

# Liquidity and Asset Prices

Raphael A. Espinoza<sup>§</sup> and Dimitrios P. Tsomocos<sup>\*</sup>

July 2008

## Abstract

We show in an exchange economy with liquidity constraints that the volume of trade and asset prices depend on both the supply of liquidity by the Central Bank and on the liquidity of assets and commodities. As a result, monetary aggregates are informative for the assessment of economic developments and the conduct of monetary policy. We also show that the positive correlation between state prices and the future spot rate generates a risk-premium in the term structure of interest rates, even in absence of aggregate uncertainty. These results do not obtain in representative agent models but hold in any monetary economy with heterogeneous agents and short-term liquidity effects, where monetary costs act as transaction costs and the quantity theory of money is verified.

**Keywords:** liquidity ; cash-in-advance constraints ; term structure of interest rates

**JEL Classification:** E43; G12

---

<sup>§</sup>Department of Economics and Christ Church, University of Oxford, email: raphael.espinoza@chch.oxon.org

<sup>\*</sup>Saïd Business School and St. Edmund Hall, University of Oxford

We are grateful for suggestions and comments from Valpy FitzGerald, John Geanakoplos, Charles Goodhart, Pete Kyle, Herakles Polemarchakis, Tano Santos, Oren Sussman, and from participants at the 3rd Annual CARESS-Cowles Conference, the 2007 European Winter Econometric Society Meeting, the Royal Economic Society 2007 Meeting, the XVIth European Workshop on General Equilibrium, the 5th INFINITI Conference, the 2007 EEA-ESEM Congress and seminars at Oxford and the Bank of England.

# 1 Introduction

We argue in this paper that the correlation between monetary costs and real asset payoffs in monetary models creates risk-premia in asset prices. Monetary costs generate a wedge between selling and buying prices in models where money is needed for trade, and consequently affect marginal utilities and equilibrium prices. Unlike in representative agent models, this premium exists even in absence of aggregate uncertainty (*i.e.*, when the aggregate endowment is not state-dependent), and therefore causes the term structure to lie above levels predicted by the pure expectation hypothesis.

The risk-premia generated exist in addition to the ones derived from the representative agent Euler equation, and studied in Lucas (1978), Breeden (1979), Mehra and Prescott (1985), Backus *et al.* (1989), or Grossman *et al.* (1987). This is because the risk-premia are due to the funding transaction costs that affect individuals' marginal utilities, though not the (aggregated) representative agent's utility.

We set out a monetary equilibrium with cash-in-advance constraints built along the lines of Dubey and Geanakoplos (1992, 2003, 2006). We assume complete markets since our focus is on the consequences of the monetary structure. The model transcends representative agent models in several ways. First, trade activity is endogenous - in fact, trade activity is determined by liquidity. Recall that representative agent models can be thought of as "sell-all" models, thereby determining trade *a priori*. These models have been extended by adding money (as in Lucas, 1990, and Lucas and Stokey, 1983, 1987); however, in these papers money plays a role only through income and expectations, and not through trade.

In our model trade is affected by liquidity and the monetary costs generate a strong correlation between asset payoffs and spot interest rates. We are thus able to calculate explicitly a "liquidity-risk premium" in the term structure. Bansal and Coleman II (1996) also produce a general equilibrium where transaction costs have an effect on bond prices, however, in their representative agent model, trade is forced, since the representative agent sells all of her endowments and subsequently buys it back. Furthermore, the transaction technology is exogenously specified and transaction services are generated only from bond holdings and not from asset holdings. Other papers such as Fan (2006) or Chabi-Yo *et al.* (2007) show how uninsurable risk and agent heterogeneity add an additional factor in pricing models. In our model, although asset markets are complete, monetary transaction costs add a further uninsurable risk that matters for asset prices.

We also argue that liquidity, because it is a broad concept, can not be fully described by the supply of money from the Central Bank. The word 'liquidity' corresponds to two dif-

ferent concepts in this paper. First, there is the supply of cash by the Central Bank, or by the banking system in general. Second, liquidity may also refer to the easiness with which commodities and assets are traded. Extending Grandmont and Younes (1972), we assume in the cash-in-advance constraints that a share  $\Lambda$  of the receipts of an agent’s sales can be used simultaneously to purchase assets or commodities. The (exogenous) coefficients  $\Lambda$ s capture therefore the liquidity of these assets or commodities, or in a keynesian fashion, the relative illiquidity of money. The share  $\Lambda$  can also be understood as a continuous version of the distinction between “cash goods” (when  $\Lambda = 0$ ) and “credit goods” (when  $\Lambda = 1$ ).

In our model the Central Bank policy has no effect on the liquidity of endowments, although it does affect the spot interest rate and therefore the funding cost of liquidity constraints. This assumption allows us to clearly separate the effects of liquidity that are channeled through interest rates from the direct effects of liquidity on the way trade is conducted. Demand for money becomes a function of two parameters: the spot interest rate and the liquidity of assets and endowments. Consequently, the model is able to show how trade and asset prices depend on the two concepts of liquidity. This result would hold in any monetary economy where short-run liquidity effects exist, and monetary costs act as transaction costs.

Furthermore, the model suggests that monetary aggregates do provide additional information (on economic activity, inflation and asset prices) that is not included in interest rates. This is because inflation may result from increases in money supply but also from increases in commodities and assets’ liquidity. Hence, our monetary economy has a distinct monetary flavour which cannot be reproduced in representative agent models.

The paper proceeds as follows: section 2 presents the model. Section 3 derives some properties of the monetary equilibrium, while the main results on trade and asset prices are discussed in section 4. Section 5 presents a closed-form solution in the Constant Relative Risk-Aversion case. Finally, section 6 concludes.

## 2 The Baseline Model

### 2.1 Cash-in-advance Constraints

Cash-in-advance models<sup>1</sup> aim to capture the importance of liquidity for transactions. There are many versions of cash-in-advance models in the monetary theory literature (e.g. Lucas and Stokey, 1983 and 1987; Svensson, 1985; Bloise, Drèze and Polemarchakis, 2005). We follow the model of Dubey and Geanakoplos (1992) in which, to ensure a positive nominal interest rate, a sufficient requirement is that agents hold some exogenous monetary endow-

---

<sup>1</sup>The modern treatment of cash-in-advance models dates as far back as Clower (1967).

ment. If the aggregate private monetary endowment is  $m$ , then in a one-period version of the model the short-term nominal interest rate is  $r = \frac{m}{M}$ , where  $M$  is the supply of money by the Central Bank. Hence a short-term liquidity effect is obtained in a very simple way since  $r$  is decreasing in  $M$ . Although an exogenous endowment of money can be justified in a one-period model<sup>2</sup>, this assumption is harder to explain in a multi-period setting. Dubey and Geanakoplos (2006), following Gurley and Shaw's (1960) initial statement of the difference between inside and outside money, argue that outside money is indeed a reality: for instance, when money is printed by the government to purchase real assets, commodities, or to pay for labour, it gives money to the private sector free and clear of any liability, and independently from Central Bank lending. One can also think of outside money as a compact simplification for a more general nominal friction that pins down the price of money. Default on the money market plays, for instance, the same role as outside money to ensure the existence of a positive interest rate (Shubik and Wilson, 1977; Shubik and Tsomocos, 1992; Espinoza *et al.*, 2008). In any case, what matters for the main results is the existence of the liquidity effect rather than its origins.

Positive interest rates generate both nominal determinacy and money non-neutrality. First, any model with a monetary transaction technology (*i.e.*, with an essential role for money) removes nominal indeterminacy because scaling prices up or down necessarily requires some monetary injections or withdrawals which will, in turn, alter interest rates.<sup>3</sup> Hence, for a given interest rate there can be only one price vector.<sup>4</sup> Second, nominal determinacy implies monetary non-neutrality because any non-symmetric change in either money supply, outside money or default, alters the monetary transaction cost of borrowing money. This, in turn, changes the ratio of marginal utilities of buying and selling commodities, resulting in different equilibrium allocations.

This class of models generates inefficient allocation because transaction costs create a wedge between selling and buying prices. In this setting, the optimal quantity of money that reestablishes efficiency is infinite, so as to produce zero interest rates.<sup>5</sup>

A common criticism of the cash-in-advance models is that the constraints used are *ad hoc* and do not adequately capture liquidity or collateral requirements. General specifications of cash-in-advance constraints can, however, partly answer this critique. If  $q_1$  and  $q_2$  are con-

---

<sup>2</sup>Outside money may be inherited from previous periods and free from any debt requirement outstanding.

<sup>3</sup>The only exception is when interest rates are equal to zero. Then changing prices and money supply is tantamount to changing units of account while maintaining zero interest rates.

<sup>4</sup>See Dubey and Geanakoplos (2006) and Tsomocos (2008) for a formal proof.

