ELEMENTARY STATES,

SUPERGEOMETRY AND TWISTOR THEORY

by

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ABSTRACT

It is shown that $H^{p-1}(P^+,\mathcal{O}(-m-p))$ is a Fréchet space, and its dual is $H^{q-1}(\overline{P^-},\mathcal{O}(m-q))$, where P^+ and P^- are the projectivizations of subsets of generalized twistor space ($\cong \mathbb{C}^{p+q}$) on which the hermitian form (of signature (p,q)) is positive and negative definite respectively, and $\mathcal{O}(-m-p)$ denotes the sheaf of germs of holomorphic functions homogeneous of degree $-m-p$. It is then proven, for $p = 2$ and $q = 2$, that the subspace consisting of all twistor elementary states is dense in $H^{p-1}(P^+,\mathcal{O}(-m-p))$.

A supermanifold is a ringed space consisting of an underlying classical manifold and an augmented sheaf of $\mathbb{Z}_2$-graded algebras locally isomorphic to an exterior algebra. The subcategory of the category of ringed spaces generated by such supermanifolds is referred to as the super category. A mathematical framework suitable for describing the generalization of Yang-Mills theory to the super category is given. This includes explicit examples of supercoordinate changes, superline bundles, and superconnections. Within this framework, a definition of the full super Yang-Mills equations is given and the simplest case is studied in detail. A comprehensive account of the generalization of twistor theory to the super category is presented, and it is used in an attempt to formulate a complete description of the super Yang-Mills equations. New concepts are introduced, and several ideas which have previously appeared in the literature at the level of formal calculations are expanded and explained within a consistent framework.

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INTRODUCTION

In this thesis we treat two different topics: the density of twistor elementary states, and a comprehensive generalization of twistor theory to a super category (to be defined shortly). Each chapter contains its own introduction and therefore we shall limit ourselves here to giving some general motivation for studying such topics.

Twistor theory evolved as an attempt to reformulate basic relativistic physics. Originally, one of its most outstanding features was the representation of solutions of massless free field equations in terms of contour integrals of holomorphic homogeneous twistor functions. Important aspects of quantum theory, such as the splitting of fields into positive and negative frequency parts, were then characterized by holomorphic data on twistor space. It was later discovered that such twistor functions had to be regarded as representatives of elements of sheaf cohomology groups, on appropriate subsets of projective twistor space, with coefficients on the sheaf of germs of homogeneous holomorphic functions. The isomorphism between these groups and the space of solutions of the massless free field equations now bears the name *Penrose transform* [21].

One aspect of twistor theory attempts to produce a manifestly finite theory of scattering in quantum field theory. To this end, scattering amplitudes are represented in terms of multidimensional contour integrals, whose pictorial representations are known as *twistor diagrams* [66]. The proposed theory requires the use of some simple and manageable sheaf cohomology elements, represented by rational twistor functions. These elements are known as *elementary states*. They correspond, under the Penrose transform, to massless free fields with rational singularities on the null cone of a point in complexified compactified Minkowski space. In the scalar case for example, the simplest elementary state plays a role analogous to that of a plane wave, while higher order ones correspond to Grgin's spherical harmonics [31].

The study of twistor diagrams is carried out under the assumption that it is enough to consider only elementary states, rather than more general cohomology classes. In the first part of this thesis we show that this assumption is justified by proving a density theorem.

An immediate consequence of the density theorem is a proof, independent of space-time arguments, of the positive definiteness of an U(2,2) invariant hermitian inner product on cohomology, known as the *twistor scalar product* (corresponding to what is known in Physics as the scalar product of massless fields [26a]). This has applications to the representation theory of U(2,2); and more generally U(p,q).

Twistor theory generalizes to higher dimensions. The corresponding generalization of the twistor scalar product [18], leads to a twistor realization of the ladder representations of U(p,q) - the elementary states are the U(p) × U(q)-finite vectors [18,22]. The possibility of extending these representations to limits of discrete series of arbitrary semisimple Lie groups, is being actively studied at present [18a,5a].

The methods of the Penrose transform can also be generalized to the non-linear regime. The first example was Penrose's solution of the self-dual Einstein equations [62]. This construction, known as the non-linear graviton, was followed by Ward's description of the self-dual Yang-Mills equations in terms of vector bundles on projective twistor space [90,89]. On the other hand, a twistorial interpretation of full Yang-Mills and Einstein equations has proven to be a much more difficult problem. However, an understanding of the former equations on ambitwistor space (the space of null-lines in complexified Minkowski space) has emerged [45,93,12,69,53b]; and the hope is to use it as a guiding analogy for the gravitational case [44,53,5b,15b].

The twistorial description of the full Yang-Mills equations is given in terms of extensions of vector bundles to infinitesimal neighbourhoods of ambitwistor space, in the product of projective twistor space and projective dual twistor space. The geometric insight provided by the twistorial description of the self-dual equations turns out to be useful for attaining some geometric understanding of this intricate theorem. The essential ingredient is to regard the 'physical' space as being embedded as the diagonal in the product of two 'unphysical' spaces. Then the full Yang-Mills equations on the physical space are essentially equivalent to a self-dual equation on one of the unphysical factors, and an anti-self-dual equation on the other. The idea† of constructing differential geometric structures on a space by regarding it as being embedded as the 'diagonal' in the product of two other spaces, will play an important role throughout all stages of Part II of this thesis.

In the second part of a fascinating paper [93], Witten indicated an alternative way of arriving at the twistorial interpretation of the full Yang-Mills equations, by the introduction of supersymmetry within the context of ambitwistors. To a great extent Part II of this thesis developed as an attempt to understand this alternative description within a consistent mathematical framework.

Supersymmetric gauge theories have attracted considerable interest in theoretical physics. Supersymmetry is a symmetry that relates particles with different spins; so that fermions and bosons can be regarded as alternative manifestations of a single 'super particle'. Moreover, supersymmetry unifies space-time symmetries with internal symmetries. It should be said however, that there is to date no experimental evidence for such unification of fundamental particles and interactions.

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†Due to Grothendieck [32].

With the advent of string theories new interest has emerged in Witten's [93] formal constructions, which have recently been generalized to higher dimensions [94,14]. More recently, attempts to produce a supergravity analogue of Witten's constructions have also been made [86].

The mathematical framework presented in Chapter 2 is based on the Leites [54] and Kostant [49] approach to supermanifolds. A *supermanifold* is a ringed space consisting of an underlying classical manifold and an augmented sheaf of $\mathbb{Z}_2$-graded algebras locally isomorphic to an exterior algebra. The subcategory of the category of ringed spaces generated by such supermanifolds will be referred to (throughout Part II) as the *super category*.

The framework of Chapter 2 is used in Chapter 3, to describe a natural generalization of the Yang-Mills equations to the super category.

In Chapter 4 we give a comprehensive account of the generalization of twistor theory to the super category, and use it in an attempt to give a description of super Yang-Mills theory.

PART I

CHAPTER 1: ELEMENTARY STATES

§1.0 Introduction

We start by reviewing how elementary states were originally defined [66]. The starting point will be Penrose's well-known contour integral formula for generating massless fields:

$$\phi_{\overbrace{A'\ldots L'}^{m}}(x^a) = \frac{1}{(2\pi i)^2}\oint_\Gamma \pi_{A'}\ldots\pi_{L'}f(Z^\alpha)\Big|_{\omega^A = ix^{AA'}\pi_{A'}}\pi_{m'}d\pi^{m'}, \quad (Z^\alpha = (\omega^A, \pi_{A'})).$$

$f(Z^\alpha)$ is a holomorphic function (homogeneous of degree $-m-2$ in Z^α) on some domain, U, in projective twistor space, modulo a 'gauge freedom' $f \longmapsto f + f_\alpha + f_\beta$ (where f_α and f_β are holomorphic on extended domains, U_α and U_β respectively). This gauge freedom depends upon the location of the contour Γ, which in turn can be any closed contour separating the singularities of f. We refer to Roger Penrose's article [64] for an excellent explanation of how the combined freedom of the choice of both Γ and f is best described in terms of sheaf cohomology (f represents a cocycle, while f_α and f_β represent coboundaries).

$f(Z^\alpha)$ (homogeneous of degree $-m-2$) will produce a positive frequency massless field (of helicity $\frac{m}{2}$) if its singularity set in $\mathbb{PT}^+ = \{[Z^\alpha] \in \mathbb{PT} \,|\, \bar{Z}_\alpha Z^\alpha > 0\}$ consists of the union of two components between which a contour may be threaded. The simplest way of guaranteeing this is to choose the singularity set to be the union of two planes, A and B, in $\mathbb{P}T$, whose intersection lies in $\mathbb{P}T^- = \{[Z^\alpha] \in \mathbb{P}T \; \bar{Z}_\alpha Z^\alpha < 0\}$. If we choose

$A = \{[Z^\alpha] \in \mathbb{P}T \,|\, A_\alpha Z^\alpha = 0\}$ and $B = \{[Z^\alpha] \in \mathbb{P}T \,|\, B_\alpha Z^\alpha = 0\}$,

then this is achieved by any rational function of the form

$$e \equiv \frac{(C_\gamma A^\gamma)^c (D_\delta Z^\delta)^d}{(A_\alpha Z^\alpha)^{a+1} (B_\beta Z^\beta)^{b+1}}, \tag{1.1}$$

where $a + b - c - d = m$. The corresponding space-time field was named by R. Penrose [66 , 65] an *elementary state*. Equivalently, we refer to (1.1) as a canonical representative of an elementary state. (Note that

$$e + \frac{(C_\gamma A^\gamma)^c (D_\delta Z^\delta)^d}{(A_\alpha Z^\alpha)^{a+b+2}} + \frac{(C_\gamma A^\gamma)^c (D_\delta Z^\delta)^d}{(B_\beta Z^\beta)^{a+b+2}} \tag{1.2}$$

defines the same elementary state as (1.1).)

Let L denote the intersection (in $\mathbb{P}T^-$) of A and B, i.e. $L = \{[Z^\alpha] \in \mathbb{P}T \,|\, A_\alpha Z^\alpha = 0 = B_\beta Z^\beta\}$. This projective line corresponds, under the twistor correspondence, to a point, say ℓ^a, in the backward tube of complexified Minkowski space $\mathbb{M}^-$. We shall then say, for example, that the elementary state (1.1) is based at ℓ^a, or equivalently, at L.

The objective of this chapter is to prove a density theorem for the elementary states. When stated in terms of space-time fields, this theorem is equivalent to saying that the elementary states, based on a fixed point in $\mathbb{M}^-$, are dense in the space of all positive frequency massless free fields.

In this chapter we will state the results for a generalised twistor space T of dimension p+q (equipped with a hermitian form of signature (p,q)). One of the reasons for doing this is the representation theory implications of the density theorem which are outlined at the end of the chapter. The other reason is that the various results needed for the proof of the density theorem can be stated more elegantly in the general case. In particular, we have found the study of pseudoconvexity properties of generalised twistor spaces (§2) interesting in its own right. However higher dimensional generalizations of twistor theory do not form part of the objectives of this chapter. Therefore, where descriptions or proofs are obscured by the unspecified dimensionality of the generalized twistor space, we have presented them in the familiar 4-dimensional case.

The organization of Chapter 1 is as follows. In §1.1 we briefly review the generalized twistor correspondence (as presented in [18]) stating only the essential facts. In §1.2 we use the results of Andreotti and Grauert [3] to draw conclusions concerning the finite dimensionality of the k^{th} cohomology groups on regions of generalized projective twistor space, $P_n \equiv P(T)$ $(n = p+q-1)$, for integer values of k, other than certain critical ones. These results are needed in §1.3, where we use a generalization (due to H.B. Laufer [50]) of Serre duality to show that $H^{p-1}(P^+, \mathcal{O}(-m-p))$ is a Fréchet space and that its topological dual is given by $H^{q-1}(\overline{P^-}, \mathcal{O}(m-q))$ (P^+ and P^- are appropriate open subsets of P_n).

Generalized elementary states are defined in §1.4, and the density theorem is proved (in the four dimensional case) in §1.5. The general idea in this proof is similar to that of the proof of the Runge Approximation Theorem (see for example [40]). It relies upon a corollary of the Hahn-Banach Theorem (see for example [74]) which states that if the vanishing of a linear functional f on a subspace M of a Fréchet space X implies that f is identically zero, then M is dense in X. In our case the linear functionals on the spaces of positive frequency massless fields, turn out to be massless fields of negative frequency.

In §1.6 we follow [18] and introduce the generalized twistor scalar product, and (following a suggestion of M.G. Eastwood) indicate how to establish its positive definiteness by arguing as in the density theorem of §1.5.

The role of the elementary states in representation theory is briefly outlined in §1.7. The direct proof of the positive definiteness of the twistor scalar product essentially completes the twistor construction of the ladder representations of U(p,q) (cf. [47,73]) initiated in [18].

§1.1 The generalized Penrose and twistor transforms [18]

We take *generalized twistor space* T to be a (p+q)-dimensional complex vector space equipped with a non-degenerate hermitian form Φ of signature (p,q). We denote the corresponding projective space P(T) of T by P_n where $n = p+q-1$.

Let M be the Grassmannian of p-dimensional subspaces of T and F_{1p} the flag manifold of 1-dimensional subspaces inside the p-dimensional subspaces of T. Then the generalized twistor correspondence between P_n and M is given by the double fibration

$$\begin{array}{ccccc} & & F_{1p} & & \\ & \swarrow^{\nu} & & \searrow^{\mu} & \\ & M & & P_n & \\ & \cup & & \cup & \\ & x & \longmapsto & L_x = \mu(\nu^{-1}(x)) & \\ \nu(\mu^{-1}[Z]) = & \Sigma_{[Z]} & \longleftarrow & [Z] & . \end{array} \qquad (1.3)$$

A point $x \in M$ gives rise to a (p-1)-dimensional subspace L_x of P_n. Conversely, a generalized projective twistor $[Z] \in P_n$ gives rise to a p-dimensional subspace $\Sigma_{[Z]}$ of M referred to as a *generalized* α*-plane*.

Taking (p = 2, q = 2) one obtains the ordinary twistor correspondence [21]. In this case M is complexified compactified Minkowski space (denoted by $\mathbb{M}$), P_3 is projective twistor space (denoted by $\mathbb{PT}$), and F_{12} is the projective primed spin bundle on $\mathbb{M}$ (denoted by $\mathbb{F}_{12}$ or, non-projectively, by $\mathcal{O}_{A'}$).

Let $M^+ \subset M$ denote the set of p-dimensional subspaces, $\hat{Z}$, of T such that $\Phi|_{\hat{Z}}$ is positive definite. Similarly, let $M^- = \{\hat{Z} \subset T \text{ s.t. } \Phi|_{\hat{Z}} < 0\}$, and denote the common boundary of M^+ and M^- by M^0. Then

$$x \in M^+ \iff L_x \subset P^+ = P(T^+) \text{ where } T^+ = \{Z \in T \text{ s.t. } \Phi(Z,Z) > 0\}$$

$$x \in M^0 \iff L_x \subset P^0 = P(T^0) \text{ where } T^0 = \{Z \in T \text{ s.t. } \Phi(Z,Z) = 0\}^\dagger$$

$$x \in M^- = L_x \subset P^- = P(T^-) \text{ where } T^- = \{Z \in T \text{ s.t. } \Phi(Z,Z) < 0\}^\dagger.$$

In the ordinary twistor case ($p = 2$, $q = 2$) M^+ and M^- are the forward and backward tubes in $\mathbb{M}$ denoted respectively by $\mathbb{M}^+$ and $\mathbb{M}^-$.

Theorem 1.i [18] If $p \geq 2$ and $q \geq 2$, then for each integer m, there is a canonical linear isomorphism known as the *Penrose transform*

$$P : H^{p-1}(P^+, \mathcal{O}(-m-p)) \to \ker D_m : \Gamma(M^+, \mathcal{O}(V_m)) \to \Gamma(M^+, \mathcal{O}(W_m)) \qquad (1.4)$$

where V_m and W_m are holomorphic vector bundles (see [18] for their general definition), and D_m is a linear differential operator of second order if $m = 0$ and otherwise of first order.

Note: If ($p = 2$, $q = 2$), then

$$V_m = \begin{Bmatrix} \mathcal{O}_{(A'B'\ldots C')}[-1]' & (m > 0) \\ \mathcal{O}[-1]' & (m = 0) \\ \mathcal{O}_{(AB\ldots C)}[-1]' & (m \leq 0) \end{Bmatrix}$$

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†Note that if $p > q$, then these subspaces are empty.

$$W_m = \begin{cases} \mathcal{O}_{A(B'\ldots C')}[-2]' & (m > 0) \\ \mathcal{O}[-2]'[-1] & (m = 0) \\ \mathcal{O}_{A'(B\ldots C)}[-1]'[-1] & (m < 0) \end{cases}$$

and

$$D_m = \begin{cases} \partial_A^{A'} & (m > 0) \\ \Box & (m = 0) \\ \partial_{A'}^{A} & (m < 0) \end{cases}$$

where (in local coordinates):

$$\partial_A^{A'} = -\partial/\partial x_{A'}^{A},$$

$$\Box = -\partial_A^{A'}\partial_{A'}^{A},$$

$$\partial_{A'}^{A} = -\partial/\partial x_A^{A'} . \qquad \Box$$

Let $T^* = \mathrm{Hom}_{\mathbb{C}}(T,\mathbb{C})$ denote the *generalized dual twistor space* equipped with a hermitian form Φ^* (dual to Φ). Furthermore denote by P_n^* the corresponding projective space of T^*. Then the generalized $\mathcal{P}$-transform identifies $H^{q-1}(P^{*-},\mathcal{O}(m-q))$ with solutions of differential equations on M^{*-}, where M^{*-} denotes the set of q-1 dimensional linear subspaces of T^* on which Φ^* is negative definite, and $P^{*-} = P(T^{*-})$ with $T^{*-} = \{W \in T^* \text{ s.t. } \Phi^*(W,W) < 0\}$.

It can be shown [18] that $M^+ \cong M^{*-}$ and that consequently there is a canonical linear isomorphism

$$\mathcal{T} : H^{p-1}(P^+,\mathcal{O}(-m-p)) \to H^{q-1}(P^{*-},\mathcal{O}(m-q)) \otimes \det T, \qquad (1.5)$$

known as the *twistor transform*. In fact, $\mathcal{T}$ can be realized without introducing M at all (see [83]).

The lowest dimensional case ($p = 1$, $q = 1$) is rather special. For $m \geq 0$ the general statements of the transforms (1.4) and (1.5) apply (although there are no field equations). But, for $m \leq -1$, (1.4) and (1.5) hold only if $H^0(P^+, \mathcal{O}(-m-1))$ is replaced by $H^0(P^+, \mathcal{O}(-m-1))/H^0(P_1, \mathcal{O}(-m-1))$.

Note: The Generalized Penrose transform (1.4) is still an isomorphism if P^+ and M^+ are replaced by more general subsets with certain mild topological restrictions on the fibres of μ of (1.3) (see [17]).

§1.2 Pseudoconvexity properties of twistor spaces

In this section we use the results of Andreotti and Grauert [3] (see also [1,2,37]) to draw conclusions concerning the k^{th} cohomology groups on P^+ and P^-, for integer values of k other than the critical ones: p-1 in the case of P^+ and q-1 in the case of P^-. We shall only state these results for P^+. The corresponding ones for P^- are obtained by interchanging p and q in our statements.

§1.2.1 A defining function on P_n

Suppose that Ψ is some positive definite hermitian form on T. Then $\phi = \Phi/\Psi$ is a well defined, C^∞-real-valued function on the corresponding generalized projective twistor space $P_n = P(T)$ $(n = p+q-1)$. ϕ is in fact the defining function [26] for the two relatively compact domains of P_n:

$$P^+ = \{z \in P_n \text{ s.t. } \phi(z) > 0\}$$

$$P^- = \{z \in P_n \text{ s.t. } \phi(z) < 0\} .$$

Their common boundary $P^0 = \phi^{-1}(0)$ is a smooth submanifold of real dimension 2n-1.

We will choose coordinates $Z \equiv (Z^1,\ldots,Z^{p+q})$ for T such that Φ is diagonal, i.e.

$$\Phi(Z,Z) = |Z^1|^2 + \ldots + |Z^p|^2 - |Z^{p+1}|^2 - \ldots - |Z^{p+q}|^2.$$

Let

$$R(Z,Z) = |Z^1|^2 + \ldots + |Z^p|^2$$

and

$$S(Z,Z) = |Z^{p+1}|^2 + \ldots + |Z^{p+q}|^2$$

so that

$$\Phi(Z,Z) = R(Z,Z) - S(Z,Z) ,$$

and take

$$\Psi(Z,Z) = R(Z,Z) + S(Z,Z) .$$

Then

$$\phi = \frac{R-S}{R+S} ,$$

and $-1 \leq \phi \leq 1$.

Let $[Z] \equiv [Z^1,\ldots,Z^{p+q}]$ denote the homogeneous coordinates of P_n, and consider the following (q-1)-dimensional projective subspace of P_n lying entirely within P^-:

$$L^- = \{[Z] \in P_n \text{ s.t. } Z^1 = Z^2 = \ldots = Z^p = 0\} \subset P^- .$$

Interchanging p and q, one obtains the following (p-1)-dimensional subspace within P^+:

$$L^+ = \{[Z] \in P_n \text{ s.t. } Z^{p+1} = Z^{p+n} = \ldots = Z^{p+q} = 0\} \subset P .$$

Let us now define coordinates on $\{[Z] \in P_n \text{ s.t. } Z^1 \neq 0\} \subset P_n$, by setting $z^i = Z^i/Z^1$ $(i = 2,\ldots,p+q)$. In these coordinates ϕ becomes

$$\phi(z) = \frac{1 + |z^2|^2 + \ldots + |z^p|^2 - |z^{p+1}|^2 - \ldots - |z^{p+q}|^2}{1 + |z^2|^2 + \ldots + |z^{p+q}|^2} . \quad (1.7)$$

It is useful to visualize the division of P_n as in Fig. 1.1

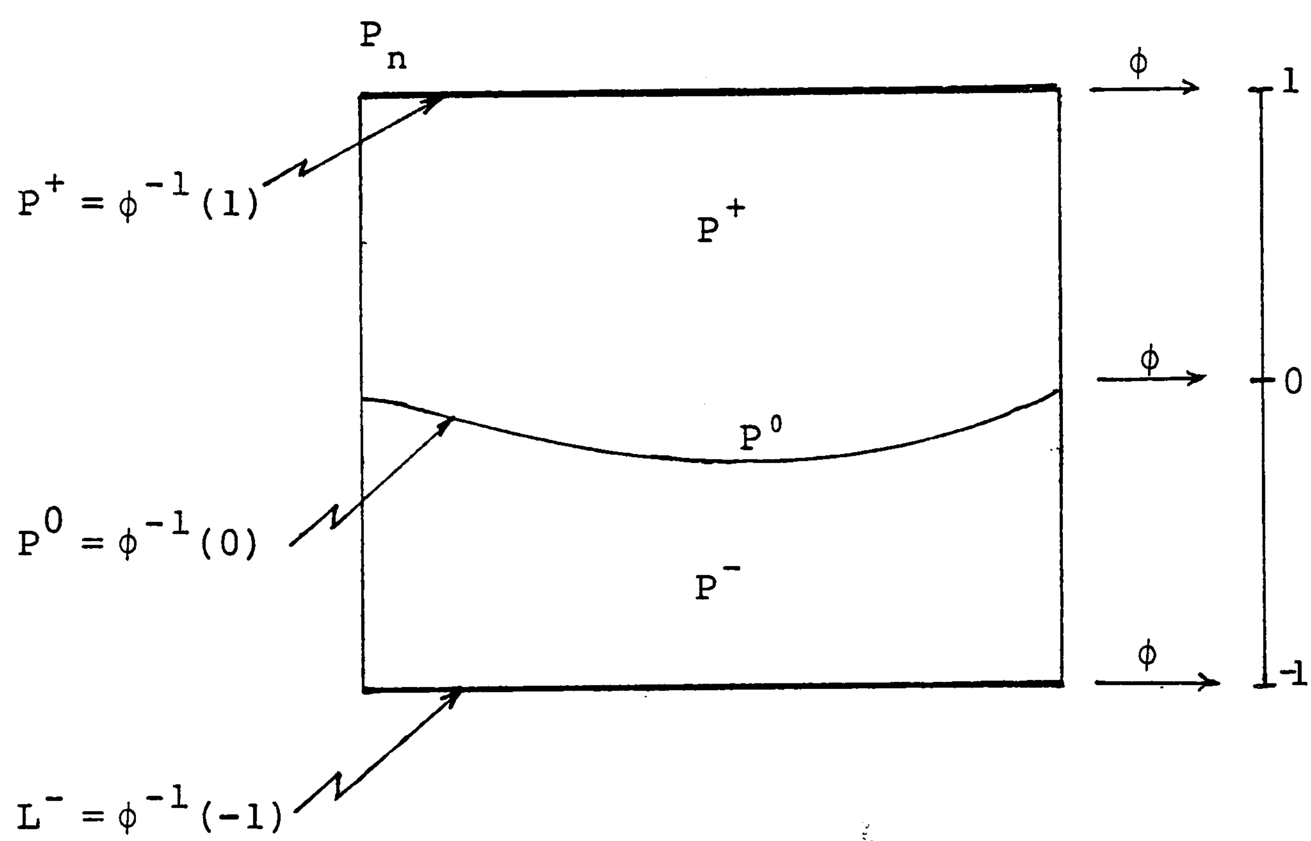


Fig. 1.1

§1.2.2 Definitions

(i) A C^∞-real-valued function on a complex n-dimensional manifold X is said to be *strictly r-plurisubharmonic*, if the complex Hessian H(f) has at least n-r positive eigenvalues at every point of X.

(ii) A domain X of a complex manifold is said to be *strictly r-pseudoconvex* if there exists a compact set $K \subset X$, and a strictly r-plurisubharmonic function f on X-K, such that the sets $B_c = \{x \in X | f(x) < c\}$ are relatively compact $\forall\ c \in \mathbb{R}$.

(iii) A domain X of a complex manifold is said to be *strictly r-pseudoconcave* if there exists a compact set $K \subset X$, and a strictly r-plurisubharmonic function f on X-K, such that the sets $B_c = \{x \in X | f(x) > c\}$ for $0 < c \in \mathbb{R}$ are relatively compact.

(iv) If in (ii), $K = \emptyset$ then f is said to be an *exhaustion function*, and X is called *r-complete*.

§1.2.3 Evaluation of the number of positive and negative eigenvalues of the complex Hessian of ϕ

In our local coordinates

$$H(\phi) = \sum_{i,j=2}^{p+q} \phi_{z^i \bar{z}^j} dz^i \otimes d\bar{z}^j ,$$

where

$$\phi_{z^i \bar{z}^j} = \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} .$$

We first evaluate ϕ_{z^i}:

$$\phi_{z^i}(z) = \frac{\operatorname{sgn}(i)\bar{z}^i(1+|z^2|^2+\ldots+|z^{p+q}|^2)-\bar{z}^i(1+|z^2|^2+\ldots+|z^p|^2-|z^{p+1}|^2-\ldots-|z^{p+1}|^2)}{(1+|z^2|^2+\ldots+|z^{p+q}|^2)^2} \tag{1.8}$$

where

$$\operatorname{sgn}(i) = \begin{cases} 1 \text{ if } 2 \le i \le p \\ \\ -1 \text{ if } p+1 \le i \le p+q \end{cases}$$

and

$$\delta^{ij} = \begin{cases} 1 \text{ if } i = j \\ \\ 0 \text{ if } i \ne j . \end{cases}$$

Then

$$\phi(z)_{z^i\bar{z}^j} = \frac{\delta^{ij}[\mathrm{sgn}(i)(1+|z^2|^2+\ldots+|z^{p+q}|^2)-(1+|z^2|^2+\ldots+|z^p|^2-|z^{p+1}|^2-\ldots-|z^{p+q}|^2)]}{(1+|z^2|^2+\ldots+|z^{p+q}|^2)^2}$$

$$+\ \frac{\bar{z}^i z^j[\mathrm{sgn}(i)-\mathrm{sgn}(j)]}{(1+|z^2|^2+\ldots+|z^{p+q}|^2)^2}$$

$$-\ \frac{2\bar{z}^i z^j[\mathrm{sgn}(i)(1+|z^2|^2+\ldots+|z^{p+q}|^2)-(1+|z^2|^2+\ldots+|z^p|^2-|z^{p+1}|^2-\ldots-|z^{p+q}|^2)]}{(1+|z^2|^2+\ldots+|z^{p+q}|^2)^3} \tag{1.9}$$

Let us now consider $H(\phi)$ restricted to a point y on the projective line

$$Y = \{z \in P_n \text{ s.t. } z^2 = z^3 = \ldots = z^{p+q-1} = 0 \ \&\ z^{p+q} = t\}.$$

Note that $\phi|_Y = \frac{1-|t|^2}{1+|t|^2}$, so Y intersects P^0 at $|t|^2 = 1$, and L^+ at $t = 0$. Then

$$\phi_{z^i\bar{z}^j}\Big|_Y = \frac{[\mathrm{sgn}(i)(1+|t|^2)-(1-|t|^2)]}{(1+|t|^2)^2} - \frac{2\bar{z}^i z^j[\mathrm{sgn}(i)(1+|t|^2)-(1-|t|^2)]}{(1+|t|^2)^3}$$

i.e.

$$[\phi_{z^i\bar{z}^j}|_Y] = \underbrace{\left(\begin{array}{ccccc|ccccc}
2|t|^2 & 0 & \cdots & & 0 & 0 & \cdots & & & 0 \\
0 & 2|t|^2 & 0 & \cdots & 0 & \vdots & & & & \vdots \\
\vdots & & \ddots & & 0 & \vdots & & \ddots & & \vdots \\
0 & 0 & \cdots & 0 & 2|t|^2 & 0 & \cdots & & & 0 \\
\hline
0 & \cdots & & & 0 & -2 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & & & \vdots & 0 & -2 & 0 & & 0 \\
\vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & & & \ddots & \vdots & 0 & & & -2 & 0 \\
0 & \cdots & & & 0 & 0 & 0 & \cdots & 0 & -2\left(\frac{1-|t|^2}{1+|t|^2}\right)
\end{array}\right)}_{p-1 \qquad\qquad q}$$

Thus $H(\phi)|_{P^{\circ}\cap Y}$ has q-1 eigenvalues strictly less than zero and p-1 ones strictly greater than zero. On the other hand, $H(\phi)|_{P^{+}\cap Y}$ has q eigenvalues strictly less than zero and p-1 eigenvalues greater than or equal to zero. The latter are actually equal to zero on L^{+}.

The above statements are independent of the particular choice of Y. To see this, note that U(p,q) acts transitively on (the homogeneous coordinates of) $\overline{P^{+}}$ preserving Φ (but not Ψ), and U(p+q) acts transitively on (the homogeneous coordinates of) P_n preserving Ψ. Thus the quotient $\frac{\Phi}{\Psi} = \frac{R-S}{R+S} = \phi$ is preserved by $U(p,q) \cap U(p+q) = U(p) \times U(q)$, where U(p) preserves R and U(q) preserves S. In fact, the orbits of $U(p) \times U(q)$ are the (2n-1)-real-dimensional level sets of ϕ. Any point with homogeneous coordinates $[z^1,\ldots,z^p,z^{p+1},\ldots,z^{p+q}]$ can be moved to another point $[a,0,\ldots,0,z^{p+1},\ldots,z^{p+q}]$ (where $a \neq 0$), by a U(p) transformation. Via a U(q) transformation, this last point may be taken to $[a,0,\ldots,0,0,\ldots,0,b]$. Dividing through by a we obtain the point $[1,0,\ldots,0,\frac{b}{a}]$ on Y as required. Therefore we conclude that $H(\phi)$ on P^{+} has q eigenvalues strictly less than zero and p-1 eigenvalues greater than or equal to zero.

§1.2.3 A plurisubharmonic function on P^+

We want to define a new function for P^+ by taking

$$\rho = \log 1/\phi \ .$$

Then

$$\rho_{z^i} = -\frac{\phi_{z^i}}{\phi} \ ,$$

and

$$\rho_{z^i \bar{z}^j} = -\frac{\phi_{z^i \bar{z}^j}\phi + 2\phi_{z^i}\phi_{\bar{z}^j}}{\phi^2} \ .$$

From (1.8):

$$\phi_{z^i}\phi_{\bar{z}^j}\Big|_Y = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & & & & & \cdot & \cdot \\ \cdot & & \cdot & & & & \cdot & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{4|t|^2}{(1+|t|^2)^4} \end{pmatrix} .$$

Therefore, applying the arguments of the previous subsection to $\phi_{z^i}\phi_{\bar{z}^j}\Big|_Y$, one concludes that on all P_n the eigenvalues of $\phi_{z^i}\phi_{\bar{z}^j}$ are greater than or equal to zero. Hence, on P^+, negative eigenvalues of $H(\phi)$ correspond to positive eigenvalues of $H(\rho)$. Therefore on P^+, $H(\rho)$ has $q = n-(p-1)$ positive eigenvalues which is equivalent to saying that ρ is strictly $(p-1)$-plurisubharmonic. Furthermore ρ lies in the range $0 \le \rho < \infty$, and the set $\{z \in P^+ \text{ s.t. } \rho(z) < c, \forall\, c \in \mathbb{R}\}$ is relatively compact in P^+. Hence P^+ is $(p-1)$-pseudoconvex. P^+ is actually $(p-1)$-complete, since ρ is strictly $(p-1)$-plurisubharmonic on all of P^+ (and not just on the complement of some compact subset of P^+).

§1.2.4 The implications of the finite dimensionality theorems of Andreotti and Grauert [3]

Theorem 14 in [3] implies that for any holomorphic vector bundle V, the dimension of $H^k(P^+,\mathcal{O}(V))$ is finite for $k > p-1$. In fact, since P^+ is p-1 complete, $H^k(P^+,\mathcal{O}(V)) = 0$ for $k > p-1$ (see Corollary in [3], page 248).

To consider the range $0 \le k < p-1$ we need to make use of the pseudoconcavity results in [3]. To do this we will define a positive function on P_n-L^- by setting

$$\psi(z) = \phi(z) + 1 \qquad (\forall\ z \in P_n).$$

Then, the set

$$B_c = \{z \in P_n-L^- \text{ s.t. } \psi(z) > c,\ 0 < c \in \mathbb{R}\}$$

is relatively compact. Furthermore, outside the compact set $\overline{P^+} \subset P_n-L^-$ (i.e. on P^--L^-), $H(\psi)$ has $p = n-(q-1)$ positive (and q-1 negative) eigenvalues. This is equivalent to saying that ψ is strictly (q-1)-plurisubharmonic on $(P_n-L^-)-\overline{P^+}$. Therefore P_n-L^- is (q-1)-pseudoconcave and, by Theorem 14 in [3], $\dim H^k(P_n-L^-,\mathcal{O}(V)) < \infty$ for $0 \le k < p-1$. Furthermore from the lemma in the proof of Theorem 14, together with Theorem 13 and Proposition 22 (in [3]), one concludes that the restriction map,

$$H^k(P_n-L^-,\mathcal{O}(V)) \to H^k(P^+,\mathcal{O}(V)) \qquad (1.12)$$

is an isomorphism for $0 \le k < p-1$.

Actually the lemma of Theorem 14 only implies that

$$H^k(P_n-L^-,\mathcal{O}(V) \cong H^k(B_c,\mathcal{O}(V)) \qquad (1.13)$$

for $c < \inf\limits_{P^+} \psi = 1$.

However, one can always enlarge P^+ by a finite amount by redefining its boundary. For example, replace $\phi = \frac{R-S}{R+S}$ by $\hat{\phi} = \frac{\hat{R}-\hat{S}}{\hat{R}+\hat{S}}$ where $\hat{R} = (1-\alpha)R$ and $\hat{S} = (1+\alpha)S$ $(0 < \alpha < 1)$, so that $\phi = 0 \Longleftrightarrow \hat{\phi} = -\alpha$. This in turn can be achieved by rescaling the homogeneous coordinates in the following way:

$$[\hat{z}^1,\ldots,\hat{z}^p,\hat{z}^{p+1},\ldots,\hat{z}^{p+q}] = [\sqrt{1-\alpha}\, z^1,\ldots,\sqrt{1-\alpha}\, z^p,\sqrt{1+\alpha}\, z^{p+1},\ldots,\sqrt{1+\alpha}\, z^{p+q}].$$

Then $P^+ = \{\hat{z} \in P_n \text{ s.t. } \phi(\hat{z}) > -\alpha\}$ and $P^- = \{\hat{z} \in P_n \text{ s.t. } \phi(\hat{z}) < -\alpha\}$ and hence $B_{1-\alpha} = P^+$, and the right hand side of (1.13) becomes $H^k(P^+,\mathcal{O}(V))$.

