j,k}}{\partial q_1}.$$

As another check one can observe that $\frac{\partial}{\partial q_1}\left(\frac{1}{16\sqrt{|\det C|}}\right)=B(m,q)$, where $B(m,q)$ is computed as in the Lemma 3.2.3.

3.3 The triangle graph: motives and coaction

In this section we study the motives and coaction of the triangle graph, both with non-vanishing and vanishing masses.

3.3.1 The triangle graph with non-vanishing masses

3.3.1.1 Motive of the triangle graph with non-vanishing masses

In the case of the triangle graph we must work with the full graph motive because there is a linear part as well as the quadric in the polar locus of the integrand for

$d = 4$ dimensions. We start with the case when $F = M = N_G = 3$, i.e., all masses and momenta are non-vanishing. In this case the motive of interest is:

$$mot_G = H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta_G \setminus (Q \cup L) \cap \Delta_G)_{/S}$$

We will need the following lemma to compute the semi-simplification of mot_G. We will make use of the *Gysin spectral sequence* [23]. Let D be a simple normal crossing divisor in X, and let $D_I = D_{i_1} \cap ... \cap D_{i_k}$ where D_i are the irreducible components of D, and $I = \{i_1, \ldots, i_k\}$, finally let $D_\varnothing = X$. Then we have the following spectral sequence:

$$E_1^{-p,q} = \bigoplus_{|I|=p} H^{q-2p}(D_I)(-p) \Rightarrow H^{-p+q}(X \setminus D) \tag{3.43}$$

where the $d_1^{-p,q} : E_1^{-p,q} \to E_1^{-p+1,q}$ is the alternating sum of Gysin homomorphisms

$$H^{q-2p}(D_I)(-1) \to H^{q-2p+2}(D_{I \setminus \{i\}})$$

for each $i \in I$, multiplied by the appropriate sign.

Lemma 3.3.1. *Let $Q \subset \mathbb{P}^2$ be a smooth quadric, and $L \subset \mathbb{P}^2$ be a projective line. Then*

$$H^1(\mathbb{P}^2 \setminus (Q \cup L)) \cong \mathbb{Q}(-1) \quad \textit{and} \quad H^2(\mathbb{P}^2 \setminus (Q \cup L)) \cong \mathbb{Q}(-2).$$

Proof. Next we need to compute $H^1(\mathbb{P}^2 \setminus (Q \cup L))$ and $H^2(\mathbb{P}^2 \setminus (Q \cup L))$, which we can do using the Gysin spectral sequence. Let us compute the relevant part of the first page. We will need the following elements:

$$\begin{aligned}
E_1^{-2,4} &\cong H^0(Q \cap L)(-2) \cong \mathbb{Q}(-2)^{\oplus 2} \\
E_1^{-1,4} &\cong H^2(Q)(-1) \oplus H^2(L)(-1) \cong \mathbb{Q}(-2)^{\oplus 2} \\
E_1^{0,4} &\cong H^4(\mathbb{P}^2) \cong \mathbb{Q}(-2) \\
E_1^{-1,2} &\cong H^0(Q)(-1) \oplus H^0(L)(-1) \cong \mathbb{Q}(-2)^{\oplus 2} \\
E_1^{0,2} &\cong H^2(\mathbb{P}^2) \cong \mathbb{Q}(-1) \\
E_1^{0,0} &\cong H^0(\mathbb{P}^2) \cong \mathbb{Q}(0)
\end{aligned} \tag{3.44}$$

From this we can see that the first page of the spectral sequence is

$$\begin{array}{ccccc}
0 & \to & \mathbb{Q}(-2)^{\oplus 2} & \to & \mathbb{Q}(-2)^{\oplus 2} \to \mathbb{Q}(-2) \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow 0 \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q}(-1)^{\oplus 2} \to \mathbb{Q}(-1) \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \mathbb{Q}(0)
\end{array} \tag{3.45}$$

Let v_L and v_Q be generators of $H^0(L)(-1)$ and $H^0(Q)(-1)$ respectively. Then on the second row we have the Gysin morphisms

$$H^0(L)(-1) \xrightarrow{Gysin} H^2(\mathbb{P}^2) \quad \text{, and} \quad H^0(Q)(-1) \xrightarrow{Gysin} H^2(\mathbb{P}^2),$$

the images of which are the fundamental classes $G(v_L) = [L]$, $G(v_Q) = [Q]$, respectively. Moreover, $[L]$ generates $H^2(\mathbb{P}^2)$, and $[Q] = 2[L]$, since Q intersects L in two points, generically. Therefore, the morphism $\mathbb{Q}(-1)^{\oplus 2} \to \mathbb{Q}(-1)$ is surjective, and the kernel is generated by $(2v_L, -v_Q)$. Hence $gr_2^W H^2(\mathbb{P}^2 \setminus Q \cup L) \cong 0$ and $gr_2^W H^1(\mathbb{P}^2 \setminus Q \cup L) \cong \mathbb{Q}(-1)$.

Finally, $gr_4^W H^2(\mathbb{P}^2 \setminus Q \cup L)$ is computed by the kernel of the map $E^1_{-2,4} \to E^1_{-1,4}$. The kernel of $H^0(Q \cap L)(-2) \to H^2(L)(-1)$ is one-dimensional because $H^2(L \setminus Q \cap L) = 0$, and similarly the kernel of the other Gysin map $H^0(Q \cap L)(-2) \to H^2(Q)(-1)$ is one-dimensional. Hence $gr_4^W H^2(\mathbb{P}^2 \setminus Q \cup L) \cong \mathbb{Q}(-2)$. □

Lemma 3.3.2. *The weight graded pieces of the one-loop triangle graph motive mot_G with non-vanishing masses and momenta are*

$$gr^W mot_G \cong \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 5} \oplus \mathbb{Q}(0)_{/S}$$

Proof. It is enough to prove the claim over any fibre $t \in S(\mathbb{C})$. Let $\Delta'_I = \Delta_I \setminus (Q \cup L) \cap \Delta_I$. Then the E_1 page of the relative cohomology spectral sequence reads

$$\begin{aligned}
&H^2(\mathbb{P}^2 \setminus (Q \cup L)) \longrightarrow \oplus_{|I|=1} H^2(\Delta'_I) \longrightarrow \oplus_{|I|=2} H^2(\Delta'_I) \\
&H^1(\mathbb{P}^2 \setminus (Q \cup L)) \longrightarrow \oplus_{|I|=1} H^1(\Delta'_I) \longrightarrow \oplus_{|I|=2} H^1(\Delta'_I) \\
&H^0(\mathbb{P}^2 \setminus (Q \cup L)) \longrightarrow \oplus_{|I|=1} H^0(\Delta'_I) \longrightarrow \oplus_{|I|=2} H^0(\Delta'_I)
\end{aligned} \tag{3.46}$$

To compute this, we note that

$$H^1(\Delta_i \setminus (Q \cup L) \cap \Delta_i) \cong H^1(\mathbb{P}^1 \setminus \{u_{1_i}, u_{2_i}, l_{1_i}\}) \cong \mathbb{Q}(-1)^{\oplus 2},$$

where $u_{1_i}, u_{2_i}, l_{1_i}$ are the points of intersection of the quadric Q and the line L with the face $\Delta_i \hookrightarrow \Delta_G$. From the previous lemma we get the leftmost column, and note that $E_1^{1,2} = E_1^{2,1} = E_1^{1,2} = E_1^{2,2} = 0$. Taking cohomology of the rows we get the result. □

3.3.1.2 Coaction on the triangle graph with non-vanishing masses and momenta

Notice from the proofs of the previous two lemmas that all the motivic periods of the triangle graph motive in the case of non-vanishing masses, except the one of weight 4 and one of weight 0, are equivalent to motivic periods of the motives associated to the faces of Δ_G via the face maps, as in (3.2.1). Note that G/e_i, for $i \in \{1,2,3\}$, are bubble graphs with non-vanishing masses. In this case we consider the full motive of the bubble graph mot_{G/e_i} (3.16) for all i. The fiber over the generic point of $(gr_2^W mot_{G/e_i})_{\mathrm{dR}}$ is of rank 2 and we can choose its basis to be the classes of the two forms:

$$\begin{aligned} \theta^1_{G/e_i} &= \frac{\alpha_k d\alpha_j - \alpha_j d\alpha_k}{\Xi_{G/e_i}} \\ \theta^2_{G/e_i} &= \frac{(x-y)(\alpha_k d\alpha_j - \alpha_j d\alpha_k)}{(\alpha_k - x\alpha_j)(\alpha_k - y\alpha_j)} \end{aligned} \tag{3.47}$$

where i,j,k are pairwise distinct and x,y are coordinates of points $u_1 \in Q \cap \Delta_I$ and $L|_{\Delta_i}$ respectively in the coordinate chart of Δ_i where $\alpha_j \neq 0$.

We can choose a basis of the fiber of $(mot_G)_{\mathrm{dR}}$ over the generic point such that it contains the class of the Feynman form of G in 4 dimensions $[\omega_G]$, and the 5 weight 2 classes denoted by $\left[\omega_i^j\right]$, which we define to be the images of $\left[\theta^j_{G/e_i}\right]$ under the de Rham component of the face map (2.4.2):

$$\begin{aligned} \Phi_{i,dR} : (mot_{G/e_i})_{\mathrm{dR}} &\to (mot_G)_{\mathrm{dR}} \\ \left[\theta^j_{G/e_i}\right] &\mapsto \left[\omega_i^j\right] \end{aligned} \tag{3.48}$$

Note that there are 6 such classes $\left[\omega_i^j\right]$, two for each face Δ_i, but there is a relation between them, as can be seen in the proof of Lemma 3.3.2.

Proposition 3.3.3. *Let G be a triangle graph with non-vanishing generic masses and momenta. Then the motivic Galois coaction on the associated motivic Feynman amplitude is:*

$$\begin{aligned} \Delta I^{\mathfrak{m}}_G = {} & I^{\mathfrak{m}}_{G/e_1}\left(\theta^1_{G/e_1}\right) \otimes \left[mot_G, \left[\omega_1^1\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} \\ & + \sum_{i=2,3}\sum_{j=1,2} I^{\mathfrak{m}}_{G/e_i}\left(\theta^j_{G/e_i}\right) \otimes \left[mot_G, \left[\omega_i^j\right]^\vee, [\omega_G]\right]^{\mathfrak{dr}} \\ & + I^{\mathfrak{m}}_G \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + 1 \otimes I^{\mathfrak{dr}}_G \end{aligned} \tag{3.49}$$

Remark 3.3.1. Both the motivic and the de Rham side of the coaction in the previous proposition can be expressed in terms of motivic and de Rham logarithms respectively, using the same techniques as in the previous section.

3.3.2 Triangle graph with vanishing masses

3.3.2.1 Motive

When one of the masses m_i vanishes, i.e., when we have $F = 3$ and $M = 2$, the quadric Q passes through a point of Δ_G defined by the vanishing of the coordinates corresponding to the other two edges $V(\alpha_j) \cap V(\alpha_k)$, where i, j, k are pairwise distinct. In the following figure, we have chosen $m_1 = 0$:

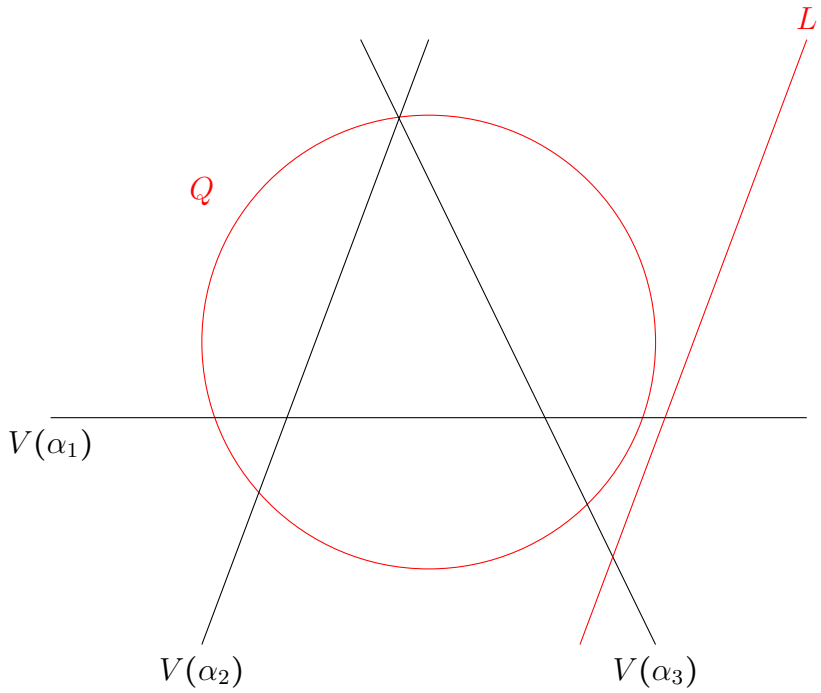


Figure 3.5: The geometry underlying the triangle graph with one vanishing mass.

This means that the poles of the integrand meet the boundary of the domain of integration, and $\Delta_G \cup Q \cup L$ is not simple normal crossing anymore, over each fiber of $K^{gen}_{F,M}$. Thus we cannot realize the relevant Feynman integral of the graph as a family of motivic periods of the motive $H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta_G \setminus (Q \cup L) \cap \Delta_G)_{/S}$. To remedy the situation we must blow up along $V(\alpha_j) \cap V(\alpha_k) \subset \mathbb{P}^2$. The point $V(\alpha_j) \cap V(\alpha_k)$ corresponds to a motic subgraph γ (see definition 2.5.6) spanned by the edges $\{e_j, e_k\}$ of our triangle graph G.

Let $\pi_G : P^G \to \mathbb{P}^2$ for the blow-up of $\mathbb{P}^2$ at the point $V(\alpha_3) \cap V(\alpha_2) = [1 : 0 : 0]$. Denote by $\widetilde{Q}, \widetilde{L}$ the strict transforms of Q and L where these are given by the vanishing of:

$$\begin{aligned} &\Psi_G = \alpha_1 + \alpha_2 + \alpha_3, \text{ and} \\ &\Xi_G(m, q) = q_1^2\alpha_2\alpha_3 + q_2^2\alpha_1\alpha_3 + q_3^2\alpha_1\alpha_2 + (m_2^2\alpha_2 + m_3^2\alpha_3)\Psi_G. \end{aligned} \tag{3.50}$$

Let $D = \pi_G^{-1}(\Delta_G)$ be the total transform of Δ_G, and denote by D_{-1} the exceptional divisor corresponding to the motic subgraph γ (see (3.6)). By the results recalled in 2.5 we get

$$[\widetilde{\sigma_G}] \in \Gamma(U_{2,3}^{gen}, (mot_G)_B^\vee), \text{ where}$$
$$mot_G = (H_B^2(P^G \smallsetminus \widetilde{Q} \cup \widetilde{L}, D \smallsetminus (\widetilde{Q} \cup \widetilde{L}) \cap D)_{/S}, H_{\mathrm{dR}}^2(P^G \smallsetminus \widetilde{Q} \cup \widetilde{L}, D \smallsetminus (\widetilde{Q} \cup \widetilde{L}) \cap D)_{/S}, c). \tag{3.51}$$

Lemma 3.3.4. *$\pi_G^*(\omega_G(m,q))$ does not have any poles along the exceptional divisor D_{-1}. Hence $\pi_G^*(\omega_G(m,q))$ is a global section of $\Omega^2_{P^G \smallsetminus \widetilde{Q} \cup \widetilde{L}/k_S}$, and defines a class $[\pi_G^*(\omega_G(m,q))] \in (mot_G)_{\mathrm{dR}}$.*

Proof. We can consider the following affine charts of P^G: let $\mathbb{A}_{23,1}$ be the affine space with the coordinate ring $\mathcal{O}(\mathbb{A}_{23,1}) = \mathbb{Z}[\beta_1^{23,1}, \beta_2^{23,1}]$, where, by abuse of notation, $\beta_1^{23,1} = \frac{\alpha_2}{\alpha_3}$, and $\beta_2^{23,1} = \alpha_3$, and $\alpha_1 = 1$. The exceptional divisor, in this affine chart, is given by $\beta_2^{23,1} = 0$. One analogously defines $\mathbb{A}_{23,2}$, by $\beta_1^{23,2} = \alpha_2$, and $\beta_2^{23,2} = \frac{\alpha_3}{\alpha_2}$, where $\alpha_1 = 1$. Two more affine spaces, given by $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, away from the exceptional divisor, complete a covering of P^G.

Let us see what $\pi_G^*(\omega_G(m,q))$ looks like in $\mathbb{A}_{23,1}$. The differential form Ω_G is pulled back to $\pi_G^*(\Omega_G) = d(\beta_1^{23,1}\beta_2^{23,1})d\beta_2^{23,1} = \beta_2^{23,1} d\beta_1^{23,1}\beta_2^{23,1}$. We also have

$$\begin{aligned} \Psi_{23} &= \pi_G^*(\Psi_G) = \beta_2^{23,1}\beta_1^{23,1} + \beta_2^{23,1} + 1 \\ \Xi_{23} &= \pi_G^*(\Xi_G) = \beta_2^{23,1}(q_1^2\beta_1^{23,1}\beta_2^{23,1} + q_2^2 + q_3^2\beta_1^{23,1} + (m_2^2\beta_1^{23,1} + m_3^2)\Psi_{23}) \end{aligned} \tag{3.52}$$

and therefore

$$\pi_G^*(\omega_G(m,q)) = \left(\frac{\beta_2^{23,1} d\beta_1^{23,1}\beta_2^{23,1}}{\Psi_{23}\Xi_{23}}\right)$$

has no poles along $\beta_2^{23,1} = 0$ since it cancels out. The result follows.

□

When either of the other two masses, m_2 or m_3, vanishes, we have to blow up points $V(\alpha_1) \cap V(\alpha_3)$, and $V(\alpha_1) \cap V(\alpha_2)$ respectively. Denote the corresponding exceptional divisors D_{-2}, D_{-3}. The proof of the previous lemma proceeds analogously, for each of the exceptional divisors. Having done this we get the associated motivic Feynman amplitude:

$$[mot_G, [\sigma_G], [\pi^*(\omega_G)]]^{\mathfrak{m}} \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, U_{k,3}^{gen}, gen},$$

where $1 \leq k < 3$ is the number of non-vanishing masses.

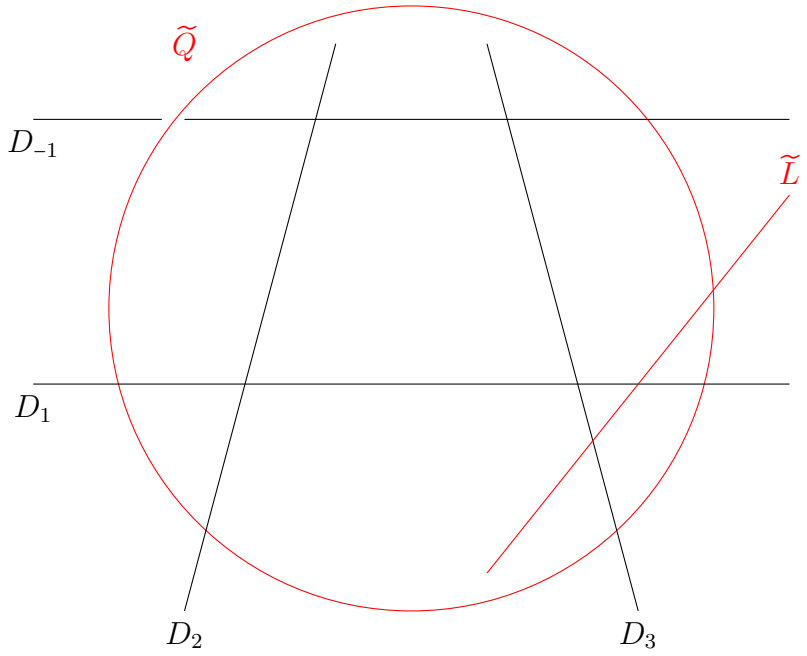


Figure 3.6: The geometry underlying the triangle graph with one vanishing mass after blowing up.

Lemma 3.3.5. *Let G be a 1-loop graph with 3 internal edges. Let $1 \leq v \leq 3$ be the number of vanishing masses. Then*

$$gr^W mot_G \cong \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 5-v} \oplus \mathbb{Q}(0)_{/S}$$

Proof. Let us see how one proceeds in the case of one vanishing mass. Without loss of generality let that mass be $m_1 = 0$.

We apply the relative cohomology spectral sequence at each fiber $t \in S(\mathbb{C})$:

$$H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}), D \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D)$$

Its first page reads:

$$\begin{aligned}
&H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^2(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^2(D_{\{i,j\}} \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_{\{i,j\}}) \\
&H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^1(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^1(D_{\{i,j\}} \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_{\{i,j\}}) \\
&H^0(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^0(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i) \longrightarrow \bigoplus_{i\in\{-1,1,2,3\}} H^0(D_{\{i,j\}} \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_{\{i,j\}})
\end{aligned} \tag{3.53}$$

We use (3.43) to compute $H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}))$ and $H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L}))$.

$$\begin{array}{l} \ldots \to H^2(\widetilde{Q})(-1) \oplus H^2(\widetilde{L})(-1) \to H^4(P^G) \\ \ldots \longrightarrow 0 \longrightarrow 0 \\ \ldots \to H^0(\widetilde{Q})(-1) \oplus H^0(\widetilde{L})(-1) \to H^2(P^G) \\ \ldots \longrightarrow 0 \longrightarrow 0 \\ \ldots \longrightarrow 0 \longrightarrow H^0(P^G) \end{array} \tag{3.54}$$

We have that $H^2(P^G)$ is generated by $[\widetilde{L}], [D_{-1}]$, and the image of the Gysin morphism $H^0(\widetilde{Q})(-1) \to H^2(P^G)$ is $[\tilde{Q}] = 2[\widetilde{L}] - [D_{-1}]$ because the intersection of $\tilde{Q} \cdot D_{-1} = 1$ and $\tilde{Q} \cdot L = 2$. The image of $H^0(\widetilde{L})(-1) \to H^2(P^G)$ is simply $[\widetilde{L}]$, so the difference of these two morphisms, which is the map, $E^1_{-1,2} \to E^1_{0,2}$, is an isomprhism. Therefore $gr_2^W H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) = 0$ and $gr_2^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) = 0$. In the top row nothing changes by blowing up, hence the computation is the same as in the previous lemma. We get $H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong \mathbb{Q}(-2)$ and $H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong 0$.

Next, we compute $\bigoplus_{i \in \{-1,1,2,3\}} H^1(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i)$. For the exceptional divisor we have $D_{-1} \cap (\widetilde{Q} \cup \widetilde{L}) = \{1 \text{ point}\}$, therefore $H^1(D_{-1} \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_{-1}) \cong H^1(\mathbb{A}^1) = 0$. For $D_1 = \widetilde{V(\alpha_1)}$, since it is away from the exceptional divisor, we have that $D_1 \cap (\widetilde{Q} \cup \widetilde{L}) \cong D_1 \cap (Q \cup L) = \{3 \text{ points}\}$, therefore $H^1(D_1 \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_1) \cong \mathbb{Q}(-1)^{\oplus 2}$. For the remaining two irreducible components D_3 and D_2 we have $D_{3/2} \cap (\widetilde{Q} \cup \widetilde{L}) = \{2 \text{ points}\}$, therefore $H^1(D_{3/2} \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_{3/2}) \cong \mathbb{Q}(-1)$. Putting this together into (3.53) we get

$$\mathrm{gr}^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}), D \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 4} \bigoplus \mathbb{Q}(0).$$

When another mass vanishes, say m_2, in the spectral sequence (3.54) we get $H^2(P^G) \cong \mathbb{Q}(-1)^{\oplus 3}$, generated by $[\widetilde{L}], [D_{-1}], [D_{-2}]$. The image of the Gysin morphism $H^0(\widetilde{Q})(-1) \to H^2(P^G)$ is generated by $2[\widetilde{L}] - [D_{-1}] - [D_{-2}]$. Therefore the rank of the cokernel of the morpshim $E^1_{-1,2} \to E^1_{0,2}$ is of rank 1, and the kernel remains trivial. Hence we have $gr_2^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong \mathbb{Q}(-1)$. Everything else remains the same as in the previous case, therefore we have $H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1)$, and $H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong 0$.

We also have

$$\bigoplus_{i \in \{-2,-1,1,2,3\}} H^1(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i) \cong \mathbb{Q}(-1)^{\oplus 2}$$

where one $\mathbb{Q}(-1)$ comes from D_3 and D_1 each. Plugging these results into the spectral sequence (3.53) we get:

$$\mathrm{gr}^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}), D \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 3} \bigoplus \mathbb{Q}(0).$$

Finally, if all three masses vanish we calculate, analogously to the previous two cases, $H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1)^{\oplus 2}$, and $H^1(P^G \setminus (\widetilde{Q} \cup \widetilde{L})) \cong 0$, where P^G is the blow up of $\mathbb{P}^2$ at 3 points. This case differs from the previous ones only in $gr_2^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}))$, which is given by the cokernel of the morphism $E^1_{-1,2} \to E^1_{0,2}$ in (3.54). We have that $E^1_{0,2} \cong H^2(P^G)$ generated by $[\widetilde{L}], [D_{-1}], [D_{-2}], [D_{-3}]$, and the image of the Gysin morphism $H^0(\widetilde{Q})(-1) \to H^2(P^G)$ is generated by $2[\widetilde{L}] - [D_{-1}] - [D_{-2}] - [D_{-3}]$. Hence the cokernel is of rank 2. Moreover, we have: $\bigoplus_{i \in \{-3,-2,-1,1,2,3\}} H^1(D_i \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D_i) \cong 0$ because we remove 1 point from each of the three exceptional divisors, its intersection with $\tilde{Q}$, and one from each of the D_i, their intersection with $\tilde{L}$, which makes D into a hexagon of affine lines. Therefore, from (3.53) we get:

$$\mathrm{gr}^W H^2(P^G \setminus (\widetilde{Q} \cup \widetilde{L}), D \setminus (\widetilde{Q} \cup \widetilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 2} \bigoplus \mathbb{Q}(0).$$

□

3.3.2.2 Coaction on the triangle graph with two or more vanishing masses

When one of the masses in the triangle graph vanishes, and other masses and momenta are generic, the proof of lemma 3.3.5 and the same argument as in 3.3.1.2 implies that the coaction in this case becomes:

$$\begin{aligned} \Delta I^{\mathfrak{m}}_G = &\left[mot''_{G/e_2}, \left[\sigma_{G/e_2}\right], \left[\theta^2_{G/e_1}\right]\right]^{\mathfrak{m}} \otimes \left[mot_G, \left[\omega_2^2\right]^\vee, \left[\omega_G\right]\right]^{\mathfrak{dr}} + \\ &+ \left[mot''_{G/e_3}, \left[\sigma_{G/e_3}\right], \left[\theta^2_{G/e_3}\right]\right]^{\mathfrak{m}} \otimes \left[mot_G, \left[\omega_3^2\right]^\vee, \left[\omega_G\right]\right]^{\mathfrak{dr}} + \\ &+ \sum_{j=1,2} I^{\mathfrak{m}}_{G/e_1}\left(\theta^j_{G/e_1}\right) \otimes \left[mot_G, \left[\omega_1^j\right]^\vee, \left[\omega_G\right]\right]^{\mathfrak{dr}} + \\ &+ I^{\mathfrak{m}}_G \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + 1 \otimes I^{\mathfrak{dr}}_G \end{aligned} \tag{3.55}$$

where we have taken, without loss of generality, $m_1 = 0$, and mot''_{G/e_2} and mot''_{G/e_3} are motives of bubble graphs with one vanishing mass (3.18), while mot_{G/e_1} is the motive of a bubble graph with non-vanishing masses (3.16).