<sup>5</sup>For this to be true it is necessary for inflation to be costless. The Lucas Parable and Calvo-pricing are two ways to generate costly distortion of relative prices because of inflation but this case is outside the scope of this paper.

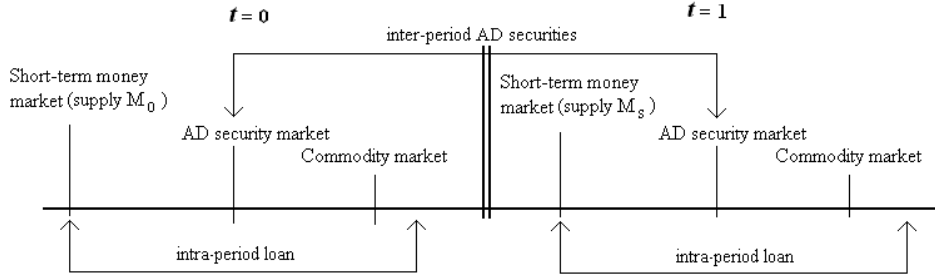


Figure 1: Timing of commodity and money markets

sumption levels in commodities 1 and 2,  $p_1$  and  $p_2$  are commodity prices, a general form of the cash-in-advance budget constraint is

$$p_1 q_1 + p_2 q_2 \leq p_1 \Lambda_1 e_1 + p_2 \Lambda_2 e_2$$

where  $e_1$  and  $e_2$  are commodity endowments and  $\Lambda_1$  and  $\Lambda_2$  liquidity parameters. The standard use of cash-in-advance constraint has been to set  $\Lambda_1 = 0$  and  $\Lambda_2 = 1$  with  $q_2 = 0$ . Hence, good 1 is the illiquid consumption good, whereas good 2 is fiat money (*i.e.*, a perfectly liquid commodity that cannot be consumed). However it is more realistic to assume that the different commodities are not all equally liquid. As long as commodities do not have the same liquidity, (otherwise, the budget constraints collapse to the standard budget constraints), money - or credit - demand is positive in order to bridge the gap between expenditures and receipts (Grandmont and Younes, 1972). Furthermore, we will show that although the liquidity parameters do not affect the link between money supply and the short-term interest rates, they do affect the way trade is conducted and therefore asset prices.

## 2.2 Structure of the Model

The model consists of two agents,  $\alpha$  and  $\beta$ , living for two periods in a complete markets economy. The first period is indexed by the subscript 0, whereas in the second period, the different states of nature are indexed by  $s \in \mathcal{F} \equiv \{1 \dots S\}$ . We also define the set  $\tilde{\mathcal{F}} \equiv \{0\} \cup \mathcal{F}$ .

The sequence of markets is shown in figure 1. The money market meets first, allowing agents to borrow cash (intra-period loans) from the Central Bank. The securities market meets next allowing agents to trade securities (inter-period loans). We assume that all the Arrow securities, paying in nominal terms, are available. The commodities market meets last and therefore money received for sales cannot be used instantaneously to purchase commodities. Finally, the money market closes forcing agents to repay the Central Bank.

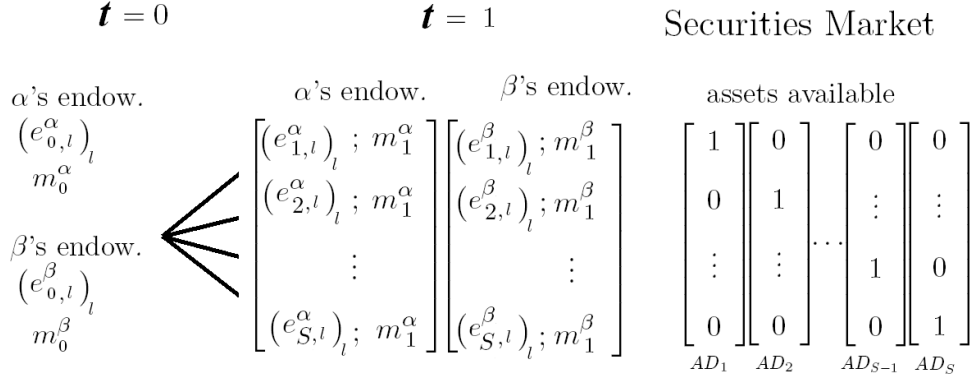


Figure 2: Time and Uncertainty Structure of the Model

Agents consume and trade  $L$  commodities indexed by  $l \in L \equiv \{1 \dots L\}$  in both periods and in all states of nature. Preferences are additively separable in commodities, periods and states of nature. The subjective probabilities of the state  $s$  occurring in the future will be written as  $\pi_s^i$ ,  $i \in \{\alpha \dots \beta\}$ . Without loss of generality, we assume that the future is not discounted.<sup>6</sup> The two agents are endowed with  $e_{s,l}^i$ ,  $s \in \tilde{\mathcal{F}}$ ,  $i \in \{\alpha, \beta\}$  and with monetary endowments free of any liability (outside money)  $m_s^i$ ,  $s \in \tilde{\mathcal{F}}$ ,  $i \in \{\alpha, \beta\}$ . Without affecting the results of the model we assume that these monetary endowments are non-stochastic, and write, therefore,  $\forall i, \forall s \neq s' \in \tilde{\mathcal{F}}, m_s^i = m_{s'}^i = m_1^i$ . The endowment structure is described in figure 2.

### 2.3 Maximisation Problem

We assume that agents live two periods and value the two periods and different states of nature with a Von Neumann-Morgenstern utility function. For an agent  $i \in \{\alpha, \beta\}$  and a commodity  $l$ , let  $u_{l,i}$  be utility derived from the consumption of good  $l$ ,  $e_l^i$  her commodity endowment, and  $m_s^i$ ,  $s \in \tilde{\mathcal{F}}$  her endowment in fiat money free of any liability. We also assume that utility is additively separable and strictly quasi-concave in the different consumption goods. We note  $L_s^i(+)$  the set of goods in state  $s$  that agent  $i$  will sell (so that consumption in that commodity is  $e_l^i - q_l^i$ ), and  $L_s^i(-)$  is the set of goods that agent  $i$  will buy (so that consumption will be  $e_l^i + q_l^i$ ). The price of commodity  $l$  in state  $s$  is written  $p_{s,l}$ . With positive transaction costs, Dubey and Geanakoplos (2006) show that there are no wash sales (no-one would both buy and sell the same commodity), *i.e.*  $\forall s \in \tilde{\mathcal{F}}, L_s^i(+)$  and  $L_s^i(-)$  form a partition of  $L$ . Furthermore, since we have only two agents the commodities sold by  $\alpha$  are the ones bought by  $\beta$  and *vice-versa*, *i.e.*  $L_s^\alpha(+)$  =  $L_s^\beta(-)$  and  $L_s^\alpha(-)$  =  $L_s^\beta(+)$ .

<sup>6</sup>Subjective probabilities can be adjusted to take into account the subjective discount rate.

Asset trade, represented by  $h_s^i$ , may be positive or negative, but all Arrow securities are in zero net supply. Markets are complete in the sense that there are  $S$  independent Arrow securities, indexed by  $s \in \{1 \dots S\}$  paying one unit of account to the owner if state  $s$  occurs, and 0 otherwise. The price of such an asset is  $\theta_s$ .

In each state  $s \in \bar{\mathcal{F}}$ , there is also a money market for intra-period loans. The money supply from the Central Bank is  $M_s$ . The money demand from agent  $i$  is  $\mu_s^i$  and the equilibrium bond price is  $\eta_s = \frac{1}{1+r_s}$  where  $r_s$  is the intra-period interest rate.

Let us write  $\mathcal{F}^i(+)$  the set of states of nature for which agent  $i \in \{\alpha, \beta\}$  buys Arrow securities, and define  $\mathcal{F}^i(-)$  the set of states of nature for which agent  $i$  sells Arrow securities.<sup>7</sup> Since we will study only interior equilibria where all markets are active, we have  $\mathcal{F}^i(+)\cup\mathcal{F}^i(-)=\mathcal{F}$ . With our two-agent structure,  $\mathcal{F}^\alpha(+)=\mathcal{F}^\beta(-)$  and  $\mathcal{F}^\alpha(-)=\mathcal{F}^\beta(+)$ . The maximisation problem of agent  $i$  is (in brackets are the Lagrangian multipliers, which will also be used to label the respective equations):

$$\begin{aligned} \max \quad & \sum_{l \in L_0^i(+)} u_{i,l}(e_{0,l}^i - q_{0,l}^i) + \sum_{l \in L_0^i(-)} u_{i,l}(e_{0,l}^i + q_{0,l}^i) \\ & + \sum_{s \in \mathcal{F}} \pi_s \left[ \sum_{l \in L_s^i(+)} u_{i,l}(e_{s,l}^i - q_{s,l}^i) + \sum_{l \in L_s^i(-)} u(e_{s,i}^i + q_{s,i}^i) \right] \end{aligned}$$

s.t.

