On the Regularity of Holonomically Constrained Minimisers in the Calculus of Variations

Christopher Hopper

A thesis submitted for the degree of Doctor of Philosophy



The Queen's College
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Trinity Term
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novum libellum parentibus meis dono
namque tu solebas meas esse aliquid putare nugas

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ABSTRACT

This thesis concerns the regularity of holonomic minimisers of variational integrals in the context of direct methods in the calculus of variations. Specifically, we consider Sobolev mappings from a bounded domain into a connected compact Riemannian manifold without boundary, to which such mappings are said to be holonomically constrained. For a general class of strictly quasiconvex integral functionals, we give a direct proof of local $C^{1,\alpha}$-Hölder continuity, for some $0 < \alpha < 1$, of holonomic minimisers off a relatively closed 'singular set' of Lebesgue measure zero. Crucially, the proof constructs comparison maps using the universal covering of the target manifold, the lifting of Sobolev mappings to the covering space and the connectedness of the covering space. A certain tangential $\mathbb{A}$-harmonic approximation lemma obtained directly using a Lipschitz approximation argument is also given. In the context of holonomic minimisers of regular variational integrals, we also provide bounds on the Hausdorff dimension of the singular set by generalising a variational difference quotient method to the holonomically constrained case with critical growth. The results are analogous to energy-minimising harmonic maps into compact manifolds, however in this case the proof does not use a monotonicity formula. We discuss several applications to variational problems in condensed matter physics, in particular those concerning the superfluidity of liquid helium-3 and nematic liquid crystals. In these problems, the class of mappings are constrained to an orbit of 'broken symmetries' or 'manifold of internal states', which correspond to a subgroup of residual symmetries.

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STATEMENT OF ORIGINALITY

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text or commonly known.

Christopher Hopper
12 September 2014

Preface

In this thesis we are concerned with the regularity properties for minimisers of variational integrals of the form

$$\int_\Omega f(x,u,Du)dx,$$

where $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ for an open $\Omega \subset \mathbb{R}^n$ with $n, N \geq 2$, subject to the subsidiary condition that the minimiser be constrained to take values in a given compact submanifold $\mathcal{M} \subset \mathbb{R}^N$. In fact, by Nash's embedding theorem it is enough to consider embedded submanifolds of $\mathbb{R}^N$ for a sufficiently large integer N. From direct method considerations in the calculus of variations, we consider the class of Sobolev mappings

$$W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M}) = \{u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\}$$

to which minimisers are said to be holonomically constrained (cf. §0.2). In this context we say $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M})$ is a local holonomic minimiser of the above functional if

$$\int_{\Omega'} f(x,u,Du)dx \leq \int_{\Omega'} f(x,v,Dv)dx$$

for all mappings $v \in (u + W^{1,p}_0(\Omega',\mathbb{R}^N)) \cap W^{1,p}(\Omega',\mathcal{M})$ and all compactly contained open subset Ω' of Ω. Note that the competitor v to the minimiser u must agree with u on the boundary in the trace sense, as well as fulfilling the manifold constraint condition.

Given this definition we show partial Hölder continuity for holonomic minimisers $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M})$ of a variational integral of the above form, under the provision that $\mathcal{M}$ be a connected compact Riemannian manifold without boundary and the integrand f satisfies natural differentiability, growth, strict quasiconvexity, u-dependence Hölder continuity and coercivity assumptions (cf. Theorem 6.1). This generalises the only other result to date, namely that of Evans and Gariepy [EG3], for holonomic minimisers of quasiconvex functionals (cf. §0.2.2). In the context of holonomic minimisers of regular variational

integrals, we also provide bounds on the Hausdorff dimension of the singular set (cf. Theorem 7.6) by generalising a variational difference quotient method from Kristensen and Mingione [KM2] to the holonomically constrained case with critical growth.

The manifold constrained regularity problem has several applications to problems in condensed matter physics (cf. Appendix A). For such applications the (unconstrained) mapping is a tensor order parameter $T = (T_{\alpha\beta})\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, invariant under a given symmetry group G, with an integral functional of the form

$$\int_\Omega f_{\mathrm{grad}}(T, \nabla T) + f_B(T)\, dx.$$

Critical points of the bulk energy f_B form an orbit of solutions corresponding to a subgroup $H \subset G$ of residual symmetries. The resulting manifold of internal states $\mathcal{M} = G/H$ describes the broken symmetries of the system; such T taking values in this group orbit form the corresponding constrained mapping case.

Of particular interest is the example of superfluid helium-3. In this case the order parameter $A = (A_{\mu j})$ is a 3×3 complex valued matrix with spin index μ and orbital index j. As a map the order parameter $A\colon \Omega \subset \mathbb{R}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3 \equiv \mathbb{R}^{18}$. The associated symmetry group $G = SO(3) \times SO(3) \times U(1)$ with a group action $*$ defined by

$$(R^{(1)}, R^{(1)}, \phi) * A_{\mu j} = e^{i\phi} R^{(1)}_{\mu\nu} R^{(2)}_{jk} A_{\nu k}$$

for a phase ϕ with rotation matrices $R^{(1)}$ and $R^{(2)}$ in the respective spin space $\boldsymbol{S}$ and orbital space $\boldsymbol{L}$. In regions near the critical temperate T_c, i.e. the Ginzburg-Landau regime, the bulk free energy f_B can be obtained from $A_{\mu j}$ by contracting all indices in second and fourth order terms that are $SO(3) \times SO(3) \times U(1)$-invariant. The resulting expansion yields

$$\begin{aligned} f_B = {} & -\alpha\, A_{\mu j}{}^* A_{\mu j} + \beta_1\, A_{\mu j}{}^* A_{\mu j}{}^* A_{\nu\ell} A_{\nu\ell} + \beta_2\, A_{\mu j}{}^* A_{\mu j} A_{\nu\ell}{}^* A_{\nu\ell} \\ & + \beta_3\, A_{\mu j}{}^* A_{\nu j}{}^* A_{\mu\ell} A_{\nu\ell} + \beta_4\, A_{\mu j}{}^* A_{\nu j} A_{\nu\ell}{}^* A_{\mu\ell} + \beta_5\, A_{\mu j}{}^* A_{\nu j} A_{\nu\ell} A_{\mu\ell}{}^* \end{aligned}$$

where the coefficients α and β_i depend on external parameters like temperature and pressure. When allowing for spatial variations in the order parameter, the gradient energy of an anisotropic superfluid in the Ginzburg-Landau regime has leading $SO(3) \times SO(3) \times U(1)$-invariant terms quadratic in the gradient of the form

$$f_{\mathrm{grad}} = \gamma_1\, \partial_j A_{\mu\ell} \partial_j A_{\mu\ell}{}^* + \gamma_2\, \partial_j A_{\mu\ell} \partial_\ell A_{\mu j}{}^* + \gamma_3\, \partial_j A_{\mu j} \partial_\ell A_{\mu\ell}{}^*$$

for positive constants γ_i. The corresponding manifold of internal states $\mathcal{M}$ can be found by considering continuous and discrete subgroups of $SO(3) \times SO(3) \times U(1)$ (cf. [BV]). For instance, in the so-called B-phase the order parameter is constrained to the manifold

of internal states

$$\tfrac{SO(3)_{\boldsymbol{L}}\times SO(3)_{\boldsymbol{S}}}{SO(3)_{\boldsymbol{L}+\boldsymbol{S}}}\times U(1)_\phi = SO(3)_{\boldsymbol{L},\boldsymbol{S}}\times S^1_\phi,$$

whereas in the dipole-free A-phase the order parameter is constrained to the manifold

$$(\tfrac{SO(3)_{\boldsymbol{S}}}{U(1)_{L_z}}\times \tfrac{SO(3)_{\boldsymbol{L}}\times U(1)_\phi}{U(1)_{L_z-\phi}})/\mathbb{Z}_2 = (S^2_{\boldsymbol{S}}\times SO(3)_{\boldsymbol{L},\phi})/\mathbb{Z}_2.$$

A similar scenario occurs also in the Landau-de Gennes theory for nematic liquid crystals. In this case the tensor order parameter Q is a traceless symmetric 3×3 real matrix, i.e. $Q\colon \Omega\subset\mathbb{R}^3\to \mathrm{Sym}^2_0(\mathbb{R}^3)\equiv\mathbb{R}^5$. The symmetry group $G=SO(3)$ with the group action $*$ defined by

$$R*Q = RQR^T$$

for any $R\in SO(3)$. The bulk free-energy expanded to fourth order takes the form

$$f_B(Q) = -\tfrac{a^2}{2}\mathrm{tr}\ Q^2 - \tfrac{b^2}{3}\mathrm{tr}\ Q^3 + \tfrac{c^2}{4}(\mathrm{tr}\ Q^2)^2,$$

where a, b and c are material and temperature dependent constants. The bulk energy f_B is invariant under the $SO(3)$-action by conjugation on the five-dimensional space of Q-tensors (cf. [LT]). Critical points of the bulk energy form an orbit of solutions in the five-dimensional space of Q-tensors. Such Q taking values in a group orbit have constant eigenvalues and constant bulk energy. We refer to such a situation as the constrained theory of nematic liquid crystals. In §6.8 we show the well-known fact that the space of constrained uniaxial Q-tensors (i.e. Q-tensor having two equal non-zero eigenvalues) is equivalent to $SO(3)/D_\infty = \mathbb{R}P^2$, with the corresponding space of constrained biaxial Q-tensors (i.e. Q-tensors having three distinct eigenvalues) equalling $SO(3)/D_4 = S^3/\mathbb{H}$.

The introduction chapter recounts the main developments of so-called direct methods in the calculus of variations and related regularity results for minimisers of variational integrals. The first part of §0.1 outlines the beginnings of the calculus of variations and, in particular, discusses the development of the minimality condition as a natural local condition. In this regard, the work of [E4] formalises this minimality condition from which necessary condition are derived. In §0.1.1, Dirichlet's principle and Weierstrass criticism of the admissibility of this principle is highlighted. Within this context Hilbert's famous 19th and 20th (variational) problems for the existence, uniqueness and regularity are introduced in §0.1.2 together with a survey of their early developments. Of particular note is the result of Weyl [W5] and the *a priori* estimates of Caccioppoli [C2, C4]. In §0.1.3 De Giorgi's solution [DG6] to Hilbert's 19th problem is present with reference to ideas from [C2, C4]. Thereafter a brief survey of counterexamples to everywhere regularity in the vectorial case is given. In §0.1.5 a review of the partial regularity results in the (unconstrained) vectorial case is presented with a particular emphasis on the quasiconvexity condition. The second part of the introduction, §0.2, introduces variational problems

with holonomic constraints. The earliest such problem for geodesics goes back to the inception of the calculus of variations. In §0.2.1 we give a survey of the regularity results for harmonic maps (as a natural generalisation of the geodesic problem), beginning with the original result due to Morrey [M16]. The seminal work of [SU1] on energy-miniminising harmonic maps is in particular highlighted. The last section §0.2.2 sketches the partial regularity proof of [EG3] by an indirect blow argument for constrained minimisers of quasiconvex functionals.

The preliminaries of Chapter 1 establish notation and basic results. We remark that the given proof in §1.3 of the local form of Gehring's lemma uses a maximal function argument that is not readily available in the literature. In particular, we show the role of the Calderon-Zygmund decomposition and Vitali's covering lemma for the distribution of the maximal function, when combined with a reverse Hölder-type inequality, to be key to the self-improving property. The proof of Weyl's lemma in §1.7 is done via a difference quotient method which is revisited in Chapter 7. In §1.8 we introduce, using the normal bundle, the notion of a tubular neighbourhood $\mathcal{O}$ for a compact submanifold $\mathcal{M}$ of $\mathbb{R}^N$ without boundary. The resulting retraction $\Pi\colon \mathcal{O} \to \mathcal{M}$ is C^{r-1} whenever $\mathcal{M}$ is a C^r-submanifold (cf. Lemma 1.47). Corollary 1.51 states the well known fact that the gradient $\nabla\Pi$ projects vectors onto the tangential subspace $T\mathcal{M}$. On the other hand, by working with the distance function, more general sets of so-called positive reach can be considered (e.g. manifolds with boundary) based on results by [F1]. For any such set $A \subset \mathbb{R}^N$ of positive reach, there exists $\varepsilon > 0$ such that for $0 < r < \varepsilon$ the neighbourhood $\{x : \operatorname{dist}(x, A) \leq r\}$ has a boundary of class $C^{1,1}$ with a Lipschitz continuous retraction (cf. Theorem 1.49). In §1.9 we introduce the Sobolev space of mappings into compact manifolds and state the formal definition of holonomically constrained local minimisers of integral functionals.

Chapter 2 aims to present a direct proof of the partial $C^{1,\alpha}_{\mathrm{loc}}$-continuity of (unconstrained) local minimisers for integral functionals of the form

$$\int_\Omega f(Du)dx,$$

under the assumptions the integrand f satisfies quadratic growth and strict quasiconvexity. Our proof uses an $\mathbb{A}$-harmonic approximation lemma (cf. Lemma 2.9), a linearisation strategy, Caccioppoli inequalities of the second kind and Gehring's lemma. The goal is to give a model proof in a simple and well known situation with an emphasis on the direct nature of the proof of Lemma 2.9. The proof of Lemma 2.9 gives an explicit ε-δ dependence and, moreover, affords a regularity result obtained entirely on the level of the gradient. It is based on the the general Sobolev-Orlicz space proof of [DLSV] using a Lipschitz truncation that goes back to [AF3]. The proof by direct means of the regularity

result in the unconstrained case is subsequently adapted to the manifold constrained case in Chapter 6.

Chapter 3 discusses the distinguished case of minimising harmonic maps into compact manifolds as well as the related higher order problem for biharmonic maps. We remark that the original ε-regularity result of [SU1] (i.e. Theorem 3.1) has been proved by [S11] using a harmonic approximation lemma (cf. Lemma 3.3), similar to that of the previous chapter, with a 'technical regularity lemma' (cf. Lemma 3.4). Of particular interest is the need in Lemma 3.4 for Caccioppoli inequalities of the first kind. Lemma 3.5 establishes such inequalities under the bounded excess assumption (3.3). Our proof here of Lemma 3.5 differs from the original of [S11, §2.8] and is along the lines of a hole-filling argument. We remark that the hole-filling proof of Lemma 3.5 involves several important non-standard technical points that need to be emphasised. Firstly, in seeking a suitable comparison map which satisfies the manifold constraint condition, a lemma of Luckhaus (cf. Chapter 4) is employed. The resulting hole-filling construction of Corollary 4.9 affords a bound on the distance of the perturbation from the manifold. To ensure the perturbed mapping remains close to the manifold, the monotonicity formula (3.2) together with the bounded excess assumption (3.3) are needed. The resulting distance bound (3.7) from the manifold can be made small provided the quotient $\frac{r}{s}$ is fixed and strictly less than 1. As the standard iteration lemma (i.e. Lemma 0.7) is no longer viable, only a finite number of iterations (with the ratio fixed) are performed. Then a covering argument of [S12] is used to get the desired result. The implications of Lemma 3.5 show that only partial higher integrability is possible as the bounded excess assumption cannot be removed (cf. Corollary 3.8). The following §3.2 give a derivation of the monotonicity formula for harmonic maps based on the stress-energy tensor. The derivation method is then applied to more general integrands before concluding such signed quantities are elusive in general.

The second part of Chapter 3 is devoted to biharmonic maps. In particular, we derive an analogous ε-regularity result for minimising biharmonic maps into a compact manifold. The difficulty here comes with the realisation that the Luckhaus construction of Chapter 4, due to problems in implementing the homogeneous degree zero extension (cf. Remark 4.7), no longer passes over to the higher order case. Nonetheless we show the desired result (i.e. Theorem 3.20) follows along the lines of the original proof of [SU1], using a smooth mollification together with a monotonicity formula due to [A5] (cf. Theorem 3.17). No other minimality proof of the ε-regularity result has yet been given in the literature for the biharmonic case, however there are several equivalent partial regularity results for stationary solutions (cf. Remark 3.24). The final §3.4 surveys various regularity results in the weakly harmonic map case. In the special case of weakly harmonic maps

into the sphere S^k, the right-hand side of (3.35) can be seen to have an additional divergence structure. This can then be used to show everywhere local continuity for domains $\Omega \subset \mathbb{R}^2$ (cf. Theorem 3.27). In the case of compact manifolds the moving frame method of Hélein recovers conservation laws for the Coulomb frame analogous to the divergence structure of the spherical case. The technical aspect of Lemma 3.28 shows the method to be perturbative and inapplicable to locally energy minimising harmonic maps.

The results of Chapter 4 address the non-trivial difficulties in finding appropriate comparison maps that satisfy the non-linear manifold constraints. The idea, which goes back to results by [L21, L22], attempts to exploit the continuity properties of the Sobolev embeddings. By taking a triangulation of the domain, we restrict the given Sobolev mappings to lower-dimensional skeletons for which continuity prevails. In such cases a simple convex combination yields the desired construction. Then to extend the constructed mapping from these lower-dimensional skeletons, a homogeneous degree zero extension is used. The resulting Lemma 4.1 obtains the desired construction, however the distance bound (4.1) can only be controlled exactly when a monotonicity formula is available (cf. Lemma 3.5).

Chapter 5 is focused on removing the technical difficulties of Chapter 4. The model case here is an extension lemma due to [HKL1, Lem. A.1] for Sobolev mappings into the sphere S^k (cf. Lemma 5.30). The result [HL2, Lem. 6.2] generalises the latter construction to compact ℓ-connected manifolds for some integer $\ell \geq 1$ (cf. Lemma 5.33). Note that the sphere S^k is $(k-1)$-connected for $k \geq 2$. The proof of the extension-type Lemma 5.33 relies upon the locally Lipschitz retraction of Lemma 5.32 which is proved entirely by elementary means. By working with the universal covering we argue the connectedness condition is canonical for the covering space. From the paper [BC1] we also recall the Sobolev lifting results (cf. Theorem 5.20) which appeals, by an induction argument, to the continuous lifting criteria of Proposition 5.8 along line segments. In particular, Theorem 5.20 shows any Sobolev mapping $u \in W^{1,p}(\Omega, \mathcal{M})$, for $p \geq 2$, into a compact manifold $\mathcal{M}$ has a lifting $\varphi \in W^{1,p}(\Omega, \widetilde{\mathcal{M}})$ with $|Du| = |D\varphi|$ a.e. that factors through the universal covering $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ as regular as u permits. The final §5.5 constructs comparison maps that satisfy a hole-filling construction for $2 \leq p < \ell + 2$ by combines the extension result of Lemma 5.33 with the lifting result of Theorem 5.20. Lemma 5.34 provides an interior result while Lemma 5.35 gives an up to the boundary construction. In both cases the elementary convex combination of a bump function used in the unconstrained case (cf. equation (2.12)) is now lifting upstairs.

In Chapter 6 we give a partial regularity proof (by direct means) for holonomic minimisers $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ of integral functionals of the form

$$\int_\Omega f(x, u, Du)dx,$$

under the structural assumption that the integrand f satisfies (h0), (h1), (h2), (h3) and (h4) (cf. §6.1 for details). The manifold $\mathcal{M}$ is taken to be compact and without boundary together with the assumption that the universal covering space is compact and ℓ-connected for some integer $1 \leq \ell \leq n-2$. Note that the covering space is always 1-connected by definition. For $2 \leq p < \ell + 2$ we use the construction in §5.5 to prove local higher integrability and global higher integrability up to the boundary. The construction is subsequently adapted in §6.6 to provide reverse Hölder-type inequalities using strict quasiconvexity and minimality. It is important to stress the absence of any additional smallness conditions in the results of the comparison maps constructed in §5.5. In §6.3 we use the standard technique of Ekeland's variational principle to show that an holonomic minimiser of the full problem almost minimises the so-called frozen problem. In §6.4 a new tangential $\mathbb{A}$-harmonic approximation lemma is proved by direct means, based on the model solution of §2.2 and the results of §6.3. The resulting excess improvement of §6.7 subsequently yields the desired partial regularity result.

In Chapter 7 we give upper bounds on the Hausdorff dimension of the singular set for holonomic minima of regular variational integrals of the form

$$\int_\Omega F(x, u, Du)dx,$$

under the structural assumptions that the integrand F satisfies the assumptions of (7.6). Our approach generalises the unconstrained case of [KM2] by utilising the the higher integrability of §6.2. The key point here is to compare constrained minimisers of the full problem with constrained minimisers of the frozen problem. In §7.1 we apply a difference quotient method, under the assumption of higher integrability, to the constrained frozen problem for which stationary solutions have critical growth. The desired bounds in Theorem 7.6 on the singular set are then obtained by an iterative procedure.

Contents

Preface xi

Notation and list of symbols xxiii

Introduction 1
0.1. An historical overview to regularity theory in the calculus of variations . 1
0.1.1. Development of the direct method 8
0.1.2. Hilbert's problems 12
0.1.3. The solution of De Giorgi 16
0.1.4. Counterexamples to everywhere regularity 19
0.1.5. Partial regularity in the vectorial case 21
0.2. The problem of mapping into a manifold: regularity with holonomic constraints 25
0.2.1. Harmonic mappings into compact manifolds 27
0.2.2. Partial regularity results for holonomic minimisers of quasiconvex functionals 31

1. Preliminaries 35
1.1. Slicing inequalities 36
1.2. Maximal function 37
1.3. Gehring's lemma 39
1.4. Modulus of continuity 44
1.5. Hölder continuous spaces 46
1.6. Sobolev spaces 48
1.6.1. Traces 51
1.6.2. Difference quotients 54
1.7. Weyl's lemma 55
1.8. Submanifolds, projections and the distance function 58
1.8.1. Tubular neigbourhoods 59
1.8.2. Sets of positive reach and the regularity of the squared distance function 60
1.9. Sobolev mappings into manifolds 62
1.10. Constrained stationary variations 64

1.11. Triangulations 65

2. Partial regularity for unconstrained minimisers of quasiconvex functionals in the quadratic growth case 69
2.1. Excess-improvement and iteration 71
2.2. The $\mathbb{A}$-harmonic approximation lemma 72
2.3. Approximate harmonicity 75
2.4. Caccioppoli inequality of the second kind 76

3. Regularity aspects of harmonic and biharmonic maps 81
3.1. Caccioppoli inequalities of the first kind and partial higher integrability for harmonic maps 84
3.2. Monotonicity 88
3.3. Partial regularity for minimising biharmonic maps 93
3.3.1. A biharmonic ε-regularity result 97
3.4. Remarks concerning Hélein's moving frame method and the regularity of weakly harmonic maps 105

4. Luckhaus' lemma 115
4.1. Proof of Luckhaus' Lemma 118
4.2. Applications to the hole-filling technique 121

5. Extensions and the lifting of Sobolev mappings into compact manifolds 125
5.1. Brief review of homotopy groups 127
5.2. Covering spaces 129
5.3. The lifting problem 132
5.4. Extension-type results for Sobolev mappings 135
5.5. Comparison maps into arbitrary compact manifolds 144

6. Partial regularity for holonomic minimisers of general quasiconvex functionals 147
6.1. Statement of main result 147
6.2. Higher integrability 151
6.3. An application of Ekeland's variational principle 155
6.4. The tangential $\mathbb{A}$-harmonic approximation lemma 158
6.5. Approximate tangential harmonicity 163
6.6. Reverse Hölder-type inequalities 168
6.7. Excess-improvement and iteration 174
6.8. Applications to problems in condensed matter physics 176

7. Singular set bounds for holonomic minimisers of integral functionals 181
7.1. Revisting the difference quotient method for holonomic minimisers of regular variational integrals 185

A. Examples of ordered media with broken symmetries 195
A.1. Ferromagnetism 196
A.2. Uniaxial nematic liquid crystals 196
A.3. Biaxial nematic liquid crystals 197

A.4. Superfluidity of liquid helium-3 201

Bibliography 209

Notation and list of symbols

$\mathbb{R}^n$	Euclidean n-space, $n \geq 2$, with points $x = (x^1, \ldots, x^n)$, $x^i \in \mathbb{R}$ (real numbers); norm $\lvert x\rvert = ((x^1)^2 + \cdots + (x^n)^2)^{\frac{1}{2}}$.
$\mathbb{R}^{Nn}$	Identified as the space of real $N \times n$ matrices for integers $N, n \geq 2$.
S^n, B^n	The unit sphere in $\mathbb{R}^{n+1}$, all vectors of length 1; the unit disk or ball in $\mathbb{R}^n$, all vectors of length ≤ 1. The boundary of the n-ball $\partial B^n = S^{n-1}$.
$B(x,r) = B_r(x)$	An open ball of radius r in $\mathbb{R}^n$ centred at x.
Ω	A proper open subset of $\mathbb{R}^n$; Ω is a *domain* if it is also connected.
∂U, $\bar{U}$, $U \backslash V$	Boundary of a set U; $\bar{U} = U \cup \partial U$ denotes the closure; $U \backslash V = \{x \in U : x \notin V\}$.
$V \subset\subset U$	V compactly contained in U, i.e. $V \subset \bar{V} \subset U$ with $\bar{V}$ compact.
$\lvert E\rvert$, $\mathcal{L}^n(E)$	Volume of $E \subset \mathbb{R}^n$, i.e. the Lebesgue measure of $E \subset \mathbb{R}^n$.
$\mathcal{H}^k(E)$, $\dim_{\mathcal{H}}(E)$	k-dimensional Hausdorff measure of $E \subset \mathbb{R}^n$ defined by $\mathcal{H}^k(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$, where $$\mathcal{H}^k_\delta(E) = \inf\Big\{ \sum_{j=0}^{\infty} \omega_k \rho_j^k : E \subset \bigcup_{j=0}^{\infty} B_{\rho_j}(x_j), \rho_j \leq \delta, x_j \in \mathbb{R}^n \Big\}$$ and $\omega_k = \mathcal{L}^k(B(0,1))$; the Hausdorff dimension of E defined by $\dim_{\mathcal{H}}(E) = \inf\{k \geq 0 : \mathcal{H}^k(E) = 0\}$.
$\mathbf{1}_E$	Characteristic function, i.e. $\mathbf{1}_E(x) = 1$ if $x \in E$; $\mathbf{1}_E(x) = 0$ if $x \notin E$.
supp u	Support of u, i.e. the closer of the set on which $u \neq 0$.
$\Delta^h u$	The difference quotient $\Delta^h u(x) = u(x+h) - u(x)$.
$\lfloor x \rfloor$	Floor function taking the largest integer no greater than x, i.e. $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

$C(\Omega)$	Set of continuous functions on Ω.
$C_c(\Omega)$, $C_0(\Omega)$	Set of continuous functions whose support is compact; the closure of $C_c(\Omega)$ in the sup-norm.
$C^k(\Omega)$, $C^k_c(\Omega)$	Set of functions having derivatives of order $\leq k$ continuous in Ω (k = integer ≥ 0 or $k = \infty$); compactly supported $C^k(\Omega)$ functions.
$C^k(\bar{\Omega})$	Set of functions in $C^k(\Omega)$ whose derivatives of order $\leq k$ have continuous extensions to $\bar{\Omega}$.
$C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$	Hölder spaces with $0 < \alpha \leq 1$ whose k-th order partial derivatives are α-Hölder continuous; the set of functions belonging to $C^{k,\alpha}(\Omega')$ for $\Omega' \subset\subset \Omega$.
$L^p(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$	L^p-space, for $p \geq 1$, consisting of measurable functions on Ω such that norm $\|u\|_{p;\Omega} = \|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{p}}$ is finite; the set of functions belonging to $L^p(\Omega')$ for $\Omega' \subset\subset \Omega$.
$L^\infty(\Omega)$	The space of bounded measurable functions on Ω with the norm $\|u\|_{\infty;\Omega} = \|u\|_{L^\infty(\Omega)} = \sup_\Omega |u|$.
$L^p(\Omega, \mathbb{R}^N)$	Vector valued functions $u = (u^1, \dots, u^N)\colon \Omega \to \mathbb{R}^N$ such that $u^i \in L^p(\Omega)$.
$(u)_\Omega$	The average $(u)_\Omega = \fint_\Omega u\,dx = \frac{1}{\lvert\Omega\rvert}\int_\Omega u\,dx$ for any $u \in L^1(\Omega)$.
$(u)_r$, $(u)_{x,r}$	The average, or mean value, of $u \in L^1(\Omega)$ on the ball $B_r(x)$.
$W^{s,p}(\Omega)$, $W^{s,p}_{\mathrm{loc}}(\Omega)$	Sobolev space with norm $\|\cdot\|_{s,p;\Omega} = \|\cdot\|_{W^{s,p}(\Omega)}$; the set of functions belonging to $W^{s,p}(\Omega')$ for $\Omega' \subset\subset \Omega$.
$W^{s,p}_0(\Omega)$	The closure of $C^\infty_c(\Omega)$ in the $\|\cdot\|_{W^{s,p}(\Omega)}$-norm.
$W^{s,p}(\Omega, \mathbb{R}^N)$	Vector valued functions $u = (u^1, \dots, u^N)\colon \Omega \to \mathbb{R}^N$ such that $u^i \in W^{s,p}(\Omega)$.
$\mathfrak{R}(\cdot)$	The trace operator $\mathfrak{R}\colon W^{1,p}(\Omega, \mathbb{R}^N) \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N)$.
$W^{s,p}(\Omega, \mathcal{M})$	Space of Sobolev mappings into a submanifold $\mathcal{M} \subset \mathbb{R}^N$.
$\Pi\colon \mathcal{O} \to \mathcal{M}$	The retraction from a tubular neighbourhood $\mathcal{O}$ onto a submanifold $\mathcal{M} \subset \mathbb{R}^N$.
$\nabla\Pi(\theta)$	The gradient of the retraction Π at a point $\theta \in \mathcal{M}$ that projection of vectors onto the tangent space $T_\theta\mathcal{M}$.
$\pi_n(X, x_0)$, $\pi_n(X)$	n-th homotopy group of a topological space X at base point x_0.

For $u = (u^1, \dots, u^N)\colon \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$ open, the derivative of u is a map $Du\colon \Omega \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ such that $Du(x) : \mathbb{R}^n \to \mathbb{R}^N$ is a linear map with

$$Du(x)(e_j) = \lim_{t\to 0} \frac{u(x + te_j) - u(x)}{t},$$

provided the derivative exists; for brevity it is also denoted by $D_j u = u_{x_j} = \frac{\partial u}{\partial x^i}$. When $N = 1$ the gradient $Du = (D_1 u, \dots, D_n u)$ and the Hessian matrix

$$D^2 u = (D^2_{ij} u)_{n\times n} = \Big(\frac{\partial^2 u}{\partial x^i \partial x^j}\Big).$$

For $N > 1$ and integers $k \geq 1$,

$$D^k u(x)(e_{\beta_1}, \dots, e_{\beta_k}) = \Big(\frac{\partial^k u^1}{\partial x_1^{\beta_1} \cdots \partial x_1^{\beta_k}}, \dots, \frac{\partial^k u^N}{\partial x_1^{\beta_1} \cdots \partial x_1^{\beta_k}}\Big)$$

for multi-index $\beta = (\beta_1, \dots, \beta_k)$, $\beta_j =$ integers ≥ 0, with $|\beta| = \beta_1 + \ldots + \beta_k = k$, and j-th coordinate vector $e_j = (0, \dots, 0, 1, 0, \dots, 0)$.

If $f \in C^k(\Omega, \mathbb{R}^N)$ Taylor's formula implies

$$\begin{aligned} f(a+h) = f(a) + Df(a)h + \frac{1}{2}D^2 f(a)(h,h) + \cdots + \frac{1}{k!}D^k f(a)(h, \dots, h) \\ + \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\big(D^k f(a+th) - D^k f(a)\big)(h, \dots, h)dt \end{aligned}$$

for any $a \in \Omega$ and $h \in \mathbb{R}^n$ with $a + h \in \Omega$. In particular, when $k = 2$ we have

$$f(a+h) = f(a) + Df(a)(h) + \int_0^1 (1-t)D^2 f(a+th)(h,h)dt.$$

Introduction

We give an account of the regularity theory emerging from the development of so-called direct methods in the calculus of variations with a particular emphasis on results pertaining to the minima of variational integrals. In §0.2 we turn our attention to variational problems with holonomic constraints to which the thesis is primarily devoted.

0.1. An historical overview to regularity theory in the calculus of variations

The art of optimisation, whether it is maximising or minimising, and the description of the solutions' essential properties reside in the field of mathematical analysis aptly named the calculus of variations. In a broad sense such optimisation principles have been known since antiquity. For instance, the second century BC astronomer Zenodorus is best known for his now lost treatise *On isometric figures*, which examines a number of different isoperimetric problems (e.g. proving the area of any polygon is smaller than the area of a circle of equal perimeter).[1] Such problems, appealing because they are easy to formulate and understand but much less so to solve, have proved invaluable for the development of analysis and geometry. The appeal can also be seen in natural philosophy's amicable relationship with optimum principles, so much so that many laws of nature can be posed in the form of a variational principle. The first known example in

[1]Several propositions from this treatise are preserved by Theon of Alexandria in his commentary on Book I of Ptolemy's *Magna Syntaxis* known from the Arabic as the *Almagest*. Theon's commentary first survives in the Greek *editio princeps* of 1538, and Zenodorus's propositions can be found in [PA, pp. 11–17]. An annotated copy of the print by Henry Savile includes notes on the text (cf. Bodleian library MSS Savile 26–31). A parallel French–Greek edition of the passages in question can be found in [H2, pp. 33–49]. Hultsch translated the passages into Latin in his edition of *Pappus* [H22, vol. 3, pp. 1189–1211] for the purpose of comparison with Pappus' own exposition of the same propositions. A further Greek edition can be found in [R5, pp. 355–63].

the latter context appears in Heron of Alexandria's *Catoptrica* (theory of mirrors), which deduces the laws of reflection for mirrors from the postulate that light takes the shortest path [SN]. The first contribution to make use of the calculus in some form is attributed to Pierre de Fermat's influential work [F2], which deduced the laws of refraction from the postulate that light traverses an optical medium in the least possible time. Another early noteworthy variational problem, in fact the earliest known modern formulation, is that of Isaac Newton's determination of the solid of revolution of least resistance to rectilinear motion in a uniform fluid [W9, pp. 456–480]. The problem appears, albeit tersely and without proof, in the scholium to Proposition 34, Book 2, of the *Philosophiae Naturalis Principia Mathematica* (1726, 3rd ed.) and was met with incomprehension by Newton's contemporaries who sought to replicate the calculations.[2]

The seminal event in the early development of the calculus of variations took place when Johann Bernoulli challenged 'the sharpest mathematicians' to solve the so-called Brachistochrone problem (from βράχιστος, *brachistos*—the shortest; χρόνος, *chronos*—time). First announced in *Acta Eruditorum* (June 1696), the problem invites one to find the curve between two points in a vertical plane such that a point mass, under the influence of its own gravity, descends along the curve in the quickest possible time.[3] In terms of the calculus it can be recast as that of finding a curve $y = y(x)$, $0 \leq x \leq a$, such that the total time of descent

$$\int_0^a \sqrt{\frac{1 + (\frac{dy}{dx})^2}{2y}}\, dx$$

is least. After a further explanation of the problem was published at Groningen, a compilation of solutions was printed in *Acta Eruditorum* (May 1697), containing articles by Johann Bernoulli and his brother Jacob, as well as contributions from Leibniz, l'Hôpital, Tschirnhaus and Newton (as an excerpt reprinted from the *Philosophical Transactions* [N6]). All found the cycloid as the solution.

The self-evident difficulty of this toy problem, having stimulated such mathematical interest, lies in the fact that it seeks to find the form of a curve. This distinguishes the Brachistochrone problem, and similar such types, from those with only a finite degree

[2]David Gregory, then Savilian Professor of Astronomy at Oxford, visited Newton in 1694 to appeal personally to the scholium's author for enlightenment. The impulse lead Newton to rework his prior analysis, the results of which were passed on to Gregory [T13, pp. 323-5, pp. 375–383]. They were eventually reprinted by Andrew Motte [N7, vol. 2, app. pp. v–viii] since 'the demonstration of these curious theorems being omitted by the author, the analysis thereof communicated by a friend, is added at the end of this volume'.

[3]Galileo Galilei had also considered such a problem in his *Discorsi e dimostrazioni matematiche, intorno à due nuove scienze* (1638). By arguing, without use of the calculus, that the arc of a circle is faster than any number of its chords, he incorrectly concluded 'the path of quickest descent [lationem omnium velocissiman] from one point to anther is not the shortest path, namely, a straight line, but the arc of a circle' (cf. 3rd Day, Theor. 22, Prop. 36).

of freedom. Consequently there remained a desire to develop, in an analogous way, a more advanced type of infinitesimal analysis for finding such curves. A simple premise was found to be key, namely that *any curve which minimises a given integral must have sub-arcs minimising the same integral.* This natural idea (which, in one form or another, was stated by all of Johann's contemporaries[4]) implies that the *necessary* condition for a minimiser has to be a local condition, and it has to involve all that is local at a point. Thus they sought a relation involving all the differentials from which they hoped, by integrating, the desired curve can be found.

Subsequent to this early period of research sees the contributions of Leonhard Euler recast the subject as a distinct branch of analysis. Euler's early efforts culminated in the seminal monograph *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti* (1744), which shifts focus—upon the realisation that seemingly disparate earlier problems were all instances of a single form—towards a discussion of general approaches to classes of integral variational problems.[5] The monograph's basic organisation and direction represent a substantial break from the then established tradition, including Euler's own previous papers on the subject. One should note that the historically prominent isoperimetric problems occupy a secondary position in Euler's arrangement. Moreover, there are two analytical features of Euler's approach that distinguished it from the work of earlier efforts: firstly, the variational problem is formulated in terms of a general integrand of the form $Z(x, y, p)$ and, secondly, the derivation of the necessary condition involves the differential of Z and is explicitly carried out in terms of the partial derivatives of this function.

We recount the details of Euler's important analytical developments as follows, beginning first in [E4, Chap. 1] with:

> **Hypothesis I.** 16. In this tract we always designate the abscissa, to which we refer all curves, by the letter x and the ordinate by the letter y. Then indeed, with equal elements of the abscissas taken, we always write $dy = p\,dx$; $dp = q\,dx$; etc.

This is followed by the comment in Scholion 37. that 'in the following the maximum or minimum formula will always be indicated by $\int Z\,dx$. While Z will be a function not only of the quantities x and y, but also of p, q, etc.'

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[4]Jacob's approach, in particular, explicitly states this principle (cf. Jacob's solution [B2] as well as the related paper [B4] and the reworked analysis of the latter paper by Johann [B6]). The memoir [B6] is significant as it represents the most advanced thinking of the period on isoperimetric problems and serves as the point of departure for Euler's monograph [E4].

[5]In fact Carathéodory's introduction to *Opera Omnia* (ser. 1, vol. 24) lists one hundred problems, classified into eleven different classes, which Euler considered within his publications on the subject [C13, lvi–lxii].

Then a refinement of the local condition of Jacob Bernoulli is stated:

Proposition II. Theorem. 38. *If amz is a curve* [Fig. 1] *in which the value of the formula* $\int Z dx$ *is a maximum or a minimum, and* Z *is an algebraic or determinate function of* x, y, p, q, *etc., then any portion* mn *of the same curve enjoys the same properties that, for the part related to the abscissa* MN, *the value of* $\int Z dx$ *is also a maximum or minimum.*

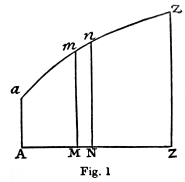


Fig. 1

The discussion then establishes the following notation:

Hypothesis II. 48. If the abscissa AZ of a curve [Fig. 2] is divided infinitely into innumerable small elements IK, KL, LM, etc., all equal to one another, and some portion AM is denoted by x, to which some variable function F corresponds, then we shall denote the same function F, just as it may refer to the points of the abscissa either following N, O, P, Q etc. or preceding L, K, I etc. thus, so that the value of this function, which for the point M is $= F$, by F', F'', F''', etc. for the following points of the abscissa N, O, P, Q, etc. and by $F_{\prime}$, $F_{\prime\prime}$ $F_{\prime\prime\prime}$, etc. the preceding points of the abscissa L, K, I, etc. Thus, without prolix writing differentials, the value of any variable function, which is held at any point of the abscissa, can be indicated conveniently.

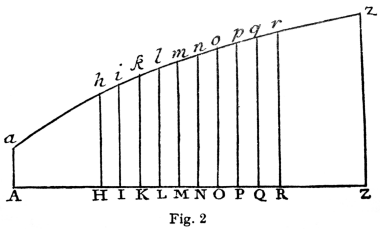


Fig. 2

With such a construction the identities $F' = F + dF$, $F'' = F' + dF'$, $F''' = F'' + dF''$, etc. and $F = F_{\prime} + dF_{\prime}$, $F_{\prime} = F_{\prime\prime} + dF_{\prime\prime}$, $F_{\prime\prime} = F_{\prime\prime\prime} + dF_{\prime\prime\prime}$, etc. immediately follow. Moreover, if we denote the ordinates Mm, Nn, Oo, Pp, etc. by y, y', y'', y''' etc. and $L\ell$, Kk, Ii, etc. by $y_{\prime}$, $y_{\prime\prime}$, $y_{\prime\prime\prime}$, etc. one also has $p = \frac{y'-y}{dx}$, $p' = \frac{y''-y'}{dx}$, $p'' = \frac{y'''-y''}{dx}$, etc. and $p_{\prime} = \frac{y_{\prime\prime}-y_{\prime}}{dx}$, $p_{\prime\prime} = \frac{y_{\prime\prime\prime}-y_{\prime\prime}}{dx}$, etc. by Hypothesis I. So if the expression $\int Z dx$ refers to the abscissa $AM = x$ the value corresponding to the next element $MN = dx$, as argued in Corollaries VI–VII, must be $Z dx$. In a similar way the values of $\int Z dx$ belonging to the elements MN, MO, OP, etc. are indicated by $Z dx$, $Z' dx$, $Z'' dx$, etc. Thus if the expression $\int Z dx$ refers to the abscissa $AM = x$ the value belonging to the abscissa AZ equals $\int Z dx + Z dx + Z' dx + Z'' dx +$ etc. until we arrive at the point Z. In which case if we are to find the curve for which the value of $\int Z dx$, with reference to the given abscissa AZ, has a maximum or minimum, we must obtain, when the abscissa $AM = x$, a maximum or minimum of the corresponding expression $\int Z dx + Z dx + Z' dx + Z'' dx +$ etc.

Having elucidated this construction there follows:

Definition V. 62. The *differential value* of a formula corresponding to a given maximum or minimum is the difference between the values, which this formula obtains on the sought curve as well as on the same changed by an infinitely small amount.

Given this notation, Chapter 2 commences with the following:

Proposition I. Problem. 1. *If in some curve amz* [Fig. 4] *any ordinate Nn may be augmented by an infinitely small part nν; to find the increments or decrements, which the individual determinate quantities pertaining to the curve will hence receive.*

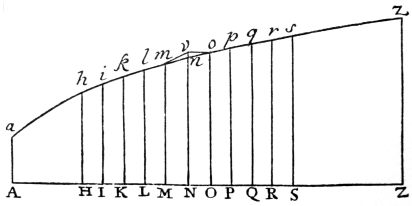


Fig. 4

To obtain the solution from the above construction all quantities depending on y' are changed; the others remain fixed. As the quantity y' increases by $n\nu$ it follows that $p = \frac{y'-y}{dx}$ becomes $+\frac{n\nu}{dx}$ and $p' = \frac{y''-y'}{dx}$ becomes $-\frac{n\nu}{dx}$. We then finally consider:

Proposition III. Problem. 21. *If Z is a determinate function of x, y and p so that $dZ = M\,dx + N\,dy + P\,dp$; to find among all curves corresponding to the same abscissa that curve for which $\int Z dx$ is a maximum or minimum.*

Solution.[6] Let amz be the sought curve, and suppose some ordinate $Nn = y'$ is increased by the small amount $n\nu$, then the differential value of the formula $\int Z dx$, or of the quantity equivalent to this, namely $Z dx + Z' dx + Z'' dx + Z''' dx +$ etc. together with $Z_{,} dx + Z_{,,} dx + Z_{,,,} dx +$ etc., must be $= 0$. Therefore the differential value of the whole quantity $\int Z dx$, resulting from the translation of the point n to ν will be obtained, if the differential values of the individual terms are sought, insofar as they have been affected by the translation and combined into a sum. But from the translation of the point n to ν only those terms in which the quantities y', p and p' are present will undergo change, therefore only the terms $Z dx$ and $Z' dx$; for as Z is a function of y and p besides x, so too Z is a function of y and p likewise. Accordingly these terms must be differentiated, and in place of the differentials dy', dp and dp' one substitutes the above-mentioned values $+n\nu$, $+\frac{n\nu}{dx}$ and $-\frac{n\nu}{dx}$. But just as $dZ = M dx + N dy + P dp$ so there will be $dZ' = M' dx + N' dy' + P' dp'$. Hence the differential value of Z will be $P \cdot \frac{n\nu}{dx}$ and that of Z' will be $N' \cdot n\nu - P' \cdot \frac{n\nu'}{dx}$; thus both terms $Z dx + Z' dx$, hence the whole formula $\int Z dx$, will have a differential value $= n\nu \cdot (P + N' dx - P')$. But $P' - P = dP$ and in place of N' there can be written N; so that the differential value will be $= n\nu \cdot (N dx - dP)$. Therefore, since the differential value of the formula $\int Z dx$ made equal to zero provides the equation for the sought curve, this equation will be $0 = N dx - dP$ or $N - \frac{dP}{dx} = 0$, which expresses the nature of the sought curve. *Q.E.I.*

[6]Although Euler's derivation predates modern notions of formal analysis, Adolf Kneser [K3] has shown this approach to be perfectly sound and indeed can easily be made rigorous.

Having established this result several examples are given after which Euler makes the following observation in Scholion III. 39. with regards to the above construction:

> Hence to resolve problems, in which a curve is sought having a maximum or minimum value of the formula $\int Z dx$, where $dZ = M dx + N dy + P dp$, the value of Z itself must be differentiated and for the differential $M dx + N dy + P dp$ one must put 0 in place of $M dx$, leave $N dy$ unchanged, and write $-p\,dP$ in place of $P dp$; and that which results must be set to equal zero. Then in this way we obtain $N dy - p\,dP = 0$; this equation, on account of $dy = p\,dx$, will change into $N - \frac{dP}{dx} = 0$, which is the one we have found. Therefore, one is looking to derive a method free from a geometric solution, by means of which it should be evident that in such a search for the maximum and the minimum, $-p\,dP$ should be written in lieu of $P dp$.

This invitation to find a simpler method free from a geometric solution lead the young Joseph-Louis Lagrange to propose, in a letter [L4] of 1755 to Euler, a new construct (that was subsequently published in [L1]). Lagrange was perspicacious enough to recognise the differential d used in Euler's above derivation was employed in two distinctively different ways, and so formally introduced a separate δ symbol to distinguish the two.[7] The resulting δ-algorithm of Lagrange was subsequently adopted by Euler who, in an exposition [E5] of Lagrange's elegant approach, coined the term *calculi variationum* to describe the subject founded on this new method (as a set of algorithmic rules, i.e. a calculus, allowing for algebraic procedures which are as much as possible independent of the geometric construct). In a later paper [E6] of 1772, Euler went further and presented what would become the standard interpretation of the δ-method: by considering an admissible one-parameter family of curves $y = y(x,t)$, the variation δy of the desired curve $y(x) = y(x,0)$ embedding in this family can be defined by $\delta y = \frac{\partial y(x,t)}{\partial t}\big|_{t=0}$. Euler also suggested the explicit construct $y(x,t) = X(x) + t\,V(x)$, where $y(x) = X(x)$ is the given curve and $V(x)$ is some comparison (so that $\delta y = V$).

In modern notation this fundamental necessary condition, for an integral of the type $\int_a^b f(x,y,p)dx$, takes the form $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial p} = 0$. The equation is appropriately known as the Euler-Lagrange equation. Solutions to said equation are referred to as extremal or

[7] By introducing δ with properties analogous to the usual d of the differential calculus (so that δ is interchangeable with d and the integral $\int$ together with the rules $\delta(x+y) = \delta x + \delta y$ and $\delta(xy) = x\delta y + y\delta x$) the following algebraic argument was put forward (cf. [E5, Prob. 3, 35]): for the equation of the maximum or minimum it is necessary that $\delta \int Z\,dx = 0$ or equivalently the condition $\int \delta Z\,dx = 0$. Applying the operator δ to the integrand gives $\delta Z = N\delta y + P\delta p$ and as $\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}$ we have

$$\int P\,\delta p\,dx = P\,\delta y|_A^Z - \int \frac{dP}{dx}\delta y\,dx.$$

Since δy is taken to vanish at the end points $x = A, Z$ we conclude $\int (N - \frac{dP}{dx})\delta y\,dx = 0$ from which we infer the desired equation $N - \frac{dP}{dx} = 0$. Hence Euler's remark in [E4, Chap. 2, Scholion III. 39] manifests itself, with hindsight, simply as a desire for integration by parts.

stationary solutions. Written out in full (with $f_y = \frac{\partial f}{\partial y}$, $f_p = \frac{\partial f}{\partial p}$, etc.) gives

$$f_{pp} \cdot y'' + f_{py} \cdot y' + f_{px} = f_y \tag{0.1}$$

and shows the equation to be quasi-linear of second order. Evidently the equation is singular when f_{pp} vanishes, hence regular variational problems are those for which $f_{pp} > 0$. In the general case, when $x = (x^\alpha)_{\alpha=1}^n$ and $y = (y^i)_{i=1}^N$ is a vector valued function, the corresponding equations are $f_{y^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha} f_{p^i_\alpha} = 0$. In the expanded form they can be expressed as

$$\sum_j \sum_{\alpha,\beta} f_{p^i_\alpha p^j_\beta} y^j_{,\alpha\beta} + \sum_j \sum_\alpha f_{p^i_\alpha y^j} y^j_{,\alpha} + \sum_\alpha f_{p^i_\alpha x^\alpha} = f_{y^i} \tag{0.2}$$

and become evidently singular whenever the quadratic form $\sum_{i,j} \sum_{\alpha,\beta} f_{p^i_\alpha p^j_\beta} \lambda^i_\alpha \lambda^j_\beta$ in $\lambda = (\lambda^i_\alpha)$ is degenerate.

The question of sufficiency, i.e. whether or not a solution to the Euler-Lagrange equation is indeed a maximia or minimia, was largely not addressed until the 19th century. Nonetheless it is curious to note the largely forgotten 'paradox' of Euler [E7], originally written in 1779, which considered an integral of the form

$$\int \sqrt{x(1+p^2)}\, dx.$$

For this problem the necessary condition $Ndx = dP$ yields a parabola $y = 2\sqrt{ax - a^2} + b$. To see the supposed 'paradox' let the abscissa $AX = x$ and $XY = y$ in Fig. 2. By setting $AE = a$ and $b = 0$ we see the solution is a curve EY such that the value of the integral along EY is equal to $\frac{2}{3}(2a + x)\sqrt{x - a}$. However the value of the integral along EA is $\frac{2}{3}a\sqrt{a}$, it vanishes along AV, and along VY it is $\frac{2}{3}x\sqrt{x}$. Thus the total value of the integral along $EAVY$ is equal to $\frac{2}{3}(a\sqrt{a} + x\sqrt{x})$. So, for example, if $a = 1$ and $x = 100$ the value of the integral along EY is $\frac{2}{3} \cdot 100\sqrt{99}$ but along $EAVY$ it is $\frac{2}{3} \cdot 1001$. By approximating $\sqrt{99} \approx 10 - \frac{1}{21}$ the value of the integral along EY is $\approx \frac{2}{3} \cdot 1015$, hence the 'paradox' that curve given by the necessary condition does not yield a true minima.

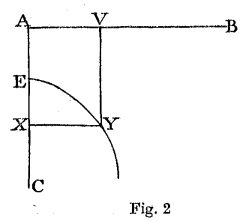


Fig. 2

In another direction Carathéodory discovered the paper of Johann Bernoulli [B6] (which sought to demonstrate the minimum of Brachistochrone problem is really attained for the cycloid) contained something of what was to become a field theory and made, in the appendix to his dissertation [C10, pp. 69–79], an elegant generalisation of Johann's method. The first systematic study of the sufficiency question to be influential was initiated by Legendre, who in [L13] developed the idea of studying second variations for deciding such questions (stating the necessary $f_{pp} \geq 0$ and the sufficient $f_{pp} > 0$ conditions). Despite

drawing criticism from Lagrange (cf. [L2, pp. 305ff.]) for containing errors, Legendre's ideas proved to be fruitful when Jacobi resumed the problem in his short paper [J] that sketched an entire theory for second variations including his celebrated theory of conjugate points. This was subsequently developed into what is known as the classical indirect approach; the details of which, for instance, can be found in the monograph [GH]. In what follows, however, we will focus on so-called direct methods for obtaining solutions that are indeed minima with a particular emphasis on higher dimensional problems.

0.1.1. Development of the direct method. One of the great successes of the Euler-Lagrange equation(s) is that it allows one to find an explicit solution to the variational problem in question—particularly when arriving at the ODE equation (0.1) which could be integrated explicitly (perhaps through additional symmetry considerations). However when passing to higher dimensional integrals, the resulting PDE, i.e. equation (0.2), shows evident limitations due to the difficulty of finding solutions (if they exist) to all but the simplest geometric cases.

In the investigation of theses problems one common flawed reasoning lay in the subtle distinction between the notions of infimum and minimum. An example of this from Gauss's work on potential theory reads as follows:

> It is self-evident that whenever a mass M is *uniformly* distributed across a surface the resulting overall positive potential V will be greater at every point of the surface, as $\frac{M}{r}$, where r is the largest distance between two points on the surface: this value itself could be the potential only at one endpoint of the line r if the total mass was concentrated at the other endpoint, a case that is not of concern when speaking of continuous distributions, whereby each element of the surface ds has an infinitely small mass $m\,ds$. The integral $\int Vm\,ds$ over the entire surface, is in any case greater than $\int \frac{M}{r} m\,ds$ or $\frac{MM}{r}$, and so there must exist a distribution [of charge] for which this integral has a minimum value. [G3, §30, p. 232]

Nevertheless a better understanding of analysis opened such arguments to criticism. Fundamental in the founding of the calculus of variations as a rigorous field—conforming to the modern standard of analysis—was the contributions made by Karl Weierstrass. His lectures at the University of Berlin succeeded in removing all the difficulties of the classical indirect approach that were contained in the prior investigations of Euler, Lagrange, Legendre and Jacobi simply by stating precisely and analysing carefully the problem at hand. For instance a clear distinction between necessary and sufficient conditions was made as well as the careful specification of continuity properties that must be satisfies by functions and their variations. Although the original notes [W3] were published much too late in 1927 to be of any influence, the dissemination of his ideas was mostly made

through the lecture notes of his students (cf. [K2] and [B18]). Yet there remained several outstanding issues.

The most interesting of which surrounds the ideas put forth by Bernhard Riemann who, in his inaugural dissertation *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (1851), strived to construct holomorphic functions $w = u + iv$ of a complex variable $z = x + iy$ (which satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) via the application of variational methods. The introduction of his argument reads as follows:

> **Theorem.** Let α and β be arbitrary functions of x, y for which the integral
> $$\int \Big[\Big(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y}\Big)^2 + \Big(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\Big)^2\Big] dT$$
> taken over any surface T splayed out in the plane A, has a finite value; when varying α by continuous functions, or functions discontinuous only at isolated points, that equal zero on the boundary, the integral takes a minimum value for one of these functions; and, when we exclude discontinuities that can be removed by changing functions at isolated points, this function is unique.
>
> We denote by λ an unspecified function vanishing on the boundary, either continuous or possessing only discontinuities at isolated points, for which the integral
> $$L = \int \Big[\Big(\frac{\partial \lambda}{\partial x}\Big)^2 + \Big(\frac{\partial \lambda}{\partial y}\Big)^2\Big] dT$$
> taken over the whole surface is finite. Denote by ω the unspecified function $\alpha + \lambda$, and finally denote by Ω the integral
> $$\int \Big[\Big(\frac{\partial \omega}{\partial x} - \frac{\partial \beta}{\partial y}\Big)^2 + \Big(\frac{\partial \omega}{\partial y} + \frac{\partial \beta}{\partial x}\Big)^2\Big] dT$$
> over the whole surface. The set of functions λ forms a connected domain, in that any function passes continuously into any other, while no limit of the functions is discontinuous along a line unless L simultaneously becomes infinite (§17); now, if we set $\omega = \alpha + \lambda$ the integral Ω, for such function λ, has a finite value which tends to infinity with L and varies continuously with the form of λ yet can never be less than zero; consequently, for at least one form of the function ω the integral Ω attains a minimum value.
>
> To prove the second part of our theorem, denote by u one of the functions ω for which Ω attains its minimum value, let h be an unspecified constant, so that $u + h\lambda$ satisfies the prescribed conditions for the function ω. For $\omega = u + h\lambda$, we write the value of Ω as
> $$\begin{aligned} &= \int \Big[\Big(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y}\Big)^2 + \Big(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x}\Big)^2\Big] dT \\ &\quad + 2h \int \Big[\Big(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y}\Big)\frac{\partial \lambda}{\partial x} + \Big(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x}\Big)\frac{\partial \lambda}{\partial y}\Big] dT \\ &\quad + h^2 \int \Big[\Big(\frac{\partial \lambda}{\partial x}\Big)^2 + \Big(\frac{\partial \lambda}{\partial y}\Big)^2\Big] dT = M + 2Nh + L\,h^2, \end{aligned}$$
> say. This must be at least M for every λ (by definition of the minimum), provided h is sufficiently small. Now it follows that $N = 0$ for each λ; because

> otherwise
> $$2Nh + Lh^2 = Lh^2(1 + \tfrac{2N}{Lh})$$
> would be negative if h is opposite in sign to N and of absolute value $< \frac{2N}{L}$. The value of Ω for $\omega = u + \lambda$, a form which obviously contains all possible values of ω, is thus $M + L$. Since L is essentially positive, Ω cannot attain, for any form of the function ω, a value smaller than what is attained for $\omega = u$.
>
> [R2, §16, pp. 30ff.]

Thereafter an argument is given for the uniqueness of the statement. Then in [R2, §18, p. 33] an application of Gauss's theorem (on an open domain that excludes the discontinuities of u, β and λ) yields $N = -\int \lambda\,(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})dT - \int(\frac{\partial u}{\partial p} + \frac{\partial \beta}{\partial s})\lambda\, ds$. As this holds for all λ which vanish on the boundary, the second term vanishes and one concludes $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Now the subtle point of the above argument is that it does not, in fact, prove the existence of such functions and, as a result, elicited justified criticism thereof—which incidentally dismembered the very basis of Riemann's foundation of complex analysis. In Riemann's subsequent paper *Theorie der Abel'schen Functionen* (1857) the above 'theorem' morphed into what he called Dirichlet's principle, since:

> there is a principle, which Dirichlet applied to a function of three variables satisfying Laplace's partial differential equation—probably inspired by a similar idea of Gauss—in his lectures, over several years, on forces obeying an inverse square law.[8] [R2, §3, p. 90]

The idea of the principle is to seek, for given bounded domain $\Omega \subset \mathbb{R}^n$ and $u_0 \in C(\partial\Omega)$, a function $u_{\min}$ in the class

$$\mathscr{C}(\Omega; u_0) = \{u \in C^2(\Omega) \cap C(\bar{\Omega}) : u = u_0 \text{ on } \partial\Omega\}$$

which minimises the corresponding Dirichlet energy

$$E(u) = \int_\Omega |Du|^2 dx$$

among all functions in that class. If one can prove the existence of such a minimiser $u_{\min}$ then one automatically solves the associated Dirichlet problem $\Delta u = 0$ in Ω with $u = u_0$ on $\partial\Omega$ (since $u_{\min}$ is a stationary point of E by virtue of its minimality).

Despite its elegant simplicity Weierstrass, nevertheless, gave a critical re-evaluations of the admissibility of this 'principle'. In particular he constructed an example of a variational problem that did not admit a minimum solution. By minimising the variational

[8]Statements by Dirichlet can be found in [D5, §32, pp. 127-8]. Gauss also had applied similar conclusions in his work on forces proportional to the inverse square of the distance [G3, §30–34]. Similarly, William Thomson (Lord Kelvin) made use of this principle in [T1].

integral

$$J(u) = \int_{-1}^{1} x^2 u'(x)^2 dx,$$

over the class

$$\mathcal{C} = \{u \in C^1[-1,1] : u(-1) = a,\ u(1) = b\},$$

the paper [W2] showed the *infimum* of J in the class $\mathcal{C}$ to vanish despite the value not being attained in that class.[9] To see this, Weierstrass considered the family of comparison functions $u_\varepsilon(x) = \frac{a+b}{2} + \frac{b-a}{2}\frac{\arctan x/\varepsilon}{\arctan 1/\varepsilon}$, $\varepsilon > 0$, which of course belong to $\mathcal{C}$. Now as

$$0 \le J(u_\varepsilon) < \int_{-1}^{1} (x^2 + \varepsilon^2) u_\varepsilon'(x)^2 dx$$

and $u_\varepsilon'(x) = \frac{b-a}{2\arctan 1/\varepsilon}\frac{\varepsilon}{x^2+\varepsilon^2}$ it follows that $J(u_\varepsilon) < \frac{\varepsilon}{2}\frac{(b-a)^2}{\arctan 1/\varepsilon}$ from which we conclude $\inf_{u\in\mathcal{C}} J(u) = 0$. The contradiction arises from the observation that $J(u_{\min}) = 0$ implies $u_{\min}(x) = \text{const.}$ for all $x \in [-1,1]$ so that $u_{\min}(-1) = u_{\min}(1)$. Hence $u_{\min} \notin \mathcal{C}$.

In a similar vein one can do likewise for the Dirichlet problem. By considering the punctured disk $\Omega = B\backslash\{0\}$, $B = B(0,1)$, and the class

$$\mathscr{C} = \{u \in C^2(\Omega) \cap C(\bar{\Omega}) : u(0) = 1,\ u|_{\partial B} = 0,\ E(u) < +\infty\},$$

we clearly have $E(u) \ge 0$ for all $u \in \mathscr{C}$. To determine the infimum, fix a cutoff function $\eta \in C^\infty(\mathbb{R})$ which is $\eta(x) = 1$ for $x \le 1$ and $\eta(x) = 0$ for $x \ge 2$. For $0 < \varepsilon < 1$ define $u_\varepsilon(x) = 1 - \eta\left(\frac{\log|x|}{\log\varepsilon}\right)$ if $0 < |x| \le 1$ and $u_\varepsilon(x) = 1$ if $x = 0$. Then $u_\varepsilon(x) = 1$ for $0 \le |x| \le \varepsilon^2$ and $u_\varepsilon(x) = 0$ for $\varepsilon \le |x| \le 1$. Hence $u_\varepsilon \in \mathscr{C}$ as u_ε is smooth with compact support in B. Noting $\frac{\partial u_\varepsilon}{\partial x^i}(x) = -\frac{1}{\log\varepsilon} D\eta\Big(\frac{\log|x|}{\log\varepsilon}\Big)\frac{x^i}{|x|^2}$ for $\varepsilon^2 \le |x| \le \varepsilon$, with $\frac{\partial u_\varepsilon}{\partial x^i}(x) = 0$ otherwise, yields

$$0 \le E(u_\varepsilon) = \int_B |Du_\varepsilon|^2 \le \sup|D\eta|^2 \frac{4|B|}{(\log\varepsilon)^2}\int_{\varepsilon^2}^{\varepsilon} r^{n-3} dr.$$

Thus $E(u_\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ so that $\inf_{u\in\mathscr{C}} E(u) = 0$. Nonetheless there does not exist $u_{\min} \in \mathscr{C}$ such that $E(u_{\min}) = 0$ (since the minimum would be a constant and could not obey the given boundary conditions).

Weierstrass's critique of Dirichlet's principle threw the calculus of variations into a *Grundlagenkrise* for some 30 years during which alternative existence proofs for the Dirichlet problem were obtained (for instance the integral equation method of C. Neumann [N5], the alternating method of H. Schwarz [S6] and the sweeping-out process of Poincaré [P1, P2] among others). It took the combined efforts of many outstanding mathematicians several decades to resurrect the beauty of the idea from the above truths.

[9] It seems that Riemann knew of Weierstrass's criticism well before the latter's publication in 1870.

0.1.2. Hilbert's problems. One of the earliest justifications of Dirichlet's principle via a direct approach was made (after some unsuccessful attempts by Cesare Arzelà [A6]) by David Hilbert, who in [H15, H16] gave a rigorous proof of the existence of a minimiser for the one-dimensional Dirichlet problem.[10] This result, although 'classical' in nature, marks the beginning of the so-called direct methods in the calculus of variations (in the sense that one takes a minimising sequence and proves an appropriate modified subsequence converges uniformly to a minimiser). Nonetheless the treatment of the minimising problem for a general functional of the form

$$\mathfrak{F}(u;\Omega) = \int_\Omega f(x,u,Du)dx,$$

under suitable assumptions on the integral f, remained open.

It is within this context that Hilbert, looking back to Riemann's ideas in complex analysis, proposed the following variational problems during his now famous lecture on 'the future problems of mathematics' at the second International Conference of Mathematics held in Paris [H17]. Speaking at the Sorbonne on the 8 August 1900, he asked:[11]

Problem 19. *Are the solutions of regular problems in the calculus of variations always necessarily analytic?*

Problem 20. *Has not every regular variation problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of a solution shall be suitably extended?*

Specifically, Problem 19 concerns the theme of *regularity* to which we will be primarily concerned with. The related Problem 20 concerns the question of existence of solutions to boundary value problems and, in particular, the existence of solutions to variational problems.

Early progress on Problem 20 was made by Leonida Tonelli in the papers [T3, T4] on the existence of minimisers for one-dimensional variational problems. This work stressed the notion of semicontinuity (upon realising the Arzelà-Ascoli compactness theorem and Baire's semicontinuity idea can be transferred from real valued functions to variational integrals), however the existence was achieved via the local solvability of the Euler-Lagrange

[10]Other early contributions can be found in [L9, C11, L19, F6, L10]. Of particular importance are the papers by Lebesgue [L9, L10] which show the area functional not to be continuous but merely semicontinuous with respect to the uniform convergence of surfaces.

[11]In passing we remark that it appears that Hilbert conceived his lecture as a counter to the rather bland advocacy of the importances of applied mathematics made by Poincaré [P3] at the first International Conference of Mathematics held in Zürich. His former graduate student Otto Blumenthal remarked in [B16, p. 405] that 'quite a few [problems] stem from the general situations of mathematics or from the problems-context of other researchers'.

equations which, as a result, hid the underlying premise. The first lucid presentation that captures the essence of the direct method was Tonelli's paper [T5]. His theory was subsequently revised in [T6, T7] from which a systematic exposition of the one-dimensional variation problem was given in his monograph [T8]. Later attempts by Tonelli [T9, T10] at generalising to the two-dimensional case were mired by fact that he was working with absolutely continuous functions (believing this was the appropriate class to work with).

From the 1920s onwards the notion of generalised or weak solution emerged, in part, motivated by direct method considerations. In fact it was Beppo Levi [L19] who first observed a general minimising sequence for the Dirichlet integral is Cauchy in the Dirichlet norm, and therefore converges in an appropriate completion space (with respect to the Dirichlet norm) to a generalised function. This lead to the work of Morrey [M14, M15] in which the class of absolutely continuous functions was replaced with a class that is equivalent to what is now know as Sobolev spaces.[12] Building on Tonelli's ideas, Morrey formulated the modern direct approach to the existence and regularity of minimisers of multi-dimensional variational problems (cf. [M20]).

The overall idea of the direct method can be outlined as follows:[13] first fix the class $\mathscr{C}$ of 'admissible functions' with a suitable topology τ (so that one has a notion of sequential convergence). Let $\mathfrak{F}\colon \mathscr{C} \to \mathbb{R} \cup \{+\infty\}$ be a given functional defined on said class. Then we seek to:

(1) Show $\mathfrak{F}$ is well defined on $\mathscr{C}$ and bounded from below, i.e. $\inf_{\mathscr{C}} \mathfrak{F} > -\infty$. From this we can take a minimising sequence $(u_j) \subset \mathscr{C}$, i.e. $\lim_{j\to\infty} \mathfrak{F}(u_j) = \inf_{\mathscr{C}} \mathfrak{F}$.
(2) Show (u_j) admits a converging subsequence (u_{j_k}) with respect to a topology τ on $\mathscr{C}$, i.e. $u_{j_k} \to u_0 \in \mathscr{C}$ in τ. To do so we need some compactness condition for the existence of a subsequence and a closedness conditions to ensure $u_0 \in \mathscr{C}$.
(3) Show $\mathfrak{F}$ is sequentially lower semicontinuous with respect to the given topology τ, i.e. $\mathfrak{F}(u_0) \leq \liminf_{j\to\infty} \mathfrak{F}(u_j)$ whenever $u_j \to u_0$ in τ.

From this we infer $\inf_{\mathscr{C}} \mathfrak{F} \leq \mathfrak{F}(u_0) \leq \liminf_{j\to\infty} \mathfrak{F}(u_j) \leq \inf_{\mathscr{C}} \mathfrak{F}$ and conclude $\mathfrak{F}(u_0) = \inf_{\mathscr{C}} \mathfrak{F}$. In this programme it should be noted that there are a number of implicitly conflicting and competing requirements. To ensure existence, it was recognised that lower semicontinuity of the functional $\mathfrak{F}$ was required and that the class $\mathscr{C}$ of functions be compact in some sense. For semicontinuity one prefers a relatively strong topology but for compactness the opposite is preferable: the weaker the topology the easier it is for sequences to converge. Moreover, the requirement for the class $\mathscr{C}$ to have a topology that is sufficiently coarse enough to ensure the compactness is in conflict with the need

[12] A concise account of the development of the theory of Sobolev spaces can be found in [N2].
[13] In a rough sense the outline aims to generalise the Weierstrass extreme value theorem for continuous real valued functions on a closed bounded space of finite-dimension.

for one to be able to make sense of the related Euler-Lagrange equations (which desire a stronger differentiability than that needed to ensure the existence of a solution). The result of the need to balance these requirements leads to a very general existence theorem in which a solution is known to exist only in a suitably general class where continuity, let alone C^2-differentiability, is not known. Consequently, the latter differentiability problem leads to Hilbert's related former problem.

One of the first solutions to Hilbert's 19th problem was that of Bernstein [B8] who showed C^3-solutions of non-linear elliptic equations in two-variables to be analytic (further results for non-linear equations in the two-dimensional case were subsequently developed in [B9, B10, B11, B12]). A similar result of Lichtenstein [L20] showed C^2-solutions of a regular double integral to be of class C^3 (and so also analytic by Bernstein's first result). The same conclusion holds for solutions of class C^1 with Hölder continuous derivatives by Eberhard Hopf [H20] and by Morrey [M13] if solution is Lipschitz. Let use also mention the results of Schauder [S1, S2], that made use of some of Bernstein's methods, and the subsequent simplifications thereof by Caccioppoli [C2, C3]. The method in Caccioppoli's note open the way to new developments; so much so that Leray [L17, L18], combining Caccioppoli's method with that of Bernstein, characterised a large class of non-linear elliptic equations. The first result for the Dirichlet problem in $n > 2$ variables was given by Leray and Schauder [LS] in the Hölder class. At about the same time Caccioppoli [C1] stated a similar results, without details, under a less restrictive hypothesis (which was later elaborated by Miranda in [M10, M11]). The note of Caccioppoli represents the most advanced point reached in the theory of non-linear differential equations with $n > 2$ variables until new ground was broken in the 1950s.

In the special case of linear problems, weak solutions are in fact classical. This full regularity property was first enunciated by Hermann Weyl in his celebrated paper [W5]. By replacing Dirichlet's minimum principle with the construction of orthogonal projections in a suitable Hilbert space (and by making extensive use of Lebesgue's integration theory) Weyl presented an elegant solution to the Dirichlet problem. The result of which yields the following:

Theorem 0.1 (Weyl's lemma, [W5])**.** *Suppose $u \in L^1_{\mathrm{loc}}(\Omega)$ and satisfies*

$$\int_\Omega u(x)\Delta v(x)\,dx = 0$$

for all $v \in C^\infty_c(\Omega)$. Then $u \in C^\infty(\Omega)$ is equivalent to a harmonic function.

As we shall see, this important property of harmonic functions forms the basis of future regularity results for non-linear problems—since one attempts to transfer, via

some comparison argument, the 'good' regularity properties enjoyed by linear problems to the non-linear setting.

In addition to this special case, the *a priori* estimates attributed to Caccioppoli [C4] are also of fundamental importance. The insight of Caccioppoli's idea seeks to bound the derivative by the function which is opposite, or in reverse, to that of the inequality of Poincaré and more generally those of Sobolev. The original basis for achieving this rests upon the following lemma that was first introduced in [C2] (with proofs given in [L17] and [C4]).

Lemma 0.2. *If the function $\varphi(\rho)$ satisfies, for $0 < \rho \leq r$, the inequality*

$$\omega(\rho)\varphi(\rho) - \lambda \int_0^\rho \varphi(t)^2 dt + \delta(\rho) \geq 0$$

where λ is a positive constant and $\omega(\rho)$, $\delta(\rho)$ are non-negative functions, the second non-decreasing, one has

$$\int_0^\rho \varphi(t)^2 dt \leq \frac{\delta(r)}{\lambda} + \frac{1}{\lambda^2(r-\rho)^2} \int_0^r \omega(t)^2 dt.$$

Proof. Suppose for $0 < \rho \leq r$ that $\int_0^\rho \varphi(t)^2 dt > \frac{\delta(r)}{\lambda}$, otherwise the conclusion is trivial. Set

$$\psi(\rho) = \int_0^\rho \varphi(t)^2 dt - \frac{\delta(r)}{\lambda}$$

for such $0 < \rho \leq r$. By the hypothesis $\psi'(\rho)\psi(r)^{-2} \geq \frac{\lambda^2}{\omega(\rho)^2}$ from which integrating gives

$$\frac{1}{\lambda^2}\Big(\frac{1}{\psi(\rho)} - \frac{1}{\psi(r)}\Big) > \int_\rho^r \frac{dt}{\omega(t)^2} \geq \frac{(r-\rho)^2}{\int_\rho^r \omega(t)^2 dt},$$

where the last inequality follows by Cauchy-Schwarz. It then follows

$$\psi(\rho) < \frac{1}{\lambda^2(r-\rho)^2} \int_\rho^r \omega(t)^2 dt < \frac{1}{\lambda^2(r-\rho)^2} \int_0^r \omega(t)^2 dt. \qquad \square$$

With this in hand Caccioppoli [C4, §4] derived *a priori* estimates for linear elliptic equations of n-variables in the form

$$L(u) = \sum_{i,k=1}^n A_{ik} \frac{\partial^2 u}{\partial x^i \partial x^k} + \sum_{i=1}^n B_i \frac{\partial u}{\partial x^i} + Cu = f,$$

where A_{ik}, B_i, C, f are functions of $x = (x^i)_{i=1}^n$ and the quadratic form $A = (A_{ik})$ has norm $\|A\| = 1$ satisfying $\sum_{i,k} A_{ik}\lambda_i\lambda_k \geq \alpha|\lambda|^2$ for some positive constant α. In which case on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ we have

$$\int_\Omega L(u)u\, dx = \int_\Omega f u\, dx.$$

Integrating by parts then yields

$$\sum_{i,k}\int_\Omega [A_{ik}p_ip_k - (B_ip_i + Cu - f - p_kD_iA_{ik})u]dx = \sum_{i,k}\int_{\partial\Omega} uA_{ik}p_k\nu_i d\sigma,$$

where $p_i = \frac{\partial u}{\partial x^i}$, $p_{ik} = \frac{\partial^2 u}{\partial x^i \partial x^k}$ and ν_i are the components of the outward pointing normal to $\partial\Omega$. By an application of Cauchy-Schwarz we get

$$\begin{aligned}\alpha\int_\Omega |p|^2dx \le a\Big(\int_{\partial\Omega}|u|^2d\sigma\Big)^{\frac12}\Big(\int_{\partial\Omega}|p|^2d\sigma\Big)^{\frac12} + b\Big(\int_\Omega |u|^2dx\Big)^{\frac12}\Big(\int_\Omega |p|^2dx\Big)^{\frac12}\\ + c\int_\Omega |u|^2dx + \Big(\int_\Omega |u|^2dx\Big)^{\frac12}\Big(\int_\Omega |f|^2dx\Big)^{\frac12}\end{aligned}$$

for positive constants a, b and $c = \max|C|$. Then by Cauchy's inequality we have

$$\begin{aligned}0 \le \Big(\int_{\partial\Omega}|u|^2d\sigma\Big)^{\frac12}\Big(\int_{\partial\Omega}|p|^2d\sigma\Big)^{\frac12}\\ - \lambda\int_\Omega |p|^2dx + \mu\Big(\int_\Omega |u|^2dx\Big)^{\frac12}\Big[\Big(\int_\Omega |u|^2dx\Big)^{\frac12} + \Big(\int_\Omega |f|^2dx\Big)^{\frac12}\Big]\end{aligned}$$

for positive constants λ, μ. Now if we take $\Omega = B_\rho(x_0)$, $\rho > 0$, an application of Lemma 0.2 gives

$$\int_{B_\rho}|Du|^2 \le \gamma\Big[\frac{1}{(r-\rho)^2}\Big(\int_{B_r}|u|^2\Big)^{\frac12} + \Big(\int_{B_r}|f|^2\Big)^{\frac12}\Big]\Big(\int_{B_r}|u|^2\Big)^{\frac12}$$

for some $\gamma > 0$ and any $r > \rho > 0$. In particular if $f = 0$ we simply get

$$\int_{B_\rho}|Du|^2 \le \gamma\int_{B_r}\Big|\frac{u}{r-\rho}\Big|^2 \tag{0.3}$$

which, as desired, bounds the integral of the derivative by that of the function.

0.1.3. The solution of De Giorgi. For over half a century after Hilbert's proposed 19th problem there remained a gap between the results of two distinct research directions. On the one hand the above mentioned differentiability and analyticity results for less and less regular extremals was developed in the context of 'classical' spaces. On the other, direct methods in the calculus of variations ensures the existence of solutions among very wide classes of functionals, where the solution *a prior* has very weak differentiability properties (e.g. Stampacchia [S19] proved the existence of solutions with second order derivatives in L^2 satisfying the weak Euler-Lagrange equations almost everywhere). The problem was that the solutions constructed in the latter case were known to have L^2-derivatives but were not quite strong enough to feed into the machinery of the former results—which need at least some continuity of first derivatives to prove analyticity. This gap was finally filled by Ennio De Giorgi's 1957 paper [DG6]—by doing so he completed the last and hardest part of Hilbert's 19th problem.

The initial motivation of De Giorgi's solution arose out of a discussion with Guido Stampacchia in August of 1955. By working from earlier results on the perimeters of general sets in $\mathbb{R}^n$, and his proof of the isometric inequalities for these (cf. [DG4, DG5]), De Giorgi was able to extract a solution in less than two months. The insight of [DG6] centres on the introduction of the following class of functions.

Definition 0.3. Let $\mathcal{B}(\Omega;\gamma)$ denote the class of functions $u \in W^{1,2}(\Omega)$ for which there is a constant $\gamma > 0$ such that the inequality

$$\int_{A(k)\cap B_\rho(x_0)} |Du|^2 \leq \frac{\gamma}{(R-\rho)^2} \int_{A(k)\cap B_R(x_0)} |u-k|^2$$

holds for all $x_0 \in \Omega$, $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$, where $A(k) = \{x \in \Omega : u(x) > k\}$.

From this starting point De Giorgi was able to prove the following non-trivial result.

Theorem 0.4. *If $u, -u \in \mathcal{B}(\Omega;\gamma)$ then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $0 < \alpha < 1$. Moreover for $0 < \rho < R$ with $B_R(x_0) \subset\subset \Omega$ the oscillation of u satisfies*

$$\operatorname{osc}(u;\rho) \leq c\left(\frac{\rho}{R}\right)^\alpha \operatorname{osc}(u;R)$$

for some $c > 0$ independent of x_0, ρ and R, where $\operatorname{osc}(u;\rho) := \sup_{B_\rho} u - \inf_{B_\rho} u$.

Given this, a simple application of the above ideas of Caccioppoli yields the following:

Theorem 0.5. *Let $u \in W^{1,2}(\Omega)$ be a weak solutions of*

$$\operatorname{div}(a_{ij}(x) D_j u) = 0$$

under the assumptions that

$$|a_{ij}(x)| \leq L, \qquad a_{ij}(x)\lambda^i\lambda^j \geq \ell|\lambda|^2$$

for a.e. $x \in \Omega$ and all $\lambda \in \mathbb{R}^n$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha = \alpha(n, \frac{L}{\ell}) > 0$.

Note that the achievement here lies in the fact that the coefficients a_{ij} are *a priori* only measurable functions. We give a proof based on the translation in [DG9, pp. 149–166].

Proof. Fix a point $y \in \Omega$ and a real number k, and set

$$\varphi(x) = \begin{cases} u(x) - k & \text{if } x \in A(k) \\ 0 & \text{if } x \in \Omega\backslash A(k). \end{cases}$$

It is easily checked that $\varphi \in W^{1,2}(\Omega)$ as well. Now take $0 < r < \operatorname{dist}(y, \partial\Omega)$ and a smooth radial function η such that $\eta = 0$ on $[r, \infty)$ and set $\chi(x) = \eta(|x|)$. The function $\chi\varphi$ belongs to $W^{1,2}(\Omega)$ and vanishes on $\Omega\backslash B_r(y)$; from the given weak solution it follows $\int_\Omega a_{ij}(x) D_j u D_i(\chi\varphi) dx = 0$. Since χ vanishes identically on $\Omega\backslash B_r(y)$ it follows by Fubini's

theorem that

$$\int_0^r d\rho\Big[\eta'(\rho)\int_{\partial B_\rho(y)} a_{ij}\varphi D_j u\,\nu_i d\sigma + \eta(\rho)\int_{\partial B_\rho(y)} a_{ij}D_i\varphi D_j u\,d\sigma\Big] = 0,$$

where ν_i are the components of the outward pointing normal to $\partial B_r(y)$. Then integrating by parts yields

$$\int_0^r d\rho\,\eta'(\rho)\Big[\int_{\partial B_\rho(y)} a_{ij}\varphi D_j u\,\nu_i d\sigma - \int_0^\rho ds\int_{\partial B_s(y)} a_{ij}D_i\varphi D_j u\,d\sigma\Big] = 0.$$

By the arbitrariness of r and the function χ, together with the definition of φ, we obtain $\int_{A(k)\cap\partial B_\rho(y)} a_{ij}D_j u\,(u-k)\nu_i d\sigma = \int_{A(k)\cap B_\rho(y)} a_{ij}D_i u D_j u\,dx$ for a.e. $0<\rho<\operatorname{dist}(y,\partial\Omega)$. From the given elliptic conditions this implies

$$\ell\int_{A(k)\cap B_\rho(y)} |Du|^2 \le L\int_{A(k)\cap\partial B_\rho(y)} (u-k)|Du|.$$

Now set $\psi_1(\rho) = \int_{A(k)\cap\partial B_\rho(y)} |w-k|^2$ and $\psi_2(\rho) = \int_{A(k)\cap\partial B_\rho(y)} |Du|^2$ so that an application of Cauchy-Schwarz implies $\frac{\ell}{L}\int_0^\rho \psi_2(t)dt \le \sqrt{\psi_1(\rho)\psi_2(\rho)}$. It then follows by Lemma 0.2, with $\gamma = (\frac{L}{\ell})^2$, that

$$\int_0^{\rho_1}\psi_2(t)dt \le \frac{\gamma}{(\rho_2-\rho_1)^2}\int_0^{\rho_2}\psi_1(t)dt,$$

for $0<\rho_1<\rho_2<\operatorname{dist}(y,\partial\Omega)$. Thus the required inequality in Definition 0.3 is satisfied for u. Moreover, whenever u satisfies the hypotheses of the statement the same holds for $-u$. Hence $u\in\mathcal{B}(\Omega;\gamma)$. □

The application of the above theorem to the minima of functionals is immediate since minimisers in the class $W^{1,2}_{\rm loc}(\Omega)$ of a regular variational problem of the form

$$\int_\Omega f(Du)dx,$$

with a strictly convex function $f\colon\mathbb{R}^n\to\mathbb{R}$ satisfying a uniform ellipticity condition of the form

$$\frac{1}{c}|\lambda|^2 \le D^2 f(Du)(\lambda,\lambda) \le c|\lambda|^2, \tag{0.4}$$

for a.e. $x\in\Omega$ and all $\lambda\in\mathbb{R}^n$, will fulfil the necessary condition

$$\int_\Omega Df(Du)(D\phi)\,dx = 0$$

for all $\phi\in C^\infty_c(\Omega)$. By an approximation argument it is possible to show $w=Du$ satisfies $\int_\Omega D^2 f(Du)(Dw,D\phi)\,dx = 0$ for all $\phi\in C^\infty_c(\Omega)$. Therefore w is locally Hölder continuous and as a consequence $u\in C^{1,\alpha}_{\rm loc}$. The information that $Du\in C^\alpha_{\rm loc}$ is fundamental since it makes the coefficients $a_{ij}(x) = D^2_{ij}f(Du(x))$ more regular than was known *a priori*. Furthermore, if $f\in C^\infty$ then $u\in C^\infty_{\rm loc}$ by [M18, Theor. 9.2] (which ensures the

existence and the continuity of all the derivatives for C^1-extremals) and by the results of Stampacchia and Hopf (cf. [S19, teor. VII]).

