Fibrewise CoHopf Spaces

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Thesis submitted Trinity Term, 1992, in support of application to supplicate for the degree of D. Phil.

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Abstract

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A fibrewise coHopf space X over a base B is a sectioned space for which the diagonal map $X \to X \times_B X$ may be compressed into $X \vee_B X$ up to fibrewise pointed homotopy. Such spaces have been investigated by I. M. James in the case where X is a sphere bundle over a sphere. The purpose of this thesis is to demonstrate some of the properties of fibrewise coHopf spaces over more general bases. Particular attention is given to sphere bundles and fibrations with spherical fibre.

The fibrewise reduced suspension of a sectioned fibrewise space with closed section is fibrewise coHopf with associative comultiplication (up to fibrewise pointed homotopy) and a fibrewise inversion. Examples of fibrewise coHopf spaces not of this form are exhibited, and sufficient conditions are given to ensure that a fibrewise coHopf space has the primitive fibrewise pointed homotopy type of a fibrewise reduced suspension, in terms of the dimension and connectivity of the space, its base and the fibres. It is shown that these conditions may be relaxed if the fibrewise coHopf structure on the space is assumed to be homotopy-associative. An example of a non-associative fibrewise coHopf sphere bundle is given.

It is shown that, if $q > 1$ is odd, a sectioned orientable q-sphere bundle over a finite connected complex is fibrewise coHopf if and only if its fibrewise localisation at the prime 2 is fibrewise coHopf. Moreover, the fibrewise rationalisation of an odd-dimensional sphere bundle over a finite polyhedron whose fibrewise unreduced suspension is fibrewise coHopf is shown to be a trivial fibration. As an application, it is shown that new fibrewise coHopf spherical fibrations may be constructed by mixing.

The Thom space is used to determine the cohomology ring of the total space of a fibrewise coHopf sphere bundle in terms of that of its base, and a generalised Hopf invariant is constructed which vanishes on fibrewise coHopf sphere bundles.

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Contents

Chapter 1
Introduction

1.1 CoHopf Spaces

We open with a resumé of the main results concerning coHopf spaces in ordinary homotopy theory. Recall that a coHopf space is a space X equipped with a map $\mu\colon X \to X \vee X$ such that $q_i \circ \mu \simeq \mathrm{id}_X$, where $q_i\colon X \vee X \to X$ is the projection map, for $i = 1, 2$. Equivalently, if $j\colon X \vee X \to X \times X$ is the inclusion, then $j \circ \mu \simeq \Delta\colon X \to X \times X$. Examples of coHopf spaces include spheres S^q $(q \geqslant 1)$ and other reduced suspensions ΣY, where Y is any pointed Hausdorff space.

The elementary properties of such spaces are well-known, and are explained, for example, in pp121–127 of [34]. Most work has concentrated on three problems:

(i) Which spaces are coHopf?

(ii) When is a coHopf space associative up to homotopy?

(iii) When does a coHopf space (X, μ) have the primitive homotopy type of a suspension?

Included in the first of these problems is the *Ganea Conjecture:* that every connected coHopf space has the homotopy type of $Z \vee B$, where Z is simply connected, and B is a bouquet of circles. This conjecture has been proved for homotopy-associative coHopf CW-complexes by Berstein and Dror [2] and for coloops which are CW-complexes by Hilton, Mislin and Roitberg [13] (a coloop is a coHopf space (X, μ) for which $(\mathrm{id}_X \vee \nabla) \circ (\mu \vee \mathrm{id}_X)$ and $(\nabla \vee \mathrm{id}_X) \circ (\mathrm{id}_X \vee \mu)$ are homotopy equivalences $X \vee X \to X \vee X$). It is shown in [12] that the fundamental group of a coHopf space X is free, by consideration of the commutative diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[dr, "\Delta"'] & \pi_1(X \vee X) \arrow[r, "\cong"] \arrow[d] & \pi_1(X) * \pi_1(X) \arrow[d, "p"] \\
& \pi_1(X \times X) \arrow[r, "\cong"] & \pi_1(X) \times \pi_1(X),
\end{tikzcd}$$

where p is the projection homomorphism which sends all commutators to the identity.

More frequently, two- and three-cell complexes have been studied in the search for coHopf structures. The two-cell case was settled by Berstein and Hilton, who proved [4] that if $m-1 \geqslant q \geqslant 2$ and f is a map $S^{m-1} \to S^q$, then $S^q \cup_f e^m$ is coHopf if and only if f commutes up to homotopy with the coHopf structure maps on S^{m-1} and S^q (ie. f is primitive). In addition they produced a generalised Hopf invariant which vanishes if and only if f is primitive. Using this, they exhibited in [5] a space $X_3 = S^3 \cup_\alpha e^7$, where α is an element of order 3 in $\pi_6(S^3)$. This space is coHopf but does not have the homotopy type of a suspension.

The related question of whether or not coHopf spaces of this form are homotopy associative has been studied in some detail. Berstein [1] has proved that, for any odd prime p, and any $\alpha \in \pi_{2p}(S^3)$ of order p, that no coHopf structure on $X_p = S^3 \cup_\alpha e^{2p+1}$ is homotopy associative, by investigating properties of the Pontrjagin algebra $H_*(\Omega X; \mathbb{Z}_p)$. The associativity question has been settled for two-cell complexes by Chan [7, 8], who produced an invariant, the vanishing of which is both necessary and sufficient to prove the existence of a homotopy associative coHopf structure. Examples of homotopy-associative coHopf spaces which are not suspensions have been found by Berstein and Harper [3]. These spaces are of the form $S^5 \cup_\epsilon e^{35}$ (where ϵ is a generator of the 3-primary part of $\pi_{34}(S^5)$) and complexes with cells in dimensions 5, $6p-2$ and $2p+3$, where p is an odd prime.

Problem (iii) has been studied mainly by restricting the connectivity and dimensions of the spaces considered. As pointed out by Ganea [11], a $(q-1)$-connected CW-complex of dimension $\leqslant 2q-1$ has the homotopy type of a suspension. Ganea offers no proof. There follows a simple proof in the spirit of [11]:

Theorem 1.1.1. *Suppose that $q \geqslant 1$, and that X is a $(q-1)$-connected CW-complex of dimension $\leqslant 2q-1$. Then $X \simeq \Sigma Y$ for some CW-complex Y.*

PROOF. Assume first that $q > 1$. Ganea shows that the retraction $p\colon \Sigma\Omega X \to X$ is $(2q-1)$-connected. Since $H_{2q-1}(X)$ is free, we apply Theorem 2.1 of [5] to find a CW-complex K and a map $v\colon K \to \Omega X$ such that

$$v_*\colon H_i(K) \xrightarrow{\cong} H_i(\Omega X) \cong H_{i+1}(\Sigma\Omega X) \cong H_{i+1}(X) \quad \text{for } i < 2q-2;$$

$$H_{2q-2}(K) \xrightarrow{v_*} H_{2q-2}(\Omega X) \cong H_{2q-1}(\Sigma\Omega X) \xrightarrow{p_*} H_{2q-1}(X) \quad \text{is an isomorphism;}$$

$$H_i(K) = 0 \quad \text{for } i > 2q-2.$$

Then $\Sigma K \simeq X$, since $p \circ \Sigma v$ is a homology equivalence of simply-connected CW-complexes.

The case where $q = 1$ is easily proved, since a 0-connected 1-dimensional complex has the homotopy type of a bouquet of circles. □

More can be proved if the space is required to be coHopf. For example, Theorem A of [5] states:

Theorem 1.1.2. *If X is a $(q-1)$-connected space of the based homotopy type of a CW-complex, $q \geqslant 1$, with locally finitely generated homology,* *and if* $\dim X \leqslant 3q-3$*, then any coHopf structure* (X,μ) *is equivalent to a suspension structure.*

Thus, not only does X have the homotopy type of a suspension, but the homotopy equivalence may be chosen to be primitive, with respect to the suspension coHopf structure and the given map μ. In particular, (X,μ) is homotopy associative and admits a homotopy inversion. The example X_3 demonstrates, at least for $q=3$, that the above bound is the best possible. Under the assumption that the coHopf space (X,μ) is homotopy associative, we can strengthen the result, as in Corollary 3.5 of [11]:

Theorem 1.1.3. *Suppose that the CW-complex* X *is a* $(q-1)$*-connected coHopf space and that* $\dim X \leqslant 4q-5$*, where* $q \geqslant 2$*. Then* X *has the primitive homotopy type of a suspension if and only if it is homotopy associative.*

Given the subject matter of this thesis, some mention should be made of the paper of Woo and Kim [35], in which the definition of a coHopf space is extended to encompass ex-spaces (fibrewise pointed spaces). This is perhaps the earliest published paper referring to fibrewise coHopf spaces. Unfortunately the main results of this paper were anticipated and proved in greater generality in [20].

1.2 Fibrewise CoHopf Spaces

We now introduce the fibrewise analogue of the coHopf space:

Definition 1.2.1. *Let* X *be a fibrewise pointed space over* B*. A fibrewise comultiplication on* X *is a fibrewise pointed map* $\sigma\colon X \to X \vee_B X$*. A* *fibrewise coHopf structure on X is a fibrewise comultiplication σ such that the triangle*

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Delta"'] & X \vee_B X \arrow[d, "j"] \\ & X \times_B X \end{tikzcd}$$

commutes up to fibrewise pointed homotopy, where $\Delta\colon X \to X \times_B X$ is the fibrewise diagonal map and $j\colon X \vee_B X \hookrightarrow X \times_B X$ is the inclusion. If X admits a fibrewise coHopf structure, we say that X is fibrewise coHopf over B.

These definitions differ from those of Ganea [11]; what Ganea calls a "comultiplication" is what we call a "coHopf structure".

If the base space B is a point, then these definitions reduce to the standard (non-fibrewise) definitions. A trivial bundle $B \times F \to B$ admits a fibrewise coHopf structure (with respect to the section s defined by $s(b) = (b, *)$) if and only if the fibre F is a coHopf space. In particular the bundle $B \times S^1$ is fibrewise coHopf with respect to such a section. Much of the remainder of this section is devoted to proving various elementary results in the theory of fibrewise coHopf spaces, most of which reduce to well-known results when the base space is a point.

Initially we prove that our definition is equivalent to that of James (see p131 of [21]).

Lemma 1.2.2. *Let σ be a comultiplication on the fibrewise pointed space X over B. Then σ is a fibrewise coHopf structure if and only if $q_i \circ \sigma$ is fibrewise pointed homotopic to the identity on X for $i = 1, 2$, where $q_1, q_2\colon X \vee_B X \to X$ are the fibrewise projections.*

PROOF. Necessity is easily seen from the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Delta"'] & X \vee_B X \arrow[d, "j"] \arrow[dr, "q_i"] & \\ & X \times_B X \arrow[r, "q_i"] & X, \end{tikzcd}$$

which commutes up to fibrewise pointed homotopy. To prove sufficiency, choose fibrewise pointed homotopies $H^{(i)}: X \times I \to X$ from $q_i \circ \sigma$ to id_X (for $i = 1,2$). Then, by the universal property of the fibrewise product (see [21], p2), there is a fibrewise pointed homotopy $H: X \times I \to X \times_B X$ from $j \circ \sigma$ to the diagonal map $\Delta: X \to X \times_B X$. □

Lemma 1.2.3. *Let X be a fibrewise coHopf space over B with structure map σ and closed section. Suppose that $f: B' \to B$ is a continuous map. Then the pullback space $f^*(X)$ over B' is a fibrewise coHopf space.*

PROOF. First recall that, as on p92 of [21], there is a natural fibrewise pointed topological equivalence $f^*(X) \vee_{B'} f^*(X) \to f^*(X \vee_B X)$. The commutative square

$$\begin{tikzcd} f^*(X) \arrow[r, "\sigma\varphi"] \arrow[d] & X \vee_B X \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

defines a fibrewise pointed map $\theta: f^*(X) \to f^*(X \vee_B X)$, where $\varphi: f^*(X) \to X$ is the canonical projection. This is a fibrewise coHopf structure on $f^*(X)$, as may be seen from the diagram

$$\begin{tikzcd}
f^*(X) \arrow[rr, "\varphi"] \arrow[d, "\theta"'] & & X \arrow[d, "\sigma"] \\
f^*(X \vee_B X) \arrow[rr] \arrow[dr, "\sim"] \arrow[dd, "q_i"'] & & X \vee_B X \arrow[dd, "q_i"] \\
& f^*(X) \vee_{B'} f^*(X) \arrow[dl, "q_i"'] & \\
f^*(X) \arrow[rr, "\varphi"] \arrow[d] & & X \arrow[d] \\
B' \arrow[rr, "f"] & & B
\end{tikzcd}$$

which commutes up to fibrewise pointed homotopy. □

Lemma 1.2.4. *Let X be a fibrewise coHopf space over B with structure map σ, and let Y be a fibrewise pointed space over B. Suppose that the sections of X and Y are closed. Then the fibrewise smash product $X \wedge_B Y$ admits a fibrewise coHopf structure over B.*

PROOF. The fibrewise pointed maps σ and id_Y induce a fibrewise pointed map

$$\theta\colon X \wedge_B Y \to (X \vee_B X) \wedge_B Y.$$

The range of this map is equivalent, by Proposition (6.1) of [21], to the fibrewise space $(X \wedge_B Y) \vee_B (X \wedge_B Y)$. Postcomposition by the fibrewise projection maps

$$q_i\colon (X \wedge_B Y) \vee_B (X \wedge_B Y) \to (X \wedge_B Y) \qquad (\text{for } i = 1, 2)$$

gives maps which are fibrewise pointed homotopic to the identity on $(X \wedge_B Y)$ by naturality. □

In particular, the fibrewise reduced suspension of any fibrewise pointed space with closed section is fibrewise coHopf.

Lemma 1.2.5. *Let X be a fibrewise coHopf space over B with structure map σ, and suppose that the fibrewise pointed space Y over B is fibrewise pointed dominated by X. Then Y admits a fibrewise pointed coHopf structure. In particular, if X and Y are fibrewise pointed homotopy equivalent, then Y is fibrewise coHopf if and only if X is.*

PROOF. Let $f\colon X \to Y$, $g\colon Y \to X$ be fibrewise pointed maps such that $f \circ g$ is fibrewise pointed homotopic to id_Y. Define $\theta = (f \vee_B f) \circ \sigma \circ g\colon Y \to Y \vee_B Y$. Then, for $i = 1, 2$,

$$\begin{aligned} q_i \circ \theta &= q_i \circ (f \vee_B f) \circ \sigma \circ g \\ &= f \circ q_i \circ \sigma \circ g \\ &\simeq_B^B f \circ g \\ &\simeq_B^B \mathrm{id}_Y, \end{aligned}$$

and so θ is a fibrewise coHopf structure on Y.

The most conspicuous application of (non-fibrewise) coHopf structures is in the construction of the multiplication on the homotopy groups of a space using the suspension comultiplication on a sphere. This generalises to the fibrewise category:

Lemma 1.2.6. *Let $\sigma\colon X \to X \vee_B X$ be a fibrewise coHopf structure on the fibrewise pointed space X over B. Suppose that the section of X is closed, and that X is fibrewise compact and fibrewise regular. Then, if Y is a fibrewise pointed space over B, σ induces a fibrewise Hopf structure* *on the fibrewise pointed mapping space* $M = \mathrm{Maps}_B^B(X,Y)$, *and hence a multiplication on the set* $[X,Y]_B^B$ *of fibrewise pointed homotopy classes of fibrewise maps* $X \to Y$. *These constructions are natural with respect to fibrewise pointed maps defined on* Y. *Conversely, if there is a multiplication on* $[X,Y]_B^B$ *for all fibrewise pointed* Y *over* B, *natural with respect to fibrewise pointed maps defined on* Y, *which has a two-sided unit given by the fibrewise pointed homotopy class* $[c]$ *of the fibrewise constant map* $c\colon X \to Y$, *then* X *is fibrewise coHopf.*

PROOF. We first define a fibrewise multiplication $\mu\colon M \times_B M \to M$ by the formula

$$\mu(f,g) = \nabla \circ (f \vee_B g) \circ \sigma.$$

Continuity of μ follows from the continuity of the fibrewise map η from $M \times_B M$ to the space of fibrewise pointed maps $X \vee_B X \to X \vee_B X$ defined by $\eta(f,g) = f \vee_B g$. (The map η is continuous by Proposition (9.19) of [21].) Naturality of μ follows immediately from the naturality of the folding map ∇; if $\psi\colon Y \to Z$ is a fibrewise pointed map over B, then $\mu(\psi \circ f, \psi \circ g) = \psi \circ \mu(f,g)$.

It remains to show that μ defines a fibrewise Hopf structure on M. Suppose that $H\colon X \times I \to X \times_B X$ is a fibrewise pointed homotopy from $j \circ \sigma$ to Δ. We define a fibrewise selfhomotopy $\bar{H}$ of M by the formula

$$\bar{H}(f,t) = f \circ q_1 \circ H_t',$$

Then $\bar{H}$ is a continuous fibrewise pointed map such that

$$\bar{H}(f,0) = f \circ q_1 \circ j \circ \sigma = f \circ q_1 \circ \sigma = \mu(f,c) \quad \text{and}$$

$$\bar{H}(f,1) = f \circ q_1 \circ \Delta = f.$$

Thus if $i_1 : M \to M \times_B M$ is the fibrewise inclusion into the first coördinate, then $\mu \circ i_1 \simeq^B_B \mathrm{id}_M$. The proof that $\mu \circ i_2 \simeq^B_B \mathrm{id}_M$ is similar.

Since X is fibrewise compact and fibrewise regular, we may apply Proposition (3.99) of [20] to identify $[X,Y]^B_B$ with the set of vertical homotopy classes of sections of M, thus obtaining a multiplication on $[X,Y]^B_B$.

Now suppose that for all fibrewise pointed spaces Y, $[X,Y]^B_B$ admits a multiplication with two-sided identity $[c]$, natural with respect to fibrewise pointed maps defined on Y. Then there is a fibrewise pointed map $\theta\colon X \to X \vee_B X$, which is the product of the maps $i_1, i_2\colon X \to X \vee_B X$. The compositions $q_i \circ \theta$ are fibrewise pointed homotopic to the identity on X, by naturality. Thus X is fibrewise coHopf over B. □

In fact, it is not necessary to assume fibrewise compactness and fibrewise regularity, merely to obtain a multiplication on $[X,Y]^B_B$. The formula

$$\mu_*([f],[g]) = [\nabla \circ (f \vee_B g) \circ \sigma]$$

defines a multiplication on fibrewise pointed homotopy classes for all fibrewise coHopf spaces (X,σ).

Dually, a fibrewise Hopf structure on a space Y over B induces a multiplication on $[X,Y]^B_B$ for any fibrewise pointed space X over B. In the case where X is fibrewise coHopf and Y is fibrewise Hopf, the following result holds:

Lemma 1.2.7. *Suppose that X, Y are fibrewise pointed spaces over B, such that X is fibrewise coHopf and Y is fibrewise Hopf, with structure maps $\sigma\colon X \to X \vee_B X$ and $\mu\colon Y \times_B Y \to Y$ respectively. Then the two* *induced multiplications on $G = [X,Y]^B_B$ agree and are commutative and associative.*

PROOF. Let $f, g\colon X \to Y$ be fibrewise pointed maps. Then the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Delta"'] & X \vee_B X \arrow[r, "f \vee g"] \arrow[d] & Y \vee_B Y \arrow[d] \arrow[dr, "\nabla"] & \\ & X \times_B X \arrow[r, "f \times g"] & Y \times_B Y \arrow[r, "\mu"] & Y \end{tikzcd}$$

commutes up to fibrewise pointed homotopy. Hence the two multiplications on G agree.

Now consider the postcomposition function $\mu_*\colon [X, Y \times_B Y]^B_B \to [X,Y]^B_B$. By naturality, this is a homomorphism. Let φ be the natural isomorphism

$$[X,Y]^B_B \times [X,Y]^B_B \to [X, Y \times_B Y]^B_B$$

(see p127 of [21]). Then $\mu_* \circ \varphi$ defines a homomorphism from $G \times G$ to G. But $\mu_* \circ \varphi([f],[g]) = [\mu \circ (f \times_B g) \circ \Delta]$, and so $\mu_* \circ \varphi$ is the multiplication on G induced by μ and σ. The condition that this is a homomorphism is precisely that, for all classes α, β, γ, δ in $[X,Y]^B_B$, $(\alpha\beta)(\gamma\delta) = (\alpha\gamma)(\beta\delta)$. Putting $\alpha = \delta = [c]$, the class of the fibrewise constant map, we see that the multiplication is commutative. Putting $\gamma = [c]$, we see that it is associative. □

Definition 1.2.8. *Let X, Y be fibrewise pointed spaces over B, admitting fibrewise coHopf structures σ, τ respectively. Let $f\colon X \to Y$ be a fibrewise pointed map. We say that f is primitive (with respect to σ and τ) if the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"] \\ X \vee_B X \arrow[r, "f \vee f"] & Y \vee_B Y \end{tikzcd}$$

commutes up to fibrewise pointed homotopy.

We will also refer to a primitive fibrewise map f as a homomorphism, or a co-H-map.

Definition 1.2.9. *Let X be a fibrewise pointed space over B admitting a fibrewise coHopf structure σ. We say that (X, σ) is fibrewise homotopy commutative if the diagram*

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\sigma"] & \\ X \vee_B X \arrow[rr, "\tau"] & & X \vee_B X \end{tikzcd}$$

commutes up to fibrewise pointed homotopy, where τ is the fibrewise switching map. We say that (X, σ) is fibrewise homotopy associative if the square

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\sigma"] & \\ X \vee_B X \arrow[dr, "\sigma \vee \mathrm{id}"'] & & X \vee_B X \arrow[dl, "\mathrm{id} \vee \sigma"] \\ & X \vee_B X \vee_B X & \end{tikzcd}$$

commutes up to fibrewise pointed homotopy. We say that (X, σ) admits a fibrewise homotopy inversion if there is a fibrewise pointed map $\eta \colon X \to X$ such that the diagram

$$\begin{tikzcd} & X \vee_B X \arrow[r, "\mathrm{id} \vee \eta"] & X \vee_B X \arrow[dr, "\nabla"] & \\ X \arrow[ur, "\sigma"] \arrow[rrr, "c"] \arrow[dr, "\sigma"'] & & & X \\ & X \vee_B X \arrow[r, "\eta \vee \mathrm{id}"] & X \vee_B X \arrow[ur, "\nabla"'] & \end{tikzcd}$$

commutes up to fibrewise pointed homotopy, where $c\colon X \to X$ is the fibrewise constant map. A fibrewise cogroup (or fibrewise cogroup-like space) is a fibrewise homotopy associative fibrewise coHopf space with a fibrewise homotopy inversion.