Finally, we can now use the cohomological generalization of Riemann's Second Removable Singularities Theorem due to G. Scheja:

Theorem 1.ii [77]. Let K be a compact submanifold of a complex manifold P. Then

$$H^k(P,\mathcal{O}(V)) \to H^k(P-K,\mathcal{O}(V))$$

is an isomorphism for $0 \le k < \operatorname{codim} K - 1$. □

In our case $K = L^-$ and $\operatorname{codim} L^- = n-(q-1) = p$. Therefore for $0 \le k < p-1$ we have

$$\begin{tikzcd}[column sep=small] H^k(P_n,\mathcal{O}(V)) \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & H^k(P^+,\mathcal{O}(V)) \arrow[dl, "\simeq"] \\ & H^k(P_n-L^-,\mathcal{O}(V)) & \end{tikzcd} \qquad (1.14)$$

The groups $H^k(P_n,\mathcal{O}(V))$ may be computed by Bott's generalization of the Borel-Weil theorem [8].

In particular, we will be interested in cases where V is a homogeneous line bundle such as the r^{th} power of the dual of the hyperplane section bundle over P_n : $(L^*)^r \equiv L^{-r}$, whose sheaf of germs of sections is denoted, as usual, by $\mathcal{O}(r)$.

$H^k(P_n,\mathcal{O}(r)) = 0$ unless either (i) $k = 0$ and $r \geq 0$ in which case $H^0(P_n,\mathcal{O}(r))$ (hence also $H^0(P^+,\mathcal{O}(r))$) is isomorphic to the space of homogeneous polynomials of degree r on P_n, or (ii) $k = n$ and $r \leq -n-1$ in which case $H^n(P_n,\mathcal{O}(r))$ (hence also $H^n(P^+,\mathcal{O}(r))$) is isomorphic to the dual of $H^0(P_n,\mathcal{O}(-r-n-1))$.

Finally note that from Theorem 5 of [2] we have for $k \neq p-1$

$$H^k(\overline{P^+},\mathcal{O}(V)) \cong H^k(P^+,\mathcal{O}(V)).$$

Therefore the conclusions of this section for $k \neq p-1$ also apply if we replace P^+ by $\overline{P^+}$. However one could deduce this directly since, by definition

$$H^k(\overline{P^+},\mathcal{O}(V)) = \lim_{\rightarrow} H^k(U_\varepsilon,\mathcal{O}(V))$$

(where the direct limit is taken over open sets U_ε containing $\overline{P^+}$ of the form $\{z \in P_n \text{ s.t. } \phi(z) > -\varepsilon\}$ $(\varepsilon > 0)$.)

Note that in [2] there is also a theorem (Theorem 7, page 804) which states that $H^{p-1}(P^+,\mathcal{O}(V))$ and $H^{q-1}(P^-,\mathcal{O}(V))$ are infinite dimensional. However we can also deduce this via the generalized Penrose transform.

§1.3 Topologies on cohomology

§1.3.1 Description of the various topologies

Let V be a holomorphic vector bundle of rank r over an n-dimensional complex manifold X. The cohomology groups $H^k(X,\mathcal{O}(V))$ can be regarded as linear topological spaces by endowing them with certain topologies. The various possibilities are discussed by H.G. Laufer [50].

A particularly natural topology (originally due to J.P. Serre [80]) can be described as follows. Consider the Dolbeault resolution of $\Omega^j(V)$:

$$0 \to \Omega^j(V) \to \xi^{j,1}(V) \to \xi^{j,2}(V) \to \dots \to \xi^{j,n} \to 0 .$$

Now choose a local trivialization and endow the spaces of smooth (j,k)-forms with the topology of uniform convergence on compact sets for all derivatives of the coefficients. These coefficients form a system of

$$r\,\frac{n!}{j!(n-j)!}\;\frac{n!}{k!(n-k)!}$$

smooth functions. For example, if $\omega \in \xi^{j,k}(V)$, then (locally)

$$\omega = \omega_{\alpha_1\dots\alpha_n\bar{\alpha}_1\dots\bar{\alpha}_n}\, dz^{\alpha_1}\wedge\dots\wedge dz^{\alpha_n}\wedge d\bar{z}^{\bar{\alpha}_1}\wedge\dots\wedge d\bar{z}^{\bar{\alpha}_n} \equiv \omega_{J\bar{J}}\, dz^J\wedge d\bar{z}^{\bar{J}},$$

where $\omega_{J\bar{J}} \equiv \omega_{\alpha_1\dots\alpha_n\bar{\alpha}_1\dots\bar{\alpha}_n} \in \mathcal{O}(V)$.

Let ϕ be a holomorphic coordinate mapping from a neighbourhood U of a compact set $K \subset X$, to an open set in $\mathbb{C}^n$. And, let $\psi : \pi^{-1}(U) \to U\times\mathbb{C}^r$ be the local trivialization. Then, the seminorms

$$\rho_{K,\phi,\psi,L\bar{L}}(\omega) = \sup_{z\in\phi(K)} \sup_{J\bar{J}} \left|D^{L\bar{L}}\psi.\omega_{J\bar{J}}\right|$$

$$(D^{J\bar{J}} \equiv \frac{\partial^{\beta_1+\ldots+\beta_n+\bar{\beta}_1+\ldots+\bar{\beta}_n}}{\partial z^{\beta_1}\ldots\partial z^{\beta_n}\partial\bar{z}^{\bar{\beta}_1}\ldots\partial\bar{z}^{\bar{\beta}_n}})$$

define a Fréchet space topology on the spaces of (j,k)-forms. (This topology is generated by sets of the form $\{\omega \in \xi^{j,k}(V)$ s.t. $\rho_{K,\phi,\psi,L\bar{L}}(\omega-a) < \varepsilon,\ \forall\ K,\phi,\psi,L\bar{L},(\varepsilon > 0)\}$.)

The $\bar{\partial}_k$ operator is then a continuous linear operator between Fréchet spaces. Therefore ker $\bar{\partial}_k$ is a closed subspace of a Fréchet space, and hence it too is a Fréchet space. However, the image of $\bar{\partial}_{k-1}$ is not necessarily closed. Therefore, if one gives to

$$H^k(X,\Omega^j(V)) = \ker\bar{\partial}_k/\mathrm{Im}\ \bar{\partial}_{k-1}$$

the quotient topology, it will not in general be Hausdorff. However, if the topology is Hausdorff then $H^k(X,\Omega^j(V))$ is a Fréchet space.

Laufer also considers cohomology calculated using currents (differential forms with distributions as coefficients). The natural topology for currents is the strong topology as the dual of the spaces of Dolbeault forms (equipped with the topology of uniform convergence on compact sets).

There is also a natural topology for cohomology with compact supports, $H^k_*(X,\Omega^j(V))$. The only difference here is that one starts with the Schwartz topology for the (j,k)-forms (or currents) with compact support.

The last possibility is to use the Čech definition of cohomology with respect to a Leray covering and endow the k-cochains with the topology of uniform convergence on compact sets within the intersections of finitely many open sets of the cover. One then takes the direct product with the direct product topology and gives the k-cocycles the subspace topology. $H^k(X,\Omega^j(V))$ then receives the quotient topology.

Let δ_k be the coboundary operator. Analogously to the Dolbeault case, the space of cocycles, ker δ_k, is closed and therefore a Fréchet space. However, the coboundaries are not necessarily closed so that $H^k(X,\Omega^j(V))$ is not necessarily Hausdorff. If $\mathrm{Im}\delta_{k-1}$ is closed, then $H^k(X,\Omega^j(V))$ is Fréchet.

§1.3.2 Generalization of Serre duality

Theorem 1.iii [50] The strong topologies for $H^k(X,\Omega^j(V))$ and $H_*^k(X,\Omega^j(V))$ with respect to a Leray cover, Dolbeault forms, and currents, all coincide. □

Serre duality states that if X is compact, then

$$[H^k(X,\Omega^j(V))]^* \cong H_*^{n-k}(X,\Omega^{n-j}(V^*)), \tag{1.15}$$

where '[ ]*' denotes the topological dual. The pairing

$$H^k(X,\Omega^j(V)) \otimes_{\mathbb{C}} H_*^{n-k}(X,\Omega^{n-j}(V^*)) \to \mathbb{C}$$

may be given by the wedge product followed by integration over X of the Dolbeault representatives.

In [80], J.P. Serre also established that provided that ∂_k has closed range (i.e. provided that $H^{k+1}(X,\Omega^j(V))$ is Hausdorff) then the isomorphism (1.15) holds even when X is not compact.

Armed with Theorem 1.iii, Laufer identifies the obstructions to $\bar{\partial}_k$ having closed range, and gives formulae that hold when X is not compact:

Theorem 1.iv.a [50]

$$[H^k(X,\Omega^j(V))]^* \oplus R^{k+1}(X,\Omega^j(V)) \cong H_*^{n-k}(X,\Omega^{n-j}(V^*),$$

where $R^{k+1}(X,\Omega^j(V)) = 0$ if and only if $H^{k+1}(X,\Omega^j(V))$ has Hausdorff topology. □

Theorem 1.iv.b [50]

$$[H_*^k(X,\Omega^j(V))]^* \oplus R_*^{k+1}(X,\Omega^j(V)) \cong H^{n-k}(X,\Omega^{n-j}(V^*)),$$

where $R_*^{k+1}(X,\Omega^j(V)) = 0$ if and only if $H_*^{k+1}(X,\Omega^j(V))$ has Hausdorff topology. □

R^{k+1} and R_*^{k+1} have indiscrete (trivial) topologies. R^{q+1} measures the extent to which $\bar{\partial}_{k+1}$ is a relative open map.

Finally, we shall also need to make use of the fact that if $H^k(X,\Omega^j(V))$ is of finite dimension, then it has Hausdorff topology. This follows from Proposition 6 in [80].

§1.3.3 Identification of the linear functionals on $H^{p-1}(P^+,\mathcal{O}(-m-p))$

Theorem 1.v Suppose $p \geq 2$ and $q \geq 2$. Then $H^{q-1}(\overline{P^-},\mathcal{O}(m-q))$ is isomorphic to the topological dual of the Fréchet space $H^{q-1}(P^+,\mathcal{O}(-m-p))$.

Proof We need to make use of the following exact sequence for cohomology with compact supports [29]

$$\ldots \to H_*^{k-1}(A) \to H_*^k(X-A) \to H_*^k(X) \to H_*^k(A) \to \ldots \tag{1.17}$$

where A is any closed subset of X. ($H_*^k(Y) = H^k(Y)$ if Y is compact.)

To prove that $H^{p-1}(P^+,\mathcal{O}(-m-p))$ is Fréchet, from Theorem 1.iv.a we need to show that

$$[H^{p-2}(P^+,\mathcal{O}(-m-p))]^* \cong H_*^{q+1}(P^+,\Omega^n(m+p)) \ . \tag{1.18}$$

This is done by first noting that Ω^n is identified with $\mathcal{O}(-n-1)$ simply by choosing an element of det T, so we have the isomorphism

$$\Omega^n(m+p) \cong \mathcal{O}(m-q) \ . \tag{1.19}$$

Secondly, from (1.14), we know that

$$H^{p-2}(P^+,\mathcal{O}(-m-p)) \cong H^{p-2}(P_n,\mathcal{O}(-m-p)) \ . \tag{1.20}$$

And finally, following also from the conclusions of §1.2.4, $H^q(\overline{P^-},\mathcal{O}(m-q))$ and $H^{q+1}(\overline{P^-},\mathcal{O}(m-q))$ can be put equal to zero in the exact sequence (1.17) (with $A = \overline{P^-}$ and $X = P_n$) to obtain

$$H_*^{q+1}(P^+,\mathcal{O}(m-q)) \cong H^{q+1}(P_n,\mathcal{O}(m-q)). \tag{1.21}$$

The right hand side of both (1.20) and (1.21) are at most finite dimensional and, in any case, Serre duality formula holds, i.e. $\forall$ $p,q \geq 2$ and $\forall$ m, we have

$$[H^{p-2}(P_n,\mathcal{O}(-m-p))]^* \cong H^{q+1}(P_n,\mathcal{O}(m-q)) \ , \tag{1.22}$$

establishing (1.18).

Since $H^p(P^+,\mathcal{O}(-m-p)) = 0$, we immediately have

$$[H^{p-1}(P^+,\mathcal{O}(-m-p))]^* \cong H^q_*(P^+,\Omega^n(m+p)) \ . \tag{1.23}$$

Using (1.19) and substituting

$$H^{q-1}(P_n,\mathcal{O}(m-q)) \cong H^q(P_n,\mathcal{O}(m-q)) \cong 0$$

into the exact sequence (1.17), one obtains the identification required:

$$H^q_*(P^+,\Omega^n(m+p)) \cong H^{q-1}(\overline{P^-},\mathcal{O}(m-q)) . \ \square \tag{1.24}$$

The pairing

$$H^{q-1}(\overline{P^-},\mathcal{O}(m-q)) \otimes_{\mathbb{C}} H^{p-1}(P^+,\mathcal{O}(-m-p)) \to \mathbb{C}$$

is provided by the *dot product*. This process was originally introduced in [28a]. It consists of first taking the cup product and then applying the Mayer-Vietoris connecting homomorphism. For example, if $f \in H^{q-1}(P^-_\varepsilon,\mathcal{O}(m-q))$ (P^-_ε is some neighbourhood of $\overline{P^-}$) and $g \in H^{p-1}(P^+,\mathcal{O}(-m-p))$, then

$$\begin{array}{ccccc} H^{q-1}(P_\varepsilon^-,\mathcal{O}(m-q)) \otimes_{\mathbb{C}} H^{p-1}(P^+,\mathcal{O}(-m-p)) & \xrightarrow{\cup} & H^{n-2}(P_\varepsilon^- \;\; P^+,\mathcal{O}(-p-q)) & \xrightarrow{\delta} & H^{n-1}(P_n,\mathcal{O}(-n-1)) \cong \mathbb{C}. \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (f\ ,\ g) & \longmapsto & f \cup g & \longmapsto & f \cdot g \end{array} \tag{1.25}$$

Note that, for the last identification with $\mathbb{C}$ we have made use of Serre duality.

Corollary 1.vi Suppose $p \geq 2$ and $q \geq 2$. Then

$$[H^{p-1}(P^+,\mathcal{O}(-m-p))]^{**} \cong H^{p-1}(P^+,\mathcal{O}(-m-p))$$

Proof

$$[H^{q-1}(\overline{P^-},\mathcal{O}(m-q))]^* \cong [H^q_*(P^+,\Omega^n(m+p))]^* \cong H^{p-1}(P^+,\mathcal{O}(-m-p))$$

The first isomorphism is clear from (1.24) in the proof of Theorem 1.v. The second follows immediately from Theorem 1.iv.b, provided that $H^{q+1}_*(P^+,\Omega^n(m+p))$ is Hausdorff which is evidently true by (1.21). □

§1.3.4 The case ($p = 1$, $q = 1$)

Theorem 1.v and Corollary 1.vi do not apply, as they stand, to the case ($p=1$, $q=1$). It will be seen that this is essentially because the gauge freedom is not automatically taken care of by the cohomology in this dimension.

For $m \geq 2$ we have, just as before

$$[H^0(P^+,\mathcal{O}(-m-1))]^* \cong H^1_*(P^+,\mathcal{O}(m-1)).$$

However, the identification of $H^1_*(P^+,\mathcal{O}(m-1))$ is now different. From the vanishing of $H^0_*(P^+,\mathcal{O}(V))$ (use uniqueness of analytic continuation) and the exact sequence (1.17) one concludes that for $m \geq 2$

$$H^1_*(P^+,\mathcal{O}(m-1)) \cong H^0(\overline{P^-},\mathcal{O}(m-1))/H^0(P_1,\mathcal{O}(m-1)).$$

(Note that the right hand side above is isomorphic to $H^0(\overline{P^{*+}},\mathcal{O}(-m-1))$ under the twistor transform.)

For $m \leq -1$, just as for the $\mathcal{P}$-transform, one must factor out $H^0(P^+,\mathcal{O}(-m-1))$ by the constant functions, i.e.

$$H^0(P^+,\mathcal{O}(-m-1))/H^0(P_1,\mathcal{O}(-m-1)).$$

The above quotient is a Fréchet space whose topological dual is $H^0(\overline{P^-},\mathcal{O}(-m-1))$.

§1.4 Elementary states

§1.4.1 Generalized definition of elementary states

Consider a Leray cover of $P_n - L^-$ by Stein sets $U_j = P_n - H_j$ where $H_j = \{[Z^1,\dots,Z^{p+q}] \in P_n$ s.t. $Z^j = 0\}$ for $0 \le j \le p\}$, i.e. $U_j = \{[Z^1,\dots,Z^{p+q}] \in P_n$ s.t. $Z^j \neq 0$ for $1 \le j \le p\}$. Following H.B. Laufer [51], (see also [20a]), a (p cochain, $f_{12\dots p} \in \Gamma(U_1 \cap U_2 \cap \dots \cap U_p, \mathcal{O}(r))$, can be represented by the unique Laurent series (convergence in this sense is assumed)

$$f_{12\dots p} = \sum_{\nu_1,\dots,\nu_p \in \mathbb{Z}^p} Q_{\nu_1\dots\nu_p}(Z^{p+1},\dots,Z^{p+q})(Z^1)^{\nu_1}\dots(Z^p)^{\nu_p}, \tag{1.27}$$

whose coefficients $Q_{\nu_1\dots\nu_p}(Z^{p+1}\dots Z^{p+q})$ are polynomials (in the variables $Z^{p+1},\dots,Z^{p+q}$) homogeneous of degree $r - \nu_1 - \dots - \nu_p$, i.e. $Q_{\nu_1\dots\nu_p} \in \Gamma(P_n, \mathcal{O}(r-\nu_1-\dots-\nu_p))$. All $f_{12\dots p}$ represent cocycles, and $f_{12\dots p}$ is a coboundary if

$$f_{12\dots p} = \sum_{k=1}^{p} (-1)^{k-1} g_{12\dots\hat{k}\dots p} \tag{1.28}$$

where $g_{12\dots\hat{k}\dots p} \in \Gamma(U_1 \cap U_2 \cap \dots \cap U_{k-1} \cap U_{k+1} \cap \dots \cap U_p, \mathcal{O}(r))$ ($g_{j\hat{k}\ell\dots m}$ meaning $g_{j\ell\dots m}$). If we now expand each $g_{12\dots\hat{k}\dots p}$ (in (1.28)) in a unique Laurent series, then all coboundaries are represented by

$$\sum_{k=1}^{p} (-1)^{k-1}\Big[\sum_{\nu_k \ge 0} R_{\nu_1\dots\hat{\nu}_k\dots\nu_p}(Z^{p+1}\dots Z^{p+q})(Z^1)^{\nu_1}\dots(Z^{k-1})^{\nu_{k-1}}(Z^{k+1})^{\nu_{k+1}}\dots(Z^p)$$

where $R_{\nu_1\dots\hat{\nu}_k\dots\nu_p}$ is a homogeneous polynomial of degree $r-\nu_1-\dots-\nu_{k-1}-\nu_{k+1}-\dots-\nu_p$. Thus the cohomology class represented by $f_{12\dots p}$ corresponds to the Laurent series

(1.27) with $\nu_1,\ldots,\nu_p < 0$. In other words, if $f \in H^{p-1}(P_n-L^-,\mathcal{O}(r))$, then

$$f = \sum_{\mu_1,\ldots,\mu_p>0} \frac{Q_{\mu_1\ldots\mu_p}(Z^{p+1}\ldots Z^{p+q})}{(Z^1)^{\mu_1}\ldots(Z^p)^{\mu_p}}, \tag{1.29}$$

where $Q_{\mu_1\ldots\mu_p}$ is a homogeneous polynomial of degree $\mu_1 + \ldots + \mu_p + r$ in $Z^{p+1},\ldots,Z^{p+q}$.

Each term on the expansion (1.29) is included in $H^{p-1}(P_n-L^-,\mathcal{O}(r))$ as a Čech representative of a cohomology class with respect to our Leray cover. Each cohomology class represented by the terms of (1.29) is called an *elementary state based at L^-*.

Let

$$H^{p-1}_{\ell}(P_n-L^-,\mathcal{O}(r)) \cong \left\{ \sum_{\substack{\mu_1,\ldots,\mu_p>0 \\ \text{s.t.}\,\mu_1+\ldots+\mu_p=\ell+1}} \frac{Q_{\mu_1\ldots\mu_p}(Z^{p+1},\ldots,Z^{p+q})}{(Z^1)^{\mu_1}\ldots(Z^p)^{\mu_1}} \right\} \tag{1.30}$$

where ℓ is a positive integer. We will refer to elements of these finite dimensional cohomology groups as *ℓ^{th} order elementary states*. Let $H^{p-1}_{\leq\ell}(P_n-L^-,\mathcal{O}(r))$ denote (1.30) with $\mu_1 + \ldots + \mu_p \leq \ell+1$ instead of $\mu_1 + \ldots + \mu_p = \ell+1$. Then there is a filtration

$$H^{p-1}_{\leq 1}(P_n-L^-,\mathcal{O}(r)) \subset H^{p-1}_{\leq 2}(P_n-L^-,\mathcal{O}(r)) \subset \ldots \subset H^{p-1}_{\leq \ell}(P_n-L^-,\mathcal{O}(r)) \subset \ldots \subset H^{p-1}(P_n-L^-,\mathcal{O}(r)).$$

The dimensionality of these groups may be computed using the methods of [20a].

§1.4.2 An orthogonal basis in the 4-dimensional case

In the usual 4-dimensional twistor case, instead of choosing coordinates it is customary to take planes in $\mathbb{P}T$ defined by, say: $\bar{A}_\alpha, \bar{B}_\beta, \bar{C}_\sigma, \bar{D}_\delta \in \mathbb{P}T^*$ such that

$$L^+ = \{[Z^\alpha] \in \mathbb{P}T \text{ s.t. } \bar{C}_\alpha Z^\alpha = 0 = \bar{D}_\alpha Z^\alpha\} \subset \mathbb{P}T^+$$

and

$$L^- = \{[Z^\alpha] \in \mathbb{P}T \text{ s.t. } \bar{A}_\alpha Z^\alpha = 0 = \bar{B}_\alpha Z^\alpha\} \subset \mathbb{P}T^-.$$

Let $(A^\alpha, B^\beta, C^\gamma, D^\delta)$ be a basis for T for which the hermitian form is diagonal, i.e.

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Then the associated basis for T^* can be written as

$$(\bar{A}_\alpha = \frac{\varepsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta}{\Delta}, \quad \bar{B}_\beta = \frac{\varepsilon_{\alpha\beta\gamma\delta} A^\alpha C^\gamma D^\delta}{\Delta},$$

$$\bar{C}_\gamma = \frac{\varepsilon_{\alpha\beta\delta\gamma} A^\alpha B^\beta D^\delta}{\Delta}, \quad \bar{D}_\delta = \frac{\varepsilon_{\alpha\beta\delta\gamma} A^\alpha B^\beta C^\gamma}{\Delta}),$$

where $\Delta \equiv \varepsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta$ ($\varepsilon_{\alpha\beta\gamma\delta}$ is a totally antisymmetric tensor in 4 dimensions). If (Z^1, Z^2, Z^3, Z^4) and (W_1, W_2, W_3, W_4) are the coordinates (with respect to the above basis) of a twistor and a dual twistor respectively, then the rule for twistor complex conjugation is

$$\bar{Z}_1 = \overline{Z^1}, \quad \bar{Z}_2 = \overline{Z^2}, \quad \bar{Z}_3 = -\overline{Z^3}, \quad \bar{Z}_4 = -\overline{Z^4} .$$

With this basis, the projective lines L^+ and L^- correspond to complex conjugate points $\bar{y}^a \in \mathbb{M}^+$ and $y^a \in \mathbb{M}^-$ respectively. In the dual picture, the points $\bar{y}^a$ and y^a correspond to the following projective lines respectively:

$$L_\perp^- = \{W_\alpha \in \mathbb{PT}^* \text{ s.t. } W_\alpha A^\alpha = W_\alpha B^\alpha = 0\} \subset \mathbb{PT}^{*-}$$

and

$$L_\perp^+ = \{[W_\alpha] \in \mathbb{PT}^* \text{ s.t. } W_\alpha C^\alpha = W_\beta D^\beta = 0\} \subset \mathbb{PT}^{*+} .$$

Fig. 1.2 below illustrates the situation.

With respect to this basis, an element of $H^1(\mathbb{PT}-L^-, \mathcal{O}(-m-2))$ may be represented by the (suitably convergent) Laurent series

$$\sum_{\substack{a_1,a_2,a_3,a_4\geq 0 \\ \text{s.t. } a_1+a_2-a_3-a_4=m}} a_1!a_2! \frac{(\bar{C}_\gamma Z^\gamma)^{a_3}(\bar{D}_\delta Z^\delta)^{a_4}}{(\bar{A}_\alpha Z^\alpha)^{a_1+1}(\bar{B}_\beta Z^\beta)^{a_2+1}} .$$

Let the general term in the above series be denoted by

$$e^a_{L^-} = a_1!a_2! \frac{(\bar{C}_\gamma Z^\gamma)^{a_3}(\bar{D}_\delta Z^\delta)^{a_4}}{(\bar{A}_\alpha Z^\alpha)^{a_1+1}(\bar{B}_\beta Z^\beta)^{a_2+1}} \in H^1_{a_1+a_2+1}(\mathbb{PT}-L^-, \mathcal{O}(-m-2)) \quad (1.3$$

where a is the multi-index (a_1,a_2,a_3,a_4) with $a_1+a_2-a_3-a_4 = m$. An elementary twistor theory calculation (see [68] for explicit details) shows that the twistor transform of (1.31a) is given by

$$\hat{e}^a_{L_\perp^+} \equiv \mathcal{T}(e^a_{L^-}) = \Delta a_2!a_3! \frac{(W_\alpha A^\alpha)^{a_1}(W_\beta B^\beta)^{a_2}}{(W_\gamma C^\gamma)^{a_3+1}(W_\delta D^\delta)^{a_4+1}} \in H^1_{a_3+a_4+1}(\mathbb{PT}^*-L_\perp^+, \mathcal{O}(m-2$$

(1.3

(Recall that $L^- \subset \mathbb{PT}^-$ corresponds to $L_\perp^+ \in \mathbb{PT}^{*+}$.) Finally, with our choice of basis, complex conjugation of (1.31b) gives

$$\bar{e}^a_{L^+} \equiv \overline{\hat{e}^a_{L_\perp^+}} = \bar{\Delta} a_2!a_3! \frac{(Z^\alpha\bar{A}_\alpha)^{a_1}(Z^\beta\bar{B}_\beta)^{a_2}}{(Z^\gamma\bar{C}_\gamma)^{a_3+1}(Z^\delta\bar{D}_\delta)^{a_4+1}} \in H^1_{a_3+a_4+1}(\mathbb{PT}-L^+, \mathcal{O}(m-2))$$

(1.3

i.e. an elementary state based at $L^+ \subset \mathbb{PT}^+$.

Note that these general elementary state representatives can all be generated from the first order elementary state by repeated application of the following differential operators:

$$\bar{C}_\alpha \frac{\partial}{\partial \bar{A}_\alpha}\ ,\quad \bar{D}_\beta \frac{\partial}{\partial \bar{A}_\beta}\ ,\quad \bar{C}_\gamma \frac{\partial}{\partial \bar{B}_\gamma}\ ,\quad \bar{D}_\delta \frac{\partial}{\partial \bar{B}_\delta} \tag{1.32}$$

or

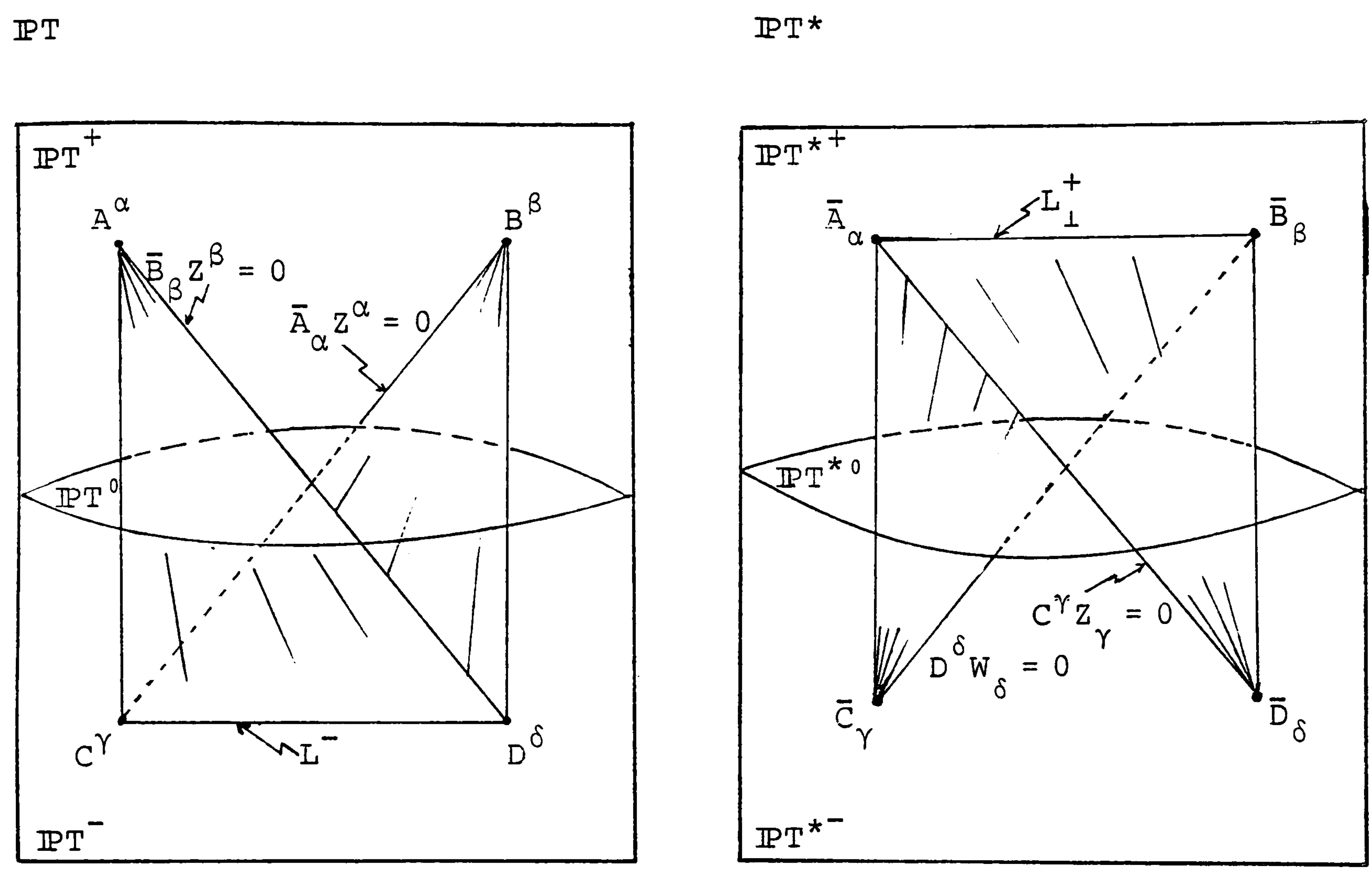
$$A^\alpha \frac{\partial}{\partial C^\alpha}\ ,\quad B^\beta \frac{\partial}{\partial C^\beta}\ ,\quad A^\gamma \frac{\partial}{\partial D^\gamma}\ ,\quad B^\delta \frac{\partial}{\partial D^\delta} \tag{1.32'}$$

For example, if m = 0, then $e^{a}_{L^-}$ can be generated from $e^{(0,0,0,0)}_{L^-}$ as follows

$$e^{a}_{L^-} \equiv \mathbf{D}^{a}_{y} e^{(0,0,0,0)}_{L^-} = \left\{ \begin{array}{ll} \left(\bar{C}_\sigma \frac{\partial}{\partial \bar{A}_\sigma}\right)^{a_3} \left(\bar{D}_\delta \frac{\partial}{\partial \bar{A}_\delta}\right)^{a_1 - a_3} \left(\bar{D}_\gamma \frac{\partial}{\partial \bar{B}_\gamma}\right)^{a_2} & (a_3 \geq a_1) \\ & \\ \left(\bar{C}_\sigma \frac{\partial}{\partial \bar{A}_\sigma}\right)^{a_1} \left(\bar{C}_\delta \frac{\partial}{\partial \bar{B}_\delta}\right)^{a_3 - a_1} \left(\bar{D}_\gamma \frac{\partial}{\partial \bar{B}_\gamma}\right)^{a_4} & (a_3 \geq a_1) \end{array} \right\} \frac{1}{(A_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)} \tag{1.33}$$

($\left(\bar{C}_\sigma \frac{\partial}{\partial \bar{A}_\sigma}\right)^{r}$ meaning $\bar{C}_\sigma \frac{\partial}{\partial \bar{A}_\delta}$ applied r times).

Fig. 1.2



$$\bar{A}_\alpha A^\alpha = \bar{B}_\beta B^\beta = 1 \ , \quad \bar{C}_\gamma C^\gamma = \bar{D}_\delta D^\delta = -1$$

$$\bar{A}_\beta B^\beta = \bar{A}_\gamma C^\gamma = \bar{A}_\delta D^\delta = \bar{B}_\mu C^\mu = \bar{C}_\nu D^\nu = 0$$

$\mathbb{M}^I$		$\mathbb{P}T$		$\mathbb{P}T^*$
y^a	$\longleftrightarrow$	$L^- = C^{[\gamma}D^{\delta]}$	$\longleftrightarrow$	$L_\perp^+ = \bar{A}_{[\alpha}\bar{B}_{\beta]}$
$\bar{y}^a$	$\longleftrightarrow$	$L^+ = A^{[\alpha}B^{\beta]}$	$\longleftrightarrow$	$L_\perp^- = \bar{C}_{[\gamma}\bar{D}_{\delta]}$
p^a	$\longleftrightarrow$	$A^{[\alpha}C^{\gamma]}$	$\longleftrightarrow$	$\bar{B}_{[\beta}\bar{D}_{\delta]}$
$\bar{p}^a$	$\longleftrightarrow$	$B^{[\beta}D^{\delta]}$	$\longleftrightarrow$	$\bar{A}_{[\alpha}\bar{C}_{\gamma]}$
q^a	$\longleftrightarrow$	$A^{[\alpha}D^{\delta]}$	$\longleftrightarrow$	$\bar{B}_{[\beta}\bar{C}_{\gamma]}$
$\bar{q}^a$	$\longleftrightarrow$	$B^{[\beta}C^{\gamma]}$	$\longleftrightarrow$	$\bar{A}_{[\alpha}\bar{D}_{\delta]}$

$$\Delta = \varepsilon_{\alpha\beta\gamma\delta}A^\alpha B^\beta C^\gamma D^\delta \ , \quad \bar{\Delta} = \varepsilon^{\alpha\beta\gamma\delta}\bar{A}_\alpha\bar{B}_\beta\bar{C}_\gamma\bar{D}_\delta$$

§1.5 The density theorem

Theorem 1.vii For fixed $L^- \subset P^-$, the elementary states based at L^- are dense in $H^{p-1}(P^+, \mathcal{O}(-m-p))$.

Proof (case (p = 2, q = 2) only).

From Theorem 1.v we know that $H^{p-1}(P^+, \mathcal{O}(-m-p))$ is a Fréchet space and that linear functionals on it may be identified with $H^{q-1}(\overline{P^-}, \mathcal{O}(m-q))$.

Corollary of the Hahn-Banach Theorem Let M be a subspace of a Fréchet space X, and suppose that F is a continuous linear functional on X. If $F(x) = 0 \ \forall \ x \in M \Longrightarrow F \equiv 0$, then M is dense in X.

Proof $F(x) = 0 \ \forall \ x \in M \Longrightarrow F(x) = 0 \ \forall \ x \in \bar{M}$ by continuity. Suppose $\bar{M} \neq X$, so $z \in X - \bar{M}$. Define a linear functional on the linear span of $\bar{M}$ and z by: $F(x + \lambda z) = \lambda$ where λ is a non-zero scalar. Now extend (using the Hahn-Banach theorem) to a linear functional on X. This gives rise to a contradiction. □

Let $F \in H^{q-1}(\overline{P^-}, \mathcal{O}(m-q))$ be our linear functional on $H^{p-1}(P^+, \mathcal{O}(-m-p))$. We want to show that if F vanishes on the set of all elementary states $H_\infty(L^-) \equiv \bigoplus_{\ell=1}^{\infty} H_\ell^{p-1}(P_n - L^-, \mathcal{O}(-m-p))$, then F is identically zero. That is we want to show that

$$F \cdot e_{L^-} = 0 \ \forall \ e_{L^-} \in H_\infty(L^-) \Longrightarrow F \equiv 0 \ . \tag{1.42}$$

We shall prove this in the 4-dimensional case where we can make use of familiar twistor notation. The generalization for all p and q requires the use of representation theory notation, which we will not introduce here.

