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SOME TOPICS

IN

FUNCTIONAL - DIFFERENTIAL

EQUATIONS

A thesis submitted for the degree of

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by

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A B S T R A C T

This thesis is concerned with the asymptotic behaviour of solutions of the differential-difference equation:-

$$w'(s) = g(s)[w(s - 1) - w(s)]$$

where $g(s)$ is a continuous real-valued function. $g(s)$ is assumed to have one of the following asymptotic behaviours

algebraic

exponential algebraic

constant

zero

periodic

Chapter 2 covers the behaviour of solutions as $s \to +\infty$ for $g(s) \geq 0$.

Chapter 3 covers the behaviour of solutions as $s \to +\infty$ for $g(s) \leq 0$.

Chapter 4 covers $s \to -\infty$ and $g(s) \geq 0$.

Chapter 5 covers $s \to -\infty$ and $g(s) \leq 0$.

A C K N O W L E D G E M E N T S

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C O N T E N T S

Chapter 1

The functional-differential equation

$$w'(s) = g(s)\,[w(s-1) - w(s)] \qquad 1$$

Chapter 2

Behaviour of solutions as $s \to +\infty$ for $g \geq 0$ 4

Chapter 3

Behaviour of solutions as $s \to +\infty$ for $g \leq 0$ 53

Chapter 4

Behaviour of solutions as $s \to -\infty$ for $g \geq 0$ 83

Chapter 5

Behaviour of solutions as $s \to -\infty$ for $g \leq 0$ 97

References 115

C H A P T E R 1

The functional differential equation $w'(s) = g(s)[w(s - 1) - w(s)]$

This thesis is concerned with the asymptotic behaviour of solutions of the differential-difference equation:-

$$w'(s) = g(s)[w(s - 1) - w(s)] \qquad (1.1)$$

where $g(s)$ is a continuous real-valued function.

Although some delay-differential equations are encountered even in the work of Euler, the systematic study of such equations started only in the twentieth century. The original motivation was the needs of the applied sciences, especially the theory of automatic control. In the last twenty years, the area of application has expanded greatly.

The equation (1.1), with certain particular values of $g(s)$, has application as a mathematical idealisation and simplification of an industrial problem involving wave motion in the overhead supply line to an electrified railway system [7, 11]. Another particular case arises in a partitioning problem in number theory [12].

The basic theory of differential-difference equations is covered by the book of Bellman and Cooke [2] and by that of El'sgol'ts [6]. Results from these works are assumed without direct reference.

By a solution of (1.1) we mean a complex-valued continuous function $w(s)$ defined in some sub-interval of $0 \leq s < \infty$ and satisfying (1.1). Existence of such solutions is trivial since any constant is a solution. A solution can always be extended to the right: we solve (1.1) as a differential equation for $w(s)$ in the interval $[s_0, s_0 + 1]$, $w(s - 1)$ being known. We demand continuity at $s = s_0$

so that the solution is unique. If the solution is merely continuous in $[s_o - 1, s_o]$, it will be continuously differentiable in $[s_o, s_o + 1]$ since integration is involved. The process can thus be continued indefinitely, the solution becoming progressively more differentiable. It thus makes sense to try to discuss the asymptotic behaviour of all solutions at $+\infty$.

Extending to the left, however, the solution loses degrees of differentiability. It is therefore more reasonable to show that, given a specific asymptotic behaviour at $-\infty$, there is a solution of (1.1) which exhibits this behaviour.

In Chapter 2, we consider the behaviour of solutions of (1.1) as $s \to +\infty$ for functions $g(s)$ which are everywhere positive. To obtain more detailed results, $g(s)$ is restricted to the following basic cases:-

asymptotically algebraic

asymptotically exponential algebraic

~~[i.e. ... exp(polynomial)]~~

(1.2)

asymptotically constant

tends to zero

periodic.

In general terms, all solutions and their derivatives can be shown to be bounded, under various conditions.

Chapter 3 deals with the behaviour of solutions as $s \to +\infty$ with $g(s)$ everywhere negative. The cases listed in (1.2) are covered for g. Solutions are generally $O(\exp[-\int^s g(t)dt])$. Chapters 4 and 5 cover the behaviour of solutions as $s \to -\infty$ for $g(s)$ positive and negative respectively. This is a more difficult problem than that of Chapters 2 and 3 as the equation is now equivalent to one of

advanced type. Starting from any Hölder-continuous function of period 1, say $f(s)$, we can, in general, find a solution of (1.1) which behaves like $f(s)$ as $s \to -\infty$.

The assumptions that $g(s)$ is positive or negative need not hold for all s; it suffices to assume that g is positive/negative for $s >$ some s_1. For simplicity, we require g positive/negative everywhere.

My work is based on that of Kato and McLeod [11]. They consider the equation

$$y'(x) = ay(\lambda x) + by(x) \tag{1.3}$$

Under the transformation

$$x = e^s \qquad \lambda = e^c \qquad y(x) = z(s)$$

(1.3) becomes

$$e^{-s}z'(s) = az(s + c) + bz(s)$$

which overlaps (1.1) in several cases.

C H A P T E R 2

Behaviour of solutions as $s \to +\infty$ for $g \geq 0$

In this chapter we consider the behaviour of solutions of

$$w'(s) = g(s)[w(s-1) - w(s)] \qquad (2.1)$$

as $s \to +\infty$, for $g(s)$ continuous and positive for all s. All theorems in this chapter, except in § 3.1, assume that $g(s)$ is positive whether explicitly stated or not. One result can be proved in this full generality.

Theorem 1

If $g(s)$ is continuous and $\geq 0 \ \forall s$ and $w(s)$ is any solution of (2.1), then $w(s)$ is bounded as $s \to +\infty$.

Proof

Let I_n denote $[s_o + n, s_o + n + 1]$ where s_o is a fixed value of s. Suppose $|w(s)|$ has least upper bound K_n in I_n. We can write the equation as

$$\frac{d}{ds}\left[\exp\left\{\int^s g(z)dz\right\} w(s)\right] = \exp\left[\int^s g(z)dz\right] g(s)\, w(s-1)$$

And so for $s \in I_{n+1}$

$$\left[w(t) \exp\left\{\int^t g(z)dz\right\}\right]\Bigg|_{s_o+n+1}^{s} = \int_{s_o+n+1}^{s} \exp\left[\int^t g(z)dz\right] g(t)\, w(t-1)\, dt$$

$|w| < K_n$ in I_n and thus

$$\left|w(s) \exp\left[\int^s g(z)dz\right]\right| \leq K_n \exp\left[\int^s g(z)dz\right]$$

Thus $|w(s)| \leq K_n$ for $s \in I_{n+1}$

$$\Rightarrow \qquad K_{n+1} \leq K_n$$

$$\Rightarrow \qquad |w(s)| \leq K_o$$

for $s \in I_n \ \forall\, n$, i.e. $w(s)$ is bounded as $s \to +\infty$.

To proceed further we need to be more specific about the asymptotic behaviour of $g(s)$. It is crucial whether $g(s)$ is unbounded, bounded or has no limit at infinity. § 1, 2, 3 deal with these cases.

§ 1 $g(s)$ unbounded as $s \to +\infty$.

The two basic cases covered in this category are $g(s)$ a polynomial and $g(s) = \exp[h(s)]$, where $h(s)$ is a polynomial. However the results which pertain to these cases can be proved under more general conditions and this is done. The case $g(s) \sim As$ is particularly interesting and is dealt with separately in § 1.2.

§ 1.1 $g(s)$ asymptotically algebraic in s.

In this section, we consider the basic case $g(s)$ a polynomial in s. However, we need impose only the following conditions on $g(s)$:-

Conditions I $\exists\ k, \beta, m$ such that

$g(s)$ is m times continuously differentiable

$$g(s) = O(s^k)$$

$$\frac{1}{g(s)} = O(s^{-\beta}) \qquad \text{as } s \to +\infty$$

$$\frac{g^{(r)}(s)}{g(s)} = O(s^{-r}) \qquad \text{for } r = 1, 2, \ldots, m$$

and where $\beta > 0$ and $1 > (m - 1)(k - \beta)$

Clearly, if Conditions I hold with m = N then they hold with m = N - 1. Since $g(s) = O(s^k)$ and $\frac{1}{g(s)} = O(s^{-\beta})$, we must have $k \geq \beta$.

Theorem 2

If Conditions I hold with m = 2 and w(s) is a solution of (2.1) then w'(s) is bounded as $s \to +\infty$.

Proof

Let I_n be as before and let $|w'(s)|$ have least upper bound J_n in I_n. Differentiating (2.1) we obtain

$$w''(s) = g(s)[w'(s-1) - w'(s)] + g'(s)[w(s-1) - w(s)]$$

and so

$$w''(s) = g(s)w'(s-1) - w'(s)[g(s) - \frac{g'(s)}{g(s)}] \qquad (2.2)$$

Re-arranging, we have

$$\frac{d}{ds}\{\frac{w'(s)}{g(s)}\exp[\int^{s} g(z)dz] = \exp[\int^{s} g(z)dz]\, w'(s-1)$$

and $s \geq s_0$, for $s \in I_{n+1}$

$$[\exp\{\int^{s} g(z)dz\}\frac{w'(s)}{g(s)}]\Big|_{s_0+n+1}^{s} = \int_{s_0+n+1}^{s} \exp[\int^{t} g(z)dz]\, w'(t-1)\, dt$$

$$\Rightarrow$$

$$|\frac{w'(s)}{g(s)}\exp[\int^{s} g(z)dz]| \leq J_n\{\frac{\exp[\int^{s_0+n+1} g(z)dz]}{g(s_0+n+1)} + \int_{s_0+n+1}^{s} \exp[\int^{t} g(z)dz]dt\} \qquad (2.3)$$

Now consider separately the integral in (2.3)

$$\int_{s_0+n+1}^{s} \exp[\int^{t} g(z)dz] = \frac{\exp[\int^{t} g]}{g(t)}\Big|_{s_0+n+1}^{s} + \int_{s_0+n+1}^{s} \frac{g'(t)\exp[\int^{t} g]}{g^2(t)}\, dt$$

$$= \exp[\int^{t} g]\,[\frac{1}{g(t)} + \frac{g'(t)}{g^3(t)}]\Big|_{s_0+n+1}^{s} - \int_{s_0+n+1}^{s} \exp[\int^{t} g]\frac{d}{dt}[\frac{g'(t)}{g^3(t)}]$$

However,

$$\int_{S_0+n+1}^{s} \exp\Big[\int^{t} g\Big] \frac{d}{dt}\Big[\frac{g'(t)}{g^3(t)}\Big]\,dt$$

$$= \int_{S_0+n+1}^{s} \exp\Big[\int^{t} g\Big]\Big[\frac{g''(t)}{g^3(t)} - \frac{3[g'(t)]^2}{g^4(t)}\Big]\,dt$$

Thus returning to (2.3), we have

$$\Big|\frac{w'(s)}{g(s)} \exp\Big[\int^{s} g\Big]\Big| \le J_n \Big\{ \frac{\exp\big[\int^{S_0+n+1} g\big]}{g(s_0+n+1)} + \exp\Big[\int^{s} g\Big]\Big[\frac{1}{g(s)} + \frac{g'(s)}{g^3(s)}\Big]$$

$$- \exp\Big[\int^{S_0+n+1} g\Big]\Big[\frac{1}{g(s_0+n+1)} + \frac{g'(s_0+n+1)}{g^3(s_0+n+1)}\Big]$$

$$+ \int_{S_0+n+1}^{s} \exp\Big[\int^{t} g\Big]\Big[\frac{3[g'(t)]^2}{g^4(t)} - \frac{g''(t)}{g^3(t)}\Big]\,dt \Big\}$$

$$\Rightarrow \quad |w'(s)| \le J_n \Big\{1 + \frac{g'(s)}{g^2(s)} - \exp\Big[-\int_{S_0+n+1}^{s} g\Big] \frac{g'(s_0+n+1)\,g(s)}{g^3(s_0+n+1)}$$

$$+ g(s) \int_{S_0+n+1}^{s} \exp\Big[-\int_{t}^{s} g\Big]\Big[\frac{3[g'(t)]^2}{g^4(t)} - \frac{g''(t)}{g^3(t)}\Big]\,dt \Big\} \qquad (2.4)$$

Since

$$g(s) = O(s^k) \qquad \frac{1}{g(s)} = O(s^{-\beta})$$

$$\frac{g'(s)}{g(s)} = O\Big(\frac{1}{s}\Big) \qquad \frac{g''(s)}{g(s)} = O\Big(\frac{1}{s^2}\Big)$$

the third term on the right hand side is

$$O\big[\exp[-B(s_0+n+1)^{\beta}]\; s^k\; (s_0+n+1)^{-1-2\beta}\big] \qquad \beta > 0$$

We can estimate the integral term in (2.4) by

$$g(s)\int_{s_0+n+1}^{s}\left|\frac{3[g'(t)]^2}{g^4(t)}-\frac{g''(t)}{g^3(t)}\right|dt$$

$$\leq g(s)\int_{s_0+n+1}^{s}\left[\frac{A}{t^{2+2\beta}}+\frac{B}{t^{2+2\beta}}\right]dt$$

$$\leq \frac{g(s)\,C}{(s_0+n+1)^{2+2\beta}} \qquad \text{where A, B and C are constants.}$$

Hence

$$|w'(s)| \leq J_n\left\{1+O\left(\frac{1}{s^{1+\beta}}+C\frac{g(s)}{(s_0+n+1)^{2+2\beta}}\right)\right\}$$

$$= J_n\left\{1+O([s_0+n]^{-\delta})\right\}$$

where $\delta = \min[1+\beta,\ 2+2\beta-k]$

$\prod_{r=1}^{\infty}(1+n^{-r})$ converges for $r>1$ and so it remains to show that $\delta > 1$.

We have $k \geq \beta$ and $1+\beta > k$. If $\delta = 1+\beta$, trivially $\delta > 1$. If $\delta = 2+2\beta-k$ i.e. $1+\beta \geq 2+2\beta-k$ then $k \geq 1+\beta$ which is a contradiction. Thus $\delta > 1$ and we can choose s_0 in such a way that

$$\prod_{n=0}^{\infty}(1+O([s_0+n]^{-\delta}) \leq 2$$

and so

$$J_n \leq 2J_0 \qquad \forall\ n$$

thus

$$|w'(s)| \leq 2J_0$$

for $s \in I_n \quad \forall\ n$.

We can now use an induction argument to show that, when Conditions I hold for some m, then $w^{(m-1)}(s)$ is bounded.

Theorem 3

If Conditions I hold for $m = M + 1$ and $w(s)$ is a solution of (2.1) then $w^{(M)}(s)$ is bounded for all s, $M \geq 1$.

Proof

From Theorems 1 and 2 we know that $w(s)$ and $w'(s)$ are bounded under these conditions and so we may assume $M \geq 2$. Recalling equations (2.1) and (2.2), we make the induction hypotheses that

$w'(s)$, $w''(s)$,...., $w^{(N-1)}(s)$ are bounded

and that

$$w^{(N)}(s) = g(s)\{w^{(N-1)}(s-1) - w^{(N-1)}(s)\} + (N-1)\frac{g'(s)}{g(s)}w^{(N-1)}(s) + O(\frac{1}{s^2})$$

These hypotheses hold for n = 1, 2. Assume that they hold for all $N \leq M$.

$M \geq 2$ and so by Leibnitz' formula

$$w^{(M+1)}(s) = g(s)[\, w^{(M)}(s-1) - w^{(M)}(s)\,] + \sum_{r=1}^{M} \binom{M}{r} g^{(r)}(s)\,[\, w^{(M-r)}(s-1) - w^{(M-r)}(s)]$$

By hypothesis

$$w^{(M+1)}(s) = g(s)[\, w^{(M)}(s-1) - w^{(M)}(s)] + \sum_{r=1}^{M} \binom{M}{r} \frac{g^{(r)}(s)}{g(s)}\,[\, w^{(M-r+1)}(s) - (M-r)\frac{g'(s)}{g(s)}w^{(M-r)}(s) + O(\frac{1}{s^2})]$$

Using Conditions I, we find that

$$w^{(M+1)}(s) = g(s)[\, w^{(M)}(s-1) - w^{(M)}(s)] + M\frac{g'(s)}{g(s)}w^{(M)}(s) + O(\frac{1}{s^2}) \qquad (2.5)$$

Thus we have established the second hypothesis for the case $N = M + 1$.

It remains to show that $w^{(M)}(s)$ is bounded.

From (2.5)

$$w^{(M+1)}(s) + [g(s) - M\frac{g'(s)}{g(s)}]\, w^{(M)}(s)$$

$$= g(s)\, w^{(M)}(s-1) + O(\frac{1}{s^2})$$

$$\Rightarrow \quad \frac{d}{ds}\left\{\frac{w^{(M)}(s)}{g^M(s)} \exp[\int^s g]\right\} = \exp[\int^s g]\, g^{-M+1}(s)\, w^{(M)}(s-1)$$

$$+ O(\frac{\exp[\int^s g]}{s^2 g^M(s)})$$

Let I_n be $[s_o + n,\ s_o + n + 1]$ as before. Assume that $|w^{(M)}(s)|$ has least upper bound $J_{n,M}$ in I_n.

For $s \in I_{n+1}$

$$\left|\frac{w^{(M)}(s)}{g^M(s)} \exp[\int^s g]\right| \le J_{n,M}\left\{\frac{\exp[\int^{s_o+n+1} g]}{g^M(s_o+n+1)} + \int_{s_o+n+1}^{s} \frac{\exp[\int^t g]}{g^{M-1}(t)}\, dt\right.$$

$$\left. + O(\int_{s_o+n+1}^{s} \frac{\exp[\int^t g]}{t^2\, g^M(t)}\, dt)\right\} \qquad (2.6)$$

$$\int_{s_o+n+1}^{s} \frac{\exp[\int^t g]}{g^{M-1}(t)}\, dt = \frac{\exp[\int^t g]}{g^M(t)}\Big|_{s_o+n+1}^{s} - \int_{s_o+n+1}^{s} \exp[\int^t g]\frac{d}{dt}(\frac{1}{g^M(t)})\, dt$$

$$= \left\{\exp[\int^t g]\,[\frac{1}{g^M(t)} + \frac{Mg'(t)}{g^{M+2}(t)}]\Big|_{s_o+n+1}^{s}\right.$$

$$\left. - M\int_{s_o+n+1}^{s} \exp[\int^t g]\frac{d}{dt}[\frac{g'(t)}{g^{M+2}(t)}]\, dt\right\}$$

This last integral is

$$- M \int_{S_0+n+1}^{s} \exp\Big[\int^{t} g \Big] \Big[\frac{g''(t)}{g^{M+2}(t)} - (M+2) \frac{[g'(t)]^2}{g^{M+3}(t)} \Big] dt$$

Returning to (2.6) we have

$$\Big| \frac{w^{(M)}(s)}{g^{M}(s)} \exp\Big[\int^{s} g \Big] \Big| \le J_{n,M} \Big\{ \frac{\exp[\int^{s} g]}{g^{M}(s)} + \frac{\exp[\int^{t} g]\, M\, g'(t)}{g^{M+2}(t)} \Big|_{S_0+n+1}^{s}$$

$$- M \int_{S_0+n+1}^{s} \exp\Big[\int^{t} g \Big] \Big[\frac{g''(t)}{g^{M+2}(t)} - (M+2) \frac{[g'(t)]^2}{g^{M+3}(t)} \Big] dt \Big\}$$

$$+ O\Big(\int_{S_0+n+1}^{s} \frac{\exp[\int^{t} g]}{t^2\, g^{M}(t)} dt \Big)$$

$$| w^{(M)}(s) | \le J_{n,M} \Big\{ 1 + M \frac{g'(s)}{g^2(s)}$$

$$- \frac{\exp[-\int_{S_0+n+1}^{s} g]\, M\, g'(s_0 + n + 1)\, g^{M}(s)}{g^{M+2}(s_0 + n + 1)}$$

$$+ O\Big(g^{M}(s)\, M \int_{S_0+n+1}^{s} \Big[\frac{A}{t^{2+(M+1)\beta}} \Big] dt + g^{M}(s) \int_{S_0+n+1}^{s} \frac{1}{t^{2+M\beta}} dt \Big)$$

The third term is of smaller order than the first two since $[g(s)]^{-1} = O(s^{-\beta})$ and so we may incorporate it into the error term giving

$$O\Big(\frac{C\, s^{kM} [1 + (s_0 + n + 1) e^{-B(s_0+n+1)^{\beta}}]}{(s_0 + n + 1)^{2+(M+1)\beta}} + \frac{s^{kM}}{(s_0 + n + 1)^{2 + M\beta}} \Big)$$

whence

$$| w^{(M)}(s) | \le J_{n,M} \Big\{ 1 + O(s^{-1-\beta} + E(s_0 + n + 1]^{-\gamma}) \Big\}$$

where $\gamma = 2 + M\beta - kM$. Let $\delta = \min[1 + \beta, \gamma]$

$$|w^{(M)}(s)| \le J_{n,M} \Big\{ 1 + O([s_0 + n]^{-\delta}) \Big\}$$

It remains to show that $\delta > 1$.

Since $\beta > 0$, $1 + \beta > 1$, and so if $\delta = 1 + \beta$, then $\delta > 1$.

If $\delta = \gamma = 2 + M\beta - kM$, since Conditions I hold for $m = M + 1$, we have $2 + M\beta - kM > 1$ i.e. $\delta > 1$.

Thus we have established both induction hypotheses and so $w^{(M)}(s)$ is bounded.

If we strengthen our assumptions, these two theorems enable us to prove an interesting result concerning the limit function of $w^{(M)}(s)$.

Theorem 4

If Conditions I hold for $m = M + 2$ and $\beta > 1$ ($\Rightarrow k > 1$), then if n is an integer, $\lim_{n\to\infty} w^{(M)}(s + n)$ exists ($= f_M(s)$ say) and is a continuous function which is periodic with period 1. Also, if $\beta > 2$,

$$f_M(s) = f_o^{(M)}(s)$$

Proof

If $M = 0$, from (2.1) we have

$$g(s) [w(s - 1) - w(s)] = O(1)$$

$$w(s - 1) - w(s) = O([g(s)]^{-1})$$

If $M > 0$, from the induction hypotheses of Theorem 3, which were established under these conditions

$$g(s) [w^{(M)}(s - 1) - w^{(M)}(s)] + O(\frac{1}{s^2}) = O(1)$$

and thus

$$w^{(M)}(s - 1) - w^{(M)}(s) = O(\frac{1}{g(s)}) \qquad (2.7)$$

From above, (2.7) also holds for $M = 0$.

Now we replace s successively by $s + 1$, $s + 2$,...., $s + n$ and sum the equations to give

$$w^{(M)}(s + n) = w^{(M)}(s) - \sum_{r=1}^{n} O\left(\frac{1}{g(s + r)}\right)$$

Letting $n \to \infty$, we see that $\lim_{n\to\infty} w^{(M)}(s + n)$ exists provided that

$$\sum_{r=1}^{\infty} O\left(\frac{1}{g(s + r)}\right) < \infty \qquad (2.8)$$

There exists N, a constant, such that

$$\frac{1}{g(s + r)} \leq \left(\frac{A}{(s + r)}\right)^{\beta} \quad \text{for} \quad s + r > N$$

and so the sum (2.8) exists if $\sum_{r=N}^{\infty} \frac{1}{r^{\beta}}$ exists. ~~We can slightly amend the proof of the standard integral test to give the result that we need.~~

Thus we have convergence of the sum (2.8) if

$$\int^{\infty} \frac{dx}{x^{\beta}}$$

exists, i.e. if $\beta > 1$.