In the proof of Lemma 3.3.5 we saw that when 2 masses of the triangle graph vanish we have

$$gr^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1).$$

Similarly when all 3 masses vanish we get

$$gr^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1)^{\oplus 2}.$$

Note that for the motive of the triangle graph with generic non-vanishing masses and momenta all motivic periods of weight 2 are equivalent to motivic periods of the motives $H^1(\Delta_i \setminus (Q \cup L) \cap \Delta_i, \Delta_i \cap (\Delta_j \bigcup \Delta_K))$, where $1 \leq i \leq 3$, and i, j, k pairwise distinct, via the face maps. This is crucial in relating the motivic side of the coaction to the motivic periods associated to motives of quotient graphs. However, when two or more masses vanish, this is not the case any longer.

We can still compute the coaction, as we now show in the case of all three masses vanishing and three non-trivial external momenta, by using the residue. Consider the following hyperplanes:

$$F_1 = V(\alpha_2 + \alpha_3), \quad F_2 = V(\alpha_1 + \alpha_3)$$

and denote by F_i their strict transforms as well. Note that we still consider all schemes base changed to $K_{F,M}$. Let D_{-i} be the exceptional divisor over the point $V(\alpha_j) \cap V(\alpha_k)$ for $1 \leq i, j, k \leq 3$ pairwise distinct. Denote the following intersection points:

$$D_i \cap \tilde{L} = l_i, \quad D_{-i} \cap \tilde{Q} = u_i, \quad F_i \cap D_{-i} = g_i, \quad F_i \cap D_i = t_i, \quad \text{for } i = 1, 2$$

and

$$D_{-i} \cap D_j = p_0^i, \quad D_{-i} \cap D_k = p_1^i, \quad i = 1, 2, \quad i, j, k \text{ pairwise distinct}$$

as well as

$$\tilde{Q} \cap \tilde{L} = \{f_0, f_1\}, \quad D_i \cap \tilde{L} = d_i \quad i = 1, 2.$$

Theorem 3.3.6. *Let G be the triangle graph with all internal masses vanishing, and non-trivial momenta. Then the coaction on the associated motivic Feynman amplitude is:*

$$\begin{aligned}
\Delta I_G^{\mathfrak{m}} = {} & c_1 \log^{\mathfrak{m}}([g_1 u_1 | p_0^1 p_1^1]) \otimes \log^{\mathfrak{dr}}(f_0 f_1 | d_1 d_2) \mathbb{L}^{\mathrm{dR}} + \\
& + c_2 \log^{\mathfrak{m}}([g_2 u_2 | p_0^2 p_1^2]) \otimes \log^{\mathfrak{dr}}(f_0 f_1 | d_1 d_2) \mathbb{L}^{\mathrm{dR}} + \\
& + c_3 \log^{\mathfrak{m}}([g_1 u_1 | p_0^1 p_1^1]) \otimes \log^{\mathfrak{dr}}(f_0 f_1 | d_1 d_3) \mathbb{L}^{\mathrm{dR}} + \\
& + c_4 \log^{\mathfrak{m}}([g_2 u_2 | p_0^2 p_1^2]) \otimes \log^{\mathfrak{dr}}(f_0 f_1 | d_1 d_3) \mathbb{L}^{\mathrm{dR}} + \\
& + I_G^{\mathfrak{m}} \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + 1 \otimes I_G^{\mathfrak{dr}}
\end{aligned}$$

where a_i, b_i are undetermined constants in k_S. The cross-ratios evaluate to:

$$\log^{\mathfrak{m}}([e_i u_i | p_0^i p_1^i]) = \log^{\mathfrak{m}}\left(\frac{q_j^2}{q_k^2}\right), \quad i = 1,2, \quad i,j,k \text{ pairwise distinct},$$

$$\log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_2]) = \log^{\mathfrak{dr}}\left(\frac{\left(q_1^2 + q_2^2 - q_3^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}\right)^2}{4q_1^2q_2^2}\right),$$

and

$$\log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_3]) = \log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_2]) + \log^{\mathfrak{dr}}\left(\frac{q_1^2 + q_3^2 - q_2^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}}{q_1^2 + q_3^2 - q_2^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_3^2 - 2q_2^2q_3^2}}\right).$$

Proof. From Lemma 3.3.5 we have that $\mathrm{gr}^W mot_G \cong \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 2} \oplus \mathbb{Q}(0)_{/S}$ and we can choose a basis $\{e_0, e_1, e_2, \omega_G\}$ of $(mot_G)_{\mathrm{dR}}$ such that e_0 spans $W_0(mot_G)_{\mathrm{dR}}$, and e_0, e_1, e_2 span $W_2(mot_G)_{\mathrm{dR}}$. We may take e_0 to be the dual of 2.20. From the definition of the coaction we have

$$\Delta I_G^{\mathfrak{m}} = I_G^{\mathfrak{m}} \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + \sum_{i=1,2} [mot_G, [\sigma_G], e_i]^{\mathfrak{m}} \otimes [mot_G, e_i^\vee, [\omega_G]]^{\mathfrak{dr}} + 1 \otimes I_G^{\mathfrak{dr}},$$

We restrict to the fiber over the generic point. To identify the de Rham side we consider the residue morphism along $\tilde{L}$

$$\mathrm{Res}_{\tilde{L}} : H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \to H^1(\tilde{L} \setminus \tilde{L} \cap \tilde{Q}, \tilde{L} \cap D)(-1),$$

where the target is isomorphic to $H^1(\mathbb{P}^1 \setminus \{f_0, f_1\}, \{d_1, d_2, d_3\})(-1)$. A basis of its de Rham realization can be chosen to be $\{[res(\omega_G)(-1)], [dh_1(-1)], [dh_2(-1)]\}$, where h_1 is a meromorphic function on $\mathbb{P}^1$ such that $h_1(d_1) - h_1(d_2) = 1$, and similarly h_2 is a meromorphic function on $\mathbb{P}^1$ such that $h_1(d_2) - h_1(d_3) = 1$. By definition we have

$$\log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_2]) = [H^1(\mathbb{P}^1 \setminus \{f_0, f_1\}, \{d_1, d_2, d_3\}), [dh_1]^\vee, [res(\omega_G)]]^{\mathfrak{dr}}.$$

We can use the residue homomorphism to write an equivalence of de Rham periods

$$[mot_G, e_i^\vee, [\omega_G]]^{\mathfrak{dr}} = \left[H^1(\mathbb{P}^1 \setminus \{f_0, f_1\}, \{d_1, d_2, d_3\})(-1), \eta_i^\vee, [res(\omega_G)(-1)]\right]^{\mathfrak{dr}},$$

where $\eta_i^\vee$ is such that $res^\vee(\eta_i^\vee) = e_i^\vee$. We can express $\eta_i^\vee$ in terms of the dual basis to our chosen basis $\{[res(\omega_G)(-1)], [dh_1(-1)], [dh_2(-1)]\}$. Therefore we obtain

$$[mot_G, e_i^\vee, [\omega_G]]^{\mathfrak{dr}} = b_{i,1} \log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_2])\mathbb{L}^{\mathfrak{dr}} + b_{i,2} \log^{\mathfrak{dr}}([f_0 f_1 | d_1 d_3])\mathbb{L}^{\mathfrak{dr}}.$$

Here we assume that we have chosen e_i such that the coefficient with

$$[H^1_{\mathrm{dR}}(\mathbb{P}^1 \setminus \{f_0, f_1\}, \{d_1, d_2, d_3\})(-1), [res(\omega_G)(-1)]^\vee, [res(\omega_G)(-1)]] = (\mathbb{L}^{\mathfrak{dr}})^2$$

vanishes.

To determine the motivic periods in the coaction consider the pullback morphism:

$$i^* : H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \to H^2(P^G \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2,3} F_i), D \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i))$$

Computing the Gysin spectral sequence (3.43) and the relative cohomology spectral sequence for the motive on the right we get that:

$$H^2(P^G \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i) \cong \mathbb{Q}(-2)$$

and

$$H^1(D_j \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i)) \cong \mathbb{Q}(-1) \quad \text{for } j = \{-1, -2\}$$

and

$$H^1(D_j \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i)|_{D_j}) \cong 0 \quad \text{for } j = \{1, 2\},$$

therefore, since the pullback preserves the weight, the class of $i^*(e_i)$ vanishes in $H^2(P^G \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2,3} F_i)$. We can therefore write it as the differential of a 1-form which in turn restricts to a k_S-linear combination of non-exact differential forms on $D_j \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i)|_{D_j}$ for $j = \{-2, -1\}$. We get an equivalence of classes of differential forms:

$$\begin{aligned}[(i^*(e_i), 0, \dots, 0)] = a_{i,1}[(0, \theta_1, 0, \dots, 0)] + \\ + a_{i,2}[(0, 0, \theta_2, 0, \dots, 0)]\end{aligned} \tag{3.56}$$

where θ_l are forms of the form (3.19) with simple poles at the points u_l and g_l, for $l = 1, 2$. We denote by $E_j = H^1(D_j \setminus (\tilde{Q} \cup \tilde{L} \cup \cup_{i=1,2} F_i)|_{D_j}, D_j \cap F_k, D_j \cap F_l)$ for $j = \{-2, -1\}$ and l, k, j pairwise distinct. We get an equivalence of motivic periods:

$$\begin{aligned}[mot_G, [\sigma_G], e_i]^{\mathfrak{m}} = a_{i,1}[E_1, [\partial\sigma_G \cap D_{-1}], [(0, \theta_1, 0, \dots, 0)]]^{\mathfrak{m}} + \\ + a_{i,2}[E_2, [\partial\sigma_G \cap D_{-2}], [(0, 0, \theta_2, \dots, 0)]]^{\mathfrak{m}}\end{aligned} \tag{3.57}$$

Now notice that $[E_l, [\partial\sigma_G \cap D_{-l}], [(0, \dots, \theta_l, 0, \dots, 0)]]^{\mathfrak{m}} = \log^{\mathfrak{m}}(g_l u_l | p_0^l p_1^l)$, and collect all the constants to obtain the result. It is a standard exercise to put coordinates on a blow up of $\mathbb{P}^2$ at a point (see proof of Lemma 3.3.4), which in turn enables us to compute the intersection points and the cross ratios, obtaining the result. □

Remark 3.3.2. The constants c_i, $1 \leq i \leq 4$ depend on the choice of a basis of $(mot_G)_{\mathrm{dR}}$. In order to determine them one would need to write down such a basis explicitly, and follow the recipe in the proof. An example of this, done in detail, can be found in the next chapter, where we study the sunrise graph. See proof of Theorem 4.2.7.

Remark 3.3.3. In order to produce a graphical interpretation of the motivic side of the coaction in a case such as the one of the massless triangle graph we should regularize the motivic periods of the bubble graphs which are obtained by contracting an edge of the triangle graph. Conjecture 1 in [12] predicts that, after including regularized motivic Feynman periods, the Galois conjugates would be motivic periods of subquotient graph motives with the number of edges of the subquotient graph controlled by the weight of the conjugate.

3.4 Twisted cohomology and dimensional regularization

Motivated by the central conjecture in [1] about the interplay of the motivic Galois coaction and dimensional regularization, as well as the work of Brown and Dupont on the Lauricella hypergeometric functions and their coactions [15], we study in this section the integrals of the shape

$$\int_\sigma f^\epsilon \omega_G, \text{ where } f = \frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}}, \tag{3.58}$$

for G a Feynman graph, and $\Psi_G, \Xi_G, \omega_G, \sigma$ defined in 2.5. Following [15] we will fix ϵ to be a generic complex number, and represent the integral (3.58) as a matrix coefficient of a comparison isomorphism of cohomology with coefficients in a certain local system defined in terms of the function f. We will use the 'global' Tannakian framework, as well as certain results on twisted cohomology from [15] which we recall next as they apply in our setting. We will show how the coaction of the ring of functions on the resulting Tannaka group reflects the expected results of [1].

3.4.1 Twisted motives of Feynman graphs

Tannakian category We will use a pared down version of the category of realizations we used so far in this chapter. Let $k_{\mathrm{dR}}, \mathbb{Q}_{\mathrm{B}} \subset \mathbb{C}$ be two subfields of the complex numbers, and let $\mathcal{T}$ be the category whose objects are triples $(V_{\mathrm{dR}}, V_{\mathrm{B}}, c)$,

where V_{dR} is a finite-dimensional k_{dR}-vector space, V_{B} is a finite-dimensional $\mathbb{Q}_{\mathrm{B}}$-vector space, and $c : V_{\mathrm{dR}} \otimes_{k_{\mathrm{dR}}} \mathbb{C} \xrightarrow{\sim} V_{\mathrm{B}} \otimes_{\mathbb{Q}_{\mathrm{dR}}} \mathbb{C}$ is an isomorphism we refer to as the comparison isomorphism. The category comes with two fiber functors $\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}$ sending a triple to it first and second entry respectively. As before we have the ring of 'motivic periods' $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}))$, as well as the ring of 'de Rham periods' $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}} = \mathcal{O}(\mathrm{Aut}^{\otimes}_{\mathcal{T}}(\omega_{\mathrm{dR}}))$. The ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ is a $(\mathbb{Q}_{\mathrm{B}}, k_{\mathrm{dR}})$-bimodule spanned by equivalence classes $[V, \sigma, \omega]^{\mathfrak{m}}$, where $V \in \mathrm{ob}(\mathcal{T})$, $\sigma \in V^{\vee}_{\mathrm{B}}$, and $\omega \in V_{\mathrm{dR}}$. The equivalence is defined in the same way as in 2.2 with respect to homomorphisms in $\mathcal{T}$. Similarly, $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$ is spanned by equivalence classes $[V, f, \omega]$ where $f \in V^{\vee}_{\mathrm{dR}}$, and $\omega \in V_{\mathrm{dR}}$. The ring $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ is equipped with the period homomorphism

$$\begin{aligned} \mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} &\to \mathbb{C} \\ [V, \sigma, \omega]^{\mathfrak{m}} &\mapsto \sigma(c(\omega)). \end{aligned} \tag{3.59}$$

We also get a coaction

$$\Delta_{\mathcal{T}} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \otimes \mathcal{P}^{\mathfrak{dr}}_{\mathcal{T}}$$

given by

$$\Delta_{\mathcal{T}}([V, \sigma, \omega]^{\mathfrak{m}}) = \sum_i [V, \sigma, e_i]^{\mathfrak{m}} \otimes [V, e_i^{\vee}, \omega]^{\mathfrak{dr}}, \tag{3.60}$$

where $\{e_i\}$ is a basis of V_{dR}, and $\{e_i^{\vee}\}$ its associated dual basis.

A vector bundle and its associated local system. Let G be a Feynman graph, and fix the masses and momenta $m_e^2, q_{e'}^2 \in k \subset \mathbb{C}$ where $e \in E_G, e' \in E_G^{ext}$. We make the following definitions:

$$f = \frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}}, \ \epsilon \in \mathbb{C}, \ \epsilon \notin \frac{1}{h_G}\mathbb{Z}, \ \epsilon \notin \frac{1}{h_G+1}\mathbb{Z}, \ \mathbb{Q}_{\mathrm{B}} = \mathbb{Q}(e^{2\pi i \epsilon}), \ k_{\mathrm{dR}} = k(\epsilon).$$

Consider the rank one vector bundle

$$(\mathcal{O}_{\mathbb{P}^{N_G-1} \setminus X_G}, \nabla_\epsilon), \ \text{where } \nabla_\epsilon = d + \epsilon \log(f)$$

on the base change $(\mathbb{P}^{N_G-1} \setminus X_G) \times_k k_{\mathrm{dR}}$. Let $\mathcal{L}_\epsilon$ be the local system on the complex points of $\mathbb{P}^{N_G-1} \setminus X_G$ defined by

$$\mathcal{L}_\epsilon = \mathbb{Q}_{\mathrm{B}} f^{-\epsilon}.$$

We have that

$$\mathcal{L}_\epsilon \otimes_{\mathbb{Q}_{\mathrm{B}}} \mathbb{C} \cong (\mathcal{O}^{an}_{\mathbb{P}^{N_G-1} \setminus X_G})^{\nabla_\epsilon}.$$

Definition 3.4.1. As before we have $X_G = V(\Xi_G) \bigcup V(\Psi_G) \subset \mathbb{P}^{N_G-1}$ and Δ_G the union of coordinate hyperplanes in $\mathbb{P}^{N_G-1}$. Let $\Delta'_G = \Delta_G \setminus X_G$. We define the following object, denoted $mot_{G,\mathcal{T}}$, of the category $\mathcal{T}$:

$$\left(H^{N_G-1}_{\mathrm{dR}}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \nabla_\epsilon), H^{N_G-1}_{\mathrm{B}}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \mathcal{L}_\epsilon), c_{\mathrm{B,dR}}\right),$$

where $H^{N_G-1}_{\mathrm{dR}}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \nabla_\epsilon)$ is the twisted algebraic de Rham cohomology, and $H^{N_G-1}_{\mathrm{B}}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \mathcal{L}_\epsilon)$ is the singular cohomology with coefficients in a local system. The comparison $c_{\mathrm{B,dR}}$ is provided by a relative version of Deligne's comparison for twisted cohomology [22, §6].

Remark 3.4.2. The local sections of $\mathcal{L}_\epsilon$ are determinations of $f^{-\epsilon}$. Therefore, a class in Betti homology $H^{\mathrm{B}}_{N_G-1}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \mathcal{L}_{-\epsilon})$ is represented by an $N_G - 1$-cycle σ in $\mathbb{P}^{N_G-1} \setminus X_G(\mathbb{C})$ with its boundary in Δ'_G, loaded with a determination of f^ϵ. We denote such a class by

$$[\sigma \otimes f^\epsilon].$$

Recall that the domain of integration of the Feynman integral associated to G is

$$\sigma = \{[\alpha_1 : \ldots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G-1}(\mathbb{R}).$$

It can happen that it intersects the graph hypersurfaces X_G along the faces of σ, as discussed in 2.5.2. For this reason we considered blow-ups of $\mathbb{P}^{N_G-1}$ when working with cohomology with coefficients in $\mathbb{Q}$. However, when we allow coefficients in the local system $\mathcal{L}_\epsilon$ we can work without blowing up by using the following lemma.

Lemma 3.4.1. *Let $\epsilon \in \mathbb{C}, \epsilon \notin \frac{1}{h_G}\mathbb{Z}$, and $\epsilon \notin \frac{1}{h_G+1}\mathbb{Z}$, as defined previously. Then we have*

$$H^\bullet_{\mathrm{B}}(\mathbb{P}^{N_G-1} \setminus X_G; \mathcal{L}_\epsilon) \cong H^\bullet_c(\mathbb{P}^{N_G-1} \setminus X_G; \mathcal{L}_\epsilon),$$

and

$$H^\bullet_{\mathrm{B}}(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \mathcal{L}_\epsilon) \cong H^\bullet_c(\mathbb{P}^{N_G-1} \setminus X_G, \Delta'_G; \mathcal{L}_\epsilon),$$

where on the right-hand side we have compactly supported cohomology with coefficients in a local system.

Proof. Let $j : \mathbb{P}^{N_G-1} \setminus X_G \subset \mathbb{P}^{N_G-1}$. Our constraints on ϵ imply that the monodromy operators corresponding to the connected components of X_G (the two graph hypersurfaces) don't have 1 as an eigenvalue. The following isomorphism in the bounded derived category of sheaves follows [28, Theorem 3.4.4.]

$$j_!\mathcal{L}_\epsilon \cong Rj_*\mathcal{L}_\epsilon.$$

We know that

$$H_c^\bullet(\mathbb{P}^{N_G-1}; j_!\mathcal{L}_\epsilon) \cong H_c^\bullet(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon),$$

and

$$H_{\mathrm{B}}^\bullet(\mathbb{P}^{N_G-1}; Rj_*\mathcal{L}_\epsilon) \cong H_{\mathrm{B}}^\bullet(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon),$$

from which the first claim follows. As Δ'_G does not intersect X_G by definition the second claim follows from the analogous argument for relative cohomology. □

Corollary 3.4.2. *In homology we have the isomorphism*

$$H_\bullet^{lf}(\mathbb{P}^{N_G-1} \smallsetminus X_G, \Delta'_G; \mathcal{L}_{-\epsilon}) \cong H_\bullet^{\mathrm{B}}(\mathbb{P}^{N_G-1} \smallsetminus X_G, \Delta'_G; \mathcal{L}_{-\epsilon}),$$

where on the left we have locally finite homology, represented by locally finite chains with boundary in Δ'_G loaded, as before, with determinations of f^ϵ.

We state another corollary of the lemma which will be of use later.

Corollary 3.4.3. *The cohomology $H_{\mathrm{B}}^\bullet(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon)$ is concentrated in the middle degree $N_G - 1$. The rank of $H_{\mathrm{B}}^{N_G-1}(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon)$ is computed by the Euler characteristic $\chi(\mathbb{P}^{N_G-1} \smallsetminus X_G)$.*

Proof. The first claim follows from the fact that $\mathbb{P}^{N_G-1} \smallsetminus X_G$ is affine, and therefore $H_{\mathrm{B}}^k(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon) = 0$ for $k > N_G - 1$ by Artin vanishing. Poincare duality gives

$$H_c^k(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon)^\vee \cong H^{2(N_G-1)-k}(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_{-\epsilon})$$

and by the previous lemma the left hand side is isomorphic to $H_{\mathrm{B}}^k(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon)^\vee$, hence it vanishes for $k > N_G - 1$. The first claim follows.

The second claim follows from

$$\dim\left(H_{\mathrm{B}}^{N_G-1}(\mathbb{P}^{N_G-1} \smallsetminus X_G; \mathcal{L}_\epsilon)\right) = \chi(\mathbb{P}^{N_G-1} \smallsetminus X_G, \mathcal{L}_\epsilon) = \chi(\mathbb{P}^{N_G-1} \smallsetminus X_G),$$

where the first equality follows from the first claim, and the second equality follows from the fact that the local system $\mathcal{L}_\epsilon$ is of rank 1 and the general result for any X a finite CW-complex and local system $\mathcal{L}$ of rank r [28, Proposition 2.5.4.]

$$\chi(X, \mathcal{L}) = r\chi(X).$$

□

Definition 3.4.3. Define the following matrix coefficient of $mot^{\epsilon}_{G} \in \mathrm{Ob}(\mathcal{T})$

$$I^{\mathfrak{m}}_{G,\mathcal{T}} = \left[mot_{G,\mathcal{T}}, \left[\sigma \otimes \left(\frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}} \right)^{\epsilon} \right], [\omega_G] \right]^{\mathfrak{m}},$$

where σ is viewed as a locally finite chain on $\mathbb{P}^{N_G-1} \smallsetminus X_G(\mathbb{C})$ corresponding to $\{[\alpha_1 : \ldots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G-1}(\mathbb{R})$, and ω_G is the Feynman integrand defined in 2.5.

The image under the period homomorphism of the matrix coefficient $I^{\mathfrak{m}}_{G,\mathcal{T}}$ is the dimensionally-regularized Feynman integral (3.58), specialized at the given generic value of ϵ.