We will consider the scalar case (i.e. m = 0) only, but there is no difficulty in treating other homogeneities. Let ϕ denote the negative frequency scalar field corresponding to $F \in H^1(\overline{\mathbb{PT}^-}, \mathcal{O}(-2))$ under the Penrose transform. We use the orthogonal basis of §1.4.2 and denote a representative of a first order elementary state based at

$$L^- = \{[Z^\alpha] \in \mathbb{PT} \text{ s.t. } \bar{A}_\alpha Z^\alpha = 0 = \bar{B}_\beta Z^\beta\} \subset \mathbb{PT}^- ,$$

by

$$e_{L^-}^{(0,0,0,0)} = \frac{1}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)} .$$

It was shown in [20] that $F \cdot e_{L^-}^{(0,0,0,0)}$ is equivalent to the Penrose transform of F evaluated at L^-. That is

$$F \cdot e_{L^-}^{(0,0,0,0)} \cong \phi(y) ,$$

where $\phi(y)$ denotes the field evaluated at the point y in $\mathbb{M}^-$, corresponding to $L^- \subset \mathbb{PT}^-$. Note that $\phi(y)$ may be regarded as a function of $\bar{A}_{[\alpha}\bar{B}_{\beta]}$ (see [65]), where $\bar{A}_\alpha$ and $\bar{B}_\beta$ define the generators of the null cone with vertex at y in $\mathbb{M}^-$.

We will now consider a representative of one of the second order elementary states, say

$$e_{L^-}^{(0,0,1,0)} = \frac{(\bar{C}_\gamma Z^\gamma)}{(\bar{A}_\alpha Z^\alpha)^2(\bar{B}_\beta Z^\beta)} . \tag{1.43}$$

Making use of the differential operators (1.32) we have

$$F\cdot e_{L^-}^{(0,0,1,0)} = -F\cdot\bar{C}_\sigma \frac{\partial}{\partial\bar{A}_\sigma}\left(\frac{1}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)}\right) \tag{1.44}$$

$$= -\bar{C}_\sigma \frac{\partial}{\partial\bar{A}_\sigma}(F\cdot e_{L^-}^{(0,0,0,0)}) \cong -\bar{C}_\sigma \frac{\partial}{\partial\bar{A}_\sigma}(\phi(y)).$$

Thus, if we write the general elementary state representative $e_{L^-}^a$ (using (1.33)) as $\mathbb{D}_y e_{L^-}^{(0,0,0,0)}$, then $\forall$ a

$$F\cdot e_{L^-}^a = 0 \Longrightarrow \mathbb{D}_y^a(F\cdot e_{L^-}^{(0,0,0,0)}) = 0 \Longrightarrow \mathbb{D}_y^a(\phi(y)) = 0.$$

The initial data for an analytic massless free field is given on a null hypersurface [63], and $\mathbb{D}_y^a$ represents all derivatives along the generators of one such hypersurface. Therefore it follows that if all these derivatives vanish then the field itself must be identically zero.

An alternative way of arriving at this conclusion is to note that $e_{L^-}^{(0,0,1,0)}$ can also be written in terms of both helicity raising and helicity lowering operators (see [16,65]) as follows:

$$e_{L^-}^{(0,0,1,0)} = (\bar{C}_\gamma Z^\gamma)A^\delta \frac{\partial}{\partial Z^\delta}\left(\frac{1}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)}\right).$$

If we now choose

$$\bar{C}_\gamma = (0,c^{C'}),\quad \bar{D}_\gamma = (0,d^{D'}),\quad A^\alpha = (a^A,0),\quad B^\beta = (b^B,0) \tag{1.45}$$

and write Z^α in terms of the usual spinors $(\omega^A,\pi_{A'})$, then $F\cdot e_{L^-}^{(0,0,1,0)}$ can be rewritten as

$$F\cdot(C^{D'}\pi_{D'})a^E \frac{\partial}{\partial\omega^E} e_{L^-}^{(0,0,0,0)} .$$

Integrating by parts (which is a well defined operation on cohomology), we obtain

$$-a^E c^{D'}\pi_{D'} \frac{\partial}{\partial\omega^E} (F\cdot e_{L^-}^{(0,0,0,0)}) . \qquad (1.46)$$

Finally, under the Penrose transform,(1.46) becomes (see [16])

$$-ia^A c^{A'}(\partial_{AA'}\phi)(y) ,$$

i.e. the derivative of the field at y.

From [16] it is not difficult to see that in general $\forall$ a

$$F\cdot e_{L^-}^a = 0 \Longrightarrow (\partial_{(A\ldots}^{(A'} \partial_{E)}^{E')}\phi)(y) = 0 .$$

By virtue of the field equations, all derivatives of the field are determined by the symmetrized spinor derivatives [63]. Thus, by analyticity $\phi \equiv 0$ and hence F is identically zero.

Note: Our choice (1.45) corresponds to choosing non-standard coordinates for $\mathbb{M}$ such that y is the origin (which can be moved to $\mathbb{M}^-$ by a complex translation). This is a convenient choice which simplifies the Penrose transform of (1.44') (see [16]). □

The same argument can be used for $H^{p-1}(\overline{P^+},\mathcal{O}(-m-p))$ with the strong topology of the dual of a Fréchet space. Recall that, by Corollary 1.vi, we can identify the dual of $H^{p-1}(\overline{P^+},\mathcal{O}(-m-p))$ with $H^{q-1}(P^-,\mathcal{O}(m-q))$. Thus we also have the following result.

Corollary 1.viii For fixed $L^- \subset P^-$, the elementary states based at L^- are dense in $H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p))$. □

Finally, note that if $p = 1$ and $q = 1$ then L^- is a point and $P_1 - L^- \cong \mathbb{C}$. Thus, in this degenerate case, Theorem 1.vii is simply stating the well known fact that the polynomial functions on the disc are dense (with the topology of uniform convergence on compact sets).

§1.6 The scalar product

We now follow [18] and define a hermitian inner product on $H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p)) \subset H^{p-1}(P^+, \mathcal{O}(-m-p))$ so that the Hilbert space completion (which lies between $H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p))$ and $H^{p-1}(P^+, \mathcal{O}(-m-p))$) supports a unitary representation of $U(p,q)$.

Firstly we need a definition of complex conjugation of cohomology classes. This is provided by the following conjugate linear isomorphism

$$\begin{array}{ccc} H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p)) & \longrightarrow & H^{q-1}(\overline{P^-}, \mathcal{O}(m-q)) \; , \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \bar{f} \end{array} \tag{1.47}$$

defined by

$$\bar{f}(z) = \overline{T(f(z))} \; .$$

In this way the generalized twistor scalar product can be regarded as the continuous functional

$$< | > \; : H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p)) \times H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p)) \to \mathbb{C} \; ,$$

given by [18]

$$<f|g> = \bar{f}\cdot g \in H^{p+q-1}(P_n, \mathcal{O}(-p-q)) \otimes \det T^* \cong \det T \otimes \det T^* \cong \mathbb{C} \, .$$

Continuity of the scalar product (with respect to the topology on $H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p))$ described in §1.3 follows from the continuity of the dot product and the twistor transform. The former is immediate from its Čech definition (which corresponds to multiplying the appropriate cocycles together [28b]). The latter is established from the following fact:

Proposition 1.ix The Penrose transform is a topological (as well as algebraic) isomorphism.

Sketch of proof The spaces of sections of vector bundles over M, $\Gamma(M, \mathcal{O}(V_m))$, are Fréchet spaces with the topology of uniform convergence on compact sets of all derivatives (see [34]). The spaces of generalized fields form closed subspaces of $\Gamma(M, \mathcal{O}(V_m))$ and hence they too are Fréchet spaces. Uniform convergence on a compact set $K \subset M$ corresponds, under the Penrose transform, to uniform convergence on the corresponding compact set K' of P_n where $K' \cong K \times P_{p-1}$. Therefore $\mathcal{P}$ is continuous. And, since $\mathcal{P}$ is also surjective, by the Open Mapping Theorem (see for example [34]), $\mathcal{P}$ is a topological isomorphism. □

Hermiticity of the scalar product is clear from the definition of the twistor transform in terms of the propagator [20,18].

U(p,q) invariance of the scalar product is manifest since its definition is coordinate free.

Positive definiteness of the scalar product can now be proven directly from the density of the elementary states:

Theorem 1.x $\langle h|h\rangle \geq 0 \ \forall \ h$ and $\langle h|h\rangle = 0 \Longleftrightarrow h = 0$.

Proof It is straight forward to verify by direct calculation that the elementary states are suitably orthogonal with respect to $\langle\,|\,\rangle$ and that $\langle\,|\,\rangle$ is positive definite on each elementary state. As an example, consider the $(p = 2,\ q = 2)$ case. Let $e^a_{L^-}, e^b_{L^-} \in H^1(\mathbb{P}T - L^-, \mathcal{O}(-m-2))$ be two general representatives such as (1.31a). Then

$$\langle e^b_{L^-} | e^a_{L^-}\rangle = \bar{e}^b_{L^+} \cdot e^a_{L^-} \,, \tag{1.49}$$

where $\bar{e}^b_{L^+}$ is given by (1.31c). With this choice of basis, (1.49) becomes [68]

$$\langle e^b_{L^-} | e^a_{L^-}\rangle = a_1!a_2!a_3!a_4!\,\delta_{ab} \,,$$

where

$$\delta_{ab} = \begin{cases} 1 & \text{if } a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,\ a_4 = b_4 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle\ |\ \rangle$ is continuous and the elementary states are dense, evidently $\langle\ |\ \rangle$ cannot be negative. To show that it is actually positive definite one must check that the orthogonal complement of the space of elementary states is empty. To do this take $f \in H^{p-1}(\overline{P^+}, \mathcal{O}(-m-p))$ and suppose that $f \not\equiv 0$ and that $\langle f|f\rangle = 0$. Let e_{L^-} be an elementary state based at $L^- \subset P^-$. Then $\forall\ \lambda \in \mathbb{C}$

$$0 \le \langle\lambda f+e_{L^-}|\lambda f+e_{L^-}\rangle = \langle\lambda f|\lambda f\rangle + \langle\lambda f|e_{L^-}\rangle + \langle e_{L^-}|\lambda f\rangle + \langle e_{L^-}|e_{L^-}\rangle$$
$$= \lambda\bar{\lambda}\langle f|f\rangle + \bar{\lambda}\overline{\langle e_{L^-}|f\rangle} + \lambda\langle e_{L^-}|f\rangle + \langle e_{L^-}|e_{L^-}\rangle$$
$$= \qquad \mathrm{Re}\{\lambda\langle e_{L^-}|f\rangle\} + \langle e_{L^-}|e_{L^-}\rangle.$$

Since Reλ may be negative, we must have $\langle e_{L^-}|f\rangle = 0$ for all elementary states e_{L^-}.

Now

$$\langle e_{L^-}|f\rangle = \bar{e}_{L^+}\cdot f ,$$

where $\bar{e}_{L^+}$ is the elementary state based at $L^+ \subset P^+$ corresponding to $\bar{y} \in \mathbb{M}^+$ (i.e. to the complex conjugate point of $y \in \mathbb{M}^-$). Therefore, arguing as in the proof of Theorem 1.vi we conclude that $\bar{e}_{L^+}\cdot f = 0 \Longrightarrow f \equiv 0$. This gives rise to a contradiction. □

§1.7 The role of the elementary states in representation theory

§1.7.1 The general case

$H^{p-1}(P^+,\mathcal{O}(-m-p))$ supports an infinite dimensional representation of $U(p,q)$ which may be defined as follows. Write the homogeneous coordinates $Z \equiv [Z^1,\ldots,Z^{p+q}]$ as a column vector and let $M \in U(p,q)$ act on them by matrix multiplication. If f is a representative of a cohomology class in $H^{p-1}(P^+,\mathcal{O}(-m-p))$, then under the action of M, $f \mapsto f_M$ where $f_M(Z) = f(M^{-1}Z)$.

For each ℓ, the spaces $H^{p-1}_{\ell}(P_n-L^-,\mathcal{O}(r))$ support a finite dimensional irreducible representation of $U(p)\times U(q)$. Since $U(p)\times U(q)$ is the maximal compact subgroup of $U(p,q)$, this is equivalent to saying that the space of elementary states, $H_\infty(L^-) \equiv \bigoplus_{\ell=1}^{\infty} H^{p-1}_{\ell}(P_n-L^-,\mathcal{O}(-m-p))$, is the space of *$U(p)\times U(q)$-finite vectors* (cf. [78]).

$U(q)$ acts on the numerator variables of (1.30) and corresponding representation on the homogeneous polynomials of degree $\ell+1+r$, $Q_{\mu_1\ldots\mu_p}(Z^{p+1},\ldots,Z^{p+q})$, is that of the totally symmetric dual representation $(\mathrm{Sym}^{\ell+1+m})^*$. $U(p)$ acts on the denominator variables (of (1.30)) and the corresponding representation on $\dfrac{1}{(Z^1)^{\mu_1}\ldots(Z^p)^{\mu_p}}$ is that of the totally symmetric representation tensored with the determinant representation $\mathrm{Sym}^{\ell+1-p} \otimes \mathrm{Det}$. The representation of $U(p)\times U(q)$ is the tensor product of the above two representations. Note that the action of $U(p)$ is more complicated since it involves changing the Čech cover (which is *not* invariant, cf. [64]).

From the density theorem of §1.5 one concludes that the representation of $U(p,q)$ supported by the Fréchet space $H^{p-1}_{\xi}(P^+, \mathcal{O}(-m-p))$ is irreducible, otherwise the closure of $H_\infty(L^-)$ would provide a proper invariant subspace.

There are abstract means of dealing with the above finite dimensional irreducible representations (see [5a,18a,22]). However these will not be discussed here. We shall limit ourselves to illustrating the action of $U(2)\times U(2)$ on classical elementary states with the aid of a simple example.

§1.7.2 An example

Take $r = -2$. Then a representative of a first order elementary state (i.e. an element of $H^1_1(\mathbb{PT}-L^-,\mathcal{O}(-2))$ can be written as $\frac{1}{(Z^1)(Z^2)}$. Let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$. Then under the action of $A \in U(2)$

$$\frac{1}{(Z^1)(Z^2)} \longmapsto \frac{1}{(aZ^1 + bZ^2)(cZ^1 + dZ^2)}.$$

Suppose $\left|\frac{bc}{da}\right| < 1$. Then one may rewrite the right hand side above and expand it in a Laurent series (convergent within the range $\left|\frac{c}{d}\right| < \left|\frac{Z^2}{Z^1}\right| < \left|\frac{a}{b}\right|$) as follows:

$$\frac{1}{a(Z^1)[1 + \frac{bZ^2}{aZ^1}]} \; \frac{1}{(Z^2)[1 + \frac{cZ^1}{dZ^2}]}$$

$$= \frac{1}{a(Z^1)d(Z^2)} [1-(\frac{bZ^2}{aZ^1})+(\frac{bZ^2}{aZ^1})^2-(\frac{bZ^2}{aZ^1})^3+\ldots][1-(\frac{cZ^1}{dZ^2})+(\frac{cZ^1}{dZ^2})^2-(\frac{cZ^1}{dZ^2})^3+\ldots]$$

$$= \frac{1}{a(Z^1)d(Z^2)} [1+(\frac{bc}{da})+(\frac{bc}{da})^2+(\frac{bc}{da})^3+\ldots] + \text{(cross terms)}$$

$$= \frac{1}{ad(Z^1)(Z^2)} [\frac{1}{1-\frac{bc}{ad}}] + \text{(cross terms)}$$

$$= \frac{1}{(Z^1)(Z^2)} \frac{1}{\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}} + \text{(cross terms)}$$

$$= \frac{1}{(Z^1)(Z^2)} \det A + \text{(cross terms)}$$

It is not difficult to convince oneself that the cross terms are in fact coboundaries and may be discarded.

We now consider the second order elementary states, i.e. elements of $H^1_2(\mathbb{P}T-L^-,\mathcal{O}(-2))$. A typical representative can be written as

$$[(Z^3)+(Z^4)]\,[\frac{1}{(Z^1)^2(Z^2)}+\frac{1}{(Z^1)(Z^2)^2}]\,. \tag{1.51}$$

Suppose $B^{-1}\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in U(2)$. Then under the action of $B\in U(2)$ the left hand factor of (1.51) transforms as

$$\begin{pmatrix}(Z^3)\\(Z^4)\end{pmatrix}\longmapsto\underbrace{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}_{B^{-1}}\begin{pmatrix}(Z^3)\\(Z^4)\end{pmatrix}.$$

Now consider the right hand factor of (1.51) under the action of $A\in U(2)$. Discarding coboundary terms, a straightforward calculation (similar to that for the first order case above) leads to

$$\begin{pmatrix}\dfrac{1}{(Z^1)^2(Z^2)}\\[2ex] \dfrac{1}{(Z^1)(Z^2)^2}\end{pmatrix}\longmapsto\underbrace{\frac{1}{\left(\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)^2}\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}}_{(\det A)A^T}\begin{pmatrix}\dfrac{1}{(Z^1)^2(Z^2)}\\[2ex] \dfrac{1}{(Z^1)(Z^2)^2}\end{pmatrix}.$$

As a last example consider the representative of $H^1_3(\mathbb{P}T-L^-,\mathcal{O}(-2))$ given by

$$[(Z^3)^2+(Z^3)(Z^4)+(Z^4)^2][\frac{1}{(Z^1)^3(Z^2)}+\frac{1}{(Z^1)^2(Z^3)^2}+\frac{1}{(Z^1)(Z^3)^3}]\,.$$

Discarding coboundary terms, similar calculations (omitted here) lead to

$$\begin{pmatrix} (Z^3)^2 \\ (Z^3)(Z^4) \\ (Z^4)^2 \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}}_{\odot^2 B^{-1}} \begin{pmatrix} (Z^3)^2 \\ (Z^3)(Z^4) \\ (Z^4)^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dfrac{1}{(Z^1)^3(Z^2)} \\ \dfrac{1}{(Z^1)^2(Z^2)^2} \\ \dfrac{1}{(Z^1)(Z^2)^3} \end{pmatrix} \longmapsto \underbrace{\frac{1}{\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^3} \begin{pmatrix} d^2 & -cd & c^2 \\ -2bd & ad+bc & -2ac \\ b^2 & -ba & a^2 \end{pmatrix}}_{(\det A)\odot^2 A^T} \begin{pmatrix} \dfrac{1}{(Z^1)^3(Z^2)} \\ \dfrac{1}{(Z^1)^2(Z^2)^2} \\ \dfrac{1}{(Z^1)(Z^2)^3} \end{pmatrix}.$$

PART II

CHAPTER 2: SUPERGEOMETRY

§2.0 Introduction

A mathematical formulation of a supermanifold theory was initiated, independently, by Kostant [49] and Leites [54]. This formulation is based on powerful and fruitful ideas (originally due to Grothendieck [32]) which lie at the heart of analytic and algebraic geometry. Within the spirit of these subjects one regards the sheaf of functions on a manifold as the primary object. All fundamental constructions of differential geometry are then defined in terms of it; and the underlying manifold plays a secondary role.

A supermanifold is an ordinary manifold with its structure sheaf augmented so as to contain anticommuting elements. This is achieved by demanding that locally the augmented sheaf be isomorphic to the sheaf of sections of an exterior algebra of $\mathbb{R}^N$ or $\mathbb{C}^N$. In the smooth category, the global structure of such objects was investigated by M. Batchelor [6]. She proved that a supermanifold contained no more information than that of the exterior algebra of the sheaf of sections of a smooth vector bundle (called the underlying vector bundle of the supermanifold). However, in the holomorphic category, the situation is much more interesting. P. Green [30] showed that there is an additional structure over and above that of the exterior algebra of the underlying vector bundle. We shall deal entirely with the latter category.

It should be mentioned at this point that the approach of Kostant and Leites is by no means the only approach to supermanifolds. In fact, for reasons which are not altogether clear to us, most authors prefer to adopt a very different approach. This latter approach starts by mimicking the usual differential geometric definition of a manifold; replacing $\mathbb{R}^n$ (or $\mathbb{C}^n$) by a so-called super Euclidean space $\mathbb{E}_L^{n|N} \cong (\Lambda_{\bar{0}})^n \oplus (\Lambda_{\bar{1}})^N$, where $(\Lambda_{\bar{0}})^n$ denotes n copies of the even part of the exterior algebra of $\mathbb{R}^L$ (or $\mathbb{C}^L$), and $(\Lambda_{\bar{1}})^N$ denotes N copies of the odd part. The number L is usually taken to be very large, and sometimes even infinite. In order to have a definition of a manifold, one further needs to give a topology to $\mathbb{E}_L^{n|N}$, and to give some sort of 'super smooth' or 'super holomorphic' criteria. This is where the various authors differ. On the one hand, $\mathbb{E}_L^{n|N}$ has a natural Banach space topology by virtue of it being a $2^{L-1}(n+N)$-dimensional vector space. This is the topology used by A. Rogers [75], A. Jadzyck and K. Pilch [46], and C. Boyer and S. Gitler [9]. However, each of these authors has a different definition of super smoothness and super analyticity. The alternative topology for $\mathbb{E}_L^{n|N}$ is the (non-Hausdorff) coarser topology, also known as the DeWitt topology [15]: 'A set U is open in $\mathbb{E}_L^{n|N}$ if and only if $U = \varepsilon^{-1}(V)$ for some V open in $\mathbb{R}^n$ (or $\mathbb{C}^n$) (ε is the canonical projection of $\mathbb{E}_L^{n|N}$ onto $\mathbb{R}^n$ (or $\mathbb{C}^n$), induced by the canonical augmentation $\Lambda^0\mathbb{R}^L \to \mathbb{R}$ (or $\Lambda^0\mathbb{C}^L \to \mathbb{C}$))'.

Using the DeWitt topology, Batchelor [7] claimed to have found a suitable definition of super smoothness so as to prove an equivalence between the supermanifolds obtained in this way and Kostant-Leites supermanifolds. However the definition of super smoothness needed to prove such an equivalence was rather restrictive, and did not seem to interest the various authors who prefer to work with more general definitions such as those of Rogers [75].

In this thesis we shall adopt the approach of Kostant and Leites. We start by giving, in §2.1, a brief review of some standard constructions in analytic geometry. We hope that this review would make some of the concepts involved in the theory of supermanifolds appear more natural to the twistor theorist.

The organization of §2.1 is as follows. Subsections §2.1.1 to §2.1.3 deal with ringed spaces and $\mathbb{C}$-analytic spaces. In §2.1.4 we give standard algebraic definitions of tangent spaces, cotangent spaces and jets at a point. In §2.1.5 we follow D.C. Spencer [87] and describe the algebraic constructions of the tangent, cotangent and jet bundles. These constructions are based on an idea of Grothendieck [32] which consists of regarding the topological space of interest as being embedded as the diagonal in the product of two copies of itself. (This idea will play a crucial role in Chapters 3 and 4.) In §2.1.6, following [27,12,24] we give a brief description of infinitesimal neighbourhoods which will be used throughout the rest of the thesis. Finally, in §2.1.7 we state the definition of complex manifold thickenings [24], which is completely analogous to the definition of a supermanifold (§2.3.1).

Following D.A. Leites [54], in §2.2 we give the various definitions of super linear algebra which will be needed in latter chapters. They include: vector superspaces, superalgebras, supermatrices, A-supermodules and A-superalgebras (A is a $\mathbb{Z}_2$-graded commutative superalgebra). The structure of A-supermodules, discussed in §2.2.4, will play a crucial role in understanding supervector bundles (§2.3.6).

§2.3 deals with supermanifolds. The definition of a Kostant-Leites supermanifold is given in §2.3.1. Filtrations of supermanifolds are defined in §2.3.2. We then follow [24] and discuss (in §2.3.3) their global structure using the results of Batchelor [6] and Green [30]. In §2.3.4 we give the definition of supercoordinates and discuss the various ways in which they may be viewed. (Equivalent definitions of supercoordinates (as superfunctions) are given in [49] and [54].) Furthermore, using the results of [6,30,24] (outlined in §2.3.3), we define (in §2.3.5) what we mean by 'changes of supercoordinates' and illustrate this with a simple example (in §2.3.6). Our discussion here is essential to understanding the supercoordinate dependent constructions of Chapters 3 and 4.

Finally, §2.3.6 to §2.3.10 deal with supervector bundles and superconnections. Note that these are by no means the only possible definitions. In particular, there are more general definitions of $\mathbb{Z}_2$-graded vector bundles and connections due to Quillen [71]. However our definitions have been chosen so as to agree with what is required to interpret super Yang-Mills theory, which is the subject of the next chapter. Indeed, the 'super abelian version' of super Yang-Mills theory provided us with a test for our superline bundle definition of §2.3.7.

Super derivations and differential forms on supermanifolds have been discussed by Kostant [49] and Leites [54]. Consequently, the original contributions in §2.3.8 and §2.3.9 merely consist of interpreting their constructions in terms of our supervector bundle definitions.

§2.1 Some standard constructions in analytic geometry [27,87,24]

§2.1.1 Ringed spaces

A *ringed space* is a pair $(X,\mathcal{A})$ consisting of a topological space, X, and a sheaf of rings on $(X,\mathcal{A})$. A *morphism of ringed spaces* $\phi : (X,\mathcal{A}) \to (Y,\mathcal{B})$ consists of a continuous map $\phi : X \to Y$ and a homomorphism of sheaves of rings $\tilde{\phi} : \mathcal{B} \to \phi_* \mathcal{A}$, i.e. a homomorphism of rings $\tilde{\phi} : \Gamma(U,\mathcal{B}) \to \Gamma(\phi^{-1}(U),\mathcal{A})$ $\forall$ U open in Y.

A very important example of a ringed space is the pair consisting of an ordinary complex manifold and its sheaf (of germs) of holomorphic functions, $\mathcal{O}$. All fundamental concepts of differential geometry involve constructions which can be realized from knowledge of $\mathcal{O}$ alone. This is indeed the underlying philosophy of analytic geometry which regards $\mathcal{O}$ as the primary object.

A supermanifold is also another example of a ringed space where the sheaf of rings in question is $\mathbb{Z}_2$-graded. However, before giving their precise definition we shall recall some familiar constructions of analytic geometry. The concepts involved here are very closely related to those needed in the study of supermanifolds and in some cases we shall use them as guiding analogies.

§2.1.2 Analytic spaces

A ringed space (X,A) is called a *$\mathbb{C}$-ringed space* if A is a sheaf of $\mathbb{C}$-algebras, and if A_x (the stalk of A over $x \in X$) is a local ring [4] such that for every $x \in X$, $A_x/M_x \cong \mathbb{C}$ is an isomorphism of $\mathbb{C}$-algebras, where M_x denotes the unique maximal ideal of A_x. If furthermore: (i) X is Hausdorff and, (ii) for every $x \in X$, there exists an open neighbourhood U of x such that $(U,A|_U)$ is isomorphic to some local model, then (X,A) is said to be a *$\mathbb{C}$-analytic space.* (A *local model* is a ringed space of the form $(\mathrm{supp}\,0/I,(0/I)|_{\mathrm{supp}\,0/I})$ where 0 (temporarily) denotes the sheaf of germs of holomorphic functions on an open subset of $\mathbb{C}^n$, I denotes a coherent ideal in 0, and 'supp' means the point set where the sheaf is non-zero.)

A $\mathbb{C}$-analytic space (X,A) is called *non-singular* at $x \in X$ if there exists an open neighbourhood U of x such that $(U,A|_U)$ is isomorphic to some local model of the form $(U,0_W)$, where $W \subset \mathbb{C}^n$ is an open subset. If it is non-singular for every $x \in X$ and X is paracompact, then (X,A) is called a complex manifold.

We will often abuse notation and omit restriction symbols on sheaves. For example if A is a sheaf on X and $U \subset X$, we will often (though not always) write $(U,A|_U)$ as (U,A). From now on when $U = \mathrm{supp}A \subset X$, the restriction symbol will always be omitted.

An *analytic sheaf* S on a $\mathbb{C}$-analytic space (X,A) is a sheaf of A-modules. Furthermore, S is said to be *coherent* if for each $x \in X$, there exists an open neighbourhood U of x such that there is an exact sequence

$$0 \to A^p|_U \to A^q|_U \to S_U \to 0.$$

Let (X,A) be a $\mathbb{C}$-analytic space and suppose that X is a complex manifold with sheaf (of germs) of holomorphic functions denoted by $\mathcal{O}$. Then there is a short exact sequence

$$0 \to I \to A \overset{\varepsilon}{\to} \mathcal{O} \to 0,$$

where I is a coherent sheaf of ideals in A, and ε is a canonical homomorphism (given by $f \mapsto \tilde{f}$, with $\tilde{f}$ defined by $\tilde{f}(x) := f(x)\ \forall\ x \in X$) called the *augmentation map*. The ideal sheaf I is usually called the *ideal sheaf of nilpotents*, or the *nilradical*. (X,A) is said to be a *reduced* $\mathbb{C}$-analytic space if and only if $(X,A) \cong (X,A/I)$, otherwise it is called *unreduced*.

§2.1.3 Vector bundles on analytic spaces

A holomorphic *vector bundle* E of rank r on (X,A) is defined, via its transition functions, as an element of $H^1(X,GL(r,A))$.

The familiar one-to-one correspondence between vector bundles and locally free analytic sheaves holds for $\mathbb{C}$-analytic spaces. If E is a vector bundle on (X,A), then $A(E)$ will denote its corresponding locally free sheaf of A-modules. Conversely, we will sometimes define vector bundles on $\mathbb{C}$-analytic spaces (X,A) as locally free sheaves of A-modules and make no notational distinction between the two concepts.

Note: If $\mathcal{O}$ denotes the sheaf of holomorphic functions on X, then $GL(r;\mathcal{O}) \cong \mathcal{O}(GL(r,\mathbb{C}))$. This isomorphism no longer holds if $\mathcal{O}$ is replaced by a more general sheaf, say A.

§2.1.4 Algebraic definitions of tangent spaces, cotangent spaces and jets at a point

A *derivation* or *tangent vector* to (X,A) at a point $x \in X$ is a $\mathbb{C}$-linear map $D_x : A_x \to \mathbb{C}$, satisfying

$$D_x(f\cdot g) = (Df)\cdot g(x) + f(x)\cdot D(g) \quad \forall\ f,g \in A\ . \qquad (2.2)$$

It is not difficult to show [27] that condition (2.2) is equivalent to the statement that D_x annihilates: (i) the constant elements of A_x, i.e. $D_x(\mathbb{C}\cdot 1) = 0$, and (ii) the ideal of functions vanishing at x to second order, i.e. $D_x(M_x^2) = 0$.

The restriction of derivations (at $x \in X$) to M_x induces a $\mathbb{C}$-linear map $M_x/M_x^2 \to \mathbb{C}$. In this way one identifies $(M_x/M_x^2)^*$ and M_x/M_x^2 with the tangent and cotangent spaces (at a point $x \in X$) respectively.

These constructions may be generalized to vector bundles E simply by replacing M_x by $M_x(E) := M_x \otimes_A A(E)$.

Another closely related concept is that of jets [87]. The *k-jet* of E at $x \in X$ is defined as

$$J_k(E)_x = A(E)_x / M^{k+1}(E)_x\ , \qquad (2.3)$$

where one sets $J_o(E)_x = E_x$, i.e. $A(E)_x/M(E)_x$ is identified with the fibre of E at x (recall that $A_x/M_x \cong \mathbb{C}$). Thus, for the 1-jet of E one has the exact sequence

$$0 \to M(E)_x/M_x^2(E) \to J_1(E)_x \to A(E)_x/M(E)_x \to 0. \qquad (2.4)$$

§2.1.5 Global constructions

The constructions of §2.1.4 are local ones. Taking the disjoint union of the above vector spaces for all $x \in X$, in general fails to give rise to vector bundles. We will summarize (following D.C. Spencer [87]), a different point of view originally adopted by B. Malgrange [56], and based on ideas of A. Grothendieck [32].

Let X be a differentiable manifold embedded as the diagonal in $X \times X$ (i.e. $X \xhookrightarrow{\Delta} X \times X$), and let $\mathcal{O}_{X \times X}$ be the structure sheaf on $X \times X$. Also let I denote the ideal sheaf consisting of those elements of $\mathcal{O}_{X \times X}$ which vanish on the diagonal. Then the structure sheaf of X (as the diagonal) is given by

$$A := \mathcal{O}_{X \times X}/I .$$

One may now give an invariant algebraic characterization of the cotangent bundle, T*, of X by taking the *sheaf of differential 1-forms* to be

$$\Omega^1 := I/I^2 \qquad (2.5)$$

This is a locally free sheaf with support on X as the diagonal in $X \times X$.

Let π_1 and π_2 respectively denote the projections of the first and second factors of $X \times X$ onto X, and denote their pull backs by $i \equiv \pi_1^* : A \to \mathcal{O}_{X\times X}$ and $j \equiv \pi_2^* : A \to \mathcal{O}_{X\times X}$ respectively. Then the *exterior derivative* may be characterized as follows: take $(\zeta,\eta) \in X \times X$ and define

$$d : A \xrightarrow{i-j} \mathcal{O}_{X\times X} \longrightarrow \mathcal{I}/\mathcal{I}^2 \tag{2.6}$$

by $f \longmapsto f(\zeta)-f(\eta) \equiv df(\zeta,\eta)$. Clearly df is linear and vanishes on the diagonal, so $df \in \mathcal{I}/\mathcal{I}^2$. To check that d is a derivation, suppose $f,g \in \mathcal{O}_{X\times X}$, then

$$\begin{aligned} fg \longmapsto fg(\zeta) - fg(\eta) &= f(\zeta)g(\zeta) - f(\eta)g(\eta) \\ &= f(\zeta)[g(\zeta)-g(\eta)]+[f(\zeta)-f(\eta)]g(\eta) \\ &= f(\zeta)dg(\zeta,\eta) + df(\zeta,\eta)g(\eta), \end{aligned}$$

where dg and df both vanish on the diagonal as required.

Finally, *k-jets* are now defined as the quotient

$$J_k := \mathcal{O}_{X\times X}/\mathcal{I}^{k+1} , \tag{2.7}$$

with $J_0 = A$.

§2.1.6 Infinitesimal neighbourhoods

The next examples of analytic spaces are of great importance in twistor theory. They will also play a fundamental role in the construction of the non-trivial supermanifolds of 'physical' interest.

Let $(Y,\mathcal{O}_Y)$ be a $\mathbb{C}$-analytic space, and let $X = \mathrm{supp}(A/I)$ where I is the coherent sheaf of ideals consisting of functions vanishing on X. Then $X_{(m)} \equiv (X,A_{(m)})$ with $A_{(m)} = \mathcal{O}_Y/I^{m+1}$, is called the m^{th} *infinitesimal neighbourhood of X in Y.* Note that there is a filtration

$$(X,A) \equiv X_{(0)} \subset X_{(1)} \subset X_{(2)} \subset \ldots \subset X_{(m)} \subset \ldots ,$$

with $X_{(m)} \subset X_{(m+1)}$ defined by

$$0 \to I^{m+1}/I^{m+2} \to A_{(m+1)} \to A_{(m)} \to 0 . \qquad (2.10)$$

If (X,A) and $(Y,\mathcal{O}_Y)$ are non-singular, then

$$\tilde{I}^{m+1} \equiv I^{m+1}/I^{m+2}$$

is a locally free analytic sheaf on X. In fact, it is the sheaf of germs of sections of the $m+1^{\text{th}}$ symmetric tensor power of the conormal bundle of X holomorphically embedded as a closed submanifold in Y, i.e.

$$\tilde{I}^{m+1} \cong \odot^{m+1}N^*$$

where

$$N^* \cong T^*Y|_X/T^*X.$$

A section of $A_{(m)}$ is determined by the value of the function on X together with all the $(m+1)^{\text{th}}$ normal derivatives.

§2.1.7 Thickenings of complex manifolds

The definition of infinitesimal neighbourhoods depended upon representing X within a larger topological space Y. If X is a complex manifold it is possible to give a local definition of thickenings of X which does not require any mention of an ambient manifold.

A *thickening of a complex manifold* X [24] is a $\mathbb{C}$-analytic space, denoted by $X_{(m)} \equiv (X, A_{(m)})$, such that: (i) $\exists$ an augmentation map $\varepsilon : A_{(m)} \to A$, and (ii) for every $x \in X$ $\exists$ an open neighbourhood U of x such that

$$A_{(m)}|_U \cong A|_U[t]/(t^{m+1})$$

is an isomorphism of sheafs of $\mathbb{C}$-algebras augmented over $\mathbb{C}$.

Informally, one may define (following [24]) 'local coordinates on $X_{(m)}$' as the pair (x,t), where x denotes the local coordinates on an open subset U of X, and t denotes the nilpotent such that $A_{(m)}|_U \cong A|_U[t]/(t^{m+1})$.

In the next chapter (§3.1.3) we shall come across an unreduced $\mathbb{C}$-analytic space whose definition is very similar to that of a $\mathbb{C}$-manifold thickening. It is formally constructed by adding a certain nilpotent variable with some further structure. In other words, it is defined exactly as $\mathbb{C}$-manifold thickening except that the ideal (t^{m+1}) is replaced by a more complicated one.

§2.1.8 Composition series notation

The symbol '+' between sheaves has the following meaning.

$$H = A + B \iff 0 \to B \to H \to A \to 0.$$

$$H = A + B + C \iff \text{either (i) } 0 \to C \to H \to A + B \to 0$$

$$\text{and } 0 \to B \to A + B \to A \to 0$$

$$\text{or equivalently (ii) } 0 \to B + C \to H \to A \to 0$$

$$\text{and } 0 \to C \to B + C \to B \to 0$$

$$H = A + B + C + D \iff \ldots$$

etc.