We have established that, under the conditions given:-

$$\lim_{n\to\infty} w^{(M)}(s + n) = f_M(s)$$

It is clear from (2.7) that $f_M(s)$ is periodic of period 1 and so we need only consider the interval [0, 1)

$$f_M(s) = w^{(M)}(s) - \sum_{n=1}^{\infty} O\left(\frac{1}{g(s + n)}\right)$$

$\sum_{n=1}^{\infty} O\left(\frac{1}{g(s + n)}\right)$ is uniformly convergent on [0, 1) and so $f_M(s)$ is continuous.

From above we find

$$\sum_{n=1}^{\infty} O\left(\frac{1}{g(s+n)}\right) = O(s^{1-\beta})$$

and so, integrating, we have

$$\int^{s} f_M(t)\,dt = w^{(M-1)}(s) + O(s^{2-\beta})$$

$w^{(M-1)}(s) \longrightarrow f_{M-1}(s)$ and thus, if $\beta > 2$

$$\tilde{f}_{M-1}(s) = \int^{s} f_M(t)\,dt$$

i.e. $f_{M-1}'(s) = f_M(s)$

Repeating this, we find

$$f_o^{(M)}(s) = f_M(s)$$

Theorem 4 associates with each solution of (2.1), the periodic function $f_o(s)$ ($= f(s)$ say). If we strengthen the conditions on $g(s)$, we can show that this association is a one-to-one correspondence

$$w(s) \rightleftarrows f(s)$$

This work is covered by a paper of de Bruijn [3]. He deals with the equation

$$q(x)w'(x) + p(x)w(x) - w(x-1) = 0$$

under the following conditions.

For any $n \geq 0$ $\exists$ B_n such that

$q(x)$, $p(x)$ are continuous $x \geq 1$

$q(x) > 0 \quad x \geq 1$

$p(x) > \frac{1}{2} \quad x \geq B_n + 1$

$| p(x) - 1 | < B_n \phi(x)$

$q(x)$, $p(x)$ have continuous 1st derivatives $x \geq 2$

$q(x)$, $p(x)$ have continuous 2nd derivatives $x \geq 3$

. . . . . .

$q(x)$, $p(x)$ have continuous nth derivatives $x \geq n + 1$

such that

$$| q^{(k)}(x) | < B_n \Phi(x)$$
$$| p^{(k)}(x) | < B_n \phi(x) \qquad \text{for } x \geq k + 1 \quad k = 1, 2, \ldots . n$$

$$\{q(x)\}^2 < B_n \phi(x) \quad x \geq 1$$

Here $\phi(x)$ is an arbitrary decreasing positive function for $x \geq 1$ with convergent integral

$$\int_1^\infty \Phi(x)\, dx$$

If these conditions are satisfied, the equation is said to belong to the class $B_\infty(\Phi)$.

However, de Bruijn's completeness and uniqueness theorems ([3] Theorems 6 and 7) can be made to apply to (2.1). Any solution $w(s)$ of (2.1) gives rise to a continuous function $f(s)$ of period 1, by virtue of Theorem 4, when Conditions I hold for $m = 2$. The correspondence $w \rightarrow f$ defines a linear operator, T, $f = Tw$; the domain of T, D, is the set of all solutions of (2.1) (under Conditions I, $m = 2$), the range R is the set of all $f = Tw$, $w \in D$. We denote the set of all continuous functions of period 1 by R_o.

Theorem 5

If Conditions II, below, hold, then R is dense in R_o, i.e. if $\chi \in R$ and $\varepsilon > 0$ are given, then a function $w \in D$ can be found such that

$$|| \chi - Tw || < \varepsilon$$

$$[\ ||\ f\ ||\ = \max_{0 \leq x \leq 1} |\ f(x)\ |\ ]$$

Conditions II

i) $\{[\ g\ (s)]^{-1}\}^{(n)}$ is continuous $n = 0, 1, 2, \ldots \quad s \geq 1$

ii) $\exists$ positive constants B and ρ , $\rho > 1$ such that

$$|\ [\ \frac{1}{g(s)}\ ]^{(n)}\ |\ <\ B^{n+1}\ n^{n}\ s^{-n-\rho} \qquad \text{for } n = 0, 1, 2, \ldots$$

Proof

De Bruijn [3] section 4. It is not difficult to show that Conditions II imply that the equation (2.1) belongs to $B_{\infty}\ (\ s^{-\rho}\ )$ in de Bruijn's notation.

Theorem 6

If Conditions II hold and w(s) is a solution of (2.1) such that $\lim_{n \to \infty} w(s + n) = 0$ then $w(s) \equiv 0$.

Proof

De Bruijn [3], Theorem 7. Conditions II are a translation of de Bruijn's conditions into our notation. The proof is by contradiction. We assume $w(s) \not\equiv 0$, then $\exists$ a,b such that $w(s) \neq 0$ for $a \leq s \leq b$ where $0 \leq a < b \leq a + 1$. We take $1 < \eta < \rho$ and build up a solution of the adjoint equation:-

$$\left\{\frac{1}{g(s)}\ y(s - 1)\right\}^{(1)} - y(s - 1) + y(s) = 0$$

Define

$$y_0(s) = \exp\left\{-(s - a)^{(\eta - 1)^{-1}} - (b - s)^{(\eta - 1)^{-1}}\right\} \qquad a < s < b$$

$$y_0(a) = 0$$

$$y_0(s) = 0 \quad b \leq s \leq a + 1$$

$$y_n(s) = (1 - \Omega_n)\, y_{n-1}(s)$$

where

$$\Omega_n \Phi = \left\{ \frac{\Phi}{g(s + n)} \right\}^{(1)}$$

If we write

$$y(s + n) = y_n(s) \quad a \leq s \leq a + 1 \quad n = 0, 1, 2, \ldots$$

then $y(s)$ is a solution of the adjoint equation for $s \geq a + 1$.

It can be shown that $| y(s) | < C$ for $s \geq a$ where C is a constant.

We can also show that

$$(w(s), y(s)) = \int_{S-1}^{S} w(t)\, y(t)\, dt + \frac{w(s)\, y(s - 1)}{g(s)}$$

is independent of s for $s \geq a + 1$ and thus, since $w(s) \to 0$ and $| y(s) | < C$ as $s \to \infty$, we have $(w, y) = 0$.

But

$$(w, y) = \int_{a}^{b} w(t)\, y(t)\, dt$$

and w is continuous and non-zero for $a \leq s \leq b$,

$$y(t) \geq 0 \quad y(t) \not\equiv 0 \quad \Rightarrow \quad (w, y) \neq 0$$

which is a contradiction, thus $w(s)$ is identically zero.

Linking Theorems 5 and 6 we have a one-to-one correspondence

$$w \rightleftarrows f$$

It is not easy to link Conditions I and II; however, we can show that, if $g(s)$ is a polynomial in s of degree greater than 1, then both sets of conditions hold. Trivially Conditions I hold with $\beta = k$ and $m = \infty$.

Lemma 7

Conditions II hold if $g(s)$ is a polynomial of degree greater than 1.

Proof

Since $g(s)$ is a polynomial, it has a finite number of zeroes. Thus, by changing the axes, we can assume that all zeroes of $g(s)$ have non-positive real part. We may also assume, without any loss of generality, that the leading term of $g(s)$ has coefficient 1.
We consider first $g(s) = s^k$, $k > 1$.

$$\left[\frac{1}{g(s)} \right]^{(n)} = (-1)^n k (k+1) \ldots (k+n-1) s^{-k-n}$$

and this is clearly continuous for $s \geq 1$. Taking $\rho = k$, B can always be chosen so that

$$k (k+1) \ldots (k+n-1) < B^{n+1} n^n \qquad \forall n$$

Thus Conditions II hold for $g(s) = s^k$, when $k > 1$.

It remains to show that

$$g(s) = s^k + a_1 s^{k-1} + \ldots + a_k$$

satisfies the conditions.

Hypothesis: $n \geq 2$

$$\left(\frac{1}{g(s)} \right)^{(n-1)} = \frac{(-1)^{n-1} k (k+1) \ldots (k+n-2) s^{(k-1)(n-1)}}{g^n(s)} + O(s^{-n-k+\varepsilon}) \qquad (2.9)$$

$\forall \varepsilon > 0$. Here the O-term is uniform in s and n.

If $n = 2$

$$\left[\frac{1}{g(s)} \right]^{(1)} = - \frac{g'(s)}{g^2(s)} = - k s^{k-1} [g(s)]^{-2} + O(s^{-k-2})$$

Thus the hypothesis holds for $n = 2$. Assume that it holds for n.

Thus

$$\left[\frac{1}{g(s)} \right]^{(n-1)} = \frac{(-1)^{n-1} k (k+1) \ldots (k+n-2) s^{(k-1)(n-1)}}{g^n(s)} + O(s^{-n-k+\varepsilon})$$

$g(s)$ is a polynomial and hence $\left[\frac{1}{g(s)} \right]^{(n-1)}$ has an asymptotic inverse power series expansion for all n. Thus

$$\left[\frac{1}{g(s)} \right]^{(n-1)} = \frac{(-1)^{n-1} k (k+1) \ldots (k+n-2) s^{(k-1)(n-1)}}{g^n(s)} + h(s) \tag{2.10}$$

where

$$h(s) \sim \frac{a_o}{s^{n+k}} + \frac{a_1}{s^{n+k+1}} + \ldots + \frac{a_N}{s^{n+k+N}}$$

for some N. Since $\left[\frac{1}{g(s)} \right]^{(n)}$ also has an asymptotic inverse power series expansion, we have justified differentiation of $h(s)$ to give

$$h'(s) \sim \frac{a_o(-n-k)}{s^{n+k+1}} + \frac{a_1(-n-k-1)}{s^{n+k+2}} + \ldots + \frac{a_N(-n-k-N)}{s^{n+k+N+1}}$$

Thus we can differentiate (2.10) to give

$$\left[\frac{1}{g(s)} \right]^{(n)} = \frac{(-1)^{n-1} k(k+1)\ldots(k+n-2)(k-1)(n-1) s^{kn-n-k}}{g^n(s)} + \frac{(-1)^n k(k+1)\ldots(k+n-2) s^{(k-1)(n-1)} n g'(s)}{g^{n+1}(s)} + h'(s)$$

$$= \frac{(-1)^n k(k+1)\ldots(k+n-3)(k+n-2)[-(k-1)(n-1) s^{kn-n} + nk s^{kn-n}]}{g^{n+1}(s)} + O(s^{-n-k-1+\varepsilon})$$

$$= \frac{(-1)^n k(k+1)\ldots(k+n-2)(k+n-1) s^{kn-n}}{g^{n+1}(s)} + O(s^{-n-k-1+\varepsilon})$$

Thus the result holds for n + 1 and the induction hypothesis is

established. Immediately, we find

$$\left[\frac{1}{g(s)} \right]^{(n)} = \left(s^{-k} \right)^{(n)} + O\left(s^{-n-k-1+\varepsilon} \right)$$

and we have the result of the theorem, since

$$\left| \left[\frac{1}{g(s)} \right]^{(n)} \right| < C \left(s^{-k} \right)^{(n)} < C B^{n+1} n^n s^{-n-k}$$

$$< (CB)^{n+1} n^n s^{-n-k} \qquad (C > 1)$$

Theorems 1 to 6 give the most important results in this section. However, there are a number of other interesting results which hold under very general conditions. Theorem 8 requires some terminology from the theory of partitions.

Let π_n denote a partition of the integer n into integer parts. Define $m_\pi(i)$ as the number of "i"s in π i.e. the number of times the integer i occurs in the partition π. Define C_π^n as the number of ways of partitioning a set of n different objects so that the result is a partition of type π. Define $r_n(\pi)$ as the weight of π (i.e. the total number of integers (including multiplicities) occurring in the partition π). Thus

$$r_n(\pi) = \sum_{i=1}^{n} m_\pi(i)$$

Theorem 8

If w(s) is a solution of (2.1) then, for s sufficiently large,

$$\left| w^{(n)} \right| \leq \sum_{\substack{\text{all} \\ \text{partitions} \\ \pi \\ \text{of} \\ n}} |g(s)|^{m(1)} \ldots |g^{(r)}(s)|^{m(r+1)} C_\pi^n 2^{r_n(\pi)} K$$

for $n = 0, 1, \ldots\ldots$ and where $|w| \leq K$

Proof

To simplify the details, we write

$$r(s) = w(s-1) - w(s)$$

Using Leibnitz' formula

$$w^{(n)}(s) = g^{(n-1)}(s)\, r(s) + \ldots + \binom{n-1}{t} g^{(n-t-1)}(s)\, r^{(t)}(s) + \ldots + r^{(n-1)}(s)\, g(s) \qquad (2.11)$$

Let

$$B_n(s) = \sum_{\substack{\text{all partitions} \\ \pi \\ \text{of} \\ n}} |h(s)|^{m(1)} |h'(s)|^{m(2)} \ldots |h^{r}(s)|^{m(r+1)}\, {}^{n}C_{\pi} \qquad (2.12)$$

where $h = 2g$.

We proceed by induction:-

Hypothesis $\quad |w^{(n)}(s)| \leq B_n(s)\, K$

If $n = 1$

$$w'(s) = g(s)\,[w(s-1) - w(s)]$$

$$|w'(s)| \leq 2\,|g|\,K = K\,|h|$$

$B_1 = |h|$ and so the hypothesis holds for $n = 1$. Assume that it holds for n. From (2.11)

$$|w^{(n+1)}(s)| \leq |g^{(n)}|\,|r| + \binom{n}{1} |g^{(n-1)}|\,|r'| + \ldots + \binom{n}{t} |g^{(n-t)}|\,|r^{(t)}| + \ldots + |r^{(n)}|\,|g|$$

By hypothesis

$$|r^{(t)}| = |w^{(t)}(s-1) - w^{(t)}(s)|$$

$$\leq 2\, B_t(s)\, K$$

Thus

$$\left| \frac{w^{(n+1)}(s)}{K} \right| \leq 2 \left| g^{(n)} \right| + 2 \binom{n}{1} \left| g^{(n-1)} \right| |h| + \cdots$$

$$+ 2 \binom{n}{t} \left| g^{(n-t)} \right| \sum_{\substack{\text{partitions} \\ \pi_t \\ \text{of} \\ t}} |h|^{m(1)} |h'|^{m(2)} \cdots C_{\pi_t}^{t}$$

$$+ \cdots + |g| \sum_{\substack{\text{partitions} \\ \pi_n \\ \text{of} \\ n}} |h|^{m(1)} |h'|^{m(2)} \cdots C_{\pi_n}^{n} \qquad (2.13)$$

This gives

$$\left| \frac{w^{(n+1)}(s)}{K} \right| \leq \sum_{\substack{\text{partitions} \\ \pi_{n+1} \\ \text{of} \\ n+1}} |h|^{m(1)} |h'|^{m(2)} \cdots C_{\pi_{n+1}}^{n+1} \qquad (2.14)$$

since every term in (2.13) has the form

$$\left| h^{(n-t)} \right| \sum_{\substack{\text{partitions} \\ \pi_t \\ \text{of} \\ t}} |h(s)|^{m(1)} |h'(s)|^{m(2)} \cdots |h^{r}(s)|^{m(r+1)} C_{\pi_t}^{t}$$

$$= \sum_{\substack{\text{partitions} \\ \pi_t \\ \text{of} \\ t}} |h(s)|^{m(1)} |h'(s)|^{m(2)} \cdots |h^{(n-t)}|^{m(n-t+1)+1} \cdots C_{\pi_t}^{t}$$

and so corresponds to taking a partition of t and adding the integer $n - t + 1$ to it, thus making a partition of $n + 1$. It remains to consider the coefficient of any given term. Let us fix one partition of $n + 1$, say π. We obtain terms corresponding to π from that term in (2.12) with $\left| g^{(n-t)} \right|$ outside the summation if and only if

$n - t + 1$ occurs in π. The coefficient of the contribution from this term is clearly

$$\binom{n}{t} C^{t}_{\pi(t)}$$

where $\pi(t)$ is the partition of t obtained by removing one integer $n - t + 1$ from π. The total coefficient of the term corresponding to π is thus

$$\sum_{t} \binom{n}{t} C^{t}_{\pi(t)} \qquad (2.15)$$

It remains to show that the expression in (2.15) is

$$C^{n+1}_{\pi}$$

We consider the partition π from a different viewpoint. π is a partition of $n + 1$ objects. Select one object; call it A. Let t be an integer occurring in π. We pose the question - in how many ways can A occur in a group of size $n - t + 1$ whilst still maintaining the partition π ? If there is one possible way in which this can happen, then we can choose the other $n - t$ objects in the group in $\binom{n}{n-t}$ ways. There are t objects left from the $n + 1$ original objects and these can be partitioned in $C^{t}_{\pi(t)}$ ways (by definition). Thus the number of ways A can occur in a group of size $n - t + 1$ in π is

$$\binom{n}{n-t} C^{t}_{\pi(t)} = \sum_{t} \binom{n}{t} C^{t}_{\pi(t)}$$

However this number of ways is also C^{n+1}_{π} by definition. Hence

$$C^{n+1}_{\pi} = \sum_{t} \binom{n}{t} C^{t}_{\pi(t)} \qquad (2.16)$$

Using (2.16) in (2.13), we obtain (2.14) and so we have established

the induction hypothesis and thus Theorem 8.

Ideally we would like to obtain an inverse power series expansion for a solution of (2.1). This does not seem to be possible in general.

§1.2 $g(s) \sim As$

For the equation (2.1) with $g \geq 0 \ \forall \ s$ and $g(s) \sim As$ as $s \to \infty$, it seems more difficult to prove results. An equation which can be made to fall into this category is

$$F'(x) = e^{\alpha x+\beta} F(x-1) \tag{2.17}$$

which can be transformed to

$$y'(z) = a\, y(\lambda z) \tag{2.18}$$

(2.17) was discussed in 1953 in a paper by de Bruijn [5]. (2.18) is obtained from (2.17) by writing

$$a = e^{\beta}/\alpha \qquad e^{\alpha x} = z$$

$$F(x) = y(e^{\alpha x}) \qquad \lambda = e^{-\alpha}$$

(2.18) is one of the equations discussed by Kato and McLeod in [11].

To obtain an equation of the form (2.1) from (2.18) is more complicated. We let

$$t = \log z \qquad c = \log \lambda$$

$$y(z) = \phi(t)\, \psi(t) \tag{2.19}$$

where

$$\phi(t) = e^{kt}\, t^{h} \exp\left[-\frac{1}{2c}(t - \log t)^2\right]$$

with

$$k = \frac{1}{2} - \frac{1}{c} - \frac{1}{c}\log(-ac)$$

and

$$h = -1 + \frac{1}{c}\log(-ac)$$

Here 'log' denotes the principal branch of the logarithm.

Calculation shows that

$$\frac{\phi'(t)}{\phi(t)} = \frac{1}{c}\left(- t + \log t + 1 + ck - \frac{\log t}{t} + \frac{hc}{t} \right)$$

$$a\, e^{t}\, \frac{\phi(t+c)}{\phi(t)} = -\frac{t}{e\,c}\left(1 + \frac{c}{t}\right)^{\left[\frac{t}{c} + \frac{1}{c}\log\left(\frac{-ac}{t^{\frac{1}{2}}(t+c)^{\frac{1}{2}}}\right)\right]}$$

$$= -\frac{1}{c}\left\{ t - \log t + (h + \tfrac{1}{2})c + O\left(\frac{[\log t]^2}{t}\right) \right\}$$

$$\text{as } t \to \infty$$

Thus, for large t,

$$\psi'(t) \sim \left[- \frac{t}{c} + (\log t)/c - h - \tfrac{1}{2} + o(1) \right]\left[\psi(t+c) - \psi(t)\right]$$

Now let

$$t = -cs \qquad \text{and} \qquad \psi(t) = w\left(-\frac{t}{c}\right).$$

Then, assuming that $c < 0$ (i.e. $0 \leq \lambda \leq 1$)

$$w'(s) \sim \left[- cs - \log(-cs) - \tfrac{1}{2}c + \log(-ac) + o(1) \right] \times \left[w(s-1) - w(s) \right]$$

i.e. (2.1) with $g(s)$ the particular function found from the transformation (2.19). It is such that

$$g(s) \sim -cs - \log(-cs)$$

Call this function $G(s)$.

De Bruijn derives many results about (2.17) which can be translated into results about (2.1) with $g(s) = G(s)$. We quote one example:-

Theorem 9

If $w(s)$ is a solution of

$$w'(s) = G(s)\,[w(s-1) - w(s)]$$

then

i) $w(s) = O(1)$

ii) no solution $w(s)$, other than the identically zero solution, is $o(1)$ as $s \to \infty$.

iii) there exists a solution with the form

$$w(s) \sim u(s - \log s) + o(1)$$

where

$$u(t) = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left(-\frac{2n\pi i t}{c}\right)$$

with

$$\gamma_n = 0\left\{\exp\left(\frac{\pi^2 |n|}{c} + \frac{(\log|n|)^2}{2c} + C\log|n|\right\}\right.$$

iv) all solution have the asymptotic behaviour described in iii) for some such u.

My attempts to prove a generalisation of Theorem 9 for any function $g(s) \sim As$ have met with limited success. The conditions of Theorem 1 are still fulfilled for general g and so $w(s) = O(1)$. Conditions I hold for m if

$$\frac{g^{(r)}(s)}{g(s)} = O\left(\frac{1}{s^r}\right) \qquad \frac{1}{g(s)} = O\left(\frac{1}{s^\beta}\right) \quad \beta > 0$$

$r = 1, 2, \ldots, m$

$1 > (m-1)(1-\beta)$

These are satisfied by the basic case

$$g(s) = As + B$$

and so Theorems 2 and 3 are still applicable. Theorem 4 however does not apply. A similar but weaker result may be obtained. It was first published in 1949 by de Bruijn [3].

Theorem 10

Let $g(s)$ be such that

$\frac{1}{g(s)}$ is continuous for $s \geq 1$

$\frac{1}{g(s)}$ has continuous first derivative for $s \geq 2$

$\frac{1}{g(s)}$ has continuous 2nd derivative for $s \geq 3$

.. ..

$\frac{1}{g(s)}$ has continuous nth derivative for $s \geq n + 1$

and $\exists$ B_n such that

$$\left| \left[\frac{1}{g(s)} \right]^{(k)} \right| < B_n s^{-2} \qquad s \geq k + 1 \quad k = 1, 2, \ldots, n$$

$$\left[g(s) \right]^{-2} < B_n s^{-2} \qquad s \geq 1$$

Let $w(s)$ be a solution of (2.1), then there exists a function $\psi(s)$, which is continuous and has period 1, with n derivatives, such that

$$w^{(k)}(s) - \psi^{(k)}\left(s - \int_{B_n+1}^{s} \frac{1}{g(t)}\, dt\right) \to 0 \quad \text{as} \quad s \to \infty$$

for $k = 0, 1, 2, \ldots, n$

Proof

Translation and slight adaptation of Theorem 4 of [3].

Note

The conditions on $g(s)$ are trivially satisfied for $n = \infty$ when

$$g(s) = As + B$$

No uniqueness theorem would appear to hold in this case, but Theorem 8 still applies.