3.4.1.1 Massless triangle example

Let G be the triangle graph 3.1.4.1, with vanishing masses, i.e., $m_1 = m_2 = m_3 = 0$, and generic momenta. We set $d = 4 - 2\epsilon$. As discussed in 3.3.2.2, in the case when all three masses vanish we have that $Q = V(\Xi_G)$ passes through the three corners of the simplex σ, i.e. $\Delta_1 \cap \Delta_2, \Delta_1 \cap \Delta_3$, and $\Delta_2 \cap \Delta_3$, where $\Delta_i = V(\alpha_i)$ as before. Since the local system $\mathcal{L}_\epsilon$ has no poles along $\Delta'_G = \Delta_G \smallsetminus X_G$ we can apply the relative cohomology spectral sequence as in the previous section, with the difference that here we take coefficients in the local system $\mathcal{L}_\epsilon$. We get

$$\begin{array}{ccccc} H^2(\mathbb{P}^2 \smallsetminus (Q \cup L); \mathcal{L}_\epsilon) & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & \oplus_{1\leq i\leq 3} H^1(\Delta'_i; \mathcal{L}_\epsilon|_{\Delta'_G}) & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \oplus_{1\leq i\leq j\leq 3} H^0(\Delta'_i \cap \Delta'_j; \mathcal{L}_\epsilon|_{\Delta'_i \cap \Delta'_j}). \end{array}$$

We get the zero entries by the vanishing argument in 3.4.3. Next we note that the bottom right corner $H^0(\Delta'_i \cap \Delta'_j; \mathcal{L}_\epsilon|_{\Delta'_i \cap \Delta'_j}) = 0$ for all i, j since $\Delta_i \cap \Delta_j$ are precisely the points of intersection of Δ_G and Q, which we remove since we remove Q. We can notice that

$$H^1(\Delta'_i; \mathcal{L}_\epsilon|_{\Delta'_i}) \cong H^1(\mathbb{P}^1 \smallsetminus \{p_1, p_2, p_3\}; \mathcal{L}_\epsilon|_{\Delta'_i}),$$

where we define the points p_1, p_2, to be $\Delta'_i \cap Q$ and p_3 to be $\Delta'_i \cap L$. These points in $\mathbb{P}^1$ are $[1 : 0], [0 : 1]$, and $[-1 : 1]$ respectively. This, along with its de Rham counterpart, makes up a rank one object in the category $\mathcal{T}$. We also note that this is the object mot^{ϵ}_{G/e_i} associated to the bubble graph 3.1.4.2 with vanishing masses obtained by contracting the edge e_i of the massless triangle graph. Let $i = 3$ without

loss of generality. We take the Feynman integrand of the bubble graph in $d = 4$ (see 2.17):

$$\omega_{G/e_3} = \frac{\Omega_{G/e_3}}{(\alpha_1+\alpha_2)^2}.$$

We can integrate it against the cycle

$$\left[\sigma \otimes \left(\frac{(\alpha_1+\alpha_2)^2}{q_3^2\alpha_1\alpha_2}\right)^{\epsilon}\right] \in H_1^{lf}(\mathbb{P}^1 \setminus \{p_1,p_2,p_3\}; \mathcal{L}_{-\epsilon}|_{\Delta_3'}),$$

where σ is a path between p_1 and p_2 in $\mathbb{P}^1 \setminus \{p_3\}(\mathbb{R})$. This computes the image under the period homomorphism of $I^{\mathfrak{m}}_{G/e_3,\mathcal{T}}$, i.e.:

$$\begin{aligned} \mathrm{per}(I^{\mathfrak{m}}_{G/e_3,\mathcal{T}}) &= \int_0^{\infty} \left(\frac{(\alpha_1+1)^2}{q_3^2\alpha_1}\right)^{\epsilon} \frac{-d\alpha_1}{(\alpha_1+1)^2} \\ &= -q_3^{-2\epsilon} \int_0^1 u^{-\epsilon}(1-u)^{-\epsilon} du = -q_3^{-2\epsilon}\beta(-\epsilon+1,-\epsilon+1) \end{aligned} \tag{3.61}$$

where we write the integral on the right hand side in the affine open $\alpha_2 \neq 0$, and we have used the change of variables $u = \frac{\alpha_1}{1+\alpha_1}$. The 'motivic lift' of the Euler beta function, denoted $\beta^{\mathfrak{m}}(s,t)$ was defined previously in [15] as the matrix coefficient of the object in $\mathcal{T}$ given by the twisted cohomology of $\mathbb{P}^1$ minus 3 points. From the previous computation we see that

$$I^{\mathfrak{m}}_{G/e_3,\mathcal{T}} = -q_3^{-2\epsilon}\beta^{\mathfrak{m}}(-\epsilon+1,-\epsilon+1).$$

The fact that the period is non-trivial also tells us that $[\omega_{G/e_i}]$ generates $(mot^{\epsilon}_{G/e_i})_{\mathrm{dR}}$. Let ϕ^{ϵ} be a version of face maps 2.4.2 defined by inclusion of double complexes which compute twisted cohomology (see (3.64) below). Then we have in particular $\phi_i : mot^{\epsilon}_{G/e_i} \to mot_{G,\mathcal{T}}$, for $1 \leq i \leq 3$. The images of the forms of bubble graphs under the de Rham components of the face maps $\phi^{\epsilon}_{i,\mathrm{dR}}([\omega_{G/e_i}])$, along with the Feynman form of the triangle graph $[\omega_G]$, form a basis of $(mot_{G,\mathcal{T}})_{\mathrm{dR}}$. Then we can write the coaction on $I^{\mathfrak{m}}_{G,\mathcal{T}}$ as

$$\begin{aligned} \Delta(I^{\mathfrak{m}}_{G,\mathcal{T}}) &= I^{\mathfrak{m}}_{G,\mathcal{T}} \otimes f^{\mathfrak{dr}}_{G,\mathcal{T}} + \sum_{i=1}^{3} I^{\mathfrak{m}}_{G/e_i,\mathcal{T}} \otimes f^{\mathfrak{dr}}_{G/e_i,G,\mathcal{T}} \\ &= I^{\mathfrak{m}}_{G,\mathcal{T}} \otimes f^{\mathfrak{dr}}_{G,\mathcal{T}} + \sum_{i=1}^{3} -q_i^{-2\epsilon}\beta^{\mathfrak{m}}(-\epsilon+1,-\epsilon+1) \otimes f^{\mathfrak{dr}}_{G/e_i,G,\mathcal{T}}. \end{aligned} \tag{3.62}$$

To write the above formula we have used the equality of motivic periods in the category $\mathcal{T}$

$$\left[mot_{G,\mathcal{T}}, [\sigma \otimes f^{\epsilon}], \phi_{i,\mathrm{dR}}([\omega_{G/e_i}])\right]^{\mathfrak{m}} = I^{\mathfrak{m}}_{G/e_i,\mathcal{T}}$$

which is established via the face map — see 3.23 and the following discussion to see how this works in the case of constant coefficients. The same argument applies here. We define the de Rham periods on the right hand side of the coaction

$$f^{\mathfrak{dr}}_{G/e_i,G,\mathcal{T}} = [mot_{G,\mathcal{T}}, \phi_{i,\mathrm{dR}}([\omega_{G/e_i}])^\vee, [\omega_G]]^{\mathfrak{dr}},$$

and we do not discuss these further here. They are predicted to be related to cut Feynman graphs in [1].

Remark 3.4.4. The reader is invited to compare the twisted coaction formula above with the computations [1, §3.3], in particular eq. 3.35 therein which is formulated based on the first few terms in the expansion in ϵ. We can also compare this to the previous section where we worked with constant coefficients 3.3.2.2. As we can see, for the massless triangle graph, when we work with twisted cohomology we can express all motivic periods in the coaction in terms of quotient graphs with strictly fewer edges except one, which corresponds to the triangle graph itself.

3.4.1.2 Twisted cohomology groups associated to one-loop graphs in dimension $d - 2\epsilon$ and reduction to $d + 1$ edges

Let G the a one-loop Feynman graph with N_G edges in d dimensions, where d is even. We can write its second Symanzik polynomial as follows

$$\Xi_G(m,q) = \sum_{1\le i<j\le N_G} \alpha_i\alpha_j(\mu_i-\mu_j)^2 + \sum_{1\le i\le N_G}(m_i^2\alpha_i)\left(\sum_{1\le i\le N_G}\alpha_i\right),$$

where $\mu_i \in \mathbb{R}^d$ such that $q_i = \mu_{i+1} - \mu_i$ for all $q_i \in E^{ext}_G$. It is a quadratic form with the associated matrix $C_G(m,\mu)$, i.e. a matrix $C_G(m,\mu)$ such that

$$\Xi_G = (\alpha_1,\dots,\alpha_{N_G})C_G(m,\mu)(\alpha_1,\dots,\alpha_{N_G})^t.$$

Denote the rank of $C_G(m,\mu)$ by $r_G(m,\mu)$. Similarly, we consider the matrix $C'_G(\mu)$ associated to the form $\Xi_G(m,q)|_L$, where $L = V(\Psi_G) = V(\sum_{1\le i\le N_G}\alpha_i)$ as before. Note that it does not depend on the masses, and denote its rank by $r'_G(\mu)$.

Lemma 3.4.4. *Let G be a one-loop Feynman graph with N_G edges. Then*

$$\chi(\mathbb{P}^{N_G-1}\setminus(Q\cup L)) = \begin{cases}\pm 1 & , r_G(m,\mu) - r'_G(\mu) \equiv 1 \mod 2\\ 0 & , \textit{otherwise}\end{cases} \tag{3.63}$$

i.e. the Euler characteristic is ± 1 if $r_G(m,\mu) - r'_G(\mu)$ have opposite parity and 0 otherwise.

Proof. We have that

$$\chi(\mathbb{P}^{N_G-1} \setminus (Q \cup L)) = \chi(\mathbb{P}^{N_G-1} \setminus Q) - \chi(L \setminus Q \cap L),$$

and we must therefore compute $\chi(\mathbb{P}^{N_G-1} \setminus Q)$ and $\chi(L \setminus Q \cap L)$. By the same Leray spectral sequence argument as the one we used in the proof of proposition 3.1.2 we have that $H^k(\mathbb{P}^{N_G-1} \setminus Q) \cong H^k(\mathbb{P}^{r_G(m,\mu)-1} \setminus Q_0)$ for all $k \geq 0$, where Q_0 is a smooth quadric. From lemma 3.1.1 we know that the cohomology of the complement of a smooth quadric is of rank one for $k = r_G(m,\mu) - 1$ odd, and for $k = 0$, while it is zero otherwise. Hence we have

$$\chi(\mathbb{P}^{N_G-1} \setminus Q) = \chi(\mathbb{P}^{r_G(m,\mu)-1} \setminus Q_0) = \begin{cases} 1 & , r_G(m,\mu) \text{ even} \\ 0 & , \text{otherwise.} \end{cases}$$

Note that the same calculation gives us $\chi(L \setminus Q \cap L)$ with $r_G(m,\mu)$ replaced by $r'_G(\mu)$, since $L \cong \mathbb{P}^{N_G-2}$. This concludes the proof. □

Proposition 3.4.5. *Let G be a one-loop Feynman graph with N_G edges in dimension $d - 2\epsilon$, where d is even, with generic momenta. We have*

$$\dim(H^{N_G-1}(\mathbb{P}^{N_G-1} \setminus (Q \cup L); \mathcal{L}_\epsilon)) = \begin{cases} 0 & , N_G \geq d+2 \\ 1 & , 2 \leq N_G \leq d+1 \end{cases}$$

Proof. As we saw in corollary 3.4.3 the dimension of these twisted cohomology groups is computed by $\chi(\mathbb{P}^{N_G-1} \setminus (Q \cup L))$. By the previous lemma it is enough to understand the ranks $r_G(m,\mu)$ and $r'_G(\mu)$ of the quadric Q and its restriction to L. Related results exist in the literature, see for example [7, §6] for the case $d = 4$, $N_G = 6$. We can write the matrix $C_G(m,\mu)$ as

$$C_G(m,\mu) = (m_i^2 + m_j^2 + (\mu_i - \mu_j)^2)_{1 \leq i,j \leq N_G} = M_1 + M_2 - 2N,$$

where

$$M_1 = (m_i^2 + \mu_i^2)_{1 \leq i,j \leq N_G}, M_2 = (m_j^2 + \mu_j^2)_{1 \leq i,j \leq N_G}, N = (\mu_i \cdot \mu_j)_{1 \leq i,j \leq N_G},$$

and $\mu_i \cdot \mu_j$ is the Euclidean product in $\mathbb{R}^d$. This follows by expanding

$$(\mu_i - \mu_j)^2 = \mu_i^2 + \mu_j^2 + 2\mu_i \cdot \mu_j.$$

The matrix $(\mu_i \cdot \mu_j)_{i,j}$ is of rank $\leq d$, and for generic momenta its rank is exactly d. Note that the matrix M_1 has equal columns and M_2 is its transpose, hence their ranks are equal to one, therefore $r_G(m,\mu)$ is at most $d+2$. One can check that for

a graph G' with $d+2$ edges and generic momenta we have $\det(C'_G(0,\mu)) \neq 0$, see [7, Lemma 6.1]. In other words the matrix $C_G(m,\mu)$ for a graph G with $\geq d+2$ edges and generic momenta has a non-zero minor of order $d+2$. It follows that

$$r_G(m,\mu) = \begin{cases} d+2 & , N_G \geq d+2 \\ N_G & , \text{otherwise.} \end{cases}$$

The matrix of the restriction $Q|_L$ is given by $U^t C_G(m,\mu)U$, for a matrix U with N_G rows and $N_G - 1$ columns:

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Note that $M_1 U = 0$ and $U^t M_2 = 0$, and we are left with $U^t C_G(m,\mu)U = 2U^t NU$. The condition that N has no non-zero minors of order d is a closed condition, as is the condition that $U^t NU$ has no non-zero minors of order d. Since our momenta are generic we obtain:

$$r'(\mu) = \begin{cases} d & , N_G \geq d+1 \\ N_G - 1 & , \text{otherwise.} \end{cases}$$

This, along with (3.63) gives the result. □

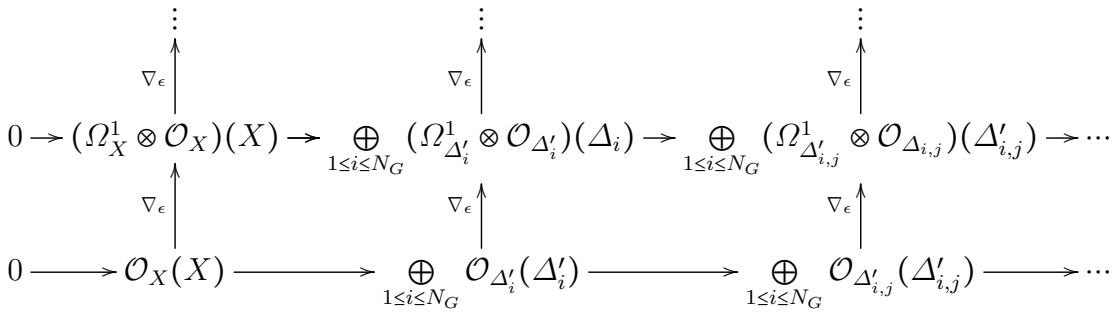
From the previous result we get the twisted version of proposition 3.1.4.

Corollary 3.4.6. *Let G be a one-loop Feynman graph with $N_G \geq d+2$ edges in dimension $d-2\epsilon$, where d is even, with generic momenta. Then we have*

$$I^{\mathfrak{m}}_{G,\mathcal{T}} = \sum_{|I|=N_G-d-1} a_I \cdot I^{\mathfrak{m}}_{G/e_I,\mathcal{T}},$$

where $a_I \in k(\epsilon)$.

Proof. The proof is exactly the same as the proof of proposition 3.1.4, except that we must replace the complexes of global sections of algebraic differential forms in the double complex (3.8) with the global sections of the twisted de Rham double complex:



(3.64)

where $X = \mathbb{P}^{N_G-1} \setminus Q \cup L$, and Δ'_I for $I \subset \{1, \ldots, N_G\}$ are given by the vanishing of coordinates, all of which are affine so the twisted de Rham cohomology is computed by the complex of global sections above.

The previous proposition tells us that for $N_G \geq d+2$ the Feynman form $\omega_G = \nabla_\epsilon(\omega')$, for some $\omega' \in \Omega^{N_G-2}_X \otimes \mathcal{O}_X$. We can proceed by restricting and taking primitives as in 3.1.4 until we have a $k(\epsilon)$-linear combination of d-forms on $\mathbb{P}^d \setminus Q \cup L$. Each of the d-dimensional faces of Δ'_i corresponds to a graph with $d+1$ edges obtained from G by contracting edges. Its twisted de Rham cohomology has rank one and is spanned by the Feynman form of the graph. This follows for example from the comparison isomorphism as we know that the associated Feynman integrals are non-zero. The remainder of the argument is as in 3.1.4. □

Remark 3.4.5. Note that the previous proposition and corollary do not depend on the masses being generic, only the momenta.

3.4.1.3 Coaction

Having reduced the integrals associated to graphs with $\geq d+2$ edges in $d-2\epsilon$ dimensions to those associated to graphs with $d+1$ edges we now compute the coaction on a one-loop graph with $d+1$ edges and generic momenta. We will also see how its twisted cohomology, and consequently the coaction, depends on vanishing masses.

Theorem 3.4.7. *Let G be a graph as above. The coaction, with respect to cohomology with twisted coefficients, is*

$$\Delta_{\mathcal{T}}(I^{\mathfrak{m}}_{G,\mathcal{T}}) = \sum_{I\in\mathbb{I}} I^{\mathfrak{m}}_{G/e_I,\mathcal{T}} \otimes f^{\mathfrak{d}\mathfrak{r}}_{G/e_I,G,\mathcal{T}}, \tag{3.65}$$

where $\mathbb{I}$ is a set of subsets I of $\{1, \ldots, N_G\}$ such that $0 \leq |I| < N_G$, excluding those I such that G/e_I is a massless tadpole, i.e. a massless self-edge graph.

Proof. Since $\mathcal{L}_\epsilon$ extends over all faces of the divisor Δ_G we may apply the relative cohomology spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\Delta_I \setminus (Q \cup L) \cap \Delta_I; \mathcal{L}_\epsilon) \Rightarrow H^{p+q}(\mathbb{P}^{N_G-1} \setminus (Q \cup L), \Delta_G \setminus (Q \cup L) \cap \Delta_G; \mathcal{L}_\epsilon).$$

From proposition 3.4.5 we know that the only terms on the left hand side for $q > 0$ which are non-zero are the middle dimensional ones:

$$H^q(\Delta_I \setminus (Q \cup L) \cap \Delta_I; \mathcal{L}_\epsilon), \text{ where } |I| = N_G - q.$$

These are of rank 1 and in de Rham spanned by the class of the Feynman form of the graph G/e_I. It follows that we have a basis element of $(mot_{G,\mathcal{T}})_{\mathrm{dR}}$, denoted ω_G^I, corresponding to each quotient graph with more than one edge via the de Rham face map

$$\begin{aligned} \phi_{I,\mathrm{dR}} : (mot^\epsilon_{G/e_I})_{\mathrm{dR}} &\to (mot_{G,\mathcal{T}})_{\mathrm{dR}} \\ [\omega_{G/e_I}] &\mapsto [\omega_G^I], \end{aligned} \tag{3.66}$$

given by inclusion of complexes (3.64). It remains to see what happens on the row $q = 0$ of the spectral sequence. The only potentially non-zero components are

$$H^0(\Delta_I \setminus (Q \cup L) \cap \Delta_I; \mathcal{L}_\epsilon), \text{ where } |I| = N_G - 1.$$

The faces Δ_I with $|I| = N_G - 1$ correspond to quotient graphs with all but one edge contracted referred to as tadpoles, or self-edge graphs. If a mass $m_i = 0$, for some $1 \leq i \leq N_G$ the quadric Q passes through the corresponding corner of the coordinate simplex in $\mathbb{P}^{N_G-1}$ given by the intersection of $\alpha_j = 0$ for $1 \leq j \leq N_G, j \neq i$ — see figure 3.5 for the case $N_G = 3, m_1 = 0$ with $m_2 \neq 0, m_3 \neq 0$. Hence the corresponding term

$$H^0(\Delta_I \setminus (Q \cup L) \cap \Delta_I; \mathcal{L}_\epsilon), \text{ where } I = \{1, \ldots, N_G\} \setminus \{i\}$$

vanishes. Alternatively, if $m_i \neq 0$ we get a rank 1 contribution to the spectral sequence and a corresponding de Rham basis element as in (3.66). The corresponding period of the tadpole is

$$\mathrm{per}(I^{\mathfrak{m}}_{G/e_I,\mathcal{T}}) = (m_i^2)^{-\epsilon}.$$

Applying the general formula for the coaction we get

$$\Delta_{\mathcal{T}}(I^{\mathfrak{m}}_{G,\mathcal{T}}) = \sum_I \left[mot^\epsilon_G, \left[\sigma \otimes \left(\frac{\Psi_G^{h_G+1}}{\Xi_G^{h_G}} \right)^\epsilon \right], [\omega_G^I] \right]^{\mathfrak{m}} \otimes \left[mot^\epsilon_G, [\omega_G^I]^\vee, [\omega_G] \right]^{\mathfrak{dr}}, \tag{3.67}$$

where I runs over all subsets of $\{1, \ldots, N_G\}$ except the singleton sets $\{i\}$ for all i such that $m_i = 0$ and the whole set $\{1, \ldots, N_G\}$. By the equivalence of motivic periods via the corresponding face maps we obtain the statement of the theorem. □

Chapter 4

The sunrise, motivic Galois coaction and integrals of modular forms

Next, we will study the motive of the *sunrise* Feynman graph, and its associated periods. It is known that the Feynman integral associated to the sunrise in $d = 2$ dimensions of space-time can be expressed in terms of elliptic dilogarithms and integrals of modular forms [8, 6]. In this chapter we will compute the motivic Galois coaction on the motivic Feynman amplitude associated to the sunrise graph, which is the first such computation for a (motivic) elliptic polylogarithm. We introduce the operation of subdividing edges of a graph and show how it can be used to complete a basis of the de Rham realization of a Feynman motive when face maps cannot be used for this purpose. We also show that the sunrise Feynman integral can be expressed in terms of integrals of modular forms in a different way to those in [8, 6], by expressing it as a period of the relative completion of the torsor of paths on $X_1(6)$.

4.1 The motive $H^1(\mathcal{E}, \{P, Q\})$

Let $k \subset \mathbb{C}$ be a field, and let $\mathcal{E} \subset \mathbb{P}^2$ be an elliptic curve defined over k. Let P and Q be two k-points on $\mathcal{E}$. We consider the category of realisations of objects over k,

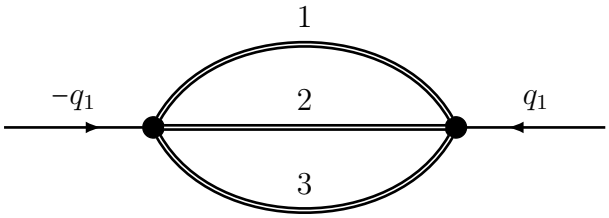


Figure 4.1: The sunrise Feynman graph.

denoted $\mathcal{H}_k$, consisting of triples $((H_{\mathrm{B},\varphi})_\varphi, H_{\mathrm{dR}}, (c_\varphi)_\varphi)$ where $(H_{\mathrm{B},\varphi})_\varphi$ are $\mathbb{Q}$-vector spaces for every embedding $\varphi : k \hookrightarrow \mathbb{C}$, H_{dR} is a finite dimensional k-vector space, and $c_\varphi : H_{\mathrm{dR}} \otimes_{k,\varphi} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{B},\varphi} \otimes_{\mathbb{Q}} \mathbb{C}$ is an isomorphism for every φ. This category is a special case of $\mathcal{H}(S)$, defined in 2.1, where $S = \mathrm{Spec}(k)$. We will consider the following object

$$H^1(\mathcal{E}, \{P, Q\}) = ((H_{\mathrm{B},\sigma})_\sigma, H_{\mathrm{dR}}, (c_\varphi)_\varphi) \in \mathrm{Ob}(\mathcal{H}_k),$$

where

$$H_{\mathrm{dR}} = H^1_{\mathrm{dR}}(\mathcal{E}, \{P, Q\}) \text{ and } H_{\mathrm{B},\varphi} = H^1_B(\mathcal{E}_\varphi(\mathbb{C}), \{P, Q\}; \mathbb{Q})$$

are algebraic de Rham and singular cohomology respectively, and $\mathcal{E}_\varphi = \mathcal{E} \times_{k,\varphi} \mathbb{C}$ for every embedding $\varphi : k \hookrightarrow \mathbb{C}$[1]. Note that the object $H^1(\mathcal{E}, \{P, Q\})$ sits in a long exact sequence of mixed Hodge structures

$$0 \to \widetilde{H}^0(\{P, Q\}) \to H^1(\mathcal{E}, \{P, Q\}) \to H^1(\mathcal{E}) \to 0 \tag{4.1}$$

where on the left we have reduced cohomology.

4.1.1 de Rham realization, Betti realization and periods

4.1.1.1 de Rham realization

We can push out (4.1) along the isomorphism $\phi : H^1(\mathcal{E}) \xrightarrow{\cong} H^1(\mathcal{E} \smallsetminus [0:1:0])$ to an equivalent extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{H}^0(\{P, Q\}) & \longrightarrow & H^1(\mathcal{E}, \{P, Q\}) & \longrightarrow & H^1(\mathcal{E}) & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \widetilde{H}^0(\{P, Q\}) & \longrightarrow & E & \longrightarrow & H^1(\mathcal{E} \smallsetminus [0:1:0]) & \longrightarrow & 0 \end{array}$$

we can then represent a basis of the de Rham realization of E by global algebraic differential forms on the punctured elliptic curve since it is affine. As an example we may take $\mathcal{E}$ given by the Weierstrass equation

$$y^2 z = x^3 + axz^2 + bz^3,$$

for some $a, b \in k$. Then we may consider the following differential forms

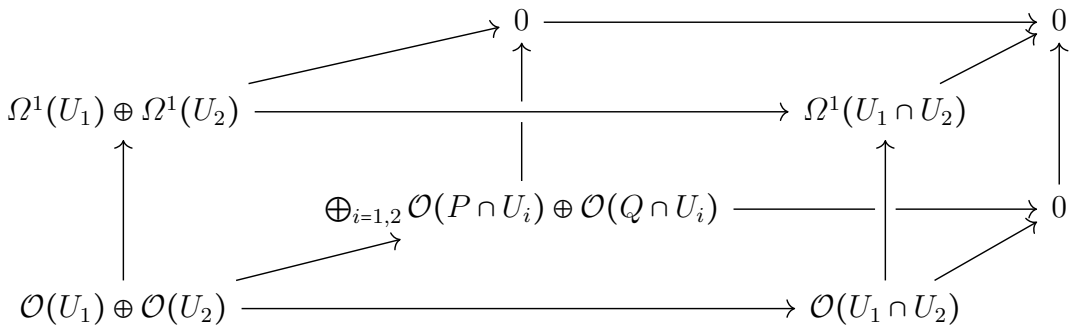
$$\omega = \frac{dx}{2y}, \eta = \frac{xdx}{2y}, \text{ and } \xi_{P,Q} = df,$$

[1] Note that the embedding data is necessary to define the comparison isomorphism since H_{dR} is a k-vector space which we must complexify, which depends on the embedding. Similarly, in order to speak of homology cycles we must take the complex points of $\mathcal{E}$, which depends on the embedding. This subtlety does not arise when the variety is defined over $\mathbb{Q}$ since then we have a unique embedding. For its role in the study of Artin motives and Grothendieck's version of classical Galois theory see [14, §5.1]

where ω is a differential form of the first kind, η is a differential form of the second kind with a double pole at $[0:1:0]$ and vanishing residue, and $\xi_{P,Q}$ is the differential of a meromorphic function f on $\mathcal{E}$ such that $f(P) - f(Q) = 1$. The classes $[\omega]$ and $[\eta]$ map onto generators of $H^1_{\mathrm{dR}}(\mathcal{E})$ via the map $H^1_{\mathrm{dR}}(\mathcal{E}, \{P,Q\}) \to H^1_{\mathrm{dR}}(\mathcal{E})$, while $[\xi_{P,Q}]$ generates the image of the injective map $\widetilde{H}^0_{\mathrm{dR}}(\{P,Q\}) \to H^1_{\mathrm{dR}}(\mathcal{E}, \{P,Q\})$.

Remark 4.1.1. We could also use our previous definition of relative de Rham cohomology (2.9) in the case of $H^1_{\mathrm{dR}}(\mathcal{E}, \{P,Q\})$ directly, instead of using the isomorphism $\phi : H^1(\mathcal{E} \setminus \{[0:1:0]\}) \to H^1(\mathcal{E})$. Since we are not in the affine case we can use the Čech resolution of each sheaf in the double complex (2.8).