Alternative proofs of Theorem 0.5 have been proposed by Moser [M22, M23] and more recently by Tilli [T2]. De Giorgi's work was also extended to linear equations by Morrey [M19], Stampacchia [S20] and to quasilinear equations in divergence form by Ladyzhenskaya and Ural′tseva [LU]. We also remark that Nash [N1] independently obtained an equivalent result for parabolic equations, however Nash's ideas were not as fruitful as De Giorgi's. Moreover, as Caffarelli [C5, p. 43] remarks: 'Besides providing the solution to Hilbert's 19th problem, De Giorgi's paper [DG6] has an enormous impact in the theory of non-linear elliptic equations, in homogenization theory, in the theory of free boundary problems and in the theory of degenerate diffusions, just to quote a few important research areas. This is due to the fact that the ideas underlying De Giorgi's regularity theorem have a universality that goes far beyond Hilbert's problem.'

0.1.4. Counterexamples to everywhere regularity. For a long-time it was hoped that the above regularity result of De Giorgi could pass over to the vectorial case (i.e. when $u = (u^\alpha)\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ for $N, n > 1$). These hopes were dashed when De Giorgi [DG8] himself constructed a surprising counterexample of a second order elliptic system, with bounded measurable coefficients, having a discontinuous solution of the form $x/|x|^\alpha$, $\alpha = \alpha(n) > 1$, for $n \geq 3$. Examples of higher order elliptic equations, with real analytic coefficients, having discontinuous solutions were also constructed by Maz′ya [M2]. The result was also extended by Giusti and Miranda [GM6] who gave an example of a non-linear system with regular coefficients (as opposed to De Giorgi's which where singular at the origin).

Given these examples one could ask if Theorem 0.5 extends to a *partial* regularity result for systems under the same assumptions, i.e. if solutions to an elliptic system of the form

$$D_i(A^{ij}_{\alpha\beta}(x)D_j u^\beta) = 0,$$

are in fact partially regular (in the sense they are locally Hölder continuous away from point singularities or some closed subset) under the assumption that

$$\ell|\xi|^2 \leq A^{ij}_{\alpha\beta}(x)\xi^\alpha_i\xi^\beta_j \leq L|\xi|^2$$

for all $\xi \in \mathbb{R}^{Nn}$, $x \in \Omega$ a.e., some $0 < \ell \leq L$ with $i, j = 1, \dots, n$ and $\alpha, \beta = 1, \dots, N$. According to the classical results of [M18, DN] every weak solution of this system is locally Hölder continuous provided $A^{ij}_{\alpha\beta}$ are continuous on Ω. Nevertheless without continuity the question was settled in the negative by Souček [S16] who constructed an elliptic system, for $n = N \geq 3$, with bounded weak solutions that are discontinuous on a dense set. Building on Souček's ideas John, Malý and Stará [JMS] constructed, for any given

F_σ-set $F \subset \Omega$, an elliptic system that admitted a bounded solution $u \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ which is essentially discontinuous on F and essentially continuous on its complement $\Omega\backslash F$, provided $\frac{\ell}{L} < K(n) = \big(\sqrt{1+\frac{(n+2)^2}{n-1}} - 1\big)/\big(\sqrt{1+\frac{(n+2)^2}{n-1}} + 1\big)$ with $n = N \geq 3$. It is intriguing to note that if the elliptic ratio $\frac{\ell}{L}$ satisfies $\frac{\ell}{L} > K(n)$ a result of Koshelev [K5] shows that any weak solution of the above elliptic system to be Hölder continuous.

Looking back to Hilbert's 19th problem there is also the question of the existence of singularities for *minimisers* of regular variational integrals of the form

$$\int_\Omega f(Du)dx$$

with, for instance, a strictly convex real analytic integrand f satisfying the uniform ellipticity condition (0.4). The first result in this direction was achieved by Nečas [N4], who provided just such an example in which the minimiser has a discontinuous (but still bounded) gradient. By starting with the prospective minimiser $x \mapsto x^i x^j/|x|$ Nečas sought to construct, for large n, an integrand which would yields the prospective as the (unique) minimiser. It is interesting to note the resulting integrand has an intrinsic tensorial nature which, incidentally, is similar in flavour to the expanded gradient terms found in the examples of Appendix A. In the lower dimension $n \geq 5$ case an example of this tensorial construction can be seen explicitly in the paper of Nečas, Hao and Leonardi [HLN]. By constructing a functional of the form

$$\begin{aligned}\mathfrak{F}(u) = \frac{\alpha_0}{4}\int_{B^n} |\nabla u|^4 + \frac{\alpha_1}{2}\int_B |\nabla u|^2 + \frac{\alpha_2}{2}\int_{B^n} \nabla_m u \nabla_m u \nabla_{ijk} u \nabla_{ijk} u \\ + \frac{\alpha_3}{4}\int_{B^n} (\nabla_{ijk} u \nabla_{ijk} u)^2 + \frac{\alpha_4}{4}\int_{B^n} (\nabla_i u \nabla_i u)^2 \\ + \alpha_5 \int_{B^n} \nabla_i u \nabla_j u \nabla_k u \nabla_{ijk} u + \alpha_6 \int_{B^n} (\nabla_i u)(\partial_i u^{jj}) + \frac{\alpha_7}{2}\int_{B^n} (\partial_i u^{jj})^2,\end{aligned}$$

where $\alpha_i = \alpha_i(n)$, $N = n^2$ and

$$\begin{aligned}\nabla_i u &= \frac{1}{2}(\partial_j u^{ij} + \partial_j u^{ji} + \partial_i u^{jj}) \\ \nabla_{ijk} u &= \frac{1}{6}(\partial_k u^{ij} + \partial_j u^{ki} + \partial_i u^{jk} + \partial_k u^{kj} + \partial_j u^{ik}) \\ &\quad + \frac{2n-1}{3(n^2-1)}(\delta_{ij}\nabla_k u + \delta_{ki}\nabla_j u + \delta_{jk}\nabla_i u),\end{aligned}$$

they managed to show the corresponding global $\mathfrak{F}$-minimiser $u\colon B^n \to \mathbb{R}^{n^2}$ over the class $u + W^{1,4}_0(B^n, \mathbb{R}^{n^2})$ is given by (the Lipschitz continuous but not C^1) map

$$u^{ij}(x) = \frac{x^i x^j}{|x|} - \frac{|x|}{n}\delta^{ij}.$$

Some two decades after Nečas original example, the novel and substantial work of Šverák and Yan [ŠY1,ŠY2] managed to construct minimisers of functionals that are non-Lipschtiz for $n = 3$, $N = 5$ and even unbounded for $n = 5$, $N = 14$ (note that when $n = 2$ the minimiser is smooth and for $n \leq 4$ the Sobolev embeddings imply that every solution must be Hölder continuous). In looking back to Nečas original example, Šverák and Yan looked to a candidate mapping $u = (u^{ij})\colon B^n \to \mathbb{R}^N$ of the form

$$u^{ij}(x) = \frac{1}{|x|^\varepsilon}\Big(\frac{x^i x^j}{|x|} - \frac{|x|}{n}\delta^{ij}\Big).$$

From this they constructed an integrand for which u, as a mapping into the symmetric trace-free $n \times n$ matrices (so that $N = \frac{1}{2}n(n+1) - 1$), is a minimiser. Further details on Nečas' examples can be found in [KM4].

The importance of these examples here lies in the fact that it shows the irregularity of minima to be a peculiar feature of the vectorial case and is not due to the presence of coefficients.

0.1.5. Partial regularity in the vectorial case. In the scalar case it is known, by a result of Tonelli [T6], that a functional

$$\mathfrak{F}(u; a, b) = \int_a^b f(x, u(x), u'(x))dx,$$

for a C^2-function $f\colon \mathbb{R}^3 \to \mathbb{R}$ with $u \in W^{1,1}(a, b)$, is lower semicontinuous in the L^∞ topology if and only if f is convex in the last variable. When u is vector valued, i.e. if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ for some open $\Omega \subset \mathbb{R}^n$, the convexity hypothesis on the integrand is sufficient but too strong to be necessary for lower semicontinuity. Thus, in order to prove the existence of minimisers for general integral functionals subject to given boundary data, Morrey [M17] introduced a notion of *quasiconvexity* which isolated a property of the integrand that, under appropriate growth conditions, is necessary and sufficient for the weak sequential lower semicontinuity of the functional. Namely, $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is said to be quasiconvex if and only if

$$\fint_{B^n} f(A + D\phi(y))dy \geq f(A)$$

for all $A \in \mathbb{R}^{Nn}$ and every $\phi \in C_c^\infty(B^n, \mathbb{R}^N)$. Here the unit ball B^n can be replaced with any other open subset $\Omega \subset \mathbb{R}^n$ by a covering argument. We remark that quasiconvex functions satisfy Jensen's inequality for a class of probability measures that give the distribution of certain gradients. A nice duality can be seen with gradient Young measures which, more or less, satisfy Jensen's inequality for all quasiconvex functions (cf. [KP]). The inherent difficulty of this condition is due to the non-local nature of quasiconvexity;

a result of Kristensen [K6] shows there is no local condition characterising quasiconvexity. As a generalisation of the scalar case we have, in particular, the following:

Lemma 0.6. *Let $p \geq 1$ and $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ be a quasiconvex function of class C^1 satisfying*

$$|f(\xi)| \leq L(1 + |\xi|^p) \tag{0.5}$$

for some positive constant L. Then for every $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ the functional $w \mapsto \int_\Omega f(Dw)$ is sequentially lower semicontinuous on the Dirichlet class $u + W_0^{1,p}(\Omega, \mathbb{R}^N)$ endowed with the weak topology of $W^{1,p}$.

For a continuous integrand in the higher order case the paper of Meyers [M4] shows quasiconvexity to be necessary and sufficient for lower semi-continuity. When the continuity hypothesis is dropped the seminal paper of Acerbi and Fusco [AF1] shows a non-negative Carathéodory integrand with p-growth is weakly sequentially lower semicontinuous on the $W^{1,p}$-Sobolev space if and only if the integrand is quasiconvex.

As to the regularity considerations for quasiconvex functionals, the work of Evans [E8] showed a slightly strengthened version of quasiconvexity forces partial regularity. This is done by considering the *uniformly strict quasiconvex* condition

$$\gamma \int_{B^n} (1 + |A|^2 + |D\phi(y)|^2)^{\frac{p-2}{2}} |D\phi(y)|^2 dy \leq \int_{B^n} f(A + D\phi(y)) - f(A)\, dy$$

for some positive γ, all $A \in \mathbb{R}^{Nn}$ and all $\phi \in C_c^\infty(B^n, \mathbb{R}^N)$. The results of [E8, EG1] then proved partial $C^{1,\alpha}_{\text{loc}}$-regularity for minimisers u of a functional $\mathfrak{F}(u;\Omega) = \int_\Omega f(Du)dx$ (i.e. local Hölder continuity of the derivative Du outside a set of Lebesgue measure zero) if the integrand f is of class C^2, strictly quasiconvex in the above sense with the growth condition (0.5) for $p \geq 2$, and satisfies an additional bound on the second derivative $D^2 f$. Subsequently this result was generalised to integrals with x- and u-dependences in [FH, GM4] and the bound on $D^2 f$ was removed in [AF2]. Further results for quasiconvex integrand in a general form can be found in [H18, G7, H3, H4]. An extension to the subquadratic case $1 < p < 2$ has been obtained in [CFM]. For the higher order case see [G13, K7]. A partial regularity result for a class of strong local minimisers of quasiconvex functionals can be found in [KT]. Also the note [EG2] shows that 'uniform strict quasiconvexity damps out oscillations in the gradients' of minimising sequences at least away from $\partial\Omega$, i.e. if the integrand f is strictly quasiconvex and satisfies (0.5) for $p > 1$ then any $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathfrak{F}(u_k) \to \mathfrak{F}(u)$ implies $u_k \to u$ strongly in $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

A basic technical observation of [E8] shows the strict quasiconvexity condition permit the proof of Caccioppoli inequalities for minimisers. In this context the original Lemma 0.2 is not used. Instead, for $\frac{R}{2} \leq r < s \leq R$ a cut-off function $\eta \in C_c^\infty(\Omega)$ is

chosen such that $\eta = 1$ on B_r, $\eta = 0$ on $\Omega\backslash B_s$, $0 \le \eta \le 1$ and $|D\eta| \le C/(s-r)$. Then with $\phi = \eta(u-a)$ the above uniformly strict quasiconvex condition can be shown to yield, for each $\Lambda > 0$, an inequality of the form

$$\int_{B_r} |Du - A|^2 \le C \int_{B_s \backslash B_r} |Du - A|^p + \Big|\frac{u-a}{s-r}\Big|^p,$$

for some constant $C = C(\Lambda) > 0$, where a denotes the affine function $x \mapsto \xi + A \cdot (x - x_0)$ with $\xi \in \mathbb{R}^N$ and $A \in \mathbb{R}^{Nn}$ such that $|A| \le \Lambda$ (cf. Lemma 2.13). Then by Widman's hole-filling trick [W10] we add $C \int_{B_r} |Du - Da|^p$ to both side and apply the below 'simple, but fundamental [iteration] lemma' [GG2, §1] to conclude

$$\int_{B_{R/2}} |Du - A|^p \le C \int_{B_R} \Big|\frac{u-a}{R}\Big|^p.$$

Here we say the Caccioppoli inequality is of the first kind if a is a constant and of the second kind if a is an affine function.

Lemma 0.7. *Let $\phi(t)$, $0 \le R_0 \le t \le R_1$, be a non-negative bounded function. Suppose for every $R_0 \le r < s \le R_1$ we have*

$$\phi(r) \le \theta\phi(s) + A + B(s-r)^{-\alpha}$$

where $A, B, \alpha \ge 0$ and $0 < \theta < 1$ are constants. Then there exists a constant $c = c(\alpha, \theta)$ such that for every $R_0 \le \rho < R \le R_1$ we have

$$\phi(\rho) \le c\,(A + B(R-\rho)^{-\alpha}).$$

Proof. Consider the sequence (t_j) defined by

$$t_0 = \rho, \qquad t_{j+1} - t_j = (1-\lambda)\lambda^j(R-\rho) \tag{0.6}$$

for some $0 < \lambda < 1$ of our choosing. By iteration

$$\phi(t_0) \le \theta^k \phi(t_k) + \Big[A + \frac{B}{(1-\lambda)^\alpha}(R-\rho)^{-\alpha}\Big] \sum_{j=0}^{k-1} \Big(\frac{\theta}{\lambda^p}\Big)^j.$$

Now set λ such that $\theta\lambda^{-\alpha} < 1$ (i.e. $\theta < \lambda^\alpha < 1$) and send $j \to \infty$ to get the desired inequality with $c = (1-\lambda)^{-\alpha}(1-\theta\lambda^{-\alpha})^{-1}$. □

To obtain partial regularity a linearisation strategy is often used to transfer the 'good' regularity properties of the linearised elliptic system to the non-linear problem. The technique of harmonic type approximations, in particular, has proved both a popular and elemental way of obtaining partial regularity for solutions to non-linear elliptic systems and minimisers of particular integral functionals (cf. [DM2]). The underlying premise of this method goes back to ideas introduced by De Giorgi [DG7] in his treatment of minimal

surfaces. Specifically he establishes 'some properties of regular surfaces approximating minimal surfaces; these properties will be obtained by a comparison argument with the graph of suitable harmonic functions' [DG9, p. 244]. In the context of regularity for elliptic systems it was utilised by Simon [S11, §1.6] to provide a neat and elementary proof of partial regularity. In the context of quasiconvex integrals the technique was initiated by Duzaar, Gastel and Grotowski [DGG]. This approach, popularised in geometric measure theory (cf. [A3]), is typically achieved through an indirect compactness argument. However, a version of this lemma using 'a direct approach based on the Lipschitz truncation technique which requires no contradiction argument' has recently been proved by [DLSV]. Other linearisation techniques include the indirect blow-up method, going back to results by [M21] for weak solutions to non-linear elliptic systems and by [GM5, G9] for the case of quasilinear systems in divergence form, as well as a direct approach going back to [GG1, GG2] and [GM3, GM2].

Aside from partial regularity the only global regularity property that remains in the vectorial case is the so-called 'higher integrability'. In this regard the importance of Caccioppoli inequalities becomes paramount, since they afford $Du \in L^q_{\mathrm{loc}}$ for some $q > p$ via Gehring's lemma [G5] and the Poincaré-Sobolev inequality (for details see §1.3). We remark the higher integrability exponent q is explicitly computable, however the optimal higher integrability exponents for minimisers of, say, regular variational integrals remains largely an open problem (e.g. the paper [KM1] gives preliminary results in this direction by proving dimension-free higher integrability bounds, however precise control on the higher integrability exponents remains elusive).

A further question of interest, once partial regularity is obtained, is to ask if the singular set is not just Lebesgue measure zero but in some sense smaller if not empty at all. For regular variational problems of the form $\int_\Omega f(Du)dx$, or the equivalent elliptic systems case, a standard measure density result of [G8] together with higher integrability and higher differentiability shows the Hausdorff dimension of the singular set is strictly less than $n-2$ (see for instance [G6, pp. 98ff.]). By showing u belongs to a fractional Sobolev space the results of [M8, M7] generalises the singular set bound to non-differentiable elliptic systems with x- and u-dependences. For the minima of functionals of the form $\int_\Omega f(x, u, Du)dx$ in the convex case, results by [KM2] afford bounds on the singular set via a variational difference quotient method. For the minima of quasiconvex functionals it is an open problem as to the Hausdorff dimension bounds; to date only the result of [KM3] shows that at least for Lipschitz continuous minimisers of quasiconvex integrals the dimension of the singular set is strictly smaller than n. For further details on other aspects (including boundary regularity, degenerate cases, conditions for everywhere regularity, ω-minima and (p, q)-growth conditions) can be found in the survey [M9].

0.2. The problem of mapping into a manifold: regularity with holonomic constraints

In the development and application of the calculus of variations one is often interested in imposing subsidiary conditions, or constraints, on the class of admissible functions. Such constraints are often either of a differential or integral form. For instance one might seek minimisers $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of a variational integral

$$\int_\Omega f(x,u,Du)dx$$

subject to a subsidiary condition of the kind $\int_\Omega g(u,u,Du)dx = k$ for some constant k. Conditions of this form are referred to as *isoperimetric constraints* from which one obtains the collection of so-called isoperimetric problems. In another direction one could consider subsidiary conditions of the kind

$$G(x,u(x)) = 0 \tag{0.7}$$

or of the more general form

$$G(x,u(x),Du(x)) = 0, \tag{0.8}$$

where G denotes either a scalar or vector valued function. Following the terminology introduced by Hertz [H12, p. 91], conditions of the former type are called *holonomic constraints* (from ὅλος, *holos*—whole, entire, complete in all its parts; νόμος, *nomos*—law <taken to mean 'integrable' via the Latin *integer*—complete, whole, entire, intact>) whereas those of the latter type are known as *nonholonomic constraints*.

Example 0.8. In [E4, Chap. 3] Euler considered integral functionals of the form

$$\int_a^b f(x,y,y',z)dx,$$

where the variable z in the integrand itself is subject to a nonholonomic constraint of the kind $z' - g(x,y,y') = 0$. Using multipliers Euler showed stationary solutions satisfy

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big) + \frac{\partial g}{\partial y}\int_x^b \frac{\partial f}{\partial z}dz - \frac{d}{dx}\Big(\frac{\partial g}{\partial y'}\int_x^b \frac{\partial f}{\partial z}dx\Big) = 0.$$

One such example is that of the brachistochrone in a resisting medium: the time of descent is proportional to $\int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{z}}dx$ subject to the speed satisfying the relation $z' = \kappa - R(z)\sqrt{1-y'^2}$, where κ is the acceleration due to gravity and R is some function of z measuring resistance (cf. [E4, Chap. 3, Exemplum II, 46]).[14]

[14]Notwithstanding the later general synthesis of Lagrange [L3], the analysis of the constraint problems in [E4, Chap. 3] is said to be 'a first-rate result which even an Euler was not all too often successful' [C13, §19, p. xxii].

We remark that every holonomic constraint of the form (0.7) can, indeed, be transformed by differentiation into the nonholonomic constraint

$$\sum_{i=1}^{n} \frac{\partial G}{\partial x^i} dx^i + \sum_{\alpha=1}^{N} \frac{\partial G}{\partial u^\alpha} du^\alpha = 0.$$

Conversely, a differential relation of the form

$$\sum_{i=1}^{n} A_i(x,u) dx^i + \sum_{\alpha=1}^{N} B_\alpha(x,u) du^\alpha = 0$$

can be transformed into the form (0.7) if the integrability conditions of Frobenius' theorem are satisfied. For the purposes of this dissertation nonholonomic constraints in the strict sense will not be considered; we will exclusively focus on holonomic constraints of the form

$$G(y) = 0,$$

where $G = (G^1, \dots, G^k)$ for some $1 \leq k < N$. Such a condition is, up to a technicality, equivalent to defining an $(N-k)$-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^N$. Given this, the primary question of interest centres around minimising an integral functional of the form

$$\int_\Omega f(x, u, Du) dx$$

for manifold valued mappings $u : \Omega \to \mathcal{M}$ (or equivalently vector valued minimisers of the functional under the subsidiary condition $G(u(x)) = 0$).[15] The archetypical problem is of course:

Example 0.9. To find the shortest line, or geodesic, between two points P_1 and P_2 on a given surface $\mathcal{M} = \{y \in \mathbb{R}^3 : G(y) = 0\}$, as a mapping $u : [0,1] \to \mathbb{R}^3$, which minimises the length functional

$$\mathfrak{L}(u) = \int_0^1 |\dot{u}(t)| dt$$

subject to the holonomic constraints $G(u) = 0$ (e.g. if $G(y) = |y|^2 - 1$ the minimiser must be an arc of a great circle of S^2).

The geodesic problem in fact goes back to the early development of the calculus of variations: Johann Bernoulli proposed just such a problem, as the first of six in his *Problêmes à resoudre*, particularly in the cases of the spheroid, the conoid and the cylindroid [B5]. In the form of a parametric surface $\mathcal{M} = \{x(u,v) \in \mathbb{R}^3 : (u,v) \in \Omega \subset \mathbb{R}^2\}$ one can

[15]Note that it suffices to consider submanifolds of $\mathbb{R}^N$, since every Riemannian manifold $(\mathcal{M}^m, g)$, with a C^r-metric g for $r \geq 3$, can be isometrically embedded into $\mathbb{R}^N$ for a sufficiently large integer N by the Nash embedding theorem (cf. [A4]). For example the original result of Nash has $N = m(3m+11)/2$ and when $m = 2$ Gromov [G12] showed every compact surface isometrically embeds into $\mathbb{R}^5$—which cannot be improved in general.

phrase the problem as that of finding a path $t \mapsto x(u(t), v(t))$, from a point P_1 at $t = t_1$ to a point P_2 at $t = t_2$ on the given surface $\mathcal{M}$, that minimises the arc length

$$\int_{t_1}^{t_2} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt,$$

where E, F and G are defined by the first fundamental form (i.e. the induced metric g on the surface) given by

$$g = Edu^2 + 2Fdudv + Gdv^2. \tag{0.9}$$

The article [B3] of Jacob Bernoulli takes up a discussion of his brother's problem for various surfaces of revolution. Johann too also address the question in his collected works [B7, Problema, p. 108] and in addition considers the problem

> [of finding], for a shortest line on a given surface, the inclination of the osculating plane at any point to the tangential plane of the given surface at the same point. Moreover, we call the *osculating plane* that through which three point on the sought curve are infinitely near to each other.
>
> [B7, Problema, p. 113]

The investigations of Carathéodory [C12] note the latter proposition was first discussed by Johann with Leibniz in a correspondence of 1698 (cf. [LB, pp. 389–394], [L16, N. 219, p. 877]); Johann concluded the two planes meet perpendicularly. The geodesic problem was also later proposed by Johann to his then pupil Euler. The resulting memoir [E3] represents Euler's first publication on a calculus of variations topic. The concurrent work of Clairaut [C14] independently considers the problem of geodesics on a surface of revolution. Nevertheless the generic conception of geodesics on surfaces embedded in $\mathbb{R}^3$ does not appear until the seminal work of Gauss [G4] (whose ideas on curvature and geodesics initiated the development of modern differential geometry).

0.2.1. Harmonic mappings into compact manifolds. If $x\colon Q \subset \mathbb{R}^2 \to \mathbb{R}^N$ is a vector valued map from a domain Q in the Euclidean (u, v)-space, minimisers of the Dirichlet energy can be seen to be harmonic (in the sense they satisfy $\Delta x = 0$). If one constrains $x = (x^\mu)_{\mu=1}^m$ to a Riemannian manifold $(\mathcal{M}^m, g)$, the canonical generalisation, originally put forth by Bochner [B17], seeks minimisers of

$$D(x; Q) = \int_Q (E + G)dudv, \tag{0.10}$$

where from (0.9) we have $E = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial u}$, $F = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial v}$, and $G = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial v}\frac{\partial x^\nu}{\partial v}$. Solutions of the corresponding Euler-Lagrange equations

$$\Delta x^\lambda + \Gamma^\lambda_{\mu\nu}(x)\Big(\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial u} + \frac{\partial x^\mu}{\partial v}\frac{\partial x^\nu}{\partial v}\Big) = 0,$$

where the non-linear first order terms $\Gamma^{\lambda}_{\mu\nu}$ denotes the usual Christoffel symbols, are said to be *harmonic surfaces* (the above is an extension, from one to two parameters, of the geodesic equation $\frac{d^2x^\lambda}{ds^2} + \Gamma^{\lambda}_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$).

Due to the presence of the additional non-linearity from the Lagrange multipliers the analysis of minimisers in this new context is decidedly more difficult than in the unconstrained case. The question of regularity for minimisers in this constrained context was first obtained by Morrey [M16] in connection with his analysis of Plateau's problem. In order to use the machinery of direct methods Morrey found it expedient to carry over the theory of $W^{1,2}$-functions to vector functions with values in a Riemannian manifold $\mathcal{M}$. His approach, as seen in [M16, Chap. 2], is as follows. Firstly, a mapping $x\colon Q \to \mathcal{M}$ is said to be *measurable* if for every open set U in $\mathcal{M}$, the pre-image under x is measurable in Q. If x_1 and x_2 are two such mappings the geodesic distance $|x_1 - x_2|$ is measurable on Q also. In which case a vector valued function $x = x(u, v)$ into $\mathcal{M}$ is said to be of *class* L^p if and only if x is measurable and $|x(u, v) - z_0| \in L^p(Q)$ for each $z_0 \in \mathcal{M}$. Then a function $x = x(u, v)$ into $\mathcal{M}$ is of *class* $W^{1,2}$ if and only if $x \in L^p$; for almost all u_0 and v_0 the functions $x(u_0, \cdot)$ and $x(\cdot, v_0)$ are absolutely continuous (A.C.); and, the partial derivatives $\partial_u x$, $\partial_v x$, i.e. the derivatives of the arc-length with respect to u (or v) along $u = \text{const.}$ (or $v = \text{const.}$) respectively, are in L^p also. For such a class of function the existence of a minimising $x = x(u, v)$ then follows by direct methods (cf. [M20, Lem. 9.4.14, 9.4.15, 9.4.16]). To solve the regularity problem for minimisers of (0.10) Morrey appeals to the following lemma of Courant under the assumption $\mathcal{M}$ is 'homogeneously regular' (i.e. there exists positive numbers ℓ and L such that for each point of $\mathcal{M}$ there is a coordinate chart with domain $B(0, 1)$ for which $\ell|\xi|^2 \leq g_{\mu\nu}(x)\xi^\mu\xi^\nu \leq L|\xi|^2$ for any $x \in B(0, 1)$ and ξ arbitrary).

Lemma 0.10 ([C15], [M16, Lem. 3.1])**.** *Suppose ξ is A.C. on $\partial B(x_0, R)$ with*

$$\int_0^{2\pi} |\xi'(\theta)|^2 d\theta \leq \frac{\ell}{\pi}.$$

Then there exists a function $x(u, v)$ of class $W^{1,2}$ into $\mathcal{M}$ which coincides with ξ on $\partial B(x_0, R)$ for which

$$D(x; B(x_0, R)) \leq \frac{L}{\ell} \int_0^{2\pi} |\xi'(\theta)|^2 d\theta.$$

Local Hölder continuity then follows by an application of this lemma together with the minimality condition (cf. [M16, p. 840-1]). In addition, higher regularity is considered for manifolds that are sufficiently smooth; the outcome of which is the following regularity result (cf. [M20, Theor. 9.4.2]).

Theorem 0.11 (Morrey)**.** *Let $\mathcal{M}$ be a C^3-Riemannian manifold that is homogeneously regular. Then a minimiser $x = x(u, v)$ of (0.10) is of the class $C^{2,\alpha}_{\mathrm{loc}}$ for any $0 < \alpha < 1$. If $\mathcal{M}$ is of class $C^{r,\beta}$, $r \geq 3$, $0 < \beta < 1$, then x is of the class $C^{r,\beta}_{\mathrm{loc}}$. If $\mathcal{M}$ is smooth then x is locally also.*[16]

When considering domains $\Omega \subset \mathbb{R}^n$, for $n > 2$, the A.C. boundary property seen in Lemma 0.10 no longer holds. Thus in order to discerning the regularity for constrained mappings $u\colon \Omega \to \mathcal{M}$ the notion of a suitable Sobolev space $W^{1,2}(\Omega, \mathcal{M})$ different to that of the above needs to be chosen. The difficulty of doing so emerges with the realisation that as soon as $n > 2$ maps in $W^{1,2}(\Omega, \mathcal{M})$ need not be continuous. Thus even if $W^{1,2}(\Omega, \mathcal{M})$ makes sense there is no reason for maps to take values in an open subset—thereby ruling out the possibility of using local charts on the target manifold $\mathcal{M}$. As is outlined in §1.9 it is expedient to avoid these difficulties by using Nash's embedding theorem to view the target manifold $\mathcal{M}$ extrinsically with the additional assumption that $\mathcal{M}$ is compact.[17] In which case there exists an isometric embedding $(\mathcal{M}, g) \hookrightarrow (\mathbb{R}^N, \langle\cdot,\cdot\rangle)$ so that for any domain $\Omega \subset \mathbb{R}^n$ we define

$$W^{1,2}(\Omega, \mathcal{M}) = \{u \in W^{1,2}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\}.$$

In fact an argument of [S11, §4.8] or [HL2, §7] shows the replacement of the domain as bounded open subset with a compact manifold to be inconsequential in the harmonic map case. Taking these considerations into account we say $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ is a minimiser of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx$$

if and only if $E(u) \leq E(v)$ for all bounded open $\Omega \subset \mathbb{R}^n$ and all $v \in W^{1,2}(\Omega, \mathcal{M})$ such that $v - u \in W^{1,2}_0(\Omega, \mathbb{R}^N)$. Now in this framework the seminal paper of Schoen and Uhlenbeck [SU1] established the following partial regularity result that generalised the above theorem of Morrey.

Theorem 0.12 ([SU1])**.** *For any E-minimising $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ into a compact manifold $\mathcal{M}$ there exists a closed subset $\mathrm{Sing}(u) \subset \Omega$ such that $\mathcal{H}^{n-3}(\mathrm{Sing}(u)) = 0$ with $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega\backslash\mathrm{Sing}(u), \mathcal{M})$ for some $0 < \alpha < 1$.*

A discussion of the details of this regularity result, including the biharmonic generalisation of Schoen and Uhlenbeck's proof, is given in Chapter 3. Similar results for the case in which the image lies in a single coordinate chart were obtained by Giaquinta and

[16] From the work of Hélein [H10] the same result goes through for (weakly) stationary solutions of (0.10) also (cf. §3.4 in Chapter 3).

[17] The above result of Morrey [M16] had to work without the Nash embedding theorem as it was not yet proved.

Giusti [GG2, GG3] (and by [HW2, HKW] when the image of the map is contained in a convex ball of $\mathcal{M}$). In the p-harmonic case Hardt and Lin [HL2] showed that p-energy minimisers, for $1 < p < +\infty$, are $C^{1,\alpha}_{\mathrm{loc}}$ for some $0 < \alpha < 1$ except possibly on a closed set of Hausdorff dimension at most $n - \lfloor p \rfloor - 1$. Further partial regularity results in the p-harmonic case can be found in [F7, L22, DM1] for maps into a compact target manifold and by [U2, U1] in the unconstrained case. The question of minimising in an appropriate homotopy class has also been studies by White [W6, W7]. Luckhaus [L21] has proved similar regularity results for a more general class of convex functionals with appropriately well-behaved blow-ups. Further results in this direction for convex functionals with p-growth can also be found in [HKL1, HL2]. In particular reverse Hölder inequalities are obtained under the assumption $\mathcal{M}$ is $(\lfloor p \rfloor - 1)$-connected via a retraction of $\mathbb{R}^N \backslash K$ onto $\mathcal{M}$ for some polyhedron K in $\mathbb{R}^N \backslash \mathcal{M}$. This kind of retraction was previously used in (the domain) by [W6] for the existence of finite energy extensions of Lipschitz maps. In fact the idea of using such a retraction with analytic estimates goes back to the proof in [FF] of isoperimetric inequalities. We will use a version of this retraction in Chapter 5.

A particular feature of harmonic maps sees the minimality naturally create singularities. In the case of the p-energy

$$E_p(u) = \int_{B^n} |Du|^p dx$$

on the unit n-ball a result of [H19] shows the map $\frac{x}{|x|} \colon B^n \to S^{n-1}$ to be an energy minimiser in the sense that $E_p(\frac{x}{|x|}) \leq E_p(u)$ for any $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$ on ∂B^n. Likewise the paper [HLW] shows the map $x \mapsto \frac{x}{|x|}$ to be p-energy minimising for $n - 1 \leq p < n$ in the Sobolev class $W^{1,p}(B^n, S^{n-1})$ with respect to its boundary values. Using the Hopf map $H \colon S^3 \to S^2$ a similar result by [CG] shows the the homogeneous degree zero extension $u_0 \colon B^8 \to S^4$, defined by $u_0(x) = H(\frac{x}{|x|})$, is energy minimising in the sense that $E_2(u_0) \leq E_2(u)$ for any $u \in H + W^{1,2}_0(B^4, S^2)$. An equivalent result is also obtained for the Hopf map $H \colon S^7 \to S^4$. In fact a result of [BCL] shows minimising harmonic maps from B^3 to S^2 can only have singularity of degree ± 1. Namely, the only homogeneous minimising maps from $B^3 \to S^2$ sending $x \mapsto \psi(x/|x|)$ are of the form $\psi(x/|x|) = \pm R \frac{x}{|x|}$ where $R \in SO(3)$.

Contrasting the minimising case sees stationary solutions $u \in W^{1,2}(B^3, S^2)$ of the Dirichlet energy have an Euler-Lagrange equation of the form $-\Delta u = u|Du|^2$ weakly. Under the hypothesis that u belongs to $L^\infty \cap W^{1,2}$ there are no general results concerning regularity or partial regularity of weak solutions. Indeed:

Theorem 0.13 ([R3])**.** *For any non-constant map $\varphi \in C^\infty(\partial B^3, S^2)$ there exists an everywhere discontinuous weakly harmonic map $u \in W^{1,2}(B^3, S^2)$ with $u = \varphi$ on ∂B^3.*

In other words the singular set $\mathrm{Sing}(u)$, denoting the largest open set where u is C^∞, coincides with the closed ball $\bar{B}^3$. In fact the target manifold S^2 in the result of [R3] can be replaced by an arbitrary manifold Σ homeomorphic to S^2. We remark that Riviére's proof is based on the relaxed Dirichlet energy introduced by [BBC] and on a subtle dipole construction derived from [BCL] and [HLP].

Further details on harmonic maps can be found in the surveys [HW1] and [EL1, EL2].

0.2.2. Partial regularity results for holonomic minimisers of quasiconvex functionals. Aside from the result in [L21, HKL1, HL2] for convex functionals, the only result for holonomic minimisers of quasiconvex functionals to date is that of the Evans and Gariepy paper [EG3]. By appealing to the blow-up technique introduced in [EG1], the paper [EG3] considers a functional of the form

$$\mathfrak{F}(u;\Omega) = \int_\Omega f(Du)dx$$

under the structural hypothesis on f that:

(H1) There exists some constant $C > 0$ such that $|D^2 f(\xi)| \le C$ for all $\xi \in \mathbb{R}^{Nn}$.

(H2) For some constant $\gamma > 0$,

$$\int_{B(x,r)} f(A) + \gamma|D\varphi|^2 \le \int_{B(x,r)} f(A + D\varphi)$$

for all $B(x,r) \subset \Omega$, $A \in \mathbb{R}^{Nn}$ and $\varphi \in C_0^1(B(x,r), \mathbb{R}^N)$.

Then for holonomic minimisers $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ of $\mathfrak{F}$, into a compact Riemannian submanifold $\mathcal{M} \subset \mathbb{R}^N$ without boundary, we have:

Theorem 0.14 ([EG3])**.** *Suppose f is a C^2-function that satisfies (H1) and (H2). If $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ is an holonomic minimiser of $\mathfrak{F}$, then there exists a relatively open subset $\Omega_0 \subset \Omega$ of Lebesgue measure zero and an $0 < \alpha < \frac{1}{2}$ such that $Du \in C^\alpha(\Omega_0, \mathbb{R}^{Nn})$.*

The idea of proof focuses on decay estimates for the mean-square oscillation

$$\mathcal{E}(x,r) = \fint_{B(x,r)} |Du - (Du)_{x,r}|^2.$$

Namely, we want to show:

Proposition 0.15. *There exists $0 < \tau < \frac{1}{2}$ such that for each $\Lambda > 0$ there is a constant $\varepsilon = \varepsilon(\Lambda) > 0$ such that if*

$$|(Du)_{x,r}| \le \Lambda \qquad \text{and} \qquad \mathcal{E}(x,r) < \varepsilon$$

on some $B(x,r) \subset \Omega$, then either

$$\mathcal{E}(x,r) \le r \qquad \text{or} \qquad \mathcal{E}(x,\tau r) \le \frac{1}{2}\mathcal{E}(x,r).$$

Sketch of proof. Firstly, fix $\Lambda > 0$ and suppose the result were false. Then there exists $B(x_j, r_j) \subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \leq \Lambda \qquad \text{and} \qquad \lambda_j^2 = \mathcal{E}(x_j, r_j) \to 0$$

as $j \to \infty$ while

$$\mathcal{E}(x_j, \tau r_j) > \frac{1}{2}\lambda_j^2 \qquad \text{and} \qquad \mathcal{E}(x_j, r_j) > r_j. \tag{0.11}$$

By denoting $(u)_j = (u)_{B_j}$, $(Du)_j = (Du)_{B_j}$ and $B_j = B(x_j, r_j)$ we observe that

$$\operatorname{dist}((u)_j, \mathcal{M}) \leq C r_j.$$

In which case, using the retraction $\Pi\colon \mathcal{O} \to \mathcal{M}$ from a tubular neighbourhood onto $\mathcal{M}$, the projections $a_j = \Pi((u)_j)$ and $A_j = \nabla\Pi(a_j)(Du)_j$ are well defined for sufficiently large j. A straightforward calculation also shows

$$|a_j - (u)_j| \leq C\lambda_j r_j \qquad \text{and} \qquad |A_j - (Du)_j| \leq C\lambda_j.$$

By rescaling, the blow-up sequence defined by

$$v_j(z) = \frac{u(x_j + r_j z) - a_j - r_j A_j z}{\lambda_j r_j},$$

for $z \in B^n$ in the unit ball, is bounded in the sense that

$$⨍_{B^n} |v_j|^2 + |Dv_j|^2 \leq C.$$

So by compactness there exists a subsequence, also denoted by (v_j), for which $v_j \to v$ strongly in $L^2(B^n, \mathbb{R}^N)$, $Dv_j \to Dv$ weakly in $L^2(B^n, \mathbb{R}^{Nn})$, $a_j \to a$ on $\mathcal{M}$ and $A_j \to A$ in $\mathbb{R}^{Nn}$. It is then possible to show the limit v is a weak solution to a uniformly elliptic system with constant coefficients. In particular,

$$⨍_{B_\tau(0)} |Dv - (Dv)_{0,\tau}|^2 \leq \frac{1}{8} \tag{0.12}$$

for $\tau > 0$ sufficiently small independent of Λ.

We are now in a position to find a contradiction between the good decay estimate (0.12) for the limit v and the initial assumptions. In particular, we want to show that $Dv^j \to Dv$ strongly in $L^2_{\text{loc}}(B^n, \mathbb{R}^{Nn})$. Indeed, if the latter convergence holds then

$$⨍_{B_\tau(0)} |Dv^j - (Dv^j)_{0,\tau}|^2 \leq \frac{1}{4}$$

for sufficiently large j. So by a change of variables $\mathcal{E}(x_j, \tau r_j) \leq \frac{1}{4}\lambda_j^2$ which contradicts (0.11).

To show $Dv^j \to Dv$ we consider, for $0 < s < 1$, the rescaled functional

$$\mathcal{F}_{s,j}(w) = \int_{B_s(0)} f_j(Dw)dx,$$

where

$$f_j(\xi) = \frac{f(A_j + \lambda_j\xi) - f(A_j) - Df(A_j)\lambda_j\xi}{\lambda_j^2}$$

for all $\xi \in \mathbb{R}^{Nn}$. Then $f_j(0) = 0$, $Df_j(0) = 0$ and $D^2f_j(\xi) = D^2f(A_j + \lambda_j\xi)$. Moreover, f_j satisfies (H1), (H2) and

$$\mathcal{F}_{s,j}(v_j) \le \mathcal{F}_{s,j}(w)$$

for all $w \in v_j + W_0^{1,2}(B_s(0), \mathbb{R}^N)$ with $a_j + r_jA_jz + r_j\lambda_jw(z) \in \mathcal{M}$ for a.e. $z \in B_s(0)$. From the unconstrained case of [EG1, EG2] the uniform strictly quasiconvexity condition implies

$$\frac{\gamma}{2} \limsup_{j\to\infty} \int_{B_s(0)} |Dv^j - Dv|^2 \le \limsup_{j\to\infty} \big(\mathcal{F}_{s,j}(v_j) - \mathcal{F}_{s,j}(v)\big)$$

On the other hand, a lemma of Luckhaus (cf. Chapter 4) constructs a competitor $w_j \in v_j + W_0^{1,2}(B_s(0), \mathbb{R}^N)$ such that for any $0 < r < s$ we have $w_j = v$ on $B_r(0)$ with

$$\int_{B_s(0)\setminus B_r(0)} |Dw_j|^2 \le C \int_{B_s(0)\setminus B_r(0)} |Dv|^2 + |Dv_j|^2 + \Big|\frac{v - v_j}{s - r}\Big|^2$$

and

$$\operatorname*{ess\,sup}_{z\in B_s(0)} \operatorname{dist}(a_j + r_jA_jz + r_j\lambda_jw_j(z), \mathcal{M}) \to 0$$

as $j \to \infty$. With this comparison map it is then possible to show

$$\limsup_{j\to\infty} \big(\mathcal{F}_{s,j}(v^j) - \mathcal{F}_{s,j}(v)\big) \le C(s - r)$$

by the minimality condition. Then by letting $r \nearrow s$ we get the desired result. □

Remark 0.16. A major difficulty in extending the regularity results from §0.1.5 to the manifold constrained case is finding appropriate comparison maps which satisfy the nonlinear constraints. The approach of [EG3] uses a construction derived from a particular extension lemma of Luckhaus. The method works in the blow-up case since one sending $j \to \infty$ first then lets $r \to s$ in the above proof. In Chapter 6 a direct proof is given which avoids this construction.

Chapter 1

Preliminaries

Let $L^p(\Omega)$, for $p \geq 1$, denote the classical Banach space of measurable functions on a bounded domain $\Omega \subset \mathbb{R}^n$ such that norm

$$\|u\|_{p;\Omega} = \|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx \Big)^{\frac{1}{p}}$$

is finite. For $p = \infty$ we denote by $L^\infty(\Omega)$ the space of bounded measurable functions on Ω with the norm

$$\|u\|_{\infty;\Omega} = \|u\|_{L^\infty(\Omega)} = \sup_\Omega |u|.$$

As per usual, measurable functions are identified on sets of measure zero. Any pointwise property attributed to a measurable function is understood to hold for some function in its equivalence class. The supremum and infimum of such functions are understood as the essential supreumum and infimum. In the following we shall use $\|u\|_p$ for $\|u\|_{L^p(\Omega)}$ when there is no ambiguity.

Recall Hölder's inequality

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

for functions $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore one has

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q(\Omega)}$$

for $u \in L^q(\Omega)$ with $p \leq q$ so that $L^1(\Omega) \supset L^p(\Omega) \supset L^q(\Omega) \supset L^\infty(\Omega)$ for $1 \leq p \leq q \leq \infty$. When $p = 2$ Hölder's inequality reduces to the well known Cauchy–Schwarz inequality; in fact $L^2(\Omega)$ is a Hilbert space with inner product $(u, v) = \int_\Omega uv\, dx$. For $1 < p < \infty$ the space $L^p(\Omega)$ is reflexive. Finally, we denote by $L^p_{\mathrm{loc}}(\Omega)$ the set of functions belonging to $L^p(\Omega')$ for $\Omega' \subset\subset \Omega$.

1.1. Slicing inequalities

Suppose $0 < \delta < \varepsilon$ and $f \geq 0$ a.e. for a given measurable extended real-valued function f on a measure space (X, Σ, μ). Then with $E = \{x \in X : f(x) > \varepsilon\}$ and $F = \{x \in X : f(x) < \delta\}$ Markov's inequality implies $\mu(E) \leq \frac{1}{\varepsilon}\int_X f$ and

$$\begin{aligned}\mu(F) = \mu(F \cap (X\backslash E)) &= \mu(\{x \in X\backslash E : f(x) < \delta\}) \\ &= \mu(\{x \in X\backslash E : \varepsilon - f(x) > \varepsilon - \delta\}) \leq \frac{1}{\varepsilon - \delta}\int_{X\backslash E}(\varepsilon - f).\end{aligned}$$

Furthermore $\mu(\{x \in X : f(x) > \varepsilon \text{ or } f(x) < \delta\}) = \mu(E) + \mu(F)$ since $E \cap F = \emptyset$.

Example 1.1. For a given non-negative measurable function ψ on a ball $B_R(x_0) \subset \mathbb{R}^n$ the relation

$$\int_{\partial B_s} \psi \leq \frac{A}{R-r}\int_{B_R\backslash B_r} \psi$$

holds for a 'good set' of $s \in (r, R)$ of positive measure, where $r < R$ and the constant $A > 1$ (for if not then Markov's inequality implies the latter inequality fails to hold with a set of measure

$$\mathcal{L}^1(E) = \mathcal{L}^1\Big\{s \in (r, R) : \int_{\partial B_s} \psi > \frac{A}{R-r}\int_{B_R\backslash B_r} \psi\Big\} \leq \frac{R-r}{A}$$

by virtue of the identity $\int_{B_R\backslash B_r} \psi = \int_r^R \int_{\partial B_s} \psi \, ds$). Likewise the reverse relation

$$\frac{a}{R-r}\int_{B_R\backslash B_r} \psi \leq \int_{\partial B_s} \psi$$

holds for a 'good set' of $s \in (r, R)$ of positive measure (for if not the inequality fails to hold with a set of measure

$$\mathcal{L}^1(F) = \mathcal{L}^1\Big\{s \in (r, R) : \frac{a}{R-r}\int_{B_R\backslash B_r} \psi > \int_{\partial B_s} \psi\Big\} \leq \frac{R-r}{A-a}\Big(A - \frac{A}{R-r}\mathcal{L}^1(E)\Big)$$

which is strictly less than $R - r$, provided the constant $0 < a < 1$). Furthermore, for the selection of $0 < a < 1 < A$ the bound

$$\mathcal{L}^1(E) + \mathcal{L}^1(F) \leq \frac{R-r}{A} + \frac{R-r}{A-a}\Big(A - \frac{A}{R-r}\mathcal{L}^1(E)\Big)$$

must be strictly less than $R - r$; for if not then $\frac{1}{A} + \frac{1}{A-a}(A - \frac{A}{R-r}\mathcal{L}^1(E)) = 1$ implies $\frac{A}{R-r}\mathcal{L}^1(E) = 1 + a(1 - \frac{1}{A}) > 1$ which contradicts the above estimate. Therefore there exists a 'good set' of $s \in (r, R)$ of positive measure for which

$$\frac{a}{R-r}\int_{B_R\backslash B_r} \psi \leq \int_{\partial B_s} \psi \leq \frac{A}{R-r}\int_{B_R\backslash B_r} \psi$$

with an exceptional set strictly less than $R - r$.

1.2. Maximal function

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f\colon \Omega \to \mathbb{R}$ be a measurable function. The *distribution function* of f, denoted by $\lambda_f(t)$ or simply $\lambda(f)$, is defined as the function $\lambda\colon [0,\infty) \to [-\infty,\infty]$ given by

$$\lambda(t) = |\{x \in \Omega : |f(x)| > t\}|,$$

where $|\cdot|$ denotes the Lebesgue measure. Note that $\lambda(t)$ is non-increasing; right-continuous with jumps at values of t for which $\lambda(t) - \lambda(t^-) = |\{x \in \Omega : |f(x)| = t\}| > 0$, $\lambda(t) \to 0$ as $t \to \infty$ if $f \in L^1(\Omega)$, and $\|f\|_{L^\infty(\Omega)} = \inf\{t \geq 0 : \lambda(t) = 0\}$.

Lemma 1.2. *Suppose ψ is an absolutely continuous non-negative function on $\mathbb{R}^n$ with $\psi(0^+) = 0$. Then*

$$\int_\Omega \psi(|f(x)|)dx = \int_0^\infty \psi'(t)\lambda_f(t)dt.$$

In particular if $\psi(t) = t^p$, for $p > 0$, then

$$\int_\Omega |f|^p = p\int_0^\infty t^{p-1}\lambda(t)dt.$$

Thus any quantity dealing solely with the size of a function can be expressed in terms of the distribution.

Proof. By Fubini's theorem we have

$$\begin{aligned}\int_0^\infty \psi'(t)\lambda_f(t)dt &= \int_\Omega \int_0^\infty \psi'(t)\mathbf{1}_{\{f>t\}}dt\,dx \\ &= \int_\Omega \Big(\int_0^{|f(x)|} \psi'(t)dt\Big)dx = \int_\Omega \psi(|f(x)|)dx.\end{aligned}$$

□

Let us now define, for a given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \fint_{B_r(x)} |f| \tag{1.1}$$

for all $x \in \mathbb{R}^n$. Note that the supremum could also be taken over all balls, or for that matter cubes, containing x.

Proposition 1.3 ([S21, p. 5])**.** *If $f \in L^1(\mathbb{R}^n)$ then for every $t > 0$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \frac{5^n}{t}\int_{\mathbb{R}^n} |f|.$$

Moreover if $f \in L^p(\mathbb{R}^n)$, for $p > 1$, then $Mf \in L^p(\mathbb{R}^n)$ and

$$\|Mf\|_p \leq A\|f\|_p$$

where $A = A(n,p) = O(\frac{1}{p-1})$ as $p \searrow 1$.

With this in hand one can infer, via the maximal function, Lebesgue's differentiation theorem:

Theorem 1.4. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then for a.e. $x \in \mathbb{R}^n$ we have*

$$\limsup_{r \searrow 0} \fint_{B_r(x)} |f(y) - f(x)|dy = 0. \tag{1.2}$$

Remark 1.5. A point $x \in \mathbb{R}^n$ is said to be a *Lebesgue point* of f if (1.2) holds. As

$$|f(z) - (f)_{z,r}| \leq \fint_{B_r(z)} |f - f(z)|$$

it follows that $f(x) = \lim_{r \searrow 0} \fint_{B_r(x)} f$ for all such Lebesgue points x. We define the *precise representative* f^* to be

$$f^*(x) = \begin{cases} \lim\limits_{r \searrow 0} \fint_{B_r(x)} f & \text{if } x \in \mathbb{R}^n \text{ is a Lebesgue point} \\ 0 & \text{otherwise} \end{cases}$$

Hence the precise representative selects a member from the equivalence class of f that has a maximal set of Lebesgue points. Often, by an abuse of notation, we will identify f with its precise representative f^*.

Proof. For $\alpha > 0$ let

$$E_\alpha = \{x \in \mathbb{R}^n : \limsup_{r \searrow 0} \fint_{B_r(x)} |f(y) - f(x)|dy > \alpha\}.$$

Without loss in generality consider $f \in L^1(\mathbb{R}^n)$. For fixed $\varepsilon > 0$ there exists a continuous $g \in C_c(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f - g| \leq \varepsilon$.

By estimating

$$\fint_{B_r(x)} |f(y) - f(x)|dy \leq M(f - g)(x) + |(f - g)(x)| + \operatorname*{osc}_{B_r(x)} g$$

and taking the lim sup as $r \searrow 0$ the last term vanishes so that

$$E_\alpha \subset \{M(f - g) > \tfrac{\alpha}{2}\} \cup \{|f - g| > \tfrac{\alpha}{2}\}.$$

By Markov's inequality $|\{x \in \mathbb{R}^n : |(f - g)(x)| > \frac{\alpha}{2}\}| \leq \frac{2}{\alpha}\varepsilon$ and by Proposition 1.3 we have

$$|\{x \in \mathbb{R}^n : M(f - g)(x) > \tfrac{\alpha}{2}\}| \leq \frac{2 \cdot 5^n}{\alpha}\varepsilon.$$

Hence $|E_\alpha| \leq \frac{2(1+5^n)}{\alpha}\varepsilon$ for arbitrary $\varepsilon > 0$. So $|E_\alpha| = 0$ for each $\alpha > 0$. In which case

$$|\{x \in \mathbb{R}^n : \limsup_{r \searrow 0} \fint_{B_r(x)} |f(y) - f(x)|dy > 0\}| = |\bigcup_{\alpha > 0} E_\alpha| \leq \sum_{\alpha > 0} |E_\alpha| = 0. \qquad \square$$

1.3. Gehring's lemma

The lemma of Gehring [G5], which first appeared in the context of quasiconformal maps, constitutes a general and powerful tool that is fundamental to real analysis. A local variant of Gehring's result goes back to the work of [GM3] and is often phrased as a self-improving property for reverse Hölder inequalities. For our purposes the lemma provides a means by which one obtains higher integrability for variational problems. We state a local form of the result as follows.

Proposition 1.6. *Let B denote an open ball in $\mathbb{R}^n$ and let $f, g \in L^p(\mathsf{B})$, $p > 1$, be non-negative functions. Suppose for constants $\kappa \geq 0$ and $0 < \sigma < 1$ that*

$$\Big(\fint_{\sigma B} f^p \Big)^{\frac{1}{p}} \leq \kappa \fint_B f + \Big(\fint_B g^p \Big)^{\frac{1}{p}} \tag{1.3}$$

for all balls $B \subset \mathsf{B}$. Then there exists $q_0 = q_0(n, p, \kappa) > p$ such that

$$\Big(\fint_{\tau \mathsf{B}} f^q \Big)^{\frac{1}{q}} \leq C \Big[\Big(\fint_{\mathsf{B}} f^p \Big)^{\frac{1}{p}} + \Big(\fint_{\mathsf{B}} g^q \Big)^{\frac{1}{q}} \Big]$$

for $p < q \leq q_0$ and $0 < \tau < 1$ where $C = C(n, p, \kappa, \sigma, \tau)$.[1] *In particular $f \in L^q_{\mathrm{loc}}(\mathsf{B})$ whenever $g \in L^q(\mathsf{B})$.*

Remark 1.7. It is important to note that the hypothesis (1.3) must hold *for all* balls $B \subset \mathsf{B}$. The increasing support condition is also needed as the statement with $\sigma = \tau = 1$ is much too stringent for solutions of elliptic systems (cf. [G6, pp. 134-5]). A more general statement replaces (1.3) with a reverse Hölder inequality of the form

$$\Big(\fint_{\sigma B} f^p \Big)^{\frac{1}{p}} \leq \theta \Big(\fint_B f^p \Big)^{\frac{1}{p}} + \kappa \fint_B f + \Big(\fint_B g^p \Big)^{\frac{1}{p}}$$

for some $0 < \theta < 1$. A proof in this form can be found in [G6, pp. 122ff.] (see also [GM3, Prop. 5.1] for θ small). An argument by [G10, p. 204] using Lemma 0.7 shows the latter condition can be reduced to the former (1.3).

We give a proof of Proposition 1.6 via maximal inequalities based on [S22] and [I]. The essence of our approach stems from the following two familiar inequalities for the distribution of the maximal function: namely the inequality

$$\lambda_{Mf}(t) \geq \frac{1}{2^n t} \int_{f > t} f(x) dx, \tag{1.4}$$

which follows from the Calderon-Zygmund decomposition lemma, and that of

$$\lambda_{Mf}(t) \leq \frac{2 \cdot 3^n}{t} \int_{f > \frac{t}{2}} f(x) dx, \tag{1.5}$$

[1] For instance when $n = 3$, $p = 2$ and $\kappa = 1$ our proof has $2 < q \leq 2.000\,003\,196\,917$.

which is a consequence of Vitali's covering lemma (cf. [S21, Chap. 1]). From these inequalities we establish (1.6) below—after which the above Proposition, using a localisation argument, more or less follows from.

Lemma 1.8. *Suppose ψ and f are non-negative real valued functions on $[0,\infty)$ and $\mathbb{R}^n$ respectively. Then,*

$$\int_{\mathbb{R}^n} \psi \circ f = -\int_0^\infty \frac{\psi(t)}{t} \frac{d}{dt}\Big(\int_{f>t} f\Big) dt.$$

Proof. Denote by $H(t) = \int_{-\infty}^t \delta(s)ds$ the Heaviside step function; δ is the Dirac delta function. By computing

$$\frac{d}{dt}\int_{f>t} f = -\int_{\mathbb{R}^n} f(x)H'(f(x)-t)dx = \int_{\mathbb{R}^n} f(x)\delta(f(x)-t)dx$$

we find

$$\begin{aligned}\int_0^\infty \frac{\psi(t)}{t}\Big(\int_{\mathbb{R}^n} f(x)\delta(f(x)-t)dx\Big)dt &= \int_{\mathbb{R}^n} f(x)\Big(\int_0^\infty \frac{\psi(t)}{t}\delta(f(x)-t)dt\Big)dx \\ &= \int_{\mathbb{R}^n} \psi(f(x))dx. \qquad \square\end{aligned}$$

Lemma 1.9. *Suppose Ψ is a non-negative increasing absolutely continuous function on $[0,\infty)$ with $\Psi(t) = o(t)$ as $t \searrow 0$. Then for each non-negative measurable function f on $\mathbb{R}^n$ we have*

$$\int_{\mathbb{R}^n} \Psi(f) \le 2^n \int_{\mathbb{R}^n} \Phi(Mf), \qquad \int_{\mathbb{R}^n} \Phi(Mf) \le 3^n \int_{\mathbb{R}^n} \Psi(2f),$$

where

$$\Phi(t) = \Psi(t) - \int_0^t s^{-1}\Psi(s)ds.$$

Hence if $\Psi(t) = t^p$, $p > 1$, then

$$2^{-n}\int_{\mathbb{R}^n} f^p \le (1-\tfrac{1}{p})\int_{\mathbb{R}^n} (Mf)^p \le 3^n \int_{\mathbb{R}^n} (2f)^p. \tag{1.6}$$

Proof. Assume without loss in generality that $\Psi(f)$ is integrable; otherwise the inequalities holds by trivial means. By Lemma 1.2, Lemma 1.8, (1.4) and integration by parts we get

$$\int_{\mathbb{R}^n} \Psi(f) = -\int_0^\infty \frac{\Psi(t)}{t}\frac{d}{dt}\Big(\int_{f>t} f\Big)dt \le 2^n \int_0^\infty \Phi'(t)\lambda_{Mf}(t)dt = 2^n \int_{\mathbb{R}^n} \Phi(Mf),$$

since $t^{-1}\Phi'(t) = \frac{d}{dt}(\frac{\Psi(t)}{t})$ and as Ψ is increasing

$$\frac{\Psi(t)}{t}\int_{f>t} f < \int_{f>t} \frac{\Psi(f)}{f} f = \int_{f>t} \Psi(f) \to 0$$

as $t \to \infty$. Likewise by (1.5) we get

$$\begin{aligned}\frac{1}{2\cdot 3^n}\int_{\mathbb{R}^n}\Phi(Mf) &= \frac{1}{2\cdot 3^n}\int_0^\infty \Phi'(t)\lambda_{Mf}(t)dt\\ &\le \int_0^\infty \Phi'(t)\Big(\int_{f>\frac{t}{2}} f\Big)\frac{dt}{t}\\ &= -\int_0^\infty \frac{\Psi(t)}{t}\frac{d}{dt}\Big(\int_{f>\frac{t}{2}} f\Big)dt = \frac{1}{2}\int_{\mathbb{R}^n}\Psi(2f). \qquad \square\end{aligned}$$

We can now prove the following higher integrability result for a maximal inequality of the following form.

Proposition 1.10 ([I, p. 197])**.** *Let $p>1$, $\kappa \ge \frac{1}{4}$ and suppose that*

$$(Mf^p)^{\frac{1}{p}} \le \kappa Mf + (Mg^p)^{\frac{1}{p}}$$

for non-negative $f, g \in L^q(\mathbb{R}^n)$ with $p < q \le p + \frac{p-1}{6^n(4\kappa)^{2p-1}}$. Then,

$$\int_{\mathbb{R}^n} f^q \le 2^{2p+1}6^n \int_{\mathbb{R}^n} g^q.$$

Proof. The result follows from (1.6). Indeed, for $q > p$

$$\begin{aligned}\int_{\mathbb{R}^n} f^q = \int (f^p)^{\frac{q}{p}} &\le 2^n(1-\tfrac{p}{q})\int (Mf^p)^{\frac{q}{p}}\\ &\le 2^n(1-\tfrac{p}{q})\int (\kappa Mf + (Mg^p)^{\frac{1}{p}})^q\\ &\le 2^{n+q-1}(1-\tfrac{p}{q})\kappa^q\int (Mf)^q + 2^{n+q-1}(1-\tfrac{p}{q})\int (Mg^p)^{\frac{q}{p}}\\ &\le \Big(2^{n+q-1}\frac{q-p}{q}\kappa^q\Big)\Big(3^n 2^q\frac{q}{q-1}\Big)\int_{\mathbb{R}^n} f^q\\ &\quad + \Big(2^{n+q-1}\frac{q-p}{q}\Big)\Big(3^n 2^{\frac{q}{p}}\frac{q}{q-p}\Big)\int_{\mathbb{R}^n} g^q.\end{aligned}$$

Now let $q \le p+(p-1)\lambda$ for some $0<\lambda\le 1$ of our choosing (so $p<q\le p+(p-1)\lambda \le 2p-1$, i.e. $p>1$). It then follows that

$$\Big(2^{n+q-1}\frac{q-p}{q}\kappa^q\Big)\Big(3^n 2^q\frac{q}{q-1}\Big) \le \frac{1}{2}6^n\lambda(4\kappa)^{p+(p-1)\lambda} \le \frac{1}{2}6^n\lambda(4\kappa)^{2p-1}.$$

Thus by setting $\lambda = 1/[6^n(4\kappa)^{2p-1}]$ we obtain

$$\int_{\mathbb{R}^n} f^q \le \frac{1}{2}\int_{\mathbb{R}^n} f^q + 2^{2p-\frac{1}{p}}6^n\int_{\mathbb{R}^n} g^q. \quad \square$$

The higher integrability assumption of Proposition 1.10 can be circumvented via a regularisation argument; the result of which yields the following:

Lemma 1.11. *Let f and g be non-negative functions of class $L^p(\mathbb{R}^n)$, for $p > 1$. Suppose for constants $\kappa \geq 1/4$ and $0 < \sigma < 1$ that*

$$\Big(\fint_{\sigma B} f^p \Big)^{\frac{1}{p}} \leq \kappa \fint_B f + \Big(\fint_B g^p \Big)^{\frac{1}{p}} \tag{1.7}$$

for all balls $B \subset \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} f^q \leq 2^{2p+1} 6^n \int_{\mathbb{R}^n} g^q$$

for $p < q \leq p + \frac{p-1}{6^n (4\kappa)^{2p-1}}$.

Proof. First observe that (1.7) is stable under mollification, namely:

Claim 1.12. If (1.7) holds the regularised $F = \rho_\varepsilon * f$ and $G = \rho_\varepsilon * g$ (where ρ_ε denotes the standard mollifier, cf. [GT, §7.2]) satisfy

$$\Big(\fint_{\sigma B} F^p \Big)^{\frac{1}{p}} \leq \kappa \fint_B F + \Big(\fint_B G^p \Big)^{\frac{1}{p}} \tag{1.8}$$

for all balls $B \subset \mathbb{R}^n$.

Proof of Claim. By Minkowski's inequality,

$$\begin{aligned}
\Big(\fint_{\sigma B} F^p \Big)^{\frac{1}{p}} &= \Big(\fint_{\sigma B} \Big| \int_{\mathbb{R}^n} \rho_\varepsilon(y) f(x-y) dy \Big|^p dx \Big)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^n} \rho_\varepsilon(y) \Big(\fint_{\sigma B} f(x-y)^p dx \Big)^{\frac{1}{p}} dy \\
&\leq \int_{\mathbb{R}^n} \rho_\varepsilon(y) \Big[\kappa \fint_B f(x-y) dx + \Big(\fint_B g(x-y)^p dx \Big)^{\frac{1}{p}} \Big] dy \\
&= \kappa \fint_B F + \int_{\mathbb{R}^n} \rho_\varepsilon(y) \Big(\fint_B g(x-y)^p dx \Big)^{\frac{1}{p}} dy \\
&\leq \kappa \fint_B F + \Big(\int_{\mathbb{R}^n} \rho_\varepsilon(y) \fint_B g(x-y)^p dx dy \Big)^{\frac{1}{p}} \\
&= \kappa \fint_B F + \Big(\fint_B G^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

As the mollified $F, G \in L^q(\mathbb{R}^n)$ one can apply the maximal function to (1.8) and use Proposition 1.10 to obtain

$$\int_{\mathbb{R}^n} F^q \leq 2^{2p+1} 6^n \int_{\mathbb{R}^n} G^q.$$

By applying Fatou's lemma to F and noting

$$\int_{\mathbb{R}^n} G^q \leq \int_{\mathbb{R}^n} \rho_\varepsilon(y) \Big(\int_{\mathbb{R}^n} g(x-y)^q dx \Big) dy \leq \int_{\mathbb{R}^n} g^q$$

the desired inequality follows. □

Proof of Proposition 1.6. First introduce the auxiliary functions

$$F(x) = \delta(x)f(x), \qquad G(x) = \delta(x)g(x) + \Big(\int_{\mathsf{B}} f^p\Big)^{\frac{1}{p}} \mathbf{1}_{\mathsf{B}}(x)$$

for $x \in \mathbb{R}^n$, where $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n \backslash \mathsf{B})^{\frac{n}{p}}$. Then $F, G \in L^p(\mathbb{R}^n)$ with $F = G = 0$ off the ball B. Moreover, for such F and G we claim that:

Claim 1.13.

$$\Big(\fint_{\sigma B} F^p\Big)^{\frac{1}{p}} \leq 2^{\frac{n}{p}} \kappa \fint_B F + \Big(\frac{8}{\omega_n{}^{\frac{1}{n}} \sigma(1-\sigma)}\Big)^{\frac{n}{p}} \Big(\fint_B G^p\Big)^{\frac{1}{p}} \tag{1.9}$$

for all balls $B \subset \mathbb{R}^n$, where $\omega_n = |B(0,1)|$ is the volume of the unit n-ball.

Proof of Claim. Without loss of generality let $\sigma B \cap \mathsf{B}$ be non-empty; otherwise the left-hand side is zero. With this we distinguish two cases:

`Case 1.` Suppose $3B \subset \mathsf{B}$.

Denote $\mathsf{B} = \mathsf{B}(y_0, R)$ and $B = B(y, r)$ from which one can rewrite the condition $3B \subset \mathsf{B}$ as the inequality $|y - y_0| + 3r < R$. Note also

$$\max_{\sigma B} \delta = (R + |y - y_0| + \sigma r)^{\frac{n}{p}}, \qquad \min_{B} \delta = (R - |y - y_0| - r)^{\frac{n}{p}}$$

so that

$$\min_B \delta = \Big(\frac{R - |y - y_0| - r}{R + |y - y_0| + \sigma r}\Big)^{\frac{n}{p}} \max_{\sigma B} \delta = \mu \min_B \delta.$$

It then follows that

$$\begin{aligned}\Big(\fint_{\sigma B} F^p\Big)^{\frac{1}{p}} \leq \max_{\sigma B} \delta \Big(\fint_{\sigma B} f^p\Big)^{\frac{1}{p}} &\leq \mu^{-1} \min_B \delta \Big[\kappa \fint_B f + \Big(\fint_B g^p\Big)^{\frac{1}{p}}\Big] \\ &\leq \mu^{-1}\kappa \fint_B F + \mu^{-1}\Big(\fint_B G^p\Big)^{\frac{1}{p}}\end{aligned}$$

and as $\frac{R-|y-y_0|}{r} > 3$ we also find that $\mu = \Big(\frac{\frac{R-|y-y_0|}{r} - 1}{\frac{R-|y-y_0|}{r} + \sigma}\Big)^{\frac{n}{p}} > (\frac{2}{3+\sigma})^{\frac{n}{p}} > 2^{-\frac{n}{p}}$.

`Case 2.` Suppose $3B \not\subset \mathsf{B}$.

As before, with $\mathsf{B} = \mathsf{B}(y_0, R)$ and $B = B(y, r)$, one can rewrite the conditions $3B \not\subset \mathsf{B}$ and $\sigma B \cap \mathsf{B} \neq \emptyset$ as the inequalities $3r + |y - y_0| > R$, $|y - y_0| < \sigma r + R$. As

$$\max_{\sigma B} \delta < \max_B \delta = (R + r - |y - y_0|)^{\frac{n}{p}} < (4r)^{\frac{n}{p}}$$

we have

$$\Big(\fint_{\sigma B} F^p\Big)^{\frac{1}{p}} \leq \Big(\max_B \delta^p \frac{1}{\sigma^n |B|} \int_{\mathsf{B}} f^p\Big)^{\frac{1}{p}} \leq \Big(\frac{4}{\omega_n{}^{\frac{1}{n}} \sigma}\Big)^{\frac{n}{p}} \Big(\int_{\mathsf{B}} f^p\Big)^{\frac{1}{p}}.$$

Furthermore by the definition of G we find

$$\Big(\fint_B G^p\Big)^{\frac{1}{p}} \geq \Big(\frac{|B \cap \mathsf{B}|}{|B|}\Big)^{\frac{1}{p}} \Big(\int_{\mathsf{B}} f^p\Big)^{\frac{1}{p}},$$

and as $|B \cap \mathsf{B}| > \omega_n \cdot (\frac{R+r-|x-x_0|}{2})^n$ if $B \not\subset \mathsf{B}$, we have $\frac{|B\cap\mathsf{B}|}{|B|} > (\frac{1-\sigma}{2})^n$. In which case

$$\Big(\fint_{\sigma B} F^p\Big)^{\frac{1}{p}} \leq \Big(\frac{8}{\omega_n{}^{\frac{1}{n}}\sigma(1-\sigma)}\Big)^{\frac{n}{p}}\Big(\fint_B G^p\Big)^{\frac{1}{p}}.$$

Having established (1.9) one can now apply Lemma 1.11 to get

$$\int_{\mathbb{R}^n} F^q \leq \frac{2^{2p+7n+1}\cdot 3^n}{8^{\frac{n}{p}}\big(\omega_n{}^{\frac{1}{n}}\sigma(1-\sigma)\big)^{\frac{nq}{p}}}\int_{\mathbb{R}^n} G^q$$

for $p < q \leq p + \frac{p-1}{24^n\cdot 2^{-n/p}(4\kappa)^{2p-1}}$. By the definition of F and G it follows that

$$\Big(\fint_{\mathsf{B}}(\delta f)^q\Big)^{\frac{1}{q}} \leq C\Big(\fint_{\mathsf{B}}(\delta g)^q\Big)^{\frac{1}{q}} + C\Big(\int_{\mathsf{B}} f^p\Big)^{\frac{1}{p}}$$

with $C = C(n,p,q,\sigma)$. Then as $\min_{\tau\mathsf{B}}\delta = (1-\tau)^{\frac{n}{p}}\big(\frac{|\mathsf{B}|}{\omega_n}\big)^{\frac{1}{p}}$ and $\max_{\mathsf{B}}\delta = \big(\frac{|\mathsf{B}|}{\omega_n}\big)^{\frac{1}{p}}$ for $0<\tau<1$ we finally conclude

$$\Big(\fint_{\tau\mathsf{B}} f^q\Big)^{\frac{1}{q}} \leq \frac{C}{(1-\tau)^{\frac{n}{p}}\tau^{\frac{n}{q}}}\Big[\Big(\fint_{\mathsf{B}} f^p\Big)^{\frac{1}{p}} + \Big(\fint_{\mathsf{B}} g^q\Big)^{\frac{1}{q}}\Big]. \qquad \square$$

1.4. Modulus of continuity

A *modulus of continuity* $\omega\colon [0,+\infty] \to [0,+\infty]$ measures *quantitatively* both the pointwise continuity of, and the uniform continuity of, maps between metric spaces. Formally, we we say:

Definition 1.14. A modulus of continuity is any continuous real-extended valued function ω that vanishes at 0, i.e. $\lim_{t\searrow 0}\omega(t) = \omega(0) = 0$.

If $f\colon X \to Y$ is a mapping between metric spaces then f is said to admit ω as a *modulus of continuity* if

$$d(f(x),f(y)) \leq \omega(d(x,y))$$

for all $x, y \in X$. Furthermore, if f admits a modulus of continuity as per Definition 1.14 then it also admits a modulus of continuity which is non-decreasing simply by defining $\omega^+(t) = \sup_{s\leq t}\omega(s)$ (which is of course $\omega \leq \omega^+$ and non-decreasing).

The conditions of infinitesimality at 0 of the modulus of continuity implies the uniform continuity of f. In fact:

Proposition 1.15. *A function f is uniformly continuous if and only if it admits a modulus of continuity.*

One also observes families of functions sharing the same modulus of continuity are exactly equicontinuous families. Indeed,

Proposition 1.16. *For a given family $(f_\lambda)_{\lambda\in\Lambda}$ of real-valued functions on a metric space X sharing a common modulus of continuity ω, the inferior envelope* $\inf f_\lambda$ *and the superior envelope* $\sup f_\lambda$ *are both real-valued functions with a modulus of continuity ω, provided they are of a finite valued at every point.*

For example the moduli $\omega(t) = kt$ describes the k-Lipschitz functions with Lipschitz constant k and the moduli $\omega(t) = kt^\alpha$ describes the Hölder continuity class with exponent α (cf. §1.5).

Proof. Fix $\varepsilon > 0$ and let $\underline{f} = \inf f_\lambda$. Then there exists is a λ such that $f_\lambda \le \underline{f} + \varepsilon$ so that

$$d(\underline{f}(x), \underline{f}(y)) \le d(\underline{f}(x), f_\lambda(x)) + d(f_\lambda(x), f_\lambda(y)) + d(f_\lambda(y), \underline{f}(y)) \le \omega(d(x,y)) + 2\varepsilon.$$

The claim follows as ε is arbitrary. Likewise for $\sup f_\lambda$. □

Lemma 1.17. *For a uniformly continuous function $f : X \to Y$ the minimal (or optimal) modulus of continuity*

$$\omega_f(t) = \sup\{d(f(x), f(y)) : x, y \in X,\ d(x,y) = t\}$$

is both a non-decreasing and subadditive if the domain is convex.

Proof. The first property is a direct consequence of $\omega^+(s) = \sup_{0\le t\le s} \omega_f(t) = \omega_f(s)$ by the definition of ω_f. For the second let $x' = \theta x + (1-\theta)y$ for $x, y \in X$ and $0 < \theta < 1$. Then

$$\begin{aligned}\omega_f(s+t) = \sup_{d(x,y)\le s+t} d(f(x), f(y)) &\le \sup_{\substack{d(x,x')\le s\\ d(x',y)\le t}} \Big[d(f(x), f(x')) + d(f(x'), f(y))\Big] \\ &\le \omega_f(t) + \omega_f(s).\end{aligned}$$

□

We say a *concave* modulus of continuity ϖ is the smallest concave function above a given modulus of continuity ω. Note that one does not know *a priori* if $\lim_{t\searrow 0} \varpi(t) = 0$ directly from this definition, i.e. if ϖ is well-defined. Nonetheless,

Lemma 1.18. *If a given modulus of continuity ω is both non-decreasing and subadditive, then there exists a concave modulus of continuity ϖ such that $\omega(s) \le \varpi(s) \le 2\omega(s)$ for all $s \ge 0$.*

Proof. For each s let t and integer k be such that $kt < s < (k+1)t$. By the non-decreasing and subadditivity properties of ω we have $\omega(s) \le \omega((k+1)t) \le (k+1)\omega(t) \le \omega(t) + s\frac{\omega(t)}{t}$. As $s \mapsto \omega(t) + s\frac{\omega(t)}{t}$ for fixed t is concave and above $\omega(s)$ it follows that $\varpi(s) \le \omega(t) + s\frac{\omega(t)}{t}$ by the minimality of ϖ. Setting $t = s$ yields the desired inequality which also shows $\lim_{t\searrow 0} \varpi(t) = 0$. □

Example 1.19. For a uniformly continuous function $f : X \to Y$ with a convex ball $B \subset X$ there exists $K > 0$ such that $\omega_f(d(x,y)) \leq K$ for all $x, y \in B$. For the rescaled $\omega_0 = K^{-1}\omega_f$ define

$$\omega_1(t) := \min\{\max\{\omega_0(t), t\}, 1\} \geq \omega_0(t).$$

It follows that ω_1 is both non-decreasing and subadditive (since ω_0 is by Lemma 1.17 and a quick check shows the min-max construction preserves these properties) in addition to the fact that $\omega_1(t) = 1$ for all $t \geq 1$. Then by Lemma 1.18 there exists a non-decreasing concave $\varpi : [0, +\infty) \to [0, 1]$ such that

$$d(f(x), f(y)) \leq \omega_f(d(x,y)) \leq K\varpi(d(x,y))$$

with $\lim_{t \searrow 0} \varpi(t) = 0$ and $\varpi(t) = 1$ for all $t \geq 1$.

1.5. Hölder continuous spaces

We say a continuous function u on open $\Omega \subset \mathbb{R}^n$ is Hölder continuous with exponent α if

$$[u]_{\alpha;\Omega} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \qquad 0 < \alpha \leq 1,$$

is finite. We say u is locally Hölder continuous if u is Hölder continuous on compact subsets of Ω. When $\alpha = 1$ the function u is said to be Lipschitz continuous. The notion of Hölder continuity can be seen as a natural enlargement of the differentiable spaces. In a certain sense it provides a quantitative measure of continuity which could also be viewed a type of fractional differentiability.

Example 1.20. The function $u(x) = x^\beta$ on $(0, 1)$, with $0 < \beta \leq 1$, is α-Hölder continuous for $0 < \alpha \leq \beta$ but not for $\alpha > \beta$.

The Hölder spaces $C^{k,\alpha}(\Omega)$, for non-negative integers k, are defined as the subspace of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are Hölder continuous. Likewise $C^{k,\alpha}_{\text{loc}}(\Omega)$ denotes the spaces consisting of functions whose k-th order partial derivatives are locally Hölder continuous. By setting

$$|D^k u|_{0;\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|, \qquad [D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}$$

we can define the norms

$$\|u\|_{C^k(\Omega)} = \sum_{j=0}^{k} |D^j u|_{0;\Omega}$$

$$\|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^k(\Omega)} + [D^k u]_{\alpha;\Omega}$$

on the spaces $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ respectively. With these norms $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ are Banach spaces. Note that $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$ is merely a metric space. By a notational abuse we also identify $C^{k,0}(\Omega) = C^k(\Omega)$.

For $0 \le \alpha \le \beta \le 1$ we have $C^k(\Omega) \supset C^{k,\alpha}(\Omega) \supset C^{k,\beta}(\Omega) \supset C^{k,1}(\Omega)$ and if in addition Ω is convex then $C^{k,1}(\Omega) \supset C^{k+1}(\Omega)$.

Example 1.21. For $\lambda \in (0,1]$ the function $u_\lambda(x) = \sum_{n=1}^\infty \frac{\cos(2^n x)}{2^{n\lambda}}$ is such that $u_\lambda \in C^{0,\alpha}(0,\pi)$ for any $0 < \alpha < \lambda$. Nonetheless u_λ is nowhere differentiable (cf. [H6]).

Example 1.22. The function $u(x) = -\frac{1}{\log|x|}$ if $x \neq 0$ with $u(0) = 0$ is continuous but $u \notin C^{0,\alpha}(-\frac{1}{2}, \frac{1}{2})$ for any $\alpha \in (0,1]$.

Hence the inclusions $C^0(\Omega) \supset C^{0,\alpha}(\Omega) \supset C^1(\Omega)$, for $0 < \alpha < 1$, are strict. When $\alpha = 1$ we recall:

Theorem 1.23 (Rademacher)**.** *If $u \in C^{0,1}(\Omega)$ then u is differentiable almost everywhere in Ω, i.e. the set of point at which u is not differentiable is a set of Lebesgue measure zero.*

Of fundamental importance is the following integral characterisation of Hölder continuity.

Theorem 1.24 (Campanato, [C7, teor. I.2])**.** *Let $u \in L^p(\Omega)$ for $1 \le p < \infty$ and $0 < \alpha \le 1$. Suppose there exists a constant $K > 0$ such that*

$$\Big(\fint_{B(x_0,r)} |u - (u)_{x_0,r}|^p \Big)^{\frac{1}{p}} \le K r^\alpha$$

for all $B(x_0,r) \subset \Omega$. Then the precise representative of u is in $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ and whenever $B(x_0,r) \subset \Omega$ we have

$$|u(x) - u(y)| \le cK|x-y|^\alpha$$

for every $x, y \in B(x_0, \frac{r}{4})$ and some $c = c(n,\alpha,p) > 0$.

Proof. We begin with the following estimate:

Claim 1.25. Suppose $B(x, \frac{R}{2}) \subset B(x_0,R) \subset \Omega$, then

$$|(u)_{x,\frac{R}{2}} - (u)_{x_0,R}| \le 2^{\frac{n}{p}} K R^\alpha.$$

Proof of claim. Note that $|(u)_{x,\frac{R}{2}} - (u)_{x_0,R}| \le \big(\fint_{B_{R/2}(x)} |u - (u)_{x_0,R}|^p \big)^{\frac{1}{p}} \le 2^{\frac{n}{p}} K R^\alpha$.

Let $x \in \Omega$ be a Lebesgue point of u (cf. Remark 1.5). We find that

$$|(u)_{x,2^{-j}r} - (u)_{x,r}| \le \sum_{i=1}^j |(u)_{x,2^{-i}r} - (u)_{x,2^{-i+1}r}| \le \frac{2^{\frac{n}{p}-\alpha}}{1 - 2^{-\alpha}} K r^\alpha$$

whenever $B(x,r) \subset \Omega$. So by sending $j \to \infty$ we conclude

$$|u(x) - (u)_{x,r}| \leq cKr^\alpha$$

for some $c = c(n, \alpha, p)$, where $u(x) = u^*(x)$ is the precise representative. Thus if $B(x, \frac{r}{2}) \subset B(x_0, r) \subset \Omega$ the above implies

$$\begin{aligned} |u(x) - (u)_{x_0,r}| &\leq |u(x) - (u)_{x,\frac{r}{2}}| + |(u)_{x,\frac{r}{2}} - (u)_{x_0,r}| \\ &\leq cK(\tfrac{r}{2})^\alpha + 2^{\frac{n}{p}} Kr^\alpha \leq cKr^\alpha. \end{aligned} \tag{1.10}$$

Now let $B(y,R) \subset \Omega$. If $x_1, x_2 \in B(y, \frac{R}{4})$ are Lebesgue points, set $\bar{x} = \frac{1}{2}(x_1 + x_2)$ and $r = |x_1 - x_2|$. In which case (1.10) implies $|u(x_1) - u(x_2)| \leq |u(x_1) - (u)_{\bar{x},r}| + |(u)_{\bar{x},r} - u(x_2)| \leq cKr^\alpha$. In other words,

$$|u(x_1) - u(x_2)| \leq cK|x_1 - x_2|^\alpha \tag{1.11}$$

whenever $x_1, x_2 \in B(y, \frac{R}{4})$ are Lebesgue points. In fact we claim that all points in $B(y, \frac{R}{4})$ are Lebesgue points: to see this fix any $z \in B(y, \frac{R}{4})$. Then there exists $s > r > 0$ such that $B(y,r) \subset B(z,s) \subset B(y, \frac{R}{4})$. Select a Lebesgue point $x' \in B(z, \frac{r}{2}) \subset B(z, \frac{s}{2})$. Then (1.10) implies that $|(u)_{z,s} - (u)_{z,r}| \leq |(u)_{z,s} - u(x')| + |u(x') - (u)_{z,r}| \leq cKr^\alpha$, so $\{(u)_{z,r}\}_{r>0}$ is in fact Cauchy. Therefore (1.11) implies $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$. □

1.6. Sobolev spaces

Let $u \in L^1_{\text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, and α any multi-index. A locally integrable function $v \in L^1_{\text{loc}}(\Omega)$ is said to be the α-th *weak derivative* of u if it satisfies

$$\int_\Omega \varphi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \, dx$$

for all $\varphi \in C^\infty_c(\Omega)$. For such a function we write $v = D^\alpha u$ and note that it is determined up to a set of measure zero. We say u is k times *weakly differentiable* if all its weak derivatives exist for orders up to and including k.