§1.3 $g(s) = \exp[h(s)]$, $h(s)$ asymptotically algebraic in s.

If we consider the behaviour of solutions of (2.1) with $g(s) = \exp[h(s)]$, $h(s) \sim As^k$ with $h(s)$ real and continuous, results can be obtained similar to those in § 1.1; however in some cases separate proofs are required. Obviously, Theorem 1 still holds and we have $w(s)$ bounded. Equally immediately, Conditions I do not hold for any m, nor does there seem to be any obvious analogue. However, we can still establish, under certain conditions, that $w^{(m)}(s)$ is bounded.

$$w'(s) = \exp[h(s)][w(s-1) - w(s)] \qquad (2.20)$$

Conditions III $\exists$ k, m such that

$h(s)$ is m-times continuously differentiable

$h(s) \sim As^k \quad A > 0, k > 0$

$h^{(r)}(s) = O(s^{\alpha}) \qquad r = 1, 2, \ldots, m$

We can assume that $\alpha \geq 0$. We notice that, if Conditions III hold with $m = N$, then they hold with $m = N - 1$.

Theorem 11

If Conditions III hold with $m = 2$ and $w(s)$ is a solution of (2.20) then $w'(s)$ is bounded.

Proof

Let I_n be $[s_o + n, s_o + n + 1]$ and let $| w'(s) |$ have least upper

bound J_n in I_n. We have

$$w''(s) = e^{h(s)} w'(s-1) - [e^{h(s)} - h'(s)] w'(s) \qquad (2.21)$$

Thus, proceeding exactly as in Theorem 2, we have (equivalent to (2.4) of Theorem 2)

$$\begin{aligned} |w'(s)| \le J_n \Big\{ 1 &+ h'(s) e^{-h(s)} \\ &- \exp\Big[-\int_{s_0+n+1}^{s} e^{h} \Big] h'(s_0+n+1) e^{h(s) - 2h(s_0+n+1)} \\ &+ e^{h(s)} \int_{s_0+n+1}^{s} \exp\Big[-\int_{t}^{s} e^{h} \Big] \Big[\frac{2[h'(t)]^2}{e^{2h(t)}} - \frac{h''(t)}{e^{2h(t)}} \Big] dt \Big\} \end{aligned} \qquad (2.22)$$

Since $h(s) \sim As^k$, the third term on the right hand side is

$$O\left(e^{-A(s_0+n+1)^{k-\varepsilon}} \right) \quad \text{for all } \varepsilon > 0.$$

We also have

$$h'(s) = O(s^{\alpha}) \qquad h''(s) = O(s^{\alpha})$$

As before we estimate the integral term:-

$$\begin{aligned} & e^{h(s)} \int_{s_0+n+1}^{s} \exp\Big[-\int_{t}^{s} e^{h} \Big] \Big[2\frac{[h'(t)]^2}{e^{2h(t)}} - \frac{h''(t)}{e^{2h(t)}} \Big] dt \\ &= O\Big(e^{h(s)} \int_{s_0+n+1}^{s} \exp\Big[-\int_{t}^{s} e^{h} \Big] t^{2\alpha} e^{-2h(t)} dt \Big) \\ &= O\left(e^{h(s)} s^{2\alpha} e^{-2h(s_0+n+1)} \right) \\ &= O\left([s_0+n+2]^{2\alpha} e^{-A(s_0+n+1)^{k-\varepsilon}} \right) \end{aligned}$$

for any $\varepsilon > 0$. Returning to (2.22), we find

$$\begin{aligned} |w'(s)| &\le J_n \Big\{ 1 + O\left(s^{\alpha} e^{-As^{k-\varepsilon}} + (s_0+n+2)^{2\alpha} e^{-A(s_0+n+1)^{k-\varepsilon}} \right) \Big\} \\ &= J_n \Big\{ 1 + O\left([s_0+n+2]^{2\alpha} e^{-A(s_0+n+1)^{k-\varepsilon}} \right) \Big\} \end{aligned}$$

$$\prod_{n=0}^{\infty} (1 + n^{2\alpha} e^{-An^{k-\varepsilon}}) < \infty \qquad \text{for } k - \varepsilon > 0$$

and so, choosing s_o such that

$$\prod_{n=0}^{\infty} [1 + 0([s_o + n + 2]^{2\alpha} e^{-A(s_o+n+1)^{k-\varepsilon}})] \leq 2$$

and choosing ε sufficiently small we have

$$J_n \leq 2 J_o \qquad \forall\, n$$

and the result holds.

The proof of the result that $w^{(m)}(s)$ is bounded is more complicated in detail, but is similar to Theorem 3 in principle.

Theorem 12

If Conditions III hold for $m = M + 1$ and $w(s)$ is a solution of (2.20) then $w^{(M)}(s)$ is bounded.

Proof

From Theorems 1 and 11, $w(s)$ and $w'(s)$ are bounded ($M \geq 1$). The proof is by induction.

Hypothesis $w^{(r)}(s)$ is bounded $r = 1, 2, \ldots, M - 1$ and

$$w^{(M)}(s) = e^{h(s)} [w^{(M-1)}(s - 1) - w^{(M-1)}(s)] + \\ + (M - 1) h'(s) w^{(M-1)}(s) + 0(s^{(M-1)\alpha}) \qquad (2.23)$$

To start the induction, consider $M = 2$; $w'(s)$ is bounded. From (2.21), (2.23) holds for $M = 2$. Assume that the hypothesis holds for M. Differentiating M times,

$$w^{(M+1)}(s) = e^{h(s)} \{w^{(M)}(s - 1) - w^{(M)}(s)\} + \\ + \sum_{r=1}^{M} \binom{M}{r} (e^{h(s)})^{(r)} [w^{(M-r)}(s - 1) - w^{(M-r)}(s)]$$

(2.23) holds for all $m \leq M$ and thus

$$w^{(M+1)}(s) = e^{h(s)}\{w^{(M)}(s-1) - w^{(M)}(s)\} +$$

$$+ \sum_{r=1}^{M} \binom{M}{r} [e^{h(s)}]^{(r)} e^{-h(s)} [\, w^{(M-r+1)}(s) - (M-r)\, h'(s)\, w^{(M-r)}(s$$

$$+ O(s^{(M-r)\alpha})]$$

Consider $[e^{h(s)}]^{(r)} e^{-h(s)} = H_r(s)$, say; then, since $h^{(r)}(s) = O(s^{\alpha})$ for $r = 1, 2, \ldots, M$, $H_r(s)$ is dominated by the term $(h'(s))^r$ and so

$$H_r(s) = O(s^{\alpha r})$$

$$w^{(M+1)}(s) = e^{h(s)}\{w^{(M)}(s-1) - w^{(M)}(s)\} +$$

$$+ M\, h'(s)\, w^{(M)}(s) + O(s^{M\alpha})$$

since $w^{(M-r+1)}(s)$ is bounded for $r = 2, 3, \ldots, M+1$.

To establish the induction hypothesis, it remains to show that $w^{(M)}(s)$ is bounded.

$$w^{(M+1)}(s) + [e^{h(s)} - M\, h'(s)]\, w^{(M)}(s)$$

$$= e^{h(s)}\, w^{(M)}(s-1) + O(s^{M\alpha})$$

$$\frac{d}{ds}\Big[\, w^{(M)}(s) \exp\Big\{\int^{s} e^{h(z)}\, dz - Mh(s)\Big\}\Big]$$

$$= \exp\Big\{\int^{s} e^{h(z)}\, dz - (M-1)\, h(s)\Big\}\, w^{(M)}(s-1)$$

$$+ O(s^{M\alpha} e^{-Mh(s) + \int^{s} e^{h}})$$

Let I_n be as before and assume $|\, w^{(M)}\, |$ has least upper bound J_n in I_n. For $s \in I_{n+1}$

$$| w^{(M)}(s)\, e^{-Mh(s)} + \int^{s} e^{h} | \leq J_n \Big\{ e^{-Mh(s_0+n+1)} + \int^{s_0+n+1} e^{h} + \int_{s_0+n+1}^{s} e^{(1-M)h(z) + \int^{z} e^{h}}\, dz + O\Big(\int_{s_0+n+1}^{s} z^{M\alpha}\, e^{-Mh} + \int^{z} e^{h}\, dz \Big) \Big\} \tag{2.24}$$

Consider separately the integral terms on the right-hand side of (2.24)

$$I_1 = \int_{s_0+n+1}^{s} \exp\Big\{(1-M)h(z) + \int^{z} e^{h}\Big\} dz$$

$$= \frac{e^{(1-M)h(z) + \int^{z} e^{h}}}{(1-M)h'(z) + e^{h(z)}}\Bigg|_{s_0+n+1}^{s}$$

$$+ \int_{s_0+n+1}^{s} \exp\Big\{(1-M)h(z) + \int^{z} e^{h}\Big\} \frac{[(1-M)h''(z) + h'(z)e^{h(z)}]}{[(1-M)h'(z) + e^{h(z)}]^2}\, dz$$

$$= e^{-Mh(z)} + \int^{z} e^{h} [1 + (1-M)h'\, e^{-h}]^{-1} \Big|_{s_0+n+1}^{s}$$

$$+ e^{(1-M)h(z) + \int^{z} e^{h}} \frac{[(1-M)h''(z) + h'(z)\, e^{h(z)}]}{[(1-M)h'(z) + e^{h(z)}]^3} \Bigg|_{s_0+n+1}^{s}$$

$$+ \int_{s_0+n+1}^{s} \exp\Big\{(1-M)h(z) + \int^{z} e^{h}\Big\} \frac{d}{dz}\Big[\frac{(1-M)h'' + h'e^{h}}{[(1-M)h' + e^{h}]^3} \Big]\, dz$$

Recalling Conditions III, we have

$$I_1 = e^{-Mh} + \int^{z} e^{h} [1 + O(h'\, e^{-h})] \Big|_{s_0+n+1}^{s}$$

$$+ e^{(1-m)h + \int^{z} e^{h} - 2h(z)}\, h'(z)\, (1 + O(h'e^{-h})) \Big|_{s_0+n+1}^{s}$$

$$+ \int_{s_0+n+1}^{s} \frac{e^{(1-M)h + \int e^{h}}}{[(1-m)h' + e^{h}]^4} [e^{2h}(h'' - 2\{h'\}^2) + O(e^{h}\{h'\}^3)]\, dz$$

The remaining integral term in I_1 can be estimated by

$$\int_{S_{0+n+1}}^{S} e^{(1-M)h + \int e^{h}} e^{-4h} \left\{ e^{2h}[h'' - 2(h')^2] + O(e^{h}[h']^3) \right\} [1 + O(h'e^{-h})]dz$$

$$= \int_{S_{0+n+1}}^{S} e^{-(M+1)h + \int e^{h}} [h'' - 2(h')^2 + O(e^{-h}(h')^3)]dz$$

$$= O\left(e^{-(M+1)h(s) + \int_{s}^{S} e^{h}} s^{2a} \right)$$

Now consider the other integral term in (2.24)

$$I_2 = \int_{S_{0+n+1}}^{S} z^{Ma} e^{-Mh(z) + \int^{z} e^{h}} dz$$

$$= \frac{e^{-Mh(z) + \int^{z} e^{h}} z^{Ma}}{[-Mh'(z) + e^{h(z)}]} \Bigg|_{S_{0+n+1}}^{S}$$

$$- \int_{S_{0+n+1}}^{S} e^{-Mh(z) + \int^{z} e^{h}} \frac{[(-Mh' + e^{h}) z^{Ma-1} Ma]}{[-Mh'(z) + e^{h(z)}]^2} dz$$

$$- \int_{S_{0+n+1}}^{S} e^{-Mh(z) + \int^{z} e^{h}} \frac{[z^{Ma}(-Mh'' + h'e^{h})]}{[-Mh'(z) + e^{h(z)}]^2} dz$$

$$= e^{-Mh + \int^{z} e^{h}} z^{Ma} e^{-h} (1 + O(h'e^{-h})) \Big|_{S_{0+n+1}}^{S}$$

$$+ \int_{S_{0+n+1}}^{S} z^{Ma-1} e^{-(M+2)h + \int e^{h}} (1 + O(h'e^{-h}))(zh'e^{h} + O(e^{h})) dz$$

$$= O\left(e^{-(M+1)h + \int e^{h}} s^{Ma} \right) + O\left(\int_{S_{0+n+1}}^{S} z^{Ma} e^{-(M+1)h + \int e^{h}} h'dz \right)$$

$$= O\left(e^{-(M+1)h + \int e^{h}} s^{Ma} + e^{-(M+1)h + \int e^{h}} s^{Ma} + a \right)$$

Here we have repeatedly used Conditions III to neglect terms.

Collecting together in (2.24) we find

$$| w^{(M)}(s)\, e^{-Mh(s)} + \int^{s} e^{h} | \leq J_n \Big\{ e^{-Mh} + \int^{s} e^{h} + O\Big(h' e^{-h}\, e^{-Mh} + \int^{z} e^{h} \Big|_{S_o+n+1}^{s} + e^{-(M+1)h(s)} + \int^{s} e^{h}\, h'(s) + e^{-(M+1)h(s)} + \int^{s} e^{h}\, s^{2\alpha} + e^{-(M+1)h} + \int^{s} e^{h}\, s^{M\alpha+\alpha}\Big)\Big\}$$

$$\Rightarrow \quad | w^{(M)}(s) | \leq J_n \Big\{ 1 + O\Big(h'(s)e^{-h(s)} - h'(s_o+n+1)\, e^{-(M+1)h(s_o+n+1)} + Mh(s) - \int_{S_o+n+1}^{s} e^{h} + e^{-h(s)}\, s^{2\alpha} + e^{-h(s)}\, s^{M\alpha+\alpha}\Big)\Big\}$$

$$= J_n \Big\{ 1 + O\Big([s_o + n + 2]^{M\alpha+\alpha}\, e^{-A(s_o+n+1)^{k-\varepsilon}}\Big)\Big\}$$

for any $\varepsilon > 0$.

Now $\prod_{n=0}^{\infty}(1 + n^{\alpha}\, e^{-An^{k-\varepsilon}})$ converges for $A > 0$ and $k - \varepsilon > 0$ and thus we have, as before

$$J_n \leq 2J_o \qquad \forall\, n$$

and the induction is established.

Having established the boundedness of $w^{(M)}(s)$ under Conditions III, we can now state the theorems equivalent to 4, 5, 6, 7 and 8 under Conditions III. In most cases the proofs are identical since the only essential condition for these theorems is that $w^{(M)}(s)$ be bounded.

Theorem 13

If Conditions III hold for $m = M + 2$ then $\lim_{n\to\infty} w^{(M)}(s + n)$ exists ($= f_M(s)$ say) and is a continuous function which is periodic, of

period 1, and satisfies $f_M(s) = f_o^{(M)}(s)$.

Proof

If M = 0, from (2.20) we have

$$w(s-1) - w(s) = O(e^{-h(s)})$$

If M > 0, from the induction hypothesis of Theorem 12, established under these conditions:-

$$e^{h(s)}\,[w^{(M)}(s-1) - w^{(M)}(s)] = O(s^{\alpha}) + O(s^{M\alpha}) + O(1)$$

$$\Rightarrow \quad w^{(M)}(s-1) - w^{(M)}(s) = O(s^{M\alpha}\, e^{-h(s)}) \qquad (2.25)$$

Thus (2.25) also holds for M = 0.

We can now proceed exactly as in Theorem 4, noting that

$$\int^{\infty} \frac{s^{M\alpha}}{e^{h(s)}}\,ds < \infty$$

for all finite values of M.

Theorems 5 and 6 are still relevant under the translation of Conditions II,

Conditions II (translation)

i) $(e^{-h(s)})^{(n)}$ is continuous for n = 0, 1, 2,.... $s \geq 1$

ii) $\exists$ positive constants B and ρ, with $\rho > 1$ such that

$$|\,(e^{-h(s)})^{(r)}\,| < B^{n+1}\, n^n\, s^{-n-\rho} \quad \text{for } n = 0, 1, 2,...$$

Thus again, under Conditions II, we have a one-to-one correspondence between solutions w(s) of (2.20) and $f_o(s)$

$$w(s) \rightleftarrows f_o(s)$$

We can easily prove that Conditions II hold when h(s) is a polynomial.

Lemma 14

Conditions II hold whenever h(s) is a polynomial of positive degree.

Proof

Trivially $(e^{-h(s)})^{(n)}$ is continuous when h(s) is a polynomial. We need to show that $\exists$ B, ρ such that

$$| [e^{-h}]^{(n)} s^n + \rho | < B^{n+1} n^n$$

with $\rho > 1$. By an argument similar to that used in Lemma 7, we see that

$$| [e^{-h(s)}]^{(n)} | < C [h'(s)]^n e^{-h(s)}$$

when h(s) is a polynomial of degree n. Thus,

$$\begin{aligned} | [e^{-h(s)}]^{(n)} s^{n+\rho} | &< C s^{kn+\rho} e^{-h(s)} \\ &< D e^{-h(s)} + s^{\varepsilon} \qquad \text{where } k > \varepsilon > 0 \\ &< E \\ &< B^{n+1} n^n \end{aligned}$$

for any n by choosing B suitably.

Theorem 8 is immediately applicable.

§ 2 g(s) bounded as $s \to \infty$.

In this section we consider the behaviour of solutions of (2.1), with $g(s) \geq 0$ $\forall$ s and bounded, such that $\lim_{s\to\infty} g(s)$ exists. There are two cases:- g(s) asymptotically constant and $g(s) \to 0$. Theorem 1 is still applicable and so we have that all solutions are bounded. Theorem 8 is also applicable.

We assume that the following conditions hold

Conditions IV $\exists$ m such that

g(s) is m-times continuously differentiable.

$$g^{(r)}(s) = O(1) \qquad r = 0, 1, \ldots, m \qquad s \to \infty$$

Trivially these conditions cover g(s) = A.

Theorem 15

If Conditions IV hold for m = M, then $w^{(M+1)}(s)$ is bounded.

Proof

Under these conditions w(s) is bounded, say by K, and

$$w'(s) = g(s)\,[w(s-1) - w(s)]$$

Thus, for large s,

$$|\, w'(s) \,| \leq 2A_0 \,|\, w \,| \leq 2\,A\,K$$

where $|g(s)| \leq A_0$

We proceed by induction.

Hypothesis $w^{(M)}(s)$ is bounded

The hypothesis holds for m = 1; assume that it holds for m $\leq$ M

By Leibnitz

$$w^{(M+1)}(s) = \sum_{r=0}^{M} \binom{M}{r} g^{(r)}(s)\,[w^{(M-r)}(s-1) - w^{(M-r)}(s)]$$

Thus

$$|\, w^{(M+1)}(s) \,| \leq \sum_{r=0}^{M} \binom{M}{r} A_r\, 2\, W_{M-r} = W_{r+1} \text{ say}$$

where $|\, g^{(r)}(s) \,| \leq A_r$ and $|\, w^{(M-r)}(s) \,| \leq W_{M-r}$

$$r = 0, 1, \ldots, M$$

Hence the induction is established and $w^{(M+1)}(s)$ is bounded.

Thus, under Conditions IV, $w^{(M)}(s)$ is bounded.

We can prove a result about $\lim_{n\to\infty} w(s+n)$. It appears in work of de Bruijn [4].

Theorem 16

If $w(s)$ is a solution of (2.1) and if

$$\sum_{n=1}^{\infty} \exp\left[- \int_{n-1}^{n} g(t)\, dt \right] = \infty \qquad (2.26)$$

then $\lim_{s\to\infty} w(s)$ exists (= A say) and

$$| w(s) - A | \leq C \prod_{\nu=1}^{n} \left\{ 1 - \exp\left[- \int_{\nu}^{\nu+1} g(t)\, dt \right] \right\} \qquad (2.27)$$

$$n \leq s \leq n+1$$

C is a constant.

Proof

De Bruijn [4] §2 and §3. The proof involves solving (2.1) as a linear differential equation with $w(s-1)$ assumed known.

Remarks

i) The condition (2.26) holds if

$$\frac{1}{g(s)} \geq \frac{1}{\log s} \qquad \text{for large } s$$

which clearly includes $g(s) \sim a_o s^{\alpha}$ $\alpha \leq 0$ and $a_o > 0$.

ii) (2.27) can be used to find the asymptotic form in specific cases. For example, if $g(t) = t^{-1}$ (2.27) implies

$$| w(s) - A | \leq \frac{C}{(n+1)!} \qquad n \leq s \leq n+1$$

Thus $w(s) = A + O(\left[\Gamma(s+1)\right]^{-1})$

§2.1 g(s) asymptotically constant.

The case covered by this section is simple in conception and there is considerable literature on the subject of linear differential

difference equations with asymptotically constant coefficients. The results most relevant to this thesis are those of Bellman and Cooke, Wright, Yates and Hale [1, 9, 14, 15, 16]. Yates deals with the special case

$$g(s) = \frac{as + b}{cs + d} \qquad c \neq 0 \qquad \frac{d}{c} > 0$$

and improves slightly for this special case on the results which follow.

It has been known for many years (see for example Wright [15] 1949) that the asymptotics of solutions for $g(s) \sim A$ depend crucially on the zeroes of the characteristic function, $\hat{h}(s)$, of (2.1)

$$\hat{h}(s) = s + A - A\, e^{-s} \tag{2.28}$$

It is well known (see for instance Wilder [13]) that the zeroes of $\hat{h}(s)$ are asymptotically spaced along $A = | s\, e^{s} |$ and are such that

$$| s_m | \sim \pm (2m + \tfrac{1}{2})\, \pi$$

where s_m is a zero of $\hat{h}(s)$ of large modulus. Hence the roots of large modulus lie on a curve which is symmetric about the real axis and is similar to an exponential curve for large $| s |$. As $| s | \to \infty$, with s on the curve, the curve becomes more and more nearly parallel to the imaginary axis and $\mathrm{Re}\,(s) \to -\infty$. The roots are either real or occur in conjugate pairs, and on any vertical line there lie at most two roots. It is clear that we can arrange the zeroes of $\hat{h}(s)$ in a sequence $\{ s_m \}$ such that

$$| \mathrm{Im}\,(s_m) | \leq | \mathrm{Im}\,(s_{m+1}) | \tag{2.29}$$

Trivially, all zeroes of $\hat{h}(s)$ are simple. In what follows, we are interested in the zero of $\hat{h}(s)$ with the largest real part.

Lemma 17

$s = 0$ is the zero of $\hat{h}(s) = s + A - A\, e^{-s}$ with largest real part.

Proof

$s = 0$ is a solution of $\hat{h}(s) = 0$. Let $s = \sigma + it$ be a zero of $\hat{h}(s)$ such that $\sigma \geq 0$ for σ, t real.

$$A\, e^{-s} = s + A$$

Taking real and imaginary parts

$$\begin{aligned} A\, e^{-\sigma} \cos t &= \sigma + A \\ A\, e^{-\sigma} \sin t &= -t \end{aligned} \qquad (2.30)$$

From the first equation, using $A, \sigma \geq 0$

$$e^{-\sigma} \cos t \geq 1$$

But $\cos t \leq 1$, $e^{-\sigma} \leq 1$ and so we must have equality throughout.

$$e^{-\sigma} = 1 \qquad \sigma = 0$$

$$\cos t = 1 \qquad t = n\pi \qquad n = 0, 1, 2, \ldots$$

The second equation of (2.30) $\Rightarrow$ $t = 0$

Hence result.