If a fibrewise coHopf space over B with closed section admits a fibrewise homotopy commutative coHopf structure, then the induced coHopf structure on $X \wedge_B Y$ is also fibrewise homotopy commutative for any fibrewise pointed space Y over B with closed section. Analogous results hold for fibrewise associativity and for the existence of a fibrewise inversion. In particular, the fibrewise reduced suspension of a fibrewise space with closed section is a fibrewise cogroup.

Lemma 1.2.10. *Suppose that (X, σ) is a fibrewise coHopf space over B, and that Y is a fibrewise pointed space over B. Then the induced multiplication on $[X, Y]^B_B$ is associative if (X, σ) is fibrewise homotopy associative, commutative if (X, σ) is fibrewise homotopy commutative, and admits inverses if (X, σ) admits a fibrewise homotopy inversion.*

In particular, if (X, σ) is a fibrewise cogroup, then $[X, Y]^B_B$ is a group. In this case we write it as $\pi^B_B(X, Y)$. We will call such a group a generalised fibrewise homotopy group.

Lemma 1.2.11. *Let X and Y be fibrewise compact and fibrewise regular fibrewise coHopf spaces over B with closed sections. Then there are two fibrewise coHopf structures on $X \wedge_B Y$, that obtained from X and that obtained from Y. These two structures are fibrewise pointed homotopic, and are fibrewise associative and commutative.*

PROOF. Let $Z = (X \wedge_B Y) \vee_B (X \wedge_B Y)$, and consider the set of fibrewise pointed homotopy classes of maps from $X \wedge_B Y$ to Z. There is a homomorphism

$$[X \wedge_B Y, Z]_B^B \to [X, \mathrm{Maps}_B^B(Y, Z)]_B^B,$$

which is an isomorphism by Corollary (9.26) of [21] and Proposition (3.99) of [20]. However, the multiplication on $[X, \mathrm{Maps}_B^B(Y, Z)]_B^B$ is both associative and commutative, by (1.2.7), and the multiplications induced by the fibrewise coHopf structure of X and the fibrewise Hopf structure of $\mathrm{Maps}_B^B(Y, Z)$ coincide. The latter multiplication is equivalent under the above isomorphism to that induced on $[X \wedge_B Y, Z]_B^B$ by the coHopf structure on Y.

Each comultiplication is fibrewise pointed homotopic to the product of the injections $i_1, i_2 \colon X \wedge_B Y \to Z$, and they are therefore fibrewise pointed homotopic to each other. Fibrewise homotopy commutativity and associativity follow from the commutativity and associativity of the product on $[X \wedge_B Y, Z]_B^B$. □

Well-sectioned spaces. Before proceeding further, we should note that many of the spaces that we shall work with are fibrewise well-pointed with closed sections. We need to prove that the class of such spaces is closed under some standard constructions:

Lemma 1.2.12. *Let X be a fibrewise well-pointed space over B with closed section. Then $X \vee_B X$, $X \times_B X$, $\Sigma_B^B X$, $\Omega_B^B X$ are all fibrewise well-pointed with closed section.*

PROOF. The square

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "s"'] & X \arrow[d, "i_1"] \\ X \arrow[r, "i_2"] & X \vee_B X \end{tikzcd}$$

is a fibrewise pushout. Apply Proposition (20.2) of [21] to show that $X \vee_B X$ is fibrewise well-pointed. Closure of the section follows from the definition of the quotient topology.

Since (X, B) is a closed fibrewise cofibred pair, by Corollary (20.20) of [21] the pair $(\Sigma_B X, \Sigma_B B)$ is closed and fibrewise cofibred, and we may collapse $\Sigma_B B$ to deduce that $\Sigma_B^B X$ is fibrewise well-pointed with closed section.

We will construct fibrewise Strøm structures for $(\Omega_B^B X, B)$ and $(X \times_B X, B)$ from one on X. Embed B in X via the section. Recall that a fibrewise Strøm structure on (X, B) is a map $\alpha\colon X \to I$ which is zero on B, and a fibrewise homotopy $H\colon I \times X \to X$ rel B such that $H_0 = \mathrm{id}_X$ and $H(t,x) \in B$ whenever $t > \alpha(x)$.

A Strøm structure on $(X \times_B X, B)$ may be given by the formulæ

$$\alpha'(x,y) = \max(\alpha(x), \alpha(y))$$
$$H'_t(x,y) = (H_t(x), H_t(y)).$$

Define $\alpha''\colon \Omega_B^B X \to I$ by $\alpha''(\omega) = \sup\{\alpha(\omega(t)) : t \in I\}$. Then the fibrewise homotopy $H''\colon I \times \Omega_B^B X \to \Omega_B^B X$ defined by $H''_t(\omega)(s) = H_t(\omega(s))$ is a fibrewise homotopy of $id_{\Omega_B^B X}$ rel B. Moreover, if $t > \alpha''(X)$ then $t > \alpha(\omega(s))$ for all $s \in I$. Hence $H''(t, \omega(s)) \in B$ for all $s \in I$, and so $H''(t,\omega)$ is the basepoint in the fibre of ω. Thus (Ω_B^B, B) admits a fibrewise Strøm structure, and so $\Omega_B^B X$ is fibrewise well-pointed with closed section. □

We will wish to use $SO(n)$ sphere bundles as examples. Indeed, in the later sections they will be the prime objects of study. From Proposition III.4.8 of [26], we know that any such bundle admits a section if and only if it is an unreduced fibrewise suspension of a $SO(n-1)$ sphere bundle. Therefore, all such bundles are fibrewise well-pointed.

The main reason for using the fibrewise well-pointedness of our spaces is the following lemma, which is a special case of Proposition (20.12) of [21]:

Lemma 1.2.13. *Let $\varphi\colon X \to Y$ be a fibrewise pointed map, where X and Y are fibrewise well-pointed spaces over B. If φ is a fibrewise homotopy equivalence then φ is a fibrewise pointed homotopy equivalence.*

We are now able to prove that, for X in a certain class of fibrewise coHopf spaces, the condition that X admit an inversion is automatically satisfied.

Lemma 1.2.14. *Suppose (X, σ) is a fibrant fibrewise coHopf space over B, and suppose further that X is fibrewise well-pointed, has simply connected fibres of the pointed homotopy type of a CW-complex, and that B admits a numerable categorical covering. Then X admits a left and a right fibrewise inversion. Moreover, if the coHopf structure on X is homotopy associative, then the two fibrewise inversions are fibrewise pointed homotopic.*

PROOF. As usual, we embed B in X via the section. We prove that X admits a right fibrewise inversion. Define the fibrewise pointed map k to be the composition

$$X \vee_B X \xrightarrow{\mathrm{id} \vee \sigma} X \vee_B X \vee_B X \xrightarrow{\nabla \vee \mathrm{id}} X \vee_B X.$$

We wish to show that k is a fibrewise homotopy equivalence. Let V be a member of the numerable covering of B. Then X_V is a trivial fibration over V, and so therefore is $(X \vee_B X)_V$. Then the induced homomorphism

$$k_*\colon H_*(X_V \vee_V X_V) \to H_*(X_V \vee_V X_V)$$

is an isomorphism, and so, since the fibres have the homotopy type of simply-connected CW-complexes, k is a fibrewise homotopy equivalence over each such V. By Satz 9.1 of [31] we see that k is a fibrewise homotopy equivalence. Since $X \vee_B X$ is fibrewise well-pointed, we may apply (1.2.13) to show that k is a fibrewise pointed homotopy equivalence. Let φ be a fibrewise pointed homotopy inverse for k. Let $j = q_1 \circ \varphi \circ i_2$:

$$X \xrightarrow{i_2} X \vee_B X \xrightarrow{\varphi} X \vee_B X \xrightarrow{q_1} X.$$

We claim that j is a right inversion for the coHopf structure σ.

The diagrams

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "i_1"'] & X \vee_B X \arrow[d, "k"] \arrow[dr, "\mathrm{id}"] & & \\
& X \vee_B X \arrow[r, "\varphi"] & X \vee_B X \arrow[r, "q_1"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "i_2"] \arrow[dr, "\sigma"'] & X \vee_B X \arrow[d, "k"] \arrow[dr, "\mathrm{id}"] & \\
& X \vee_B X \arrow[r, "\varphi"] & X \vee_B X
\end{tikzcd}$$

commute up to fibrewise pointed homotopy, and the composition $X \to X$ of the first is fibrewise pointed homotopic to the identity. We wish to show that $\nabla \circ (\mathrm{id}_X \vee_B j) \circ \sigma$ is fibrewise pointed nulhomotopic. The following diagram commutes up to fibrewise pointed homotopy:

$$\begin{tikzcd}
X \arrow[r, "\sigma"] & X \vee_B X \arrow[r, "(\mathrm{id}_X \vee_B j)"] \arrow[dr, "i_1 \vee_B i_2"'] & X \vee_B X \arrow[r, "\nabla"] & X \\
& & (X \vee_B X) \vee_B (X \vee_B X) \arrow[u, "(q_1 \varphi) \vee_B (q_1 \varphi)"'] \arrow[r, "\nabla"] & X \vee_B X \arrow[u, "q_1 \varphi"']
\end{tikzcd}$$

and therefore $\nabla \circ (\mathrm{id}_X \vee_B j) \circ \sigma$ is fibrewise pointed homotopic to $q_1 \circ \varphi \circ \sigma$, and so to $q_1 \circ i_2$, which is fibrewise pointed nulhomotopic, as required. The left fibrewise inversion follows by symmetry.

Suppose j_L, j_R are fibrewise left and right inversions for an associative fibrewise coHopf space X over B. Then, writing $*$ for the (unique) fibrewise pointed constant map $X \to X$, we have

$$\begin{aligned}
j_L &\simeq^B_B \nabla(j_L \vee_B *)\sigma \\
&\simeq^B_B \nabla(j_L \vee_B (\nabla(\mathrm{id}_X \vee_B j_R)\sigma))\sigma \\
&\simeq^B_B \nabla(\nabla(j_L \vee_B \mathrm{id}_X)\sigma \vee_B j_R)\sigma \\
&\qquad \text{(since } \sigma \text{ is fibrewise homotopy associative)} \\
&\simeq^B_B \nabla(* \vee_B j_R)\sigma \\
&\simeq^B_B j_R.
\end{aligned}$$

□

The fibrewise pointed map $k\colon X \vee_B X \to X \vee_B X$ defined above is called the *fibrewise coshear map.*

It is known (see [4]) that there exist coHopf spaces which do not have the homotopy type of a suspension. These may certainly be viewed as fibrewise coHopf spaces over a point which are not fibrewise pointed homotopy equivalent to fibrewise reduced suspensions of any fibrewise pointed space. However, from the point of view of fibrewise homotopy theory, this is not in itself of great interest. What we will seek in later chapters are fibrewise coHopf spaces whose fibres are suspensions, but which do not have the fibrewise pointed homotopy type of fibrewise reduced suspensions. In particular, we will investigate the case where the fibre is a sphere.

The first examples of such fibrewise coHopf spaces may be constructed using the main result of [14], which characterises the fibrewise coHopf sectioned S^{q-1}-bundles over a sphere S^n as follows:

Proposition 1.2.15. *(James) Let E be the sectioned S^q-bundle over S^n which has classifying map $\alpha \in \pi_{n-1}(SO(q))$. Then E is fibrewise coHopf if and only if $\Sigma^q_* \rho_* \alpha = 0$, where $\rho\colon SO(q) \to S^{q-1}$ is the evaluation map.*

Corollary 1.2.16. *(James) If E is a fibrewise coHopf S^q-bundle over S^n and q is even then E has the fibrewise pointed homotopy type of a reduced suspension.*

Corollary 1.2.17. *(James) If E is a fibrewise coHopf S^q-bundle over S^n and $n < 2q-2$ then E has the fibrewise pointed homotopy type of a reduced suspension.*

In fact, the argument given in [14] shows, in conjunction with Corollary 1.7 of [16], that the conclusions of (1.2.17) apply when $n \leqslant 2q$, provided that $n \neq 2q-2$.

The following results will be used when we wish to work with sphere bundles:

Lemma 1.2.18. *Let E be an n-sphere bundle over the complex B. Then E is fibrewise closed, fibrewise compact and fibrewise regular over B.*

PROOF. Fibrewise closure follows from local triviality and Proposition (1.8) of [21]. Fibrewise compactness follows from Proposition (3.2), while fibrewise regularity follows from local triviality and the regularity of the fibre S^n. □

Lemma 1.2.19. *The unstable J-homomorphism*

$$J\colon \pi_3(SO(3)) \to \pi_6(S^3)$$

is an epimorphism.

Proof. Identify S^3 with the group of quaternions of absolute value 1. As shown on p115 of [28], the homotopy class $[\rho]$ of the homomorphism $\rho\colon S^3 \to SO(3)$ defined by $\rho(q)(r) = qrq^{-1}$ generates the infinite cyclic group $\pi_3(SO(3))$. It is shown in [33] that $J[\rho]$ is a generator for $\pi_6(S^3) \cong \mathbb{Z}_{12}$. □

We may therefore exhibit our first sphere bundle which is fibrewise coHopf but does not have the fibrewise pointed homotopy type of a fibrewise reduced suspension:

Example. Let α be a generator of $\pi_3(SO(3))$, and let E_3 be a sectioned S^3-bundle over S^4 formed by the clutching construction on 4α. Embed B in E_3 via the section. Then $HJ\alpha = 0$, and so E_3 is a fibrewise coHopf space. However, if E_3 were to have the fibrewise pointed homotopy type of a fibrewise reduced suspension, the space E_3/B would have the homotopy type of a suspended space. But this space is just the non-suspended coHopf space X_3 that we met earlier. Hence E_3 does not have the fibrewise pointed homotopy type of a fibrewise reduced suspension.

The theory of fibrewise coHopf spaces has strong links with the theory of fibrewise pointed Lusternik-Schnirelmann category. The following results are due to James and Morris (Propositions 6.1 and 6.2 of [22]):

Proposition 1.2.20. *(James/Morris) Let X be a fibrewise pointed space over B. Suppose that X admits a fibrewise pointed categorical neighbourhood of B. If, for some $n \geqslant 1$, the diagonal*

$$\Delta\colon X \to \Pi_B^n X$$

can be compressed into $\Pi_B^n(X,B)$ *by a fibrewise pointed homotopy, then* $\mathrm{cat}_B^B X \leqslant n$.

Proposition 1.2.21. *(James/Morris) Let* X *be a fibrewise space over* B *with closed section such that* X*, as a space, is fully normal. If* $\mathrm{cat}_B^B X \leqslant n$*, for some* $n \geqslant 1$*, then the diagonal*

$$\Delta : X \to \Pi_B^n X$$

can be compressed into $\Pi_B^n(X,B)$ *by a fibrewise pointed homotopy.*

In particular, putting $n = 2$, then Δ can be compressed into $\Pi_B^2(X,B)$ by a fibrewise pointed homotopy if and only if X is fibrewise coHopf. We may therefore use the following result from §7 of [22] to deduce that certain sectioned fibre bundles are fibrewise coHopf for dimensional reasons.

Lemma 1.2.22. *(James/Morris) Let* X *be a sectioned fibre bundle over the base* B *with fibre* F*. Suppose that* F *is* $(r-1)$*-connected, where* $r > 1$*, and that* (X,B) *is a CW-pair. Then if* $\dim X < nr$ *then* $\mathrm{cat}_B^B X \leqslant n$.

We conclude this section with a result showing that, for spaces lying in a certain class, the question of the existence of a fibrewise coHopf structure may be posed as an extension problem. The proof is essentially that of James (Proposition (2.2) of [14]).

Lemma 1.2.23. *Suppose that* B *is a connected space admitting a numerable categorical cover (for example a paracompact locally contractible space). Let* X *be a fibrewise non-degenerate fibrewise pointed space over* B *such that the projection* $p: X \to B$ *is a fibration. Suppose that there is a* *coHopf structure on a fibre X_b which can be extended to a fibrewise pointed map $\theta\colon X \to X \vee_B X$. Then X admits a fibrewise coHopf structure.*

PROOF. Let q_1, q_2 be the fibrewise projections $X \vee_B X \to X$. By Theorem (6.3) of [10] the compositions $q_i \circ \theta\colon X \to X$ are fibrewise homotopy equivalences for $i = 1, 2$, and are thus fibrewise pointed homotopy equivalences. Choose fibrewise pointed homotopy inverses α_1, α_2. Then $(\alpha_1 \vee_B \alpha_2) \circ \theta\colon X \to X \vee_B X$ is a fibrewise coHopf structure on X. □

1.3 A Universal Fibrewise CoHopf Sphere Bundle

We now attempt to construct a universal fibrewise coHopf n-sphere bundle. Let $\pi\colon E(n) \to BO(n)$ be the universal sectioned n-sphere bundle. Without loss of generality, we may assume that $BO(n)$ is a CW-complex. Define, for a sectioned fibrewise space X over B, the space

$$\mathrm{coH}_B(X) = \left\{ f \in \mathrm{Maps}_B^B(X, X \vee_B X) : f \text{ is a coHopf structure map} \right\}.$$

Now consider the fibrewise projection map

$$\pi_2\colon E(n) \times_{BO(n)} \mathrm{coH}_{BO(n)}(E(n)) \to \mathrm{coH}_{BO(n)}(E(n))$$

onto the second coördinate. This is induced by the pullback

$$\begin{array}{ccc} & & E(n) \\ & & \downarrow \\ \mathrm{coH}_{BO(n)}(E(n)) & \longrightarrow & BO(n) \end{array}$$

and is therefore a sectioned n-sphere bundle. Since $\mathrm{coH}_{BO(n)}(E(n))$ is a bundle over $BO(n)$ with fibre a path-component of $\Omega^n(S^n \vee S^n)$, it is paracompact and locally contractible and has the homotopy type of a CW-complex. We claim that

$$E_c(n) = E(n) \times_{BO(n)} \mathrm{coH}_{BO(n)}(E(n))$$

is fibrewise coHopf over $B_c(n) = \mathrm{coH}_{BO(n)}(E(n))$ with respect to the projection π_2 and the section induced by the section of $E(n)$ over $BO(n)$.

There is a map $\Theta\colon E_c(n) \to E_c(n) \vee_{B_c(n)} E_c(n)$ induced by the square

$$\begin{tikzcd}
E(n) \times_{BO(n)} \mathrm{coH}_{BO(n)}(E(n)) \arrow[r, "\mathrm{eval}"] \arrow[d, "\pi_2"'] & E(n) \vee_{BO(n)} E(n) \arrow[d] \\
\mathrm{coH}_{BO(n)}(E(n)) \arrow[r] & BO(n)
\end{tikzcd}$$

which is a coHopf structure on each fibre. Recall that sectioned sphere bundles are fibrewise well-pointed and hence fibrewise non-degenerate. It follows from (1.2.23) that there is a fibrewise coHopf structure $\bar{\Theta}$ on $E_c(n)$ over $B_c(n)$ of the form $(\alpha_1 \vee_{B_c(n)} \alpha_2) \circ \Theta$, for a suitable pair of fibrewise pointed homotopy equivalences $\alpha_1, \alpha_2\colon E_c(n) \to E_c(n)$ over $B_c(n)$.

Now let X be a sectioned n-sphere bundle over the connected complex B with fibrewise coHopf structure map $\theta\colon X \to X \vee_B X$. From (1.2.18), X is fibrewise compact and fibrewise regular over B. Recall that X is bundle equivalent to a pullback $\varphi^*(E(n))$, for some map $\varphi\colon B \to BO(n)$. By Proposition (3.99) of [20], the map θ induces a map

$$B \to \mathrm{coH}_B(X) \simeq_B^B \mathrm{coH}_B(\varphi^*(E(n))) \simeq_B^B \varphi^*(\mathrm{coH}_{BO(n)}(E(n))),$$

and hence a map $B \to \mathrm{coH}_{BO(n)}(E(n))$ such that Θ pulls back to give θ. In particular, putting $B = B_c(n)$, there is a homotopy equivalence $\epsilon\colon B_c(n) \to B_c(n)$

such that $(\alpha_1 \vee_B \alpha_2) \circ \Theta$ is the pullback of Θ via ϵ. By Theorem 6.21 of [31] and (1.2.13), this map ϵ is a fibrewise pointed homotopy equivalence over $BO(n)$, since sectioned sphere bundles are fibrant and fibrewise well-pointed. Thus φ lifts to a map ψ:

$$\begin{tikzcd} & B_c(n) \arrow[d] \\ B \arrow[r, "\varphi"] \arrow[ur, dashed, "\psi"] & BO(n) \end{tikzcd}$$

such that $(\psi^*(E_c(n)), \psi^*\bar{\Theta})$ is primitively fibrewise pointed homotopy equivalent to (X, θ).

We have therefore proved

Proposition 1.3.1. *There is a universal fibrewise coHopf n-sphere bundle*

$$\begin{tikzcd} S^n \arrow[r] & E_c(n) \arrow[d] \\ & B_c(n) \end{tikzcd}$$

such that any fibrewise coHopf n-sphere bundle over a connected complex B is primitively induced by a map $\psi\colon B \to B_c(n)$. We may view the classifying map $B_c(n) \to BO(n)$ for this bundle as a fibrewise structure on $B_c(n)$ over $BO(n)$ such that homotopies of a map $X \to B_c(n)$ correspond to primitive bundle self-equivalences of the pullback bundle, while fibrewise pointed vertical homotopies over $BO(n)$ induce fibrewise homotopies of the fibrewise coHopf structure on the pullback over X.