Remark 1.26. It is important to distinguish between weak and distributional derivatives. Any L^1-function possesses distributional derivatives of any order, however for weak derivatives this is not the case (e.g. the Heaviside step function has no weak derivative but its distributional derivative is the Dirac delta distribution).

Using the standard approximation argument by mollification one also has:

Theorem 1.27. *Let $u, v \in L^1_{\text{loc}}(\Omega)$. Then $v = D^\alpha u$ if and only if there exists a sequence $(u_j) \subset C^\infty(\Omega)$ that converge to u in $L^1_{\text{loc}}(\Omega)$ whose derivatives $D^\alpha u_j$ converge to v in $L^1_{\text{loc}}(\Omega)$.*

This equivalent characterisation of weak derivatives could also be taken as their definition. Many results from the classical differential calculus extend to weak derivatives via this approximation.

Lemma 1.28. *Let $f \in C^1(\mathbb{R})$ such that $f' \in L^\infty(\Omega)$. Then the composition $f \circ u$ of any weakly differentiable function u on Ω is weakly differentiable with $D(f \circ u) = f'(u)Du$.*

Definition 1.29. We denote the Sobolev space, for $p \geq 1$ and k non-negative integer, by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

where $D^\alpha u$ denote the weak derivative of u. The space is equipped with the norm

$$\|u\|_{k,p;\Omega} = \|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \Big)^{\frac{1}{p}}.$$

When there is no ambiguity we shall write $\|u\|_{k,p}$ for $\|u\|_{k,p;\Omega}$.[2]

Remark 1.30. Recall that a continuous function $f\colon [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{j=1}^k |f(I_j)| < \varepsilon$ whenever $I_1, \ldots, I_k$ are pairwise disjoint intervals in $[a,b]$ with $\sum_j^k |I_j| < \delta$. Absolutely continuous functions are differentiable almost everywhere. The space $ACL(\Omega)$ is defined to be that of measurable functions defined on a domain Ω which are absolutely continuous on $L \cap \Omega$ for almost every line L parallel to any of the coordinate axes. If we define $ACL^p(\Omega)$ to be the class $ACL(\Omega) \cap L^p(\Omega)$ then $ACL^p(\Omega) = W^{1,p}(\Omega)$ in the sense that any function in $ACL^p(\Omega)$ lies in an equivalence class in $W^{1,p}(\Omega)$ while every element of $W^{1,p}(\Omega)$ has a representative in $ACL^p(\Omega)$ (cf. [Z, Theor. 2.1.4]). The class of such functions goes back to the work of Beppo Levi and Tonelli (cf. §0.1.2).

One can verify $W^{k,p}(\Omega)$ is a Banach space.[3] The space $W^{k,p}_{\text{loc}}(\Omega)$ denotes the set of functions belonging to $W^{k,p}(\Omega')$ for $\Omega' \subset\subset \Omega$. We let $W^{k,p}_0(\Omega)$ denote the closure of $C^\infty_c(\Omega)$ in the $\|\cdot\|_{k,p}$-norm. We say $v \in u + W^{k,p}_0(\Omega)$ if $u, v \in W^{k,p}(\Omega)$ and $v - u \in W^{k,p}_0(\Omega)$. The spaces $W^{k,p}(\Omega)$, $W^{k,p}_0(\Omega)$ are separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. In the case $p = \infty$ the Sobolev and Lipschitz spaces are related: $W^{k,\infty}_{\text{loc}}(\Omega) = C^{k-1,1}_{\text{loc}}(\Omega)$ for arbitrary Ω and $W^{k,\infty}(\Omega) = C^{k-1,1}(\Omega)$ for Ω with, say, smooth boundary. We define $W^{k,p}(\Omega, \mathbb{R}^N)$ to be the set of vector valued functions $u = (u^1, \ldots, u^N)\colon \Omega \to \mathbb{R}^N$ with $u^i \in W^{k,p}(\Omega)$ for each $i = 1, \ldots, N$ and likewise for $W^{k,p}_0(\Omega, \mathbb{R}^N)$.

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[2] An equivalent the norm takes the form $\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}$.

[3] Note that the Sobolev spaces $W^{k,p}(\Omega)$ to a large extent resemble their Lebesgue counterparts $L^p(\Omega)$ with the additional benefit of being able to interact nicely with (weak) derivatives: the derivative Du of a function $u \in L^p$ could leave all Lebesgue spaces whereas the derivative Du of a function $u \in W^{k,p}$ will be in $W^{k-1,p}$.

Remark 1.31. The chain rule of Lemma 1.28 extends to the spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$. As a corollary elements of $W^{1,p}(\Omega)$ remain invariant under the operator of truncation: for if $u \in W^{1,p}(\Omega)$ then $u^+ = \max\{u, 0\} \in W^{1,p}(\Omega)$ with $Du^+ = Du$ if $u > 0$ with $Du^+ = 0$ if $u \leq 0$. Nonetheless this feature, in general, is no longer true for $W^{k,p}(\Omega)$ with $k \geq 2$. The note of [D3] shows that if $f \in C^\infty(\mathbb{R})$ and $f \circ u \in W^{k,p}(\mathbb{R}^n)$ for all $u \in W^{k,p}(\mathbb{R}^n)$, with $1 \leq p < \frac{n}{k}$ for $k \geq 3$ or $1 < p < \frac{n}{2}$ for $k = 2$, then $f \circ u = cu$ for some $c \in \mathbb{R}$.

A convolution and partition of unity argument shows that:

Theorem 1.32. *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$, for $1 \leq p < \infty$ and k a non-negative integer, is dense in $W^{k,p}(\Omega)$. Moreover, if Ω has, say, Lipschitz boundary then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

By apply fundamental theorem of calculus, together with density and completeness, there exists $C = C(n, p)$ such that for any $u \in W_0^{1,p}(\Omega)$

$$\begin{aligned} \|u\|_{\frac{np}{n-p}} &\leq C\|Du\|_p && \text{for } p < n \\ \sup_\Omega |u| &\leq C|\Omega|^{\frac{1}{n}-\frac{1}{p}}\|Du\|_p && \text{for } p > n. \end{aligned}$$

As a continuous embedding (i.e. bounded linear injective mapping) of Banach spaces one can show, by iterating k times, the well known *Sobolev embeddings*: $W_0^{k,p}(\Omega) \subset L^{\frac{np}{n-kp}}(\Omega)$ for $kp < n$; $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $kp = n$, $1 \leq q < \infty$; and $W_0^{k,p}(\Omega) \subset C^{k-\frac{n}{p}}(\Omega)$ for $kp > n$, where $C^{k-\frac{n}{p}}(\Omega) = C^{\ell,k-\frac{n}{p}-\ell}(\Omega)$ if $0 \leq \ell < k - \frac{n}{p} < \ell + 1$. In fact when $kp = n$ one embeds into the Orlicz spaces (cf. [T12]). The spaces $W_0^{k,p}(\Omega)$ can only be replaced with $W^{k,p}(\Omega)$ for certain domains Ω that, for instance, have Lipschitz boundary (see [D2, Def. 1.41] for details on the description of the boundary). For such domains one also has the continuous embedding $W^{k,p}(\Omega) \subset W^{\ell,q}(\Omega)$ for $1 \leq p < q \leq \infty$ and $k > \ell$ if $\ell > k - \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{k-\ell}{n}$. On the other hand if $0 \leq \ell < k - \frac{n}{p} < \ell + 1$ then $W^{k,p}(\Omega) \subset C^{\ell,\alpha}(\Omega)$ with $\alpha = k - \frac{n}{p} - \ell$. The results go back to the work of [S14, S15], [N10] and that of [M13, M15] in the Hölder case.

The Rellich-Kondrashov theorem states that these embeddings are also compact (i.e. the embedding operator between Banach spaces is compact) under further restrictions, namely $W^{k,p}(\Omega) \subset W^{\ell,q}(\Omega)$ is compactly embedded for $\frac{1}{q} > \frac{1}{p} - \frac{k-\ell}{n}$ if $\ell > k - \frac{n}{p}$ and $W^{k,p}(\Omega) \subset C^{\ell,\alpha}(\Omega)$ is compactly embedded for $\alpha < k - \frac{n}{p} - \ell$ if $0 \leq \ell < k - \frac{n}{p} < \ell + 1$. The theorem is originally due to [R1] when $p = 2$ and [K4] for the general case.

Often it suffices to apply the weaker *Poincaré inequalities* which for $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, takes the form

$$\|u\|_p \leq \operatorname{diam}\Omega \|Du\|_p \tag{1.12}$$

while for $u \in W^{1,p}(\Omega)$, with convex Ω, we have

$$\|u - (u)_\Omega\|_p \leq c \operatorname{diam}\Omega \|Du\|_p \tag{1.13}$$

for some $c = c(n,p)$. Note also the following variant which states

$$\|u - u^*\|_p \leq c \operatorname{diam}\Omega \|Du\|_p \tag{1.14}$$

for $u^* = \int_\Omega \varphi u$ with $\int_\Omega \varphi = 1$, $\varphi \in L^\infty(\Omega)$ (cf. [MZ, Lem. 1.50]). By combining the above one obtains for any $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, Lipschitz domain Ω, the *Sobelov-Poincaré inequality*

$$\|u - (u)_\Omega\|_p \leq c\|Du\|_{p_*} \tag{1.15}$$

for some $c = c(n,p,\Omega)$ with $p_* = \frac{np}{n+p} < p$. When $\Omega = B_R$ is a ball of radius R we can take $c(n,p,\Omega) = c(n,p)R$.

1.6.1. Traces. We examine how one can make sense of assigning boundary values to a function. If $u \in C(\bar{\Omega})$ then it defines a unique function $U\colon \partial\Omega \to \mathbb{R}$ such that $U(x) = u(x)$ for all $x \in \partial\Omega$. The map U is said to be the restriction of u to $\partial\Omega$ and henceforth is denoted by $u|_{\partial\Omega}$. For Sobolev functions, which are defined up to sets of measure zero, there is no direct meaning one can assign to the expression of u restricted to $\partial\Omega$. Nonetheless, by using density and completeness, one can obtain:

Theorem 1.33. *Let Ω be a Lipschitz domain. For $p \geq 1$ there exists a bounded linear operator*

$$\mathfrak{R}\colon W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{1.16}$$

such that $\|\mathfrak{R}\, u\|_{L^p(\partial\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)}$ for some $c = c(p,\Omega)$ and $\mathfrak{R}\, u = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Furthermore, for all $u \in W^{1,p}(\Omega)$ and $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\int_\Omega u \operatorname{div}\varphi \, dx = -\int_\Omega Du \cdot \varphi + \int_{\partial\Omega} (\varphi \cdot \nu)\, \mathfrak{R}\, u \, d\mathcal{H}^{n-1}$$

where ν denotes the unit outer normal to $\partial\Omega$.

The function $\mathfrak{R}\, u$ is said to be the *trace* of u on $\partial\Omega$ and is uniquely defined up to sets of $\mathcal{H}^{n-1}$-measure zero.

Corollary 1.34. *Let Ω be a Lipschitz domain and $p \geq 1$. Suppose $u \in W^{1,p}(\Omega)$ then $u \in W_0^{1,p}(\Omega)$ if and only if $\mathfrak{R}\, u = 0$ on $\partial\Omega$.*

Proof of the above theorem can be found in [EG4, pp. 133-5]; for the corollary one could consult [E10, pp. 259-60].

The following example show the trace, as an operator $\mathfrak{R}\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, not to be surjective.

Example 1.35 (Hadamard, [H1]). Let $\Omega = B(0,1)$ be the unit disk in $\mathbb{R}^2$. Consider a function f defined on $\partial\Omega$ which corresponds to the Fourier series

$$f(\theta) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos n\theta + b_n \sin n\theta).$$

A harmonic function u in Ω equalling f on $\partial\Omega$ is given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n\geq 1} r^n(a_n \cos n\theta + b_n \sin n\theta) = \frac{c_0}{2} + \sum_{n\geq 1} c_n r^n \cos(n\theta + \alpha_n)$$

with $a_n = c_n \cos\alpha_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos\theta\, d\theta$ and $b_n = c_n \sin\alpha_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin\theta\, d\theta$. Using polar coordinates and Parseval's formula yields $\int_\Omega |Du|^2 = \pi\sum_{n\geq 0} n c_n^2$. Thus it is enough, for example, to take $c_n = 2^{-j}$ for $n = 2^{2j}$ with $c_n = 0$ for $n \neq 2^{2j}$ so that $u \notin W^{1,2}(B(0,1))$. On the other hand $u = f \in L^p(\partial\Omega)$ for all $p \geq 1$, since the uniformly converging series on $\bar{\Omega}$ defines a continuous function on this set.[4]

To determine the image of the trace operator let us introduce, for $s > 0$ and $p \geq 1$, the Sobolev-Slobodeckij space

$$W^{s,p}(\Omega) = \{u \in W^{\lfloor s\rfloor,p}(\Omega) : \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{\frac{n}{p}+(s-\lfloor s\rfloor)}} \in L^p(\Omega\times\Omega) \text{ for all } |\alpha| = \lfloor s\rfloor\}$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\sum_{|\alpha|\leq\lfloor s\rfloor}\int_\Omega |D^\alpha u|^p dx + \sum_{|\alpha|=\lfloor s\rfloor}\int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p(s-\lfloor s\rfloor)}} dxdy\Big)^{\frac{1}{p}}.$$

By utilising this norm, together with Hardy's inequality, the result of Gagliardo [G1] characterises the trace as follows.

Theorem 1.36 ([G1])**.** *Let Ω be Lipschitz domain and $p > 1$. Then there exists a unique continuous linear mapping $\mathfrak{R}\colon W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ such that $\mathfrak{R}\, u = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega)\cap C(\bar{\Omega})$. Moreover, there exists a continuous linear mapping $T\colon W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ such that if $v \in W^{1-\frac{1}{p},p}(\partial\Omega)$, and $Tv = u$, we have $v = \mathfrak{R}\, u$ on $\partial\Omega$.*

Now if Ω is a domain with $C^{k-2,1}$ boundary for integers $k \geq 2$ and $p \geq 1$ then there also exists a unique continuous linear mapping

$$\mathfrak{R}_k\colon W^{k,p}(\Omega) \to W^{k-1,p}(\partial\Omega)$$

[4]Hadamard's example is a response to Hilbert's work and shows it is not possible to extend any boundary function to the whole of the domain in such a way the Dirichlet energy is finite for the extension. It is clear that this property is necessary for the applicability of of Dirichlet's principle. Similar observations can also be found in [P5].

such that $\mathfrak{R}_k\, u = u|_{\partial\Omega}$ for all $u \in W^{k,p}(\Omega) \cap C^{k-1}(\bar{\Omega})$ and if $\mathfrak{R}$ denotes the map from (1.16) then we also have $D^\alpha(\mathfrak{R}_k\, u) = \mathfrak{R}(D^\alpha u)$ a.e. on $\partial\Omega$ for all $u \in W^{k,p}(\Omega)$ and every multi-index α for which $|\alpha| \leq k-1$ (cf. [KJF, p. 329], [N3, Rem. 4.6, p. 90]).

If Ω is a domain with Lipschitz boundary then the (outward) unit normal ν to $\partial\Omega$ exists almost everywhere on $\partial\Omega$. For $u \in W^{k,p}(\Omega) \cap C^{k-1}(\bar{\Omega})$ we define the exterior normal derivatives of order $\ell \leq k-1$ to be

$$\frac{\partial^\ell u}{\partial \nu^\ell} = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \nu^\alpha D^\alpha u \quad \text{a.e.} \quad \partial\Omega.$$

When $k = 1$ we simply have $\frac{\partial u}{\partial \nu} = \nu \cdot Du$. With this established the work of Nečas [N3] generalises Theorem 1.36 as follows.

Theorem 1.37 ([N3, p. 99])**.** *Let $p > 1$ and k be a positive integer. Suppose Ω is a domain with $C^{k-1,1}$ boundary. Then there exists a continuous linear mapping*

$$\mathfrak{R}_k \colon W^{k,p}(\Omega) \to W^{k-\frac{1}{p},p}(\partial\Omega) \times W^{k-1-\frac{1}{p},p}(\partial\Omega) \times \cdots \times W^{1-\frac{1}{p},p}(\partial\Omega)$$

such that

$$\mathfrak{R}_k\, u = \Big(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}, \ldots, \frac{\partial^{k-1} u}{\partial \nu^{k-1}} \Big)$$

for all $u \in W^{k,p}(\Omega) \cap C^{k-1}(\bar{\Omega})$.

Theorem 1.38 ([N3, p. 104])**.** *Let $p > 1$ and k be a positive integer. Suppose Ω is a domain with $C^{k,1}$ boundary. Then there exists a continuous linear mapping*

$$T_k \colon W^{k-\frac{1}{p},p}(\partial\Omega) \times W^{k-1-\frac{1}{p},p}(\partial\Omega) \times \cdots \times W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{k,p}(\Omega)$$

such that if $(u_0, u_1, \ldots, u_{k-1}) \in \prod_{\ell=0}^{k-1} W^{k-\ell-\frac{1}{p},p}(\partial\Omega)$ with $T_k(u_0, u_1, \ldots, u_{k-1}) = v$ then

$$v = \mathfrak{R}_k\, u \quad \text{on} \quad \partial\Omega$$

for $\ell = 0, 1, \ldots, k-1$.

Further trace results for the Sobolev-Slobodeckij spaces $W^{s,p}(\Omega)$ can be found in [M1] when $s - \frac{1}{p}$ is not an integer and in [K1] when $s - \frac{1}{p}$ is an integer.

Remark 1.39. In the higher order case there is a degree of subtlety with regard to the regularity of the boundary $\partial\Omega$. When $k = 1$ the result of Theroem 1.38 does not agree with the second part of Theorem 1.36 as the boundary needs to be $C^{1,1}$ instead of $C^{0,1}$. Similarly, the result of [N3, Theor. 4.13] states that if Ω is a domain with $C^{k,1}$ boundary and $u \in W^{k,p}(\Omega)$ with $\mathfrak{R}_k\, u = 0$ on $\partial\Omega$ then $u \in W_0^{k,p}(\Omega)$. However when $k = 2$ this statement goes through for Ω with only $C^{0,1}$ boundary (cf. [N3, Theor. 4.12]).

1.6.2. Difference quotients. When applying direct methods to multiple integral variational problems one considers admissible class of functions which *a priori* might only have weak or distributional derivatives. Often solutions obtained by direct means satisfy only a very weakened form of the Euler-Lagrange equations. A simple technique for establishing further differentiability properties, particularly in the context of linear PDE, is to consider the difference quotients of the function in question: i.e. we define, for a given function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, the difference quotient

$$\Delta^h u(x) = u(x+h) - u(x) \tag{1.17}$$

for all $x \in \Omega_h = \{y \in \Omega : y + h \in \Omega\}$ with $h \neq 0$. The method originated in the work of [S10] and [N9].

The following results relates to the application of difference quotients in the context of Sobolev spaces.

Lemma 1.40. *Let $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$. Then $\Delta^h u \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ satisfying $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ and we have*

$$\|\Delta^h u\|_{L^p(\Omega')} \leq |h| \|Du\|_{L^p(\Omega)}.$$

Proof. First assume $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$. Then $\Delta^h u(x) = \int_0^1 Du(x+th)\cdot h\, dt$ so that

$$\int_{\Omega'} |\Delta^h u|^p \leq \int_{\Omega'} \int_0^1 |Du(x+th)\cdot h|^p dt dx \leq |h|^p \int_{\Omega} |Du|^p.$$

The extension to functions in $W^{1,p}$ follows by Theorem 1.32. □

Lemma 1.41. *Let $u \in L^p(\Omega)$ for $1 < p < \infty$. If there exists a constant K such that $\Delta^h u \in L^p(\Omega')$ and*

$$\|\Delta^h u\|_{L^p(\Omega')} \leq K|h|$$

for all $|h| > 0$ and $\Omega' \subset\subset \Omega$ satisfying $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, then $u \in W^{1,p}(\Omega')$ with

$$\|Du\|_{L^p(\Omega')} \leq K$$

and $\Delta^h u \to Du$ as $h \to 0$ in $L^p(\Omega')$.

Proof. Suppose $\|\Delta^h u\|_{L^p(\Omega')} \leq K$ holds for all $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. For any $\phi \in C_c^1(\Omega')$ we have

$$\int_{\Omega'} u\, \Delta^h \phi = -\int_{\Omega'} \Delta^{-h} u\, \phi.$$

By reflexivity of L^p there exists $v \in L^p(\Omega')$ such that $\Delta^{-h_j} u \rightharpoonup v$ in $L^p(\Omega')$ as $h_j \to 0$. Thus

$$\int_{\Omega'} u\, D\phi = \lim_{h_j \to 0} \int_{\Omega'} u\, \Delta^{h_j} \phi = -\lim_{h_j \to 0} \int_{\Omega'} \Delta^{-h_j} u\, \phi = -\int_{\Omega'} v\, \phi$$

so $v = Du$ in the sense of distributions. Furthermore $\|Du\|_p \leq \liminf \|\Delta^{-h_j} u\|_p \leq K$. To see that $\Delta^h u \to Du$ is strong in $L^p(\Omega')$ take any $\varphi \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ so that $\Delta^h u - Du = \Delta^h(u-\varphi) + \Delta^h\varphi - D\varphi + D(\varphi - u)$. Then by Lemma 1.40 we have $\|\Delta^h u - Du\|_{L^p(\Omega')} \leq \|\Delta^h\varphi - D\varphi\|_{L^p(\Omega')} + c\|D(\varphi-u)\|_{L^p(\Omega)}$ where $c = c(n,p)$. By Theorem 1.32 the second term on the right-hand side can be made arbitrarily small while the first term goes to 0 as $h \to 0$ since $\Delta^h\varphi \to D\varphi$ on compact sets. □

1.7. Weyl's lemma

A symmetric bilinear form $\mathbb{A}$ on $\mathbb{R}^{Nn}$ is said to be (strongly) *elliptic in the sense of Legendre-Hadamard*, or simply elliptic, if for all $\eta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^n$ there holds

$$\mathbb{A}(\eta \otimes \xi, \eta \otimes \xi) \geq \ell |\eta|^2 |\xi|^2 \tag{1.18}$$

for some $\ell > 0$. We call ℓ an ellipticity constant associated to $\mathbb{A}$. Note that $\mathbb{A}$ is bounded (in the sense that $\mathbb{A}(\zeta_1, \zeta_2) \leq L|\zeta_1||\zeta_2|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^{Nn}$) for free by finite dimensionality. A classical result, proved by use of the Fourier transformation, shows (1.18) to be equivalent to the strong ellipticity condition

$$\int_\Omega \mathbb{A}(D\phi, D\phi) \geq \ell \int_\Omega |D\phi|^2 \tag{1.19}$$

for all $\phi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ (cf. [G10, Lem. 5.1] and [D1, Theor. 5.3]).

Definition 1.42. We say $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is an $\mathbb{A}$-*harmonic map* if $\mathbb{A}$ satisfies the strong ellipticity condition (1.19) and

$$\int_\Omega \mathbb{A}(Du, D\phi) = 0$$

for all $\phi \in C_c^1(\Omega, \mathbb{R}^N)$.

The great advantage of $\mathbb{A}$-harmonic maps is that they enjoy full regularity as attested by the following *a priori* estimates.

Lemma 1.43 (Weyl)**.** *Let $v \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that*

$$\int_\Omega \mathbb{A}(Dv, D\phi) = 0$$

for any $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, where $\mathbb{A}$ satisfies the strong ellipticity condition with elliptic constant ℓ and upper bound L. Then $v \in C^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and for any $B_R \subset\subset \Omega$ we have

$$\sup_{B_{R/2}} |D^k v| \leq \frac{c}{R^{k-1}} \fint_{B_R} |Dv|$$

for some $c = c(n, N, k, \frac{L}{\ell})$ and any integer $k \geq 1$.

This semi-classical result goes back to work of Weyl [W5] and is closely related to the mean value property of harmonic functions (cf. [M20, §2.3]). We give a proof based on [CFM, pp. 150-1] using the difference quotient method of §1.6.2.

Proof. Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of

$$D_i(\mathbb{A}^{ij}_{\alpha\beta} D_j u^\beta) = 0, \qquad \alpha = 1, \dots, N. \tag{1.20}$$

Take a function $\eta \in W_0^{1,\infty}(B_{(1+\delta)R})$, $0 < \delta \le \frac{1}{3}$, satisfying $0 \le \eta \le 1$. For $0 < |h| \le \delta R$ set $\phi = \Delta^{-h}(\eta^2 \Delta^h u)$. As $\phi \in W_0^{1,2}(B_{(1+2\delta)R})$ it follows that

$$0 = \int_\Omega \mathbb{A}(Du, D\phi) = \int_\Omega \mathbb{A}(\Delta^h Du, \eta^2 \Delta^h Du + \Delta^h u \otimes D(\eta^2)) = I + II.$$

Using Cauchy's inequality and the strong ellipticity condition we bound

$$\begin{aligned} I &= \int_\Omega \mathbb{A}(D(\eta \Delta^h u), D(\eta \Delta^h u)) \\ &\quad - \int_\Omega 2\mathbb{A}(\eta \Delta^h Du, \Delta^h u \otimes D\eta) + \mathbb{A}(\Delta^h u \otimes D\eta, \Delta^h u \otimes D\eta) \\ &\ge \frac{\ell}{2} \int_\Omega |\eta \Delta^h Du|^2 - (L + 8(1 + \tfrac{L}{\ell})^2 \ell) \int_\Omega |\Delta^h u \otimes D\eta|^2 \end{aligned}$$

and

$$II = 2 \int_\Omega \mathbb{A}(\eta \Delta^h Du, \Delta^h u \otimes D\eta) \ge -\frac{\ell}{4} \int_\Omega |\eta \Delta^h Du|^2 - \frac{16 L^2}{\ell} \int_\Omega |\Delta^h u \otimes D\eta|^2.$$

Since $I + II = 0$ we find that

$$\int_\Omega |\eta \Delta^h Du|^2 \le c \int_\Omega |\Delta^h u \otimes D\eta|^2$$

for some $c = c(\frac{L}{\ell}) > 0$. The function η may now be chosen as a cut-off function such that $\eta = 1$ on B_R with $\eta = 0$ off $B_{(1+\delta)R}$ and $|D\eta| \le \frac{2}{\delta R}$. It then follows that $\|\Delta^h Du\|_{L^2(B_R)} \le c|h|(\delta R)^{-1} \|Du\|_{L^2(B_{(1+2\delta)R})}$. By Lemma 1.41 we obtain $u \in W^{2,2}(B_R, \mathbb{R}^N)$ with $\int_{B_R} |D^2 u|^2 \le \frac{c}{(\delta R)^2} \int_{B_{(1+2\delta)R}} |Du|^2$. Since higher order derivatives of u are also solutions of (1.20), an iteration argument combined with Caccioppoli's inequality (0.3) yields

$$\int_{B_R} |D^k u|^2 \le \frac{c}{(\delta R)^{2k}} \int_{B_{(1+3\delta)R}} |u|^2 \tag{1.21}$$

for integers $k \ge 1$. In which case $u \in W^{k+1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all integers $k \ge 1$.

By the Sobolev embeddings [M20, Theor. 3.5.1] we have

$$\sup_{B_R} |u| \le c(n, N, k) \sum_{j=0}^{k} R^j \Big(\fint_{B_R} |D^j u|^2 \Big)^{\frac{1}{2}}$$

provided $k > \frac{n}{2}$. So if we take $k = n$ it follows from (1.21) that

$$\sup_{B_R} |u| \le \frac{c}{\delta^n R^{\frac{n}{2}}} \Big(\int_{B_{(1+3\delta)R}} |u|^2 \Big)^{\frac{1}{2}} \le \frac{c}{\delta^n R^{\frac{n}{2}}} \Big(\sup_{B_{(1+3\delta)R}} |u| \Big)^{\frac{1}{2}} \Big(\int_{B_{(1+3\delta)R}} |u| \Big)^{\frac{1}{2}}.$$

Using Cauchy's inequality we obtain

$$\sup_{B_R} |u| \le \frac{1}{2} \sup_{B_{(1+3\delta)R}} |u| + \frac{c}{\delta^{2n} R^n} \int_{B_{2R}} |u|.$$

Then by appealing to Lemma 0.7 we conclude

$$\sup_{B_R} |u| \le c \fint_{B_{2R}} |u|.$$

Applying this estimate to the derivatives $D^k u$ yields the desired inequality.

Now let $v \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that

$$\int_\Omega \mathbb{A}(Dv, D\phi) = 0$$

for all $\phi \in C_c^1(\Omega, \mathbb{R}^N)$. If ρ denotes a mollifier, the regularisation of v defined by the convolution $v_\varepsilon(x) = \int_{|y|<1} \rho(z) u(x + \varepsilon y) dy$ for $x \in \Omega_\varepsilon = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) < \varepsilon\}$ can readily be shown to satisfy

$$\int_{\Omega_\varepsilon} \mathbb{A}(Dv_\varepsilon, D\varphi) = 0$$

for all $\varphi \in C_c^1(\Omega_\varepsilon, \mathbb{R}^N)$. Then from the above analysis it follows that

$$\sup_{B_R} |D^k v_\varepsilon| \le \frac{c}{R^{k-1}} \fint_{B_{2R}} |Dv_\varepsilon|$$

for any $B_{2R} \subset \Omega_\varepsilon$ and integers $k \ge 1$. Since v_ε converges to v in L^1_{loc} the desired conclusion follows. □

Remark 1.44. As $|Dv(y) - (Dv)_r| \le 2r \sup_{B_r} |D^2 v|$ for $y \in B_r(x)$ and $0 < r < R$, the above estimates imply that $|Dv(y) - (Dv)_r| \le C \fint_{B_{2R}} |\frac{r}{R} Dv|$. Now if one replaces v by $\bar{v} = v - a$, for some affine function a, we have $D\bar{v} - (D\bar{v})_r = Dv - (Dv)_r$ and if $Da = (Dv)_{2R}$ we get the interior estimate

$$\fint_{B_r} |Dv - (Dv)_r|^2 \le C \Big(\frac{r}{R} \Big)^2 \fint_{B_{2R}} |Dv - (Dv)_{2R}|^2.$$

Furthermore if one prescribes boundary data $f \in W^{1,2}$, the strong ellipticity condition (1.19) yields

$$\ell \int_\Omega |Df - Dv|^2 \le \int_\Omega \mathbb{A}(Df, Df - Dv),$$

since $f - v \in W^{1,2}(\Omega, \mathbb{R}^N)$. As $\mathbb{A}$ is bounded, Hölder's inequality then implies the boundary estimate $\|Dv\|_{2;\Omega} \le c \|Df\|_{2;\Omega}$ for some $c = c(n, N, \frac{L}{\ell})$.

Let us also recall the following L^p-result of Campanato and Stampacchia [CS] (who in fact consider a more general case with Hölder continuous coefficients).

Lemma 1.45. *For $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ and $f \in L^p(\Omega, \mathbb{R}^{Nn})$, $p > 1$, the Dirichlet problem*

$$-D_i(\mathbb{A}^{ij}_{ab} D_j v^b) + D_i f^i_a = 0 \tag{1.22}$$

for $v \in g + W^{1,p}_0(\Omega, \mathbb{R}^N)$ has a unique weak solution such that

$$\|Dv\|_{L^p(\Omega,\mathbb{R}^{Nn})} \le c\big(\|Dg\|_{L^p(\Omega,\mathbb{R}^{Nn})} + \|f\|_{L^p(\Omega,\mathbb{R}^{Nn})}\big)$$

for some c is independent of v, f and g, provided Ω is smooth (e.g. $C^{1,1}$) bounded open set in $\mathbb{R}^n$, $n \ge 2$, and $\mathbb{A}$ is a real symmetric strongly elliptic bilinear form of constant coefficients.

Sketch of proof. Replacing v by $v - g$ and f^i_a by $f^i_a + \mathbb{A}^{ij}_{ab} D_j g^b$ reduces the problem to the inhomogeneous case with vanishing boundary data. Now consider the operator $T\colon f \mapsto Du$. When $p = 2$ the existence of v follows by the direct method in the calculus of variations where the coercivity condition is provided by (1.19); it also implies that $T\colon L^2 \to L^2$ is bounded. For $p \ge 2$ a result of Campanato shows that $\|Dv\|_{\mathscr{L}^{2,n}(\Omega,\mathbb{R}^{Nn})} \le c\|f\|_{\mathscr{L}^{2,n}(\Omega,\mathbb{R}^{Nn})}$ whenever $f \in \mathscr{L}^{2,n}$ (cf. [C9, p. 364], [G6, p. 84]). From this it follows by an interpolation theorem of Stampacchia that $T\colon L^p \to L^p$ is bounded for $p \ge 2$ (cf. [S18], [G6, p. 75]). The uniqueness of v (modulo constant) follows easily from the $p = 2$ case. A duality argument ends the proof for $1 < p < 2$ as the system is real and symmetric. □

1.8. Submanifolds, projections and the distance function

Let $\mathcal{M}$ be an $(N-k)$-dimensional embedded submanifold of a Riemannian manifold (X^N, g) without boundary and $\boldsymbol{i}\colon \mathcal{M} \to X$ be an embedding. Denote by ∇ the Levi-Civita connection of X with respect to g and for any vector field v at $\boldsymbol{i}(x)$ write the components of ∇v as $v^a{}_{;b}$ with respect to a local frame (ξ_a) for TX in a neighbourhood of $\boldsymbol{i}(x)$.

The image of $T_x\mathcal{M}$ in $T_{\boldsymbol{i}(x)}X$ under the differential $\boldsymbol{i}_*$ is an $(N-k)$-dimensional plane through the origin. Thus there exist non-zero 1-forms $n^{\mu|} \in T^*_{\boldsymbol{i}(x)}X$ for $\mu = 1, \dots, k$ such that $n^{\mu|}(\boldsymbol{i}_*(v)) = 0$ for all $v \in T_x\mathcal{M}$. The forms $n^{\mu|}$ are unique up to sign and normalisation factor; here we will assume without loss of generality that they are orthonormal, i.e. $\mathsf{g}^{ab} n^{\mu|}{}_a n^{\nu|}{}_b = \delta^{\mu\nu}$. Now define the symmetric 2-tensor

$$P_{ab} = \mathsf{g}_{ab} - \sum_{\mu=1}^{k} n^{\mu|}{}_a n^{\mu|}{}_b. \tag{1.23}$$

The tensor $P^a{}_b = \mathsf{g}^{ac}P_{cb}$ is a projection operator, i.e. $P^a{}_bP^b{}_c = P^a{}_c$. It projects vectors $v \in T_{\boldsymbol{i}(x)}X$ onto the subspace $\boldsymbol{i}_*(T_x\mathcal{M})$ of $T_{\boldsymbol{i}(x)}X$ tangent to $\boldsymbol{i}(\mathcal{M})$ so that

$$v^a = P^a{}_bv^b + \sum_{\mu=1}^{k} n^{\mu|a}n^{\mu|}{}_bv^b,$$

where the second term represents the part of v orthogonal to $\boldsymbol{i}(\mathcal{M})$. Similarly, one can extend $P^a{}_b$ in the usual way by taking duals and tensor products to project an (r,s)-tensor T at $\boldsymbol{i}(x)$ onto $\otimes^r H^*_{\boldsymbol{i}(x)} \otimes^s H_{\boldsymbol{i}(x)}$, where $H = \boldsymbol{i}_*(T\mathcal{M})$, i.e. the part orthogonal to $\operatorname{span}\{n^{\mu|}\}$ on all indices.

If $\bar{n}^{\mu|}$ is any extension of the unit normal $n^{\mu|}$ onto an open neighbourhood of $\boldsymbol{i}(\mathcal{M})$, we call the tensor A defined on $\boldsymbol{i}(\mathcal{M})$ by

$$A_{abc} = \sum_{\mu=1}^{k} P^d{}_aP^e{}_b\bar{n}^{\mu|}{}_{d;e}\bar{n}^{\mu|}{}_c \tag{1.24}$$

the *second fundamental form* of $\mathcal{M}$ (cf. [HE, §2.7]). We remark that the definition is independent of the extension, since the projection $P^a{}_b$ restricts the covariant derivatives to directions tangent to $\boldsymbol{i}(\mathcal{M})$. As $\nabla\mathsf{g} = 0$ and $P^d{}_a\bar{n}^{\mu|}{}_d = 0$ it follows from (1.23) that the expression is equivalent to

$$A_{abc} = -P^d{}_aP^e{}_bP_{cd;e}. \tag{1.25}$$

In expressing $\boldsymbol{i}(\mathcal{M})$ locally by the equations $\{f^\mu = 0\}_{\mu=1}^k$, where $df^\mu \neq 0$, the fields $\bar{n}^{\mu|} = df^\mu$ locally so that $f^\mu{}_{;ab} = f^\mu{}_{;ba}$ and $f^\mu{}_{;a}P^a{}_b = 0$; thus A_{abc} is symmetric in the first two indices.

1.8.1. Tubular neigbourhoods. For an embedded submanifold $\mathcal{M}$ in $\mathbb{R}^N$ without boundary, and its normal bundle $\pi\colon (T\mathcal{M})^\perp \to \mathcal{M}$, define the 'endpoint map' $E\colon (T\mathcal{M})^\perp \to \mathbb{R}^N$ by sending $(x,v) \mapsto x + v$. A *tubular neighbourhood* of $\mathcal{M}$ is an open neighbourhood $\mathcal{O}$ of $\mathcal{M}$ in $\mathbb{R}^N$ that is the diffeomorphic image under the map E of an open neighbourhood $\mathscr{O} \subset (T\mathcal{M})^\perp$ of the zero section $0\colon \mathcal{M} \to (T\mathcal{M})^\perp$. As the differential E_* is surjective along this zero section, E is a local diffeomorphism by the inverse function theorem. One then has the following existence result:

Theorem 1.46. *Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^N$ without boundary, then there exists a tubular neighbourhood $\mathcal{O}$ of $\mathcal{M}$ in $\mathbb{R}^N$.*

By identifying $\mathcal{M}$ with its image in $\mathbb{R}^N$ we say that a neighbourhood $\mathcal{O}$ of $\mathcal{M}$ has the *unique nearest point property* if for every $x \in \mathcal{O}$ there is a unique point $\Pi(x) \in \mathcal{M}$ such that

$$\operatorname{dist}(x,\mathcal{M}) = |x - \Pi(x)|.$$

The map

$$\Pi\colon \mathcal{O} \to \mathcal{M} \tag{1.26}$$

is called the *retraction* onto $\mathcal{M}$.

Lemma 1.47. *Let $\mathcal{M}$ be a C^r-submanifold of $\mathbb{R}^N$ without boundary for $r \geq 2$. Then there exists a neighbourhood $\mathcal{O}$ of $\mathcal{M}$ with the unique nearest point property and the retraction $\Pi\colon \mathcal{O} \to \mathcal{M}$ is C^{r-1}.*

Proof. The existence of $\mathcal{O}$ follows from Theorem 1.46. As the normal bundle $\pi\colon (T\mathcal{M})^\perp \to \mathcal{M}$ is C^{r-1} it follows that E is a C^{r-1}-diffeomorphism. Hence the map $\Pi = \pi \circ E^{-1}$ is C^{r-1}. □

A *focal point* y at $x \in \mathcal{M}$ with multiplicity ν occurs if $y = E(x, v)$, where v is normal to $\mathcal{M}$ at x, and the differential $E_*|_{(x,v)}$ is a critical point with nullity $\nu > 0$; they are points where nearby normals intersect. By [M6, Lem. 6.3] these focal points are precisely of the form $x + \kappa_i^{-1} v$, where κ_i are the principal curvatures of $\mathcal{M}$ at x in the unit normal direction v.

Remark 1.48. As $W^{r-1,\infty}_{\mathrm{loc}}(\mathcal{O}, \mathbb{R}^N) = C^{r-2,1}(\mathcal{O}, \mathbb{R}^N)$ are the locally Lipschitz functions, it follows for each compact set K there exists $C = C(K)$ such that

$$|\nabla^{(j)}\Pi(x) - \nabla^{(j)}\Pi(y)| \leq C|x - y|$$

for all $x, y \in K$ where $j = 0, \ldots, r-2$ (cf. [EG4, §4.2.3]). For instance one could take K to be the closure of the tubular neighbourhood $\{x + v : x \in \mathcal{M}, v \in (T_x\mathcal{M})^\perp, |v| < \varepsilon\}$, for $\varepsilon > 0$ sufficiently small, so as to remain 'strictly away from focal points'. Furthermore, each $\nabla^{(j)}\Pi$ are uniformly bounded and uniformly continuous on compact sets for $j = 0, \ldots, r-1$.

Note that the boundary of a manifold cannot have a tubular neighbourhood in the above sense. However, there is a retraction of an open neighbourhood onto $\partial\mathcal{M}$. This is obviously possible locally, and two local retractions into coordinate domains can be glued together with a bump function.

1.8.2. Sets of positive reach and the regularity of the squared distance function. For a subset $A \subset \mathbb{R}^N$ let *distance function* $\delta\colon \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\delta(x) = \operatorname{dist}(x, A) = \inf\{|x - y| : y \in A\}.$$

Furthermore, let $\operatorname{Unp}(A)$ be the set of all point $x \in \mathbb{R}^N$ for which there exists a unique point of A nearest to x and, as in the previous section, let the map $\Pi\colon \operatorname{Unp}(A) \to A$ associate to $x \in \operatorname{Unp}(A)$ the unique $a \in A$ such that $\delta(x) = |x - a|$.

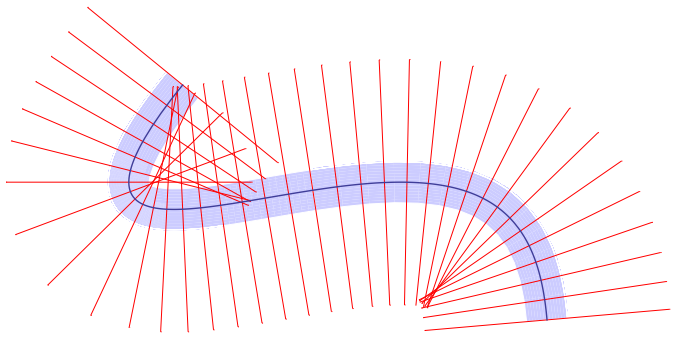


Figure 1.1. A manifold represented as a curve with normal lines and a shaded tubular neighbourhood.

If $a \in A$ then $\operatorname{reach}(A, a)$ is the supremum of the set of all numbers r for which $\{x : |x - a| < r\} \subset \operatorname{Unp}(A)$. We then define

$$\operatorname{reach}(A) = \inf\{\operatorname{reach}(A, a) : a \in A\}.$$

If $\operatorname{reach}(A) > 0$ then it is said that A has *positive reach*. A well known characterisation of convexity (going back to independent results by Bunt, Kritikos and Motzkin) states that a subset $A \subset \mathbb{R}^N$ has $\operatorname{reach}(A) = +\infty$ if and only if A is closed and convex; for a detailed account see [D4] and references therein.

By introducing the squared distance function $\eta(x) = \frac{1}{2}\delta(x)^2$ we have:

Theorem 1.49 ([F1, 4.8])**.** *For every nonempty closed subset A of $\mathbb{R}^N$ the following statements hold with $U = \operatorname{Unp}(A)$.*

(1) $|\delta(x) - \delta(y)| \le |x - y|$ *for all* $x, y \in \mathbb{R}^N$.

(2) If $x \in \mathbb{R}^N \backslash A$ *and* δ *is differentiable at* x*, then* $x \in U$ *and*

$$\nabla\delta(x) = \frac{x - \Pi(x)}{\delta(x)}.$$

(3) Π *is continuous.*

(4) δ *is continuously differentiable on* $\operatorname{Int}(U \backslash A)$ *and* η *is continuously differentiable on* $\operatorname{Int}(U)$ *with*

$$\nabla\eta(x) = x - \Pi(x)$$

for $x \in \operatorname{Int}(U)$.

(5) If for $0 < r < s < +\infty$ *and* $x, y \in U$ *we have* $\delta(x) \le r$, $\delta(y) \le r$ *and* $\operatorname{reach}(A, \Pi(x)) \ge s$, $\operatorname{reach}(A, \Pi(y)) \ge s$, *then*

$$|\Pi(x) - \Pi(y)| \le \frac{s}{s - r}|x - y|.$$

(6) If $0 < q < r < \operatorname{reach}(A)$ *then* $\nabla\delta$ *is Lipschitzian on* $\{x : q \le \delta(x) \le r\}$ *and* $\nabla\eta$ *is Lipschitzian on* $\{x : \delta(x) \le r\}$.

Remark 1.50. It follows that the boundary $\{x : \delta(x) = r\}$ is an $(N-1)$-dimensional $C^{1,1}$-manifold with Lipschitzian normal (cf. [F1, F5]).

For a compact C^r-submanifold $\mathcal{M} \subset \mathbb{R}^N$ without boundary, $r \geq 2$, the distance function δ is C^{r-1} near $\mathcal{M}$—since it is continuous and can be written in terms of directions normal to $\mathcal{M}$—and in fact $\eta \in C^r(U)$ (cf. [GT, p. 354], [F3]). Furthermore, by approximating $\mathcal{M}$ locally at a point x as the graph of a quadratic polynomial given by the second fundamental form, one can show for any $z \in \mathbb{R}^N$ with $0 < |z| \ll 1$ sufficiently small, using the elementary geometry of the normal line at $\Pi(x+z)$ through $x+z$ and the normal line at x along the projection of z onto $(T_x\mathcal{M})^\perp$, that

$$\eta(x+z) = \frac{1}{2}|Qz|^2 + O(|z|^3),$$

where the matrix $Q = (Q_{ab})$ represents the orthogonal projection onto the normal space of $\mathcal{M}$ at the point x. On the other hand, the Taylor series of η is

$$\eta(x+z) = \frac{1}{2}(\nabla^2\eta)(x)(z,z) + O(|z|^3),$$

since $\eta(x) = 0$ and $\nabla\eta(x) = 0$ due to the projection $\Pi(x) = x$. It therefore follows that $\nabla^2\eta$ represents the orthogonal projection onto the normal space of $\mathcal{M}$. Combining this fact with (1.23) gives

$$P_{ab} = \delta_{ab} - \eta_{;ab}. \tag{1.27}$$

Then by taking a further derivative of (4) in Theorem 1.49 we conclude:

Corollary 1.51. *The tensor*

$$P^a{}_b = \Pi^a{}_{;b}$$

projects (r,s)-tensors at each point $x \in \mathcal{M}$ onto the corresponding tangential subspace.

Example 1.52. For the n-sphere $S^n \subset \mathbb{R}^{n+1}$ the retraction Π is given by $x \mapsto \frac{x}{|x|}$ for $x \neq 0$. So from (1.25) and Corollary 1.51 one finds at a point $u \in S^n$ that

$$P^a{}_b = \Pi^a{}_{;b} = \delta^a_b - u^a u_b$$
$$A_{ab}{}^c = u^c(\delta_{ab} - u_a u_b).$$

1.9. Sobolev mappings into manifolds

Let $\mathcal{M}$ be a closed connected C^r-Riemannian submanifold of $\mathbb{R}^N$. We define the Sobolev space of mappings into $\mathcal{M}$ by

$$W^{1,p}(\Omega, \mathcal{M}) = \{u \in W^{1,p}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\}.$$

The set $W^{1,p}(\Omega,\mathcal{M})$ inherits strong and weak topologies from $W^{1,p}(\Omega,\mathbb{R}^N)$. The definition is also independent of the isometric embedding: for if $\boldsymbol{i}_1\colon(\mathcal{M},g)\to\bar{\mathcal{M}}_1\subset\mathbb{R}^{N_1}$ and $\boldsymbol{i}_2\colon(\mathcal{M},g)\to\bar{\mathcal{M}}_2\subset\mathbb{R}^{N_2}$ are two isometric embeddings then there exists $\Psi:\mathbb{R}^{N_2}\to\mathbb{R}^{N_1}$, coinciding with $\boldsymbol{i}_2\circ\boldsymbol{i}_1^{-1}$ on $\mathcal{M}_1$, such that $u\in W^{1,p}(\Omega,\mathcal{M}_1)$ if and only if $\Psi\circ u\in W^{1,p}(\Omega,\mathcal{M}_2)$ with $\|Du\|_{2;\Omega}=\|D(\Psi\circ u)\|_{2;\Omega}$ (cf. [H11, Lem. 1.4.3]). The local space $W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M})$ is defined to consist of maps belonging to $W^{1,p}(\Omega',\mathcal{M})$ for all $\Omega'\subset\subset\Omega$.

Given a functional of the form

$$\mathfrak{F}(v;\Omega)=\int_\Omega f(x,v,Dv)dx$$

we say:

Definition 1.53 (Holonomically constrained minimisers)**.** A map $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M})$ is an (holonomically constrained) *local minimiser* of $\mathfrak{F}$, or simply an holonomic $\mathfrak{F}$-minimiser, if

$$\mathfrak{F}(u;\Omega')\le\mathfrak{F}(v;\Omega')$$

for all

$$v\in(u+W^{1,p}_0(\Omega',\mathbb{R}^N))\cap W^{1,p}(\Omega',\mathcal{M})$$

and all open $\Omega'\subset\subset\Omega$.

Note that the competitor v to the minimiser u must equal u on the boundary in the trace sense of §1.6.1 as well as fulfilling the manifold constrained condition.

A distinguishing feature of the class $W^{1,p}(\Omega,\mathcal{M})$ is the problem of density of smooth maps. Unlike the unconstrained case (cf. Theorem 1.32) density no longer holds in general. Of course we do have density from Sobolev embeddings in the special case $p>n$. In fact in the borderline case:

Theorem 1.54 ([SU2, p. 267])**.** *Let Σ^2 be a compact surface with possibly empty C^1 boundary. Let $\mathcal{M}$ be a compact manifold without boundary. Then $C^\infty(\Sigma,\mathcal{M})$ is dense in $W^{1,2}(\Sigma,\mathcal{M})$.*

However, in general:

Theorem 1.55 ([B13], [HL1])**.** *For $1\le p<n$ smooth maps $C^\infty(\Omega,\mathcal{M})$ are dense in $W^{1,p}(\Omega,\mathcal{M})$ if and only if the homotopy group $\pi_{\lfloor p\rfloor}(\mathcal{M})=0$ and Ω satisfies the $(\lfloor p\rfloor-1)$-extension property with respect to $\mathcal{M}$.*

Theorem 1.56 ([B13], [HL1])**.** *For $1\le p<n$, the class of maps $u\in W^{1,p}(\Omega,\mathcal{M})$ such that u is smooth off a singular set $\Sigma(u)=\cup_{j=1}^r\Sigma_i$, for some integer $r\ge1$, such that Σ_i is a subset of a linear subspace of $\mathbb{R}^n$ of dimension $n-\lfloor p\rfloor-1$ with smooth boundary $\partial\Sigma_i$, is dense in $W^{1,p}(\Omega,\mathcal{M})$.*

Let us remark also that [SU2, §4] showed $C^\infty(B^3, S^2)$ is not dense in $W^{1,2}(B^3, S^2)$, e.g. the projection $B^3 \to S^2$ given by $x \to \frac{x}{|x|}$ cannot be approximated by smooth maps.

In addition the following result shows the closure of $C^\infty(\Omega, \mathcal{M})$ in the $W^{1,p}$-norm to be less appropriate for the calculus of variations than the above definition of $W^{1,p}(\Omega, \mathcal{M})$, since:

Theorem 1.57 ([B13, p. 155])**.** *If $\pi_{\lfloor p \rfloor}(\mathcal{M}) \neq 0$ and p is not an integer, then smooth maps are not sequentially dense for the weak topology in $W^{1,p}(\Omega, \mathcal{M})$. Moreover every map in $W^{1,p}(\Omega, \mathcal{M})$ which is a weak limit of smooth maps is also a strong limit of smooth maps in $W^{1,p}(\Omega, \mathcal{M})$. If p is an integer and $\pi_{\lfloor p \rfloor}(\mathcal{M}) \neq 0$ then smooth maps are dense in $W^{1,p}(\Omega, \mathcal{M})$ for the weak topology.*

1.10. Constrained stationary variations

Consider the functional

$$\mathfrak{F}(v; \Omega) = \int_\Omega f(Dv) dx,$$

where $v \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, and let $\mathcal{M}$ be a closed connected C^3-Riemannian submanifold of $\mathbb{R}^N$. We derive the Euler-Lagrange equations for holonomically constrained minimisers (cf. Definition 1.53), using the retraction Π of (1.26) from a tubular neighbourhood onto $\mathcal{M}$, as follows.

Lemma 1.58. *Suppose the integrand $f \colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a C^1-function that for $p > 1$ satisfies*

$$|Df(\xi)| \leq C(1 + |\xi|^{p-1})$$

for some $C > 0$ and all $\xi \in \mathbb{R}^{Nn}$. Then for any holonomic $\mathfrak{F}$-minimiser $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ we have

$$\int_\Omega \frac{\partial f}{\partial \xi^a_i}(Du) D_i(\Pi^a{}_{;b}(u)\phi^b) = 0$$

for all $\phi \in W^{1,p}_0(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

Proof. Let $\mathcal{O}_\rho$ be a tubular neighbourhood of $\mathcal{M}$ with positive reach ρ. Note that $u(x) + \varepsilon\phi(x) \in \mathcal{O}_\rho$ for a.e. $x \in \Omega$ provided $|\varepsilon| < \rho \|\phi\|^{-1}_{L^\infty}$. For such $\varepsilon \neq 0$ the difference quotient

$$\begin{aligned}\frac{\mathfrak{F}(\Pi(u + \varepsilon\phi); \Omega) - \mathfrak{F}(u; \Omega)}{\varepsilon} &= \frac{1}{\varepsilon} \int_\Omega f(D(\Pi(u + \varepsilon\phi))) - f(Du) \\ &= \frac{1}{\varepsilon} \int_\Omega \int_0^\varepsilon \frac{d}{dt} f(D(\Pi(u + t\phi))) dt \\ = \frac{1}{\varepsilon} \int_\Omega \int_0^\varepsilon \frac{\partial f}{\partial \xi^a_i}(D(\Pi(u + t\phi)))&(\Pi^a{}_{;bc}(u + t\phi)(u^b_i + t\phi^b_i)\phi^c + \Pi^a{}_{;b}(u + t\phi)\phi^b_i).\end{aligned}$$

By the growth condition it follows that

$$\frac{f(D(\Pi(u+\varepsilon\phi))) - f(Du)}{\varepsilon} \leq C(1 + |Du|^p + |D\phi|^p)$$

where the bound is integrable and independent of ε. Thus by applying the dominated convergence theorem together with minimality yields

$$\delta_\phi \mathfrak{F}(u) = \frac{d}{d\varepsilon}\mathfrak{F}(\Pi(u+\varepsilon\phi))\Big|_{\varepsilon=0} = \int_\Omega \frac{\partial f}{\partial \xi_i^a}(Du)(\Pi^a{}_{;bc}(u)u_i^b\phi^c + \Pi^a{}_{;b}(u)\phi_i^b) = 0. \quad \square$$

Example 1.59. When considering the Dirichlet energy, i.e. $f(\xi) = \frac{1}{2}|\xi|^2$, the previous lemma implies

$$\delta_\phi \mathfrak{F}(u) = \int_\Omega u_i^a(\Pi^a{}_{;b}(u)\phi_i^b + \Pi^a{}_{;bc}(u)u_i^b\phi^c) = 0.$$

As $\Pi(u) = u$ one has that $\Pi^a{}_{;b}(u)u_i^b = u_i^a$. So by (1.25), (1.27) and Corollary 1.51 it follows that $u_i^a\Pi^a{}_{;b}(u)\phi_i^b = u_i^a\phi_i^a$ and $u_i^a\Pi^a{}_{;bc}(u)u_i^b\phi^c = -A_{abc}u_i^a u_i^b\phi^c$. Hence we conclude that

$$\Delta u + A(u)(Du, Du) = 0 \tag{1.28}$$

weakly, where A is the second fundamental form of the embedding of $\mathcal{M}$ into $\mathbb{R}^N$. Such solutions to (1.28) are said to be *weakly harmonic maps*. It is clear that minimisers of the Dirichlet energy are weakly harmonic maps.

1.11. Triangulations

The notion of a triangulation of a manifold was first introduced by Poincaré in his *Analysis Situs* series [P4] where he made a rather unconvincing attempt to justify that every differentiable manifold has a polyhedral subdivision; this fact was first proved by Cairns in [C6]. Thereafter J. H. C. Whitehead [W8] extended Cairns' result to a 'preferred class' of triangulations, referred to as C^1-triangulations of C^1-complexes. Before stating Whitehead's result, let us recall some basic concepts of rectilinear cell complexes and simplicial complexes that for instance can be found in [M24]:

Definition 1.60. A (rectilinear) *cell* Δ is a bounded subset of $\mathbb{R}^n$ consisting of all points x satisfying a system of *finite* linear inequalities $\ell_i(x) = \sum a_{ij}x_j \geq b_i$; it is a compact convex subset of $\mathbb{R}^n$.

A *simplex* σ is the convex hull of its $n+1$ vertices. The *dimension* of Δ is the dimension of the smallest dimensional plane $\mathfrak{P}$ containing Δ. So a m-cell Δ contains $m+1$ independent points but not $m+2$. Since Δ must contain the simplex spanned by these $m+1$ points it must have interior points as a subset of $\mathfrak{P}$. Moreover an m-cell Δ is homeomorphic to the m-ball. Now if Δ is an m-cell, $\partial\Delta$ (the boundary of Δ) is the

union of a finite number of $(m-1)$-cells, each the intersection of an $(m-1)$-plane with Δ. These cells are uniquely determined by Δ. Each of the $(m-1)$-cells Σ_i into which $\partial\Delta$ is decomposed is called a *face* of Δ; each of the $(m-2)$-faces of Σ_i is called an $(m-2)$-face of Δ. As Δ is given by a system of linear inequalities, replacing an inequality by equality determines a face of Δ, and conversely.

Definition 1.61. A (rectilinear) *cell complex* K is a collection of cells in $\mathbb{R}^n$ such that:

(1) Each face of a cell in K is in K.
(2) The intersection of two cells in K is either empty or a face of each of them.
(3) Each point of $|K|$ has a neighbourhood intersecting finitely many cells of K (where the *polyhedron*, denoted by $|K|$, is the union of all cells of K).

A simplicial complex is a cell complex where the cells are simplicies.

The *dimension* of K is the maximal dimension of any cell in K. A *subcomplex* K' is a subset of K which itself is a complex; to verify a subcomplex is indeed a complex it is clear that one only needs to check that faces of each cell in K' are in K. In particular the *k-skeleton* of K is the subcomplex of K consisting of all cells of dimension at most k. A *subdivision* K' of K is a complex such that $|K'| = |K|$ and each cell of K' is contain in a cell of K.

Definition 1.62. Let K be a complex. The map $f\colon |K| \to \mathcal{M}$ is said to be *differentiable* of class C^r relative to K if $f|_\Delta$ is of class C^r for each $\Delta \in K$. In short we say $f\colon K \to \mathcal{M}$ is of class C^r. The map f is said to be *non-degenerate* if $\operatorname{rank}(f|_\Delta) = \dim \Delta$ for all $\Delta \in K$.

We seek to define a *C^r-triangulation* of $\mathcal{M}$ to be a suitable sort of immersion $f\colon K \to \mathcal{M}$ which is also homeomorphic onto. By [M24, 8.4] we can define:

Definition 1.63. A C^r-triangulation of $\mathcal{M}$ to be a pair (K, f) where K is a cell complex and $f\colon K \to \mathcal{M}$ is C^r non-degenerate map which is a homeomorphism of $|K|$ onto $\mathcal{M}$.

Given this we now state the following triangulation theorem:

Theorem 1.64 ([W8])**.** *Any closed C^r-manifold $\mathcal{M}$ has a C^r-triangulation.*

Remark 1.65. The result is one of the most useful tools of differential topology and while the statement seems intuitive enough, its proof is in fact surprisingly involved; an excellent exposition of this can be found in [M24, Chap. 2] who intern follows [W8] with some modifications.

For the purposes here it allows us restrict Sobolev mappings to lower-dimensional skeletons as follows. First let $u, v\colon |K| \to \mathbb{R}$ be two everywhere defined Borel measurable

functions. We say u and v are equivalent if for any $\Delta \in K$ we have $u|_\Delta = v|_\Delta$ $\mathcal{H}^{\dim \Delta}$-a.e. on Δ. An equivalence class of Borel functions on K is formed by identify two such function in this way. Following [HL1, §3] we define the space $W^{1,p}(K,\mathbb{R})$ to be the set of all Borel functions $u : |K| \to \mathbb{R}$ such that for each cell Δ the restriction $u|_\Delta$ belongs to $W^{1,p}(\Delta,\mathbb{R})$ with its trace $\mathfrak{R}(u|_\Delta) = u|_{\partial\Delta}$. The accompanying norm is given by $\|u\|_{W^{1,p}(K,\mathbb{R})} = \sum_{\Delta\in K} \|u|_\Delta\|_{W^{1,p}(\Delta,\mathbb{R})}$. Likewise for

$$K^{(d)} = \bigcup\{\Delta : \Delta \in K,\ \dim(\Delta) = d\}$$

we say $u \in W^{1,p}(K^{(d)},\mathbb{R})$ if $u\colon K^{(d)} \to \mathbb{R}$ is a Borel function such that $u|_\Delta \in W^{1,p}(\Delta,\mathbb{R})$ for any $\Delta \in K$ with $\dim(\Delta) = d$ and norm

$$\|u\|_{W^{1,p}(K^{(d)},\mathbb{R})} = \sum_{\substack{\Delta\in K\\ \dim\Delta=d}} \|u|_\Delta\|_{W^{1,p}(\Delta,\mathbb{R})}$$

that also satisfies $\mathfrak{R}(u|_{\Delta_1})|_\Sigma = \mathfrak{R}(u|_{\Delta_2})|_\Sigma$ for any $\Sigma \in K$ with $\dim(\Sigma) = d-1$, $\Sigma \subset \partial\Delta_i$, $\dim\Delta_i = d$ for $i = 1, 2$. Finally, if in addition $\mathcal{N} \subset \mathbb{R}^k$ is a submanifold we define

$$W^{1,p}(K,\mathcal{N}) = \{u \in W^{1,p}(K,\mathbb{R}^k) : u(x) \in \mathcal{N}\ \mathcal{H}^{\dim\Delta}\text{-a.e. for all } \Delta \in K\}.$$

It is clear that $u \circ f \in W^{1,p}(K,\mathcal{N})$ for any $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ where $f\colon K \to \mathcal{M}$ is a triangulation of the domain manifold.

Chapter 2

Partial regularity for unconstrained minimisers of quasiconvex functionals in the quadratic growth case

In this chapter let us consider functionals of the form

$$\mathfrak{F}(v;\Omega) = \int_\Omega f(Dv)dx,$$

where $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a given C^2-function and $v\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ with $n, N \geq 2$. In the absence of constraints we say:

Definition 2.1. A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is an (unconstrained) *local minimiser* of $\mathfrak{F}$ if $\mathfrak{F}(u;\Omega') \leq \mathfrak{F}(v;\Omega')$ for all $v \in u + W^{1,2}_0(\Omega', \mathbb{R}^N)$ and all open $\Omega' \subset\subset \Omega$.

For such minimisers we give a direct proof of the below partial regularity result which, furthermore, will serve as a model case for the constrained case considered in Chapter 6. To do so we impose the following structural assumptions on the integrand, namely we suppose:

(H1) f satisfies the quadratic growth condition

$$|f(\xi)| \leq \Gamma(1 + |\xi|^2)$$

for some $\Gamma > 0$ and all $\xi \in \mathbb{R}^{Nn}$.

(H2) f satisfies Morrey's quasiconvexity condition in the following strong form

$$\gamma \int_{\Omega'} |D\phi|^2 \leq \int_{\Omega'} f(\xi + D\phi) - f(\xi)$$

for some $\gamma > 0$, every $\xi \in \mathbb{R}^{Nn}$, every bounded $\Omega' \subset \mathbb{R}^n$ with $|\partial\Omega'| = 0$ and every $\phi \in C^\infty_c(\Omega', \mathbb{R}^N)$.

Note that there is no growth condition placed on the second order derivatives of f. We merely have, for any given $\Lambda > 0$, the existence of a finite K_Λ such that

$$\sup_{|\xi|\leq\Lambda} |D^2 f(\xi)| \leq K_\Lambda. \tag{2.1}$$

There are also no global uniform continuity assumptions imposed on $D^2 f$. One only has that $D^2 f$ is uniformly continuous on compact subsets. In fact, associated to $\Lambda > 0$, a min-max construction (cf. Example 1.19) implies the existence of a non-decreasing concave modulus of continuity $\omega : [0, +\infty) \to [0, 1]$, with $\lim_{t\searrow 0} \omega(t) = 0$ and $\omega(t) = 1$ for all $t \geq 1$, such that

$$|D^2 f(\xi_1) - D^2 f(\xi_2)| \leq c\,\omega(|\xi_1 - \xi_2|) \tag{2.2}$$

for any $|\xi_i| \leq \Lambda + 1$. In a different direction it is easy to show that the growth condition (H1) together with the quasiconvexity of (H2) is enough (in fact f need only to be rank-one convex) for the derivative Df to satisfy the growth condition

$$|Df(\xi)| \leq c(1 + |\xi|). \tag{2.3}$$

Taking these considerations into account we seek to prove the following:

Theorem 2.2. *Suppose the integrand f satisfies (H1) and (H2). For any local minimiser $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of $\mathfrak{F}$ there is a relatively closed subset $\operatorname{Sing} u \subset \Omega$ of Lebesgue measure zero such that*

$$u \in C^{1,\beta}_{\mathrm{loc}}(\Omega \backslash \operatorname{Sing} u, \mathbb{R}^N)$$

for any $0 < \beta < 1$.

Our treatment here gives an entirely direct proof on the level of the gradient via the so-called $\mathbb{A}$-harmonic approximate lemma (cf. §2.2) together with Caccioppoli inequalities using only minimality, the quadratic growth of (H1) and the strong quasiconvexity of (H2). These estimates allow sufficient decay of the quadratic excess functions $\mathcal{E}$ given by

$$\mathcal{E}(r) = \mathcal{E}(x, r) = \Big(\fint_{B_r(x)} |Du - (Du)_{x,r}|^2 \Big)^{\frac{1}{2}}.$$

The ensuing regularity follows via Theorem 1.24 with the singular set characterised by

$$\operatorname{Sing} u = \big\{ x \in \Omega : \limsup_{r\searrow 0} |(Du)_{x,r}| = +\infty \quad \text{or} \quad \liminf_{r\searrow 0} \mathcal{E}(x, r) > 0 \big\}.$$

An overview of various partial regularity results in the unconstrained setting is discussed in §0.1.5.

Remark 2.3. If we take a point $x \in \Omega \backslash \operatorname{Sing} u$ it follows that $\limsup |(Du)_{x,r}| < +\infty$ and $\liminf \mathcal{E}(x, r) = 0$ by definition. Then there exists $\Lambda > 0$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ such that $|(Du)_{x,r}| < \Lambda$ for all $0 < r < R$. Similarly, for each $\varepsilon > 0$ there exists

$0 < R < \operatorname{dist}(x, \partial\Omega)$ such that $\mathcal{E}(x, R) < \varepsilon$. Now *if* for each $\Lambda > 0$ we can find an $\varepsilon = \varepsilon(\Lambda) > 0$ sufficiently small and an $R = R(\Lambda) \in (0, \operatorname{dist}(x, \partial\Omega))$ sufficiently small such that

$$\mathcal{E}(x, R) < \varepsilon \tag{2.4}$$

then $x \in \Sigma_{\Lambda,\varepsilon,R}$, where $\Sigma_{\Lambda,\varepsilon,R} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R,\ \mathcal{E}(x, R) < \varepsilon,\ |(Du)_{x,R}| < \Lambda\}$ is open (since for fixed $0 < r \ll 1$ the maps $x \mapsto |(Du)_{x,r}|$ and $x \mapsto \mathcal{E}(x, r)$ are continuous on $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$). Hence we need to show for each $x \in \Sigma_{\Lambda,\varepsilon,R}$ and $0 < \beta < 1$ that

$$\mathcal{E}(x, r) \leq K r^{\beta}, \qquad 0 < r < R, \tag{2.5}$$

where $K = K(\beta, \varepsilon, \Lambda, R)$, in order to conclude $Du \in C^{0,\beta}_{\mathrm{loc}}(\Sigma_{\Lambda,\varepsilon,R}, \mathbb{R}^{Nn})$ by Theorem 1.24 of Campanato's characterisation. Consequently Du is $C^{0,\beta}_{\mathrm{loc}}$-continuous on

$$\operatorname{Reg} u = \bigcup_{\Lambda>0} \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R = R(\Lambda),\ \mathcal{E}(x, R) < \varepsilon = \varepsilon(\Lambda),\ |(Du)_{x,R}| < \Lambda\},$$

of which $\operatorname{Sing} u$ is the complement in Ω, provided one has (2.4) and can establish the excess-decay (2.5).

2.1. Excess-improvement and iteration

The above Theorem 2.2 follows once we have established the following excess-improvement:

Proposition 2.4. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimiser of $\mathfrak{F}$. If for any given $\Lambda > 0$ and any $\varepsilon \in (0, 1)$ the smallness conditions*

$$|(Du)_{x_0,R}| \leq \Lambda, \quad \mathcal{E}(x_0, R) \leq \varepsilon$$

hold for some $B(x_0, R) \subset\subset \Omega$, then for each $0 < \tau \leq \frac{1}{4}$ we have

$$\mathcal{E}(x_0, \tau R)^2 \leq C(\tau^2 + \tau^{-n}\widetilde{\omega}(\varepsilon))\mathcal{E}(x_0, R)^2$$

for some $C = C(n, N, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 0$ and a modulus of continuity $\widetilde{\omega}$, in the sense of Definition 1.14, independent of τ, R and ε.

Before outlining the proof of this proposition let us first show how it implies the desired decay estimate (2.5): first fix $0 < \beta < 1$ before making judicious choices of the parameters τ and ε such that both

$$C\tau^2 \leq \tfrac{1}{2}\tau^{2\beta} \qquad \text{and} \qquad C\tau^{-n}\widetilde{\omega}(\varepsilon) \leq \tfrac{1}{2}\tau^{2\beta}$$

hold. As the first inequality is equivalent to choosing $0 < \tau \leq (2C)^{-1/(2-2\beta)} = \tau_0$ we must also require $0 < \widetilde{\omega}(\varepsilon) \leq \frac{1}{2C}\tau^{n+2}$. Then for such (fixed) τ we can decrease ε even further if necessary to obtain:

Corollary 2.5. *For each $\Lambda > 0$ there exist $\varepsilon = \varepsilon(\tau, \Lambda) \in (0, 1]$ such that*

$$\mathcal{E}(\tau R)^2 \leq \tau^{2\beta} \mathcal{E}(R)^2,$$

whenever $|(Du)_R| \leq \Lambda$ and $\mathcal{E}(R) \leq \varepsilon$.

By iterating this result we find for each $\Lambda > 0$ there exist $\varepsilon = \varepsilon(\beta, \tau, \Lambda + 1) \in (0, 1]$ such that

$$\mathcal{E}(\tau^j R)^2 \leq \tau^{2\beta j} \mathcal{E}(R)^2$$

for integers $j \geq 1$, whenever $|(Du)_R| \leq \Lambda$ and $\mathcal{E}(R) \leq \varepsilon$.

Then for each $x_0 \in \operatorname{Reg} u$ and $0 < \beta < 1$ it easily follows that there exists $\Lambda > 0$, an ε sufficiently small and an $0 < R < \operatorname{dist}(x_0, \partial\Omega)$ sufficiently small such that

$$\mathcal{E}(x_0, r)^2 \leq 2\tau^{-n} \Big(\frac{r}{R}\Big)^{2\beta} \mathcal{E}(x_0, R)^2, \qquad 0 < r < R, \tag{2.6}$$

whenever $|(Du)_{x_0,R}| \leq \Lambda$ and $\mathcal{E}(x_0, R) \leq \varepsilon$ hold on some $B(x_0, R) \subset\subset \Omega$, which by Remark 2.3 is what one seeks to demonstrate.

2.2. The $\mathbb{A}$-harmonic approximation lemma

For a symmetric bilinear form $\mathbb{A}$ on $\mathbb{R}^{Nn}$ we say:

Definition 2.6. A map $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a *δ-approximate $\mathbb{A}$-harmonic map* if $\mathbb{A}$ satisfies the strong ellipticity condition (1.19) and for some $\delta > 0$ we have

$$\Big| \fint_\Omega \mathbb{A}(Du, D\phi) \Big| \leq \delta \Big(\fint_\Omega |Du - (Du)_\Omega|^2 \Big)^{\frac{1}{2}} \|D\phi\|_\infty$$

for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^N)$.

Note that in order to make sense of the right-hand side of the above inequality the test functions ϕ are not taken to be in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ but merely in the dense subspace $C_c^\infty(\Omega, \mathbb{R}^N)$.

By using the Hardy-Littlewood maximal function of (1.1) we establish the following standard pointwise approximation of $W^{1,1}$-functions that goes back to [AF3].

Lemma 2.7. *Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ then there exists a set $\Sigma \subset \mathbb{R}^n$ of Lebesgue measure zero and a constant $c = c(n) > 0$ such that*

$$|f(x) - f(y)| \leq c|x - y|(M(Df)(x) + M(Df)(y))$$

for all $x, y \in \mathbb{R}^n \backslash \Sigma$.

Proof. Let Σ be the complement of the set of Lebesgue points of f. Then for each $x \in \mathbb{R}^n \backslash \Sigma$ denote $B_i = B(x, r/2^i)$ so that

$$|f(x) - (f)_{x,r}| \leq \sum_{i=0}^{\infty} |(f)_{B_{i+1}} - (f)_{B_i}| \leq 2^n \sum_{i=0}^{\infty} \fint_{B_i} |f - (f)_{B_i}| \leq crM(Df)(x).$$

If $y \in B(x,r)\backslash\Sigma$ then $B(x,r) \subset B(y,2r)$ and

$$\begin{aligned} |f(y) - (f)_{x,r}| &\leq |f(y) - (f)_{y,2r}| + |(f)_{y,2r} - (f)_{x,r}| \\ &\leq crM(Df)(y) + \fint_{B(x,r)} |f - (f)_{y,2r}| \leq crM(Df)(y). \end{aligned}$$

Therefore for $x, y \in \mathbb{R}^n \backslash \Sigma$ set $r = 2|x-y|$ so that $y \in B(x,r)\backslash\Sigma$ and hence

$$|f(x) - f(y)| \leq |f(x) - (f)_{x,r}| + |f(y) - (f)_{x,r}| \leq cr(M(Df)(x) + M(Df)(y)). \quad \square$$

Lemma 2.8. *Let $v \in W_0^{1,p}(\Omega)$ for $p \geq 1$ and $\Omega \subset \mathbb{R}^n$. Then for each $\lambda > 0$ there exists $v_\lambda \in W_0^{1,\infty}(\Omega)$ with $\|v_\lambda\|_{1,\infty;\Omega} \leq \lambda$ and*

$$\begin{aligned} \|v_\lambda\|_{1,p;\Omega} &\leq c\|v\|_{1,p;\Omega} \\ |\{v_\lambda \neq v\}| &\leq c\lambda^{-p} \int_\Omega |v|^p + |Dv|^p \end{aligned}$$

for some $c = c(n,p) > 0$.

Proof. Extend $v \in W_0^{1,p}(\Omega)$ to $\mathbb{R}^n$ by zero. Using the maximal function let

$$S_0 = \{x \in \mathbb{R}^n : |v| > \lambda\} \qquad \text{and} \qquad S_1 = \{x \in \mathbb{R}^n : M(Dv) > \theta\lambda\}$$

for $0 < \theta \ll 1$ to be of our choosing. By Markov's inequality

$$|S_0| \leq \lambda^{-p} \int_{\mathbb{R}^n} |v|^p \leq c\lambda^{-p} \|v\|_{1,p;\Omega}^p$$

and by Proposition 1.3 we have

$$|S_1| \leq c(\theta\lambda)^{-p} \int_{\mathbb{R}^n} |Dv|^p \leq c(\theta\lambda)^{-p} \|v\|_{1,p;\Omega}^p.$$

Now set $S = S_0 \cap S_1$ and let Σ denote the complement of the set of Lebesgue points for v. By Lemma 2.7 we have $|v(x) - v(y)| \leq c\lambda\theta|x-y| \leq \lambda|x-y|$ for any $x, y \in \mathbb{R}^n \backslash (S \cup \Sigma)$, where θ is small enough so that $c\theta < 1$. Therefore by the Kirszbraun extension theorem (cf. [S5, p. 21]) there exists a Lipschitz function v_λ on $\mathbb{R}^n$ such that $\|v_\lambda\|_{1,\infty} \leq \lambda$ with $v_\lambda = v$ on $\mathbb{R}^n \backslash (S \cup \Sigma)$ and moreover $v_\lambda = 0$ a.e. on $\mathbb{R}^n \backslash \operatorname{supp} v$, since $S \subset \operatorname{supp} v$. In which case the above estimates imply $|\{v_\lambda \neq v\}| \leq S \leq c\lambda^{-p}\|v\|_{1,p}^p$ and

$$\|v_\lambda\|_{1,p}^p = \int_{\mathbb{R}^n \backslash S} |v|^p + |Dv|^p + \int_S |v_\lambda|^p + |Dv_\lambda|^p \leq \|v\|_{1,p}^p + 2\lambda^p |S| \leq c\|v\|_{1,p}^p. \quad \square$$

We now give a direct proof of the following:

Lemma 2.9 ($\mathbb{A}$-harmonic approximation)**.** *Let $B_R \subset\subset \Omega$ be a ball, $\alpha > 1$ and $L \geq \ell > 0$. For all $\varepsilon \in (0,1]$ there exists $\delta = \delta(\varepsilon, n, N, \alpha, \frac{L}{\ell}) > 0$ such that if $u \in W^{1,2}(B_R, \mathbb{R}^N)$ is any δ-approximate $\mathbb{A}$-harmonic map, where $\mathbb{A}$ is a strongly elliptic bilinear form with ellipticity constant ℓ and an upper bound L, then the unique $\mathbb{A}$-harmonic map $h \in u + W_0^{1,2}(B_R, \mathbb{R}^N)$ satisfies*

$$\fint_{B_R} |Du - Dh|^2 \leq \varepsilon \Big(\fint_{B_R} |Du - (Du)_R|^{2\alpha} \Big)^{\frac{1}{\alpha}}.$$

Proof. By a rescaling set $\ell = 1$, $L = \frac{L}{\ell}$ and take $B_R = B$ to be the unit ball without loss of generality. Fix $\phi \in W_0^{1,2}(B, \mathbb{R}^N)$, set $A = (Du)_B$ and denote $\Lambda^2 = \fint_B |D\phi|^2$. By Lemma 2.8 there is a Lipschitz approximation $\phi_\lambda \in W_0^{1,\infty}(B, \mathbb{R}^N)$ with $\lambda \in (k\Lambda, k^2\Lambda)$ for some integer $k > 1$. In which case

$$\fint_B \mathbb{A}(Du, D\phi) = \fint_B \mathbb{A}(Du, D\phi_\lambda) + \fint_B \mathbb{A}(Du - A, D\phi - D\phi_\lambda) = I + II.$$

For the second term we estimate using Lemma 2.8, recalling $\alpha > 1$,

$$\begin{aligned} II &\leq L \fint_B |Du - A||D\phi - D\phi_\lambda| \mathbf{1}_{\{\phi \neq \phi_\lambda\}} \\ &\leq c \fint_B |Du - A|^2 \mathbf{1}_{\{\phi \neq \phi_\lambda\}} + \frac{1}{4} \fint_B |D\phi|^2 \\ &\leq c \Big(\fint_B |Du - A|^{2\alpha} \Big)^{\frac{1}{\alpha}} \Big(\frac{|\{\phi \neq \phi_\lambda\}}{|B|} \Big)^{1 - \frac{1}{\alpha}} + \frac{1}{4} \fint_B |D\phi|^2 \end{aligned}$$

for some $c = c(n, N, L)$. Again, using the estimates from Lemma 2.8 we bound

$$\frac{|\{\phi_\lambda \neq \phi\}|}{|B|} \leq \frac{c}{k^2}.$$

Thus for each $\varepsilon > 0$ choose $k = k(\varepsilon, \alpha) > 0$ sufficiently large so that

$$II \leq \frac{\varepsilon}{2} \Big(\fint_B |Du - A|^{2\alpha} \Big)^{\frac{1}{\alpha}} + \frac{1}{4} \fint_B |D\phi|^2.$$

Furthermore, as u is a δ-approximate $\mathbb{A}$-harmonic map and $\|D\phi_\lambda\|_\infty \leq \lambda \leq k^2\Lambda$, we have

$$I \leq \delta \Big(\fint_B |Du - A|^2 \Big)^{\frac{1}{2}} \|D\phi_\lambda\|_\infty \leq k^2 \delta \Big(4 \fint_B |Du - A|^2 + \frac{1}{4} \fint_B |D\phi|^2 \Big).$$

By choosing δ sufficiently small so that $4k^2\delta \leq \frac{\varepsilon}{2}$, i.e. for a δ satisfying

$$\delta \leq c\, \varepsilon^{2 + \frac{1}{\alpha - 1}} \tag{2.7}$$

where $c = c(n, N, \alpha, L) > 0$, we bound

$$\fint_B \mathbb{A}(Du, D\phi) \leq \varepsilon \Big(\fint_B |Du - A|^{2\alpha} \Big)^{\frac{1}{\alpha}} + \frac{1}{2} \fint_B |D\phi|^2.$$

Then as ϕ is arbitrary we have

$$\begin{aligned}\varepsilon\Big(\fint_B |Du-A|^{2\alpha}\Big)^{\frac{1}{\alpha}} &\geq \sup_{\phi\in W_0^{1,2}}\Big\{\fint_B \mathbb{A}(Du,D\phi)-\frac{1}{2}|D\phi|^2\Big\}\\ &\geq \fint_B \mathbb{A}(Du-Dh,Du-Dh)-\frac{1}{2}|Du-Dh|^2\\ &\geq \frac{1}{2}\fint_B |Du-Dh|^2,\end{aligned}$$

since h is an $\mathbb{A}$-harmonic map with $u-h\in W_0^{1,2}(B,\mathbb{R}^N)$ and $\mathbb{A}$ satisfies the ellipticity condition (1.19). □

Remark 2.10. The technique of harmonic-type approximations often goes via an indirect compactness argument (cf. [DM2]). Our proof avoids a *reductio ad absurdum* argument and is based on the the general Sobolev-Orlicz space proof of [DLSV]. Note also that the direct approach introduces higher integrability and need for Gehring's lemma.

2.3. Approximate harmonicity

Now to prove Proposition 2.4 we first need to establish the approximate harmonicity of the local minimiser u as per Definition 2.6. To do so we first note, for any $\varphi\in C_c^\infty(B_R,\mathbb{R}^N)$ and any matrix $A\in\mathbb{R}^{Nn}$, the quasiconvexity of (H2) implies

$$\begin{aligned}&\epsilon\int_{B_R} Df(A)(D\varphi)+\epsilon^2\int_{B_R}\int_0^1 (1-t)D^2f(A+t\epsilon D\varphi)(D\varphi,D\varphi)dt\\ &\quad=\int_{B_R} f(A+D\phi)-f(A)\geq\gamma\epsilon^2\int_{B_R}|D\varphi|^2.\end{aligned}$$

So by sending $\epsilon\to 0$ the latter inequality becomes

$$\int_{B_R} D^2f(A)(D\varphi,D\varphi)\geq 2\gamma\int_{B_R}|D\varphi|^2. \tag{2.8}$$

Thus for each given $\Lambda>0$ the bilinear form $\mathbb{A}=D^2f((Du)_R)$ is elliptic in the sense of Legendre-Hadamard, i.e. (1.18), with an ellipticity constant $\ell=2\gamma$ and from (2.1) an upper bound $L=K_\Lambda$, whenever $|(Du)_R|\leq\Lambda$. With this one can use a linearisation strategy to prove the following approximate harmonicity result.

Lemma 2.11. *Let $B_R\subset\subset\Omega$ and $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a local minimiser of $\mathfrak{F}$. Then for each $\Lambda>0$ there exists $C=C(n,N,\Lambda,K_\Lambda)>0$ such that*

$$\Big|\fint_{B_R} D^2f((Du)_R)(Du,D\phi)\Big|\leq C\Big(\omega(\mathcal{E}(R))^{\frac{1}{2}}+\mathcal{E}(R)\Big)\mathcal{E}(R)\|D\phi\|_\infty$$

for all $\phi\in C_c^\infty(B_R,\mathbb{R}^N)$, whenever $|(Du)_R|\leq\Lambda$.