We expect considerably stronger results than those obtained in the general case when we restrict $g(s)$ to $g(s) = A$, constant, and this indeed happens. We discuss this case first.

Wright [15] and Bellman and Cooke [1] are relevant here, but the work of Bellman and Cooke is covered by that of Wright.

Theorem 18

If $w(s)$ is a solution of

$$w'(s) = A[\, w(s - 1) - w(s)\,] \qquad A > 0$$

then

$$w(s) = \sum_{m=1}^{\infty} P_0(m)\, e^{s_m s} \qquad (2.31)$$

The series is uniformly convergent in any finite interval $1 + \delta < s < C$. Here $P_o(m)$ is the residue of

$$\frac{w(1) + e^s(A + s) \int_o^1 w(u)\, s^{-su}\, du}{-A + (A + s)\, e^s}$$

at the zero s_m of $\hat{h}(s)$.

Proof

Adaptation of Wright [15], Theorem 1.

If we now return to the original general assumption $g(s) \sim A$, two weaker results can be obtained. The first deals with the behaviour of all solutions. We define η as the set of real parts of the zeroes of $\hat{h}(s)$ together with its limit points and $-\infty$. Thus η is bounded above. We next define a measure of the behaviour of a function at infinity as follows. If f is square integrable over the interval (x_o, x) for all finite $x > x_o$, then define $\omega(f)$ by the condition that $f\, e^{-\sigma x}$ is in $L^2(x_o, +\infty)$ $\forall \sigma > \omega(f)$ but not for any $\sigma < \omega(f)$. If $fe^{-\sigma x}$ is in $L^2(x_o, +\infty)$ $\forall \sigma$, we take $\omega(f) = -\infty$; if $fe^{-\sigma x}$ is not $L^2(x_o, +\infty)$ for any σ, we take $\omega(f) = +\infty$

Theorem 19

If w(s) is a solution of

$$w'(s) = [A + G(s)][w(s - 1) - w(s)]$$

where $A + G(s) \geq 0$ $\forall s$ and $G(s) \to 0$ as $s \to \infty$, then either

$$\max(\omega(w), \omega(w')) \in \eta$$

or w is the zero solution.

Proof

Wright [14], Theorem 2, suitably translated.

Although this is a very elegant theorem, it is difficult to obtain concrete information from it. The worst possible value for $\max [\omega(w), \omega(w')]$. is 0. This means that $w^{(\beta)} e^{-\sigma s}$, $\sigma > 0$, is square integrable to ∞ but $w^{(\beta)} e^{\sigma s}$ is not for any $\sigma > 0$, where $\beta = 0$ or $\beta = 1$. Certainly w and w' are $o(e^{\sigma s}) \quad \forall \sigma > 0$ i.e.

$$w' = o(e^{\sigma s}) \quad \forall \sigma > 0$$

Under the condition that

$$G(s) = 0(1)$$

which is not very restrictive, we already have that $w = 0(1)$, $w' = 0(1)$ and so Theorem 19 is not very informative.

Work of Bellman and Cooke [1] refined by Hale [9] gives information about specific solutions. If s_m is a root of $\hat{h}(s)$ we define

$$u_m(s) = s_m s + \frac{s_m}{A(1 + s_m + A)} \int^s G(t)\, dt$$

Theorem 20

If either (i) or (ii) holds where

(i) $\int^{\infty} | G(s) | ds < \infty$

(ii) $G(s) \to 0$ monotonically as $s \to \infty$

$$G'(s) = o(G(s))$$

$$\int^{\infty} G^2(s)\, ds < \infty$$

$$\int^{\infty} | G'(s) | ds < \infty$$

$$\int^{\infty} \left| \frac{G''(s)}{G(s)} \right| ds < \infty$$

$$\lim_{s \to \infty} \frac{G(s - l)}{G(s)} = 1 \qquad \text{for } 0 \leq l \leq 1$$

then

$$w'(s) = [A + G(s)][w(s - 1) - w(s)] \tag{2.32}$$

has a solution of the form

$$w(s) = e^{u_m(s)} [1 + o(1)] \quad \text{as } s \to \infty$$

for each m.

If G(s) has an asymptotic power series expansion of the form

$$G(s) \sim \sum_{n=1}^{\infty} g_n s^{-n}$$

and G'(s) and G"(s) exist and have asymptotic power series, then $\exists$ a solution of (2.32) for each m, with asymptotic power series expansion of the form

$$w(s) \sim e^{s_m s} s^{r_m} \sum_{n=0}^{\infty} w_n s^{-n}$$

where w_n is constant and $w_0 \neq 0$ and

$$r_m = \frac{g_1 s_m}{A(1 + A + s_m)}$$

Proof

Bellman and Cooke [1], Theorems 4 and 5.

The condition marked * in (ii) is shown to be unnecessary by Hale [9].

§ 2.2 $g(s) \to 0$ as $s \to \infty$.

This section also deals with a case for which results have been known for many years. Theorem 16 is relevant here. Conditions IV are still a plausible assumption and so we still have $w^{(m)}(s)$ bounded under these conditions.

Most of the results of § 2.1 collapse when we assume that $g(s) \to 0$. However, the work of Wright [14] described in Theorem 19

still has meaning. The set η reduces to $\{0, -\infty\}$ and the remarks following Theorem 19 still apply. Theorem 16 still applies.

§ 3 g(s) periodic.

This section deals with the case in which $\lim_{s\to\infty} g(s)$ does not exist but g(s) is periodic. There are two distinct sub-cases:-

i) the period of g(s), ω, is k^{-1} where k is an integer.

ii) $\omega \neq 1/k$ for any $k \in \mathbb{Z}$.

In the first case, the periodic behaviour of g(s) reinforces the unit one time lag in the equation and we can prove some interesting results; in the second, the periodicity seems to be destroyed by the lag and I can find no results which use the periodicity of g(s).

§3.1 $\omega = 1/k \qquad k \in \mathbb{Z}^+$

In this section, we can relax slightly the assumption that g(s) is positive. It is necessary only that

$$B = \int_0^\omega g(t)\, dt \geq 0$$

The results depend on the work of Hahn [8]. We state the basic theorem in a form suitably translated from Hahn's work.

Theorem 21

Let g(s) be a periodic continuous function of period k^{-1}, where k is a positive integer. We define

$$w_i(s) = \exp[(\mu_i - 1) \int_0^s g(t)\, dt] \qquad (2.33)$$

where μ_i is a root of

$$y\, e^{Bky} = e^{Bk} \tag{2.34}$$

and

$$B = \int_o^\omega g(t)\, dt$$

If $B > 0$, then all solutions of

$$w'(s) = g(s)[w(s-1) - w(s)]$$

can be written as absolutely convergent series of the form

$$\sum_i c_i\, w_i(s) \tag{2.35}$$

where c_i are constants.

If $B = 0$, then the only solutions are constants.

Proof

See Hahn [8]. The proof runs along the following lines. If we have a solution $w(s)$, then, since $g(s)$ is periodic, we have another solution

$$\tilde{w}(s) = w(s+\omega)$$

Thus, the substitution $s \to s + \omega$ induces a linear operator on the space of the solutions

$$\tilde{w} = Lw$$

If $z(s)$ is a solution of the adjoint equation

$$z'(s) = -g(s)\, z(s+1) + g(s)\, z(s)$$

then denote by L^* the corresponding operator generated by $s \to s - \omega$.

L and L^* have the same eigenvalues and these can be shown to be the μ_i. We define a scalar product of solutions by

$$(w, z) = w(s)\, z(s) + \int_{s-1}^{s} w(t+1)\, g(t)\, z(t)\, dt$$

and this can be shown to be independent of s. Also

$$(Lw,\ z) \ = \ (w,\ L^*z)$$

It can now be shown that the eigenvectors of L and L* form a biorthogonal system with respect to the scalar product. Consider those special solutions which have the value 0 in $s_o - 1 \leq s \leq s_o$. It can be shown that any solution can be represented as a product of its restriction to $s_o - 1 \leq s \leq s_o$ and these special solutions. However, the special solutions can be expanded as a series in the eigenfunctions of L and L*. These series are uniformly convergent and we may apply the scalar product term-by-term to obtain the result.

Thus, we need to know about the roots of

$$y\, e^{Bky} \ = \ e^{Bk}$$

when $B > 0$. To simplify the calculation, we write

$$x \ = \ Bk\, y \qquad \text{and } \alpha \ = \ Bk\, e^{Bk}$$

and the equation (2.34) becomes

$$x \ = \ \alpha\, e^{-x}$$

with $\alpha > 0$.

Lemma 22

The complex roots in the upper half plane (lower half plane by symmetry) of the equation

$$x \ = \ \alpha\, e^{-x}$$

for $\alpha > 0$, lie one in each horizontal strip

$$(2p + 1)\,\pi \ < \ v \ < \ 2(p + 1)\,\pi \qquad\qquad p \ = \ 0,\ 1,\ 2,\ldots.$$

where $x = u + iv$, at the intersection of

$$v \ = \ \pm\,(\alpha^2\, e^{-2u} - u^2)^{\frac{1}{2}} \qquad (2.36)$$

and

$$u = - v \cot v$$

The only real root lies at the intersection of

$$v = \pm (\alpha^2 e^{-2u} - u^2)^{\frac{1}{2}}$$

with the positive real axis i.e. at u = Bk and this is the root with largest real part.

Proof

For complex roots, we write

$$x = r e^{i\theta} = u + iv$$

then $r = \alpha e^{-u}$ $\qquad \theta = - v + 2n\pi$

Hence

$$v = \pm (\alpha^2 e^{-2u} - u^2)^{\frac{1}{2}}$$

Thus the roots occur when

$$u = - v \cot v$$

$$v = \pm (\alpha^2 e^{-2u} - u^2)^{\frac{1}{2}}$$

Consider the strips

$$2p\pi < v < (2p + 1)\pi \qquad p = 0, 1, 2, ...$$

Here v and sin v have the same sign, but $v = r \sin \theta$, and so $\theta \neq - v$ and thus there are no roots in these strips. However, there are roots at the intersections of (2.36) in the strips

$$(2p + 1)\pi < v < 2(p + 1)\pi \qquad p = 0, 1, 2, ...$$

Now we consider real roots. From first equation of (2.36)

$$\frac{dv}{du} = \mp \frac{(\alpha^2 e^{-2u} + u)^{\frac{1}{2}}}{(\alpha^2 e^{-2u} - u^2)^{\frac{1}{2}}}$$

which is finite except when $u = \pm \alpha e^{-u}$. This is a root if v = 0 and the positive sign is taken, and then

$$u = \lim_{v \to 0} \left[- \frac{v}{\tan v} \right] = \frac{-1}{\sec^2 0} = - 1$$

$$\Rightarrow \quad \alpha = - e^{-1}$$

But $\alpha > 0$ and so there are no double roots.

There is obviously a real root when

$$u = \alpha e^{-u}$$

and, since $\alpha = Bk\ e^{Bk}$, this is when

$$u = Bk\ e^{Bk - u} \qquad B > 0 \tag{2.37}$$

Thus $u = Bk$ is a root; since the left hand side of (2.37) is an increasing function and the right hand side is decreasing there can be only one solution. Since

$$v^2 = \alpha^2 e^{-2u} - u^2$$

$$\frac{dv^2}{du} = - 2 \alpha^2 e^{-2u} - 2u$$

which is clearly negative for $u > 0$. Thus v^2 is a decreasing function of u, for $u > 0$. When $u = Bk$, $v = 0$ and so $u = Bk$ is the foot with largest real part.

Hence Lemma 22 is established. The positions of the roots are indicated in Fig 1, for various values of α.

We also notice that, under the relaxed condition in force in this section i.e.

$$B = \int_0^\omega g(t)\ dt \geq 0$$

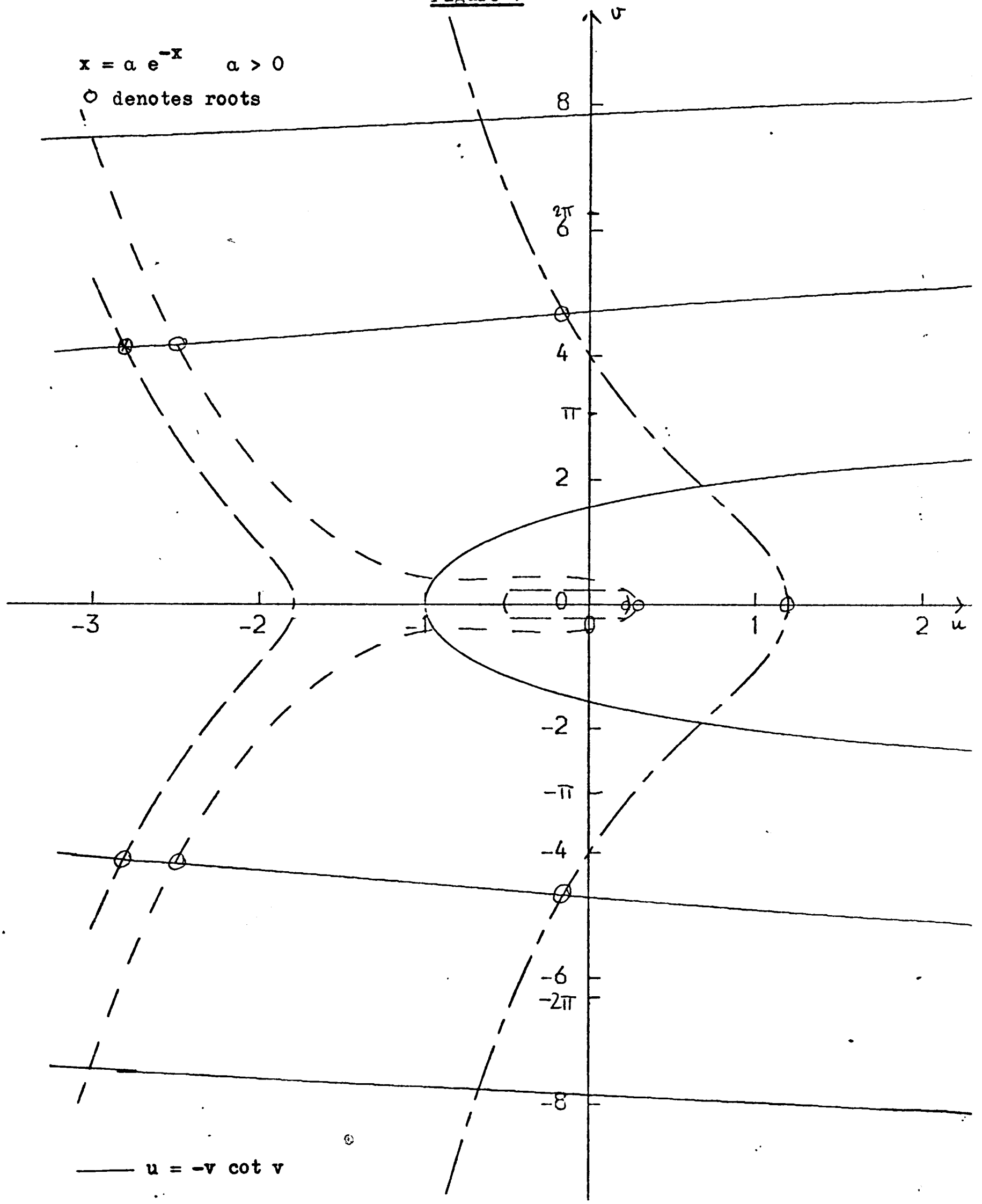
we can assume that $\int_0^S g(t)\ dt \geq 0$ for s sufficiently large, except when $B = 0$.

Lemma 23

For s > some S

$$\int_0^S g(t)\ dt \geq 0$$

Figure 1



—— $u = -v \cot v$

— —

— — $v = \pm(a^2 e^{-2u} - u^2)^{\frac{1}{2}}$

—— - ——

a = 0.3

a = 0.4

a = 4.0

Graph produced by the Oxford University ICL 1906A computer.

if $B > 0$.

Proof

$$\int_0^s g(t)\,dt = \left(\int_0^{n\omega} + \int_{n\omega}^{s} \right) (g(t)\,dt)$$

where n is a positive integer such that

$$n\omega \leq s < (n+1)\omega$$

Thus

$$\int_0^s g(t)\,dt = nB + \int_{n\omega}^{s} g(t)\,dt \qquad (2.38)$$

g is periodic and continuous and so is bounded. On $[n\omega, s]$, $|g(t)| \leq M$, say, where M is independent of n. Thus,

$$\left| \int_{n\omega}^{s} g(t)\,dt \right| < M\omega \qquad \forall n \quad \forall s \text{ s.t. } n\omega \leq s \leq (n+1)\omega$$

Take $K = M/B$ and then for $s > K$ the second term in (2.38) is less in modulus than the first and the result holds.

Now we apply the results of these last two lemmas to Theorem 21. From figure 1, the roots of $x = \alpha e^{-x}$ which have small modulus can have either positive or negative real part whilst roots of large modulus have negative real part. Thus roots of $y e^{Bky} = e^{Bk}$ have a similar distribution. $\int_0^s g(t)\,dt$ has positive sign for sufficiently large s and thus, from Theorem 21, the dominating solutions $w_i(s)$ are those with μ_i having positive real part. Thus we are interested in the solutions of $x = \alpha e^{-x}$ with positive real part, and it is clear that the dominant solution for large s is that corresponding to $x = Bk$ i.e. $\mu_1 = 1$ i.e. the dominating solution is

$$w_1(s) \sim 1$$

Clearly, in a more specific example, much more detailed information on the behaviour of solutions can be obtained.

§ 3.2 g(s) having general period ω.

In this case, it is unreasonable to expect results which depend on the periodicity of g, as the lag and period do not reinforce. There is a great deal of theory (see for example Hale [10] which contains an exhaustive bibliography), in terms of the characteristic multipliers of the equation, which are defined as follows.

Let C[-1, 0] be the space of continuous complex valued functions on [-1, 0] with norm $|| \zeta(\alpha) || = \sup_{-1 \le \alpha \le 0} (| \zeta(\alpha) |)$. Let $w_\zeta(s)$ be a solution which has initial function $\zeta(s)$ in [-1, 0]. Then, for $s > 0$

$$T(s)\,\zeta(\alpha) = w_\zeta(s + \alpha)$$

defines a linear map T(s) of C[-1, 0] into itself. $U = T(\omega)$ is a completely continuous map of C[-1, 0] into itself. If we consider the equation

$$(U - \lambda I)\,\zeta(\alpha) = 0$$

where I denotes the identity map, then the nonzero λ for which this holds are known as the characteristic multipliers.

In our case it does not seem possible to show that such multipliers even exist. However, the theory for g(s) bounded is obviously applicable. From § 2, Theorem 17, we have that, if $g^{(r)}(s) = 0(1)$ $r = 1, 2, \ldots, m$ and w(s) is a solution of (2.1), then $w^{(m+1)}(s)$ is bounded. From the results of § 3.1, it would seem that we should be able to show that w(s) and w'(s) are bounded assuming merely that

$$\int_0^s g(t)\,dt > 0$$

with g(s) periodic, but this does not seem possible by an extension of the methods already used.

C H A P T E R 3

Behaviour of solutions as $s \to \infty$ for $g \leq 0$

This chapter deals with the behaviour of solutions of

$$w'(s) = g(s)\,[w(s-1) - w(s)]$$

at $+\infty$ when $g(s)$ is negative for all s. We write $j(s) = -g(s)$ and consider

$$w'(s) = -j(s)\,[w(s-1) - w(s)] \qquad (3.1)$$

at $+\infty$ for $j(s)$ positive for all s. All theorems in this chapter, with the exception of those in § 3, assume that $j(s) \geq 0\ \forall s$ and that $j(s)$ is continuous. There is no general result corresponding to Theorem 1 and we must be more specific about $j(s)$ immediately, apart from remarking that Theorem 8 still holds as its proof does not depend upon the sign of g.

§ 1 $j(s)$ unbounded as $s \to \infty$.

As in Chapter 2, there are two basic cases in this section; $j(s)$ a polynomial and $j(s) = \exp\{h(s)\}$, where $h(s)$ is a polynomial. The results however are proved in greater generality. Intuitively, we do not expect the solutions of (3.1) to be as 'small' as those of (2.1) and this is indeed the case.

§ 1.1 $j(s)$ asymptotically algebraic in s.

Conditions V

$j(s)$ is continuous

$$j(s) = O(s^k) \quad k > 0$$

$$\frac{1}{j(s)} = O(s^{-\beta}) \quad \beta > 0$$

Trivially, these conditions are satisfied by $g(s) = -\, j(s)$ a polynomial of degree k, with $\beta = k$.

Theorem 24

If w(s) is a solution of (3.1) and j(s) satisfies Conditions V then

$$w(s) \exp\{-\int^{s} j(t)\, dt\}$$

is bounded.

Proof

Let I_n denote $[s_o + n, s_o + n + 1]$ and let $| w(s)\, e^{-\int^{s} j} |$ have least upper bound J_n in I_n.

Thus

$$\frac{d}{ds}\left[w(s) \exp\{-\int^{s} j\} \right] = -\exp\{-\int^{s} j\}\, j(s)\, w(s-1)$$

If $s \in I_{n+1}$

$$\exp\left[-\int^{t} j\right] w(t) \Big|_{s_o+n+1}^{s} = -\int_{s_o+n+1}^{s} \exp\left[-\int^{t} j\right] w(t-1)\, j(t)\, dt$$

Thus

$$\left| \exp\left[-\int^{s} j\right] w(s) \right| \leq J_n\left(1 + \int_{s_o+n+1}^{s} j(t) \exp\left[-\int_{t-1}^{t} j\right] dt\right) \ldots$$

Using Conditions V we find that

$$\int_{s_o+n+1}^{s} j(t) \exp\left[-\int_{t-1}^{t} j(r)\, dr\right] dt = O\left(\int_{s_o+n+1}^{s} t^k\, e^{-At^{\beta-\varepsilon}}\, dt\right)$$

with $A > 0$ and $\forall \varepsilon > 0$.

Thus

$$| w e^{-\int^s j} | \leq J_n [1 + O(s^k e^{-A(s_o+n+1)^{\beta-\varepsilon}})]$$

$$= J_n[1 + O([s_o + n + 2]^k e^{-A(s_o+n+1)^{\beta-\varepsilon}})]$$

Now $\prod_n (1 + n^k e^{-An^{\beta-\varepsilon}})$ converges for $\beta - \varepsilon > 0$ and thus, as ε is arbitrarily small, the product converges for $\beta > 0$. Choosing s_o such that

$$\prod_n [1 + O([s_o + n + 2]^k e^{-A(s_o+n+1)^{\beta-\varepsilon}})] < 2$$

we have

$$J_{n+1} \leq 2 J_o$$

for $s \in I_{n+1}$. Thus the result holds.

We have established that $w(s) = O(e^{\int^s j})$ and our aim is now to show that there is a solution which is as large as $e^{\int^s j}$ i.e. such that

$$w(s) \sim L e^{\int^s j}$$

To this end we prove Theorem 25.

Conditions VI

$j(s)$ is continuously differentiable

$j(s) = p(s)(1 + o(\frac{1}{s}))$ where $p(s)$ is a polynomial with leading term As^k $A>0$ $k>0$

$j'(s) = p'(s)(1 + o(\frac{1}{s}))$

These conditions are stronger than Conditions V but, although I believe the following theorem to be true under these weaker conditions, the details of the proof would become considerably more complicated.