$$\sum_{l \in L_0^i(-)} p_{0,l} q_{0,l}^i + \sum_{s \in \mathcal{F}^i(-)} \theta_s h_s^i \leq \sum_{l \in L_0^i(+)} \Lambda_{0,l} p_{0,l} q_{0,l}^i + \sum_{s \in \mathcal{F}^i(+)} \theta_s h_s^i + \eta_0 \mu_0^i + m_0^i \quad (\varphi^i)$$

$$\mu_0^i \leq \sum_{l \in L_0^i(+)} (1 - \Lambda_{0,l}) p_{0,l} q_{0,l}^i \quad (\xi^i)$$

$$\forall s \in \mathcal{F}^i(+), \quad \sum_{l \in L_s^i(-)} p_{s,l} q_{s,l}^i + h_s^i \leq \sum_{l \in L_s^i(+)} \Lambda_{s,l} p_{s,l} q_{s,l}^i + \eta_s \mu_s^i + m_s^i \quad (\Psi_s^i)$$

$$\forall s \in \mathcal{F}^i(+), \quad \mu_s^i \leq \sum_{l \in L_s^i(+)} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l}^i \quad (\chi_s^i)$$

$$\forall s \in \mathcal{F}^i(-), \quad \sum_{l \in L_s^i(-)} p_{s,l} q_{s,l}^i \leq \sum_{l \in L_s^i(+)} \Lambda_{s,l} p_{s,l} q_{s,l}^i + \lambda_s h_s^i + \eta_s \mu_s^i + m_s^i \quad (\Psi_s^i)$$

$$\forall s \in \mathcal{F}^i(-), \quad \mu_s^i \leq \sum_{l \in L_s^i(+)} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l}^i + (1 - \lambda_s) h_s^i \quad (\chi_s^i)$$

<sup>7</sup>Of course these two sets are only defined in equilibria - the same applies to the sets  $L_s^i(\pm)$  - and it is, therefore, a notation abuse to write the maximisation problem as if the sets were known *ex ante*.

The originality in this otherwise standard maximisation problem comes from the inclusion of cash-in-advance constraints with liquidity parameters  $\lambda_s$  and  $\Lambda_s$ . Agents cannot use the receipts of their commodities sales to purchase commodities and securities because of illiquidity in the goods market (represented by the fact that only a share  $\Lambda_s$  of the receipts of sales in state  $s$  can be used for simultaneous trade). As a result they first borrow from the money markets  $\eta_s \mu_s^i$  and repay to the Central Bank  $\mu_s^i$  at the end of the period after having received the whole benefits from their sales. The situation is similar for the receipts of asset payoffs, since only a share  $\lambda_s$  of the payoffs can be used at the time the commodity markets meet.<sup>8</sup> Hence, in anticipation of late receipts agents borrow in the short-term money market (intra-period market) and repay their debt when they finally get the asset payoffs, at the end of each period.

## 2.4 Definition of a Monetary Equilibrium

$\left( (p_{s,l})_{l \in \{1 \dots L\}}, (q_{s,l}^\alpha)_{l \in \{1 \dots L\}}, (q_s^\beta)_{l \in \{1 \dots L\}}, \eta_s, \mu_s^\alpha, \mu_s^\beta, \theta_s, h_s^\alpha, h_s^\beta \right)_{s \in \bar{\mathcal{F}}} \in \mathbb{R}^{(L \times (S+1))^3 \times (S+1)^3 \times S^3}$  is a **Monetary Equilibrium**<sup>9</sup> if and only if:

- (i)  $(q_{s,l}^\alpha, \mu_s^\alpha, h_s^\alpha)_{s \in \bar{\mathcal{F}}} \in \text{Argmax} \{ U_\alpha : (\varphi^\alpha), (\xi^\alpha), \forall s, (\Psi_s^\alpha), (\chi_s^\alpha) \text{ hold} \}$
- (ii)  $(q_{s,l}^\beta, \mu_s^\beta, h_s^\beta)_{s \in \bar{\mathcal{F}}} \in \text{Argmax} \{ U_\beta : (\varphi^\beta), (\xi^\beta), \forall s, (\Psi_s^\beta), (\chi_s^\beta) \text{ hold} \}$
- (iii) Commodity markets clear, *i.e.*<sup>10</sup>

$$q_{s,l}^\alpha(p_{s,l}) = q_{s,l}^\beta(p_{s,l}) \equiv q_{s,l} \quad \forall s \in \bar{\mathcal{F}}, \quad \forall l \in \{1 \dots L\}$$

- (iv) Money and Arrow security markets clear, *i.e.*

$$\begin{aligned} \mu_s^\alpha + \mu_s^\beta &= (1 + r_s)M_s = M_s/\eta_s & \forall s \in \bar{\mathcal{F}} \\ h_s^\alpha(\theta_s) &= h_s^\beta(\theta_s) & \forall s \in \bar{\mathcal{F}}, \end{aligned}$$

Importantly, we restrict the study to interior equilibria where all financial markets are active and, as a result (see below), no money is carried over.

<sup>8</sup>For the sake of simplicity, we did not add a cash-in-advance constraint at the time of the purchase of assets in equation  $(\varphi_i)$ , but the constraint holds at the time when asset payoffs are received: in equation  $(\Psi_s^i)$  only a share  $\lambda_s$  of asset payoffs can be used simultaneously to purchase commodities.

<sup>9</sup>The existence of a Monetary Equilibrium is ensured when the Gains-from-Trade Hypothesis (Dubey and Geanakoplos, 1992, 2003) is satisfied.

<sup>10</sup>The demand and supply for commodities and assets depend on all the prices and the other exogenous parameters. For the sake of exposition, we only write the dependence on the respective price of goods or assets.

### 3 Properties of a Monetary Equilibrium

In the discussion of the implications of cash-in-advance models, we explained that given the existence of outside money (default can play the same role, see Shubik and Tsomocos, 1992, or Espinoza *et al.*, 2008), interest rates are positive and nominal determinacy obtains. This section confirms these results and shows the consequences for trade and asset prices. Proposition 1 shows that short-term interest rates are determined in a way that is independent from real demand functions. A positive value of money ensures that the demand for money is determined and this pins down the nominal value of trade (Proposition 2).

#### Proposition 1: Short-Term Interest Rates

If  $m_s^\alpha + m_s^\beta > 0$ ,  $\forall s \in \bar{\mathcal{F}}$

$$r_s = \frac{m_s^\alpha + m_s^\beta}{M_s}$$

#### Proof

For  $s = 0$ , we sum equations  $(\varphi^\alpha)$ ,  $(\varphi^\beta)$ ,  $(\xi^\alpha)$ ,  $(\xi^\beta)$  and, since all summations cancel out, we find that

$$\mu_0^\alpha + \mu_0^\beta = \eta_0(\mu_0^\alpha + \mu_0^\beta) + m_0^\alpha + m_0^\beta$$

From the money market equilibrium condition,  $\mu_0^\alpha + \mu_0^\beta = M_0(1 + r_0) = M_0/\eta_0$ , we deduce  $M_0(1 + r_0) = M_0 + m_0^\alpha + m_0^\beta$ .

The proof is similar for  $s \in \mathcal{F}^\alpha(+) = \mathcal{F}^\beta(-)$ , *i.e.*, we sum equations  $(\Psi_s^\alpha)$ ,  $(\Psi_s^\beta)$ ,  $(\chi_s^\alpha)$ ,  $(\chi_s^\beta)$ . The same is also true for  $s \in \mathcal{F}^\alpha(-) = \mathcal{F}^\beta(+)$ .

□

Proposition 1 replicates an argument already presented by Gurley and Shaw (1960) and reformulated by Dubey and Geanakoplos (1992, 2003, 2006) according to which a positive value for money is possible even in simple finite horizon models with cash-in-advance as long as agents are endowed with some money free of liability (the outside money endowments  $m^\alpha$  and  $m^\beta$ ). Proposition 1 also shows that the model is able to replicate in a very simple way a liquidity effect, *i.e.* the fact that an increase in money supply is accompanied by a fall in the short-term interest rate.

## Proposition 2: Quantity Theory of Money and Liquidity

In an interior equilibrium,<sup>11</sup>

$$(i) \sum_{l \in L} (1 - \Lambda_{0,l}) p_{0,l} q_{0,l} = M_0 + m_0^\alpha + m_0^\beta \quad (1)$$

$$(ii) \forall i \in \{\alpha, \beta\} \forall s \in \mathcal{F}^i(+), \sum_{l \in L_s^i(+)} [(1 - \Lambda_{s,l}) + (1 - \lambda_s)(\Lambda_{s,l} + (1 - \Lambda_{s,l})\eta_s)] p_{s,l} q_{s,l} \\ + \sum_{l \in L_s^i(-)} (\lambda_s - \Lambda_{s,l}) p_{s,l} q_{s,l} = M_s + \lambda_s m_s^\alpha + m_s^\beta \quad (2)$$

### Proof

In an interior equilibrium, all the budget constraints are binding.