We may use this result to demonstrate the existence of bundles with more than one fibrewise pointed homotopy class of fibrewise coHopf structure. For example, let E be the fibrewise suspension of a trivial S^1-bundle over S^1, and give E the polar section. Then fibrewise pointed homotopy classes of fibrewise coHopf structures on E are in bijective correspondence with fibrewise pointed vertical homotopy classes of liftings

$$\begin{tikzcd} & B_c(1) \arrow[d] \\ S^1 \arrow[ur, dashed] \arrow[r, "c"] & BO(1), \end{tikzcd}$$

where c is the constant map to the basepoint of $BO(1)$, and are therefore classified by elements of $\pi_1(\mathrm{coH}(S^2)) \cong \pi_1(\Omega^2(S^2 \vee S^2)) \cong \pi_3(S^2 \vee S^2)$. By the Hilton-Milnor theorem this is a non-trivial group, and so there exist distinct classes of fibrewise coHopf structure on E.

Indeed, if we examine the homotopy sequence of the fibration $B_c(1) \to BO(1)$:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2(BO(1)) \arrow[r] \arrow[d, "\cong"'] & \pi_1(\mathrm{coH}(S^2)) \arrow[r] \arrow[d, "\cong"'] & \pi_1(B_c(1)) \arrow[r] & \pi_1(BO(1)) \\ & 0 & \mathbb{Z} \oplus \mathbb{Z} \oplus (\text{extra terms}) & & \end{tikzcd}$$

it is apparent that we may choose classifying elements in $\pi_1(\mathrm{coH}(S^2))$ which have different images in $\pi_1(B_c(1))$. Any two such elements induce fibrewise pointed homotopy classes θ_1, θ_2 of fibrewise coHopf structure on E such that there is no primitive fibrewise pointed bundle equivalence $(E, \theta_1) \to (E, \theta_2)$.

We may also consider the case where B is a cogroup-like space (for example, a suspension).

Corollary 1.3.2. *Let (E, θ) be a fibrewise coHopf n-sphere bundle over a cogroup-like CW-complex B. Then the set of fibrewise pointed homotopy classes of fibrewise coHopf structures on E is in bijective correspondence with the image of the homomorphism*

$$i_* \colon [B, \mathrm{coH}(S^n)] \to [B, B_c(n)]$$

induced by the inclusion of a fibre.

PROOF. The bundle E is induced by a map $\varphi\colon B \to BO(n)$, and the fibrewise pointed homotopy class of θ by a fibrewise pointed vertical homotopy class $[\psi]$ of liftings $B \to B_c(n)$. Since B is a cogroup-like space, we may form the class $[\psi] + i_*[\alpha] \in [B, B_c(n)]$ for any $\alpha \in [B, \mathrm{coH}(S^n)]$. This class is sent by π_* to the class $[\varphi] \in [B, BO(n)]$, and therefore contains a lifting of φ and all liftings which are fibrewise pointed vertically homotopic to it, since $\pi\colon B_c(n) \to BO(n)$ is a fibration.

Conversely, if $[\psi_0] \in [B, B_c(n)]$ is such that $\pi_*[\psi_0] = [\varphi]$, we may construct the class $[\psi_0] - [\psi] \in [B, B_c(n)]$. The image of this under π_* is the zero element of $[B, BO(n)]$. Since π is a fibration, we may lift a corresponding nulhomotopy $B \times I \to BO(n)$ to demonstrate that $[\psi_0] - [\psi]$ contains a map whose image lies entirely within a fibre. Thus $[\psi_0] - [\psi]$ lies in the image of i_*. $\square$

Chapter 2

Fibrewise CoHopf Spaces and Dimension

As we stated in Chapter 1, the reduced fibrewise suspension $\Sigma_B^B X$ of any fibrewise pointed space X over B with closed section admits a fibrewise coHopf structure. We will be seeking, *inter alia*, constraints on a fibrewise coHopf space E which ensure that it has the fibrewise pointed homotopy type of a fibrewise reduced suspension. Our aim in this chapter is to present such constraints in the form of conditions on the dimension of E.

2.1 Fibrewise Coretractions

Recall (see [15]) that a pointed CW-complex X is a Hopf space if and only if it is a retract of $\Omega\Sigma X$ with respect to the canonical inclusion $X \to \Omega\Sigma X$. The dual of this result appears in Theorem 1.1 of Ganea's paper [11], which states that the set of coHopf structures on a pointed CW-complex X is in bijective correspondence with the set of right homotopy inverses of the canonical projection $\Sigma\Omega X \to X$. As Ganea points out, the correspondence between homotopy classes of Hopf structures and homotopy classes of retractions $\Omega\Sigma X \to X$ is not necessarily bijective; S^3 admits 12 Hopf structures up to homotopy, but there are infinitely many homotopy classes of retractions $\Omega S^4 \to S^3$. We will prove a fibrewise version of Ganea's result in this section.

Definition 2.1.1. *Let X be a fibrewise pointed space over B. There is a natural fibrewise pointed map $p\colon \Sigma_B^B \Omega_B^B X \to X$, which is the adjoint of the identity map on $\Omega_B^B X$. A* fibrewise coretraction *is a fibrewise pointed map $\gamma\colon X \to \Sigma_B^B \Omega_B^B X$ such that $p \circ \gamma \simeq_B^B id_X$.*

The fibrewise map p is continuous by Corollary (9.24) of [21]. If X is a fibrewise reduced suspension $\Sigma_B^B D$, then the fibrewise suspension of the canonical inclusion $D \to \Omega_B^B \Sigma_B^B D$ is a fibrewise coretraction.

The purpose of this section is therefore to show the connection between fibrewise coHopf structures on a space X and fibrewise coretractions $X \to \Sigma_B^B \Omega_B^B X$. Before we can do this, however, we must prove some technical results.

Let X be a fibrewise well-pointed space over B with projection p_X and closed section, and embed B in X via the section. We identify $\mathrm{Maps}_B(I \times B, X)$ with the subspace of $\mathrm{Maps}(I, X)$ which consists of paths ω such that $p_X \circ \omega$ is a constant map.

Lemma 2.1.2. *The inclusion*

$$u\colon \Omega_B^B X \hookrightarrow P_B^B X = \{\, \omega \in \mathrm{Maps}_B(I \times B, X) : \omega(0) \in B \,\}$$

is a closed fibrewise cofibration, and hence a fibrewise pointed cofibration.

Proof. We prove that u is a closed fibrewise cofibration by exhibiting a fibrewise Strøm structure on the pair $(P_B^B X, \Omega_B^B X)$. Since X is fibrewise well-pointed with closed section, there is a fibrewise Strøm structure on (X, B), i.e. a map $\alpha\colon X \to I$ such that $\alpha(B) = 0$, and a fibrewise homotopy $H\colon X \times I \to X$ rel B of id_X such that $H_t(x) \in B$ whenever $t > \alpha(x)$. As on p138 of [21], we assume that $H_t(x) \in B$

whenever $t \geqslant \alpha(x)$. Let $p\colon P_B^B X \to X$ be the map which evaluates a path at 1. The following diagram commutes:

$$\begin{tikzcd}
P_B^B X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}\times 0"'] & P_B^B X \arrow[d, "p"] \\
P_B^B X \times I \arrow[r, "H\circ(p\times\mathrm{id})"] & X.
\end{tikzcd}$$

Since p is a fibrewise fibration, there is a lifting $H'\colon P_B^B X \times I \to P_B^B X$ over B. We now define $\alpha'(\omega) = \alpha(p\omega)$, and

$$F_t(\omega) = \begin{cases} H'_t(\omega) & \text{for } t \leqslant \alpha(p\omega), \\ H'_{\alpha(p\omega)}(\omega) & \text{for } t \geqslant \alpha(p\omega). \end{cases}$$

For all $\omega \in \Omega_B^B X$, we see that $\alpha'(\omega) = 0$. Also, F is a fibrewise homotopy of the identity rel $\Omega_B^B X$. If $t > \alpha'(\omega)$, then

$$\begin{aligned} pF_t(\omega) &= pH'_{\alpha(p\omega)}(\omega) \\ &= H \circ (p \times \mathrm{id})(\omega, \alpha(p\omega)) \\ &= H_{\alpha p(\omega)}(p(\omega)) \\ &\in B, \end{aligned}$$

and so $F_t(\omega)$ lies in $\Omega_B^B X$. Thus (α', F) defines a fibrewise Strøm structure on the pair $(P_B^B X, \Omega_B^B X)$, and u is a closed fibrewise cofibration. □

The following result is proved in the non-fibrewise case in Proposition (6.7) of [20]:

Lemma 2.1.3. *Suppose $Z = C_1 \cup C_2$ is a fibrewise pointed space over B, where the inclusion $Y = C_1 \cap C_2 \hookrightarrow C_1$ is a fibrewise pointed cofibration. Then Z has the fibrewise pointed homotopy type of the reduced fibrewise double mapping cylinder $X = C_1 \cup_{Y\times\{0\}} (Y \tilde{\times} I) \cup_{Y\times\{1\}} C_2$.*

Proof. Let $\lambda\colon X \to Z$ be the canonical map, and let $\tilde{M}$ be the reduced fibrewise mapping cylinder of the inclusion $u\colon Y \hookrightarrow C_1$. As in §21 of [21], the diagram

$$\begin{tikzcd} & Y \arrow[dl, "i_0"'] \arrow[dr, "u"] & \\ I\tilde{\times}Y \arrow[dr, "\mathrm{id}\times u"'] & & C_1 \arrow[dl, "i_0"] \\ & I\tilde{\times}C_1 & \end{tikzcd}$$

induces a fibrewise pointed map $k\colon \tilde{M} \to I\tilde{\times}C_1$, which by Proposition (21.3) of [21] admits a left inverse $l\colon I\tilde{\times}C_1 \to \tilde{M}$. Define $\mu\colon Z \to X$ by setting $\mu = l_0$ on C_1 and $\mu = \mathrm{id}$ on C_2. Then $\lambda\mu = \mathrm{id}$, while $\mu\lambda \simeq^B_B \mathrm{id}$, using the fibrewise pointed homotopy defined by l. □

Corollary 2.1.4. *Let X be a fibrewise well-pointed space over B with closed section, and let V be the subspace of $\mathrm{Maps}_B(I \times B, X)$ consisting of paths ω such that $\omega(1)$ or $\omega(0)$ (or both) lie in the image of the section. Then V has the fibrewise pointed homotopy type of $\Sigma^B_B \Omega^B_B X$.*

Proof. Embed B in X via the section, and write $\Omega = \Omega^B_B X$. By (2.1.2) the inclusion $u\colon \Omega \hookrightarrow P^B_B X = \{\omega \in \mathrm{Maps}_B(I \times B, X) : \omega(0) \in B\}$ is a fibrewise pointed cofibration. Now define

$$C_1 = P^B_B X \qquad \text{and} \qquad C_2 = \{\omega \in \mathrm{Maps}_B(I \times B, X) : \omega(1) \in B\}.$$

It follows from (2.1.3) that $V = C_1 \cup_\Omega C_2$ has the fibrewise pointed homotopy type of $U = C_1 \cup_{\Omega\tilde{\times}\{0\}} (\Omega\tilde{\times}I) \cup_{\Omega\tilde{\times}\{1\}} C_2$. It remains only to show that $(U, C_1 \vee_B C_2)$ is a closed fibrewise pointed cofibred pair, for then U has the fibrewise pointed homotopy type of $\Sigma_B^B\Omega_B^B X$ by Proposition (21.6) of [21], since C_1 and C_2 are fibrewise pointed contractible.

By the definitions of the product and quotient topologies, $C_1 \vee_B C_2$ is closed in U. We prove that $(U, C_1 \vee_B C_2)$ is a fibrewise cofibred pair (and hence a fibrewise pointed cofibred pair) by exhibiting a fibrewise Strøm structure on it. By (1.2.12) there is a fibrewise Strøm structure (β, H) on (Ω, B). Define

$$\alpha(u) = \begin{cases} 0 & \text{for } u \in C_1 \\ \min(2 - |4s-2|, \beta(\omega)) & \text{for } u = (\omega, s) \in \Omega\tilde{\times}I \\ 0 & \text{for } u \in C_2 \end{cases}$$

and

$$H_t'(u) = \begin{cases} u & \text{for } u \in C_1 \vee_B C_2 \\ (H_t(\omega), \max(0, s - \frac{1}{4}t)) & \text{for } u = (\omega, s) \in \Omega\tilde{\times}[0, \frac{1}{4}] \\ (H_t(\omega), (1-t)s + t(2s - \frac{1}{2})) & \text{for } u = (\omega, s) \in \Omega\tilde{\times}[\frac{1}{4}, \frac{3}{4}] \\ (H_t(\omega), \min(1, s + \frac{1}{4}t)) & \text{for } u = (\omega, s) \in \Omega\tilde{\times}[\frac{3}{4}, 1]. \end{cases}$$

Then the pair (α, H') is a fibrewise Strøm structure, and the result follows. □

Armed with this model for $\Sigma_B^B\Omega_B^B X$, we are able to prove the following proposition, which appears in a non-fibrewise form in [11]. If X is fibrewise Hausdorff, recall from p75 of [21] that $\Omega_B^B X$ is also fibrewise Hausdorff, and therefore has closed section by Proposition (2.9) of [21]. Hence, by (1.2.4), $\Sigma_B^B\Omega_B^B X$ is fibrewise coHopf.

Proposition 2.1.5. *Suppose X is a fibrewise Hausdorff space over B which is fibrewise well-pointed, and let*

$$S\colon \Sigma^B_B\Omega^B_B X \to \Sigma^B_B\Omega^B_B X \vee_B \Sigma^B_B\Omega^B_B X$$

be the fibrewise suspension comultiplication. Then the fibrewise map

$$(p \vee_B p)\circ S\colon \Sigma^B_B\Omega^B_B X \to X \vee_B X$$

induces monomorphisms of generalised fibrewise homotopy groups and a bijection between fibrewise pointed homotopy classes of fibrewise coretractions $X \to \Sigma^B_B\Omega^B_B X$ and fibrewise pointed homotopy classes of fibrewise coHopf structures $X \to X \vee_B X$.

Proof. Let W be the fibrewise homotopy pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle\Delta} \\ X \vee_B X & \xrightarrow{j} & X \times_B X. \end{array}$$

More explicitly, define $W \subseteq X \times_B \mathrm{Maps}_B(I \times B, X \times_B X) \times_B (X \vee_B X)$ by

$$W = \{(x,\lambda,y) : \Delta(x) = \lambda(0), \lambda(1) = j(y)\}.$$

We wish to show that the fibrewise projections $f\colon W \to X$ and $g\colon W \to X \vee_B X$ are fibrewise fibrations.

Since $(I \times B, \dot{I} \times B)$ is a closed fibrewise cofibred pair and $I \times B$ is fibrewise locally compact and fibrewise regular, we may apply Proposition 23.3 of [21] to deduce that the restriction map

$$\mathrm{Maps}_B(I \times B, X \times_B X) \to \mathrm{Maps}_B(\dot{I} \times B, X \times_B X) = (X \times_B X) \times_B (X \times_B X)$$

is a fibrewise fibration. Hence its pullback over $\operatorname{Im}\Delta \times_B (X \vee_B X)$ (by the inclusion) is a fibrewise fibration, whence f and g are fibrewise fibrations, being compositions of fibrewise fibrations.

Define a map $\varphi\colon V \to W$, where

$$V = \{\, \xi \in \mathrm{Maps}_B(I \times B, X) : \xi(0) = * \,\} \cup \{\, \xi \in \mathrm{Maps}_B(I \times B, X) : \xi(1) = * \,\},$$

by

$$\varphi(\xi) = \left(\xi(\tfrac{1}{2}), \lambda, (\xi(1), \xi(0))\right), \quad \text{where} \quad \lambda(t) = \left(\xi\left(\frac{1+t}{2}\right), \xi\left(\frac{1-t}{2}\right)\right)$$

Then φ is a fibrewise pointed topological equivalence, and so $f \circ \varphi$ and $g \circ \varphi$ are fibrewise fibrations. Clearly $(g \circ \varphi)^{-1}(*) = \Omega_B^B X \subseteq V$, and the inclusion map $\Omega_B^B X \to V$ is fibrewise pointed nulhomotopic.

Let A be a fibrewise pointed space over B. Suppose that $\alpha\colon A \to V$ is such that the composition $g \circ \varphi \circ \alpha$ is fibrewise pointed nulhomotopic, and suppose that $H\colon A \times I \to X \vee_B X$ is a fibrewise pointed nulhomotopy of $g \circ \varphi \circ \alpha$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i_0"'] & V \arrow[d, "g\circ\varphi"] \\ A \times I \arrow[r, "H"] & X \vee_B X \end{tikzcd}$$

Then H lifts to a homotopy $H'\colon A \times I \to V$, such that $H'_0 = \alpha$ and $\operatorname{Im}(H'_1)$ lies in $\Omega_B^B X \subseteq V$. Hence α is fibrewise pointed nulhomotopic, and so $g \circ \varphi$ induces monomorphisms of generalised fibrewise homotopy groups.

Now consider the map $\varepsilon\colon \Sigma_B^B \Omega_B^B X \to V$ given by

$$\varepsilon\,\langle s, \omega \rangle = \begin{cases} \omega_{0,2s} & \text{for } 0 \leqslant 2s \leqslant 1 \\ \omega_{2s-1,1} & \text{for } 1 \leqslant 2s \leqslant 2, \end{cases}$$

where $\omega_{a,b}(t) = \omega(a(1-t)+bt)$. We may regard $\Sigma^B_B\Omega^B_BX$ as the union of two fibrewise pointed contractible subsets $\Sigma^B_B\Omega^B_BX^+$, $\Sigma^B_B\Omega^B_BX^-$ with intersection Ω^B_BX, such that

$$\varepsilon\left(\Sigma^B_B\Omega^B_BX^+\right) \subseteq \{\,\xi \in \mathrm{Maps}_B(I\times B, X) : \xi(0) = *\,\} = V_0$$
$$\varepsilon\left(\Sigma^B_B\Omega^B_BX^-\right) \subseteq \{\,\xi \in \mathrm{Maps}_B(I\times B, X) : \xi(1) = *\,\} = V_1$$

and such that ε restricts to the inclusion $\Omega^B_BX \hookrightarrow V$. Since the inclusions of Ω^B_BX into $\Sigma^B_B\Omega^B_BX^+$, $\Sigma^B_B\Omega^B_BX^-$, V_0, V_1 are fibrewise pointed cofibrations by (2.1.2) and Proposition (21.3) of [21], and ε restricts to fibrewise pointed homotopy equivalences $\Sigma^B_B\Omega^B_BX^+ \to V_0$, $\Sigma^B_B\Omega^B_BX^- \to V_1$, by Proposition 21.10 of [21] ε is a fibrewise pointed homotopy equivalence. In addition, $g\circ\varphi\circ\varepsilon = (p\vee_B p)\circ S$ and $f\circ\varphi\circ\varepsilon = p$. Hence $(p\vee_B p)\circ S$ induces monomorphisms of generalised fibrewise homotopy groups.

For any fibrewise coretraction γ, the diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[dr, "\Delta"] & \Sigma^B_B\Omega^B_BX \arrow[r, "S"] \arrow[dr, "\Delta"] & \Sigma^B_B\Omega^B_BX \vee_B \Sigma^B_B\Omega^B_BX \arrow[r, "p\vee_B p"] \arrow[d, "i"] & X \vee_B X \arrow[d, "i"] \\
 & X \times_B X \arrow[r, "\gamma\times_B\gamma"] & \Sigma^B_B\Omega^B_BX \times_B \Sigma^B_B\Omega^B_BX \arrow[r, "p\times_B p"] & X \times_B X
\end{tikzcd}$$

commutes up to fibrewise pointed homotopy, and so $(p\vee_B p)\circ S\circ\gamma$ is a fibrewise coHopf structure on X. Conversely, let σ be a fibrewise coHopf structure on X. Then, because W is a fibrewise homotopy pullback, σ induces a fibrewise section s_σ of $f\circ\varphi$, and hence (via ε) a fibrewise coretraction γ such that $(p\vee_B p)\circ S\circ\gamma \simeq^B_B \sigma$. Finally, if γ_1, γ_2 are coretractions such that $(p\vee_B p)\circ S\circ\gamma_1 \simeq^B_B (p\vee_B p)\circ S\circ\gamma_2$, then by the first part of the lemma $\gamma_1\circ p \simeq^B_B \gamma_2\circ p$, and so

$$\gamma_1 \simeq^B_B \gamma_1\circ p\circ\gamma_1 \simeq^B_B \gamma_2\circ p\circ\gamma_1 \simeq^B_B \gamma_2.$$

□

Indeed, this result may be used to characterise the fibrewise primitive maps between two suitable fibrewise spaces:

Lemma 2.1.6. *Let $f\colon X \to Y$ be a fibrewise pointed map between two fibrewise Hausdorff, fibrewise well-pointed fibrewise coHopf spaces. Then f is fibrewise primitive with respect to fibrewise coHopf structures σ, τ if and only if the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \Sigma^B_B\Omega^B_B X \arrow[r, "\Sigma\Omega f"] & \Sigma^B_B\Omega^B_B Y, \end{tikzcd}$$

commutes up to fibrewise pointed homotopy, where φ, ψ are the corresponding fibrewise coretractions.

PROOF. Assume that the diagram above commutes up to fibrewise pointed homotopy. Since $f \circ p = p \circ \Sigma^B_B\Omega^B_B f$ and $(\Sigma^B_B\Omega^B_B f \vee_B \Sigma^B_B\Omega^B_B f) \circ S = S \circ \Sigma^B_B\Omega^B_B f$, we deduce that

$$\begin{aligned}(f \vee_B f) \circ \sigma &\simeq^B_B (p \vee_B p) \circ S \circ \Sigma^B_B\Omega^B_B f \circ \varphi \\ &\simeq^B_B (p \vee_B p) \circ S \circ \psi \circ f \\ &\simeq^B_B \tau \circ f,\end{aligned}$$

and so f is fibrewise primitive. Conversely, if f is fibrewise primitive, then

$$\begin{aligned}(p \vee_B p) \circ S \circ \Sigma^B_B\Omega^B_B f \circ \varphi &\simeq^B_B (p \vee_B p) \circ (\Sigma^B_B\Omega^B_B f \vee_B \Sigma^B_B\Omega^B_B f) \circ S \circ \varphi \\ &\simeq^B_B (f \vee_B f) \circ (p \vee_B p) \circ S \circ \varphi \\ &\simeq^B_B (p \vee_B p) \circ S \circ \psi \circ f.\end{aligned}$$

But X is fibrewise pointed dominated by the fibrewise cogroup-like space $\Sigma^B_B\Omega^B_B X$, and so the diagram above commutes up to fibrewise pointed homotopy, by (2.1.5).