Theorem 25

If $w(s)$ is a solution of (3.1) and Conditions VI hold, then, if

$w(s) = o(\exp[\int^{s} j])$, $w(s) = O(1)$.

Proof

We have

$$| w(s) | \leq M \exp[\int^{s} j] \qquad \text{for } s > A'$$

M, A' constants and, since $j(t)\exp[-\int^{t} j]w(t-1)$ is $O(e^{-t^{e-\varepsilon}})$ $\forall \varepsilon > 0$, its integral to ∞ exists and

$$w(s) = -\exp[\int^{s} j] \int_{s}^{\infty} j(t) \exp[-\int^{t} j]\, w(t-1)\, dt \qquad (3.2)$$

Thus, for $s > A' + 1$

$$| w(s) | \leq M \exp[\int^{s} j] \int_{s}^{\infty} j(t) \exp[-\int_{t-1}^{t} j]\, dt$$

$$| w | \leq e^{\int^{s} j} M \left[\frac{e^{-\int_{t-1}^{t} j}\, j(t)}{-j(t) + j(t-1)} \Big|_{s}^{\infty} - \int_{s}^{\infty} e^{-\int_{t-1}^{t} j} \frac{d}{dt}\left[\frac{j(t)}{-j(t)+j(t-1)}\right] dt \right]$$

From conditions VI,

$$\frac{d}{dt}\left[\frac{j(t)}{j(t) - j(t-1)}\right] = \frac{1}{k}\,(1 + o(1))$$

Thus, substituting and integrating by parts,

$$| w | \leq \frac{M\, e^{\int^{s-1} j}\, j(s)}{j(s) - j(s-1)} + M\, e^{\int^{s} j} \int_{s}^{\infty} e^{-\int_{t-1}^{t} j}\, O(1)\, dt$$

$$= \frac{M\, e^{\int^{s-1} j}\, j(s)}{j(s) - j(s-1)} \left(1 + O\left(\frac{1}{j(s)} \right) \right)$$

Thus

$$| w | \leq M\, e^{\int^{s-1} j}\, \frac{s}{k}\,(1 + O(\tfrac{1}{s})) \qquad s > A' + 1 \qquad (3.3)$$

We make the induction hypothesis

$$| w | \leq M\, e^{\int^{s-n} j}\, \frac{s^{n}}{k^{n}\, n!}\,(1 + O(\tfrac{1}{s})) \qquad \text{for } s > A' + n \qquad (3.4)$$

By (3.3) the hypothesis holds for n = 1. Assuming that (3.4) is true for n:-

$$| w | \leq \frac{AM}{k^n\, n!}\, e^{\int^{s} j}\,(1 + O(\tfrac{1}{s})) \int_{s}^{\infty} e^{-\int_{t-n-1}^{t} j}\, t^k\,(t-1)^n\, dt$$

for $s > A' + n + 1$

$$\int_{s}^{\infty} e^{-\int_{t-n-1}^{t} j}\, t^k\,(t-1)^n\, dt$$

$$= \frac{e^{-\int_{t-n-1}^{t} j}\, t^k\,(t-1)^n}{-j(t) + j(t-n-1)} \Big|_{s}^{\infty} + \int_{s}^{\infty} e^{-\int_{t-n-1}^{t} j}\, \frac{d}{dt}\Big[\frac{t^k\,(t-1)^n}{j(t)-j(t-n-1)}\Big]\, dt$$

$$= \frac{e^{-\int_{s-n-1}^{s} j}\, s^k\,(s-1)^n}{j(s) - j(s-n-1)} + \int_{s}^{\infty} e^{-\int_{t-n-1}^{t} j}\, t^n\,(1 + o(1))A^{-1}\, k^{-1}\,(n+1)^{-1}\, dt$$

$$= \frac{e^{-\int_{s-n-1}^{s} j}\, s^k\,(s-1)^n}{A\,k\,(n+1)\, s^{k-1}}\,(1 + O(\tfrac{1}{s}))$$

Thus

$$| w | \leq \frac{M\, s\,(s-1)^n}{(n+1)!\, k^{n+1}}\, e^{\int_{s-n-1}^{s-n-1} j}\,(1 + O(\tfrac{1}{s}))$$

$$\leq \frac{M\, s^{n+1}}{(n+1)!\, k^{n+1}}\, e^{\int_{s-n-1}^{s-n-1} j}\,(1 + O(\tfrac{1}{s}))$$

and the induction hypothesis (3.4) is established. From (3.4), for $A' + n + 1 > s > A' + n$

$$| w | \leq \frac{M}{n!}\,(\tfrac{s}{k})^n\, e^{\int_{A+1}^{s} j}\,(1 + O(\tfrac{1}{s}))$$

$$| w | \leq N\, e^{s/k} \qquad \text{for } s > B \qquad (3.5)$$

We return to (3.2) to obtain

$$| w(s) | \leq e^{\int^s j} N \int_s^\infty e^{-\int^t j} j(t)\, e^{(t-1)/k}\, dt$$

$$\leq N\, e^{\int^s j} \left[- e^{-\int^t j + (t-1)/k} \Big|_s^\infty + \frac{1}{k} \int_s^\infty e^{-\int^t j + (t-1)/k}\, dt \right]$$

$$\leq N\, e^{(s-1)/k} (1 + o(1)) \quad \text{for } s > B + 1$$

Repeating this

$$| w(s) | \leq N\, e^{(s-n)/k} (1 + o(1)) \qquad \text{for } B + n + 1 > s > B + n$$

This gives

$$| w(s) | \leq N\, e^{(B+1)/k} (1 + o(1)) \qquad \forall n$$

$$\leq P$$

Thus w(s) is bounded $\forall$ s.

By using the work of de Bruijn [3] as in Chapter 2 § 1.1, we can prove a theorem analogous to Theorem 6, in this case.

Theorem 26

If w(s) is a solution of (3.1) and j(s) satisfies Conditions II (see Chapter 2) then, if w(s) = o(1), it is identically zero.

We can now show that under certain conditions there is a solution $w(s) \sim L\, e^{\int^s j}$

Theorem 27

If j(s) satisfies Conditions VI and $j'(s) \geq 0$ for $s \geq s_0$ for some fixed point s_0, then there is a solution of (3.1) such that

$$w(s) \sim L\, e^{\int^s j}$$

Proof

Consider the interval $[s_o, s_o + 1]$. Take any solution $w(s)$ such that both $w'(s) > 0$ and $w''(s) > 0$ in that interval. Let $w''(s)$ vanish for the first time (for $s \geq s_o$) at $s = s_1$. Since $w'' > 0$ in $[s_o, s_1)$,

$$w'(s_1) > w'(s_1 - 1) > 0$$

Thus

$$j(s_1)\, w'(s_1) > j(s_1)\, w'(s_1 - 1) > 0$$

But

$$w''(s_1) = \left[j(s_1) + \frac{j'(s_1)}{j(s_1)}\right] w'(s_1) - j(s_1)\, w'(s_1 - 1)$$

By assumption $\dfrac{j'(s_1)}{j(s_1)}\, w'(s_1) \geq 0$. Thus $w''(s_1) > 0$, a contradiction. Hence $w''(s) > 0$ for all $s \geq s_o$ and so $w'(s)$ is positive increasing. Thus $w(s)$ is unbounded and so $w(s) \neq O(1)$. By Theorem 25, if $w(s) \neq O(1)$ then $w(s) \neq o(e^{\int^s j})$.

Fron Theorem 24, we know that $w(s) = O(e^{\int^s j})$. It remains to show that $w(s)\, e^{-\int^s j}$ tends to a definite limit. From Theorem 24, we have

$$e^{-\int^s j}\, w(s) \Big|_{s_1}^{s_2} = O\left(\int_{s_1}^{s_2} e^{-\int_{s-1}^{s} j}\, j(s)\, ds \right) \qquad (3.6)$$

as $s_1, s_2 \to \infty$

$$= O\left(\int_{s_1}^{s_2} e^{-As^\beta}\, s^k\, ds \right)$$

Given $\beta > \varepsilon > 0$, $\exists$ S s.t. $s > S$

$$e^{-A s^\beta}\, s^k < e^{-A s^{\beta-\varepsilon}}\, s^{\beta-\varepsilon-1}$$

Thus, for $s_1, s_2 > S$,

$$e^{-\int^s j}\, w(s) \Big|_{s_1}^{s_2} = O\left(\int_{s_1}^{s_2} e^{-A s^{\beta-\varepsilon}}\, s^{\beta-\varepsilon-1}\, ds \right)$$

$$= O(e^{-A s^{\beta - \epsilon}} \Big|_{s_1}^{s_2})$$

which is clearly arbitrarily small. By the Cauchy principle

$$w(s)\, e^{-\int^{s} j} \longrightarrow L \quad \text{as} \quad s \to \infty$$

and so we have found a solution such that

$$w(s) \sim L\, e^{\int^{s} j}$$

These are the main results obtainable in this case. We notice that, in this chapter, the results are applicable in the case $k = 1$, so that the theorems do apply to the case $j(s) = s$.

§1.2 $g(s) = -\exp[h(s)]$, $h(s)$ asymptotically algebraic.

As before, we write $g(s) = -j(s)$, and consider $j(s) \geq 0$.

This section deals with the case in which $j(s) = \exp[h(s)]$ and $h(s)$ is a polynomial. Again the results are proved under slightly more general conditions.

Conditions VII

$h(s)$ is continuous

$$h(s) = O(s^{k}) \qquad k > 0$$

$$\frac{1}{h(s)} = O(s^{-\beta}) \qquad \beta > 0$$

Theorem 28

If $w(s)$ is a solution of (3.1) and $j(s) = \exp[h(s)]$ where $h(s)$ satisfies Conditions VII then

$$w(s) \exp\Big\{ -\int^{s} j(t)\, dt \Big\}$$

is bounded.

Proof

We proceed as in Theorem 24 to obtain

$$| e^{-\int_{s_0}^{s} j}\, w(s) | \leq J_n\Big(1 + \int_{s_0+n+1}^{s} j(t)\, e^{-\int_{t-1}^{t} j}\, dt\Big)$$

for $s \in I_{n+1}$, where J_n is the least upper bound of $| w(s)\, e^{-\int^{s} j} |$ in $I_n \equiv [s_0 + n, s_0 + n + 1]$. Under Conditions VII, for s sufficiently large, there exists a non-zero constant K such that

$$j(s) - j(s-1) > K\, j(s)$$

Thus
$$\int_{s_0+n+1}^{s} j(t)\, e^{-\int_{t-1}^{t} j}\, dt = O\Big(\int_{s_0+n+1}^{s} [e^{h(t)} - e^{h(t-1)}]\, e^{-\int_{t-1}^{t} e^{h(z)}\, dz}\, dt\Big)$$

$$= O\Big(\exp\Big[-\int_{s_0+n}^{s_0+n+1} e^{h(z)}\, dz\Big]\Big)$$

$$= O\Big(e^{-e^{A(s_0+n)^{\beta-\varepsilon}}}\Big)$$

for all $\varepsilon > 0$ and where A is a positive constant. Now

$$\prod_{n=1}^{\infty} \Big(1 + O\Big(e^{-e^{A(s_0+n)^{\beta-\varepsilon}}}\Big)\Big) < \infty$$

and so $J_{n+1} \leq 2\, J_0$

since s_0 can be chosen such that

$$\prod_{n} [1 + O(e^{-e^{A(s_0+n)^{\beta-\varepsilon}}})] < 2$$

As in §1.1 of this chapter, we now show that there is a solution $w(s) \sim L\, e^{\int^{s} j}$.

Theorem 29

If $w(s)$ is a solution of (3.1) and $j(s) = \exp[h(s)]$ where $h(s)$ satisfies Conditions VII; then, if $w(s) = o(\exp[\int^{s} j])$, $w(s) = O(1)$.

Proof

As in the proof of Theorem 25, we find that

$$w(s) = -e^{\int^{s} j} \int_{s}^{\infty} j(t)\, e^{-\int^{t} j}\, w(t-1)\, dt \qquad (3.7)$$

and there exist constants M and A such that

$$|w(s)| \leq M e^{\int^{s} j} \int_{s}^{\infty} j(t)\, e^{-\int_{t-1}^{t} j}\, dt \quad \text{for } s > A + 1$$

As in Theorem 28, for s sufficiently large, there exists a nonzero constant K such that

$$j(s) - j(s-1) > K\, j(s)$$

Clearly $K < 1$. Thus

$$|w(s)| \leq \frac{M}{K} e^{\int^{s} j} \int_{s}^{\infty} [j(t) - j(t-1)]\, e^{-\int_{t-1}^{t} j}\, dt$$

$$= \frac{M}{K} e^{\int^{s-1} j} \quad \text{for } s > A + 1$$

Substituting in (3.7) and repeating this process, we find that

$$|w(s)| \leq \frac{M}{K^n} e^{\int^{s-n} j} \quad \text{for } A + n + 1 \geq s > A + n$$

$$\leq \frac{M}{K^n} e^{\int^{A+1} j}$$

Thus $|w(s)| \leq \frac{M R}{K^n}$ $\quad A + n + 1 \geq s > A + n$

$$|w(s)| \leq M R' \exp\{s[-\log K]\}$$

Since $K < 1$, $-\log K > 0$

Thus $| w(s) | \leq N e^{Cs}$ for $s > B$ (3.8)

where $C > 0$.

We establish the induction hypothesis:-

$$| w(s) | \leq N e^{C(s-n)} [1 + O(\frac{1}{j(s-n)})] \tag{3.9}$$

for $B + n + 1 \geq s > B + n$

By (3.8), the hypothesis holds for $n = 0$. Assume that (3.9) holds for n. Then, for $s > B + n + 1$,

$$| w(s) | \leq N e^{\int_s^s j} \int_s^\infty j(t) e^{-\int_s^t j + C(t-n-1)} \{1 + O(\frac{1}{j(t-n-1)})\}$$

$$\leq N (1 + O(\frac{1}{j(s-n-1)})) e^{\int_s^s j} \int_s^\infty j(t) e^{-\int_s^t j} e^{C(t-n-1)} dt$$

$$\int_s^\infty j(t) e^{-\int_s^t j} e^{C(t-n-1)} dt$$

$$= e^{-\int_s^s j} e^{C(s-n-1)} + C \int_s^\infty e^{-\int_s^t j} e^{C(t-n-1)} dt$$

$$= e^{-\int_s^s j} e^{C(s-n-1)} [1 + O(\frac{1}{j(s)})]$$

by integration by parts. Thus for $s > B + n + 1$

$$| w(s) | \leq N (1 + O(\frac{1}{j(s-n-1)})) e^{C(s-n-1)} (1 + O(\frac{1}{j(s)}))$$

$$= N e^{C(s-n-1)} (1 + O(\frac{1}{j(s-n-1)}))$$

and the induction is established. From (3.9) we find

$$| w(s) | \leq N e^{C(B+1)} [1 + O(\frac{1}{j(B)})]$$

$$\leq P \qquad \forall\ s > B$$

Hence w(s) is bounded for all s.

As in §1.1, provided that $j(s)$ satisfies Conditions II, a uniqueness theorem analogous to Theorem 6 holds. We now show that there is a solution $w(s) \sim L\, e^{\int^s j}$.

Theorem 30

If $j(s) = \exp[h(s)]$ and $h(s)$ satisfies Conditions VII with $h'(s) \geq 0$ for $s \geq s_0$ for some fixed point s_0, then there is a solution of (3.1) $w(s) \sim L\, e^{\int^s j}$.

Proof

The proof is the same as that of Theorem 27 with the following modification. To show that $w(s)\, e^{-\int^s j}$ tends to a definite limit, we proceed as follows:-

From Theorem 28

$$w(s)\, e^{-\int^s j} \int_{S_1}^{S_2} \leq J_n \int_{S_1}^{S_2} e^{-\int_{t-1}^{t} j}\, j(t)\, dt$$

$$\leq \frac{J_n}{K} \left[e^{-\int_{S_1-1}^{S_1} j} - e^{-\int_{S_2-1}^{S_2} j} \right]$$

$$= O\left(e^{-e^{s_1}}{}^{\beta-\varepsilon} \right)$$

for $\beta > \varepsilon > 0$ and thus this can be made as small as we please by taking s_1 sufficiently large.

§2 $g(s)$ bounded as $s \to \infty$.

In this section we consider the behaviour of solutions of (3.1) with $-g(s) = j(s) \geq 0$ $\forall s$, bounded, and such that $\lim_{s\to\infty} j(s)$ exists, where $j(s)$ is continuous. There are two cases to be considered: $j(s)$ asymptotically constant and $j(s)$ tending to 0.

§2.1 j(s) asymptotically constant.

The case covered by this section is very similar to § 2.1 of Chapter 2 and the general remarks and references which appear at the beginning of that section apply here.

$$w'(s) = -j(s)\,[w(s-1) - w(s)] \tag{3.10}$$

As before the solutions of (3.10) depend on the zeroes of the characteristic function. For $j(s) \sim A$, this characteristic function is given by

$$\hat{h}(s) = s - A + A\,e^{-s} \tag{3.11}$$

The same general remarks about the positioning and ordering of the zeroes of $\hat{h}(s)$ as appear in Chapter 2 § 2.1 are relevant. One difference is that for $\hat{h}(s)$ as given in (3.11), a double root occurs at $s = 0$ if $A = 1$. Thus $A = 1$ must be considered separately.

We shall be interested in the solution of $\hat{h}(s) = 0$ with largest real part.

Lemma 31

Let s_m, $m = 0, \pm 1, \ldots\ldots$, be the zeroes of $\hat{h}(s) = s - A + A\,e^{-s}$ (3.12)

then

$$\text{Re } s_m < A \qquad \forall\ m$$

If $A \leq 1$, this bound can be sharpened to

$$\text{Re } s_m < (1 - e^{-A})\,A \qquad \forall\ m$$

Proof

Set Re $s = k\,A$ Im $s = t$

From (3.12) we find

$$e^{kA}(k-1) = -\cos t \tag{3.13}$$

$$t = A\,e^{-kA} \sin t \tag{3.14}$$

Since $-1 \leq \cos t \leq 1$, we have

$$1 - e^{-kA} \leq k \leq 1 + e^{-kA} \tag{3.15}$$

$k = 0$, $t = 0$ gives one zero and so we know that K, the maximum possible value of k, is ≥ 0. From (3.15) we find $K < 2$, since $A > 0$. It is now easier to deal with the cases $A > 1$ and $A \leq 1$ separately.

If $A \leq 1$, then, from (3.14), we have

$$t \leq \sin t$$

and so $t = 0$ is the only possibility. Thus from (3.13),

$$e^{kA}(1 - k) = 1 \tag{3.16}$$

whence $k < 1$.

From (3.16)

$$k - 1 < -e^{-A}$$

$$k < 1 - e^{-A}$$

Hence $$K < 1 - e^{-A} \tag{3.17}$$

If $A > 1$, we first try to find a value of $k > 1$, a solution of (3.13) and (3.14). We already know that $K < 2$ and so (3.13) implies that $|t| > \pi/2$ and so from (3.14),

$$|A e^{-kA}| > \pi/2 \tag{3.18}$$

We now consider $A e^{-kA}$ as a function of A with k fixed as K. The maximum of this function occurs when

$$(1 - KA) e^{-KA} = 0$$

Since A is finite, the maximum is at $A = 1/K$ and this is a global maximum. However, this point is outside our range. As $A \to \infty$, $A e^{-KA} \to 0$ and so it is clear that the maximum value of $A e^{-KA}$ for $A > 1$ occurs at $A = 1 + \varepsilon$ for $\varepsilon > 0$ and sufficiently small.

Thus A = 1 will give an upper bound for $A\ e^{-KA}$. This bound is e^{-K}. Therefore, from (3.18), we need

$$e^{-K} > \pi/2$$

This is a contradiction since $K > 1$. Thus $K \leq 1$.

The case K = 1 can also be eliminated. From (3.13) we find

$$t = (2n + 1)\frac{\pi}{2}$$

and thus

$$(2n + 1)\frac{\pi}{2} = (-1)^n A\ e^{-A} \tag{3.19}$$

Proceeding as above, we show that the maximum value of $A\ e^{-A}$ is 1/e and thus 1/e is the maximum modulus of the right hand side of (3.19). The minimum modulus of the left hand side is $\pi/2$.

$$\frac{\pi}{2} > \frac{1}{e}$$

and so K = 1 is not a possible solution. Thus K < 1.

We now restrict to the case j(s) = A. Some fairly strong results are obtained.

Theorem 32

If w(s) is a solution of

$$w'(s) = A[w(s) - w(s - 1)]$$

then

$$w(s) = \sum_{m=1}^{\infty} P_o(m)\ e^{s_m s}$$

Here $P_o(m)$ is the residue of

$$\frac{w(1) + e^s\ (s - A) \int_0^1 w(u)\ e^{-su}\ du}{A + (s - A)\ e^s} \tag{3.20}$$

at the zero s_m of $\hat{h}(s)$. The series is uniformly convergent in any finite interval $1 + \delta < s < C$.

Proof

Adaptation of Wright [15] Theorem 1.

If we now return to the original general assumption $j(s) \sim A$, two rather weaker results can be obtained. Define η and $\omega(f)$ as in Chapter 2 § 2.1.

Theorem 33

If $w(s)$ is a solution of

$$w'(s) = [A + J(s)] [w(s) - w(s - 1)]$$

where $A > 0$ and $J(s) \to 0$ as $s \to \infty$, then either

$$\max[\omega(w), \omega(w')] \in \eta$$

or w is the zero solution.

Proof

Wright [14] Theorem 2.

It is even more difficult to obtain concrete information from this theorem than from Theorem 19. In this case η contains elements as large as KA and so this is the worst possible value for $\max[\omega(w), \omega(w')]$. However the theorem does give information on the behaviour of all solutions.

The work of Bellman and Cooke [1] and Hale [9] gives information about specific solutions.

Theorem 34

If $J(s)$ satisfies the conditions i) or ii) defined in Theorem 20 for $G(s)$, then

$$w'(s) = [A + J(s)]\,[w(s) - w(s-1)] \qquad (3.21)$$

has a solution of the form

$$w(s) = e^{u_m(s)}\,[1 + o(1)]$$

as $s \to \infty$, corresponding to each simple zero of $\hat{h}(s)$.

$u_m(s)$ is defined by

$$u_m(s) = s_m s + \frac{s_m}{A(1 - A + s_m)} \int^{S} J(t)\,dt$$

where s_m, $m = 0, \pm 1, \ldots$ are the simple zeroes of $\hat{h}(s) = s - A + A e^{-s}$

If $J(s)$ has an asymptotic power series expansion of the form

$$J(s) \sim \sum_{n=1}^{\infty} j_n s^{-n}$$

and $J'(s)$ and $J''(s)$ exist and have asymptotic power series, then there exists a solution of (3.21) for each m, with an asymptotic power series expansion

$$w(s) \sim e^{s_m s}\, s^{r_m} \sum_{n=0}^{\infty} w_n s^{-n}$$

where each w_n is a constant and $w_o \neq 0$

$$r_m = \frac{j_1 s_m}{A(1 - A + s_m)}$$

Proof

Bellman and Cooke [1] suitably adapted.

All zeroes of $\hat{h}(s)$ are simple, except when $A = 1$. In this case $s = 0$ is a double zero of $\hat{h}(s)$. The following theorem deals

with this case.