Example: (2.10) may be rewritten as

$$A_{(m)} = A_{(m-1)} + \tilde{I}^m$$

$$= A + \tilde{I}^1 + \tilde{I}^2 + \ldots + \tilde{I}^m .$$

Note that a splitting of a short exact sequence converts the composition series symbol '+' into a direct sum symbol '⊕'.

§2.2 Super linear algebra [54]

§2.2.1 Superspaces and superalgebras

A *vector superspace*, M, is a $\mathbb{Z}_2$-graded vector space. That is, a vector space M admitting a $\mathbb{Z}_2 = \{\bar{0},\bar{1}\}^{\dagger}$-decomposition into two homogeneous (sometimes also referred to as pure) components (i.e. $M = M_{\bar{0}} \oplus M_{\bar{1}}$). Thus any element v, in M, can be written as $v = v_{\bar{0}} + v_{\bar{1}}$, where $v_{\bar{0}} \in M_{\bar{0}}$ and $v_{\bar{1}} \in M_{\bar{1}}$ are called, respectively, the even and odd parts of v. A pure element v of M is said to be even if and only if $v_{\bar{1}} = 0$ and odd if $v_{\bar{0}} = 0$. The $\mathbb{Z}_2$-*grading* of a pure element w is denoted by $\bar{w}$. The *dimension* of M is denoted by $r|s$, where $r = \dim M_{\bar{0}}$ and $s = \dim M_{\bar{1}}$.

If M and N are vector superspaces one can form new vector superspaces, $M \oplus N$, $M \otimes N$, Hom(M,N), by setting (see [55a])

$$(M \oplus N)_{\bar{a}} = M_{\bar{a}} \oplus N_{\bar{b}} \tag{2.11a}$$

$$(M \otimes N)_{\bar{a}} = \bar{\bigoplus_{\bar{b}+\bar{c}=\bar{a}}} M_{\bar{b}} \otimes M_{\bar{c}} \tag{2.11b}$$

$$\mathrm{Hom}(M,N)_{\bar{a}} = \{F \in \mathrm{Hom}(M,N) \text{ s.t. } F(M_{\bar{b}}) \subset N_{\bar{b}+\bar{a}}\} \tag{2.11c}$$

A vector superspace M is called a *superalgebra* (i.e. a $\mathbb{Z}_2$-*graded algebra*), if M is also an associative algebra (with identity: $1 \in M_{\bar{0}}$) such that $M_{\bar{a}}M_{\bar{b}} \subseteq M_{\bar{a}+\bar{b}}$. It is further called *supercommutative* (or $\mathbb{Z}_2$-graded commutative), if $\forall$ homogeneous elements α and β, $\alpha\beta = (-1)^{\bar{\alpha}\bar{\beta}}\beta\alpha$.

For any superalgebra M $\exists$ a canonical projection $\varepsilon : M \to M/(M_{\bar{1}})$, where $(M_{\bar{1}})$ denotes the ideal generated by all elements of $M_{\bar{1}}$. If M is supercommutative, then any element $m \in M$ is *invertible* if and only if $\varepsilon(M)$ is invertible.

---

†<u>*Note:*</u> $\bar{0} + \bar{0} = \bar{0}$, $\bar{0} + \bar{1} = \bar{1}$, $\bar{1} + \bar{0} = \bar{1}$, $\bar{1} + \bar{1} = \bar{0}$

$\bar{0}\,\bar{0} = \bar{0}$, $\bar{0}\,\bar{1} = \bar{0}$, $\bar{1}\,\bar{0} = \bar{0}$, $\bar{1}\,\bar{1} = \bar{1}$.

The tensor product of two superalgebras M and N, is the vector superspace $M \otimes N$ with superalgebra structure given by

$$(m_1 \otimes n_1)(m_2 \otimes n_2) = (-1)^{\bar{n}_1 \bar{m}_2} m_1 m_2 \otimes n_1 n_2 \tag{2.12}$$

$\forall\, m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Any $\mathbb{Z}$-graded algebra admits a decomposition into even and odd parts, and therefore it is also $\mathbb{Z}_2$-graded. Suppose that the typical example, $\otimes \mathbb{C}^N$, is generated by elements v_i for $i = 1, \ldots, N$. Then, by taking the quotient by homogeneous ideals of $\otimes \mathbb{C}^N$ (i.e. those generated by homogeneous (or pure) elements), one can construct more interesting examples of superalgebras:

(i) Exterior algebra: $\Lambda \mathbb{C}^N = \otimes \mathbb{C}^N / (v_i \otimes v_j \oplus v_j \otimes v_i)$ (2.13a)

(ii) Symmetric algebra: $\odot \mathbb{C}^N = \otimes \mathbb{C}^N / (v_i \otimes v_j - v_j \otimes v_i)$ (2.13b)

(iii) Clifford algebra:

$$\mathrm{Cliff}(\mathbb{C}^N) = \otimes \mathbb{C}^N / (v_i \otimes v_j + v_j \otimes v_i - g_{ij}) \tag{2.13c}$$

Examples (i) and (ii) are both $\mathbb{Z}$ and $\mathbb{Z}_2$-graded. However, example (iii) is only $\mathbb{Z}_2$-graded as the ideal $(v_i \otimes v_j + v_j \otimes v_i - g_{ij})$ is not $\mathbb{Z}$-homogeneous (since $g_{ij} \in \mathbb{C}$ while $v_i \otimes v_j \in \otimes^2 \mathbb{C}$). Note that one normally omits the tensor product symbols and writes $v_i \otimes v_j$ as $v_i v_j$, so statements such as

$$v_i v_j + v_j v_i - g_{ij} = 0 \,, \tag{2.14}$$

are to be interpreted as the equivalence relation on the quotient.

For our purposes we shall be mainly interested in exterior algebras (also known as Grassmann algebras). These supercommutative superalgebras have the important property that any element $\kappa \in \Lambda^{\bullet}\mathbb{C}^N$ can be uniquely written as a finite sum

$$\kappa = \sum_{\mu \in \Theta} b_\mu v^\mu ,$$

where μ denotes the set of finite sequences $(\mu_1, \ldots, \mu_N)$, $\mu_j = 0$ or 1, and $v^\mu \equiv v_1^{\mu_1} \ldots v_N^{\mu_N}$. (We will also use other notations for this type of series expansion.)

§2.2.2 A-Supermodules

Let A be a superalgebra. A left A-*supermodule*, M, is a vector superspace M together with a left action of a superalgebra: $A \times M \to M$ satisfying, $\forall\, a_1, a_2 \in A$ and $m, m' \in M$

(i) $a_1(a_2m) = (a_1a_2)m$

(ii) $(a_1 + a_2)m = a_1m + a_2m$, $a(m + m') = am + am'$

(iii) $1 \cdot m = m$

(iv) If a and m are homogeneous, then $\overline{am} = \bar{a} + \bar{m}$.

If A is supercommutative, then a left A-supermodule becomes a bi-module with left and right multiplication compatible through the rule

$$am = (-1)^{\bar{a}\bar{m}} ma \qquad \forall\, a \in A,\ m \in M . \tag{2.15}$$

From now on, given an A-supermodule, we shall assume that A is supercommutative.

An *A-superalgebra* is an A-supermodule which is also a superalgebra, i.e. satisfying the following additional requirement

(v) $(am)m' = a(mm')$ $\quad \forall a \in A$ and $m, m' \in M$.

§2.2.3 Supermatrices

A *supermatrix* is a matrix with a $\mathbb{Z}_2$-grading assigned to each row and column (denoted $\overline{\text{row}(i)}$ and $\overline{\text{col}(j)}$). It is usual to arrange the rows and columns of a supermatrix in such a way that the even ones come first and the odd ones second. Thus, for example, an $(r|s)\times(r'|s')$-supermatrix has r and r' even rows and columns and s and s' odd rows and columns. The overall grading of a supermatrix X, with entries X_{ij} in a superalgebra A, is given as follows

$$\begin{aligned} \bar{X} &= \bar{0} \text{ if } \overline{X_{ij}} + \overline{\text{row}(i)} + \overline{\text{col}(j)} = \bar{0} \quad \forall\, i,j \\ \bar{X} &= \bar{1} \text{ if } \overline{X_{ij}} + \overline{\text{row}(i)} + \overline{\text{col}(j)} = \bar{1} \quad \forall\, i,j\,. \end{aligned} \tag{2.16}$$

Alternatively, if X is written in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C and D are matrices of sizes $r\times r$, $s\times r$, $r\times s$ and $s\times s$, then

$$\begin{aligned} \bar{X} &= \bar{0} \text{ if } \overline{A_{ij}} = \overline{D_{ij}} = \bar{0} \text{ and } \overline{B_{ij}} + \overline{C_{ij}} = \bar{1} \\ \bar{X} &= \bar{1} \text{ if } \overline{A_{ij}} = \overline{D_{ij}} = \bar{1} \text{ and } \overline{B_{ij}} + \overline{C_{ij}} = \bar{0}\,. \end{aligned} \tag{2.17}$$

The product of two supermatrices Y and Z is given by $(YZ)_{ij} = \sum_k Y_{ik}Z_{kj}$. This product is homogeneous if and only if the grading of the columns of Y agrees with the grading of the corresponding rows of Z.

The vector superspace of $(r|s)\times(r'|s')$-supermatrices over a (supercommutative) superalgebra A is given the structure of an A-supermodule by setting:

$$(Xa)_{ij} = (-1)^{\bar{a}\,\overline{\mathrm{col}(j)}} X_{ij}a \;, \qquad (2.20a)$$

$$(aX)_{ij} = (-1)^{\bar{a}\,\overline{\mathrm{row}(i)}} aX_{ij} \;. \qquad (2.20b)$$

Square (i.e. $(r|s)\times(r|s)$) supermatrices over A form an associative superalgebra denoted by $Mat(r|s;A)$. Thus (2.20) is equivalent to defining a homomorphism of superalgebras, $A \to Mat_{r|s}(A)$, given by $a \longmapsto [a_{ij}]$ where $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = (-1)^{\bar{a}\,\overline{\mathrm{row}(i)}}a$.

The transpose of a supermatrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is defined as follows:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{ST} = \begin{bmatrix} A^T & C^T \\ -B^T & D^T \end{bmatrix} \text{ iff } \bar{X} = \bar{0}, \text{ and}$$
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix} \text{ iff } \bar{X} = \bar{1}. \qquad (2.21)$$

It can be shown that

$$(YZ)^{ST} = (-1)^{\bar{Y}\bar{Z}} Z^{ST} Y^{ST} \;. \qquad (2.22)$$

Supertraces are defined on $Mat_{r|s}(A)$ as follows:

$$\mathrm{str}X \equiv \mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}A - \mathrm{tr}D \text{ iff } \bar{X} = \bar{0}$$
$$= \mathrm{tr}A + \mathrm{tr}D \text{ iff } \bar{X} = \bar{1} \;. \qquad (2.23)$$

Note: The motivation for the above definitions will become apparent in the next subsection when we discuss the A-supermodule basis and coordinates.

Proposition 2.i [54] $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (Mat(r|s;A))_{\bar{0}}$ is invertible if and only if A and D are invertible. □

Let $GL(r|s;A)$ denote the multiplicative group of all even invertible elements of $Mat\,(r|s;A)$. The *Berezinian* function (sometimes referred to as the *superdeterminant*) is a homomorphism

$$\mathrm{Ber} : GL(r|s;A) \to GL(1|0;A) \equiv A^*_{\bar{0}} \ , \tag{2.24}$$

defined by

$$X \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \mathrm{Ber}X = \frac{\det(A-BD^{-1}C)}{\det D} \ . \tag{2.25}$$

Since, by 2.i, D must be invertible, the determinants involved are well defined. It is not difficult to check that BerX is an even invertible element of A.

The *inverse of the Berezinian* is given by

$$(\mathrm{Ber}X)^{-1} = \frac{\det(D-CA^{-1}B)}{\det A} \ , \tag{2.26}$$

This is also well defined since A must be invertible by Proposition 2.i.

The Berezinian function has the fundamental multiplicative property:

Theorem 2.ii [54a] If $Y,Z \in GL(r|s;A)$, then $\mathrm{Ber}(YZ) = (\mathrm{Ber}Y)(\mathrm{Ber}Z)$. □

§2.2.4 Supermodule basis and coordinates

A *basis* for an A-supermodule M is a set of elements $m^i \in M$, with $m^i \in M_{\bar{0}}$ for $i = 1,\ldots,r$ and $m^i \in M_{\bar{1}}$ for $i = r+1,\ldots,r+s$, such that every $m \in M$ can be expressed as a unique finite sum $m = \mu^L_i m^i$ (using the Einstein summation convention), where the coefficients, $\mu^L_i \in A$, are not all zero. M is said to be *free* if it has such a basis, and in this case:

$$M \cong A \otimes M_{\bar{0}} \oplus A \otimes M_{\bar{1}} .$$

From now on we shall assume that M is free, and denote its dimension by $\dim M = r|s$.

Let M and N be A-supermodules of dimensions $r|s$ and $r'|s'$ respectively. An *A-homomorphism* from M to N is a linear mapping $F : M \to N$ such that $F(ma) = (F(m))a$ $\forall a \in A$ and $m \in M$. Suppose m^j and n^i are bases of M and N respectively. Then to any homogeneous A-homomorphism F there corresponds a $(\dim N)\times(\dim M)$-supermatrix (over A), $[F_i{}^j]$, of the same grading, defined by

$$F(m^j) = n^i F_i{}^j \tag{2.28}$$

where

$$\overline{\mathrm{row}(i)} = \overline{n^i} \quad \text{and} \quad \overline{\mathrm{col}(j)} = \overline{m^j} . \tag{2.29}$$

Therefore, the vector superspace of A-homomorphisms from M to N, $\mathrm{Hom}_A(M,N)$, has a natural A-supermodule structure given by (2.20). We denote the resulting A-supermodule by

$Hom_A(M,N)$. Furthermore if $M = N$, then the above correspondence defines an isomorphism of A-superalgebras

$$Hom_A(M,N) \equiv End_A(M) \xrightarrow{\simeq} \mathrm{Mat}(r|s;A), \quad (r|s = \dim M). \qquad (2.30)$$

Suppose $m = \mu^L_i m^i \in M$. Then the set $\{\mu^L_i\}$ will be called the *left coordinates* of M. The element $m \in M$ can now be written as the row vector $(\mu^L_1,\ldots,\mu^L_{r+s})$ by defining $\overline{\mathrm{col}(i)} = \overline{m^i}$. Alternatively, m may be expressed in terms of *right coordinates* $\{\mu^R_i\}$ as $m = m^i\mu^R_i$. In this case, we may also write m as the column vector $\begin{pmatrix}\mu^R_1\\ \vdots\\ \mu^R_{r+s}\end{pmatrix}$ by defining $\overline{\mathrm{row}(i)} = \overline{m^i}$. One can check that the left coordinates are related to the right coordinates by the supertransposition rule (2.22).

If $\{m^i\}$ is a basis of M, then its associated dual basis $\{m^*_j\}$ of $M^* = Hom_A(M,A)$ is given by $m^*_j(m^i) = \delta_j{}^i$. Thus if $m^* = a^i m^*_i$ and $m = \mu_i m^i$, then $a^i = m^*(m^i)$, $\mu_i = m^*_i(m)$, and $m^*(m) = a^i\mu_i$.

The even automorphisms of an $r|s$-dimensional A-supermodule, M, are precisely the group $GL(r|s;A)$. By even automorphisms we mean those which preserve the overall Z_2-grading of $M = M_{\bar{0}} \oplus M_{\bar{1}}$, where

$$M_{\bar{0}} = [A_{\bar{0}} \otimes M_{\bar{0}} \oplus A_{\bar{1}} \otimes M_{\bar{1}}] \text{ and } M_{\bar{1}} = [A_{\bar{0}} \otimes M_{\bar{1}} \oplus A_{\bar{1}} \otimes M_{\bar{0}}]. \qquad (2.31)$$

Unless otherwise stated, from now on we shall be dealing entirely with vector superspaces and superalgebras over $\mathbb{C}$. We will frequently use the notation $\mathbb{C}^{r|s} \equiv \mathbb{C}^r \oplus \mathbb{C}^s$ for the trivial $\mathbb{C}$-vector superspace. Furthermore, $r|s$-dimensional A-supermodules over $\mathbb{C}$ will sometimes be denoted by $A^{r|s} \equiv A \otimes_{\mathbb{C}} \mathbb{C}^{r|s}$.

§2.2.5 Lie superalgebras

An A-superalgebra, G, is a Lie superalgebra if its multiplication '[,)' satisfies $\forall$ u,v,w $\in G$

(i) $[u,v) = -(-1)^{\bar{u}\bar{v}}[v,u)$

(ii) $(-1)^{\bar{u}\bar{v}}[u,[v,w)) + (-1)^{\bar{v}\bar{w}}[v,[w,u)) + (-1)^{\bar{w}\bar{u}}[w,[u,v)) = 0.$

The multiplication '[,)' is referred to as the super-commutator, and (ii) is known as the *super Jacobi identity*.

In fact to specify a Lie superalgebra, G, it is sufficient to define the following:

(i) a Lie algebra $G_{\bar{0}}$.

(ii) A representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ defined by [,), i.e. a $G_{\bar{0}}$-module, $G_{\bar{1}}$, s.t. $[u,v] = -[v,u]$ $\forall u \in G_{\bar{1}}$ and $v \in G_{\bar{0}}$.

(iii) A symmetric bilinear mapping $G_{\bar{1}} \times G_{\bar{1}} \to G_{\bar{0}}$ (given by [,)) that is a homomorphism of $G_{\bar{0}}$-modules.

A typical example of a Lie superalgebra is $\mathrm{Der}(C) = \{D \in \mathrm{End}(C) \text{ s.t. } D(ab) = (Da)\, b + (-1)^{\bar{D}\bar{a}} aD(b)\}$, where C is any superalgebra.

§2.3 Super differential geometry

§2.3.1 Definition of a supermanifold (cf. §2.1.5)

A complex $(n|N)$-dimensional *supermanifold*, $X_{[N]}$, is a ringed space $(X, \mathcal{O}_{[N]})$, consisting of a complex n-dimensional classical manifold X, whose structure sheaf is denoted by $\mathcal{O}$, and a sheaf of $\mathbb{Z}_2$-graded commutative $\mathbb{C}$-superalgebras over X, $\mathcal{O}_{[N]}$ (sometimes called the *superstructure sheaf* or the *sheaf of superfunctions*), satisfying the following:
(i) $\exists$ a surjective homomorphism of sheaves of $\mathbb{C}$-superalgebras $\varepsilon : \mathcal{O}_{[N]} \to \mathcal{O}$ called the augmentation map, and (ii) $\exists$ an open cover $\{U_i\}$ of X and isomorphisms of sheaves of $\mathbb{C}$-superalgebras augmented over $\mathcal{O}$:

$$\mathcal{O}_{[N]}|_{U_i} \cong \Lambda^{\bullet}(\mathcal{O} \otimes_{\mathbb{C}} V^*)|_{U_i} \qquad (2.32)$$

for some N-dimensional $\mathbb{C}$-vector space V^*, (where

$$\mathcal{O} \otimes_{\mathbb{C}} V^* \cong \underbrace{\mathcal{O} \oplus \ldots \oplus \mathcal{O}}_{N}),$$

and ε restricts to the projection onto the degree zero part of $\Lambda^{\bullet}(\mathcal{O} \otimes_{\mathbb{C}} V^*)$, i.e.

$$\varepsilon : \mathcal{O}_{[N]}|_{U_i} \cong \Lambda^{\bullet}(\mathcal{O} \otimes_{\mathbb{C}} V^*)|_{U_i} \to \mathcal{O}|_{U_i}. \qquad (2.33)$$

(This definition is equivalent to Kostant and Leites definition [49,54].)

Maps between supermanifolds are special cases of ringed space morphisms defined in §2.1.1. -that is, a mapping of the underlying topological space, (which is a complex manifold in this case), and a homomorphism of the superstructure sheaves in the opposite direction. More precisely, suppose (X,S) and (Y,F) are holomorphic supermanifolds. Then a *supermanifold mapping* $\phi : (X,S) \to (Y,F)$ is a pair consisting of: (i) a holomorphic mapping of the underlying manifolds $\phi : X \to Y$, and (ii) a homomorphism of sheaves of $\mathbb{C}$-superalgebras augmented over $\mathbb{C}$, $\tilde{\phi} : F \to \phi_* S$, (i.e. a homomorphism of rings $\tilde{\phi} : \Gamma(U,F) \to \Gamma(\phi^{-1}(U),S)$ $\forall$ U open in Y).

If $X_{[N]}$ is a supermanifold, then to every open subset $U \subset X$ the corresponding supermanifold $U_{[N]} = (U, \mathcal{O}_{[N]}|_U)$ will be called an *open subsupermanifold* of $X_{[N]}$.

A supermanifold $X_{[N]} = (X, \mathcal{O}_{[N]})$ is called *connected* if X is connected and is called *compact* if X is compact.

The definition of a smooth supermanifold is completely analogous to the above. However there is a fundamental difference between smooth supermanifolds and holomorphic ones.

§2.3.2 The global structure of supermanifolds

In the smooth category, M. Batchelor [6] showed that there exists a rank N vector bundle, E, such that $\Lambda^{\cdot} C(E)$ is isomorphic to $\mathcal{O}_{[N]}$. Consequently up to isomorphism, vector bundles over a smooth manifold X are in one-to-one correspondence with smooth supermanifolds over X.

Let A be the sheaf of automorphisms of $\Lambda^{\cdot}(\mathcal{O} \otimes_{\mathbb{C}} V^*)$ as a $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra augmented over $\mathcal{O}$, i.e. $\mathbb{Z}_2$-grading preserving automorphisms which are the identity on $\mathcal{O}$. Isomorphism classes of supermanifolds $X_{[N]}$, over X, are classified by $H^1(X,A)$. (This is entirely analogous to the classification of vector bundles (of fixed rank) over X by $H^1(X,\mathcal{B})$, where $\mathcal{B}$ is the sheaf of automorphisms of $\mathcal{O} \otimes_{\mathbb{C}} V^*$ as a $\mathcal{O}$-module.) Let

$$A_j = \{R \in A \text{ s.t. } RF = F \ \forall \ F \in \Lambda^i(\mathcal{O} \otimes_{\mathbb{C}} V^*) \text{ for } i < j\} ,$$

so there is a filtration of sheaves (of normal subgroups)

$$\mathrm{Id} = A_{N+1} \leq A_N \leq \ldots \leq A_2 \leq A_1 = A . \qquad (2.34)$$

<u>*Theorem 2.iii*</u> [6,30,24]

$$A_j/A_{j+1} \cong \begin{cases} G\ell(N;\mathcal{O}), \text{ for } j = 1 \\ \mathcal{D}er\,\mathcal{O} \otimes_{\mathcal{O}} \Lambda^j(\mathcal{O} \otimes_{\mathbb{C}} V^*), \text{ for } j \text{ even} \\ \mathcal{H}om\,(\mathcal{O}\otimes_{\mathbb{C}} V^*, \Lambda^j(\mathcal{O} \otimes_{\mathbb{C}} V^*), \text{ for } j \text{ odd and } j \neq 1, \end{cases} \qquad (2.35)$$

where $G\ell(N;\mathcal{O})$ denotes the sheaf of invertible N×N matrices with entries in $\mathcal{O}$, $\mathcal{D}er\,\mathcal{O}$ denotes the sheaf of derivations on X (i.e. the tangent bundle), and $\mathcal{H}om\,(\mathcal{O}\otimes_{\mathbb{C}} V^*, \Lambda^j(\mathcal{O} \otimes_{\mathbb{C}} V^*))$ denotes the sheaf of homomorphisms from $\mathcal{O} \otimes_{\mathbb{C}} V^*$ to $\Lambda^j(\mathcal{O} \otimes_{\mathbb{C}} V^*)$. (See Fig. 2.1 in §2.3.4.)□

A supermanifold gives rise to an *underlying vector bundle* via the map

$$H^1(X,A_1) \to H^1(X,A_1/A_2) \cong H^1(X,G\ell(N;\mathcal{O})) . \qquad (2.36)$$

An $(n|N)$-dimensional supermanifold $(X,\mathcal{O}_{[N]})$ is said to be *trivial* if there exists a rank N vector bundle E over X such that

$$(X,\mathcal{O}_{[N]}) \cong (X,\Lambda^{\bullet}\mathcal{O}(E)) .$$

There is an obstruction theory for supermanifolds completely analogous to that for thickenings of complex manifolds:

Theorem 2.iv [24] Given a rank N vector bundle E over X, the obstructions to $\Lambda^{\bullet}\mathcal{O}(E)$ being the only possible, superstructure on X, lie in

$$\begin{cases} H^2(X, \mathcal{D}er\,\mathcal{O} \otimes_{\mathcal{O}} \Lambda^j\mathcal{O}(E)) & \text{if } j(\geq 2) \text{ is even} \\ \\ H^2(X, \mathcal{H}om\,(\mathcal{O}(E), \Lambda^j\mathcal{O}(E))) & \text{if } j(\geq 3) \text{ is odd .} \end{cases} \tag{2.37a}$$

Moreover, if $\mathcal{O}_{[N]}$ is a superstructure, then

$$\begin{cases} H^1(X, \mathcal{D}er\,\mathcal{O} \otimes_{\mathcal{O}} \Lambda^j\mathcal{O}(E)) & \text{if } j(\geq 2) \text{ is even} \\ \\ H^1(X, \mathcal{H}om\,(\mathcal{O}(E), \Lambda^j\mathcal{O}(E))) & \text{if } j(\geq 3) \text{ is odd} \end{cases} \tag{2.37b}$$

acts freely to produce all possible superstructures. □

§2.3.3 Fibration of supermanifolds

Let $X_{[N]} = (X,S)$ and $Y_{[M]} = (Y,T)$ be trivial supermanifolds with underlying vector bundles V and W respectively. Then $X_{[N]} \times Y_{[M]} \equiv (X \times Y,\ S \otimes T)$ is another trivial supermanifold, with underlying vector bundle $\pi_X^*V \oplus \pi_Y^*W$, i.e. $S \otimes T \cong \Lambda^\cdot(\pi_X^*V \oplus \pi_Y^*W))$ where π_X^* and π_Y^* are the pull backs of the projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

If $X_{[N]}$ and $Y_{[N]}$ are non-trivial, one can adopt this as the local definition of $X_{[N]} \times Y_{[M]}$ and check that it patches correctly.

A *fibration of supermanifolds* is a pair of supermanifolds $X_{[N]}$ and $Y_{[M]}$ and a supermanifold mapping $\pi : Y_{[M]} \to X_{[N]}$, such that for every $x \in X$ there exists an open neighbourhood U of x and an open subsupermanifold $U_{[N]}$ in $X_{[N]}$ such that

$$\pi^{-1}(U_{[N]}) \cong F \times U_{[N]} ,$$

for some fixed subsupermanifold F of $Y_{[N]}$ called the *fibre of the fibration.*

§2.3.4 Supercoordinates

A *local supercoordinate patch* on $(X, \mathcal{O}_{[N]})$ is;
(i) a suitable classical coordinate patch on X, i.e. an open subset U of X together with a family of homeomorphisms $\xi = (\xi^1, \ldots, \xi^n) : U \to \mathbb{C}^n$, which are holomorphic.
(ii) a choice of isomorphism (of $\mathbb{C}$-superalgebras augmented over $\mathcal{O}$) over U

$$\alpha : \mathcal{O}_{[N]} \cong \Lambda^{\cdot}(\mathcal{O} \otimes_{\mathbb{C}} V^*) = \mathcal{O} \oplus \mathcal{O} \otimes_{\mathbb{C}} V^* \oplus \Lambda^2(\mathcal{O} \otimes_{\mathbb{C}} V^*) \oplus \ldots \oplus \Lambda^N(\mathcal{O} \otimes_{\mathbb{C}} V^*). \quad (2.38)$$

(Note that if $\hat{\alpha}$ is another choice of isomorphism over an open subset $\hat{U}$, then on the overlap $U \cap \hat{U}$, the map

$$\alpha(\mathcal{O}_{[N]}) \to \hat{\alpha}(\mathcal{O}_{[N]}) \quad (2.39)$$

is an element of A.)

<u>*Note:*</u> If S is a sheaf, then from now on we will abuse the notation, and write $s \in S$, meaning that s is a local section of S.

Consider (with respect to a suitable basis for $\mathbb{C}^n$ and V^*)

$$\xi^{\alpha} \in \mathcal{O} \subset \Lambda^{\cdot}(\mathcal{O} \otimes_{\mathbb{C}} V^*) \quad (2.40a)$$

and

$$e^j \equiv \begin{array}{c} \\ j\{ \\ \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{O} \otimes_{\mathbb{C}} V^* \subset \Lambda^{\cdot}(\mathcal{O} \otimes_{\mathbb{C}} V^*) \ , \quad (2.40b)$$

for each $\alpha = 1, \ldots, n$ and $j = 1, \ldots, N$; and denote their image under the local (over U) isomorphism α^{-1}, by x^{α} and ζ^j respectively The superfunctions

$x^\alpha \in \mathcal{O}_{[N]\bar{0}}$ $(\alpha = 1,\ldots,n)$ and $y^j \in \mathcal{O}_{[N]\bar{1}}$ $(j = 1,\ldots,N)$, are called respectively the *even* and *odd supercoordinates* of this coordinate patch. Alternatively, the supercoordinates may be regarded as the collections $(x^1,\ldots,x^n) \in \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{[N]\bar{0}}$ and $(y^1,\ldots,y^N) \in V \otimes_{\mathbb{C}} \mathcal{O}_{[N]\bar{1}}$, where $\dim_{\mathbb{C}} V = N$.

By virtue of the anticommuting nature of the odd supercoordinates, a superfunction $F \in \mathcal{O}_{[N]}$ can always be expanded locally as a finite sum

$$F(x^\alpha;y^j) \equiv f(x^\alpha) + f_j(x^\alpha)y^j + f_{jk}(x^\alpha)y^jy^k + \ldots + f^{(x^\alpha)}_{jk\ldots\ell}y^jy^k\ldots y^\ell, \tag{2.41}$$

where $f^{(x^\alpha)}_{jk\ldots\ell} = f_{jk\ldots\ell}(\xi^\alpha)$. Thus, equivalently, a superfunction $F \in \mathcal{O}_{[N]}$ may be thought of as a collection of classical functions on X, and denoted by

$$F \equiv [f, f_j, f_{jk}, \ldots, f_{jk\ldots\ell}]. \tag{2.42}$$

In this way, for each α and j, x^α and y^j may be regarded respectively as the superfunctions:

$$x^\alpha \equiv [\delta_\beta{}^\alpha, 0,\ldots,0] \quad (\text{i.e. } x^\alpha(x^\beta;y^k) \equiv \delta_\beta{}^\alpha x^\beta)$$

and (2.43)

$$y^j \equiv [0, \delta_k{}^j, 0,\ldots,0] \quad (\text{i.e. } y^j(x^\beta;y^k) \equiv \delta_k{}^j y^k).$$

We will normally abuse notation and make no distinction between the superfunctions x^α and the coordinates of the underlying manifold, and between the superfunctions y^j and the e^j's.

A *change of supercoordinates* is a different choice of both (i) and (ii) (in the definition of the local supercoordinates patch), giving rise to a different collection of even-odd supercoordinates, say $\hat{x}^\alpha$ and $\hat{y}^j$. Hence, in order to study the admissible changes of supercoordinates one must have a better understanding of the automorphism sheaf, A. This sheaf was investigated in [24] by representing elements of $\alpha(\mathcal{O}_{[N]})$ as column vectors, so that elements of A could be identified with matrices with zeroes above the diagonal (since the augmentation must be preserved), and zeroes in alternating positions below the diagonal (in order to preserve the $\mathbb{Z}_2$-grading). The situation is illustrated in Fig. 2.1

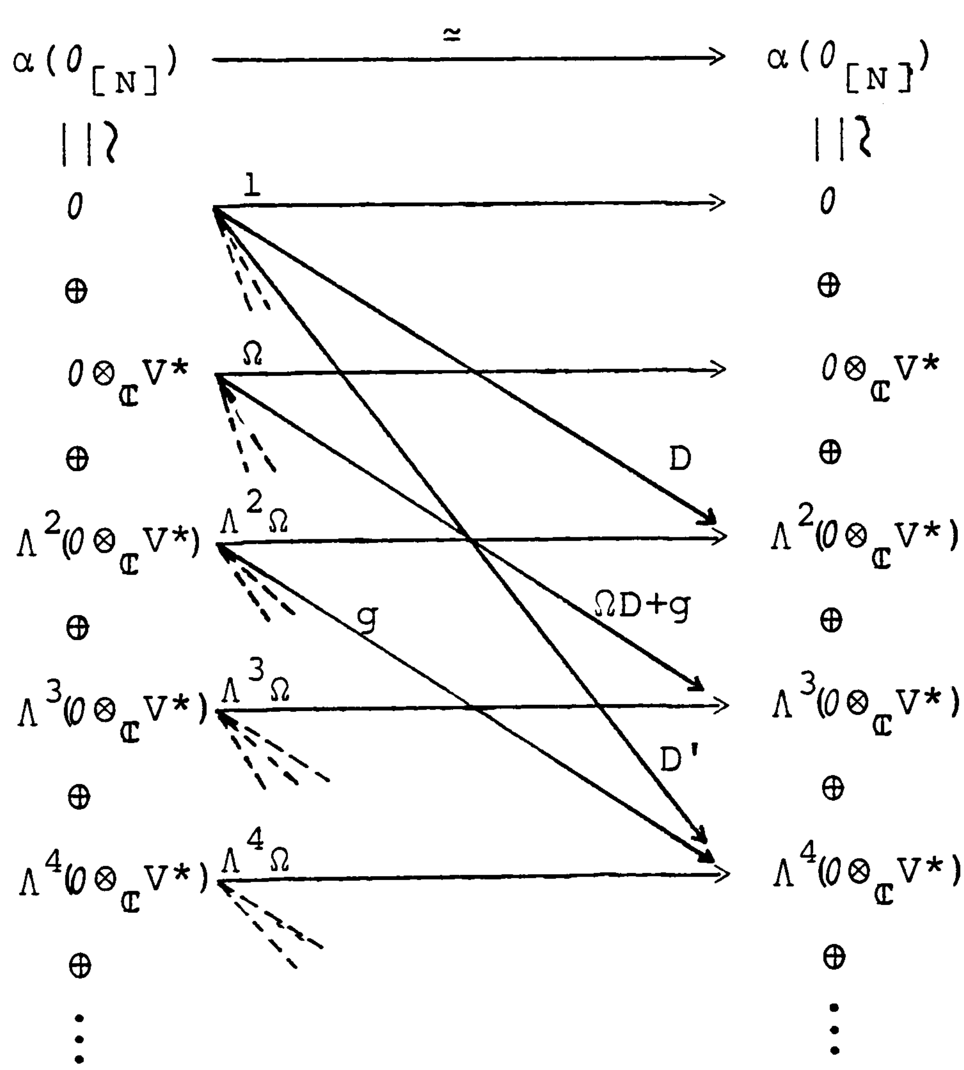


Fig. 2.1

where $\Omega \in G\ell(N;\mathcal{O})$, $D \in \mathcal{D}er\,\mathcal{O} \otimes \Lambda^2(\mathcal{O}\otimes_{\mathbb{C}} V^*)$, $D' \in \mathcal{D}er\,\mathcal{O} \times \Lambda^4(\mathcal{O}\otimes_{\mathbb{C}} V^*)$, $g \in \mathcal{H}om\,(\mathcal{O}^N, \Lambda^3(\mathcal{O}\otimes_{\mathbb{C}} V^*))$, etc. The particular form of these maps can be deduced either from (2.35), or by demanding that the collection of maps, as a whole, form a ring homomorphism (see [24]).

§2.3.5 An example

Changes of supercoordinates look hopelessly complicated. We shall try to clarify the situation with the following simple example:

$$\begin{aligned} \hat{x}^1 &= x^1 + y^2y^3 & \hat{y}^1 &= ay^1 + by^1y^2y^3 \\ \hat{x}^2 &= x^2 & \hat{y}^2 &= y^2 \\ &\vdots & &\vdots \\ \hat{x}^n &= x^n & \hat{y}^N &= y^N \end{aligned} \tag{2.45}$$

This is a genuine change of supercoordinates . Clearly, in the definition of a local supercoordinate patch (§2.3 4), (i) has remained unchanged, while (ii) has indeed changed. Since the mapping D in Fig.2.1 is a derivation, it is clear that any superfunction whose expansion in terms of the odd supercoordinates is of the form, say

$$f(x^1,\ldots,x^n) + f_1(x^1,\ldots,x^n)y^1 ,$$

under the change of supercoordinates (2.45), becomes

$$\begin{aligned} &f(x^1+y^2y^3,x^2,\ldots,x^n) + f_1(x^1+y^2y^3,x^2,\ldots,x^n)(ay^1+by^1y^2y^3) \\ &\quad = f(x^1,\ldots,x^n) + \frac{\partial f}{\partial x^1}(x^1,\ldots,x^n)y^2y^3 \\ &\quad + f_1(x^1,\ldots,x^n)a\,y^1 \\ &\quad + \left\{\left(\frac{\partial}{\partial x^1} f_1(x^1,\ldots,x^n)\right)a + b\, f_1(x^1,\ldots,x^n)\right\}y^1y^2y^3. \end{aligned} \tag{2.47}$$

Clearly, supercoordinates are *not* coordinates in any classical sense. Nevertheless it is tempting, and sometimes even useful, to think of the supercoordinate pair, $(x^\alpha;y^j)$, as a 'point in the supermanifold'. However, we do not know how to build a non-trivial supermanifold from 'collections of all such points'. The only geometric notion which can be associated to a 'point in a supermanifold' is that of the underlying classical point. Note that essentially the same features are common to $\mathbb{C}$-analytic spaces, and in particular, to thickenings of complex manifolds (see §2.1.7).