Proof. From the Euler-Lagrange equations for $\mathfrak{F}$ we observe, with $A = (Du)_R$, that

$$\begin{aligned}
&\fint_{B_R} D^2 f(A)(Du, D\phi) \\
&\quad = \fint_{B_R} D^2 f(A)(Du - A, D\phi) + \big(Df(A) - Df(Du)\big)(D\phi) \qquad (2.9)\\
&\quad = \fint_{B_R} \int_0^1 \big[D^2 f(A) - D^2 f(A + t(Du - A))\big](Du - A, D\phi)dt. \qquad (2.10)\\
&\quad = \fint_{B_R} \mathbf{1}_{\{|Du-A|\le 1\}} G + \mathbf{1}_{\{|Du-A|>1\}} G,
\end{aligned}$$

where

$$G = \int_0^1 [D^2 f(A) - D^2 f(A + t(Du - A))](Du - A, D\phi)dt.$$

Now on $B_R \cap \{|Du - A| > 1\}$ equations (2.1) and (2.3) imply, via (2.9), that

$$\begin{aligned}
|G| &\le K_\Lambda |Du - A||D\phi| + c(1 + |Du - A|)|D\phi| \\
&\le c|Du - A||D\phi| \le c|Du - A|^2|D\phi|.
\end{aligned}$$

Moreover, on $B_R \cap \{|Du - A| \le 1\}$ equation (2.10) and the modulus of continuity (2.2) imply $|G| \le c\,\omega(|Du - A|)|Du - A||D\phi|$. Hence

$$\begin{aligned}
\Big|\fint_{B_R} \mathbf{1}_{\{|Du-A|\le 1\}} G\Big| &\le c\Big(\fint_{B_R} \omega(|Du - A|)^2\Big)^{\frac{1}{2}} \mathcal{E}(R)\|D\phi\|_\infty \\
&\le c\Big[\omega\Big(\fint_{B_R} |Du - A|\Big)\Big]^{\frac{1}{2}} \mathcal{E}(R)\|D\phi\|_\infty \\
&\le c\,\omega(\mathcal{E}(R))^{\frac{1}{2}} \mathcal{E}(R)\|D\phi\|_\infty,
\end{aligned}$$

where the second last inequality follows by the fact that ω is concave and at most 1 and the last inequality follows by Cauchy-Schwarz and the fact that ω is non-decreasing. □

Remark 2.12. Note that the approximate harmonicity only uses (H1); the fact that $u \in W^{1,2}_{\mathrm{loc}}$ is an extremal of $\mathfrak{F}$ in the sense that div $Df(Du) = 0$ weakly; the rank-one convexity of $D^2 f(A) - \ell|\cdot|^2$ (cf. (2.8)); and, the fact that f is a C^2-function.

2.4. Caccioppoli inequality of the second kind

In addition to the approximate harmonicity we establish Caccioppoli inequalities of the second kind using quadratic growth, strict quaisconvexity and minimality together with Gehring's lemma before moving on to prove Proposition 2.4.

Lemma 2.13. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimiser of $\mathfrak{F}$. Then for each $\Lambda > 0$ there exists $C = C(n, N, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 0$ such that*

$$\int_{B_{R/2}(x_0)} |Du - A|^2 dx \leq C \int_{B_R(x_0)} \Big| \frac{u - \xi - A \cdot (x - x_0)}{R} \Big|^2 dx$$

for any $B_R(x_0) \subset\subset \Omega$, $\xi \in \mathbb{R}^N$ and $A \in \mathbb{R}^{Nn}$ with $|A| \leq \Lambda$.

Proof. Let $\frac{R}{2} \leq r < s \leq R$ and choose a cut-off function $\eta \in C^\infty_c(\Omega)$ so that $\eta = 1$ on B_r, $\eta = 0$ on $\Omega \backslash B_s$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s - r)$. Define

$$\phi_1 = \eta(u - a) \qquad \text{and} \qquad \phi_2 = (1 - \eta)(u - a),$$

where a denotes the affine function $x \mapsto \xi + A \cdot (x - x_0)$, so that $D\phi_1 + D\phi_2 = Du - A$. Using (H2) we obtain

$$\begin{aligned} \gamma \int_{B_s} |D\phi_1|^2 &\leq \int_{B_s} f(Du - D\phi_2) - f(Du) \\ &\quad + \int_{B_s} f(Du) - f(A + D\phi_2) \\ &\quad + \int_{B_s} f(A + D\phi_2) - f(A) \\ &= I + II + III. \end{aligned} \tag{2.11}$$

By introducing the auxiliary functional

$$\mathfrak{F}_s(w) = \int_{B_s} \mathsf{F}(Dw) dx,$$

with integrand $\mathsf{F}(\xi) = f(\xi + A) - f(A) - Df(A)(\xi)$, one notes that

$$\begin{aligned} II &= \int_{B_s} f(Du) - f(A) - Df(A)(D\phi_1 + D\phi_2) \\ &\quad + \int_{B_s} \int_0^1 \big(Df(A) - Df(A + \lambda D\phi_2) \big) \cdot D\phi_2 \, d\lambda \\ &= \mathfrak{F}_s(u - a) + II' \end{aligned}$$

since $\phi_1 = 0$ on ∂B_s. Now as

$$I + III = \int_{B_s} \int_0^1 (Df(A + \lambda D\phi_2) - Df(Du - \lambda D\phi_2)) \cdot D\phi_2 \, d\lambda$$

it follows that $I + II + III = \mathfrak{F}_s(u - a) + III'$, where

$$III' = \int_{B_s} \int_0^1 (Df(A) - Df(Du - \lambda D\phi_2)) \cdot D\phi_2 \, d\lambda.$$

Looking to the auxiliary functional we have

$$\begin{aligned}\mathfrak{F}_s(u-a) &= \int_{B_s} f(Du) - f(A) - Df(A)(Du-A)\\ &\le \int_{B_s} f(Du-D\phi_1) - f(A) - Df(A)(Du-D\phi_1-A)\\ &= \mathfrak{F}_s(u-\phi_1-a),\end{aligned}$$

since u minimises $\mathfrak{F}$ on B_s. Furthermore, using (H1) and (2.3) one easily bounds $\mathsf{F}(\xi) \le C|\xi|^2$ (since for $|\xi|>1$, $\mathsf{F}(\xi)\le C|\xi|^2$ and for $|\xi|\le 1$ write $\mathsf{F}(\xi) = \int_0^1 (1-\lambda)D^2 f(A+\lambda\xi)(\xi,\xi)d\lambda$ so that $|\mathsf{F}(\xi)| \le K_{\Lambda+1}|\xi|^2$). Noting

$$u-\phi_1 = (1-\eta)u + \eta a \tag{2.12}$$

it follows that

$$\mathfrak{F}_s(u-\phi_1-a) = \mathfrak{F}_s((1-\eta)(u-a)) \le C\int_{B_s} |D\phi_2|^2.$$

To estimate III' we note that the integrand of III' on $B_s \cap \{|Du-A|+|D\phi_2|>1\}$ can be bounded, using (2.3) and the fact that $|A|\le\Lambda$, by

$$|Df(A) - Df(Du-\lambda D\phi_2)||D\phi_2| \le C(|Du-A|+|D\phi_2|)|D\phi_2|.$$

On $B_s\cap\{|Du-A|+|D\phi_2|\le 1\}$ we use (2.1), noting that $|Du-\lambda D\phi_2|\le\Lambda+1$, to obtain

$$|Df(A) - Df(Du-\lambda D\phi_2)||D\phi_2| \le K_{\Lambda+1}(|Du-A|+|D\phi_2|)|D\phi_2|.$$

Now as $D\phi_2 = (1-\eta)(Du-A) - (u-a)\otimes D\eta$ we have $|D\phi_2| \le C(|Du-A| + |\frac{u-a}{s-r}|)$. Thus (2.11) together with the fact that $\phi_1 = u-a$, $\phi_2 = 0$ on B_r yields

$$\int_{B_r} |Du-A|^2 \le C\int_{B_s\setminus B_r} |Du-A|^2 + \Big|\frac{u-a}{s-r}\Big|^2$$

for some $C = C(n,N,\gamma,\Gamma,\Lambda,K_{\Lambda+1})$. Then by Widman's hole filling trick we obtain

$$\int_{B_r} |Du-A|^2 \le \chi\int_{B_s} |Du-A|^2 + \frac{1}{(s-r)^2}\int_{B_R} |u-a|^2$$

where $\chi = \frac{C}{C+1} < 1$. The result now follows by an application of Lemma 0.7. □

Having established the above Caccioppoli inequalities we can now set $\xi = (u)_{x_0,R}$ and apply the Sobelov-Poincaré inequality of (1.15) to get

$$\fint_{B_{R/2}} |Du-A|^2 \le C\Big(\fint_{B_R} |Du-A|^{2_*}\Big)^{2/2_*},$$

where $2_* = 2n/(n+2) < 2$. Then by applying Gehring's lemma from §1.3 we conclude:

Corollary 2.14. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimiser of $\mathfrak{F}$. Then for each $\Lambda > 0$ there exists $q = q(n, N, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 2$ such that*

$$\Big(\fint_{B_{R/2}} |Du - A|^q\Big)^{1/q} \leq C\Big(\fint_{B_R} |Du - A|^2\Big)^{1/2}$$

for any $B_R \subset\subset \Omega$ and $A \in \mathbb{R}^{Nn}$ with $|A| \leq \Lambda$.

With this we can now finish by giving a proof of the excess-improvement estimate.

Proof of Proposition 2.4. Fix $\Lambda > 0$, $\varepsilon \in (0,1)$ and assume the smallness conditions $|(Du)_R| < \Lambda$ and $\mathcal{E}(R) < \varepsilon$ hold for some $B_R(x_0) \subset\subset \Omega$. It follows by (2.1) and (2.8) that $\mathbb{A} = D^2 f((Du)_R)$ is an elliptic bilinear form in the sense of (1.19) with ellipticity constant $\ell = 2\gamma$ and an upper bound $L = K_\Lambda$.

Now set $\alpha = q/2$ from Corollary 2.14 and let

$$\widetilde{\omega}(t) = (\omega(t)^{\frac{1}{2}} + t)^{\frac{\alpha-1}{2\alpha-1}},$$

where ω is the modulus of continuity given by (2.2). Then for each $\varepsilon_0 = \widetilde{\omega}(\varepsilon)$ the approximate harmonicity of Lemma 2.11 shows u to be a δ-approximate $\mathbb{A}$-harmonic map with $\delta = \delta(\varepsilon_0) = C\varepsilon_0^{\frac{2\alpha-1}{\alpha-1}}$, since $\delta = C(\omega(\varepsilon)^{\frac{1}{2}} + \varepsilon) \geq C(\omega(\mathcal{E}(R))^{\frac{1}{2}} + \mathcal{E}(R))$. As the choice of δ also satisfies the growth rate (2.7) it follows by Lemma 2.9 that there exists a unique $\mathbb{A}$-harmonic map $h \in u + W^{1,2}_0(B_R, \mathbb{R}^N)$ with

$$\fint_{B_R} |Dh - Du|^2 \leq \varepsilon_0\Big(\fint_{B_R} |Du - (Du)_R|^q\Big)^{\frac{2}{q}}.$$

So for any $0 < \tau \leq \frac{1}{2}$ and any matrix $A \in \mathbb{R}^{Nn}$,

$$\mathcal{E}(\tau R)^2 \leq 4\fint_{B_{\tau R}} |Du - A|^2 \leq c\fint_{B_{\tau R}} |Dh - A|^2 + c\fint_{B_{\tau R}} |Du - Dh|^2.$$

By setting $A = (Dh)_{\tau R}$ the interior and boundary estimates of Remark 1.44 imply

$$\fint_{B_{\tau R}} |Dh - (Dh)_{\tau R}|^2 \leq C\tau^2\fint_{B_R} |Dh - (Dh)_R|^2 \leq C\tau^2\fint_{B_R} |Du - (Du)_R|^2,$$

since $(Dh)_R = (Du)_R$ by the divergence theorem. Furthermore the conclusion of Lemma 2.9 together with Corollary 2.14 yield

$$\begin{aligned}\fint_{B_{\tau R}} |Du - Dh|^2 \leq \tau^{-n}\fint_{B_R} |Du - Dh|^2 &\leq \tau^{-n}\varepsilon_0\Big(\fint_{B_R} |Du - (Du)_R|^q\Big)^{2/q} \\ &\leq C\tau^{-n}\varepsilon_0\fint_{B_{2R}} |Du - (Du)_{2R}|^2.\end{aligned}$$

Thus $\mathcal{E}(\tau R)^2 \leq C(\tau^2 + \tau^{-n}\widetilde{\omega}(\varepsilon))\mathcal{E}(2R)^2$ for each $0 < \tau \leq \frac{1}{2}$. □

Chapter 3

Regularity aspects of harmonic and biharmonic maps

The notion of harmonic maps between Riemannian manifolds and, for that matter, various generalisation thereof, has proved to be of considerable mathematical interest. As one the most natural problems one can pose for mappings between manifolds, there are (not surprisingly) many varied examples that are of both of a geometric and physical interest (cf. [HW1]). In analytical terms the problem can be seen as a model case from which more general integrands can be developed (cf. Chapter 6). Looking to §0.2.1 we consider holonomic minimisers of the Dirichlet energy

$$E(u;\Omega) = \frac{1}{2}\int_{\Omega} |Du|^2 dx, \tag{3.1}$$

with the class of manifold valued mappings

$$W^{1,2}_{\mathrm{loc}}(\Omega,\mathcal{M}) = \{u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\},$$

for a bounded open $\Omega \subset \mathbb{R}^n$ and a connected compact C^3-Riemannian submanifold $\mathcal{M}$ of $\mathbb{R}^N$ without boundary. A map $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathcal{M})$ is a local minimiser of the the Dirichlet energy if

$$\int_{\Omega'} |Du|^2 dx \leq \int_{\Omega'} |Dv|^2 dx$$

for every open $\Omega' \subset\subset \Omega$ and every $v \in W^{1,2}(\Omega,\mathcal{M})$ such that $v - u \in W^{1,2}_0(\Omega,\mathbb{R}^N)$ (cf. Definition 1.53). Such minimisers can be seen to satisfy the necessary condition

$$\Delta u + A(u)(Du,Du) = 0$$

weakly, where A is the second fundamental form of the embedding $\mathcal{M} \subset \mathbb{R}^N$ (cf. Example 1.59). The nonlinearity appearing in the Lagrange multiplier term—which can be

seen geometrically as the need for $-\Delta u \perp T_u\mathcal{M}$ weakly (cf. [H11, p. 38])—is the reason why most of analytical properties valid for unconstrained harmonic functions are lost: existence, regularity, uniqueness and minimality may fail in general unless some extra hypotheses are added.

The existence question for harmonic maps between manifolds is a delicate matter. When considering Dirichlet boundary conditions we recall the direct method as outlined in steps (1), (2) and (3) from §0.1.2: for a given $\phi \in W^{1,2}(\Omega, \mathcal{M})$ let the class of admissible function be

$$\mathscr{C} = (\phi + W_0^{1,2}(\Omega, \mathbb{R}^N)) \cap W^{1,2}(\Omega, \mathcal{M}).$$

As $0 \leq \inf_{\mathscr{C}} E \leq E(\phi) < +\infty$ there exists a minimising sequence $(u_j) \subset \mathscr{C}$. By the Poincaré inequality we have

$$\|u_j\|_{W^{1,2}} \leq c_1 \|\phi\|^2_{W^{1,2}} + c_2 E(u_j)^{\frac{1}{2}}$$

for some $c_1, c_2 > 0$. As (u_j) is a minimising sequence $E(u_j)$ is bounded, hence by the latter inequality $\|u_j\|_{W^{1,2}}$ is also. Then by the Banach-Alaoglu compactness theorem there exists $u_0 \in \phi + W_0^{1,2}(\Omega, \mathbb{R}^N)$ and subsequence $u_{j_k} \rightharpoonup u_0$ weakly in $W^{1,2}$. By Rellich-Kondrashov there exists a further subsequence (u_{j_k}) that converges strongly in L^p, for $p < \frac{2n}{n-2}$, and a.e. on Ω to u_0 (from which we conclude $u_0 \in \mathcal{M}$ a.e.). Furthermore, the sequentially lower semicontinuity goes through as per usual, since $W^{1,p}(\Omega, \mathcal{M})$ inherits strong and weak topologies from $W^{1,p}(\Omega, \mathbb{R}^N)$. Thus there exists an holonomic E-minimiser $u_0 \in (\phi + W_0^{1,2}(\Omega, \mathbb{R}^N)) \cap W^{1,2}(\Omega, \mathcal{M})$. Cautiously, however, there are examples in the manifold constrained case where the closedness condition of step (2) from §0.1.2 to ensure $u_0 \in \mathscr{C}$ no longer holds (i.e. when $\mathscr{C} \subset W^{1,2}(\Omega, \mathcal{M})$ is not closed for the weak topology). For example there may not exist a minimiser amongst mappings from S^n to itself of a given degree k when $n \geq 3$.[1] For additional existence and non-existence results we refer the reader to the survey found in [HW1, §5]. Moreover, the desire for this closedness condition highlights the reason why (by the density result of Theorem 1.57) one does not define the class of admissible functions to be the closure of $C^\infty(\Omega, \mathcal{M})$ in the $W^{1,p}$-norm.

In examining the regularity properties of harmonic maps, the 1948 result of Morrey [M16] (cf. Theorem 0.11) shows minimisers to exhibit full regularity when $n = 2$. When $n > 2$ only partial regularity carries over by the 1982 seminal result of Schoen and Uhlenbeck [SU1] (cf. Theorem 0.12). The following theorem, an ε-regularity result, show energy minimising harmonic maps to be regular in a neighbourhood of every point in which the excess is suitably small.

[1] In this regard direct methods seem to lead to severe difficulties—which is one of the reasons why, for instance, the Eells and Sampson paper [ES1] looked to pursue gradient-flow techniques.

Theorem 3.1 (ε-Regularity)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(B_1(0), \mathcal{M})$ be a local minimiser of the Dirichlet energy where $\mathcal{M}$ is a compact Riemannian manifold without boundary. Then there exists $\varepsilon > 0$ and $\sigma \in (0,1)$ depending on $\mathcal{M}$ such that if for some ball $B_R(x_0) \subset\subset B_1(0)$ we have*

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 \leq \varepsilon,$$

then $u \in C^{0,\alpha}(B_{\sigma R}(x_0), \mathcal{M})$ for some $\alpha \in (0,1)$.

An immediate consequence of the ε-regularity shows the $(n-2)$-dimensional measure of the singular set is zero, namely:

Corollary 3.2. *For any energy minimising harmonic map $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$, $\Omega \subset \mathbb{R}^n$, there exists a closed subset $\mathrm{Sing}(u) \subset \Omega$ characterised by*

$$\mathrm{Sing}(u) = \Big\{ x \in \Omega : \limsup_{R \searrow 0} R^{2-n} \int_{B_R(x)} |Du|^2 > 0 \Big\}$$

for which $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$ and in particular $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n-2$.

In fact a further dimension reduction argument of Federer shows the Hausdorff dimension of the singular set to be bounded by $n-3$ (cf. [SU1, §5]); a detailed study of the singular set can be found in [S11, Chap. 3 and 4]. We also remark that the original proof of Schoen and Uhlenbeck does not make use of Caccioppoli-type inequalities. Instead a mollification argument is used together with the following monotonicity property for the excess:

$$\rho^{2-n} \int_{B_\rho(x_0)} |Du|^2 < r^{2-n} \int_{B_r(x_0)} |Du|^2 \tag{3.2}$$

which holds for every $x_0 \in B_1(0)$ and all $0 < \rho < r < \mathrm{dist}(x_0, \partial B_1(0))$. On the other hand, when looking to the techniques for unconstrained minimisers (cf. §0.1.5), it is desirable to have Caccioppoli-type inequalities at one's disposal. Unfortunately this is generally not possible using standard methods, nonetheless the elegant treatment of Simon [S11, Chap. 1 and 2] affords a proof of Theorem 3.1 via direct PDE arguments. The approach centres on combining the following harmonic approximation lemma with a certain 'technical regularity lemma' stated thereafter.

Lemma 3.3 (Harmonic approximation)**.** *Let $B = B_1(0) \subset \mathbb{R}^n$. For each $\varepsilon > 0$ there exists $\delta = \delta(n, \varepsilon) > 0$ such that if $f \in W^{1,2}(B)$ with $\int_B |Df|^2 \leq 1$ and*

$$\Big| \int_B Df \cdot D\phi \Big| \leq \delta \sup_B |D\phi|$$

for all $\phi \in C^\infty_c(B)$, then there exists a harmonic $u \in C^\infty(B)$ such that $\int_B |Du|^2 \leq 1$ and $(\int_B |f-u|^2)^{\frac{1}{2}} \leq \varepsilon$.

We have given proof of this lemma—which typically follows by indirect compactness arguments (cf. [DM2])—by direct means in §2.2. A variant of this technique will be later revisited in §6.4.

Lemma 3.4 ([S11, p. 13])**.** *Suppose* $0 < \alpha < 1$ *and* $\beta \geq 1$ *are given. There exists* $\delta = \delta(n, \alpha, \beta) > 0$ *such that if the map* $u \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ *for some ball* $B_R(x_0) \subset\subset \Omega$ *satisfies*

$$\Big(\fint_{B_R(x_0)} |u - (u)_{x_0,R}|^2 \Big)^{\frac{1}{2}} \leq \delta,$$

is a weak solution to the equation

$$\Delta u = F$$

on $B_R(x_0)$*, where* $F \in L^1(B_R(x_0), \mathbb{R}^N)$ *with* $|F| \leq \beta |Du|^2$ *a.e. on* $B_R(x_0)$*, and satisfies*

$$\fint_{B_{r/2}(y)} |Du|^2 \leq \beta \fint_{B_r(y)} \Big| \frac{u - (u)_{y,r}}{r} \Big|^2$$

for all $B_r(y) \subset B_R(x_0)$*, then* $u \in C^{1,\alpha}(B_{R/4}(x_0))$*.*

One should note that this lemma highlights the key regularity issues: namely, the small excess assumption, the critical growth of the nonlinear term and the need for Caccioppoli inequalities of the first kind. It also shows the desired regularity follows once the Caccioppoli inequalities are verified. A generalisation of Lemmas 3.3 and 3.4 to the degenerate $p \neq 2$ case has also been given in [DM1].

3.1. Caccioppoli inequalities of the first kind and partial higher integrability for harmonic maps

In this section we derive Caccioppoli inequalities of the first kind for energy minimising maps under the assumption of bounded excess. Originally due to [S11, §2.8] the proof is based on a lemma of Luckhaus that goes back to an extension-type theorem of [SU1, Lem. 4.3]; a separate treatment of Luckhaus' result and proof is discussed in Chapter 4.

The approach taken here pursues a hole filling argument together with a suitably adapted covering lemma of Simon (Lemma 3.6 below). It should be noted that the standard iteration lemma (i.e. Lemma 0.7) cannot be used with the hole filling argument as the construction of the comparison map derived from from Luckhaus' lemma (i.e. Corollary 4.9) prevents one from sending $r \to s$ (which is needed in the proof of the standard iteration lemma).[2] To compensate, only a finite number of iterations are performed.

[2]A similar scenario also occurs for strong local minimisers [KT, p. 77] however one is only able to get a pre-Caccioppoli inequality in that case.

Lemma 3.5. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be an holonomic E-minimiser. For each $\Lambda > 0$ there exists $C = C(n, N, \Lambda) > 0$ such that if*

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 \leq \Lambda \tag{3.3}$$

on some ball $B_R(x_0) \subset\subset \Omega$, then

$$r^{2-n} \int_{B_{r/2}(y)} |Du|^2 \leq C r^{-n} \int_{B_r(y)} |u - (u)_{y,r}|^2 \tag{3.4}$$

for all $y \in B_{R/2}(x_0)$ and $r \leq R/4$.

Proof. Fix $\frac{1}{2} < \theta < 1$ to be of our choosing. For an arbitrary $\sigma_0 \leq R/4$ and $y_0 \in B_{R/2}(x_0)$ let $B_s(y) \subset B_\sigma(y) \subset B_{\sigma_0}(y_0)$ where $\frac{\sigma}{2\theta} \leq s \leq \sigma$ and set $r = \theta s$ (so that $\frac{\sigma}{2} \leq r = \theta s < s \leq \sigma$). By the monotonicity formula (3.2) we have

$$s^{2-n} \int_{B_s(y)} |Du|^2 \leq (\tfrac{R}{4})^{2-n} \int_{B_{\frac{R}{4}}(y)} |Du|^2 \leq 4^{n-2} R^{2-n} \int_{B_R(x_0)} |Du|^2 \leq 4^n \Lambda \tag{3.5}$$

where the last inequality follows by the assumption (3.3).

Without loss of generality assume

$$s^{-n} \int_{B_s(y)} |u - (u)_{y,s}|^2 \leq \delta^2 \tag{3.6}$$

for $0 < \delta < 1$ sufficiently small independent of u and the choice of balls (for if (3.6) does not hold then by (3.5) we would trivially have

$$\Big(s^{2-n} \int_{B_s(y)} |Du|^2\Big)\Big(s^{-n} \int_{B_s(y)} |u - (u)_{y,s}|^2\Big)^{-1} \leq \frac{4^n \Lambda}{\delta^2}$$

which gives (3.4) with $C = 4^n \Lambda / \delta^2$).

Now by Corollary 4.9 there exists $w \in W^{1,2}(B_s(y), \mathbb{R}^N)$ such that $w - u \in W^{1,2}_0(B_s(y), \mathbb{R}^N)$ and $w|_{B_t} = (u)_{y,s}$ for some $r < t < (1 + \frac{\varepsilon}{4})r$ with $\varepsilon = \theta^{-1} - 1$ (i.e. $\theta = \frac{1}{1+\varepsilon}$). Moreover,

$$\operatorname{dist}((u)_{y,s}, \mathcal{M}) \leq \Big(\fint_{B_s(y)} |u - (u)_{y,s}|^2 \Big)^{\frac{1}{2}} \leq c_1 \delta,$$

where $c_1 = c_1(n)$, and by (3.5), (3.6), (4.5) we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{B_s \setminus B_t} \operatorname{dist}(w, \mathcal{M}) &\leq c_2 \Big((s-r)^{2-n} \int_{B_s \setminus B_r} |Du|^p + (s-r)^{-n} \int_{B_s \setminus B_r} |u - v|^2 \Big)^{\frac{3}{8}} \\
&\quad \times \Big((s-r)^{-n} \int_{B_s \setminus B_r} |u - (u)_{y,s}|^2 \Big)^{\frac{1}{8}} \\
&\leq c_2 (1-\theta)^{-\frac{n}{2}} (4^n \Lambda)^{\frac{3}{8}} \delta^{\frac{1}{4}} \\
&\leq \frac{c_2}{(1-\theta)^{\frac{n}{2}}} \delta^{\frac{1}{4}}
\end{aligned} \tag{3.7}$$

where $c_2 = c_2(n, N, \Lambda)$. As the manifold $\mathcal{M} \subset \mathbb{R}^N$ has a tubular neighbourhood $\mathcal{O}_\rho$ with positive reach $\rho > 0$ and a retraction $\Pi\colon \mathcal{O}_\rho \to \mathcal{M}$ we require

$$\delta = \delta(\theta) = \min\{c_1^{-1}\rho, c_2^{-4}(1-\theta)^{2n}\rho^4\}.$$

In which case one can define a comparison map $\Pi \circ w \in (u + W_0^{1,2}(B_s(y), \mathbb{R}^N)) \cap W^{1,2}(B_s(y), \mathcal{M})$. Then by minimality we have

$$\int_{B_s(y)} |Du|^2 \leq \int_{B_s(y)} |D(\Pi \circ w)|^2$$

from which it follows by Remark 1.48 and (4.6) that

$$\int_{B_r(y)} |Du|^2 \leq C \int_{B_s(y)\setminus B_r(y)} |Du|^2 + \Big|\frac{u-(u)_{y,s}}{s-r}\Big|^2$$

for some $C = C(n, N, \Lambda) > 0$. Therefore by Widman's hole filling trick we conclude

$$\int_{B_{\theta s}(y)} |Du|^2 \leq \chi \int_{B_s(y)} |Du|^2 + E(\theta)\sigma^{-2} \tag{3.8}$$

where $\chi = \frac{C}{1+C} \in (\frac{1}{2}, 1)$ is independent of the choice of the balls and

$$E(\theta) = \frac{4}{(1-\theta)^2} \int_{B_{\sigma_0}(y_0)} |u - (u)_{y_0,\sigma_0}|^2,$$

provided $\delta = \delta(n, N, \Lambda, \theta, \rho)$ is sufficiently small.

Now fix θ to be $\chi < \theta < 1$ and let $B_s(y) \subset B_\sigma(y)$ with $\frac{\sigma}{2\chi} \leq s \leq \sigma$. By iterating (3.8) k-times for $k \leq \log_\theta \chi$ $(= \log\chi/\log\theta)$, where s is replaced by $s^{(k)} = \theta^{k-1}s$ and r is replaced by $r^{(k)} = \theta s^{(k)}$, we find that

$$\int_{B_{\theta^k s}(y)} |Du|^2 \leq \chi^k \int_{B_s(y)} |Du|^2 + \frac{E(\theta)\sigma^{-2}}{1-\chi}.$$

Note that $r^{(k)} = \theta^k s \geq \frac{\sigma}{2}$ by the choice of s. Furthermore, as the ratio of $r^{(k)}$ to $s^{(k)}$ remains the constant and equal to θ and

$$⨍_{B_{s^{(k)}}(y)} |u - (u)_{y,s^{(k)}}|^2 \leq 2^{kn+1} ⨍_{B_s(y)} |u - (u)_{y,s}|^2$$

the bound (3.7) remains unchanged provided

$$\delta = \delta(k, \theta) = 2^{-kn-1}\min\{c_1^{-1}\rho, c_2^{-4}(1-\theta)^{2n}\rho^4\}$$

is taken to be sufficiently small. This ensures the construction of w for each $r^{(k)}$ and $s^{(k)}$ remains within a tubular neighbourhood of the manifold $\mathcal{M}$ with $\frac{\sigma}{2} \leq r^{(k)} < s^{(k)} \leq \sigma$.

Finally by choosing $\theta = \chi^{1/k_0}$ for an integer $k_0 \gg 0$ sufficiently large and setting $k = k_0 = \log_\theta \chi$ we conclude that

$$\sigma^2 \int_{B_{\sigma/2}(y)} |Du|^2 \le \chi^k \sigma^2 \int_{B_\sigma(y)} |Du|^2 + \frac{E(\chi^{1/k})}{1-\chi}$$

for all balls $B_\sigma(y) \subset B_{\sigma_0}(y_0)$ (where $\sigma_0 \le R/4$ and $y_0 \in B_{R/2}(x_0)$). The desired result now follows by invoking Lemma 3.6 below. □

Lemma 3.6 ([S12, p. 398])**.** *Let $\alpha \ge 0$, $\lambda \in (0, \frac{1}{2}]$, $\chi \in (\frac{1}{2}, 1)$ and S be a monotone subadditive function on the convex subsets of an n-ball $B = B(x_0, R_0)$.*[3] *If for some $k > 0$ we have*

$$\sigma^\alpha S(B_{\lambda\sigma}(x)) \le \chi^k \sigma^\alpha S(B_\sigma(x)) + E$$

for all balls $B_\sigma(x) \subset B$, then there exists $C = C(n, \alpha, \lambda)$ such that

$$R^\alpha S(B_{R/2}(x)) \le CE$$

for any ball $B_R(x) \subset B$, provided we choose $k \gg 0$ sufficiently large with $k = k(n, \alpha, \lambda, \chi)$.

Proof. Let

$$Q = \sup_{B_\sigma(x) \subset B} \sigma^\alpha S(B_{\sigma/2}(x)) < +\infty.$$

In light of the given inequality one has

$$2^{-\alpha} \sigma^\alpha S(B_{\lambda\sigma/2}(x)) \le \chi^k Q + E \tag{3.9}$$

for each $B_\sigma(x) \subset B$.

For arbitrary $B_\sigma(x) \subset B$ select points $x_j \in B_{\sigma/2}(x)$ such that $B_{\sigma/2}(x) \subset \bigcup_{j=1}^N B_{\lambda\sigma/4}(x_j)$ where we estimate $N \le c$ for some $c = c(n, \lambda)$. As each $B_{\sigma/2}(x_j) \subset B$ we can apply (3.9) with $B_{\sigma/2}(x_j)$ in place of $B_\sigma(x)$ and use the subadditivity of S to estimate

$$\sigma^\alpha S(B_{\sigma/2}(x)) \le \sum_{j=1}^N \sigma^\alpha S(B_{\lambda\sigma/4}(x_j)) \le C(\chi^k Q + E)$$

where $C = C(n, \alpha, \lambda)$. Now take k sufficiently large so that $C\chi^k < \frac{1}{2}$ (i.e. $k > \log \frac{1}{2C} / \log \chi$) and note that this choice is independent of E. In which case

$$\sigma^\alpha S(B_{\sigma/2}(x)) \le \tfrac{1}{2} Q + CE$$

and as the right-hand side is independent of the choice of $B_\sigma(x)$ we take the supremum over all such balls to conclude

$$Q \le 2CE.$$

In particular $R^\alpha S(B_{R/2}(x)) \le 2CE$ for any ball $B_R(x) \subset B$. □

[3] That is, $S(A) \le \sum_{j=1}^N S(A_j)$ whenever A_j are convex subsets of B and $A \subset \bigcup_{j=1}^N A_j$.

Remark 3.7. Note that the choice of $k \gg 0$ is equivalent to choosing $0 \ll \theta < 1$, where $\theta = \theta(n, \alpha, \lambda, \chi)$ is related to k via $k = \log_\theta \chi$. Let us also draw the reader's attention to the following remarks of Simon [S12, p. 399]: 'Special cases of the above lemma are commonly used in many different contexts—it is therefore somewhat surprising that it seems not to be very widely known that a general lemma of the above type is available. A crucial point of the above lemma is that ε [$= \chi^k$ here] can be chosen independent of E; this is important because in many applications (and in the present application) E depends on ε.'

Having established Lemma 3.5 one can easily obtain the following partial higher integrability result via the Sobelov-Poincaré inequality of (1.15) and Gehring's lemma of §1.3. Note that the assumption (3.3) cannot be removed as it is essential for ensuring the well defined nature of the comparison map as constructed via Luckhaus' lemma.

Corollary 3.8. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be an holonomic E-minimiser. For each $\Lambda > 0$ there exists $C = C(n, N, \Lambda) > 0$ and $q = q(n, N, \Lambda) > 2$ such that if*

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 \leq \Lambda$$

on some ball $B_R(x_0) \subset\subset \Omega$, then $Du \in L^q_{\mathrm{loc}}(B_R(x_0), \mathbb{R}^{Nn})$ and

$$\Big(\fint_{B_{r/2}(y)} |Du|^q \Big)^{\frac{1}{q}} \leq C \Big(\fint_{B_r(y)} |Du|^2 \Big)^{\frac{1}{2}}$$

for all $y \in B_{R/2}(x_0)$, $r \leq R/4$.

3.2. Monotonicity

The regularity properties for harmonic maps (and, in fact, its higher order generalisations) are distinguished by the presence of useful monotonicity formulae. In light of its important role in the above regularity proof for harmonic maps we give a derivation (emphasising the role of the stress-energy tensor as outlined in [HE, pp. 66-7]) in an attempt to see if it is whether or not possible to extract useful signed expressions to more general integrands by the same technique.

To do so let us first consider the general case of C^∞-mappings $f : \mathcal{M} \to \mathcal{M}'$ between Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ of dimensions n and m respectively with $\mathcal{M}$ compact, oriented and without boundary. For such an f we define the energy

$$E(f) = \int_M e(f) d\mu_g,$$

where the energy density $e(f) = \frac{1}{2}g^{ij}(g'_{\alpha\beta})_f f_i^\alpha f_j^\beta$ in local coordinates and $d\mu_g$ is the volume form on $\mathcal{M}$ with respect to the metric g (e.g. in local coordinates the volume form can be written as $d\mu_g = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$). Here the differential f_* is a smooth section of $T^*\mathcal{M} \otimes f^*T\mathcal{M}'$ over $\mathcal{M}$ that is equal to $f_* = f_i^\alpha dx^i \otimes (\partial_\alpha)_f$, where $f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}$, in local coordinates (x^i) about $p \in \mathcal{M}$ and local coordinates (y^α) about $f(p) \in \mathcal{M}'$. Using the notation of [ES1] the space of all vector fields along f is taken to be the set of all smooth sections of the pull-back bundle $f^*T\mathcal{M}'$ over $\mathcal{M}$. For any such vector field v along f the directional derivative of E in the direct v is given by $\delta_v E(f) = \frac{d}{dt}E(f_t)|_{t=0}$, where $f_t(p) = \exp_{f(p)}(tv(p))$ is the endpoint of the geodesic segment in $\mathcal{M}'$ starting at $f(p)$ and determined in length and direction by the vector $tv(p) \in T_{f(p)}\mathcal{M}'$. By the divergence theorem one can then show

$$\delta_v E(f) = -\int_{\mathcal{M}} \langle \tau(f), v\rangle \, d\mu(g),$$

where $\tau(f) = g^{ij}(\nabla_i f_*)(\partial_j)$ denotes the *tension field* of the map f that is locally given by

$$\tau(f)^\gamma = g^{ij}(f_{ij}^\gamma - \Gamma_{ij}^k f_k^\gamma + (\Gamma'^\gamma_{\alpha\beta})_f f_i^\alpha f_j^\alpha).$$

In which case we say $f \in C^\infty(\mathcal{M}, \mathcal{M}')$ is a (stationary) *harmonic map* if $\tau(f) = 0$.

In the higher order generalisation, the concept of polyharmonic maps between Riemannian manifolds was first introduced in [ES2]. Here we restrict our attention to biharmonic maps with an associated functional

$$E_2(f) = \frac{1}{2}\int_{\mathcal{M}} |\tau(f)|^2 d\mu(g). \tag{3.10}$$

By a similar calculation to that of the above one has

$$\delta_v E_2(f) = \int_{\mathcal{M}} \langle \tau_2(f), v\rangle \, d\mu(g),$$

where $\tau_2(f) = \Delta\tau(f) + \operatorname{tr} R^{\mathcal{M}'}(f_*, \tau(f))f_*$. In which case a smooth map f satisfying $\tau_2(f) = 0$ is said to be *biharmonic*.

The derivation of the so-called stress-energy tensor can be seen to germinate from the diffeomorphism invariance of an integral $\int_{\mathcal{M}} \omega$ of a differential n-forms ω over a manifold $\mathcal{M}$. For example if the n-form $\omega = e(f)d\mu$ and $\Phi\colon \mathcal{M} \to \mathcal{M}$ is an orientation preserving diffeomorphism then

$$\int_{\mathcal{M}} e(f)d\mu = \int_{\Phi(\mathcal{M})} e(f)d\mu = \int_{\mathcal{M}} \Phi^*(e(f)d\mu),$$

from which we have

$$\int_{\mathcal{M}} (e(f)d\mu - \Phi^*(e(f)d\mu)) = 0.$$

If the diffeomorphism is generated by a vector field X on $\mathcal{M}$ (i.e. $\Phi = \Phi_t$ is the flow generated by X) we conclude

$$\int_{\mathcal{M}} \mathcal{L}_X(e(f)d\mu) = 0, \tag{3.11}$$

where $\mathcal{L}_X$ denotes the Lie derivative. In order to expand (3.11) we first note that

$$\mathcal{L}_X(\tfrac{1}{2}|f_*|^2 d\mu) = \langle f_*, \nabla_X f_* \rangle \, d\mu + \tfrac{1}{2}|f_*|^2 \mathcal{L}_X d\mu$$

and $\mathcal{L}_X d\mu = \frac{1}{2}\langle g, \mathcal{L}_X g\rangle \, d\mu$. For the first term one can show, using symmetry and compatibility, that $\langle f_*, \nabla_X f_*\rangle = \langle f_*, \nabla(f_* X)\rangle - \frac{1}{2}\langle f^* g', \mathcal{L}_X g\rangle$ where $(\mathcal{L}_X g)_{ij} = g_{kj}\nabla_i X^k + g_{ik}\nabla_j X^k$. Furthermore, as

$$\nabla_j(g^{ij}(g'_{\alpha\beta})_f f_i^\alpha X^k f_k^\beta) = \left\langle g^{ij}\nabla_j f_i, f_* X\right\rangle + \langle f_*, \nabla(f_* X)\rangle$$

it follows by the divergence theorem that

$$0 = \int_{\mathcal{M}} \mathcal{L}_X(\tfrac{1}{2}|f_*|^2 d\mu) \tag{3.12}$$

$$= -\int_{\mathcal{M}} \langle \tau(f), f_* X\rangle \, d\mu + \frac{1}{2}\int_{\mathcal{M}} \left\langle \tfrac{1}{2}|f_*|^2 g - f^* g', \mathcal{L}_X g\right\rangle d\mu. \tag{3.13}$$

In which case we naturally define the stress-energy tensor

$$S = \tfrac{1}{2}|f_*|^2 g - f^* g' \tag{3.14}$$

associated to the energy functional E. Moreover

$$\text{div } S = -\langle \tau(f), f_*\rangle, \tag{3.15}$$

since for arbitrary X the divergence theorem implies

$$\frac{1}{2}\int_{\mathcal{M}} \langle S, \mathcal{L}_X g\rangle \, d\mu = \int_{\mathcal{M}} S_i{}^k \nabla_k X^i d\mu = -\int_{\mathcal{M}} X^i \nabla_k S_i{}^k d\mu.$$

Consequently if f is harmonic then S is conservative—in the sense that div $S = 0$. Likewise from the paper of [LMO] the stress-energy tensor associated to the bienergy functional E_2 is given by

$$\begin{aligned} S_2(X,Y) = \frac{1}{2}|\tau(f)|^2 \langle X, Y\rangle + \langle f_*, \nabla\tau(f)\rangle \langle X, Y\rangle \\ - \langle f_*(X), \nabla_Y \tau(f)\rangle - \langle f_*(Y), \nabla_X \tau(f)\rangle \end{aligned}$$

for vector fields X and Y on $\mathcal{M}$ with the further property that

$$\text{div } S_2 = -\langle \tau_2(f), f_*\rangle.$$

Example 3.9. To derive the monotonicity formula (3.2) for harmonic mappings we compute

$$\frac{d}{dr}\Big(r^{2-n}\int_{B_r(x_0)}|Du|^2dx\Big) = (2-n)r^{1-n}\int_{B_r(x_0)}|Du|^2dx + r^{2-n}\int_{\partial B_r(x_0)}|Du|^2d\sigma(x).$$

Then to evaluate the second term let the tangent vector field $Y = \langle S, X\rangle$ on $B_r(x_0)$ be the pairing of the stress-energy tensor $S_{ij} = \frac{1}{2}|Du|^2\delta_{ij} - u_i^\alpha u_j^\alpha$ from (3.14) with the vector field X given by $X(x) = x - x_0$. By Stokes' theorem we get

$$\int_{\partial B_r(x_0)}\langle Y,\nu\rangle\, d\sigma(x) = \int_{B_r(x_0)}\operatorname{div} Y dx = \int_{B_r(x_0)} S_j^i\partial_i X^j dx$$

where the second equality is due to div $S = 0$ weakly by (3.15). It then follows that $r\int_{\partial B_r(x_0)}\frac{1}{2}|Du|^2 - |Du\cdot\nu|^2 = (\frac{n}{2}-1)\int_{B_r(x_0)}|Du|^2$ from which we conclude

$$\frac{d}{dr}\Big(r^{2-n}\int_{B_r(x_0)}|Du|^2dx\Big) = r^{2-n}\int_{\partial B_r(x_0)}2|Du\cdot\nu|^2d\sigma \tag{3.16}$$

is non-negative.

Let us now consider computing an analogous expression for integrands of the form

$$\mathfrak{F}(u;\Omega) = \int_\Omega F(Du)dx$$

for holonomic minimiser $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathcal{M})$ into a compact manifold $\mathcal{M}$ with an integrand $F \in C^2(\mathbb{R}^{Nn})$. By performing a 'domain variation' argument—equivalent to that of the Lie derivative expression (3.11)—we derive the following stress-energy tensor associated to the functional $\mathfrak{F}$.

Lemma 3.10. *For an holonomic $\mathfrak{F}$-minimiser $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathcal{M})$ we have*

$$\int_{B_R(x_0)}\Big(F(Du)\delta_{ij} - \frac{\partial F}{\partial\xi_i^a}(Du)D_ju^a\Big)D_i\xi^j dx = 0$$

for every test function $\xi \in C_c^\infty(B_R(x_0),\mathbb{R}^n)$.

Proof. Let $\xi \in C_c^\infty(B_R(x_0),\mathbb{R}^n)$ so the map $\Phi_t\colon B_R(x_0)\to B_R(x_0)$ given by $\Phi_t(x) = x + t\,\xi(x)$ is a diffeomorphism of $B_R(x_0)$ onto itself for $|t|$ sufficiently small. Set $U_t(x) = u(\Phi_{-t}(x))$ then as $\Phi_{-t}(x) = \Phi_t(x)^{-1}$ we find by change of variables that

$$\begin{aligned}\int_{B_R(x_0)}F(DU_t(x))dx &= \int_{\Phi_t(B_R(x_0))}F(DU_t(x))dx\\ &= \int_{B_R(x_0)}F(Du(x)D\Phi_{-t}(\Phi_t(x)))\det D\Phi_t(x)dx\\ &= \int_{B_R(x_0)}F(Du(x))dx + t\int_{B_R(x_0)}(F(Du)\operatorname{div}\xi - DF(Du)(Du\cdot D\xi))dx + o(t)\end{aligned}$$

since $D\Phi_{-t}(\Phi_t(x)) = I - tD\xi(x) + o(t)$ and $\det D\Phi_t(x) = 1 + t\,\mathrm{div}\,\xi(x) + o(t)$. Hence

$$\begin{aligned} 0 = &\int_{B_R(x_0)} (F(DU_t) - F(Du))dx \\ &- t\int_{B_R(x_0)} (F(Du)\mathrm{div}\,\xi - DF(Du)(Du\cdot D\xi))dx + o(t) \end{aligned}$$

from which the desired conclusion follows by minimality of u together with the fact that $U_t = u$ on $\partial B_R(x_0)$. □

Lemma 3.11. *Suppose the integrand F satisfy the quadratic growth condition*

$$F(\xi) \le \Gamma(1 + |\xi|^2)$$

for some $\Gamma > 0$ and all $\xi \in \mathbb{R}^{Nn}$. Then for an holonomic $\mathfrak{F}$-minimiser $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ we have

$$\begin{aligned} \frac{d}{dR}\Big[R^{2-n}\int_{B_R(x_0)} F(Du)dx\Big] = &R^{2-n}\int_{\partial B_R(x_0)} \langle DF(Du)\cdot\nu, Du\cdot\nu\rangle\, d\sigma(x) \\ &+ R^{1-n}\int_{B_R(x_0)} (2F(Du) - \langle Du, DF(Du)\rangle)dx \end{aligned}$$

for any $x_0 \in \Omega$ and a.e. $0 < R < \mathrm{dist}(x_0, \partial\Omega)$.

Note that when $F(\xi) = \frac{1}{2}|\xi|^2$ the right-hand side of the reduces to the expression $r^{2-n}\int_{\partial B_r} |Du\cdot\nu|^2$ as expected in the harmonic case (cf. (3.16)). However in the general case one seems to lack a priori any useful signed quantities.

Proof. Fix a smooth radial cut-off function $\eta = \eta(r)$, $r = |x - x_0|$, with $\mathrm{supp}\,\eta \subset B_R(x_0)$ and $0 \le \eta \le 1$. By choosing $\xi(x) = \eta(r)(x - x_0)$ Lemma 3.10 implies that

$$\begin{aligned} &\int_{B_R} (nF(Du) - \langle DF(Du), Du\rangle)\eta(r)dx + \int_{B_R} F(Du)r\eta'(r)dx \\ &\quad = \int_{B_R} \frac{\partial F}{\partial \xi^a_i} D_j u^a \frac{(x^i - x^i_0)(x^j - x^j_0)}{|x - x_0|}\eta'(r)dx. \end{aligned}$$

Then using integration by parts we find that

$$\begin{aligned} &\int_{B_R} (nF(Du) - \langle DF(Du), Du\rangle)\eta(r)dx \\ &\quad = \int_0^R dr\,\eta(r)\frac{d}{dr}\Big[r\int_{\partial B_r(x_0)} F(Du)d\sigma(x)\Big] \\ &\qquad - \int_0^R dr\,\eta(r)\frac{d}{dr}\Big[r\int_{\partial B_r(x_0)} \langle DF(Du)\cdot\nu, Du\cdot\nu\rangle\, d\sigma(x)\Big], \end{aligned}$$

where ν is the unit normal given by $x^i - x_0^i = r\nu^i$. By choosing a sequence (η_j) suitably approximating the characteristic function $\mathbf{1}_{B_R(x_0)}$ we find in the limit that

$$\int_{B_R} (nF(Du) - \langle DF(Du), Du\rangle) = R\int_{\partial B_R} (F(Du) - \langle DF(Du)\cdot\nu, Du\cdot\nu\rangle)$$

for a.e. $0 < R < \operatorname{dist}(x_0, \partial\Omega)$. □

Remark 3.12. In [A2] Allard gives a monotonicity formula for the area functional in the context of Geometric Measure Theory (see also [S13]). Also in a largely expository paper [E11] Evans discusses the possibility of useful monotonicity formulae for the singular quasiconvex functional $\int_\Omega (\frac{1}{2}|Du|^2 + \frac{1}{(\det Du)^\gamma})dx$ when $n = N > 1$ and $\gamma > 0$.

3.3. Partial regularity for minimising biharmonic maps

When considering higher order generalisations of the Dirichlet energy one typically looks to the Hessian energy functional

$$E_2(u;\Omega) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx$$

and the tension field energy functional

$$T(u;\Omega) = \frac{1}{2}\int_\Omega |\Delta u + A(u)(Du, Du)|^2 dx$$

with the class of manifold valued mappings

$$W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M}) = \{u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\}.$$

We remark that E_2 is a generalisation of biharmonic functions and T is equivalent to the intrinsically defined (3.10), where A is the second fundamental form given by (1.24). By construction the functional T is independent of the isometric embedding of the target manifold, however E_2 is dependent on the embedding. Stationary points of E_2 are called *extrinsic* biharmonic maps whereas stationary points of T are called *intrinsic* biharmonic maps. In particular:

Lemma 3.13. *For an holonomic E_2-minimiser $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ we have*

$$\int_\Omega \Big[\Delta u^a \Delta\phi^a + \Delta u^a D_i(\Pi^a{}_{;bc}(u)u_i^b))\phi^c + \big(2\Delta u^a \Pi^a{}_{;bc}(u)u_i^b + D_i(\Pi^c{}_{;bd}(u)u_j^b u_j^d)\big)\phi_i^c\Big]dx$$
$$= \int_\Omega \Delta u^a \Delta(\Pi^a{}_{;b}(u)\phi^b)dx = 0$$

for all $\phi \in W^{2,2}_0(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

Likewise, using the method of Lemma 1.58 and the orthogonality, one also gets the following Euler-Lagrange equations for the intrinsic case.

Lemma 3.14. *For any T-minimising $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ we have*

$$-\int_\Omega \langle \Delta u, \Delta(\nabla\Pi(u)\phi)\rangle \, dx = \int_\Omega A(u)(Du,Du)\Big[(\nabla A)(u)(Du,Du)(\nabla\Pi(u)\phi) + 2A(u)(Du, \nabla(\nabla\Pi(u)\phi))\Big]dx$$

for all $\phi \in W^{2,2}_0(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

The biharmonic case, like the harmonic case, is also distinguished by a monotonicity formula. The stress-energy tensor for the E_2-functional is given by:

Lemma 3.15. *If $u \in W^{2,2}(B_R(x_0), \mathcal{M})$ is a stationary (extrinsic) biharmonic map, then*

$$\int_{B_R(x_0)} \Big[\Big(\frac{1}{2}|\Delta u|^2\delta_{ij} - 2u_{ij}\Delta u\Big)\xi^j_i - u_i \Delta u \Delta \xi^i\Big]dx = 0$$

for every test function $\xi \in C^\infty_c(B_R(x_0), \mathbb{R}^n)$.

The following monotonicity formula for (extrinsically) stationary biharmonic maps goes back to the statement of [CWY, Prop. 3.2] and was rigorously proved from Lemma 3.15 by Angelsberg [A5].

Proposition 3.16 (Monotonicity)**.** *For an (extrinsically) stationary biharmonic map $u \in W^{2,2}(B_\rho, \mathcal{M})$ it holds that*

$$\sigma(R) - \sigma(r) = 4\int_{B_R\setminus B_r} \Big(\frac{|u_j + x^j u_{ij}|^2}{|x|^{n-2}} + (n-2)\frac{|x^j u_j|^2}{|x|^n}\Big)dx$$

for a.e. $0 < r < R \le \frac{\rho}{2}$, where

$$\sigma(r) = \sigma_1(r) + \sigma_2(r) \tag{3.17}$$

with

$$\sigma_1(r) = r^{4-n}\int_{B_r} |\Delta u|^2 + r^{3-n}\int_{\partial B_r} |Du|^2$$
$$\sigma_2(r) = r^{3-n}\int_{\partial B_r} 2x^i u_i u_{ij} + 3|Du|^2 - 4r^{-2}|x^i u_i|^2.$$

Using this monotonicity formula it is possible to prove the following result based on the approach by [S23, App. B] which, in contrast to [CWY, Lem. 4.8] or [W1, Lem. 5.3], does not use the biharmonic map equation.

Theorem 3.17. *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be a stationary (extrinsic) biharmonic map. Suppose there exists $\varepsilon > 0$ such that*

$$R^{4-n} \int_{B_R(x_0)} |D^2 u|^2 + R^{-2}|Du|^2 \leq \varepsilon \tag{3.18}$$

on some $B_R(x_0) \subset \Omega$. Then there exists a constant $C > 0$ such that

$$r^{4-n} \int_{B_{\frac{r}{4}}(x_1)} |D^2 u|^2 + r^{-2}|Du|^2 \leq C\varepsilon \tag{3.19}$$

for all $0 < r < \frac{R}{2}$ and all $x_1 \in B_{\frac{R}{2}}(x_0)$.

Proof. Let (3.18) hold on a ball $B_R(x_0)$. To establish (3.19) it suffices to show for any $0 < r < \frac{R}{2}$ and every Lebesgue point $x_1 \in B_{\frac{R}{2}}(x_0)$ of the function $|D^2u|^2 + |Du|^2$ that

$$\rho^{4-n} \int_{B_\rho(x_1)} |\Delta u|^2 + \rho^{3-n} \int_{\partial B_\rho(x_1)} |Du|^2 \leq C\varepsilon \tag{3.20}$$

for some $\frac{r}{2} < \rho < r$ and a constant $C > 0$ independent of u, r, R and ε.

To see (3.20) fix a Lebesgue point $x_1 \in B_{\frac{R}{2}}(x_0)$ and after a shift of coordinates also assume $x_1 = x_2 = 0$ without loss in generality. From Example 1.1 there exists $s \in (\frac{R}{2}, R)$ such that

$$\begin{aligned} |\sigma(s)| &\leq Cs^{4-n} \int_{B_s} |\Delta u|^2 + Cs^{5-n} \int_{\partial B_s} |D^2 u|^2 + s^{-2}|Du|^2 \\ &\leq CR^{4-n} \int_{B_R} |D^2 u|^2 + R^{-2}|Du|^2 \leq C\varepsilon \end{aligned}$$

by the initial hypothesis (3.18). Likewise

$$\liminf_{t \searrow 0} |\sigma(t)| = \lim_{t \searrow 0} \inf_{\varrho > \frac{t}{4}} |\sigma(\varrho)| \leq C \lim_{t \searrow 0} t^2 \fint_{B_{\frac{t}{2}}} |D^2 u|^2 + |Du|^2 = 0$$

by the Lebesgue point assumption on $|D^2u|^2 + |Du|^2$ (cf. Remark 1.5). Hence there exists $0 < t < \frac{r}{2}$ such that $|\sigma(t)| \leq \varepsilon$.

By a further application of Example 1.1 there exists $\frac{r}{2} < \rho < r$ such that

$$\rho^{3-n} \int_{\partial B_\rho} |u_j + x^j u_{ij}|^2 + 4\rho^{-2}|x^j u_j|^2 \leq C \int_{B_r \setminus B_{\frac{r}{2}}} \Big(\frac{|u_j + x^j u_{ij}|^2}{|x|^{n-2}} + (n-2)\frac{|x^j u_j|^2}{|x|^n} \Big) dx.$$

By estimating

$$2x^i u_{ij} u_j + 3|Du|^2 = 2(u_j + x^i u_{ij})u_j + |Du|^2 \geq -|u_j + x^i u_{ij}|^2$$

we bound

$$\sigma_2(\rho) \geq -\rho^{3-n} \int_{\partial B_\rho} |u_j + x^j u_{ij}|^2 + 4\rho^{-2}|x^j u_j|^2.$$

Furthermore, using the values $\frac{R}{2} < s < R$ and $0 < t < \frac{r}{2}$ together with the monotonicity formula of Proposition 3.16, we bound

$$4\int_{B_s\setminus B_t}\Big(\frac{|u_j + x^j u_{ij}|^2}{|x|^{n-2}} + (n-2)\frac{|x^j u_j|^2}{|x|^n}\Big)dx \le |\sigma(s)| + |\sigma(t)| \le C\varepsilon.$$

Then by the positivity of the monotonicity formula we have $0 \le \sigma(s) - \sigma(\rho)$ which together with the above estimates implies

$$\sigma_1(\rho) \le \sigma(s) - \sigma_2(\rho) \le |\sigma(s)| - \sigma_2(\rho) \le C\varepsilon.$$

Thus the desired (3.20) holds.

Finally, it follows by potential estimates of Lemma 3.18 below that

$$r^{4-n}\int_{B_{\frac{r}{4}}} |D^2u|^2 + r^{-2}|Du|^2 \le C\rho^{4-n}\int_{B_\rho}|\Delta u|^2 + Cr^{2-n}\int_{B_r}|Du|^2 \le C\varepsilon$$

where the last inequality following from (3.20) and Example 1.1 in the reverse. □

Lemma 3.18. *For any $u \in W^{2,2}(B_R(x))$ we have*

$$\int_{B_{R/2}(x)} |D^2u|^2 \le C\int_{B_R(x)} |\Delta u|^2 + R^{-2}|Du|^2$$

for some $C > 0$ independent of u and the ball $B_R(x)$.

Proof. First recall the identity

$$\|D^2v\|_{2;B_R} = \|\Delta v\|_{2;B_R} \tag{3.21}$$

for any $v \in W_0^{1,2}(B_R)$ (cf. [GT, Corol. 9.10]). Then for $0 < \sigma < 1$ introduce a cut-off function $\eta \in C_0^2(B_R)$ satisfying $0 \le \eta \le 1$ with $\eta = 1$ on $B_{\sigma R}$, $\eta = 0$ for $|x| \ge \sigma' R$, $\sigma' = \frac{1+\sigma}{2}$, and $|D\eta| \le \frac{4}{(1-\sigma)^2}$, $|D^2\eta| \le \frac{16}{(1-\sigma)^2R^2}$. The desired inequality follows by setting $v = \eta(u - (u)_R)$ and calculating $\|D^2u\|_{2;B_{\sigma R}} \le \|\Delta v\|_{2;B_R} \le C\|\Delta u\|_{2;B_R} + \frac{C}{(1-\sigma)R}\|Du\|_{2;B_R} + \frac{C}{(1-\sigma)^2R^2}\|u - (u)_R\|_{2;B_R}$. □

In addition let us also recall the following Gagliardo–Nirenberg interpolation inequality.

Lemma 3.19 ([N10], [G2])**.** *Let $u \in L^q(\Omega)$ and $D^mu \in L^r(\Omega)$ where m is positive integer, $1 \le q, r \le \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. For the derivatives D^ju, $0 \le j < m$, the following inequality holds*

$$\|D^ju\|_p \le c_1\|D^mu\|_r^\alpha\|u\|_q^{1-\alpha} + c_2\|u\|_{\tilde q}$$

for constant c_1, c_2 depend on the domain, where $\frac{1}{p} = \frac{j}{n} + \alpha(\frac{1}{r} - \frac{m}{n}) + (1-\alpha)\frac{1}{q}$ for all $\frac{j}{m} \le \alpha \le 1$ and any $\tilde q > 0$.

In particular if $u \in W^{2,2}(B_R(x), \mathcal{M})$ we have

$$\int_{B_R(x)} |Du|^4 \leq C\|u\|_\infty^2 \int_{B_R(x)} |D^2u|^2 + r^{-2}|Du|^2 \tag{3.22}$$

for some constant C independent of r. Note that $\|u\|_\infty \leq C(\mathcal{M})$ as $\mathcal{M}$ is compact.

3.3.1. A biharmonic ε-regularity result. Having established Theorem 3.17 we now look to prove the following ε-regularity theorem for minimising (extrinsic) biharmonic maps based on the original harmonic map proof of [SU1].

Theorem 3.20. *Let $u \in W^{2,2}_{\mathrm{loc}}(B_1(0), \mathcal{M})$ be an E_2-minimiser and suppose the Riemannian manifold $\mathcal{M}$ is compact without boundary. Then there exists $\varepsilon > 0$ and $\sigma \in (0,1)$ depending on $\mathcal{M}$ such that if for some $B_R(x_0) \subset\subset B_1(0)$ we have*

$$R^{4-n} \int_{B_R(x_0)} |D^2u|^2 + R^{-2}|Du|^2 \leq \varepsilon \tag{3.23}$$

then $u \in C^{0,\alpha}(B_{\sigma R}(x_0), \mathcal{M})$ for some $\sigma \in (0,1)$.

The result follows once the following decay estimate is establish.

Proposition 3.21. *There exists $\varepsilon > 0$ and $\tau \in (0,1)$ such that if*

$$\int_{B_1(0)} |D^2u|^2 + |Du|^2 \leq \varepsilon$$

then

$$\tau^{4-n} \int_{B_\tau(0)} |D^2u|^2 + \tau^{-2}|Du|^2 \leq \frac{1}{2} \int_{B_1(0)} |D^2u|^2 + |Du|^2. \tag{3.24}$$

From the ε-regularity theorem we also have the following characterisation and bound on the singular set.

Corollary 3.22. *For any E_2-minimiser $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$, $\Omega \subset \mathbb{R}^n$, there exists a closed subset $\mathrm{Sing}(u) \subset \Omega$ characterised by*

$$\mathrm{Sing}(u) = \left\{ x \in \Omega : \limsup_{R \searrow 0} R^{4-n} \int_{B_R(x)} |D^2u|^2 + R^{-2}|Du|^2 > 0 \right\}$$

for which $\mathcal{H}^{n-4}(\mathrm{Sing}(u)) = 0$ and in particular $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n-4$.

Proof of Theorem 3.20. Let u satisfy (3.23) and assume without loss of generality that $x_0 = 0$ and $R = 1$. By Proposition 3.21 for $u\colon B_1(0) \to \mathcal{M}$ we have

$$\tau^{4-n} \int_{B_\tau(0)} |D^2u|^2 + \tau^{-2}|Du|^2 \leq \frac{1}{2} \int_{B_1(0)} |D^2u|^2 + |Du|^2.$$

As the rescaled map $u_\tau(x) = u(\tau x)$ satisfies the hypothesis of Proposition 3.21 we also have

$$\int_{B_1(0)} |D^2 u_\tau|^2 + |Du_\tau|^2 = \tau^{4-n} \int_{B_\tau(0)} |D^2 u|^2 + \tau^{-2}|Du|^2 \leq \frac{1}{2} \int_{B_1(0)} |D^2 u|^2 + |Du|^2 \leq \varepsilon.$$

Therefore $\tau^{4-n} \int_{B_\tau(0)} |D^2 u_\tau|^2 + \tau^{-2}|Du_\tau|^2 \leq \frac{1}{2} \int_{B_1(0)} |D^2 u_\tau|^2 + |Du_\tau|^2$ too, or equivalently $(\tau^2)^{4-n} \int_{B_{\tau^2}(0)} |D^2 u|^2 + \tau^{-4}|Du|^2 \leq (\frac{1}{2})^2 \int_{B_1(0)} |D^2 u|^2 + |Du|^2$. By iteration we then find

$$(\tau^j)^{4-n} \int_{B_{\tau^2}(0)} |D^2 u|^2 + (\tau^j)^{-2}|Du|^2 \leq \Big(\frac{1}{2}\Big)^j \int_{B_1(0)} |D^2 u|^2 + |Du|^2$$

for any positive integer j. Now given any $0 < r < 1$ choose j such that $\tau^{j+1} \leq r \leq \tau^j$ and set $\alpha = \frac{1}{2} \log 2 / \log \tau^{-1}$. Then by the above iteration

$$\begin{aligned} r^{4-n} \int_{B_r(0)} |D^2 u|^2 + r^{-2}|Du|^2 &\leq \tau^{4-n}(\tau^j)^{4-n} \int_{B_{\tau^j}(0)} |D^2 u|^2 + \tau^{-2}(\tau^j)^{-2}|Du|^2 \\ &\leq \tau^{2-n}(\tau^j)^{2\alpha} \int_{B_1(0)} |D^2 u|^2 + |Du|^2 \\ &\leq \tau^{2-n}\Big(\frac{r}{\tau}\Big)^{2\alpha} \int_{B_1(0)} |D^2 u|^2 + |Du|^2 \\ &\leq \Big(\tau^{2-n-2\alpha} \int_{B_1(0)} |D^2 u|^2 + |Du|^2\Big) r^{2\alpha}. \end{aligned}$$

The desired result now follows by Theorem 1.24, since

$$r^{-n} \int_{B_r(0)} |u - (u)_r|^2 \leq C r^{4-n} \int_{B_r(0)} |D^2 u|^2 + r^{-2}|Du|^2$$

by the Poincaré inequality. □

Proof of Proposition 3.21. We follow the original ideas from [SU1] which approximate u by a smooth mollification. By comparing the mollification to an (unconstrained) biharmonic function one can obtained estimates similar to (3.24). The desired result follows by construction a comparison function from the mollification in such a way it agrees with the u on the boundary.

1. Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}^+$ with $\operatorname{supp} \varphi \subset B_1(0)$ and $\int_{B_1(0)} \varphi(x)dx = 1$ be a smooth radial mollifier. Denote $u_{x,h}(y) = u(x - hy)$ then by Theorem 3.17 there exists $C > 0$ such that

$$\begin{aligned} &\int_{B_1(0)} |D^2 u_{x,h}(y)|^2 + |Du_{x,h}(y)|^2 dy \\ &\qquad = h^{4-n} \int_{B_h(x)} |D^2 u(y)|^2 + h^{-2}|Du(y)|^2 dy \leq C\varepsilon \end{aligned} \tag{3.25}$$

for all $0 < h < \frac{1}{8}$ and $x \in B_{\frac{1}{2}}(0)$. Now define the h-mollification

$$u^{(h)}(x) = (\varphi^{(h)} * u)(x) = \int_{B_h(x)} \varphi^{(h)}(x-y)u(y)dy = \int_{B_1(0)} u_{x,h}(y)\varphi(y)dy$$

for all $x \in B_{\frac{1}{2}}(0)$ and $0 < h < \frac{1}{8}$, where $\varphi^{(h)}(x) = h^{-n}\varphi(x/h)$. By the Poincaré inequality (1.14) we have

$$h^{-n}\int_{B_h(x)} |u(y) - u^{(h)}(x)|^2 dy = \int_{B_1(0)} |u_{x,h}(y) - u^{(h)}(x)|^2 dy \le c\int_{B_1(0)} |Du_{x,h}|^2$$

from which in conjunction with (3.25) it follows

$$\begin{aligned}\operatorname{dist}(u^{(h)}(x), \mathcal{M}) &\le \Big(\fint_{B_h(x)} |u(y) - u^{(h)}(x)|^2 dy\Big)^{\frac{1}{2}} \\ &\le C\Big(\int_{B_1(0)} |Du_{x,h}(y)|^2 dy\Big)^{\frac{1}{2}} \\ &\le C\Big(\int_{B_1(0)} |D^2 u_{x,h}|^2 + |Du_{x,h}|^2\Big)^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}}\end{aligned}$$

for all $x \in B_{\frac{1}{2}}(0)$ and $0 < h < \frac{1}{8}$. As the manifold $\mathcal{M} \subset \mathbb{R}^N$ has a tubular neighbourhood $\mathcal{O}_\rho$ with positive reach $\rho > 0$ and a retraction $\Pi\colon \mathcal{O}_\rho \to \mathcal{M}$ we define the map $u_h\colon B_{\frac{1}{2}}(0) \to \mathcal{M}$ to be

$$u_h = \Pi \circ u^{(h)} \tag{3.26}$$

for sufficiently small $\varepsilon > 0$.

2. Let $v \in C^\infty(\bar{B}_{\frac{1}{2}}(0), \mathbb{R}^N)$ be the solution to

$$\Delta^2 v = 0$$

in $B_{\frac{1}{2}}(0)$ with $v = u^{(h)}$ and $\frac{\partial v}{\partial \nu} = \frac{\partial u^{(h)}}{\partial \nu}$ on $\partial B_{\frac{1}{2}}(0)$. By subtracting $u^{(h)}(0)$ and applying the maximum principle from [A1] we get $\|v - u^{(h)}(0)\|_{1,\infty;B_{\frac{1}{2}}(0)} \le c\|u^{(h)} - u^{(h)}(0)\|_{1,\infty;\partial B_{\frac{1}{2}}(0)}$ from which we bound $\|v - u^{(h)}\|_{1,\infty;B_{\frac{1}{2}}(0)} \le c\|u^{(h)} - u^{(h)}(0)\|_{1,\infty;\bar{B}_{\frac{1}{2}}(0)}$. To estimate the right-hand side note that

$$Du^{(h)}(x) = D(\varphi^{(h)} * u)(x) = \varphi^{(h)} * Du(x) = \int_{B_h(x)} \varphi^{(h)}(x-y)Du(y)dy.$$

Then by Jensen's inequality $|Du^{(h)}(x)|^2 \le \varphi^{(h)} * |Du|^2(y) \le ch^{-n}\int_{B_h(x)} |Du|^2$ and likewise $|D^2u^{(h)}(x)|^2 \le ch^{-n}\int_{B_h(x)} |D^2u|^2$. Consequently as

$$\sup_{x\in\bar{B}_{\frac{1}{2}}(0)} |u^{(h)}(x) - u^{(h)}(0)|^2 \le c \sup_{\bar{B}_{\frac{1}{2}}(0)} |Du^{(h)}|^2$$

$$\sup_{x\in\bar{B}_{\frac{1}{2}}(0)} |Du^{(h)}(x) - Du^{(h)}(0)|^2 \le c \sup_{\bar{B}_{\frac{1}{2}}(0)} |D^2u^{(h)}|^2$$

we bound

$$\|u^{(h)} - u^{(h)}(0)\|^2_{1,\infty;\bar{B}_{\frac{1}{2}}(0)} \leq Ch^{-n} \int_{B_h(x)} |D^2u|^2 + |Du|^2 \leq C\frac{\varepsilon}{h^4} \tag{3.27}$$

by the monotonicity (3.25).

3. Recalling (3.26) we note

$$Du_h = \nabla\Pi \circ u^{(h)} \cdot Du^{(h)}, \quad D^2u_h = \nabla^2\Pi \circ u^{(h)}(Du^{(h)}, Du^{(h)}) + \nabla\Pi \circ u^{(h)} D^2u^{(h)}.$$

By Remark 1.48 it follows for any $0 < \theta < \frac{1}{4}$ that

$$\begin{aligned}\theta^{4-n} \int_{B_\theta(0)} |D^2u_h|^2 + \theta^{-2}|Du_h|^2 &\leq C\theta^{4-n} \int_{B_\theta(0)} |D^2u^{(h)}|^2 + |Du^{(h)}|^4 + \theta^{-2}|Du^{(h)}|^2 \\ &\leq C\theta^{4-n} \int_{B_\theta(0)} |D^2u^{(h)}|^2 + \theta^{-2}|Du^{(h)}|^2\end{aligned}$$

where the second inequality follows by (3.22). To estimate the latter we note that

$$\begin{aligned}\theta^{4-n} \int_{B_\theta(0)} |D^2u^{(h)}|^2 + \theta^{-2}|Du^{(h)}|^2 &\leq C\theta^{4-n} \int_{B_\theta(0)} |D^2v|^2 + \theta^{-2}|Dv|^2 \\ &\quad + C\theta^{4-n} \int_{B_\theta(0)} |D^2u^{(h)} - D^2v|^2 + \theta^{-2}|Du^{(h)} - Dv|^2 \ = I + II.\end{aligned}$$

Using the interior regularity estimates from Lemma 1.43 we have

$$\sup_{B_{\frac{1}{4}}(0)} |Dv|^2 \leq c \fint_{B_{\frac{1}{2}}(0)} |Dv|^2, \qquad \sup_{B_{\frac{1}{4}}(0)} |D^2v|^2 \leq c \fint_{B_{\frac{1}{2}}(0)} |D^2v|^2.$$

With these estimates, the boundary data for v, the Poincaré inequality and the identity (3.21) we estimate

$$\begin{aligned}I &\leq C\theta^4 \int_{B_{\frac{1}{2}}(0)} |D^2v|^2 + \theta^{-2}|Dv|^2 \\ &\leq C\theta^2 \int_{B_{\frac{1}{2}}(0)} |\Delta v - \Delta u^{(h)}|^2 + C\theta^4 \int_{B_{\frac{1}{2}}(0)} |D^2u^{(h)}|^2 + \theta^{-2}|Du^{(h)}|^2 \\ &\leq C\theta^2 \int_{B_{\frac{1}{2}}} |D^2u^{(h)}|^2 + |Du^{(h)}|^2 \\ &\leq C\theta^2 \int_{B_1(0)} |D^2u|^2 + |Du|^2 \end{aligned} \tag{3.28}$$

where the second last inequality is due to the minimality of v, i.e.

$$\int_{B_{\frac{1}{2}}(0)} |\Delta v|^2 \leq \int_{B_{\frac{1}{2}}(0)} |\Delta u^{(h)}|^2,$$

and the last inequality is via the bounds

$$\begin{aligned}\int_{B_{\frac{1}{2}}(0)} |Du^{(h)}|^2 &\leq c\int_{B_{\frac{1}{2}}(0)} dx \int_{B_h(x)} \varphi^{(h)}(x-y)|Du|^2(y)dy \\ &\leq c\int_{B_1(0)} dy\, |Du|^2(y)\int_{\mathbb{R}^n} \varphi^{(h)}(x-y)dx \\ &\leq c\int_{B_1(0)} |Du|^2, \end{aligned} \tag{3.29}$$

since $\int_{B_1(0)} \varphi(z)dz = 1$, with an identical calculation also showing

$$\int_{B_{\frac{1}{2}}(0)} |D^2u^{(h)}|^2 \leq c\int_{B_1(0)} |D^2u|^2. \tag{3.30}$$

Similarly, using integration by parts and the properties of v, we bound the second term by

$$\begin{aligned} II &\leq C\theta^{2-n}\int_{B_{\frac{1}{2}}(0)} |\Delta u^{(h)} - \Delta v|^2 = C\theta^{2-n}\int_{B_{\frac{1}{2}}(0)} \left\langle u^{(h)} - v, \Delta^2 u^{(h)}\right\rangle \\ &\leq C\theta^{2-n}\sup_{B_{\frac{1}{2}}(0)} |u^{(h)} - v| \int_{B_{\frac{1}{2}}(0)} |\Delta^2 u^{(h)}|. \end{aligned}$$

Now by Green's formula

$$\begin{aligned}\Delta^2 u^{(h)}(x) = \int_{\mathbb{R}^n} \Delta_x^2\varphi^{(h)}(x-y)u(y)dy &= \int_{\mathbb{R}^n} \Delta_y^2\varphi^{(h)}(x-y)u(y)dy \\ &= \int_{\mathbb{R}^n} \Delta_y\varphi^{(h)}(x-y)\Delta u(y)dy\end{aligned}$$

Thus if $\psi(y) = \varphi^{(h)}(x-y)$ we have

$$|\Delta^2 u^{(h)}(x)| \leq \Big|\int_{B_h(x)} \langle \Delta\psi, \Delta u\rangle\, dy\Big|.$$

By the stationary biharmonic map equation of Lemma 3.13 it follows that

$$\begin{aligned}\Big|\int_{B_h(x)} \langle \Delta u, \Delta\psi\rangle\, dy\Big| &\leq c\int_{B_h(x)} (|D^2u|^2 + |Du|^4)|\psi| \\ &\quad + c\int_{B_h(x)} (|D^2u||Du| + |Du|^3)|D\psi|.\end{aligned}$$

Since $D\varphi^{(h)} = h^{-1}(D\varphi)^{(h)}$ calculations identical to (3.29) and (3.30) imply

$$\begin{aligned}\int_{B_{\frac{1}{2}}(0)} |\Delta^2 u^{(h)}(x)|dx &\leq C\int_{B_1(0)} (|D^2u|^2 + |Du|^4) \\ &\quad + Ch^{-1}\int_{B_1(0)} (|D^2u||Du| + |Du|^3).\end{aligned}$$

Then by (3.22) we conclude

$$\int_{B_{\frac{1}{2}}(0)} |\Delta u^{(h)}| \leq Ch^{-1} \int_{B_1(0)} |D^2u|^2 + |Du|^2.$$

By combining this with the estimate (3.27) we finally bound

$$II \leq C\theta^{2-n}\varepsilon^{\frac{1}{2}}h^{-3} \int_{B_1(0)} |D^2u|^2 + |Du|^2. \tag{3.31}$$

4. We seek to define a competitor to u by modifying u_h so that it agrees with u on $\partial B_1(0)$. To do so first fix $0 < \alpha < 1$ to be of our choosing then set $\tau = \varepsilon^\gamma$ where $\gamma = \gamma(n) \in (0, \alpha]$ is to be chosen to depend only on n. Without loss of generality assume $\tau < \frac{1}{8}$ and that $\varepsilon^\alpha < \frac{1}{6}$ for ε sufficiently small. Let k be the greatest positive integer less than or equal to $\frac{\tau}{3\varepsilon^{2\alpha}}$ and write $[\tau, \tau + 3k\varepsilon^{2\alpha}] = \bigcup_{i=1}^k I_i$ where $I_i = [\tau + 3(i-1)\varepsilon^{2\alpha}, \tau + 2i\varepsilon^{2\alpha}]$. Then by the slicing technique of §1.1 there exists $1 \leq j \leq k$ such that

$$\int_{|x|\in I_j} |Du|^2 \leq \frac{1}{k} \int_{B_{\tau+3k\varepsilon^{2\alpha}} \setminus B_\tau} |Du|^2 \leq 6\varepsilon^\alpha \int_{B_1(0)} |Du|^2 \tag{3.32}$$

and

$$\int_{|x|\in I_j} |D^2u|^2 \leq \frac{1}{k} \int_{B_{\tau+3k\varepsilon^{2\alpha}} \setminus B_\tau} |D^2u|^2 \leq 6\varepsilon^\alpha \int_{B_1(0)} |D^2u|^2, \tag{3.33}$$

since $k > \frac{\tau}{3\varepsilon^{2\alpha}} - 1 > \frac{1}{3\varepsilon^\alpha} - 1 > \frac{1}{6\varepsilon^\alpha}$.

Now define θ to be such that $I_j = [\theta - \varepsilon^{2\alpha}, \theta + 2\varepsilon^{2\alpha}]$ and set $\bar{h} = \varepsilon^{3\alpha}$. Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function that satisfies

$$h(r) = \begin{cases} \bar{h} & \text{if } r \leq \theta + \frac{1}{3}\varepsilon^{2\alpha} \\ 0 & \text{if } r \geq \theta + \frac{2}{3}\varepsilon^{2\alpha} \end{cases}$$

with $|h'(r)| \leq c\bar{h}\varepsilon^{-2\alpha} = c\varepsilon^\alpha$ and $|h''(r)| \leq c\bar{h}^2\varepsilon^{-4\alpha} = c\varepsilon^{2\alpha}$. Then set

$$u^{(h(r))}(x) = \varphi^{(h(r))} * u(x) = \int_{B_1(0)} \varphi^{(h(r))}(x-y)u(y)dy, \qquad r = |x|,$$

and define $u_0 = \Pi \circ u^{(h(r))}$. By construction

$$u_0(x) = \begin{cases} u_{\bar{h}}(x) & \text{if } |x| \leq \theta \\ u(x) & \text{if } |x| \geq \theta + \varepsilon^{2\alpha} \end{cases}$$

so that $u \in W^{2,2}(B_{\theta+2\varepsilon^{2\alpha}}(0), \mathcal{M})$ with $u - u_0 \in W^{2,2}(B_{\theta+\varepsilon^{2\alpha}}(0), \mathbb{R}^N)$. In which case we have

$$\int_{B_{\theta+\varepsilon^{2\alpha}}} |\Delta u|^2 \leq \int_{B_{\theta+\varepsilon^{2\alpha}}} |\Delta u_0|^2$$

by minimality. As $\tau < \theta < 2\tau$ the latter inequality implies

$$\tau^{4-n}\int_{B_\tau}|D^2u|^2+\tau^{-2}|Du|^2 \le \tau^{4-n}\int_{B_{\theta+\varepsilon^{2\alpha}}}|D^2u|^2+\tau^{-2}|Du|^2$$
$$\le C\tau^{4-n}\int_{B_{\theta+\varepsilon^{2\alpha}}}|D^2u_0|^2+\tau^{-2}|Du_0|^2,$$

since by Poincaré

$$\int_{B_{\theta+\varepsilon^{2\alpha}}}|Du|^2 \le C(2\tau)^2\int_{B_{\theta+\varepsilon^{2\alpha}}}|D^2u-D^2u_0|^2+\int_{B_{\theta+\varepsilon^{2\alpha}}}|Du_0|^2$$

and by (3.21) we have

$$\int_{B_{\theta+\varepsilon^{2\alpha}}}|D^2u|^2 \le C\int_{B_{\theta+\varepsilon^{2\alpha}}}|\Delta u|^2+|D^2u_0|^2.$$

Furthermore, from (3.28), (3.31), the below Claim 3.23 and (3.32), (3.33) we conclude

$$\tau^{4-n}\int_{B_{\theta+\varepsilon^{2\alpha}}}|D^2u_0|^2+\tau^{-2}|Du_0|^2 \le C\theta^{4-n}\int_{B_\theta}|D^2u_{\bar h}|^2+\theta^{-2}|Du_{\bar h}|^2$$
$$+C\tau^{4-n}\int_{B_{\theta+\varepsilon^{2\alpha}}\setminus B_\theta}|D^2u_0|^2+\tau^{-2}|Du_0|^2$$
$$\le C\Big(\tau^2+\tau^{2-n}(\varepsilon^\alpha+\varepsilon^{\frac12}\bar h^{-3})\Big)\int_{B_1(0)}|D^2u|^2+|Du|^2.$$

Setting $\alpha=\frac{1}{36}$ and $\gamma=\min\{\frac{\alpha}{2(n-2)},\frac{\alpha}{4}\}$ yields

$$\tau^{4-n}\int_{B_\tau}|D^2u|^2+\tau^{-2}|Du|^2 \le C\varepsilon^{2\gamma}\int_{B_1(0)}|D^2u|^2+|Du|^2$$

whence (3.24), provided ε is small enough so that $C\varepsilon^{2\gamma}\le\frac12$.

It remains to prove the following:

Claim 3.23. There exists $c>0$ such that

$$\int_{B_{\theta+\varepsilon^{2\alpha}}\setminus B_\theta}|D^2u_0|^2+\tau^{-2}|Du_0|^2 \le c\int_{B_{\theta+2\varepsilon^{2\alpha}}\setminus B_{\theta-\varepsilon^{2\alpha}}}|Du|^2+\tau^{-2}|Du|^2.$$

To see this we note, by the boundedness of $\nabla\Pi$ and $\nabla^2\Pi$, that it is enough to prove the claim for $u^{(h)}=u^{(h(|x|))}(x)$, instead of u_0. As

$$u^{(h(|x|))}(x)=\int_{B_1(0)}\varphi(y)u(x-h(|x|)y)dy$$

it is clear that $u^{(h)}$ is smooth whenever u is smooth. Thus in the first instance let us first consider a smooth $u\colon B_{\frac12}(0)\to\mathbb{R}^N$. Now if Ω is a domain compactly contained in

$B_{1-\varepsilon^{2\alpha}}(0)$ we find that

$$\int_\Omega |Du^{(h)}|^2 \le c\int_\Omega \int_{B_1(0)} |Du(x-h(|x|)y)|^2 dydx$$
$$\int_\Omega |D^2u^{(h)}|^2 \le c\int_\Omega \int_{B_1(0)} (|D^2u(x-h(|x|)y)|^2 + |h''(|x|)|^2|Du(x-h(|x|)y)|^2)dydx.$$

For $x\in\Omega$ let $z = x-h(|x|)y$ defines a map $F_y\colon \Omega\to\Omega_{\varepsilon^{2\alpha}}$, where $\Omega_{\varepsilon^{2\alpha}} = \{x : \mathrm{dist}(x,\Omega) < \varepsilon^{2\alpha}\}$. By the definition of $h(|x|)$ we see that F_y is a diffeomorphism onto $F_y(\Omega)$ with a Jacobian approximately one. Thus by a change of variables we can estimate

$$\int_\Omega |Du|^2(x-h(|x|)y)dx \le c\int_{\Omega_{\varepsilon^{2\alpha}}} |Du|^2$$
$$\int_\Omega |D^2u|^2(x-h(|x|)y)dx \le c\int_{\Omega_{\varepsilon^{2\alpha}}} |D^2u|^2$$

so that

$$\int_\Omega |Du^{(h)}|^2 \le c\int_{\Omega_{\varepsilon^{2\alpha}}} |Du|^2, \qquad \int_\Omega |D^2u^{(h)}|^2 \le c\int_{\Omega_{\varepsilon^{2\alpha}}} |D^2u|^2. \tag{3.34}$$

Now if u_j is a sequence of smooth maps from $B_{\frac34}(0)\to\mathbb{R}^N$ converging strongly to u in $W^{2,2}(B_{\frac34}(0),\mathcal{M})$ then the latter inequality implies

$$\int_{B_{\frac12}(0)} |Du_i^{(h)} - Du_j^{(h)}|^2 \le c\int_{B_{\frac34}(0)} |Du_i - Du_j|^2$$
$$\int_{B_{\frac12}(0)} |D^2u_i^{(h)} - D^2u_j^{(h)}|^2 \le c\int_{B_{\frac34}(0)} |D^2u_i - D^2u_j|^2$$

and hence $\{u_j^{(h)}\}$ is Cauchy in $W^{2,2}(B_{\frac12}(0),\mathbb{R}^N)$. It then follows that $\lim_{j\to\infty} u_j^{(h)} = u^{(h)}$ and $u^{(h)} = u$ on $B_{\frac12}(0)\backslash B_{\theta+\varepsilon^{2\alpha}}$. We can then apply (3.34) with $\Omega = B_{\theta+\varepsilon^{2\alpha}}\backslash B_{\varepsilon^{2\alpha}}$ to get the desired conclusion. □

Remark 3.24. In the literature most proofs of Theorem 3.20 are for stationary solutions. The first of which can be found in [CWY], using techniques from work of F. Hélein (cf. §3.4) that exploits the special structure of the nonlinear terms, for mappings into a sphere. A follow up paper [W1] again looks at partial regularity in the stationary case for both extrinsic and intrinsic biharmonic maps $u\in W^{2,2}(\Omega,\mathcal{M})$ into a compact manifold $\mathcal{M}$ and concludes u is smooth away from a closed set of $(n-4)$-dimensional Hausdorff measure zero. The paper also states the monotonicity formulae for stationary intrinsic biharmonic maps takes the same form as that for stationary extrinsic biharmonic maps, i.e. Proposition 3.16. The work of [S23], extending previous results from [RS] for harmonic maps, proves an ε-regularity theorem for stationary biharmonic maps into arbitrary targets via a gauge theory.