Theorem 35

Consider equation (3.21) with A = 1 and J(s) twice continuously differentiable. Define the condition (*) by

f satisfies (*) if

$$\lim_{t\to\infty} \frac{f(t-1)}{f(t)} = 1 = \lim_{t\to\infty} \frac{f'(t-1)}{f'(t)}$$

for all 1 such that $0 \leq 1 \leq 1$.

Define r(s) and t(s) by

$$r(s) = \exp \int^{s} J(\alpha)\, d\alpha$$

$$t(s) = -J'(s) + J^2(s)$$

Then

i) if t(s) is such that

$$s^2\, t(s) = o(1)$$

$$t'(s) = o(t(s))$$

$$\int^{\infty} \alpha \mid t(\alpha) \mid d\alpha < \infty$$

t(s) satisfies (*)

$$J(s) = O(\frac{1}{s})$$

$$J'(s) = O(\frac{1}{s^2})$$

J(s) satisfies (*)

then there exist two solutions of

$$w'(s) = [1 + J(s)]\,[w(s) - w(s-1)] \qquad (3.22)$$

having asymptotic forms

$$w(s) = s\, r(s)\,[1 + o(1)]$$

and $$w(s) = r(s)\,[1 + o(1)]$$

ii) if $t(s) = c_o\, s^{-2}[1 + O(\frac{1}{s})]$

$t(s)$ satisfies (*)

$J(s)$ satisfies (*)

$$J(s) = O(\frac{1}{s})$$

$$J'(s) = O(\frac{1}{s^2})$$

then there exist two solutions of (3.22) of the form

$$c_o \neq -\frac{1}{4}$$

$$w(s) = r(s)\,|\,t(s)\,|^{-1/4} \exp\left\{\pm \int^{s} (1 + \frac{1}{4c_o})^{\frac{1}{2}}\, t^{\frac{1}{2}}(a)\, da\right\}(1 + o(1))$$

$$c_o = -\frac{1}{4}$$

$$w(s) = r(s)\,|\,t(s)\,|^{-1/4} \log s\,(1 + o(1))$$

$$w(s) = r(s)\,|\,t(s)\,|^{-1/4}\,(1 + o(1))$$

iii) if $t(s) = o(1)$

$t(s) \neq 0$ for $s \geq s_o$

$$t'(s) = o(t^{3/2}(s))$$

$$t''(s) = o(t^2(s))$$

$$\int^{\infty} |\,t(a)\,|^{\frac{1}{2}}\, da = \infty$$

$$\int^{\infty} |\,t(a)\,|\, da < \infty$$

$$\int^{\infty} \frac{|\,t'(a)\,|^2}{|\,t(a)\,|^{5/2}}\, da < \infty$$

$$\int^{\infty} \frac{|\, t''(a)\,|^2}{|\, t(a)\,|^{3/2}}\, da < \infty$$

$J(s)$ and $t(s)$ satisfy (*)

$$J(s) = O(|\, t(s)\,|^{\frac{1}{2}})$$

$$J'(s) = O(|\, t(s)\,|)$$

then there exist two solutions of (3.22) of the form

$$w(s) = |\, t(s)\,|^{-1/4}\, r(s) \exp\left\{\pm \int^{s} t^{\frac{1}{2}}(a)\, da\right\} (1 + o(1))$$

iv) if $t(s) = o(1)$

$t(s) \neq 0 \quad s \geq s_0$

$$t'(s) = O(t^2(s))$$

$$t''(s) = O(t^3(s))$$

$$\int^{\infty} |\, t(s)\,|^{3/2}\, ds < \infty$$

$$\int^{\infty} |\, t(s)\,|\, ds = \infty$$

$t(s)$ and $J(s)$ satisfy (*)

$$J(s) = O(|\, t(s)\,|)$$

$$J'(s) = O(|\, t(s)\,|^{3/2})$$

then there exist two solutions of (3.22) of the form

$$w(s) = |\, t(s)\,|^{-1/4}\, r(s) \exp\left\{\pm \int^{s} t(a)^{\frac{1}{2}}\, da\right\} \quad (1 + o(1))$$

Proof

Bellman and Cooke [1]

This theorem can be made clearer if we assume that $J(s)$ has an inverse power series

$$J(s) = s^{-\delta} \sum_{n=0}^{\infty} J_n s^{-n}$$

with $J_0 \neq 0$. The conditions are then satisfies as follows

i) $\delta > 1$ ii) $\delta = 1$ Iii) $1/2 < \delta < 1$ iv) $1/3 < \delta \leq 1/2$

If we assume further that

$$J(s) \sim \sum_{n=1}^{\infty} \bar{J}_n s^{-n}$$

asymptotically and if $J'(s)$ and $\bar{J}''(s)$ exist and have asymptotic power series expansions, then these results simplify to:-

$$w'(s) = [1 + J(s)] [w(s) - w(s - 1)]$$

has solutions of the form

$$w(s) = s^{2\bar{J}_1 + 1} (1 + o(1)) \quad \text{if } \bar{J}_1 \neq 0, -\tfrac{1}{2}, -1$$
$$\text{and} \quad w(s) = 1 + o(1)$$

$$w(s) = s\,(1 + o(1)) \quad \text{if } \bar{J}_1 = 0$$
$$\text{and} \quad w(s) = 1 + o(1)$$

$$w(s) = \log s\,(1 + o(1)) \quad \text{if } \bar{J}_1 = -\tfrac{1}{2}$$
$$\text{and} \quad w(s) = 1 + o(1)$$

$$w(s) = 1 + o(1) \quad \text{if } \bar{J}_1 = -1$$
$$\text{and} \quad w(s) = s^{-1}(1 + o(1))$$

§ 2.2 $j(s) \to 0$ as $s \to \infty$.

This section deals with the case $j \to 0$ at infinity. Some

results can be obtained by letting $A \to 0$ in the previous section. The work of Wright [14] covered by Theorem 33 still has meaning. The set η reduces to

$$\{0, -\infty\}$$

and the remarks following Theorem 19 are again relevant.

However, a direct approach yields some interesting results. We work under the following conditions:-

Conditions VIII

$j(s)$ is continuous

i) $\quad j(s) = 0(s^{-k}) \quad k > 1$

or

ii) $\quad j(s) = \exp[- h(s)] \qquad h(s) = 0(s^{k}) \quad k > 0$

Theorem 36

If $w(s)$ is a solution of

$$w'(s) = j(s)[w(s) - w(s - 1)] \tag{3.23}$$

where $j(s)$ satisfies Conditions VIII, then $w(s)$ is bounded.

Proof

Let I_n be $[s_o + n, s_o + n + 1]$ and let $| w(s) |$ have least upper bound J_n in I_n. If $s \in I_{n+1}$

$$e^{-\int^{t} j} w(t) \Big|_{s_o+n+1}^{s} = -\int_{s_o+n+1}^{s} e^{-\int^{t} j} j(t)\, w(t - 1)\, dt$$

$$| w(s) | \leq e^{\int^{s} j} J_n \left(2 e^{-\int^{s_o+n+1} j} - e^{-\int^{s} j}\right)$$

$$= J_n[1 + 0(e^{\int_{s_o+n+1}^{s} j} - 1)]$$

If Conditions VIII i) apply

$$| w(s) | \leq J_n(1 + O([s_o + n + 1]^{-k+\varepsilon}))$$

where ε is arbitrarily small.

$$\overline{|_n|}(1 + O([s_o + n + 1]^{-k+\varepsilon})) < \infty$$

for $k > 1$ and ε sufficiently small, and so $w(s)$ is bounded.

If Conditions VIII ii) apply

$$| w(s) | \leq J_n[1 + O(e^{-A(s_o+n+1)^{k-\varepsilon}})]$$

for ε arbitrarily small. The relevant infinite product converges and so again $w(s)$ is bounded.

Theorem 37

If $w(s)$ is a solution of (3.23) and $j(s)$ is n-times continuously differentiable such that $j^{(r)}(s) = O(1)$ $r = 0, 1, \ldots, n$ and if $j(s)$ satisfies Conditions VIII, then $w^{(n+1)}(s)$ is bounded.

Proof

Method as for Theorem 15.

Conditions VIII do not cover the case $j(s) = \frac{A}{s}$. As yet I am unable to prove results for this case. The work of de Bruijn [4] and Yates [16] although seemingly relevant to this case, either collapses to the statement that any constant is a solution of (3.23) or is inapplicable. It seems that only the results of Wright [14] above can be applied.

§ 3 j(s) periodic

Here we cover cases in which $\lim_{s\to\infty} j(s)$ does not exist but where $j(s)$ is periodic. As in § 3 of Chapter 2, there are two distinct cases dependent on whether the period, ω, of $j(s)$ is the inverse of an integer.

If $\omega \neq 1/k$ for $k \in \mathbb{Z}$, we do not expect that the periodicity of j will greatly affect the solutions as its effect is 'destroyed' by the lag. In general, the remarks in Chapter 2 § 3.1 apply.

We now turn to the case where $\omega = 1/k$ $k \in \mathbb{Z}$. We impose the condition that

$$B = \int_0^\omega j(t)\,dt \geq 0 \tag{3.24}$$

Our results again depend on the work of Hahn [8].

Theorem 38

Let $j(s)$ be a periodic continuous function of period k^{-1} where k is a positive integer. We define

$$w_i(s) = \exp[(1 - \mu_i)\int_0^s j(t)\,dt\,] \tag{3.25}$$

where μ_i is a root of

$$y\,e^{Bk} = e^{Bk\,y}$$

If $B > 0$ and $B \neq \omega$ then all solutions of

$$w'(s) = j(s)[w(s) - w(s-1)] \tag{3.26}$$

can be written as an absolutely convergent series of the form

$$\sum_i c_i\, w_i(s) \tag{3.27}$$

where c_i are constants.

If $B = \omega$, then in the sum (3.27), the system of functions

$\{w_i(s)\}$ must be extended by means of a finite number of functions of the form

$$s^{r_j} \; w_i(s)$$

where the r_j are non-negative integers.

Proof

Hahn [8]

Thus, in this case, we need information on the roots of

$$y\, e^{Bk} = e^{Bk\, y}$$

when $B > 0$. We write

$$x = Bk\, y \qquad \text{and} \qquad \beta = Bk\, e^{-Bk}$$

to give

$$x = \beta\, e^{x}$$

Since $B, k > 0$ there are three distinct cases

i) $Bk > 1$ since $Bk < e^{Bk-1}$ we have $0 < \beta < e^{-1}$

ii) $Bk = 1$ $\beta = e^{-1}$

iii) $0 < Bk < 1$ $0 < \beta < e^{-1}$

Lemma 39

The complex roots in the upper half plane (lower half plane by symmetry) of the equation

$$x = \beta\, e^{x}$$

for $e^{-1} \geq \beta > 0$, lie one in each strip

$$2p\pi < v < (2p + 1)\,\pi \qquad p = 1, 2, \ldots$$

when $x = u + iv$, at the intersection therein of the appropriate branches of the curves

$$v = \pm (\beta^2 e^{2u} - u^2)^{\frac{1}{2}} \tag{3.28}$$

and

$$u = v \cot v \tag{3.29}$$

The real roots occur as follows:-

If $\beta = e^{-1}$, there is a double real root at s = 1 and all other roots are complex.

If $0 < \beta < e^{-1}$, there are two real roots, one at each of the intersections of (3.28) with the positive real axis; one with real part greater than 1 and the other with real part less than 1.

[One of these roots is always at Bk, where $\beta = Bk\ e^{-Bk}$]

The real roots are those with smallest real part.

Proof

For the complex roots, set

$$x = r e^{i\theta} = u + iv$$

Thus $r = \beta e^{u}$ $\theta = v$

and

$$v = \pm (\beta^2 e^{2u} - u^2)^{\frac{1}{2}} \qquad \tan\theta = \frac{v}{u}$$

and the roots occur when

$$v = \pm (\beta^2 e^{2u} - u^2)^{\frac{1}{2}}$$

$$u = v \cot v$$

We consider the strips

$$(2p + 1)\pi < v < 2(p + 1)\pi \qquad p = 0, 1, 2, \ldots$$

Here v and sin v have opposite sign. $v = r \sin\theta$ and so $\theta \neq v$. Thus there are no roots in these strips. There are roots at the intersections of (3.28) and the branches of (3.29) in the strips

$$2p\pi < v < (2p + 1)\pi \qquad p = 1, 2, \ldots$$

and possibly in $0 < v < \pi$.

If $x = \sigma + it$ is a root with $0 < t < \pi$,

$$t = \beta e^{t \cot t} \sin t$$

$$t \cot t = \log\left[\frac{t}{\beta \sin t}\right]$$

Let $\gamma(t) = t \cot t - \log\left[\frac{t}{\beta \sin t}\right]$

$$\gamma'(t) = \frac{(\sin t \cos t - t)}{\sin^2 t} + \frac{(t - \tan t)}{t \tan t}$$

$$< 0$$

Thus $\gamma(t)$ decreases as t increases in $0 < t < \pi$. $\gamma(\pi) < 0$ $\forall$ finite positive β. $\gamma(0) = 1 + \log \beta$ which is positive only if $\beta > e^{-1}$. However, $\beta < e^{-1}$ and so there is no root in

$$0 < t < \pi$$

For the real roots, on (3.28)

$$\frac{dv}{du} = \pm \frac{\beta^2 e^{2u} - u}{(\beta^2 e^{2u} - u^2)^{\frac{1}{2}}}$$

which is finite except when $u = \pm \beta e^{u}$. Only the positive sign will do and so

$$v = 0 \qquad u = \lim_{v \to 0} \frac{v}{\tan v} = 1$$

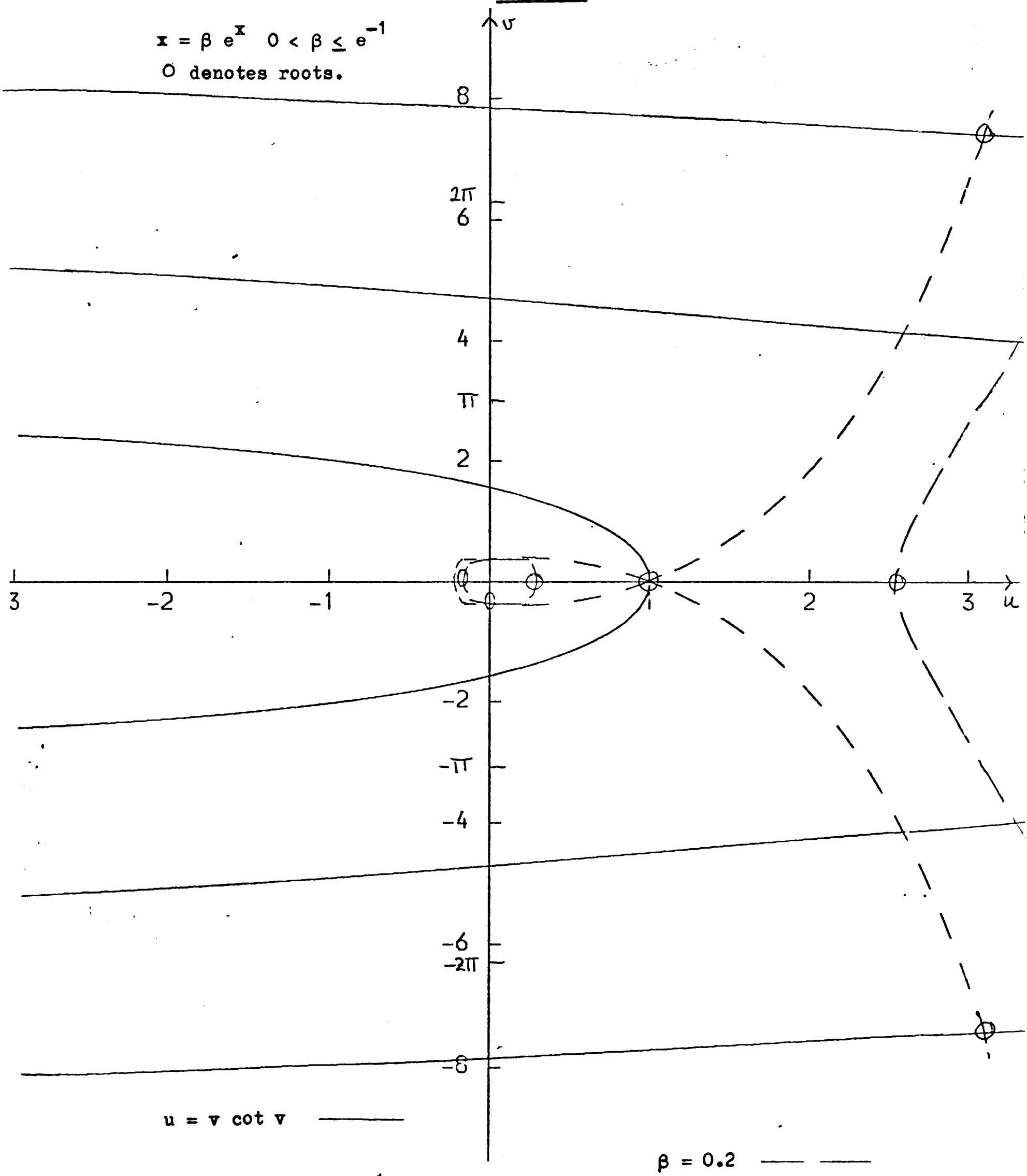
$$\Rightarrow \quad \beta = e^{-1}$$

Thus there is a double real root at $u = 1$ $v = 0$ when $\beta = e^{-1}$. Otherwise, there are obviously real roots when

$$v = \pm (\beta^2 e^{2u} - u^2)^{\frac{1}{2}}$$

cuts the positive real axis. $u = Bk$ is always a solution. Consideration of $u = \beta e^{u}$, $0 < \beta < e^{-1}$ shows that one root has

Figure 2



$u = v \cot v$ ———

$v = \pm (\beta^2 e^{2u} - u^2)^{\frac{1}{2}}$

$\beta = 0.2$ —— ——

$\beta = e^{-1}$ — — —

Graph produced by the Oxford University ICL 1906A computer.

$u > 1$ and the other has $u < 1$. By considering the equations, it is clear that the real roots are those with largest real part. The positions of the roots are indicated in Figure 2 for various values of β.

In this case, we can again base assumptions about the sign of $\int_0^s j(t)\,dt$ on the sign of B.

Lemma 40

For s > some S

$$\int_0^s j(t)\,dt \geq 0$$

if B > 0.

Proof

As in Lemma 23.

We can now apply the results of these lemmas to Theorem 38. From Figure 2, all roots of $x = \beta e^x$ have positive real part when $0 < \beta \leq e^{-1}$. Thus roots of

$$y = Bk\, e^{Bk}\, y\, e^{-Bk}$$

will also have positive real part.

$\int_0^s j(t)\,dt$ has positive sign for sufficiently large s and so, from Theorem 38, the dominating solutions $w_i(s)$ are those with μ_i having small real part. The μ_i with smallest real part is given by the smaller real solution. Call this solution μ_1. It can be identified as follows

i) $Bk > 1$ μ_1 is the real root which is not 1.

ii) $Bk = 1$ $\mu_1 = 1$

iii) $0 < Bk < 1$ $\mu_1 = 1$

In case i) μ_1 can easily be estimated numerically.

If $Bk > 1$, the dominating $w_i(s)$ (say $w_1(s)$) is such that

$$w_1(s) \sim \exp\left\{ \lambda \int_0^s j(t)\, dt \right\}$$

where $\lambda = 1 - \mu_1$ and so $1 > \lambda > \frac{1}{Bk}$

If we let

$$M = \sup_s | j(s) |$$

then for s sufficiently large

$$\int_0^s j(t)\, dt < s\, M$$

thus $w_1(s) = O(e^{\lambda M s}) = O(e^{M s})$ and so all solutions of (3.26) are $O(e^{Ms})$. For any solution $w(s)$ which is $o(e^{Ms})$, by choosing the root with next smallest real part μ_2, say, we can estimate $w(s)$ more accurately. In this case, $\mu_2 = 1$ and so from (3.27)

if $w(s) = o(e^{Ms})$ then $w(s) = O(1)$.

In the cases ii) and iii) we have immediately that all solutions are $O(1)$.

C H A P T E R 4

Behaviour of solutions as $s \to -\infty$ for $g(s) \geq 0$

This chapter covers the behaviour of solutions of

$$w'(s) = g(s)\,[w(s-1) - w(s)] \tag{4.1}$$

as $s \to -\infty$ with $g(s)$ continuous and positive for all s. This problem is equivalent to determining the asymptotic behaviour for large t of solutions of the advanced equation

$$y'(t) = -j(t)\,[y(t+1) - y(t)] \tag{4.2}$$

with $j(t) \geq 0 \ \forall\ t$. As this equation is advanced, we do not expect to be able to prove such strong results as is possible in Chapters 2 and 3. One result can be proved in complete generality.

Theorem 4.1

If $y(t)$ is a solution of (4.2) and $y(t)$ is $o(1)$ as $t \to \infty$, then

$y(t) \equiv 0$

Proof

By extending to the left, we have solutions defined in intervals of the form $[0, R)$ $\quad 0 < R \leq \infty$

Assume $y(t)$ is a solution of (4.2) such that $y(t) = o(1)$.

From the equation (4.2)

$$\frac{d}{dt}\Big[\, y(t)\, \exp\Big\{-\int^{t} j\Big\}\Big] = -j(t)\, e^{-\int^{t} j}\, y(t+1)$$

Since $y(t) = o(1)$, for t sufficiently large, $y(t+1)\, e^{-\int^{t} j}\, j(t) \leq e^{-\int^{t} j}\, j(t)$ and so its integral at ∞ exists and

$$y(t) = e^{\int^{t} j} \int_{t}^{\infty} y(t+1)\, e^{-\int^{t} j}\, j(t)\, dt$$

For each $R \geq 1$ define

$$K(R) = \sup_{t \geq R} | y(t) |$$

$y(t) = o(1)$ and so $K(R) \to 0$ as $R \to \infty$. Let $t \geq R$

$$| y(t) | \leq e^{\int_t^t j} \int_t^\infty | y(x+1) | e^{-\int_t^x j} | j(x) | dx$$

$$\leq e^{\int_t^t j} K(R+1) e^{-\int_t^t j} = K(R+1)$$

Thus

$$K(R) \leq K(R+1)$$

Repeating this we have

$$K(R) \leq K(R+n)$$

and now letting $n \to \infty$, $K(R+n) \to 0$. Thus $K(R) = 0$ and so $y(t) = 0$.

To make further progress we require to specify $j(t)$ more precisely. As before there are two basic cases; $j(t)$ bounded or unbounded at infinity.

§ 1 $j(t)$ unbounded as $t \to \infty$.

For this class of functions $j(t)$, we are able to show, under some restrictions on $j(t)$, that, given any periodic function with period 1 which is Hölder-continuous, then there is a solution of (4.2) which behaves like this function at infinity. By restricting $j(t)$ to two more specific classes, we can be more definite.

§ 1.1 $j(t)$ asymptotically algebraic in t

This section deals mainly with the case where $j(t)$ is a

polynomial, but the conditions are slightly more general.

Theorem 42

Let $f(t)$ be a Hölder-continuous function (with exponent θ $1 \geq \theta > 0$) of period 1 and let $j(t)$ be continuous and such that $j(t) = O(t^k)$ $k > 0$ $\frac{1}{j(t)} = O(t^{-\beta})$ where $\beta(\theta + 1) > 1 + k$, then there is a solution of (4.2) [and, by Theorem 41, only one solution] such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(t^{-(\theta+1)\beta + k + 1})$$

Proof

We proceed by constructing such a solution.