§2.3.6 Supervector bundles

Recall that a vector bundle on a $\mathbb{C}$-analytic space (X,A) can be defined as a sheaf of locally free A-modules. Equipped with the supermodule definitions of §2.2, the obvious generalization to the super category now suggests itself.

A *supervector bundle of rank* $r|s$ on a supermanifold $(X,\mathcal{O}_{[N]})$, is a sheaf E of locally free $\mathcal{O}_{[N]}$-supermodules of dimension $r|s$. Thus for every point $x \in X$, $\exists$ an open neighbourhood U_α of x such that

$$E|_{U_\alpha} \cong \mathcal{O}^{r|s}_{[N]}|_{U_\alpha} \equiv \mathcal{O}_{[N]}|_{U_\alpha} \otimes \mathbb{C}^{r|s}.$$

A *trivial supervector bundle* is one where $E \cong \mathcal{O}^{r|s}_{[N]}$ globally.

If one chooses a basis $(m^1,\ldots,m^{r+s})$ for $\mathbb{C}^{r|s}$, then, locally, a section F of a supervector bundle is an $(r|s)$-vector valued superfunction (e.g. using left $\mathcal{O}_{[N]}$-supermodule coordinates: $F = \sum_{j=1}^{r+s} f_j m^j$, where $f_j \in \mathcal{O}_{[N]}$). Choosing a different trivialization (over, say, $U_\beta \subset X$), corresponds to choosing a different supermodule basis. Thus the transition functions of a supervector bundle are supermatrix valued superfunctions $g_{\alpha\beta} \in H^1(X,GL(r|s;\mathcal{O}_{[N]})$ (see [16c]), satisfying the usual cocycle condition

Note that for every point x of the underlying classical manifold X, there are associated $(r|s)$-dimensional vector superspaces, which we refer to as the *fibres of the supervector bundle*. However, the analogy between ordinary vector bundles and supervector bundles does not go any further because a supervector bundle cannot be reconstructed from the fibres alone. The isomorphism $\mathcal{O}(GL(r;\mathbb{C})) \cong GL(r,\mathcal{O})$ has no analogue in the super category. (Recall that $GL(r|s;\mathcal{O}_{[N]})$ mixes the $\mathbb{Z}_2$-grading of $\mathcal{O}_{[N]}$ and $\mathbb{C}^{r|s}$, preserving $(\mathcal{O}_{[N]} \otimes_{\mathbb{C}} \mathbb{C}^{r|s})_{\bar{0}}$.) Note that vector bundles on $\mathbb{C}$-analytic spaces essentially share the same features.

§2.3.7 Superline bundles

It is not immediately clear what one should call a superline bundle. The justification for our choice ultimately lies in the fact that it provides the right framework for the description of the super abelian version of Super Yang-Mills theory (see Chapter 3).

In the ordinary (commuting) case, there is the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

($\mathcal{O}^*$ denotes the sheaf of germs of nowhere vanishing holomorphic functions), and one can construct line bundles according to the homomorphism

$$\exp: H^1(X,\mathcal{O}) \to H^1(X,GL(1;\mathbb{C})). \tag{2.48}$$

A similar construction applies in the super category, where there is a homomorphism of sheaves of groups

$$\mathcal{O}_{[N]} \to GL(1|1;\mathcal{O}_{[N]}). \tag{2.49}$$

($\mathcal{O}_{[N]}$ is an additive group whereas $GL(1|1;\mathcal{O}_{[N]})$ is a multiplicative group.)

One way of defining the homomorphism (2.49) is the following. Let $\Lambda = \Lambda^{\bullet}\mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, and write elements of Λ formally as $f = g + h\tau$ (for $g,h \in \mathbb{C}$) with the algebra structure given by $\tau^2 = 0$ (i.e. $\Lambda \cong \mathbb{C}[\tau]/\tau^2$). Let Λ act on $\mathbb{C}^{1|1} = \mathbb{C} \oplus \mathbb{C}$ by identifying $\mathbb{C}^{1|1}$ with Λ, and multiplying on the right, i.e. if (m^1,m^2) is a basis for $\mathbb{C}^{1|1}$, then[†]

$$(\alpha m^1+\beta m^2)f = \alpha g m^1 + (\alpha h + \beta g)m^2 \quad \forall\ \alpha,\beta \in \mathbb{C}.$$

Now let F_0 and F_1 denote respectively the even and odd parts of $F \in \mathcal{O}_{[N]}$. We define expF as follows:

$$\exp F \equiv e^{F_0 + F_1\tau} = (e^{F_0})(1 + F_1\tau) \in \mathcal{O}_{[N]} \otimes_{\mathbb{C}} \Lambda. \tag{2.50}$$

(Note that had we defined $\exp F = e^{F_0 + F_1}$, this would *not* be a group homomorphism.) Hence if $F \in \mathcal{O}_{[N]}$, then expF acts on $\mathcal{O}^{1|1}_{[N]} \equiv \mathcal{O}_{[N]} \otimes_{\mathbb{C}} \mathbb{C}^{1|1}$, preserving the *overall* even-odd grading of $\mathcal{O}^{1|1}_{[N]}$, according to

$$(Am^1 + Bm^2)(e^{F_0})(1 + F_1\tau) = Ae^{F_0}m^1 + (Be^{F_0} + Ae^{F_0}F_1)m_2 \tag{2.51}$$

$\forall\ A,B \in \mathcal{O}_{[N]}$.

The effect of τ is to change the ring structure of $\mathcal{O}_{[N]}$, making it into a commutative ring so that the exponential may be safely taken. Note that τ was merely a device introduced in order to motivate the definition of exp in this context; and, clearly, (2.51) could be written without introducing τ at all. For, writing $Am^1 + Bm^2 \in \mathcal{O}^{1|1}_{[N]}$ as a row vector (A B), then the action of $\mathcal{O}_{[N]} \otimes_{\mathbb{C}} \Lambda$ on it becomes

---

[†] *Note:* $m^1\tau = \tau$ and $m^2\tau = 0$.

$$(A\ B) \longmapsto (A\ B)\begin{pmatrix} e^{F_0} & e^{F_0}F_1 \\ 0 & e^{F_0} \end{pmatrix}. \tag{2.52}$$

This action preserves the overall grading of $\mathcal{O}^{1|1}_{[N]}$, as required.

Let

$$\mathcal{O}^*_{[N]} = \text{Im exp} : \mathcal{O}_{[N]} \to \mathcal{O}_{[N]} \otimes_{\mathbb{C}} \Lambda \cong GL(1|1;\mathcal{O}_{[N]}). \tag{2.53}$$

We define a *superline bundle* to be an element of

$$H^1(X;\mathcal{O}^*_{[N]}) \subset H^1(X,GL(1|1;\mathcal{O}_{[N]})). \tag{2.54}$$

In other words, superline bundles are constructed according to the composition

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}_{[N]}) & \xrightarrow{\simeq} & H^1(X,\mathcal{O}^{1|1}_{[N]\bar{0}}) & \to & H^1(X,\mathcal{O}^*_{[N]}) & \xrightarrow{\simeq} & H^1(X,GL(1|1;\mathcal{O}_{[N]})) \\ \ni & & \ni & & \ni & & \ni \\ F_0 + F_1 & \longmapsto & F_0 + F_1\tau & \longmapsto & e^{F_0+F_1\tau} & \longmapsto & e^{F_0}\begin{pmatrix} 1 & F_1 \\ 0 & 1 \end{pmatrix}. \end{array}$$

§2.3.8 The tangent bundle of a supermanifold

Let $Z^A \equiv (x^\alpha;y^j)$ ($\alpha = 1,\ldots,n$ and $j = 1,\ldots,N$) be the even-odd supercoordinates of $X_{[N]} \equiv (X,\mathcal{O}_{[N]})$. We will define partial derivatives by their action on terms of the power series expansion of a superfunction (cf. analogous definitions for thickenings in [24]), as follows

$$\frac{\partial}{\partial x^\alpha}\,(f(y^1)^{\nu_1}\ldots(y^r)^{\nu_r}) = \frac{\partial f}{\partial x^\alpha}(y^1)^{\nu_1}\ldots(y^r)^{\nu_r} \tag{2.56a}$$

$$\frac{\partial}{\partial y^j}\,(f(y^1)^{\nu_1}\ldots(y^r)^{\nu_r}) = (-1)^{\nu_1+\ldots+\nu_{j-1}} f(y^1)^{\nu_1}\ldots(y^{j-1})^{\nu_{j-1}}(y^{j+1})^{\nu_{j+1}}\ldots(y^r)^{\nu_r}, \tag{2.56b}$$

where $\nu_j = 0$ or 1 and $f = f(x^\alpha)$. These partial derivatives satisfy the graded Leibniz rule. That is, given homogeneous elements $F,G \in \mathcal{O}_{[N]}$:

$$\frac{\partial}{\partial x^\alpha}\,(FG) = \frac{\partial F}{\partial x^\alpha}\,G + F\,\frac{\partial G}{\partial x^\alpha} \tag{2.57a}$$

and

$$\frac{\partial}{\partial y^j}\,(FG) = \frac{\partial F}{\partial y^j}\,G + (-1)^{\bar F}\,F\,\frac{\partial G}{\partial y^j}\,, \tag{2.57b}$$

or equivalently

$$\frac{\partial}{\partial Z^A}\,(FG) = \frac{\partial F}{\partial Z^A}\,G + (-1)^{\overline{Z^A}\bar F}\,F\,\frac{\partial G}{\partial Z^A}\,. \tag{2.57}$$

The Lie superalgebra of derivations, $\mathcal{D}er\ \mathcal{O}_{[N]}$ (with product given by the supercommutator [,)), has the structure of a $\mathcal{O}_{[N]}$-supermodule given by

$$(FD)G = F(D(G)) \quad \forall\ F,G \in \mathcal{O}_{[N]},\quad D \in \mathcal{D}er\ \mathcal{O}_{[N]}. \tag{2.58}$$

Moreover, $\mathcal{D}er\ \mathcal{O}_{[N]}$ is locally free [54] (its basis is given by

$$\left\{\frac{\partial}{\partial Z^A}\right\} \equiv \left\{\frac{\partial}{\partial x^\alpha}\ ,\ \frac{\partial}{\partial y^j}\right\}).$$

Therefore $\mathcal{D}er\ \mathcal{O}_{[N]}$ gives rise to a supervector bundle of rank $n|N$, which we call the *tangent bundle of the supermanifold* $X_{[N]}$.

§2.3.9 Differential forms on a supermanifold

In our local even-odd supercoordinates, *differential forms on a supermanifold* $X_{[N]}$ are objects of the form

$$\omega = \omega_{A_1 \ldots A_p} dZ^{A_1} \mathbin{\hat{\wedge}} \ldots \mathbin{\hat{\wedge}} dZ^{A_p} , \tag{2.59}$$

where the *super wedge product* is defined by:

$$A \mathbin{\hat{\wedge}} B = \begin{cases} A \odot B & \text{if } \bar{A} = \bar{B} = \bar{1} \\ \\ A \wedge B & \text{otherwise.} \end{cases} \tag{2.60}$$

Thus, the symmetries of the coefficients $\omega_{A_1 \ldots A_p}$ vary accordingly. We denote these (super)symmetries by

$$\omega_{A_1 \ldots A_p} = \omega_{[A_1 \ldots A_p)} .$$

Denote the sheaf of germs of differential p-forms on $X_{[N]}$, by Ω^p and set $\Omega^0 = \mathcal{O}_{[N]}$. Note that Ω^p is $(\mathbb{Z} \oplus \mathbb{Z}_2)$-bigraded. In particular, ω above has $(\mathbb{Z} \oplus \mathbb{Z}_2$-bigrading: $(p, \overline{\omega_{A_1 \ldots A_p}} + \overline{Z^{A_1}} + \ldots + \overline{Z^{A_p}})$.

The existence of a differential operator d, satisfying $d^2 = 0$, was shown by Kostant [49]. In our local even-odd super-coordinates d, on $F \in \mathcal{O}_{[N]}$, is given by

$$dF = dZ^A \frac{\partial F}{\partial Z^A} \equiv dx^a \frac{\partial F}{\partial x^a} + dy^j \frac{\partial F}{\partial y^j} . \tag{2.61}$$

Note that d has $(\mathbb{Z}, \mathbb{Z}_2)$ bigrading: $(1, \bar{0})$.

It can be shown [49] that Ω^p is isomorphic to $\mathit{Hom}(\Lambda^p \mathit{Der}\,\mathcal{O}_{[N]}, \mathcal{O}_{[N]})$. In this way one identifies one-forms as sections of the dual of the tantent bundle of the supermanifold.

The deRham sequence on a supermanifold can be written as

$$\begin{array}{ccccccccc}
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \longrightarrow & \cdots \\
\Vert\wr & & \Vert\wr & & \Vert\wr & & \Vert\wr & & \\
\mathcal{O}_{[N]} & \xrightarrow{\partial_A} & \mathcal{O}_{[N]A} & \xrightarrow{\partial_B} & \mathcal{O}_{[N][AB)} & \xrightarrow{\partial_C} & \mathcal{O}_{[N][ABC)} & \longrightarrow & \cdots,
\end{array} \tag{2.62}$$

where

$$\begin{aligned}
&\partial_A \equiv \frac{\partial}{\partial Z^A} \equiv \left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^j}\right\} \qquad (\alpha = 1,\ldots,n\ ;\ j = 1,\ldots,N) \\
&\mathcal{O}_{[N]A} = \mathcal{O}_{[N]\alpha} \oplus \mathcal{O}_{[N]j} \\
&\mathcal{O}_{[N][AB)} = \mathcal{O}_{[N][\alpha\beta]} \oplus \mathcal{O}_{[N]\alpha j} \oplus \mathcal{O}_{[N](jk)} \\
&\mathcal{O}_{[N][ABC)} = \mathcal{O}_{[N][\alpha\beta\gamma]} \oplus \mathcal{O}_{[N][\alpha\beta]j} \oplus \mathcal{O}_{[N]\alpha(jk)} \oplus \mathcal{O}_{[N](jk\ell)}
\end{aligned} \tag{2.63}$$

etc.

The deRham sequence on $X_{[N]}$ can be regarded a product of the ordinary deRham sequence in the category of convergent power series on X, with an analogous construction for anticommuting variables (see [16b]). For example, an element of the former complex can be written as (assuming convergence)

$$f_{(\alpha\beta\ldots\gamma)[\sigma\mu\ldots\rho]} \in \odot\, \mathbb{C}_\alpha \otimes_{\mathbb{C}} \Lambda\mathbb{C}_\sigma . \tag{2.64}$$

An element of the latter is obtained simply by interchange of symmetrization with antisymmetrization

$$f_{[jk\ldots\ell](mn\ldots p)} \in \Lambda\mathbb{C}_j \otimes_{\mathbb{C}} \odot\, \mathbb{C}_m \tag{2.65}$$

(no assumption of convergence is necessary here).
Tensoring (2.64) and (2.65) one obtains

$$\odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} \Lambda\mathbb{C}_j \longrightarrow \begin{array}{c} (\odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} \mathbb{C}_\sigma) \otimes_{\mathbb{C}} \Lambda\mathbb{C}_j \\ \oplus \\ \odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} (\Lambda\mathbb{C}_j \otimes_{\mathbb{C}} \mathbb{C}_m) \end{array} \longrightarrow \begin{array}{c} (\odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} \Lambda^2\mathbb{C}_\sigma) \otimes_{\mathbb{C}} \Lambda\mathbb{C}_j \\ \oplus \\ (\odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} \mathbb{C}_\sigma) \otimes (\Lambda\mathbb{C}_j \otimes_{\mathbb{C}} \mathbb{C}_m) \\ \oplus \\ \odot\mathbb{C}_\alpha \otimes (\Lambda\mathbb{C}_j \otimes \odot^2\mathbb{C}_m) \end{array} \rightarrow \cdots \tag{2.66}$$

Identifying, in (2.66), $\mathcal{O}_{[N]}$ with $\odot\mathbb{C}_\alpha \otimes_{\mathbb{C}} \Lambda\mathbb{C}_j$, precisely gives the deRham sequence (2.62).

Thus, exactness of the deRham sequence for supermanifolds follows immediately from the exactness of (2.66). The latter is a consequence of the fact that the tensor product of two exact complexes is exact (by Künneth formula, see for example [85]).

§2.3.10 Superconnection

We will now consider differential forms with values on a supervector bundle E, of rank $(r|s)$. Denote them by $\Omega^p(E) = \Omega^p \otimes_{\mathcal{O}_{[N]}} E$. Recall that, locally

$$E \cong \mathcal{O}^{r|s}_{[N]} \equiv \mathcal{O}_{[N]} \otimes \mathbb{C}^{r|s}.$$

Thus the modified deRham sequence becomes

$$\begin{array}{ccccccccc} \Omega^0(E) & \xrightarrow{\nabla} & \Omega^1(E) & \xrightarrow{\nabla} & \Omega^2(E) & \xrightarrow{\nabla} & \Omega^3(E) & \rightarrow & \cdots \\ \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\ \mathcal{O}^{r|s}_{[N]} & \xrightarrow{\nabla_A} & \mathcal{O}^{r|s}_{[N]A} & \xrightarrow{\nabla_B} & \mathcal{O}^{r|s}_{[N][AB)} & \xrightarrow{\nabla_C} & \mathcal{O}^{r|s}_{[N][ABC)} & \rightarrow & \cdots , \end{array} \tag{2.67}$$

where

$$\mathcal{O}^{r|s}_{[N]A} \cong \mathcal{O}_{[N]A} \otimes_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]}$$

$$\mathcal{O}^{r|s}_{[N][AB)} \cong \mathcal{O}_{[N][AB)} \otimes_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]}$$

$$\mathcal{O}^{r|s}_{[N][ABC)} \cong \mathcal{O}_{[N][ABC)} \otimes_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]} .$$

A *superconnection* ∇ is a $(1,\bar{0})$-bigraded operator (i.e. a $\mathbb{Z}_2$-grading preserving operator), defined by analogy with the ordinary connection. Namely, if $F \in \mathcal{O}_{[N]}$ and $\xi \in \Omega^{0}(\mathcal{E})$ then $\nabla(F\xi) = dF \otimes \xi + F\nabla\xi$. If one chooses a frame (i.e. a basis for $\mathbb{C}^{r|s}$) $\xi = (m^1,\ldots,m^{r+s})$, then the superconnection is defined by $\nabla(m^a) = m^a\phi_a{}^b$, with

$$\phi \equiv [\phi_a{}^b] \in \Omega^1(\mathit{End}_{\mathcal{O}_{[N]}}(\mathcal{O}^{r|s}_{[N]})).$$

It transforms under changes of frame

$$\xi \longmapsto \xi' = \xi g \tag{2.68}$$

(for some $g \in GL(r|s;\mathcal{O}_{[N]})$), as

$$\phi \longmapsto g^{-1}(\phi + d)g. \tag{2.69}$$

In local supercoordinates, we will usually suppress the internal supermatrix indices and write the superconnection as $\nabla = dZ^A\nabla_A = dx^\alpha\nabla_\alpha + dy^j\nabla_j$, where

$$\nabla_\alpha = \frac{\partial}{\partial x^\alpha} + \Phi_\alpha \text{ and } \nabla_j = \frac{\partial}{\partial y^j} + \Phi_j .$$

With respect to a local trivialization of $\mathcal{E}$, for each α and j:

$$\Phi_\alpha \in (\mathit{End}_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]}))_{\bar{0}}$$

and

$$\Phi_j \in (\mathit{End}_{\mathcal{O}_{[N]}} (\mathcal{O}^{r|s}_{[N]})_{\bar{1}} .$$

Note: If we set all anticommuting variables to zero, a supervector bundle becomes an ordinary $\mathbb{Z}_2$-graded vector bundle (i.e. the total space still admits a $\mathbb{Z}_2$-graded decomposition). Our definition of a superconnection is a special case of Quillen's definition [71]. The latter is more general because it does not preserve the $\mathbb{Z}_2$-grading of the vector bundle.

CHAPTER 3: SUPER MINKOWSKI SPACE AND SUPER YANG-MILLS THEORY

§3.0 Introduction

In field theory one usually deals with fields that transform under some representation of a Lie algebra whose generators are represented by differential operators (vector fields). For example, the Poincaré algebra has generators $\partial^a = \dfrac{\partial}{\partial x^a}$ and $J^{ab} = x^a\partial^b - x^b\partial^a$, satisfying the usual commutation relations (see for example [72]). Another example is the Lorentz algebra, with only the six rotations J^{ab} $(= -J^{ba})$ as generators. The vector fields ∂^a and J^{ab} can be exponentiated to a Lie group; in these particular cases, to the identity connected components of Poincaré and Lorentz groups, denoted by $P_+^\uparrow(1,3)$ and $O_+^\uparrow(1,3)$ respectively.

Affine Minkowski space can be defined as the representation space for the group of translations, $P_+^\uparrow(1,3)/O_+^\uparrow(1,3)$, generated by exponentiating the infinitesimal translation operators ∂_a.

Affine super Minkowski space is usually introduced as the representation space for the Lie superalgebra generated by the 4 infinitesimal translation operators ∂_a together with 4N 'odd translation' operators, ∂_A^j and $\partial_{A'j}$, satisfying the following super commutation relations

$$\begin{aligned}
&[\partial_a, \partial_b] = 0, \quad (\partial_A^j, \partial_B^k) = 0, \quad (\tilde{\partial}_{A'j}, \tilde{\partial}_{B'k}) = 0 \\
&[\partial_a, \partial_B^j] = 0, \quad [\partial_a, \tilde{\partial}_{B'k}] = 0 \quad (\partial_A^j, \tilde{\partial}_{A'k}) = 2\delta_k^j\partial_a.
\end{aligned} \tag{3.1}$$

In the Physics literature it is standard practice to assume that these Lie superalgebra (usually regarded as the quotient of the Poincaré superalgebra by the Lorentz superalgebra [74a]) generators can be exponentiated to a 'Lie supergroup'. Multiplication by the 'Lie supergroup' elements is thought to generate finite transformations on the parameter space: labelled by $(x^{AA'}, \theta^A_j, \tilde{\theta}^{A'j})$, where θ^A_j and $\tilde{\theta}^{A'k}$ are 'anticommuting spinors'. Such transformations are called 'supersymmetry transformations'; and all superfield equations are then required to be 'invariant under supersymmetry transformations'. These superfield equations, are usually written in terms of the differential operators

$$\partial_a = \frac{\partial}{\partial x^a} \;, \quad \partial^j_A = \frac{\partial}{\partial \theta^A_j} + \tilde{\theta}^{A'} \frac{\partial}{\partial x^{AA'}} \;, \quad \tilde{\partial}_{A'k} = \frac{\partial}{\partial \tilde{\theta}^{A'k}} + \theta^A_k \frac{\partial}{\partial x^{AA'}} \qquad (3.2)$$

(One can easily verify that these operators satisfy (3.1).)

The informal constructions outlined above are commonly used in the Physics literature as motivation for introducing super Minkowski space as the 'space of all $(x^{AA'}, \theta^A_j, \tilde{\theta}^{A'k})$', [74a,28].

Our starting point for introducing super Minkowski space will be quite different; and the above motivation is presented for the purposes of comparison only. The fundamental obstacle for using this geometrically appealing motivation is that we have no satisfactory definition of a 'Lie supergroup'. (B. Kostant [49] has a definition of a Lie supergroup in terms of Hopf algebras. However this definition is not suitable for our purposes.)

At the Lie superalgebra level there are no difficulties in making sense of supersymmetry transformations within the Kostant-Leites formalism (see [13]). Moreover the concept of 'invariance under supersymmetry transformations' can be taken to mean 'supercoordinate independence': a perfectly well defined concept within the Kostant-Leites formalism of Chapter 2 (see §2.3.4 and §2.3.5). The extent to which this interpretation of invariance under supersymmetry transformations is compatible with the physical aspects of supersymmetry will not be discussed.

The lack of a suitable Lie supergroup definition within the Kostant-Leites supermanifold formalism is an unsatisfactory feature of the theory. The effect of this deficiency on the original motivations of supersymmetry will not be discussed.

The objective here is to define super Minkowski space $\mathbb{M}_{[N]}$ as a non-trivial supermanifold by regarding it as the 'diagonal' in the product of two trivial supermanifolds (antichiral super Minkowski space and chiral super Minkowski space). We will not attempt a full justification of this as a good definition, except in as much that this is a suitable definition for the twistor correspondences which we will be discussing in Chapter 4. We anticipate that superambitwistor space (a non-trivial supermanifold) is defined in a completely analogous way to $\mathbb{M}_{[N]}$, namely as the 'diagonal' in the product of two trivial supermanifolds: projective supertwistor space and projective dual supertwistor space.

The organization of §3.1 is as follows. We begin in §3.1.1 by defining Minkowski space $\mathbb{M}$ as the diagonal in an 8-dimensional space $\mathbb{M} \times \mathbb{M}$ (cf. [93]). In §3.1.2 we follow suggestions of M.G. Eastwood and use the constructions of §3.1.1 as guiding analogy to define the superstructure sheaf of super Minkowski space. This is done by regarding $\mathbb{M}_{[N]}$ as being embedded as the diagonal in a trivial supermanifold $(\mathbb{M} \times \mathbb{M})_{[N]}$. (The case N = 1 is discussed in detail in §3.1.3.) Finally in §3.1.4, $(\mathbb{M} \times \mathbb{M})_{[N]}$ is identified with the product of antichiral super Minkowski space and chiral super Minkowski space: ${}^{-}\mathbb{M}_{[N]} \times {}^{+}\mathbb{M}_{[N]}$. The constructions are then summarized in Fig. 3.2 (cf. Fig. 4.1 in Chapter 4).

The second part of this chapter (§3.2) deals with super Yang-Mills theory: a natural generalization of the classical Yang-Mills equations to the super category. Superfield equations referred to as the super Yang-Mills equations have been arrived at from physical considerations within a Lagrangian formalism [11,25a,28c]. However, the physical motivations for studying super Yang-Mills theory will not be discussed.

There is some uncertainty (particularly for N > 1) regarding what one should adopt as a definition of the super Yang-Mills equations. We present here what we consider to be a natural definition.

The N = 1 super Yang-Mills theory is studied in detail. We found such a study essential in order to attempt to give a supertwistor interpretation of these superfield equations. Originally, we carried out the calculations at the formal level of θ-expansions (standard practice in the Physics literature), and found that in order to obtain a 'reasonable' system of classical equations some of the coefficients (called the *odd coefficients*) of the θ-expansions of superpotentials and curvature components, had to anticommute with an odd number of θ's. Moreover, when reducing to the super abelian case, we expected commutators and anticommutators of superfields to vanish. This could not happen unless the anticommutators of the odd coefficients also vanished. From the formalism of Chapter 2 we hope to convince the reader that there is nothing mysterious about these odd coefficients; and that both of the above mentioned 'anticommuting properties' are merely a consequence of the $\mathcal{O}_{[N]}$-supermodule structure ($\mathcal{O}_{[N]}$ denotes the superstructure sheaf of $\mathbb{M}_{[N]}$) of the Lie superalgebra of supermatrices over $\mathcal{O}_{[N]}$.

We have organized §3.2 in the following way. In §3.2.1 we consider an $r|s$ supervector bundle, E on $\mathbb{M}_{[N]}$, and define the corresponding superpotentials and super Yang-Mills fields. With respect to some choice of trivialization for E, these objects are regarded respectively as 1-forms and 2-forms on $\mathbb{M}_{[N]}$ with values in $End_{\mathcal{O}_{[N]}} E$. The special case of a superline bundle is investigated in more detail. Following M. Sohnius [84], in §3.2.2 we state, and briefly discuss, the Bianchi identities and constraint equations. These results will be needed in §3.2.3 where we give a definition of the super Yang-Mills equations. The commuting and anticommuting properties of the coefficients of the θ-expansions of superpotentials and curvature components are discussed in §3.2.4, where several examples of the super abelian case are given. In §3.2.5 we investigate the $N = 1$ super Yang-Mills theory. The relevant calculations are given in the appendix. Finally, in §3.2.6 we state the classical equations implied by our definition of the super Yang-Mills equations for $N = 3$, and compare them with the $N = 3$ constraint equations of Harnad et al. [35].

§3.1 Super Minkowski space

§3.1 Infinitesimal neighbourhoods of $\mathbb{M}$ [19]

Consider $\mathbb{M}$ embedded as the diagonal in $\mathbb{M}\times\mathbb{M}$, i.e. $\mathbb{M}\hookrightarrow\mathbb{M}\times\mathbb{M}$. Let π_1 and π_2 denote respectively the projection onto $\mathbb{M}$ of the first and second factors of $\mathbb{M}\times\mathbb{M}$. Then $\pi_1^*\mathcal{O}_{A'}$ and $\pi_2^*\mathcal{O}_A$ denote the pull-backs to $\mathbb{M}\times\mathbb{M}$ of the dual primed and unprimed spin bundles on $\mathbb{M}$ (denoted by $\mathcal{O}_{A'}$ and $\mathcal{O}_A$ respectively). When restricted to the diagonal these bundles $(\pi_1^*\mathcal{O}_{A'}$ and $\pi_2^*\mathcal{O}_A)$ are canonically isomorphic to $\mathcal{O}_{A'}$ and $\mathcal{O}_A$ respectively.

Let u^a and v^a be the coordinates on each factor of $\mathbb{M}\times\mathbb{M}$, and let K and L be the 2-dimensional subspaces of twistor space T associated with the point (u^a,v^a) in $\mathbb{M}\times\mathbb{M}$. Then $\pi_1^*\mathcal{O}_{A'}$ and $\pi_2^*\mathcal{O}^A$ have fibres (over points in $\mathbb{M}\times\mathbb{M}$) isomorphic to K and T/L respectively. And so

$$\mathcal{O}^a \equiv \pi_1^*\mathcal{O}^{A'} \otimes \pi_2^*\mathcal{O}^A \tag{3.3}$$

has fibre $K^* \otimes_{\mathbb{C}} T/L \cong \mathrm{Hom}_{\mathbb{C}}(K,T/L)$ over points in $\mathbb{M}\times\mathbb{M}$.

Let z^a denote the element of $\mathrm{Hom}_{\mathbb{C}}(K,T/L)$ (i.e. a section of $\mathcal{O}^a$), given by the composition

$$K \hookrightarrow T \twoheadrightarrow T/L\ .$$

Then $z^a = 0$ if and only if $K = L$. Intuitively z^a is the off-diagonal coordinate $z^a = u^a-v^a$, and one may think of $x^a = u^a+v^a$ as the coordinate on the diagonal.

Let I be the ideal of functions on $\mathbb{M}\times\mathbb{M}$ vanishing on the diagonal. It is defined by the image of the map $\mathcal{O}_a \xrightarrow{\times z^a} \mathcal{O}$, where $\mathcal{O}_a$ is the dual of $\mathcal{O}^a$ (see (3.3)) and $\mathcal{O}$ denotes the structure sheaf of $\mathbb{M}\times\mathbb{M}$. Thus, the exact sequence

$$(\mathbb{M}\times\mathbb{M}): \quad \mathcal{O}_a \xrightarrow{\times z^a} \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 , \tag{3.4}$$

defines a sheaf $\mathcal{O}_\Delta$ (supported on the diagonal) which we refer to as the structure sheaf of $\mathbb{M}$.

The k^{th} infinitesimal neighbourhood of $\mathbb{M}$ in $\mathbb{M}\times\mathbb{M}$, $\mathbb{M}_{(k)} \equiv (\mathbb{M},\mathcal{O}_{(k)})$, is defined by setting

$$\mathcal{O}_{(k)} = \mathcal{O}/I^{k+1} \tag{3.5}$$

with $\mathcal{O}_{(0)} = \mathcal{O}_\Delta$.

§3.1.3 The superstructure on $\mathbb{M}$

Let V be an N-dimensional complex vector space and denote its dual by V*. Define (using abstract indices)

$$\mathcal{O}_{A'k} = V^* \otimes_{\mathbb{C}} \pi_1^*\mathcal{O}_{A'} \ \& \ \mathcal{O}_A^j = V \otimes \pi_2^*\mathcal{O}_A \quad ,$$

where $\pi_1^*\mathcal{O}_{A'}$ and $\pi_2^*\mathcal{O}_A$ denote the pull-back to $\mathbb{M}\times\mathbb{M}$ of the dual primed spin bundle (on $\mathbb{M}$) and the unprimed spin bundle (on $\mathbb{M}$) respectively.

Following suggestions of M.G. Eastwood we define an $(8|4N)$-dimensional trivial supermanifold

$$(\mathbb{M}\times\mathbb{M})_{[N]} \equiv (\mathbb{M}\times\mathbb{M},\ \Lambda^\bullet(\mathcal{O}_{B'k} \oplus \mathcal{O}_B{}^\ell)) \ , \tag{3.6}$$

where

$$\Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B}) = \mathcal{O} \oplus \mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B} \oplus \mathcal{O}_{B'C'jk} \oplus \mathcal{O}_{BB'j}{}^{k} \oplus \mathcal{O}^{\ell m}_{BC} \oplus \ldots \quad (3.6')$$

Let $(u^a, v^a; \tilde{\theta}^{A'j}, \theta^A{}_j)$ be supercoordinates of $(\mathbb{M} \times \mathbb{M})_{[N]}$, with

$$\tilde{\theta}^{A'j} \in \mathcal{O}^{A'j} \otimes_{\mathcal{O}} \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B})_{\bar{1}} \quad (3.7a)$$

$$\theta^A_j \in \mathcal{O}^A_j \otimes_{\mathcal{O}} \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B})_{\bar{1}} \quad (3.7b)$$

$$(u^a, v^b) \in (\mathcal{O}^a \oplus \mathcal{O}^b) \otimes_{\mathcal{O}} \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B})_{\bar{0}} \, . \quad (3.7c)$$

Recall that (see §2.3.4, §2.3.5 and §2.3.6) u^a and v^a can be thought of as classical coordinates on each factor of $\mathbb{M} \times \mathbb{M}$.

Alternatively, since any superfunction $F \in \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^{\ell}_{B})$ is of the form $F \equiv [f, f_{B'k}, f^j_B, f_{BB'k}{}^{j}, \ldots]$, one may wish to think of each individual $\tilde{\theta}^{A'j}$ and θ^A_j as

$$\tilde{\theta}^{A'j} \equiv [0, \varepsilon_{B'}{}^{A'} \delta_k{}^j, 0, \ldots, 0], \text{ i.e. } \tilde{\theta}^{A'j}(u^b, v^b; \tilde{\theta}^{B'k}, \theta^B_\ell) \equiv \varepsilon_{B'}{}^{A'} \delta_k{}^j \tilde{\theta}^{B'k}$$

$$\theta^A_j \equiv [0, 0, \varepsilon_B{}^A \delta_j{}^k, 0, \ldots, 0], \quad \text{i.e. } \theta^A_j(u^b, v^b; \tilde{\theta}^{B'\ell}, \theta^B_k) \equiv \varepsilon_B{}^A \delta_j{}^k \theta^B_k \, .$$

In these supercoordinates, the differential operators (3.2) become

$$\partial_a = \frac{1}{2}\left[\frac{\partial}{\partial u^a} + \frac{\partial}{\partial v^a}\right] \quad (3.8a)$$

$$\partial^j_A = \frac{\partial}{\partial \theta^A_j} + \tilde{\theta}^{A'j} \frac{\partial}{\partial u^{AA'}} \quad (3.8b)$$

$$\tilde{\partial}_{A'k} = \frac{\partial}{\partial \tilde{\theta}^{A'k}} + \theta^A_j \frac{\partial}{\partial v^{AA'}} \quad (3.8c)$$

One can easily check that they also satisfy the supercommutation relations (3.1).