□

2.2 Fibrewise Coalgebras

Suppose that X is a fibrewise coHopf, fibrewise well-pointed, fibrewise Hausdorff space over B. We will find it convenient to express properties of the fibrewise coHopf structure of X in terms of the homotopy properties of the corresponding fibrewise coretraction. In particular, we will be interested in fibrewise cogroups; that is, spaces carrying a fibrewise homotopy associative fibrewise coHopf structure with a fibrewise inversion. These correspond to the following definition:

Definition 2.2.1. *Let X be a fibrewise space over B with a fibrewise coretraction γ. We say that (X,γ) is a* fibrewise homotopy coalgebra *if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & \Sigma^B_B\Omega^B_B X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\Sigma^B_B e} \\ \Sigma^B_B\Omega^B_B X & \xrightarrow{\Sigma\Omega\gamma} & \Sigma^B_B\Omega^B_B\Sigma^B_B\Omega^B_B X \end{array}$$

commutes up to fibrewise pointed homotopy, where $e\colon X \to \Omega^B_B\Sigma^B_B X$ is the fibrewise adjoint of the identity on $\Sigma^B_B X$.

The aim of this section is to explain the nature of this correspondence between fibrewise cogroups (X,σ) and fibrewise coalgebras (X,γ).

Let $k \geqslant 2$ and let $\{X_i : i = 1,\ldots,k\}$ be a set of k copies of X. The fibrewise path-space fibration over X pulls back via the folding map $X_1 \vee_B \cdots \vee_B X_k \to X$

to a fibrewise fibration $\pi\colon W \to X_1 \vee_B \cdots \vee_B X_k$. We define a fibrewise map

$$\Phi_k\colon \Sigma_B^B\Omega_B^B X_1 \vee_B \cdots \vee_B \Sigma_B^B\Omega_B^B X_{k-1} \to X_1 \vee_B \cdots \vee_B X_k$$

by

$$\Phi_k(*,\ldots,\langle s,\omega_i\rangle,\ldots,*) = \begin{cases} (*,\ldots,\omega_i(2s),\ldots,*) & \text{for } 0 \leqslant 2s \leqslant 1 \\ (*,\ldots,*,\omega_k(2-2s)) & \text{for } 1 \leqslant 2s \leqslant 2, \end{cases}$$

where ω_k is ω_i composed with the equivalence $X_i \to X_k$.

Lemma 2.2.2. *There is a fibrewise pointed homotopy equivalence*

$$\theta\colon \Sigma_B^B\Omega_B^B X_1 \vee_B \cdots \vee_B \Sigma_B^B\Omega_B^B X_{k-1} \to W$$

such that $\pi \circ \theta = \Phi_k$. Moreover, Φ_k induces monomorphisms of generalised fibrewise homotopy groups.

PROOF. Embed B in X via the section, and let

$$P_B^B X = \{\, \xi \in \mathrm{Maps}_B(I \times B, X) : \xi(0) \in B \,\}.$$

Write $W = \bigcup\{\, W_i : 1 \leqslant i \leqslant k \,\}$, where

$$W_i = \left\{\, ((*,\ldots,x_i,\ldots,*),\xi) \in (X_1 \vee_B \cdots \vee_B X_k) \times_B P_B^B X : x_i \in X_i, x_i = \xi(1) \,\right\}.$$

Each W_i is fibrewise topologically equivalent to $P_B^B X$ for all i, and is therefore fibrewise contractible. In addition, if $i \neq j$ then $W_i \cap W_j = B \times_B \Omega_B^B X$. For a fibrewise pointed space A over B, let $C_B^B A$ be the reduced fibrewise cone on A;

$$C_B^B A = (A \times I)/_B(A \times \{0\} \cup B \times I),$$

(regarding B as a subspace of A via the section) and define

$$V = \left(C_B^B\Omega_B^B X_1 \amalg \cdots \amalg C_B^B\Omega_B^B X_k\right) / \sim,$$

where $(\omega_i, 1) \sim (\omega_j, 1)$ for all i, j. For any path ω in X and any scalar $s \in I$, define the path $\omega^{(s)}$ by $\omega^{(s)}(t) = \omega(st)$. Then the fibrewise map

$$\varepsilon\colon V \to W \qquad \text{given by} \qquad \varepsilon(s, \omega_i) = \left((*, \ldots, \omega_i(s), \ldots, *), \omega_i^{(s)}\right)$$

restricts to fibrewise pointed homotopy equivalences $W_i \to C_B^B\Omega_B^B X_i$ which fix the points of $\Omega_B^B X$. Since the inclusions $\Omega_B^B X \hookrightarrow W_i$ and $\Omega_B^B X \hookrightarrow C_B^B\Omega_B^B X_i$ are fibrewise pointed cofibrations, we may apply Proposition 21.10 of [21] to show that ε restricts to fibrewise pointed homotopy equivalences

$$C_B^B\Omega_B^B X_i \simeq_B^B W_i \text{ rel } \bigcap_i C_B^B\Omega_B^B X_i.$$

Hence ε is a fibrewise pointed homotopy equivalence $V \simeq_B^B W$.

Notice that the fibrewise quotient map $V \to V/_B(C_B^B\Omega_B^B X_k)$ is a fibrewise pointed homotopy equivalence, with fibrewise homotopy inverse

$$\psi\colon \Sigma_B^B\Omega_B^B X_1 \vee_B \cdots \vee_B \Sigma_B^B\Omega_B^B X_{k-1} \to V$$

defined by

$$\psi\left(*, \ldots, \langle s, \omega_i\rangle, \ldots, *\right) = \begin{cases} (2s, \omega_i) \in C_B^B\Omega_B^B X_i & 0 \leqslant 2s \leqslant 1 \\ (2(1-s), \omega_k) \in C_B^B\Omega_B^B X_k & 1 \leqslant 2s \leqslant 2, \end{cases}$$

where ω_k is as before. Define $\theta = \varepsilon \circ \psi$. Then, since

$$\pi\left((*, \ldots, x_i, \ldots, *), \xi\right) = (*, \ldots, x_i, \ldots, *),$$

the fibrewise pointed homotopy equivalence θ behaves as required.

To prove that Φ_k induces monomorphisms of generalised fibrewise homotopy groups, it is sufficient to show that π has the same property. Suppose that A is a fibrewise cogroup-like space over B, and suppose that $f\colon A \to W$ is a fibrewise pointed map such that $\pi \circ f$ is fibrewise pointed nulhomotopic via the fibrewise pointed homotopy $H\colon A \times I \to X \vee_B \cdots \vee_B X$. Since π is a fibrewise fibration, there is a lifting

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & W \arrow[r] \arrow[d, "\pi"] & P_B^B X \arrow[d] \\
A \times I \arrow[r, "H"] \arrow[ur, dashed] & X_1 \vee_B \cdots \vee_B X_k \arrow[r, "\nabla"] & X,
\end{tikzcd}$$

and so f is fibrewise pointed homotopic to a fibrewise pointed map whose image lies entirely in $\pi^{-1}(B)$. The existence of a cross-section of the fibrewise folding map implies that this inverse image is fibrewise pointed contractible in W, and so f is fibrewise pointed nulhomotopic. □

We are now able to use this result, as in [11], to prove the following proposition. The proof is a modification of Ganea's.

Proposition 2.2.3. *Suppose that X is a fibrewise Hausdorff space over B which is fibrewise well-pointed. Let $\gamma\colon X \to \Sigma_B^B \Omega_B^B X$ be a fibrewise coretraction, and let $\sigma\colon X \to X \vee_B X$ be the corresponding fibrewise coHopf structure. Then the following are equivalent:*

(i) σ *is fibrewise homotopy associative and admits a fibrewise inversion* η*;*

(ii) γ *is primitive with respect to* σ *and* S*;*

(iii) (X, γ) *is a fibrewise coalgebra.*

PROOF. (i)$\Rightarrow$(ii):

In the diagram

$$\begin{tikzcd}
 & X & \\
\Sigma_B^B\Omega_B^B X \arrow[ur, "p"] \arrow[r, "\Phi_2"] & X \vee_B X \arrow[u, "\pi_1"] \arrow[r, "\nabla"] & X \\
X \arrow[u, dashed, "\psi"] \arrow[r, "\sigma"] & X \vee_B X \arrow[u, "\mathrm{id} \vee_B \eta"] &
\end{tikzcd} \tag{*}$$

the composition $\nabla \circ (\mathrm{id}_X \vee_B \eta) \circ \sigma$ is fibrewise pointed nulhomotopic by definition. A fibrewise pointed nulhomotopy of this map induces a fibrewise pointed lifting $X \to P_B^B X$ in the diagram

$$\begin{tikzcd}
 & W \arrow[r] \arrow[d] & P_B^B X \arrow[d] \\
X \arrow[r, "(\mathrm{id} \vee_B \eta)\sigma"] & X \vee_B X \arrow[r, "\nabla"] & X.
\end{tikzcd}$$

Hence, by (2.2.2), there is a fibrewise pointed map ψ such that the square in (*) commutes up to fibrewise pointed homotopy. Since $\pi_1 \circ (\mathrm{id}_X \vee_B \eta) \circ \sigma \simeq_B^B \mathrm{id}_X$, the map ψ is a fibrewise coretraction. In the diagram

$$\begin{tikzcd}
\Sigma_B^B\Omega_B^B X \arrow[r, "S"] & \Sigma_B^B\Omega_B^B X \vee_B \Sigma_B^B\Omega_B^B X \arrow[dr, "\mathrm{id} \vee_B J"] & & X \vee_B X \vee_B X \\
X \arrow[u, "\psi"] \arrow[r, "\sigma"] & X \vee_B X \arrow[u, "\psi \vee_B \psi"] & \Sigma_B^B\Omega_B^B X \vee_B \Sigma_B^B\Omega_B^B X \arrow[ur, "\Phi_3"] &
\end{tikzcd} \tag{†}$$

let J be the fibrewise suspension inversion on $\Sigma_B^B \Omega_B^B X$, and let

$$E\colon X \vee_B X \to X \vee_B X \vee_B X, \qquad T\colon X \vee_B X \to X \vee_B X$$

be inclusion into the end terms and the switching map respectively. Since

$$\Phi_3 = (\mathrm{id}_X \vee_B \mathrm{id}_X \vee_B \nabla) \circ (\mathrm{id}_X \vee_B T \vee_B \mathrm{id}_X) \circ (\Phi_2 \vee_B \Phi_2) \text{ and } \Phi_2 \circ J = T \circ \Phi_2,$$

we see that

$$\begin{aligned}\Phi_3 \circ (\mathrm{id}_X \vee_B J) &= (\mathrm{id}_X \vee_B \mathrm{id}_X \vee_B \nabla) \circ (\mathrm{id}_X \vee_B T \vee_B \mathrm{id}_X) \circ (\Phi_2 \vee_B (T \circ \Phi_2)) \\ &= (\mathrm{id}_X \vee_B T) \circ (\mathrm{id}_X \vee_B \nabla \vee_B \mathrm{id}_X) \circ (\Phi_2 \vee_B \Phi_2).\end{aligned}$$

Let $\varphi = (\mathrm{id}_X \vee_B \eta) \circ \sigma$. Then $\varphi \simeq_B^B \Phi_2 \circ \psi$, and so

$$\begin{aligned}\Phi_3 \circ (\mathrm{id} \vee_B J) \circ S \circ \psi &\simeq_B^B (\mathrm{id}_X \vee_B T) \circ (\mathrm{id}_X \vee_B \nabla \vee_B \mathrm{id}_X) \circ (\Phi_2 \vee_B \Phi_2) \circ S \circ \psi \\ &\simeq_B^B (\mathrm{id}_X \vee_B T) \circ E \circ \Phi_2 \circ \psi \\ &\qquad \text{(since } S \text{ is fibrewise homotopy associative)} \\ &\simeq_B^B (\mathrm{id}_X \vee_B T) \circ E \circ \varphi \\ &\simeq_B^B (\mathrm{id}_X \vee_B T) \circ (\mathrm{id}_X \vee_B \nabla \vee_B \mathrm{id}_X) \circ (\varphi \vee_B \varphi) \circ \sigma \\ &\qquad \text{(since } \sigma \text{ is fibrewise homotopy associative)} \\ &\simeq_B^B \Phi_3 \circ (\mathrm{id} \vee_B J) \circ (\psi \vee_B \psi) \circ \sigma.\end{aligned}$$

Since X is fibrewise pointed dominated by a fibrewise reduced suspension and $\mathrm{id}_X \vee_B J$ is a fibrewise pointed self homotopy equivalence of $X \vee_B X$, we may apply (2.2.2) and deduce that $S \circ \psi \simeq_B^B (\psi \vee_B \psi) \circ \sigma$, and so the square in (†) commutes up to fibrewise pointed homotopy. Hence

$$\begin{aligned}(p \vee_B p) \circ S \circ \gamma &\simeq_B^B \sigma \quad \text{since } \sigma \text{ is the fibrewise comultiplication corresponding to } \gamma \\ &\simeq_B^B (p \vee_B p) \circ (\psi \vee_B \psi) \circ \sigma \qquad \text{since } \psi \text{ is a fibrewise coretraction} \\ &\simeq_B^B (p \vee_B p) \circ S \circ \psi\end{aligned}$$

and so by (2.1.5), we have $\gamma \simeq_B^B \psi$. It follows that the square

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\gamma"'] & X \vee_B X \arrow[d, "\gamma \vee_B \gamma"] \\ \Sigma_B^B \Omega_B^B X \arrow[r, "S"] & \Sigma_B^B \Omega_B^B X \vee_B \Sigma_B^B \Omega_B^B X \end{tikzcd}$$

commutes up to fibrewise pointed homotopy.

(ii)⇒(i):

Fibrewise homotopy associativity follows from the fibrewise primitivity of γ. A fibrewise inversion is given by $p \circ J \circ \gamma$.

(ii)⇔(iii):

Recall that $S\colon \Sigma_B^B \Omega_B^B X \to \Sigma_B^B \Omega_B^B X \vee_B \Sigma_B^B \Omega_B^B X$ is the comultiplication corresponding to $\Sigma_B^B e\colon \Sigma_B^B \Omega_B^B X \to \Sigma_B^B \Omega_B^B \Sigma_B^B \Omega_B^B X$, and apply (2.1.6) to deduce that each of the diagrams

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \Sigma_B^B \Omega_B^B X \arrow[d, "\Sigma_B^B e"] \\ \Sigma_B^B \Omega_B^B X \arrow[r, "\Sigma\Omega\gamma"] & \Sigma_B^B \Omega_B^B \Sigma_B^B \Omega_B^B X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\sigma"'] & \Sigma_B^B \Omega_B^B X \arrow[d, "S"] \\ X \vee_B X \arrow[r, "\gamma \vee_B \gamma"] & \Sigma_B^B \Omega_B^B X \vee_B \Sigma_B^B \Omega_B^B X \end{tikzcd}$$

commutes (up to fibrewise pointed homotopy) if and only if the other does. □

Corollary 2.2.4. *The map $\Sigma_B^B e\colon \Sigma_B^B X \to \Sigma_B^B \Omega_B^B \Sigma_B^B X$ is a fibrewise coalgebra structure on $\Sigma_B^B X$ for any fibrewise Hausdorff, fibrewise well-pointed space X over B.*

Moreover, we have the following corollary:

Corollary 2.2.5. *Let X be a fibrewise cogroup over the base B, and let $f\colon X \to Y$ be a fibrewise pointed map. Suppose that X is fibrewise Hausdorff and fibrewise well-pointed. Then there is a primitive fibrewise pointed map $f'\colon X \to \Sigma_B^B \Omega_B^B Y$, unique up to fibrewise pointed homotopy, such that $p \circ f' \simeq_B^B f$.*

PROOF. Let γ be a primitive coretraction for X, and define $f' = \Sigma_B^B \Omega_B^B f \circ \gamma$. Then $p \circ f' \simeq_B^B f$. Further, if $h\colon X \to \Sigma_B^B \Omega_B^B Y$ is a primitive fibrewise pointed map such that $p \circ h \simeq_B^B f$, then by (2.1.6) the diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma_B^B \Omega_B^B Y \arrow[d, "\Sigma_B^B e"] \\
\Sigma_B^B \Omega_B^B X \arrow[r, "\Sigma_B^B \Omega_B^B h"] & \Sigma_B^B \Omega_B^B \Sigma_B^B \Omega_B^B Y
\end{tikzcd}$$

commutes up to fibrewise pointed homotopy, and so

$$\begin{aligned}
h &\simeq_B^B \Sigma_B^B \Omega_B^B p \circ \Sigma_B^B e \circ h \\
&\simeq_B^B \Sigma_B^B \Omega_B^B f \circ \gamma \\
&\simeq_B^B f'.
\end{aligned}$$

2.3 Weakly Fibrant Fibrewise Spaces

In general, the fibres of a fibrewise space may have different homotopy types and connectivities. In order to obtain the dimensional results that we seek, it will be convenient to consider *fibrant* fibrewise spaces; that is, those whose projection map is a fibration. This enables us meaningfully to discuss the relationships between the dimension of the base, the connectivity of the fibres, and the global behaviour of the fibrewise space. We recall from (I.7.13) of [34] that a fibre bundle over a paracompact base is fibrant. However, the class of fibrant spaces is too restrictive for our purposes, and I propose to use the following definition:

Definition 2.3.1. *A fibrewise space X over B is said to be* weakly fibrant *if there is a fibrant space $\bar{X}$ over B and a fibrewise homotopy equivalence $\varphi_X \colon X \simeq_B \bar{X}$.*

Weakly fibrant spaces share many of the homotopy-theoretic properties of fibrant spaces; for example, the fibres of a weakly fibrant space over a path-connected base are of the same homotopy type. Accordingly, we prove results concerning fibrant spaces first, and then generalise to weakly fibrant spaces.

Lemma 2.3.2. *Let X, Y be fibrant fibrewise spaces with simply connected fibres over the simply connected space B. Let $f \colon X \to Y$ be a fibrewise map. Then the connectivity of f, considered as a non-fibrewise map, is greater than or equal to the connectivity of f restricted to any fibre.*

PROOF. Let n be the connectivity of f restricted to a fibre F_X. The long exact sequence of a fibration is functorial, so there is a commutative diagram

$$\begin{tikzcd}
\pi_{i+1}(X) \arrow[r] \arrow[d, "f_*"] & \pi_{i+1}(B) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_i(F_X) \arrow[r] \arrow[d, "(f|_{F_X})_*"] & \pi_i(X) \arrow[r] \arrow[d, "f_*"] & \pi_i(B) \arrow[d, "\mathrm{id}"] \\
\pi_{i+1}(Y) \arrow[r] & \pi_{i+1}(B) \arrow[r] & \pi_i(F_Y) \arrow[r] & \pi_i(Y) \arrow[r] & \pi_i(B),
\end{tikzcd}$$

where $(f|_{F_X})_*$ is an isomorphism $\pi_i(F_X) \to \pi_i(F_Y)$ for $i \leqslant n$. Since B, F_X, F_Y are simply connected, all the fundamental groups in the diagram are trivial. Applying the five lemma, we deduce that f_* is an isomorphism $\pi_i(X) \to \pi_i(Y)$ for $i \leqslant n$, and so f is an n-connected map. □

Corollary 2.3.3. *Let X, Y be weakly fibrant fibrewise spaces with simply connected fibres over the simply connected space B. Let $f\colon X \to Y$ be a fibrewise map. Then the connectivity of f, considered as a non-fibrewise map, is greater than or equal to the connectivity of f restricted to any fibre.*

PROOF. Let $\varphi_X\colon X \simeq_B \bar{X}$, $\varphi_Y\colon Y \simeq_B \bar{Y}$ be fibrewise homotopy equivalences of X and Y to fibrant spaces over B. Then there is a fibrewise map $\bar{f}\colon \bar{X} \to \bar{Y}$ such that the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "\varphi_Y"] \\
\bar{X} \arrow[r, "\bar{f}"] & \bar{Y}
\end{tikzcd}$$

commutes up to fibrewise homotopy. The connectivity of f is equal to that of $\bar{f}$, since φ_X, φ_Y are homotopy equivalences, considered as maps of spaces. Restricting to a single fibre, we obtain a homotopy-commutative square

$$\begin{tikzcd}
F_X \arrow[r, "f|_{F_X}"] \arrow[d, "\simeq"'] & F_Y \arrow[d, "\simeq"] \\
F_{\bar{X}} \arrow[r, "\bar{f}|_{F_{\bar{X}}}"] & F_{\bar{Y}},
\end{tikzcd}$$

and so the connectivity of $f|_{F_X}$ is equal to that of $\bar{f}|_{F_X}$. □

Lemma 2.3.4. *Let A, C, D be fibrant fibrewise spaces over B. Then for fibrewise maps $f\colon A \to D$, $g\colon C \to D$, the fibrewise homotopy pullback*

$$
\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\psi"'] & A \arrow[d, "f"] \\
C \arrow[r, "g"] & D
\end{tikzcd}
$$

is fibrant.

Proof. Let $\omega\colon K \to X$ be a map, with $H\colon K \times I \to B$ a homotopy of $p_X \circ \omega$.

$$
\begin{tikzcd}
 & & A \arrow[dr] \arrow[dd] & \\
K \arrow[r, "\omega"] \arrow[dd, "i_0"'] & X \arrow[ur] \arrow[r] \arrow[ddr] & C \arrow[r] \arrow[d] & D \arrow[dl] \\
 & & & \\
K \times I \arrow[rr, "H"] & & B &
\end{tikzcd}
$$

There are liftings $\alpha\colon K \times I \to A$, $\gamma\colon K \times I \to C$ of the homotopy H, so that

$$
\alpha \circ i_0 = \varphi \circ \omega \qquad p_A \circ \alpha = H
$$

$$
\gamma \circ i_0 = \psi \circ \omega \qquad p_C \circ \gamma = H.
$$

Recall (see p194 of [20]) that the unbased fibrewise mapping space $\mathrm{Maps}_B(B \times I, D)$ is fibrant. Since

$$
(K \times I \times I, (K \times \{0\} \times I) \cup (K \times I \times \{0\}) \cup (K \times I \times \{1\}))
$$

is homeomorphic to $(K \times I \times I, K \times \{0\} \times I)$, the problem of lifting a homotopy

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Maps}_B(B \times I, D) \arrow[d] \\ K \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

with specified endpoints is equivalent to the standard homotopy lifting problem for fibrations. Choose a solution $\Lambda_{k,t}$ with endpoints $\alpha_{k,t}$ and $\gamma_{k,t}$.