(i) We sum equations  $(\varphi^\alpha)$  and  $(\varphi^\beta)$ .

$$\sum_{l \in L_0^\alpha(-) \cup L_0^\beta(-)} p_{0,l} q_{0,l}^\alpha + \sum_{s \in \mathcal{F}^\alpha(-) \cup \mathcal{F}^\beta(-)} \theta_s h_s^\alpha = \sum_{l \in L_0^\alpha(+)} \Lambda_{0,l} p_{0,l} q_{0,l}^\alpha \\ + \sum_{s \in \mathcal{F}^\alpha(+)} \theta_s h_s^\alpha + \eta_0 (\mu_0^\alpha + \mu_0^\beta) + m_0^\alpha + m_0^\beta \quad (3)$$

Since  $\forall s \in \mathcal{F}$ ,  $L_s^\alpha(-) = L_s^\beta(+)$  and  $L_s^\beta(-) = L_s^\alpha(+)$ , and since  $\mathcal{F}^\alpha(+)$  and  $\mathcal{F}^\beta(-)$  are symmetric, the previous result can be simplified:

$$\sum_{l \in L_0^\alpha(-) \cup L_0^\alpha(+)} (1 - \Lambda_{0,l}) p_{0,l} q_{0,l}^\alpha = \eta_0 (\mu_0^\alpha + \mu_0^\beta) + m_0^\alpha + m_0^\beta = M_0 + m_0^\alpha + m_0^\beta$$

where the last equality comes from the money market clearing condition.

(ii) Similarly, we sum equations  $(\Psi_s^\alpha)$  for a state  $s \in \mathcal{F}^\alpha(+)$  and equation  $(\Psi_s^\beta)$  for a state  $s \in \mathcal{F}^\beta(-) = \mathcal{F}^\alpha(+)$  and prove the second part of the proposition for  $i = \alpha$  (the proof is symmetric for  $\beta$ ):

$$\sum_{l \in L_s^\alpha(-)} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l}^\alpha + \sum_{l \in L_s^\alpha(+)} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l}^\beta + h_s^\alpha = \\ \lambda_s h_s^\beta + \eta_s (\mu_s^\alpha + \mu_s^\beta) + m_s^\alpha + m_s^\beta \quad (4)$$

<sup>11</sup>Interior solutions are guaranteed by the preference specification we have adopted.

From the asset market equilibrium condition and from equation  $(\Psi_s^\alpha)$ , we have

$$h_s^\alpha - \lambda_s h_s^\beta = (1 - \lambda_s) h_s^\beta = (1 - \lambda_s) \sum_{l \in L_s^\alpha(+)} \Lambda_{s,l} p_{s,l} q_{s,l}^\alpha - (1 - \lambda_s) \sum_{s \in L_s^\alpha(-)} p_{s,l} q_{s,l}^\alpha + (1 - \lambda_s) (\eta_s \mu_s^\alpha + m_s^\alpha)$$

and since

$$\mu_s^\alpha = \sum_{l \in L_s^\alpha(+)} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l}^\alpha$$

we have

$$\begin{aligned} & \sum_{l \in L_s^\alpha(+)} [(1 - \Lambda_{s,l}) + (1 - \lambda_s)(\Lambda_{s,l} + (1 - \Lambda_{s,l})\eta_s)] p_{s,l} q_{s,l} \\ & + \sum_{l \in L_s^\alpha(-)} (\lambda_s - \Lambda_{s,l}) p_{s,l} q_{s,l} = M_s + \lambda_s m_s^\alpha + m_s^\beta \end{aligned}$$

□

Proposition 2 mirrors the canonical Quantity Theory of Money  $MV = PQ$ , the modern formulation of which is due to Irving Fisher (1911). This is clear for period 0, and can be seen more easily in state  $s$  under two special, albeit reasonable, assumptions that will also be needed in the rest of the paper.

(i) Assume that  $\forall s \in \mathcal{F}$ ,  $\lambda_s = 1$ , *i.e.*, assets are totally liquid which means that their payoffs can be used immediately for purchase. Then, Proposition 2 collapses to

$$\forall s \in \mathcal{F} \quad \sum_{l \in L} (1 - \Lambda_{s,l}) p_{s,l} q_{s,l} = M_s + m_s^\alpha + m_s^\beta$$

(ii) Assume instead that  $\forall i \in \{\alpha, \beta\}$ ,  $\forall s \in \mathcal{F}^i(+)$ ,  $L_s^i(-) = \emptyset$ , *i.e.* if an agent is borrowing with repayment conditional on state  $s$  (selling Arrow securities), it is because she is relatively rich in all commodities in state  $s$  and hence will never need to buy commodities in that state. Then, Proposition 2 collapses to

$$\sum_{l \in L} [(1 - \Lambda_{s,l}) + (1 - \lambda_s)(\Lambda_{s,l} + (1 - \Lambda_{s,l})\eta_s)] p_{s,l} q_{s,l} = M_0 + \lambda_s m_s^\alpha + m_s^\beta$$

In the original Quantity Theory of Money, which can be re-written  $\sum_{l \in L} p_l q_l / V = M$ , the velocity of money  $V$  denotes the number of times that money is used for transactions within the period in consideration. In our model,  $V$  was implicitly set to one, as only one trade in commodities or asset was allowed per period. Hence, the coefficients in Proposition 2 do not parametrize velocity of money; rather, they capture the fact that money is used in an economy where pure credit is unable to finance trade (if  $\Lambda_{s,l} < 1$ ) or where assets payoffs are not settled instantaneously (if  $\lambda_s < 1$ ). There is also an indirect link with the keynesian

concept of liquidity  $L$  where  $M/p = L(Y)$ . In our model the parameters  $1 - \Lambda_s$  and  $1 - \lambda_s$  refer to the illiquidity of commodities and assets, which could also be interpreted as the relative liquidity of money. One interesting point to make is that higher asset (or commodity) liquidity generates higher nominal income, part of which will be reflected in higher activity (see Proposition 3), and part of it in higher inflation.

Our liquidity parameters also have consequences for the importance of financing costs, since more liquid assets and commodities require less (costly) use of money. The following two propositions, that generalise the results from Espinoza *et al.* (2008), illustrate that, when liquidity is uncertain (whether this means the supply of liquidity or the easiness with which assets and commodities are sold) trade becomes more volatile and this adds a risk-premium in the term structure of interest rates, for a given distribution of aggregate endowments. Hence, the model is able to generate an upward term structure in equilibrium lying above what would be predicted by a representative agent model.

## 4 Trade and Asset Prices

In this section, we solve for the equilibrium state prices. Monetary costs matter for transactions and hence allocations are affected by the short-term interest rate or by asset and endowment liquidity (Proposition 3). As a result, marginal utilities, and therefore asset prices, are also functions of liquidity (Proposition 4).

### Proposition 3 : Non-Neutrality of Money

Assume there is no aggregate uncertainty.

#### Case (i)

Let  $\forall s \in \mathcal{F}$ ,  $\lambda_s = 1$ . Then,  $\forall s, s' \in \mathcal{F}^\alpha(-), \forall l_0 \in L_s^\alpha(+) \cap L_{s'}^\alpha(+)$ ,

$$r_s \geq r_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \implies q_{s,l_0}^\alpha \leq q_{s',l_0}^\alpha \quad (5)$$

#### Case (ii)

Let  $\forall i \in \{\alpha, \beta\}$ ,  $\forall s \in \mathcal{F}^i(+)$ ,  $L_s^i(-) = \emptyset$ . Then,  $\forall s, s' \in \mathcal{F}^\alpha(-), \forall l_0 \in L_s^\alpha(+) \cap L_{s'}^\alpha(+)$ :

$$r_s \geq r_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \text{ and } \lambda_s \leq \lambda_{s'} \implies q_{s,l_0}^\alpha \leq q_{s',l_0}^\alpha \quad (6)$$

Note that by symmetry, the proposition also holds for  $\beta$ .