As for results using minimality, the paper of [HW3] proves a partial regularity result for minimising biharmonic maps into a sphere. Unlike the stationary case the singular set is shown to have a Hausdorff dimension is at most $n-5$ via Federer's dimension deduction argument. A further result of [S3] looks at dimension reduction for minimising (extrinsic) biharmonic maps into a compact manifold. The approach is based on the analysis of defect measures using the tools of geometric measure theory. Of interest is Scheven's remark that Luckhaus' construction of a comparison map, via an inductive cell decomposition of the domain, seems to be cumbersome in the setting of $W^{2,2}$-maps (cf. Remark 4.7). Our proof of Theorem 3.20 shows the original technique of Schoen and Uhlenbeck [SU1] to be robust enough to give a partial regularity result for minimising biharmonic mappings into an arbitrary compact manifold.

3.4. Remarks concerning Hélein's moving frame method and the regularity of weakly harmonic maps

As we have seen earlier in this chapter, a direct consequence of the partial regularity theorem of Schoen and Uhlenbeck [SU1] shows 2-dimensional locally energy minimising harmonic maps to be Hölder continuous, and in fact smooth, everywhere on the domain $\Omega \subset \mathbb{R}^2$ (cf. Corollary 3.2). The question as to whether or not this holds for 2-dimensional weakly harmonic maps, i.e. solutions to (1.28), is not immediately clear. Nonetheless the work of Hélein [H11] shows this result to be true. The approach of Hélein places a strong emphasis on the role of symmetries in variational problems principally through the use of Noether's theorem: if a variational problem is invariant under the action of a continuous group of symmetries, one can associate to each solution of this variational problem a system of conservation laws, i.e. one or several divergence-free vector fields, defined on the source domain with the number of independent conservation laws equalling the dimension of the group of symmetries.

The prototypical example is illustrated by weakly harmonic maps into the sphere S^k. In this case (1.28) reduces, via Example 1.52, to

$$-\Delta u = |Du|^2 u. \tag{3.35}$$

With this the idea is to exploit the special structure of the right-hand side in order to 'compensate' for the fact $|Du|^2 u \in L^1_{\mathrm{loc}}$ *a priori*. To achieve this note that the condition $|u| = 1$ implies $\sum_{b=1}^{k+1} u^b Du^a \cdot Du^b = Du^a \cdot D(\frac{1}{2}|u|^2) = 0$ from which we rewrite (3.35) as

$$-\Delta u^a = \sum_{b=1}^{k+1} (u^a Du^b - u^b Du^a) \cdot Du^b.$$

In this form we observe

$$u^a Du^b - u^b Du^a$$

is divergence free. In terms of symmetry groups acting on the target manifold (cf. [H11, §1.3]) we recall the group of rotations $SO(k+1)$ acts on S^k via

$$SO(k+1) \times S^k \to S^k; \quad (A, y) \mapsto A.y$$

where $A.y$ is defined by the matrix product. The set of Killing vector fields on S^k can be identified with the Lie algebra $\mathfrak{so}(k+1) = \{X \in \mathfrak{gl}(k+1) : X^T = -X\}$ of skew-symmetric matrices associated to $SO(k+1)$. To each element $A \in \mathfrak{so}(k+1)$ we associate the tangent vector field on S^k given by $y \mapsto A.y$ from which all Killing vector fields can be obtained. Then the theorem of Noether [H11, Theor. 1.3.1] states that the Noether currents

$$J = \sum_{i=1}^{n} J^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \Big\langle A.u, \frac{\partial u}{\partial x^i} \Big\rangle \frac{\partial}{\partial x^i} \tag{3.36}$$

are divergence free, i.e. we have the conservation law div $J = 0$. Note that the number of independent conservation laws equalling the dimension of the group of symmetries.

Example 3.25. In the special situation when $k = 2$ we note that any $A \in \mathfrak{so}(3)$ can be written as

$$A = \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix} = aR_x + bR_y + cR_z,$$

where R_x, R_y, R_z are $\frac{\pi}{2}$-rotations about the x, y and z axis given by

$$R_x = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & 1 \\ & -1 & 0 \end{array}\right), \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_z = \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 0 \end{array}\right).$$

As R_x, R_y and R_z forms a basis for $\mathfrak{so}(3)$ it follows from (3.36) that the vector of currents

$$\Big(u^2 \frac{\partial u^3}{\partial x^i} - u^3 \frac{\partial u^2}{\partial x^i}, u^3 \frac{\partial u^1}{\partial x^i} - u^1 \frac{\partial u^3}{\partial x^i}, u^1 \frac{\partial u^2}{\partial x^i} - u^2 \frac{\partial u^1}{\partial x^i}\Big)$$

is divergence free so that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x^i}\Big(u \times \frac{\partial u}{\partial x^i}\Big) = 0.$$

Having expounded the symmetry group for S^k we can, using the language of differential forms, define

$$\beta^{ab} = *(u^a du^b - u^b du^a)$$

where $*$ denotes the Hodge star operator. Then the conservation law can be written as $d\beta^{ab} = 0$ and (3.35) can be rewritten as

$$-\Delta u^a \, dx^1 \wedge \cdots \wedge dx^n = \sum_{b=1}^{k+1} du^b \wedge \beta^{ab}.$$

By the Poincaré lemma every closed form is exact on a contractible domain so there exists $\alpha \in W^{1,2}_{\text{loc}}$ such that $\beta = d\alpha$. Thus when $n = 2$ we conclude that

$$-\Delta u^a = \sum_{b=1}^{k+1} \Big(\frac{\partial \alpha^{ab}}{\partial x} \frac{\partial u^b}{\partial y} - \frac{\partial \alpha^{ab}}{\partial y} \frac{\partial u^b}{\partial x} \Big).$$

In this form the right-hand side is, in terms of integrability, still no better than L^1_{loc} however it now presents the same structure as in the following:

Theorem 3.26 (Wente, [W4]). *Let $A, B \in W^{1,2}(D^2)$ be two functions on the unit disk $D^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ and let $u \in W^{1,1}_0(D^2)$ solve*

$$\begin{aligned} -\Delta u &= \{A, B\} \quad && \text{in } D^2 \\ u &= 0 && \text{on } \partial D^2 \end{aligned}$$

where

$$\{A, B\} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}.$$

Then $u \in C^0(\bar{D}^2) \cap W^{1,2}(D^2)$ and

$$\|u\|_{L^\infty} + \|Du\|_{L^2} \leq C \|DA\|_{L^2} \|DB\|_{L^2}$$

where the constant $C > 0$ is independent of A and B.

From this compensated regularity result it follows that $u \in C^0_{\text{loc}}(D^2, S^k)$ which together with Schauder estimates gives the desired:

Theorem 3.27 (Hélein, [H9]). *If $u \in W^{1,2}_{\text{loc}}(\Omega, S^k)$ is a weakly harmonic map on a 2-dimensional domain $\Omega \subset \mathbb{R}^2$ then $u \in C^\infty(\Omega, S^k)$.*

The self evident limitation of the above method lies in the fact that symmetric variational problems are exceptions. The challenge to prove the same result for harmonic maps into a non-symmetric m-dimensional manifold $\mathcal{M}$ requires new techniques. The approach of Hélein, first outlined in [H10], employs the idea of moving frames, i.e. to each point y on $\mathcal{M}$ one assigns an orthonormal basis of the tangent space $T_y\mathcal{M}$ that depends smoothly on y. The method goes back to Gaston Darboux's work on the study of surfaces and was later developed by Elie Cartan in the context of Riemannian Geometry. For the purposes here the choice of orthonormal frame fields allows one to select

moving frames that are harmonic sections of the fibre bundle. Since the rotation group $SO(m)$ is a symmetry group for that bundle, and for the associated harmonic section, conservation laws reappear as 'infinitesimal symmetries' (in contrast to the divergence form structure, say, for the sphere which is an artefact of the global symmetries of the target manifold). The resulting moving frame, inspired by the analogy with the use of Coulomb gauges by physicists for gauge theories, is called a *Coulomb frame*. The upshot of such a selection of privileged coordinates leads to the appearance of structures similar to that of Theorem 3.26.

When implementing Hélein's approach on a technical level one first would like to guarantee the existence of an orthonormal frame field $\widetilde{e}$ over the manifold $\mathcal{M}$. However due to topological reasons it is not always possible to do so since there are non-parallelisable manifolds like the sphere S^k (excluding the cases S^1, S^3 and S^7) which do not permit such structures.[4] This is in contrast to, say, a Lie group, the k-dimensional torus, or for that matter any orientable 3-manifold which are all parallelisable. The purpose of the following lemma is to surmount this difficulty by taking further embedding $\boldsymbol{i}$ of the given manifold $\mathcal{M}$ into a manifold $\widehat{\mathcal{M}}$ in such a way that a weakly harmonic map u into $\mathcal{M}$ will yield a weakly harmonic map $\boldsymbol{i} \circ u$ with values in $\widehat{\mathcal{M}}$ which in addition is constructed so that $\widehat{\mathcal{M}}$ is parallelisable.

Lemma 3.28 ([H11, Lem. 4.1.2])**.** *Let $\mathcal{M}$ be a compact m-dimensional submanifold without boundary that is C^r-embedded in $\mathbb{R}^N$ for $r \geq 4$. Then there exists an N-dimensional submanifold $\widehat{\mathcal{M}}$ which is C^{r-1}-isometrically embedded in $\mathbb{R}^{\widehat{N}}$ and a C^{r-1}-embedding $\boldsymbol{i}$ of $\mathcal{M}$ in $\widehat{\mathcal{M}}$ such that*

(1) $\boldsymbol{i}\colon \mathcal{M} \to \widehat{\mathcal{M}}$ is an isometric embedding.
(2) $\widehat{\mathcal{M}}$ is diffeomorphic to the torus $\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{N\text{-}times}$.
(3) If $u \in W^{1,2}(\Omega, \mathcal{M})$ is a weakly harmonic map then $\boldsymbol{i} \circ u \in W^{1,2}(\Omega, \widehat{\mathcal{M}})$ is also.

Remark 3.29. One of the advantages of this construction is that it is valid for all dimensions; on the other hand one needs to assume $\mathcal{M}$ is at least C^r for $r \geq 4$. From the perturbation argument of the below proof one can also see the inapplicable of this lemma to locally energy minimising harmonic maps.

Proof. As $\mathcal{M}$ is compact there exists an N-cube $Q \subset \mathbb{R}^N$ which contains $\mathcal{M}$. By identifying opposite faces of Q we obtain an N-dimensional flat torus T^N with the Euclidean

[4] A manifold is said to be *parallelisable* if there exists a global basis of tangent vectors for the tangent space.

metric $\langle \cdot, \cdot \rangle$. Indifferently we consider $\mathcal{M}$ as a submanifold of Q, T^N or $\mathbb{R}^N$. Let

$$\mathcal{O} = \mathcal{O}_{2\varepsilon}(\mathcal{M}) = \{z \in Q : \operatorname{dist}(z, \mathcal{M}) < 2\varepsilon\}$$

be a tubular neighbourhood of $\mathcal{M}$. As the retraction $P\colon \mathcal{O} \to \mathcal{M}$ is a surjective submersion there exists fibration of the tubular neighbourhood, i.e. for each $x \in \mathcal{M}$ the fiber over x, denoted $\mathcal{O}_x$, is the inverse image $P^{-1}(x) \subset \mathcal{O}$; it is a closed embedded submanifold by the implicit function theorem. As $\mathcal{O}$ has the Riemannian metric $\langle \cdot, \cdot \rangle$, the tangent space $T_z\mathcal{O}$ each point $z \in \mathcal{O}$ decomposes into an orthogonal direct sum

$$T_z\mathcal{O} = H_z \oplus V_z$$

where $V_z = \ker P_* = T_z\mathcal{O}_{P(z)}$ is the vertical space and $H_x = V_x^{\perp}$ is the horizontal space. Moreover, as g is the Riemannian metric on $\mathcal{M}$ we also have that $\langle X, Y \rangle = g(P_*X, P_*Y)$ whenever X and Y are horizontal. Any vector field W on $\mathcal{O}$ can be written uniquely as $W = W^H + W^V$, where W^H is horizontal and W^V is vertical. If X is vector field on $\mathcal{M}$ there exists a unique horizontal vector field $\widetilde{X}$ on $\mathcal{O}$, called the horizontal lift of X, that is P-related to X (i.e. $P_*\widetilde{X}_z = X_{P(z)}$ for each $z \in \mathcal{O}$).

Now let $\eta \in C_c^\infty(\mathcal{O}_{2\varepsilon}(\mathcal{M}))$ such that $\eta = 1$ on $\mathcal{O}_\varepsilon(\mathcal{M})$ with $\eta = 0$ outside $\mathcal{O}_{2\varepsilon}(\mathcal{M})$. We can then define the metric

$$h = \eta \langle \cdot, \cdot \rangle_{\mathcal{O}} + (1 - \eta) \langle \cdot, \cdot \rangle_{\mathbb{R}^N}$$

on the cube Q, where the metric on $\mathcal{O} = \mathcal{O}_{2\varepsilon}(\mathcal{M})$, due to the fibration, is given by

$$\langle X_z, Y_z \rangle_{\mathcal{O}} = g_{P(z)}(P_*X_z, P_*Y_z) + \langle X_z^V, Y_z^V \rangle_{\mathbb{R}^N}$$

for all $z \in \mathcal{O}$ and $X, Y \in T_z\mathcal{O}$. Hence we have a Riemannian manifold (Q, h).

Using the Nash embedding theorem we may isometrically embed (Q, h) into the Euclidean space $\mathbb{R}^{\widehat{N}}$ for some positive integer $\widehat{N}$ sufficiently large. Let $\boldsymbol{i}$ denote the embedding and let $\widehat{\mathcal{M}} = \boldsymbol{i}(Q)$ so that $\mathcal{M} \subset Q$ with $\boldsymbol{i}(\mathcal{M}) \subset \widehat{\mathcal{M}}$. Again, we consider the tubular neighbourhood

$$\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_\delta(\widehat{\mathcal{M}}) = \{z \in \mathbb{R}^{\widehat{N}} : \operatorname{dist}(z, \widehat{\mathcal{M}}) < \delta\}$$

with a retraction $\widehat{P}\colon \widehat{O} \to \widehat{\mathcal{M}}$. Also let

$$\widehat{\mathcal{O}}_\delta(\mathcal{M}) = \{z \in \mathbb{R}^{\widehat{N}} : \operatorname{dist}(z, \boldsymbol{i}(\mathcal{M})) < \delta\}$$

and suppose $\delta > 0$ is sufficiently small so that $\widehat{\mathcal{O}}_\delta(\mathcal{M}) \cap \widehat{\mathcal{M}} \subset \boldsymbol{i}(\mathcal{O}_\varepsilon(\mathcal{M}))$. Then, since $\boldsymbol{i}$ is an embedding, we can define the inverse diffeomorphism $\boldsymbol{j}\colon \widehat{\mathcal{O}}_\delta(\mathcal{M}) \cap \widehat{\mathcal{M}} \to \mathcal{O}_\varepsilon(\mathcal{M})$ such that $\boldsymbol{i} \circ \boldsymbol{j} = \mathrm{id}_{\widehat{\mathcal{O}}_\delta(\mathcal{M}) \cap \widehat{\mathcal{M}}}$ and $\boldsymbol{j} \circ \boldsymbol{i} = \mathrm{id}_{\mathcal{O}(\mathcal{M})}$.

Now for $u \in W^{1,2}(\Omega, \mathcal{M})$ we define

$$E(u) = \frac{1}{2}\int_\Omega \left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^i} \right\rangle_{\mathbb{R}^N} dx$$

and for $v \in W^{1,2}(\Omega, \widehat{\mathcal{M}})$ let

$$\widehat{E}(v) = \frac{1}{2}\int_\Omega \left\langle \frac{\partial v}{\partial x^i}, \frac{\partial v}{\partial x^i} \right\rangle_{\mathbb{R}^{\widehat{N}}} dx.$$

To check property (3) we suppose $u \in W^{1,2}(\Omega, \mathcal{M})$ is a weakly harmonic map. For a test function $\phi \in W_0^{1,2}(\Omega, \mathbb{R}^{\widehat{N}}) \cap L^\infty(\Omega, \mathbb{R}^{\widehat{N}})$ set $v = \boldsymbol{i} \circ u \in W^{1,2}(\Omega, \widehat{\mathcal{M}})$ and let

$$w_t = \boldsymbol{j} \circ \widehat{P}(v + t\phi)$$

for $0 < |t| < \delta \|\phi\|_{L^\infty}^{-1}$. As $\boldsymbol{j}$ is an isometry

$$\widehat{E}(\widehat{P}(v + t\phi)) = \frac{1}{2}\int_\Omega h_{w_t}\Big(\frac{\partial w_t}{\partial x^i}, \frac{\partial w_t}{\partial x^i}\Big) dx.$$

By expanding w_t to first order we have

$$w_t = \boldsymbol{j} \circ \widehat{P}(v) + t\psi_t = u + t\psi_t$$

where $\psi_t = \int_0^1 \frac{\partial \boldsymbol{j} \circ \widehat{P}}{\partial y^j}(v + st\phi)\phi^j ds$. Since $w_t \in \mathcal{O}_\varepsilon(\mathcal{M})$ the fibration implies

$$h_{w_t}\Big(\frac{\partial w_t}{\partial x^i}, \frac{\partial w_t}{\partial x^i}\Big) = g_{P(w_t)}\Big(\frac{\partial P(w_t)}{\partial x^i}, \frac{\partial P(w_t)}{\partial x^i}\Big) + \left\langle \Big(\frac{\partial w_t}{\partial x^i}\Big)^V_{w_t}, \Big(\frac{\partial w_t}{\partial x^i}\Big)^V_{w_t} \right\rangle_{\mathbb{R}^N}.$$

Noting $(\frac{\partial w_t}{\partial x^i})^V_{w_t} = (\frac{\partial u}{\partial x^i} + t\frac{\partial \psi_t}{\partial x^i})^V_{u+t\psi_t} = O(t)$ we conclude

$$\widehat{E}(\widehat{P}(v + t\phi)) = E(P(u + t\psi_t)) + O(t^2).$$

As u is weakly harmonic it follows that

$$\lim_{t\to 0} \frac{\widehat{E}(\widehat{P}(v + t\phi)) - \widehat{E}(v)}{t} = 0. \qquad \square$$

As a result of Lemma 3.28 it is possible to lift, if need be, a harmonic map to another which takes values in a parallelisable manifold $\mathcal{M}^m$ (diffeomorphic to a torus). In which case we have available a C^1-orthonormal frame field $\widetilde{e}(y) = (\widetilde{e}_1(y), \ldots, \widetilde{e}_m(y))$ over such a manifold. Let $\mathcal{F}$ denote the fibre bundle of orthonormal frame fields over $\mathcal{M}$. If $u \in W^{1,2}(\Omega, \mathcal{M})$ denotes the harmonic map, the pull-back bundle $u^*\mathcal{F}$ by u is the frame bundle over Ω whose fibre over a point $x \in \Omega$ is the set of orthonormal frames for $T_{u(x)}\mathcal{M}$ depending on x in a $W^{1,2}$-fashion. The gauge group

$$W^{1,2}(\Omega, SO(m)) = \{R \in W^{1,2}(\Omega, \mathbb{R}^{n\times n}) : R \in SO(n) \text{ a.e.}\}$$

on $u^*\mathcal{F}$ acts by sending $e_a \mapsto \sum_{b=1}^m e_b R_a^b$ for any $R = (R_b^a) \in W^{1,2}(\Omega, SO(m))$. As $\widetilde{e} \circ u$ is a section of $u^*\mathcal{F}$ we consider, for each $R \in W^{1,2}(\Omega, SO(m))$, the frame $e = (e_1, \dots, e_m)$ defined by

$$e_a(x) = \sum_{b=1}^m \widetilde{e}_b(u(x)) R_a^b(x)$$

for a.e. $x \in \Omega$. By choosing an R that minimises the functional

$$\mathfrak{F}(R) = \frac{1}{2} \int_\Omega |^t e de|^2 dx,$$

where

$$|^t e de|^2 = \sum_{i=1}^n \sum_{a,b=1}^m \left\langle e_a, \frac{\partial e_b}{\partial x^i} \right\rangle^2,$$

we obtain a so-called Coulomb frame $e = (e_1, \dots, e_m)$ that satisfies the conditions of the following:

Lemma 3.30 ([H11, Lem. 4.1.3])**.** *Let $\mathcal{M}$ be an m-dimensional compact manifold without boundary that is C^2-embedded in $\mathbb{R}^N$. Let $u \in W^{1,2}(\Omega, \mathcal{M})$ and $\widetilde{e}$ be a C^1-section of $\mathcal{F}$. Then there exists a Coulomb frame $e = (e_1, \dots, e_m)$ over Ω associated to u, i.e. a finite energy section of $u^*\mathcal{F}$, such that*

$$\sum_{i=1}^n \frac{\partial}{\partial x^i} \left\langle e_a, \frac{\partial e_b}{\partial x^i} \right\rangle = 0 \quad \text{in } \Omega \tag{3.37a}$$

$$\sum_{i=1}^n \left\langle e_a, \frac{\partial e_b}{\partial x^i} \right\rangle \nu^i(x) = 0 \quad \text{on } \partial\Omega \tag{3.37b}$$

where ν is the normal vector on $\partial\Omega$. Moreover,

$$\|de\|_{L^2(\Omega)} \leq C \|du\|_{L^2(\Omega)}$$

where the constant $C > 0$ depends only on $\mathcal{M}$.

We remark that (3.37) reproduces the conservation law for the currents (3.36) derived for S^k-valued harmonic maps with the same symmetry group (in fact (3.37) is the Euler-Lagrange equation for the minimiser of $\mathfrak{F}$). Thus the moving frame method allows for the recovery of a conservation law that now accompanies the Coulomb frame.

Having established this framework thus far one can proceed by using the weakly harmonic equations for u together with the spaces of Hardy and Lorentz to produce a gain in regularity. The details of which are outlined in [H10] and [H11, Chap. 4] from which we obtain:

Theorem 3.31 (Hélein, [H10]). *Let $\mathcal{M}$ be a compact Riemannian C^r-manifold without boundary for $r \geq 4$. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ is a weakly harmonic map on a 2-dimensional domain $\Omega \subset \mathbb{R}^2$ then $u \in C^\infty(\Omega, \mathcal{M})$.*

Let us also mention the more recent work of Rivière [R4] (and the related [S4]) that gives an alternative proof for weakly harmonic maps into an arbitrary target manifold without the introduction of the indirect moving frame method of Hélein. While Hélein seeks divergence free quantities, the observation of Rivière shows the antisymmetry to be sufficient for regularity.

Theorem 3.32 ([R4]). *Let $\Omega \in L^2(D^2, \bigwedge^1 \mathfrak{so}(k) \otimes \mathbb{R}^2)$ and suppose $u \in W^{1,2}_{\mathrm{loc}}(D^2, \mathbb{R}^N)$ satisfies*

$$-\Delta u = \Omega \cdot Du$$

on a 2-dimensional domain $D^2 \subset \mathbb{R}^2$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(D^2, \mathbb{R}^N) \cap W^{1,p}_{\mathrm{loc}}(D^2, \mathbb{R}^N)$ for every $p \geq 1$ and $\alpha \in [0, 1)$.

Remark 3.33. Note that

$$(\Omega \cdot Du)^a = \sum_{b=1}^{N} \sum_{i=1}^{2} \Omega^a_{bi} u^b_i$$

where $\Omega^a_b = (\Omega^a_{bi})_{i=1,2}$ is such that $\Omega^a_b = -\Omega^b_a$. We recall the weakly harmonic equation $-\Delta u^c = A_{ab}{}^c(u) u^a_i u^b_i$ from (1.28). As the second fundamental form $A(u)$ is orthogonal to $T_u\mathcal{M}$ we have $\sum_{c=1}^{N} A_{cb}{}^a(u) Du^a = 0$ from which we conclude

$$-\Delta u^c = \sum_{i=1}^{2} (A_{ab}{}^c(u) - A_{cb}{}^a(u)) u^b_i u^a_i = \Omega^c_a \cdot Du^a,$$

where

$$\Omega^c_a = (A_{ab}{}^c(u) - A_{cb}{}^a(u)) Du^b \in L^2(D^2, \textstyle\bigwedge^1 \mathfrak{so}(k) \otimes \mathbb{R}^2).$$

Thus weakly harmonic maps into arbitrary manifolds satisfy the assumptions of this theorem.

In the general case when considering domains $D \subset \mathbb{R}^n$ one also has, as an analogy to Theorem 3.1, the following partial regularity result.

Theorem 3.34. *Let $u \in W^{1,2}_{\mathrm{loc}}(D, \mathcal{M})$ be a weakly harmonic map from a domain $D \subset \mathbb{R}^n$ into a compact Riemannian manfiold $\mathcal{M}$ without boundary. Then there exists $\varepsilon > 0$, $\sigma \in (0, 1)$ depending on $\mathcal{M}$ such that if for some ball $B_R(x_0) \subset\subset D$ we have*

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 dx \leq \varepsilon,$$

then $u \in C^{0,\alpha}(B_{\sigma R}(x_0), \mathcal{M})$ for some $\alpha \in (0, 1)$.

The result was first proved by Evans [E9] when $\mathcal{M} = S^k$. The work of Béthuel [B14] generalised the result to arbitrary closed manifolds via the moving frame technique. A subsequent proof by Rivière and Struwe [RS], based on observation from [R4], proved the result without use of the moving frames. In fact the regularity follows from the following more general result (cf. [GM1, §10.5]).

Proposition 3.35 ([RS])**.** *For positive integers n, k there exists $\varepsilon = \varepsilon(n,k) > 0$ such that for every $\Omega \in L^2(B, \bigwedge^1 \mathfrak{so}(k) \otimes \mathbb{R}^n)$ and for every weak solution $u \in W^{1,2}(B, \mathbb{R}^k)$ of*

$$-\Delta u = \Omega \cdot Du$$

on $B = B(0,1) \subset \mathbb{R}^n$ satisfying the growth assumptions

$$\sup_{B_r(x) \subset B} r^{2-n} \int_{B_r(x)} |\Omega|^2 dx \leq \varepsilon$$

and

$$\sup_{B_r(x) \subset B} r^{2-n} \int_{B_r(x)} |Du|^2 dx < +\infty,$$

we have $u \in C^{0,\alpha}_{\mathrm{loc}}(B, \mathbb{R}^k)$ for some $\alpha \in (0,1)$.

Chapter 4

Luckhaus' lemma

In seeking to prove the regularity of vector-valued maps which minimise an integral functional subject to a family of smooth nonlinear (manifold) constraints, a major technical difficulty resides in finding comparison maps which satisfy the manifold constraints (particularly since one cannot localise the problem in the image for $n > 2$). One idea in overcoming this difficulty rests upon implementing a particular lemma attributed to S. Luckhaus. Originally proved in [L21, Lem. 1], and revised in [L22, Lem. 3], it can been seen as an improvement on the extension lemma of [SU1, Lem. 4.3]. Alternative renditions can also be found in [H5, Lem. 2.2] (which is based on estimates from [HL1, §3]) and in the book [S11, §2.12.2]. Our version here is reminiscent of [L22, Lem. 3] although uses ideas alluded to in the approach of [H5, HL1] (which slices Sobolev mappings over lower-dimensional skeletons).

Lemma 4.1 (Luckhaus)**.** *Let $f_1 \in W^{1,p}(\mathcal{M},\mathcal{N})$, $f_2 \in W^{1,p}(\mathcal{M},\mathcal{O})$ and $f_3 \in W^{1,p}(\mathcal{M},\mathbb{R}^k)$ for $p \geq 2$ and $0 < \lambda < 1$, where $\mathcal{M} \subset \mathbb{R}^\ell$ is a closed Riemannian C^r-manifold of dimension $m \geq 2$ and $\mathcal{N} \subset \mathbb{R}^k$ is a closed subset of positive reach with a neighbourhood $\mathcal{O}$. Then there exists a function $f \in W^{1,p}(\mathcal{M} \times (0,\lambda),\mathbb{R}^k)$ with*

$$f(\cdot,0) = f_1, \qquad f(\cdot,\lambda) = f_2$$

and constants $c, C > 0$ depending on p, k, ℓ, $\mathcal{M}$ such that

$$\operatorname*{ess\,sup}_{\mathcal{M}\times(0,\lambda)} \operatorname{dist}(f,\mathcal{N}) \leq c\lambda^{1-\frac{2m+1}{2p}} A^{1-\frac{1}{2p}} \|f_1 - f_2\|_p^{\frac{1}{2p}} \tag{4.1}$$

where $A^p = \int_{\mathcal{M}}(|Df_1|^p + |Df_2|^p + |\frac{f_1-f_2}{\lambda}|^p)$ and

$$\int_{\mathcal{M}\times(0,\lambda)} |\nabla f - (Df_3, 0))|^p \leq C\lambda \int_{\mathcal{M}} |Df_1 - Df_3|^p + |Df_2 - Df_3|^p + |\tfrac{f_1-f_2}{\lambda}|^p. \tag{4.2}$$

Remark 4.2. One can draw parallels between the above statement and Whitney's approximation theorem on manifolds—which states that a continuous map $f_0 \colon \mathcal{M} \to \mathcal{N}$ is homotopic to a smooth map $f_1 \colon \mathcal{M} \to \mathcal{N}$, where the homotopy is a retraction of a convex combination of f_0 and f_1 (cf. [L12, Theor. 6.26]). The construction of f_1 follows by the continuity of f_0 together with a partition of unity argument. The difficulty here is the loss of continuity.

Remark 4.3. To prove this lemma a 'first attempt' might be to construct

$$f(x,s) = f_2(x) + \psi(s)(f_1(x) - f_2(x))$$

where $(x,s) \in \mathcal{M} \times (0,\lambda)$ and $\psi \in C^\infty(0,\lambda)$ is a cut-off function that is $\psi = 1$ in a neighbourhood of 0 and $\psi = 0$ in a neighbourhood of λ with $|\psi'(s)| \leq C\lambda^{-1}$. Then

$$\operatorname{dist}(f(x,s),\mathcal{N}) \leq 2\|f_1 - f_2\|_{L^\infty}$$

and

$$|\nabla f| \leq |Df_1| + |Df_2| + C\lambda^{-1}|f_1 - f_2|,$$

where $\nabla = (D, \partial_s)$ on $\mathcal{M} \times (0,\lambda)$. The problem with this construction is that it affords only L^∞ control on the distance of f from $\mathcal{N}$ despite the fact that the given functions are in L^p. As we desire this distance to be sufficiently small given only L^p control a triangulation of the domain manifold $\mathcal{M}$ is used so that for lower-dimensional simplexes the Sobolev embeddings can be employed then, for simplexes where the embeddings no longer hold, we simply perform a homogeneous degree zero extension. By doing so this 'first attempt' can be recovered via this slicing over lower-dimensional skeletons and homogeneous degree zero extension argument; the end result of which is given by (4.1).

Lemma 4.4 (Morrey-type embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded convex domain and $u, v \in W^{1,1}(\Omega)$. Then for $p' > n$ and any $q \geq 1$,*

$$|u(x) - v(x)|^q \leq |\Omega|^{-\frac{1}{p'}} \|(u-v)^q\|_{p'} + C|\Omega|^{\frac{1}{n}-\frac{1}{p'}} \||u-v|^{q-1}|Du - Dv|\|_{p'}$$

a.e. (Ω) for some $C = C(n,p',q) > 0$.

Proof. From [GT, Lem. 7.16 and 7.12] we have that

$$|u(x) - (u)_\Omega| \leq C|\Omega|^{\frac{1}{n}-\frac{1}{p'}} \|Du\|_{p'} \quad \text{a.e. } (\Omega)$$

and so $|u(x) - u(y)| \leq |u(x) - (u)_\Omega| + |u(y) - (u)_\Omega| \leq c\|Du\|_{p'}$. Now there exists a y such that

$$\begin{aligned} |u(x) - v(x)|^q &\leq \Big| |u-v|^q(x) - |u-v|^q(y) \Big| + \fint |u-v|^q \\ &\leq c\|D|u-v|^q\|_{p'} + |\Omega|^{-1}\|(u-v)^q\|_1 \end{aligned}$$

$$\leq c\||u-v|^{q-1}|D(u-v)|\|_{p'} + |\Omega|^{-\frac{1}{p'}}\|(u-v)^q\|_{p'}$$

since $|D|u|^q| \leq q|u|^{q-1}|Du|$ and $\|u\|_1 \leq |\Omega|^{1-\frac{1}{p'}}\|u\|_{p'}$. □

Corollary 4.5. *Suppose $u, v, \in W^{1,1}(B)$ on a ball $B \subset \mathbb{R}^n$ of radius $r > 0$. Then for $p > n + \frac{1}{2}$,*

$$|u(x) - v(x)| \leq Cr^{\alpha}\Big(\int |Du - Dv|^p + \frac{|u-v|^p}{r^p}\Big)^{\frac{1}{p}(1-\frac{1}{2p})}\Big(\int |u-v|^p\Big)^{\frac{1}{p}\frac{1}{2p}}$$

a.e. (B) for some $C > 0$ where $\alpha = (1 - \frac{2n}{2p-1})(1 - \frac{1}{2p})$.

Proof. For any $p > p'$, Lemma 4.4 and Hölder's (with $\bar{p} = p/p'$ and $\bar{q} = p/(p-p')$) implies that

$$\begin{aligned}
|u(x) - v(x)|^q &\leq Cr^{1-\frac{n}{p'}}\||Du - Dv||u-v|^{q-1}\|_{p'} + Cr^{-\frac{n}{p'}}\||u-v||u-v|^{q-1}\|_{p'} \\
&\leq Cr^{1-\frac{n}{p'}}\Big(\|Du - Dv\|_p + \|r^{-1}|u-v|\|_p\Big)\Big(\int |u-v|^{\frac{(q-1)pp'}{p-p'}}\Big)^{\frac{p-p'}{pp'}} \\
&\leq Cr^{1-\frac{n}{p'}}\Big(\int |Du - Dv|^p + \frac{|u-v|^p}{r^p}\Big)^{\frac{1}{p}}\Big(\int |u-v|^{\frac{(q-1)pp'}{p-p'}}\Big)^{\frac{p-p'}{pp'}}.
\end{aligned}$$

By setting $p' = p - 1/2$ and $q = 2p/(2p-1)$ the desired result follows. □

Lemma 4.6 (Extension)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded convex domain, $p \geq 2$ and $u \in W^{1,p}(\partial\Omega)$. If $\check{u}$ denotes the homogeneous degree 0 extension of u into Ω then for $p < n$ we have*

$$\int_\Omega |D\check{u}|^p \leq C \operatorname{diam}\Omega \int_{\partial\Omega} |Du|^p$$

for some $C = C(n,p) > 0$.

Proof. First consider the case when $\Omega = B_R(0)$. The radial homogeneous degree 0 extension $\check{u}$ is given by $\check{u}(x) = u(R\frac{x}{|x|})$ for $x \in B_R(0)$. In which case $D\check{u}(x) = Du(R\frac{x}{|x|})(I - \frac{x}{|x|} \otimes \frac{x}{|x|})\frac{R}{|x|}$ for $x \in B_R(0)$. It then follows that

$$\begin{aligned}
\int_{B_R(0)} |D\check{u}|^p &= \int_0^R \Big(\int_{\partial B_r(0)} |D\check{u}(x)|^p d\sigma(x)\Big)dr \\
&\leq C\int_0^R \Big(\int_{\partial B_r(0)} |\tfrac{R}{r}Du(R\tfrac{x}{|x|})|^p d\sigma(x)\Big)dr \\
&\leq C\int_0^R \Big(\frac{R}{r}\Big)^{p+1-n} dr \int_{\partial B_R(0)} |Du(R\tfrac{x}{|x|}))|^p d\sigma(x) \leq CR\int_{\partial B_R(0)} |Du|^p
\end{aligned}$$

since $n - p > 0$.

For a general bounded convex domain Ω, there exists a bi-Lipschitz $\phi\colon B_R(0) \to \Omega$. In which case we extend u from $\partial\Omega$ into Ω by defining $(\check{u} \circ \phi)(x) = (u \circ \phi)(R\frac{x}{|x|})$ for

$x \in B_R(0)$. By Rademacher's theorem ϕ is differentiable a.e. and so by the chain rule $D(\check{u} \circ \phi) = ((D\check{u}) \circ \phi) \cdot D\phi$ hence integration by substitution and the bi-Lipschitzness of ϕ (i.e. $c \leq |\det D\phi| \leq C$) implies that

$$\begin{aligned}\int_\Omega |D\check{u}|^p &\leq \int_{B_R(0)} |D(\check{u} \circ \phi)|^p |(D\phi^{-1}) \circ \phi|^p |\det D\phi| \\ &\leq C \int_{B_R(0)} |D(\check{u} \circ \phi)|^p \\ &\leq CR \int_{\partial B_R(0)} |D(u \circ \phi)|^p \leq C \operatorname{diam} \Omega \int_{\partial\Omega} |Du|^p\end{aligned}$$

where the second last inequality follows from the previous step. □

Remark 4.7. If one was to generalise this homogeneous degree 0 extension lemma to the higher order case the need in Theorem 1.37 for Ω to have additional boundary regularity would be incompatible with the Lipschitz regularity of a rectilinear cell complex of Definition 1.60. In addition the estimate (4.1) (and (4.5) below) arising from Lemma 4.4 seems to naturally desires a monotonicity formula to control the bound.

4.1. Proof of Luckhaus' Lemma

With reference to §1.11 let $h\colon K \to \mathcal{M}$ be a triangulation and assume the cells have diameter at most λ without loss of generality. Let $K^{(d)} = \bigcup_j \Delta_j^d$, for $d = 0, \dots, m$, denote the union of all cells of dimension d and let $\bar{K}^{(d)} = \bigcup_{e \leq d} \bigcup_j \Delta_j^e$ denote the d-skeleton of all cells of dimension less than or equal to d, where Δ_j^e denotes a e-cell indexed by j. Let $U = \{x \in \mathbb{R}^\ell : \operatorname{dist}(x, \mathcal{M}) < 2\varepsilon\}$, for some $\varepsilon > 0$, be a tubular neighbourhood of $\mathcal{M}$ so that the retraction $\Pi\colon U \to \mathcal{M}$ is differentiable.

Define $H\colon |K| \times B_\varepsilon^\ell(0) \to \mathcal{M}$ by $H(x, \xi) = H_\xi(x)$ where $H_\xi(x) = \Pi(h(x) + \xi)$. Assume $\varepsilon > 0$ is sufficiently small so that the map H_ξ is a diffeomorphism for each $\xi \in B_\varepsilon^\ell(0)$.[1] It follows that there exists a constant $c_0 > 0$ such that $|\det DH(x, \xi)| \geq c_0$. By the co-area formula (cf. [BZ2, 13.4.6], [HL1, Lem. 3.3]) we have, for any measurable $\psi\colon \mathcal{M} \to \mathbb{R}$, that

$$\begin{aligned}\int_{B_\varepsilon^\ell(0)} d\mathcal{H}^\ell(\xi) \int_{K^{(d)}} &\psi(H(\xi(x)) d\mathcal{H}^d(x) \\ &\leq c_0^{-1} \int_{K^{(d)} \times B_\varepsilon^\ell(0)} \psi(H(x, \xi)) \, |\det DH(x, \xi)| d\mathcal{H}^{d+\ell}(x, \xi) \\ &\leq c_0^{-1} \int_M \Big(\int_{H^{-1}(y)} \psi(H(x, \xi)) d\mathcal{H}^{d-m+\ell}(x, \xi) \Big) d\mathcal{H}^m(y)\end{aligned}$$

[1] Note that for each $x \in \mathcal{M}$ there exists a unique point $y = y(x)$ which is the intersection of $(T_x\mathcal{M})^\perp \cap (x + B_{2\varepsilon}^\ell(0))$ with $\mathcal{M} + \xi$, provided $|\xi| \leq \varepsilon$ for $\varepsilon > 0$ sufficiently small (cf. §1.8.1). In which case we can write $H_\xi^{-1}(x) = h^{-1}(y(x) - \xi)$ for $x \in \mathcal{M}$.

$$= c_0^{-1} \int_{\mathcal{M}} \psi(y) \mathcal{H}^{d-m+\ell}(H^{-1}(y)) d\mathcal{H}^m(y)$$
$$\leq c\lambda^{d-m} \int_{\mathcal{M}} \psi$$

since we can cover $H^{-1}(y) \subset \bigcup_j \Delta_j^d \times B_\varepsilon^\ell(0)$ where $\operatorname{diam}(\Delta_j^d \times B_\varepsilon^\ell(0)) \sim \lambda + \varepsilon$ and $(\lambda+\varepsilon)^{d-m+\ell} \leq (1+\varepsilon)^\ell \lambda^{-(m-d)}$. In which case there exists ξ by the slicing inequality (cf. Example 1.1) such that

$$\int_{K^{(d)}} \psi(H_\xi(x)) d\mathcal{H}^d(x) \leq c\lambda^{d-m} \int_{\mathcal{M}} \psi \tag{4.3}$$

holds. We now proceed by an induction argument over d.

1. For $d = 1, 2, \ldots, \lfloor p-1 \rfloor$ apply Corollary 4.5 with $u = f_1 \circ H_\xi$, $v = f_2 \circ H_\xi$ and $\Omega = \Delta_j^d$ together with (4.3) to bound

$$|f_1 \circ H_\xi(x) - f_2 \circ H_\xi(x)|$$
$$\leq C\lambda^\beta \Big(\int_{\mathcal{M}} |Df_1 - Df_2|^p + \Big| \frac{f_1 - f_2}{\lambda} \Big|^p \Big)^{\frac{1}{p}(1-\frac{1}{2p})} \Big(\int_{\mathcal{M}} |f_1 - f_2|^p \Big)^{\frac{1}{p}\frac{1}{2p}}$$

for $\mathcal{H}^d$-a.e. $x \in \Delta_j^d$, $\beta = 1 - \frac{2m+1}{2p}$, since $d + \frac{1}{2} < p$ (due to $p \geq 2$) and $\operatorname{diam} \Omega \sim \lambda$. As the right-hand side of the latter inequality is independent of Δ_j^d we conclude

$$\operatorname*{ess\,sup}_{K^{(d)}} |f_1 \circ H_\xi - f_2 \circ H_\xi| \leq c\lambda^\beta A^{1-\frac{1}{2p}} \|f_1 - f_2\|_p^{\frac{1}{2p}}. \tag{4.4}$$

By defining

$$F(x, \theta) = \frac{\theta}{\lambda} f_1 \circ H_\xi(x) + \frac{\lambda - \theta}{\lambda} f_2 \circ H_\xi(x)$$

for $(x, \theta) \in K^{(d)} \times (0, \lambda)$ and letting $\nabla = (D, \frac{\partial}{\partial \theta})$ denote the gradient on $K^{(d)} \times (0, \lambda)$ one notes that

$$|\nabla F - (D(f_3 \circ H_\xi), 0)| \leq |D(f_1 \circ H_\xi) - D(f_3 \circ H_\xi)|$$
$$+ |D(f_2 \circ H_\xi) - D(f_3 \circ H_\xi)| + \frac{1}{\lambda} |f_1 \circ H_\xi - f_2 \circ H_\xi|.$$

By integrating this over $K^{(d)} \times (0, \lambda)$ we find

$$\int_{K^{(d)} \times (0,\lambda)} |\nabla F - (D(f_3 \circ H_\xi), 0)|^p$$
$$\leq c\lambda \int_{K^{(d)}} |(Df_1 - Df_3) \circ H_\xi|^p |DH_\xi|^p$$
$$+ c\lambda \int_{K^{(d)}} |(Df_2 - Df_3) \circ H_\xi|^p |DH_\xi|^p$$

$$+ c\lambda \int_{K^{(d)}} \Big| \frac{(f_1 - f_2) \circ H_\xi}{\lambda} \Big|^p$$
$$\leq c\lambda^{d+1-m} \int_{\mathcal{M}} |Df_1 - Df_3|^p + |Df_2 - Df_3|^p + |\tfrac{f_1 - f_2}{\lambda}|^p$$

where in the last inequality follows via (4.3).

2. Now consider $d > \lfloor p-1 \rfloor$. For such integers we:

Claim 4.8. For an extended F,

$$\int_{K^{(d)} \times (0,\lambda)} |\nabla F - (D(f_3 \circ H_\xi), 0)|^p$$
$$\leq c\lambda^{d+1-m} \int_{\mathcal{M}} |Df_1 - Df_3|^p + |Df_2 - Df_3|^p + |\tfrac{f_1 - f_2}{\lambda}|^p.$$

Proof of claim. By the previous step the claim holds for $d = \lfloor p-1 \rfloor$ if $p \geq 2$ (or $d = 1$ if $1 < p < 2$). Assume now the claim holds for $d \geq \lfloor p-1 \rfloor$ if $p \geq 2$ (or $d \geq 1$ if $1 < p < 2$). Let $G = F - f_3 \circ H_\xi$ on $K^{(d)} \times (0, \lambda)$ and note that

$$\partial(\Delta_j^{d+1} \times (0,\lambda)) = \Delta_j^{d+1} \times 0 \cup \Delta_j^{d+1} \times \lambda \cup \partial\Delta_j^{d+1} \times (0,\lambda).$$

Suppose $G(x,\theta)$ is defined on all d-cells $\Delta_j^d \times (0,\lambda)$ with $G(x,0) = (f_1 - f_3) \circ H_\xi(x)$ and $G(x,\lambda) = (f_2 - f_3) \circ H_\xi(x)$ for $x \in \Delta_j^{d+1}$. Thus $G(x,\theta)$ is defined $\mathcal{H}^{d+1}$-a.e. on $\partial(\Delta_j^{d+1} \times (0,\lambda))$. As $\Delta_j^{d+1} \times (0,\lambda)$ is convex in the $(d+2)$-plane we can apply Lemma 4.6 with $\Omega = \Delta_j^{d+1} \times (0,\lambda)$ so that G is degree 0 homogeneously extended into Ω with

$$\int_{\Delta_j^{d+1} \times (0,\lambda)} |\nabla \check{G}|^p \leq c\lambda \int_{\Delta_j^{d+1} \times 0} |D(f_1 \circ H_\xi) - D(f_3 \circ H_\xi)|^p$$
$$+ c\lambda \int_{\Delta_j^{d+1} \times \lambda} |D(f_2 \circ H_\xi) - D(f_3 \circ H_\xi)|^p$$
$$+ c\lambda \int_{\partial\Delta_j^{d+1} \times (0,\lambda)} |\nabla G|^p,$$

since $d + 2 \geq \lfloor p-1 \rfloor + 2 > p$. So if we denote $\partial\Delta_j^{d+1} = \bigcup_k \Delta_{j,k}^d$ and $F = \check{G} + f_3 \circ H_\xi$ then

$$\int_{K^{(d+1)} \times (0,\lambda)} |\nabla F - (D(f_3 \circ H_\xi), 0)|^p = \sum_j \int_{\Delta_j^{d+1} \times (0,\lambda)} |\nabla \check{G}|^p$$
$$\leq c\lambda^{d+2-m} \int_{\mathcal{M}} \big(|Df_1 - Df_3|^p + |Df_2 - Df_3|^p \big) + c\lambda \int_{K^{(d)} \times (0,\lambda)} |\nabla G|^p$$
$$\leq c\lambda^{d+2-m} \int_{\mathcal{M}} |Df_1 - Df_3|^p + |Df_2 - Df_3|^p + |\tfrac{f_1 - f_2}{\lambda}|^p$$

where the first inequality follows from (4.3) with the previous estimate and the second inequality follows by the induction hypothesis.

Furthermore for $d > \lfloor p-1 \rfloor$ the homogeneous degree 0 extension of F implies that

$$\operatorname*{ess\,sup}_{K^{(d)}\times(0,\lambda)} \operatorname{dist}(F,\mathcal{N}) = \operatorname*{ess\,sup}_{K^{(\lfloor p-1\rfloor)}\times(0,\lambda)} \operatorname{dist}(F,\mathcal{N}) \le \operatorname*{ess\,sup}_{K^{(\lfloor p-1\rfloor)}\times(0,\lambda)} |F - f_1\circ H_\xi| \le c\lambda^\beta A^{1-\frac{1}{2p}} \|f_1 - f_2\|_p^{\frac{1}{2p}},$$

where the first inequality is due to the condition $f_1 \circ H_\xi$ take values in $\mathcal{N}$ a.e. and the second follows by (4.4).

3. Now define $f\colon \mathcal{M}\times(0,\lambda)\to\mathbb{R}^k$ by $f(x,\theta) = F(H_\xi^{-1}(x),\theta)$ so that

$$\operatorname*{ess\,sup}_{\mathcal{M}\times(0,\lambda)} \operatorname{dist}(f,\mathcal{N}) = \operatorname*{ess\,sup}_{|K|\times(0,\lambda)} \operatorname{dist}(F,\mathcal{N}).$$

This together with the above estimates imply (4.1) since $|K| = \bigcup_{d=0}^m K^{(d)}$. Furthermore, as $H_\xi\colon |K|\to\mathcal{M}$ is a diffeomorphism integration by substitution yields

$$\begin{aligned}\int_{\mathcal{M}\times(0,\lambda)} |\nabla f - (Df_3,0)|^p &\le \int_{|K|\times(0,\lambda)} |\nabla F - (D(f_3\circ H_\xi),0)|^p |DH_\xi^{-1}|^p |\det DH_\xi| \\ &\le c\int_{|K|\times(0,\lambda)} |\nabla F - (D(f_3\circ H_\xi),0)|^p.\end{aligned}$$

By the slicing inequality we observe that

$$\int_{|K|\times(0,\lambda)} |\nabla G|^p \, d\mathcal{H}^{m+1} \le c\,\mathcal{H}^{m-d}\Big(\bigcup_{e\neq d}\Delta_j^e\Big)\int_{K^{(d)}\times(0,\lambda)} |\nabla G|^p\, d\mathcal{H}^{d+1}.$$

Thus

$$\begin{aligned}\int_{|K|\times(0,\lambda)} |\nabla G|^p \, d\mathcal{H}^{m+1} &\le c\sum_{d=0}^m \lambda^{m-d}\int_{K^{(d)}\times(0,\lambda)} |\nabla G|^p\, d\mathcal{H}^{d+1} \\ &\le c\lambda\int_{\mathcal{M}} |Df_1 - Df_3|^p + |Df_2 - Df_3|^p + |\tfrac{f_1-f_2}{\lambda}|^p\end{aligned}$$

from which (4.2) now follows.

4.2. Applications to the hole-filling technique

For a given functional $\mathfrak{F}$ and closed submanifold $\mathcal{M}\subset\mathbb{R}^N$ we seek to construct a suitable comparison map ϕ to a local minimiser $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathcal{M})$ which according to Definition 1.53 must satisfy $\mathfrak{F}(u;B)\le\mathfrak{F}(\phi;B)$ on a ball $B\subset\mathbb{R}^n$ with the requirements

$\phi \in W^{1,p}(B,\mathcal{M})$ and $\phi - u \in W^{1,p}_0(B,\mathbb{R}^N)$. As $\mathcal{M}$ has a tubular neighbourhood $\mathcal{O}_\rho$ of (finite) positive reach ρ with retraction $\Pi\colon \mathcal{O}_\rho \to \mathcal{M}$ it is possible to build ϕ from a map $w \in W^{1,p}(B,\mathbb{R}^N)$ by setting $\phi = \Pi(w)$ provided one can show $\operatorname{dist}(w,\mathcal{M})$ is sufficiently small together with the required boundary condition. Moreover it is desired, particularly in the context of hole filling arguments, for w to be equal to a different function v in on a ball strictly in interior of B.

The following seeks to quantify the distance bound for w in the desired setting by appealing to polar coordinates and Luckhaus' Lemma on the $(n-1)$-sphere. The result of which allows for an annular interpolation so that one can construct comparison maps that would have otherwise been achieved via an elementary convex combination had not the manifold constraints been imposed.

Corollary 4.9. *Let $u \in W^{1,p}(B_{(1+\varepsilon)r}\backslash B_r,\mathcal{M})$ and $v \in W^{1,p}(B_{(1+\varepsilon)r}\backslash B_r,\mathcal{O})$ for $p \geq 2$ and $0 < \varepsilon < 1$, where the ball $B_r = B_r(x_0) \subset \mathbb{R}^n$ and $\mathcal{M} \subset \mathbb{R}^N$ is a closed submanifold with a tubular neighbourhood $\mathcal{O}$ of $\mathcal{M}$ in $\mathbb{R}^N$. Then there exists $w \in W^{1,p}(B_{(1+\varepsilon)r}\backslash B_r,\mathbb{R}^N)$ with*

$$w|_{B_t\backslash B_r} = v, \qquad w|_{B_{(1+\varepsilon)r}\backslash B_{(1+\frac{15}{16}\varepsilon)r}} = u$$

for some $r < t < (1+\frac{\varepsilon}{4})r$ and constants $c, C > 0$ depending on p, n and N such that

$$\operatorname*{ess\,sup}_{B_{(1+\varepsilon)r}\backslash B_t} \operatorname{dist}(w,\mathcal{M}) \leq c\Big((\varepsilon r)^{p-n}\int_{B_{(1+\varepsilon)r}\backslash B_r} |Du|^p + |Dv|^p + \Big|\frac{u-v}{\varepsilon r}\Big|^p\Big)^{\frac{1}{p}(1-\frac{1}{2p})} \times \Big((\varepsilon r)^{-n}\int_{B_{(1+\varepsilon)r}\backslash B_r} |u-v|^p\Big)^{\frac{1}{2p^2}} \tag{4.5}$$

and

$$\int_{B_{(1+\varepsilon)r}\backslash B_r} |Dw|^p \leq C\int_{B_{(1+\varepsilon)r}\backslash B_r} |Du|^p + |Dv|^p + \Big|\frac{u-v}{\varepsilon r}\Big|^p. \tag{4.6}$$

Proof. By Example 1.1 there exists $r < t < (1+\frac{\varepsilon}{4})r$ such that

$$\int_{\partial B_t} \psi \leq \frac{c}{\varepsilon r}\int_{B_{(1+\varepsilon)r}\backslash B_r} \psi$$

for $\psi = |\frac{u-v_1}{\varepsilon r}|^p$, $\psi = |Du|^p$, and $\psi = |Dv|^p$.

In the trace sense let

$$\widehat{u}(x) = u|_{\partial B_t}(x_0 + tx) \qquad \text{and} \qquad \widehat{v}(x) = v_i|_{\partial B_t}(x_0 + tx)$$

for $x \in S^{n-1}$ denote a rescaling. Now apply Lemma 4.1 to get a function f on $S^{n-1}\times(0,\frac{\varepsilon}{2})$ with $f|_{S^{n-1}\times\{0\}} = \widehat{v}_1$, $f|_{S^{n-1}\times\{\frac{\varepsilon}{2}\}} = \widehat{u}$ and constants $c, C > 0$ depending on p, n, N,

S^{n-1} such that

$$\int_{S^{n-1}\times(0,\frac{\varepsilon}{2})} |\nabla f|^p \le C\varepsilon \int_{S^{n-1}} |D\widehat{u}|^p + |D\widehat{v}|^p + \Big|\frac{\hat{u}-\hat{v}}{\varepsilon}\Big|^p$$

and

$$\operatorname*{ess\,sup}_{S^{n-1}\times(0,\frac{\varepsilon}{2})} \operatorname{dist}(f,\mathcal{M}) \le c\,\varepsilon^{1-\frac{2n-1}{2p}} \Big(\int_{S^{n-1}} |D\widehat{u}|^p + |D\widehat{v}|^p + \Big|\frac{\hat{u}-\hat{v}}{\varepsilon}\Big|^p \Big)^{\frac{1}{p}(1-\frac{1}{2p})} \Big(\int_{S^{n-1}} |\widehat{u}-\widehat{v}|^p \Big)^{\frac{1}{p}\frac{1}{2p}}$$

Now define $w\colon B_{(1+\varepsilon)r}(x_0)\backslash B_r(x_0) \to \mathbb{R}^N$ by

$$w(x) = \begin{cases} v_1(x) & \text{if } r \le |x-x_0| \le t \\ f(\frac{x-x_0}{|x-x_0|}, \frac{|x-x_0|}{t} - 1) & \text{if } t \le |x-x_0| \le (1+\frac{\varepsilon}{2})t \\ u(x_0 + \eta(|x-x_0|)\frac{x-x_0}{|x-x_0|}) & \text{if } (1+\frac{\varepsilon}{2})t \le |x-x_0| \le (1+\varepsilon)r \end{cases}$$

where the smooth cut-off function $\eta\colon [(1+\frac{\varepsilon}{2})t, (1+\varepsilon)r] \to [t,(1+\varepsilon)r]$ is such that

$$\eta(z) = t \qquad \text{for} \qquad (1+\tfrac{\varepsilon}{2})t \le z \le (1+\tfrac{7}{8}\varepsilon)r$$
$$\eta(z) = z \qquad \text{for } (1+\tfrac{15}{16}\varepsilon)r \le z \le (1+\varepsilon)r$$

with $|\eta'(z)| \le c/(1-\frac{\varepsilon}{2})$.

Using the transformation $T\colon B_{(1+\frac{\varepsilon}{2})t}\backslash B_t(x_0) \to S^{n-1}\times(0,\frac{\varepsilon}{2})$ given by

$$T(x) = (\tfrac{x-x_0}{|x-x_0|}, \tfrac{|x-x_0|}{t} - 1)$$

the the chain rule implies

$$\begin{aligned} \int_{B_{(1+\frac{\varepsilon}{2})t}\backslash B_t} |Dw|^p &\le \int_{B_{(1+\frac{\varepsilon}{2})t}\backslash B_t} |\nabla f(T(x))|^p |DT(x)|^p dx \\ &\le C \int_t^{(1+\frac{\varepsilon}{2})t} ds\, s^{n-1} \int_{S^{n-1}} |t^{-1}\nabla f(\Omega, \tfrac{s}{t} - 1)|^2 d\Omega \\ &\le C(1+\tfrac{\varepsilon}{2})^{n-1} t^n \int_{S^{n-1}\times(0,\frac{\varepsilon}{2})} |t^{-1}\nabla f|^p, \end{aligned}$$

since $DT(x) = (\frac{1}{|x-x_0|}(I - \frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}), \frac{1}{t}\frac{x-x_0}{|x-x_0|})$ so that $|DT(x)| \le c\,t^{-1}$, and likewise

$$\int_{B_{(1+\frac{\varepsilon}{2})r}\backslash B_{(1+\frac{\varepsilon}{2})t}} |Dw|^p \le C \int_{B_{(1+\frac{\varepsilon}{2})r}\backslash B_{(1+\frac{\varepsilon}{2})t}} |Du|^p.$$

The desired inequality now follows from the fact that $1 < \frac{t}{r} < \frac{5}{4}$ and the rescalings

$$\int_{S^{n-1}} \Big|\frac{\hat{u}-\hat{v}}{\varepsilon t}\Big|^p = t^{1-n} \int_{\partial B_t} \Big|\frac{u-v}{\varepsilon t}\Big|^p, \qquad \int_{S^{n-1}} |t^{-1}D\widehat{u}|^p = t^{1-n}\int_{\partial B_t} |Du|^p.$$

The essential supremum bound follows from the fact that

$$\operatorname*{ess\,sup}_{B_{(1+\varepsilon)r}\setminus B_t} \operatorname{dist}(w,\mathcal{M}) = \operatorname*{ess\,sup}_{S^{n-1}\times(0,\frac{\varepsilon}{2})} \operatorname{dist}(f,\mathcal{M}),$$

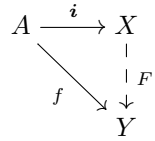
since u takes values in $\mathcal{M}$ a.e. and $T(B_{(1+\frac{\varepsilon}{2})t}\setminus B_t) = S^{n-1}\times(0,\frac{\varepsilon}{2})$, together with the above estimates. □

Chapter 5

Extensions and the lifting of Sobolev mappings into compact manifolds

As seen in [B13, BL, HL1] there is a fascinating relationship between homotopy theory and the study of Sobolev mappings into compact manifolds. By looking at the so-called *extension problem* (a general type of topological problem of fundamental importance in homotopy theory) we seek to construct comparison maps which satisfy the manifold constraints of the target that, unlike the results in Chapter 4, do so without the need for so-called 'smallness conditions'.

To illustrate the extension problem let $F\colon X \to Y$ be a continuous mapping between topological spaces X and Y. For a subspace A of X the map F defines a unique continuous map $f\colon A \to Y$ such that $f(x) = F(x)$ for each $x \in A$. The map f is said to be the *restriction* of F to A and is denoted by $f = F\big|_A$. The map F will be called a (continuous) *extension* of f over X. If $\boldsymbol{i}\colon A \to X$ denotes the *inclusion map*, defined by $\boldsymbol{i}(a) = a \in X$ for every $a \in A$, the relation $f = F\big|_A$ is equivalent to the commutativity of the following diagram:



The *extension problem* asks whether or not a given $f\colon A \to Y$ on a given subspace A of a topological space X has an extension F over X, i.e. if a given $A \subset X$ and f can be completed to a commutative diagram of the above form within the category Top of topological spaces. Of particular interest here is the question of whether such a problem can be solved in the context of Sobolev mappings into a compact manifold, i.e. in the

above we replace topological spaces by smooth manifolds and continuous maps by Sobolev maps.

In the situation that $Y = A$ and $\boldsymbol{i} = \mathrm{id}_A$ is the identity map on A, we obtain an important special case of the extension problem, namely the *retraction problem*. If id_A has a (continuous) extension $r\colon X \to A$ then A is said to be a *retract* of X and r is said to be the (continuous) *retraction* of X onto A.

Example 5.1. Every C^r-submanifold $\mathcal{M} \subset \mathbb{R}^N$, for $r \geq 2$, has a tubular neighbourhood $\mathcal{O}$ of M such that the metric projection $\Pi\colon \mathcal{O} \to M$ is a C^{r-1}-retraction of $\mathcal{O}$ onto M (cf. Lemma 1.47).

Example 5.2. The identity map from S^{n-1} to itself has no continuous extension over $\mathbb{R}^n$. Nonetheless if one deletes from $\mathbb{R}^n$, say, the origin 0 then S^{n-1} is a retract of $\mathbb{R}^n\backslash\{0\}$ with a continuous retraction $r\colon \mathbb{R}^n\backslash\{0\} \longrightarrow S^{n-1}$ that sends $x \mapsto \frac{x}{|x|}$.

Observe that if A is a retract of X then the extension problem becomes trivial regardless of Y. Indeed,

Lemma 5.3. *A is a retract of X if and only if, for any topological space Y, every continuous map $f\colon A \to Y$ has a continuous extension over X.*

Proof. If A is a retract of X with a retraction $r\colon X \to A$, then $f \circ r\colon X \to Y$ is an extension of f. Conversely, if the latter condition holds then with $Y = A$ the identity map on A has an extension $r\colon X \to A$. □

For a given continuous surjective mapping $p\colon E \to B$ from a topological space E onto a topological space B, and a continuous map $f\colon X \to B$ from a given topological space X, the *lifting problem* asks to find a continuous $F\colon X \to E$ such that $p \circ F = f$, i.e. if

$$\begin{array}{ccc} & & E \\ & \overset{F}{\nearrow} & \big\downarrow p \\ X & \xrightarrow[f]{} & B \end{array}$$

can be completed to a commutative diagram. The map F is called a (continuous) *lifting of f* (relative to p). This problem is dual to that of the above extension problem.

By examining the extension and lifting problems for Sobolev mappings, together with the retraction problem onto a compact manifold, we construct comparison maps (cf. §5.5) that require no need for a priori smallness conditions (and so by implication dispenses with, for instance, the need for monotonicity formulae that were required in Chapter 3). The idea for this construction is, once again, based on recovering the convex combination used for the comparison function in the unconstrained case, cf. (2.12). The following

example provides a model case in which the elementary convex combination is now applied upstairs in the lifting.

Example 5.4. For $u \in W^{1,2}(B_R(x_0), S^1)$ there exists $\varphi \in W^{1,2}(B_R(x_0), \mathbb{R})$ such that $u = e^{i\varphi}$ by Theorem 5.18 below. If we let $B_s(x) \subset B_R(x_0)$, $0 < r < s < \operatorname{dist}(x, \partial B_R(x_0))$ and $\eta \in C_c^\infty(B_R(x_0))$ be a cut-off function such that $\eta = 1$ on $B_r(x)$, $\eta = 0$ on $B_R(x_0)\backslash B_s(x)$, $0 \le \eta \le 1$ and $|D\eta| \le C/(s-r)$, then by setting

$$\psi = (1-\eta)\varphi + \eta(\varphi)_{B_s}$$

we have $\psi = \varphi$ on $\partial B_s(x)$ so that $v = e^{i\psi} \in (u + W_0^{1,2}(B_s(x), \mathbb{R}^2)) \cap W^{1,2}(B_s(x), S^1)$ and by construction

$$\begin{aligned}\int_{B_s} |Dv|^2 = \int_{B_s} |D\psi|^2 &\le C \int_{B_s\backslash B_r} |D\varphi|^2 + C \int_{B_s} \Big| \frac{\varphi - (\varphi)_{B_s}}{s-r} \Big|^2 \\ &\le C \int_{B_s\backslash B_r} |Du|^2 + C \frac{s^{n+2}}{(s-r)^2} \Big(\fint_{B_s} |Du|^{\frac{2n}{n+2}} \Big)^{\frac{n+2}{n}}.\end{aligned}$$

5.1. Brief review of homotopy groups

Let X and Y be topological spaces and $f, g\colon X \to Y$ be continuous maps. We say a *homotopy* from f to g is a continuous map $h_t\colon X \to Y$, $0 \le t \le 1$, such that $h_0 = f$ and $h_1 = g$. Given a topological space X and a point $x_0 \in X$, a *loop* at x_0 is a continuous function $f\colon [0,1] \to X$ such that $f(0) = x_0 = f(1)$. Given two loops f and g with the same base point their *product*, denoted by $f \cdot g$, is the loop that runs them one after another in double speed, i.e. $(f \cdot g)(t) = f(2t)$ for $t \in [0, 1/2]$ and $(f \cdot g)(t) = g(2t-1)$ for $t \in [1/2, 1]$. Two loops f and g are *equivalent*, written $f \sim g$, if there is a homotopy $h_t\colon [0,1] \to X$, $0 \le t \le 1$, such that $h_0 = f$, $h_1 = g$ and $h_t(0) = x_0 = h_t(1)$ for every t. This relation $\sim$ is reflexive, symmetric and transitive. It is easy to see that homotopies between loops at a given base point divide loops into homotopy equivalence classes. Let $\pi_1(X, x_0)$ denote the set of all these classes and $[f]$ denote the class containing the loop f at x_0. The loop f is said to be a representative for the class $[f]$. We define multiplication in $\pi_1(X, x_0)$ by setting $[f][g] = [f \cdot g]$ so that $\pi_1(X, x_0)$ forms into a group which we refer to as the *fundamental group* of X at x_0. If $f\colon X \to Y$ is a continuous map between topological spaces we define $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ by sending $[\phi_{x_0}] \mapsto [f \circ \phi_{x_0}]$ for loops ϕ_{x_0} based at x_0. One can check f_* is an homomorphism and satisfies the functoriality conditions $(f \circ g)_* = f_* \circ g_*$, $(f^{-1})_* = f_*^{-1}$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$.

A topological space X is said to be *disconnected* if it is the union of two disjoint non-empty open sets, otherwise X is said to be *connected*. A space X is said to be *pathwise connected* if every pair of points in X can be connected by a path in X. A space X is said

to be *locally pathwise connected* if it admits a basis of pathwise connected open sets. For a pathwise connected space X the fundamental group $\pi_1(X, x_0)$ at various base points $x_0 \in X$ are all isomorphic, hence we can simply write $\pi_1(X)$.

Definition 5.5. A topological space X is *simply connected* if it is pathwise connected and if the fundamental group of that space is trivial (i.e., all loops are homotopic to the constant loop).

For a pair (X, x_0) consisting of a topological space X and a point x_0 in X we let $\pi_0(X, x_0)$ denote the set of all path components of X. For each integer $n > 1$, we define the n-th *homotopy group* $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f\colon S^n \to X$ that map a given base point $s_0 \in S^n$ to the base point $x_0 \in X$. Equivalently, we can define $\pi_n(X, x_0)$ to be the group of homotopy classes of maps $f\colon [0,1]^n \to X$ from the unit n-cube to X that take the boundary of the n-cube to x_0. The group $\pi_n(X, x_0)$ is abelian when $n \geq 2$.

Lemma 5.6 ([H7, p. 346])**.** *For a topological space X with base point x_0 the following conditions are equivalent:*

(1) Every continuous map $S^n \to X$ is homotopic to a constant map.
(2) Every continuous map $S^n \to X$ extends to a map $B^{n+1} \to X$.
(3) $\pi_n(X, x_0) = 0$ for all $x_0 \in X$.

A topological space X is said to be *n-connected* if its first n homotopy groups vanish identically, i.e.

$$\pi_0(X) = \pi_1(X) = \ldots = \pi_{n-1}(X) = \pi_n(X) = 0. \tag{5.1}$$

A non-empty space X is 0-connected if and only if it is pathwise connected; a space X is 1-connected if and only if it is simply connected.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} + \mathbb{Z}_{12}$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}_2 + \mathbb{Z}_{24}$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

Table 5.1. Homotopy groups of spheres, cf. [S7,S8,S9]. Note that $\pi_{k+r}(S^k) = 0$ for $r < 0$ and $\pi_k(S^k) = \mathbb{Z}$.

5.2. Covering spaces

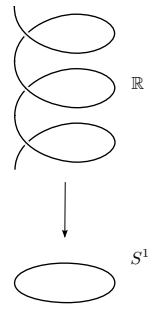
For a given connected and locally pathwise connected topological space B, a connected space E is said to be a *covering space* over B relative to a continuous map $\pi\colon E \to B$ if the following conditions are satisfied:

(i) π maps E onto B.

(ii) For each $b \in B$ there is a connected open neighbourhood U of b in B such that each component of $\pi^{-1}(U)$ is open in E and is mapped homeomorphically onto U by π.

It immediately follows that E is locally pathwise connected and that $\pi\colon E \to B$ is an open map. In fact the combination of connectedness and local pathwise connectedness implies that E is actually pathwise connected. We call π a covering map and B the base of the covering.

Example 5.7. The real line $\mathbb{R}$ as a covering space over the circle S^1 with respect to the exponential map $\exp\colon \mathbb{R} \to S^1$, i.e.



We recall the lifting criterion for a continuous map $f\colon X \to B$ of a connected and locally pathwise connected space X into the base space B of a covering space E relative to a projection $\pi\colon E \to B$:[1]

Proposition 5.8 (Lifting criterion)**.** *Let $x_0 \in X$ and e_0 be a point of E with $\pi(e_0) = f(x_0) = b_0$. Then there exists a unique map $F\colon X \to E$ such that $F(x_0) = e_0$ and $\pi \circ F = f$ if and only if the image of $f_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is contained in that of $\pi_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$.*

The criterion for deciding when two covering spaces are the same is as follows.

Proposition 5.9. *Let E and E' be two covering spaces over the same base space B relative to $\pi\colon E \to B$ and $\pi'\colon E' \to B$ respectively. If for $b_0 \in B$ points $e_0 \in E$ and $e_0' \in E'$ are such that $\pi(e_0) = b_0 = \pi'(e_0')$ and $\pi_*(\pi_1(E, e_0)) \subset \pi'_*(\pi_1(E', e_0'))$, then there exists a unique continuous map $\phi\colon E \to E'$ such that $\phi(e_0) = e_0'$ and $\pi' \circ \phi = \pi$. Moreover, E is a covering space over E' relative to ϕ.*

[1]Details of the proofs in this section and additional facts on covering spaces may be found, for example, in [H21, S17, L11].

We say a covering space E over B relative to $\pi\colon E \to B$ is *universal* if, for every covering space E' over B relative to $\pi'\colon E' \to B$, there exists a map $\phi\colon E \to E'$ such that $\pi' \circ \phi = \pi$ and that E is a covering space over E' relative to ϕ.

By introducing the property of semi-locally simply connectedness, which states that every point has a neighborhood with the property that every loop in that neighbourhood can be contracted to a single point, we have:

Proposition 5.10 (Existence)**.** *Suppose B is connected, locally pathwise connected and semi-locally simply connected. Then for every subgroup G of $\pi_1(X, x_0)$ there is a covering space E over B relative to a projection $\pi\colon E \to B$ and a point $e_0 \in \pi^{-1}(b_0)$ such that G is the image of the induced homomorphism $\pi_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$.*

In particular, if G is the trivial subgroup of $\pi_1(B, b_0)$ then E is a simply connected covering space over B and so by Proposition 5.9 it follows that E is a universal covering space over B. In fact the converse hold so that the covering of B which corresponds to the identity element in $\pi_1(B, b_0)$ is referred to as *the universal covering space of B.*

Theorem 5.11. *Every connected, locally pathwise connected and semi-locally simply connected topological space admits a universal covering space. Moreover, a covering space is universal if and only if it is simply connected.*

It is possible for a connected locally pathwise connected space to have a universal covering space that is not simply connected (e.g. the Hawaiian earring), however every manifold is locally simply connected (in the sense that it admits a basis of simply connected open subsets), since it has a basis of coordinate balls, and every locally simply connected space is semi-locally simply connected.

Corollary 5.12. *Every connected manifold admits a universal covering by a simply connected manifold.*

Corollary 5.13. *A connected compact manifold has a compact universal cover if and only if the fundamental group of the base space is finite.*

Remark 5.14. In the manifold case the above condition (ii) strengthens so that $\pi^{-1}(U)$ is mapped diffeomorphically onto U by smooth π for any connected open neighbourhood U. In addition, if we have Riemannian manifolds $(\mathcal{M}, g)$ and $(\widetilde{\mathcal{M}}, \widetilde{g})$ then we say $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ is a Riemannian covering if π is both a smooth covering and a local isometry, i.e. $\widetilde{g} = \pi^* g$.

Proposition 5.15. *Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a Riemannian manifold and G is a free and proper group of isometries of $(\widetilde{\mathcal{M}}, \widetilde{g})$. Then there exists on the quotient manifold $\mathcal{M} = \widetilde{\mathcal{M}}/G$ a unique Riemannian metric g such that the canonical projection $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ is a Riemannian covering map.*

For example the antipodal projection $\pi\colon S^n \to \mathbb{R}P^n$ onto the real projective space $\mathbb{R}P^n$ with a canonical metric is a Riemannian covering map with corresponding isometry group $G = \{\mathrm{Id}, -\mathrm{Id}\}$.

In the context of Lie groups we recall:

Proposition 5.16. *Each connected Lie group has a simply connected covering space which is itself a Lie group such that the covering map is a Lie group homomorphism.*

In fact we have:

Proposition 5.17. *Let G and H be connected Lie groups and $\phi\colon G \to H$ be a homomorphism. Then ϕ is a covering map if and only if $d\phi\colon T_eG \to T_eH$ is an isomorphism.*

Now let G be compact Lie group and $\mathfrak{g}$ be the Lie algebra of G. By the structural theory of compact Lie algebras

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell,$$

where $\mathfrak{c}$ is the centre and $\mathfrak{g}_j$ are simple ideas of $\mathfrak{g}$. If we let $k = \dim_{\mathbb{R}} \mathfrak{c}$ and G_j be the simply connected simple compact Lie groups associated to the Lie algebras $\mathfrak{g}_j$, then for any connected compact Lie group G the universal covering (cf. [HM, Theor. 6.6])

$$\widetilde{G} = \mathbb{R}^k \times G_1 \times \cdots \times G_\ell. \tag{5.2}$$

Note also that $\pi_2(G) = 0$ for any connected compact Lie group G (cf. [BtD, p. 225]) and that $\pi_3(G) = \mathbb{Z}$ for any simply connected simple compact Lie group G (cf. [B19]). Examples of some compact Lie group and their coverings are listed in Table 5.2.

G	π_0	π_1	UC	$\dim_{\mathbb{R}}$	
$S^1 = U(1)$	0	$\mathbb{Z}$	$\mathbb{R}$	1	Circle Group
$S^3 = \mathrm{Sp}(1)$	0	0		3	Quaternions of abs. value 1
$O(n)$, $n > 2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$n(n-1)/2$	Orthogonal Group
$SO(n)$, $n > 2$	0	$\mathbb{Z}_2$	$\mathrm{Spin}(n)$	$n(n-1)/2$	Special Orthogonal Group
$U(n)$, $n > 1$	0	$\mathbb{Z}$	$\mathbb{R} \times SU(n)$	n^2	Unitary Group
$SU(n)$	0	0		$n^2 - 1$	Special Unitary Group
$\mathrm{Spin}(n)$, $n > 2$	0	0		$n(n-1)/2$	Spin Group
$\mathrm{Sp}(n)$	0	0		$n(2n+1)$	Compact Symplectic Group

Table 5.2. Some real compact Lie groups. When the group is not simply connected the universal cover, UC, is given. Note that S^1 is isomorphic to $U(1)$, $SO(2)$, $\mathrm{Spin}(2)$ and $\mathbb{R}\backslash\mathbb{Z}$. Furthermore $\mathrm{Spin}(3) = SU(2) = \mathrm{Sp}(1)$, $\mathrm{Spin}(4) = SU(2) \times SU(2)$, $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ and $\mathrm{Spin}(6) = SU(4)$.

5.3. The lifting problem

In this section we seek to address the lifting problem for Sobolev spaces. Specifically, for a given connected compact Riemannain manifold $\mathcal{M}$ without boundary and an open simply connected subset $\Omega \subset \mathbb{R}^n$ we ask if any mapping $u \in W^{s,p}(\Omega, \mathcal{M})$ factors through the universal covering $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ as regular as u permits, i.e. if

$$\begin{array}{ccc} & & \widetilde{\mathcal{M}} \\ & \overset{\varphi}{\nearrow} & \big\downarrow \pi \\ \Omega & \xrightarrow[u]{} & \mathcal{M} \end{array}$$

can be completed to a commutative diagram with $\varphi \in W^{s,p}(\Omega, \widetilde{\mathcal{M}})$. Of course, if $\mathcal{M}$ is simply connected then $\widetilde{\mathcal{M}} = \mathcal{M}$ and π is the identity map, hence we are concerned with the non-trivial case of $\pi_1(\mathcal{M}) \neq 0$.

The first result in this direction is for mappings $u = u_1 + iu_2 \in W^{1,p}(\Omega, S^1)$. If we assume φ is known, i.e. $u = e^{i\varphi}$, then by differentiation $D\varphi = u_1 Du_2 - u_2 Du_1$. In solving the latter equation we have:

Theorem 5.18 ([BZ1])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth bounded domain which is simply connected. Then every $u \in W^{1,p}(\Omega, S^1)$ for $p \geq 2$ can be written as*

$$u = e^{i\varphi}$$

for some $\varphi \in W^{1,p}(\Omega, \mathbb{R})$.

Note that u is bounded in L^∞ but that this might not be true for φ, nonetheless $|Du| = |D\varphi|$ a.e. The restriction of $p \geq 2$ is optimal in any dimension $n \geq 2$ (for if $p < 2$ there is some $u \in W^{1,p}$ which cannot be lifted by a $\varphi \in W^{1,p}$). In fact the lifting problem for $W^{s,p}$-Sobolev maps into S^1 is completely solved:

Theorem 5.19 ([BBM])**.** *Assume $n \geq 2$, $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain which is simply connected. If $0 < s < 1$ then the answer to the lifting question in $W^{s,p}(\Omega, S^1)$ is positive if $sp < 1$, negative if $1 \leq sp < n$ and positive if $sp \geq n$. On the other hand, if $1 \leq s < \infty$ then the answer to the lifting question is negative if $sp < 2$ and positive if $sp \geq 2$.*

Here 'positive' means that every $u \in W^{s,p}(\Omega, S^1)$ may be written as $u = e^{i\varphi}$ for some $\varphi \in W^{s,p}(\Omega, \mathbb{R})$ and 'negative' means that for some u there exists no such φ such that $u = e^{i\varphi}$.

In the general case we have the following result of [BC1].

Theorem 5.20 (Lifting)**.** *Let $\mathcal{M}$ be a smooth connected compact Riemannian manifold without boundary with a universal covering $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ and $\Omega \subset \mathbb{R}^n$ be an open simply connected subset. For every $2 \le p \le \infty$ and any $u \in W^{1,p}(\Omega, \mathcal{M})$ there exists $\varphi \in W^{1,p}(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$. Moreover, φ is unique up to the action of an element of $\pi_1(\mathcal{M})$ and satisfies $|Du| = |D\varphi|$ for a.e. $x \in \Omega$.*

Note that $\pi_1(\mathcal{M})$, which naturally acts isometrically on $\widetilde{\mathcal{M}}$ and $\mathcal{M}$, can be isometrically identified with the coset $\widetilde{\mathcal{M}}/\pi_1(\mathcal{M})$ (as $\widetilde{\mathcal{M}}$ is simply connected).

We give a proof following the original result of [BC1] with some minor differences starting first with the one dimensional case.

Lemma 5.21. *Let $[a,b] \subset \mathbb{R}$ be a segment and $1 \le p \le \infty$. Then for any $u \in W^{1,p}([a,b], \mathcal{M})$ there exists a lifting $\varphi \in W^{1,p}([a,b], \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$. Moreover, φ is unique up to the action of an element of $\pi_1(\mathcal{M})$ and satisfies $|\varphi'| = |u'|$.*

Proof. As $u \in W^{1,p} \subset C^0$ is continuous and $\pi_1(\Omega) = 0$ the standard lifting criterion (cf. Propostion 5.8) implies u has a continuous lifting $\varphi \in C^0([a,b], \widetilde{M})$ that is unique up to the action of an element of $\pi_1(\mathcal{M})$. Moreover, since π is a local isometry, one may write locally $\varphi = \sigma \circ u$ for a local section σ of π. In which case $|\varphi'| = |u'| \in L^p$ and so $\varphi \in W^{1,p}([a,b], \widetilde{\mathcal{M}})$. □

Proof of Theorem 5.20. It suffices to consider the case when Ω is the unit n-cube $[0,1]^n$.

The case $n = 2$. The idea is to slice $[0,1]^2$ into horizontal and vertical directions with u chosen to be the precise representative.[2] To do so let

$$\Theta^h = \{t \in [0,1] : u(\cdot, t) \in W^{1,p}([0,1] \times \{t\}, \mathcal{M})\}$$
$$\Theta^v = \{s \in [0,1] : u(s, \cdot) \in W^{1,p}(\{s\} \times [0,1], \mathcal{M})\}$$

which, by Fubini's theorem, are both of unit measure. Without loss of generality assume $0 \in \Theta^h \cap \Theta^v$. Then by Lemma 5.21 there exists a lifting φ_0 of u restricted to $([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$.

Now for each $t \in \Theta^h$ there exists a lifting $\varphi_t(\cdot)$ of $u(\cdot, t)$ on $[0,1] \times \{t\}$ with $|\varphi_t(\cdot)'| = |D_1 u(\cdot, t)| \in L^p$ by Lemma 5.21. In which case we can construction a map φ^h on $\bigcup_{t \in \Theta^h} [0,1] \times \{t\}$ by setting $\varphi^h(x,t) = \varphi_t(x)$ for each $x \in [0,1]$ and $t \in \Theta^h$. As $\varphi^h = \varphi_0$ on $\{0\} \times \Theta^h$ we have $|\varphi^h(x,t) - \varphi_0(0,t)| \le \int_0^1 |D_1 u(\lambda x, t)| d\lambda$. Therefore $\varphi^h \in L^p$ and $|D_1 \varphi^h| = |D_1 u| \in L^p$. Likewise we can define a lifting φ^v on $\bigcup_{s \in \Theta^h} \{s\} \times [0,1]$ such that $\varphi^v = \varphi_0$ on $\Theta^v \times \{0\}$ with $\varphi^v \in L^p$ and $|D_2 \varphi^v| = |D_2 u| \in L^p$.