Write

$$X = \{(x, \lambda, y) : x \in A, y \in C, \lambda \in \mathrm{Maps}_B(B \times I, D), \lambda(0) = f(x), \lambda(1) = g(y)\},$$

and define $H' \colon K \times I \to X$ by the formula

$$H'(k,t) = (\alpha(k,t), \Lambda_{k,t}, \gamma(k,t)).$$

By construction,

$$p_A(\alpha(k,t)) = H(k,t) = p_C(\gamma(k,t)).$$

Hence p_X is a fibration. □

Corollary 2.3.5. *Let A, C, D be weakly fibrant fibrewise spaces over B. Then for fibrewise maps $f \colon A \to D$, $g \colon C \to D$, the fibrewise homotopy pullback*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, dashed] & A \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is weakly fibrant.

Proof. Let $\varphi_A \colon A \simeq_B \bar{A}$, $\varphi_C \colon C \simeq_B \bar{C}$, $\varphi_D \colon D \simeq_B \bar{D}$ be fibrewise homotopy equivalences of A, C and D to fibrant spaces over B. Suppose that, for some K over B, the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \bar{A} \arrow[r, "\simeq_B"] \arrow[d, "\bar{f}"] & A \arrow[dd, "f"] \\
\bar{C} \arrow[r, "\bar{g}"] \arrow[d, "\simeq_B"'] & \bar{D} \arrow[dr, "\simeq_B"] & \\
C \arrow[rr, "g"] & & D
\end{tikzcd}$$

commutes up to fibrewise homotopy, and so there is a fibrewise map $K \to \bar{X}$ making the diagram

$$\begin{tikzcd}
K \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& \bar{X} \arrow[r] \arrow[d] & \bar{A} \arrow[d] \\
& \bar{C} \arrow[r] & \bar{D}
\end{tikzcd}$$

commute up to fibrewise homotopy. The result now follows from the uniqueness of fibrewise homotopy pullbacks up to fibrewise homotopy equivalence. □

Corollary 2.3.6. *Suppose that X is a sectioned weakly fibrant space over B. Then the based loop space $\Omega_B^B(X)$ is weakly fibrant. Moreover, if X is fibrant, then $\Omega_B^B X$ is fibrant.*

Proof. The fibrewise homotopy pullback of the diagram

$$\begin{tikzcd}
& B \arrow[d] \\
B \arrow[r] & X
\end{tikzcd}$$

is $\Omega_B^B X$. □

Lemma 2.3.7. *Let X be a fibrant, fibrewise compact and fibrewise Hausdorff space over B. Then $\Sigma_B X$ is weakly fibrant.*

PROOF. It is shown on p186 of [20] that $\Sigma_B X$ is fibrant with the coarse topology. But Theorem 1 of [27] shows that the fibrewise homotopy type of $\Sigma_B X$ is independent of whether the coarse or fine topology is chosen. □

Corollary 2.3.8. *If X is as above, but is also sectioned and fibrewise non-degenerate, then $\Sigma^B_B X$ is weakly fibrant over B.*

PROOF. Let $\check{X}_B$ be the fibrewise mapping cylinder of the section. Then by definition, the projection $\rho\colon \check{X}_B \to X$ is a fibrewise pointed homotopy equivalence, and is well-pointed. Thus $\Sigma_B \check{X}_B \simeq_B \Sigma^B_B \check{X}_B$, as explained on p148 of [21]. But then $\Sigma_B \check{X}_B \simeq_B \Sigma_B X$ and $\Sigma^B_B \check{X}_B \simeq_B \Sigma^B_B X$, whence $\Sigma^B_B X \simeq_B \Sigma_B X$. Hence $\Sigma^B_B X$ is weakly fibrant. □

2.4 Cogroups and Suspensions

We are now able to obtain a partial answer to the problem of determining when a fibrewise coHopf structure on a given space is induced by a fibrewise suspension structure. We write $\dim X \leqslant r$ to indicate that X has the homotopy type of an r-dimensional CW-complex.

Proposition 2.4.1. *Let X be a fibrant, fibrewise well-pointed, fibrewise compact and fibrewise Hausdorff space over a simply connected CW-complex B, such that the fibres of X are $(n-1)$-connected. Suppose further that (X, γ) is a fibrewise coalgebra, that $\dim X \leqslant 4n-5$, where $n \geqslant 2$, and that $\dim B < 4n-7$. Then there is a fibrewise pointed homotopy* *equivalence between X and a fibrewise suspension which is primitive with respect to the suspension comultiplication.*

PROOF. Set $D_1 = \Omega_B^B X$, $\gamma_1 = \gamma$. Consider inductively

$$\begin{tikzcd} D_{i+1} \arrow[r, "h_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\star"] & \Omega_B^B X \arrow[d, "\Omega_B^B \gamma_i"] \\ D_i \arrow[r, "e_i"] & \Omega_B^B \Sigma_B^B D_i \end{tikzcd}$$

$$\begin{tikzcd} \Sigma_B^B D_{i+1} \arrow[r, "\Phi_i"] & W_i \arrow[r, "\theta_i"] \arrow[d, "\psi_i"'] \arrow[dr, phantom, "\star"] & \Sigma_B^B \Omega_B^B X \arrow[r, "p"] \arrow[d, "\Sigma_B^B \Omega_B^B \gamma_i"] & X \arrow[d, "\gamma_i"] \\ & \Sigma_B^B D_i \arrow[r, "\Sigma_B^B e_i"] & \Sigma_B^B \Omega_B^B \Sigma_B^B D_i \arrow[r, "p_i"] & \Sigma_B^B D_i, \end{tikzcd}$$

where $e_i = e$, $p_i = p$, the starred squares are fibrewise homotopy pullbacks, and Φ_i is induced by the maps $\Sigma_B^B h_i$ and $\Sigma_B^B g_i$. There exists a fibrewise pointed map $\varepsilon\colon X \to W_1$ such that $\psi_1 \circ \varepsilon = \gamma_1 = \theta_1 \circ \varepsilon$. On each fibre, the argument of Ganea in [11] shows that Φ_1 is a $(4n-5)$-connected map. Notice that since D_2 is fibrewise well-pointed, we may use $\Sigma_B D_2$ as our fibrant model for $\Sigma_B^B D_2$, and obtain a fibrewise pointed homotopy equivalence between the two, rather than the usual fibrewise homotopy equivalence. Using this fact, with Theorem 6.3 of [17], and by (2.3.5), we deduce that ε lifts to a fibrewise pointed map $\gamma_2\colon X \to \Sigma_B^B D_2$ with $\Phi_1 \circ \gamma_2 \simeq_B^B \varepsilon$. Ganea's argument now shows that $p \circ \theta_2 \circ \Phi_2$ restricts to a $(4n-5)$-connected map on the fibres, and hence (by (2.3.3)) is $(4n-5)$-connected on $\Sigma_B^B D_3$.

A (non-fibrewise) reduced suspension may be thought of as the union of two contractible subspaces. In the same way, we may consider $\Sigma_B^B X$ as the union of two subspaces, each with the fibrewise homotopy type of B. Applying the Mayer-Vietoris sequence for singular homology, we obtain:

$$\cdots \to H_i(B) \oplus H_i(B) \to H_i(\Sigma_B^B X) \to H_{i-1}(X) \to H_{i-1}(B) \oplus H_{i-1}(B) \to \cdots$$

For $i \geqslant 4n-6$, we have $H_{i-1}(B)=0$, so $H_i(\Sigma_B^B X) \cong H_{i-1}(X)$. In addition, since $\dim X \leqslant 4n-5$, we see that $H_{4n-5}(X)$ is free, and the composition

$$H_{4n-6}(D_3) \xrightarrow{\cong} H_{4n-5}(\Sigma_B^B D_3) \xrightarrow{(p\circ\theta_2\circ\Phi_2)_*} H_{4n-5}(X)$$

is surjective, by the Whitehead Theorem. Hence, by Theorem 2.1 of [5], there is a connected CW-complex Y and a map $f\colon Y \to D_3$ such that

$$\begin{aligned} f_*\colon H_q(Y) &\cong H_q(D_3) \qquad (q < 4n-6)\\ (p\circ\theta_2\circ\Phi_2\circ\Sigma_B^B f)_*\colon H_{4n-5}(\Sigma_B^B Y) &\cong H_{4n-5}(X)\\ H_q(Y) &= 0 \qquad\qquad (q \geqslant 4n-5). \end{aligned}$$

We equip Y with the projection to B that makes f into a fibrewise map. (The section of Y is obtained by compressing the section of D_3 into the $(4n-7)$-skeleton to give a map $s'\colon B \to Y$ such that $f\circ s'$ is homotopic to the section of D_3. We may then replace Y by the mapping path space of f and solve the lifting problem

$$\begin{tikzcd} B \arrow[r, "s'"] \arrow[d, "i_0"'] & Y \arrow[d, "f"] \\ B\times I \arrow[r] \arrow[ur, dashed] & D_3 \end{tikzcd}$$

to obtain a section for Y such that f is a fibrewise pointed map.) Applying the five lemma to the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_{q+1}(B)\oplus H_{q+1}(B) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{q+1}(\Sigma_B^B Y) \arrow[r] \arrow[d, "(\Sigma f)_*"] & H_q(Y) \arrow[r] \arrow[d, "f_*"] & H_q(B)\oplus H_q(B) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\ \cdots \arrow[r] & H_{q+1}(B)\oplus H_{q+1}(B) \arrow[r] & H_{q+1}(\Sigma_B^B D_3) \arrow[r] & H_q(D_3) \arrow[r] & H_q(B)\oplus H_q(B) \arrow[r] & \cdots \end{tikzcd}$$

we deduce that $\Sigma_B^B f$ induces isomorphisms $H_q(\Sigma_B^B Y) \to H_q(\Sigma_B^B D_3)$ for $q < 4n-6$, and also for $q = 4n-6$ by the dimension of B. Hence

$$p \circ \theta_2 \circ \Phi_2 \circ \Sigma_B^B f \colon \Sigma_B^B Y \to X$$

is a homology equivalence. Let M be the mapping path space of the composition $p_{D_3} \circ f$. Then M has the homotopy type of Y, and is a fibrant fibrewise pointed space. The unreduced fibrewise suspension of M has the fibrewise homotopy type of X, by Theorem 6.21 of [31]. Since $\Sigma_B M$ is simply connected (by the long exact sequence for a fibration) and fibrewise well-pointed, by (1.2.13) there is a fibrewise pointed homotopy equivalence $\Sigma_B M \to X$.

Now let $\check{M}_B$ be the fibrewise mapping cylinder of the section $B \to M$. Then the map $\Sigma_B \check{M}_B \to X$ is a fibrewise pointed homotopy equivalence, as is the projection $\Sigma_B \check{M}_B \to \Sigma_B^B \check{M}_B$ (recall that $\check{M}_B$ is fibrewise non-degenerate). Hence there is a fibrewise pointed homotopy equivalence $F \colon \Sigma_B^B \check{M}_B \to X$, such that $F \simeq_B^B p \circ \theta_2 \circ \Phi_2 \circ \Sigma_B^B q$ for some map $q \colon \check{M}_B \to D_3$.

It remains only to show that F is primitive. Notice that $\theta_1 \circ \Phi_1 = \Sigma_B^B h_1$ and that $\psi_2 \circ \Phi_2 = \Sigma_B^B g_2$. Hence

$$\begin{aligned}
\gamma \circ p \circ \theta_2 \circ \Phi_2 \circ \Sigma_B^B q &\simeq_B^B \theta_1 \circ \Phi_1 \circ \gamma_2 \circ p \circ \theta_2 \circ \Phi_2 \circ \Sigma_B^B q \\
&\simeq_B^B \Sigma_B^B h_1 \circ p_2 \circ \Sigma_B^B e_2 \circ \psi_2 \circ \Phi_2 \circ \Sigma_B^B q \\
&\simeq_B^B \Sigma_B^B h_1 \circ p_2 \circ \Sigma_B^B e_2 \circ \Sigma_B^B g_2 \circ \Sigma_B^B q \\
&\simeq_B^B \Sigma_B^B (h_1 \circ g_2 \circ q),
\end{aligned}$$

and is thus a primitive map. Thus $\gamma \circ F$ and $\Sigma_B^B \Omega_B^B F \circ \Sigma_B^B e$ are primitive maps $\Sigma_B^B \check{M}_B \to \Sigma_B^B \Omega_B^B X$ such that $p \circ \gamma \circ F \simeq_B^B p \circ \Sigma_B^B \Omega_B^B F \circ \Sigma_B^B e$. Recall that $\Sigma_B^B \check{M}_B$ is fibrewise Hausdorff, and is fibrewise well-pointed. Applying (2.2.5) we deduce that F is primitive with respect to the given fibrewise comultiplications. $\square$

This result enables us to investigate, for example, some associativity properties of comultiplications on sectioned orthogonal sphere bundles over spheres.

Corollary 2.4.2. *The bundle E_3 which we encountered in Chapter 1 admits no coHopf structure which is fibrewise homotopy associative.*

PROOF. Suppose that σ were a fibrewise homotopy associative fibrewise coHopf structure on E_3. Then by (1.2.14), E would admit a fibrewise inversion $\eta\colon E \to E$. By (2.2.3) there would be a coalgebra structure on E, and so E would have the fibrewise pointed homotopy type of a fibrewise reduced suspension. We know this cannot happen, and so conclude that there are no associative fibrewise coHopf structures on E_3. □

There is also an analogous result, related to that of Berstein and Hilton [5] which may be obtained in the same way as that above, and which does not require that the fibrewise coHopf structure induce a fibrewise coalgebra structure.

Proposition 2.4.3. *Let X be a fibrant, fibrewise well-pointed, fibrewise compact and fibrewise Hausdorff sectioned space over a simply connected CW-complex B, such that the fibres of X are $(n-1)$-connected. Suppose further that (X, γ) is a fibrewise coHopf space, that $\dim X \leqslant 3n-3$, where $n \geqslant 2$, and that $\dim B < 3n-5$. Then there is a fibrewise pointed homotopy equivalence between X and a fibrewise suspension, which is primitive with respect to the suspension comultiplication.*

PROOF. As before, Φ_1 is $(4n-5)$-connected, and so $p \circ \theta_1 \circ \Phi_1$ is $(3n-3)$-connected. We use the homology section result from [5] to obtain a CW-complex

Y and a map $f\colon Y \to X$ such that

$$\begin{aligned} f_*\colon H_q(Y) &\cong H_q(D_2) \quad (q < 3n-4) \\ (p \circ \theta_1 \circ \Phi_1 \circ \Sigma_B^B f)_*\colon H_{3n-3}(\Sigma_B^B Y) &\cong H_{3n-3}(X) \\ H_q(Y) &= 0 \qquad\qquad (q > 3n-4). \end{aligned}$$

We may assume as before, without affecting the homotopy type of Y, that Y is a fibrewise well-pointed, fibrewise non-degenerate weakly fibrant space over B and that f is a fibrewise map. Then $p \circ \theta_1 \circ \Phi_1 \circ \Sigma_B^B f$ is a fibrewise pointed homotopy equivalence. The proof of primitivity is as before. □

This enables us to characterise those simply connected smooth manifolds whose tangent sphere bundles admit fibrewise coHopf structure, with the exception of those manifolds of dimension smaller than 5. The tangent sphere bundle of any manifold admitting two linearly independent vector fields is a fibrewise reduced suspension, and is thus fibrewise coHopf.

Corollary 2.4.4. *A simply connected smooth manifold M of dimension greater than 4 has a fibrewise coHopf tangent sphere bundle if and only if it admits two linearly independent vector fields.*

PROOF. Let X be the tangent sphere bundle to M, and let $n = \dim M - 1$. Suppose that X is fibrewise coHopf. Then X is a fibrant, fibrewise well-pointed, fibrewise compact and fibrewise Hausdorff space over M with $(n-1)$-connected fibres. Since $\dim X = 2n+1 \leqslant 3n-3$ and $\dim M = n+1 < 3n-5$, we may apply (2.4.3) to deduce that X is fibrewise pointed homotopy equivalent to the fibrewise reduced suspension of a sectioned spherical fibration. Applying Lemma (3.2) of [30] to the square

$$\begin{tikzcd} BSO(n-2) \arrow[r] \arrow[d] & BF(n-2) \arrow[d] \\ BSO(n-1) \arrow[r] & BF(n-1) \end{tikzcd}$$

(adopting Sutherland's notation) we see that X admits two linearly independent sections. $\square$

Necessary and sufficient conditions for X to admit two linearly independent sections are given in [32], in the case where M is compact. Notice that the above theorem is true for compact manifolds M such that $\dim M < 4$, since all compact orientable smooth manifolds of dimension 3 admit two linearly independent vector fields, as do those of dimension 2 with vanishing Euler characteristic.

Chapter 3
Localisation and Mixing

3.1 Localisation

Let X be a fibrant space over a path-connected complex B with simply connected fibre F. Suppose that T is a set of primes. We may construct, as in the proof of Theorem 4.2 of [29], a fibrant space X^f_T over B and a fibrewise map $\ell\colon X \to X^f_T$ such that ℓ restricts to the localisation map $F \to F_T$ on each fibre.

It is a trivial consequence of the functoriality of localisation that if Y is a simply connected (non-fibrewise) coHopf space and T is a set of primes, then the localisation Y_T of Y at T is also a coHopf space. Similarly, fibrewise localisations of fibrant fibrewise coHopf spaces with simply connected fibre are fibrewise coHopf. We might reasonably ask whether we may deduce that a fibrant space X over B is fibrewise coHopf, given that $X^f_{(p)}$ is fibrewise coHopf for some set of primes p. The dual question has been partially answered by Cook in his thesis [9] with the following result:

Theorem 3.1.1. *(Cook) Let E be a sectioned orientable q-sphere bundle over a connected finite complex B, where q is odd. Then E is fibrewise Hopf if and only if the fibrewise localisation of E at the prime 2 is fibrewise Hopf.*

We aim in this section to prove a statement dual to Cook's theorem. We begin with a definition.

Definition 3.1.2. *A fibrewise pointed map $\theta\colon E \to E$ of sectioned fibrant spaces over a base B with fibre S^q is said to be of* degree n *if the restriction of θ to each fibre is a map $S^q \to S^q$ of degree n. A fibrewise pointed map $\theta'\colon E \to E \vee_B E$ is said to be of* bidegree (m,n) *if the compositions $p_i \circ \theta'\colon E \to E$ for $i = 1,2$ are of degree m, n respectively, where $p_i\colon E \vee_B E \to E$ are the fibrewise projections onto the coördinates.*

The non-fibrewise analogue of the following lemma is trivial:

Lemma 3.1.3. *Let E be a well-sectioned fibrant space over B with fibre S^q, such that B admits a numerable categorical covering $\mathscr{U}$. Suppose that there is a fibrewise pointed map $\varphi\colon E \to E \vee_B E$ of bidegree $(1,1)$. Then E is fibrewise coHopf.*

PROOF. Let $\theta_i = p_i \circ \varphi\colon E \to E$, for $i = 1,2$. Then the θ_i are fibrewise pointed. Over each element of the covering $\mathscr{U}$, E is trivial, and so the θ_i are fibrewise homotopy equivalences, by Theorem 3.3 of [10]. Since E is well-sectioned, the θ_i are fibrewise pointed homotopy equivalences, by (1.2.13). Choose fibrewise pointed homotopy inverses τ_1, τ_2. The diagram

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[dr, "\theta_i"'] & E \vee_B E \arrow[r, "\tau_1 \vee_B \tau_2"] \arrow[d, "p_i"] & E \vee_B E \arrow[d, "p_i"] \\
& E \arrow[r, "\tau_i"] & E
\end{tikzcd}$$

commutes up to fibrewise pointed homotopy for $i = 1,2$, and so $(\tau_1 \vee_B \tau_2) \circ \varphi$ is a fibrewise coHopf structure on E.

Our main tool in this section will be the following proposition, which relates coHopf structures on localisations of bundles to fibrewise maps defined on the bundle itself.

Proposition 3.1.4. *Let E be a sectioned orientable q-sphere bundle over a finite connected complex B, where q is odd and greater than 1. Let T be a set of primes. Then the fibrewise T-localisation of E is a fibrewise coHopf space over B if and only if there exists a fibrewise pointed map $E \to E \vee_B E$ of bidegree (a, b), where a and b are units in $\mathbb{Z}_T$.*

PROOF. Suppose E^f_T admits a fibrewise coHopf structure σ. We proceed by induction on the number of cells in B.