## Proof

Let  $\mathcal{L}$  be the lagrangian associated with  $\alpha$ 's maximisation problem set in section 2.3. For any state  $s \in \bar{\mathcal{F}}$ , let the index  $(s, 1) \in L^\alpha(-)$  denote a good bought by agent  $\alpha$ , and the index  $(s, l_0) \in L^\alpha(+)$  a good sold by  $\alpha$ . The first order conditions are, for agent  $\alpha$ :

$$\begin{aligned}
\text{For } s = 0, \quad & \frac{\partial \mathcal{L}}{\partial q_{0,1}^\alpha} = u'_{\alpha,1}(e_{0,1}^\alpha + q_{0,1}^\alpha) - p_{0,1}\varphi^\alpha = 0 \\
& \frac{\partial \mathcal{L}}{\partial q_{0,l_0}^\alpha} = -u'_{\alpha,l_0}(e_{0,l_0}^\alpha - q_{0,l_0}^\alpha) + \Lambda_{0,l_0}p_{0,l_0}\varphi^\alpha + (1 - \Lambda_{0,l_0})p_{0,l_0}\xi^\alpha = 0 \\
\forall s \in \mathcal{F} \quad & \frac{\partial \mathcal{L}}{\partial q_{s,1}^\alpha} = \pi_s u'_{\alpha,1}(e_{s,1}^\alpha + q_{s,1}^\alpha) - p_{s,1}\Psi_s^\alpha = 0 \\
& \frac{\partial \mathcal{L}}{\partial q_{s,l_0}^\alpha} = -\pi_s u'_{\alpha,l_0}(e_{s,l_0}^\alpha - q_{s,l_0}^\alpha) + \Lambda_{s,l_0}p_{s,l_0}\Psi_s^\alpha + (1 - \Lambda_{s,l_0})p_{s,l_0}\chi_s^\alpha = 0 \\
& \frac{\partial \mathcal{L}}{\partial \mu_0^\alpha} = \eta_0\varphi^\alpha - \xi^\alpha = 0 \\
& \frac{\partial \mathcal{L}}{\partial \mu_s^\alpha} = \eta_s\Psi_s^\alpha - \chi_s^\alpha = 0 \\
\forall s \in \mathcal{F}^\alpha(+), \quad & \frac{\partial \mathcal{L}}{\partial h_s^\alpha} = \theta_s\varphi^\alpha - \Psi_s^\alpha = 0 \\
\forall s \in \mathcal{F}^\alpha(-), \quad & \frac{\partial \mathcal{L}}{\partial h_s^\alpha} = -\theta_s\varphi^\alpha + \lambda_s\Psi_s^\alpha + (1 - \lambda_s)\chi_s^\alpha = 0
\end{aligned}$$

The First Order Conditions are the same for agent  $\beta$  except that a good  $(s, 1)$  becomes a good  $(s, l_0)$  and *vice-versa*.

Step 1: Using the First-Order Conditions for a good  $(s, l_0)$  sold by  $\alpha$ , we have, since  $\chi_s^\alpha = \eta_s\Psi_s^\alpha$ ,

$$\pi_s u'_{\alpha,l_0}(e_{s,l_0}^\alpha - q_{s,l_0}^\alpha) = \Lambda_{s,l_0}p_{s,l_0}\Psi_s^\alpha + (1 - \Lambda_{s,l_0})p_{s,l_0}\chi_s^\alpha \quad (7)$$

$$= p_{s,l_0}\Psi_s^\alpha(\Lambda_{s,l_0} + (1 - \Lambda_{s,l_0})\eta_s) \quad (8)$$

a) For a state of nature  $s \in \mathcal{F}^\alpha(+)$ , since  $\Psi_s^\alpha = \theta_s\varphi^\alpha$ , we have

$$\frac{\pi_s u'_{\alpha,l_0}(e_{s,l_0}^\alpha - q_{s,l_0}^\alpha)}{p_{s,l_0}\theta_s(\Lambda_{s,l_0} + (1 - \Lambda_{s,l_0})\eta_s)} = \varphi^\alpha \quad (9)$$

b) For a state of nature  $s \in \mathcal{F}^\alpha(-)$ , since  $\lambda_s\Psi_s^\alpha + (1 - \lambda_s)\eta_s\Psi_s^\alpha = \theta_s\varphi^\alpha$ , we have

$$\frac{\pi_s u'_{\alpha,l_0}(e_{s,l_0}^\alpha - q_{s,l_0}^\alpha)(\lambda_s + (1 - \lambda_s)\eta_s)}{p_{s,l_0}\theta_s(\Lambda_{s,l_0} + (1 - \Lambda_{s,l_0})\eta_s)} = \varphi^\alpha \quad (10)$$

Step 2: Using the First-Order Conditions for  $\beta$ , who buys the commodity  $(s, l_0)$ , we have

$$\pi_s u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta) = p_{s, l_0} \Psi_s^\beta \quad (11)$$

a) For a state of nature  $s \in \mathcal{F}^\beta(-) = \mathcal{F}^\alpha(+)$ , since  $\lambda_s \Psi_s^\beta + (1 - \lambda_s) \eta_s \Psi_s^\beta = \theta_s \varphi^\beta$ , we get

$$\pi_s u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta) = \frac{p_{s, l_0} \theta_s \varphi^\beta}{\lambda_s + (1 - \lambda_s) \eta_s} \implies \frac{\pi_s u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta)(\lambda_s + (1 - \lambda_s) \eta_s)}{p_{s, l_0} \theta_s} = \varphi^\beta \quad (12)$$

b) For a state of nature  $s \in \mathcal{F}^\beta(+)$ , since  $\Psi_s^\beta = \theta_s \varphi^\beta$ , we find

$$\pi_s u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta) = p_{s, l_0} \theta_s \varphi^\beta \implies \frac{\pi_s u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta)}{p_{s, l_0} \theta_s} = \varphi^\beta \quad (13)$$

Step 3: Dividing  $\alpha$ 's First-Order Conditions by  $\beta$ 's First-Order Conditions:

a) For a state of nature  $s \in \mathcal{F}^\beta(-) = \mathcal{F}^\alpha(+)$

$$\frac{u'_{\alpha, l_0}(e_{s, l_0}^\alpha - q_{s, l_0}^\alpha)}{u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta)(\lambda_s + (1 - \lambda_s) \eta_s)(\Lambda_{s, l_0} + (1 - \Lambda_{s, l_0}) \eta_s)} = \frac{\varphi^\alpha}{\varphi^\beta} \quad (14)$$

b) For a state of nature  $s \in \mathcal{F}^\beta(+)$ , since  $\Psi_s^\beta = \theta_s \varphi^\beta$ , we find

$$\frac{u'_{\alpha, l_0}(e_{s, l_0}^\alpha - q_{s, l_0}^\alpha)(\lambda_s + (1 - \lambda_s) \eta_s)}{u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta)(\Lambda_{s, l_0} + (1 - \Lambda_{s, l_0}) \eta_s)} = \frac{\varphi^\alpha}{\varphi^\beta} \quad (15)$$

**Case (i)** Assume  $\forall s \in \bar{\mathcal{F}}, \lambda_s = 1$ , then both equations (14) and (15) collapse to provide the same result, *i.e.*

$$\frac{u'_{\alpha, l_0}(e_{s, l_0}^\alpha - q_{s, l_0}^\alpha)}{u'_{\beta, l_0}(e_{s, l_0}^\beta + q_{s, l_0}^\beta)(\Lambda_{s, l_0} + (1 - \Lambda_{s, l_0}) \eta_s)} = \frac{\varphi^\alpha}{\varphi^\beta} \quad (16)$$

Remember that the market clearing condition is  $q_{s, l_0}^\alpha = q_{s, l_0}^\beta$ . Also, note that for any utility functions such that  $u'_{i, l} > 0$  and  $u''_{i, l} < 0$ ,  $i \in \{\alpha, \beta\}$ ,  $l_0 \in \{1 \dots L\}$ , the function

$$\frac{u'_{\alpha, l_0}(e_{l_0}^\alpha - q_{s, l_0}^\alpha)}{u'_{\beta, l_0}(q_{s, l_0}^\beta + q_{s, l_0}^\alpha)}$$

is increasing in  $q_{s, l_0}^\alpha$ . Finally, note that the right-hand side of equation (14), and therefore of equation (16), is independent of  $s$ . Hence, since  $\eta_s = \frac{1}{1 + r_s}$ ,  $\forall s, s' \in \mathcal{F}^\alpha(-), \forall l_0 \in L_s^\alpha(+)$  (i.e. for a commodity  $l_0$  that is sold by  $\alpha$  in two different states of nature  $s, s'$  in which  $\alpha$  is relatively rich):

$$r_s \geq r_{s'} \text{ and } \Lambda_{s, l_0} \leq \Lambda_{s', l_0} \implies q_{s, l_0}^\alpha \leq q_{s', l_0}^\alpha \quad (17)$$

**Case (ii)** Assume  $\forall i \in \{\alpha, \beta\}, \forall s \in \mathcal{F}^i(+), L_s^i(-) = \emptyset$  (*i.e.* assume that if an agent  $i$  is borrowing with repayment conditional on state  $s$ , it is because she is relatively rich in all commodities in state  $s$  and therefore will never buy commodities in that state), then case  $b)$  is never relevant since in a state  $s \in \mathcal{F}^\beta(+)=\mathcal{F}^\alpha(-)$ , commodity  $(s, l_0)$  (which was assumed to lie in  $L^\alpha(+)=L^\beta(-)$ ) does not exist. Hence only case  $a)$  applies, and we deduce that

$$r_s \geq r_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \text{ and } \lambda_s \leq \lambda_{s'} \implies q_{s,l_0}^\alpha \leq q_{s',l_0}^\alpha \quad (18)$$

□

Proposition 3 suggests that liquidity in commodity and asset markets is important for trade activity. Liquidity (broadly defined) increases trade activity. Furthermore, a policy maker who would want to have more information on markets' liquidity to assess future economic developments would need to look at monetary aggregates. Indeed, knowing past real variables and prices is not enough to deduce the  $\lambda_s$  and  $\Lambda_{s,l}$ , and supplementary information on nominal income  $\sum_{l \in L} p_{s,l} q_{s,l}$  and money supply  $M_s$  is needed to determine the current liquidity stance. Therefore, the model has a distinct monetary flavour in that monetary aggregates provide information that is not captured by economic variables (interest rates, prices and real activity).