Consider a covering $\mathcal{U}$ of $\mathbb{P}^2$ given by three affine pieces: U_1 given by $z \neq 0$, U_2 given by $y \neq 0$ and U_3 given by $x \neq 0$. In fact it will be enough to consider $U_1 \cup U_2$ since its complement in $\mathbb{P}^2$ is $(1:0:0)$ and this does not lie on our curve. Let the coordinates on U_1 be denoted by $(x, y, 1)$, and coordinates on U_2 by $(u, 1, v)$. The relative algebraic Čech-de Rham triple complex, which computes algebraic de Rham cohomology of the pair $(\mathcal{E}, \{P,Q\})$ has the form



An element of $H^1_{dR}(\mathcal{E}, \{P,Q\})$ is a class represented by a tuple in

$$\Omega^1(U_1) \oplus \Omega^1(U_2) \oplus \mathcal{O}(U_1 \cap U_2) \oplus \bigoplus_{i=1,2} \mathcal{O}(P \cap U_i) \oplus \mathcal{O}(Q \cap U_i) \tag{4.2}$$

which satisfies the cocycle condition in the total complex of the above triple complex.

From the short exact sequence (4.1) we know that the motive $H^1(\mathcal{E}, \{P,Q\})$ has rank 3. A basis of de Rham cohomology is given by the classes of the following cocycles:

$$\begin{aligned}
&\left(\frac{dx}{2y}, \frac{du}{1 - 2auv - 3bv^2}, 0, \ldots, 0\right) \\
&\left(\frac{xdx}{2y}, \frac{udu}{v(1 - 2auv - 3bv^2)} - d\left(\frac{1}{u^2}\right), \frac{1}{u^2}, 0 \ldots, 0\right) \\
&(df, 0, f|_{U_1 \cap U_2}, 0, \ldots, 0),
\end{aligned} \tag{4.3}$$

where in the last tuple $f \in \mathcal{O}(U_1)$ such that $f(P) - f(Q) = 1$. For example we can take $f = \frac{x-x(Q)}{x(P)-x(Q)}$ if the points P, Q have different x coordinates. The cocycle $\xi_{P,Q}$ is cohomologous to $(0,0,0,-1,0,0,0)$ where the -1 is at the position corresponding to $\mathcal{O}(P \cap U_1)$. The morphism $\phi : H^1_{dR}(\mathcal{E}) \to H^1_{dR}(\mathcal{E} \setminus [0:1:0])$ is induced by sending a tuple in (4.2) to its first entry.

Remark 4.1.2. A few things should be noted about our choice of basis of the de Rham realization. The space of global 1-forms $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/k})$, which is 1-dimensional over k, injects into $H^1_{\mathrm{dR}}(\mathcal{E})$, hence our choice of basis element $[\omega]$ is unique up to a multiplicative constant in k.

To complete a basis of $H^1_{\mathrm{dR}}(\mathcal{E})$ we must choose a differential of the second kind η on $\mathcal{E}$. Here we have the freedom to add any multiple of the global form ω to η since this again gives a form of the second kind. We can see, for example from the second line of (4.3) that it must be a form of the second kind, as it will necessarily have a pole in one of the elements of the affine cover which needs to be cancelled by an exact term, which can, in turn, only be a differential of an algebraic form if the residue at the pole is 0. The choice of a form of the second kind corresponds to choosing a splitting of the Hodge filtration

$$0 \subset F^1 H^1_{\mathrm{dR}}(\mathcal{E}) \subset F^0 H^1_{\mathrm{dR}}(\mathcal{E}) = H^1_{\mathrm{dR}}(\mathcal{E}).$$

There is no canonical splitting of the Hodge filtration for (families of) elliptic curves, and we will make a choice for each elliptic curve where we wish to work with a basis of de Rham cohomology explicitly.

In our basis of $H^1_{\mathrm{dR}}(\mathcal{E}, \{P,Q\})$ we also have an element $[\xi_{P,Q}]$ which is unique up to a multiplicative constant since we have an injection $\widetilde{H}^0(\{P,Q\}) \hookrightarrow H^1(\mathcal{E}, \{P,Q\})$ and we may take the image of a generator of the 1-dimensional space $\widetilde{H}^0(\{P,Q\})$.

4.1.1.2 Betti realization

A set of generators of the dual of the Betti realization $H_{B,\sigma}$, i.e. the relative singular homology $H_1^{\mathrm{B}}(\mathcal{E}(\mathbb{C}), \{P,Q\}; \mathbb{Q})$, is given by the classes

$$[\alpha], [\beta], \text{ and } [\gamma_{P,Q}],$$

where α and β are the two standard closed loops on the elliptic curve $\mathcal{E}$, and $\gamma_{P,Q}$ is a path from P to Q on the elliptic curve. Note that $[\gamma_{P,Q}]$ is a chosen lift of a generator of $\tilde{H}_0^{\mathrm{B}}(\{P,Q\}; \mathbb{Q})$ in

$$0 \to H_1^{\mathrm{B}}(\mathcal{E}(\mathbb{C}); \mathbb{Q}) \to H_1^{\mathrm{B}}(\mathcal{E}(\mathbb{C}), \{P,Q\}; \mathbb{Q}) \xrightarrow{\partial} \tilde{H}_0^{\mathrm{B}}(\{P,Q\}; \mathbb{Q}) \to 0.$$

As such it is unique up to adding rational multiples of α and β.

4.1.1.3 Periods

Consider the function

$$\begin{aligned} \exp_{\mathcal{E}} : \mathbb{C} &\longrightarrow \mathcal{E}(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto [\wp(z) : \wp'(z) : 1], \end{aligned}$$

Note that $\exp_{\mathcal{E}}^*\left(\frac{dx}{2y}\right) = dz$ and $\exp_{\mathcal{E}}^*\left(\frac{xdx}{2y}\right) = -\wp(z)dz = d\zeta(z)$, where $\zeta(z)$ is the Weierstrass zeta function.

The comparison isomorphism c_σ is given by integration, which in our case produces the period matrix

$$P_{\mathcal{E},P,Q} = \begin{matrix} [\omega] \\ \omega_1 \\ \omega_2 \\ z_1 - z_2 \end{matrix} \begin{matrix} [\eta] \\ \eta_1 \\ \eta_2 \\ \zeta(z_1) - \zeta(z_2) \end{matrix} \begin{matrix} [\xi_{P,Q}] \\ 0 \\ 0 \\ 1 \end{matrix} \begin{matrix} \\ [\alpha] \\ [\beta] \\ [\gamma_{P,Q}], \end{matrix} \tag{4.4}$$

where ω_1 and ω_2 are the numbers classically known as the periods of the elliptic curve $\mathcal{E}$, while η_1 and η_2 are usually called the quasi-periods of $\mathcal{E}$. The morphsim $\exp_{\mathcal{E}}$ restricts to a complex analytic isomorphism on $\mathbb{C}/\Lambda$, where Λ is the lattice spanned the periods. We define z_1, z_2 to be the points in the preimage of P and Q respectively under this isomorphism.

4.1.2 Motivic periods, de Rham periods, and the coaction

Definition 4.1.3. To an elliptic curve $\mathcal{E}$ over k, with a choice of a form of the first kind ω, and form of the second kind η, along with a basis $\{[\alpha], [\beta]\}$ of the homology $H_1(\mathcal{E}; \mathbb{Q})$, we may associate the following motivic periods (see 2.4 and assume $S = \mathrm{Spec}(k)$ therein):

$$K_1^{\mathfrak{m}} = \left[H^1(\mathcal{E}), [\alpha], [\omega]\right]^{\mathfrak{m}}, \quad K_2^{\mathfrak{m}} = \left[H^1(\mathcal{E}), [\beta], [\omega]\right]^{\mathfrak{m}}, \tag{4.5}$$

as well as

$$E_1^{\mathfrak{m}} = \left[H^1(\mathcal{E}), [\alpha], [\eta]\right]^{\mathfrak{m}}, \quad E_2^{\mathfrak{m}} = \left[H^1(\mathcal{E}), [\beta], [\eta]\right]^{\mathfrak{m}}. \tag{4.6}$$

Furthermore, given two k-points P, Q, and a path $\gamma_{P,Q}$ between then, we define the following two motivic periods of $H^1(\mathcal{E}, \{P, Q\})$

$$F_{P,Q}^{\mathfrak{m}} = \left[H^1(\mathcal{E}, \{P, Q\}), [\gamma_{P,Q}], [\omega]\right]^{\mathfrak{m}}, \quad E_{P,Q}^{\mathfrak{m}} = \left[H^1(\mathcal{E}, \{P, Q\}), [\gamma_{P,Q}], [\eta]\right]^{\mathfrak{m}}. \tag{4.7}$$

We refer to $K_1^{\mathfrak{m}}, K_2^{\mathfrak{m}}$ as *motivic complete elliptic integrals of the first kind*, $E_1^{\mathfrak{m}}, E_2^{\mathfrak{m}}$ as *motivic complete elliptic integrals of the second kind*, and $F_{P,Q}^{\mathfrak{m}}$ and $E_{P,Q}^{\mathfrak{m}}$ as *motivic incomplete elliptic integrals of the first and second kind* respectively.

Definition 4.1.4. We also associate certain de Rham periods to an elliptic curve $\mathcal{E}$ with a choice of basis $\{[\omega],[\eta]\}$ of $H^1_{\mathrm{dR}}(\mathcal{E})$. Denote the dual basis $\{[\omega]^\vee,[\eta]^\vee\}$. Then define

$$\begin{aligned} K_1^{\mathfrak{dr}} &= \left[H^1(\mathcal{E}),[\omega]^\vee,[\omega]\right]^{\mathfrak{dr}}, \quad K_2^{\mathfrak{dr}} = \left[H^1(\mathcal{E}),[\eta]^\vee,[\omega]\right]^{\mathfrak{dr}} \\ E_1^{\mathfrak{dr}} &= \left[H^1(\mathcal{E}),[\omega]^\vee,[\eta]\right]^{\mathfrak{dr}}, \quad E_2^{\mathfrak{dr}} = \left[H^1(\mathcal{E}),[\eta]^\vee,[\eta]\right]^{\mathfrak{dr}} \end{aligned} \tag{4.8}$$

Furthermore, given two k-points P,Q we can complete the basis of $H^1_{\mathrm{dR}}(\mathcal{E},\{P,Q\})$ by adding $[df]$, where f is a meromorphic function on $\mathcal{E}$ such that $f(P)-f(Q)=1$, to the classes $[\omega],[\eta]$. We denote the dual basis associated to it by $\{[\omega]^\vee,[\eta]^\vee,[df]^\vee\}$. Then define

$$F_{P,Q}^{\mathfrak{dr}} = \left[H^1(\mathcal{E},\{P,Q\}),[df]^\vee,[\omega]\right]^{\mathfrak{dr}}, \quad E_{P,Q}^{\mathfrak{dr}} = \left[H^1(\mathcal{E},\{P,Q\}),[df]^\vee,[\eta]\right]^{\mathfrak{dr}}. \tag{4.9}$$

We refer to $K_1^{\mathfrak{dr}},K_2^{\mathfrak{dr}}$ as *de Rham complete elliptic integrals of the first kind*, $E_1^{\mathfrak{dr}},E_2^{\mathfrak{dr}}$ as *de Rham complete elliptic integrals of the second kind*, and $F_{P,Q}^{\mathfrak{dr}}$ and $E_{P,Q}^{\mathfrak{dr}}$ as *de Rham incomplete elliptic integrals of the first and second kind* respectively.

Remark 4.1.5. The above definitions are stated for an elliptic curve over k, i.e. $\mathcal{E}\to\mathrm{Spec}(k)$, and two k-points on it. They naturally generalize to families of elliptic curves $\mathcal{E}\to S$ over a more general base scheme S, along with a family of divisors $\{P,Q\}$. See 2.2, 2.4, and 2.3 for general definitions of families of motivic and de Rham periods. When we have such a dependence on parameters we denote it by

$$K_1^{\mathfrak{m}}(t_1,\dots,t_n) = \left[H^1(\mathcal{E})_{/S},[\alpha],[\omega]\right]^{\mathfrak{dr}}, \quad K_2^{\mathfrak{m}}(t_1,\dots,t_n) = \left[H^1(\mathcal{E})_{/S},[\beta],[\omega]\right]^{\mathfrak{dr}},$$

and analogously for the other motivic and de Rham periods in the previous two definitions. Here $(t_1,\dots,t_n)$ are coordinates on an affine chart of S. When we work with a fixed elliptic curve it will be clear in context which elliptic curve the motivic and de Rham periods defined above are associated to, and we suppress it from the notation to avoid cluttering.

We can apply the general coaction formula 2.3 to the previously defined motivic periods, e.g.

$$\begin{aligned} \Delta(K_1^{\mathfrak{m}}) &= K_1^{\mathfrak{m}}\otimes K_1^{\mathfrak{dr}} + E_1^{\mathfrak{m}}\otimes K_2^{\mathfrak{dr}}, \\ \Delta(F_{P,Q}^{\mathfrak{m}}) &= F_{P,Q}^{\mathfrak{m}}\otimes K_1^{\mathfrak{dr}} + E_{P,Q}^{\mathfrak{m}}\otimes K_2^{\mathfrak{dr}} + 1\otimes F_{P,Q}^{\mathfrak{dr}}. \end{aligned} \tag{4.10}$$

4.1.3 Single-valued periods

4.1.3.1 Defining single-valued periods over a general base

We enrich the category $\mathcal{H}(S)$, defined in 2.1 with a isomorphism of local systems

$$F_\infty : \mathbb{V}_{\mathrm{B}} \xrightarrow{\sim} \sigma^* \mathbb{V}_{\mathrm{B}}$$

where $\sigma : S(\mathbb{C}) \xrightarrow{\sim} S(\mathbb{C})$ is induced by complex conjugation. Let $\mathcal{O}_{\overline{S}}$ be the sheaf of antiholomorphic functions on S^{an}. Then we require that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} & \xrightarrow{c} & \mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}} \\ \downarrow{\scriptstyle id\otimes\rho} & & \downarrow{\scriptstyle F_\infty\otimes\rho} \\ \mathcal{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^{an}} & \xrightarrow{\overline{c}} & \sigma^*\mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{\overline{S}^{an}} \end{array} \tag{4.11}$$

where $\rho : f \mapsto \overline{f}$, and $\overline{c}$ is the pullback of the comparison isomorphism by σ.

The morphism F_∞ induces an isomorphism of rings of motivic periods

$$F_\infty : \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m},\overline{X},Y}_{\mathcal{H}(\overline{S})},$$

where $\mathcal{H}(\overline{S})$ is the category defined in the same way as $\mathcal{H}(S)$ but the comparison isomorphism is antiholomorphic

$$\overline{c} : \mathcal{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^{an}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{\overline{S}^{an}}.$$

The composition of morphisms $\sigma^* F_\infty$ defines an isomorphism

$$\sigma^* F_\infty : \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(\overline{S})},$$

which sends

$$[\mathcal{V}, \gamma, \omega] \mapsto \left[\overline{\mathcal{V}}, \gamma \circ \sigma^* F_\infty, \omega\right],$$

where $\overline{\mathcal{V}} = (\sigma^*\mathbb{V}_{\mathrm{B}}, \mathcal{V}_{\mathrm{dR}}, \overline{c})$. Recall that the period map on $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}$ takes values in $M_{X,Y}(S(\mathbb{C}))$ which is the ring of multivalued meromorphic functions on $S(\mathbb{C})$ with a prescribed branch on X. Similarly the period map on $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(\overline{S})}$ takes values in $\overline{M}_{X,Y}(S(\mathbb{C}))$ which is the ring of quotients of antiholomorphic functions on $S(\mathbb{C})$. We have $\mathrm{per}(\sigma^* F_\infty \xi) = \overline{\mathrm{per}(\xi)}$, where ξ is a motivic period over S.

Denote by $\mathcal{P}$ the ring $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(\overline{S})}$, and by f the morphism $\sigma^* F_\infty$ composed with $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(\overline{S})} \to \mathcal{P}$ sending $x \mapsto 1 \otimes x$. Let ι the morphism $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \to \mathcal{P}$ sending $x \mapsto x \otimes 1$. We have two $\mathcal{P}$-points

$$\iota, f \in \mathrm{Hom}\left(\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}, \mathcal{P}\right) = \mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}}, \omega^X_{\mathrm{B}})(\mathcal{P}),$$

of the scheme $\mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}},\omega^X_{\mathrm{B}})$, which is a torsor over $\mathrm{Aut}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}})$, hence we have

$$\mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}},\omega^X_{\mathrm{B}})(\mathcal{P})\times \mathrm{Aut}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}})(\mathcal{P})\to \mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{\mathrm{dR}},\omega^X_{\mathrm{B}})(\mathcal{P}).$$

Therefore there exists a unique $\mathcal{P}$-point of $G^{\mathfrak{dr},Y}_{\mathcal{H}(S)}$, i.e. a morphism $\mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)}\to\mathcal{P}$, which we denote by $\mathrm{s}^{\mathfrak{m}}$ and which satisfies

$$f\circ \mathrm{s}^{\mathfrak{m}}=\iota.$$

We call it the *single-valued morphism*.

The morphism $\mathrm{s}^{\mathfrak{m}}$ is computed for each object $\mathcal{V}\in\mathrm{Ob}(\mathcal{H}(S))$ by the composition

$$\omega^Y_{\mathrm{dR}}(\mathcal{V})\otimes\mathcal{P}\xrightarrow{c^{\mathfrak{m}}_{\mathcal{V}}}\omega^X_{\mathrm{B}}(\mathcal{V})\otimes\mathcal{P}\xrightarrow{f^{\infty}_{\mathcal{V}}}\omega^X_{\mathrm{B}}(\mathcal{V})\otimes\mathcal{P}\xrightarrow{(c^{\mathfrak{m}}_{\mathcal{V}})^{-1}}\omega^Y_{\mathrm{dR}}(\mathcal{V})\otimes\mathcal{P},$$

where $c^{\mathfrak{m}}_{\mathcal{V}}$ is the extension of scalars of the comparison isomorphism c to $\mathcal{P}$, and the morpshim $f^{\infty}_{\mathcal{V}}$ sends $x\otimes y\otimes 1\mapsto x\otimes 1\otimes\sigma^*F_\infty y$, where $x\in\omega^Y_{\mathrm{dR}}(\mathcal{V})$, $y\in\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}$, $\sigma^*F_\infty y\in\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(\bar{S})}$. If $C^{\mathfrak{m}}_{\mathcal{V}}$ is the *universal period matrix*, i.e. the matrix representing $c^{\mathfrak{m}}_{\mathcal{V}}$, then we have $\mathrm{s}^{\mathfrak{m}}_{\mathcal{V}}$ is represented by

$$\left(\sigma^*F_\infty C^{\mathfrak{m}}_{\mathcal{V}}\right)^{-1}C^{\mathfrak{m}}_{\mathcal{V}}.$$

Applying the period morphism, and defining $\mathrm{s}=\mathrm{per}(\mathrm{s}^{\mathfrak{m}})$, we get that s is represented by

$$\mathrm{s}_{\mathcal{V}}=\left(\overline{C_{\mathcal{V}}}\right)^{-1}C_{\mathcal{V}},\tag{4.12}$$

where $C_{\mathcal{V}}$ is the matrix representing the comparison isomorphism for the object $\mathcal{V}\in\mathrm{Ob}(\mathcal{H}(S))$. Note that this map is indeed invariant under the change of Betti basis, as this amounts to replacing $C_{\mathcal{V}}$ with $PC_{\mathcal{V}}$ for some $P\in\mathrm{GL}(\omega^X_{\mathrm{B}}(\mathcal{V});\mathbb{Q})$, and complex conjugation acts trivially on the coefficients of P as they are rational.

Example 4.1.6. Continuing the example of the motivic logarithm from 2.4.2 we can write down the universal period matrix associated to $\mathcal{V}\in\mathrm{Ob}(\mathcal{H}(S))$ (defined in 2.4.2):

$$c^{\mathfrak{m}}_{\mathcal{V}}=\begin{pmatrix}\mathbb{L}^{\mathfrak{m}} & 0\\ \log^{\mathfrak{m}}(x) & 1\end{pmatrix}\tag{4.13}$$

Computing $(\overline{c^{\mathfrak{m}}_{\mathcal{V}}})^{-1}c^{\mathfrak{m}}_{\mathcal{V}}$ we get

$$\mathrm{s}^{\mathfrak{m}}(\log^{\mathfrak{dr}}(x))=\log^{\mathfrak{m}}(x)+\overline{\log^{\mathfrak{m}}}(x).$$

Applying the period map to the right hand side we get $2\log|x|$. More generally one can define motivic multiple polylogarithms and their de Rham versions as motivic

and de Rham periods of the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ (see [14, §10.6.2]). An analogous computation to the one for the logarithm then leads to

$$\mathrm{s}^{\mathfrak{m}}(\mathrm{Li}_2^{\partial_{\mathfrak{r}}}(x)) = \mathrm{Li}_2^{\mathfrak{m}}(x) - \overline{\mathrm{Li}_2^{\mathfrak{m}}}(x) + (\log^{\mathfrak{m}}(x) + \overline{\log^{\mathfrak{m}}}(x))\overline{\mathrm{Li}_1^{\mathfrak{m}}}(x).$$

The image of the right hand side under the period homomorphism is $2i$ times the Bloch-Wigner dilogarithm.

Remark 4.1.7. For some general results on the single-valued pairing (over a fixed subfield of $\mathbb{C}$) see [16]. In its sequel [17] the authors use these results to prove a conjecture of Stieberger that relates open and closed string amplitudes at tree level via the single-valued morphism, as well as to show that the KLT formula expressing closed string amplitudes as quadratic expressions in open string amplitudes follows from a variant of the single-valued formalism for cohomology with coefficients in a local system.

4.1.3.2 Single-valued periods and elliptic curves

Let us now consider the single-valued periods associated to the de Rham periods defined in 4.1.4, where we let $\mathcal{E} \to \mathcal{M}_{1,1}$ be the universal elliptic curve. Let $P_{\mathcal{E}}$ be the period matrix of $H^1(\mathcal{E})$ with respect to some chosen basis $[\omega],[\eta]$ of $H^1_{\mathrm{dR}}(\mathcal{E})$ and $\{[\alpha],[\beta]\}$ of $H_1(\mathcal{E};\mathbb{Q})$.

$$\begin{matrix} & [\omega] & [\eta] \\ \begin{matrix}\omega_1 \\ \omega_2\end{matrix} & \begin{pmatrix}\omega_1 & \eta_1 \\ \omega_2 & \eta_2\end{pmatrix} & \begin{matrix}[\alpha] \\ [\beta]\end{matrix} \end{matrix}. \tag{4.14}$$

Let $\tau = \frac{\omega_2}{\omega_1}$, $\lambda = \frac{\omega_1}{2\pi i}$, and let $\mathbb{G}_2^*(\tau) = \mathbb{G}_2(\tau) + \frac{1}{8\pi\,\mathrm{Im}\,\tau}$, where

$$\mathbb{G}_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1}\sigma_1(n)q^n.$$

The function $\mathbb{G}_2^*(\tau)$ is the modified real analytic version of the Eisenstein series of weight 2, which transforms like a modular form of weight 2. Here σ_1 is the divisor sum function. As pointed out in [16, §6.2] we obtain the single-valued period matrix of $H^1(\mathcal{E})$

$$\overline{P_{\mathcal{E}}}^{-1}P_{\mathcal{E}} = \begin{pmatrix} f_{1,1}(\tau) & f_{1,2}(\tau) \\ f_{2,1}(\tau) & f_{2,2}(\tau)\end{pmatrix},$$

where the entries of the matrix are given by

$$\begin{aligned} f_{1,1}(\tau) &= -\lambda\overline{\lambda}^{-1}8\pi\,\mathrm{Im}(\tau)\overline{\mathbb{G}_2^*(\tau)} \\ f_{1,2}(\tau) &= \lambda^{-1}\overline{\lambda}^{-1}(4\pi\,\mathrm{Im}\,\tau)^{-1}((-8\pi\,\mathrm{Im}(\tau))^2\mathbb{G}_2^*(\tau)\overline{\mathbb{G}_2^*(\tau)} - 1) \\ f_{2,1}(\tau) &= -\lambda\overline{\lambda}4\pi\,\mathrm{Im}\,\tau, \quad f_{2,2}(\tau) = -\lambda^{-1}\overline{\lambda} - 8\pi\,\mathrm{Im}(\tau)\mathbb{G}_2^*(\tau). \end{aligned} \tag{4.15}$$

This follows from the relations shown by Fricke and Legendre respectively

$$\mathbb{G}_2(\tau) = -\frac{1}{2}\frac{\omega_1\eta_1}{(2\pi i)^2} \quad \text{and} \quad \omega_1\eta_2 - \eta_1\omega_2 = 2\pi i.$$

By the definition of the single-valued map we have

$$\begin{aligned} \mathrm{s}(K_1^{\mathfrak{dr}}(\tau)) &= f_{1,1}(\tau), \quad \mathrm{s}(K_2^{\mathfrak{dr}}(\tau)) = f_{1,2}(\tau) \\ \mathrm{s}(E_1^{\mathfrak{dr}}(\tau)) &= f_{2,1}(\tau), \quad \mathrm{s}(E_2^{\mathfrak{dr}}(\tau)) = f_{2,2}(\tau). \end{aligned} \tag{4.16}$$

Let and r, s be the powers of the prefactors λ and $\overline{\lambda}$ in (4.15) respectively. Then these functions transform as

$$f(\gamma\tau) = (c\tau + d)^{-r}(c\overline{\tau} + d)^{-s} f(\tau),$$

for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Remark 4.1.8. The above single-valued functions are the starting point for Brown's theory of real analytic modular forms with two weights — see [11] and sequels. These are functions on the upper half plane which satisfy a modular transformation property such as the one in the previous equation, for arbitrary (r, s), and which have an expansion of the form

$$f(\tau) = \sum_{k=-N}^{N} \sum_{m,n\geq 0} a_{m,n}^{(k)} (-2\pi \operatorname{Im}\tau)^k q_\tau^m \overline{q_\tau}^n,$$

where $q_\tau = e^{2\pi i\tau}$. They are generalized by the single-valued functions we define next.

Now let us consider the motive $H^1(\mathcal{E}, \{P, Q\})$, and its period matrix $P_{\mathcal{E},P,Q}$. We have two de Rham periods: $F_{P,Q}^{\mathfrak{dr}}(\tau)$ and $E_{P,Q,\eta}^{\mathfrak{dr}}(\tau)$, which do not appear as de Rham periods of $H^1(\mathcal{E})$. We summarise the properties of their single-valued periods.