The action of the differential operators (3.8) on the off-diagonal coordinates $z^a = u^a - v^a$ is given by

$$\partial_a z^b = 0 \tag{3.9a}$$

$$\partial^j_A z^b = \tilde{\theta}^{B'j} \varepsilon_A{}^B \tag{3.9b}$$

$$\partial_{A'k} z^b = -\theta^B_k \varepsilon_{A'}{}^{B'} \quad . \tag{3.9c}$$

The right hand sides of (3.9) above are precisely what one obtains by replacing z^b (on the left hand sides) by the quantity $\sigma^b \equiv \theta^B_j \tilde{\theta}^{B'j}$. Hence, the vector fields $\partial_{\mathcal{A}} \equiv (\partial_a, \partial^j_A, \tilde{\partial}_{A'k})$ annihilate $z^b - \sigma^b \in \mathcal{O}^b \otimes_{\mathcal{O}} \Lambda^{\bullet}(\mathcal{O}_{A'k} \oplus \mathcal{O}^j_A)_{\bar{0}}$, and therefore are well defined on the zero set of the section $z^b - \sigma^b$. Note that each individual σ^a can be regarded as the superfunction

$$\sigma^a \equiv [0,0,0,\delta_j{}^k \varepsilon_B{}^A \varepsilon_{B'}{}^{A'},0,\ldots,0] \ ,$$

i.e. $\sigma^a(u^b, v^b; \tilde{\theta}^{B'j}, \theta^B_k) \equiv \delta_j{}^k \varepsilon_B{}^A \varepsilon_{B'}{}^{A'} \theta^B_k \tilde{\theta}^{B'j}$.

By analogy with the previous subsection, it is now clear how to proceed to define the superstructure sheaf of $\mathbb{M}_{[N]}$ supported on the diagonal, namely through the exact sequence (c f. (3.4)) [19]

$$(\mathbb{M}\times\mathbb{M}): \quad \mathcal{O}_a \otimes_{\mathcal{O}} \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^j_B) \xrightarrow{\times (z^a - \sigma^a)} \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}^j_B) \to \mathcal{O}_{[N]} \to 0 \ . \tag{3.10}$$

Equivalently

$$\mathcal{O}_{[N]} \cong \Lambda^{\bullet}(\mathcal{O}_{B'k} \oplus \mathcal{O}_B{}^j)/\mathcal{I}_{[N]}$$

where $I_{[N]}$ denotes the ideal generated by $z^a - \sigma^a$, i.e. $I_{[N]}$ is the image of the map $\times(z^a - \sigma^a)$ in (3.10). The $(4|4N)$-dimensional supermanifold $\mathbb{M}_{[N]} \equiv (\mathbb{M}, \mathcal{O}_{[N]})$ will be called *super Minkowski space*.

The ideal $I_{[N]}$ is not $\mathbb{Z}$-homogeneous (cf. the Clifford algebra example of a superalgebra (2.13c)) and therefore $\mathcal{O}_{[N]}$ is $\mathbb{Z}_2$-graded only. This reflects the fact that the superstructure of $\mathbb{M}_{[N]}$ is non-trivial.

§3.1.3 The simplest case (N = 1)

One could, in principle, write $\mathcal{O}_{[N]}$ in terms of the composition series notation of §2.1.8. However, its general form for all N is very complicated because of the symmetry constraints in the spinor decomposition of the supercoordinate σ^a. We will investigate $\mathcal{O}_{[1]}$ in detail and write it in terms of this notation. The cases N = 2 and N = 3 would of course require more effort but they should not prove intractable.

We start by substituting (3.6') (for N = 1) in the defining sequence (3.10) to obtain

$$\left.\begin{tikzcd}[row sep=tiny]
\mathcal{O}_a \arrow[r, "\times z^a"] \arrow[dddddddddddddddddr] \arrow[ddddddr, "-\delta^a_b"] & \mathcal{O} \\
\oplus & \oplus \\
\mathcal{O}_{Ba} \arrow[r, "\times z^a"] \arrow[ddddddddddr, "-\delta^a_b"] & \mathcal{O}_B \\
\oplus & \oplus \\
\mathcal{O}_{B'a} \arrow[r, "\times z^a"] \arrow[ddddddddddr, "-\delta^a_b"] & \mathcal{O}_{B'} \\
\oplus & \oplus \\
\mathcal{O}_{ba} \arrow[r, "\times z^a"] \arrow[ddddddddddr, "-\delta^a_b"] & \mathcal{O}_b \\
\oplus & \oplus \\
\mathcal{O}_a[-1] \arrow[r, "\times z^a"] & \mathcal{O}[-1] \\
\oplus & \oplus \\
\mathcal{O}_a[-1]' \arrow[r, "\times z^a"] & \mathcal{O}[-1]' \\
\oplus & \oplus \\
\mathcal{O}_{Ba}[-1]' \arrow[r, "\times z^a"] & \mathcal{O}_B[-1]' \\
\oplus & \oplus \\
\mathcal{O}_{B'a}[-1] \arrow[r, "\times z^a"] & \mathcal{O}_{B'}[-1] \\
\oplus & \oplus \\
\mathcal{O}_a[-1][-1]' \arrow[r, "\times z^a"] & \mathcal{O}[-1][-1]'
\end{tikzcd}\right\} \to \mathcal{O}_{[1]} \to 0. \qquad (3.11)$$

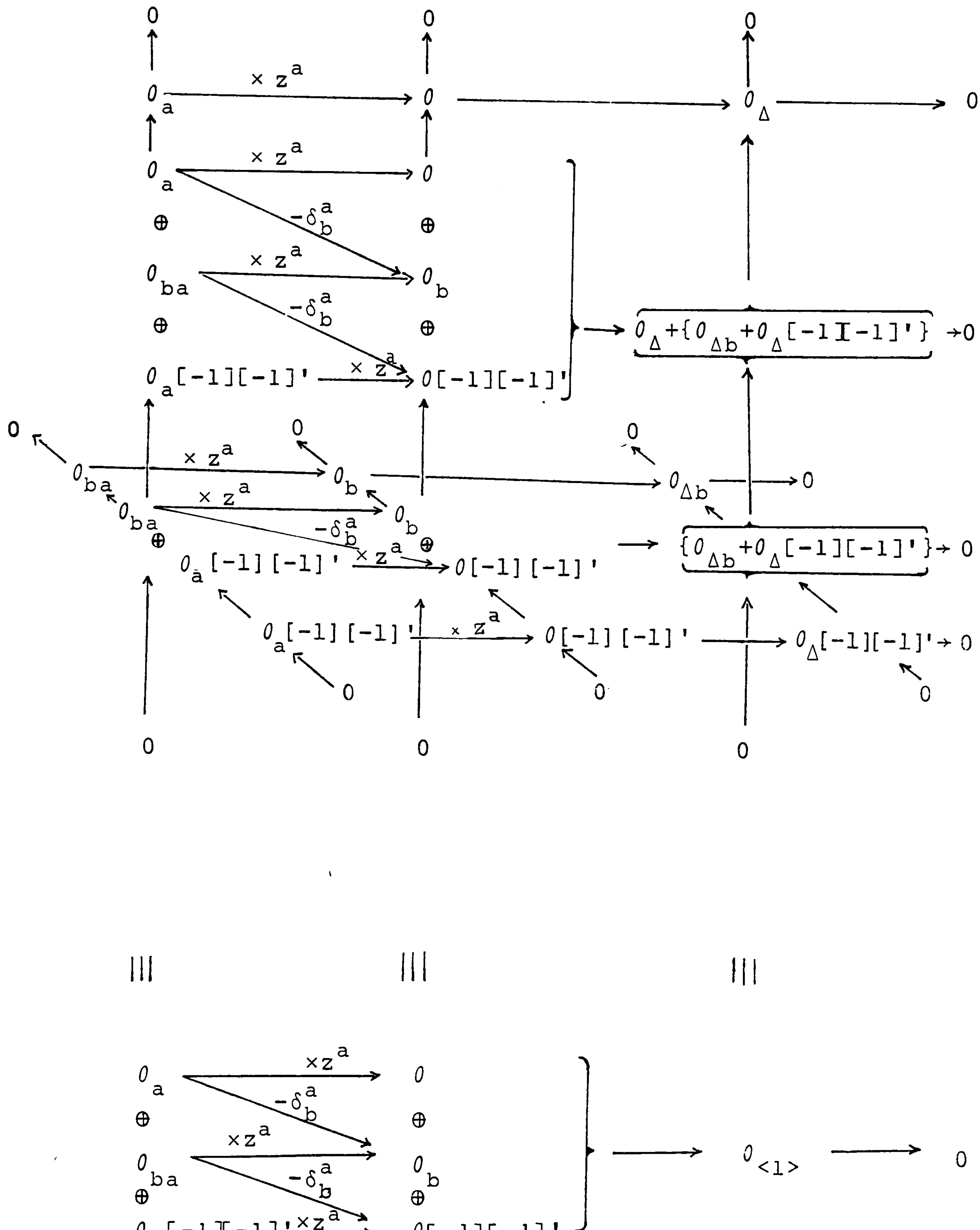
We now isolate the following part of (3.11)

$$\left.\begin{tikzcd}[row sep=tiny]
\mathcal{O}_a \arrow[r, "\times z^a"] \arrow[ddr, "-\delta^a_b"] & \mathcal{O} \\
\oplus & \oplus \\
\mathcal{O}_{ba} \arrow[r, "\times z^a"] \arrow[ddr, "-\delta^a_b"] & \mathcal{O}_b \\
\oplus & \oplus \\
\mathcal{O}_a[-1][-1]' \arrow[r, "\times z^a"] & \mathcal{O}[-1][-1]'
\end{tikzcd}\right\} \to \mathcal{O}_{<1>} \to 0\,; \qquad (3.12)$$

and, using trivial extensions, rewrite it as quotients of exact complexes as illustrated in Fig. 3.1.

Note: The diagonal maps correspond to the action of $\sigma^a \equiv \theta^A\tilde{\theta}^{A'}$.

Fig. 3.1



It is not difficult to convince oneself (from Fig. 3.1) that (3.12) can be written as $\mathcal{O}_{<1>} = \mathcal{O}_{\Delta} + \mathcal{O}_{\Delta b} + \mathcal{O}_{\Delta}[-1][-1]'$.†

Similarly, one can separately investigate the other simpler parts of (3.11). The answer is

$$\mathcal{O}_{[1]} \cong \underbrace{\{\mathcal{O} + \mathcal{O}_b + \mathcal{O}\,[-1][-1]'\} \oplus \mathcal{O}\,[-1] \oplus \mathcal{O}\,[-1]'}_{\bar{0}} \;\oplus\; \underbrace{\begin{pmatrix} \{\mathcal{O}_B + \mathcal{O}_{B'}[-1]\} \\ \oplus \\ \{\mathcal{O}_{B'} + \mathcal{O}_B[-1]'\} \end{pmatrix}}_{\bar{1}} \tag{3.13}$$

Note that as we anticipated, $\mathcal{O}_{[1]}$ is no longer $\mathbb{Z}$-graded, but $\mathbb{Z}_2$-graded only. The diagonal maps induced by σ^a preserve the $\mathbb{Z}_2$-grading but do not respect the $\mathbb{Z}$-grading.

The sheaf

$$\mathcal{O}_{<1>} \equiv \mathcal{O} + \mathcal{O}_b + \mathcal{O}\,[-1][-1]' ,$$

defines a non-singular unreduced $\mathbb{C}$-analytic space $\mathbb{M}_{<1>} \equiv (\mathbb{M}, \mathcal{O}_{<1>})$, whose local model (cf. §2.17) is given by $\mathcal{O}[z^a]/(z^a z^b z^c, z^{B(B'}z^{C')C})$.

Setting $\sigma^a = \theta^A \theta^{A'}$ gives rise to a surjective map of ringed spaces, $\mathbb{M}_{[1]} \twoheadrightarrow \mathbb{M}_{<1>}$, which is entirely determined by the inclusion $\mathcal{O}_{<1>} \hookrightarrow \mathcal{O}_{[1]}$ (since the map of underlying manifolds is the identity).

More generally, one has the inclusion $\mathcal{O}_{<N>} \cong \mathcal{O}/\mathcal{I}_{<N>} \hookrightarrow \mathcal{O}_{[N]}$ where $\mathcal{I}_{<N>}$ is the ideal generated by the irreducible components of $z^a z^b \ldots z^c$ which correspond to those irreducible components of $\sigma^a \sigma^b \ldots \sigma^c$ which vanish when

†From now on $\mathcal{O}_{\Delta}$ will also be denoted by $\mathcal{O}$.

we set $\sigma^a = \theta^A_j\theta^{A'j} = \theta^A_1\tilde{\theta}^{A'1} + \ldots + \theta^A_N\tilde{\theta}^{A'N}$. The $\mathbb{C}$-analytic space $\mathbb{M}_{<N>}$ was referred to as *thick Minkowski space* by M.G. Eastwood [16b]. However, note that it is not a thickening of a complex manifold as defined in §2.1.7.

Locally one can always choose a splitting and write superfunctions on $\mathbb{M}_{[N]}$ as a series in terms of $\tilde{\theta}^{A'j}$ and θ^A_k. For example, for N = 1

$$\Psi(x^a;\tilde{\theta}^{A'},\theta^A) \equiv \phi(x^a)+\theta^B\tilde{\theta}^{B'}\phi'_{BB'}(x^a)+\theta^2\tilde{\theta}^2\phi''(x^a)+\theta^2\gamma(x)+\tilde{\theta}^2\tilde{\gamma}(x^a)$$

$$+\theta^B\psi_B(x^a)+\theta^2\tilde{\theta}^{B'}\psi'_{B'}(x^a)+\theta^{B'}\tilde{\psi}_{B'}(x^a)+\theta^B\tilde{\theta}^2\tilde{\psi}'_B(x^a) \tag{3.14}$$

$(\theta^2 \equiv \varepsilon_{AB}\theta^A\theta^B$ and $\tilde{\theta}^2 \equiv \varepsilon_{A'B'}\tilde{\theta}^{A'}\tilde{\theta}^{B'})$.

The relationships between the coefficients $(\phi,\phi'_{BB'},\phi'')$, $(\psi_B,\psi'_{B'})$ and $(\tilde{\psi}_{B'},\tilde{\psi}'_B)$ are dictated by the choice of splitting of the relevant exact sequences.

§3.1.4 Chiral and antichiral superspaces

We define *antichiral super Minkowski space* and *chiral super Minkowski space* respectively by

$$^-\mathbb{M}_{[N]} \equiv (\mathbb{M},{}^-\mathcal{O}_{[N]}) \quad \text{and} \quad {}^+\mathbb{M}_{[N]} \equiv (\mathbb{M},{}^+\mathcal{O}_{[N]}) ,$$

where

$$^-\mathcal{O}_{[N]} = \{\Phi \in \mathcal{O}_{[N]} \text{ s.t. } \partial^j_A\Phi = 0\} \tag{3.15a}$$

$$^+\mathcal{O}_{[N]} = \{\Phi \in \mathcal{O}_{[N]} \text{ s.t. } \tilde{\partial}_{A'k}\Phi = 0\}. \tag{3.15b}$$

It is easy to check that if (for example) $\Phi \in {}^{-}\mathcal{O}_{[1]}$, then Φ has local series expansion:

$$[\phi(x^a)-\theta^B\tilde{\theta}^{B'}\partial_{BB'}\phi(x^a) - \frac{\theta^2\tilde{\theta}^2}{8}\,\Box\,\phi(x^a)]$$
$$+\tilde{\theta}^{B'}[\psi_{B'}(x^a)-\theta^B\tilde{\theta}^{B'}\partial_{BB'}\psi_{B'}(x^a)]+\tilde{\theta}^2\gamma(x^a)\ , \qquad (3.16)$$

where $(x^a;\theta^A,\tilde{\theta}^{A'})$ are the supercoordinates of $\mathbb{M}_{[1]}$ and

$$\partial_{BB'} = \frac{\partial}{\partial x^{BB'}}\ , \quad \Box = \frac{\partial}{\partial x_{BB'}}\frac{\partial}{\partial x^{BB'}}\ ,$$

Note that we can now rewrite (3.16) as

$$\Phi(x_-^a;\tilde{\theta}^{A'}) \equiv \phi(x_-^a) + \tilde{\theta}^{B'}\psi_{B'}(x_-^a) + \tilde{\theta}^2\gamma(x_-^a) \qquad (3.17a)$$

where $(x_-^{AA'} = x^{AA'} - \theta^A\tilde{\theta}^{A'};\tilde{\theta}^{A'},\theta^A)$ are genuine new supercoordinates for $\mathbb{M}_{[1]}$ (cf. §2.3.5), called the *antichiral supercoordinates*.

Similarly, note that $\tilde{\Phi} \in {}^{+}\mathcal{O}_{[1]}$ can be written as

$$\tilde{\Phi}(x^a;\theta^A) \equiv \tilde{\phi}(x_+^a) + \theta^B\tilde{\psi}_B(x_+^a) + \theta^2\tilde{\gamma}(x_+^a)\ , \qquad (3.17b)$$

where $(x_+^{AA'} = x^{AA'} + \theta^A\tilde{\theta}^{A'};\tilde{\theta}^{A'},\theta^A)$ are genuine new supercoordinates for $\mathbb{M}_{[1]}$, called the *chiral supercoordinates*.

In general, for all N one sets $x_-^{AA'} = x^{AA'} - \theta_j^A\tilde{\theta}^{A'j}$ and $x_+^{AA'} = x^{AA'} + \theta_j^A\tilde{\theta}^{A'j}$. Then $(x_-^{AA'};\tilde{\theta}^{A'j})$ and $(x_+^{AA'};\theta_j^A)$ are taken as the supercoordinates for the $(4|2N)$-dimensional supermanifolds ${}^{-}\mathbb{M}_{[N]}$ and ${}^{+}\mathbb{M}_{[N]}$ respectively.

From the generalization (for all N) of (3.17a) and (3.17b) it is clear that, at least locally

$$^{-}\mathcal{O}_{[N]} \cong \Lambda^{\bullet}\mathcal{O}_{A'k} \tag{3.18a}$$

and

$$^{+}\mathcal{O}_{[N]} \cong \Lambda^{\bullet}\mathcal{O}^{j}_{A} \quad , \tag{3.18b}$$

so $\mathcal{O}_{A'k}$ and $\mathcal{O}^{j}_{A}$ are the underlying vector bundles for $^{-}\mathbb{M}_{[N]}$ and $^{+}\mathbb{M}_{[N]}$ respectively.

In fact, the only possible superstructures with either $\mathcal{O}_{A'k}$ or $\mathcal{O}^{j}_{A}$ as underlying vector bundles are the trivial ones. (Note that this is certainly not the case if instead of $\mathcal{O}_{A'k}$ or $\mathcal{O}^{j}_{A}$ one takes their direct sum as the underlying vector bundle.) This can be deduced by computing the cohomology groups of Theorem 2.iv. (The computations have been carried out by M.G. Eastwood [19].) Hence

$$^{-}\mathbb{M}_{[N]} = (\mathbb{M},\Lambda^{\bullet}\mathcal{O}_{A'k}) \text{ and } ^{+}\mathbb{M}_{[N]} = (\mathbb{M},\Lambda^{\bullet}\mathcal{O}^{j}_{A}).$$

Furthermore, since

$$\Lambda^{\bullet}\mathcal{O}_{A'k} \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}^{j}_{A} \cong \Lambda^{\bullet}(\mathcal{O}_{A'k} \oplus \mathcal{O}^{j}_{A}) \tag{3.19}$$

we can conclude that

$$^{-}\mathbb{M}_{[N]} \times {}^{+}\mathbb{M}_{[N]} = (\mathbb{M}\times\mathbb{M}, \Lambda^{\bullet}\mathcal{O}_{A'k} \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}^{j}_{A}) \cong (\mathbb{M}\times\mathbb{M})_{[N]} \tag{3.20}$$

Finally we will determine the form of the differential operators ∂_a, ∂^{j}_{A} and $\tilde{\partial}_{A'k}$ in chiral and antichiral supercoordinates. Consider the chiral case first

$$\frac{\partial}{\partial\theta^{A}_{j}} = \frac{\partial\theta_{k}^{B}}{\partial\theta^{A}_{j}}\frac{\partial}{\partial\theta^{B}_{k}} + \frac{\partial x^{BB'}_{+}}{\partial\theta^{A}_{j}}\frac{\partial}{\partial x^{BB'}_{+}} = \frac{\partial}{\partial\theta^{A}_{j}} + \tilde{\theta}^{B'j}\frac{\partial}{\partial x^{AB'}_{+}}$$

$$\frac{\partial}{\partial\tilde{\theta}^{A'j}} = \frac{\partial\theta^{B}_{k}}{\partial\tilde{\theta}^{A'j}}\frac{\partial}{\partial\theta^{B}_{k}} + \frac{\partial x^{BB'}_{+}}{\partial\tilde{\theta}^{A'j}}\frac{\partial}{\partial x^{BB'}_{+}} = -\theta^{A}_{j}\frac{\partial}{\partial x^{AA'}_{+}}$$

$$\frac{\partial}{\partial x^{AA'}} = \frac{\partial\theta^{B}_{k}}{\partial x^{AA'}}\frac{\partial}{\partial\theta^{B}_{k}} + \frac{\partial x^{BB'}_{+}}{\partial x^{AA'}}\frac{\partial}{\partial x^{BB'}_{+}} = \frac{\partial}{\partial x^{AA'}_{+}} \ .$$

Hence

$$(\partial_a, \partial^j_A, \tilde{\partial}_{A'k}) \mapsto (\partial^+_a = \frac{\partial}{\partial x^a}\,,\ \partial^{+j}_A = \frac{\partial}{\partial \theta^A_j} + 2\tilde{\theta}^{A'j}\,\frac{\partial}{\partial x^{AA'}_+}\,,\ \tilde{\partial}^+_{A'k} = \frac{\partial}{\partial \tilde{\theta}^{A'k}})\,. \tag{3.22}$$

Similarly, for the antichiral case

$$(\partial_a, \partial^j_A, \tilde{\partial}_{A'k}) \mapsto (\partial^-_a = \frac{\partial}{\partial x^a_-}\,,\ \partial^{-j}_A = \frac{\partial}{\partial \theta^A_j}\,,\ \partial^-_{A'k} = \frac{\partial}{\partial \tilde{\theta}^{A'k}} - 2\theta^A{}_j\,\frac{\partial}{\partial x^{AA'}_-})\,. \tag{3.23}$$

§3.1.5 A summary

The constructions of §3.1 may be illustrated by the diagram below

$$\begin{tikzcd}
(u^a,v^a) & & (u^a,v^a;\tilde{\theta}^{A'j},\theta^A_j) & & (x^a_-;\tilde{\theta}^{A'j}) & & (x^a_+;\theta^A_j) \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\
\mathbb{M}\times\mathbb{M} \arrow[rr, hook, "\varepsilon"] & & (\mathbb{M}\times\mathbb{M})_{[N]} & \cong & {}^-\mathbb{M}_{[N]} & \times & {}^+\mathbb{M}_{[N]} \\
\mathbb{M}_{\langle N\rangle} \arrow[u, hook] & & \mathbb{M}_{[N]} \arrow[u, hook] \arrow[ll, twoheadrightarrow, "\sigma^a = \theta^A_j\tilde{\theta}^{A'j}"'] \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(x^a,\sigma^a) & & (x^{AA'};\theta^A_j,\tilde{\theta}^{A'j}) \\
& \mathbb{M} \arrow[uuul, hook, "\varepsilon"] \arrow[uuur, hook, "\varepsilon"'] \\
& \rotatebox{90}{$\in$} \\
& x^a
\end{tikzcd}$$

Fig. 3.2

The maps denoted by ε are the supermanifold augmentation maps, obtained by setting the nilpotent elements of the superstructure sheaves to zero.

§3.2 Super Yang-Mills theory

§3.2.1 The superpotentials and the super Yang-Mills field

The $(4|4N)$-vector fields $\partial_A \equiv (\partial_a, \partial^j_A, \tilde{\partial}_{A'k})$ provide a frame for the tangent bundle $\mathcal{D}er\,\mathcal{O}_{[N]}$ of $\mathbb{M}_{[N]}$ (i.e. $(4|4N)$-linearly independent sections which provide a basis for the $\mathcal{O}_{[N]}$-supermodule $\mathcal{D}er\,\mathcal{O}_{[N]}$). From these differential operators one can construct the $(1,\bar{0})$-bigraded (w.r.t. $(\mathbb{Z},\mathbb{Z}_2)$) operator

$$d \equiv dx^{AA'}\partial_{AA'} + d\theta^A_j\partial^j_A + d\tilde{\theta}^{A'k}\tilde{\partial}_{A'k}$$

satisfying $d^2 = 0$.

Consider the deRham sequence on $\mathbb{M}_{[N]}$; and note that elements

$$X \in \mathcal{O}_{[N]\mathcal{A}} = \mathcal{O}_{[N]a} \oplus \mathcal{O}_{[N]A}^{\;j} \oplus \mathcal{O}_{[N]A'j}$$

and

$$Y \in \mathcal{O}_{[N]\mathcal{AB}} = \mathcal{O}_{[N]ab} \oplus \mathcal{O}_{[N]aB}^{\;k} \oplus \mathcal{O}_{[N]aB'k} \oplus \mathcal{O}_{[N]AB}^{\;jk} \oplus \mathcal{O}_{[N]AB'k}^{\;j} \oplus \mathcal{O}_{[N]A'B'jk}$$

can be written as supermatrices with the following (super)symmetries:

$$X = (A \,\vdots\, B \quad C) \tag{3.24a}$$

$$Y = \left(\begin{array}{c|cc} F & G & H \\ \hline -G & I & J \\ -H & J & K \end{array}\right) \tag{3.24b}$$

where

$$A \in \mathcal{O}_{[N]a}, \quad B \in \mathcal{O}_{[N]A}^{\;j}, \quad C \in \mathcal{O}_{[N]A'j}$$

$F \in \mathcal{O}_{[N]ab}$, $\quad G \in \mathcal{O}_{[N]aB}{}^{k}$, $\quad H \in \mathcal{O}_{[N]aB'k}$

$I \in \mathcal{O}_{[N]AB}{}^{jk}$, $\quad J \in \mathcal{O}_{[N]AB'k}{}^{j}$, $\quad K \in \mathcal{O}_{[N]A'B'jk}$.

Let E be a supervector bundle on $\mathbb{M}_{[N]}$ of rank $(r|s)$, i.e. locally $E \cong \mathcal{O}^{r|s}_{[N]} \equiv \mathcal{O}_{[N]} \otimes_{\mathbb{C}} \mathbb{C}^{r|s}$. In our local supercoordinates, a superconnection on E can be written as

$$\nabla = dZ^{A}\nabla_{A} = dx^{a}\nabla_{a} + d\theta^{A}_{j}\nabla^{j}_{A} + d\tilde{\theta}^{A'k}\tilde{\nabla}_{A'k} , \qquad (3.25)$$

with

$$\nabla_{a} = \partial_{a} + \Phi_{a}, \quad \nabla^{j}_{A} = \partial^{j}_{A} + \Psi^{j}_{A}, \quad \tilde{\nabla}_{A'k} = \tilde{\partial}_{A'k} + \tilde{\Psi}_{A'k}, \qquad (3.26)$$

where Φ_{a}, Ψ^{j}_{A} and $\tilde{\Psi}_{A'k}$ will be referred to as the *superpotentials*. With respect to a trivialization of E, for each of the 4+4N values of their indices, these superpotentials can be regarded as the following homogeneous elements of $End_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]}$:

$$\Phi_{a} \in (End_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s})_{\bar{0}}, \quad \Psi^{j}_{A} \in (End_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s})_{\bar{1}}, \quad \tilde{\Psi}_{A'k} \in (End_{\mathcal{O}_{[N]}} \mathcal{O}^{r|s}_{[N]})_{\bar{1}} .$$

We shall frequently write the collection of superpotentials in the form (3.24a):

$$\phi_{A} \equiv (\Phi_{a} \quad \Psi^{j}_{A} \quad \tilde{\Psi}_{A'k})$$

Under a change of frame (i.e. a gauge transformation), generated by $g \in GL(r|s;\mathcal{O}_{[N]})$, superpotentials transform as

$$(\Phi_{a} \quad \Psi^{j}_{A} \quad \tilde{\Psi}_{A'k}) \longmapsto (g^{-1}(\partial_{a}+\Phi_{a})g \quad g^{-1}(\partial^{j}_{A}+\Psi^{j}_{A})g \quad g^{-1}(\tilde{\partial}_{A'k}+\tilde{\Psi}_{A'k})g) . \qquad (3.27)$$

As a particular example of such a superconnection, consider the special case of a superline bundle as obtained earlier (in §2.3.7) using

$$\exp : H^1(\mathbb{M}, \mathcal{O}_{[N]}) \to H^1(\mathbb{M}, GL(1|1;\mathcal{O}_{[N]})).$$

An element $h \in H^1(\mathbb{M},\mathcal{O}_{[N]})$ is represented by a Čech cocycle $h_{\alpha\beta}$ satisfying (the cocycle condition) $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$. A *superconnection on* h is a collection of 1-forms $\check{\phi}_\alpha$, on $\mathbb{M}_{[N]}$, such that under changes of frame: $\check{\phi}_\alpha \longmapsto \check{\phi}_\beta - dh_{\alpha\beta}$, i.e. in our local supercoordinates

$$(\check{\Phi}_a \quad \check{\Psi}^j_A \quad \check{\tilde{\Psi}}_{A'k}) \longmapsto (\check{\Phi}_a - \partial_a h \quad \check{\Psi}^j_A - \partial^j_A h \quad \check{\tilde{\Psi}}_{A'k} - \tilde{\partial}_{A'k} h),$$

where $\check{\phi}_A \equiv (\check{\Phi}_a \quad \check{\Psi}^j_A \quad \check{\tilde{\Psi}}_{A'k}) \in \mathcal{O}_{[N]A}$.

From $\check{\phi}_\alpha$ we can now construct the corresponding *super connection on exp* $h \in H^1(\mathbb{M}, \mathcal{O}^*_{[N]}) \subset H^1(\mathbb{M}, GL(1|1,\mathcal{O}_{[N]}))$ which we denote by ϕ_α. (Note that the transition functions of *exp* h satisfy $(\exp h_{\alpha\beta})(\exp h_{\beta\gamma})(\exp h_{\gamma\alpha}) = 1$, and that a section g_α of *exp* h, over the open set U_α, is related to g_β, over another open set U_β, by $g_\alpha = \exp h_{\alpha\beta} g_\beta$.)
Following from §2.3.7, suppose that $s = Am^1 + Bm^2$ $(A,B \in \mathcal{O}_{[N]})$ is a section of exp h. Let the superconnection ϕ_α act on $\mathcal{O}^{1|1}_{[N]} \equiv \mathcal{O}_{[N]} \otimes_{\mathbb{C}} \mathbb{C}^{1|1}$ by identifying $\mathbb{C}^{1|1}$ with $\Lambda^\bullet\mathbb{C} \cong \mathbb{C}[\tau]/(\tau^2)$ and multiplying on the right. Then, in our local supercoordinates, ∇s becomes

$$\begin{aligned}\nabla_a(Am^1 + Bm^2) &= \partial_a(Am^1 + Bm^2) + (Am^1 + Bm^2)(\check{\Phi}_{a_{\bar{0}}} + \check{\Phi}_{a_{\bar{1}}}\tau) \\ &= (\partial_a A + A\check{\Phi}_{a_{\bar{0}}})m^1 + (\partial_a B + B\check{\Phi}_{a_{\bar{0}}} + A\check{\Phi}_{a_{\bar{1}}})m^2 \qquad (3.28)\end{aligned}$$

(Recall that $m^1\tau = m^2$ and $m^2\tau = 0$.)

$$\nabla^j_A(Am^1+Bm^2) = \partial^j_A(Am^1+Bm^2)+(Am^1+Bm^2)(\check{\psi}^j_{A_{\bar{1}}}+\check{\psi}^j_{A_{\bar{0}}}\tau)$$
$$= (\partial^j_A A + A\check{\psi}^j_{A_{\bar{1}}})m^1 + (\partial^j_A B - B\check{\psi}^j_{A_{\bar{1}}} + A\check{\psi}^j_{A_{\bar{0}}})m^2 \tag{3.28b}$$

$$\tilde{\nabla}_{A'k}(Am^1+Bm^2) = \tilde{\partial}_{A'k}(Am^1+Bm^2)+(Am^1+Bm^2)(\check{\tilde{\Psi}}_{A'k_{\bar{1}}}+\check{\tilde{\Psi}}_{A'k_{\bar{0}}}\tau)$$
$$= (\tilde{\partial}_{A'k}A + A\check{\tilde{\Psi}}_{A'k_{\bar{1}}})m^1 + (\tilde{\partial}_{A'k}B - B\check{\tilde{\Psi}}_{A'k_{\bar{1}}} + A\check{\tilde{\Psi}}_{A'k_{\bar{0}}})m^2, \tag{3.28c}$$

where $(\check{\Phi}_{a_{\bar{0}}} \quad \check{\psi}^j_{A_{\bar{0}}} \quad \check{\tilde{\Psi}}_{A'k_{\bar{0}}}) \in \mathcal{O}_{[N]A_{\bar{0}}}$ and $(\check{\Phi}_{a_{\bar{1}}} \quad \check{\psi}^j_{A_{\bar{1}}} \quad \check{\tilde{\Psi}}_{A'k_{\bar{1}}}) \in \mathcal{O}_{[N]A_{\bar{1}}}$. Equivalently, writing $Am^1 + Bm^2$ as the row vector $(A \quad B)$, (3.28) may be rewritten in supermatrix form as

$$(A \quad B) \xmapsto{\Phi_a} (A \quad B)\begin{pmatrix} \check{\Phi}_{a_{\bar{0}}} & \check{\Phi}_{a_{\bar{1}}} \\ 0 & \check{\Phi}_{a_{\bar{0}}} \end{pmatrix} \tag{3.29a}$$

$$(A \quad B) \xmapsto{\psi^j_A} (A \quad B)\begin{pmatrix} \check{\psi}^j_{A_{\bar{1}}} & \check{\psi}^j_{A_{\bar{0}}} \\ 0 & -\check{\psi}^j_{A_{\bar{1}}} \end{pmatrix} \tag{3.29b}$$

$$(A \quad B) \xmapsto{\tilde{\Psi}_{A'k}} (A \quad B)\begin{pmatrix} \check{\tilde{\Psi}}_{A'k_{\bar{1}}} & \check{\tilde{\Psi}}_{A'k_{\bar{0}}} \\ 0 & -\check{\tilde{\Psi}}_{A'j_{\bar{1}}} \end{pmatrix}. \tag{3.29c}$$

Thus, ϕ_α can be expressed in local supercoordinates as

$$dx^a\Phi_a + d\theta^A_j\psi^j_A + d\tilde{\theta}^{A'k}\tilde{\Psi}_{A'k}$$

where $(\Phi_a \quad \psi^j_A \quad \tilde{\Psi}_{A'k})$ represent the $(1|1)\times(1|1)$-supermatrices of (3.29).

The superconnection induces a curvature $F \in \Omega^2(End_{0_{[N]}} E)$. In local supercoordinates

$$F = dZ^A \wedge dZ^B F_{AB} , \tag{3.30}$$

where F_{AB} , known as the *super Yang-Mills field*, is given by

$$F_{ab} \equiv [\nabla_a,\nabla_b] = \partial_a\Phi_b - \partial_b\Phi_a + [\Phi_a,\Phi_b] \tag{3.31a}$$

$$F_{aB}^{\ j} \equiv [\nabla_a,\nabla_B^j] = \partial_a\Psi_B^j - \partial_B^j\Phi_a + [\Phi_a,\Psi_B^j] \tag{3.31b}$$

$$F_{aB'j} \equiv [\nabla_a,\tilde\nabla_{B'j}] = \partial_a\tilde\Psi_{B'j} - \tilde\partial_{B'j}\Phi_a + [\Phi_a,\tilde\Psi_{B'j}] \tag{3.31c}$$

$$F_{AB}^{ik} \equiv (\nabla_A^j,\nabla_B^k) = \partial_A^j\Psi_B^k + \partial_B^k\Psi_A^j + (\Psi_A^j,\Psi_B^k) \tag{3.31d}$$

$$F_{A'B'jk} \equiv (\tilde\nabla_{A'j},\tilde\nabla_{B'k}) = \tilde\partial_{A'j}\tilde\Psi_{B'k} + \tilde\partial_{B'k}\tilde\Psi_{A'j} + (\tilde\Psi_{A'j},\tilde\Psi_{B'k}) \tag{3.31e}$$

$$F_{AA'k}^{j} \equiv (\nabla_A^j,\tilde\nabla_{A'k})-2\delta_k^j\nabla_{AA'} = \partial_A^j\tilde\Psi_{A'k}+\tilde\partial_{A'k}\Psi_A^j+(\Psi_A^j,\tilde\Psi_{A'k})-2\delta_k^j\Phi_{AA'} , \tag{3.31f}$$

where the commutators and anticommutators on the right-hand sides above are given by supermatrix multiplication. Note in the case of a superline bundle, where the superpotentials are $(1|1)\times(1|1)$-supermatrices of the form given in (2.29), all these commutators and anticommutators vanish. Thus, in this special case we will say that F_{AB} is *super abelian*.

It turns out to be convenient (in §3.2.3) to have the super Yang-Mills fields (3.31) written in the form (3.24b) as follows

$$F_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} F_{ab} & F_{aB}{}^{k} & F_{aB'k} \\ F^{j}_{Ab} & F^{jk}_{AB} & F^{j}_{AB'k} \\ F_{A'bj} & F_{A'Bj}{}^{k} & F_{A'B'jk} \end{pmatrix} . \qquad (3.32)$$

(Each entry in the above supermatrix is itself a supermatrix.)