Define

$$y_0(t) = f(t)$$

$$y_1(t) = e^{\int_t^\infty j} \int_t^\infty e^{-\int_s^\infty j} j(s)\,[y_0(s) - y_0(t)]\,ds$$

$$y_{n+1}(t) = e^{\int_t^\infty j} \int_t^\infty e^{-\int_s^\infty j} j(s)\, y_n(s+1)\,ds \qquad n = 1, 2, \ldots \tag{4.3}$$

Inductively $y_n(t)$ is bounded and so our definition of $y_{n+1}(t)$ is valid. Differentiating, we have

$$y'_{n+1}(t) = -j(t)\,[y_n(t+1) - y_{n+1}(t)] \tag{4.4}$$

$$y_1'(t) + y_0'(t) = j(t)\, y_1(t) \tag{4.5}$$

We write

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

and sum (4.4) over n. Using (4.5) we find

$$y'(t) = -j(t)\,[y(t+1) - y(t)]$$

since $y_o(t)$ is periodic.

It remains to show the following:-

i) $y(t)$ has the correct asymptotic behaviour

and

ii) the series for $y(t)$ is absolutely and uniformly convergent.

We consider first $y_1(t)$

$$y_o(s) - y_o(t) = f(s) - f(t)$$

When s is 'near' t

$$| f(s) - f(t) | < K_1 | s - t |^{\theta}$$

since $f(s)$ is Hölder-continuous. For s 'far' from t

$$| f(s) - f(t) | < K_2$$

since $f(s)$ is periodic and Hölder-continuous. Thus

$$| y_1(t) | \leq e^{\int_t^t j} \int_t^{\infty} e^{-\int_t^s j}\, j(s)\, | y_o(s) - y_o(t) |\, ds$$

$$\leq K_1 \int_t^{t+\delta} e^{-\int_t^s j}\, j(s)\, (s-t)^{\theta}\, ds$$

$$+ K_2 \int_{t+\delta}^{\infty} e^{-\int_t^s j}\, j(s)\, ds$$

where $\frac{1}{2} < \delta < 1$.

$$\int_{t+\delta}^{\infty} e^{-\int_t^s j}\, j(s)\, ds = e^{-\int_t^{t+\delta} j(s)\, ds}$$

$$= O(e^{-t^{\beta-\varepsilon}}) \qquad (4.6)$$

for ε arbitrarily small.

$$\int_t^{t+\delta} e^{-\int_t^s j} j(s)\,(s-t)^{\theta}\,ds$$

$$= \int_0^{\delta} e^{-\int_t^{t+u} j} j(t+u)\,u^{\theta}\,du$$

$$\leq K_3\, t^k \int_0^{\delta} e^{-uDt^{\beta}}\, u^{\theta}\,du \qquad (4.7)$$

for $D > 0$.

Putting $uDt^{\beta} = x$

$$\int_t^{t+\delta} e^{-\int_t^s j} j(s)\,(s-t)^{\theta}\,ds \leq K_3 \frac{t^k}{D^{\theta+1}\, t^{\beta(\theta+1)}} \int_0^{t^{\beta}\delta D} e^{-x}\, x^{\theta}\,dx \qquad (4.8)$$

Since $\int_0^{\infty} e^{-x}\, x^{\theta}\,dx$ exists we find, on combining (4.6), (4.7) and (4.8),

$$| y_1(t) | \leq \frac{K}{t^{\beta(\theta+1)-k}}$$

From (4.3), it is now easy to show by induction that

$$| y_n(t) | \leq \frac{K}{(t+n-1)^{\beta(\theta+1)-k}}$$

for $n = 1, 2, \ldots$

We now consider

$$| y(t) - y_0(t) | \leq \sum_{n=1}^{\infty} | y_n(t) |$$

$$\leq K \sum_{n=1}^{\infty} \frac{1}{(t+n-1)^{\beta(\theta+1)-k}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(t+n-1)^{\beta(\theta+1)-k}}$$

$$\leq \int_1^{\infty} (t+x-1)^{-\beta(\theta+1)+k}\,dx$$

$$= \frac{t^{-\beta(\theta+1)+k+1}}{1-(\theta+1)\beta+k}$$

Thus $y(t)$ has the correct asymptotic behaviour. Immediately, the series is uniformly and absolutely convergent.

§ 1.2 $j(t) = \exp[h(t)]$ $h(t)$ asymptotically algebraic in t

A similar theorem to Theorem 42 can be proved in the case where $j(t)$ is the exponential of a polynomial.

Theorem 43

Let $f(t)$ be as in Theorem 42, with $\theta \geq 0$, and let $j(t) = \exp[h(t)]$ $h(t) \sim A\, t^k$ where $k > 0$ $A > 0$ and $h(t)$ is continuous, then there is a solution $y(t)$ of (4.2) [and, by Theorem 41, only one solution] such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O\left(e^{-A\theta\, t^{k-\varepsilon}} \right)$$

for all $\varepsilon > 0$.

Proof

The proof follows the lines of that of Theorem 42. We define $\{ y_n(t) \}$ as before. It remains to show that this series gives $y(t)$ the correct asymptotic behaviour and that the series for y is uniformly and absolutely convergent. We proceed as before to find

$$| y_1(t) | \leq K_1 \int_t^{t+\delta} e^{-\int_t^s j} j(s) (s-t)^\theta \, ds + K_2 \int_{t+\delta}^{\infty} e^{-\int_t^s j} j(s) \, ds$$

$$\int_{t+\delta}^{\infty} e^{-\int_t^s j} j(s)\,ds = e^{-\int_t^{t+\delta} j(s)\,ds}$$

$$= O\left(e^{-e^{At^{k-\varepsilon}}} \right) \qquad (4.9)$$

for ε arbitrarily small.

$$\int_t^{t+\delta} e^{-\int_t^s j} j(s)\,(s-t)^{\theta}\,ds = \int_0^{\delta} e^{-\int_t^{t+u} j} j(t+u)\,u^{\theta}\,du$$

$$\leq K_4\, e^{At^{k+\varepsilon}} \int_0^{\delta} e^{-ue^{At^{k-\varepsilon}}} u^{\theta}\,du$$

$$\leq K_5\, e^{At^{\varepsilon}} - A\theta t^{k-\varepsilon} \int_0^{\delta e^{At^{k-\varepsilon}}} e^{-x} x^{\theta}\,dx$$

$$\leq \frac{K}{e^{A\theta t^{k-\varepsilon'}}}$$

for ε' arbitrarily small.

Thus, as previously,

$$| y_n(t) | \leq \frac{K}{e^{A\theta (t+n-1)^{k-\varepsilon'}}}$$

$$| y(t) - y_o(t) | \leq K \sum_{n=1}^{\infty} \frac{1}{e^{A\theta (t+n-1)^{k-\varepsilon'}}}$$

$$\leq K_6\, e^{-A\theta t^{k-\varepsilon''}} \sum_{r=0}^{\infty} e^{-A\theta t^{k-\varepsilon''-1}}\, r(k-\varepsilon')$$

$$= \frac{K_7\, e^{-A\theta t^{k-\varepsilon''}}}{1 - e^{A(k-\varepsilon'')\theta t^{k-\varepsilon''-1}}}$$

$$\leq K_8\, e^{-A\theta t^{k-\varepsilon''}}$$

for ε" arbitrarily small. Thus $y(t)$ has the correct asymptotic behaviour and the series is absolutely and uniformly convergent, by inspection.

§2 $j(t)$ bounded as $t \to \infty$.

This section deals with the cases in which $\lim_{t\to\infty} j(t)$ exists. We consider first the case $j(t) \to A \neq 0$.

§2.1 $j(t)$ asymptotically constant.

The results in this section are not as strong as those of Chapter 2. Intrinsically, we cannot expect such good results, but the superficial reason is difficulty with the roots of the characteristic equation. If $j(t) \to A$ as $t \to \infty$, then the associated characteristic function is

$$\hat{h}(s) = s - A + A\, e^{s} \tag{4.10}$$

Lemma 44

All zeroes of $s - A + A\, e^{s}$ have Re $s > 0$, except for $s = 0$.

Proof

Let $s = \sigma + it$ be a zero of $\hat{h}(s)$

$$\begin{aligned} \sigma - A + A\, e^{\sigma} \cos t &= 0 \\ t &= -A\, e^{\sigma} \sin t \end{aligned} \tag{4.11}$$

If we assume that $\sigma \leq 0$, we have $e^{\sigma} \cos t \geq 1$. Since $e^{\sigma} \leq 1$ and $\cos t \leq 1$, we thus have $\sigma = 0$ and $\cos t = 1$. From the second equation of (4.11) we find $t = 0$. Thus $\sigma > 0$ except for $s = 0$.

It is clear that the zeroes of large modulus can be ordered,

and that they are spaced along $|s| = \frac{1}{A} e^{s}$ and satisfy

$$|s_m| \sim \pm (2m + \tfrac{1}{2})\pi \qquad m = 0, \pm 1, \ldots.$$

The work of Wright [14] can again be used but the results are not very informative. We define

η = {real parts of the zeroes of (4.9) and their limit points} $\cup$ {$+\infty$}

and use the notation of Chapter 2 §2.1.

Theorem 45

If $y(t)$ is a solution of (4.2) with $j(t) \to A$ as $t \to \infty$, then either

$$\max [\omega(y), \omega(y')] \in \eta$$

or y is the zero solution.

Proof

Wright [14].

As η has no upper bound this theorem tells us little. If we restrict ourselves to the case $j(t) = A$, a more specific result can be obtained.

Theorem 46

Let the zeroes of $s - A + Ae^{s}$ be denoted s_m, $m = 0, \pm 1, \ldots.$ All functions of the form

$$f(t) = \sum_m P_m e^{s_m t}$$

where the P_m are constants, are solutions of

$$y'(t) = -A[y(t + 1) - y(t)]$$

Proof

$f'(t) = \sum_m s_m P_m e^{s_m t}$ $\qquad$ $f(t + 1) = \sum_m e^{s_m} P_m e^{s_m t}$

Thus

$$f'(t) + A\, f(t+1) - A\, f(t) = \sum_m P_m e^{s_m t}[s_m + A\, e^{s_m} - A] = 0$$

§ 2.2 $j(t) \to 0$ as $t \to \infty$.

Results similar to those of § 1 of this chapter can be established when $j(t) \to 0$. The work of § 2.1 gives us one result when we set A to zero. We find that

$$\mathcal{M} = \{0, \infty\}$$

and, from Theorem 45, we find

$$\max\{\omega(y), \omega(y')\} \in \{0, \infty\}$$

Thus the worst case occurs when

$$\max\{\omega(y), \omega(y')\} = \infty$$

i.e. for any $\sigma > 0$, one of $y\, e^{-\sigma t}$ and $y'\, e^{-\sigma t}$ is not of integrable square over $[t_0, \infty)$.

We now prove results analogous to Theorem 43.

Theorem 47

Let $f(t)$ be a Hölder-continuous function (with exponent θ, $1 \geq \theta > 0$) of period 1. Let $j(t)$ be continuous and such that $j(t) = O(t^{-k})$ $\frac{1}{j(t)} = O(t^{\beta})$ $\beta > 0$, then there is a solution $y(t)$ of (4.2) [and, by Theorem 41, only one solution] such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(t^{-k+\theta+2})$$

provided that $k > \theta + 2$

Proof

We define the following series.

$$
\begin{aligned}
y_0(t) &= f(t) \\
y_1(t) &= e^{\int^t j} \int_t^\infty e^{-\int^s j}\, j(s) \left\{ y_0(s) - y_0(t) \right\} ds \\
y_{n+1}(t) &= e^{\int^t j} \int_t^\infty e^{-\int^s j}\, j(s)\, y_n(s+1)\, ds \\
& \qquad\qquad n = 1, 2, \ldots\ldots
\end{aligned}
\tag{4.12}
$$

Inductively, $y_n(t)$ is bounded, and so the definition is valid.

We define

$$
y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{4.13}
$$

and immediately

$$
y'(t) = - j(t) \left\{ y(t+1) - y(t) \right\}
$$

It remains to show that the series for $y(t)$ is absolutely and uniformly convergent and that it has the correct asymptotic behaviour.

We consider $y_1(t)$:-

$$
| y_0(s) - y_0(t) | = | f(s) - f(t) | < B | s - t |^{\Theta} \tag{4.14}
$$

Thus

$$
\begin{aligned}
| y_1(t) | &\leq B \int_t^\infty e^{\int_s^t j}\, j(s)\, (s-t)^{\Theta}\, ds \\
&\leq C \int_t^\infty s^{-k+\Theta}\, ds \\
&\leq D\, t^{-k+\Theta+1}
\end{aligned}
$$

since $k > \Theta + 2$

We continue by induction.

Hypothesis

$$| y_n(t) | \leq D (t + n - 1)^{-k+\theta+1}$$

The hypothesis is true for n = 1. Then, assuming the hypothesis for n,

$$| y_{n+1}(t) | \leq e^{\int^t j} D \int_t^\infty e^{-\int^s j} j(s) (s + n)^{-k+\theta+1} ds$$

$$\leq D (t + n)^{-k+\theta+1}$$

Since $k > \theta + 1$, the series for y(t) is thus absolutely and uniformly convergent.

$$| y(t) - f(t) | \leq \sum_{n=1}^{\infty} D | (t + n - 1)^{-k+\theta+1} |$$

$$\leq D \int_1^\infty (t + x - 1)^{-k+\theta+1} dx$$

$$= \frac{D}{(k - \theta - 2)} t^{-k+\theta+2}$$

Thus y(t) has the correct asymptotic behaviour.

Theorem 48

Let f(t) be as in Theorem 47. Let $j(t) = \exp[-h(t)]$ where h(t) is continuous and $h(t) \sim At^k$ with $A > 0$. Then there is a solution y(t) of (4.2) [and, by Theorem 41, only one solution] such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(e^{-At^{k-\varepsilon}})$$

provided that $k > 0$, for all $\varepsilon > 0$.

Proof

We proceed as in the last theorem to equation (4.14). When s is

'near' t

$$| f(s) - f(t) | < B | s - t |^{\theta}$$

and when s is 'far' from t

$$| f(s) - f(t) | < C$$

since f is periodic and Hölder-continuous.

Thus

$$| y_1(t) | \leq B \int_t^{t+\delta} e^{\int_s^t j} j(s) (s - t)^{\theta} \, ds + C \int_{t+\delta}^{\infty} e^{-\int_t^s j} j(s) \, ds \tag{4.15}$$

for some $0 < \delta < 1$. Now

$$\int_{t+\delta}^{\infty} e^{\int_s^t j} j(s) \, ds = e^{\int_{t+\delta}^t j} - e^{\int_{\infty}^t j}$$

$$\leq \frac{C}{t^{k-1}} e^{-A(t+\delta)^{k-\varepsilon}} \tag{4.16}$$

after some manipulation, while ε is arbitrarily small.

$$\int_t^{t+\delta} e^{\int_s^t j} j(s) (s - t)^{\theta} \, ds = \int_0^{\delta} e^{\int_{t+u}^t j} j(t + u) u^{\theta} \, du$$

$$\leq K e^{-At^{k-\varepsilon'}} \int_0^{\delta} u^{\theta} \, du$$

$$\leq M e^{-At^{k-\varepsilon'}} \tag{4.17}$$

where ε' is arbitrarily small.

Thus from (4.15), (4.16) and (4.17),

$$| y_1(t) | \leq D e^{-At^{k-\varepsilon'}}$$

Inductively,

$$| y_n(t) | \leq D e^{-A(t+n-1)^{k-\varepsilon'}}$$

As before, the series is uniformly and absolutely convergent for $k > 0$. To check the asymptotic behaviour

$$| y(t) - f(t) | \leq \sum_{n=1}^{\infty} D e^{-A(t+n-1)^{k-\varepsilon'}}$$

$$\leq D e^{-t^{k-\varepsilon'}}$$

Hence the result is established.

CHAPTER 5

Behaviour of solutions as $s \to -\infty$ for $g(s) \leq 0$

In this chapter, we consider the behaviour of solutions of

$$w'(s) = g(s)\,[w(s-1) - w(s)] \tag{5.1}$$

as $s \to -\infty$ with $g(s)$ continuous and ≤ 0 for all s. The problem is equivalent to determining the asymptotic behaviour as $t \to +\infty$ of solutions of the advanced equation:-

$$y'(t) = k(t)\,[y(t+1) - y(t)] \tag{5.2}$$

where $k(t)$ is continuous and $k(t) \geq 0 \;\forall\, t$. As in Chapter 4, the equation is advanced and so the strong results of Chapter 3 cannot be expected. We must immediately be specific about the asymptotic behaviour of $k(t)$.

§1 $k(t)$ unbounded as $t \to \infty$.

In this section we show that, if $k(t)$ is unbounded at ∞, then there is a basic set of solutions which decay as $\exp[-\int^t k]$ and that all solutions which are $o(1)$ are multiples of this basic set. We also find that, given any Hölder-continuous function $f(t)$ of period 1, then, in general, there is a solution of (5.2) which behaves like $f(t)$ at infinity. This solution is unique up to the addition of constant multiples of members of the basic set. It makes computation easier if we continue to consider separately the two cases:- $k(t)$ algebraic and exponential algebraic.

§ 1.1 k(t) asymptotically algebraic in t

Theorem 49

If $k(t) \sim M\, t^{\alpha}$, $M > 0$, $\alpha > 0$ and $k(t)$ is continuous, then there is a solution of (5.2) that decays as $L\, e^{-\int^t k}$.

Proof

We construct such a solution:-

Define

$$y_0(t) = L\, e^{-\int^t k}$$

$$y_{n+1}(t) = -\, e^{-\int^t k} \int_t^{\infty} e^{\int^s k}\, k(s)\, y_n(s+1)\, ds \qquad (5.3)$$

$$n = 0, 1, 2, \ldots$$

Thus

$$y_{n+1}'(t) = k(t)\,[y_n(t+1) - y_{n+1}(t)] \qquad (5.4)$$

Inductively, $y_n(t)$ is bounded and so the definition of $y_{n+1}(t)$ is valid. We let

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

and, summing (5.4) over n shows that $y(t)$ is a solution of (5.2). It remains to show that this series for y is absolutely and uniformly convergent and that $y(t)$ has the correct asymptotic behaviour.

Consider $y_1(t)$

$$y_1(t) = -\, L\, e^{-\int^t k} \int_t^{\infty} k(s) \exp[-\int_s^{s+1} k]\, ds$$

Thus

$$| y_1(t) | \leq A\, e^{-\int^t k}\, t^{1+\varepsilon}\, e^{-Mt^{\alpha-\varepsilon}}$$

after some manipultation; here ε is arbitrarily small.

We make the induction hypothesis on y_n that:-

$$| y_n | \leq \frac{A^n}{n!} e^{-\int^t k - nM t^{\alpha-\varepsilon}} t^{n(1+\varepsilon)} \qquad \forall \varepsilon > 0.$$

This holds for n = 1; assume that it holds for n. Then, from (5.3),

$$| y_{n+1} | \leq e^{-\int^t k} \int_t^\infty e^{\int^s k} k(s) \frac{A^n}{n!} e^{-\int^{s+1} k - nM(s+1)^{\alpha-\varepsilon}} (s+1)^{n(1+\varepsilon)} ds$$

We proceed by integrating by parts several times to obtain

$$| y_{n+1} | \leq \frac{A^{n+1}}{(n+1)!} e^{-\int^t k - (n+1)Mt^{\alpha-\varepsilon}} t^{(n+1)(1+\varepsilon)}$$

and the induction is established. The series is absolutely and uniformly convergent.

$$| y(t) - y_o(t) | \leq \sum_{n=1}^{\infty} | y_n(t) |$$

$$\leq \sum_{n=1}^{\infty} \frac{A^n}{n!} t^{n(1+\varepsilon)} e^{-nMt^{\alpha-\varepsilon} - \int^t k}$$

$$\leq B e^{-\int^t k + At^{1+\varepsilon}} e^{-Mt^{\alpha-\varepsilon}}$$

Thus

$$y(t) \sim L e^{-\int^t k} [1 + O(e^{At^{1+\varepsilon}} e^{-Mt^{\alpha-\varepsilon}})]$$

and y(t) has the correct asymptotic behaviour. We denote these solutions by $y_L(t)$.

Lemma 50

If y(t) is a solution of (5.2) and if $k(t) \sim Mt^\alpha$ $\alpha > 1$, $M > 0$ and k(t) is continuous, and if $y(t) = o(1)$ then $y(t) = O(e^{-\int^t k})$.

Proof

From (5.2), we find

$$y(t) = e^{-\int_{t_0}^{t} k} \int_{t_0}^{t} k(s)\, y(s+1)\, e^{\int_{s}^{t} k}\, ds + e^{-\int_{t_0}^{t} k}\, y(t_0) \tag{5.5}$$

where $t \geq t_0 \geq 0$. Define

$$K(R) = \sup_{t \geq R} |\, y(t)\, |$$

for each $R \geq 0$. $K(R)$ decreases to 0 as $R \to \infty$ since $y(t) = o(1)$.

$$|\, y(t)\, | \leq K(t_0)\, e^{-\int_{t_0}^{t} k} + e^{-\int_{t_0}^{t} k}\, K(t_0+1) \int_{t_0}^{t} k(s)\, e^{\int_{s}^{t} k}\, ds$$

$$\leq K(t_0)\, e^{-\int_{t_0}^{t} k} + K(t_0+1) \tag{5.6}$$

If $t \geq t_1 \geq t_0 \geq 0$, then, taking the supremum of (5.6) for $t \geq t_1$, we find that

$$K(t_1) \leq C\, e^{-\int_{t_0}^{t_1} k} + K(t_0+1)$$

for some constant C. Put $t_1 = t_0 + \frac{1}{2}$

$$K(t_1) \leq C\, e^{-\int_{t_1-\frac{1}{2}}^{t_1} k} + K(t_1 + \tfrac{1}{2}) \tag{5.7}$$

for $t_1 \geq \frac{1}{2}$.

Thus

$$K(t_1 + \tfrac{1}{2}) \leq C\, e^{-\int_{t_1}^{t_1+\frac{1}{2}} k} + K(t_1+1) \tag{5.8}$$

and so, using (5.8) in (5.7),

$$K(t_1) \leq C\, e^{-\int_{t_1-\frac{1}{2}}^{t_1} k} + C\, e^{-\int_{t_1}^{t_1+\frac{1}{2}} k} + K(t_1+1)$$

We make the induction hypothesis

$$K(t_1) \le C \sum_{n=1}^{m} \exp\Big[-\int_{t_1 + n/2 - 1}^{t_1 + n/2 - 1/2} k(s)\, ds\Big] + K(t_1 + \tfrac{m}{2}) \qquad (5.9)$$

for $m = 1, 2, \ldots$ It is true for $m = 1,2$ and we assume that it holds for m. Thus replacing t_1 by $t_1 + \frac{1}{2}$ in (5.9) we find

$$K(t_1 + \tfrac{1}{2}) \le C \sum_{n=1}^{m} \exp\Big[-\int_{t_1 + \frac{n}{2} - \frac{1}{2}}^{t_1 + n/2} k\Big] + K(t_1 + \tfrac{m+1}{2}) \qquad (5.10)$$

and using (5.10) in (5.7) we find

$$K(t_1) \le C\, e^{-\int_{t_1 - \frac{1}{2}}^{t_1} k} + C \sum_{n=1}^{m} \exp\Big[-\int_{t_1 + \frac{n-1}{2}}^{t_1 + \frac{n}{2}} k\Big] + K(t_1 + \tfrac{m+1}{2})$$

$$K(t_1) \le C \sum_{n=1}^{m+1} \exp\Big[-\int_{t_1 + \frac{n}{2} - 1}^{t_1 + \frac{n}{2} - \frac{1}{2}} k\Big] + K(t_1 + \tfrac{m+1}{2})$$

and the induction is established. Letting $m \to \infty$ in (5.9), we find

$$K(t_1) \le C \sum_{n=1}^{\infty} \exp\Big[-\int_{t_1 + \frac{n}{2} - 1}^{t_1 + \frac{n}{2} - \frac{1}{2}} k(s)\, ds\Big]$$

Thus

$$K(t_1) \le C\, e^{-\int_{t_1 - \frac{1}{2}}^{t_1} k(s)\, ds} [1 + O(e^{-Ma\, t_1^{a-1-\varepsilon}/4})]$$

for all $\varepsilon > 0$.