Proposition 4.1.1. *The single-valued periods associated to $F_{P,Q}^{\mathfrak{dr}}(\tau)$ and $E_{P,Q,\eta}^{\mathfrak{dr}}(\tau)$ are*

$$\begin{aligned} \mathrm{s}\left(F_{P,Q}^{\mathfrak{dr}}(\tau)\right) &= \overline{(\zeta(z_1) - \zeta(z_2))} f_{2,1}(\tau) + (\overline{z_1} - \overline{z_2}) f_{1,1}(\tau) + 2\pi i(z_1 - z_2) \\ \mathrm{s}\left(E_{P,Q,\eta}^{\mathfrak{dr}}(\tau)\right) &= \overline{(\zeta(z_1) - \zeta(z_2))} f_{2,2}(\tau) + (\overline{z_1} - \overline{z_2}) f_{1,2}(\tau) + 2\pi i(\zeta(z_1) - \zeta(z_2)). \end{aligned} \tag{4.17}$$

Denote the two functions by $f_{3,1}(\tau, z_1, z_2)$ and $f_{3,2}(\tau, z_1, z_2)$ respectively. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then these functions transform as

$$\begin{aligned} f_{3,1}\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}\right) &= (c\tau + d)^{-1} f_{3,1}(\tau, z_1, z_2) \\ f_{3,2}\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}\right) &= (c\tau + d) f_{3,2}(\tau, z_1, z_2). \end{aligned} \tag{4.18}$$

They have expansions of the form

$$f(\tau, z_1, z_2) = \sum_{-N\le k_1,k_2,k_3\le N} \sum_{\substack{m_i,n_i\ge 0\\ 1\le i\le 3}} a_{m,n}^{(k_1,k_2,k_3)} z_1^{k_1} z_2^{k_2} (-2\pi \operatorname{Im}\tau)^{k_3} q_i^{m_i} \overline{q_i}^{n_i},$$

where $q_1 = q_\tau = e^{2\pi i\tau}, q_2 = e^{2\pi i z_1}, q_3 = e^{2\pi i z_2}$.

Proof. This follows from (4.12) by computing the matrix $(\overline{P_{\mathcal{E},P,Q}})^{-1} P_{\mathcal{E},P,Q}$, where $P_{\mathcal{E},P,Q}$ is defined in (4.4), and noting that the Weierstrass zeta function transforms as

$$\zeta\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)\zeta(\tau, z).$$

We have assumed so far that the Weierstrass zeta function is defined with respect to the lattice generated by $(\tau, 1)$, and here we have written the dependence explicitly. Moreover, z_1, z_2 are also defined with respect to the lattice generated by $(\tau, 1)$, and correspond to $\frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}$ with respect to the generators of the lattice given by $\gamma(\tau, 1)^T$. The expansion follows from the previous computation for $f_{1,1}, f_{1,2}, f_{2,1}$ and $f_{2,2}$ as well as the well known expansion of the Weierestrass zeta function (see [36, Ch. XVIII §3])

$$\zeta(\tau, z) = \eta_2 z + \pi i \frac{q_z+1}{q_z-1} + 2\pi i \sum_{n\ge 1} \frac{q_\tau^n/q_z}{1-q_\tau q_z} - \frac{q_\tau^n q_z}{1-q_\tau^n q_z},$$

where $q_z = e^{2\pi i z}$, where we then expand the terms $\frac{-1}{1-q_z}, \frac{1}{1-q_\tau q_z}$, and $\frac{1}{1-q_\tau^n q_z}$. □

4.2 The sunrise

We denote the sunrise graph (figure 4) by G. Its first and second Symanzik polynomials are

$$\begin{aligned} \Psi_G &= \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3, \text{ and} \\ \Xi_G(m, q) &= q_1^2\alpha_1\alpha_2\alpha_3 + (m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3)\Psi_G. \end{aligned} \tag{4.19}$$

We note that the zero locus of the second Symanzik polynomial, along with a choice of a rational point, defines an elliptic curve $\mathcal{E} = V(\Xi_G(m, q))$ over a field k for fixed values of $m_1, m_2, m_3, q_1^2 \in k \subset \mathbb{C}$. We will allow the masses and momenta to vary, and work with the family of elliptic curves $\mathcal{E} \to S$, where $S = K_{1,3}^{gen} \setminus L_G$ is the complement of the Landau variety of G denoted L_G (see definition 2.5.7) in the space of generic kinematics $K_{1,3}^{gen}$ (see definition 2.5.5). Note that we have the points $[1:0:0], [0:1:0], [0:0:1]$ on $\mathcal{E}$ for any value of $m_1, m_2, m_3, q_1^2 \in S(\mathbb{C})$.

4.2.1 The motive

We define the (restricted) motive of the sunrise graph according to definition 2.5.8. There are three motic subgraphs of G, spanned by any of the three pairs of internal edges, which implies that we must blow-up $\mathbb{P}^2$ at $\{(1:0:0),(0:1:0),(0:0:1)\}$ (see 2.5.6 and the discussion thereafter). Denote the blow-up by $\pi_G : P^G \to \mathbb{P}^2$, and let, by abuse of notation, $\mathcal{E}$ denote the strict transform of the vanishing locus of the second Symanzik polynomial. Denote the total transform of $\Delta_G = \bigcup_{e\in E_G} V(\alpha_e)$ in $\mathbb{P}^2$ by D. The restricted motive of the sunrise graph is

$$mot'_G = H^2(P^G \smallsetminus \mathcal{E}, D \smallsetminus \mathcal{E} \cap D)_{/S}. \tag{4.20}$$

We denote the exceptional divisors by D_{-1}, D_{-2}, D_{-3}, their union by D_E, and the strict transforms of the faces of the coordinate simplex in $\mathbb{P}^2$ by D_1, D_2, D_3. Then geometrically we have the following picture.

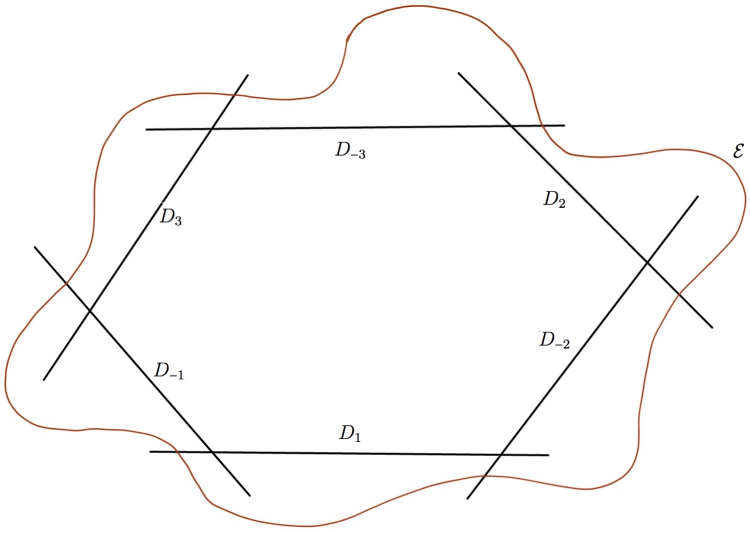


Figure 4.2: The geometry underlying the sunrise graph after blowing up.

The following proposition holds in the category $\mathcal{H}_k$ for any fiber M of mot'_G over $\mathcal{S}(\mathbb{C})$ defined over $k \subset \mathbb{C}$.

Proposition 4.2.1. *The weight-graded (semi-simple) object associated to M is*

$$gr^W M = H^1(\mathcal{E})(-1) \bigoplus \mathbb{Q}(-1)^{\oplus 3} \bigoplus \mathbb{Q}(0)$$

Proof. Since $\mathcal{E}$ is smooth the residue/Gysin long exact sequence applies to

$$M = H^2(P^G \smallsetminus \mathcal{E}, D \smallsetminus \mathcal{E} \cap D)$$

so that we have

$$0 \to H^2(P^G, D) \to M \xrightarrow{res} H^1(\mathcal{E}, D \cap \mathcal{E})(-1) \xrightarrow{Gysin} H^3(P^G, D). \tag{4.21}$$

We can apply the relative cohomology long exact sequence to $H^2(P^G, D)$, part of which is

$$H^1(P^G) \to H^1(D) \to H^2(P^G, D) \to H^2(P^G) \to H^2(D).$$

On the right we have $H^2(D) = \mathbb{Q}(-1)^{\oplus 6}$ since D is a union of 6 projective lines — the three exceptional divisors and the three strict transforms of the coordinate hyperplanes. We have $H^2(P^G) = \mathbb{Q}(-1)^{\oplus 4}$, where one of the classes corresponds to $H^2(\mathbb{P}^2)$ and is represented by a general projective line in $\mathbb{P}^2$, while the remaining three correspond to the three exceptional divisors. The rightmost arrow is therefore injective. Furthermore, we have $H^1(P^G) = 0$, and $H^1(D) = \mathbb{Q}(0)$ since D is a union of 6 projective lines such that each two meet in a single point. It follows that $H^2(P^G, D) = \mathbb{Q}(0)$.

We can apply the relative cohomology long exact sequence to $H^1(\mathcal{E}, D \cap \mathcal{E})(-1)$ to get

$$0 \to \widetilde{H}^0(D \cap \mathcal{E})(-1) \to H^1(\mathcal{E}, D \cap \mathcal{E})(-1) \to H^1(\mathcal{E})(-1) \to 0$$

where on the left hand side we have reduced cohomology. There are six points of intersection $\mathcal{E} \cap D$ so the rank of $\widetilde{H}^0(D \cap \mathcal{E})(-1)$ is 5.

Finally, we need to know the image of the residue map in (4.21), or equivalently the kernel of the Gysin morphism. From the relative cohomology long exact sequence we have that

$$H^2(P^G) \to H^2(D) \to H^3(P^G, D) \to H^3(P^G).$$

Here $H^3(P^G) = 0$, while $H^2(P^G)$ and $H^2(D)$ have ranks 4 and 6 respectively. It follows that $H^3(P^G, D)$ has rank 2. One can see that the Gysin map in (4.21) is surjective, for example by showing that $H^3(P^G \setminus \mathcal{E}, D \setminus \mathcal{E} \cap D) = 0$ using the relative cohomology long exact sequence. Therefore we coclude that

$$gr^W \mathrm{Im}(res) = H^1(\mathcal{E})(-1) \bigoplus \mathbb{Q}(-1)^{\oplus 3}.$$

The result follows. □

4.2.2 Basis of de Rham cohomology

4.2.2.1 The Feynman integrand and the holomorphic form on $\mathcal{E}$

We know from 2.5.2 that $[\pi_G^*(\omega_G(m,q))]$, where $\omega_G(m,q) = \frac{\Omega_G}{\Xi_G}$ is the Feynman integrand for the sunrise graph in $d = 2$ space-time dimensions, gives a section of $(mot'_G)_{\mathrm{dR}}$ over the complement of the Landau variety. Alternatively we can check directly that the form $\pi_G^*(\omega_G(m,q))$ does not have any poles along the exceptional divisors D_{-1}, D_{-2}, D_{-3} by an analogous argument to the one in the proof of 3.3.4. We also note that we have a commutative diagram

$$\begin{array}{ccc} H^2(\mathbb{P}^2 \setminus \mathcal{E}) & \longrightarrow & H^2(P^G \setminus \mathcal{E}) \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ H^1(\mathcal{E})(-1) & \xrightarrow{=} & H^1(\mathcal{E})(-1) \end{array} \tag{4.22}$$

The map on the top row is induced by $i \circ p^{-1}$

$$P^G \setminus \mathcal{E} \xleftarrow{i} P^G \setminus (\mathcal{E} \cup D_E) \xrightarrow[\cong]{p} \mathbb{P}^2 \setminus \mathcal{E},$$

where on the right we have the restriction of the projection π_G to the complement of the exceptional divisors where it is an isomorphism. A well known theorem of Griffiths [30, 8.6] tells us that, for $X \subset \mathbb{P}^n$ a smooth projective hypersurface, we have an isomorphism induced by the residue map

$$A_k^n(X)/dA_{k-1}^{n-1}(X) \underset{\cong}{\overset{res}{\to}} F^{n-k}H^{n-1}(X;\mathbb{C})_{prim}$$

where on the left hand side we have $A_k^n(X)$ rational n-forms on $\mathbb{P}^n$ with poles of order k along X, and on the right hand side we have the primitive part of the singular cohomology of X[2]. When n is even, as is our case, all cohomology of X is primitive [30, Lemma 8.20]. Using this result in the case $n = 2, k = 1$ and $X = \mathcal{E}$, along with the commutativity of the diagram (4.22), we obtain that $res((\pi_G^*(\omega_G(m,q)))$ is the holomorphic differential on $\mathcal{E}$, or, in other words,

$$res((\pi_G^*(\omega_G(m,q))) \in F^1H^1_{\mathrm{dR}}(\mathcal{E}) = F^2H^1_{\mathrm{dR}}(\mathcal{E})(-1).$$

We also note that, since res is a homomorphism of mixed Hodge structures, $[\pi_G^*(\omega_G(m,q))]$ is of weight 3. This will be used later when we complete the de Rham basis and show that the Feynman integral associated to the sunrise graph can be expressed in terms of integrals of modular forms.

[2]For $X \subset \mathbb{P}^n$ a smooth projective variety, and $\xi \in H^2(X;\mathbb{C})$ a hyperplane section the primitive cohomology of X is defined as $H^{n-k}(X;\mathbb{C})_{prim} = \ker\left(\cup\xi^{k+1} : H^{n-k}(X;\mathbb{C}) \to H^{n+k+2}(X;\mathbb{C})\right)$.

4.2.2.2 Weight 0

Lemma 4.2.2. *Let ν_0 be the integrand of a bubble graph (3.2) obtained by deleting an edge of the sunrise graph G in $d = 4$ dimensions. We consider a face map (2.4.2)*

$$\phi : mot^{\natural}_{G\smallsetminus e_i} \to mot'_G$$

where $mot^{\natural}_{G\smallsetminus e_i}$ denotes the motive $H^1(\mathbb{P}^1 \smallsetminus [1:-1], \{[1:0],[0:1]\})$[3]. Then $[\phi_{\mathrm{dR}}(\nu_0)]$ generates $W_0(mot'_G)_{\mathrm{dR}}$, where ϕ_{dR} is the face map ϕ in the de Rham realization.

Proof. Without loss of generality we delete the edge e_3 of G. In $d = 4$ the Feynman integrand of $G \smallsetminus e_3$ is

$$\nu_0 = \frac{\alpha_1 d\alpha_2 - \alpha_2 d\alpha_1}{(\alpha_1+\alpha_2)^2}.$$

Let $\mathbb{A}_{12,2} = \mathrm{Spec}(k[u,v])$ be an affine open in P^G where we have $(\pi_G|_{\mathbb{A}_{12,2}})^*(\alpha_1) = u$ and $(\pi_G|_{\mathbb{A}_{12,2}})^*(\alpha_2) = uv$, and let us consider the form

$$\omega_0 = \frac{m_3^4\alpha_3^4(\alpha_1 d\alpha_2 - \alpha_2 d\alpha_1)}{\Xi_G(m,q)^2}.$$

Note that

$$\pi_G^*(\omega_0)|_{\mathbb{A}_{12,2}} = \frac{m_3^4 dv}{(uv + (m_1^2 u + m_2^2 uv + m_3^2)(uv+v+1))^2},$$

When this is restricted the exceptional divisor in $\mathbb{A}_{12,2}$, which is given by $u = 0$ in this chart, we get

$$\pi_G^*(\omega_0)|_{\mathbb{A}_{12,2}\cap\{u=0\}} = \frac{dv}{(1+v)^2}.$$

We can see then that we have

$$\pi_G^*(\omega_0)|_{\mathbb{A}_{12,2}\cap\{u=0\}} = \phi_{\mathrm{dR}}|_{\alpha_1\neq 0}(\nu_0|_{\alpha_1\neq 0}).$$

An analogous computation, where we replace $\mathbb{A}_{12,2}$ by $\mathbb{A}_{12,1}$ where $(\pi_G|_{\mathbb{A}_{12,1}})^*(\alpha_1) = uv$ and $(\pi_G|_{\mathbb{A}_{12,1}})^*(\alpha_2) = v$, and $\alpha_1 \neq 0$ by $\alpha_2 \neq 0$, implies that $[\pi_G^*(\omega_0)|_{D_{-3}}] = [\phi_{\mathrm{dR}}(\nu_0)]$. The fact that the restriction of $\pi_G^*(\omega_0)$ to the exceptional divisor is not exact implies that its differential defines a non-trivial class in relative cohomology

$$[\phi_{\mathrm{dR}}(\nu_0)] = [\pi_G^*(\omega_0)|_{D_{-3}}] = [d(\pi_G^*(\omega_0))] \in (mot'_G)_{\mathrm{dR}}$$

This class spans the weight zero subspace $W_0(mot'_G)_{\mathrm{dR}}$. □

[3] This motive can be viewed as another restricted motive of the bubble graph where we consider the complement of the vanishing locus of the first Symanzik polynomial. Compare with the motives defined in 3.1.4.2.

4.2.2.3 Subdivision of edges

The sunrise graph is similar to the triangle graph with all three vanishing masses (see 3.3.5, and 3.3.2.2) in that only a 1-dimensional subspace of the de Rham realization of its motive is spanned by the image under the face map 2.4.2 of the de Rham class of the Feynman integrand of a subquotient graph with a strictly smaller number of edges. This follows from the relative cohomology long exact sequence

$$0 \to H^1(D \setminus \mathcal{E} \cap D) \to mot'_G \to H^2(P^G \setminus \mathcal{E}) \to 0,$$

where on the left hand side we have $H^1(D \setminus \mathcal{E} \cap D) = \mathbb{Q}(0)$. The reader is invited to compare this to the case of the four-edge one-loop graph with all three non-vanishing masses (see (3.22) and (3.23)), where the images of the de Rham classes of Feynman integrands of subquotient graphs with strictly fewer edges, along with the Feynman integrand of the four-edge one-loop graph itself, form a basis of the de Rham realization of its motive.

We will now show that we can relate certain elements in the de Rham realization of a graph motive, which cannot be realized as images under face maps, to classes of Feynman integrands (2.17) of Feynman graphs obtained by subdividing an edge of a given graph (Figure 4.3). The construction is general, and holds for any (connected) Feynman graph G. We will use it in the rest of this section to complete the basis of the de Rham realization of the sunrise graph.

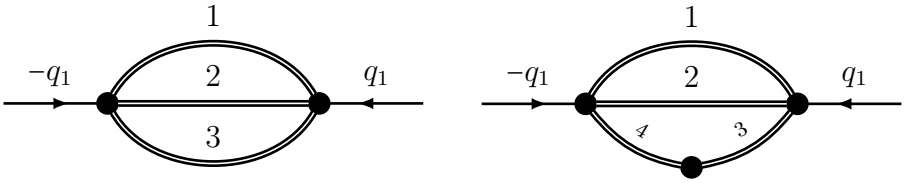


Figure 4.3: Example of obtaining a graph from a given graph by subdividing an edge.

Definition 4.2.1. Given a Feynman graph G and an internal edge $e \in E_G$ connected to vertices $v_{e,1}, v_{e,2} \in V_G$, we denote by $G_{s(e)}$ the graph obtained by adding a vertex v and two edges e_1, e_2 to $G \setminus e$ such that e_1 is connected to $v_{e,1}$ and v, and e_2 is connected to v and $v_{e,2}$. If e has a non-zero mass m_e in G then both edges e_1 and e_2 have mass m_e in $G_{s(e)}$, otherwise e_1 and e_2 have zero mass. We say that $G_{s(e)}$ is obtained by subdividing e in G. Define $\Gamma_{s(I)}$, where $I = \{e_1^{k_1}, \ldots, e_{N_G}^{k_{N_G}}\}$ for each $k_i \geq 0$ an integer, to be the graph obtained by subdividing k_i times the edge e_i for $1 \leq i \in N_G$.

We will order the edges of G so that the parameter α_{N_G} corresponds to the edge e in E_G, and that $\alpha_{N_G}, \alpha_{N_G+1}$ correspond to the edges e_1, e_2 in $E_{G_{s(e)}}$ obtained by subdividing e. For a given graph G we will consider the following morphism

$$\begin{aligned} \rho : \mathbb{A}^1 \times \mathbb{P}^{N_G-1} &\longrightarrow \mathbb{P}^{N_G} \\ (y, \alpha_1, \ldots, \alpha_{N_G}) &\mapsto (\alpha_1, \ldots, y\alpha_{N_G}, (1-y)\alpha_{N_G}). \end{aligned} \tag{4.23}$$

Lemma 4.2.3. *Let $X_{\Xi_G} = V(\Xi_G)$ and $X_{\Psi_G} = V(\Psi_G)$ be the graph hypersurfaces associated to G. Then*

$$\rho^*(X_{\Xi_{G_{s(e)}}}) = X_{\Xi_G} \quad \text{and} \quad \rho^*(X_{\Psi_{G_{s(e)}}}) = X_{\Psi_G}.$$

Moreover, we have

$$\rho^*\left(\omega_{G_{s(e)}}\right) = -\frac{\alpha_{N_G}\Psi_G}{\Xi_G}\omega_G \wedge dy.$$

Proof. The first statement follows simply from the fact that we obtain the polynomials $\Xi_{G_{s(e)}}$ and $\Psi_{G_{s(e)}}$ from the polynomials Ξ_G and Ψ_G by replacing α_{N_G} by $\alpha_{N_G} + \alpha_{N_G+1}$ in the latter. This follows from their definition 2.5.2 in terms of spanning trees of G and $G_{s(e)}$. For each spanning tree T of G which includes the edge e we obtain a spanning tree of $G_{s(e)}$ spanned by $E_T \cup \{e_1, e_2\}$, where E_T are the edges of T. For each spanning tree T of G which didn't include e we get two spanning trees of $G_{s(e)}$, one spanned by $E_T \cup e_1$ and the other $E_T \cup e_2$, as we must attach the new vertex to the rest of the spanning tree. This accounts for all spanning trees of $G_{s(e)}$. Each spanning 2-tree, which are used to define $\Xi_{G_{s(e)}}$, contains either one or both of the new edges e_1, e_2 and is obtained from spanning 2-trees of G analogously to the previous case of spanning trees. There is no term in $\Xi_{G_{s(e)}}$ corresponding to a spanning 2-tree that contains only the new vertex in $G_{s(e)}$ as one of its connected components as there is no incoming momentum to the new vertex.

We will now show that $\rho^*(\Omega_{G_{s(e)}}) = -\alpha_{N_G}\Omega_G \wedge dy$, where Ω_G is defined in (2.18). This, combined with the first statement and the definition of the Feynman integrand ω_G of a graph G (2.17) — the extra factor $\frac{\Psi_G}{\Xi_G}$ comes from $G_{s(e)}$ having one more edge than G but the same number of loops — then implies the second statement of the lemma.

For $i \notin \{N_G, N_G + 1\}$ we have

$$\begin{aligned} &\rho^*((-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_{N_G} \wedge d\alpha_{N_G+1}) \\ &= (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d(y\alpha_{N_G}) \wedge d((1-y)\alpha_{N_G}) \\ &= (-1)^{i+1} \alpha_{N_G} \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots \wedge d\alpha_{N_G} \wedge dy. \end{aligned} \tag{4.24}$$

For $i = N_G$ we have

$$\begin{aligned}\rho^*((-1)^{N_G}\alpha_{N_G}d\alpha_1\wedge\ldots\wedge\widehat{d\alpha_{N_G}}\wedge d\alpha_{N_G+1}) &= (-1)^{N_G}y\alpha_{N_G}d\alpha_1\wedge\ldots\wedge d((1-y)\alpha_{N_G})\\ &= (-1)^{N_G+1}y\alpha_{N_G}^2 d\alpha_1\wedge\ldots\wedge d\alpha_{N_G-1}\wedge dy + (-1)^{N_G}y\alpha_{N_G}d\alpha_1\wedge\ldots\wedge d\alpha_{N_G}+\\ &+ (-1)^{N_G+1}y^2\alpha_{N_G}d\alpha_1\wedge\ldots\wedge d\alpha_{N_G}.\end{aligned} \tag{4.25}$$

Finally for $i = N_G + 1$ we have

$$\begin{aligned}&\rho^*((-1)^{N_G+1}\alpha_{N_G+1}d\alpha_1\wedge\ldots\wedge d\alpha_{N_G}) = (-1)^{N_G+1}(1-y)\alpha_{N_G}d\alpha_1\wedge\ldots\wedge d(y\alpha_{N_G})\\ &= (-1)^{N_G+1}y\alpha_{N_G}d\alpha_1\wedge\ldots d\alpha_{N_G} + (-1)^{N_G+1}\alpha_{N_G}^2 d\alpha_1\wedge\ldots\wedge d\alpha_{N_G-1}\wedge dy+\\ &+ (-1)^{N_G+2}y^2\alpha_{N_G}d\alpha_1\wedge\ldots\wedge d\alpha_{N_G} + (-1)^{N_G+2}y\alpha_{N_G}^2 d\alpha_1\wedge\ldots\wedge d\alpha_{N_G-1}\wedge dy.\end{aligned} \tag{4.26}$$

Adding the last two equations together we obtain the claim. □

We will show that the morphism ρ lifts to a morphism of the blow-ups $P^G \to \mathbb{P}^{N_G-1}$ and $P^{G_{s(e)}} \to \mathbb{P}^{N_G}$ associated to the graphs G and $G_{s(e)}$ (see 2.5.2 for definitions). We will need the following lemma.