[2]The precise representative u of a $W^{1,p}$-function has the property that, on a cube $[a_1, b_1] \times \cdots \times [a_n, b_n]$, it is an L^p-function such that the restrictions $[a_j, b_j] \ni x^j \mapsto u(x^1, \ldots, x^{j-1}, x^j, x^{j+1}, \ldots, x^n)$ are absolutely continuous for $\mathcal{H}^{n-1}$-a.e. $(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n)$, cf. Remark 1.30.

To construct the desired φ let $s \in \Theta^v$ and $t \in \Theta^h$ be strictly positive and consider the rectangle $R = (0,s) \times (0,t)$. Since $u \in W^{1,p}(R, \mathcal{M})$ and $p \geq 2$ a standard density result (cf. Theorem 1.54) yields the existence of a sequence $(u_k) \subset C^\infty(\bar{R}, \mathcal{M})$ such that $u_k \to u$ in $W^{1,p}(R, \mathcal{M})$. In fact for k sufficiently large (u_k) converges uniformly on any line segment to the absolutely continuous u. Thus we may write $u_k = \pi \circ \varphi_k$ for smooth $\varphi_k \colon \bar{R} \to \widetilde{M}$ by the lifting criterion for continuous maps. In which case $\varphi_k \to \varphi_0$ uniformly on $([0,s] \times \{0\}) \cup (\{0\} \times [0,t])$, $\varphi_k \to \varphi^v$ uniformly on $\{s\} \times [0,t]$, and $\varphi_k \to \varphi^h$ uniformly on $[0,s] \times \{t\}$. Now as s and t are arbitrary we conclude $\varphi^h = \varphi^v$ a.e. on $\bar{R}$ by the uniqueness of the lifting on ∂R where u is continuous. Hence there exists $\varphi \in W^{1,p}([0,1]^2, \widetilde{\mathcal{M}})$ with $u = \pi \circ \varphi$ such that $|D_1 \varphi| = |D_1 \varphi^h| = |D_1 u| \in L^p$ and $|D_2 \varphi| = |D_2 \varphi^h| = |D_2 u| \in L^p$.

The case $n \geq 3$. We proceed by induction. To do so denote $H_s = H_s^j = [0,1]^{j-1} \times \{s\} \times [0,1]^{n-j}$ for fixed $1 \leq j \leq n$. Let

$$\Theta = \{s \in [0,1] : u\big|_{H_s} \in W^{1,p}(H_s, \mathcal{M})\}$$

which, by Fubini's theorem, is of unit measure. Assume $0 \in \Theta$ without loss of generality. By the induction hypothesis one has a lifting $\varphi_{H_s} \in W^{1,p}$ of $u\big|_{H_s}$ for each $s \in \Theta$. Thus we construct a lifting φ on $\bigcup_{s \in \Theta} H_s$ in which $\varphi = \varphi_{H_0}$ a.e. on H_0 by letting $\varphi(x^{j-1}, s, x^{n-j}) = \varphi_{H_s}(x)$ for all $x = (x^{j-1}, x^{n-j}) = (x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n) \in [0,1]^{n-1}$ and $s \in \Theta$. Furthermore note $|D_k \varphi| = |D_k \varphi_{H_s}| = |D_k u\big|_{H_s}| \in L^p$ for any $k \neq j$ and as $\varphi_{H_0} \in L^p$ we have, as in the above, $\varphi \in L^p$ too. To cover the remaining directional derivative we note, by Lemma 5.21 and the absolute continuity of the precise representative u along lines, that one has a lifting $\varphi_{L_x} \in W^{1,p}$ of $u(x^{j-1}, \cdot, x^{n-j})$ for a.e. $x = (x^{j-1}, x^{n-j}) \in [0,1]^{n-1}$, where $L_x = x^{j-1} \times [0,1] \times x^{n-j}$. Moreover $|\varphi'_{L_x}| = |D_j u| \in L^p$ and similarly one can define $\psi(x^{j-1}, s, x^{n-1}) = \varphi_{L_x}(x)$ for $s \in [0,1]$ and a.e. $x \in [0,1]^{n-1}$. We claim $\varphi = \psi$ a.e. $[0,1]^n$. To see this fix $s \in \Theta$ and $x \in [0,1]^{n-1}$ off a null set. Then consider the rectangle R defined by the vertices $(x^{j-1}, 0, 0)$, $(x^{j-1}, 0, x^{n-j})$, (x^{j-1}, s, x^{n-1}) and $(x^{j-1}, s, 0)$ lying in the affine 2-plane $\{(x^{j-1}, \lambda, \mu x^{n-j}) \in \mathbb{R}^n : \lambda, \mu \in \mathbb{R}\}$. From the density argument as in the above on R we conclude $\varphi_{H_s}(x) = \varphi_{L_x}(s)$. Hence there exists a $\varphi \in W^{1,p}([0,1]^n, \widetilde{\mathcal{M}})$ such that $\varphi(x^{j-1}, s, x^{n-j}) = \varphi_{H_s}(x) = \varphi_{L_x}(s)$ a.e. with $|D_j \varphi| = |\varphi'_{L_s}| = |D_j u| \in L^p$ and $|D_k \varphi| = |D_k \varphi_{H_s}| = |D_k u| \in L^p$ for $k \neq j$.

To check the uniqueness first suppose there are two different liftings φ^1 and φ^2 of u on $[0,1]^n$. If so one can then find an $x \in [0,1]^n$ such that $\varphi^1(x) \neq \varphi^2(x)$. By taking an $(n-1)$-plane H intersecting x the uniqueness from the induction hypothesis implies $\varphi^1\big|_H$ and $\varphi^2\big|_H$ (which are both liftings of $u\big|_H$) must be equal. Contradiction. $\square$

The question of uniqueness of the lifting is treated as follows:

Lemma 5.22. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected domain. Let $0 < s_1, s_2 < \infty$ and $1 \le p_1, p_2 \le \infty$ be such that $s_1 p_1 \ge 1$, $s_2 p_2 \ge 1$. If $\varphi_1 \in W^{s_1,p_1}(\Omega, \widetilde{\mathcal{M}})$ and $\varphi_2 \in W^{s_2,p_2}(\Omega, \widetilde{\mathcal{M}})$ satisfy $\pi \circ \varphi_1 = \pi \circ \varphi_2$ then φ_1 and φ_2 differ from the action of an element of $\pi_1(\mathcal{M})$, i.e. there exists $h \in \pi_1(\mathcal{M})$ such that $\varphi_1 = h \cdot \varphi_2$ in Ω. In particular $\varphi_1, \varphi_2 \in (W^{s_1,p_1} \cap W^{s_2,p_2})(\Omega, \widetilde{\mathcal{M}})$.*

In addition to Theorem 5.20 we also have the following negative result for a smooth connected compact Riemannian manifold $\mathcal{M}$ without boundary with a universal covering $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ and an open simply connected subset $\Omega \subset \mathbb{R}^n$.

Theorem 5.23. *Let $1 \le s < \infty$ and $1 < p < \infty$. If $sp < 2$ and $\pi_1(\mathcal{M}) \neq 0$ then there exists some $u \in W^{s,p}(\Omega, \mathcal{M})$ such that there does not exists $\varphi \in W^{s,p}(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$. On the other hand, if $sp \ge 2$ then for any $u \in W^{s,p}(\Omega, \mathcal{M})$ there exists $\varphi \in (W^{s,p} \cap W^{1,sp})(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$.*

For $0 < s < 1$ we have, in the case $\widetilde{\mathcal{M}}$ is compact, the following:

Theorem 5.24. *Let $0 < s < 1$ and $1 < p < \infty$. If $1 \le sp < 2$ and $\pi_1(\mathcal{M}) \neq 0$ then there exists some $u \in W^{s,p}(\Omega, \mathcal{M})$ such that there does not exist $\varphi \in W^{s,p}(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$. On the other hand, if $sp < 1$ or $sp \ge N$ then for any $u \in W^{s,p}(\Omega, \mathcal{M})$ there exists $\varphi \in W^{s,p}(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$.*

Note that the assumption of simply connectedness of Ω is not necessary if $sp < 1$. It is an open problem as to what happens in the range $2 \le sp < N$, $0 < s < 1$ when $\widetilde{\mathcal{M}}$ is compact. Proofs of these results can be found in [BC1].

5.4. Extension-type results for Sobolev mappings

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and a compact submanifold $\mathcal{M} \subset \mathbb{R}^N$ without boundary we have

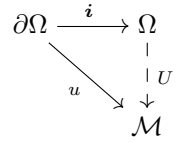
$$W^{s,p}(\Omega, \mathcal{M}) = \{u \in W^{s,p}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{M} \text{ a.e. } x \in \Omega\}.$$

By §1.6.1 the trace maps

$$\mathfrak{R}\colon W^{1,p}(\Omega, \mathbb{R}^N) \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N)$$

so that any map in $W^{1,p}(\Omega, \mathcal{M})$ has a trace in $W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ by definition. The question of interest is to determine under what conditions a map in $W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ is the trace

of a map in $W^{1,p}(\Omega, \mathcal{M})$, i.e. for a given $u \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ can



be completed to a commutative diagram with $U \in W^{1,p}(\Omega, \mathcal{M})$. Following [BD] we denote the subspace $T^p(\partial\Omega, \mathcal{M})$ of extendible maps in $W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ by

$$T^p(\partial\Omega, \mathcal{M}) = \{u \in W^{1-\frac{1}{p},p}(\Omega, \mathcal{M}) : \text{ there exists } U \in W^{1,p}(\Omega, \mathcal{M}) \text{ s.t. } \mathfrak{R}U = u\}.$$

Hence we seek to find conditions under which $T^p(\partial\Omega, \mathcal{M})$ coincides with $W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$.

In the case of continuous maps we say that property $\mathscr{P}(\Omega, \mathcal{M})$ holds if for any continuous map $u\colon \partial\Omega \to \mathcal{M}$ there is some continuous extension $U\colon \Omega \to \mathcal{M}$ such that $U = u$ on $\partial\Omega$ (i.e. if the above diagram can be completed to a commutative diagram in the continuous setting). When $\Omega = B^n$ is an n-ball and $\mathcal{M}$ is pathwise connected a standard result in homotopy theory (cf. [H7, Lem. 4.7, p. 348] and [H21, Lem. 3.2, p. 177]) states that:

Lemma 5.25. *$\mathscr{P}(B^n, \mathcal{M})$ holds if and only if $\pi_{n-1}(\mathcal{M}) = 0$.*

Similarly, Sobolev mappings $W^{1,p} \subset C^{0,1-\frac{n}{p}}$ when $p > n$. Thus by a p-harmonic extension argument and continuity we have:

Theorem 5.26 ([BD])**.** *For $p \geq n$ we have $T^p(\partial\Omega, \mathcal{M}) = W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ if any only if $\mathscr{P}(\Omega, \mathcal{M})$ holds.*

In the more complicated $p < n$ case we first note that the condition $\pi_{\lfloor p \rfloor - 1}(\mathcal{M}) = 0$ is necessary for Sobolev extensions to exist.

Theorem 5.27 ([BD])**.** *If $\pi_{\lfloor p \rfloor - 1}(\mathcal{M}) \neq 0$ for $1 < p < n$, then we have*

$$T^p(\partial\Omega, \mathcal{M}) \neq W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M}).$$

Proof. If $\pi_{\lfloor p \rfloor - 1}(\mathcal{M}) \neq 0$ then by a result of [B15] there exists $u \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathcal{M})$ which cannot be approximated by smooth maps in $C^\infty(\partial\Omega, \mathcal{M})$ with the $W^{1-\frac{1}{p},p}$ norm. Now assume there exists $U \in W^{1,p}(\Omega, \mathcal{M})$ such that $U|_{\partial\Omega} = u$. Then by Theorem 1.56 U can be approximated by functions u_j which are smooth outside a singular set $\Sigma(u)$ of dimension $n - \lfloor p \rfloor - 1$. One has $\Sigma(u) \subset \operatorname{int}(\Omega)$ and that the u_j are smooth on $\partial\Omega$. By continuity of the trace map $\mathfrak{R}$ we obtain $u_j|_{\partial\Omega} \to u|_{\partial\Omega}$ in $W^{1-\frac{1}{p},p}$ and hence a contradiction. □

As we have seen in Chapter 4 it is possible to obtain an extension-type result, under additional smallness conditions, for Sobolev mappings into a compact manifolds without boundary. In the application to harmonic maps such results can be used to find, provided one has a monotonicity formula, suitable comparisons to the minimiser that satisfy the nonlinear manifold constraints. With such comparison maps it is then possible to prove Caccioppoli inequalities near points with small scale energy (cf. §3.1). The original extension lemma of [SU1] reads as follows.

Lemma 5.28 ([SU1, Lem. 4.3])**.** *Let $\mathcal{M} \subset \mathbb{R}^N$ be a compact submanifold without boundary. For $n \geq 2$ there exists $\delta = \delta(n, \mathcal{M})$ and a constant $k = k(n)$ such that if for given $0 < \varepsilon < 1$, $\xi \in \mathbb{R}^N$ and $v \in W^{1,2}(\partial B_r, \mathcal{M})$ we have*

$$r^2 \fint_{\partial B_r} |D_{\tan} v|^2 \cdot \fint_{\partial B_r} |v - \xi|^2 \leq \delta^2 \varepsilon^k,$$

then there exists $w \in W^{1,2}(B_r, \mathcal{M})$ with $w|_{\partial B_r} = v$ such that

$$\int_{B_r} |Dw|^2 \leq C\varepsilon r \int_{\partial B_r} |D_{\tan} v|^2 + C\varepsilon^{-k} r^{-1} \int_{\partial B_r} |v - \xi|^2.$$

By contrast the paper of [HKL1] shows the extension-type result holds in the special case of the 2-sphere, crucially, *without* the smallness assumption needed in the last lemma (or for that matter the need for (4.5) in Corollary 4.9).

Lemma 5.29 ([HKL1, Lem. 2.1])**.** *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain If $u \in W^{1,2}(\Omega, S^2)$ and $x_0 \in \Omega$, then for a.e. $0 < r < \operatorname{dist}(x_0, \partial\Omega)$ there is a function $w \in W^{1,2}(B_r(x_0), S^2)$ such that*

$$w = u \qquad \text{on } \partial B_r(x_0)$$

and

$$\int_{B_r(x_0)} |Dw|^2 \leq C\Big(\int_{\partial B_r(x_0)} |D_{\tan} u|^2 \cdot \int_{\partial B_r(x_0)} |u - \xi|^2 \Big)^{\frac{1}{2}}$$

where $\xi \in \mathbb{R}^3$ is arbitrary and C is an absolute constant.

To prove this result first note that $\int_{\partial B_r(x_0)} |Du|^2 < +\infty$ for almost every $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. Then for any such r let v be the harmonic extension of u given by

$$\begin{aligned} \Delta v &= 0 \qquad \text{in } B_r(x_0) \\ v &= u \qquad \text{on } \partial B_r(x_0). \end{aligned}$$

As v is harmonic it follows that

$$\int_{\partial B_r(x_0)} \Big| \frac{\partial v}{\partial \nu} \Big|^2 \leq C \int_{\partial B_r(x_0)} |D_{\tan} v|^2$$

for some $C = C(n) > 0$, where ν denotes the outward unit normal. Then by Green's formula

$$\int_{B_r(x_0)} |Dv|^2 = \int_{\partial B_r(x_0)} \frac{\partial v}{\partial \nu} \cdot (v - \xi) \leq \Big(\int_{\partial B_r(x_0)} \Big| \frac{\partial v}{\partial \nu} \Big|^2 \Big)^{\frac{1}{2}} \Big(\int_{\partial B_r(x_0)} |v - \xi|^2 \Big)^{\frac{1}{2}}$$
$$\leq C \Big(\int_{\partial B_r(x_0)} |D_{\tan} u|^2 \Big)^{\frac{1}{2}} \Big(\int_{\partial B_r(x_0)} |u - \xi|^2 \Big)^{\frac{1}{2}}.$$

With this the above result now follows via the following:

Lemma 5.30 ([HKL1, Lem. A.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. For any $v \in W^{1,p}(\Omega, \mathbb{R}^{k+1})$, $1 < p < k+1$, with $|v| = 1$ on $\partial\Omega$, there exists a function $w \in W^{1,p}(\Omega, S^k)$, $k > 1$, such that*

$$w = v \qquad \text{on } \partial\Omega$$

and

$$\int_\Omega |Dw|^p \leq C \int_\Omega |Dv|^p$$

for some positive constant $C = C(n, k, p)$.

Proof. For $|a| < 1$ consider the function

$$w_a(x) = \frac{v(x) - a}{|v(x) - a|}, \qquad x \in \Omega,$$

of which the gradient is

$$Dw_a(x) = \Big(I - \frac{v - a}{|v - a|} \otimes \frac{v - a}{|v - a|} \Big) \frac{Dv}{|v - a|}.$$

Thus $|Dw_a| \leq C|v - a|^{-1}|Dv|$ in Ω for some $C = C(k) > 0$. By Fubini's theorem

$$\int_{B_{\frac{1}{2}}(0)} \int_\Omega |Dw_a|^p dx\, da = \int_\Omega \int_{B_{\frac{1}{2}}(0)} |Dw_a|^p da\, dx$$
$$\leq C \int_\Omega \int_{B_{\frac{1}{2}}(0)} |v - a|^{-p} |Dv|^p da\, dx$$
$$\leq C \int_\Omega |Dv|^p,$$

since there exists some $C = C(k, p) > 0$ such that

$$\int_{B_{\frac{1}{2}}(0)} |v - a|^{-p} da \leq C < +\infty,$$

provided $1 < p < k + 1$. Moreover, for some $K > 0$ we have

$$\int_\Omega |Dw_a|^p \leq \frac{K}{|B_{\frac{1}{2}}(0)|} \int_{B_{\frac{1}{2}}(0)} \int_\Omega |Dw_a|^p$$

for a 'good set' of $a \in B_{\frac{1}{2}}(0)$ (if not then Markov's inequality implies the latter fails with an exceptional set less than or equal to $K^{-1}|B_{\frac{1}{2}}(0)|$). Therefore there exists an a_0 with $|a_0| \leq \frac{1}{2}$ such that

$$\int_\Omega |Dw_{a_0}|^p \leq C \int_\Omega |Dv|^p.$$

Now let $P_a \colon \mathbb{R}^{k+1} \backslash \{a\} \to S^k$ send

$$\xi \mapsto \frac{\xi - a}{|\xi - a|}$$

and for any $|a| \leq \frac{1}{2}$ let $(P_a|_{S^k})^{-1} \colon S^k \to S^k$ send

$$\eta \mapsto a + [-a \cdot \eta + ((a \cdot \eta)^2 + 1 - |a|^2)^{\frac{1}{2}}]\eta.$$

Note that $(P_a|_{S^k})^{-1} \circ P_a(\xi) = \xi$ and $D[(P_a|_{S^k})^{-1} \circ P_a](\xi) \colon T_\xi S^k \to T_\xi S^k$ is the identity map for any $|\xi| = 1$. Since $|D(P_a|_{S^k})^{-1}(\eta)| \leq C$ uniformly independent of a the desired conclusion follows by setting

$$w = (P_a|_{S^k})^{-1} \circ P_a \circ v \in W^{1,p}(\Omega, S^k). \qquad \square$$

It is important to note that the contrast between the former result of Lemma 5.28 (that need an additional smallness condition) and the latter result of Lemma 5.30 (that does without) essentially is reflected in the use of the two different retractions of Example 5.1 and Example 5.2. The need for suitable smallness conditions is a result of the need to stay in the tubular neighbourhood, where as in the case of the sphere one can retract all but origin. It turns out that the latter retraction onto S^k is inherently topological and can be generalised to any compact ℓ-connected manifold for some integer $\ell \geq 1$ (note that S^k is $(k-1)$-connected). The resulting retraction result of Lemma 5.32 below was first proved in [HL2, Lem. 6.1] using several results from homotopy theory (similar results can be found in [BPVS1] and [BPVS2]). Our proof here differs from that given in [HL2] and is by entirely elementary means via the following non-trivial result of White [W6].

Before stating White's result we remark that a triangulation $f \colon K \to \mathcal{N}$ of a given manifold $\mathcal{N}$ can be such that the complex K, as defined by Definition 1.61, consists of cubes each of which is isometric to the standard cube $[-1,1]^m \subset \mathbb{R}^m$, i.e. K is a 'cubication' of $\mathcal{N}$. For a cube in $\mathbb{R}^m$ centred at the origin with faces parallel to the coordinate axes, the dual skeleton of dimension j is the set of points in the cube which have at least $m-j$ components equal to zero. By using an isometry the dual skeleton of a cube in $\mathbb{R}^m$ of dimension m can be defined in a general position. Thus the j-dimensional dual skeleton L^j of a complex K of cubes is simply the union of the dual skeletons of dimension j of each cube in the complex (cf. [BPVS1, §2.1]).

Lemma 5.31 ([W6, p. 130])**.** *Let K be a triangulation of a given manifold $\mathcal{N}^n$ and $L^{n-\ell-1}$ be the $(n-\ell-1)$-dimensional dual skeleton of K for some integer $0 \leq \ell \leq n-1$. Then there exists a locally Lipschitz continuous function*

$$\psi\colon K^n \backslash L^{n-\ell-1} \to K^\ell$$

from the complement of the dual skeleton $L^{n-\ell-1}$ onto the skeleton K^ℓ such that

$$|D\psi(x)| \leq \frac{C}{\operatorname{dist}(x, L^{n-\ell-1})}$$

for some C depending on n, ℓ and K.

Proof. Without loss in generality let the triangulation $f\colon K \to \mathcal{N}$ of the manifold $\mathcal{N}$ be such that the complex K consists of cubes each of which is isometric to the standard cube $[-1,1]^m \subset \mathbb{R}^m$. Also, let K^ℓ denote the ℓ-dimensional skeleton as the collection of all cubes of dimension at most ℓ. Now for a cube $[-1,1]^n$ define the function $|\cdot|_\ell\colon [-1,1]^n \to \mathbb{R}$ by

$$|x|_\ell = \min_{\substack{S \subset \{1,\dots,n\} \\ |S| = \ell+1}} \max_{i \in S} |x_i|.$$

Then if $L^{n-\ell-1}$ denotes the $(n-\ell-1)$-dual skeleton of $[-1,1]^n$ we have $x \in L^{n-\ell-1}$ if and only if $|x|_\ell = 0$. Moreover, the mapping $[-1,1]^n \ni x \mapsto |x|_\ell$ is Lipschitz continuous with Lipschitz constant 1. We can then define the mapping $\phi_\ell\colon [-1,1]^n \backslash L^{n-\ell-1} \to K^\ell$ by $\phi_\ell(x) = (y_1, \dots, y_n)$, where

$$y_i = \begin{cases} \operatorname{sgn} x_i & \text{if } |x_i| \geq |x|_\ell \\ x_i/|x|_\ell & \text{if } |x_i| < |x|_\ell. \end{cases}$$

Observe that $|D\phi_\ell(x)| \leq C/|x|_\ell$ for some $C > 0$ also. Given this, we can then perform the the same construction for every cube in K. If two cubes have a non-empty intersection the corresponding maps $\phi_{\ell,1}$ and $\phi_{\ell,2}$ coincide on the common face. Hence we can glue together the locally Lipschitz continuous maps obtained for each of the cubes in the complex K so as to obtain a global map ψ with the desired properties. □

Lemma 5.32. *Let $\mathcal{M} \subset \mathbb{R}^k$, $k \geq 3$, be a compact ℓ-connected submanifold contained in a sufficiently large k-dimensional cube Q for some integer $1 \leq \ell \leq k-2$. Then there exists a closed $(k-\ell-2)$-dimensional Lipschitz polyhedron $X \subset Q \backslash \mathcal{M}$ and a locally Lipschitz retraction $r\colon Q \backslash X \to \mathcal{M}$ such that for any $x \in Q \backslash X$,*

$$|Dr(x)| \leq \frac{C}{\operatorname{dist}(x, X)}$$

for some $C > 0$ depending on k, ℓ and $\mathcal{M}$.

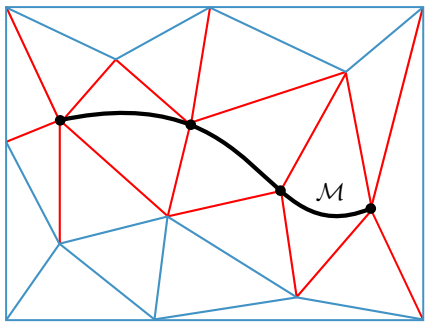


Figure 5.1. An extension of a triangulation of a manifold $\mathcal{M} \subset Q$ to a triangulation of the cube Q. Simplexes intersecting with $\mathcal{M}$ are shown in red whereas ones that are not are in blue.

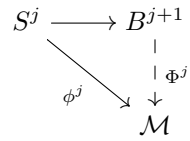
Proof. Let $f\colon T \to \mathcal{M}$ be a triangulation of the manifold $\mathcal{M}^m$. As $\mathcal{M} \subset Q$ is contained in a sufficiently large cube Q we can extend the triangulation T of $\mathcal{M}$ to a triangulation of the cube $Q \subset \mathbb{R}^k$ which we denote by K.

Now consider all edges Δ^1 of the complex K that meet $\mathcal{M}$ at a vertex such that the edge is not contained in $\mathcal{M}$ (cf. Figure 5.1). If for such Δ^1 we send the vertex not in $\mathcal{M}$ to any point in $\mathcal{M}$ we have a mapping $\phi^0\colon S^0 \to \mathcal{M}$. Since $\pi_0(\mathcal{M}) = 0$ there exists $\Phi^0\colon B^1 \to \mathcal{M}$ that extends ϕ^1 by Lemma 5.6. Hence we can map $\Delta^1 \to \mathcal{M}$ for all edges Δ^1 that meet $\mathcal{M}$ at a vertex but do not lie in $\mathcal{M}$.

For faces Δ^2 of the complex K with one edge contained in $\mathcal{M}$ such that the face does not lie in $\mathcal{M}$ we can use the previous step: by noting the edges of such a face Δ^2 can be mapped into $\mathcal{M}$ (since one vertex lies in $\mathcal{M}$) and as $\partial\Delta^2 \simeq S^1$ in a bi-Lipschitz fashion we have a mapping $\phi^1\colon S^1 \to \mathcal{M}$. Since $\pi_1(\mathcal{M}) = 0$ there exists an extension $\Phi^1\colon B^2 \to \mathcal{M}$ by Lemma 5.6. Hence we can map $\Delta^2 \to \mathcal{M}$ for all faces Δ^2 that meet $\mathcal{M}$ at one edge but do not lie in $\mathcal{M}$. Furthermore, for faces Δ^2 that not have an edge in $\mathcal{M}$ (i.e. those that only meet $\mathcal{M}$ at a vertex or those that do not meet $\mathcal{M}$ at all) we can extend $\partial\Delta^2 \to \mathcal{M}$ into the interior by face contraction (for instance if one considers a face in the 2-plane with vertices $(-1,0)$, $(1,0)$ and $(0,1)$ such that there a mapping $\phi\colon E \to \mathcal{M}$ along the edges $E = \langle(-1,0),(0,1)\rangle$ and $E = \langle(1,0),(0,1)\rangle$ then we can extend ϕ by face contraction alone lines $L = \{(x,y) : 0 \le y \le 1 - \operatorname{sgn}(x)x\}$, $-1 \le x \le 1$, where $(x,y) \mapsto \phi(x, 1-\operatorname{sgn}(x)x)$ take the values along the edges). As faces with common edges will share the same value we can map $K^2 \cup T^m \to \mathcal{M}$ for the 2-dimensional skeleton K^2.

Proceeding inductively, we now suppose $K^j \cup T^m \to \mathcal{M}$ for some integer $1 \le j \le \ell$. Then for any $(j+1)$-simplex Δ^{j+1} with one j-dimensional face meeting $\mathcal{M}$ such that the simplex is not contained in $\mathcal{M}$ we have a map $\phi^j\colon S^j \to \mathcal{M}$ by the induction hypothesis

and the fact that $\partial\Delta^{j+1} \simeq S^j$. As $\pi_j(\mathcal{M}) = 0$ it follows by Lemma 5.6 that



can be completed to a commutative diagram. Hence we can map $\Delta^{j+1} \to \mathcal{M}$ for all $(j+1)$-simplexes Δ^j that meet $\mathcal{M}$ at one j-dimensional face but do not lie in $\mathcal{M}$. Then as before for any $(j+1)$-simplexes Δ^{j+1} that does not have a j-dimensional face in $\mathcal{M}$ we can extend $\partial\Delta^{j+1} \to \mathcal{M}$ into the interior by face contraction. Hence we have a map $K^{j+1} \cup T^m \to \mathcal{M}$ for the $(j+1)$-dimensional complex K^{j+1}.

As the obstruction of $\pi_{\ell+1}(\mathcal{M}) \neq 0$ we have a Lipschitz retraction $\phi\colon K^{\ell+1} \cup T^m \to \mathcal{M}$ by the above construction. Furthermore, by Lemma 5.31 there is a locally Lipschitz retraction $\psi\colon (K^k \backslash L^{k-\ell-2}) \cup T^m \to K^{\ell+1} \cup T^m$ such that

$$|D\psi(x)| \leq \frac{C}{\operatorname{dist}(x, L^{k-\ell-2})}$$

for every $x \in (K^k \backslash L^{k-\ell-2}) \cup T^m$. The desired conclusion now follows by setting $X = \overline{L^{k-\ell-2} \backslash T^m}$ with the locally Lipschitz retraction $r = \phi \circ \psi$. □

With the retraction result of the previous lemma established it is now possible to generalise Lemma 5.30 to the following result that goes back to [HL2, Lem. 6.2].

Lemma 5.33 (Extension-type)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\widetilde{\mathcal{M}} \subset \mathbb{R}^k$, $k \geq 3$, be a compact ℓ-connected submanifold for some integer $1 \leq \ell \leq k-2$. For $1 < p < \ell + 2$ let $\phi \in (W^{1,p} \cap L^\infty)(\Omega, \mathbb{R}^k)$ such that the trace $\mathfrak{R}\,\phi \in W^{1-\frac{1}{p},1}(\partial\Omega, \widetilde{\mathcal{M}})$. Then there exists $\psi \in (\phi + W_0^{1,p}(\Omega, \mathbb{R}^k)) \cap W^{1,p}(\Omega, \widetilde{\mathcal{M}})$ and a constant $C = C(n,p,k,\ell) > 0$ such that*

$$\int_\Omega |D\psi|^p \leq C \int_\Omega |D\phi|^p$$

and

$$\int_\Omega |x-y|^{1-n} |D\psi(y)| dy \leq C \int_\Omega |x-y|^{1-n} |D\phi(y)| dy$$

for a.e. $x \in \Omega$.

Proof. By Theorem 1.49 there exists a neighbourhood $U = \{\xi \in \mathbb{R}^k : \operatorname{dist}(\xi, \widetilde{\mathcal{M}}) \leq \rho$, for some $0 < \rho < \operatorname{reach}(\widetilde{\mathcal{M}})$, with the unique nearest point property such that the map $\Pi\colon U \to \widetilde{\mathcal{M}}$, which associates to each $\xi \in U$ the unique $\xi_0 \in \widetilde{\mathcal{M}}$ such that $\operatorname{dist}(\xi, \widetilde{\mathcal{M}}) = |\xi - \xi_0|$, is Lipschitz continuous. As $\phi \in L^\infty$ and $\widetilde{\mathcal{M}}$ is compact there exists a sufficiently large k-dimensional cube Q such that $\widetilde{\mathcal{M}} \subset U \subset Q$ and $\operatorname{dist}(\phi, \mathcal{M}) \leq \frac{1}{2}\operatorname{dist}(\widetilde{\mathcal{M}}, \partial Q)$

a.e. As the boundary ∂U is a $C^{1,1}$-manifold by Remark 1.50 we have a locally Lipschitz retraction $r\colon Q\backslash X \to \partial U$ for some $(k-\ell-2)$-dimensional Lipschitz polyhedron $X \subset Q\backslash U$ by Lemma 5.32 since ∂U is ℓ-connected by the continuity of Π. In combining these two maps we then have

$$P = \Pi \circ r\colon Q\backslash X \to \widetilde{\mathcal{M}}.$$

Furthermore we may assume, after a bounded change of metric, that X is a (linear) polyhedron. Then for every $\xi \in Q$ we have $\operatorname{dist}(\xi, X) = \operatorname{dist}(\xi, Z)$ for some affine space Z generated by the faces of X. It then follows that $\int_Q \operatorname{dist}(\xi, Z)^{-p} d\xi$ is finite, since $\dim Z \leq k-\ell-2$. Hence there exists a positive constant $C = C(p,k,\ell)$ such that

$$\int_Q |\nabla P(\xi)|^2 d\xi \leq C \int_Q \operatorname{dist}(\xi, X)^{-p} d\xi \leq C < +\infty. \tag{5.3}$$

Now for $\sigma < \min\{\rho, \frac{1}{2}\operatorname{dist}(\widetilde{\mathcal{M}}, \partial Q)\}$ and $a \in B^k_\sigma = \{\xi \in \mathbb{R}^k : |\xi| < \sigma\}$ denote the translations $Q_a = \{\xi + a : \xi \in Q\}$ and $X_a = \{\xi + a : \xi \in X\}$ so that one has the retraction $P_a\colon Q_a\backslash X_a \to \widetilde{\mathcal{M}}$ given by $P_a(\xi) = P(\xi - a)$. Then by Fubini's theorem and (5.3) there exists some $C = C(p,k,\ell) > 0$ such that

$$\begin{aligned}
\int_{B^k_\sigma}\int_\Omega |D(P_a \circ \phi)(x)|^p dx\, da &\leq \int_\Omega \int_{B^k_\sigma} |\nabla P_a(\phi(x))|^p |D\phi(x)|^p da\, dx \\
&= \int_\Omega |D\phi(x)|^p \Big(\int_{B^k_\sigma} |(\nabla P)(\phi(x) - a)|^p da \Big) dx \\
&\leq \int_\Omega |D\phi(x)|^p \Big(\int_Q |(\nabla P)(\xi)|^p d\xi \Big) dx \\
&\leq C \int_\Omega |D\phi|^p
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B^k_\sigma}\int_\Omega |x-y|^{1-n} |D(P_a \circ \phi)(y)| dy\, da \\
&\qquad \leq \int_\Omega |x-y|^{1-n} |D\phi(y)| \Big(\int_{B^k_\sigma} |(\nabla P)(\phi(y) - a)| da \Big) dy \\
&\qquad \leq \int_\Omega |x-y|^{1-n} |D\phi(y)| \Big(\int_Q |(\nabla P)(\xi)| d\xi \Big) dy \\
&\qquad \leq C \int_\Omega |x-y|^{1-n} |D\phi(y)| dy
\end{aligned}$$

for a.e. $x \in \Omega$. Furthermore, for some $K > 0$ we have

$$\int_\Omega |D(P_a \circ \phi)|^p \leq \frac{K}{|B^k_\sigma|} \int_{B^k_\sigma}\int_\Omega |D(P_a \circ \phi)|^p$$

and

$$\int_\Omega |x-y|^{1-n}|D(P_a\circ\phi)(y)|dy \le \frac{K}{|B^k_\sigma|}\int_{B^k_\sigma}\int_\Omega |x-y|^{1-n}|D(P_a\circ\phi)(y)|dy\,da$$

for a 'good set' of $a\in B^k_\sigma$ (if not then Markov's inequality implies the latter fails with an exceptional set less than or equal to $K^{-1}|B^k_\sigma|$). Therefore there exists an a_0 with $|a_0|\le\sigma$ such that

$$\int_\Omega |D(P_{a_0}\circ\phi)|^p \le C\int_\Omega |D\phi|^p$$

and

$$\int_\Omega |x-y|^{1-n}|D(P_{a_0}\circ\phi)(y)|dy \le C\int_\Omega |x-y|^{1-n}|D\phi(y)|dy.$$

for a.e. $x\in\Omega$. We also note that $\sup_{a\in B^k_\sigma}\mathrm{Lip}(P_a|_{\widetilde{\mathcal{M}}})^{-1}$ is finite for sufficiently small σ, by the construction of P and the inverse function theorem. Thus by setting $\psi = (P_{a_0}|_{\widetilde{\mathcal{M}}})^{-1}\circ P_{a_0}\circ\phi\in W^{1,p}(\Omega,\widetilde{\mathcal{M}})$ the desired conclusion follows. □

5.5. Comparison maps into arbitrary compact manifolds

In considering the conditions of the above Lemma 5.33 it is typically not the case for a compact manifold to be ℓ-connected for some $\ell\ge 1$. For instance the real projective space $\mathbb{R}P^n$, $n\ge 2$, has $\pi_0(\mathbb{R}P^n)=0$ and $\pi_r(\mathbb{R}P^n)=\pi_r(S^n)$ for $r\ge 2$ but $\pi_1(\mathbb{R}P^n)=\mathbb{Z}_2$. Nevertheless from Corollary 5.12 we know the universal covering has a covering space that is indeed 1-connected. Hence the connectedness condition is canonical not for the base space but rather for the covering space. This is exemplified in particular by the list of examples in Table A.1. Thus by appealing to the lifting result of Theorem 5.20 we can construct suitable comparison maps, based on the model of Example 5.4, by applying the extension result of Lemma 5.33 in the covering.

Lemma 5.34. *Let $\mathcal{M}\subset\mathbb{R}^N$ be a connected compact submanifold without boundary and $\pi\colon\widetilde{\mathcal{M}}\to\mathcal{M}$ be a universal cover such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell\ge 1$. If $u\in W^{1,p}(\Omega,\mathcal{M})$ for $2\le p<\ell+2$ on a smooth bounded domain $\Omega\subset\mathbb{R}^n$, then on a ball $B_s\subset\subset\Omega$ there exists $v\in(u+W^{1,p}_0(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$ and a constant $C=C(n,N,p,\ell)>0$ such that for any $0<r<s$,*

$$\int_{B_s}|Dv|^p \le C\int_{B_s\setminus B_r}|Du|^p + \frac{C}{(s-r)^p}\int_{B_s}|\varphi-(\varphi)_{B_s}|^p$$

and

$$|u(x)-v(x)| \le C\int_{B_s}|x-y|^{1-n}|Du(y)|dy + \frac{C}{s-r}\int_{B_s}|x-y|^{1-n}|\varphi(y)-(\varphi)_{B_s}|dy$$

for a.e. $x\in B_s$, where $\varphi\in W^{1,p}(B_s,\widetilde{\mathcal{M}})$ is such that $u=\pi\circ\varphi$.

Proof. For any $u \in W^{1,p}(\Omega, \mathcal{M})$, $p \geq 2$, there exists $\varphi \in W^{1,p}(\Omega, \widetilde{\mathcal{M}})$ such that $u = \pi \circ \varphi$ and $|Du| = |D\varphi|$ a.e. by Theorem 5.20. Suppose $\widetilde{\mathcal{M}} \subset \mathbb{R}^k$ for some $k \geq 3$ without loss of generality. Also let $B_s \subset\subset \Omega$, $0 < r < s$ and $\eta \in C_c^\infty(\Omega)$ be a cut-off function such that $\eta = 1$ on B_r, $\eta = 0$ on $\Omega \backslash B_s$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s-r)$. Thus if we set

$$\phi = (1-\eta)\varphi + \eta(\varphi)_{B_s}$$

then $\phi \in (W^{1,p} \cap L^\infty)(B_s, \mathbb{R}^k)$ with $\mathfrak{R}\,\phi = \mathfrak{R}\,\varphi \in W^{1,1-\frac{1}{p}}(\partial B_s, \widetilde{\mathcal{M}})$. Hence by Lemma 5.33 there exists $\psi \in (\varphi + W_0^{1,p}(B_s, \mathbb{R}^k)) \cap W^{1,p}(B_s, \widetilde{\mathcal{M}})$, for $2 \leq p < \ell + 2$, such that

$$\int_{B_s} |D\psi|^p \leq C \int_{B_s} |(1-\eta)D\varphi - D\eta \otimes (\varphi - (\varphi)_{B_s})|^p.$$

Then by defining $v = \pi \circ \psi$ we have $\mathfrak{R}\,v = \pi \circ \mathfrak{R}\,\varphi = \mathfrak{R}\,u$ from which the desired result now follows, since $|Dv| = |D\psi|$ a.e. and by standard potential estimates (cf. [GT, (7.37)])

$$|u(x) - v(x)| \leq C \int_{B_s} |x-y|^{1-n} |Du(y) - Dv(y)| dy. \qquad \square$$

Let us also consider the following up-to-the-boundary variant of the previous comparison lemma.

Lemma 5.35. *Let $\mathcal{M} \subset \mathbb{R}^N$ be a connected compact submanifold without boundary and $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ be a universal cover such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \geq 1$. Let $u_1, u_2 \in W^{1,p}(\mathsf{B}, \mathcal{M})$ for $2 \leq p < \ell + 2$ on a fixed ball $\mathsf{B} \subset \mathbb{R}^n$ such that $u_1 - u_2 \in W_0^{1,p}(\mathsf{B}, \mathbb{R}^N)$. If we consider a ball $B_s(x_0)$ such that $B_s(x_0) \cap \mathsf{B} \neq \emptyset$, $B_s(x_0) \cap (\mathbb{R}^n \backslash \mathsf{B}) \neq \emptyset$ then there exists $v \in (u_1 + W_0^{1,p}(B_s(x_0) \cap \mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(B_s(x_0) \cap \mathsf{B}, \mathcal{M})$ and a constant $C = C(n, N, p, \ell) > 0$ such that*

$$\int_{B_s(x_0) \cap \mathsf{B}} |Dv|^p \leq C \int_{(B_s(x_0) \backslash B_t(x_0)) \cap \mathsf{B}} |Du_1|^p + C \int_{B_s(x_0) \cap \mathsf{B}} |Du_2|^p + \Big| \frac{\varphi_1 - h \cdot \varphi_2}{s-t} \Big|^p$$

for any $0 < t < s$, where $\varphi_1, \varphi_2 \in W^{1,p}(\mathsf{B}, \widetilde{\mathcal{M}})$ are such that $u_1 = \pi \circ \varphi_1$, $u_2 = \pi \circ \varphi_2$ with $\varphi_1 - h \cdot \varphi_2 \in W_0^{1,p}(\mathsf{B}, \mathbb{R}^k)$ for some $h \in \pi_1(\mathcal{M})$.

Proof. Firstly, by Theorem 5.20 there exists $\varphi_1, \varphi_2 \in W^{1,p}(\mathsf{B}, \widetilde{\mathcal{M}})$ such that $u_1 = \pi \circ \varphi_1$, $u_2 = \pi \circ \varphi_2$. As $u_1 = u_2$ on $\partial\mathsf{B}$ it follows that $\pi \circ \varphi_1 = \pi \circ \varphi_2$. Hence by Lemma 5.22 there exists $h \in \pi_1(\mathcal{M})$ such that $\varphi_1 = h \cdot \varphi_2$ on $\partial\mathsf{B}$.

Now consider a ball $B_s(x_0)$ such that $B_s(x_0) \cap \mathsf{B} \neq \emptyset$, $B_s(x_0) \cap (\mathbb{R}^n \backslash \mathsf{B}) \neq \emptyset$ and for $t < s$ let $\eta \in C_0^\infty(\mathsf{B})$ be a cut-off function such that $\eta = 1$ on $B_t(x_0) \cap \mathsf{B}$, $\eta = 0$ on $\mathsf{B} \backslash B_s(x_0)$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s-t)$. If in Lemma 5.33 we set

$$\phi = (1-\eta)\varphi_1 + \eta(h \cdot \varphi_2)$$

then $\phi \in (W^{1,p} \cap L^\infty)(B_s \cap \mathsf{B}, \mathbb{R}^k)$ and $\mathfrak{R}\,\phi = \mathfrak{R}\,\varphi_1 \in W^{1,1-\frac{1}{p}}(\partial(B_s \cap \mathsf{B}), \widetilde{\mathcal{M}})$. So there exists $\psi \in (\varphi_1 + W_0^{1,p}(B_s \cap \mathsf{B}, \mathbb{R}^k)) \cap W^{1,p}(B_s \cap \mathsf{B}, \widetilde{\mathcal{M}})$ such that

$$\int_{B_s \cap \mathsf{B}} |D\psi|^p \le C \int_{B_s \cap \mathsf{B}} |(1-\eta)D\varphi_1 + \eta(h \cdot D\varphi_2) - D\eta \otimes (\varphi_1 - h \cdot \varphi_2)|^p.$$

The desired result now follows by setting $v = \pi \circ \psi$, since $|Dv| = |D\psi|$ and $|h \cdot D\varphi_2| = |Du_2|$ a.e. $\square$

Chapter 6

Partial regularity for holonomic minimisers of general quasiconvex functionals

In this chapter we turn our attention to partial regularity results for holonomic minimisers $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ of a general class of functional of the form

$$\int_\Omega f(x, u, Du)dx,$$

where the integrand has an explicit 'u-dependence'. In such a setting partial regularity—based on the unconstrained proof given in Chapter 2—is recovered (cf. Theorem 6.1) via the extension and lifting results of Chapter 5.

6.1. Statement of main result

Consider the functional

$$\mathfrak{F}(v; \Omega) = \int_\Omega f(x, v, Dv)dx$$

where $v \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ with $n, N \geq 2$. Let the integrand $f \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$, for $p \geq 2$, satisfies the following structural assumptions, namely:

(h0) The function $\xi \mapsto f(x, u, \xi)$ is C^2 and $(x, u, \xi) \mapsto D^2_{\xi\xi} f(x, u, \xi)$ is jointly continuous.

(h1) f satisfies the p-growth condition

$$|f(x, u, \xi)| \leq \Gamma(1 + |\xi|^p)$$

for some constant $\Gamma > 0$ and all (x, u, ξ).

(h2) f satisfies Morrey's quasiconvexity condition in the following strong form

$$\gamma \int_{\Omega'} (1+|\xi|^2+|D\phi(y)|^2)^{\frac{p-2}{2}} |D\phi(y)|^2 dy \leq \int_{\Omega'} f(x,u,\xi+D\phi(y)) - f(x,u,\xi)\, dy$$

for some constant $\gamma > 0$, every (x,u,ξ), every bounded open $\Omega' \subset \mathbb{R}^n$ with $|\partial\Omega'| = 0$ and every $\phi \in C_c^\infty(\Omega', \mathbb{R}^N)$.

(h3) f satisfies

$$|f(x,u,\xi) - f(y,v,\xi)| \leq \Gamma\, \omega_\mu(|x-y| + |u-v|)(1+|\xi|^p)$$

where $\omega_\mu(t) = \min(1, t^\mu)$ for a given $0 < \mu \leq 1$.

(h4) There exists a continuous $\psi : \mathbb{R}^{Nn} \to \mathbb{R}$ satisfying

$$f(x,u,\xi) \geq \psi(\xi) \qquad \text{and} \qquad \int_\Omega \psi(D\varphi) \geq \int_\Omega \psi(0) + \gamma |D\varphi|^p$$

for every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$.

We remark that (h4) is obviously satisfied if $f(x,u,\xi) \geq |\xi|^p$ but also allows for integrands with varying sign. It is easy to show that the growth condition (h1) together with the quasiconvexity of (h2) is enough (in fact f need only to be rank-one convex) for the derivative $D_\xi f$ to satisfy the growth condition

$$|D_\xi f(x,u,\xi)| \leq c(1+|\xi|^{p-1}) \tag{6.1}$$

and

$$|f(x,u,\xi+\eta) - f(x,u,\xi)| \leq c(1+|\xi|^{p-1}+|\eta|^{p-1})|\eta|. \tag{6.2}$$

As remarked in [AF2] there is no growth condition placed on the second order derivatives of $\xi \mapsto f(x,u,\xi)$. We merely have, for any given $\Omega' \subset\subset \Omega$ and any given $\Lambda > 0$, the existence of a finite K_Λ such that

$$\sup_{x\in\bar{\Omega}'} \sup_{|u|,|\xi|\leq\Lambda} |D^2_{\xi\xi} f(x,u,\xi)| \leq K_\Lambda. \tag{6.3}$$

There are also no global uniform continuity assumptions imposed on $D^2_{\xi\xi} f(x,u,\xi)$. One only has $(u,\xi) \mapsto D^2_{\xi\xi} f(x,u,\xi)$ uniformly continuous on compact subsets. In fact, associated to $\Lambda > 0$, a min-max construction (cf. Example 1.19) implies the existence of a non-decreasing concave modulus of continuity $\omega : [0,+\infty) \to [0,1]$, with $\lim_{t\searrow 0} \omega(t) = 0$ and $\omega(t) = 1$ for all $t \geq 1$, such that

$$|D^2_{\xi\xi} f(x,u_1,\xi_1) - D^2_{\xi\xi} f(x,u_2,\xi_2)| \leq c\,\omega(|u_1-u_2| + |\xi_1-\xi_2|) \tag{6.4}$$

for any $|u_i|, |\xi_i| \leq \Lambda + 1$.

As to the holonomic constraint we recall that any given Riemannian manifold $(\mathcal{M}, g)$ with a C^r-metric g ($r \geq 3$) can—by the Nash embedding theorem—be isometrically

embedded into $\mathbb{R}^N$ for a sufficiently large integer N. In fact a result of Gromov [G12] shows that every compact surface isometrically embeds into $\mathbb{R}^5$ (the infimum dimension of target Euclidean space cannot in general be improved for this case). Indeed, the minimal isometric embedding of the real projective space is $\mathbb{R}P^2 \to S^4(\frac{1}{\sqrt{3}}) \subset \mathbb{R}^5$. Also, as many examples can be considered as homogeneous spaces, e.g. $S^n = SO(n+1)/SO(n)$ and $\mathbb{R}P^n = SO(n+1)/O(n)$, a result of Moore [M12] shows any compact Riemannian homogeneous $\mathcal{M} = G/H$ space possesses an embedding in some Euclidean space which is both equivariant and isometric.

In considering the above geometric and analytic aspects we seek to prove (with respect to the holonomic minimality condition of Definition 1.53) the following:

Theorem 6.1. *Let $\mathcal{M}$ be a connected compact C^3-Riemannian manifold without boundary with a universal covering $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$. Suppose the fundamental group $\pi_1(\mathcal{M})$ is finite, $\widetilde{\mathcal{M}}$ is ℓ-connected for some integer $1 \leq \ell \leq n-2$, and the integrand f satisfies (h0), (h1), (h2), (h3) and (h4) for $2 \leq p < \ell + 2$. Then for any holonomic $\mathfrak{F}$-minimiser $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ there is a relatively closed subset $\operatorname{Sing} u \subset \Omega$ of Lebesgue measure zero such that*

$$u \in C^{1,\beta}_{\mathrm{loc}}(\Omega \backslash \operatorname{Sing} u, \mathcal{M})$$

for some $0 < \beta < 1$.

We remark that the finiteness condition of $\pi_1(\mathcal{M})$ is equivalent to assuming the covering space $\widetilde{\mathcal{M}}$ is compact (cf. Corollary 5.13). Moreover the covering space is always simply connected by Theorem 5.11, hence $\widetilde{\mathcal{M}}$ is at least 1-connected. In fact:

(i) If $\mathcal{M} = S^k$ for $k \geq 2$ then $\pi_1(\mathcal{M}) = 0$ is trivial so that $\pi = \mathrm{Id}_{\mathcal{M}}$, $\widetilde{\mathcal{M}} = \mathcal{M}$ is $(k-1)$-connected and $2 \leq p < k+1$.

(ii) If $\mathcal{M} = \mathbb{R}P^k$ for $k \geq 2$ then $\pi_1(\mathcal{M}) = \mathbb{Z}_2$ is finite, $\widetilde{\mathcal{M}} = S^k$ is $(k-1)$-connected and $2 \leq p < k+1$.

(iii) If $\mathcal{M} = G$ is a connected compact Lie group the finiteness of $\pi_1(\mathcal{M})$ can be relaxed as the simply connected covering group splits into $\widetilde{\mathcal{M}} = \mathbb{R}^k \times G_1 \times \cdots \times G_\ell$, where G_j are in fact 2-connected, so that $2 \leq p < 4$, cf. (5.2).

Here we focuses on the case $p < n$; if $p \geq n$ the Sobolev embeddings imply the extension result of Theorem 5.26 and the regularity argument would be in the single coordinate chart case. Our proof of Theorem 6.1 is direct via an application of Ekeland's variational principle; a tangential $\mathbb{A}$-harmonic approximate lemma; Caccioppoli inequalities using only minimality, the p-growth of (h1) and the strong quasiconvexity of (h2); the Sobolev lifting result of Theorem 5.20; and, the extension-type result of Lemma 5.33 using the ℓ-connectedness of the compact covering space $\widetilde{\mathcal{M}}$. The resulting estimates allow for a

sufficient decay of the quadratic excess functions $\mathcal{E}$ given by

$$\mathcal{E}(r) = \mathcal{E}(x,r) = \Big(\fint_{B_r(x)} |V(Du - (Du)_{x,r})|^2 \Big)^{\frac{1}{2}}$$

where $V(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}}\xi$. Regularity follows via Campanato's integral characterisation of Hölder continuity (cf. Theorem 1.24) with the singular set characterised by

$$\begin{aligned} \operatorname{Sing} u = \big\{ x \in \Omega : \limsup_{r \searrow 0} |(u)_{x,r}| = +\infty \quad &\text{or} \quad \limsup_{r \searrow 0} |(Du)_{x,r}| = +\infty \\ &\text{or} \quad \liminf_{r \searrow 0} \mathcal{E}(x,r) > 0 \big\}. \end{aligned} \tag{6.5}$$

In fact Schauder estimates give $C^{1,\alpha}_{\mathrm{loc}}$ for $\alpha < \mu$. A model proof by direct means on the level of the gradient in the unconstrained setting is given in Chapter 2 with $p = 2$ and an integrand of the form $f = f(Dv)$.

Remark 6.2. If we take a point $x \in \Omega \backslash \operatorname{Sing} u$ it follows that $\limsup |(u)_{x,r}| < +\infty$, $\limsup |(Du)_{x,r}| < +\infty$ and $\liminf \mathcal{E}(x,r) = 0$ by definition. Then there exists $\Lambda > 0$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ such that $|(u)_{x,r}| < \Lambda$ and $|(Du)_{x,r}| < \Lambda$ for all $0 < r < R$. Similarly, for each $\varepsilon > 0$ there exists $0 < R < \operatorname{dist}(x, \partial\Omega)$ such that $\mathcal{E}(x,R) < \varepsilon$. Now *if* for each $\Lambda > 0$ we can find $\varepsilon = \varepsilon(\Lambda) > 0$ and $R = R(\Lambda) \in (0, \operatorname{dist}(x, \partial\Omega))$ both sufficiently small such that

$$\mathcal{E}(x,R) < \varepsilon \tag{6.6}$$

then $x \in \Sigma_{\Lambda,\varepsilon,R}$, where $\Sigma_{\Lambda,\varepsilon,R} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R,\ \mathcal{E}(x,R) < \varepsilon,\ |(Du)_{x,R}| < \Lambda,\ |(u)_{x,R}| < \Lambda\}$ is open (since for fixed $0 < r \ll 1$ the maps $x \mapsto |(u)_{x,r}|$, $x \mapsto |(Du)_{x,r}|$ and $x \mapsto \mathcal{E}(x,r)$ are continuous on $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$). Hence we need to show for some $0 < \beta < 1$ and each $x \in \Sigma_{\Lambda,\varepsilon,R}$ that

$$\mathcal{E}(x,r) \leq K r^{\beta}, \qquad 0 < r < R, \tag{6.7}$$

where $K = K(\beta, \varepsilon, \Lambda, R)$, in order to conclude $Du \in C^{0,\beta}_{\mathrm{loc}}(\Sigma_{\Lambda,\varepsilon,R}, \mathbb{R}^{Nn})$ by Theorem 1.24. Consequently Du is $C^{0,\beta}_{\mathrm{loc}}$-continuous on

$$\operatorname{Reg} u = \bigcup_{\Lambda > 0} \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R(\Lambda),\ \mathcal{E}(x,R) < \varepsilon(\Lambda),\ |(Du)_{x,R}| < \Lambda,\ |(u)_{x,R}| < \Lambda\}$$

of which $\operatorname{Sing} u$ is the complement in Ω, provided one has (6.6) and can establish the excess-decay (6.7).

The auxiliary functions $V \colon \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}}\xi \tag{6.8}$$

for every $\xi \in \mathbb{R}^k$ is used in the definition of the excess. Note that $\mathbb{R}^+ \ni t \mapsto |V(t)|$ is convex and $|\xi|^2, |\xi|^p \le |V(\xi)|^2 \le c(p)(|\xi|^2 + |\xi|^p)$ when $p \ge 2$. Also, from the proof in [DGK, §4] we have the Sobolev-Poincaré-type inequality

$$\fint_{B_R} |V(\tfrac{v-(v)_R}{R})|^2 \le c\Big(\fint_{B_R} |V(Dv)|^{2_*}\Big)^{2/2_*} \tag{6.9}$$

for any $v \in W^{1,p}(B_R, \mathbb{R}^N)$, where $B_R \subset \mathbb{R}^n$, $c = c(n, N, p)$, $n \ge 2$ and $2_* = \frac{2n}{n+2} < 2$. The following properties of (6.8) will also be called upon.

Lemma 6.3 ([CFM, p. 143], [B1, p. 800])**.** *Let $p \ge 2$ and $V\colon \mathbb{R}^k \to \mathbb{R}^k$ be as in* (6.8)*. Then for any $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ there holds:*

$$\max\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le 2^{\frac{p-2}{4}} \max\{|\xi|, |\xi|^{\frac{p}{2}}\} \tag{6.10a}$$

$$|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)| \tag{6.10b}$$

$$|V(\xi + \eta)| \le c(p)(|V(\xi)| + |V(\eta)|) \tag{6.10c}$$

$$c(p)|\xi - \eta| \le \frac{|V(\xi) - V(\eta)|}{(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}} \le C(k, p)|\xi - \eta| \tag{6.10d}$$

$$|V(\xi) - V(\eta)| \le c(k, p, \Lambda)|V(\xi - \eta)|, \quad \text{whenever } |\eta| \le \Lambda \tag{6.10e}$$

$$|V(\xi - \eta)| \le c(p)|V(\xi) - V(\eta)|. \tag{6.10f}$$

6.2. Higher integrability

From the lifting and extension results of Chapter 5 we obtain Caccioppoli inequalities of the first kind using only the growth (h1), (6.2) and the coercivity (h4). By an application of Gehring's lemma from §1.3 this implies local higher integrability for the gradient Du as follows.

Lemma 6.4. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \ge 1$. Suppose the integrand f satisfies (h1), (h4) and* (6.2) *for $2 \le p < \ell + 2$. Then for any holonomic $\mathfrak{F}$-minimiser $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ there exist $C = C(n, N, p, \gamma, \Gamma) > 0$, $q = q(n, N, p, \gamma, \Gamma) > p$ such that $Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ and*

$$\Big(\fint_{B_{R/2}(x_0)} |Du|^q\Big)^{1/q} \le C\Big(\fint_{B_R(x_0)} 1 + |Du|^p\Big)^{1/p}$$

for any $B_R(x_0) \subset\subset \Omega$.

Proof. For $B_R(x_0) \subset\subset \Omega$ let $\frac{R}{2} \le r < s \le R$ and choose a cut-off function $\eta \in C^\infty_c(\Omega)$ such that $\eta = 1$ on $B_r(x_0)$, $\eta = 0$ on $\Omega \backslash B_s(x_0)$, $0 \le \eta \le 1$ and $|D\eta| \le C/(s - r)$. Define

$\phi_1 = \eta(u-(u)_{x_0,R})$ and $\phi_2 = (1-\eta)(u-(u)_{x_0,R})$. Using (h4) we get

$$\int_{B_s} \gamma|D\phi_1|^p + \psi(0) \le \int_{B_s} \psi(D\phi_1) \le \int_{B_s} f(x,u,D\phi_1) = \int_{B_s} f(x,u,Du-D\phi_2). \quad (6.11)$$

Furthermore, by (6.2) we estimate

$$\begin{aligned}\int_{B_s} f(x,u,Du-D\phi_2) - f(x,u,Du) &\le C\int_{B_s} (1+|Du|^{p-1}+|D\phi_2|^{p-1})|D\phi_2| \\ &\le C\int_{B_s\setminus B_r} 1+|Du|^p + \Big|\frac{u-(u)_R}{s-r}\Big|^p,\end{aligned}$$

since $D\phi_2 = (1-\eta)Du - (u-(u)_R)\otimes D\eta$. Also, by the minimality of u we have

$$\int_{B_s} f(x,u,Du) \le \int_{B_s} f(x,v,Dv)$$

for any $v \in (u+W^{1,p}_0(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$. Now by the lifting result of Theorem 5.20 there exists $\varphi \in W^{1,p}(B_s,\widetilde{\mathcal{M}})$ such that $u = \pi\circ\varphi$ with $|D\varphi| = |Du|$ a.e. Then from Lemma 5.34 one constructs a suitable competitor $v \in (u+W^{1,p}(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$ such that

$$\begin{aligned}\int_{B_s}|Dv|^p &\le C\int_{B_s\setminus B_r}|Du|^p + C\int_{B_s}\Big|\frac{\varphi-(\varphi)_R}{s-r}\Big|^p \\ &\le C\int_{B_s\setminus B_r}|Du|^p + C\frac{R^{n+p}}{(s-r)^p}\Big(\fint_{B_R}|Du|^{p_*}\Big)^{p/p_*}\end{aligned}$$

by the Sobelov-Poincaré inequality (1.15). Then by (6.11) and (h1) we obtain

$$\begin{aligned}\int_{B_r}|Du|^p \le \int_{B_s}|D\phi_1|^p &\le \frac{1}{\gamma}\int_{B_s} f(x,u,Du-D\phi_2) + |\psi(0)| \\ &\le CR^n + C\int_{B_s\setminus B_r}|Du|^p + C\frac{R^{n+p}}{(s-r)^p}\Big(\fint_{B_R}|Du|^{p_*}\Big)^{p/p_*}\end{aligned}$$

for some $C = C(n,N,p,\gamma,\Gamma)$. So by Widman's hole filling trick we get

$$\int_{B_r}|Du|^p \le \chi\int_{B_s}|Du|^p + R^n + \frac{R^{n+p}}{(s-r)^p}\Big(\fint_{B_R}|Du|^{p_*}\Big)^{p/p_*}$$

where $\chi = \frac{C}{1+C} < 1$. Then by the standard iteration of Lemma 0.7 we conclude that

$$\fint_{B_{R/2}(x_0)}|Du|^p \le C + C\Big(\fint_{B_R(x_0)}|Du|^{p_*}\Big)^{p/p_*} \quad (6.12)$$

for all $B_R(x_0)\subset\Omega$. In which case Gehring's result of Proposition 1.6, with $f = |Du|^{p_*}$ and $\alpha = p/p_* > 1$, shows that there exists $\beta > \alpha$ such that $(\fint_{B_{R/2}} f^\beta)^{1/\beta} \le C + C(\fint_{B_R} f^\alpha)^{1/\alpha}$

for all $B_R \subset\subset \Omega$. Then by defining $q = \frac{\beta}{\alpha} p > p$ we obtain

$$\Big(\fint_{B_{R/2}} |Du|^q \Big)^{1/q} \leq C \Big(\fint_{B_R} 1 + |Du|^p \Big)^{1/p}$$

for all $B_R \subset\subset \Omega$. In particular $Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$. □

We can also get higher integrability up to the boundary by adapting the proof of the previous lemma using standard up to the boundary arguments (cf. [AF4, Lem. 3.5], [G6, p. 152]).

Lemma 6.5. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \geq 1$. Suppose the integrand f satisfy (h1), (h4) and (6.2) for $2 \leq p < \ell + 2$. Fix $(x_0, u_0) \in \Omega \times \mathbb{R}^N$ and on a given ball $\mathsf{B} \subset \Omega$ assume $u \in W^{1,q_0}(\mathsf{B}, \mathcal{M})$ for some $q_0 > p$. If $\widehat{u} \in (u + W^{1,p}_0(\mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(\mathsf{B}, \mathcal{M})$ satisfies*

$$\int_{\mathsf{B}} f(x_0, u_0, D\widehat{u}) \leq \int_{\mathsf{B}} f(x_0, u_0, Dv) \tag{6.13}$$

for all $v \in (u + W^{1,p}_0(\mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(\mathsf{B}, \mathcal{M})$, then there exists $q \in (p, q_0]$ such that $\widehat{u} \in W^{1,q}(\mathsf{B}, \mathcal{M})$ and

$$\Big(\fint_{\mathsf{B}} |D\widehat{u}|^q \Big)^{1/q} \leq C \Big(\fint_{\mathsf{B}} |D\widehat{u}|^p \Big)^{1/p} + C \Big(\fint_{\mathsf{B}} 1 + |Du|^{q_0} \Big)^{1/q_0}.$$

Proof. For interior estimates fix $B_r(x_0) \subset \mathsf{B}$, let $\frac{r}{2} \leq t < s \leq r$ and choose a cut-off function $\eta \in C^\infty_c(\mathsf{B})$ such that $\eta = 1$ on $B_t(x_0)$, $\eta = 0$ on $\mathsf{B} \backslash B_s(x_0)$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s-t)$. By defining $\phi_1 = \eta(\widehat{u} - (\widehat{u})_{x_0,r})$ and $\phi_2 = (1-\eta)(\widehat{u} - (\widehat{u})_{x_0,r})$ together with (h4) yields

$$\int_{B_s} \gamma |D\phi_1|^p + \psi(0) \leq \int_{B_s} \psi(D\phi_1) \leq \int_{B_s} f(x_0, u_0, D\phi_1) = \int_{B_s} f(x_0, u_0, D\widehat{u} - D\phi_2).$$

By the condition (6.13) and the construction given in Lemma 5.34 we have $v \in (u + W^{1,p}_0(\mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(\mathsf{B}, \mathcal{M})$ such that $v = \widehat{u}$ on $\mathsf{B} \backslash B_s(x_0)$ with $v = \text{const.}$ on $B_t(x_0)$. Thus $\int_{\mathsf{B}} f(x_0, u_0, D\widehat{u}) \leq \int_{B_s} f(x_0, u_0, Dv) + \int_{\mathsf{B}\backslash B_s} f(x_0, u_0, D\widehat{u})$ and

$$\int_{B_s} |Dv|^p \leq C \int_{B_s \backslash B_t} |D\widehat{u}|^p + C \frac{r^{n+p}}{(s-t)^p} \Big(\fint_{B_r} |D\widehat{u}|^{p_*} \Big)^{p/p_*}.$$

With this, the procedure outlined in Lemma 6.4, together with Widman's hole filling and Lemma 0.7, yields

$$\fint_{B_{r/2}(x_0)} |D\widehat{u}|^p \leq C \Big(\fint_{B_r(x_0)} 1 + |D\widehat{u}|^{p_*} \Big)^{p/p_*} \tag{6.14}$$

for any $B_r(x_0) \subset \mathsf{B}$.

For estimates near the boundary assume $B_r(x_0) \cap \mathsf{B} \neq \emptyset$, $B_r(x_0) \cap (\mathbb{R}^n \backslash \mathsf{B}) \neq \emptyset$. Let $\frac{r}{2} \leq t < s \leq r$ and choose a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\eta = 1$ on $B_t(x_0) \cap \mathsf{B}$, $\eta = 0$ on $\mathsf{B} \backslash B_s(x_0)$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s-t)$. Define $\phi_1 = \eta(\widehat{u} - u)$ and $\phi_2 = (1-\eta)(\widehat{u} - u)$. Since $\widehat{u} - u \in W_0^{1,p}(\mathsf{B}, \mathbb{R}^N)$ we have $\phi_1 \in W_0^{1,p}(B_s(x_0) \cap \mathsf{B}, \mathbb{R}^N)$. By (h4) it follows that

$$\begin{aligned}\int_{B_s \cap \mathsf{B}} \gamma |D\phi_1|^p + \psi(0) \leq \int_{B_s \cap \mathsf{B}} \psi(D\phi_1) &\leq \int_{B_s \cap \mathsf{B}} f(x_0, u_0, D\phi_1) \\ &= \int_{B_s \cap \mathsf{B}} f(x_0, u_0, D\widehat{u} - Du - D\phi_2).\end{aligned}$$

Now for any $\varepsilon > 0$ of our choosing we bound

$$\begin{aligned}&\int_{B_s \cap \mathsf{B}} f(x_0, u_0, D\widehat{u} - Du - D\phi_2) - f(x_0, u_0, D\widehat{u}) \\ &\quad \leq c \int_{B_s \cap \mathsf{B}} (1 + |D\widehat{u}|^{p-1} + |Du + D\phi_2|^{p-1}) |Du + D\phi_2| \\ &\quad \leq C_\varepsilon \int_{B_s \cap \mathsf{B}} (1 + |Du|^p + |D\phi_2|^p) + \varepsilon \int_{B_s \cap \mathsf{B}} |D\widehat{u} - Du|^p \\ &\quad \leq C_\varepsilon \int_{B_s \backslash B_t \cap \mathsf{B}} |D\widehat{u} - Du|^p + C_\varepsilon \int_{B_s \cap \mathsf{B}} 1 + |Du|^p + \Big|\frac{\widehat{u} - u}{s-t}\Big|^p + \varepsilon \int_{B_s \cap \mathsf{B}} |D\widehat{u} - Du|^p\end{aligned}$$

by (6.2), since $D\phi_2 = (1-\eta)(D\widehat{u} - Du) - D\eta \otimes (\widehat{u} - u)$. From the condition (6.13) we have

$$\int_{\mathsf{B}} f(x_0, u_0, D\widehat{u}) \leq \int_{B_s \cap \mathsf{B}} f(x_0, u_0, Dv) + \int_{\mathsf{B} \backslash B_s} f(x_0, u_0, D\widehat{u})$$

for any $v \in (\widehat{u} + W_0^{1,p}(\mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(\mathsf{B}, \mathcal{M})$ with $v = \widehat{u}$ a.e. on $\mathsf{B} \backslash B_s$. Furthermore, by Lemma 5.35 there exists $v \in (\widehat{u} + W_0^{1,p}(B_s \cap \mathsf{B}, \mathbb{R}^N)) \cap W^{1,p}(B_s \cap \mathsf{B}, \mathcal{M})$ such that

$$\int_{B_s \cap \mathsf{B}} |Dv|^p \leq C \int_{B_s \backslash B_t \cap \mathsf{B}} |D\widehat{u} - Du|^p + C \int_{B_s \cap \mathsf{B}} |Du|^p + \Big|\frac{\widehat{\varphi} - h \cdot \varphi}{s-t}\Big|^p,$$

where $\varphi, \widehat{\varphi} \in W^{1,p}(\mathsf{B}, \widetilde{\mathcal{M}})$ are such that $u = \pi \circ \varphi$, $\widehat{u} = \pi \circ \widehat{\varphi}$ with $\widehat{\varphi} - h \cdot \varphi \in W_0^{1,p}(\mathsf{B}, \mathbb{R}^k)$ for some $h \in \pi_1(\mathcal{M})$ by the uniqueness of the lifting. In putting these estimate together we find

$$\begin{aligned}\int_{B_t \cap \mathsf{B}} |D\widehat{u} - Du|^p \leq C \int_{B_s \backslash B_t \cap \mathsf{B}} |D\widehat{u} - Du|^p \\ + C \int_{B_s \cap \mathsf{B}} 1 + |Du|^p + \Big|\frac{\widehat{\varphi} - h \cdot \varphi}{s-t}\Big|^p + \Big|\frac{\widehat{u} - u}{s-t}\Big|^p.\end{aligned}$$

Then by Widman's hole filling and the iteration of Lemma 0.7 we obtain

$$\int_{B_{r/2} \cap \mathsf{B}} |D\widehat{u} - Du|^p \leq C \int_{B_r \cap \mathsf{B}} (1 + |Du|^p) + Cr^{-p} \int_{B_r \cap \mathsf{B}} |\widehat{\varphi} - h \cdot \varphi|^p + |\widehat{u} - u|^p.$$

As $\widehat{u} - u$ can be extended by zero outside B and the volume of $B_{2r} \cap \mathsf{B}$ is greater than the volume of B_r we may apply a modified Sobolev-Poincaré inequality to estimate

$$\int_{B_r \cap \mathsf{B}} |\widehat{u} - u|^p \leq c\Big(\int_{B_{2r} \cap \mathsf{B}} |D\widehat{u} - Du|^{p_*} \Big)^{p/p_*}$$

and, likewise

$$\int_{B_r \cap \mathsf{B}} |\widehat{\varphi} - h \cdot \varphi|^p \leq c\Big(\int_{B_{2r} \cap \mathsf{B}} |Du|^{p_*} + |D\widehat{u}|^{p_*} \Big)^{p/p_*}$$

since $|Du| = |h \cdot D\varphi|$ and $|D\widehat{u}| = |D\widehat{\varphi}|$. We then conclude

$$\int_{B_{r/2} \cap \mathsf{B}} |D\widehat{u}|^p \leq C \int_{B_{2r} \cap \mathsf{B}} (1 + |Du|^p) + Cr^{-p} \Big(\int_{B_{2r} \cap \mathsf{B}} |D\widehat{u}|^{p_*} \Big)^{p/p_*}. \tag{6.15}$$

Now by defining $f(x) = |D\widehat{u}(x)|^{p_*}$ for $x \in \mathsf{B}$ with $f(x) = 0$ for $x \notin \mathsf{B}$ and $g(x) = (1 + |Du(x)|^p)^{p_*/p}$ for $x \in \mathsf{B}$ with $g(x) = 0$ for $x \notin \mathsf{B}$ it follows by (6.14) and (6.15) that

$$⨍_{B_{r/2}} f^\alpha \leq C\Big(⨍_{B_{2r}} f \Big)^\alpha + C ⨍_{B_{2r}} g^\alpha$$

for all balls $B_r \subset \mathbb{R}^n$, where $\alpha = p/p_*$. Then by Gehring's lemma there exists $\beta \in (\frac{p}{p_*}, \frac{q_0}{p_*}]$ such that $(⨍_{B_{r/2}} f^\beta)^{1/\beta} \leq C(⨍_{B_{2r}} f^\alpha)^{1/\alpha} + C(⨍_{B_{2r}} g^\beta)^{1/\beta}$. Thus with $q = p_*\beta \in (p, q_0]$ we obtain

$$\Big(⨍_{\mathsf{B}} |D\widehat{u}|^q \Big)^{1/q} \leq C\Big(⨍_{\mathsf{B}} |D\widehat{u}|^p \Big)^{1/p} + C\Big(⨍_{\mathsf{B}} 1 + |Du|^q \Big)^{1/q}. \qquad \square$$

6.3. An application of Ekeland's variational principle

In this section we use the following lemma of Ekeland together with ideas from Acerbi and Fusco [AF2] to show an holonomic $\mathfrak{F}$-minimiser almost minimises the so-called frozen problem. This result is then subsequently utilised in §6.5 in order to show approximate tangential harmonicity via a linearisation strategy applied to the frozen integrand.

Lemma 6.6 ([E1])**.** *Let (X, d) be a complete metric space and $\mathcal{J} \colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional that is bounded from below but not identically $+\infty$. For $\epsilon > 0$ and a point $u \in X$ such that*

$$\mathcal{J}(u) < \epsilon + \inf \mathcal{J}$$

there exists some point $v \in X$ such that $d(u, v) \leq 1$ and

$$\mathcal{J}(v) \leq \mathcal{J}(w) + \epsilon\, d(v, w)$$

for every $w \in X$.

To satisfy the lower semicontinuity requirement we recall the following:

Lemma 6.7. *Let the integrand f satisfy (h1), (h2) and fix $(x_0, u_0) \in \Omega \times \mathbb{R}^N$. If $B_R \subset \Omega$ is a ball and $u \in W^{1,p}(B_R, \mathcal{M})$ the functional $w \mapsto \int_{B_R} f(x_0, u_0, Dw)$ is sequentially weakly lower semicontinous on $(u + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,p}(B_R, \mathcal{M})$ endowed with the weak topology and satisfies*

$$\int_{B_R} f(x_0, u_0, Dw) \geq \gamma \int_{B_R} |Dw|^p - c \int_{B_R} (1 + |Du|^p). \tag{6.16}$$

Proof. As the quasiconvexity of (h2) and (h1) imply semicontinuity (cf. Lemma 0.6) and as $W^{1,p}(B_R, \mathcal{M})$ inherits its weak topology from $W^{1,p}(B_R, \mathbb{R}^N)$ the desired semicontinuity follows therefrom. To see the second part let $u_{\text{ext}} \in (u)_R + W_0^{1,p}(B_{2R}, \mathbb{R}^N)$ be an extension of $u \in W^{1,p}(B_R, \mathcal{M})$ such that $\int_{B_{2R}} |Du_{\text{ext}}|^p \leq c \int_{B_R} |Du|^p$. Given $w \in (u + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,p}(B_R, \mathcal{M})$ define $\widetilde{w} \in (u)_R + W_0^{1,p}(B_{2R}, \mathbb{R}^N)$ such that $\widetilde{w} = w$ on B_R with $\widetilde{w} = u_{\text{ext}}$ on $B_{2R} \backslash B_R$. Then by (h2) we have

$$\begin{aligned} \int_{B_{2R}} \gamma |D\widetilde{w}|^p + f(x_0, u_0, 0) &\leq \int_{B_{2R}} f(x_0, u_0, D\widetilde{w}) \\ &= \int_{B_R} f(x_0, u_0, Dw) + \int_{B_{2R} \backslash B_R} f(x_0, u_0, Du_{\text{ext}}) \end{aligned}$$

from which the desired (6.16) follows by (h1). □

For a given $u \in W^{1,p}_{\text{loc}}(\Omega, \mathcal{M})$ and fixed ball $B_{2R}(x_0) \subset \Omega$ with $0 < R < 1$ let $\widehat{u} \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ be the minimiser of the the corresponding frozen problem:

$$\int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, D\widehat{u}) \leq \int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dw) \tag{6.17}$$

for all $w \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$.

By defining

$$\mathcal{J}(w) = \fint_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dw) dx$$

we see any holonomic $\mathfrak{F}$-minimiser u of the 'full problem' is an almost minimiser of the 'frozen problem' in the following sense.

Proposition 6.8. *Under the assumptions of Lemma 6.4 and Lemma 6.5 for an integrand f satisfying (h1), (h2), (h3) and (h4) there exists a constant $0 < \beta < 1$ such that if for each $K > 0$ the map $u \in W^{1,p}_{\text{loc}}(\Omega, \mathcal{M})$ is an holonomic $\mathfrak{F}$-minimiser with*

$$\Big(\fint_{B_{2R}(x_0)} |Du|^p \Big)^{\frac{1}{p}} \leq K$$

then there exists $c_1 = c_1(K) > 0$ such that

$$\mathcal{J}(u) \leq \mathcal{J}(\widehat{u}) + c_1 R^\beta.$$

Proof. First note that

$$\begin{aligned}\mathcal{J}(u)-\mathcal{J}(\widehat{u}) &\le \fint_{B_R} f(x,u,Du)-f(x,\widehat{u},D\widehat{u})\\ &\quad+\fint_{B_R} f(x_0,(u)_R,Du)-f(x,u,Du)\\ &\quad+\fint_{B_R} f(x,\widehat{u},D\widehat{u})-f(x_0,(u)_R,D\widehat{u}) = I+II+III.\end{aligned}$$

By the minimality of u we have $I\le 0$. Looking to II we find, using the local higher integrability of Lemma 6.4 for some exponent $q>p$, that

$$\begin{aligned}&\fint_{B_R}\omega_\mu(|x-x_0|+|u-(u)_R|)|Du|^p\\ &\qquad\le c\Big(\fint_{B_R}\omega_\mu(|x-x_0|+|u-(u)_R|)^{\frac{q}{q-p}}\Big)^{1-\frac{p}{q}}\Big(\fint_{B_{2R}}1+|Du|^p\Big).\end{aligned}$$

Moreover

$$\Big(\fint_{B_R}\omega_\mu(|x-x_0|+|u-(u)_R|)^{\frac{q}{q-p}}\Big)^{1-\frac{p}{q}}\le c\,\omega_\mu(1+2^{\frac{n}{p}}K)^{1-\frac{p}{q}}R^{\mu(1-\frac{p}{q})},$$

since ω_μ is concave and at most 1. Therefore by (h3) we bound

$$II\le\fint_{B_R}\omega_\mu(|x-x_0|+|u-(u)_R|)(1+|Du|^p)\le c_1R^{\mu(1-\frac{p}{q})}$$

for some $c_1=c_1(K)$. Looking to III we see that

$$\begin{aligned}&\fint_{B_R}\omega_\mu(|x-x_0|+|\widehat{u}-(u)_R|)|D\widehat{u}|^p\\ &\qquad\le c\Big(\fint_{B_R}\omega_\mu(|x-x_0|+|\widehat{u}-(u)_R|)^{\frac{q}{q-p}}\Big)^{1-\frac{p}{q}}\Big(\fint_{B_R}|D\widehat{u}|^q\Big)^{\frac{p}{q}}.\end{aligned}$$

Using the higher integrability up to the boundary result of Lemma 6.5 we get

$$\Big(\fint_{B_R}|D\widehat{u}|^q\Big)^{\frac{p}{q}}\le c\fint_{B_R}|D\widehat{u}|^p+c\Big(\fint_{B_R}1+|Du|^q\Big)^{\frac{p}{q}}\le c\fint_{B_{2R}}1+|Du|^p,$$

where the last inequality follows from Lemma 6.4 and (6.16) together with the minimality (6.17) of $\widehat{u}$. Furthermore, the Poincaré inequality of (1.12) implies

$$\begin{aligned}\fint_{B_R}|\widehat{u}-(u)_R| &\le \fint_{B_R}|\widehat{u}-u|+|u-(u)_R|\\ &\le c\,R\Big(\fint_{B_R}|Du|^p\Big)^{\frac{1}{p}}+c\,R\Big(\fint_{B_R}|Du-D\widehat{u}|^p\Big)^{\frac{1}{p}}\\ &\le c\,(1+K^p)^{\frac{1}{p}}R,\end{aligned}$$

where the last inequality also follows from (6.16) together with the minimality (6.17) of $\widehat{u}$. In which case one finds, in a similar fashion to that of the II estimates, that $III \leq c_1(K)R^{\mu(1-\frac{p}{q})}$ also. The result now follows with $\beta = \mu(1-\frac{p}{q})$. □

By endowing the space $(u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ with the metric

$$d(v_1, v_2) = c_1(K)^{-1} R^{-\frac{\beta}{2}} \Big(\fint_{B_R(x_0)} |Dv_1 - Dv_2|^p \Big)^{\frac{1}{p}}, \qquad \beta = \mu(1-\tfrac{p}{q}),$$

the functional $\mathcal{J}$ is lower semicontinuous (cf. Lemma 6.7) and clearly

$$\inf \mathcal{J} = \mathcal{J}(\widehat{u})$$

by definition. So by the lemma of Ekeland (cf. Lemma 6.6) and the result of Proposition 6.8 there exists $v \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ satisfying

$$\Big(\fint_{B_R(x_0)} |Dv - Du|^p \Big)^{\frac{1}{p}} \leq c_1(K) R^{\beta/2} \tag{6.18}$$

which also is a minimiser in the sense that

$$\mathcal{J}(v) \leq \mathcal{J}(w) + R^{\beta/2} \Big(\fint_{B_R(x_0)} |Dv - Dw|^p \Big)^{\frac{1}{p}} \tag{6.19}$$

among all $w \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$. Moreover, by following the proof of Lemma 1.58 we also obtain the following:

Corollary 6.9. *For any minimiser $v \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ of (6.19) we have*

$$\Big| \fint_{B_R(x_0)} Df^0(Dv) D(\nabla \Pi(v)\phi) \Big| \leq R^{\beta/2} \Big(\fint_{B_R(x_0)} |D(\nabla \Pi(v)\phi)|^p \Big)^{\frac{1}{p}}$$

for all $\phi \in W_0^{1,p}(B_R(x_0), \mathbb{R}^N) \cap L^\infty(B_R(x_0), \mathbb{R}^N)$, where Π is a retraction from a tubular neighbourhood onto $\mathcal{M}$, provided the integrand $f^0(\xi) = f(x_0, (u)_{x_0,R}, \xi)$ satisfies (h1).

6.4. The tangential $\mathbb{A}$-harmonic approximation lemma

Let $\mathbb{A}$ be a given symmetric bilinear form on $\mathbb{R}^{Nn}$. Looking to [EG3, pp. 83-5] and §1.7 we say:

Definition 6.10. A map $h \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a *tangential $\mathbb{A}$-harmonic map* about a given point $\theta \in \mathcal{M}$ if $\mathbb{A}$ satisfies the strong ellipticity condition (1.19) and

$$\int_\Omega \mathbb{A}(Dh, D\phi) = 0$$

for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^N)$ such that $\nabla\Pi(\theta)D\phi = D\phi$ and $\nabla\Pi(\theta)Dh = Dh$ a.e. (Ω).