Base step: Consider E_0, the pullback of E to the 0-skeleton $B^{(0)}$ of B. This is a fibrewise space over a discrete collection of points. Hence E_0 is a disjoint union of copies of S^q, and $E \vee_B E$ is a disjoint union of copies of $S^q \vee S^q$. By Theorem V.3.6 of [34] it is sufficient to consider the case where $B^{(0)}$ is a single point. By Theorem 1.1 of [11] coHopf structures on a space X are in bijective correspondence with coretractions $X \to \Sigma\Omega X$ (with respect to the canonical projection $p\colon \Sigma\Omega X \to X$). In the case $X = S^q$, we know that p induces an isomorphism $p_*\colon \pi_q(\Sigma\Omega S^q) \cong \pi_q(S^q)$. Hence coretractions on S^q induce an inverse for p_*. Similarly, the coretraction r corresponding to a coHopf structure σ_0 on S^q_T induces an isomorphism $\pi_q(S^q_T) \cong \pi_q(\Sigma\Omega S^q_T)$. The map $r \circ \ell$ induces an element γ of $\pi_q(\Sigma\Omega S^q_T) \cong \mathbb{Z}_T$ which projects via the isomorphism p_* to $[\ell] \in \pi_q(S^q_T)$, and is thus a unit, considered as an element of the ring $\mathbb{Z}_T$. Let N be the smallest positive integer such that $N\gamma$ is an integer. Then N is a unit in $\mathbb{Z}_T$, and there is a map $r'\colon S^q \to \Sigma\Omega S^q$ such that the diagram

$$\begin{tikzcd} S^q \arrow[r, "r'"] \arrow[d, "\ell"'] & \Sigma\Omega S^q \arrow[d, "\ell"] \\ S^q_T \arrow[r, "Nr"] & \Sigma\Omega S^q_T, \end{tikzcd}$$

commutes up to homotopy, where Nr is defined using the coHopf structure on S^q_T which is the localisation of the standard coHopf structure on S^q. Hence $r'_T \circ \ell$ is homotopic to $\ell \circ r'$, and hence to $Nr \circ \ell$. But

$$\ell^*\colon [S^q_T, \Sigma\Omega S^q_T] \to [S^q, \Sigma\Omega S^q_T]$$

is an isomorphism by the definition of ℓ, and so r'_T is homotopic to Nr. The element $p_* r'_*(1)$ of $\pi_q(S^q) \cong \mathbb{Z}$ induced by $p \circ r'$ localises to N, and is therefore a unit in $\mathbb{Z}_T$. It follows, writing the suspension comultiplication on $\Sigma\Omega S^q$ as S, that the composition

$$S^q \xrightarrow{r'} \Sigma\Omega S^q \xrightarrow{S} \Sigma\Omega S^q \vee \Sigma\Omega S^q \xrightarrow{p\vee p} S^q \vee S^q$$

has the required bidegree. Denoting this map by τ, we see that the diagram

$$\begin{tikzcd} S^q \arrow[r, "\tau"] \arrow[d, "\ell"'] & S^q \vee S^q \arrow[d, "\ell\vee\ell"] \\ S^q_T \arrow[r, "N\sigma_0"] & S^q_T \vee S^q_T \end{tikzcd}$$

commutes up to homotopy. Note that $N\sigma_0$ extends to a fibrewise pointed map $E^f_T \to E^f_T \vee_B E^f_T$, since E admits a fibrewise pointed map of degree N, by Corollary 1.9 of [25]. This completes the base step.

Inductive step: Suppose now that A is a subcomplex of B, such that B is obtained by adding a single cell to A. Suppose further that there is a diagram, commutative up to fibrewise pointed homotopy:

$$\begin{tikzcd} E_A \arrow[r, "\sigma_A"] \arrow[d, "\ell"'] & E_A \vee_A E_A \arrow[d, "\ell\vee_A\ell"] \\ E^f_T|_A \arrow[r, "\sigma'"] & E^f_T|_A \vee_A E^f_T|_A, \end{tikzcd}$$

where σ' extends to a fibrewise pointed map over the whole of B, and σ_A is a fibrewise pointed map with bidegree (a, b), where a, b are units in $\mathbb{Z}_T$. (We do not assume that σ' is a fibrewise coHopf structure.) The map σ_A induces a partial section s of the pullback bundle

$$\begin{tikzcd} \pi^*(E \vee_B E) \arrow[r] \arrow[d] & E \vee_B E \arrow[d] \\ E \arrow[r, "\pi"] & B \end{tikzcd}$$

defined over E_A. Since σ_A is a fibrewise pointed map, we may regard s as being defined on the complex $Z = (E_A \amalg B)/\sim$, where $A \subset B$ is attached to E_A by the section. Then E is obtained from Z by attaching a $(q+n)$-cell e^{q+n}. The obstruction z to extending s over e^{q+n} lies in $\pi_{q+n-1}(S^q \vee S^q)$.

The diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "s"'] & E \arrow[d, "s'"] \\ \pi^*(E \vee_B E) \arrow[r, "\ell_E"] & \pi^*(E_T^f \vee_B E_T^f) \end{tikzcd}$$

commutes up to fibrewise pointed homotopy, where j is the inclusion map, s' is the section of $\pi^*(E_T^f \vee_B E_T^f)$ corresponding to the extension of σ', and ℓ_E is fibrewise localisation over E. It follows from the naturality of the obstruction that $\ell_* z = 0$, and so there is a positive integral unit $d \in \mathbb{Z}_T$ such that $dz = 0$. By Corollary 1.9 of [25], or by the remarks following this proof, there is a fibrewise pointed map $f\colon E \to E$ of degree d, since q is odd. By the corollary to Theorem 2.1 of [29] there is a positive integer M such that $(\widetilde{f_y})_*^M = 0$ on the d-torsion of $\pi_i(S^q)$ for all $i \leqslant q+n-1$ (adopting the notation on p61 of [9]). By the Hilton-Milnor theorem, we may write $\pi_{q-n+1}(S^q \vee S^q)$ as a direct sum

$$\pi_{q-n+1}(S^q \vee S^q) \cong \pi_{q-n+1}(S^{p_1}) \oplus \pi_{q-n+1}(S^{p_2}) \oplus \cdots \oplus \pi_{q-n+1}(S^{p_N})$$

for some sequence of integers (p_i). The inclusions $\pi_{q-n+1}(S^{p_i}) \to \pi_{q-n+1}(S^q \vee S^q)$ are given by composition with iterated Whitehead products $w_i(\iota_1, \iota_2)$. Since

$$((\widetilde{f \vee f})_y)^M \circ w_i(\iota_1, \iota_2) \simeq w_i(\iota_1 \circ (\tilde{f})^M_y, \iota_2 \circ (\tilde{f})^M_y)$$

by naturality, we may apply the biadditivity of the Whitehead product to deduce that $((\widetilde{f \vee f})_y)^M \circ w_i(\iota_1, \iota_2) \simeq d^{MN_i}.w_i(\iota_1, \iota_2)$, where the addition operation is that of the homotopy group $\pi_{p_i}(S^q \vee S^q)$, and N_i is some integer. It follows that the diagram

$$\begin{tikzcd}
\pi_{q+n-1}(S^{p_i}) \arrow[r, "w_i(\iota_1,\iota_2)"] \arrow[d, "(d^{MN_i})_*"'] & \pi_{q+n-1}(S^q \vee S^q) \arrow[d, "((\widetilde{f\vee f})_y)^M_*"] \\
\pi_{q-n+1}(S^{p_i}) \arrow[r, "w_i(\iota_1,\iota_2)"] & \pi_{q+n-1}(S^q \vee S^q)
\end{tikzcd}$$

commutes, where the vertical homomorphism on the left is induced by a pointed map $S^{p_i} \to S^{p_i}$ of degree d^{N_i}. For sufficiently large M, this is zero on the d-torsion of $\pi_{q+n-1}(S^{p_i})$ for all i, and so the obstruction to an extension of $(f \vee f)^M \circ s$ vanishes. Thus there exists a section of $\pi^*(E \vee_B E)$ over E, and hence a fibrewise map $E \to E \vee_B E$. The bidegree of this map is a pair of integers, both of which are units in $\mathbb{Z}_T$.

To prove the converse, let $\theta\colon E \to E \vee_B E$ be the given map. The base B is a finite complex, and therefore admits a numerable categorical covering $\mathscr{U}$, by Proposition 6.7 of [10]. Observe that selfmaps of S^q whose degrees are units in $\mathbb{Z}_T$ localise to selfmaps of S^q_T which induce isomorphisms in homology, and hence are homotopy equivalences. The fibrewise pointed maps $E^f_T \to E^f_T \vee_B E^f_T \xrightarrow{q_i} E^f_T$, $(i = 1, 2)$ thus restrict to fibrewise homotopy equivalences over each element of $\mathscr{U}$, and are therefore fibrewise homotopy equivalences by Theorem 3.3 of [10]. By (1.2.13), they admit fibrewise pointed homotopy inverses $\alpha_i\colon E^f_T \to E^f_T$, and so the map $(\alpha_1 \vee_B \alpha_2) \circ \theta$ is a fibrewise coHopf structure on E^f_T. □

Definition 3.1.5. *We say that a sectioned fibrant space E over B is a* fibrewise coHopf space mod p *if the fibrewise localisation $E^f_{(p)}$ is fibrewise coHopf, with the section given by the composition of the section of E with the fibrewise localisation map $\ell\colon E \to E^f_{(p)}$.*

Let $\Sigma_B E$ be a sectioned S^q-bundle over B with projection π and section s given by the suspension structure, and suppose that f is a fibrewise map $E \to E$ (note that we do not require that f or E be fibrewise pointed). We define a fibrewise pointed map $P(f)\colon \Sigma_B E \to \Sigma_B E \vee_B \Sigma_B E$ as follows:

$$P(f)\langle x,t\rangle = \begin{cases} (\langle x,4t\rangle, *) & 0 \leqslant t \leqslant \frac{1}{4} \\ (\langle -x, 2-4t\rangle, *) & \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ (*, \langle f(x), 2t-1\rangle) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where $*$ denotes the basepoint in the appropriate fibre of $\Sigma_B E$. Note that the bidegree of $P(f)$ is $\left(1+(-1)^{q+1}, \deg f\right)$. In particular, if q is odd then $P(\mathrm{id})$ is of bidegree $(2,1)$, and $P(q_1 \circ P(\mathrm{id}))$ is of bidegree $(2,2)$. This construction affords us a particularly simple inductive demonstration that $\Sigma_B E$ admits fibrewise pointed selfmaps of arbitrary non-negative degree when q is odd, since then $s \circ \pi$ and the identity on $\Sigma_B E$ are fibrewise pointed maps of degree 0 and 1 respectively, while $\nabla \circ P(f)$ is a fibrewise pointed map of degree $\deg f + 2$, for any fibrewise map $f\colon E \to E$.

Corollary 3.1.6. *Let E be a sectioned orientable q-sphere bundle over a finite connected complex, where q is odd and greater than 1. Then E is a fibrewise coHopf space mod p for each odd prime p.*

PROOF. As we have seen in the above discussion, there is a map of bidegree $(2,2)$. Apply (3.1.4). □

Proposition 3.1.7. *Let E be a sectioned orientable q-sphere bundle over a finite connected complex B, where q is odd and greater than 1. If E is a mod 2 fibrewise coHopf space over B then E is a fibrewise coHopf space over B.*

PROOF. Write $E = \Sigma_B E'$, where E' is a S^{q-1}-bundle over B, and where the section on E is that induced by the suspension structure. Recall that by Proposition 6.7 of [10], B admits a numerable categorical covering. By (3.1.4) there is a map $\tau \colon E \to E \vee_B E$ of bidegree (α, β), where α, β are odd. The composition

$$E \xrightarrow{\tau} E \vee_B E \xrightarrow{\tau \vee \tau} (E \vee_B E) \vee_B (E \vee_B E) \xrightarrow{q_2 \vee q_1} E \vee_B E$$

has bidegree $(\alpha\beta, \alpha\beta)$. We may therefore assume that $\alpha = \beta = 2k+1$ without loss of generality. Let $m_3 \colon E \to E$ be a fibrewise map of degree 3 (For example, let $m_3 = \nabla \circ P(\mathrm{id})$). Consider the composition

$$E \xrightarrow{\tau} E \vee_B E \xrightarrow{\tau \vee \tau} (E \vee_B E) \vee_B (E \vee_B E) \xrightarrow{q_2 \vee q_1} E \vee_B E \xrightarrow{m_3 \vee m_3} E \vee_B E,$$

which has bidegree $(3(4k^2+4k+1), 3(4k^2+4k+1))$. Call it φ.

We have seen that there is a map $E \to E \vee_B E$ of bidegree $(2,2)$. By iterating this map and folding we may construct a map $\psi \colon E \to E \vee_B E$ of bidegree $(2(3k^2+3k+1), 2(3k^2+3k+1))$. Let $(-1) \colon E' \to E'$ denote the fibrewise map which sends each point to its antipodal point. Then the composition

$$E \xrightarrow{P(-1)} E \vee_B E \xrightarrow{\psi \vee \varphi} (E \vee_B E) \vee_B (E \vee_B E) \xrightarrow{\nabla} E \vee_B E$$

is of bidegree $(1,1)$. Hence E admits a fibrewise coHopf structure by (3.1.3). □

Corollary 3.1.8. *Let E be a sectioned orientable S^q-bundle over a simply connected finite complex B, where q is odd and greater than 1. Write $\mathbb{Z}_{(2)} = \{\, a/b \in \mathbb{Q} : b \text{ is odd} \}$. Suppose further that $\widetilde{H}_r(B) \otimes \mathbb{Z}_{(2)}$ is trivial for all $r \geqslant 0$. Then E is fibrewise coHopf.*

PROOF. By Theorem 4.1 of [24], sectioned orientable $S^q_{(2)}$-fibrations over B are classified by $[B, BG(q)_{(2)}]$, where $G(q)$ is the topological monoid of based homotopy equivalences of S^q. By the definition of localisation, this is in bijective correspondence with $[B_{(2)}, BG(q)_{(2)}]$. But $B_{(2)}$ is contractible. Hence $E^f_{(2)}$ is a trivial fibration, and so E is fibrewise coHopf. □

Examples of such bases include spaces of the form $S^n \cup_p e^{n+1}$, where p is an odd prime and $n \geqslant 2$. Sectioned S^q-bundles over such spaces are also fibrewise Hopf, if $q \in \{1, 3, 7\}$.

We may ask whether (3.1.7) may be extended to sectioned sphere bundles with even-dimensional fibre. That it may not is shown using the following result:

Lemma 3.1.9. *Let E be a sectioned bundle over a sphere $B = S^n$ with fibre S^{2q}. Suppose that α is the element of $\pi_{n-1}(SO(2q))$ classifying E. Let $\rho \colon SO(2q) \to S^{2q-1}$ be the evaluation map. If $\Sigma^{2q}_* \rho_* \alpha$ is of odd order then $E^f_{(2)}$ is fibrewise coHopf.*

PROOF. Recall (see [14]) that $\Sigma^{2q}_* \rho_* \alpha = HJ\alpha$ vanishes if and only if E is fibrewise coHopf. Let d be the order of $\Sigma^{2q}_* \rho_* \alpha$. Then $HJ(d\alpha) = 0$, and so $HJ(\alpha)$ lies in the kernel of the mod 2 localisation map $\ell \colon \pi_{n+2q-1}(S^{2q-1}) \to \pi_{n+2q-1}(S^{2q-1})_{(2)}$. But $\Sigma^{2q}_* \ell \rho_*(\alpha)$ is the obstruction to a fibrewise coHopf structure on $E^f_{(2)}$, by naturality of localisation. □

Now consider the case where $q = 3$ and $n = 9$. The long exact sequence of the fibration $SO(5) \to SO(6) \xrightarrow{\rho} S^5$ contains the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_8(SO(5)) \arrow[r] \arrow[d, "\cong"] & \pi_8(SO(6)) \arrow[r, "\rho_*"] \arrow[d, "\cong"] & \pi_8(S^5) \arrow[r] \arrow[d, "\cong"] & \pi_7(SO(5)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
& 0 & \mathbb{Z}_{24} & \mathbb{Z}_{24} & \mathbb{Z} &
\end{tikzcd}$$

and so ρ_* is an isomorphism. Let E be the sectioned S^6-bundle over S^9 corresponding to an element α of order 3 in $\pi_8(SO(6))$. Since $\rho_*(\alpha)$ is in the stable range, the element $\Sigma_*^{2q}\rho_*\alpha$ is of odd order. Hence, by the above lemma, $E^f_{(2)}$ is a fibrewise coHopf space. But E is not a fibrewise coHopf space, since $\Sigma_*^{2q}\rho_*\alpha$ is non-zero.

3.2 Mixing of Types

So far, the only examples of fibrewise coHopf spherical fibrations that we have seen have lain in a few comparatively restrictive classes. Clearly, it is desirable to develop techniques which enable us to construct more examples. Fibrewise Zabrodsky mixing is one such technique. Using it, we attempt to construct a sectioned spherical fibration $E \to B$ with odd-dimensional fibre, such that the fibrewise localisation of E at the prime 2 is fibrewise coHopf, while the fibrewise localisation away from 2 is not a fibrewise reduced suspension.

Suppose that q is odd and greater than 2. Let E be an orthogonal S^{q-1}-bundle over a finite complex B. Partition the set $\mathscr{P}$ of primes into disjoint subsets S, T. Then there are fibrewise localisations E^f_S, E^f_T, $E^f_{(0)}$, which are fibrations over B with fibres S^{q-1}_S, S^{q-1}_T, $S^{q-1}_{(0)}$ respectively. Consider the fibrewise homotopy pullback

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E^f_T \arrow[d] \\
E^f_S \arrow[r] & E^f_{(0)},
\end{tikzcd}$$

where the maps to $E^f_{(0)}$ are the fibrewise rationalisation maps. The composition $X \to E^f_{(0)} \to B$ expresses X as a space over B. By Lemma (2.3.4), X is fibrant over B with fibre S^{q-1}.

Lemma 3.2.1. *In the above situation, $\Sigma_B E \simeq^B_B \Sigma_B X$.*

PROOF. Notice that $\Sigma_B E$ and $\Sigma_B X$ are fibrewise well-pointed, and that B admits a numerable categorical covering $\mathscr{U}$. The fibrewise pointed homotopy commutative square

$$\begin{tikzcd} \Sigma_B E \arrow[r] \arrow[d] & \Sigma_B E^f_T \arrow[d] \\ \Sigma_B E^f_S \arrow[r] & \Sigma_B E^f_{(0)} \end{tikzcd}$$

induces a fibrewise pointed map $\Sigma_B E \to \Sigma_B X$ which is a homotopy equivalence over each element of $\mathscr{U}$. This map is a fibrewise pointed homotopy equivalence, by Theorem 3.3 of [10] and (1.2.13). □

This result enables us prove the following proposition, which offers hope of finding new examples of fibrewise coHopf fibrations with spherical fibre which are not suspensions. Let T be the set of all odd primes.

Proposition 3.2.2. *Suppose B is a finite complex, that q is odd, and that E and F are orientable $(q-1)$-sphere bundles over B satisfying the following conditions:*

(i) *$\Sigma_B E$ is a fibrewise coHopf space;*

(ii) *F^f_T admits no section;*

(iii) *The classifying maps $\varphi_E, \varphi_F \colon B \to BSO(q)$ are homotopic when composed with the rationalisation map $BSO(q) \to BSO(q)_{(0)}$.*

Then $E^f_{(0)} \simeq_B F^f_{(0)}$, and the fibrewise homotopy pullback $\Sigma_B X$ of the diagram

$$\begin{tikzcd} & \Sigma_B F^f_T \arrow[d] \\ \Sigma_B E^f_{(2)} \arrow[r] & \Sigma_B E^f_{(0)} \end{tikzcd}$$

is a sectioned spherical fibration which is fibrewise coHopf but is not the fibrewise suspension of a sectioned spherical fibration.

PROOF. Let $F(q)$ be the monoid of based homotopy equivalences of S^q. The homotopy-commutative diagram

$$\begin{tikzcd} B \arrow[r, "\varphi_E"] \arrow[d, "\varphi_F"'] & BSO(q)_{(2)} \arrow[r] \arrow[d] & BF(q)_{(2)} \arrow[dd] \\ BSO(q)_T \arrow[r] \arrow[d] & BSO(q)_{(0)} \arrow[dr] & \\ BF(q)_T \arrow[rr] & & BF(q)_{(0)} \end{tikzcd}$$

induces a map $B \to BSO(q)$ such that the induced sectioned S^q-bundle over B is fibrewise pointed homotopic to $\Sigma_B X$.

By uniqueness of fibrewise localisation, $\Sigma_B X^f_{(2)}$ is fibrewise pointed homotopy equivalent to $\Sigma_B E^f_{(2)}$, and $\Sigma_B X^f_T$ is fibrewise pointed homotopy equivalent to $\Sigma_B F^f_T$. Hence $\Sigma_B X$ is fibrewise coHopf, since its fibrewise 2-localisation is. However, it admits only one desuspension, since a double desuspension would induce a section of F^f_T. □

Chapter 4

Some Calculations

4.1 The Thom Space

Let X be a sectioned space over B. Embed B in X via the section. We may define the pointed space $F(X)$ to be the quotient space X/B. This construction defines a functor F from the category of sectioned spaces over B and fibrewise pointed maps to the category of pointed topological spaces and pointed maps. This functor sends fibrewise pointed homotopy classes to pointed homotopy classes. Moreover, for sectioned spaces X and Y over B, $F(X \vee_B Y)$ is naturally homeomorphic to $F(X) \vee F(Y)$, and F respects the fibrewise folding map $\nabla\colon X \vee_B X \to X$, the fibrewise switching map $\tau\colon X \vee_B X \to X \vee_B X$, and the fibrewise projections $q_i\colon X \vee_B X \to X$, for $i = 1, 2$.

Lemma 4.1.1. *Suppose that X is a fibrewise coHopf space over B, with fibrewise comultiplication $\theta_X\colon X \to X \vee_B X$. Then $F(X)$ is a coHopf space, with comultiplication $F(\theta_X)$.*

PROOF. Apply F to the diagram

$$\begin{tikzcd} X \arrow[r, "\theta_X"] \arrow[dr, "\mathrm{id}_X"'] & X \vee_B X \arrow[d, "q_i"] \\ & X, \end{tikzcd}$$

which commutes up to fibrewise pointed homotopy, for $i = 1, 2$. □

We may prove in a similar fashion that if (X, θ) is fibrewise homotopy commutative then $(F(X), F(\theta))$ is homotopy commutative, that if (X, θ) is fibrewise homotopy associative then $(F(X), F(\theta))$ is homotopy associative, and that if (X, θ) admits a fibrewise inversion then $(F(X), F(\theta))$ admits an inversion.

Now let E be a sectioned S^{q-1}-bundle over the CW-complex B. We may write E as the fibrewise unreduced suspension of a (not necessarily sectioned) S^{q-2}-bundle $\pi\colon E' \to B$:

$$E \simeq (E' \times I) \amalg B_{(0)} \amalg B_{(1)} / \sim,$$

where $(f, 0) \sim \pi(f)_{(0)}$ and $(f, 1) \sim \pi(f)_{(1)}$, and where the section is induced by the homeomorphism $B \to B_{(1)}$.