Lemma 4.2.4. *For each motic subgraph of G containing the edge e there is exactly one motic subgraph of $G_{s(e)}$ containing both e_1 and e_2. For each motic subgraph of G which does not contain e there is exactly one motic subgraph of $G_{s(e)}$ containing neither e_1 nor e_2.*

Proof. A subgraph γ of G can be motic because it is mass-momentum spanning and each of its subgraphs is not mass-momentum spanning. If such γ contains e then the corresponding subgraph in $G_{s(e)}$ must contain both e_1, e_2 because they both have the same mass as e in G. Alternatively, a subgraph γ of G can be motic because all of its subgraphs have a lower loop order than itself. If such γ contains e in G then the corresponding subgraph in $G_{s(e)}$ must contain both e_1, e_2, otherwise it would have a lower loop order than γ. Finally we cannot have an additional motic subgraph $G_{s(e)}$ containing only one of e_1, e_2 because adding one of them to a subgraph cannot make it mass-momentum spanning (since the other has the same mass), or increase the loop order of a subgraph. □

Proposition 4.2.5. *There exists a morphism $\tilde{\rho}$ such that the following diagram commutes*

$$\begin{array}{ccc}\mathbb{A}^1\times P^G & \xrightarrow{\tilde{\rho}} & P^{G_{s(e)}}\\ \downarrow{\scriptstyle id\times\pi_G} & & \downarrow{\scriptstyle \pi_{G_{s(e)}}}\\ \mathbb{A}^1\times\mathbb{P}^{N_G-1} & \xrightarrow{\rho} & \mathbb{P}^{N_G}\end{array}$$

The morphism $\tilde{\rho}$ induces a morphism of motives

$$mot_{G_{s(e)}} \xrightarrow{\tilde{\rho}^*} H^{N_G-1}(\mathbb{A}^1 \times P^G, \{0,1\} \times D_G \setminus Y_G \cap D_G) \cong mot_G \otimes \mathbb{Q}(0).$$

Proof. In order to show that the morphism $\tilde{\rho}$ is well defined we will consider compatible coverings of the spaces P^G and $P^{G_{s(e)}}$. For the definition of the covering of P^G for any graph G we follow [12, §5.3]. Let $S_G = \{1, \dots, N_G\}$. Denote by B_G the set of subsets of S_G where each element $I \in B_G$ corresponds to a linear subspace $L_I = \cap_{i \in I} V(\alpha_i) \subset \mathbb{P}^{N_G-1}$ which we blow up to define P^G. Note that each I corresponds to a motic subgraph of G.

For each flag

$$\mathcal{F} : \varnothing = I_0 \subsetneq I_1 \dots I_k \subsetneq I_{k+1} = S_G$$

and a choice of elements

$$c : j_n \in I_n \setminus I_{n-1}\,, \text{ for } 1 \le n \le k+1$$

which cannot be made larger (i.e., there is no I and $1 \le i \le k+1$ such that $I_i \subsetneq I \subsetneq I_{i+1}$ and $j_{i+1} \notin I$) we define an affine open $\mathbb{A}^{\mathcal{F},c} = \mathrm{Spec}(\mathbb{Z}[\beta_1, \dots, \widehat{\beta_{j_{k+1}}}, \dots, \beta_{N_G}])$, and a projection morphism $\pi_{\mathcal{F},c} : \mathbb{A}^{\mathcal{F},c} \to \mathrm{Spec}(\mathbb{Z}[\alpha_1, \dots, \widehat{\alpha_{j_{k+1}}}, \dots, \alpha_{N_G}])$. Note that the target of the morphism is the open in $\mathbb{P}^{N_G-1}$ where $\alpha_{j_{k+1}} \neq 0$. It is defined by

$$\pi^*_{\mathcal{F},c}(\alpha_i) = \begin{cases} \beta_i \beta_{j_n} \dots \beta_{j_{k+1}}, & \text{for } i \in I_n \setminus I_{n-1} \cup \{j_n\} \text{ for some } 1 \le n \le k+1 \\ \beta_{j_n} \dots \beta_{j_{k+1}}, & \text{for } i = j_n \text{ for some } 1 \le n \le k+1 \end{cases}. \tag{4.27}$$

Its inverse is defined on the subset of $\mathbb{P}^{N_G-1}$ where $\alpha_{j_n} \neq 0$ for $1 \le n \le k+1$ by

$$(\pi^{-1}_{\mathcal{F},c})^*(\beta_i) = \begin{cases} \frac{\alpha_i}{\alpha_{j_n}}, & \text{for } i \in I_n \setminus I_{n-1} \cup \{j_n\} \text{ for some } 1 \le n \le k+1 \\ \frac{\alpha_{j_n}}{\alpha_{j_{n+1}}}, & \text{for } i = j_n \text{ for some } 1 \le n \le k \\ \alpha_{j_{k+1}}, & i = j_{k+1} \end{cases}. \tag{4.28}$$

The affine opens $\mathbb{A}^{\mathcal{F},c}$ glue together to form P^G (see [12, §5.3]).

Recall N_G, N_G+1 are the indices corresponding to the edges obtained by subdividing of e in G according to our chosen ordering. It follows from the previous lemma each element of $B_{G_{s(e)}}$ contains either both N_G and N_G+1 or neither. Therefore for each flag $\mathcal{F}$ of elements of B_G there is a unique flag

$$\mathcal{F}' : \varnothing = I'_0 \subsetneq I'_1 \dots I'_k \subsetneq I'_{k+1} = S_{G_{s(e)}}$$

where $I_n \in B_{G_{s(e)}}$ are defined by

$$I'_n = \begin{cases} I_n \cup \{N_G+1\}, & \text{if } N_G \in I_n \\ I_n, & \text{otherwise} \end{cases}.$$

If a choice c of elements in B_G associated to $\mathcal{F}$ does not contain $j_n = N_G$ for any $1 \leq n \leq k+1$ then we can consider the affine open in $P^{G_{s(e)}}$ defined by the pair $\mathcal{F}', c$. In this case, it follows from (4.27), (4.28) and the defintion of the morphism ρ (4.23) that

$$\left(\pi^{-1}_{\mathcal{F},c} \circ \rho|_{\mathbb{A}^{\mathcal{F},c}} \circ \pi_{\mathcal{F}',c}\right)^* (\beta_i) = \begin{cases} \beta_i, & \text{if } i \notin \{N_G, N_G+1\} \\ y\beta_{N_G}, & \text{if } i = N_G \\ (1-y)\beta_{N_G}, & \text{if } i = N_G+1 \end{cases}.$$

We can see that this is well defined on the whole open $\mathbb{A}^{\not\Vdash} \times \mathbb{A}^{\mathcal{F},c}$.

If a choice c of elements in B_G associated to $\mathcal{F}$ contains $j_r = N_G$ for some $1 \leq r \leq k+1$ then we can consider two affine opens in $P^{G_{s(e)}}$: one defined by the pair $\mathcal{F}', c$ and another $\mathcal{F}', c'$ where c' is the same as c except for $j_r = N_G + 1$. We have

$$\left(\pi^{-1}_{\mathcal{F},c} \circ \rho|_{\mathbb{A}^{\mathcal{F},c}} \circ \pi_{\mathcal{F}',c}\right)^* (\beta_i) = \begin{cases} \beta_i, & \text{if } i \in I_n \setminus I_{n-1} \text{ and } n \neq r \\ \frac{\beta_i}{y}, & \text{if } i \in I_r \setminus I_{r-1} \text{ and } i \neq N_G+1 \\ \frac{1-y}{y}, & \text{if } i = N_G+1 \\ y\beta_{N_G}, & \text{if } i = N_G \end{cases},$$

and we can see that this is well defined on $\{y \neq 0\} \times \mathbb{A}^{\mathcal{F},c}$. Finally for $\mathbb{A}^{\mathcal{F}',c'}$ we have that $\left(\pi^{-1}_{\mathcal{F},c} \circ \rho|_{\mathbb{A}^{\mathcal{F},c}} \circ \pi_{\mathcal{F}',c'}\right)^*$ acts in the same way as the previous case where we swap N_G and N_G+1, as well as y and $(1-y)$. It is defined on $\{y \neq 1\} \times \mathbb{A}^{\mathcal{F},c}$. Along with the previous case we have covered $\mathbb{A}^1 \times \mathbb{A}^{\mathcal{F},c}$. Therefore we have shown that ρ extends to the blow-up, and we denote the extension $\tilde{\rho}$.

To show that it induces a morphism of motives, the second claim of the proposition, we have to check that

$$\tilde{\rho}\left(\{0,1\} \times D_G \setminus Y_G \cap D_G\right) \subset D_{G_{s(e)}} \setminus Y_{G_{s(e)}} \cap D_{G_{s(e)}}.$$

By the commutativity of the diagram we have just shown it is enough to check

$$\rho\left(\{0,1\} \times \Delta_G \setminus X_G \cap \Delta_G\right) \subset \Delta_{G_{s(e)}} \setminus X_{G_{s(e)}} \cap \Delta_{G_{s(e)}}.$$

In fact it is clear from the definition of ρ that $\rho\left(\{0,1\} \times \Delta_G\right) \subset \Delta_{G_{s(e)}}$ and we have already shown that it send graph hypersurfaces of G to graph hypersurfaces of $G_{s(e)}$. □

By the commutativity of the diagram in the previous proposition and lemma 4.2.3 we have

$$\tilde{\rho}^*\left(\left[\pi^*_{G_{s(e)}}\left(\omega_{G_{s(e)}}\right)\right]\right) = \left[\pi^*_G\left(\frac{\alpha_{N_G}\Psi_G}{\Xi_G}\omega_G\right) \wedge (-dy)\right].$$

In fact more is true. We have an equality of motivic periods

$$I^{\mathfrak{m}}_{G_{s(e)}} = -\left[mot_G, [\sigma_G], \left[\pi_G^*\left(\frac{\alpha_{N_G}\Psi_G}{\Xi_G}\omega_G\right)\right]\right]^{\mathfrak{m}} \tag{4.29}$$

since $\tilde{\rho}_*\left(\left[\gamma_{0,1}\times\sigma_{G_{s(e)}}\right]\right) = [\sigma_G]$, where $\gamma_{0,1}$ is a path from 0 to 1 in $\mathbb{A}^1$, and

$$\mathrm{per}\left(\left[H^1(\mathbb{A}^1,\{0,1\},[\gamma_{0,1}],[-dy])\right]^{\mathfrak{m}}\right) = -1.$$

4.2.2.4 Completing the de Rham basis

We will use differential forms associated to Feynman graphs obtained from the sunrise by iterative subdivision of edges to complete $\{\phi_{\mathrm{dR}}(\nu_0), [\pi_G^*(\omega_G)]\}$ to a basis of $(mot'_G)_{\mathrm{dR}}$. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(D\smallsetminus\mathcal{E}\cap D) & \longrightarrow & H^2(P^G\smallsetminus\mathcal{E}, D\smallsetminus\mathcal{E}\cap D) & \longrightarrow & H^2(P^G\smallsetminus\mathcal{E}) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle res} & & & & \\ 0 & \longrightarrow & \widetilde{H}^0(\mathcal{E}\cap D)(-1) & \xrightarrow{p} & H^1(\mathcal{E},\mathcal{E}\cap D)(-1) & \longrightarrow & H^1(\mathcal{E})(-1) & \longrightarrow & 0 \end{array}$$

The object on the top left is isomorphic to $\mathbb{Q}(0)$, and the image of its de Rham realization is spanned by $\phi_{\mathrm{dR}}(\nu_0)$ 4.2.2.2. The form $\pi_G^*(\omega_G)$ lifts the holomorphic differential on $\mathcal{E}$. We now want to choose a lift of a differential of the second kind on $\mathcal{E}$, and the three classes in $\mathrm{Im}(p)\cap\mathrm{Im}(res)$ via the residue.

Proposition 4.2.6. *Consider the differential form*

$$\eta_G = \pi_G^*\left(\frac{-\alpha_1\Psi_G}{\Xi_G}\omega_G\right),$$

such that $\eta_G\wedge dy$ corresponds via the pullback by $\tilde{\rho}$, in $d=2$ space-time dimensions, to the Feynman integrand associated to the graph $G_{s(e_1)}$. The form η_G is sent to a form of the second kind on $\mathcal{E}$ by the residue morphism. Moreover consider the forms

$$\nu_1 = \pi_G^*\left(\frac{-\alpha_1\alpha_2^2}{\Xi_G}\omega_G\right),\quad \nu_2 = \pi_G^*\left(\frac{-\alpha_1^2\alpha_3}{\Xi_G}\omega_G\right),\quad \nu_3 = \pi_G^*\left(\frac{-\alpha_2\alpha_3^2}{\Xi_G}\omega_G\right),$$

such that $\nu_i\wedge\wedge_{j=1}^3 dy_j$, for $1\le i\le 3$ correspond via $\tilde{\rho}$, in $d=4$ space-time dimensions, to the Feynman integrands associated to the graphs

$$G_{s(e_1,e_2^2)}, G_{s(e_1^2,e_3)}, G_{s(e_2,e_3^2)}$$

respectively (see figure 4.4). The images of $\omega_G,\eta_G,\nu_1,\nu_2,\nu_3$ under the residue map are linearly independent over $k(m,q)$. Hence their classes, along with the class of $\phi_{\mathrm{dR}}(\nu_0)$, form a basis of $(mot'_G)_{\mathrm{dR}}$.

Proof. By the same argument as in 4.2.2.1 the de Rham cohomology class of $res(\eta_G)$ is an element of $F^0H^1_{\mathrm{dR}}(\mathcal{E})$ since it has a pole of order 2 along $\mathcal{E}$. Moreover, since the numerator $\alpha_1\Psi_G$ of η_G is not the Jacobian ideal of Ξ_G (i.e. the ideal generated by its partial derivatives with respect to $\alpha_i, 1 \leq i \leq 3$) it follows that we cannot reduce the order of the pole along $\mathcal{E}$ by [30, Proposition 4.6], so the class of its residue in $H^1_{\mathrm{dR}}(\mathcal{E})$ is not a $k(m,q)$-multiple of the class of a holomorphic form on $\mathcal{E}$.

Let us order the points $\mathcal{E} \cap D$ and denote $P_{2i-1} = D_{-i} \cap \mathcal{E}$ and $P_{2i} = D_i \cap \mathcal{E}$ for $1 \leq i \leq 3$. As discussed in 4.1.1.1 a basis of algebraic de Rham cohomology of the motive $H^1(\mathcal{E}, D \cap \mathcal{E})$ is given by a differential form of the first kind on $\mathcal{E}$, a differential form of the second kind on $\mathcal{E}$ and images of classes in $\tilde{H}^0(D \cap \mathcal{E})$ under the map

$$\tilde{H}^0_{\mathrm{dR}}(D \cap \mathcal{E}) \to H^1_{\mathrm{dR}}(\mathcal{E}, D \cap \mathcal{E}),$$

which are given by the classes $[df_i]$ where f_i is a meromorphic function on $\mathcal{E}$ which takes the value 1 at the point P_i and 0 at P_j for $j \neq i$, for $1 \leq i \leq 6$. They satisfy the relation $[df_1] + \ldots + [df_6] = 0$.

We will consider the basis

$$\{[df_2], \ldots, [df_6], [res(\eta_G)], [res(\omega_G)]\} \text{ of } H^1_{dR}(\mathcal{E}, D \cap \mathcal{E}),$$

and write $[res(\nu_i)]$, for $1 \leq i \leq 3$ as a linear combination

$$[res(\nu_i)] = \sum_{j=1}^{5} a_{i,j}[df_{j+1}] + a_{i,6}[res(\eta_G)] + a_{i,7}[res(\omega_G)].$$

We can determine the $a_{i,j}$'s by considering the affine covering we used in the proof of proposition 4.2.5. It has six affine open sets, each of which contains two of the points P_k, $1 \leq k \leq 6$. Denote the affine open containing the points P_k, P_l by $\mathbb{A}_{k,l}$. We can compose the residue with the pullback along $r_{k,l} : \mathbb{A}_{k,l} \cap \mathcal{E} \to \mathcal{E}$ to obtain the commutative diagram

$$\begin{array}{ccc} \tilde{H}^0_{\mathrm{dR}}(D \cap \mathcal{E}) & \longrightarrow & H^1_{\mathrm{dR}}(\mathcal{E}, D \cap \mathcal{E}) \\ \downarrow & & \downarrow r^*_{k,l} \\ \tilde{H}^0_{\mathrm{dR}}(\{P_k, P_l\}) & \longrightarrow & H^1_{\mathrm{dR}}(\mathbb{A}_{k,l} \cap \mathcal{E}, \{P_k, P_l\}) \end{array}$$

Computing explicitly on the bottom row we can write

$$r^*_{k,l}(res(\nu_i)) = a_{i,6}r^*_{k,l}(res(\omega_G)) + a_{i,7}r^*_{k,l}(res(\eta_G)) + dG_{i;k,l}$$

for a polynomial $G_{i;k,l}$. To obtain it we can reduce the pole order of $r^*_{k,l}(res(\nu_i))$ analogous to the proof of lemma 3.26. This can always be done as $\mathcal{E}$ is smooth and so

the partial derivatives of Ξ_G and Ξ_G itself don't have any common zeroes, so by weak Nullstellensatz the ideal generated by them will contain any numerator, in particular that of $r^*_{k,l}(res(\nu_i))$. We obtain equations $a_{i,k} - a_{i,l} = G_{i;k,l}(P_k) - G_{i;k,l}(P_l)$. We can repeat the process for $1 \leq k \leq 5, l = k+1$ and solve the resulting system of equations to obtain all $a_{i,j}$'s. They can be computed algorithmically, and are given explicitly in Appendix A. This allows us to check that they are linearly independent. □

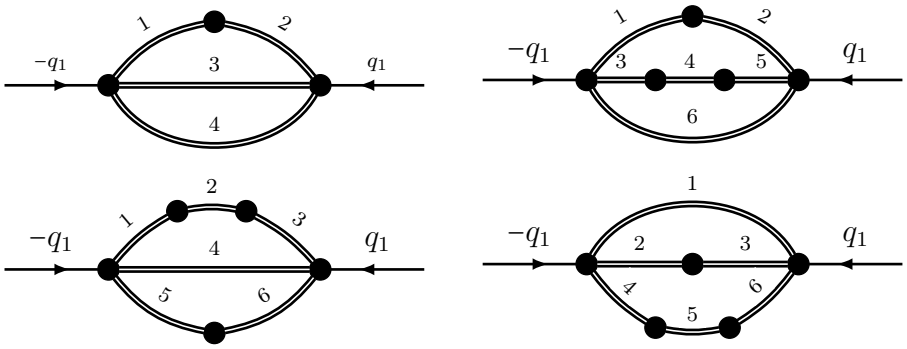


Figure 4.4: Graphs with subdivided edges corresponding to the elements of the chosen de Rham basis.

4.2.3 The motivic Galois coaction

4.2.3.1 Arbitrary masses

We write the coaction for the sunrise graph in terms of motivic periods associated to graphs obtained from it by deleting and subdividing edges defined in the previous chapter. In the following statement we drop the notation (m, q) for the dependence on masses and momenta for brevity. We remind the reader of the definitions of de Rham periods $K_1^{\mathfrak{dr}}, K_2^{\mathfrak{dr}}, F_{P_i,P_j}^{\mathfrak{dr}}$ in 4.1.4, where these are applied with respect to the elliptic curve $\mathcal{E}$ and the basis of $H^1_{\mathrm{dR}}(\mathcal{E})$ given by $\{[res(\omega_G)], [res(\eta_G)]\}$. We further define

$$F^{\mathfrak{dr}}_{\underline{b}} = \sum_{1\leq k\leq 5} b_i F^{\mathfrak{dr}}_{P_1,P_{i+1}} \tag{4.30}$$

where $\underline{b} = (b_1, \ldots, b_5)$ is a vector with coefficients in k_S, i.e. rational functions of masses and momenta. Finally we define a de Rham Feynman period

$$I^{\mathfrak{dr}}_{G,G\setminus e_3} = \left[mot'_G, [\phi_{\mathrm{dR}}(\nu_0)]^\vee, [\pi^*_G(\omega_G)]\right]^{\mathfrak{dr}}$$ [4]

[4]Here we have defined a de Rham Feynman amplitude in relation to a chosen weight 0 basis element, in our case coming from the bubble graph obtained by deleting the edge e_3 from the sunrise. In the one loop case, when our graph motive was mixed Tate, this was done without a choice of a weight 0 basis element since the weight filtration is split by the Hodge filtration in the de Rham realization – see (2.20).

Theorem 4.2.7. *The motivic Galois coaction on the motivic Feynman amplitude associated to the sunrise Feynman graph with respect to the de Rham basis defined in Proposition 4.2.6 is*

$$\begin{aligned}\Delta(I_G^{\mathfrak{m}}) = I_G^{\mathfrak{m}} \otimes K_1^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I_{G_{s(e_1)}}^{\mathfrak{m}} \otimes K_2^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I_{G\smallsetminus e_3}^{\mathfrak{m}} \otimes I_{G,G\smallsetminus e_3}^{\mathfrak{dr}} + \\ + I_{G_{s(e_1,e_2^2)}}^{\mathfrak{m}} \otimes F_{\underline{b_1}}^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I_{G_{s(e_1^2,e_3)}}^{\mathfrak{m}} \otimes F_{\underline{b_2}}^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I_{G_{s(e_2,e_3^2)}}^{\mathfrak{m}} \otimes F_{\underline{b_3}}^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}}\,,\end{aligned} \tag{4.31}$$

where $\underline{b_i} = (b_{i,1}, \dots, b_{i,5})$ *for* $1 \le i \le 3$*, and* $b_{i,j}$ *are defined in* (4.33)*.*

Proof. The general formula for the coaction (2.4) applied with respect to the basis of $(mot'_G)_{\mathrm{dR}}$ from proposition 4.2.6 gives

$$\begin{aligned}\Delta(I_G^{\mathfrak{m}}) = \left[mot'_G, [\sigma_G], [\pi_G^*(\omega_G)]\right]^{\mathfrak{m}} \otimes \left[mot'_G, [\pi_G^*(\omega_G)]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} + \\ + \left[mot'_G, [\sigma_G], [\eta_G]\right]^{\mathfrak{m}} \otimes \left[mot'_G, [\eta_G]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} + \\ + \sum_{i=1}^{3} \left[mot'_G, [\sigma_G], [\nu_i]\right]^{\mathfrak{m}} \otimes \left[mot'_G, [\nu_i]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} + \\ + \left[mot'_G, [\sigma_G], [\phi_{\mathrm{dR}}(\nu_0)]\right]^{\mathfrak{m}} \otimes \left[mot'_G, [\phi_{\mathrm{dR}}(\nu_0)]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}}.\end{aligned} \tag{4.32}$$

On the left hand side of the tensor products we have $\left[mot'_G, [\sigma_G], [\pi_G^*(\omega_G)]\right]^{\mathfrak{m}} = I_G^{\mathfrak{m}}$ by definition, and $\left[mot'_G, [\sigma_G], [\eta_G]\right]^{\mathfrak{m}} = I_{G_{s(e_1)}}^{\mathfrak{m}}$ via the subdivision morphsim $\tilde{\rho}$ (4.29). Similarly, the remaining motivic periods are identified with motivic Feynman amplitudes of graphs obtained by subdividing edges, except for $\left[mot'_G, [\sigma_G], [\phi_{\mathrm{dR}}(\nu_0)]\right]^{\mathfrak{m}} = I_{G\smallsetminus e_3}^{\mathfrak{m}}$. The latter equivalence is realised via the face map ϕ (see 4.2.2.2).

It remains to identify the de Rham periods. As in proposition 4.2.6 we consider the basis $\{[df_2], \dots, [df_6], [res(\eta_G)], [res(\omega_G)]\}$ of $H^1_{\mathrm{dR}}(\mathcal{E}, D \cap \mathcal{E})$. Let $\{e_i : 1 \le i \le 6\}$ denote the basis $\{[\phi_{\mathrm{dR}}(\nu_0)], [\nu_1], [\nu_2], [\nu_3], [\eta_G], [\pi_G^*(\omega_G)]\}$ of $(mot'_G)_{\mathrm{dR}}$ in that order, and by $\{e_i^\vee : 1 \le i \le 6\}$ the associated dual basis. We can use the residue morphism

$$mot'_G = H^2(P^G \smallsetminus \mathcal{E}, D \smallsetminus \mathcal{E} \cap D) \xrightarrow{res} H^1(\mathcal{E}, D \cap \mathcal{E})(-1)$$

to realise an equivalence of de Rham periods

$$\left[mot'_G, e_i^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} = \left[H^1(\mathcal{E}, D \cap \mathcal{E})(-1), \xi_i^\vee, [res(\omega_G)(-1)]\right]^{\mathfrak{dr}},$$

for $2 \le i \le 6$, where $\xi_i^\vee \in H^1(\mathcal{E}, D \cap \mathcal{E})(-1)^\vee$ is a functional such that $res^\vee(\xi_i^\vee) = e_i^\vee$. To determine $\xi_i^\vee$ we write it as a linear combination in terms of the dual basis to our chosen basis of $H^1(\mathcal{E}, D \cap \mathcal{E})(-1)$

$$\xi_i^\vee = \sum_{k=2}^{6} b_{i,k-1} [df_k(-1)]^\vee + b_{i,6} [res(\eta_G)(-1)]^\vee + b_{i,7} [res(\omega_G)(-1)]^\vee.$$

Now note that we must have

$$res^\vee\left(\xi_i^\vee\right)(e_j) = \xi_i^\vee(res(e_j)) = \delta_{ij}.$$

It follows that each coefficient $b_{i,k}$ is the (k,i) entry of the matrix

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,7} \\ \vdots & \ddots & \\ a_{5,1} & \dots & a_{5,7} \end{pmatrix}^{-1}, \tag{4.33}$$

where the first three rows are the coefficients of $res(\nu_i)$, $1 \le i \le 3$, in Appendix A, and the last two rows are $(0,\dots 0,1,0), (0,0,0,\dots,0,1)$. Note that $b_{i,6} = b_{i,7} = 0$ for $1 \le i \le 3$. Hence, for $1 \le i \le 3$, we have

$$\left[mot'_G, e_i^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} = \sum_{k=1}^{5} b_{i,k}\left[H^1(\mathcal{E}, D\cap\mathcal{E})(-1), [df_{k+1}]^\vee, [res(\omega_G)]\right]^{\mathfrak{dr}} \mathbb{L}^{\mathfrak{dr}}. \tag{4.34}$$

Note that the $\mathbb{L}^{\mathfrak{dr}}$ accounts for the Tate twist. Furthermore we have

$$\left[mot'_G, [\eta_G]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} = \left[H^1(\mathcal{E}, D\cap\mathcal{E})(-1), [res(\eta_G)]^\vee, [res(\omega_G)]\right]^{\mathfrak{dr}} \mathbb{L}^{\mathfrak{dr}}.$$

The latter is equal to $K_2^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}}$, by definition. Similarly we have,

$$\left[mot'_G, [\pi_G^*(\omega_G)]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} = \left[H^1(\mathcal{E}, D\cap\mathcal{E})(-1), [res(\omega_G)]^\vee, [res(\omega_G)]\right]^{\mathfrak{dr}} \mathbb{L}^{\mathfrak{dr}}.$$

The latter is equal to $K_1^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}}$.