Definition 6.11. A map $u \in W^{1,p}(B_R, \mathcal{M})$ is a *δ-approximate tangential $\mathbb{A}$-harmonic map* about a point $\theta \in \mathcal{M}$ with exponents $\alpha > 1$, $\beta_0 \in (0,1]$ if $\mathbb{A}$ satisfies the strong ellipticity condition (1.19) and for some $\delta > 0$ we have

$$\fint_{B_R} |V(Du - \nabla\Pi(\theta)Du)|^2 \leq \delta^2 R + \delta^{2(1-\frac{1}{\alpha})}\Big(\fint_{B_R} |V(Du - (Du)_R)|^{2\alpha}\Big)^{\frac{1}{\alpha}}$$

and

$$\Big|\fint_{B_R} \mathbb{A}(Du, \nabla\Pi(\theta)D\phi)\Big| \leq \delta\Big(R^{\beta_0} + \fint_{B_R} |V(Du - (Du)_R)|^2\Big)^{\frac{1}{2}} \|D\phi\|_\infty$$

for all $\phi \in C_c^\infty(B_R, \mathbb{R}^N)$.

Here $\Pi\colon \mathcal{O} \to \mathcal{M}$ is a retraction from a tubular neighbourhood $\mathcal{O}$ onto $\mathcal{M}$ (cf. §1.8.1) and $\nabla\Pi(\theta)$ is a projection onto the tangent space $T_\theta\mathcal{M}$ (cf. Corollary 1.51). A geometric interpretation can be sought by introducing the map $\mathsf{C}_\theta\colon \mathbb{R}^N \to T_\theta\mathcal{M}$ that sends

$$\mathsf{C}_\theta\colon v \mapsto \theta + \nabla\Pi(\theta)(v - \theta).$$

As $\exp_\theta\colon T_\theta\mathbb{R}^N \to \mathbb{R}^N$ is given by $\exp_\theta(v) = \theta + v$ the map C_θ projecting $v \in \mathbb{R}^N$ onto $T_\theta\mathcal{M}$ can be expressed as the conjugation of the projection tensor $\nabla\Pi(\theta)$ by $\exp_\theta$, i.e.

$$\mathsf{C}_\theta = \exp_\theta \circ \nabla\Pi(\theta) \circ \exp_\theta^{-1}.$$

With this any $v\colon \Omega \to \mathbb{R}^N$ is said to be tangential about $\theta \in \mathcal{M}$ if $\mathsf{C}_\theta(v) = v$, i.e. $v\colon \Omega \to T_\theta\mathcal{M}$, or alternatively $Dv = \nabla\Pi(\theta)Dv$ on the level of the gradient.

The following *a priori* estimates for tangential $\mathbb{A}$-harmonic maps are obtained by a change of orthonormal basis. This reduces the problem to an unconstrained elliptic system from which the result is already established (cf. §1.7).

Lemma 6.12 (Weyl-type)**.** *Let $h \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that*

$$\int_\Omega \mathbb{A}(Dh, D\phi) = 0$$

for any $\phi \in C_c^1(\Omega, \mathbb{R}^N)$ with $\nabla\Pi(\theta)D\phi = D\phi$ and $\nabla\Pi(\theta)Dh = Dh$ a.e. about a point $\theta \in \mathcal{M}$, where $\mathbb{A}$ is strongly elliptic with elliptic constant ℓ and upper bound L. Then $h \in C^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and for any $B_R \subset\subset \Omega$ we have

$$\sup_{B_{R/2}} |D^k h| \leq \frac{c}{R^{k-1}} \fint_{B_R} |Dh|$$

for some $c = c(n, N, k, \frac{L}{\ell}) > 0$ and any integer $k \geq 1$.

Proof. As $\nabla\Pi(\theta)$ is a projection onto the tangent space $T_\theta\mathcal{M}$ (cf. Corollary 1.51) there exists an $N\times N$ orthogonal matrix R such that

$$R\nabla\Pi(\theta)R^T = I_m \oplus 0_{N-m} \tag{6.20}$$

where $m = \dim\mathcal{M}$. Since $\nabla\Pi(\theta)Dh = Dh$ and $\nabla\Pi(\theta)D\phi = D\phi$ a.e. it follows from (6.20) that $D(R\phi) = SD(R\phi)$ and $D(Rh) = SD(Rh)$, where $Su = (u^1,\dots,u^m,0,\dots,0)$. Thus $(R\phi)^a_i = (Rh)^a_i = 0$ for $a = m+1,\dots,N$ and $i = 1,\dots,n$.

Let $\mathbb{B}$ be the bilinear form on $\mathbb{R}^{Nn}$ defined by $\mathbb{B}_{ij} = S^TR\mathbb{A}_{ij}R^TS$. With this one finds that

$$\begin{aligned} 0 = \int_\Omega (D_ih)^T\cdot\mathbb{A}_{ij}\cdot(D_j\phi) &= \int_\Omega (D_i(Rh))^T\cdot\mathbb{B}_{ij}\cdot(D_j(R\phi))^T \\ &= \int_\Omega (D_i(Rh))^T\cdot\mathbb{M}_{ij}\cdot(D_j(R\phi)), \end{aligned}$$

where $\mathbb{M}_{ij}$ denote the upper left $m\times m$ submatrix of the $N\times N$ matrix $\mathbb{B}_{ij}$. Likewise,

$$\int_\Omega (D_i(R\phi))^T\cdot\mathbb{M}_{ij}\cdot(D_j(R\phi)) \geq \ell\int_\Omega |D\phi|^2 = \ell\int_\Omega (D_i(R\phi))^T\cdot D_i(R\phi).$$

Thus $H = ((Rh)^1,\dots,(Rh)^m)$ is $\mathbb{M}$-harmonic in the sense that $\int_\Omega \mathbb{M}(DH,D\varphi) = 0$ with $\int_\Omega \mathbb{M}(D\varphi,D\varphi) \geq \ell\int_\Omega |D\varphi|^2$ for all $\varphi\in C^1_c(\Omega,\mathbb{R}^m)$ (cf. Definition 1.42). In which case H is a solution to an (unconstrained) linear elliptic system with constant coefficients satisfying the strong ellipticity condition (1.19). Therefore $H\in C^\infty_{\rm loc}$ by Lemma 1.43. □

Remark 6.13. As $|Dh-(Dh)_r| \leq 2r\sup_{B_r}|D^2h|$ on B_r and $0<r<R$, the above estimates imply $|Dh-(Dh)_r| \leq C\fint_{B_{2R}}|\frac{r}{R}Dh|$. Since $|V(\cdot)|^2$ is a convex function for $p\geq 2$ it follows (cf. Remark 1.44) that $\fint_{B_r}|V(Dh-(Dh)_r)|^2 \leq C(\frac{r}{R})^2\fint_{B_{2R}}|V(Dh-(Dh)_{2R})|^2$.

By appealing to the same change of variables argument we also obtain the following tangential version of Lemma 1.45 for the L^p-estimates of Campanato and Stampacchia [CS].

Lemma 6.14. *Let $F\in L^p(\Omega,\mathbb{R}^{Nn})$ such that $\nabla\Pi(\theta)F = F$ a.e. for a given point $\theta\in\mathcal{M}$ and $p>1$. If $v\in W^{1,p}_0(\Omega,\mathbb{R}^N)$ is a solution to*

$$\int_\Omega \mathbb{A}(Dv,D\phi) = \int_\Omega F\cdot D\phi$$

for any $\phi\in C^\infty_c(\Omega,\mathbb{R}^N)$ such that $\nabla\Pi(\theta)D\phi = D\phi$ and $\nabla\Pi(\theta)Dv = Dv$ a.e., where $\mathbb{A}$ is strongly elliptic of constant coefficients, then $Dv\in L^p(\Omega,\mathbb{R}^{Nn})$ and

$$\|Dv\|_{L^p(\Omega,\mathbb{R}^{Nn})} \leq c\|F\|_{L^p(\Omega,\mathbb{R}^{Nn})}$$

for some positive c independent of v and F, provided Ω is smooth (e.g. $C^{1,1}$) bounded open set in $\mathbb{R}^n$, $n\geq 2$.

Looking to §2.2 we now give a direct proof of the following:

Lemma 6.15 (tangential 𝔸-harmonic approximation)**.** *Let $B_R \subset\subset \Omega$ be a ball with $0 < R < 1$ and suppose $p \geq 2$, $\alpha > 1$, $\beta_0 \in (0,1]$, $L \geq \ell > 0$. For all $\varepsilon \in (0,1]$ there exists $\delta = \delta(\varepsilon, n, N, p, \alpha, \ell, L) > 0$ such that if $u \in W^{1,p}(B_R, \mathcal{M})$ is any δ-approximate tangential $\mathbb{A}$-harmonic map about a point $\theta \in \mathcal{M}$ with exponents α and β_0, where $\mathbb{A}$ is a strongly elliptic bilinear form with an ellipticity constant ℓ and an upper bound L, then the unique tangential $\mathbb{A}$-harmonic map $h \in \mathsf{C}_\theta(u) + W_0^{1,p}(B_R, \mathbb{R}^N)$ about θ satisfies*[1]

$$\fint_{B_R} |V(Du - Dh)|^2 \leq \varepsilon \Big[R^{\beta_0} + \mathcal{E}(R;u)^2 + \mathcal{E}(R;u)^p + \Big(\fint_{B_R} |V(Du - (Du)_R)|^{2\alpha} \Big)^{\frac{1}{\alpha}} \Big].$$

Proof. We first bound

$$\begin{aligned} \fint_{B_R} |V(Du - Dh)|^2 &\leq c \fint_{B_R} |V(Du - \nabla\Pi(\theta)Du)|^2 + |V(\nabla\Pi(\theta)Du - Dh)|^2 \\ &\leq c \fint_{B_R} |V(Du - \nabla\Pi(\theta)Du)|^2 \\ &\qquad + c \fint_{B_R} |\nabla\Pi(\theta)Du - Dh|^p + |\nabla\Pi(\theta)Du - Dh|^2 \end{aligned} \tag{6.21}$$

using (6.10a) and (6.10c).

Now set $U = \mathsf{C}_\theta(u) - h \in W_0^{1,p}(B_R, \mathbb{R}^N)$ and $H = |DU|^{p-1} \frac{DU}{|DU|} \in L^q(B_R, \mathbb{R}^{Nn})$ with $q = \frac{p}{p-1}$ so that $\nabla\Pi(\theta)H = H$ a.e. Let $v \in W_0^{1,q}(B_R, \mathbb{R}^N)$ be a solution to

$$\int_{B_R} \mathbb{A}(Dv, D\varphi) = \int_{B_R} H \cdot D\varphi$$

for any $\varphi \in W_0^{1,p}(B_R, \mathbb{R}^N)$ such that $\nabla\Pi(\theta)D\varphi = D\varphi$ and $\nabla\Pi(\theta)Dv = Dv$ a.e. By Lemma 6.14 we have

$$\int_{B_R} |Dv|^q \leq C \int_{B_R} |H|^q$$

for some $C = C(n, N, q, \ell, L)$. Using the fact that $U \in W_0^{1,p}(B_R, \mathbb{R}^N)$ satisfies $\nabla\Pi(\theta)DU = DU$ a.e. and this L^q-estimate implies

$$\frac{1}{p} \int_{B_R} |DU|^p = \int_{B_R} \mathbb{A}(Dv, DU) - \frac{1}{q}|H|^q \leq \int_{B_R} \mathbb{A}(Dv, DU) - \frac{1}{q}\frac{1}{C}|Dv|^q.$$

Moreover, as h is a tangential $\mathbb{A}$-harmonic map about θ we conclude

$$\frac{1}{p} \fint_{B_R} |\nabla\Pi(\theta)Du - Dh|^p \leq \sup_{\substack{\phi \in W_0^{1,q} \\ \nabla\Pi(\theta)D\phi = D\phi}} \Big[\fint_{B_R} \mathbb{A}(\nabla\Pi(\theta)Du, D\phi) - \frac{1}{q}\frac{1}{C}|D\phi|^q \Big].$$

[1] Here $\mathcal{E}(R;u) = (\fint_{B_R} |V(Du - (Du)_R)|^2)^{\frac{1}{2}}$ denotes the excess with respect to u.

Now fix $\phi \in W^{1,q}_0(B_R,\mathbb{R}^N)$ such that $\nabla\Pi(\theta)D\phi = D\phi$ a.e., set $A = (Du)_R$ and denote $\Lambda^q = \fint_{B_R}|D\phi|^q$. By Lemma 2.8 there exists a Lipschitz approximation $\phi_\lambda \in W^{1,\infty}_0(B_R,\mathbb{R}^N)$ with $\lambda \in (k\Lambda, k^q\Lambda)$ for some integer $k>1$. In which case

$$\begin{aligned}\fint_{B_R}\mathbb{A}(\nabla\Pi(\theta)Du, D\phi) &= \fint_{B_R}\mathbb{A}(\nabla\Pi(\theta)Du - Du, D\phi)\\ &\quad+\fint_{B_R}\mathbb{A}(Du-A, \nabla\Pi(\theta)(D\phi - D\phi_\lambda))\\ &\quad+\fint_{B_R}\mathbb{A}(Du, \nabla\Pi(\theta)D\phi_\lambda) = I + II + III.\end{aligned}$$

For the second term, recalling $\alpha>1$ and using Lemma 2.8, we estimate

$$\begin{aligned}II &\le L\fint_{B_R}|Du-A||\nabla\Pi(\theta)(D\phi - D\phi_\lambda)|\mathbf{1}_{\{\phi\neq\phi_\lambda\}}\\ &\le c\fint_{B_R}|Du-A|^p\mathbf{1}_{\{\phi\neq\phi_\lambda\}} + \frac{1}{3q}\frac{1}{C}\fint_{B_R}|D\phi|^q\\ &\le c\Big(\fint_{B_R}|Du-A|^{\alpha p}\Big)^{\frac{1}{\alpha}}\Big(\frac{|\{\phi\neq\phi_\lambda\}}{|B_R|}\Big)^{1-\frac{1}{\alpha}} + \frac{1}{3q}\frac{1}{C}\fint_{B_R}|D\phi|^q\end{aligned}$$

for some $c = c(n,N,p,\ell,L)$. Again, using the estimates from Lemma 2.8 we bound $\frac{|\{\phi_\lambda\neq\phi\}|}{|B_R|}\le ck^{-q}$. Thus for each $\varepsilon>0$ we may choose $k=k(\varepsilon,\alpha)>0$ sufficiently large so that

$$II \le \frac{\varepsilon}{2}\Big(\fint_{B_R}|V(Du-A)|^{2\alpha}\Big)^{\frac{1}{\alpha}} + \frac{1}{3q}\frac{1}{C}\fint_{B_R}|D\phi|^q.$$

As u is a δ-approximate tangential $\mathbb{A}$-harmonic map about θ and $p\ge 2$, we also have

$$III \le \delta(R^{\beta_0}+\mathcal{E}(R;u)^2)^{\frac12}\|D\phi_\lambda\|_\infty \le k^q\delta\Big(c\,(R^{\beta_0}+\mathcal{E}(R;u)^2)^{\frac{p}{2}} + \frac{1}{3q}\frac{1}{C}\fint_{B_R}|D\phi|^q\Big),$$

since $\|D\phi_\lambda\|_\infty \le \lambda \le k^q\Lambda$, and

$$I \le L\fint_{B_R}|\nabla\Pi(\theta)Du - Du||D\phi| \le c\fint_{B_R}|V(\nabla\Pi(\theta)Du-Du)|^2 + \frac{1}{3q}\frac{1}{C}\fint_{B_R}|D\phi|^q,$$

where

$$\fint_{B_R}|V(\nabla\Pi(\theta)Du-Du)|^2 \le \delta^2 R + \delta^{2(1-\frac{1}{\alpha})}\Big(\fint_{B_R}|V(Du-A)|^{2\alpha}\Big)^{\frac{1}{\alpha}}.$$

By choosing sufficiently small $\delta = O(\varepsilon^{2+\frac{1}{\alpha-1}})$, i.e. for a δ satisfying

$$\delta \le c\,\varepsilon^{2+\frac{1}{\alpha-1}} \tag{6.22}$$

for some $c = c(n,N,p,\alpha,\ell,L)>0$, we can then bound

$$\fint_{B_R}\mathbb{A}(\nabla\Pi(\theta)Du, D\phi) - \frac{1}{q}\frac{1}{C}|D\phi|^q \le \varepsilon\Big[R^{\beta_0}+\mathcal{E}(R;u)^p + \Big(\fint_{B_R}|V(Du-A)|^{2\alpha}\Big)^{\frac{1}{\alpha}}\Big].$$

Looking back to (6.21) we similarly note the strong ellipticity condition (1.19) also implies

$$\sup_{\substack{\psi\in W_0^{1,2}\\ \nabla\Pi(\theta)D\psi=D\psi}} \Big[\fint_{B_R} \mathbb{A}(\nabla\Pi(\theta)Du, D\psi) - \frac{\ell}{2}|D\psi|^2\Big] \geq \frac{\ell}{2}\fint_{B_R} |\nabla\Pi(\theta)Du - Dh|^2,$$

since h is a tangential $\mathbb{A}$-harmonic map with $\mathsf{C}_\theta(u) - h \in W_0^{1,2}(B_R, \mathbb{R}^N)$ (cf. proof of Lemma 2.9). Now one again fixes $\psi \in W_0^{1,2}(B_R, \mathbb{R}^N)$ such that $\nabla\Pi(\theta)D\psi = D\psi$ a.e. and denotes $\Xi^2 = \fint_{B_R} |D\psi|^2$. By Lemma 2.8 there exists a Lipschitz approximation $\psi_\mu \in W_0^{1,\infty}(B_R, \mathbb{R}^N)$ with $\mu \in (j\Lambda, j^2\Lambda)$ for some integer $j > 1$. As the δ-approximate tangential $\mathbb{A}$-harmonic condition about θ yields

$$\begin{aligned}\fint_{B_R} \mathbb{A}(Du, \nabla\Pi(\theta)D\psi_\mu) &\leq \delta(R^{\beta_0} + \mathcal{E}(R;u)^2)^{\frac{1}{2}}\|D\psi_\mu\|_\infty\\ &\leq \delta j^2\Big(c\,(R^{\beta_0} + \mathcal{E}(R;u)^2) + \frac{\ell}{6}\fint_{B_R} |D\psi|^2\Big)\end{aligned}$$

we can, by following the previous estimates, choose an integer $j = j(\varepsilon, \alpha) > 0$ sufficiently large and $\delta = O(\varepsilon^{2+\frac{1}{\alpha-1}})$ sufficiently small such that

$$\fint_{B_R} \mathbb{A}(\nabla\Pi(\theta)Du, D\psi) - \frac{\ell}{2}|D\psi|^2 \leq \varepsilon\Big[R^{\beta_0} + \mathcal{E}(R;u)^2 + \Big(\fint_{B_R} |V(Du - A)|^{2\alpha}\Big)^{\frac{1}{\alpha}}\Big].$$

The result now follows by combining the L^2 and L^p-versions with the initial (6.21). □

6.5. Approximate tangential harmonicity

In this section we seek to establish the approximate harmonicity of the holonomic $\mathfrak{F}$-minimiser $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ as per Definition 6.11. To do so we first note that the strong quasiconvexity of (h2), for any matrix $A \in \mathbb{R}^{Nn}$, any $\varphi \in C_c^\infty(B_R, \mathbb{R}^N)$ and fixed $(x_0, u_0) \in \Omega \times \mathbb{R}^N$, implies

$$\begin{aligned}&\epsilon\int_{B_R} Df^0(A)(D\varphi) + \epsilon^2\int_{B_R}\int_0^1 (1-t)D^2f^0(A + t\epsilon D\varphi)(D\varphi, D\varphi)dt\\ &\quad= \int_{B_R} f^0(A + \epsilon D\varphi) - f^0(A) \geq \gamma\epsilon^2\int_{B_R} |D\varphi|^2,\end{aligned}$$

where $f^0(\xi) = f(x_0, u_0, \xi)$. Therefore by sending $\epsilon \to 0$ the latter inequality becomes

$$\int_{B_R} D^2f^0(A)(D\varphi, D\varphi) \geq 2\gamma\int_{B_R} |D\varphi|^2. \tag{6.23}$$

So if, for any given $\Omega' \subset\subset \Omega$ and any given $\Lambda > 0$, we have $|(u)_R| \leq \Lambda$, $|(Du)_R| \leq \Lambda$ and $x_0 \in \Omega'$ the bilinear form

$$\mathbb{A} = D^2f^0((Du)_R)$$

is then both bounded with an upper bound $L = K_\Lambda$ from (6.3) and strongly elliptic in the sense of (1.19) with an ellipticity constant $\ell = 2\gamma$.

Given this, one can use a linearisation strategy to prove the below approximate harmonicity result of Proposition 6.18 via Ekeland's variational principle and higher integrability. The linearisation strategy is applied first to the frozen problem yielding the following:

Lemma 6.16. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ for $p \geq 2$, Π be a retraction from a tubular neighbourhood onto $\mathcal{M}$ and suppose the integrand f satisfies (h0), (h1), (h2) for fixed $B_R(x_0) \subset\subset \Omega$ with $0 < R < 1$. If $v \in (u + W^{1,p}_0(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ satisfies*

$$\fint_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dv) \leq \fint_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dw) + R^{\beta/2}\Big(\fint_{B_R(x_0)} |Dv - Dw|^p\Big)^{\frac{1}{p}}$$

for all $w \in (u + W^{1,p}_0(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ and some $0 < \beta < 1$, then for each $\Lambda > 0$ there exists $C = C(n, N, p, \Lambda, K_\Lambda) > 0$ such that

$$\begin{aligned}\Big|\fint_{B_R(x_0)} & D^2_{\xi\xi} f(x_0, (v)_{x_0,R}, (Dv)_{x_0,R})(Dv, \nabla\Pi(\Pi((v)_{x_0,R})D\phi)\Big| \\ &\leq C\Big(\mathcal{E}(R; v)^2 + \omega(\mathcal{E}(R; v))^{\frac{1}{2}}\mathcal{E}(R; v) + R + R^{\beta/2}(1 + \mathcal{E}(R; v)^2)^{\frac{1}{p}}\Big)\|D\phi\|_\infty\end{aligned}$$

for all $\phi \in C^\infty_c(B_R(x_0), \mathbb{R}^N)$, whenever $|(v)_{x_0,R}| \leq \Lambda$, $|(Dv)_{x_0,R}| \leq \Lambda$ and $\Pi((v)_{x_0,R})$ is well defined.[2]

Proof. Set $A = (Dv)_R$, $\theta = \Pi((v)_R)$ and $f^0(\xi) = f(x_0, (v)_{x_0,R}, \xi)$. Then write

$$\fint_{B_R} D^2 f^0(A)(Dv, \nabla\Pi(\theta)D\phi) = I + II + III$$

into the parts

$$I = \fint_{B_R} Df^0(Dv)D(\nabla\Pi(v)\phi), \quad II = \fint_{B_R} Df^0(Dv)\big(\nabla\Pi(\theta)D\phi - D(\nabla\Pi(v)\phi)\big)$$

and

$$\begin{aligned} III &= \fint_{B_R} D^2 f^0(A)(Dv - A, \nabla\Pi(\theta)D\phi) \\ &\qquad + \big(Df^0(A) - Df^0(Dv)\big)(\nabla\Pi(\theta)D\phi) \qquad (6.24)\\ &= \fint_{B_R}\int_0^1 \big[D^2 f^0(A) - D^2 f^0(A + t(Dv - A))\big](Dv - A, \nabla\Pi(\theta)D\phi)dt \qquad (6.25)\\ &= \fint_{B_R} \mathbf{1}_{\{|Dv-A|\leq 1\}}G + \mathbf{1}_{\{|Dv-A|>1\}}G \end{aligned}$$

[2]Here $\mathcal{E}(R; v) = (\fint_{B_R} |V(Dv - (Dv)_R)|^2)^{\frac{1}{2}}$ denotes the excess with respect to v.

where $G = \int_0^1 [D^2 f^0(A) - D^2 f^0(A + t(Dv - A))](Dv - A, \nabla\Pi(\theta)D\phi)dt$.

Now on $B_R \cap \{|Dv - A| > 1\}$ equations (6.3) and (6.1) imply, via (6.24), that

$$|G| \leq K_\Lambda |Dv - A||D\phi| + c(1 + |Dv - A|)|D\phi| \leq c|Dv - A||D\phi|.$$

Moreover, on $B_R \cap \{|Dv - A| \leq 1\}$ equation (6.25) and the modulus of continuity (6.4) imply $|G| \leq c\,\omega(|Dv - A|)|Dv - A||D\phi|$. Hence by (6.10a) and (6.10f) we have

$$\begin{aligned}
\Big| \fint_{B_R \cap \{|Dv-A|\leq 1\}} G \Big| &\leq c\Big(\fint_{B_R \cap \{|Dv-A|\leq 1\}} \omega(|Dv - A|)^2 \Big)^{\frac{1}{2}} \mathcal{E}(R; v)\|D\phi\|_\infty \\
&\leq c\Big[\omega\Big(\fint_{B_R \cap \{|Dv-A|\leq 1\}} |Dv - A| \Big)\Big]^{\frac{1}{2}} \mathcal{E}(R; v)\|D\phi\|_\infty \\
&\leq c\,\omega(\mathcal{E}(R; v))^{\frac{1}{2}} \mathcal{E}(R; v)\|D\phi\|_\infty,
\end{aligned}$$

where the second last inequality follows by the fact that ω is concave and at most 1 and the last inequality follows by Cauchy-Schwarz and the fact that ω is non-decreasing.

Expanding II yields

$$II = \fint_{B_R} Df^0(Dv)(\nabla\Pi(\theta)D\phi - \nabla\Pi(v)D\phi - \nabla^2\Pi(v)Dv \cdot \phi).$$

By the growth of (6.1) and the Lipschitz properties of the retraction Π (cf. Remark 1.48) we bound

$$\begin{aligned}
II &\leq c\|D\phi\|_\infty \fint_{B_R} (1 + |Dv|^{p-1})|v - (v)_R| + cR\|D\phi\|_\infty \fint_{B_R} |Dv| + |Dv|^p \\
&\leq cR\|D\phi\|_\infty (1 + \mathcal{E}(R; v)^2).
\end{aligned}$$

Finally by Corollary 6.9 we get

$$\begin{aligned}
I \leq R^{\beta/2}\Big(\fint_{B_R} |D(\nabla\Pi(v)\phi)|^p \Big)^{\frac{1}{p}} &\leq cR^{\beta/2}\|D\phi\|_\infty \Big(1 + R^p \fint_{B_R} |Dv|^p\Big)^{\frac{1}{p}} \\
&\leq cR^{\beta/2}\|D\phi\|_\infty (1 + \mathcal{E}(R; v)^2)^{\frac{1}{p}}. \qquad \square
\end{aligned}$$

Lemma 6.17. *Let $u \in W^{1,p}(B_R, \mathcal{M})$ for $p \geq 2$ with $0 < R < 1$. Let Π be a retraction from a tubular neighbourhood onto compact $\mathcal{M}$. If $\Pi((u)_R)$ is well defined then for any $\alpha > 1$ there exists $c = c(n, N, p, \alpha) > 0$ such that*

$$\begin{aligned}
&\fint_{B_R} |V(\nabla\Pi(\Pi((u)_R))Du - Du)|^2 \\
&\qquad \leq c\Big(R^2 \fint_{B_R} 1 + |Du|^p\Big)^{1-\frac{1}{\alpha}} \Big(\fint_{B_R} |V(Du - (Du)_R)|^{2\alpha}\Big)^{\frac{1}{\alpha}} + cR^2 \fint_{B_R} 1 + |Du|^p,
\end{aligned}$$

provided $|(Du)_R| \leq \Lambda$.

Proof. With $A = (Du)_R$ and $\theta = \Pi((u)_R)$ we note that

$$\nabla\Pi(\theta)Du - \nabla\Pi(u)Du = (\nabla\Pi(\theta) - \nabla\Pi(u))(Du - A) + (\nabla\Pi(\theta) - \nabla\Pi(u))A.$$

For any $\alpha > 1$ we can, using the Lipschitz properties of the retraction Π (cf. Remark 1.48), (6.9), (6.10a), (6.10c) and the fact that $u \in L^\infty$ (as $\mathcal{M}$ is compact), bound

$$\begin{aligned}
&\fint_{B_R} |V(\nabla\Pi(\theta)Du - Du)|^2 \\
&\quad\le c\fint_{B_R} (|u-\theta|^2|Du-A|^2 + |u-\theta|^p|Du-A|^p) + cR^2\fint_{B_R} |V(\tfrac{u-\theta}{R})|^2 \\
&\quad\le c\Big(\fint_{B_R} |u-(u)_R|^2 + |u-(u)_R|^p\Big)^{1-\frac{1}{\alpha}}\Big(\fint_{B_R} |V(Du-A)|^{2\alpha}\Big)^{\frac{1}{\alpha}} + cR^2\fint_{B_R} |V(Du)|^2 \\
&\quad\le c\Big(R^2\fint_{B_R} 1 + |Du|^p\Big)^{1-\frac{1}{\alpha}}\Big(\fint_{B_R} |V(Du-A)|^{2\alpha}\Big)^{\frac{1}{\alpha}} + cR^2\fint_{B_R} 1 + |Du|^p.
\end{aligned}$$

□

For $v - u \in W^{1,p}_0(B_R(x_0), \mathbb{R}^N)$ we have $(Du)_{x_0,R} = (Dv)_{x_0,R}$ by the divergence theorem. So by (6.10a) and (6.10e)

$$\mathcal{E}(R;v)^2 \le \mathcal{E}(R;u)^2 + c\fint_{B_R(x_0)} |Du-Dv|^2 + |Du-Dv|^p, \tag{6.26}$$

whenever $|(Du)_{x_0,R}| \le \Lambda$. Thus we can, by the higher integrability results of §6.2 and the Ekeland's variational principle of §6.3 (in particular, the fact that v is close to u in the sense of (6.18)), establish the following approximate tangential harmonicity for u.

Proposition 6.18. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \ge 1$. Suppose the integrand f satisfy (h0), (h1), (h2), (h3) and (h4) for $2 \le p < \ell + 2$. If for each $\Lambda > 0$ the map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ is an holonomic $\mathfrak{F}$-minimiser with*

$$|(u)_{x_0,2R}| \le \Lambda, \qquad |(Du)_{x_0,2R}| \le \Lambda, \qquad \Big(\fint_{B_{2R}(x_0)} |Du|^p\Big)^{\frac{1}{p}} \le \Lambda + 1$$

on some $B_{2R}(x_0) \subset\subset \Omega$ with $R = R(n, N, p, \Lambda)$ sufficiently small, then for any $\alpha > 1$ there exists $C = C(n, N, p, \Lambda, K_\Lambda, \alpha) > 0$ such that

$$\fint_{B_R(x_0)} |V(Du - \nabla\Pi((u)_{x_0,R})Du)|^2 \le CRe^2 + Ce^{2(1-\frac{1}{\alpha})}\Big(\fint_{B_R(x_0)} |V(Du-(Du)_{x_0,R})|^{2\alpha}\Big)^{\frac{1}{\alpha}}$$

$$\Big|\fint_{B_R(x_0)} D^2 f^0((Du)_{x_0,R})(Du, \nabla\Pi(\Pi((u)_{x_0,R}))D\phi)\Big| \le Ce(e + \omega(e) + \omega(e)^{\frac{1}{2}})\|D\phi\|_\infty$$

for all $\phi \in C^\infty_c(B_R(x_0), \mathbb{R}^N)$, where $f^0(\xi) = f(x_0, (u)_{x_0,R}, \xi)$ and $e^2 = R^{\beta/2} + \mathcal{E}(R;u)^2$ for some $0 < \beta < 1$.

Proof. From the higher integrability of Lemma 6.4 and Lemma 6.5 together with the result of Proposition 6.8 there exists $v \in (u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,p}(B_R(x_0), \mathcal{M})$ that satisfies (6.18) and is minimal in the sense of (6.19) for some $0 < \beta < 1$. Furthermore,

$$\operatorname{dist}((u)_R, \mathcal{M}) \leq \Big(\fint_{B_R} |u - (u)_R|^p \Big)^{\frac{1}{p}} \leq cR \Big(\fint_{B_R} |Du|^p \Big)^{\frac{1}{p}}$$

by the initial assumptions and from (6.18) we have

$$\operatorname{dist}((v)_R, \mathcal{M}) \leq cR(1 + \Lambda^p)^{\frac{1}{p}}$$

also. In which case the expressions $\Pi((u)_R)$ and $\Pi((v)_R)$ are well defined for $0 < R < c^{-1}(1 + \Lambda^p)^{-\frac{1}{p}}\rho$ where $\rho = \operatorname{reach}(\mathcal{M}) > 0$ denote the positive reach of $\mathcal{M}$ (cf. §1.8.2). With $\theta_u = \Pi((u)_R)$ and $\theta_v = \Pi((v)_R)$ we observe that

$$\begin{aligned}
&\fint_{B_R} D^2 f^0((Du)_R)(Du, \nabla\Pi(\theta_u)D\phi) \\
&\quad = \fint_{B_R} D^2 f^0((Dv)_R)(Dv, \nabla\Pi(\theta_v)D\phi) \\
&\qquad + \fint_{B_R} [D^2 f^0((Du)_R) - D^2 f^0((Dv)_R)](Dv, \nabla\Pi(\theta_v)D\phi) \\
&\qquad + \fint_{B_R} D^2 f^0((Du)_R)(Dv, (\nabla\Pi(\theta_u) - \nabla\Pi(\theta_v))D\phi) \\
&\qquad + \fint_{B_R} D^2 f^0((Du)_R)(Du - Dv, \nabla\Pi(\theta_u)D\phi).
\end{aligned}$$

Then using (6.3), (6.4) and the Lipschitz properties of the retraction Π (cf. Remark 1.48) we bound

$$\begin{aligned}
&\Big| \fint_{B_R} D^2 f^0((Du)_R)(Du, \nabla\Pi(\theta_u)D\phi) \Big| \\
&\quad \leq \Big| \fint_{B_R} D^2 f^0((Dv)_R)(Dv, \nabla\Pi(\theta_v)D\phi) \Big| \\
&\qquad + C\omega\Big(\fint_{B_R} |Du - Dv| \Big) \Big(\fint_{B_R} |Dv - (Du)_R| \Big) \|D\phi\|_\infty \\
&\qquad + C\Big(\fint_{B_R} |Dv - (Du)_R| \Big) \Big(\fint_{B_R} |u - v| \Big) \|D\phi\|_\infty \\
&\qquad + C\Big(\fint_{B_R} |Du - Dv|^p \Big)^{\frac{1}{p}} \|D\phi\|_\infty.
\end{aligned}$$

The result now follows via Lemma 6.17, since

$$\fint_{B_R} |Dv - (Du)_R| \leq CR^{\beta/2} + C \fint_{B_R} |V(Du - (Du)_R)|$$

and $\mathcal{E}(R; v)^2 \leq \mathcal{E}(R; u)^2 + CR^\beta$ by (6.26), (6.18). □

6.6. Reverse Hölder-type inequalities

We establish reverse Hölder-type inequalities following a Caccioppoli inequality of the second kind argument using strict quasiconvexity and minimality via the same extension results as used for the inequalities of §6.2.

Lemma 6.19. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \geq 1$. Suppose the integrand f satisfy (h0), (h1), (h2), (h3) for $2 \leq p < \ell + 2$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be an holonomic $\mathfrak{F}$-minimiser. Then for each $\Lambda > 0$ there exists $C = C(n, N, p, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 0$ and $R_1 = R_1(n, N, p, \Lambda) > 0$ such that*

$$\begin{aligned}\fint_{B_{R/2}(x_0)} |V(Du - A)|^2 \leq C\Big(\fint_{B_R(x_0)} |V(Du - A)|^{2_*}\Big)^{2/2_*} \\ + CR^2 \fint_{B_R(x_0)} (1 + |Du|^p) + C \fint_{B_R(x_0)} |V(\nabla \Pi(u)A - A)|^2 \\ + C \fint_{B_R(x_0)} \omega_\mu\Big[R + |u - u_0| + \Phi(Du) + R^{-1}\Phi(\Phi(Du))\Big](1 + |Du|^p)\end{aligned}$$

for any $B_R(x_0) \subset\subset \Omega$ with $0 < R < R_1$, any $u_0 \in \mathbb{R}^N$ with $|u_0| \leq \Lambda + 1$ and any $A \in \mathbb{R}^{Nn}$ with $|A| \leq \Lambda$.[3]

Proof. Let $\frac{R}{2} \leq r < s \leq R$ and choose a cut-off function $\eta \in C^\infty_c(\Omega)$ such that $\eta = 1$ on $B_r(x_0)$, $\eta = 0$ on $\Omega \backslash B_s(x_0)$, $0 \leq \eta \leq 1$ and $|D\eta| \leq C/(s - r)$. Define $\phi_1 = \eta \bar{u}$ and $\phi_2 = (1 - \eta)\bar{u}$, where $\bar{u}(x) = u(x) - (u)_{x_0,R} - A \cdot (x - x_0)$, so that $D\phi_1 + D\phi_2 = Du - A$. By setting $f^0(\xi) = f(x_0, u_0, \xi)$ and using the quasiconvexity of (h2) we obtain

$$\begin{aligned}\gamma \int_{B_s} (1 + |A|^2 + |D\phi_1|^2)^{\frac{p-2}{2}} |D\phi_1|^2 \leq \int_{B_s} f^0(Du - D\phi_2) - f^0(Du) \\ + \int_{B_s} f^0(Du) - f^0(A + D\phi_2) \\ + \int_{B_s} f^0(A + D\phi_2) - f^0(A) = I + II + III. \quad (6.27)\end{aligned}$$

By introducing the auxiliary functional $\mathfrak{F}_{s,A}(w) = \int_{B_s} \mathsf{F}(Dw)dx$, with integrand $\mathsf{F}(\xi) = f^0(\xi + A) - f^0(A) - Df^0(A)(\xi)$, we note that

$$\begin{aligned}II = \int_{B_s} f^0(Du) - f^0(A) - Df^0(A)(D\phi_1 + D\phi_2) \\ + \int_{B_s} \int_0^1 \big(Df^0(A) - Df^0(A + \lambda D\phi_2)\big) \cdot D\phi_2 \, d\lambda = \mathfrak{F}_{s,A}(\bar{u}) + II'\end{aligned}$$

[3]Here $\Phi(f)(x) = \int_{B_R} |x - y|^{1-n} |f(y)| dy$, $f \in L^1(B_R)$, denotes the Riesz potential.

since $\phi_1 = 0$ on ∂B_s. Now as

$$I + III = \int_{B_s} \int_0^1 (Df^0(A + \lambda D\phi_2) - Df^0(Du - \lambda D\phi_2)) \cdot D\phi_2 \, d\lambda$$

it follows that $I + II + III = \mathfrak{F}_{s,A}(\bar{u}) + III'$, where

$$III' = \int_{B_s} \int_0^1 (Df^0(A) - Df^0(Du - \lambda D\phi_2)) \cdot D\phi_2 \, d\lambda.$$

Looking to the auxiliary functional we have, by the minimality of u,

$$\begin{aligned}
\mathfrak{F}_{s,A}(\bar{u}) &= \int_{B_s} f^0(Du) - f^0(A) - Df^0(A)(Du - A) \\
&\leq \int_{B_s} f^0(Dv) - f^0(A) - Df^0(A)(Dv - A) \\
&\qquad + \int_{B_s} f^0(Du) - f(x, u, Du) + \int_{B_s} f(x, v, Dv) - f^0(Dv)
\end{aligned}$$

for any $v \in (u + W_0^{1,p}(B_s, \mathbb{R}^N)) \cap W^{1,p}(B_s, \mathcal{M})$. Furthermore, using (6.10a) one easily bounds $\mathsf{F}(\xi) \leq C|V(\xi)|^2$ (since for $|\xi| > 1$, $\mathsf{F}(\xi) \leq C|\xi|^p$ by (6.1) and for $|\xi| \leq 1$ write $\mathsf{F}(\xi) = \int_0^1 (1-\lambda) D^2 f^0(A + \lambda\xi)(\xi, \xi) d\lambda$ so that $|\mathsf{F}(\xi)| \leq K_{\Lambda+1}|\xi|^2$ by (6.3) since $|u_0| \leq \Lambda + 1$ also). It then follows by (h3) that

$$\begin{aligned}
\mathfrak{F}_{s,A}(\bar{u}) \leq C \int_{B_s} |V(Dv - A)|^2 + C \int_{B_s} \omega_\mu(|x - x_0| + |u - u_0|)(1 + |Du|^p) \\
+ C \int_{B_s} \omega_\mu(|x - x_0| + |v - u_0|)(1 + |Dv|^p).
\end{aligned}$$

To estimate III' we note that the integrand on $B_s \cap \{|Du - A| + |D\phi_2| > 1\}$ can be bounded, using (6.1) and the fact that $|A| \leq \Lambda$, by

$$|Df^0(A) - Df^0(Du - \lambda D\phi_2)||D\phi_2| \leq C(|Du - A| + |D\phi_2|)^{p-1}|D\phi_2|.$$

On $B_s \cap \{|Du - A| + |D\phi_2| \leq 1\}$ we use (6.3), noting that $|Du - \lambda D\phi_2| \leq \Lambda + 1$ and $|u_0| \leq \Lambda + 1$, to obtain

$$|Df^0(A) - Df^0(Du - \lambda D\phi_2)||D\phi_2| \leq K_{\Lambda+1}(|Du - A| + |D\phi_2|)|D\phi_2|.$$

Now as $D\phi_2 = (1-\eta)(Du - A) - \bar{u} \otimes D\eta$ we have $|V(D\phi_2)| \leq C(|V(Du - A)| + |V(\frac{\bar{u}}{s-r})|)$ by (6.10c). This together with the fact that $D\phi_2 = 0$ on B_r implies

$$III' \leq C \int_{B_s \setminus B_r} |V(Du - A)|^2 + |V(\tfrac{\bar{u}}{s-r})|^2.$$

Furthermore, we bound the left-hand side of (6.27) on B_r from below, using (6.10d) and (6.10f), by

$$\begin{aligned}\gamma(1+|A|^2+|D\phi_1|^2)^{\frac{p-2}{2}}|D\phi_1|^2 &\geq C(1+|A|^2+|Du|^2)^{\frac{p-2}{2}}|Du-A|^2\\ &\geq C|V(Du)-V(A)|^2\\ &\geq C|V(Du-A)|^2\end{aligned}$$

where $C=C(n,N,p,\gamma,\Lambda)$. In which case we conclude

$$\begin{aligned}\int_{B_r}|V(Du-A)|^2 \leq C\int_{B_s\setminus B_r}|V(Du-A)|^2 + C\int_{B_s}|V(Dv-A)|^2+|V(\tfrac{\bar{u}}{s-r})|^2\\ + C\int_{B_s}\omega_\mu(|x-x_0|+|u-u_0|)(1+|Du|^p)\\ + C\int_{B_s}\omega_\mu(|x-x_0|+|v-u_0|)(1+|Dv|^p) \qquad (6.28)\end{aligned}$$

for any $v\in(u+W^{1,p}_0(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$.

Now let Π be a retraction from a tubular neighbourhood onto $\mathcal{M}$ and let $\rho>0$ denote the positive reach of $\mathcal{M}$ (cf. §1.8.1 and §1.8.2). If we let $a\colon\mathbb{R}^n\to\mathbb{R}^N$ denote the linear map $x\mapsto A\cdot(x-x_0)$ on B_R then $\operatorname{dist}(u-a,\mathcal{M})\leq\|a\|_\infty\leq\Lambda R$, since $|A|\leq\Lambda$. In which case the expression

$$\widehat{u}=\Pi(u-a)$$

is well defined for $0<R<\Lambda^{-1}\rho$. From the lifting of Theorem 5.20 there exists $\widehat{\varphi}\in W^{1,p}(B_s,\widetilde{\mathcal{M}})$ such that $\widehat{u}=\pi\circ\widehat{\varphi}$. Then by the result of Lemma 5.34 there exists $\widehat{v}\in(\widehat{u}+W^{1,p}_0(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$ such that

$$\int_{B_s}|V(D\widehat{v})|^2\leq C\int_{B_s\setminus B_r}|V(D\widehat{u})|^2+c\frac{R^{n+p}}{(s-r)^p}\Big(\fint_{B_R}|V(D\widehat{u})|^{2_*}\Big)^{2/2_*}$$

by (6.9) with

$$|\widehat{v}(x)-\widehat{u}(x)|\leq C\int_{B_s}|x-y|^{1-n}|D\widehat{u}(y)|dy+\frac{C}{s-r}\int_{B_s}|x-y|^{1-n}|\widehat{\varphi}(y)-(\widehat{\varphi})_{B_s}|dy$$

for a.e. $x\in B_s$. Furthermore $|\widehat{\varphi}(y)-(\widehat{\varphi})_{B_s}|\leq c\int_{B_s}|y-z|^{1-n}|D\widehat{u}(z)|dz$ a.e. $y\in B_s$ by standard potential estimates (cf. [GT, Lem. 7.16]) and the fact that $|D\widehat{u}|=|D\widehat{\varphi}|$ a.e.

As $\operatorname{dist}(u-\widehat{u}+\widehat{v},\mathcal{M})\leq|u-\widehat{u}|\leq c\|a\|_\infty$ by Lipschitz properties of the retraction Π onto $\mathcal{M}$ (cf. Remark 1.48) we see the competitor $v\in(u+W^{1,p}_0(B_s,\mathbb{R}^N))\cap W^{1,p}(B_s,\mathcal{M})$ to the minimiser can be defined by setting

$$v=\Pi(u-\widehat{u}+\widehat{v})$$

for all $0 < R < (c\Lambda)^{-1}\rho = R_1$. In particular we have $v - u \in W_0^{1,p}(B_s, \mathbb{R}^N)$, since $\widehat{v} = \widehat{u}$ on ∂B_s by construction. Then by writing

$$\begin{aligned} Dv - A = \nabla\Pi(u - \widehat{u} + \widehat{v})\Big[&\big(\nabla\Pi(u) - \nabla\Pi(u-a)\big)Du \\ &+ \big(\nabla\Pi(u-a) - \nabla\Pi(u)\big)A + \nabla\Pi(u)A - A\Big] \\ &+ \Big[\nabla\Pi(u - \widehat{u} + \widehat{v}) - \nabla\Pi(u)\Big]A + \nabla\Pi(u)A - A + \nabla\Pi(u - \widehat{u} + \widehat{v})D\widehat{v} \end{aligned}$$

we can, again using Lipschitz properties of the retraction Π, bound $|Dv - A| \le c|D\widehat{v}| + c|\nabla\Pi(u)A - A| + c|\widehat{u} - \widehat{v}| + c\|a\|_\infty(1 + |Du|)$ for some $c = c(n, N, \Lambda)$. So by (6.9), (6.10b) and (6.10c) we have

$$\int_{B_s} |V(Dv - A)|^2 \le c\int_{B_s} |V(D\widehat{v})|^2 + c\int_{B_s} |V(\nabla\Pi(u)A - A)|^2 + R^2(1 + |Du|^p).$$

To estimate the final term in (6.28) we first note that $|v - u_0| \le |u - u_0| + |v - u| \le |u - u_0| + c|\widehat{u} - \widehat{v}|$ by the Lipschitz properties of Π. If we let Φ denote the Riesz potential $\Phi(f)(x) = \int_{B_R} |x - y|^{1-n}|f(y)|dy$, $f \in L^1(B_R)$, the above construction implies the pointwise a.e. estimate

$$|\widehat{v} - \widehat{u}| \le C\frac{R}{s - r}\Big(R + \Phi(Du) + R^{-1}\Phi(\Phi(Du))\Big).$$

It then follows, as $\omega_\mu(t) = \min(1, t^\mu)$ with $0 < \mu \le 1$ and $p \ge 2$, that

$$\begin{aligned} &\int_{B_s} \omega_\mu(|x - x_0| + |v - u_0|)(1 + |Dv|^p) \\ &\quad\le c\int_{B_s} |V(D\widehat{v})|^2 + c\int_{B_s} \omega_\mu(|x - x_0| + |v - u_0|)(1 + |Du|^p) \\ &\quad\le c\int_{B_s} |V(D\widehat{v})|^2 \\ &\qquad + c\Big(\frac{R}{s - r}\Big)^\mu \int_{B_s} \omega_\mu\Big[R + |u - u_0| + \Phi(Du) + R^{-1}\Phi(\Phi(Du))\Big](1 + |Du|^p). \end{aligned}$$

By putting these estimates together with (6.28) we find

$$\begin{aligned} \int_{B_r} |V(Du - A)|^2 \le{}& C\int_{B_s \setminus B_r} |V(Du - A)|^2 + C\frac{R^{n+p}}{(s - r)^p}\Big(\fint_{B_R} |V(Du - A)|^{2_*}\Big)^{2/2_*} \\ &+ C\Big(\frac{R}{s - r}\Big)^\mu \int_{B_s} \omega_\mu\Big[R + |u - u_0| + \Phi(Du) + R^{-1}\Phi(\Phi(Du))\Big](1 + |Du|^p) \\ &+ C\int_{B_R} |V(\nabla\Pi(u)A - A)|^2 + R^2(1 + |Du|^p) \end{aligned}$$

for some $C = C(n, N, p, \gamma, \Gamma, \Lambda, K_{\Lambda+1})$. The result now follows by Widman's hole filling trick and an application of the standard iteration of Lemma 0.7. □

Given the above result one can apply Gehring's lemma to obtain the following:

Corollary 6.20. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \geq 1$. Suppose the integrand f satisfy (h0), (h1), (h2) and (h3) for $2 \leq p < 2 + \ell$. If for each $\Lambda > 0$ the map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ is an holonomic $\mathfrak{F}$-minimiser with*

$$|(u)_{x_0,2R}| \leq \Lambda, \qquad |(Du)_{x_0,2R}| \leq \Lambda, \qquad \Big(\fint_{B_{2R}(x_0)} |Du|^p\Big)^{\frac{1}{p}} \leq \Lambda + 1$$

on some $B_{2R}(x_0) \subset\subset \Omega$ for $R = R(n, N, p, \Lambda)$ sufficiently small, then there exists $q = q(n, N, p, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > p$ and $C = C(n, N, p, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 0$ such that

$$\Big(\fint_{B_{\frac{R}{2}}(x_0)} |V(Du - (Du)_{x_0,\frac{R}{2}})|^{\frac{4q}{p+q}}\Big)^{\frac{p+q}{2q}} \leq CR^{\frac{\mu}{2}(1-\frac{p}{q})} + C\fint_{B_R(x_0)} |V(Du - (Du)_{x_0,R})|^2.$$

Proof. Firstly, as the result of Lemma 6.19 holds on all balls with $0 < R < R_1$, we can apply Proposition 1.6 from which there exists $\alpha_0 = \alpha_0(n, N, p, \gamma, \Gamma, \Lambda, K_{\Lambda+1}) > 1$ such that for any $1 < \alpha \leq \alpha_0$, any $u_0 \in \mathbb{R}^N$ with $|u_0| \leq \Lambda + 1$ and any $A \in \mathbb{R}^{Nn}$ with $|A| \leq \Lambda$,

$$\begin{aligned}\Big(\fint_{B_{R/2}} |V(Du - A)|^{2\alpha}\Big)^{\frac{1}{2\alpha}} &\leq C\Big(\fint_{B_R} |V(Du - A)|^2\Big)^{\frac{1}{2}} + CR\Big(\fint_{B_R} 1 + |Du|^{\alpha p}\Big)^{\frac{1}{2\alpha}} \\ &\quad + C\Big(\fint_{B_R} |V(\nabla\Pi(u)A - A)|^{2\alpha}\Big)^{\frac{1}{2\alpha}} \\ &\quad + C\Big(\fint_{B_R} \omega_\mu(\psi(x_0, R; u, u_0))^\alpha (1 + |Du|^{\alpha p})\Big)^{\frac{1}{2\alpha}} \qquad (6.29)\end{aligned}$$

on $B_R = B_R(x_0) \subset\subset \Omega$ with $0 < R \ll 1$ sufficiently small. Here we denote $\psi(x_0, R; u, u_0) = R + |u - u_0| + \Phi(Du) + R^{-1}\Phi(\Phi(Du))$.

Now from the local higher integrability of Lemma 6.4 there exists $q > p$ such that $Du \in L^q_{\mathrm{loc}}$ with

$$\int_{B_R} |Du|^q \leq c\Big(\fint_{B_{2R}} 1 + |Du|^p\Big)^{\frac{q}{p}}.$$

Thus we can set $\alpha_0 = q/p$ with out loss of generality. Then for any $1 < \alpha < q/p$ we estimate

$$\begin{aligned}&\fint_{B_R} \omega_\mu(\psi(x_0, R; u, u_0))^\alpha (1 + |Du|^{\alpha p}) \\ &\qquad \leq cR^{-\mu\frac{\alpha p}{q}} \omega_\mu\Big(\fint_{B_R} \psi(x_0, R; u, u_0)\Big) + R^{\mu(1-\frac{\alpha p}{q})} \fint_{B_R} 1 + |Du|^q, \qquad (6.30)\end{aligned}$$

since ω_μ is concave and no greater than 1.

By setting $u_0 = (u)_{x_0,R}$ and $A = (Du)_{x_0,R/2}$ we have $|A| \le 4^n(1+\Lambda)$ and $|u_0| \le c(2+\Lambda)$ by the initial assumptions. Hence the result (6.29) holds up to a rescaling of Λ. Moreover,

$$\fint_{B_R} \psi(x_0, R; u, (u)_R) \le cR\Big(\fint_{B_R} 1 + |Du|^p\Big)^{\frac{1}{p}}$$

by the standard potential estimates (cf. [GT, Lem. 7.12]). Then from (6.30) we have

$$\begin{aligned}\fint_{B_R} \omega_\mu(\psi(x_0, R; u, (u)_R))^\alpha (1 + |Du|^{p\alpha}) &\le cR^{\mu(1-\frac{\alpha p}{q})}\Big(\fint_{B_{2R}} 1 + |Du|^p\Big)^{\frac{q}{p}} \\ &\quad + cR^{\mu(1-\frac{\alpha p}{q})}\omega_\mu\Big[\Big(\fint_{B_R} 1 + |Du|^p\Big)^{\frac{1}{p}}\Big].\end{aligned}$$

In particular, if we set $\alpha = 2q/(p+q)$ the relation $1 < \alpha < q/p$ is satisfied and

$$\begin{aligned}\Big[\fint_{B_R} \omega_\mu(\psi(x_0, R; u, (u)_R))^\alpha (1 + |Du|^{p\alpha})\Big]^{\frac{1}{2\alpha}} &\le cR^{\frac{\mu}{4}(1-\frac{p}{q})}\Big(\fint_{B_{2R}} 1 + |Du|^p\Big)^{\frac{p+q}{4p}} \\ &\quad + cR^{\frac{\mu}{4}(1-\frac{p}{q})}\omega_\mu\Big[\Big(\fint_{B_R} 1 + |Du|^p\Big)^{\frac{1}{p}}\Big]^{\frac{p+q}{4p}}.\end{aligned}$$

Finally, the expression $\Pi((u)_{R/2})$ is well defined for $0 < R \ll 1$ sufficiently small, since

$$\operatorname{dist}((u)_{R/2}, \mathcal{M}) \le cR\Big(\fint_{B_{2R}} |Du|^p\Big)^{\frac{1}{p}} \le cR(1+\Lambda)$$

by the initial assumption. So by the Lipschitz properties of the retraction (cf. Remark 1.48) we have

$$\begin{aligned}|\nabla\Pi(u)(Du)_{\frac{R}{2}} - (Du)_{\frac{R}{2}}| &\le |\nabla\Pi(u)(Du)_{\frac{R}{2}} - \nabla\Pi(\Pi((u)_{\frac{R}{2}}))(Du)_{\frac{R}{2}}| \\ &\quad + |\nabla\Pi(\Pi((u)_{\frac{R}{2}}))(Du)_{\frac{R}{2}} - (Du)_{\frac{R}{2}}| \\ &\le c|u - (u)_{\frac{R}{2}}| + \fint_{B_{\frac{R}{2}}} |\nabla\Pi(\Pi((u)_{\frac{R}{2}}))Du - \nabla\Pi(u)Du| \\ &\le c|u - (u)_{\frac{R}{2}}| + c\Big(\fint_{B_{\frac{R}{2}}} |u - (u)_{\frac{R}{2}}|^{\frac{p}{p-1}}\Big)^{1-\frac{1}{p}}\Big(\fint_{B_{\frac{R}{2}}} |Du|^p\Big)^{\frac{1}{p}} \\ &\le c|u - (u)_{\frac{R}{2}}| + cR\Big(\fint_{B_{2R}} |Du|^p\Big)^{\frac{2}{p}}\end{aligned}$$

since $p \ge 2$. Then by (6.10b), (6.10c) and the fact that $|V(\xi)|^{2\alpha} \le c(1 + |\xi|^q)$ as $1 < \alpha < q/p$ we conclude

$$\Big(\fint_{B_R} |V(\nabla\Pi(u)A - A)|^{2\alpha}\Big)^{\frac{1}{2\alpha}} \le cR\Big(\fint_{B_{2R}} 1 + |Du|^p\Big)^{\frac{q}{2\alpha p}} + cR\Big(\fint_{B_{2R}} 1 + |Du|^p\Big)^{\frac{q}{\alpha p}}. \quad \square$$

6.7. Excess-improvement and iteration

We now look to prove Theorem 6.1 from the the following excess-improvement result.

Proposition 6.21. *Let $\mathcal{M}$ be a connected compact manifold without boundary with a universal cover $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is compact and ℓ-connected for some integer $\ell \geq 1$. Suppose the integrand f satisfy (h0), (h1), (h2), (h3), (h4) for $2 \leq p < \ell + 2$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be an holonomic $\mathfrak{F}$-minimiser. If for each $\Lambda > 0$ and $\varepsilon \in (0,1)$ the smallness conditions*

$$|(u)_{x_0,R}| \leq \Lambda, \qquad |(Du)_{x_0,R}| \leq \Lambda, \qquad \mathcal{E}(x_0,R) \leq \varepsilon$$

hold for some $B_R(x_0) \subset\subset \Omega$ with $R = R(n,N,p,\Lambda)$ sufficiently small, then for each $0 < \tau \leq \frac{1}{8}$ we have

$$\mathcal{E}(x_0,\tau R)^2 \leq C(\tau^2 + \tau^{-n}\widetilde{\omega}(\varepsilon+R))\mathcal{E}(x_0,R)^2 + C\tau^{-n}\widetilde{\omega}(\varepsilon+R)R^{\beta/2}$$

for some $C = C(n,N,p,\gamma,\Gamma,\Lambda,K_{\Lambda+1}) > 0$, some $\beta \in (0,1)$ and a modulus of continuity $\widetilde{\omega}$, in the sense of Definition 1.14, independent of τ, R and ε.

Proof. Fix $\Lambda > 0$, $\varepsilon \in (0,1)$ and assume the smallness conditions $|(u)_{x_0,4R}| \leq \Lambda$, $|(Du)_{x_0,4R}| \leq \Lambda$ and $\mathcal{E}(x_0,4R) \leq \varepsilon$ hold for some $B_{4R}(x_0) \subset\subset \Omega$. By (6.3) and (6.23) the bilinear form

$$\mathbb{A} = D^2_{\xi\xi} f(x_0, (u)_{x_0,R}, (Du)_{x_0,R})$$

is bounded, with an upper bound $L = K_\Lambda$, and strongly elliptic in the sense of (1.19) with an elliptic constant $\ell = 2\gamma$. From Lemma 6.4 there exists $q = q(n,N,p,\gamma,\Gamma) > p$ such that $Du \in L^q_{\mathrm{loc}}$. Now with $\alpha = \frac{2q}{p+q} \in (1, \frac{q}{p})$ and $\beta = \mu(1 - \frac{p}{q})$ let

$$\widetilde{\omega}(t) = (\omega(t^{\frac{\beta}{4}})^{\frac{1}{2}} + \omega(t^{\frac{\beta}{4}}) + t^{\frac{\beta}{4}})^{\frac{\alpha-1}{2\alpha-1}}$$

where ω is the modulus of continuity given by (6.4). Then for each $\varepsilon_0 = \widetilde{\omega}(\varepsilon + R)$, with $0 < R \ll 1$ sufficiently small, the approximate tangential harmonicity of Proposition 6.18 shows the holonomic $\mathfrak{F}$-minimiser u to be δ-approximate tangential $\mathbb{A}$-harmonic map about $\theta = \Pi((u)_{x_0,R})$ with $\delta = \delta(\varepsilon_0) = C\varepsilon_0^{\frac{2\alpha-1}{\alpha-1}}$ (since $(R+\varepsilon)^{\frac{\beta}{2}} \geq \frac{1}{2}(R^{\frac{\beta}{2}} + \varepsilon^2)$ so that

$$\delta \geq C(\omega(e)^{\frac{1}{2}} + \omega(e) + e),$$

where $e = (R^{\beta/2} + \mathcal{E}(R)^2)^{\frac{1}{2}})$. As the choice of δ also satisfies the growth rate (6.22) it follows by Lemma 6.15 that there exists a unique tangential $\mathbb{A}$-harmonic map $h \in \mathsf{C}_\theta(u) + W^{1,p}_0(B_R, \mathbb{R}^N)$ about θ satisfying

$$\fint_{B_R} |V(Du - Dh)|^2 \leq \varepsilon_0 \Big[R^{\beta/2} + \mathcal{E}(x_0,R)^2 + \Big(\fint_{B_R} |V(Du - (Du)_{x_0,R})|^{2\alpha} \Big)^{\frac{1}{\alpha}} \Big]$$

since $\mathcal{E}(x_0, R) < 1$ and $p \geq 2$.

Now for any $0 < \tau \leq \frac{1}{2}$ and any matrix $A \in \mathbb{R}^{Nn}$ we have

$$\mathcal{E}(x_0, \tau R)^2 \leq c \fint_{B_{\tau R}} |V(Du - A)|^2 \leq c \fint_{B_{\tau R}} |V(Dh - A)|^2 + c \fint_{B_{\tau R}} |V(Du - Dh)|^2.$$

Also note that $\|Dh - A\|_p \leq C\|\nabla\Pi(\theta)Du - A\|_p$ by Lemma 6.14 for any $A \in \mathbb{R}^{Nn}$ with $\nabla\Pi(\theta)A = A$ a.e. Thus by setting $A = (Dh)_{\tau R}$ the interior estimates of Remark 6.13 imply

$$\begin{aligned} \fint_{B_{\tau R}} |V(Dh - (Dh)_{x_0,\tau R})|^2 &\leq C\tau^2 \int_{B_R} |V(Dh - (Dh)_{x_0,R})|^2 \\ &\leq C\tau^2 \int_{B_R} |V(Du - (Du)_{x_0,R})|^2, \end{aligned}$$

since $(Dh)_R = \nabla\Pi(\theta)(Du)_R$ by the divergence theorem. Moreover the conclusion of Lemma 6.15 together with Corollary 6.20 implies

$$\begin{aligned} \fint_{B_{\tau R}} |V(Du - Dh)|^2 &\leq \tau^{-n} \fint_{B_R} |V(Du - Dh)|^2 \\ &\leq \tau^{-n}\varepsilon_0 \Big[R^{\beta/2} + \mathcal{E}(R)^2 + \Big(\fint_{B_R} |V(Du - (Du)_{x_0,R}|^{2\alpha} \Big)^{\frac{1}{\alpha}} \Big] \\ &\leq C\tau^{-n}\varepsilon_0 (R^{\beta/2} + \mathcal{E}(x_0, R)^2 + \mathcal{E}(x_0, 2R)^2). \end{aligned}$$

Thus $\mathcal{E}(\tau R)^2 \leq C(\tau^2 + \tau^{-n}\varepsilon_0)\mathcal{E}(4R)^2 + C\tau^{-n}\varepsilon_0 R^{\beta/2}$ for each $0 < \tau \leq 1/2$. □

Given Proposition 6.21 one can judiciously choose the parameter τ such that both

$$C\tau^2 \leq \frac{1}{2}\tau^{\beta/4} \qquad \text{and} \qquad C\tau^{-n}\widetilde{\omega}(\varepsilon + R) \leq \frac{1}{2}\tau^{\beta/4}$$

hold. As the first inequality is equivalent to choosing $0 < \tau \leq (2C)^{-1/(2-\beta/4)} = \tau_0$ we must also require $0 < \widetilde{\omega}(\varepsilon + R) \leq \frac{1}{2C}\tau^{n+2}$. Then for such (fixed) τ we can decrease ε and R even further, if necessary, to obtain:

Corollary 6.22. *For each $\Lambda > 0$ there exists $\varepsilon = \varepsilon(\tau, \Lambda) \in (0, 1)$ such that if the smallness conditions*

$$|(u)_{x_0,R}| \leq \Lambda, \qquad |(Du)_{x_0,R}| \leq \Lambda, \qquad \mathcal{E}(x_0, R) \leq \varepsilon$$

hold on some $B_R(x_0) \subset\subset \Omega$ with $0 < R \ll 1$ sufficiently small, then

$$\mathcal{E}(x_0, \tau R)^2 \leq \tau^{\beta/4}\mathcal{E}(x_0, R)^2 + R^{\beta/2}.$$

By iterating this result we:

Claim 6.23. For each $\Lambda > 0$ there exists $\varepsilon = \varepsilon(\tau, \Lambda + 1) \in (0, 1)$ such that for integers $j \geq 1$ we have

$$\mathcal{E}(x_0, \tau^j R)^2 \leq \tau^{j\beta/4}\mathcal{E}(x_0, R)^2 + R^{\beta/2}\frac{\tau^{j\beta/4} - \tau^{j\beta/2}}{\tau^{\beta/4} - \tau^{\beta/2}},$$

whenever $|(u)_{x_0,R}| \leq \Lambda$, $|(Du)_{x_0,R}| \leq \Lambda$, $\mathcal{E}(x_0, R) \leq \varepsilon$ holds on some $B_R(x_0) \subset\subset \Omega$ with $0 < R \ll 1$ sufficiently small.

Now if $r \in [\tau R, R)$ then $\mathcal{E}(x_0, r)^2 \leq \tau^{-n}\mathcal{E}(x_0, R)^2$; whereas if $r \in (0, \tau R]$ there exists an integer $j \geq 1$ such that $\tau^{j+1}R \leq r \leq \tau^j R$ from which Claim 6.23 implies

$$\mathcal{E}(x_0, r)^2 \leq \tau^{-n}\mathcal{E}(x_0, \tau^j R)^2 \leq \Big(\frac{r}{R}\Big)^{\beta/4}\tau^{-n-\beta/4}\Big[\mathcal{E}(x_0, R)^2 + \frac{R^{\beta/2}}{\tau^{\beta/4} - \tau^{\beta/2}}\Big].$$

Therefore for each $x_0 \in \operatorname{Reg} u$ there exists $\Lambda > 0$, an ε sufficiently small and an $0 < R < \operatorname{dist}(x_0, \partial\Omega)$ sufficiently small such that

$$\mathcal{E}(x_0, r)^2 \leq \Big(\frac{r}{R}\Big)^{\beta/4}\tau^{-n-\beta/4}\Big[\varepsilon^2 + \frac{R^{\beta/2}}{\tau^{\beta/4} - \tau^{\beta/2}}\Big], \qquad 0 < r < R, \tag{6.31}$$

whenever $|(u)_{x_0,R}| \leq \Lambda$, $|(Du)_{x_0,R}| \leq \Lambda$ and $\mathcal{E}(x_0, R) \leq \varepsilon$ hold on some $B(x_0, R) \subset\subset \Omega$ (which by Remark 6.2 is what one seeks to demonstrate).

6.8. Applications to problems in condensed matter physics

The above manifold constrained regularity problem has, in particular, several applications to problems in condensed matter physics (the details of which are outlined in Appendix A). For such applications the (unconstrained) mapping is a tensor order parameter $T = (T_{\alpha\beta})\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ which is invariant under a given symmetry group G with an integral functional of the form

$$\int_\Omega f_{\text{grad}}(T, \nabla T) + f_B(T)\, dx$$

Critical points of the bulk energy f_B form an orbit of solutions corresponding to a subgroup $H \subset G$ of residual symmetries. The resulting manifold of internal states $\mathcal{M} = G/H$ describes the broken symmetries of the system; such T taking values in this group orbit form the corresponding constrained mapping case. A list of the manifold of broken symmetries for various examples of ordered media can be found in Table A.1.

Example 6.24 (Superfluidity of liquid helium-3). The theory of superfluid helium-3 is characterised by an the order parameter $A = (A_{\mu j})$ as a 3×3 complex valued matrix with a spin index μ and orbital index j. Thus as a map the order parameter $A\colon \Omega \subset \mathbb{R}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3 \equiv \mathbb{R}^{18}$. The associated symmetry group $G = SO(3) \times SO(3) \times U(1)$, since the order parameter is invariant against rotation in spin space and in ordinary orbital

space separately as well as of course being gauge-invariant, i.e. invariant up to a phase factor. When allowing for spatial variations in the order parameter the gradient energy of an anisotropic superfluid in the Ginzburg-Landau regime has leading G-invariant terms quadratic in the gradient of the form [dG3, AdGR]

$$f_{\text{grad}} = \gamma_1\, \partial_j A_{\mu\ell} \partial_j A_{\mu\ell}{}^* + \gamma_2\, \partial_j A_{\mu\ell} \partial_\ell A_{\mu j}{}^* + \gamma_3\, \partial_j A_{\mu j} \partial_\ell A_{\mu\ell}{}^*.$$

The unusually large number of the internal degrees of freedom allows for a rich array of superfluid phases of liquid ^{3}He. For instance in the so-called B-phase the order parameter is constrained to the manifold of internal states $\mathcal{M}_B = SO(3) \times S^1$. By contrast the dipole-free A-phase has the order parameter constrained to $\mathcal{M}_A = (S^2 \times SO(3))/\mathbb{Z}_2$ where as in the dipole locked case $\mathcal{M}_A^d = SO(3)$.

Example 6.25 (Nematic liquid crystals). According to the Landau-de Gennes theory [dGP] the state of alignment of a nematic liquid crystal is characterised pointwise by a tensor order parameter Q as a traceless symmetric 3×3 real matrix, i.e. the (unconstrained) Q-tensor order parameter is a map $Q\colon \Omega \subset \mathbb{R}^3 \to \mathrm{Sym}_0^2(\mathbb{R}^3) \equiv \mathbb{R}^5$. The free-energy functional, taking into consideration spatial variations of the order parameter, is of the form

$$\int_\Omega f_{\text{el,N}}(Q, \nabla Q) dx$$

where $f_{\text{el,N}} : \mathbb{R}^5 \times \mathbb{R}^{5\times 3} \to \mathbb{R}$. The symmetry group under consideration is $G = SO(3)$ and the free-energy density must satisfy the $SO(3)$-invariance condition

$$f_{\text{el,N}}(Q, \nabla Q) = f_{\text{el,N}}(MQM^T, D^*)$$

for all $M \in SO(3)$, where $D^*_{ijk} = M^a_i M^b_j M^c_k Q_{ab,c}$. The expansion of the free energy in $SO(3)$-invariant powers of Q_{ij} and its derivatives up to the order $Q\,Q\,\nabla Q\,\nabla Q$ contains twenty-two terms and four surface relations (cf. [dG2, dGP] as well as [ST, BM, PS, LMT, VSV] for higher order terms). For instance the terms of the form $\nabla Q\,\nabla Q$ are

$$L_1^{(2)} Q_{ij,k} Q_{ij,k} \qquad L_2^{(2)} Q_{ij,j} Q_{ik,k} \qquad L_3^{(2)} Q_{ij,k} Q_{ik,j}$$

since there are in fact only five possible contractions of a symmetric 2-tensor in the form $\nabla Q\,\nabla Q$ (the two other remaining are $Q_{kk,j}Q_{ii,j}$ and $Q_{ii,j}Q_{jk,k}$ but are excluded since the matrix is traceless) with the corresponding surface relation

$$\partial_i (Q_{ij} Q_{jk,k} - Q_{jk} Q_{ij,k}) = I_2^{(2)} - I_3^{(2)}. \tag{6.32}$$

In addition, there are six terms $L_j^{(3)} I_j^{(3)}$ where $I_j^{(3)}$ is of the form $Q\,\nabla Q\,\nabla Q$ and thirteen terms $L_j^{(4)} I_j^{(4)}$ where $I_j^{(4)}$ is of the $Q\,Q\,\nabla Q\,\nabla Q$. Here $I_j^{(k)}$ is the tensor contraction weighted by the elastic constant $L_j^{(k)}$.

If we consider the case of a nematic phase in which $f_{\mathrm{el,N}}$ is described only by a linear combination of invariants $I_1^{(2)}$, $I_2^{(2)}$ and $I_3^{(2)}$ one can use the fact the invariants in this case form a bilinear form to check boundedness and coerciveness. Indeed, the analysis of [DG1, p. 344], [LMT, p. 781] shows the symmetric bilinear map $\mathbb{B}\colon W^{1,2}(\Omega,\mathrm{Sym}_0^2(\mathbb{R}^3))\times W^{1,2}(\Omega,\mathrm{Sym}_0^2(\mathbb{R}^3))\to\mathbb{R}$ defined by

$$\mathbb{B}(P,Q)=\int_\Omega L_1^{(2)}P_{ij,k}Q_{ij,k}+L_2^{(2)}P_{ij,j}Q_{ik,k}+L_3^{(2)}P_{ij,k}Q_{ik,j}\,dx$$

to be bounded with $\mathbb{B}(Q,Q)\geq\gamma\int_\Omega|\nabla Q|^2$ for some constant $\gamma>0$ and all $Q\in W^{1,2}(\Omega,\mathrm{Sym}_0^2(\mathbb{R}^3))$, provided one has the necessary and sufficient conditions

$$L_1^{(2)}+\tfrac{5}{3}L_2^{(2)}+\tfrac{1}{6}L_3^{(2)}\geq 0,\qquad L_1^{(2)}-\tfrac{1}{2}L_3^{(2)}\geq 0,\qquad L_1^{(2)}+L_3^{(2)}\geq 0.$$

In which case there exists constants $0<\gamma<\Gamma$ such that

$$\gamma|\nabla Q|^2\leq L_1^{(2)}Q_{ij,k}Q_{ij,k}+L_2^{(2)}Q_{ij,j}Q_{ik,k}+L_3^{(2)}Q_{ij,k}Q_{ik,j}\leq\Gamma|\nabla Q|^2$$

for all $Q\in W^{1,2}(\Omega,\mathrm{Sym}_0^2(\mathbb{R}^3))$. For further convexity analysis of $SO(3)$-invariant integrands see [D1, §5.3.3 and §5.3.4].

To address the analytic properties for the (extended) Landau-de Gennes theory, with an expansion of the integrand of the form

$$f_{\mathrm{el,N}}=\sum_{j=1}^{3}L_j^{(2)}I_j^{(2)}+\sum_{j=1}^{6}L_j^{(3)}I_j^{(3)}+\sum_{j=1}^{13}L_j^{(4)}I_j^{(4)},$$

we note that the integrand is a fourth order polynomial in the independent fields of Q_{ij} and $Q_{ij,k}$. Hence the issue reduces to a rather involved study of the properties of fourth order polynomials in the twenty-dimensional space of Q_{ij} and $Q_{ij,k}$. Using spherical representations of tensors the work of [LMT, p. 784] states the fourth order expansion is in fact coercive and polyconvex provided the elastic constants satisfy certain linear inequalities.

In the *constrained theory* a nematic liquid crystal is said to be *isotropic* if $Q=0$; *uniaxial* if the Q-tensor has two equal non-zero eigenvalues; and, *biaxial* if Q has three distinct eigenvalues.

In the uniaxial case the Q-tensor can be written in the form

$$Q=s(\boldsymbol{n}\otimes\boldsymbol{n}-\tfrac{1}{3}I)$$

for $\boldsymbol{n}\in S^2$ and constant $s\in\mathbb{R}$. In this case the isotropy subgroup of A is isomorphic to the orthogonal group $O(2)$ so that the coset space $SO(3)/O(2)=SO(3)/D_\infty$, where D_∞ is the group of all rotations about an axis and π-rotations about axes perpendicular to that axis, is diffeomorphic to $\mathbb{R}P^2$.

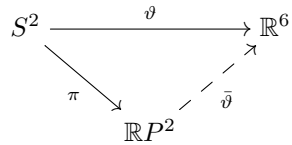
One can in fact see the identification with the real projective space explicitly as follows. Firstly, let $\boldsymbol{n} = (x, y, z) \in S^2 \subset \mathbb{R}^3$ be a unit vector and $P(\boldsymbol{n})$ be the orthogonal projection on the line $\mathbb{R}.\boldsymbol{n} = \{\lambda \boldsymbol{n} : \lambda \in \mathbb{R}\}$. As the orthogonal projection of $\boldsymbol{v} = P(\boldsymbol{n})$ is characterised by $\boldsymbol{v} = \lambda \boldsymbol{n}$ and $(\boldsymbol{v} - \boldsymbol{n}) \cdot \boldsymbol{n} = 0$ it follows that $P \colon S^2 \to \mathrm{Sym}^2(\mathbb{R}^3)$, sending $\boldsymbol{n} \mapsto \boldsymbol{n} \otimes \boldsymbol{n}$, is represented by

$$(x, y, z) \mapsto \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}.$$

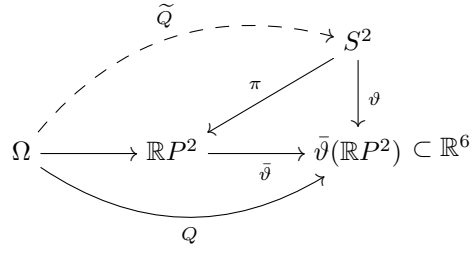
We can then define the map $\vartheta \colon S^2 \to \mathbb{R}^6$ by sending

$$(x, y, z) \mapsto (x^2, y^2, z^2, xy, xz, yz). \tag{6.33}$$

From the Jacobian matrix we see the map ϑ defines an immersion of the 2-sphere. Furthermore, as $\mathbb{R}P^2$ is compact and π is constant on the fibre $\vartheta(x, y, z) = \vartheta(-x, -y, -z)$ the map ϑ passes to the quotient, i.e. there exist $\bar{\vartheta} \colon \mathbb{R}P^2 \to \mathbb{R}^6$ such that the following



completes to a commutative diagram. To see that $\mathbb{R}P^2 = S^2/\sim$ is equal to the homogeneous space $SO(3)/O(2)$ we note the group $SO(3)$ acts transitively on S^2. This goes to the quotient to give a transitive action on $\mathbb{R}P^2$. The isotropy subgroup of an element of $\mathbb{R}P^2$ is the set of $A \in SO(3)$ such that $A\boldsymbol{n} = \pm\boldsymbol{n}$ and can be identified with the group of matrices of the form $\left(\begin{smallmatrix} \det M & 0 \\ 0 & M \end{smallmatrix}\right)$ where $M \in O(2)$. It then follows that $\mathbb{R}P^2$ is diffeomorphic to $SO(3)/O(2)$ and in particular $SO(3)/O(2)$ is an isotropy irreducible homogeneous space. Using the metric $\langle A, B\rangle = \mathrm{tr}(A^T B)$, for any 3×3 matrices A and B, we note that the map $\mathsf{C}_Q(A) = QAQ^T$, for any $Q \in SO(3)$, is an isometry. If $A = P(\boldsymbol{n})$ is the orthogonal projection onto $\mathbb{R}.\boldsymbol{n}$, $\mathsf{C}_Q(A)$ is the orthogonal projection $P(Q\boldsymbol{n})$ onto $\mathbb{R}.Q\boldsymbol{n}$. This shows C_Q preserves the image $\bar{\vartheta}(\mathbb{R}P^2)$; hence $\bar{\vartheta}$ is an isometric embedding. In which case the Q-tensor in the uniaxial case can be seen to be characterised by



where $\widetilde{Q} \colon \Omega \to S^2$ is the lifting and $\vartheta \colon S^2 \to \bar{\vartheta}(\mathbb{R}P^2) \subset \mathbb{R}^6$ is the universal covering.

In the biaxial case the matrix Q, with three distinct constant eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ independent of $x \in \Omega$, can be written in the form

$$Q = \lambda_1 \boldsymbol{n} \otimes \boldsymbol{n} + \lambda_2 \boldsymbol{m} \otimes \boldsymbol{m} + \lambda_3 \boldsymbol{\ell} \otimes \boldsymbol{\ell}$$

where $\boldsymbol{n}, \boldsymbol{m} \in S^2$ are orthogonal eigenvectors of Q with $\boldsymbol{\ell} = \boldsymbol{n} \times \boldsymbol{m} \in S^2$. We can formally identify the set of all such Q-tensors with

$$SO(3) * A = \{GAG^T \in \mathrm{Sym}_0^2(\mathbb{R}^3) : G \in SO(3)\},$$

where $A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. This is clearly the orbit of the matrix A with respect to the left action $*$ of $SO(3)$ on $\mathrm{Sym}_0^2(\mathbb{R}^3)$ given by $G * Q = GQG^T$ for $G \in SO(3)$, $Q \in \mathrm{Sym}_0^2(\mathbb{R}^3)$. Since the eigenvalues are distinct the isotropy subgroup of $SO(3)$ which fixes A, i.e. $SO(3)_A = \{G \in SO(3) : G * A = A\}$, are readily seen to be the abelian 4-element dihedral group $D_2 = \{\mathrm{diag}(1,1,1), \mathrm{diag}(-1,-1,1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1)\}$. The coset space $SO(3)/SO(3)_A = SO(3)/D_2$ can be given the structure of smooth manifold, since the bijective map $SO(3)/SO(3)_A \to SO(3) * A$ sending $[G] \mapsto GAG^T$ is a diffeomorphism. Using the 2:1 covering $\Phi\colon \mathrm{Spin}(3) \simeq S^3 \to SO(3)$ one finds the preimage of D_2 in S^3 coincides with the non-abelian 8-element quaternion group $\mathbb{H} = \langle -1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} | (-1)^2 = 1, \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \rangle$. Hence the space of all Q-tensors in the constrained biaxial nematic case can be identified with

$$S^3/\mathbb{H} = \{q\,\mathbb{H} : q \in \mathrm{Spin}(3)\}$$

endowed with a unique Riemannain structure so that the canonical projection $\Pi\colon S^3 \to S^3/\mathbb{H}$ is a Riemannian covering map (cf. [MN]).

Chapter 7

Singular set bounds for holonomic minimisers of integral functionals

It is known in the general vectorial case that both solutions to elliptic systems and minima of variational integrals may develop singularities. Hence in Chapters 2, 3 and 6 only partial regularity results were possible. Nonetheless the passage from the scalar to the vectorial case sees weaker 'higher integrability' still persisting. For constrained minimisers, in particular, *partial* higher integrability holds in the harmonic map case (cf. Corollary 3.8) via monotonicity and the extension result of Chapter 4, whereas the use of the results from Chapter 5 allows for *everywhere* higher integrability for more general holonomic minimisers (cf. Lemma 6.4) under some additional topological considerations. The need for minimality when considering the holonomically constrained problem is particularly important, since there exists everywhere discontinuous solutions $u \in W^{1,2}(B^3, S^2)$ of the harmonic map equation $-\Delta u = |Du|^2 u$ weakly, on the unit ball $B^3 \subset \mathbb{R}^3$, by a result of Rivière [R3]. Contrasting this, energy minimising harmonic maps can be seen to have a singular set with Hausdorff dimension less than or equal to $n-2$ (cf. Corollary 3.2), which in fact can be reduced to $n-3$ by a dimension reduction argument of Federer (cf. [SU1]). For minima of more general strictly quasiconvex integral functionals the result of Theorem 6.1 sees the singular set, as characterised by (6.5), to be only of Lebesgue measure zero. The objective of the following is to show the singular set, for suitably general regular variational integrals, is more than just Lebesgue measure zero (i.e. has Hausdorff dimension strictly less than n) by way of the everywhere higher integrability afforded by the minimality condition.

To get an overall idea let us first consider the harmonic map case where energy minimisers $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$, into a compact submanifold $\mathcal{M} \subset \mathbb{R}^N$, satisfy the necessary

condition

$$-\Delta u = A(u)(Du, Du)$$

weakly (cf. Example 1.59). If we assume higher integrability everywhere, that is $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{M})$ for some $q > 2$, then $-\Delta u \in L^{q/2}_{\mathrm{loc}}$ as the right-hand side has critical growth

$$|A(u)(Du, Du)| \leq c|Du|^2.$$

In which case we conclude $u \in W^{2,q/2}_{\mathrm{loc}}$ by the standard L^p-estimates[1] with $p = q/2 > 1$. Moreover, $u \in C^{0,2(1-n/q)}$ if and only if $q > n$ by the Sobolev embeddings. Nevertheless this argument does not self-improve: for as we have gained a derivative we could differentiate the above equation once more to get $-\Delta(Du) = F$, where $|F| \leq c(|Du|^3 + |D^2u|^{3/2})$. Then as $F \in L^{q/3}_{\mathrm{loc}}$ the L^p-estimates imply $u \in W^{3,q/3}_{\mathrm{loc}}$. Repeating this argument finds $u \in W^{k,q/k}_{\mathrm{loc}}$ for integers $k \geq 2$, however Hölder continuity still only holds when $n = 2$ under the assumption of everywhere higher integrability.