Assume now that there exists one commodity  $l_0$  which is never endowed to agent  $\beta$  and the price of which increases with liquidity.<sup>12</sup> In that case, Proposition 4 shows that the Arrow prices are an increasing function of spot interest rates.

#### Proposition 4: Asset Prices

Let  $s, s' \in \mathcal{F}^\alpha(-)$  and  $l_0 \in L_s^\alpha(+)\cap L_{s'}^\alpha(+)$  a commodity for which the price level is increasing with liquidity.

##### Case (i)

If  $\lambda_s = \lambda_{s'} = 1$

$$r_s \geq r_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \implies \theta_s/\pi_s \geq \theta_{s'}/\pi_{s'} \quad (19)$$

##### Case (ii)

Assume  $\forall i \in \{\alpha, \beta\}, \forall s \in \mathcal{F}^i(+), L_s^i(-) = \emptyset$ , then

$$r_s \geq r_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \text{ and } \lambda_s \leq \lambda_{s'} \implies \theta_s/\pi_s \geq \theta_{s'}/\pi_{s'} \quad (20)$$

Since the model is symmetric in  $\alpha$  and  $\beta$ , the results in the two cases are also valid for all  $s, s' \in \mathcal{F}^\beta(-)=\mathcal{F}^\alpha(+)$ .

---

<sup>12</sup>Equilibria such that prices increase with liquidity exist, as it is demonstrated in Espinoza *et al.* (2008), section 4, Theorem 2.

## Proof

For a state of nature  $s \in \mathcal{F}^\beta(-) = \mathcal{F}^\alpha(+)$ , from equation (12), applied for two states of nature  $s \neq s'$ ,

$$\frac{\pi_s u'_{\beta,l_0}(e_{s,l_0}^\beta + q_{s,l_0}^\beta)}{p_{s,l_0} \theta_s} = \frac{\pi_{s'} u'_{\beta,l_0}(e_{s',l_0}^\beta + q_{s',l_0}^\beta)}{p_{s',l_0} \theta_{s'}} \quad (21)$$

Equivalently,

$$\frac{\theta_s/\pi_s}{\theta_{s'}/\pi_{s'}} = \frac{u'_{\beta,l_0}(e_{s,l_0}^\alpha + q_{s,l_0}^\alpha)/p_{s,l_0}}{u'_{\beta,l_0}(e_{s',l_0}^\alpha + q_{s',l_0}^\alpha)/p_{s',l_0}} \quad (22)$$

Let  $r_s \geq r_{s'}$ , *i.e.*  $M_s < M_{s'}$ , and  $\lambda_s \leq \lambda_{s'}$ , and  $\Lambda_{s,l_0} \leq \Lambda_{s',l_0}$ . From Proposition 3, we know that  $q_{s,l_0}^i < q_{s',l_0}^i$ , which implies  $u'_{\beta,l_0}(q_{s,l_0}^\alpha) > u'_{\beta,l_0}(q_{s',l_0}^\alpha)$ .

Under the assumption that  $p_{s,l_0} \leq p_{s',l_0}$ , it follows that  $\frac{\theta_s}{\pi_s} > \frac{\theta_{s'}}{\pi_{s'}}$

□

The two conditions that are required for this proposition to hold may seem restrictive, but in fact they are fairly realistic. First, it is expected that some commodities are endowed specifically to one agent, in which case the set  $L_s^\alpha(+)$   $\cap$   $L_{s'}^\alpha(+)$  includes all commodities. Second, if trade in  $l_0$  is not extremely sensitive to the financing costs, then the price of  $l_0$  is likely to increase when  $M$  increases (otherwise it would mean that  $q$  increases more than proportionally to  $M$ ). A fully solved example is given in Espinoza *et al.* (2008) where it is shown that with log-utility and a simple endowment configuration that  $p_{s,l} = \frac{2M_s + r_s M_s}{e_{s,l}}$ .<sup>13</sup>

In a one-commodity world, or equivalently in a world where all commodities are traded in the same direction, there is a simpler sufficient condition, that Relative Risk-Aversion (*RRA*) be no less than 1. This is because the Euler equation can be written as

$$\frac{\theta_s/\pi_s}{\theta_{s'}/\pi_{s'}} = \frac{u'_{\beta,l_0}(q_{s,l_0}^\alpha) q_{s,l_0}^\alpha / (p_{s,l_0} q_{s,l_0}^\alpha)}{u'_{\beta,l_0}(q_{s',l_0}^\alpha) q_{s',l_0}^\alpha / (p_{s',l_0} q_{s',l_0}^\alpha)} \quad (23)$$

and  $p_{s,l_0} q_{s,l_0}^\alpha$  is increasing with  $M_s$  and decreasing with  $\Lambda_{s,l_0}$  and  $\lambda_s$ .<sup>14</sup> Furthermore  $g(q) = u'_\beta(q)q$  is decreasing in  $q$  if and only if  $u''_\beta(q)q + u'_\beta(q) \leq 0 \Leftrightarrow RRA(q) = \frac{-u''_\beta(q)q}{u'_\beta(q)} \geq 1$ . Hence, with a *RRA* higher than 1,

$$M_s \leq M_{s'} \text{ and } \Lambda_{s,l_0} \leq \Lambda_{s',l_0} \text{ and } \lambda_s \leq \lambda_{s'} \implies q_{s,l_0} < q_{s',l_0} \implies u'_{\beta,l_0}(q_{s,l_0}^\alpha) q_{s,l_0}^\alpha > u'_{\beta,l_0}(q_{s',l_0}^\alpha) q_{s',l_0}^\alpha$$

<sup>13</sup>In Espinoza *et al.*, the model is set with default instead of outside money, but this does not affect the main result.

<sup>14</sup>The Quantity Theory of Money becomes  $(1 - \Lambda_{s,l_0} + (1 + \Lambda_{s,l_0} r_s)(1 - \lambda_s)/(1 + r_s)) p_{s,l_0} q_{s,l_0} = M_s + \lambda_s m_s^\alpha + m_s^\beta$  in a one commodity economy.

and since  $p_{s,l_0} q_{s,l_0}^\alpha < p_{s',l_0} q_{s',l_0}^\alpha$  in a one commodity world, we find the same result (equation (20)) for asset prices.

## 5 Examples

The previous propositions hold in absence of aggregate uncertainty and when the subjective probability distributions of the borrower and the lender are equal. Applying a continuity argument, we show that these general results hold locally, *i.e.*, if the subjective probabilities or the endowments differ by an infinitesimal quantity (Proposition 5). We can also extend the results with non-infinitesimal differences, assuming that preferences are given by CRRA utility functions with the same coefficient of risk-aversion for both agents (Proposition 6). We provide the proofs of these propositions in the annex. Proposition 6 also helps us understand the role of money. For the sake of simplicity, we assume that there is only one commodity  $l$  and that  $\lambda_s = 1$ ,  $\Lambda_{s,l} = 0$ ,  $\forall s \in \tilde{\mathcal{F}}$ . However, similar results would hold in the general case.

### Proposition 5: Local Properties in a one-commodity economy

In addition to the hypotheses needed for Proposition 4, let  $|L| = 1$ . Assume that the endowments or the subjective probabilities of state  $s$  and  $s'$  differ by an infinitesimal quantity and  $\lambda_s = 1$ ,  $\Lambda_{s,l} = 0, \forall s \in \tilde{\mathcal{F}}$ . Then,

$$r_s > r_{s'} \implies \theta_s / \pi_s > \theta_{s'} / \pi_{s'}$$

Note that this proposition would also hold in a multi-commodity provided that the assumptions of proposition 4 hold.