Now, let ι be the inclusion of pairs $(\mathcal{E}, \{P_1, P_i\}) \to (\mathcal{E}, \{P_1, \dots, P_6\})$. It induces a morphism of motives

$$H^1(\mathcal{E}, \{P_1, \dots, P_6\}) \xrightarrow{\iota^*} H^1(\mathcal{E}, \{P_1, P_i\})$$

and an equivalence of de Rham periods

$$\left[H^1(\mathcal{E}, \{P_1, \dots, P_6\}), [df_i]^\vee, [\omega]\right]^{\mathfrak{dr}} = \left[H^1(\mathcal{E}, \{P_1, P_i\}), \left[d\tilde{f}_i\right]^\vee, [\omega]\right]^{\mathfrak{dr}} = F^{\mathfrak{dr}}_{P_i,P_1},$$

where $\tilde{f}_i$ is a meromorphic function on $\mathcal{E}$ which is 1 at P_i and 0 at P_1, and $[\omega] \in H^1_{\mathrm{dR}}(\mathcal{E})$. The equivalence holds because

$$(\iota^*)^T\left(\left[d\tilde{f}_i\right]^\vee\right)([df_k]) = \left[d\tilde{f}_i\right]^\vee(\iota^*([df_k])) = \begin{cases} 1, & k = i \\ 0, & \text{otherwise.} \end{cases}$$

The remaining de Rham period is $\left[mot'_G, [\nu_i]^\vee, [\pi_G^*(\omega_G)]\right]^{\mathfrak{dr}} = I^{\mathfrak{dr}}_{G,G\setminus e_3}$ by definition.

□

4.2.3.2 Changing the de Rham basis

Note that the explicit form of the coaction depends on the choice of the de Rham basis, and to demonstrate this we write it with respect to a different one. We once again consider the differential forms $\pi_G^*(\omega_G)$, η_G, and $\phi_{\mathrm{dR}}(\nu_0)$, as in the previous theorem, and replace the forms ν_1, ν_2, ν_3 with

$$\begin{aligned}
\mu_1 &= \frac{m_1^2 x(\partial_y \Xi_G - \partial_z \Xi_G)}{\Xi_G^2} \Omega_G, \\
\mu_2 &= \frac{m_2^2 y(\partial_z \Xi_G - \partial_x \Xi_G)}{\Xi_G^2} \Omega_G, \\
\mu_3 &= \frac{m_3^2 z(\partial_x \Xi_G - \partial_y \Xi_G)}{\Xi_G^2} \Omega_G.
\end{aligned} \tag{4.35}$$

First note that these differential forms are Feynman integrands associated to k_S-linear combinations of Feynman graphs obtained by subdivision of edges 4.2.2.3 from the sunrise graph. We can show that the motivic periods associated to μ_i and the domain of integration σ_G are motivic logarithms:

$$\begin{aligned}
[mot'_G, [\sigma_G], [\mu_1]]^{\mathfrak{m}} &= \log^{\mathfrak{m}}\left(\frac{m_3^2}{m_2^2}\right) \\
[mot'_G, [\sigma_G], [\mu_2]]^{\mathfrak{m}} &= \log^{\mathfrak{m}}\left(\frac{m_1^2}{m_3^2}\right) \\
[mot'_G, [\sigma_G], [\mu_3]]^{\mathfrak{m}} &= \log^{\mathfrak{m}}\left(\frac{m_2^2}{m_1^2}\right).
\end{aligned} \tag{4.36}$$

This follows, for example, from the fact that motivic logarithms are determined by their periods, so a computation of the periods is enough to show the equality above.

Next we wish to understand the de Rham periods $[mot'_G, [\mu_i]^\vee, [\pi_G^*(\omega_G)]]^{\mathfrak{dr}}$. We can repeat the process from the previous theorem and proposition 4.2.6 to compute the coefficients in the expression

$$[mot'_G, [\mu_i]^\vee, [\pi_G^*(\omega_G)]]^{\mathfrak{dr}} = \sum_{k=1}^{5} b'_{i,k} \left[H^1(\mathcal{E}, D \cap \mathcal{E})(-1), [df_{k+1}]^\vee, [res(\omega_G)]\right]^{\mathfrak{dr}} \mathbb{L}^{\mathfrak{dr}}.$$

In this case they are rational. Each $b'_{i,k}$ corresponds to the entry (k,i) of the matrix

$$\begin{pmatrix}
\frac{1}{6} & \frac{7}{24} & -\frac{5}{24} \\
\frac{1}{6} & -\frac{11}{24} & \frac{1}{24} \\
0 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{6} & \frac{1}{24} & -\frac{11}{24} \\
\frac{1}{6} & -\frac{5}{24} & \frac{7}{24}
\end{pmatrix}. \tag{4.37}$$

Hence we get the expression for the coaction on the sunrise

$$\begin{aligned}\Delta(I^{\mathfrak{m}}_G) = I^{\mathfrak{m}}_G \otimes K_1^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G_{s(e_1)}} \otimes K_2^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G\smallsetminus e_3} \otimes I^{\mathfrak{dr}}_{G,G\smallsetminus e_3} + \\ + \log^{\mathfrak{m}}\left(\frac{m_3^2}{m_2^2}\right) \otimes F^{\mathfrak{dr}}_{\underline{b'_1}}\mathbb{L}^{\mathfrak{dr}} + \log^{\mathfrak{m}}\left(\frac{m_1^2}{m_3^2}\right) \otimes F^{\mathfrak{dr}}_{\underline{b'_2}}\mathbb{L}^{\mathfrak{dr}} + \log^{\mathfrak{m}}\left(\frac{m_2^2}{m_1^2}\right) \otimes F^{\mathfrak{dr}}_{\underline{b'_3}}\mathbb{L}^{\mathfrak{dr}},\end{aligned} \tag{4.38}$$

where, as before, we set $\underline{b'_i} = (b'_{i,1}, \dots, b'_{i,5})$ and $F^{\mathfrak{dr}}_{\underline{b}}$ is defined in 4.30. We are also free to consider a different form on $P \smallsetminus \mathcal{E}$ the residue of which is a form of the second kind on $\mathcal{E}$, instead of η_G, as well as a different weight 0 class, instead of $[\phi_{\mathrm{dR}}(\nu_0)]$.

4.2.3.3 Equal masses

The coaction simplifies in the case when the masses are equal, i.e, $m_1^2 = m_2^2 = m_3^2$. We can observe from (4.38) that the periods of the motivic logarithms vanish in this case. In terms of mixed Hodge structures this is reflected by the splitting of the extension of $\mathbb{Q}$-mixed Hodge structures

$$0 \to H^2(P)/[\mathcal{E}] \to H^2(P \smallsetminus \mathcal{E}) \to H^1(\mathcal{E})(-1) \to 0.$$

This is shown [8, Lemma 6.16], and is a consequence of the fact that when the masses are equal the points of intersection $\mathcal{E} \cap D$ are torsion points of the elliptic curve $\mathcal{E}$ [8, Lemma 4.11]. The previous extension fits into the following diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & H^2(P)/[\mathcal{E}] & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & H^1(D \smallsetminus \mathcal{E} \cap D) & \longrightarrow & H^2(P^G \smallsetminus \mathcal{E}, D \smallsetminus \mathcal{E} \cap D) & \longrightarrow & H^2(P^G \smallsetminus \mathcal{E}) & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle res} & & \\ & & & & & & H^1(\mathcal{E})(-1) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{4.39}$$

where we write $H^2(P)/[\mathcal{E}]$ for $H^2(P)/\mathrm{Im}(Gysin)$ and $Gysin : H^0(\mathcal{E})(-1) \to H^2(P)$. As the exact sequence in the right-hand column splits canonically we get a morphism $i : H^1(\mathcal{E})(-1) \to H^2(P^G \smallsetminus \mathcal{E})$. We can pull back the short exact sequence in the middle row above via i to a short exact sequence

$$0 \to \mathbb{Q}(0) \to E \to H^1(\mathcal{E})(-1) \to 0, \tag{4.40}$$

where we plugged in the isomorphism $H^1(D \setminus \mathcal{E} \cap D) \cong \mathbb{Q}(0)$. Therefore we get an object E of $\mathcal{H}(S')$ or rank 3, where S' is the subspace of S where the masses are equal. A basis for its de Rham realization is $\{[\phi_{\mathrm{dR}}(\nu_0)], [res(\eta_G)(-1)], [res(\omega_G)(-1)]\}$, and the motivic Feynman amplitude associated to the sunrise $I^{\mathfrak{m}}_G$ is equivalent to a motivic period of E. Hence we know that the coaction formula has 3 terms as it corresponds to the rank of E. The coaction formula can be written as

$$\Delta(I^{\mathfrak{m}}_G) = I^{\mathfrak{m}}_G \otimes K_1^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G_{s(e_1)}} \otimes K_2^{\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} + I^{\mathfrak{m}}_{G\setminus e_3} \otimes I^{\mathfrak{dr}}_{G,G\setminus e_3}. \tag{4.41}$$

4.2.4 The sunrise integral and integrals of modular forms via the relative completion

In the equal-mass case the family of elliptic curves defined by the vanishing of the second Symanzik polynomial is the universal family over the modular curve $X_1(6)$ [8, §4]. We can use this fact to show that the associated period can be written in terms of integrals of Eisenstein series. This is done by relating the periods of mot'_G to periods of the relative completion of the torsor of paths $\pi_1(X_1(6), x, y)$. We start with the definition of the relative completion, its Betti and de Rham realizations, and periods. The following theory of the relative completion of fundamental groups and path torsors, and their periods in the case of modular curves, is due to Hain [31, 32] and Brown [13].

4.2.4.1 Tannakian definition of the relative completion

Let $\mathcal{T}$ be a Tannakian category with two fiber functors ω_x, ω_y, and let $\mathcal{S} \hookrightarrow \mathcal{T}$ be a full semisimple Tannakian subcategory with fiber functors obtained by restriction of ω_x, ω_y. With respect to these two categories define a third category $\mathcal{W}(\mathcal{T}, \mathcal{S}) \hookrightarrow \mathcal{T}$ such that its objects are objects $V \in \mathcal{T}$ equipped with a filtration

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

such that its graded pieces V_i/V_{i-1} are objects of $\mathcal{S}$. The category $\mathcal{W}(\mathcal{T}, \mathcal{S})$ is Tannakian with fiber functors ω_x, ω_y restricted from $\mathcal{T}$ to the subcategory. Define

$$\pi_1(\mathcal{T}, \mathcal{S}, \omega_x) = \mathrm{Aut}^{\otimes}_{\mathcal{W}(\mathcal{T},\mathcal{S})}(\omega_x), \text{ and } \pi_1(\mathcal{T}, \mathcal{S}, \omega_x, \omega_y) = \mathrm{Isom}^{\otimes}_{\mathcal{W}(\mathcal{T},\mathcal{S})}(\omega_x, \omega_y).$$

Let $S_x = \mathrm{Aut}^{\otimes}_{\mathcal{S}}(\omega_x)$, noting the different category to the one in the definition of $\pi_1(\mathcal{T}, \mathcal{S}, \omega_x)$. Then S_x is a pro-reductive affine group scheme because the category $\mathcal{S}$

is semi-simple [27, Proposition 2.23], and we get a morphism of affine group schemes $\pi_1(\mathcal{T},\mathcal{S},\omega_x) \to S_x$ induced by the inclusion of categories $\mathcal{S} \hookrightarrow \mathcal{W}(\mathcal{T},\mathcal{S})$. We get an exact sequence

$$0 \to \mathcal{U}_x \to \pi_1(\mathcal{T},\mathcal{S},\omega_x) \to S_x \to 0\,,$$

where $\mathcal{U}_x$ is a pro-unipotent group. We also write $S_{x,y} = \mathrm{Isom}^{\otimes}_{\mathcal{S}}(\omega_x,\omega_y)$, and similarly we have a morphism $\pi_1(\mathcal{T},\mathcal{S},\omega_x,\omega_y) \to S_{x,y}$.

4.2.4.2 Betti and de Rham relative completion of π_1

Let X be a smooth geometrically connected scheme over a field $k \subset \mathbb{C}$, and let $x, y \in X(k)$ be two rational points. The reader is reminded of the definitions of fiber functors in 2.1.

- **Betti:** Let $\mathcal{L}_X$ be the category of local systems of finite-dimensional k-vector spaces on X. Let $\mathcal{S}^B$ be a full semi-simple Tannakian subcategory of $\mathcal{L}_X$. Then define the *relative Betti fundamental group of* X by

$$\pi_1^{B,\mathcal{S}}(X,x) = \pi_1(\mathcal{L}_X,\mathcal{S}^B,\omega_x^B).$$

  The relative completion can be defined for any group with respect to a morphism with Zariski dense image into the k-points of a reductive algebraic group by a universal property (see [31, §1]). If we consider the topological fundamental group $\pi_1(X,x)$, and $S_x^B = \mathrm{Aut}^{\otimes}_{\mathcal{S}^B}(\omega_x^B)$, and we note that each local system on X is equivalent to a $\pi_1(X,x)$-representation, we get a Zariski dense morphism

$$\pi_1(X,x) \to S_x^B(k). \tag{4.42}$$

  Then $\pi_1^{B,\mathcal{S}}(X,x)$ is the relative completion of the group $\pi_1(X,x)$ with respect to the morphism (4.42). We also define the *relative Betti fundamental groupoid*:

$$\pi_1^{B,\mathcal{S}}(X,x,y) = \pi_1(\mathcal{L}_X,\mathcal{S}^B,\omega_x^B,\omega_y^B).$$

  It is an affine scheme over k, which also comes equipped with a morphism

$$\pi_1(X,x,y) \to \pi_1^{B,\mathcal{S}}(X,x,y)(k), \tag{4.43}$$

  which has Zariski dense image, and arises via an isomorphism of fiber functors given by the pullback along a smooth path $\gamma : [0,1] \to X(\mathbb{C})$.

- **de Rham:** Let $\mathcal{A}_X$ be the category of vector bundles on X, equipped with an integrable connection with regular singularities at infinity. Let $\mathcal{S}^{dR}$ be a full semi-simple Tannakian subcategory of $\mathcal{A}_X$. Then define the *relative de Rham fundamental group*
$$\pi_1^{dR,\mathcal{S}}(X,x) = \pi_1(\mathcal{A}_X, \mathcal{S}^{dR}, \omega_x^{dR}).$$
Similarly, define the *relative de Rham fundamental groupoid*
$$\pi_1^{dR,\mathcal{S}}(X,x,y) = \pi_1(\mathcal{A}_X, \mathcal{S}^{dR}, \omega_x^{dR}, \omega_y^{dR}).$$

- **Comparison:** The Riemann-Hilbert correspondence gives an equivalence of categories
$$\mathcal{L}_X \otimes \mathbb{C} \sim \mathcal{A}_X \otimes \mathbb{C}$$
and thus induces a comparison isomoprhism
$$c : \pi_1^{B,\mathcal{S}}(X,x,y) \otimes \mathbb{C} \xrightarrow{\sim} \pi_1^{dR,\mathcal{S}}(X,x,y) \otimes \mathbb{C}, \tag{4.44}$$
which in turn induces an isomoprhism of their affine rings.

Modular curves We are particularly interested in the previous definitions in the case when X is a modular curve X_Γ for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the category $\mathcal{S}^B$ is the category of local systems generated by $R^1 f_* \mathbb{Q}$ where $f : \mathcal{E} \to X_\Gamma$, and similarly $\mathcal{S}^{dR}$ is the category of vector bundles generated by $H^1_{\mathrm{dR}}(\mathcal{E})_{/X_\Gamma}$ with the Gauss-Manin connection. We will set one of the two base points to be a *tangential base point* [24, §15.3 - 15.12], see also [13, §4] at the cusp at infinity, denoted $\vec{1}_{i\infty}$. Setting the other base point to $\tau \in X_\Gamma(k)$ we get an object
$$\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)) = (\mathcal{O}(\pi_1^{B,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)), \mathcal{O}(\pi_1^{dR,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)) \otimes_k \mathcal{O}_{X_\Gamma}, c)$$
in the category $\mathcal{H}(k)$.

Remark 4.2.2. Note that $\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)$ is a right torsor over $\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty})$:
$$\pi_1^{rel,\mathcal{S}}(X,x,y) \times \pi_1^{rel,\mathcal{S}}(X,x) \to \pi_1^{rel,\mathcal{S}}(X,x,y), \tag{4.45}$$
and a left torsor over $\pi_1^{\mathrm{rel},\mathcal{S}}(Xy)$:
$$\pi_1^{rel,\mathcal{S}}(X,y) \times \pi_1^{rel,\mathcal{S}}(X,x,y) \to \pi_1^{rel,\mathcal{S}}(X,x,y). \tag{4.46}$$

Remark 4.2.3. We may replace the point τ and its associated fiber functors $\omega_{\mathrm{B}}^{\tau}, \omega_{\mathrm{dR}}^{\tau}$ with $\omega_{\mathrm{B}}^{Y}, \omega_{\mathrm{dR}}^{Y}$ for any $Y \subset X_\Gamma(\mathbb{C})$ simply connected and obtain a local system and vector bundle

$$\mathcal{O}(\pi_1^{B,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet)), \quad \mathcal{O}(\pi_1^{dR,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet)) \otimes_k \mathcal{O}_{X_\Gamma}.$$

Hain tells us that $\mathcal{O}(\pi_1^{B,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet))$ is an (admissible) variation of mixed Hodge structures on X_Γ [32, Theorem 7.18]. Hence we get an object in the category $\mathcal{H}(X_\Gamma)$:

$$\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet)) = (\mathcal{O}(\pi_1^{B,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet)), \mathcal{O}(\pi_1^{dR,\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \bullet)) \otimes_k \mathcal{O}_{X_\Gamma}, c).$$

We will, however, work over a fiber at τ in what follows for simplicity, and note that all results lift to families over X_Γ by replacing the fiber functors at τ with $\omega_\bullet^Y$, where $\bullet = \mathrm{B}, \mathrm{dR}$, everywhere.

4.2.4.3 Motivic periods of the relative fundamental groupoid

Our next goal is to undestand the affine rings of the relative de Rham and Betti fundamental groupoids and relate the associated periods, i.e. matrix coefficients of their comparison, to iterated integrals of modular forms.

Splittings. Let $\tau \in X_\Gamma(k)$, and consider the Tannaka group of the category $\mathcal{H}(k)$ with respect to the fiber functor $\omega_{\mathrm{dR}}^{\tau}$, which we denote by $G_{\mathcal{H}(k)}^{\mathrm{dR},\tau}$. Its action on $\mathbb{Q}(-1)$ defines a character $\chi : G_{\mathcal{H}(k)}^{\mathrm{dR},\tau} \to \mathbb{G}_m$ sending $g \in G_{\mathcal{H}(k)}^{\mathrm{dR},\tau}(R)$ to $\lambda_g \in R^\times$, for any commutative ring R, where

$$g\mathbb{L}^{\mathrm{dR}} = (1 \otimes g)\Delta\mathbb{L}^{\mathrm{dR}} = \lambda_g \mathbb{L}^{\mathrm{dR}}.$$

We get an exact sequence

$$1 \to G_{\mathcal{H}(k)}^{\mathrm{dR},\tau,ker(\chi)} \to G_{\mathcal{H}(k)}^{\mathrm{dR},\tau} \xrightarrow{\chi} \mathbb{G}_m \to 1,$$

a splitting of which gives a splitting of the W-filtration for all objects in $\mathcal{H}(k)$. Since $\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau))$ is an object of the category $\mathcal{H}(k)$ we get a splitting of the weight filtration on its de Rham realization. In turn we get a "weight 0 projection" morphism in the de Rham realization:

$$gr_0^{W,\mathrm{dR}} : \mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)) \to k.$$

Thus, choosing such a splitting is equivalent to choosing a "de Rham path" which we denote $1_\tau^{\mathrm{dR}} \in \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)(k)$, and which in turn induces an isomorphism between

the de Rham realizations of the relative torsor of paths and the relative fundamental group via the torsor structure of the former over the latter (4.45)

$$\begin{aligned} \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}) &\xrightarrow{\sim} \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau) \\ g &\mapsto 1_\tau^{\mathrm{dR}} \cdot g. \end{aligned} \tag{4.47}$$

Similarly we have

$$\begin{aligned} \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \tau) &\xrightarrow{\sim} \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau) \\ g &\mapsto g \cdot (1_\tau^{\mathrm{dR}})^{-1}. \end{aligned} \tag{4.48}$$

We also need to choose a splitting of the extension

$$0 \to U_\tau^{\mathrm{dR}} \to \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \tau) \to S_x^{\mathrm{dR}} \to 0, \tag{4.49}$$

Choosing such splittings is possible by the argument in [13, 13.9].

The unipotent and reductive paths. Let $\gamma_\tau \in \pi_1(X_\gamma, \vec{1}_{i\infty}, \tau)$ be the class of a path between $\vec{1}_{i\infty}$ and τ, and γ_τ^{B} be its image under 4.43. We can view γ_τ^{B} as an element of $\mathrm{Hom}(\mathcal{O}(\pi_1^{\mathrm{B},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)), k)$, and extend scalars to $\mathbb{C}$. Pre-composing γ_τ^{B} with the morphism of affine rings associated to the comparison isomorphism 4.44 we obtain an element

$$\gamma_\tau^{\mathrm{B,dR}} \in \pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)(\mathbb{C}) = \mathrm{Hom}(\mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)), \mathbb{C}).$$

Now consider the composition of the (inverse of) the isomorphism (4.48) between the relative de Rham torsor of paths and the relative de Rham group with the base point at τ and the morphism $\pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \tau) \to U_\tau^{\mathrm{dR}}$ which splits (4.49). The morphism of affine rings associated to this composition is a morphism $\mathcal{O}(U_\tau^{\mathrm{dR}}) \to \mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau))$. We can pre-compose $\gamma_\tau^{\mathrm{B,dR}}$ by it and we denote the resulting morphism by

$$\gamma_\tau^{\mathrm{B},u} : \mathcal{O}(U_\tau^{\mathrm{dR}}) \to \mathbb{C} \tag{4.50}$$

We may think of $\gamma_\tau^{\mathrm{B},u}$ as a formal power series the coefficients of which are periods. Choosing an element $\omega \in \mathcal{O}(U_\tau^{\mathrm{dR}})$ amounts to selecting a coefficient of this series. In this way $\gamma_\tau^{\mathrm{B},u}$ can be regarded as the modular version of the Drinfeld associator (see [10]).

Moreover, we shall need to consider the morphism obtained by pre-composing $\gamma_\tau^{\mathrm{B,dR}}$ by the morphism of affine rings associated to $\pi_1^{\mathrm{dR},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau) \to S^{\mathrm{dR}}_{\vec{1}_{i\infty},\tau}$ defined in 4.2.4.1. Denote this composition by

$$\gamma^{\mathrm{B},red}_{\vec{1}_{i\infty},\tau} : \mathcal{O}(S^{\mathrm{dR}}_{\vec{1}_{i\infty},\tau}) \to \mathbb{C}. \tag{4.51}$$

U_τ^{dR} **and modular forms.** Next, we describe the affine ring of the unipotent part of the relative de Rham fundamental group $\mathcal{O}(U_\tau^{\mathrm{dR}})$. We consider the Tannakian category $\mathcal{W}(\mathcal{L}_X, \mathcal{S}^B)$ of local systems on X equipped with a filtration with prescribed graded pieces. It has a functor to the category of local systems on X which forgets the filtration. This induces a morphism of Ext groups

$$\mathrm{Ext}^r_{\mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)}(\mathbb{Q}, \mathbb{V}) \to \mathrm{Ext}^r_{\mathcal{L}_{X_\Gamma}}(\mathbb{Q}, \mathbb{V}),$$

where $\mathbb{V} \in \mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)$. By definition $\mathrm{Ext}^r_{\mathcal{L}_{X_\Gamma}}(\mathbb{Q}, \mathbb{V}) = H^r(\Gamma; V)$ where V is the Γ-representation associated to $\mathbb{V}$, and on the left hand side we have

$$\mathrm{Ext}^r_{\mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)}(\mathbb{Q}, \mathbb{V}) = H^r(\pi_1^{\mathrm{B}, \mathcal{S}}(X_\Gamma, \tau); V).$$

This map is an isomorphism for $r = 1$, and both sides are trivial for $r \geq 2$, given that Γ is a modular group [32, §3.2, §3.4.2]. It follows that $\mathcal{O}(U_\tau^{\mathrm{dR}}) \cong T^c(H^1(U_\tau^{\mathrm{dR}}))$, where T^c stands for the tensor coalgebra. Moreover, from [14, §6] we have that

$$H^1(U_\tau^{\mathrm{dR}}) = \bigoplus_{n \geq 0} \mathrm{Ext}^1_{\mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)}(\mathbb{Q}, \mathbb{V}_n^\vee) \otimes \mathcal{V}_{n,\tau}, \tag{4.52}$$

where $\mathbb{V}_n$ is the nth symmetric power of the local system $R^1 f_* \mathbb{Q}$ for $f : \mathcal{E} \to X_\Gamma$ the universal family of elliptic curves, and $\mathcal{V}_{n,\tau}$ is the fiber at τ of the vector bundle associated to the nth symmetric power of $R^1 f_* \mathbb{Q}$ by the Riemann-Hilbert correspondence. Note that $\mathbb{V}_n$ is self-dual induced by $H^1(\mathcal{E}_\tau; \mathbb{Q})^\vee \cong H^1(\mathcal{E}_\tau; \mathbb{Q})(1)$. Since $\mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)$ is a subcategory of local systems on $S(\mathbb{C})$ we have

$$\mathrm{Ext}^1_{\mathcal{W}(\mathcal{L}_{X_\Gamma}, \mathcal{S}^B)}(\mathbb{Q}, \mathbb{V}_n^\vee) \cong H^1(\Gamma, V_n^\vee) \cong H^1(X_\Gamma, \mathbb{V}_n^\vee),$$

where in the middle we have group cohomology where $V_n^\vee$ denotes the fiber of $\mathbb{V}_n^\vee$ at $\partial/\partial q$, and on the right hand side we have singular cohomology with coefficients in the local system $\mathbb{V}_n^\vee$.

Eichler-Shimura. Grothendieck's algebraic de Rham theorem gives an isomorphism between $H^1(X_\Gamma, \mathbb{V}_n)$ and $M^!_{n+2}/\mathcal{D}^{n+1} M^!_{-n} \otimes \mathbb{C}$ where $M^!$ is the space of weakly holomorphic modular forms, and $\mathcal{D} = qdq/q$ (see [18, Corollary 1.4] for the level 1 case). When restricted to the space of holomorphic modular forms of weight $n+2$ we recover the Eichler-Shimura isomorphism. The Hodge structure of $H^1(X_\Gamma, \mathbb{V}_n)$ is known [32]. We have

$$H^1(X_\Gamma, \mathbb{V}_n) \cong \bigoplus_f M_f \oplus \bigoplus_e \mathbb{Q}(-n-1), \tag{4.53}$$

where f ranges over all cusp form of weight $n+2$ and e ranges over all Eisenstein series of weight $n+2$ for the congruence group Γ. Here M_f is the Hodge realization of the motive of the cusp form f [47]. Note that the pullback of $\mathbb{V}_n$ to the upper half-plane $\mathfrak{H}$ via $\rho : \mathfrak{H} \to X_\Gamma$ is the trivial local system whose fiber over $z \in \mathfrak{H}$ is $H^1(E_z)$, where $E_z = \mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$. Denote by X, Y the sections of $\mathbb{V}_1^\vee$ which over a point z correspond to the standard basis of $H_1(E_z)$ given by classes of paths from 0 to 1 and 0 to z, respectively. Then we have [32, Theorem 11.4] that

$$\begin{aligned} M_{n+2}(\Gamma) &\cong F^{n+1}H^1(X_\Gamma, \mathbb{V}_n) \\ f(z) &\mapsto (2\pi i)^{n+1} f(z)(zX - Y)^n dz\,, \end{aligned}$$

where $M_{n+2}(\Gamma)$ is the space of holomorphic modular forms of weight $n+2$ for the congruence group Γ.