A similar situation also occurs for biharmonic maps. In this case minimisers $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ of the energy functional $\frac{1}{2}\int_\Omega |\Delta u|^2 dx$ satisfy a stationary biharmonic map equation (cf. Lemma 3.13) of the form

$$\Delta^2 u = f_0 + \operatorname{div} f_1,$$

where $f_j = f_j(u, Du, D^2u)$ satisfy the critical growth conditions

$$|f_0| \leq c(1 + |Du|^4 + |D^2u|^2) \qquad \text{and} \qquad |f_1| \leq c(1 + |Du|^3 + |D^2u|^{3/2}).$$

Now if we assume higher integrability everywhere, so that $u \in W^{2,q}_{\mathrm{loc}}(\Omega, \mathcal{M})$ for some $q > 2$, then $f_0 \in L^{q/2}_{\mathrm{loc}}$ and $f_1 \in L^{2q/3}_{\mathrm{loc}}$. By solving $\Delta F_0 = f_0$ we see $F_0 \in W^{2,q/2}_{\mathrm{loc}}$ by L^p-estimates from which $DF \in L^{2q/3}_{\mathrm{loc}}$ by the Sobolev embeddings. In which case we can re-write the stationary biharmonic map equation as $\Delta(\Delta u) = \operatorname{div} F$, where $F \in L^{2q/3}_{\mathrm{loc}}$. Then by the divergence form L^p-estimates of Theorem 1.45 we find $\Delta(Du) \in L^{2q/3}_{\mathrm{loc}}$ which by a further application of L^p-estimates gives $u \in W^{3,2q/3}_{\mathrm{loc}}$. Having now gained an extra derivative we can continue by re-write the original equation as $\Delta^2 u = f$, where $f = f(u, Du, D^2u, D^3u)$ satisfies the growth condition $|f| \leq c(1 + |Du|^4 + |D^2u|^2 + |D^3u|^{4/3})$. Then $f \in L^{q/2}_{\mathrm{loc}}$ so that $u \in W^{4,q/2}_{\mathrm{loc}}$ by two further applications of L^p-estimates. Once again Hölder continuity is only possible for $2q > n$ and there is no self-improvement.

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[1]Namely, if $u \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ solves

$$\Delta u = f$$

in Ω for some $f \in L^p(\Omega, \mathbb{R}^N)$, where $1 < p < \infty$, then $u \in W^{2,p}(\Omega, \mathbb{R}^N)$ and

$$\|D^2u\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)}.$$

Cautiously, however, the estimates fails when $p = 1$: for if we consider the unit disk $D^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $u \in W^{1,2}_0(D^2)$ given by $u(x) = \log\log(e|x|^{-1})$ then one can check $\Delta u \in L^1(D^2)$ but that $\int_{D^2} |D^2u| = +\infty$.

We now revisit the following standard measure density result for estimating the Hausdorff dimension of the singular set that goes back to [G8].

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and λ be a monotone σ-super-additive finite set function on Ω. If $0 \leq \alpha < n$ and*

$$E_\alpha = \{x \in \Omega : \limsup_{r \searrow 0} r^{-\alpha} \lambda(B_r(x)) > 0\}$$

then $\dim_{\mathcal{H}}(E_\alpha) \leq \alpha$.

Following the observation in [M8] we can apply this result to the semi-norm for the fractional Sobolev space by setting

$$\lambda(A) = \lambda(A; u) = \int_A \int_A \frac{|Du(x) - Du(y)|^q}{|x-y|^{n+\theta q}} dxdy,$$

from which we conclude:

Corollary 7.2. *If $v \in W^{\theta,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, where $0 < \theta < 1$ and $q \geq 1$, then*

$$A = \left\{ x \in \Omega : \liminf_{r \searrow 0} \fint_{B_r(x)} |v - (v)_{x,r}|^q > 0 \right\}$$

$$B = \left\{ x \in \Omega : \limsup_{r \searrow 0} |(v)_{x,r}| = +\infty \right\}$$

has $\dim_{\mathcal{H}}(A) \leq n - \theta q$ and $\dim_{\mathcal{H}}(B) \leq n - \theta q$.

Proof. First recall the Poincaré-type inequality for functions belonging to fractional Sobolev spaces, namely if $v \in W^{\theta,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, where $q \geq 1$ and $0 < \theta < 1$, then

$$\int_{B_R} |v - (v)_R|^q \leq c(n,q) R^{\theta q} \int_{B_R} \int_{B_R} \frac{|v(x) - v(y)|^q}{|x-y|^{n+\theta q}} dxdy \tag{7.1}$$

for any $B_R \subset\subset \Omega$. So for any $\Omega' \subset\subset \Omega$ we have

$$A \cap \Omega' \subset \{x \in \Omega' : \liminf_{r \searrow 0} r^{\theta q - n} \lambda(B_r(x)) > 0\}.$$

Then by Lemma 7.1 we bound $\dim_{\mathcal{H}}(A \cap \Omega') \leq n - \theta q$ from which we conclude $\dim_{\mathcal{H}}(A) \leq n - \theta q$.

Now fix $\varepsilon > 0$ and note that $B \subset \{x \in \Omega : \limsup_{r \searrow 0} \fint_{B_r(x)} |v| = +\infty\} = C$. Suppose $x_0 \in \Omega$ and

$$\liminf_{r \searrow 0} r^{2\theta - n - \varepsilon} \lambda(B_r(x_0)) = 0$$

i.e. $x \notin E_{n+\varepsilon-2\theta}$. We claim that $x_0 \notin C$ (hence $C \subset E_{n+\varepsilon-2\theta}$ so that $\dim_{\mathcal{H}}(B) \leq n + \varepsilon - 2\theta$ for all $\varepsilon > 0$). To do so fix $B_R(x_0) \subset \Omega$, then for positive integers $k \geq 1$ we find

$$|(|v|)_{x_0, 2^{-k-1}R} - (|v|)_{x_0, 2^{-k}R}|^q \leq C \fint_{B(x_0, 2^{-k}R)} |v - (v)_{x_0, 2^{-k}R}|^q$$

$$\begin{aligned}&\leq C(2^{-k}R)^{\theta q-n}\lambda(B(x_0,2^{-k}R))\\&= C(2^{-k}R)^{\varepsilon}\Big[(2^{-k}R)^{\theta q-n-\varepsilon}\lambda(B(x_0,2^{-k}R))\Big]\\&\leq CR^{\varepsilon}(2^{-\varepsilon})^k\end{aligned}$$

so that

$$\begin{aligned}(|v|)_{x_0,2^{-j}R} &\leq (|v|)_{x_0,R}+\sum_{k=0}^{j-1}|(|v|)_{x_0,2^{-k-1}R}-(|v|)_{x_0,2^{-k}R}|\\&\leq (|v|)_{x_0,R}+\frac{CR^{\varepsilon/q}}{1-2^{-\varepsilon/q}}\end{aligned}$$

is finite. Now if $r_j \searrow 0$ there exists an integer $k_j \geq 1$ such that $2^{-k_j}R < r_j \leq 2^{-k_j+1}R$. Noting $(|v|)_{x_0,r_j} \leq 2^n(|v|)_{x_0,2^{-k+1}R}$ gives $\limsup_{r\searrow 0}(|v|)_{x_0,r} < +\infty$ as desired. □

Example 7.3. If a weakly harmonic map satisfying $-\Delta u = A(u)(Du,Du)$ is assumed to have everywhere higher integrability, i.e. $u \in W^{1,q}_{\rm loc}(\Omega,\mathcal{M})$ for some $q > 2$, then $Du \in W^{1,q/2}_{\rm loc}$ by the standard L^p-estimates, since $|A(u)(Du,Du)| \leq c|Du|^2$. As $W^{1,q/2} \subset W^{1-\varepsilon,p}$ for any $0 < \varepsilon < 1$ with $p = nq/(2n-\varepsilon q)$, Corollary 7.2 implies the Hausdorff dimension of the singular set is less than or equal to $(n-\frac{q}{2})/(1-\frac{q\varepsilon}{2n}) = n-1$ when $\varepsilon = \frac{n}{q}\frac{q-2}{n-1}$. Hence the assumption of everywhere higher integrability can be used to bound the Hausdorff dimension of the singular set. In particular, we can rule out the above mentioned everywhere discontinuous scenario of Rivière [R3] (which shows Sing $u = \bar{B}^3$ for some weakly harmonic $u \in W^{1,2}(B^3,S^2)$) whenever one has higher integrability.

We can generalise this example by recalling the following L^p-result (cf. [GT, §9.5, 9.6], [G10, §10.4]).

Proposition 7.4. *If for given $f \in L^p(\Omega,\mathbb{R}^N)$ and $\varphi \in W^{2,p}(\Omega,\mathbb{R}^N)$ on a $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$, with $1 < p < \infty$, the Dirichlet problem*

$$\begin{aligned}&A^{ij}_{\alpha\beta}D_{ij}u^\beta = f_\alpha \quad \text{in } \Omega\\&u \in \varphi + W^{1,p}_0(\Omega,\mathbb{R}^N),\end{aligned}\tag{7.2}$$

where $A^{\alpha\beta}_{ij} \in C^0(\bar{\Omega})$ satisfies $A^{ij}_{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \geq \ell|\xi|^2|\eta|^2$ for all $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^n$ and $|A^{ij}_{\alpha\beta}| \leq \Lambda$ for some $0 < \ell \leq L$, has a unique solution $u \in W^{2,p}(\Omega,\mathbb{R}^N)$ and for any $B_R \subset\subset \Omega$,

$$\|D^2u\|_{p;B_{R/2}} \leq C\|f\|_{p;B_R} + CR^{-2}\|u\|_{p;B_R}$$

for some $C = C(n,N,\ell,L) > 0$.

Now let $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a solution of (7.2) and suppose

$$|f| \leq c(1+|Du|^\sigma)$$

for some $\frac{2}{n} < \sigma \leq 2$. If $Du \in L^q_{\rm loc}(\Omega, \mathbb{R}^{Nn})$ for some $q > 2$ then $Du \in W^{1,q/\sigma}_{\rm loc}$ by Proposition 7.4 with $p = q/\sigma$. As $W^{1,\frac{q}{\sigma}} \subset W^{1-\varepsilon,\frac{nq}{n\sigma-\varepsilon q}}$, for any $0 < \varepsilon < 1$, Corollary 7.2 implies the Hausdorff dimension of the singular set is less than or equal to $(n - \frac{q}{\sigma})/(1 - \frac{q\varepsilon}{\sigma n}) = n - \frac{2}{\sigma}$ when $\varepsilon = \frac{\sigma n}{q}\frac{q-2}{\sigma n-2}$. In particular, if $|f| \leq c(1 + |Du|)$ the Hausdorff dimension of the singular set is $\leq n-2$ whereas if $|f| \leq c(1+|Du|^2)$ the bound is $\leq n - 1$. The next section seeks to replicate this singular set bound for more general elliptic systems (where the second order strong derivative is not available) via a fractional derivative argument.

7.1. Revisting the difference quotient method for holonomic minimisers of regular variational integrals

Let us first consider regular variational integrals of the form

$$\mathcal{F}(u;\Omega) = \int_\Omega f(Du)dx,$$

where $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ for $n, N \geq 2$ such that the integrand $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is a C^2-function that satisfies the growth and convexity conditions

$$\ell|\xi|^2 \leq f(\xi) \leq L(1 + |\xi|^2) \qquad \text{and} \qquad \ell|\lambda|^2 \leq D^2 f(\xi)(\lambda,\lambda) \leq L|\lambda|^2 \tag{7.3}$$

for all $\xi, \lambda \in \mathbb{R}^{Nn}$ with $L \geq \ell > 0$. If we consider a compact connected C^3-submanifold $\mathcal{M}$ of $\mathbb{R}^N$ Lemma 1.58 shows any holonomic $\mathcal{F}$-minimiser $u \in W^{1,2}_{\rm loc}(\Omega, \mathcal{M})$ satisfies

$$\int_\Omega Df(Du)\big(\nabla\Pi(u)D\phi + \nabla^2\Pi(u)Du\cdot\phi\big)dx = 0 \tag{7.4}$$

for all $\phi \in W^{1,2}_0(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$, where Π is a retraction from a tubular neighbourhood onto $\mathcal{M}$.

By revisiting the difference quotient method of §1.6.2 we obtain the following manifold constrained result under the assumption of higher integrability.

Lemma 7.5. *Let $u \in W^{1,2}_{\rm loc}(\Omega,\mathcal{M})$ solve (7.4) under the assumption the integrand satisfies (7.3). If $Du \in L^q_{\rm loc}(\Omega,\mathbb{R}^{Nn})$ for some $2 < q < 4$ then there exists a constant $C = C(n, N, q, \ell, L) > 0$ such that*

$$\int_Q |\Delta^h Du|^2 \leq C\frac{|h|^2}{|Q|^{\frac{2}{n}}}\int_{3Q}|Du - A|^2 + C|h|^{q-2}\int_{3Q} 1 + |Du|^q + C|h|^2|A|^2\int_{3Q} 1 + |Du|^2$$

for any cubes $3Q \subset\subset \Omega$, any matrix $A \in \mathbb{R}^{Nn}$ and any $0 < |h| < \frac{1}{2}|Q|^{\frac{1}{n}}$, where Δ^h denotes the difference quotient (1.17).

Proof. Let $\eta\colon \Omega \to [0,1]$ be a Lipschitz cut-off function such that $\eta = 1$ on Q, $\eta = 0$ on $\Omega\backslash(2Q)$ and $|D\eta| \leq 2/|Q|^{\frac{1}{n}}$. For $|h| < \frac{1}{2}|Q|^{\frac{1}{n}}$ set $\phi = \Delta^{-h}(\eta^2\Delta^h\bar{u})$, where $\bar{u} = u - a$ for any affine map $a\colon x \to \xi + A\cdot x$ with $\xi \in \mathbb{R}^N$ and $A \in \mathbb{R}^{Nn}$. Then $\phi \in W^{1,2}_0(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ so from (7.4) we have

$$\int_\Omega Df(Du)\nabla\Pi(u)\Delta^{-h}(D(\eta^2\Delta^h\bar{u})) + Df(u)\nabla^2\Pi(u)\cdot\Delta^{-h}(\eta^2\Delta^h\bar{u}) = 0.$$

By taking the difference quotient of the identity $Du = \nabla\Pi(u)Du$ we find that

$$\begin{aligned}
0 &= \int_\Omega \Delta^h[Df(Du)\nabla\Pi(u)]D(\eta^2\Delta^h\bar{u}) + \Delta^h[Df(Du)\nabla^2\Pi(u)Du]\cdot\eta^2\Delta^h\bar{u}\\
&= \int_\Omega \Delta^h[Df(Du)](\eta^2\Delta^h Du) - \Delta^h[Df(Du)]\Delta^h[\nabla\Pi(u)](\eta^2 Du)\\
&\quad + Df(Du)\Delta^h[\nabla\Pi(u)](\eta^2\Delta^h Du) + \Delta^h[Df(Du)\nabla\Pi(u)](2\eta D\eta\Delta^h\bar{u})\\
&\quad + \Delta^h[Df(Du)\nabla^2\Pi(u)Du]\cdot\eta^2\Delta^h\bar{u}\\
&= I + II + III + IV + V.
\end{aligned}$$

From the properties (7.3) we note that

$$\Delta^h[Df(Du)]\Delta^h Du \geq \ell|\Delta^h Du|^2, \qquad |\Delta^h[Df(Du)]| \leq L|\Delta^h Du|$$

and $|Df(Du)| \leq c(1+|Du|)$. Also, we have $|\Delta^h[\nabla\Pi(u)]| \leq c|\Delta^h u|$ by the Lipschitz properties of the retraction Π. In which case we can, for some $\varepsilon > 0$, bound the first term from below by

$$I = \int_\Omega \Delta^h[Df(Du)](\eta^2\Delta^h Du) \geq \ell\int_\Omega \eta^2|\Delta^h Du|^2,$$

with both the second and third being bounded by

$$|II| \leq \int_\Omega |\Delta^h[Df(Du)]\Delta^h[\nabla\Pi(u)](\eta^2 Du)| \leq \varepsilon\ell\int_\Omega \eta^2|\Delta^h Du|^2 + C_\varepsilon\int_\Omega \eta^2|Du|^2|\Delta^h u|^2$$

and

$$III = \int_\Omega Df(Du)\Delta^h[\nabla\Pi(u)](\eta^2\Delta^h Du) \leq \varepsilon\ell\int_\Omega \eta^2|\Delta^h Du|^2 + C_\varepsilon\int_\Omega \eta^2|Df(Du)|^2|\Delta^h u|^2.$$

For the fourth we see that

$$\begin{aligned}
IV &= \int_\Omega \Delta^h[Df(Du)]\nabla\Pi(u)(2\eta D\eta\Delta^h\bar{u}) + Df(Du)\Delta^h[\nabla\Pi(u)](2\eta D\eta\Delta^h\bar{u})\\
&\leq \varepsilon\ell\int_\Omega \eta^2|\Delta^h Du|^2 + C_\varepsilon\int_\Omega |D\eta|^2|\Delta^h\bar{u}|^2 + \eta^2|Df(Du)|^2|\Delta^h\bar{u}|^2
\end{aligned}$$

and for the fifth we have

$$\begin{aligned} V &= \int_\Omega \Delta^h[Df(Du)]\nabla^2\Pi(u)Du\cdot\eta^2\Delta^h\bar{u} \\ &\quad + \int_\Omega Df(Du)\Delta^h[\nabla^2\Pi(u)]Du\cdot\eta^2\Delta^h\bar{u} + Df(Du)\nabla^2\Pi(u)\Delta^hDu\cdot\eta^2\Delta^h\bar{u} \\ &\leq \varepsilon\ell\int_\Omega \eta^2|\Delta^hDu|^2 + C_\varepsilon\int_\Omega \eta^2|Du|^2|\Delta^h\bar{u}|^2 \\ &\quad + C_\varepsilon\int_\Omega \eta^2|Df(Du)||Du||\Delta^h\bar{u}|^2 + \eta^2|Df(Du)|^2|\Delta^h\bar{u}|^2. \end{aligned}$$

In which case we conclude

$$\int_\Omega \eta^2|\Delta^hDu|^2 \leq C\int_\Omega \eta^2(1+|Du|^2)(|\Delta^hu|^2+|\Delta^h\bar{u}|^2) + |D\eta|^2|\Delta^h\bar{u}|^2. \tag{7.5}$$

Furthermore, by the construction of the cut-off function we find that

$$\int_\Omega |D\eta|^2|\Delta^h\bar{u}|^2 \leq C\frac{|h|^2}{|Q|^{\frac{2}{n}}}\int_{3Q}|Du-A|^2,$$

since $\Delta^hu(x) = \int_0^1 Du(x+th)\cdot h\,dt$. Similarly, we note that

$$\int_\Omega \eta^2(1+|Du|^2)|\Delta^h\bar{u}|^2 \leq C|h|^2|A|^2\int_{3Q}1+|Du|^2 + \int_\Omega \eta^2(1+|Du|^2)|\Delta^hu|^2$$

and by the higher integrability

$$\begin{aligned} \int_\Omega \eta^2(1+|Du|^2)|\Delta^hu|^2 &\leq C\Big(\int_{\operatorname{supp}\,\eta}1+|Du|^q\Big)^{\frac{2}{q}}\Big(\|u\|_\infty^{\frac{4-q}{\frac{q}{2}-2}q}\int_{\operatorname{supp}\,\eta}|\Delta^hu|^q\Big)^{1-\frac{2}{q}} \\ &\leq C|h|^{q-2}\int_{3Q}1+|Du|^q. \end{aligned}$$

□

We can now look to bound the Hausdorff dimension of the singular set for holonomic minima of general variational integrals of the form

$$\mathfrak{F}(u;\Omega) = \int_\Omega F(x,u,Du)dx$$

defined for Sobolev mappings $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathcal{M})$ into a connected compact submanifold $\mathcal{M}\subset\mathbb{R}^N$. We assume $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\mathbb{R}$ is a sufficiently regular integrand that satisfies the conditions

$$\xi\mapsto F(x,u,\xi) \quad \text{is } C^2\text{-function} \tag{7.6a}$$

$$\ell|\xi|^2 \leq F(x,u,\xi) \leq L(1+|\xi|^2) \tag{7.6b}$$

$$\ell|\lambda|^2 \leq D^2_{\xi\xi}F(x,u,\xi)(\lambda,\lambda) \leq L|\lambda|^2 \tag{7.6c}$$

$$|F(x_1,u_1,\xi)-F(x_2,u_2,\xi)| \leq L\omega_\alpha(|x_1-x_2|+|u_1-u_2|)(1+|\xi|^2) \tag{7.6d}$$

for any $x_i \in \Omega$, any $u_i \in \mathbb{R}^N$, any $\xi, \lambda \in \mathbb{R}^{Nn}$, where $\omega_\alpha(t) = \min(t^\alpha, 1)$ for some fixed $\alpha \in (0,1]$. By appealing to Lemma 7.5 we obtain the following singular set bound.

Theorem 7.6. *Let $\mathcal{M}$ be a connected compact C^3-Riemannian manifold without boundary such that the fundamental group $\pi_1(\mathcal{M})$ is finite. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be a holonomic $\mathfrak{F}$-minimiser under the assumption the integrand F satisfies* (7.6)*. If $Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for some $2 < q \leq 3$ then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n - \min\{\alpha, q-2\},$$

where $\Sigma_u \subset \Omega$ is a relatively closed subset characterised by

$$\Sigma_u = \Big\{ x \in \Omega : \liminf_{r \searrow 0} \fint_{B_r(x)} |Du - (Du)_{x,r}|^2 > 0 \quad \text{or} \quad \limsup_{r \searrow 0} |(Du)_{x,r}| = +\infty \quad \text{or} \quad \limsup_{r \searrow 0} |(u)_{x,r}| = +\infty \Big\}.$$

We remark that the higher integrability assumption is fulfilled by Lemma 6.4. Our approach is based on the unconstrained case of [KM2] by appealing to Corollary 7.2 via the following result of Campanato.

Lemma 7.7 ([C8, Lem. II.3])**.** *If $v \in L^q(Q(3\rho), \mathbb{R}^N)$, $q > 1$, and for $\theta \in (0,1)$*

$$\sum_{i=1}^n \int_{-2\rho}^{2\rho} \int_{Q(\rho)} \frac{|v(x+he_i) - v(x)|^q}{|h|^{1+\theta q}} dx dh < +\infty$$

then $v \in W^{\theta,q}(Q(\rho), \mathbb{R}^N)$ and

$$\int_{Q(\rho)} \int_{Q(\rho)} \frac{|v(x) - v(y)|^q}{|x-y|^{n+\theta q}} dx dy \leq c(n,p) \sum_{i=1}^n \int_{-2\rho}^{2\rho} \int_{Q(\rho)} \frac{|v(x+he_i) - v(x)|^q}{|h|^{1+\theta q}} dx dh,$$

where $Q(3\rho) = Q(x^0, 3\rho) = \{x : |x_i - x_i^0| < 3\rho,\ i = 1, \ldots, n\} \subset \Omega$.

To do so we first look to the following comparison lemmas that relates minimisers of the full problem with those of the so-called frozen integrand.

Lemma 7.8. *Let $\mathcal{M}$ be a connected compact manifold without boundary such that the fundamental group $\pi_1(\mathcal{M})$ is finite. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{M})$ be a holonomic $\mathfrak{F}$-minimiser under the assumption the integrand F satisfies* (7.6)*. Fix $x_0 \in \Omega$ and on a given fixed ball $B_R(x_0) \subset \Omega$, with $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, assume $u \in W^{1,q}(B_R(x_0), \mathcal{M})$ for some $2 < q < 3$. If $v \in (u + W^{1,2}_0(B_R(x_0), \mathbb{R}^N)) \cap W^{1,2}(B_R(x_0), \mathcal{M})$ satisfies*

$$\int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dv) \leq \int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dw) \tag{7.7}$$

for all $w \in (u + W_0^{1,2}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,2}(B_R(x_0), \mathcal{M})$*, then there exists a constant* $c = c(n, N, q, \ell, L) > 0$ *such that*

$$\int_{B_R(x_0)} |Du - Dv|^2 \leq cR^\sigma \int_{B_R(x_0)} 1 + |Du|^q,$$

where $\sigma = \min\{\alpha, q-2\}$.

Proof. First, by the minimality condition (7.7) and the assumption (7.6b) we have

$$\int_{B_R} |Dv|^2 \leq \frac{L}{\ell} \int_{B_R} 1 + |Du|^2,$$

while by appealing to the higher integrability up to the boundary result of Lemma 6.5 we also have

$$\int_{B_R} |Dv|^q \leq C \int_{B_R} 1 + |Du|^q. \tag{7.8}$$

If we denote $F^0(\xi) = F(x_0, (u)_{x_0,R}, \xi)$ then by (7.6c) we see that

$$\frac{1}{c} \int_{B_R} |Du - Dv|^2 \leq \int_{B_R} F^0(Du) - F^0(Dv) - DF^0(Dv)(Du - Dv).$$

From the stationary equation (7.4) we note that

$$\int_{B_R} DF^0(Dv)\big(\nabla\Pi(v)D\phi + \nabla^2\Pi(v)Dv \cdot \phi)\big) = 0$$

for any $\phi \in W_0^{1,2}(B_R, \mathbb{R}^N) \cap L^\infty(B_R, \mathbb{R}^N)$. Then from the identity

$$Dv - Du = \nabla\Pi(v)(Dv - Du) + (\nabla\Pi(v) - \nabla\Pi(u))Du$$

we see that

$$\begin{aligned}\int_{B_R} DF^0(Dv)(Dv - Du) = \int_{B_R} DF^0(Dv)\big((\nabla\Pi(v) - \nabla\Pi(u))Du\big) \\ - \int_{B_R} DF^0(Dv)\big(\nabla^2\Pi(v)Dv \cdot (v - u)\big).\end{aligned}$$

Hence we can bound

$$\begin{aligned}\int_{B_R} |DF^0(Dv)(Dv - Du)| &\leq c \int_{B_R} |DF^0(Dv)|(|Dv| + |Du|)|u - v| \\ &\leq c \int_{B_R} (1 + |Dv|^2 + |Du|^2)|u - v| \\ &\leq c\Big(\fint_{B_R} 1 + |Dv|^q + |Du|^q \Big)^{\frac{2}{q}} \Big(\|u - v\|_\infty^{\frac{3-q}{q-2}} \int_{B_R} |u - v|^q \Big)^{1-\frac{2}{q}} \\ &\leq cR^{q-2} \int_{B_R} 1 + |Dv|^q + |Du|^q\end{aligned}$$

by higher integrability, since $u - v \in W_0^{1,q}(B_R, \mathbb{R}^N)$ and $\mathcal{M}$ is compact. Similarly, we have

$$\begin{aligned}\int_{B_R} F^0(Du) - F^0(Dv) \leq &\int_{B_R} F(x_0, (u)_{x_0,R}, Du) - F(x, u, Du) \\ &+ \int_{B_R} F(x, u, Du) - F(x, v, Dv) \\ &+ \int_{B_R} F(x, v, Dv) - F(x_0, (u)_{x_0,R}, Dv) \\ &= I + II + III.\end{aligned}$$

By the minimality of u we see that $II \leq 0$ and by (7.6d) we bound

$$\begin{aligned}I + III \leq L &\int_{B_R} \omega_\alpha(|x - x_0| + |u - (u)_R|)(1 + |Du|^2) \\ &+ L \int_{B_R} \omega_\alpha(|x - x_0| + |v - u| + |u - (u)_R|)(1 + |Dv|^2).\end{aligned}$$

Now with $\sigma = \min\{\alpha, q - 2\}$ we find that

$$\begin{aligned}&\int_{B_R} \omega_\alpha(|x - x_0| + |u - (u)_R|)(1 + |Du|^2) \\ &\qquad \leq cR^\sigma \int_{B_R} R^{-2-\sigma} \omega_\alpha(|x - x_0| + |u - (u)_R|)^{\frac{2+\sigma}{\sigma}} + (1 + |Du|^2)^{\frac{2+\sigma}{2}} \\ &\qquad \leq cR^\sigma \int_{B_R} R^{-2-\sigma} |u - (u)_R|^{2+\sigma} + 1 + |Du|^{2+\sigma} \\ &\qquad \leq cR^\sigma \int_{B_R} 1 + |Du|^q,\end{aligned}$$

since $\omega_\alpha(t) \leq t^\alpha$, and likewise

$$\int_{B_R} \omega_\alpha(|x - x_0| + |v - u| + |u - (u)_R|)(1 + |Du|^2) \leq cR^\sigma \int_{B_R} 1 + |Du|^q + |Dv|^q.$$

The result now follow via the up to the boundary estimate (7.8). □

Lemma 7.9. *If $v \in (u + W_0^{1,2}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,2}(B_R(x_0), \mathcal{M})$, $B_R(x_0) \subset \Omega$, satisfies*

$$\int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dv) \leq \int_{B_R(x_0)} f(x_0, (u)_{x_0,R}, Dw)$$

for all $w \in (u + W_0^{1,2}(B_R(x_0), \mathbb{R}^N)) \cap W^{1,2}(B_R(x_0), \mathcal{M})$, then there exists $C = C(n, N, \ell, L) > 0$ such that

$$\int_{B_R(x_0)} |Dv - A|^2 \leq C \int_{B_R(x_0)} |Du - A|^2$$

for any matrix $A \in \mathbb{R}^{Nn}$.

Proof. First set $f^0(\xi) = f(x_0, (u)_{x_0,R}, \xi)$ and note that $\int_{B_R(x_0)} Df^0(A)(D\phi) = 0$ for any $\phi \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N)$. In particular we have

$$\int_{B_R(x_0)} Df^0(A)(Dv - A) = \int_{B_R(x_0)} Df^0(A)(Du - A)$$

which together with the minimality of v and (7.6c) imply

$$\begin{aligned}\frac{1}{c}\int_{B_R(x_0)} |Dv - A|^2 &\le \int_{B_R(x_0)} f^0(Dv) - f^0(A) - Df^0(A)(Dv - A)\\ &\le \int_{B_R(x_0)} f^0(Du) - f^0(A) - Df^0(A)(Du - A)\\ &\le c\int_{B_R(x_0)} |Du - A|^2. \qquad \square\end{aligned}$$

Proof of Theorem 7.6. With $\sigma = \min\{\alpha, q-2\}$ we first claim:

Claim 7.10. *$Du \in W^{\theta,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for $0 < \theta < \frac{\sigma}{\sigma+2}$ and, moreover, for every couple of open subset $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ we have*

$$\int_{\Omega''}\int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}} dxdy \le C\int_{\Omega'} 1 + |Du|^q$$

for some $C = C(n, N, \sigma, \theta, \ell, L, \mathrm{dist}(\Omega'', \partial\Omega')) > 0$.

To see this, fix a couple of open subset $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ then cover $\Omega'' \subset \bigcup_j Q_j$ by a grid of non-overlapping cubes Q_j. For each cube $Q_j = Q(x_0, \frac{|h|^\beta}{3\sqrt{n}})$ take a ball $B_j = B(x, |h|^\beta)$ for some $\beta > 0$ to be specified. In which case $\Omega'' \subset \bigcup_j Q_j \subset \bigcup_j B_j \subset \Omega' \subset\subset \Omega$. Observe also that the corresponding balls B_j have the bounded overlapping properties in the sense that B_j intersects at most $(12\sqrt{n})^n$ other such balls. Consequently, if we can show

$$\int_{Q_j} |\Delta^h Du|^2 \le C|h|^{\frac{2\sigma}{\sigma+2}} \int_{B_j} 1 + |Du|^q \tag{7.9}$$

for some $C = C(n, N, \sigma, \theta, \ell, L, \mathrm{dist}(\Omega'', \partial\Omega'))$ and any

$$0 < |h| < \min\{(\tfrac{\mathrm{dist}(\Omega'', \partial\Omega')}{10^6\sqrt{n}})^{\frac{1}{\beta}}, (\tfrac{1}{3\sqrt{n}})^{\frac{1}{1-\beta}}\} = \rho_0,$$

then for any $0 < \theta < \frac{\sigma}{\sigma+2}$ we can, by Lemma 7.7, bound

$$\begin{aligned}\int_{\Omega''}\int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}} dxdy &\le \sum_j \int_{Q_j}\int_{Q_j} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}} dxdy\\ &\le C\sum_j \int_{-2\rho_0}^{2\rho_0}\int_{Q_j} \frac{|\Delta^h Du(x)|^2}{|h|^{1+2\theta}} dxdh\\ &\le C\sum_j \int_{B_j} 1 + |Du|^q\end{aligned}$$

$$\leq C \int_{\Omega'} 1 + |Du|^q,$$

since $B_j \subset \Omega'$.

Now to show (7.9) let $v \in (u + W_0^{1,2}(B_j, \mathbb{R}^N)) \cap W^{1,2}(B_j, \mathcal{M})$ be defined as in Lemma 7.8. In which case we can estimate

$$\int_{Q_j} |\Delta^h Du|^2 \leq C \int_{Q_j} |\Delta^h Dv|^2 + C \int_{B_j} |Du - Dv|^2.$$

By Lemma 7.5 with $A = 0$ and the up to the boundary estimate (7.8) we see that

$$\int_{Q_j} |\Delta^h Dv|^2 \leq C|h|^{2-2\beta} \int_{B_j} 1 + |Du|^2 + C|h|^\sigma \int_{B_j} 1 + |Du|^q$$

and by Lemma 7.8 we have

$$\int_{B_j} |Du - Dv|^2 \leq C|h|^{\sigma\beta} \int_{B_j} 1 + |Du|^q.$$

So if we set $\beta = \frac{2}{\sigma+2}$ (i.e. $2 - 2\beta = \sigma\beta$) then

$$\int_{Q_j} |\Delta^h Du|^2 \leq C|h|^{\frac{2\sigma}{\sigma+2}} \int_{B_j} 1 + |Du|^q$$

as desired.

Claim 7.11. If $Du \in W^{\theta,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for $0 < \theta < \frac{\sigma}{\sigma+2}$ and for every couple of open subset $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ there exists $C = C(n, N, \sigma, \theta, \ell, L, \mathrm{dist}(\Omega'', \partial\Omega')) > 0$ such that

$$\int_{\Omega''} \int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x - y|^{n+2\theta}} dxdy \leq C \int_{\Omega'} 1 + |Du|^q,$$

then $Du \in W^{\lambda,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for $0 < \lambda < \frac{\sigma}{\sigma+2(1-\theta)}$ and, moreover, for every couple of open subset $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ we have

$$\int_{\Omega''} \int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x - y|^{n+2\lambda}} dxdy \leq C \int_{\Omega'} 1 + |Du|^q$$

for some $C = C(n, N, \sigma, \theta, \lambda, \ell, L, \mathrm{dist}(\Omega'', \partial\Omega')) > 0$.

Proceeding as in the previous claim with the same notation we revise the estimate for the comparison function v. To do so we recall the extension operator $E\colon W^{1,q}(B_j, \mathbb{R}^N) \to W_0^{1,q}(2B_j, \mathbb{R}^N)$ such that $Ew = w$ on B_j with $\|Ew\|_{1,q;2B_j} \leq C\|w\|_{1,q;B_j}$ for all $w \in W^{1,q}(B_j)$. Then for some $\Lambda > 0$ to be specified there exists a Lipschitz approximation $U_\Lambda \in W_0^{1,\infty}(2B_j, \mathbb{R}^N)$ of $U = Eu \in W_0^{1,q}(2B_j, \mathbb{R}^N)$ by Lemma 2.8. Using the higher integrability we note that

$$\int_{2B_j} |DU - DU_\Lambda|^2 = \int_{2B_j} \mathbf{1}_{\{U \neq U_\Lambda\}} |DU - DU_\Lambda|^2 \leq C\Lambda^{-(q-2)} \int_{B_j} 1 + |Du|^q$$

by the properties of extension operator and Lemma 2.8. So from Lemma 7.5 with now $A = (DU_\Lambda)_{B_j}$ we find by (7.8) and Lemma 7.9 that

$$\begin{aligned}\int_{Q_j} |\Delta^h Dv|^2 &\le C|h|^{2-2\beta}\int_{B_j} |Du - A|^2 + C(|h|^{q-2} + |h|^2|A|^2)\int_{B_j} 1 + |Du|^q \\ &\le C|h|^{2-2\beta}\int_{B_j} |Du - (Du)_{B_j}|^2 + C|h|^{2-2\beta}\int_{B_j} |DU - DU_\Lambda|^2 \\ &\quad + C(|h|^{q-2} + |h|^2\Lambda^2)\int_{B_j} 1 + |Du|^q \\ &\le C|h|^{2-2\beta}\int_{B_j} |Du - (Du)_{B_j}|^2 \\ &\quad + C(|h|^{2-2\beta}\Lambda^{-(q-2)} + |h|^2\Lambda^2 + |h|^{q-2})\int_{B_j} 1 + |Du|^q,\end{aligned}$$

since $|A| \le \|DU_\Lambda\|_\infty \le \Lambda$. In which case if we set $\Lambda = |h|^{-\alpha}$ for some $\alpha > 0$ to be specified, then by Lemma 7.8 and the Poincaré inequality (7.1) we have

$$\begin{aligned}\int_{Q_j} |\Delta^h Du|^2 &\le C\int_{Q_j} |\Delta^h Dv|^2 + C\int_{B_j} |Du - Dv|^2 \\ &\le C|h|^{2-2\beta+2\theta\beta}\int_{B_j}\int_{B_j} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}}dxdy \\ &\quad + C(|h|^{2-2\beta+\alpha\sigma} + |h|^{2-2\alpha} + |h|^{\sigma\beta})\int_{B_j} 1 + |Du|^q.\end{aligned}$$

Now if we set $\alpha = \frac{4}{(\sigma+2)(\sigma+2(1-\theta)}$ and $\beta = \frac{2}{\sigma+2(1-\theta)}$ (i.e. if $2 - 2\beta + 2\theta\beta = \sigma\beta$ and $2 - 2\beta + \alpha\sigma = 2 - 2\alpha$) then

$$\begin{aligned}\int_{Q_j} |\Delta^h Du|^2 &\le C|h|^{\frac{2\sigma}{\sigma+2(1-\theta)}}\int_{B_j}\int_{B_j} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}}dxdy \\ &\quad + C(|h|^{2-\frac{8}{(\sigma+2)(\sigma+2(1-\theta)}} + |h|^{\frac{2\sigma}{\sigma+2(1-\theta)}})\int_{B_j} 1 + |Du|^q \\ &\le C|h|^{\frac{2\sigma}{\sigma+2(1-\theta)}}\Big(\int_{B_j}\int_{B_j} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}}dxdy + \int_{B_j} 1 + |Du|^q\Big)\end{aligned}$$

since $2 - \frac{8}{(\sigma+2)(\sigma+2(1-\theta)} > \frac{2\sigma}{\sigma+2(1-\theta)}$ if and only if $\theta < \frac{\sigma}{\sigma+2}$. So by Lemma 7.7 and the above covering argument we find that $Du \in W^{\lambda,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$, for $0 < \lambda < \frac{\sigma}{\sigma+2(1-\theta)}$, with

$$\begin{aligned}\int_{\Omega''}\int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x-y|^{1+2\lambda}}dxdy &\le C\sum_j \Big(\int_{B_j}\int_{B_j} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\theta}}dxdy + \int_{B_j} 1 + |Du|^q\Big) \\ &\le C\int_{\Omega'} 1 + |Du|^q\end{aligned}$$

by the initial hypothesis which holds for arbitrary Ω' and Ω''.

Having taken these two claims into consideration now consider the monotone increasing function $H(t) = \frac{\sigma}{\sigma+2(1-t)}$ for $0 \leq t \leq 1$ which has $H(\frac{\sigma}{2}) = \frac{\sigma}{2}$ and $t < H(t) < \frac{\sigma}{2}$ for all $0 \leq t < \frac{\sigma}{2}$. By defining the sequences $\{\theta_j\}$ and $\{\gamma_j\}$ inductively by

$$\theta_j = H(\theta_{j-1}), \qquad \theta_0 \in (0, \tfrac{\sigma}{\sigma+2})$$
$$\gamma_j = \frac{\theta_j + H(\gamma_{j-1})}{2}, \qquad \gamma_0 = \frac{\theta_0}{2} + \frac{\sigma}{2(\sigma+2)} < \frac{\sigma}{\sigma+2}$$

we see that $0 < \theta_j < \frac{\sigma}{2}$ with $\theta_j \nearrow \frac{\sigma}{2}$ and likewise $\theta_j < \gamma_j < \frac{\sigma}{2}$ with $\gamma_j \nearrow \frac{\sigma}{2}$ by induction. Next we prove inductively that $Du \in W^{\gamma_j,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for all integers $j \geq 0$ and that for every couple of open subset $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ there exists a constant $C = C(n, N, q, \ell, L, \alpha, \mathrm{dist}(\Omega'', \partial\Omega'))$ such that

$$\int_{\Omega''}\int_{\Omega''} \frac{|Du(x) - Du(y)|^2}{|x-y|^{n+2\gamma_j}} dxdy \leq C \int_{\Omega'} 1 + |Du|^q.$$

The case $j = 0$ follows immediately from Claim 7.10. If we assume the assertion holds for $j-1$ then by Claim 7.11 we have $Du \in W^{\lambda,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ for all $0 < \lambda < H(\gamma_{j-1})$. Since H is increasing it follows that $\theta_j < H(\gamma_{j-1})$ which implies $\gamma_j < H(\gamma_{j-1})$ by the definition of γ_j. Then with $\theta = \gamma_{j-1}$ and $\lambda = \gamma_j$ in Claim 7.11 together with the induction hypothesis yields the result.

The theorem now follows by Corollary 7.2, since we have shown $Du \in W^{\theta,2}_{\mathrm{loc}}$ for all $0 < \theta < \frac{\sigma}{2}$. □

Appendix A

Examples of ordered media with broken symmetries

One of the interests, in particular, for considering manifold valued mapping $u\colon \Omega \to \mathcal{M}$ that minimise an integral functional of the form

$$\int_\Omega f(x, u, Du)dx,$$

for a sufficiently general integrand f, is motivated by several problems from condensed matter physics (including ferromagnets, nematic liquid crystals and superfluid helium-3). For such problems the role of 'broken symmetries' is pivotal—as it describes the property of a macroscopic system that is in a state which does not have the full symmetry of the microscopic dynamics. From a group-theoretical point of view a 'broken symmetry' means that below a transition the lowest state of the system is no longer invariant under the full group G of the symmetry transformations but only under a subgroup $H \subset G$. The subgroup H describes the remaining (residual) symmetry of the ordered state. Hence the group G may be factorised with respect to H by

$$G = R \times H,$$

where the coset-space $R = G/H$ contains all transformations that change the ground state of the system (note that R will only be a group itself if H is normal subgroup of G). In other words R describes the 'broken symmetries' of the system and constitutes a 'manifold of internal states'. We consider the above manifold $\mathcal{M}$ as the quotient space G/H in such situations.

The concept of symmetry breaking goes back to Pierre Curie [C16] who proposed the original symmetry group of the medium must be lowered (broken, in today's terminology) to the symmetry group of the phenomenon (or to a subgroup of the phenomenon's

symmetry group) by the action of some cause in order for the occurrence of a phenomenon in a medium. In this sense symmetry breaking is what 'creates the phenomenon'. A heuristic example is provided by Abdus Salam's analogy of a dinner party at which the guests are seated around a circular table with a napkin placed between each pair of neighbours. The table setting is symmetrical until someone takes a napkin either from the left or right side. After that the symmetry is broken; the other guests can no longer choose between left or right napkins as the left-right symmetry is spontaneously broken.

A.1. Ferromagnetism

One of the best-known examples of broken symmetries is that of ferromagnetism. It was discovered by Pierre Curie that above a certain critical temperature T_c ferromagnetic materials lose their magnetic properties, i.e. the ability to have a net magnetisation in the absence of an external magnetic field. The 1928 theory of Heisenberg [H8] showed an array of spin $\frac{1}{2}$ magnetic dipoles, with spin-spin interactions between nearest neighbours favouring energetically parallel over antiparallel configurations, have a ground state in which the spins are aligned in some particular direction (i.e. a magnetisation pointing in that direction) so that the rotational symmetry of the dynamics is broken. The order parameter for a ferromagnet is the magnetisation $\boldsymbol{M}$; a real three-component vector. The bulk free energy, expanded in powers of the order parameter, is given by $F = \alpha\, \boldsymbol{M} \cdot \boldsymbol{M} + \beta\, (\boldsymbol{M} \cdot \boldsymbol{M})^2$. In the paramagnetic state the system is invariant under three-dimensional rotations in space, i.e. $G = SO(3)$, while in the ferromagnetic states only rotations around the spontaneous magnetisation $\boldsymbol{M}$, described by $H = SO(2)$ $(= U(1))$, leave the ground state invariant. In which case the broken symmetries, corresponding to any rotation changing the direction of $\boldsymbol{M}$, are $R = SO(3)/SO(2) = S^2$, i.e. the rotations represented by the 2-sphere.

A.2. Uniaxial nematic liquid crystals

A liquid crystal is a state of matter, called mesomorphic, intermediate between the ordered state of matter in crystals and the disordered state in ordinary liquids, in which the molecules retain preferential orientations relative to one another over large scales. There are three broad categories which classify liquid crystals—namely those of nematics, cholesterics and smectics phases [V]. In the constrained uniaxial nematic[1] phase the director approach to continuum modelling sees rod-like molecules represented by a unit vector $\boldsymbol{n} \in S^2$ with head-to-tail symmetry. The elastic Oseen-Frank energy associated to

[1]From νῆμα, *nēma*—that which is spun, thread, yarn.

a given configuration is given by

$$\mathcal{W}(\boldsymbol{n};\Omega)=\int_\Omega w(\boldsymbol{n},\nabla\boldsymbol{n})dx,$$

where the integrand (derived by Frank [F4] as a consequence of Galilean invariance with anticipations in the molecular theory given by Oseen [O]) is of the form

$$\begin{aligned} w(\boldsymbol{n},\nabla\boldsymbol{n}) &= \kappa_1(\operatorname{div}\boldsymbol{n})^2+\kappa_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2+\kappa_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2\\ &\quad+(\kappa_2+\kappa_4)(\operatorname{tr}(\nabla\boldsymbol{n})^2-(\operatorname{div}\boldsymbol{n})^2) \end{aligned} \tag{A.1}$$

such that the temperature dependent constants κ_i are assumed to satisfy

$$\kappa_1,\kappa_2,\kappa_3>0,\quad \kappa_2\geq|\kappa_4|,\quad 2\kappa_1\geq\kappa_2+\kappa_4.$$

Note that the respective energy contributions due to splay, twist and bend-like spatial changes of $\boldsymbol{n}$ are represented by the terms $(\operatorname{div}\boldsymbol{n})^2$, $(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2$ and $|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2$ respectively; the last term is a null-Lagrangian, i.e. $\operatorname{tr}(\nabla\boldsymbol{n})^2-(\operatorname{div}\boldsymbol{n})^2=\operatorname{div}((\nabla\boldsymbol{n})\boldsymbol{n}-(\operatorname{div}\boldsymbol{n})\boldsymbol{n})$. With this the integrand satisfies the invariance properties

$$w(\boldsymbol{n},\nabla\boldsymbol{n})=w(-\boldsymbol{n},-\nabla\boldsymbol{n}),\qquad w(\boldsymbol{n},\nabla\boldsymbol{n})=w(M\boldsymbol{n},M\nabla\boldsymbol{n}M^T) \tag{A.2}$$

for all $M\in SO(3)$. The first accounts for the lack of polarity of nematics while the second expresses the frame invariance and condition of material symmetry corresponding to the lack of chirality of nematics.[2] Existence and partial regularity of the minima in the case $\boldsymbol{n}\in S^2$ can be found in [HKL2, HKL3]. An extension of this model has been formulated by Ericksen [E2] which also takes into account a non-constant scalar order parameter—so that the state of the liquid crystal is described by a pair $(s,\boldsymbol{n})\in\mathbb{R}\times S^2$ depending on $x\in\Omega$.

A.3. Biaxial nematic liquid crystals

According to the continuum description in the Landau-de Gennes theory [dGP] the state of alignment of a nematic liquid crystal in a domain $\Omega\subset\mathbb{R}^3$ is characterised pointwise by a tensor order parameter Q as a symmetric traceless 3×3 real matrix. In a general nematic phase the tensor order parameter has five degrees of freedom (two specifying the degree order with three for the principal directions) so that

$$Q=s_1(\boldsymbol{n}\otimes\boldsymbol{n}-\tfrac{1}{3}I)+s_2(\boldsymbol{m}\otimes\boldsymbol{m}-\tfrac{1}{3}I), \tag{A.3}$$

[2] In fact, as $\boldsymbol{n}$ and $-\boldsymbol{n}$ represent the same physical state the order parameter space is taken to be $\mathbb{R}P^2=S^2/\mathbb{Z}_2$ (cf. Example 6.25).

where $s_1, s_2 \in \mathbb{R}$ and $\boldsymbol{n}, \boldsymbol{m} \in S^2$ are orthonormal eigenvectors of Q. In terms of the eigenvalues λ_1, λ_2 and λ_3 of Q the scalar order parameters s_1, s_2 are related by $s_1 = \lambda_1 - \lambda_3$, $s_2 = \lambda_2 - \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. A nematic liquid crystal is said to be isotropic if $Q = 0$; uniaxial when the Q-tensor has two equal non-zero eigenvalues; and, biaxial when Q has three distinct eigenvalues (biaxiality implies the existence of more than one preferred direction of molecular alignment). While the existence of uniaxial nematics has been known for more than a century the unambiguous experimental evidence of the existence of a biaxial nematic phase in a thermotropic liquid crystal is relatively recent [MDNS, APK, SS, MKV$^+$, FCF$^+$].

The Landau-de Gennes free-energy functional [dG2] taking into consideration spatial variations of the local order parameter is of the form

$$\mathcal{F}(Q;\Omega) = \int_\Omega \psi(Q, \nabla Q) + f_B(Q).$$

A typical expansion of elastic free-energy density yields

$$\psi(Q, \nabla Q) = L_1 I_1 + L_2 I_2 + L_3 I_3 + L_4 I_4 \tag{A.4}$$

with material constants L_i and elastic invariants I_i given by

$$I_1 = Q_{ij,j} Q_{ik,k} \qquad I_2 = Q_{ik,j} Q_{ij,k} \qquad I_3 = Q_{ij,k} Q_{ij,k} \qquad I_4 = Q_{\ell k} Q_{ij,\ell} Q_{ij,k}$$

where $Q_{ij,k} = \frac{\partial}{\partial x^k} Q_{ij}$ (as usual, summation over repeated indices is assumed). Similarly, the bulk free-energy $f_B(Q) = f_B(\mathrm{tr}(Q^2), \mathrm{tr}(Q^3))$ expanded to fourth order takes the form

$$f_B(Q) = -\tfrac{a^2}{2} \mathrm{tr}\ Q^2 - \tfrac{b^2}{3} \mathrm{tr}\ Q^3 + \tfrac{c^2}{4} (\mathrm{tr}\ Q^2)^2$$

where $a, b, c \in \mathbb{R}$ are material and temperature dependent constants. Higher order expansions of the elastic and bulk terms can be found in [ST, BM, PS, LMT, VSV].

In general the free-energy density ψ satisfies a frame invariance condition of the form

$$\psi(Q, \nabla Q) = \psi(MQM^T, D^*) \tag{A.5}$$

for all $M \in SO(3)$, where $D^*_{ijk} = M^a_i M^b_j M^c_k Q_{ab,c}$. The bulk energy f_B is invariant under the $SO(3)$-action by conjugation on the five-dimensional space of Q-tensors [LT]. Critical points of the bulk energy form an orbit of solutions in the five-dimensional space of Q-tensors. The $SO(3)$-orbit corresponding to the general case of a biaxial minimiser is a 3-manifold, while in the special case corresponding to a uniaxial minimiser the orbit reduces to a 2-manifold. Such Q taking values in a group orbit have constant scalar order parameters s_1, s_2 and constant bulk energy. We refer to such a situation as the *constrained theory of biaxial nematic liquid crystals*.

The set of all Q-tensors in the form (A.3) with three distinct constant eigenvalues $-\frac{1}{3} \leq \lambda_1 < \lambda_2 < \lambda_3 \leq \frac{2}{3}$ independent of $x \in \Omega$ can be formally identified with

$$SO(3) * A = \{GAG^T \in \mathrm{Sym}_0^2(\mathbb{R}^3) : G \in SO(3)\},$$

where $A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\mathrm{Sym}_0^2(\mathbb{R}^3)$ denotes the space of symmetric traceless 3×3 matrices (cf. [MN, N8]). This is clearly the orbit of the matrix A with respect to the left action $*$ of $SO(3)$ on $\mathrm{Sym}_0^2(\mathbb{R}^3)$ given by

$$G * Q = GQG^T$$

for $G \in SO(3)$, $Q \in \mathrm{Sym}_0^2(\mathbb{R}^3)$. Since the eigenvalues are distinct the isotropy subgroup of $SO(3)$ which fixes A, i.e.

$$SO(3)_A = \{G \in SO(3) : G * A = A\},$$

is readily seen to be the abelian 4-element dihedral group D_2. The coset space given by $SO(3)/SO(3)_A = SO(3)/D_2$ can be given the structure of smooth manifold, since the bijective map $SO(3)/SO(3)_A \to SO(3) * A$ sending $[G] \mapsto GAG^T$ is a diffeomorphism. Using the 2:1 covering $\Phi\colon \mathrm{Spin}(3) \simeq S^3 \to SO(3)$ one finds the preimage of D_2 in S^3 coincides with the non-abelian 8-element quaternion group $\mathbb{H} = \langle -1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} | (-1)^2 = 1, \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1\rangle$. Hence the space of all Q-tensors in the constrained biaxial nematic case can be identified with

$$S^3/\mathbb{H} = \{q\,\mathbb{H} : q \in \mathrm{Spin}(3)\}$$

endowed with a unique Riemannain structure so that the canonical projection $\Pi\colon S^3 \to S^3/\mathbb{H}$ is a Riemannian covering map. As S^3 is simply connected Φ is a universal covering and $\pi_1(S^3/\mathbb{H}) = \mathbb{H}$ acts isometrically on S^3.

Given this one can express ψ in terms of a map $(u, \boldsymbol{v})\colon \Omega \to S^3$, i.e. there is an $F\colon S^3 \times \mathbb{R}^{4\times 3} \to \mathbb{R}$ such that

$$\psi(Q, \nabla Q) = F((u, \boldsymbol{v}), \nabla(u, \boldsymbol{v})) \tag{A.6}$$

provided Q corresponds to $(u, \boldsymbol{v})$ in the sense that $Q = GAG^T$ with $G = \Phi(u, \boldsymbol{v})$. In which case we can define the $\mathcal{F}$-energy of a Sobolev map $Q \in W^{1,2}(\Omega, S^3/\mathbb{H})$ by the equivalent energy (A.6) of any map $(u, \boldsymbol{v}) \in W^{1,2}(\Omega, S^3)$ such that $Q = \Pi(u, \boldsymbol{v})$. The frame invariance of (A.5) is equivalent to the condition

$$F(w, M) = F(qw, L(q)M\Phi(q)^T)$$

for all $(w, M) \in S^3 \times \mathbb{R}^{4\times 3}$ and $q \in S^3$, where $L(q)$ represents the real linear maps $w \mapsto qw$ relative to the standard basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. In addition, there is also the following residual

symmetry

$$F(w, M) = F(qw, L(q)M)$$

for all $(w, M) \in S^3 \times \mathbb{R}^{4\times 3}$ and $q \in S^3$. Both this and the frame invariance condition are the counterparts to the conditions (A.2) satisfied by the energy density of uniaxial nematics in the sense of Oseen-Frank. For example the Dirichlet energy in the biaxial case can be expressed, by writing $Q = G\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)G^T$ with $G = \begin{pmatrix} \boldsymbol{n} & \boldsymbol{m} & \boldsymbol{\ell} \end{pmatrix} = \Phi(u, \boldsymbol{v})$ for $(u, \boldsymbol{v}) \in S^3$, as

$$\begin{aligned}\int_\Omega Q_{ij,k}Q_{ij,k} &= 2\int_\Omega s_1(s_1 - s_2)|\nabla \boldsymbol{n}|^2 + s_2(s_2 - s_1)|\nabla \boldsymbol{m}|^2 + s_1 s_2|\nabla \boldsymbol{\ell}|^2 \\ &= \int_\Omega F_{\mathrm{Dir}}((u, \boldsymbol{v})(x), \nabla(u, \boldsymbol{v})(x))dx,\end{aligned}$$

where $F_{\mathrm{Dir}}\colon S^3 \times \mathbb{R}^{4\times 3} \to [0, +\infty)$ is, according to [MN, §5], given by

$$F_{\mathrm{Dir}}((u, \boldsymbol{v}), H) = (k_1 + k_2 + k_3)|H|^2 - k_1A^2 - k_2B^2 - k_3C^2$$

with $A = \sum_{i=1}^3(-v_1H_i^0 + uH_1^1 + v_3H_i^2 - v_2H_i^3)^2$, $B = \sum_{i=1}^3(-v_2H_i^0 - v_3H_1^1 + uH_i^2 + v_1H_i^3)^2$, $C = C(u, \boldsymbol{v}) = \sum_{i=1}^3(-v_3H_i^0 + v_2H_1^1 - v_1H_i^2 + uH_i^3)^2$ and $k_1 = 8s_1(s_1 - s_2)$, $k_2 = 8s_2(s_2 - s_1)$, $k_3 = 8s_1s_2$.

In the constrained uniaxial case one has

$$Q = s(\boldsymbol{n} \otimes \boldsymbol{n} - \tfrac{1}{3}I)$$

for $s \in \mathbb{R}$ and $\boldsymbol{n} \in S^2$ from which one has

$$\begin{aligned} I_1 &= s^2((\mathrm{div}\ \boldsymbol{n})^2 + |\boldsymbol{n} \times \mathrm{curl}\ \boldsymbol{n}|^2) \\ I_2 &= s^2(|\boldsymbol{n} \times \mathrm{curl}\ \boldsymbol{n}|^2 + \mathrm{tr}(\nabla \boldsymbol{n})^2) \\ I_3 &= 2s^2(\mathrm{tr}(\nabla \boldsymbol{n})^2 + (\boldsymbol{n} \cdot \mathrm{curl}\ \boldsymbol{n})^2 + |\boldsymbol{n} \times \mathrm{curl}\ \boldsymbol{n}|^2) \\ I_4 &= 2s^3(\tfrac{2}{3}|\boldsymbol{n} \times \mathrm{curl}\ \boldsymbol{n}|^2 - \tfrac{1}{3}\mathrm{tr}(\nabla \boldsymbol{n})^2 - \tfrac{1}{3}(\boldsymbol{n} \cdot \mathrm{curl}\ \boldsymbol{n})^2). \end{aligned}$$

By setting $\kappa_1 = L_1s^2 + L_2s^2 + 2L_3s^2 - \frac{2}{3}L_4s^3$, $\kappa_2 = 2L_3s^2 - \frac{2}{3}L_4s^3$, $\kappa_3 = L_1s^2 + L_2s^2 + 2L_3s^2 + \frac{4}{3}L_4s^3$, $\kappa_4 = L_2s^2$ we find that

$$\psi(Q, \nabla Q) = w(\boldsymbol{n}, \nabla \boldsymbol{n}),$$

where ψ is given by (A.4) and w is given by (A.1). Furthermore, the isotropy subgroup of A is isomorphic to the orthogonal group $O(2)$ (since two of the eigenvalues λ_1, λ_2, λ_3 are equal) so that the coset space $SO(3)/O(2) = SO(3)/D_\infty$, where D_∞ is the group of all rotations about an axis and π-rotations about axes perpendicular to that axis, is diffeomorphic to $\mathbb{R}P^2$.

A.4. Superfluidity of liquid helium-3

There are two stable isotopes of the chemical element helium, namely ^{3}He and ^{4}He. At 1 atm helium-3 boils at 3.19 K compared with helium-4 at 4.23 K. Below these temperatures one finds helium liquids, unlike all other known liquids, do not solidify unless a pressure of around 30 bar is applied—under less pressure the small atomic mass and the weakness of the attractive interactions prevents the formation of a crystalline solid with a rigid lattice structure. This is the first indication of the remarkable macroscopic quantum effects in these systems. It is this very property of helium (staying liquid) that makes it such a valuable system for observing quantum behaviour on a macroscopic scale. Quantum effects are also responsible for the strikingly different behaviours of ^{3}He and ^{4}He at lower temperatures: the superfluid transition temperature of helium-4 is 2.17 K at 1 atm whereas for helium-3 it is below 3 mK and in fact forms several superfluid phases (when the superfluid phases of ^{3}He were finally discovered [ORL, OGRL] at temperatures of 2.6 mK and 1.8 mK respectively—in an experiment actually designed to observe a magnetic phase transition in solid ^{3}He—the results came as a great surprise).

With respect to the laws of quantum statistics the ^{4}He atom—being composed of an even number of electrons and nucleons—has spin zero and consequently obeys Bose-Einstein statistics. At low temperatures this system has a tendency to condense into the lowest-energy single-particle state (Bose-Einstein condensation). The qualitative feature of the Bose condensate is its phase rigidity, i.e. the fact that it is energetically favourable for the particles to condense into a single-particle state of fixed quantum-mechanical phase such that the global gauge symmetry is spontaneously broken. As a consequence the macroscopic flow of the condensate gives rise to the phenomenon of superfluidity. For superfluid helium-4 the order parameter space is S^1, i.e. the phase factor.

By contrast the total spin of the ^{3}He atom is $\frac{1}{2}$ hence liquid helium-3 obeys Fermi-Dirac statistics. In a Fermi system the Pauli exclusion principle allows only single occupation of fermion states. As predicted by Landau [L5, L6, L7] the properties of helium-3 well below its Fermi temperature (of approximately 1 K) are similar to those of a degenerate Fermi gas. In particular the formation of a phase-rigid condensate is not possible in this framework, so much so that until the mid 1950s superfluid phases of liquid helium-3 were thought to be ruled out. Nonetheless the property of superfluidity was in fact first discover experimentally in a Fermi system, namely that of the 'liquid' of conduction electrons in a superconducting metal [KO]. The subsequent theory of superconductivity [BCS] sees the formation of 'Cooper pairs', i.e. pairs of electrons with opposite momentum and spin projection, as central to the description. In superconductors the Cooper pairing occurs in a state with spins $S = 0$ and angular momenta $\ell = 0$ known as a 'spin-singlet s-wave

state'. By contrast the superfluid phases of helium-3 are characterised by a condensation of Cooper pairs into a state with relative angular momentum $\ell = 1$ and total spin $S = 1$, i.e. the so-called 'spin-triplet p-wave pairing'. This implies an anisotropic pair wave function and, generally, leads to anisotropic properties of the system.

Quantum mechanically a spin-triplet configuration $S = 1$ of two particles has three substates corresponding to the three values of spin projection S_z. These are represented by $|\uparrow\uparrow\rangle$ with $S_z = +1$, $\frac{1}{\sqrt{2}}|\uparrow\downarrow + \downarrow\uparrow\rangle$ with $S_z = 0$ and $|\downarrow\downarrow\rangle$ with $S_z = -1$. The pair wave function Φ is in general a linear superposition of all three spin substates, i.e.

$$\Phi = \varphi_{1,+}|\uparrow\uparrow\rangle + \varphi_{1,0}|\uparrow\downarrow + \downarrow\uparrow\rangle + \varphi_{1,-}|\downarrow\downarrow\rangle$$

where $\varphi_{1,+}$, $\varphi_{1,0}$, $\varphi_{1,-}$ are the complex-valued amplitudes of the respective states. The Fourier transforms $F_{\boldsymbol{k}\uparrow\uparrow}$, $F_{\boldsymbol{k}\uparrow\downarrow}$, $F_{\boldsymbol{k}\downarrow\downarrow}$ of the amplitudes can be regarded as the components of a symmetric matrix $F_{\boldsymbol{k}\alpha\beta}$ $(\alpha, \beta = \uparrow, \downarrow)$ which can be expressed as the anomalous averages of annihilation operators $a_{\boldsymbol{k}\alpha}$, with momentum $\boldsymbol{k}$ and spin α, according to $F_{\boldsymbol{k}\alpha\beta} = \langle a_{-\boldsymbol{k}\alpha} a_{\boldsymbol{k}\beta}\rangle$. The correlator $F_{\boldsymbol{k}\alpha\beta}$ is the order parameter of superfluid in a Fermi system. The 'gap' parameter $\Delta_{\boldsymbol{k}\alpha\beta}$ which describes the quasiparticle energy spectrum is expressed through $F_{\boldsymbol{k}\alpha\beta}$ and a pair potential. Both $F_{\boldsymbol{k}\alpha\beta}$ and $\Delta_{\boldsymbol{k}\alpha\beta}$ are symmetric matrices determined by three independent elements corresponding to the projections of $S_z = -1, 0, 1$. Rather than takes the amplitudes of $\Delta_{\boldsymbol{k}}$ and $F_{\boldsymbol{k}}$ corresponding to these projections it is more convenient to combine the three spin components into a complex vector $\boldsymbol{d}(\boldsymbol{k})$ in spin space. This is achieved by expanding $\Delta_{\boldsymbol{k}}$ in the three symmetric 2×2 matrices $i\sigma_\mu\sigma_2$ $(\mu = 1, 2, 3)$ by letting

$$\Delta_{\boldsymbol{k}\alpha\beta} = \Delta \sum_\mu d_\mu(\boldsymbol{k})(i\sigma_\mu\sigma_2)_{\alpha\beta},$$

where σ_μ are the Pauli matrices. We note that the vector $\boldsymbol{d}(\boldsymbol{k})$ can be represented by a linear combination of the components $\hat{k}_j$ of the unit vector $\hat{\boldsymbol{k}}$ in momentum space:

$$d_\mu(\boldsymbol{k}) = A_{\mu j}\hat{k}_j.$$

The resulting nine complex quantities $A_{\mu j}$ (for given spin index μ and orbital index j), considered as 3×3 matrix, imply the order parameter has 18 real degrees of freedom *a priori*. Clearly, the description of the system in terms of 18 real numbers is exceedingly heavy.

In regions near the critical temperate T_c, i.e. the Ginzburg-Landau regime, the bulk free energy f_B can be obtained from $A_{\mu j}$ by contracting all indices in second and fourth order terms that are invariant against rotation in spin space and in ordinary orbital space separately as well as of course being gauge-invariant, i.e. invariant up to a phase factor. In other words f_B expanded to fourth order must be invariant with respect to the symmetry

group

$$G = SO(3)_{\boldsymbol{S}} \times SO(3)_{\boldsymbol{L}} \times U(1)_{\phi} \tag{A.7}$$

where $\boldsymbol{S}$, $\boldsymbol{L}$, ϕ indicate the spin space, orbital space, and gauge respectively. The resulting expansion [MS] is given by

$$\begin{aligned} f_B = &-\alpha\, A_{\mu j}{}^* A_{\mu j} + \beta_1\, A_{\mu j}{}^* A_{\mu j}{}^* A_{\nu\ell} A_{\nu\ell} + \beta_2\, A_{\mu j}{}^* A_{\mu j} A_{\nu\ell}{}^* A_{\nu\ell} \\ &+ \beta_3\, A_{\mu j}{}^* A_{\nu j}{}^* A_{\mu\ell} A_{\nu\ell} + \beta_4\, A_{\mu j}{}^* A_{\nu j} A_{\nu\ell}{}^* A_{\mu\ell} + \beta_5\, A_{\mu j}{}^* A_{\nu j} A_{\nu\ell} A_{\mu\ell}{}^* \end{aligned}$$

where the coefficients α and β_i depend on external parameters like temperature and pressure. Note that if $A_{\mu j}$ minimises f_B then also

$$A'_{\mu j} = e^{i\varphi} R^{(1)}_{\mu\nu} R^{(2)}_{jk} A_{\nu k},$$

where $R^{(1)}$ and $R^{(2)}$ are rotation matrices in the respective spin and orbital spaces. In addition, the condensate at the same time will also oppose any spatial variations of its long-range order. Any 'bending' of the order parameter field will therefore increase the energy, thus giving an internal stiffness or rigidity to the system that is in opposition to the local orientational effects. Hence if one allows spatial variations in the order parameter the gradient energy of an anisotropic superfluid has leading terms quadratic in the gradient that also satisfy the above invariance (A.7) of the form

$$f_{\text{grad}} = \gamma_1\, \partial_j A_{\mu\ell} \partial_j A_{\mu\ell}{}^* + \gamma_2\, \partial_j A_{\mu\ell} \partial_\ell A_{\mu j}{}^* + \gamma_3\, \partial_j A_{\mu j} \partial_\ell A_{\mu\ell}{}^* \tag{A.8}$$

where γ_i are positive constants [dG3, AdGR]. Note that these terms are not quite independent in the sense that the second term can be written in terms of the third plus a divergence terms (cf. (6.32)) which is often ignored as integration would yield a surface term which would vanish under appropriate boundary conditions.

As to the order parameter structure, the unusually large number of the internal degrees of freedom allows for a rich array of superfluid phases of liquid ^{3}He. In fact they are the most complicated known vacuum states of condensed matter known so far—incorporating the behaviour of liquid crystals, magnetically ordered states, and superfluids all at the same time—in which many symmetries are simultaneously broken. Fortunately it is possible to formulate a systematic approach to the investigation of all possible symmetry breakings. The classification of all the continuous and discrete subgroups of (A.7) can be obtained using group-theoretic methods: the continuous subgroups under which the order parameter $A_{\mu j}$ is invariant are classified via an infinitesimal transformation (note that the order parameter $A_{\mu j}$ has the transformation properties of a bivector, i.e. it transforms like a vector both with respect to its spin index and its orbital index) whereas the discrete subgroups of $SO(3)$ and $U(1)$ are well known [LL, Chap. XII]. The infinitesimal generators of G are the spin operators S_x, S_y, S_z; the orbital angular momentum operators L_x, L_y,

L_z; and, the differential operator $-i\frac{d}{d\phi}$. A presentation of all the subgroups of (A.7) can be found in [BV].

It turns out that the continuous subgroup with the highest remaining symmetry corresponds to the so-called B-phase of [ORL] which is stable down to the lowest temperatures attained so far. In this phase the order parameter is invariant under *joint* rotations of $\boldsymbol{L}$ and $\boldsymbol{S}$ represented by the subgroup $H_B = SO(3)_{\boldsymbol{L}+\boldsymbol{S}}$. The corresponding broken symmetries are

$$R_B = \frac{SO(3)_{\boldsymbol{L}} \times SO(3)_{\boldsymbol{S}}}{SO(3)_{\boldsymbol{L}+\boldsymbol{S}}} \times U(1)_\phi = SO(3)_{\boldsymbol{L},\boldsymbol{S}} \times S^1_\phi$$

with the fundamental homotopy group of this space $\pi_1(R_B) = \mathbb{Z}_2 + \mathbb{Z}$. The general form of the order parameter is given by

$$A_{\mu j} = \tfrac{1}{\sqrt{3}} e^{i\phi} R_{\mu j}$$

where ϕ is an overall phases variable and $R_{\mu j} = R_{\mu j}(\hat{\boldsymbol{n}}, \theta)$ is a rotation matrix describing the relative rotations of spin and orbital spaces with a rotation axis $\hat{\boldsymbol{n}}$ and a rotation angle θ (so that the universal covering $\mathbb{R} \times S^3 \to S^1 \times SO(3)$). The order parameter is invariant under simultaneous rotations described by a matrix $O_{\ell j}$ in the orbit space and $(ROR^{-1})_{\nu\mu}$ in spin space. The explicit phase factor $e^{i\phi}$ is due to the separately broken gauge invariance. Here the relative spin-spin symmetry (as well as the gauge symmetry) is broken. This is precisely the 'spontaneously broken spin-orbit symmetry' (SBSOS) introduced by Leggett [L14] who first realise the existence of this particular symmetry breaking for ^{3}He-B. For the weak-coupling Ginzburg-Landau regime, i.e. $\gamma_1 = \gamma_2 = \gamma_3$, the gradient term (A.8) can be written as

$$\begin{aligned} f^B_{\text{grad}} = K\Big(&5(\nabla\phi)^2 + 4(\nabla\theta)^2 - 2(\hat{\boldsymbol{n}}\cdot\nabla\theta)^2 \\ &+ 2(1-\cos\theta)\Big\{4(\nabla\times\hat{\boldsymbol{n}})^2 - (1-\cos\theta)[\hat{\boldsymbol{n}}\cdot(\nabla\times\hat{\boldsymbol{n}})]^2 \\ &\quad + (3-\cos\theta)(\nabla\cdot\hat{\boldsymbol{n}})^2 - 2\sin\theta(\nabla\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}\cdot(\nabla\times\hat{\boldsymbol{n}}) \\ &\quad + (\nabla\theta)\cdot(\nabla\times\hat{\boldsymbol{n}}) - 2(\hat{\boldsymbol{n}}\cdot\nabla\theta)\hat{\boldsymbol{n}}\cdot(\nabla\times\hat{\boldsymbol{n}}) \\ &\quad + 3\nabla[\hat{\boldsymbol{n}}\cdot\nabla)\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}(\nabla\cdot\hat{\boldsymbol{n}})]\Big\} \\ &- 2\sin\theta[(\hat{\boldsymbol{n}}\cdot\nabla\theta)\nabla\cdot\hat{\boldsymbol{n}} + (\hat{\boldsymbol{n}}\times\nabla\theta)\cdot(\nabla\times\hat{\boldsymbol{n}})]\Big) \end{aligned}$$

for some constant K (cf. [VW, p. 199]).

On the other hand if one is confined to finite temperature range and above a critical pressure of about 21 bar one enters a different phases; the so-called A-phase of [ORL]. In this ^{3}He-A phase Cooper pairing occurs in the state with the projection $S_z = 0$ on the spin quantisation axis (which we denote by the real unit vector $\hat{\boldsymbol{d}}$) and with orbital momentum projection $L_z = 1$ on the orbital quantisation axis (denoted by the real

unit vector $\hat{\boldsymbol{\ell}}$). These quantisation axes are independent if one neglects the spin-orbital (dipole) interactions. In order to express $A_{\mu j}$ in the frame with arbitrary $\hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{\ell}}$ one must introduce two further unit vectors in the orbital frame, say $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$, so that $\hat{\boldsymbol{\ell}} = \hat{\boldsymbol{m}} \times \hat{\boldsymbol{n}}$. With this the general A-phase order parameter is of the form

$$A_{\mu j} = \tfrac{1}{\sqrt{2}} \hat{d}_\mu (\hat{m}_j + i \hat{n}_j).$$

Note that no explicit phase appears: the phase is implicitly contained in the orientation of the triad $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{\ell}})$ along $\hat{\boldsymbol{\ell}}$. The gauge transformation $A_{\mu j} \mapsto e^{i\phi} A_{\mu j}$ introduces an additional phase factor. On the other hand, a rotation of the orbital component of the order parameters about $\hat{\boldsymbol{\ell}}$ by an angle ψ leads to $\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}} \mapsto e^{i\psi}(\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}})$. Hence transformations of the gauge variable and of the orbital part around the z-axis with $\psi = -\phi$ cancel each other out. In addition the order parameter remains invariant under rotations of the spin about $\hat{\boldsymbol{d}}$. Together these invariances correspond to the continuous subgroup $U(1)_{S_z} \times U(1)_{L_z - \phi}$ of rotations about one direction (the z-direction) in spin space and under a joint transformation of the phase and a rotation around L_z. Furthermore there is an additional discrete $\mathbb{Z}_2$ symmetry under the replacement $\hat{\boldsymbol{d}} \mapsto -\hat{\boldsymbol{d}}$ which also removes from the group $SO(3)$ all rotations about the vector $\hat{\boldsymbol{\ell}}$ by $\pm\pi$-rotations. The resulting subgroup of residual symmetries is $H_A = U(1)_{L_z - \phi} \times U(1)_{S_z} \times \mathbb{Z}_2$. The corresponding space of broken symmetries is the five dimensional manifold

$$R_A = \left(\tfrac{SO(3)_{\boldsymbol{S}}}{U(1)_{L_z}} \times \tfrac{SO(3)_{\boldsymbol{L}} \times U(1)_\phi}{U(1)_{L_z - \phi}}\right)/\mathbb{Z}_2 = (S^2_{\boldsymbol{S}} \times SO(3)_{\boldsymbol{L},\phi})/\mathbb{Z}_2$$

with the fundamental homotopy group of this space $\pi_1(R_A) = \mathbb{Z}_4$. With the order parameter in the form for the A-phase, the gradient energy can be expressed as

$$\begin{aligned} f^A_{\mathrm{grad}} = {} & \rho_0 (\hat{\boldsymbol{\ell}} \times (\hat{\boldsymbol{m}} \cdot \nabla \hat{\boldsymbol{n}}))^2 + \rho_1 (\hat{\boldsymbol{\ell}} \cdot (\hat{\boldsymbol{m}} \cdot \nabla \hat{\boldsymbol{n}}))^2 \\ & + C_0 (\hat{\boldsymbol{m}} \cdot \nabla \hat{\boldsymbol{n}}) \cdot (\nabla \times \hat{\boldsymbol{\ell}}) + C_1 (\hat{\boldsymbol{m}} \cdot \nabla \hat{\boldsymbol{n}}) \hat{\boldsymbol{\ell}} \cdot (\nabla \times \hat{\boldsymbol{\ell}}) \\ & + K_s (\nabla \cdot \hat{\boldsymbol{\ell}})^2 + K_t (\hat{\boldsymbol{\ell}} \cdot (\nabla \times \hat{\boldsymbol{\ell}}))^2 + K_b |\hat{\boldsymbol{\ell}} \times (\nabla \times \hat{\boldsymbol{\ell}})|^2 \\ & + K_0 \nabla [(\hat{\boldsymbol{\ell}} \cdot \nabla) \hat{\boldsymbol{\ell}} - \hat{\boldsymbol{\ell}} (\nabla \cdot \hat{\boldsymbol{\ell}})] \\ & + \kappa_0 (\hat{\boldsymbol{\ell}} \times \nabla \hat{d}_\alpha) \cdot (\hat{\boldsymbol{\ell}} \times \nabla \hat{d}_\alpha) + \kappa_1 (\hat{\boldsymbol{\ell}} \cdot \nabla) \hat{d}_\alpha) \cdot (\hat{\boldsymbol{\ell}} \cdot \nabla) \hat{d}_\alpha) \end{aligned}$$

where the last two terms relate to the spin variable with the others relate to the orbit variable and the coefficients K_s, K_t, and K_b indicate the respective energy contributions are due to a splay, twist and bending spatial change of $\hat{\boldsymbol{\ell}}$ analogous to that of the above Oseen-Frank free energy density seen in the uniaxial nematic liquid crystal case (cf. [BC2], [VW, p. 196]).

If one further considers an external magnetic field $\boldsymbol{H}$, an additional direction is defined which also breaks the symmetry with respect to rotations in the spin space. The corresponding symmetry group is then $G_{\mathrm{mag}} = U(1)_{S_z} \times SO(3)_{\boldsymbol{L}} \times U(1)_\phi$. In a presence

of a strong magnetic field $\hat{\boldsymbol{d}}$ is locked in a plane by the magnetic field, i.e. $\hat{\boldsymbol{d}} \perp \boldsymbol{H}$. The space of broken symmetries is then

$$R_A^H = (S^1_{\boldsymbol{S}} \times SO(3)_{\boldsymbol{L},\phi})/\mathbb{Z}_2$$

with the fundamental homotopy group $\pi_1(R_A^H) = \mathbb{Z} \times \mathbb{Z}_4$. If one considers only dipole interactions the vector $\hat{\boldsymbol{d}}$ tends to orient parallel with $\hat{\boldsymbol{\ell}}$ (a behaviour refereed to as 'dipole locking', i.e. $\hat{\boldsymbol{d}} \,||\, \hat{\boldsymbol{\ell}}$). Only the space of rotation of the orbital frame survives so that the space of broken symmetries is

$$R_A^d = SO(3)_{\boldsymbol{L}}$$

with the fundamental homotopy group $\pi_1(R_A^d) = \mathbb{Z}_2$. If a weak magnetic field is further considered so that $\hat{\boldsymbol{d}} \,||\, \hat{\boldsymbol{\ell}} \perp \boldsymbol{H}$ the space of broken symmetries is

$$R_A^{dH} = S^1 \times S^1$$

where one circle is for the overall phase ϕ while the other is for joint rotations of $\hat{\boldsymbol{d}} \,||\, \hat{\boldsymbol{\ell}}$ in the plane. The fundamental homotopy group of this space $\pi_1(R_A^{dH}) = \mathbb{Z} + \mathbb{Z}$.

$\mathcal{M}$	π_1	UC	$\dim_{\mathbb{R}}$	
S^1	$\mathbb{Z}$	$\mathbb{R}$	1	planar spins/superfluid ^{4}He
$S^1 \times S^1$	$\mathbb{Z} + \mathbb{Z}$	$\mathbb{R}^2$	2	dipole-locked ^{3}He-A in weak magnetic field
S^2	0	S^2	2	ordinary spins/ferromagnet
$\mathbb{R}P^2$	$\mathbb{Z}_2$	S^2	2	uniaxial nematics
$S^3/\mathbb{H}$	$\mathbb{H}$	S^3	3	biaxial nematics
$\mathbb{R}P^3 = SO(3)$	$\mathbb{Z}_2$	S^3	3	dipole-locked ^{3}He-A
$S^1 \times SO(3)$	$\mathbb{Z} + \mathbb{Z}_2$	$\mathbb{R} \times S^3$	4	^{3}He-B
$(S^1 \times SO(3))/\mathbb{Z}_2$	$\mathbb{Z} + \mathbb{Z}_4$	$\mathbb{R} \times S^3$	4	dipole-free ^{3}He-A in strong magnetic field
$(S^2 \times SO(3))/\mathbb{Z}_2$	$\mathbb{Z}_4$	$S^2 \times S^3$	5	dipole-free ^{3}He-A

Table A.1. The manifold of broken symmetries $\mathcal{M}$ and universal cover (UC) of some examples of ordered media. Note that $\mathbb{R}P^3$ is diffeomorphic to $SO(3)$; the covering map $S^3 \to \mathbb{R}P^3$ is a map of groups $\mathrm{Spin}(3) \to SO(3)$, where $\mathrm{Spin}(3) = SU(2) = S^3$ and $\mathrm{Spin}(2) = U(1) = S^1$.

Remark A.1. For further information on broken symmetries with regards to gradient, dipole and magnetic interactions see [SV] as well as the book [VW] and [BV]. For a survey on symmetry breaking and the topological theory of defects using homotopy theory see [VM, M3, M5, T11, KL]. Note that an isolated d-dimensional topologically stable defects in a n-dimensional ordered phase can be classified by the homotopy groups $\pi_k(R)$ of the manifold of internal states R with $k = n - d - 1$ (cf. [KMT]). Thus for uniaxial nematics $\pi_2 = \mathbb{Z}$, $\pi_1 = \mathbb{Z}_2$, $\pi_0 = 0$ compared with biaxial nematics $\pi_2 = 0$, $\pi_1 = \mathbb{H}$, $\pi_0 = 0$

classify the respective point, line and surface defects in $\mathbb{R}^3$. For a review of the theory of anisotropic superfluid phases of helium-3 see [L15].[3]

Remark A.2. The notion of symmetries and symmetry breaking also plays a fundamental role in high-energy physics. In particular the idea of spontaneous symmetry breaking in relativistic quantum field theory was introduced on the grounds of an analogy with that of the 1957 superconductivity theory of [BCS]. Here the ground state of condensed matter physics becomes the vacuum state and the spontaneously broken symmetry is gauge invariance. As it turns out, a number of field theories display spontaneous breaking of internal symmetries.

An example of a globally broken symmetry can be found in the chiral symmetry breaking of a gauge theory such as quantum chromodynamics. If one considers an isodoublet of quarks with two flavours, namely an up and down quark (u, d), initially there is a chiral invariance between left-handed (L) and right-handed (R) states. The corresponding symmetry is decomposed as $G = SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$. The condensation between quarks (u, d) and antiquarks $(\bar{\mathrm{u}}, \bar{\mathrm{d}})$ similar to that of Cooper pairing spontaneously breaks the $SU(2)_L \times SU(2)_R$ down to a remaining $SU(2)_{L+R}$ in analogy to the B-phase of superfluid ^{3}He. For three quarks $(\mathrm{u}, \mathrm{d}, \mathrm{s})$ the flavour-chiral symmetries correspond to $G = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$.

In the case of a *global* continuous symmetry, massless bosons (known as Goldstone bosons) appear with the spontaneous breaking [G11, GSW]. The presence of these massless bosons—first seen as a serious problem since no particles of the sort had been observed in the context considered—was in fact the basis for the solution of another similar problem: the fact that the 1954 Yang–Mills theory of non-abelian gauge fields predicted unobservable massless particles, i.e. the gauge bosons (massless because of the gauge invariance). This solution manifested itself as the so-called Higgs mechanism (established in a general way independently by [H13, H14], [EB] and [GHK]) in that the internal symmetry is promoted to a local one, i.e. if a continuous global symmetry is changed to a *local* one, the Goldstone boson disappears and the gauge bosons acquire a mass; or, in other words, the Goldstone bosons are 'eaten up' to give mass to the (otherwise massless) gauge bosons, and this happens without (explicitly) breaking the gauge invariance of the theory.

[3]We would like to also mention that the 1996 Nobel Prize in Physics was awarded to D. M. Lee, D. D. Osheroff and R. C. Richardson for their discovery of superfluidity in helium-3. For pioneering contributions to the theory of superconductors and superfluids A. A. Abrikosov, V. L. Ginzburg and A. J. Leggett where awarded the 2003 Nobel prize in Physics. Also P.-G. de Gennes was awarded the 1991 Nobel prize for discovering that methods developed for studying order phenomena in simple systems can be generalised to more complex forms of matter, in particular to liquid crystals and polymers.

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22 May 2015, 8:27pm