### Proposition 6: CRRA Example in a one-commodity economy

In addition to the hypotheses needed for Proposition 4, let  $|L| = 1$ ,  $e_0^\alpha = 0$ ,  $\lambda_s = 1$ ,  $\Lambda_{s,l} = 0, e_s^\beta = 0, \forall s \in \tilde{\mathcal{F}}$ , and that agents' preferences are given by a CRRA utility function with a common constant coefficient of risk-aversion  $\rho$ . Then  $\forall s, s' \in \tilde{\mathcal{F}}$

$$\frac{q_s^\alpha / e_s^\alpha}{q_{s'}^\alpha / e_{s'}^\alpha} = \frac{\left( \frac{\pi_{s'}}{\pi_{s'}} (1 + r_{s'}) (1 + r_0) \right)^{1/\rho} + \frac{e_0^\beta}{q_0^\beta} - 1}{\left( \frac{\pi_s}{\pi_s} (1 + r_s) (1 + r_0) \right)^{1/\rho} + \frac{e_0^\beta}{q_0^\beta} - 1} \quad (24)$$

and

$$\frac{\theta_s}{\theta_{s'}} = \frac{\pi_s^\beta}{\pi_{s'}^\beta} \left( \frac{e_{s'}^\alpha}{e_s^\alpha} \right)^{\rho-1} \frac{M_{s'} + m_{t+1}^\alpha + m_{t+1}^\beta}{M_s + m_{t+1}^\alpha + m_{t+1}^\beta} \left( \frac{\left( \frac{\pi_{s'}}{\pi_s} (1 + r_s) (1 + r_0) \right)^{1/\rho} + \frac{e_0^\beta}{q_0^\beta} - 1}{\left( \frac{\pi_{s'}}{\pi_{s'}} (1 + r_{s'}) (1 + r_0) \right)^{1/\rho} + \frac{e_0^\beta}{q_0^\beta} - 1} \right)^{\rho-1} \quad (25)$$

Proposition 6 allows us to summarize the core of the model in only two equations, however, at the expense of forcing risk aversion to be the same for the two agents. The first result tells us that the proportion of endowments traded is greater the lower the interest rate. This result is therefore akin to Proposition 3 in the case of aggregate uncertainty. However, in this general framework with different subjective probabilities, the relative importance of state  $s$  for agents  $\alpha$  and  $\beta$  also matters. For example, if  $\beta$  (who is buying securities) gives more weight (relatively to  $\alpha$ ) to state  $s$ , there will be more trade in state  $s$ .

The second equation provides the intuition behind Proposition 4. The first two terms are characteristic of any asset pricing equation embedded in a general equilibrium model. If subjective probabilities (of the asset's buyer) are higher for state  $s$ , this will increase the state price. Furthermore, if aggregate endowment is lower in state  $s$ , this will also increase the state price because lower aggregate endowment implies lower consumption and therefore higher marginal utility. The last two terms are, however, special to our model. The ratio of money supplies matters because the state prices are prices of assets whose payoffs are set in nominal terms. Since a higher money supply implies a higher price level (Proposition 2), the value of any asset today is a decreasing function of next period's price level. Finally, the last term is the key general equilibrium effect. A higher spot interest rate  $r_s$  tends to lower trade activity and therefore lowers consumption of the agent who buys in the future (*i.e.*, agent  $\beta$ ). But agent  $\beta$  is also the one who is buying securities, and since she is willing to pay a higher price for assets which pay off when her consumption is low, she is therefore ready to pay more for securities that pay off when interest rates are high. Hence, states with higher interest rates also have higher state prices.

This result is not an application of the risk-premium found in pure exchange general equilibrium models with heterogeneous agents or in a representative agent model (Lucas, 1978; Breeden, 1979 ; Cox, Ingersoll and Ross, 1985 ; Backus *et al.*, 1989). Indeed, even when the endowment risk-premium has been removed as in Proposition 3, state prices in Proposition 4 are still a function of money. In the CRRA closed-form extension, it is clear how the Lucas model and our financing cost model are related. The additional risk-premium comes from the effect of money on trade and therefore on marginal utilities.

Note that when  $r_s$  is low, because  $q_s$  is high, agent  $\alpha$ 's marginal utility is low but agent  $\beta$ 's marginal utility is high. Therefore, a close link cannot be established between representative agent marginal utilities and state prices in this heterogeneous model.

The upshot of our argument is that uncertainty in aggregate production or in aggregate consumption is only one part of uncertainty in agents' marginal utilities. Transaction costs also

generate variability of marginal utilities (and thus of asset demands) in the future. Therefore, any model of risk-premium that attempts to proxy welfare by production or consumption will underestimate the risk-premium. This is especially important for the term-structure risk premium since the spot interest rate has an effect both on the asset price and on the transaction cost. Hence, the correlation between the marginal utilities and asset prices is likely to be high. The risk-premium would, in that case, be erroneously underestimated.

The model can be applied to show the existence of a ‘*liquidity-risk* premium’ for bonds with long-term maturity. The liquidity premium is due to the additional costs incurred by investors (and priced in the term structure) which an uncertain money supply generates when liquidity is constrained. Note that the level of money supply is irrelevant in the long run - although we do not analyse the long run in this model. If prices adjust to money supply, constraints on liquidity do not have real effects. However, the variance (or risk) of liquidity still has effects. This is exactly what is captured in this model, where we show that larger liquidity risks generate higher long-term interest rates. *Stricto sensu*, this is a model of the liquidity-risk premium. This liquidity-risk premium is deduced from the Proposition 4, since risk-neutral probabilities are high when the interest rates are high.

## 6 Concluding Remarks

In a state of nature with low liquidity trade has to be low and in order to induce consumers to trade at a low level, the opportunity cost of transferring money to this state must be high. This opportunity cost is the state price. Therefore, state prices and risk-neutral probabilities are higher in states with higher interest rates (and in general with low liquidity). It is important to stress that this result is explained by the interaction of the monetary technology with the exchange economy (through the Euler equations) and therefore cannot be found in a purely financial model or in a general equilibrium without money. Ultimately, it is the risk of variations in the supply of money or in markets’ liquidity that matters to determine the risk in trade values. Liquidity shocks are crucial to understanding the upward sloping term structure because two phenomena push in the same direction: first, the futures spot interest rates are affected ; second, the risk-neutral probabilities are modified. The interaction of these two effects pushes long-term rates above the historical average of future spot rates, even with nonexistent aggregate real risk. That is, the more uncertainty in the future spot rates, the higher the long-term rates. Stability of monetary policy is therefore, required to maintain flat yield curves.

This connects to another subject of discussion in the current yield curve literature: the fact that the term structure changes are not always linked to the fundamentals predicted by the

standard theory (inflation, inflation risk, macroeconomic volatility, risk-aversion, fiscal policy). Interpretations relating to the development of financial markets and liquidity risks are also needed. We argued in this paper that risk in the functioning of credit markets and in the liquidity of assets may generate term premia, and that these premia could explain changes in the yield curve of a larger magnitude than has been explained by non-monetary models. Our Proposition 3 also relates to a monetary view of the transmission mechanism. Short-term interest rates do not convey enough information to determine activity and asset prices. Monetary aggregates are also important to assess the markets' need for liquidity and therefore the prospects for inflation and asset prices.

## References

- Backus D., A. Gregory and S. Zin (1989), "Risk Premiums in the Term Structure: Evidence from Artificial Economies", *Journal of Monetary Economics*, Vol. 24(3), pp. 371-399.
- Bansal R. and W.J. Coleman II (1996), "A Monetary Explanation of the Equity Premium, Term Premium and Risk-Free Rate Puzzles", *Journal of Political Economy*, Vol. 104(6), pp. 1135-1171.
- Bloize G., J.H. Dreze and H. Polemarchakis (2005), "Monetary Economy over Infinite Horizon", *Economic Theory*, Vol. 25(1), pp. 51-74.
- Breedon D. (1979), "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities", *Journal of Financial Economics*, Vol. 7, pp. 265-296.
- Chabi-Yo F., D. Leisen and E. Renault (2007), "Implications of Asymmetry Risk for Portfolio Analysis and Asset Pricing", Bank of Canada Working Paper 2007-47.
- Clower B.W. (1967), "A Reconsideration of the Microeconomic Foundations of Monetary Theory", *Western Economic Journal*, Vol. 6, pp. 1-8.
- Cox J., J. Ingersoll and S. Ross (1985a), "An Intertemporal General Equilibrium of Asset Prices", *Econometrica*, Vol. 53, pp. 363-384.
- Cox J., J. Ingersoll and S. Ross (1985b), "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol. 53, pp 385-407.
- Dubey P. and J. Geanakoplos (1992), "The Value of Money in a Finite-Horizon Economy:

a Role for Banks”, in P. Dasgupta, D. Gale et al. (eds), *Economic Analysis of Market and Games*, Cambridge MA: MIT Press.

Dubey P. and J. Geanakoplos (2003), “Monetary Equilibrium with Missing Markets”, *Journal of Mathematical Economics*, Vol. 39, pp. 585-618.

Dubey P. and J. Geanakoplos (2006), “Real Determinacy with Nominal Assets and Outside Money”, *Economic Theory*, Vol. 27, pp. 79-106.

Espinoza R., C. Goodhart and D. Tsomocos (2008), “State Prices, Liquidity and Default”, *Economic Theory*, in print.

Fan M. (2006), “Heterogeneous Beliefs, the Term Structure and Time-Varying Risk Premia”, *Annals of Finance*, Vol. 2, pp. 259-285.