§3.2.2 The Bianchi identities and constraint equations [84]

The curvature components (3.31) are related to each other through the super Jacobi identities

$$(-1)^{\bar{\mathcal{A}}\bar{\mathcal{B}}}[\nabla_{\mathcal{A}},[\nabla_{\mathcal{B}},\nabla_{\mathcal{C}}))+(-1)^{\bar{\mathcal{B}}\bar{\mathcal{C}}}[\nabla_{\mathcal{B}},[\nabla_{\mathcal{C}},\nabla_{\mathcal{A}}))+(-1)^{\bar{\mathcal{C}}\bar{\mathcal{A}}}[\nabla_{\mathcal{C}},[\nabla_{\mathcal{A}},\nabla_{\mathcal{B}})) = 0. \qquad (3.33)$$

(3.33) is equivalent to ten identities which, when written in terms of the super Yang-Mills curvatures, become

$$\tilde{\nabla}_{A'i}F_{B'C'jk}+\tilde{\nabla}_{B'j}F_{C'A'ki}+\tilde{\nabla}_{C'k}F_{A'B'ij} = 0 \qquad (3.34a)$$

$$\nabla^{i}_{A}F^{jk}_{BC}+\nabla^{j}_{B}F^{ki}_{CA}+\nabla^{k}_{C}F^{ij}_{AB} = 0 \qquad (3.34a')$$

$$\nabla_{A'i}F_{BB'j}{}^{k}+2\delta^{k}_{j}F_{A'BB'i}+\nabla^{k}_{B}F_{B'A'ji}+\tilde{\nabla}_{B'j}F_{A'Bi}{}^{k}+2\delta^{k}_{i}F_{B'A'Bj} = 0 \qquad (3.34b)$$

$$\nabla^{i}_{A}F^{j}_{BB'k}+2\delta^{j}_{k}F^{i}_{ABB'}+\tilde{\nabla}_{B'k}F^{ij}_{AB}+\nabla^{j}_{B}F_{B'Ak}{}^{i}+2\delta^{i}_{k}F_{BB'A} = 0 \qquad (3.34b')$$

$$\nabla_{a}F_{B'C'ij}+\tilde{\nabla}_{B'i}F_{C'AA'j}-\tilde{\nabla}_{C'j}F_{AA'B'i} = 0 \qquad (3.34c)$$

$$\nabla_{a}F^{ij}_{BC}+\nabla^{i}_{B}F^{j}_{CAA'}-\nabla^{j}_{C}F_{AA'B}{}^{i} = 0 \qquad (3.34c')$$

$$\tilde{\nabla}_{A'i}F_{bc}+\nabla_{b}F_{cA'i}+\nabla_{c}F_{A'bi} = 0 \qquad (3.34d)$$

$$\nabla^{i}_{A}F_{bc}+\nabla_{b}F_{cA}{}^{i}+\nabla_{c}F^{i}_{Ab} = 0 \qquad (3.34d')$$

$$\nabla_{a}F^{i}_{BB'j}+2\delta^{i}_{j}F_{ab}+\nabla^{i}_{B}F_{B'AA'j}-\tilde{\nabla}_{B'j}F_{AA'B}{}^{i} = 0 \qquad (3.34e)$$

$$\nabla_{a}F_{bc}+\nabla_{b}F_{ca}+\nabla_{c}F_{ab} = 0 \qquad (3.34f)$$

where

$$\nabla_A F_{BC} \equiv \partial_A F_{BC} + [\phi_A, F_{BC}) ,$$

with

$$[\phi_A, F_{BC}) = \phi_A F_{BC} - (-1)^{\bar{A}(\bar{B}+\bar{C})} F_{BC}\phi_A$$

given by supermatrix multiplication. The system of superfield equations (3.34a) to (3.34f) will be denoted by

$$\nabla F = 0 . \qquad (3.34)$$

The superfield equations (3.34) are not all independent. M.F. Sohnius [84] investigated these equations by isolating the truly independent ones for different values of N. He then proceeded, for each N, to set equal to zero as many parts of the curvatures as possible without imposing 'equations of motion' on any of the coefficient fields of their θ's expansion. By equations of motion Sohnius simply meant equations on the space-time coordinates. These equations (which set parts of the curvatures to zero) are known as the *constraint equations:*

$$F_{AB} = F_{A'B'} = F_{AA'} = 0 \qquad (N = 1) \qquad (3.36a)$$

$$F^{jk}_{AB} = \varepsilon_{AB}\tilde{W}^{jk} \;, F_{A'jB'k} = \varepsilon_{A'B'}W_{jk} \;, F^{j}_{A'Ak} = 0 \;(N > 1) \qquad (3.36b)$$

where $\tilde{W}^{jk} = -\tilde{W}^{kj}$ and $W_{jk} = -W_{kj}$.

Note: For N = 4 the following additional equation is also imposed

$$\tilde{W}^{k\ell} = \varepsilon^{k\ell mn} W_{mn}. \qquad (3.37)$$

E. Witten [93] indicated that the superfield equations (3.36) were precisely the conditions for the superpotentials to be 'integrable on super null lines', and as such, had a natural interpretation within the context of superambitwistors. In the next chapter we shall see how to interpret this statement within the formalism of Chapter 2.

Substituting equations (3.36) into the Bianchi identities (3.34) one obtains further relationships between the various curvatures which we summarize below [84]

$$\nabla^k_A W_{ij} = 4\delta_{[i}{}^k W_{j]A} \qquad (N > 1) \qquad (3.38b)$$

$$\tilde{\nabla}_{A'k} \tilde{W}^{ij} = 4\delta_k^{[i} \tilde{W}^{j]}_{A'} \qquad (N > 1) \qquad (3.38b')$$

$$\nabla_{AA'} W_{ij} = \tilde{\nabla}_{A'i} W_{Aj} \qquad (N > 1) \qquad (3.38c)$$

$$\nabla_{AA'} \tilde{W}^{ij} = \nabla^i_A \tilde{W}^j_{A'} \qquad (N > 1) \qquad (3.38c')$$

$$\tilde{\nabla}_{A'i} f_{AB} = \nabla_{A'(A} W_{B)i} \qquad (\forall\ N) \qquad (3.38d)$$

$$\nabla^i_A \tilde{f}_{A'B'} = \nabla_{A(A'} \tilde{W}^i_{B')} \qquad (\forall\ N) \qquad (3.38d')$$

$$f_{AB} = \frac{1}{2N} \nabla^i_{(A} W_{B)i} \qquad (\forall\ N) \qquad (3.38e)$$

$$\tilde{f}_{A'B'} = -\frac{1}{2N} \tilde{\nabla}_{i(A'} \tilde{W}^i_{B')} \qquad (\forall\ N) \qquad (3.38e')$$

$$\nabla^i_A W^A_i + \tilde{\nabla}_{A'j} \tilde{W}^{A'j} = 0 \qquad (\forall\ N) \qquad (3.38e")$$

where

$$F_{ab} = \varepsilon_{A'B'}f_{AB} + \varepsilon_{AB}\tilde{f}_{A'B'} \qquad (\forall\ N) \qquad (3.38f)$$

$$F_{AA'B'j} = \varepsilon_{A'B'}W_{Aj} \qquad (\forall\ N) \qquad (3.38g)$$

$$F^{j}_{A'AB} = \varepsilon_{AB}\tilde{W}^{j}_{A'} \qquad (\forall\ N)\ . \qquad (3.38g')$$

With a little more manipulation of the super Jacobi identities one obtains the following additional relations

$$\nabla^{i}_{C}W^{C}_{j} = -[\tilde{W}^{ki},W_{kj}] + \frac{\delta^{i}_{j}}{N+1}[\tilde{W}^{k\ell},W_{k\ell}] \qquad (3.39a)$$

$$\tilde{\nabla}_{C'j}\tilde{W}^{C'i} = -[W_{kj},\tilde{W}^{ki}] + \frac{\delta^{i}_{j}}{N+1}[W_{k\ell},\tilde{W}^{k\ell}]\ . \qquad (3.39b)$$

§3.2.3 A definition of the super Yang-Mills equations

At this point we note that if one imposes the superfield equations (3.38) on our supermatrix representation of the super Yang-Mills field (3.32), then the latter splits naturally into two parts:

$$F = F^{(+)} + F^{(-)}\ , \qquad (3.40)$$

where (in local supercoordinates)

$$F^{(+)} = dZ^{\mathcal{A}} \wedge dZ^{\mathcal{B}}F^{(+)}_{\mathcal{AB}} \text{ and } F^{(-)} = dZ^{\mathcal{A}} \wedge dZ^{\mathcal{B}}F^{(-)}_{\mathcal{AB}}\ ,$$

with

$$F^{(+)}_{\mathcal{AB}} = \varepsilon_{AB}\begin{pmatrix} \tilde{f}_{A'B'} & \tilde{W}_{A'}^{k} & 0 \\ \tilde{W}^{j}_{B'} & \tilde{W}^{jk} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (3.41a)$$

and

$$F^{(-)}_{AB} = \varepsilon_{A'B'} \begin{pmatrix} f_{AB} & 0 & W_{Ak} \\ 0 & 0 & 0 \\ W_{Bj} & 0 & W_{jk} \end{pmatrix} \tag{3.41b}$$

By analogy with classical Yang-Mills theory, it is reasonable to call these parts the *self-dual and anti-self-dual parts of the super Yang-Mills field* respectively. Conversely, any super Yang-Mills field of the form (3.40) automatically satisfies the constraint equations.

To continue with the generalization of the classical case, we define a star operator by

$$*F = F^{(+)} - F^{(-)} . \tag{3.42}$$

A natural definition for the *super Yang-Mills equations* now suggests itself. Namely

$$\nabla * F = 0 , \tag{3.43}$$

which, by virtue of the (3.40), (3.42) and the Bianchi identities (3.3), it is equivalent to either

$$\nabla F^{(+)} = 0 \tag{3.44a}$$

or

$$\nabla F^{(-)} = 0 . \tag{3.44b}$$

It is now a simple matter to interpret these equations in terms of the various superfields comprising (3.41a). The easiest way to do this is to replace F by $F^{(-)}$ in the Bianchi identities (3.3). Then a careful check of the derivation of (3.38) from the Bianchi identities soon reveals that equation (3.38e") will be the only equation giving something new. Namely,

$$\nabla^i_A W^A_i = 0 \quad . \tag{3.45}$$

For N = 1, (3.45) is in agreement with the definition of the super Yang-Mills equations used in the Physics Literature (see [74a,28]). This equation is arrived at by variational techniques starting from an action of the form $\mathrm{str} W^j_A W^A_j$. The form of such action is determined entirely from dimensional considerations and from the particle spectrum of the theory.

We are uncertain regarding the situation for N > 1.

Finally we define the *self-dual super Yang-Mills equations* by

$$F^{(-)} = 0 \; . \tag{3.46}$$

From (3.41b) and (3.38) one can deduce that (3.46) is equivalent to

$$\begin{cases} W_A = 0 & \text{(if } N = 1) \quad (3.47a) \\ W_{jk} = 0 & \text{(if } N > 1) \; . \quad (3.47b) \end{cases}$$

§3.2.4 'Anticommuting coefficients'

The super Yang-Mills equations (both (3.43) and (3.46)) are non-linear equations in the superpotentials $(\Phi_a \quad \Psi^j_A \quad \tilde{\Psi}_{A'k})$. These superpotentials are $(r|s)\times(r|s)$-supermatrices with entries in $\mathcal{O}_{[N]}$, and therefore each entry can be expanded as a power series in the odd supercoordinates θ^A_j and $\tilde{\theta}^{A'k}$. If we write the coefficients (of the expansion of each entry) to the right of the θ's, then the $\mathcal{O}_{[N]}$-supermodule multiplication rule (2.20b) tells us how to remove a collection of θ's from the supermatrices Φ_a, Ψ^j_A and $\tilde{\Psi}_{A'k}$. In this way the superpotentials can be written as a power series (in θ^A_j and $\tilde{\theta}^{A'k}$) where each coefficient is a homogeneous element of $\mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s}$. The grading of each coefficient is determined by the overall grading of the particular superpotential and by the number of θ's with which it has been contracted. For example, the coefficients of $\Phi_a(x^a;\theta^A_j,\tilde{\theta}^{A'k})$ which are contracted with an even (odd) number of θ's are elements of $\mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s}$ of the form $\begin{array}{c} \\ r\{ \\ s\{ \end{array}\!\!\begin{array}{c} \overbrace{\ }^{r}\overbrace{\ }^{s} \\ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \end{array}$ $\left(\begin{array}{c} \\ r\{ \\ s\{ \end{array}\!\!\begin{array}{c} \overbrace{\ }^{r}\overbrace{\ }^{s} \\ \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right) \end{array}\right)$. On the other hand, the coefficients of $\Psi^j_A(x^a;\theta^A_j,\tilde{\theta}^{A'k})$ and $\tilde{\Psi}_{A'k}(x^a;\theta^A_j,\tilde{\theta}^{A'k})$ which are contracted with an even (odd) number of θ's are of the form $\begin{array}{c} \\ r\{ \\ s\{ \end{array}\!\!\begin{array}{c} \overbrace{\ }^{r}\overbrace{\ }^{s} \\ \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right) \end{array}$ $\left(\begin{array}{c} \\ r\{ \\ s\{ \end{array}\!\!\begin{array}{c} \overbrace{\ }^{r}\overbrace{\ }^{s} \\ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \end{array}\right)$.

These θ-expansions are very useful for computations. From now on instead of thinking of the superpotentials as elements of $\mathit{End}_{\mathcal{O}_{[N]}}(\mathcal{O}^{r|s}_{[N]})$ we shall think of them as elements $\mathcal{O}_{[N]}\otimes_{\mathbb{C}}\mathrm{End}\,\mathbb{C}^{r|s}$. The procedure for removing the θ's from the supermatrices defines the isomorphism between these two $\mathcal{O}_{[N]}$-supermodules.

The superalgebra structure of $\mathcal{O}_{[N]}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s}$ is given by the multiplication rule (2.12). As a consequence of this rule, the coefficients of the form $\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$ (i.e. elements of $(\mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s})_{\bar{1}}$) anticommute with products of an odd number of θ's. This property shows up when considering commutators and anticommutators of power series expansions of superpotentials.

The statements of this subsection will be illustrated with the following examples of the N = 1 super Yang-Mills theory in the superabelian case (see (3.29));

Example (i) Suppose $\Psi_A = \begin{pmatrix} \tilde{\theta}^{B'}\check{V}_{AB'} & 0 \\ 0 & -\tilde{\theta}^{B'}\check{V}_{AB'} \end{pmatrix}$,

$\Psi_{A'} = \begin{pmatrix} 0 & \theta^2\check{\lambda}_{A'} \\ 0 & 0 \end{pmatrix}$. Then (using (2.20b)

$$\Psi_A = \tilde{\theta}^{B'}\begin{pmatrix} \check{V}_{AB'} & 0 \\ 0 & \check{V}_{AB'} \end{pmatrix} \equiv \tilde{\theta}^{B'}V_{AB'}, \quad \Psi_{A'} = \theta^2\begin{pmatrix} 0 & \check{\lambda}_{A'} \\ 0 & 0 \end{pmatrix} \equiv \theta^2\lambda_{A'},$$

and
$$\begin{aligned}(\Psi_A,\tilde{\Psi}_{A'}) &= (\tilde{\theta}^{B'}V_{AB'},\theta^2\lambda_{A'}) \\ &= \tilde{\theta}^{B'}\theta^2[V_{AB'},\lambda_{A'}] \text{ (using (2.12))} \\ &= 0.\end{aligned}$$

Example (ii) Suppose $\Psi_A = \begin{pmatrix} \tilde{\theta}^{B'}\check{V}_{AB'} & 0 \\ 0 & -\tilde{\theta}^{B'}\check{V}_{AB'} \end{pmatrix}$,

$\tilde{\Psi}_{A'} = \begin{pmatrix} \theta^C\check{\tilde{V}}_{A'C} & 0 \\ 0 & -\theta^C\check{\tilde{V}}_{A'C} \end{pmatrix}$. Then

$$\Psi_A = \tilde{\theta}^{B'}\begin{pmatrix} \check{V}_{AB'} & 0 \\ 0 & \check{V}_{AB'} \end{pmatrix} \equiv \tilde{\theta}^{B'}V_{AB'}, \quad \tilde{\Psi}_{A'} = \theta^C\begin{pmatrix} \check{\tilde{V}}_{A'C} & 0 \\ 0 & \check{\tilde{V}}_{A'C} \end{pmatrix} \equiv \theta^C\tilde{V}_{A'C},$$

and

$$(\Psi_A,\tilde{\Psi}_{A'}) = (\tilde{\theta}^{B'}V_{AB'},\theta^C\tilde{V}_{A'C}) = \tilde{\theta}^{B'}\theta^C[V_{AB'},\tilde{V}_{A'C}] = 0.$$

Example (iii) Suppose $\Phi_{AA'} = \begin{pmatrix} \check{V}_{AA'} & 0 \\ 0 & \check{V}_{AA'} \end{pmatrix} \equiv V_{AA'}$,

$\tilde{\Psi}_{A'} = \begin{pmatrix} \theta^B \tilde{\check{V}}_{BA'} & 0 \\ 0 & -\theta^B \tilde{\check{V}}_{BA'} \end{pmatrix}$. Then

$$\tilde{\Psi}_{A'} = \theta^B \begin{pmatrix} \tilde{\check{V}}_{BA'} & 0 \\ 0 & \tilde{\check{V}}_{BA'} \end{pmatrix} \equiv \theta^B \tilde{V}_{BA'} \ ,$$

and

$$[\Phi_{AA'}, \tilde{\Psi}^{A'}] = [V_{AA'}, \theta^B \tilde{V}_B^{A'}] = \theta^B [V_{AA'}, \tilde{V}_B^{A'}] = 0$$

Example (iv) Suppose $\Phi_{AA'} = \begin{pmatrix} 0 & \tilde{\theta}^{B'} \check{\lambda}_A \varepsilon_{A'B'} \\ 0 & 0 \end{pmatrix}$,

$\tilde{\Psi}_{A'} = \begin{pmatrix} \theta^C \tilde{\check{V}}_{CA'} & 0 \\ 0 & -\theta^C \tilde{\check{V}}_{CA'} \end{pmatrix}$. Then

$$\Phi_{AA'} = \tilde{\theta}^{B'} \begin{pmatrix} 0 & \check{\lambda}_A \varepsilon_{A'B'} \\ 0 & 0 \end{pmatrix} \equiv \tilde{\theta}^{B'} \lambda_A \varepsilon_{A'B'} \ , \quad \tilde{\Psi}_{A'} = \theta^C \begin{pmatrix} \tilde{\check{V}}_{A'C} & 0 \\ 0 & \tilde{\check{V}}_{A'C} \end{pmatrix} \equiv \theta^C \tilde{V}_{A'C} \ ,$$

and

$$[\Phi_{AA'}, \tilde{\Psi}^{A'}] = [\tilde{\theta}^{B'} \lambda_A \varepsilon_{A'B'}, \theta^C \tilde{V}_C^{A'}] = -\tilde{\theta}^{B'} \theta^C \varepsilon_{A'B'} [\lambda_A, \tilde{V}_C^{A'}]$$

$$= -\tilde{\theta}^{B'} \theta^C \varepsilon_{A'B'} \left[\begin{pmatrix} 0 & \check{\lambda}_A \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \tilde{\check{V}}_C^{A'} & 0 \\ 0 & \tilde{\check{V}}_C^{A'} \end{pmatrix} \right] = 0.$$

Example (v) Suppose $\Psi_A = \begin{pmatrix} 0 & \tilde{\theta}^2 \check{\lambda}_A \\ 0 & 0 \end{pmatrix}$, $\tilde{\Psi}_{A'} = \begin{pmatrix} 0 & \theta^2 \tilde{\check{\lambda}}_{A'} \\ 0 & 0 \end{pmatrix}$.

Then

$$\Psi_A = \tilde{\theta}^2 \begin{pmatrix} 0 & \check{\lambda}_A \\ 0 & 0 \end{pmatrix} \equiv \tilde{\theta}^2 \lambda_A \quad , \tilde{\Psi}_{A'} = \theta^2 \begin{pmatrix} 0 & \tilde{\check{\lambda}}_{A'} \\ 0 & 0 \end{pmatrix} \equiv \theta^2 \tilde{\lambda}_{A'} \ ,$$

and

$$(\Psi_A, \tilde{\Psi}_{A'}) = \tilde{\theta}^2 \theta^2 (\lambda_A, \tilde{\lambda}_{A'}) = 0 \ .$$

§3.2.5 N = 1 super Yang-Mills

Theorem 3.i Let E be a supervector bundle of rank $r|s$ over $\mathbb{M}_{[1]}$; and let $(\Phi_a \quad \Psi_A \quad \tilde{\Psi}_{A'})$ be the corresponding superpotentials defined up to gauge freedom:

$$(\Phi_a \quad \Psi_A \quad \tilde{\Psi}_{A'}) \longmapsto (g^{-1}(\Phi_a+\partial_a)g \quad g^{-1}(\Psi_A+\partial_A)g \quad g^{-1}(\tilde{\Psi}_{A'}+\tilde{\partial}_{A'})g)\,, \tag{3.48}$$

for any $g \in GL(r|s;\mathcal{O}_{[1]})$. These superpotentials satisfy the $N = 1$ constraint equations (3.36a)

$$F_{AB} \equiv \partial_{(A}\Psi_{B)} + \frac{1}{2}\,(\Psi_A, \Psi_B) = 0 \tag{3.49a}$$

$$F_{A'B'} \equiv \tilde{\partial}_{(A'}\tilde{\Psi}_{B')} + \frac{1}{2}\,(\tilde{\Psi}_{A'}, \tilde{\Psi}_{B'}) = 0 \tag{3.49b}$$

$$F_{AA'} \equiv \partial_A\tilde{\Psi}_{A'} + \partial_{A'}\Psi_A + (\Psi_A, \tilde{\Psi}_{A'}) - 2\Phi_{AA'} = 0 \tag{3.49c}$$

if and only if

$$\Psi_A = \tilde{\theta}^{B'}V_{AB'} + \theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\tilde{\lambda}_{B'} - \tilde{\theta}^2\lambda_A + \theta^B\tilde{\theta}^2\{\phi_{AB} + \varepsilon_{AB}f\} + \theta^2\tilde{\theta}^2H_A \tag{3.50a}$$

$$\tilde{\Psi}_{A'} = \theta^BV_{A'B} - \theta^B\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_B - \theta^2\tilde{\lambda}_{A'} + \theta^2\tilde{\theta}^{B'}\{\tilde{\phi}_{A'B'} - \varepsilon_{A'B'}f\} + \theta^2\tilde{\theta}^2\tilde{H}_{A'} \tag{3.50b}$$

$$\begin{aligned}\Phi_{AA'} = V_{AA'} &- 2\theta^B\tilde{\theta}^{B'}\{G_{ab} + \varepsilon_{AB}\varepsilon_{A'B'}f\} + \frac{\theta^2\tilde{\theta}^2}{4}\,\{\mathring{\nabla}^b{*}G_{ab} + \frac{3}{2}\,(\lambda_A, \tilde{\lambda}_{A'})\}\\ &- \frac{3}{2}\,\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A + \frac{\tilde{\theta}^2\theta^B}{2}\,\{\varepsilon_{AB}\tilde{H}_{A'} - \frac{1}{2}\,\mathring{\nabla}_{A'(A}\lambda_{B)}\}\\ &- \frac{3}{2}\,\theta^B\varepsilon_{AB}\tilde{\lambda}_{A'} + \frac{\theta^2\tilde{\theta}^{B'}}{2}\,\{\varepsilon_{A'B'}H_A - \frac{1}{2}\,\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}\}\end{aligned} \tag{3.50c}$$

where

$$\phi_{AB} = -\frac{1}{2}\,\{\partial_{A'(A}V^{A'}_{B)} + \frac{1}{2}\,[V_{A'A}, V^{A'}_B]\} \equiv -\frac{1}{2}\,\mathring{\nabla}_{A'(A}V^{A'}_{B)}$$

$$\tilde{\phi}_{A'B'} = -\frac{1}{2}\,\{\partial_{A(A'}V^A_{B')} + \frac{1}{2}\,[V_{AA'}, V^A_{B'}]\} \equiv -\frac{1}{2}\,\mathring{\nabla}_{A(A'}V^A_{B')}$$

$$H_A = \frac{1}{4}\,\{\partial_{AA'}\tilde{\lambda}^{A'} + [V_{AA'}, \tilde{\lambda}^{A'}]\} \equiv \frac{1}{4}\,\mathring{\nabla}_{AA'}\tilde{\lambda}^{A'}$$

$$\tilde{H}_{A'} = \frac{1}{4}\{\partial_{A'A}\lambda^A + [V_{AA'},\lambda^A]\} \equiv \frac{1}{4}\mathring{\nabla}_{AA'}\lambda^A$$

$$\varepsilon_{AB}\tilde{\phi}_{A'B'} - \varepsilon_{A'B'}\phi_{AB} \equiv -G_{ab}$$

$$\varepsilon_{AB}\tilde{\phi}_{A'B'} + \varepsilon_{A'B'}\phi_{AB} \equiv -*G_{ab}$$

$$\mathring{\nabla}^b *G_{ab} = \partial^b *G_{ab} + [V^b, *G_{ab}] \ . \qquad \square$$

Theorem 3.ii The superpotentials (3.50) satisfy the self-dual $N = 1$ super Yang-Mills equations

$$W_A \equiv [\nabla_{AA'},\nabla^{A'}] = \partial_{A'}\Phi^{A'}_A + \partial_{AA'}\tilde{\Psi}^{A'} + [\Phi_{AA'},\tilde{\Psi}^{A'}] = 0$$

if and only if

$$\lambda_A = 0 \qquad (3.51a)$$

$$f = 0 \qquad (3.51b)$$

$$\mathring{\nabla}_{AA'}\tilde{\lambda}^{A'} = 0 \qquad (3.51c)$$

$$\phi_{AB} = 0 \ . \qquad \square \qquad (3.51d)$$

Theorem 3.iii The superpotentials (3.50) satisfy the full $N = 1$ super Yang-Mills equations

$$\nabla_A W^A \equiv \partial_A W^A + (\Psi_A, W^A) = 0$$

if and only if

$$\mathring{\nabla}_{AA'}\lambda^A = 0 \qquad (3.52a)$$

$$f = 0 \qquad (3.52b)$$

$$\mathring{\nabla}_{AA'}\tilde{\lambda}^{A'} = 0 \qquad (3.52c)$$

$$8\mathring{\nabla}^a *G_{ab} + 9(\lambda_B,\tilde{\lambda}_{B'}) = 0 \ , \qquad (3.52d)$$

Note: In the super abelian case equations (3.52) reduce to

$$\partial_{AA'}\lambda^{A} = 0$$

$$f = 0$$

$$\partial_{AA'}\tilde{\lambda}^{A'} = 0$$

$$\partial^{a} {*G}_{ab} = 0$$

i.e. an uncoupled system of equations. □

The laborious proofs of these theorems is given in the appendix. They consist of expanding the superpotentials in a power series in θ^{A} and $\tilde{\theta}^{A'}$ (as explained in §3.2.4) and analysing the equations order by order. An important ingredient in these proofs is the imposition of the gauge condition

$$\theta^{A}_{j}\Psi^{j}_{A} + \tilde{\theta}^{A'k}\tilde{\Psi}_{A'k} = 0 . \tag{3.53}$$

This gauge condition had already been used in [16b] and [35], where it had led to considerable computational simplifications.

Note that each of the fields involved in (3.52) can be regarded as the zero$^{\text{th}}$ order part of some superfield as follows. Let $G|_{0}$ denote the restriction of a superfield G to its zero$^{\text{th}}$ order term (i.e. the term independent of θ's). Then

$$f = \frac{1}{2}\nabla_{A}W^{A}|_{0} \equiv \overset{\circ}{f}$$

$$\lambda_{A} = \frac{1}{3}W_{A}|_{0} \equiv \frac{\overset{\circ}{W}_{A}}{3}$$

$$\tilde{\lambda}_{A'} = \frac{1}{3}\,\tilde{W}_{A'}\big|_0 \equiv \frac{\mathring{\tilde{W}}_{A'}}{3}$$

$$\phi_{AB} = -\frac{1}{8}\,\nabla_{(A}W_{B)}\big|_0 \equiv -\frac{1}{4}\,\mathring{f}_{AB}$$

$$\tilde{\phi}_{A'B'} = -\frac{1}{8}\,\nabla_{(A'}\tilde{W}_{B')}\big|_0 \equiv +\frac{1}{4}\,\mathring{\tilde{f}}_{A'B'}$$

$$G_{ab} = -\frac{1}{4}\,F_{ab}\big|_0 \equiv -\frac{1}{4}\,\mathring{F}_{ab} \equiv -\frac{1}{4}\,\{\varepsilon_{A'B'}\mathring{f}_{AB} + \varepsilon_{AB}\mathring{\tilde{f}}_{A'B'}\}$$

$$V_a = \Phi_a\big|_0 \equiv \mathring{\Phi}_a\,,$$

and (3.52) may be more elegantly rewritten as

$$\mathring{\nabla}_{AA'}\mathring{W}^A = 0 \qquad (3.52'a)$$

$$\mathring{f} = 0 \qquad (3.52'b)$$

$$\mathring{\nabla}_{AA'}\mathring{\tilde{W}}^{A'} = 0 \qquad (3.52'c)$$

$$\mathring{\nabla}^b{*}\mathring{F}_{ab} + \frac{1}{2}\,(\mathring{W}_A\,,\mathring{\tilde{W}}_{A'}) = 0\,. \qquad (3.52'd)$$

Writing the equations in this form facilitates comparison with the $N > 1$ generalizations.

Note: The classical Yang-Mills equations are included in (3.52).

§3.2.6 Super Yang-Mills for N > 1

For $N > 1$, the direct θ-expansion approach used in proving the theorems of §3.2.5 gets out of hand and some clever inductive method is needed. However, one direction is relatively easy. Namely, it is easy to deduce a system of classical field equations implied by the superfield equations. This is done by suitably manipulating the superfields using the Bianchi identities, and then taking the zeroth order part of the resulting superfield equations (see

[35] for N = 3 and [4a] for N = 4). The difficult part is to prove the converse. That is, to construct the superfield equations given a system of classical field equations. Using the gauge condition (3.53), and a set of recursive relations, J. Harnad, J. Hurtubise, M. Légaré and S. Shnider [35] established inductively the equivalence between the constraint equations for N = 3 and the following system of field equations

$$\mathring{\nabla}_{AA'}\mathring{W}^{A}_{j} - \frac{1}{2}[\mathring{\tilde{W}}^{i}_{A'},\mathring{W}_{ij}] + \frac{1}{8}[\mathring{Z}_{A'},\mathring{\tilde{W}}^{ki}]\varepsilon_{kij} = 0 \qquad (3.54a)$$

$$\mathring{\nabla}_{AA'}\mathring{\tilde{W}}^{A'j} - \frac{1}{2}[\mathring{W}_{Ai},\mathring{\tilde{W}}^{ij}] + \frac{1}{8}[\mathring{\tilde{Z}}_{A},\mathring{W}_{ki}]\varepsilon^{kij} = 0 \qquad (3.54b)$$

$$\mathring{\Box}\mathring{W}_{ij} + 2(\mathring{W}^{A}_{i},\mathring{W}_{Aj}) = (\mathring{Z}_{A'},\mathring{\tilde{W}}^{\ell A'})\varepsilon_{i\ell j} - \frac{1}{8}[\mathring{W}_{ij},[\mathring{W}_{pq},\mathring{\tilde{W}}^{pq}]] + \frac{3}{4}[\mathring{W}_{kj},[\mathring{W}_{pi},\mathring{\tilde{W}}^{pk}]] = 0 \qquad (3.54c)$$

$$\mathring{\Box}\mathring{\tilde{W}}^{ij} + 2(\mathring{\tilde{W}}^{A'i},\mathring{\tilde{W}}^{j}_{A'}) = (\mathring{\tilde{Z}}_{A},\mathring{W}^{A}_{\ell})\varepsilon^{i\ell j} - \frac{1}{8}[\mathring{\tilde{W}}^{ij},[\mathring{\tilde{W}}^{pq},\mathring{W}_{pq}]] + \frac{3}{4}[\mathring{\tilde{W}}^{kj},[\mathring{\tilde{W}}^{pi},\mathring{W}_{pk}]] = 0 \qquad (3.54d)$$

$$\mathring{\nabla}_{AA'}\mathring{Z}^{A'} + \frac{1}{2}[\mathring{W}_{iA},\mathring{W}_{jk}]\varepsilon^{ijk} = 0 \qquad (3.54e)$$

$$\mathring{\nabla}_{AA'}\mathring{\tilde{Z}}^{A} + \frac{1}{2}[\mathring{\tilde{W}}^{iA'},\mathring{\tilde{W}}^{jk}]\varepsilon_{ijk} = 0 \qquad (3.54f)$$

$$\mathring{\nabla}^{b}{*}\mathring{F}_{ab} + \frac{1}{2}(\mathring{\tilde{W}}^{j}_{A'},\mathring{W}_{Aj}) + \frac{5}{4}\{[\mathring{\tilde{W}}^{k\ell},\mathring{\nabla}_{AA'}\mathring{W}_{k\ell}]+[\mathring{\nabla}_{AA'}\mathring{\tilde{W}}^{k\ell},\mathring{W}_{k\ell}]\} + \frac{1}{12}(\mathring{Z}_{A'},\mathring{\tilde{Z}}_{A}) = 0 \qquad (3.54g)$$

where

$$\mathring{\nabla}_{a} \equiv \nabla_{a}\big|_{0}, \qquad \mathring{\Box} \equiv \nabla_{a}\nabla^{a}\big|_{0}$$

$$\varepsilon^{ijk}\mathring{\tilde{Z}}_{A} \equiv \nabla^{[i}_{A}\tilde{W}^{jk]}\big|_{0}, \qquad \varepsilon_{ijk}\mathring{Z}_{A'} \equiv \tilde{\nabla}_{A'[i}W_{jk]}\big|_{0}$$

$$\mathring{W}_{ij} \equiv W_{ij}\big|_{0}, \qquad \mathring{\tilde{W}}^{ij} \equiv \tilde{W}^{ij}\big|_{0}$$

$$\mathring{f}_{AB} \equiv \frac{1}{6} \nabla^{i}{}_{(A} W_{B)i}\Big|_0 , \quad \mathring{\tilde{f}}_{A'B'} \equiv -\frac{1}{6} \tilde{\nabla}_{j(A'} \tilde{W}^{j}_{B')}\Big|_0$$

$$\star \mathring{F}_{ab} \equiv \quad - \varepsilon_{A'B'} \mathring{f}_{AB} + \varepsilon_{AB} \mathring{\tilde{f}}_{A'B'} .$$

Harnad et al. called these equations 'the equations of motion of the supersymmetric Yang-Mills theory'. They are indeed equations of motion in the sense that they involve derivatives with respect to the space-time variables. However, our definition of the super Yang-Mills equations implies stronger equations than (3.54). Namely it implies the following

$$\mathring{\nabla}_{AA'} \mathring{W}^{A}_{j} - \frac{1}{5} [\mathring{W}^{i}_{A'}, \mathring{W}_{ij}] = 0 \tag{3.55a}$$

$$\mathring{\nabla}_{AA'} \mathring{\tilde{W}}^{A'j} - \frac{1}{5} [\mathring{W}_{Ai}, \mathring{\tilde{W}}^{ij}] = 0 \tag{3.55b}$$

$$\mathring{\Box} \mathring{W}_{ij} + 2(\mathring{W}^{A}_{i}, \mathring{W}_{Aj}) = \frac{1}{5} (\mathring{Z}_{A'}, \mathring{\tilde{W}}^{A'\ell}) \varepsilon_{i\ell j} \tag{3.55c}$$

$$\mathring{\Box} \mathring{\tilde{W}}^{ij} + 2(\mathring{\tilde{W}}^{A'i}, \mathring{\tilde{W}}^{j}_{A'}) = \frac{1}{5} (\mathring{\tilde{Z}}_{A}, \mathring{W}^{A}_{\ell}) \varepsilon^{i\ell j} \tag{3.55d}$$

$$\mathring{\nabla}_{AA'} \mathring{Z}^{A'} + \frac{1}{2} [\mathring{W}_{iA}, \mathring{W}_{jk}] \varepsilon^{ijk} = 0 \tag{3.55e}$$

$$\mathring{\nabla}_{A'A} \mathring{\tilde{Z}}^{A} + \frac{1}{2} [\mathring{\tilde{W}}^{iA'}, \mathring{\tilde{W}}^{jk}] \varepsilon_{ijk} = 0 \tag{3.55f}$$

$$\mathring{\nabla}^{b} \star \mathring{F}_{ab} + \frac{1}{2} (\mathring{\tilde{W}}^{j}_{A'}, \mathring{W}_{Aj}) + \frac{1}{12} (\mathring{Z}_{A'}, \mathring{\tilde{Z}}_{A}) = 0 . \tag{3.55g}$$

(This can be deduced by suitably manipulating the Bianchi identities as in [35], but with the left hand side of (3.39a) set to zero. This is a somewhat long calculation which we have omitted.)