Motivic periods of the unipotent part of $\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau))$. We will represent an element $\eta \in \mathcal{O}(U_\tau^{\mathrm{dR}})$ using the bar notation $[f_1|\cdots|f_k] \otimes [\xi_1|\cdots|\xi_k]$, where f_i are such that $f_i(z)(zX - Y)^n dz$ correspond to classes in $H^1(X_\Gamma, \mathbb{V}_n)$, and $\xi_i \in \mathcal{V}_{n,\tau}$. We associate to it a motivic period of $\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau))$:

$$\gamma_{\tau,u}^{\mathfrak{m}}(\eta) = \left[\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_\Gamma, \vec{1}_{i\infty}, \tau)), \gamma_\tau^{\mathrm{B},u}, \eta\right]^{\mathfrak{m}}. \tag{4.54}$$

Its image under the period isomorphism is given by the iterated integral of $f_1, \ldots, f_k$ along γ from $i\infty$ to τ, followed by a pairing of sections X, Y and the appropriate element of $\mathcal{V}_{n,\tau}$ which is given by integration.

Example 4.2.4. Let $\eta = [f(z)] \otimes [\omega_\tau]$, where $f(z)$ is modular form of weight 3 over $X_1(6)$ and $[\omega_\tau] \in F^1 H^1_{\mathrm{dR}}(\mathcal{E}_\tau)$. Then we have

$$\begin{aligned} \mathrm{per}(\gamma_{\tau,u}^{\mathfrak{m}}(\eta)) &= (2\pi i)^2 \int_\tau^{i\infty} f(z)(zX - Y)dz \otimes [\omega_\tau] \\ &= (2\pi i)^2 \int_\tau^{i\infty} f(z)(z - \tau)dz, \end{aligned} \tag{4.55}$$

Note that in the case when $f(z)$ is an Eisenstein series we have to regularize at $i\infty$ (see [13, §4]).

The reductive part. Recall that $S_{i\infty,\tau}^{\mathrm{dR}} = \mathrm{Isom}^{\otimes}_{\mathcal{W}(\mathcal{A}_X, \mathcal{S}^{\mathrm{dR}})}(\omega_{i\infty}^{\mathrm{dR}}, \omega_\tau^{\mathrm{dR}})$. Its affine ring is thus

$$\mathcal{O}(S_{i\infty,\tau}^{\mathrm{dR}}) \cong \bigoplus_{n \geq 0} \mathcal{V}_{n,\tau} \otimes \mathcal{V}_{n,i\infty}^\vee.$$

Note that $\mathcal{V}_{n,i\infty}$ is the nth symmetric power of the algebraic de Rham cohomology of the infinitesimal Tate elliptic curve $\mathcal{E}^{\times}_{i\infty}$. It's split Tate and we have

$$(H^1(\mathcal{E}^{\times}_{i\infty}))^{\vee} \cong \mathbb{Q}(0) \oplus \mathbb{Q}(1).$$

Denote the de Rham generators by X, Y, where X spans the $\mathbb{Q}(0)$ and Y the $\mathbb{Q}(1)$ piece. Similarly to de Rham periods, $\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau})$ is spanned, over k, by classes of triples, e.g.

$$[\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}), \nu, \mu]^{red},$$

where ν is a word in $\{\mathsf{X}, \mathsf{Y}\}$, and μ is a word, of the same length as ν, in $\{\omega, \eta\}$ where these form a basis of $H^1_{\mathrm{dR}}(\mathcal{E}_\tau)$. Recall that a path $\gamma_{i\infty,\tau}$ from $i\infty$ to τ gives a homomorphism which assigns to each such triple a complex number 4.51. Let us look more closely at how it acts. Consider the map

$$\mathcal{V}_{n,\tau} \tilde{\to} Sym^n H^1_{\mathrm{B}}(\mathcal{E}_\tau; \mathbb{Q}) \otimes \mathbb{C} \tilde{\to} Sym^n H^1_{\mathrm{B}}(\mathcal{E}_{i\infty}; \mathbb{Q}) \otimes \mathbb{C} \tilde{\to} \mathcal{V}_{n,i\infty} \otimes \mathbb{C}, \tag{4.56}$$

where the first arrow is induced by the comparison $H^1_{\mathrm{dR}}(\mathcal{E}_\tau) \otimes \mathbb{C} \cong H^1_{\mathrm{B}}(\mathcal{E}_\tau; \mathbb{Q}) \otimes \mathbb{C}$, the middle arrow is an isomorphism induced by the pullback along $\gamma_{i\infty,\tau}$, and the last arrow is the inverse of the comparison isomorphism at the fiber at $i\infty$. Denote the composition by $\gamma^{comp}_{i\infty,\tau}$. It's matrix representation is $P^{-1}_{i\infty} P_\tau$ where P_τ is the matrix of periods and quasi-periods of $\mathcal{E}_\tau$, while $P^{-1}_{i\infty}$ is the inverse of the matrix of period matrix of the infinitesimal Tate curve

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2\pi i} \end{pmatrix}.$$

Example 4.2.5. Let $[\omega_\tau]$ be as in the previous example. Then we have

$$\gamma^{\mathrm{B},red}_{\vec{1}_{i\infty},\tau}([\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}), \mathsf{Y}, [\omega_\tau]]^{red}) = \frac{1}{2\pi i}\omega_1.$$

Similarly,

$$\gamma^{\mathrm{B},red}_{\vec{1}_{i\infty},\tau}([\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}), \mathsf{X}, [\omega_\tau]]^{red}) = \omega_2,$$

where ω_1, ω_2 are the two periods of the elliptic curve $\mathcal{E}_\tau$.

We also note that the ring $\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau})$ is an object of the underlying category $\mathcal{H}(k)$, and hence is equipped with Hodge and weight filtrations. We have

$$F^{k_1+k_2}W_{r_1+r_2}\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}) = \langle [\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}), \nu, \mu]^{red} : \text{where } \nu \in F^{k_1}W_{r_1}\mathcal{V}_{n,i\infty} \text{ and } \mu \in F^{k_2}W_{r_2}\mathcal{V}_{n,\tau} \rangle_{k_S}. \tag{4.57}$$

In what follows we shall drop the affine ring from the notation $[\mathcal{O}(S^{\mathrm{dR}}_{i\infty,\tau}), \nu, \mu]^{red}$ and simply write $[\nu, \mu]^{red}$ where it is unambiguous what it refers to.

4.2.4.4 The sunrise Feynman integral

Let G be the sunrise Feynman graph with equal masses, and let

$$J_G(t) = \int_\sigma \frac{\Omega_G}{t\alpha_1\alpha_2\alpha_3 + (\alpha_1+\alpha_2+\alpha_3)\Psi_G}, \tag{4.58}$$

where $t = \frac{q_1^2}{m^2}$ and $m = m_1 = m_2 = m_3$. Then $I_G(m,q) = \frac{1}{m^2} J_G(t)$, where $I_G(m,q)$ is the sunrise Feynman integral. The vanishing locus of the denominator of the integrand of $J_G(t)$ corresponds to the universal family of elliptic curves $f : \mathcal{E} \to X_1(6)$. We may choose a basis of $H^1_{\mathrm{dR}}(\mathcal{E})$ which includes the integrand of $J_G(t)$ (see 4.2.2.1), and a basis of $H_1(\mathcal{E};\mathbb{Q})$. This fixes a choice of periods of the elliptic curve $\mathcal{E}$ denoted ω_1, ω_2. We can then write t as a modular function $g(\tau)$, where $\tau = \frac{\omega_1}{\omega_2}$. The cusps of $X_1(6)$ must be sent to the singular fibers of the denominator of the integrand of $J_G(t)$ which lie at $t = 0, 1, 9, \infty$, and we choose g such that it maps $\tau = i\infty$ to $t = 0$, and define $J_G(\tau) = g^* J_G(t)$. In terms of families of motivic periods we have $J^{\mathfrak{m}}_G(t) = m^2 I^m_G(m,q)$, and we may pull this back to $X_1(6)$ via g, denoting the resulting family of motivic periods by $J^{\mathfrak{m}}_G(\tau)$.

We can use the fact that the relative completion of the torsor of paths on $X_1(6)$ acts on mot'_G to deduce an ansatz for the sunrise Feynman integral in terms of integrals of modular forms.

Theorem 4.2.8. *Let $\tau \in X_\Gamma(k)$. Let $E_{3,i}$, for $1 \le i \le 4$, be a basis of Eisenstein series of weight 3 for $\Gamma_1(6)$, and $E_{2,i}$, for $1 \le i \le 3$, a basis of Eisenstein series of weight 2 for $\Gamma_1(6)$. Then*

$$J_G(\tau) = \sum_{1\le i\le 4} a_i (2\pi i)^2 \int_\tau^{i\infty} E_{3,i}(z)(z-\tau)dz + \sum_{1\le i\le 3} a_{i+4}\omega_1 2\pi i \int_\tau^{i\infty} E_{2,i}(z)dz + C,$$

where $a_i \in k$ are undetermined constants, and C is a linear combination of periods of the elliptic curve $\mathcal{E}_\tau$ with coefficients in k.

Proof. Recall that the graph motive mot'_G fits into the diagram (4.39), where the residue morphism splits canonically. We can pull-back all objects in (4.39) from $\mathcal{H}(S)$ to $\mathcal{H}(X_1(6))$ and we denote these pull-backs by a subscript "$/X_1(6)$". Let $i : H^1(\mathcal{E})(-1)_{X_1(6)} \to H^2(P^G \setminus \mathcal{E})_{X_1(6)}$ be the morphism which splits the residue. We can pull back $mot'_{G,X_1(6)}$ via i to

$$0 \to \mathbb{Q}(0)_{/X_1(6)} \to E_{/X_1(6)} \to H^1(\mathcal{E})(-1)_{/X_1(6)} \to 0. \tag{4.59}$$

The motivic period $J^{\mathfrak{m}}_G(\tau)$ is equivalent to a motivic period of $E_{/X_1(6)}$ via i, i.e. we have

$$\begin{aligned} J^{\mathfrak{m}}_G(\tau) &= m^2[(mot'_G)_{/X_1(6)}, g^t_*([\sigma_G]), g^*[\pi^*(\omega_G)]]^{\mathfrak{m}} \\ &= [E_{/X_1(6)}, g^t_*([\sigma_G]), i^*g^*[\pi^*(\omega_G)]]^{\mathfrak{m}}. \end{aligned}$$

We note that $g^t_*([\sigma_G]) = i^t_* g^t_*([\sigma_G])$, and $i^*g^*[\pi^*(\omega_G)] = res([g^*[\pi^*(\omega_G)]])$ by definition of the pull-back via i. Henceforth, we denote $g^t_*([\sigma_G])$ by $[\sigma^g_G]$, and $i^*g^*[\pi^*(\omega_G)]$ by $[\omega^g_G]$, for simplicity. Furthermore, the object $E_{/X_1(6)} \in \mathrm{Ob}(\mathcal{H}(X_1(6)))$ is equipped with a filtration

$$0 \subset \mathbb{Q}(0)_{/X_1(6)} \subset E_{/X_1(6)}$$

such that its graded quotient is $H^1(\mathcal{E})(-1)_{/X_1(6)}$. Hence

$$(E_{/X_1(6)})_{\mathrm{B}} \in \mathrm{Ob}(\mathcal{W}(\mathcal{L}_{X_1(6)}, S^B)), \quad (E_{/X_1(6)})_{\mathrm{dR}} \in \mathrm{Ob}(\mathcal{W}(\mathcal{A}_{X_1(6)}, S^{\mathrm{dR}}))^{5}.$$

For a Tannakian category $\mathcal{C}$ with fiber functors ω_x, ω_y a k-point of $\mathrm{Isom}^{\otimes}_{\mathcal{C}}(\omega_x, \omega_y)$ gives an isomorphism $\omega_x(M) \otimes k \xrightarrow{\sim} \omega_y(M) \otimes k$, for any $M \in \mathrm{Ob}(\mathcal{C})$. In particular, it follows that we have an action of the relative torsor of paths

$$\pi_1^{\mathrm{rel},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau) \times (E_{i\infty}) \to (E_\tau)$$

for any $\tau \subset X_1(6)(\mathbb{C})$. We now fix one such $\tau \in (X_1(6) \setminus C)(k)$, where C are the cusps of $X_1(6)$, bearing in mind remark 4.2.3. Dual to the action above we have the coaction

$$\rho : (E_\tau) \to (E_{i\infty}) \otimes_k \mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau)).$$

This is a morphism of mixed Hodge structures [32, Theorem 8.1]. The topological fundamental group acts on the dual of the Betti realization of $E_{/X_1(6)}$

$$\pi_1(X_1(6), i\infty, \tau) \times (E_{i\infty})^\vee_{\mathrm{B}} \to (E_\tau)^\vee_{\mathrm{B}}.$$

In particular the class of a path γ_τ from $i\infty$ to τ sends $[\sigma^g_G]_{i\infty}$ to $[\sigma^g_G]_\tau$, where $[\sigma^g_G]_\tau$ is the class of the domain of integration of the sunrise Feynman integral over τ, and $[\sigma^g_G]_{i\infty}$ is the class obtained by continuing $[\sigma^g_G]_{i\infty}$ along the path γ_τ. The image of this relation under (4.43) is

$$\rho^\vee_{\mathrm{B}}\left([\gamma^{\mathrm{B}}_\tau] \otimes [\sigma^g_G]_{i\infty}\right) = [\sigma^g_G]_\tau.$$

Hence we have an equivalence of motivic periods

$$J^{\mathfrak{m}}_G(\tau) = \sum_i a_i [(E_{i\infty}), [\sigma^g_G]_{i\infty}, [\eta_i]]^{\mathfrak{m}} \otimes [\mathcal{O}(\pi_1^{\mathrm{rel},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau)), \gamma^{\mathrm{B}}_\tau, [\omega_i]]^{\mathfrak{m}},$$

[5] See 4.2.4.1 and 4.2.4.2 for definitions of the relevant categories.

where

$$\rho_{\mathrm{dR}}([\omega_G^g]) = \sum_i a_i[\eta_i] \otimes [\omega_i], \text{ for}$$
$$[\eta_i] \in (E_{i\infty})_{\mathrm{dR}}, \text{ and } [\omega_i] \in \mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau).$$

Recall that de Rham relative torsor of paths is (non-canonically) isomorphic to the de Rham relative group based at τ via (4.48), and that $\pi_1^{\mathrm{dR},\mathcal{S}}(X_1(6),\tau) \cong U_\tau \ltimes S_\tau^{\mathrm{dR}}$ via the splitting we chose in (4.49). Via the same (non-canonical) isomoprhism we have $S_\tau^{\mathrm{dR}} \cong S_{i\infty,\tau}^{\mathrm{dR}}$. Hence, we have an isomorphism of the associated affine rings

$$\mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau)) \cong \mathcal{O}(U_\tau^{\mathrm{dR}}) \otimes \mathcal{O}(S_{i\infty,\tau}^{\mathrm{dR}}).$$

We know that $[\omega_G^g] \in F^2W_3((E_\tau)_{\mathrm{dR}})$, and that ρ_{dR} respects the filtrations. This provides constraints on the image of $[\omega_G^g]$. There are 4 Eisenstein series of weight 3 for $\Gamma_1(6)$, as well as 3 of weight 2. There are no cusp forms of weight less than 4. From (4.53) we know that

$$(2\pi i)^2 E_{3,i}(z)(zX - Y)dz \otimes [\omega_G^g] \in F^2W_3\mathcal{O}(U_\tau^{\mathrm{dR}}),$$

where $E_{3,i}(\tau)$ is an Eisenstein series of weight 3 for $1 \le i \le 4$[6]. Similarly, we have

$$2\pi i E_{2,i}(z)dz \in F^1W_2\mathcal{O}(U_\tau^{\mathrm{dR}}),$$

where $E_{2,i}(\tau)$ is an Eisenstein series of weight 2 for $1 \le i \le 3$, as well as

$$[[\mathsf{X}], [\omega_G^g]]^{red} \in F^1W_1\mathcal{O}(S_{i\infty,\tau}^{\mathrm{dR}}),$$

hence

$$2\pi i E_{2,i}(\tau)d\tau \otimes [[\mathsf{X}], [\omega_G^g]]^{red} \in F^2W_3\mathcal{O}(\pi_1^{\mathrm{dR},\mathcal{S}}(X_1(6), \vec{1}_{i\infty}, \tau)).$$

Then we have

$$\begin{aligned}\rho_{\mathrm{dR}}([\omega_G^g]) = &\sum_{1\le i\le 4} a_i 2\pi i E_{3,i}(z)(zX - Y)dz \otimes [\omega_G^g] + \\ &+ \sum_{1\le i\le 3} a_{i+4} 2\pi i E_{2,i}(z)dz \otimes [[\mathsf{X}], [\omega_G^g]]^{red} + \nu,\end{aligned} \tag{4.60}$$

where $\nu = \sum_i b_i \nu_{\infty,i} \otimes \nu_{red}$ is some k-linear combination in $F^2W_3((E_{i\infty})_{\mathrm{dR}} \otimes \mathcal{O}(S_{i\infty,\tau}^{\mathrm{dR}}))$, and where $b_i \in k$ are some undetermined constants. We may therefore write

$$\begin{aligned}J_G^{\mathfrak{m}}(\tau) = &\sum_{1\le i\le 4} a_i \gamma_{\tau,u}^{\mathfrak{m}}([E_{3,i}] \otimes [\omega_G^g]) + \\ &+ \sum_{1\le i\le 3} a_{i+4} \gamma_{\tau,u}^{\mathfrak{m}}([E_{2,i}]) \otimes [[\mathsf{X}], [\omega_G^g]]^{red} + \\ &+ \sum_i b_i [E_{i\infty}, [\sigma_G^g]_{i\infty}, \nu_{\infty,i}] \cdot \nu_{red},\end{aligned}$$

[6]Recall that $[\omega_G^g] = res(g^*[\pi^*(\omega_G)])$ which spans $F^1(H^1_{\mathrm{dR}}(\mathcal{E})) = F^2(H^1_{\mathrm{dR}}(\mathcal{E})(-1))$.

where $\gamma^{\mathfrak{m}}_{\tau,u}([E_{i,j}]\otimes[\omega_G^g])$ is defined in (4.54). Applying the period homomorphism on the unipotent part as defined in the paragraph following (4.54), and the pairing with respect to $\gamma_{i\infty,\tau}$ on the reductive part as described in (4.56) and thereafter, we obtain the result. □

Remark 4.2.6. In order to sharpen the above theorem by determining the coefficients we may expand both sides of the formula around $q = 0$.

Remark 4.2.7. It should be possible to generalize the above argument to the case of the sunrise with generic masses. To do this the relative completion of the fundamental groupoid of a modular curve with respect to the local system $R^1 f_* \mathbb{Q}$ for $f : \mathcal{E} \to X_\Gamma$ should be replaced with the relative completion of of the fundamental groupoid of $\mathcal{M}_{1,3}$ with respect to the local system the fiber of which over a point $[\mathcal{E}, P, Q]$ of $\mathcal{M}_{1,3}$ is $H^1(\mathcal{E}, \{P, Q\})$. This is discussed briefly in [32, §3.6, Example 3.7]. This would also allow us to extend the definition of the single-valued functions in 4.1.1 to fully generalize Brown's non-holomorphic bi-modular forms [11].

Appendix A

Residues of Feynman integrands of the sunrise with subdivided edges

In proposition 4.2.6 we consider a basis

$$\{[df_1],\dots,[df_5],[res(\eta_G)],[res(\omega_G)]\} \text{ of } H^1_{\mathrm{dR}}(\mathcal{E}, D\cap\mathcal{E})\,,$$

and we write the residues of forms ν_1,ν_2,ν_3, defined in the statement of the proposition, in this basis. The following are the coefficients for each residue form.

Coefficients of $res(\nu_1)$:

$$\begin{aligned}
a_{1,1} &= \frac{1}{2d_1}((m_1^2-m_3^2)(m_1^2-m_2^2+m_3^2+q_1^2)),\\
a_{1,2} &= \frac{-1}{d_1}((m_1^2-m_2^2+m_3^2+q_1^2)(-m_1^2+m_2^2+m_3^2+q_1^2)),\\
a_{1,3} &= \frac{-1}{2m_2^2q_1^2},\quad a_{1,4}=a_{1,1}-a_{1,3},\quad a_{1,5}=a_{1,2}+a_{1,3},\\
a_{1,6} &= \frac{-1}{d_1}m_1^2(m_1^6-3m_1^4m_2^2-3m_1^4m_3^2+3m_1^4q_1^2+3m_1^2m_2^4+2m_1^2m_2^2m_3^2-2m_1^2m_2^2q_1^2+\\
&\quad +3m_1^2m_3^4-2m_1^2m_3^2q_1^2+3m_1^2q_1^4-m_2^6+m_2^4m_3^2-m_2^4q_1^2+m_2^2m_3^4+10m_2^2m_3^2q_1^2+\\
&\quad +m_2^2q_1^4-m_3^6-m_3^4q_1^2+m_3^2q_1^4+q_1^6),\\
a_{1,7} &= \frac{1}{d_1}(m_1^6-2m_1^4m_2^2-2m_1^4m_3^2+2m_1^4q_1^2+m_1^2m_2^4+2m_1^2m_2^2m_3^2-2m_1^2m_2^2q_1^2+\\
&\quad +m_1^2m_3^4-2m_1^2m_3^2q_1^2+m_1^2q_1^4+4m_2^2m_3^2q_1^2),\ \text{where}\\
d_1 &= 2m_2^2q_1^2(3m_1^4-2m_1^2m_2^2-2m_1^2m_3^2+2m_1^2q_1^2-m_2^4+2m_2^2m_3^2-2m_2^2q_1^2-m_3^4\\
&\quad -2m_3^2q_1^2-q_1^4)
\end{aligned}$$

Coefficients of $res(\nu_2)$:

$$a_{2,1} = \frac{1}{2d_2}(2m_1^2 - 2m_3^2)$$

$$a_{2,2} = \frac{1}{m_1^2 d_2}(-m_1^4 - 4m_1^2q_1^2 + m_2^4 - 2m_2^2m_3^2 + 2m_2^2q_1^2 + m_3^4 + 2m_3^2q_1^2 + q_1^4)$$

$$a_{2,3} = \frac{-1}{2m_1^2q_1^2}, \quad a_{2,4} = a_{2,1} - a_{2,3}, \quad a_{2,5} = a_{2,2} + a_{2,3},$$

$$\begin{aligned} a_{2,6} = \frac{1}{d_2}(&m_1^6 - m_1^4m_2^2 - 3m_1^4m_3^2 + m_1^4q_1^2 - m_1^2m_2^4 - 2m_1^2m_2^2m_3^2 - 10m_1^2m_2^2q_1^2 + \\ &+ 3m_1^2m_3^4 + 2m_1^2m_3^2q_1^2 - m_1^2q_1^4 + m_2^6 - 3m_2^4m_3^2 + m_2^4q_1^2 + 3m_2^2m_3^4 + 2m_2^2m_3^2q_1^2 - \\ &- m_2^2q_1^4 - m_3^6 - 3m_3^4q_1^2 - 3m_3^2q_1^4 - q_1^6) \end{aligned}$$

$$a_{2,7} = \frac{-1}{d_2}(m_1^4 - 2m_1^2m_3^2 - m_2^4 - 2m_2^2q_1^2 + m_3^4 - q_1^4), \text{ where}$$

$$d_2 = 2q_1^2(3m_1^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_1^2q_1^2 - m_2^4 + 2m_2^2m_3^2 - 2m_2^2q_1^2 - m_3^4 - 2m_3^2q_1^2 - q_1^4)$$

Coefficients of $res(\nu_3)$:

$$a_{3,1} = \frac{1}{2d_3}(2m_1^4 - m_1^2m_2^2 - m_1^2m_3^2 + 3m_1^2q_1^2 - m_2^4 + 2m_2^2m_3^2 - 2m_2^2q_1^2 - m_3^4 - 2m_3^2q_1^2 - q_1^4)$$

$$a_{3,2} = \frac{1}{d_3}(-m_1^4 + 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_1^2q_1^2 - m_2^4 + 2m_2^2m_3^2 - 2m_2^2q_1^2 - m_3^4 - 2m_3^2q_1^2 - q_1^4)$$

$$a_{3,3} = \frac{-1}{2m_3^2q_1^2}, \quad a_{3,4} = a_{3,1} - a_{3,3}, \quad a_{3,5} = a_{3,2} + a_{3,3},$$

$$\begin{aligned} a_{3,6} = \frac{1}{d_3}(&m_1^2(m_1^6 - 3m_1^4m_2^2 - m_1^4m_3^2 + m_1^4q_1^2 + 3m_1^2m_2^4 - 2m_1^2m_2^2m_3^2 + 2m_1^2m_2^2q_1^2 - \\ &- m_1^2m_3^4 - 10m_1^2m_3^2q_1^2 - m_1^2q_1^4 - m_2^6 + 3m_2^4m_3^2 - 3m_2^4q_1^2 - \\ &- 3m_2^2m_3^4 + 2m_2^2m_3^2q_1^2 - 3m_2^2q_1^4 + m_3^6 + m_3^4q_1^2 - m_3^2q_1^4 - q_1^6)) \end{aligned}$$

$$a_{3,7} = \frac{-1}{d_3}(m_1^2(m_1^4 - 2m_1^2m_2^2 + m_2^4 - m_3^4 - 2m_3^2q_1^2 - q_1^4)), \text{ where}$$

$$\begin{aligned} d_3 = 2m_3^2q_1^2(&3m_1^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 + 2m_1^2q_1^2 - m_2^4 + 2m_2^2m_3^2 - 2m_2^2q_1^2 - m_3^4 \\ &- 2m_3^2q_1^2 - q_1^4) \end{aligned}$$

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