We construct the Thom space $T_{E'}$ of E':

$$T_{E'} = (E' \times I) \amalg B \amalg \{*\} / \approx,$$

where $(f, 0) \approx \pi(f)$ and $(f, 1) \approx *$. Thus $T_{E'} = F(E)$.

Corollary 4.1.2. *If E is fibrewise coHopf, then $T_{E'}$ is a coHopf space.*

If we restrict our attention to the case where B is a simply-connected sphere, we may use the Thom space to characterise the fibrewise coHopf sphere bundles in certain dimensions.

Lemma 4.1.3. *Suppose further that $B = S^n$, and that $2q - 4 > n > 1$. Then $E = \Sigma_B E'$ is fibrewise coHopf if and only if $T_{E'}$ is homotopy equivalent to a suspension ΣX, for some space X.*

PROOF. Suppose that E is classified by the element α of $\pi_{n-1}(SO(q-1))$. It is shown in [14] that E is fibrewise coHopf if and only if $HJ(\alpha)$ is zero. But the Thom space of E' is the mapping cone of $J\alpha$ (see p37 of [18]) and is thus a two-cell complex $S^{q-1} \cup_{J\alpha} e^{n+q-1}$. This is homotopy equivalent to a suspension if and only if $HJ\alpha$ vanishes, by Lemma 3.6 of [4] and the EHP-sequence, since $J\alpha \in \pi_{n+q-2}(S^{q-1})$.

□

Thus, Thom spaces of fibrewise coHopf sphere bundles over spheres in these dimensions have the homotopy type of doubly suspended spaces. Indeed, the following is true:

Proposition 4.1.4. *Suppose that $n \geqslant 2$ and $q \geqslant 3$. Let E' be the orientable S^{q-2}-bundle over $B = S^n$ classified by the element α of the group $\pi_{n-1}(SO(q-1))$. Then the following are equivalent:*

(i) $\Sigma_B E'$ is fibrewise coHopf;

(ii) $T_{E'}$ is coHopf;

(iii) $J\alpha$ is primitive with respect to the comultiplications on S^{n+q-2} and S^{q-1};

(iv) $HJ\alpha = 0$.

PROOF. The main theorem of [14] states the equivalence of (i) and (iv); since $n+q-3 \geqslant q-1 \geqslant 2$, Theorem 3.20 of [4] states the equivalence of (ii) and (iii). We have already proved that (i) implies (ii). But since the Hopf invariant H is merely the suspension of the Hopf-Hilton invariant, which vanishes if $J\alpha$ is primitive, we have proved that (iii) implies (iv). □

4.2 Stiefel-Whitney Classes

We will, for much of the remainder of this chapter, be studying some consequences of (4.1.2). Although we do not assert that the existence of a coHopf structure on the Thom space is a sufficient condition for the existence of a fibrewise coHopf structure on a sphere bundle, it is useful in the calculation of some cohomological invariants. Recall, for example, that the Stiefel-Whitney class $w_n(\xi) \in H^n(B;\mathbb{Z}_2)$ of an orthogonal n-sphere bundle $\xi\colon X \to B$ vanishes if X admits a section. Similarly, if X admits two linearly independent sections, then $w_{n-1}(\xi)$ is also zero. We strengthen this result in the following proposition:

Proposition 4.2.1. *Let R be either $\mathbb{Z}$ or $\mathbb{Z}_2$, and let $\xi\colon X \to B$ be an R-orientable $(n-2)$-sphere bundle over a connected complex B, such that $\Sigma_B X$ is fibrewise coHopf (with the section given by the suspension structure). Then the Euler class $e(\xi) \in H^{n-1}(B;R)$ vanishes.*

PROOF. Recall that the Thom space T of X is homeomorphic to $F(\Sigma_B X)$. There is a Thom class $U_\xi \in H^{n-1}(T;R)$ and a Thom isomorphism

$$\varphi\colon H^{n-1}(B;R) \to H^{2n-2}(T;R)$$

such that $e(\xi) = \varphi^{-1}({U_\xi}^2)$. We therefore wish to show that ${U_\xi}^2$ vanishes. To do this it is sufficient to show that cup squares vanish on the homology of T with coefficients in R. But T is coHopf, and so this follows from the inequality (1.3) of [19]. □

Corollary 4.2.2. *Let $\xi\colon E \to B$ be a fibrewise coHopf sectioned $(n-1)$-sphere bundle over the connected complex B. Then the Stiefel-Whitney classes $w_n(\xi)$ and $w_{n-1}(\xi)$ vanish.*

PROOF. Write $E = \Sigma_B E'$, where E' is a (possibly non-sectioned) $(n-2)$-sphere bundle. Any sphere bundle with orthogonal structure group is $\mathbb{Z}_2$-orientable, and the mod 2 Euler class of E' is $w_{n-1}(\xi)$. □

Corollary 4.2.3. *Suppose that $\xi\colon E \to B$ is an orientable S^{n-2}-bundle, such that $\Sigma_B E$ is fibrewise coHopf with respect to the section given by the suspension structure. Then there is a partial section of ξ, defined over the $(n-1)$-skeleton of B. In particular, if $\dim B < n$, then ξ admits a section.*

Of course, the given coHopf structure on $\Sigma_B E$ need not be the reduced fibrewise suspension coHopf structure induced by the second section.

4.3 Category and Fibrewise Cohomology

Let B be a compact Hausdorff space, let $k > 1$, and let $\pi\colon E \to B$ be a fibrewise coHopf $(k-1)$-sphere bundle with structure group $SO(k)$ and section $s\colon B \to E$. Embed B in E via s. (In this section we will refer to Massey's paper [23], which uses Čech-Alexander-Spanier cohomology with compact supports. Since B and E are compact Hausdorff, this is isomorphic to the singular cohomology which we are using.)

Recall that E is a fibrewise pointed retract of $\Sigma^B_B \Omega^B_B E$. Since the fibrewise pointed category of $\Sigma^B_B \Omega^B_B E$ does not exceed 2, we deduce that $\mathrm{cat}^B_B\, E \leqslant 2$. We recall also from [22] that we may define the fibrewise cohomology ring

$$H^*_B(E) = \widetilde{H}^*(E, B),$$

where B is regarded as embedded in E via the section s. In our case, since $\mathrm{cat}^B_B\, E \leqslant 2$, all cup products in $H^*_B(E)$ vanish.

The long exact sequence of the pair (E, B) may be dismembered into short exact sequences

$$0 \to H^q_B(E) \to \widetilde{H}^q(E) \xrightarrow{s^*} \widetilde{H}^q(B) \to 0$$

which are split by $\pi^*\colon H^q(B) \to H^q(E)$. We deduce that all cup products in $\ker s^*$ vanish.

Now consider the Gysin sequence associated with E. Again this falls apart into short exact sequences of groups

$$0 \to \widetilde{H}^q(B) \xrightarrow{\pi^*} \widetilde{H}^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \to 0. \tag{*}$$

which are split by $s^*\colon \widetilde{H}^q(E) \to \widetilde{H}^q(B)$. It follows that there is a unique group homomorphism

$$\varphi\colon H^{q-k+1}(B) \to \widetilde{H}^q(E)$$

such that $\psi\varphi$ is the identity on $H^{q-k+1}(B)$ and such that $s^*\varphi = 0$.

Now let $a = \varphi(1) \in \widetilde{H}^{k-1}(E)$. Then $\psi(a) = 1 \in H^0(B)$. Let w be any element of $H^{q-k+1}(B)$.

Following [23], we write

$$\begin{aligned} \psi(a.\pi^*(w)) &= (-1)^{kq}(\psi(a)).w \qquad \text{(by Lemma 1 of [23])} \\ &= (-1)^{kq}w, \end{aligned}$$

and so the map $\theta\colon H^{q-k+1}(B) \to \widetilde{H}^q(E)$ given by $\theta(x) = (-1)^{kq}a.\pi^*(x)$ splits the short exact sequence (*). Thus we see that any element $x \in \widetilde{H}^q(E)$ satisfies the identity

$$x = \pi^*s^*x + a.\pi^*\psi(x).$$

Notice that $s^*x = s^*\pi^*s^*x + s^*(a.\psi(x))$, and so $s^*(a.\psi(x)) = 0$. Write

$$a^2 = \pi^*(\alpha) + a.\pi^*(\beta)$$

where $\alpha = s^*(a^2) \in \widetilde{H}^{2k-2}(B)$ and $\beta = \psi(a^2) \in H^{k-1}(B)$. (The splitting induced by s^* expresses $H^q(E)$ as a direct sum $H^q(B) \oplus H^{q-k+1}(B)$ for all q, so α and β are the unique cohomology classes satisfying this equality.) Then Theorem III of [23] shows that the reduction of β modulo 2 is the Stiefel-Whitney class $w_{k-1} \in H^{k-1}(B;\mathbb{Z}_2)$, which we know to be zero. Notice also that

$$\begin{aligned}\alpha &= s^*(a^2)\\ &= (s^*(a))^2\\ &= (s^*(\varphi(1)))^2\\ &= 0.\end{aligned}$$

We now examine cup products of elements in $\ker s^*$. It is apparent that

$$\ker s^* = \{a.\pi^*\psi(x) : x \in H^*(E)\},$$

for if $s^*(x) = 0$, then $x = a.\pi^*\psi(x)$, while conversely,

$$\begin{aligned}s^*(a.\pi^*\psi(x)) &= s^*(x - \pi^*s^*x)\\ &= s^*x - s^*x\\ &= 0,\end{aligned}$$

for all $x \in H^*(E)$. Let u, v be elements of $H^{q-k+1}(B)$. Then

$$\begin{aligned}a.\pi^*(u).a.\pi^*(v) = 0 &\iff a^2\pi^*(uv) = 0\\ &\iff a.\pi^*(\beta uv) = 0\\ &\iff \beta uv = 0.\end{aligned}$$

We conclude that all such products βuv in $H^*(B)$ vanish. In particular, taking $u = v = 1 \in H^0(B)$, we see that $\beta = 0$. This enables us to calculate the structure of the reduced cohomology ring of E explicitly, for

$$\begin{aligned}(\pi^*u_1 + a.\pi^*v_1)(\pi^*u_2 + a.\pi^*v_2) &= \pi^*(u_1u_2) + a.\pi^*(v_1u_2 + (-1)^{|a||u_1|}u_1v_2) + \\ &\qquad + (-1)^{|a||v_1|}a.\pi^*(\beta v_1 v_2) \\ &= \pi^*(u_1u_2) + a.\pi^*(v_1u_2 + (-1)^{(k-1)|u_1|}u_1v_2).\end{aligned}$$

We have therefore proved the following result:

Proposition 4.3.1. *Let E be an orientable S^{k-1}-bundle over a compact Hausdorff space B, with section s. Suppose that E is fibrewise coHopf. Then the cohomology ring $H^*(E)$ is isomorphic to the ring*

$$H^*(B)[a]/(a^2 = 0),$$

where a is a cohomology class in $H^{k-1}(E)$. The projection homomorphism $\pi^*\colon H^*(B) \to H^*(E)$ is given by the inclusion, and the homomorphism s^* is given by $s^*(x + ay) = x$.

If k is odd, we may also deduce that the Pontrjagin class in dimension $2k - 2$ vanishes, since it is the square of the Euler class of the fibrewise desuspension of E.

Two sections of E which induce differing homomorphisms in cohomology may give differing fibrewise cohomology rings. It is possible that a bundle might be fibrewise coHopf with respect to one section but fail to satisfy our cohomology conditions with respect to another. Indeed, as is shown in §3 of [22], this situation may arise even if the bundle E is trivial. We may therefore wish to relate the element β to splittings of the cohomology ring $H^*(E)$ arising from other sections of E. Suppose that i is a preimage of $1 \in H^0(B)$ under ψ. Then $a = i - \pi^* s^* i$, and so

$$\begin{aligned}
\beta &= \psi(a^2) \\
&= \begin{cases} \psi(i^2) + \psi\pi^* s^*(i^2) - 2\psi(i.\pi^* s^* i) & (k \text{ odd}) \\ \psi(i^2) + \psi\pi^* s^*(i^2) & (k \text{ even}) \end{cases} \\
&= \begin{cases} \psi(i^2) - 2s^*(i) & (k \text{ odd}) \\ \psi(i^2) & (k \text{ even}). \end{cases}
\end{aligned}$$

This clarifies the connection between the section and the structure of the fibrewise cohomology ring.

4.4 The Action of the Steenrod Algebra

The splitting

$$H^*(E) \cong H^*(B) \oplus a.H^{*-k+1}(B)$$

works just as well if we take coefficients in $\mathbb{Z}_2$. We may therefore consider the action of the Steenrod algebra $\mathscr{A}_2$ on $H^*(E; \mathbb{Z}_2)$.

Lemma 4.4.1. *Let x be an element of $H^q(E; \mathbb{Z}_2)$, and suppose that $x = \pi^* u + a.\pi^* v$. Then the action of the Steenrod algebra is given by*

$$\mathrm{Sq}^i x = \pi^* \mathrm{Sq}^i u + \sum_{j=0}^{i} a.\pi^*(w_j \,\mathrm{Sq}^{i-j} v),$$

where the w_j are the Stiefel-Whitney classes of the bundle E.

PROOF. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & H^q(B;\mathbb{Z}_2) \arrow[r, "\pi^*"] & H^*(E;\mathbb{Z}_2) \arrow[r, "\partial"] \arrow[dr, "\psi"'] & H^{*+1}(T_E;\mathbb{Z}_2) \arrow[r] \arrow[d, "\tau"] & 0 \\ & & & H^{*-k+1}(B;\mathbb{Z}_2), & \end{tikzcd}$$

where T_E is the Thom space associated to E, and τ is the mod 2 Thom isomorphism. The horizontal sequence is exact. Let $\omega \in H^k(T_E;\mathbb{Z}_2)$ be the preimage of $1 \in H^0(B;\mathbb{Z}_2)$ under τ, so that $w_j = \tau(\mathrm{Sq}^j\,\omega)$. If $e \in H^{k-1}(E;\mathbb{Z}_2)$ is a preimage of ω under ∂, then we may write $a = e - \pi^* s^* e$. We deduce that

$$\begin{aligned} \psi\,\mathrm{Sq}^i\,a &= \tau\partial\,\mathrm{Sq}^i\,a \\ &= \tau\left(\mathrm{Sq}^i\,\partial e - \partial\pi^* s^*\,\mathrm{Sq}^i\,e\right) \\ &= \tau\,\mathrm{Sq}^i\,\partial e \\ &= \tau\,\mathrm{Sq}^i\,\omega \\ &= w_i. \end{aligned}$$

On the other hand, we know that $s^*\,\mathrm{Sq}^i\,a = \mathrm{Sq}^i\,s^* a = 0$. We conclude that $\mathrm{Sq}^i\,a = a.\pi^* w_i$. The formula required now follows from the Cartan formula. □

4.5 Rational Spherical Fibrations

Suppose that E is an orientable S^{q-1}-bundle over a finite connected complex B. We may rationalise E fibrewise, to obtain an oriented fibration with fibre $S^{q-1}_{(0)}$. These are classified up to fibrewise homotopy type (see [29] or [24]) by a single rational cohomology class α in $H^q(B;\mathbb{Q})$ (for q even) or $H^{2q-2}(B;\mathbb{Q})$ (for q odd).

Proposition 4.5.1. *In the above situation, if q is even and $\Sigma_B E$ is fibrewise coHopf over B (using the section induced by the suspension structure), then $E^f_{(0)}$ is a trivial fibration.*

PROOF. As in [24], write $SG(q)$, $SF(q)$ respectively for the monoid of homotopy equivalences of S^{q-1} homotopic to the identity and the monoid of pointed homotopy equivalences of S^q homotopic to the identity, and recall that

$$H^*(BSG(q);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}(x_q) & \text{for } q \text{ even} \\ \mathbb{Q}(x_{2q-2}) & \text{for } q \text{ odd.} \end{cases}$$

The bundle E is induced by a map $\varphi\colon B \to BSO(q)$, and the Euler class associated with E is the image of a generator of $H^q(BSO(q);\mathbb{Z}) \cong \mathbb{Z}$ under the map φ^*. This generator is not killed by rationalisation, and we deduce from the naturality of α that α is a fixed non-zero multiple of the rationalised Euler class of E. However, the Euler class of E vanishes by (4.2.1), and so therefore does α. Hence $E^f_{(0)}$ is a trivial fibration. □

We may ask whether the above result holds when q is odd. The following counterexample shows that it does not:

EXAMPLE. Let α be a generator of $\pi_3(SO(3)) \cong \mathbb{Z}$, and let $n \in \mathbb{Z}$ be even and non-zero. Then the sectioned S^3-bundle $\Sigma_B E$ over $B = S^4$ classified by $n\alpha$ is fibrewise coHopf by James' result [14], since $\pi_6(S^5)$ is isomorphic to $\mathbb{Z}_2$.

It remains only to show that $E^f_{(0)}$ is not trivial up to fibrewise homotopy. This may be seen if we recall that S^2-bundles over S^4 are classified up to fibrewise homotopy equivalence by $\pi_3(SG(2))$. Consideration of the exact sequences associated to the evaluation fibrations $SO(3) \to S^2$ and $SG(2) \to S^2$ demonstrate that the natural map $\pi_3(SO(3))_{(0)} \to \pi_3(SG(2))_{(0)}$ is an isomorphism.

4.6 Generalised Cohomology and the Euler Class

Let h be a generalised reduced cohomology theory, and suppose that E is an h-orientable $(q-1)$-sphere bundle over a complex B such that the fibrewise suspension $\Sigma_B E$ is fibrewise coHopf. There is a Thom class $U \in h^q(T)$ and an isomorphism $\psi\colon h^*(B) \to h^{*+q}(T)$, where T is the Thom space of E, and the h-Euler class e is defined to be $\psi^{-1}(U^2) \in h^q(B)$. We may use the same argument as we used for singular cohomology to prove the following proposition:

Proposition 4.6.1. *Suppose that h, E, e are as above and that $\Sigma_B E$ is fibrewise coHopf. Then $e = 0$.*

PROOF. It is clearly sufficient to prove that all products in $h^*(T)$ vanish. This follows from inequality (1.3) of [19] and the existence of a coHopf structure on T induced by the fibrewise coHopf structure on $\Sigma_B E$. □

4.7 The Hopf Construction

Consider a sectioned orthogonal $(n-1)$-sphere bundle E over a pointed connected CW complex B. As is well known, such bundles are classified by the homotopy classes of pointed maps $B \to BO(n-1)$. We may therefore consider the homotopy classes of the induced map $f\colon \Omega B \to O(n-1)$ and the associated action $A_f : \Omega B \times S^{n-2} \to S^{n-2}$. We will also use A_f to denote the pointed action $\Omega B \times S^{n-1} \to S^{n-1}$ induced by unreduced suspension.

Let P be the space of based paths in B, with the compact-open topology. Then P is contractible via the homotopy $h\colon P \times I \to P$, where $h_t(p)(s) = p(ts)$ for all $p \in P$ and all $s, t \in [0,1]$. It is a simple corollary of Theorem 23.3 of [21] (putting

$(X, A) = (B \times S^{n-1}, B \times \{*\}))$ that $\mathrm{Maps}_B^B(B \times S^{n-1}, E \vee_B E)$ is a fibrant space over B, and there is therefore a lifting

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "i_0"'] & \mathrm{Maps}_B^B(B \times S^{n-1}, E \vee_B E) \arrow[d, "\pi"] \\ P \times I \arrow[r, "e"] \arrow[ur, dashed] & B, \end{tikzcd}$$

where $e(p, s) = p(s)$ and $\varphi(p)$ is the pinching map θ, considered as a map defined on the fibre of $B \times S^{n-1}$ over the basepoint of B, for all $p \in P$. Any two such liftings restrict to vertically homotopic maps $P \times \{1\} \to \mathrm{Maps}_B^B(B \times S^{n-1}, E \vee_B E)$, by an application of Theorem (I.7.18) of [34] (taking Y to be the basepoint in P). This lifting induces a lifting in the diagram

$$\begin{tikzcd} P \times S^{n-1} \arrow[r, "\bar{\varphi}"] \arrow[d, "i_0"'] & E \vee_B E \arrow[d, "\pi"] \\ P \times S^{n-1} \times I \arrow[r, "\bar{e}"] \arrow[ur, dashed, "H"] & B, \end{tikzcd}$$

where $\bar{e}(p, f, s) = p(s)$ and $\bar{\varphi}(p, f) = \theta(f)$, considered as an element of the fibre of $E \vee_B E$ over the basepoint of B. Again, any two such liftings restrict to vertically homotopic maps $P \times S^{n-1} \times \{1\} \to E \vee_B E$.

Lemma 4.7.1. *Suppose that E is a fibrewise coHopf sectioned orthogonal S^{n-1}-bundle over the CW-complex B, with fibrewise comultiplication $\theta_E \colon E \to E \vee_B E$. Then the diagram*

$$\begin{tikzcd} \Omega B \times S^{n-1} \arrow[r, "A_f"] \arrow[d, "\mathrm{id} \times \theta"'] & S^{n-1} \arrow[d, "\theta"] \\ \Omega B \times (S^{n-1} \vee S^{n-1})) \arrow[r, "A_{f \vee f}"] & S^{n-1} \vee S^{n-1} \end{tikzcd} \qquad (*)$$

commutes up to homotopy, where

$$A_{f\vee f}(\omega,(s_1,s_2)) = (A_f(\omega,s_1),A_f(\omega,s_2))$$

for all $\omega \in \Omega B$, $(s_1,s_2) \in S^{n-1}\vee S^{n-1}$

Proof. Note that the map $A_{f\vee f}$ is well-defined since the action of $O(n-1)$ on S^{n-1} is pointed.

In the same manner as in the earlier discussion, we may find a lifting

$$\begin{tikzcd} P\times S^{n-1} \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[d, "\pi"] \\ P\times S^{n-1}\times I \arrow[r, "\bar{e}"] \arrow[ur, dashed, "G"] & B, \end{tikzcd}$$

and thus, postcomposing with θ_E, a homotopy $P\times S^{n-1}\times I \to E\vee_B E$ which lifts $\bar{e}$. The maps H and $\theta_E\circ G$ restrict to homotopic maps $\Omega B\times S^{n-1}\times\{1\}\to S^{n-1}\vee S^{n-1}$.