We now repeat the above argument, using the estimate obtained above to improve on the bound for $K(t_0)$ used in (5.6). Thus

$$K(t_1) \le C\, e^{-\int_{t_0 - \frac{1}{2}}^{t_1} k} [1 + O(e^{-Ma\, t_0^{a-1-\varepsilon}/4})] + K(t_0 + 1)$$

Repeating the argument we find

$$K(t_1) \le C \sum_{n=1}^{m} \exp\Big[-\int_{t_1 + \frac{n}{2} - \frac{3}{2}}^{t_1 + \frac{n}{2} - \frac{1}{2}} k(s)\, ds\Big][1 + O(e^{-Ma\, t_1^{a-1-\varepsilon}/4})]$$

$$+ K(t_1 + \tfrac{m}{2})$$

And so, letting $m \to \infty$, we have

$$K(t_1) \leq C\, e^{-\int_{t_1-1}^{t_1} k(s)\,ds} [1 + O(e^{-M\alpha\, t_1^{\alpha-1-\varepsilon}/4})][1 + O(e^{-M\alpha\, t_1^{\alpha-1-\varepsilon/2}})$$

Thus, repeating the entire argument r times we find

$$K(t_1) \leq C \exp[-\int_{t_1-\frac{r}{2}}^{t_1} k(s)\,ds] \prod_{\lambda=1}^{r} [1 + O(e^{-M\lambda\alpha\, t_1^{\alpha-1-\varepsilon}/4})]$$

The O-term is uniform in λ and t_1 and so, letting $r \to \infty$, since $\prod_{\mu=1}^{\infty} (1 + e^{-\mu})$ exists, we have

$$K(t_1) \leq D\, e^{-\int^{t_1} k(s)\,ds}$$

and thus

$$y(t) = O(e^{-\int^{t} k})$$

We can now show that solutions which are o(1) are a multiple of $y_L(t)$.

Theorem 51

If $y(t)$ is a solution of (5.2) where $k(t) \sim M\, t^{\alpha}$ $M > 0$ and $\alpha > 1$ and if $k(t)$ is continuous, and if $y(t) = o(1)$ then $y(t)$ is a multiple of $y_L(t)$.

Proof

We first show that

$$\gamma = \lim_{t \to \infty} e^{\int^{t} k}\, y(t)$$

exists. From (5.2) and using Lemma 50, we have

$$e^{\int^{t} k}\, y(t)\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} k(s)\, y(s+1)\, e^{\int^{s} k}\, ds$$

$$= O\Big(\int_{t_1}^{t_2} k(s) \exp\Big[- \int_{S}^{S+1} k \Big] ds \Big)$$

Thus

$$e^{\int^{t} k}\, y(t)\Big|_{t_1}^{t_2} = O\Big(\int_{t_1}^{t_2} s^{\alpha} e^{-Ms^{\alpha-\varepsilon}} ds \Big)$$

for all $\varepsilon > 0$. Given $\varepsilon' > 0$ such that $\alpha - \varepsilon > \varepsilon' > 0$, there exists S such that $s > S$

$$\Rightarrow \quad e^{-Ms^{\alpha-\varepsilon}} s^{\alpha} < e^{-Ms^{\alpha-\varepsilon-\varepsilon'}} s^{\alpha-\varepsilon-\varepsilon'-1}$$

Thus, for $t_1, t_2 > S$

$$e^{\int^{t} k}\, y(t)\Big|_{t_1}^{t_2} = O\Big(\int_{t_1}^{t_2} s^{\alpha-\varepsilon-\varepsilon'-1} e^{-Ms^{\alpha-\varepsilon-\varepsilon'}} ds \Big)$$

$$= O\Big(e^{-Ms^{\alpha-\varepsilon-\varepsilon'}} \Big|_{t_1}^{t_2} \Big)$$

which is clearly arbitrarily small.

Thus, by the Cauchy principle,

$$\gamma = \lim_{t\to\infty} e^{\int^{t} k}\, y(t)$$

exists. We now show that

$$y(t) = \frac{\gamma}{L}\, y_L(t)$$

Let $z(t) = y(t) - \frac{\gamma}{L}\, y_L(t)$. Thus

$$z(t) = o\Big(e^{-\int^{t} k} \Big)$$

Using (5.2) we find

$$z(t)\, e^{\int^{t} k} = - \int_{t}^{\infty} k(s)\, z(s+1)\, e^{\int^{s} k}\, ds$$

For each $R > 0$, we define

$$K_1(R) = \sup_{t \geq R} e^{\int^{t} k}\, |\, z(t)\, |$$

$K_1(R) \downarrow 0$ as $R \to \infty$.

$$| z(t)\, e^{\int^t k} | \leq K_1(R+1) \int_t^\infty k(s) \exp[-\int_s^{s+1} k]\, ds \qquad (5.11)$$

As in the proof of Theorem 49,

$$\int_\xi^\infty k(s) \exp[-\int_s^{s+1} k]\, ds \leq A\, t^{1+\varepsilon}\, e^{-Mt^{a-\varepsilon}}$$

for ε arbitrarily small. Taking the supremum of (5.11) for $t \geq R$, we find, for R sufficiently large,

$$K_1(R) \leq K_1(R+1)\, A\, R^{1+\varepsilon}\, e^{-MR^{a-\varepsilon}}$$

Thus

$$K_1(R) \leq A^2[R(R+1)]^{1+\varepsilon}\, e^{-M[R^{a-\varepsilon} + (R+1)^{a-\varepsilon}]}\, K_1(R+2)$$

Repeating this, we find

$$K_1(R) \leq A^n[\frac{(R+n-1)!}{(R-1)!}]^{1+\varepsilon}\, e^{-M[R^{a-\varepsilon} + (R+1)^{a-\varepsilon} + \ldots + (R+n-1)^{a-\varepsilon}]}\, K_1(R+$$

Letting $n \to \infty$ we have

$$K_1(R) = 0$$

$$\Rightarrow \quad z \equiv 0$$

$$\Rightarrow \quad y(t) = \frac{Y}{L}\, y_L(t)$$

Hence $y(t)$ is a multiple of $y_L(t)$.

Theorem 52

Let $f(t)$ be a Hölder-continuous function (of exponent θ, $1 \geq \theta > 0$) which has period 1. Then, if $k(t)$ is continuous and $k(t) \sim Mt^a$, there exists $y(t)$, a solution of (5.2) such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(t^{1-\alpha\theta})$$

provided that $\alpha\theta > 1$.

[By Theorem 51, this solution is unique up to the addition of multiples of $y_L(t)$].

Proof

As before, we construct such a solution. We define

$$y_0(t) = f(t)$$

$$y_1(t) = e^{-\int^t k} \int_\rho^t e^{\int^s k} k(s) \{ y_0(s) - y_0(t) \} \, ds$$

$$y_{n+1}(t) = e^{-\int^t k} \int_\rho^t e^{\int^s k} k(s) \, y_n(s+1) \, ds$$

$$n = 1, 2, \ldots$$

Inductively, $y_n(t)$ is bounded and so the definition is valid.

We define

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

and it is easy to show that $y(t)$ satisfies (5.2). It remains to show that the series for $y(t)$ is absolutely and uniformly convergent and that $y(t)$ has the correct asymptotic behaviour.

Consider $y_1(t)$. For $t \geq \rho$

$$| y_1 | \leq e^{-\int^t k} \int_\rho^t e^{\int^s k} k(s) \, (s - t)^\theta \, ds$$

Put $u = t - s$, then for ρ sufficiently large

$$| y_1 | \leq M \int_0^{t-\rho} e^{-But^\alpha} (t - u)^\alpha \, u^\theta \, du \qquad \text{with } B > 0$$

$$\leq \frac{K}{t^{\alpha\theta}}$$

since $\int_0^\infty e^{-x} x^\theta \, dx$ exists.

We take

$$| y_n(t) | \leq \frac{K}{(t + n - 1)^{\alpha\theta}}$$

as an induction hypothesis. We have established that the result is true for $n = 1$. Assume that it holds for n. We have

$$| y_{n+1}(t) | \leq e^{-\int^t k} \int_\zeta^t e^{\int^s k} \frac{k(s)\, K}{(s + n)^{\alpha\theta}} \, ds$$

$$\leq \frac{K}{(t + n)^{\alpha\theta}}$$

after some manipulation.

The induction is established and $y(t)$ is clearly uniformly and absolutely convergent for $\alpha\theta > 1$.

$$| y(t) - y_0(t) | \leq \sum_1^\infty K(t + n - 1)^{-\alpha\theta}$$

$$\leq K \int_1^\infty (t + x - 1)^{-\alpha\theta} \, dx$$

$$= K \frac{t^{1-\alpha\theta}}{1 - \alpha\theta}$$

and thus $y(t)$ has the correct asymptotic behaviour.

§1.2 $k(t) = \exp[h(t)]$, $h(t)$ asymptotically algebraic in t.

We now prove the equivalents of Theorem 49, Lemma 50 and Theorems 51 and 52 for the case when $k(t)$ is exponential algebraic.

Conditions IX

$k(t) = \exp[h(t)]$

$h(t) \sim Mt^{\alpha} \quad M > 0 \quad \alpha > 0$

$h(t)$ is continuous.

Theorem 53

If $k(t)$ satisfies Conditions IX, then there is a solution of (5.2) that decays as $L\, e^{-\int^t k}$.

Proof

We proceed as in Theorem 49, constructing the series (5.3) for $y(t)$ as before. Then

$$| y_1(t) | \leq L\, e^{-\int^t k} \int_t^{\infty} \exp[-\int_s^{s+1} k]\, k(s)\, ds \qquad (5.12)$$

There exists a constant K, such that, for s sufficiently large,

$$| y_1 | \leq \frac{L}{K} e^{-\int^t k} \int_t^{\infty} e^{-\int_s^{s+1} k} [k(s+1) - k(s)]\, ds$$

$$= \frac{L}{K} e^{-\int^{t+1} k}$$

We continue by induction as before, to find

$$| y_n | \leq \frac{L}{n!\, K^n} e^{-\int^{t+n} k}$$

Thus the series is absolutely and uniformly convergent. Now

$$| y(t) - y_o(t) | \leq \sum_{n=1}^{\infty} \frac{L}{n!\ K^n} e^{-\int^{t+n} k}$$

Clearly $y(t)$ has the correct asymptotic behaviour. As before, we denote this solution $y_L(t)$.

Lemma 54

If $y(t)$ is a solution of (5.2) where $k(t)$ satisfies Conditions IX, then if $y(t) = o(1)$ then $y(t) = O(e^{-\int^t k})$.

Proof

We proceed exactly as in Lemma 50, to find

$$y(t) = O(e^{-\int_{t-\frac{1}{2}}^{t} k(s)\, ds})$$

Since $k(t) = \exp[h(t)]$, $h(t) \sim Mt^{\alpha}$ with $\alpha > 0$, we can deduce from this that

$$\dot{y}(t) = O(e^{-\int^t k(s)\, ds})$$

Theorem 55

If $y(t)$ is a solution of (5.2) where $k(t)$ satisfies Conditions IX, and if $y(t) = o(1)$, then $y(t)$ is a multiple of $y_L(t)$.

Proof

$$\gamma = \lim_{t\to\infty} e^{\int^t k}\, y(t)$$

exists, since, from (5.2),

$$e^{\int^t k}\, y(t)\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} k(s)\, y(s+1)\, e^{\int^s k}\, ds$$

$$= O(\int_{t_1}^{t_2} k(s) \exp[-\int_{s}^{s+1} k]\, ds)$$

by Lemma 54.

$$= O(\int_{t_1}^{t_2} e^{Ms^{\alpha-\varepsilon}}\, e^{-e^{Ms^{\alpha-\varepsilon}}}\, ds)$$

$$= O(\int_{t_1}^{t_2} e^{Ms^{\alpha-\varepsilon}}\, s^{\alpha-\varepsilon-1}\, e^{-e^{Ms^{\alpha-\varepsilon}}}\, ds)$$

$$= O(e^{-e^{Ms^{\alpha-\varepsilon}}} \Big|_{t_1}^{t_2})$$

which is clearly arbitrarily small for t_1, t_2 sufficiently large. Thus, by the Cauchy principle, γ exists.

As in Theorem 51, we define

$$z(t) = y(t) - \frac{\gamma}{L}\, y_L(t)$$

$$K_1(R) = \sup_{t \geq R} e^{\int^t k} \mid z(t) \mid$$

From (5.2)

$$z(t)\, e^{\int^t k} = -\int_t^\infty k(s)\, z(s+1)\, e^{\int^s k}\, ds$$

$$\Rightarrow \quad \mid z(t)\, e^{\int^t k} \mid \leq \frac{e^{-\int_t^{t+1} k}}{K} K_1(R+1) \tag{5.13}$$

where K is a constant > 1.

Taking the supremum of (5.13) for $t \geq R$

$$K_1(R) \leq K_1(R+1)\, \frac{e^{-\int_R^{R+1} k}}{K}$$

Thus

$$K_1(R+1) \leq \frac{K_1(R+2)}{K}\, e^{-\int_{R+1}^{R+2} k}$$

$$K_1(R) \leq \frac{K_1(R+2)}{K^2}\, e^{-\int_R^{R+2} k}$$

Repeating this argument,

$$K_1(R) \leq \frac{K_1(R+n)}{K^n}\, e^{-\int_R^{R+n} k}$$

Letting $n \to \infty$, we find

$$K_1(R) = 0$$

and hence $z \equiv 0$ and $y(t)$ is a multiple of $y_L(t)$.

Theorem 56

Let $f(t)$ be a Hölder-continuous function (of exponent $1 \geq \theta > 0$) which has period 1. Then, if $k(t)$ satisfies Conditions IX, there exists $y(t)$, a solution of (5.2) such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(e^{-M \theta t^{a-\varepsilon}})$$

for all $\varepsilon > 0$. [By Theorem 55, this solution is unique up to the addition of multiples of $y_L(t)$.]

Proof

We proceed as in Theorem 52, constructing the sequence $y_n(t)$ for $y(t)$. We find

$$| y_1 | \leq e^{-\int_\rho^t k} \int_\rho^t e^{\int_\rho^s k} k(s) (s - t)^\theta \, ds$$

$$= \int_0^{t-\rho} e^{-\int_{t-u}^t k} k(t - u) \; u^\theta \, du$$

and, for ρ sufficiently large,

$$| y_1 | \leq N e^{Mt^{a+\varepsilon}} \int_0^{t-\rho} e^{-ue^{Mt^{a-\varepsilon}}} u^\theta \, du$$

$$\leq \frac{C}{e^{M\theta t^{a-\varepsilon}}}$$

since $\int_0^\infty e^{-x} x^\theta \, dx$ exists. Proceeding exactly as in Theorem 52, we now find

$$| y_n(t) | \leq \frac{C}{e^{M\theta (t+n-1)^{a-\varepsilon}}}$$

Thus the series for y is absolutely and uhiformly convergent and

$$| y(t) - y_o(t) | \leq C \sum_{n=1}^{\infty} e^{-M\theta (t+n-1)^{a-\varepsilon}}$$

$$\leq N e^{-M\theta t^{a-\varepsilon}}$$

for all $\varepsilon > 0$, and so y has the correct asymptotic behaviour.

§2 k(t) bounded as $t \to \infty$.

This section deals with k(t) bounded. For $k(t) \to A$, the two cases $A \neq 0$, $A = 0$ are considered separately.

§2.1 k(t) asymptotically constant.

If $k(t) \to A > 0$, the associated characteristic equation is

$$\hat{h}(s) = s + A - A e^{s} = 0$$

Clearly, the zeroes of $\hat{h}(s)$ can be ordered and if s_m is a zero of large modulus, it lies on

$$A |s| = e^{s}$$

such that

$$| s_m | \sim \pm (2m + \tfrac{1}{2}) \pi$$

Lemma 57

Let s_m, $m = 0, \pm 1, \pm 2, \ldots$, be the zeroes of $\hat{h}(s) = s + A - A e^{s}$ then $\operatorname{Re} s_m > -A \quad \forall \ m$. If $A \leq 1$, this bound can be sharpened to

$$\operatorname{Re} s_m > -(1 - e^{-A}) A \qquad \forall \ m$$

Proof

The proof follows from Lemma 31 by replacing s by - s.

We define $\mathcal{M}$ as

$\{$real parts of zeroes of $\hat{h}(s)$ and their limit points$\}$ $\cup$ $\{+\infty\}$

Wright [14] gives

Theorem 58

If $y(t)$ is a solution of (5.2) with $k(t) \to A > 0$ as $t \to \infty$ then either

$$\max[\omega(y), \omega(y')] \in \mathcal{M}$$

or y is the zero solution. [Notation as in Theorem 19.]

Thus, in the worst possible case,

$$\max[\omega(y), \omega(y')] = \infty$$

i.e. one of $y\, e^{-\sigma t}$ and $y'\, e^{-\sigma t}$ is not of integrable square over (t_0, ∞) for any $\sigma > 0$.

If we restrict to the case $k(t) = A$, it is clear that any function of the form

$$\sum_m P_m e^{s_m t}$$

is a solution of (5.2); here the P_m are constants and the s_m are such that $\hat{h}(s_m) = 0$.

§ 2.2 $k(t) \to 0$ as $t \to \infty$.

The results for this case which are obtained by putting A = 0 in § 2.1, are covered by the remarks in Chapter 4, § 2.2. A result analogous to Theorem 52 holds.

Theorem 59

Let $f(t)$ be a Hölder-continuous function (with exponent $1 \geq \theta > 0$) of period 1. If $k(t) = O(t^{-\alpha})$ $1/k(t) = O(t^{\beta})$ $\alpha, \beta > 0$ with $k(t)$ continuous, then there is a solution $y(t)$ of (5.2) such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(t^{-\alpha+\theta+2})$$

provided that $\alpha > \theta + 2$.

Proof

Define

$$y_0(t) = f(t)$$

$$y_1(t) = -e^{-\int_t^t k} \int_t^\infty e^{\int_t^s k} k(s) \{ y_0(s) - y_0(t) \} \, ds$$

$$y_{n+1}(t) = -e^{-\int_t^t k} \int_t^\infty e^{\int_t^s k} k(s) \, y_n(s+1) \, ds$$

$$n = 1, 2, \ldots$$

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

The proof follows that of Theorem 47 to obtain

$$| y_1(t) | \leq C \, t^{-\alpha+\theta+1}$$

and

$$| y_n(t) | \leq C \, (t + n - 1)^{-\alpha+\theta+1}$$

and thence the result.

Theorem 60

Let $f(t)$ be as in Theorem 59. If $k(t) = \exp[-h(t)]$ $h(t) \sim Mt^{\alpha}$ $M > 0$, $\alpha > 0$, $h(t)$ is continuous, then there is a solution $y(t)$ of (5.2) such that

$$y(t) = f(t) + z(t)$$

where

$$z(t) = O(e^{-Mt^{\alpha-\varepsilon}})$$

for all $\varepsilon > 0$.

Proof

The sequence $\{ y_n(t) \}$ is defined as in the proof of Theorem 59. The proof then follows that of Theorem 48 to give

$$| y_1(t) | \leq C e^{-Mt^{\alpha-\varepsilon}}$$

and

$$| y_n(t) | \leq C e^{-M(t+n-1)^{\alpha-\varepsilon}}$$

for all $\varepsilon > 0$, and thence the result.

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$s \to +\infty$ — $w'(s) = g(s)\,[\,w(s-1) - w(s)\,]$ — $g(s)$ is continuous

Properties	$\lvert w(s)\rvert \le M$	$\lvert w^{(n)}(s)\rvert \le M$	$\lim_{s\to\infty} w(s)$ exists	$w^{(n)}(s) \to f^{(n)}(s)$	uniqueness	$w(s) = O(e^{-\int_s^s g})$	if $w(s) = o(e^{-\int_s^s g})$ then $w(s) = o(1)$ and $\exists$ sol^n $w(s) \sim L e^{-\int_s^s g}$	$\exists$ representation $\forall$ solutions	$\max[w(w), w(w')] \in \mathcal{W}$ or $w \equiv 0$
$g(s) \ge 0$ Always	I or III or IV	I + $\beta>1$ or III or $\frac{1}{g(s)} \ge \frac{1}{\log s}$	I + $\beta>2$ or III	II			$g(s) = A$ or $g(s)$ periodic period $= 1/\text{integer}$	$g(s) \to A$ or $g(s) \to 0$
$g(s) \le 0$ VIII	VIII + $g^{(n)}(s) = O(1)$			II	V or VII	VI or VII	$g(s) = -A$ or $g(s)$ periodic period $= 1/\text{integer}$	$g(s) \to -A$ or $g(s) \to 0$

$t \to +\infty$ — $y'(t) = j(t)\,[\,y(t) - y(t+1)\,]$ — $j(t)$ is continuous

Properties	$\exists$ sol^n $y(t) \sim f(t)$	$\exists$ sol^n $y_L(t)$ s.t	if $y(t) = o(1)$	$\max[w(y), w(y')] \in \mathcal{W}$
uniqueness	$f(t)$ any Holder-cts fn, expt θ, $1 \ge \theta > 0$, of period 1	$y_L(t) \sim L\, e^{\int^t j}$	then $y(t) = M y_L(t)$	or $y \equiv 0$
$j(t) \ge 0$ Always	$j(t) = O(t^k)$ $[j(t)]^{-1} = O(t^{-\beta})$ with either i) $\beta(\theta+1) > k+1$ $k>0$ or ii) $k < -\theta-2$ or $j(t) = \exp h(t)$ $h(t) \sim At^k$ with either i) $A>0$ or ii) $A<0$			$j(t) \to A$ $j(t) \to 0$
$j(t) \le 0$	$j(t) \sim M t^\alpha$ $\alpha\theta > 1$ or $j(t) = -\exp[-h(t)]$ $h(t) \sim At^k$ $A>0$ $k>0$ or IX or $j(t) = O(t^{-k})$ $[j(t)]^{-1} = O(t^\beta)$ $k > \theta+2$	$j(t) \sim M t^\alpha$ $\alpha>0$ or IX	$j(t) \sim M t^\alpha$ $\alpha>1$ or IX	$j(t) \to -A$ $j(t) \to 0$

Roman numerals refer to Condition numbers.