Fisher I. (1911), *The Purchasing Power of Money*, New York: MacMillan.

Grandmont J.M. and Y. Younes (1972), “On the Role of Money and the Existence of a Monetary Equilibrium”, *Review of Economic Studies*, Vol. 39, pp. 355-372.

Grossman S., A. Melino and R. Shiller (1987), “Estimating the Continuous Time Consumption Based Asset Pricing Model”, *Journal of Business and Economic Statistics*, Vol. 5(3), pp. 315-327.

Gurley J.G. and E.S. Shaw (1960), *Money in a Theory of Finance*, Washington DC: The Brookings Institution.

Lucas R. (1978), “Asset Prices in an Exchange Economy”, *Econometrica*, Vol. 46(6), pp. 1429-1445.

Lucas R. and N. Stokey (1983), “Optimal Fiscal and Monetary Policy in an Economy Without Capital”, *Journal of Monetary Economics*, Vol. 12, pp. 55-93.

Lucas R. and N. Stokey (1987), “Money and Interest in a Cash-in-Advance Economy”, *Econometrica*, Vol. 55(3), pp. 491-513.

Lucas R. (1990), “Liquidity and Interest Rates”, *Journal of Economic Theory*, Vol. 50(2), pp. 237-264.

Mehra R. and E. Prescott (1985) “The Equity Premium: A Puzzle”, *Journal of Monetary Economics*, Vol. 10, pp. 335-359.

Shubik M. and D.P. Tsomocos (1992), “A Strategic Market Game with a Mutual Bank with Fractional Reserves and Redemption in Gold (A Continuum of Traders)”, *Journal of Economics*, Vol. 55(2), pp. 123-150.

Shubik M. and C. Wilson (1997), “The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money”, *Journal of Economics*, Vol. 37 (3-4), pp. 337-354.

Svensson L.E.O. (1985), “Money and Asset Prices in a Cash-in-Advance Economy”, *Journal of Political Economy*, Vol. 93 (5), pp. 919-944.

Tsomocos D.P. (2008), “Generic Determinacy and Money Non-Neutrality of International Monetary Equilibria”, *Journal of Mathematical Economics*, Vol. 44(7-8), pp. 866-887.

## Annex

### Proof of Proposition 5: Local Properties

We show here how the continuity argument works for different subjective probabilities, but the proof follows *mutatis mutandis* for different endowments. Let  $1 + \epsilon = \frac{\pi_s^\alpha / \pi_s^\beta}{\pi_{s'}^\alpha / \pi_{s'}^\beta}$ . Since monotonic transformations of the intertemporal utility functions do not affect the maximisation problems, one can normalise the subjective probabilities  $\frac{\pi_s^\alpha}{\pi_{s'}^\alpha}$  and  $\frac{\pi_s^\beta}{\pi_{s'}^\beta}$ . Therefore, without loss of generality,  $\epsilon$  can represent any difference between  $\alpha$ 's and  $\beta$ 's subjective probabilities. We then show the dependence of  $q_s^\alpha$  and  $q_{s'}^\alpha$  on the subjective probabilities of states  $s$  and  $s'$ :  $q_s^\alpha = q_s^\alpha(\pi_s^\alpha, \pi_{s'}^\alpha, \pi_s^\beta, \pi_{s'}^\beta)$  and  $q_{s'}^\alpha = q_{s'}^\alpha(\pi_s^\alpha, \pi_{s'}^\alpha, \pi_s^\beta, \pi_{s'}^\beta)$ . Thus,

$$q_s^\alpha = q_s^\alpha \left( \frac{\pi_s^\alpha}{\pi_{s'}^\alpha}, \frac{\pi_s^\beta}{\pi_{s'}^\beta} \right) = q_s^\alpha \left( \frac{\pi_s^\beta}{\pi_{s'}^\beta} (1 + \epsilon), \frac{\pi_s^\beta}{\pi_{s'}^\beta} \right)$$

$$q_{s'}^\alpha = q_{s'}^\alpha \left( \frac{\pi_s^\alpha}{\pi_{s'}^\alpha}, \frac{\pi_s^\beta}{\pi_{s'}^\beta} \right) = q_{s'}^\alpha \left( \frac{\pi_s^\beta}{\pi_{s'}^\beta} (1 + \epsilon), \frac{\pi_s^\beta}{\pi_{s'}^\beta} \right)$$

We first extend equation (16) to take into account the subjective probabilities  $\pi_s^\alpha$  and  $\pi_s^\beta$ ,  $\forall s$  and apply it to  $s$  and  $s'$  with  $\Lambda_{s,l_0} = \Lambda_{s',l_0} = 0$ :

$$\frac{\pi_s^\alpha u'_{\alpha,l_0}(e_{s,l_0}^\alpha - q_{s,l_0}^\alpha)}{\pi_s^\beta u'_{\beta,l_0}(e_{s,l_0}^\beta + q_{s,l_0}^\beta)\eta_s} = \frac{\pi_{s'}^\alpha u'_{\alpha,l_0}(e_{s',l_0}^\alpha - q_{s',l_0}^\alpha)}{\pi_{s'}^\beta u'_{\beta,l_0}(e_{s',l_0}^\beta + q_{s',l_0}^\beta)\eta_{s'}}$$

Keeping the ratio  $\frac{\pi_s^\beta}{\pi_{s'}^\beta}$  constant, only  $\epsilon$  is variable here. We also apply the market clearing condition,  $q_s^\alpha = q_s^\alpha(\epsilon)$  and  $q_{s'}^\alpha = q_{s'}^\alpha(\epsilon)$ , and, since we assume endowments are constant across states (*i.e.*,  $e_{s,l_0}^i = e_{s',l_0}^i \equiv e_{l_0}^i$ ,  $\forall i \in \{\alpha, \beta\}$ ,  $\forall s, s'$ ), we find:

$$(1 + \epsilon) \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \frac{1 + r_{s'}}{1 + r_s} < 1.$$

where  $f(q) = u'(e_{l_0}^\alpha - q)/u'(e_{l_0}^\beta + q)$  is increasing in  $q$ . Note first that

$$\lim_{\epsilon \rightarrow 0} (1 + \epsilon) \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \lim_{\epsilon \rightarrow 0} \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} + \lim_{\epsilon \rightarrow 0} \epsilon \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \frac{1 + r_{s'}}{1 + r_s} < 1$$

Furthermore,

$$\lim_{\epsilon \rightarrow 0} \epsilon \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = 0$$

because  $0 < \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} < \frac{1}{1 + \epsilon} < 1$  and hence  $\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))}$  is bounded. Hence, we have in the limit that  $\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} < 1$ , and, since  $f$  is continuous and increasing,  $q_s^\alpha(\epsilon) < q_{s'}^\alpha(\epsilon)$  as  $\epsilon \rightarrow 0$ . The proof of the second part then follows the proof of Proposition 4, in particular equation (21). □

### Proof of Proposition 6: CRRA example

The first-order conditions of agents  $\alpha$  and  $\beta$  and the market clearing conditions lead to,  $\forall s \in \mathcal{F}$

$$\frac{(q_0^\beta)^{-\rho}}{p_0} = \frac{(q_0^\alpha)^{-\rho}}{p_0} = \varphi^\alpha = \frac{\Psi_s^\alpha}{\theta_s} = \frac{\chi_s^\alpha(1 + r_s)}{\theta_s} = \pi_s^\alpha \frac{(e_s^\alpha - q_s^\alpha)^{-\rho}}{p_s \theta_s} (1 + r_s) \quad (26)$$

$$\frac{(e_0^\beta - q_0^\beta)^{-\rho}}{p_0} = \xi^\beta = \frac{\varphi^\beta}{1 + r_0} = \frac{\Psi_s^\beta}{\theta_s(1 + r_0)} = \pi_s^\beta \frac{(q_s^\alpha)^{-\rho}}{p_s \theta_s} \frac{1}{1 + r_0} \quad (27)$$

Dividing (26) by (27) yields

$$\frac{(q_0^\beta)^{-\rho}}{(e_0^\beta - q_0^\beta)^{-\rho}} = \frac{\pi_s^\alpha (e_s^\alpha - q_s^\alpha)^{-\rho}}{\pi_s^\beta (q_s^\alpha)^{-\rho}} (1 + r_s)(1 + r_0)$$

which leads to

$$\forall s \in \mathcal{F} \quad \frac{e_s^\alpha}{q_s^\alpha} - 1 = \frac{\left(\frac{\pi_s^\alpha}{\pi_s^\beta}(1+r_s)(1+r_0)\right)^{\frac{1}{\rho}}}{\frac{e_0^\beta}{q_0^\beta} - 1} \quad (28)$$

Dividing (28) applied to state of nature  $s$  by the corresponding (28) of  $s'$ , we show the first result. The second result comes from the application of (24) to (22) using the quantity theory of money.

□