It remains to identify H_1 with $\theta\circ A_f$ and $\theta_E\circ G_1$ with $A_{f\vee f}\circ(\mathrm{id}\times\theta)$, up to homotopy. The first identification follows from the properties of the universal sphere bundle $EO(n-1)\times_{O(n-1)} S^{n-1}\to BO(n-1)$; namely, that the lifting

$$\begin{tikzcd} P\,BO(n-1)\times S^{n-1} \arrow[r] \arrow[d, "i_0"'] & EO(n-1)\times_{O(n-1)} S^{n-1} \arrow[d, "\pi"] \\ P\,BO(n-1)\times S^{n-1}\times I \arrow[r] \arrow[ur, dashed] & BO(n-1) \end{tikzcd}$$

may be taken to give the action of $O(n-1)$ on S^{n-1} when it is restricted to $\Omega\,BO(n-1)\times S^{n-1}\times\{1\}$. If $f\colon B\to BO(n-1)$ is the classifying map for E, then any lifting $P\times S^{n-1}\times I\to S^{n-1}\vee S^{n-1}$ is induced up to vertical homotopy by this lifting and the map $Pf\colon P \to PBO(n-1)$. The second identification follows from the observation that we may construct a lifting of $\theta_E \circ \bar{e}$ by lifting

$$\begin{tikzcd}
P \times S^{n-1} \arrow[r, "\theta"] \arrow[d, "i_0"'] & P \times (S^{n-1} \vee S^{n-1}) \arrow[r] \arrow[d, "i_0"'] & E \vee_B E \arrow[d, "\pi"] \\
P \times S^{n-1} \times I \arrow[r, "\theta"] & P \times (S^{n-1} \vee S^{n-1}) \times I \arrow[r] \arrow[ur, dashed] & B,
\end{tikzcd}$$

and the uniqueness of such liftings up to vertical homotopy. □

NOTE. Inspecting the above proof, we see that the homotopy between $\theta \circ A_f$ and $A_{f\vee f} \circ (\mathrm{id} \times \theta)$ may be taken through maps which send $\Omega B \times \{*\}$ to the basepoint $*$ in $S^{n-1} \vee S^{n-1}$.

We define the Hopf construction associated to the action in the usual way. The reduced suspension $\Sigma(\Omega B \times S^{n-2})$ has the homotopy type of the wedge product $\Sigma\Omega B \vee \Sigma S^{n-1} \vee \Sigma(\Omega B \wedge S^{n-2})$, and the Hopf construction H_E is defined to be the composition

$$\Sigma(\Omega B \wedge S^{n-2}) \hookrightarrow \Sigma\Omega B \vee \Sigma S^{n-2} \vee \Sigma(\Omega B \wedge S^{n-2}) \xrightarrow{\simeq} \Sigma(\Omega B \times S^{n-2}) \xrightarrow{\Sigma f} \Sigma S^{n-2}.$$

If E is a fibrewise reduced suspension $\Sigma_B^B D$, where D is a sectioned orthogonal S^{n-2}-bundle, then H_E is the suspension of the Hopf construction associated to D. In fact we will extract more information from the Hopf construction. We first prove a preliminary lemma.

Lemma 4.7.2. *Let A, B, C be pointed CW-complexes. Define*

$$W = ((A \times B \times \{*\}) \cup (A \times \{*\} \times C) \cup (\{*\} \times B \times C)) \subseteq A \times B \times C.$$

Then there is a splitting

$$\Sigma(A \times B \times C) \simeq \Sigma(A \wedge B \wedge C) \vee \Sigma W.$$

PROOF. For any pointed CW-complex X, there is a cofibration sequence

$$\cdots \to [\Sigma^2 W, X] \to [\Sigma(A \wedge B \wedge C), X] \to [\Sigma(A \times B \times C), X] \xrightarrow{\Sigma i^*} [\Sigma W, X] \to \cdots$$

We first note that if π_{AB} is the projection $A \times B \times C \to A \times B \hookrightarrow W$, π_A is the projection $A \times B \times C \to A \hookrightarrow W$, and π_{BC}, π_{AC}, π_B, π_C are defined analogously, then

$$\Sigma i^* \circ (\Sigma\pi_{AB}^* + \Sigma\pi_{AC}^* + \Sigma\pi_{BC}^* - \Sigma\pi_A^* - \Sigma\pi_B^* - \Sigma\pi_C^*)$$

is the identity on $[\Sigma W, X]$. Hence the above exact sequence becomes a sequence of split short exact sequences. The sequence

$$0 \to [\Sigma(A \wedge B \wedge C), X] \to [\Sigma(A \times B \times C), X] \longrightarrow [\Sigma W, X] \to 0$$

induces maps $\Sigma(A \times B \times C) \to \Sigma W$, $\Sigma(A \times B \times C) \to \Sigma(A \wedge B \wedge C)$ which express $\Sigma(A \times B \times C)$ in the required manner, as in Proposition (6.27) of [20]. □

We now proceed to the main result:

Proposition 4.7.3. *Let E be a sectioned orthogonal S^{n-1}-bundle over the CW-complex B. Suppose that E is fibrewise coHopf. Then H_E is primitive; that is, the diagram*

$$\begin{array}{ccc}
\Sigma(\Omega B \wedge S^{n-1}) & \xrightarrow{\quad H_E \quad} & \Sigma S^{n-1} \\
\Big\downarrow{\scriptstyle S} & & \Big\downarrow{\scriptstyle \theta} \\
\Sigma(\Omega B \wedge S^{n-1}) \vee \Sigma(\Omega B \wedge S^{n-1}) & \xrightarrow{H_E \vee H_E} & \Sigma S^{n-1} \vee \Sigma S^{n-1}
\end{array}$$

commutes up to homotopy, where the vertical arrows are the canonical comultiplications on the suspensions.

Proof. Use the group structure on $[\Sigma(\Omega B \times S^{n-2}), S^{n-1} \vee S^{n-1}]$ to define the difference $D = (\Sigma A_f \vee \Sigma A_f) \circ S - (\theta \circ \Sigma A_f)$ between the compositions

$$\Sigma(\Omega B \times S^{n-2}) \xrightarrow{S} \Sigma(\Omega B \times S^{n-2}) \vee \Sigma(\Omega B \times S^{n-2}) \xrightarrow{\Sigma A_f \vee \Sigma A_f} S^{n-1} \vee S^{n-1}$$

and

$$\Sigma(\Omega B \times S^{n-2}) \xrightarrow{\Sigma A_f} S^{n-1} \xrightarrow{\theta} S^{n-1} \vee S^{n-1}$$

The restriction of D to $\Sigma(\Omega B \vee S^{n-2})$ is nulhomotopic, since the diagrams

$$\begin{tikzcd}
\Sigma\Omega B \arrow[r, "S"] \arrow[d, "\Sigma A_f(-,*)"'] & \Sigma\Omega B \vee \Sigma\Omega B \arrow[d, "\Sigma A_f(-,*) \vee \Sigma A_f(-,*)"] \\
S^{n-1} \arrow[r, "\theta"] & S^{n-1} \vee S^{n-1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma S^{n-2} \arrow[r, "\theta"] \arrow[d, "\mathrm{id}"'] & \Sigma S^{n-2} \vee \Sigma S^{n-2} \arrow[d, "\mathrm{id}"] \\
\Sigma S^{n-2} \arrow[r, "\theta"] & \Sigma S^{n-2} \vee \Sigma S^{n-2}
\end{tikzcd}$$

commute up to homotopy. Since the inclusion $\Sigma(\Omega B \vee S^{n-2}) \hookrightarrow \Sigma(\Omega B \times S^{n-2})$ is a cofibration, for any X there is an exact sequence

$$[\Sigma^2(\Omega B \vee S^{n-2}), X] \xrightarrow{0} [\Sigma(\Omega B \wedge S^{n-2}), X] \to [\Sigma(\Omega B \times S^{n-2}), X] \to \cdots$$

Hence D may be pulled back to a unique map $\Sigma(\Omega B \wedge S^{n-2}) \to S^{n-1} \vee S^{n-1}$, which is the Hopf construction applied to D. This map is nulhomotopic if and only if H_E is a co-H-map.

Now define a new "difference map" $D'\colon \Omega B \times S^{n-1} \to S^{n-1} \vee S^{n-1}$ as follows: let $\omega \in \Omega B$, $e^{\pi it} \in S^1$, $s \in S^{n-2}$. Regard S^1 as the unit circle in $\mathbb{C}$. Then

$$D'(\omega, e^{\pi it} \wedge s) = \begin{cases} A_{f\vee f}(\omega, \theta(e^{2\pi it} \wedge s)) & 0 \leqslant t \leqslant 1 \\ \theta((-1)A_f(\omega, e^{2\pi i(t-1)} \wedge s)) & 1 \leqslant t \leqslant 2, \end{cases}$$

where $(-1)\colon S^{n-1} \to S^{n-1}$ is a pointed map of degree -1. This map D' is well-defined since the action of ΩB fixes the basepoint of S^{n-1}. Moreover, since the diagram (*) in (4.7.1) commutes up to homotopy, and the homotopy may be taken to be through maps sending $\Omega B \times \{*\}$ to $\{*\}$, the map D' is nulhomotopic.

To relate D and D', notice that the diagram

$$\begin{tikzcd} \Omega B \times S^1 \times S^{n-2} \arrow[r, "q"] \arrow[d, "\mathrm{id} \times \bar{q}"] & \Sigma(\Omega B \wedge S^{n-1}) \arrow[d, "D"] \\ \Omega B \times S^{n-1} \arrow[r, "D'"] & S^{n-1} \vee S^{n-1} \end{tikzcd}$$

commutes up to homotopy, where $q\colon \Omega B \times S^1 \times S^{n-2} \to \Sigma(\Omega B \wedge S^{n-1})$, $\bar{q}\colon S^1 \times S^{n-2} \to S^{n-1}$ are the quotient maps. This implies that q^*D is the trivial homotopy class in $[\Omega B \times S^1 \times S^{n-2}, S^{n-1} \vee S^{n-1}]$. Now let

$$W = \left(S^1 \times \Omega B \times \{*\}\right) \cup \left(S^1 \times \{*\} \times S^{n-2}\right) \cup \left(\{*\} \times \Omega B \times S^{n-2}\right)$$

and consider the cofibration $W \hookrightarrow S^1 \times \Omega B \times S^{n-2}$. There is an associated long exact sequence

$$\cdots \to [\Sigma(\Omega B \times S^1 \times S^{n-1}), X] \xrightarrow{\Sigma i^*} [\Sigma W, X] \to [\Omega B \wedge S^{n-1}, X] \xrightarrow{q^*}$$
$$\to [\Omega B \times S^1 \times S^{n-2}, X] \xrightarrow{i^*} [W, X] \to \cdots$$

Notice that Σi^* is a surjection, by (4.7.2). By exactness, we deduce that q^* is injective, and so D is nulhomotopic. It follows that H_E is a co-H-map. $\square$

In the above work we have not used the fact that there is a fibrewise coHopf structure $E \to E \vee_B E$; the existence of a fibrewise pointed map extending the comultiplication on the fibre over the basepoint of B is sufficient. However, this does not strengthen the result as one might have thought, as these conditions are equivalent (see (1.2.23)).

4.8 A Generalised Hopf Invariant

In this section we find an invariant of our bundle which vanishes when the bundle is fibrewise coHopf. We adopt the notation of Berstein and Hilton (see [4]).

Let X be a homotopy-commutative cogroup (for example, a double suspension), and let Y be a coHopf space with comultiplication $\theta_Y : Y \to Y \vee Y$. There is an exact sequence of homotopy groups

$$\cdots \to [X, \Omega(Y \times Y, Y \vee Y)] \xrightarrow{\partial} [X, Y \vee Y] \xrightarrow{j_*} [X, Y \times Y] \to \cdots$$

This sequence splits; that is, there is a homomorphism $\kappa\colon [X, Y \times Y] \to [X, Y \vee Y]$ such that $j_*\kappa = 1$. If $\pi_1, \pi_2 \colon Y \times Y \to Y$ are the canonical projections, and $i_1, i_2 \colon Y \to Y \times Y$ are the axial injections, then we define $\kappa = (i_1\pi_1)_* + (i_2\pi_2)_*$. This is a homomorphism because X is a commutative cogroup. We deduce that there is a unique homomorphism $\omega\colon [X, Y \vee Y] \to [X, \Omega(Y \times Y, Y \vee Y)]$ such that $\omega\partial = 1$ and $\kappa\omega = 0$.

Definition 4.8.1. *The Hopf θ_Y-invariant of a homotopy class $\alpha \in [X, Y]$ is the class $\mathscr{H}(\alpha) = \omega(\theta_Y)_*\alpha$. The Hopf θ-invariant $\mathscr{H}(E)$ of a sectioned* *orthogonal S^{n-1}-bundle $E \to B$ is defined to be the element $\mathscr{H}(H_E)$ of $[\Sigma\Omega B \wedge S^{n-1}, \Omega(S^n \times S^n, S^n \vee S^n)]$.*

The Hopf θ_Y-invariant of a map measures the extent to which it fails to be a co-H-map. More precisely, we have the following result:

Lemma 4.8.2. *Let X, Y, θ_Y, α be as before. Then $\mathscr{H}(\alpha) = 0$ in the group $[X, \Omega(Y \times Y, Y \vee Y)]$ if and only if α is a class of co-H-maps.*

Proof. Let f be a representative of the homotopy class α. Suppose that $\mathscr{H}(\alpha) = 0$. Then $\omega(\theta_Y)_*\alpha = 0$, and so $(\theta_Y)_*\alpha$ lies in the image of κ. Hence $(\theta_Y)_*\alpha = \kappa(j_*(\theta_Y)_*\alpha)$. The following diagram commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[r, "\theta_Y"] \arrow[d, "\Delta_Y"] & Y \vee Y \arrow[dl, "j"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y, & \end{tikzcd}$$

and so, applying the naturality of κ, we see that

$$\begin{aligned} (\theta_Y)_*\alpha &= \kappa(j_*(\theta_Y)_*\alpha) \\ &= \kappa((f \times f)_*\Delta_X) \\ &= (f \vee f)_*\kappa(\Delta_X) \\ &= (f \vee f)_*\theta_X, \end{aligned}$$

where θ_X is the comultiplication on X. We deduce that f is a co-H-map, and therefore so is every element of α.

Conversely, suppose that $(\theta_Y)_*\alpha = (f \vee f)_*(\theta_X)$. Observe that θ_Y lies in the class $\kappa(\Delta_Y)$. Hence $(\theta_Y)_*\alpha = (f \vee f)_*\kappa(\Delta_X) = \kappa((f \times f)_*(\Delta_X))$. It follows that $\kappa j_*(\theta_Y)_*\alpha = (\theta_Y)_*\alpha$, and so $\partial\mathscr{H}(\alpha) = \partial\omega(\theta_Y)_*\alpha = 0$. Since ∂ is injective, we deduce that $\mathscr{H}(\alpha) = 0$. $\square$

Corollary 4.8.3. *Let E be a fibrewise coHopf sectioned orthogonal S^{n-1}-bundle over the CW-complex B. Then $\mathscr{H}(E) = 0$ as an element of the group $[\Sigma\Omega B \wedge S^{n-1}, \Omega(S^n \times S^n, S^n \vee S^n)]$.*

As an illustration, we produce an alternative proof of the following result, which is a special case of the main result of [14]:

Corollary 4.8.4. *Let E be a fibrewise coHopf sectioned orthogonal S^{n-1}-bundle with base space S^q. Suppose that $q \leqslant 4n - 5$. Then if $\alpha \in \pi_{q-1}(SO(n-1))$ is the classifying element for E, then $HJ\alpha = 0$.*

PROOF. Recall that for a CW-complex X, $\Sigma\Omega\Sigma X$ is homotopy equivalent to $\Sigma X \vee \Sigma(X \wedge X) \vee \ldots$, and the suspension coHopf structure comes from the injection into the first term. Now let $X = S^{q-1}$.

Since E is fibrewise coHopf, the Hopf invariant $\mathscr{H}(E)$ is zero. By naturality of the Hopf invariant, the Hopf invariant of the restriction g of H_E to $\Sigma X \subset \Sigma\Omega\Sigma X$ is zero in $[\Sigma^n X, \Omega(S^n \times S^n, S^n \vee S^n)]$. Postcomposing with the identification map

$$\Omega(S^n \times S^n, S^n \vee S^n) \to \Omega(S^n \wedge S^n)$$

we obtain the **crude Hopf invariant** associated with E, which lies in the group $[\Sigma^n X, \Omega S^{2n}] \cong \pi_{n+q}(S^{2n})$, and in this case is also zero. By the results of [6] this is the suspension of the James-Hopf invariant of g. Applying the Freudenthal Suspension Theorem, we see that the James-Hopf invariant of g must be zero. It remains to show that $g = J\alpha$. The following diagram commutes up to homotopy, where the vertical arrows are induced by the canonical inclusions $\Sigma e\colon \Sigma S^i \to \Sigma\Omega\Sigma S^i$ for various i:

$$\begin{tikzcd}
\Sigma S^{q-1} \arrow[r, "i"] \arrow[dd] & \begin{matrix}\Sigma S^{q-1} \vee \Sigma S^{n-1} \vee \\ \Sigma(S^{q-1} \wedge S^{n-1})\end{matrix} \arrow[r, "\cong"] \arrow[dd] & \Sigma(S^{q-1} \times S^{n-1}) \arrow[dd] \arrow[dr] & \\
 & & & \Sigma S^{n-1}. \\
\Sigma\Omega\Sigma S^{q-1} \arrow[r, "i"] & \begin{matrix}\Sigma\Omega\Sigma S^{q-1} \vee \Sigma S^{n-1} \vee \\ \Sigma(\Omega\Sigma S^{q-1} \wedge S^{n-1})\end{matrix} \arrow[r, "\cong"] & \Sigma(\Omega\Sigma S^{q-1} \times S^{n-1}) \arrow[ur] &
\end{tikzcd}$$

(The homotopy-commutativity of the triangle follows from the construction of the clutching function.) But the top row of this diagram is $J\alpha$ and the bottom row is g. □

A similar technique allows us to prove the following result:

Corollary 4.8.5. *Let X be a sectioned orthogonal S^{n-1}-bundle over the space $\mathbb{HP}(\infty)$, where $n > 1$. Suppose that X is fibrewise coHopf. Let $f\colon \mathbb{HP}(\infty) \to BO(n)$ be the classifying map for X. Then $HJ(\Omega f) = 0$, where we consider Ωf as an element of $\pi_3(O(n))$.*

Proof. Suppose X is fibrewise coHopf. Then by (4.8.3) the Hopf invariant $\mathscr{H}(X)$ of X is zero, and therefore so is the crude Hopf invariant. This crude Hopf invariant lies in $\pi_{n+4}(S^{2n})$.

By the Freudenthal Suspension Theorem, the suspension homomorphism

$$E\colon \pi_{n+3}(S^{2n-1}) \to \pi_{n+4}(S^{2n})$$

is an isomorphism for all $n > 1$. We conclude that the James-Hopf invariant vanishes, and therefore that H_X is a suspension. But by definition H_X is $J(\Omega f)$. Hence $HJ(\Omega f)$ vanishes. □

Of course, this result is trivial for $n > 4$. This line of argument does not give us any information for bundles over $\mathbb{R}P(\infty)$. To obtain more results over infinite projective spaces in this way, it is necessary to use the more sensitive Hopf invariants giving values in the homotopy groups of $\Omega(S^n \times S^n, S^n \vee S^n)$.

I am grateful to Michael Crabb for pointing out that the above corollary is of limited utility, since direct calculations may be made. In the case $n = 3$ the maps $\mathbb{H}P(\infty) \to BS^3$ are classified by the degree on H^4, which is either 0 or the square of an odd integer. Restricting to S^4, it follows that a fibrewise coHopf S^3-bundle over $\mathbb{H}P(\infty)$ must be trivial. The $n = 4$ case reduces to consideration of maps $\mathbb{H}P(\infty) \to BS^3 \times BS^3$. Restricting again to S^4, we see that a fibrewise coHopf bundle must restrict to a fibrewise reduced suspension over S^4, and must therefore be a fibrewise reduced suspension over the whole of $\mathbb{H}P(\infty)$.

Chapter 5
Conclusion

We have investigated some of the properties of fibrewise coHopf spaces, and given some necessary conditions for particular classes of space to be fibrewise coHopf. It should be made clear at this stage what we have left undone, and what remains to be done in the future.

Perhaps the most obvious shortcoming here is the cavalier fashion in which we have treated Lemma (4.1.2). More information is implied by a coHopf structure on a space than simply the vanishing of all cohomology cup products. For example, it is remarked in [19] that coHopf structure also implies the vanishing of Massey products. It seems probable that more sensitive tests for fibrewise coHopf structure could be devised using such data on the Thom spaces of sphere bundles. The question of whether (4.1.2) admits a converse remains unanswered.

Questions also remain concerning the behaviour of localised sectioned sphere bundles with even-dimensional fibre. We have seen that there are such bundles which are not fibrewise coHopf, though their fibrewise localisations at the prime 2 are. This offers little enlightenment on the question of which such bundles are fibrewise coHopf. Even in the case where the fibre is odd-dimensional, problems remain. In particular, do there exist fibrewise coHopf bundles whose fibrewise 2-localisation does not have the fibrewise pointed homotopy type of a fibrewise reduced suspension?

We have not discussed the classification of fibrewise coHopf structures. Knowledge of the cohomology of the base space of the universal fibrewise coHopf sphere bundle constructed in Chapter 1 would, one hopes, enable characteristic classes to be defined, thus allowing differing fibrewise coHopf structures to be distinguished.

Another question arises concerning the restrictions on the dimensions of the spaces in (2.4.1) and (2.4.3). We may ask whether our dimensional restrictions on the spaces given are the best possible. This is a difficult problem even for the case where the base is a point, particularly for (2.4.1). However, it seems likely to the author that the restriction we have chosen on the dimension of the base space is over-cautious, and that a fibrewise version of Theorem 2.1 of [5] would allow it to be relaxed.

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