For the sake of completeness, we state the reduction of the above system to N = 2. Setting $\mathring{\tilde{W}}^{ij} = \varepsilon^{ij} \mathring{\tilde{W}}$, and

$\overset{\circ}{W}_{ij} = \varepsilon_{ij}\overset{\circ}{W}$ $(\varepsilon_{ij} = -\varepsilon_{ji}$ & $\varepsilon^{ij} = -\varepsilon^{ji})$:

$$\overset{\circ}{\nabla}_{AA'}\overset{\circ}{W}^{A}_{i} - \frac{1}{5}[\overset{\circ}{\tilde{W}}^{j}_{A'},\overset{\circ}{W}]\varepsilon_{ji} = 0 \tag{3.56a}$$

$$\overset{\circ}{\nabla}_{AA'}\overset{\circ}{\tilde{W}}^{A'i} - \frac{1}{5}[\overset{\circ}{W}_{Aj},\overset{\circ}{\tilde{W}}]\varepsilon^{ji} = 0 \tag{3.56b}$$

$$\overset{\circ}{\Box}\overset{\circ}{\tilde{W}} + (\overset{\circ}{\tilde{W}}^{A'}_{j},\overset{\circ}{\tilde{W}}^{j}_{A'}) = 0 \tag{3.56c}$$

$$\overset{\circ}{\Box}\overset{\circ}{W} + (\overset{\circ}{W}^{A}_{i},\overset{\circ}{W}^{i}_{A}) = 0 \tag{3.56d}$$

$$\overset{\circ}{\nabla}^{b}*\overset{\circ}{F}_{ab} + \frac{1}{2}(\overset{\circ}{\tilde{W}}^{j}_{A'},\overset{\circ}{W}_{Aj}) = 0 \tag{3.56e}$$

There seems to be a certain amount of uncertainty as to what the super Yang-Mills equations for N = 3 are. The equations (3.45) are aesthetically pleasing and arise very naturally from our constructions. As we have seen these imply equations (3.55) which are somewhat simpler than the 'equations of motion' (3.54) of Harnad et al. However there are many reasons for wanting to give the name of N = 3 super Yang-Mills to the last system of equations. These will become apparent in the next chapter when we discuss their superambitwistor description.

In the super abelian case certainly the three systems of equations: (3.45), (3.55) and (3.54), are equivalent. In general, (3.45) $\Longrightarrow$ (3.55) $\Longrightarrow$ (3.54); and it is just conceivable that these equations are in fact equivalent. However, we are unable to see any way in which one may reasonably attempt to establish this.

Finally, note that the classical Yang-Mils equtions are included in (3.55), (3.56), and also in the N = 3 constraint equations of Harnad et al. (3.54).

Note §3.2 incorporated many ideas which have previously appeared scattered in the literature. The right form of the gauge freedom of superpotentials and the various super Yang-Mills curvatures have been discussed (for example) in [91a]. The Bianchi identities have been extensively studied in [84]. The representation of systems of classical equations (equivalent to superfield equations) in terms of zeroth order parts of superfields has been used in [35] (for N = 3) and [4a] (for N = 4). A system of classical equations equivalent to the N = 1 self-dual super Yang-Mills equations (3.51) was calculated (using a different gauge condition from (3.53)) in [87a]. An interesting paper dealing with the self-dual super Yang-Mills equation and a 'super version' of the Atiyah, Hitchin, Drinfeld and Manin construction of instantons [3a], has appeared in [79]. It should be mentioned at this point that in all the above references, work is carried out entirely at the level of formal calculations, i.e. authors start by defining super Minkowski space as the space spanned by the space-time variables $x^{AA'}$ and 4N 'anti-commuting variables' θ^A_j and $\tilde{\theta}^{A'k}$.

CHAPTER 4: SUPERTWISTOR THEORY

§4.0 Introduction

The first connections between supersymmetry and twistor theory were noted by Roger Penrose who pointed out in [65] that the Wess-Zummino supersymmetry generators [92] satisfy the twistor equation.

'Supertwistors' were formally introduced by A. Ferber [25]. He defined such objects as pairs (Z^{α},ζ^{j}), consisting of an ordinary twistor $Z^{\alpha} = (\omega^{A},\pi_{A'})$ and N 'anticommuting variables' ζ^{j}; giving rise to objects in super Minkowski space via the 'incidence relations':

$$\omega^{A} = (x^{AA'} - \theta^{A}_{j}\tilde{\theta}^{A'j})\pi_{A'} \tag{4.1a}$$

$$\zeta^{j} = \tilde{\theta}^{A'j}\pi_{A'} , \tag{4.1b}$$

where θ^{A}_{j} and $\tilde{\theta}^{A'k}$ are 'anticommuting spinors', and $x^{AA'}$ is a classical space-time variable. Ferber then used (4.1) to formally derive the 'superconformal algebra', and to give a contour integral formula for generating superfields from 'homogeneous holomorphic functions of supertwistors'.

A derivation of the superconformal algebra within a $\mathbb{CP}_5$ calculus approach to twistor theory was given by T. Hurd [43] (see also [41]). His treatment of conformal supersymmetry, using second quantized helicity raising and lowering operators, has many physically appealing features.

Using the same formal language as Ferber [25], E. Witten [93] defined the 'space of superambitwistors' as the 'space of all $(Z^{\alpha},\zeta^{j},W_{\alpha},\psi_{j})$ satisfying

$$W_{\alpha}Z^{\alpha} = 2\psi_{j}\zeta^{j} \text{ ',} \tag{4.2}$$

where ζ^{j} and ψ_{k} are 2N 'anticommuting variables', and $Z^{\alpha} = (\omega^{A},\pi_{A'})$ and $W_{\alpha} = (\xi^{A'},\eta_{A})$ denote a twistor and dual twistor respectively. Via super incidence relations such objects were thought to give rise to 'super light-rays' in super Minkowski space.

In this chapter we shall interpret the 'spaces of supertwistors and superambitwistors' within the Kostant-Leites formalism. This is not as straight-forward as it may appear at first sight, since the incidence relations cannot be interpreted as geometric statements in any usual sense. Objects such as supertwistors, super light-rays, etc. will be completely defined in terms of supercoordinates. We shall continue to use Ferber and Witten's terminology which seems to regard these objects as geometric entities. However from Chapter 2 one can anticipate that no geometric meaning can be assigned to them other than that of the underlying classical object. The reason for using this seemingly misleading terminology is that all constructions and formal calculations proceed as if such a geometric interpretation was possible.

We believe that the generalization of twistor geometry to the super category is of mathematical interest in its own right. Nevertheless, the power of supertwistor techniques does not become apparent until they are used to describe non-linear equations such as those of super Yang-Mills theory.

At a formal level, Witten used his 'space of superambitwistors' to give (in the second part of [93]) an alternative interpretation of what is now known as the Green-Isenberg-Yasskin-Witten Theorem (independently stated in [45] and in the first part of [93]). In carrying out his analysis, Witten gave a natural interpretation of the constraint equations (of super Yang-Mills theory) in terms of 'integrability conditions on super light-rays'. (Recall from Chapter 3 that, for N = 3, solutions of these equations include, as a special case, solutions of the classical Yang-Mills equations.) The complicated calculations involved in the formal proof (omitted by Witten) of such an interpretation have been carried out explicitly (in the abelian case) by M.G. Eastwood [16b], and inductively by J. Harnad et al. [35].

With the aid of Chapter 3 we are now in a position to give a description (within the formalism of Chapter 2) of a generalization of classical Yang-Mills theory to the super category. Although such a generalization is at present incomplete, we hope to have set a consistent framework for further research.

Supertwistor space $\mathbb{P}\mathbb{T}_{[N]}$ and dual supertwistor space $\mathbb{P}\mathbb{T}^*_{[N]}$ are defined in §4.1 as trivial supermanifolds. They are related to ${}^-\mathbb{M}_{[N]}$ and ${}^+\mathbb{M}_{[N]}$ respectively by means of a double fibration of supermanifolds. The incidence relations of Ferber are consequently interpreted as supermanifold mappings.

In §4.2 (following suggestions of M.G. Eastwood), superambitwistor space $\mathbb{A}_{[N]}$ is defined (analogously to the classical case) by regarding it as being embedded in $\mathbb{P}\mathbb{T}_{[N]} \times \mathbb{P}\mathbb{T}^*_{[N]}$. We also define infinitesimal neighbourhoods of $\mathbb{A}_{[N]}$ in $\mathbb{P}\mathbb{T}_{[N]} \times \mathbb{P}\mathbb{T}^*_{[N]}$. Like super Minkowski space, $\mathbb{A}_{[N]}$ is a non-trivial supermanifold. The superstructure sheaves of both $\mathbb{M}_{[N]}$ and $\mathbb{A}_{[N]}$ are similarly defined; the latter is expressed in terms of infinitesimal neighbourhoods of $\mathbb{A}$ in $\mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}$. The correspondence between $\mathbb{M}_{[N]}$ and $\mathbb{A}_{[N]}$ is given in terms of a double fibration of supermanifolds.

Part of the objective of this chapter is to present all aspects of supertwistor theory within a consistent framework. Therefore we have included in §4.3 a derivation of Ferber's contour integral formula from the Sparling-Ward splitting method [88], and a brief discussion of the superfield equations generated by such formulae. We have also included in §4.4 a (formal) supertwistor scalar product formula. Neither of these two sections forms part of the mainline argument.

The cohomological representation of superfields is given in §4.5. The purpose of this section is to state the results required for the description of super Yang-Mills theory, which is the subject of §4.6. This last section consists of two results concerning the super-ambitwistor description of integrability conditions on super α-planes and super light-rays in super Minkowski space; and concludes with a conjecture concerning the full super Yang-Mills equations.

§4.1 Supertwistors

§4.1.1 Supertwistor spaces and dual supertwistor space

We will start by defining *projective supertwistor space* as the $(3|N)$-dimensional trivial supermanifold $\mathbb{P}T_{[N]} = (\mathbb{P}T, \Lambda^{\bullet}(\mathcal{O}(-1) \otimes_{\mathbb{C}} V^*))$, where V^* is the dual of an N-dimensional complex vector space V. Recall that $\mathbb{P}T_{[N]}$ is said to be trivial because $\mathcal{O}(-1) \otimes_{\mathbb{C}} V^*$ is a vector bundle. It is in fact the conormal bundle to $\mathbb{P}T$ inside $\mathbb{P}(T \oplus V)$, where $T \oplus V$ will be referred to as the *vector superspace of supertwistors* (sometimes denoted by ST).

Let $\mathcal{O}(-1) \otimes_{\mathbb{C}} V^* = \mathcal{O}_j(-1)$, where j is an abstract index [67]. Then a superfunction F of $\mathbb{P}T_{[N]}$, i.e. a section of $\Lambda^{\bullet}\mathcal{O}_j(-1)$, will be written as

$$F \equiv [f, f_j, f_{jk}, \ldots, f_{\underbrace{jk\ldots p}_{N}}] \in \mathcal{O} \oplus \mathcal{O}_j(-1) \oplus \mathcal{O}_{jk}(-2) \oplus \ldots \oplus \mathcal{O}_{jk\ldots p}(-N)$$

where $f_{jk\ldots\ell} = f_{[jk\ldots\ell]} \in \mathcal{O}_{\underbrace{jk\ldots\ell}_{r}}(-r)$ (i.e. $f_{jk\ldots\ell}$ is a classical twistor function homogeneous of degree $-r$).
We will also tensor through by $\mathcal{O}(n)$ and refer to a section $F \in \mathcal{O}(n) \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}_j(-1)$ as a *supertwistor function homogeneous of degree n*, meaning that its coefficients $f_{\underbrace{jk\ldots\ell}_{r}}$ are classical twistor functions homogeneous of degree $n-r$.

Consider now the vector superspace of global sections (over $\mathbb{P}T$) of

$$\mathcal{O}(1) \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}_j(-1) \cong \mathcal{O}(1) \oplus \mathcal{O}_j \oplus \mathcal{O}_{jk}(-1) \oplus \ldots \oplus \mathcal{O}_{jk\ldots p}(1-N)$$

Note that T^* may be regarded as the vector space of global sections of $\mathcal{O}(1)$, and that $\mathcal{O}(k)$ has no global sections for $k < 0$. Thus it is natural to define the *vector superspace of dual supertwistors*, ST^*, by

$$\Gamma(\mathbb{P}T, \mathcal{O}(1)) \oplus \Gamma(\mathbb{P}T, \mathcal{O}_j) = T^* \oplus V^* \, .$$

Then, as one would expect

$$ST^* = T^* \oplus V^* \cong \mathrm{Hom}_{\mathbb{C}}(ST, \mathbb{C}) \, .$$

It is now clear how to proceed to define *projective dual supertwistor space*: namely as the $(3|N)$-dimensional supermanifold $\mathbb{P}T^*_{[N]} = (\mathbb{P}T^*; \Lambda^{\cdot}\mathcal{O}^j(-1))$, where $\mathcal{O}^j(-1) \cong \mathcal{O}(-1) \otimes_{\mathbb{C}} V$ is the conormal bundle to $\mathbb{P}T^*$ inside $\mathbb{P}(T^* \oplus V^*)$.

Note that the only possible superstructures with either $\mathcal{O}_j(-1)$ or $\mathcal{O}^k(-1)$ as underlying vector bundles, are the trivial ones: $\Lambda^{\cdot}\mathcal{O}_j(-1)$ and $\Lambda^{\cdot}\mathcal{O}^k(-1)$. This can be deduced by computing the cohomology groups (of Theorem 2.iv). The relevant calculations have been carried out by M.G. Eastwood [19] and will not be reproduced here.

We will define *non-projective supertwistor space* as the $(4|N)$-dimensional supermanifold $T_{[N]} = (T, \Lambda^{\cdot}(\mathcal{O} \otimes_{\mathbb{C}} V^*))$. Then $\mathbb{P}T_{[N]}$ is obtained from $T_{[N]}$ modulo an action of the multiplicative group $\mathbb{C}^*$ (on $T_{[N]}$) which we now proceed to describe. Let $\lambda \in \mathbb{C}^*$ be the supermanifold mapping

$$\lambda : (T-\{0\}, \Lambda^{\cdot}\mathcal{O}_j) \longleftarrow\!\!\!\supset$$

consisting of:

(i) $\quad T-\{0\} \to T-\{0\}$

given by $Z \mapsto \lambda Z \ \forall \ Z \in T$, and

(ii) $\quad \Gamma(\lambda^{-1}(U), \Lambda^{\bullet}\mathcal{O}_j) \leftarrow \Gamma(U, \Lambda^{\bullet}\mathcal{O}_j)$

(where U is an open subset of $T-\{0\}$) given by

$$\lambda.[f(\lambda^{-1}Z), f_j(\lambda^{-1}Z), f_{jk}(\lambda^{-1}Z), \ldots, f_{jk\ldots p}(\lambda^{-1}Z)] \leftarrow [f(Z), f_j(Z), f_{jk}(Z), \ldots, f_{jk\ldots p}(Z)]$$

where $\lambda \cdot f_{\overbrace{jk\ldots\ell}^{r}} := \lambda^r f_{jk\ldots\ell}$.

Similarly $\mathbb{P}T^*_{[N]}$ can be obtained from *non-projective dual supertwistor space* $T^*_{[N]} = (T^*, \Lambda^{\bullet}(\mathcal{O} \otimes_{\mathbb{C}} V))$ in an entirely analogous manner.

Note: Duality occurs only at the vector superspace level, i.e. ST^* is the dual of ST, and it is only in this sense that we may regard the corresponding supermanifolds to be dual to each other.

§4.1.2 Homogeneous supercoordinates

Let $(Z^\alpha; \zeta^j)$ be supercoordinates for $T_{[N]}$. Then, the equivalence class

$$[Z^\alpha; \zeta^j] \text{ s.t. } [\lambda Z^\alpha; \lambda\zeta^j] = [Z^\alpha; \zeta^j] \ \forall \ \lambda \in \mathbb{C}^*$$

defines *homogeneous supercoordinates* for $\mathbb{P}T_{[N]}$.

Let F be a supertwistor function homogeneous of degree n, i.e. a section of $\mathcal{O}(n) \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}_j(-1)$. Then as usual we denote its series expansion in terms of the odd supercoordinates by

$$F(Z^\alpha;\zeta^j) \equiv f(Z^\alpha) + f_j(Z^\alpha)\zeta^j + f_{jk}(Z^\alpha)\zeta^j\zeta^k + \ldots + f_{jk\ldots p}\zeta^j\zeta^k\ldots\zeta^p,$$

where $f_{\overbrace{jk\ldots\ell}^{r}}(Z^\alpha) = f_{[jk\ldots\ell]}(Z^\alpha)$ is a holomorphic function on T homogeneous of degree n-r.

Similarly, let

$$[W_\alpha;\psi_j] \text{ s.t. } [\mu W_\alpha;\mu\psi_j] = [W_\alpha;\psi_j] \ \forall\ \mu \in \mathbb{C}^*$$

define homogeneous supercoordinates for $\mathbb{P}T^*_{[N]}$. Then, a superfunction $G \in \mathcal{O}(n) \otimes_{\mathcal{O}} \Lambda^{\bullet}\mathcal{O}^j(-1)$ has series expansion

$$G(W_\alpha;\psi_j) \equiv g(W_\alpha) + g^j(W_\alpha)\psi_j + g^{jk}(W_\alpha)\psi_j\psi_k + \ldots + g^{jk\ldots p}(W_\alpha)\psi_j\psi_k\ldots\psi_\ell,$$

where $g^{\underbrace{jk\ldots\ell}_{r}}(W_\alpha) = g^{[jk\ldots\ell]}(W_\alpha)$ is a holomorphic function on T^* homogeneous of degree n-r.

§4.1.3 The supertwistor correspondences

Let $\mathbb{F}$ denote the classical flag manifold $\mathbb{F}_{12} = \{L_1 \subset L_2 \subset T\}$; and let ${}^{-}\mathbb{F}_{[N]}$ be the trivial supermanifold $(\mathbb{F}, \Lambda^{\bullet}\{\mathcal{O}_{A'j}\})$. Then the supermanifolds ${}^{-}\mathbb{F}_{[N]}$, ${}^{-}\mathbb{M}_{[N]}$ and $\mathbb{PT}_{[N]}$ are naturally linked by the double fibration

$$\begin{array}{ccc} & (x_{-}^{AA'},[\pi_{A'}];\tilde{\theta}^{A'j}) & \\ & \Cap & \\ & {}^{-}\mathbb{F}_{[N]} & \\ \swarrow^{\nu} & & \searrow^{\mu} \\ {}^{-}\mathbb{M}_{[N]} & & \mathbb{PT}_{[N]} \\ \Cup & & \Cup \\ (x_{-}^{AA'};\tilde{\theta}^{A'j}) & & [Z^{\alpha};\zeta^{j}]=[\omega^{A},\pi_{A'};\zeta^{j}] \end{array} \quad . \tag{4.4}$$

The supermanifold maps, ν and μ, consist of the classical forgetful projections (induced by the classical twistor incidence relations)

$$\begin{array}{ccc} & \mathbb{F}_{12} & \\ \swarrow^{\nu} & & \searrow^{\mu} \\ \mathbb{M} & & \mathbb{PT} \end{array} \quad , \tag{4.4a}$$

and maps of their superstructure sheaves. The easiest way to describe the latter is to use the usual expansion of superfunctions in terms of the odd supercoordinates, as follows.

Let M be an open subset of $\mathbb{M}$, then

$$\Gamma(\nu^{-1}(M),\Lambda^{\bullet}\{\mathcal{O}_{A'j}\}) \xleftarrow{\tilde{\nu}} \Gamma(M,\Lambda^{\bullet}\mathcal{O}_{A'j}) \tag{4.4b}$$

is given by

$$(\tilde{\nu}\phi)(x_-^{AA'},\pi_{A'};\tilde{\theta}^{A'j}) = \phi(x_-^{AA'};\tilde{\theta}^{A'j}) \qquad \forall\ \phi \in \Gamma(M,\Lambda^{\bullet}\mathcal{O}_{A'j}).$$

On the other hand if P is an open subset of $\mathbb{P}T$, then

$$\Gamma(\mu^{-1}(P),\Lambda^{\bullet}\{\mathcal{O}_{A'j}\}) \xleftarrow{\tilde{\mu}} \Gamma(P,\Lambda^{\bullet}\mathcal{O}_j(-1)) \tag{4.4c}$$

is given by

$$(\tilde{\mu}F)(x_-^{AA'},\pi_{A'};\tilde{\theta}^{A'j}) = F(x_-^{AA'}\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'}) \qquad \forall\ F \in \Gamma(P,\Lambda^{\bullet}\mathcal{O}_j(-1)).$$

Note: Recall that

$$F(\omega^A,\pi_{A'};\zeta^j) \equiv f(\omega^A,\pi_{A'})+f_j(\omega^A,\pi_{A'})\zeta^j+\ldots+f_{jk\ldots p}(\omega^A,\pi_{A'})\zeta^j\zeta^k\ldots\zeta^p;$$

and that the meaning of expressions such as $F(x_-^{AA'}\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'})$, where $x_-^{AA'} = x^{AA'} - \theta_j^A\tilde{\theta}^{A'j}$ denotes the antichiral even supercoordinate, was fully explained in §3.1.4. For example, if N = 1, then the coefficients of the expansion of F can be written as

$$\begin{aligned} f(x_-^{AA'}\pi_{A'},\pi_{A'}) \quad &= \quad f(x^{AA'}\pi_{A'},\pi_{A'})-\theta^B\tilde{\theta}^{B'}\partial_{BB'}f(x^{AA'}\pi_{A'},\pi_{A'}) \\ &- \frac{1}{8}\,\theta^2\tilde{\theta}^2\Box f(x^{AA'}\pi_{A'},\pi_{A'}). \end{aligned} \tag{4.5}$$

The supertwistor correspondence (4.4) will be summarized by saying that 'the correspondence is induced by the *super incidence relations*

$$\omega^A = x_-^{AA'}\pi_{A'} \tag{4.6a}$$

$$\zeta^j = \tilde{\theta}^{A'j}\pi_{A'}'. \tag{4.6b}$$

Thus, the super incidence relations consist of the classical incidence relations (describing (4.4a)) together with the supertwistor function mappings (4.4b) and (4.4c). The super incidence relations are not geometric statements in any usual sense.

Note: (4.6a) can be regarded (for N = 1, and with the obvious generalization otherwise) as a special case of (4.5) as follows:

$$\omega^A(x_-) = \omega^A(x) - \theta^B\tilde{\theta}^{B'}\partial_{BB'}\omega^A(x) \ , \tag{4.7}$$

where

$$\omega^A(x) = x^{AA'}\pi_{A'} \ .$$

The dimensions of the supermanifolds in (4.4) are the following: $\dim {}^-\mathbb{F}_{[N]} = 5|2N$, $\dim {}^-\mathbb{M}_{[N]} = 4|2N$ and $\dim \mathbb{PT}_{[N]} = 3|N$. Therefore the fibres of ν and μ (denoted by $L_{x_-,\tilde{\theta}}$ and $\sum_{[z,\zeta]}$ respectively) are of dimensions (1|0) and (2|N) respectively. Note that $L_{x_-,\tilde{\theta}} \cong (L_x, 0)$ is a classical object even though the parameter space, i.e. the 'space of all $x_-^{AA'}$ and $\tilde{\theta}^{A'j}$', is not. On the other hand $\sum_{[z,\zeta]}$ is a (2|N)-dimensional subsupermanifold of ${}^-\mathbb{M}_{[N]}$, which will be referred to as a *super* α-*plane*. All this is what one would intuitively expect from Ferber's incidence relations (4.6).

Lemma 4.i.a. Let F be a superfunction of ${}^-\mathbb{F}_{[N]}$. If $F(x_-^{AA'}, \pi_{A'}; \tilde{\theta}^{A'j})$ is of the form $f(x_-^{AA'}\pi_{A'}, \pi_{A'}; \tilde{\theta}^{A'k}\pi_{A'})$, then $d_\mu^- F = 0$ where $d_\mu^- \equiv (\pi^{A'}\partial^-_{AA'}, \partial_A^{-j}, \tilde{\partial}^-_{A'k})$.

Proof: Immediate from (4.6), since

$$\frac{\partial F}{\partial x_-^{AA'}} = \pi_{A'}\frac{\partial f}{\partial \omega^A} \text{ and } \frac{\partial F}{\partial \tilde{\theta}^{A'j}} = \pi_{A'}\frac{\partial f}{\partial \zeta^j} \ . \qquad \square$$

Consequently d_μ^- may be regarded as the relative exterior derivative along the fibres of μ. In this way, a super

α-plane may be thought of as being spanned by $\pi^{A'}\partial_{AA'}$ and $\pi^{A'}\tilde{\partial}^{-}_{A'k}$ (cf. the analogous statement concerning super light-rays [93]).

Similarly, the correspondence between dual supertwistor space and chiral super Minkowski space is given by the double fibration of supermanifolds

$$\begin{array}{ccc} & (x_+^{AA'},[\eta^A];\theta_j^A) & \\ & \rotatebox{90}{$\in$} & \\ & {}^{+}\mathbb{F}_{[N]} \quad (\cong (\mathbb{F},\Lambda^{\cdot}\{\mathcal{O}_A^j\})) & \\ & \swarrow^{\nu} \qquad \searrow^{\mu} & \\ {}^{+}\mathbb{M}_{[N]} & & \mathbb{PT}^{*}_{[N]} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x_+^{AA'};\theta_j^A) & & [W_\alpha;\psi_j]=[\xi^{A'},\eta_A;\psi_j] \end{array} \tag{4.8}$$

and summarized by the super incidence relations

$$\xi^{A'} = -x_+^{AA'}\eta_A \tag{4.9a}$$

$$\psi_j = \theta_j^A\eta_A \,. \tag{4.9b}$$

The fibres of ν and μ in (4.8) are denoted by $L^{\perp}_{x_+,\theta}$ and $\Sigma^{\perp}_{[W,\psi]}$ respectively. The latter are referred to as *super β-planes.*

Lemma 4.i.b. Let G be a superfunction of ${}^{+}\mathbb{F}_{[N]}$. If $G(x_+^{AA'},\eta_A;\theta_j^A)$ is of the form $g(x_+^{AA'}\eta_A,\eta_A;\theta_j^A\eta_A)$, then $d^+_\mu G = 0$ where $d^+_\mu \equiv (\eta^A\partial^+_{AA'},\eta^A\partial_A^{+j},\tilde{\partial}^+_{A'k})$.

Proof: Immediate from (4.9), since

$$\frac{\partial G}{\partial x_+^{AA'}} = -\eta_A\,\frac{\partial g}{\partial \xi^{A'}} \text{ and } \frac{\partial G}{\partial \theta_j^A} = \eta_A\,\frac{\partial g}{\partial \psi_j}\,. \qquad \square$$

§4.2 Superambitwistors

§4.2.1 Infinitesimal neighbourhoods of $\mathbb{A}$ [12,70,52]

Ambitwistor space $\mathbb{A}$ [15a] is naturally embedded in $\mathbb{P}T \times \mathbb{P}T^*$. In this subsection, $\mathbb{A}$ will be defined as a $\mathbb{C}$-analytic space in a manner which can be immediately generalized to the super category.

Let π_1 and π_2 denote the projections of $\mathbb{P}T \times \mathbb{P}T^*$ onto the first and second of its factors respectively. Then $\pi_1^*\mathcal{O}(p)$ and $\pi_2^*\mathcal{O}(q)$ will denote the pull-backs to $\mathbb{P}T \times \mathbb{P}T^*$ of powers of the hyperplane section bundles on $\mathbb{P}T$ and $\mathbb{P}T^*$ respectively; and we will define

$$\mathcal{O}(p,q) \equiv \pi_1^*\mathcal{O}(p) \otimes \pi_2^*\mathcal{O}(q) .$$

The structure sheaf of ambitwistor space is then defined by the exact sequence

$$(\mathbb{P}T \times \mathbb{P}T^*): \qquad \mathcal{O}(-1,-1) \xrightarrow{\times W_\alpha Z^\alpha} \mathcal{O} \to \mathcal{O}_{\mathbb{A}} \to 0 , \tag{4.9}$$

where W_α and Z^α denote the usual homogeneous coordinates of $\mathbb{P}T^*$ and $\mathbb{P}T$ respectively. Note that $\mathcal{O}_{\mathbb{A}}$ has support on $\mathbb{A} \subset \mathbb{P}T \times \mathbb{P}T^*$; and that $W_\alpha Z^\alpha = 0$ in $\mathcal{O}_{\mathbb{A}}$.

Let I be the ideal sheaf of $\mathbb{A}$, i.e. the image of the map $\mathcal{O}(-1,-1) \xrightarrow{\times W_\alpha Z^\alpha} \mathcal{O}$ in (4.9). Then the *m^{th} infinitesimal neighbourhood of $\mathbb{A}$ in $\mathbb{P}T \times \mathbb{P}T^*$*, $\mathbb{A}_{(m)} \equiv (\mathbb{A}, \mathcal{O}_{(m)})$, is defined by

$$\mathcal{O}_{(m)} = \mathcal{O}/I^{m+1}$$

with $$\tag{4.10}$$

$$\mathcal{O}_{(0)} = \mathcal{O}_{\mathbb{A}} .$$

Equivalently, $\mathcal{O}_{(m)}$ may be defined via the short exact sequence

$$0 \to \mathcal{O}_{(m-1)}(-1,-1) \to \mathcal{O}_{(m)} \to \mathcal{O}_{(m-1)} \to 0 \qquad (4.11)$$

or, using the composition series notation of §2.1.8, as

$$\mathcal{O}_{(m)} = \mathcal{O}_{(m-1)} + \mathcal{O}_{(m-1)}(-1,-1)$$

$$= \mathcal{O} + \mathcal{O}(-1,-1) + \mathcal{O}(-2,-2) + \ldots + \mathcal{O}(-m,-m) \; . \qquad (4.12)$$

Informally, $\mathbb{A}_{(m)}$ may be regarded as having 'homogeneous coordinates' (see [24])

$$[W_\alpha, Z^\alpha; \chi] \text{ s.t. } [W_\alpha, Z^\alpha; \chi] = [\mu W_\alpha, \lambda Z^\alpha; \mu\lambda\chi] \; , \qquad (4.13)$$

with $W_\alpha Z^\alpha = \chi$ and $\chi^{m+1} = 0$ in $\mathcal{O}_{(m)}$.

§4.2.2 The superstructure on $\mathbb{A}$

By analogy with the previous subsection, we expect *superambitwistor space* $\mathbb{A}_{[N]} \equiv (\mathbb{A}, \mathcal{O}_{[N]})$ to be naturally embedded in

$$\mathbb{PT}_{[N]} \times \mathbb{PT}^*_{[N]} = (\mathbb{PT} \times \mathbb{PT}^*, \Lambda^\cdot(\mathcal{O}_j(-1,0) \oplus \mathcal{O}^k(0,-1)) .$$

Following suggestions of M.G. Eastwood we define the superstructure sheaf of $\mathbb{A}_{[N]}$ by the following exact sequence on $\mathbb{PT} \times \mathbb{PT}^*$:

$$\mathcal{O}(-1,-1)\otimes_{\mathcal{O}}\Lambda^\cdot(\mathcal{O}_j(-1,0)\oplus\mathcal{O}^k(0,-1)) \xrightarrow{\times\, W_\alpha Z^\alpha - 2\psi_i\zeta^j} \Lambda^\cdot(\mathcal{O}_j(-1,0)\oplus_{\mathcal{O}}\mathcal{O}^k(0,-1)) \to \mathcal{O}_{[N]} \to 0 .$$
$$(4.14)$$

($[Z^\alpha;\zeta^j]$ and $[W_\alpha;\psi_j]$ are the homogeneous supercoordinates of $\mathbb{PT}_{[N]}$ and $\mathbb{PT}^*_{[N]}$ respectively.) Note that $\mathcal{O}_{[N]}$ has support on $\mathbb{A}$, and $W_\alpha Z^\alpha - 2\psi_j \zeta^j = 0$ in $\mathcal{O}_{[N]}$.

Note: The same notation is used for the superstructure sheaves of $\mathbb{M}_{[N]}$ and $\mathbb{A}_{[N]}$ because the context will always distinguish them.

The exact sequence (4.14) completely defines $\mathcal{O}_{[N]}$. Note that applying the canonical augmentation map to (4.14) one obtains the defining sequence for $\mathbb{A}$ (4.9). The procedure for expressing the superstructure sheaf of $\mathbb{A}_{[N]}$ in terms of the composition series notation (of §2.1.8) is completely analogous to that for $\mathbb{M}_{[N]}$.

§4.2.3 The N = 1 case (cf. §3.1.3)

Substitute

$$\Lambda^\bullet(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \cong \mathcal{O} \oplus \begin{matrix}\mathcal{O}(-1,0)\\ \oplus \\ \mathcal{O}(0,-1)\end{matrix} \oplus \mathcal{O}(-1,-1)$$

in (4.14) to obtain

$$\left.\begin{tikzcd}
\mathcal{O}(-1,-1) \arrow[r, "\times W_\alpha Z^\alpha"] \arrow[ddddddr, "-2"] & \mathcal{O} \\
\oplus & \oplus \\
\mathcal{O}(-2,-1) \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O}(-1,0) \\
\oplus & \oplus \\
\mathcal{O}(-1,-2) \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O}(0,-1) \\
\oplus & \oplus \\
\mathcal{O}(-2,-2) \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O}(-1,-1)
\end{tikzcd}\right\} \to \mathcal{O}_{[N]} \to 0 . \qquad (4.15)$$

This construction is clearly much easier than the analogous construction for $\mathbb{M}_{[N]}$. Since there are no diagonal maps between the odd parts, let us isolate the even part of (4.15) and use *trivial splittings to write*

$$
\begin{tikzcd}
0 & 0 & 0 & \\
\mathcal{O}(-1,-1) \arrow[u] \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O} \arrow[u] \arrow[r] & \mathcal{O}_{\mathbb{A}} \arrow[u, dash] \arrow[r] & 0 \\
\mathcal{O}(-1,-1) \arrow[u] \arrow[r, "\times W_\alpha Z^\alpha"] \arrow[ddr, "-2"] & \mathcal{O} \arrow[u] & & \\
\oplus & \oplus \arrow[r] & \{\mathcal{O}_{\mathbb{A}} + \mathcal{O}_{\mathbb{A}}(-1,-1)\} \arrow[uu] \arrow[r] & 0 \\
\mathcal{O}(-2,-2) \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O}(-1,-1) & & \\
\mathcal{O}(-2,-2) \arrow[u] \arrow[r, "\times W_\alpha Z^\alpha"] & \mathcal{O}(-1,-1) \arrow[u] \arrow[r] & \mathcal{O}_{\mathbb{A}}(-1,-1) \arrow[uu] \arrow[r] & 0 \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{4.15'}
$$

Thus, $\mathcal{O}_{[1]}$ can be written as

$$
\mathcal{O}_{[1]} = \mathcal{O}_{(1)} \oplus \begin{array}{c} \mathcal{O}_{\mathbb{A}}(-1,0) \\ \oplus \\ \mathcal{O}_{\mathbb{A}}(0,-1) \end{array} . \tag{4.16a}
$$

$\mathcal{O}_{[1]}$ is $\mathbb{Z}_2$-graded and not $\mathbb{Z}$-graded.

Note: From now on $\mathcal{O}_{\mathbb{A}}$ will also be denoted by $\mathcal{O}$.

§4.2.4 Generalization to N = 2 and N = 3

The generalization of (4.15') to $N > 1$ is more complicated since there are many more diagonal maps, induced by $-2\psi_j\zeta^j$, e.g.

$$
-2\delta^j_k \; : \; \mathcal{O}^k_j(-p,-q) \to \mathcal{O}(-p,-q)
$$

where $\mathcal{O}(-p,-q)$ denotes the trace of $\mathcal{O}^k_j(-p,-q)$.

It is not difficult to convince oneself that the construction works in general. The procedure for constructing $\mathcal{O}_{[N]}$ is essentially as follows. Decompose

$\Lambda^{\bullet}(\mathcal{O}_j(-1,0) \oplus \mathcal{O}^m(0,-1))$

$$\cong \mathcal{O} \oplus \begin{pmatrix} \mathcal{O}_i(-1,0) \\ \oplus \\ \mathcal{O}^m(0,-1) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{O}_{ij}(-2,0) \\ \oplus \\ \mathcal{O}_i^m(-1,-1) \\ \oplus \\ \mathcal{O}^{mn}(0,-2) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{O}_{ijk}(-3,0) \\ \oplus \\ \mathcal{O}_{ij}^m(-2,-1) \\ \oplus \\ \mathcal{O}_i^{mn}(-1,-2) \\ \oplus \\ \mathcal{O}^{mnp}(0,-3) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{O}_{ijk\ell}(-4,0) \\ \oplus \\ \mathcal{O}_{ijk}{}^m(-3,-1) \\ \oplus \\ \mathcal{O}_{ij}^{mn}(-2,-2) \\ \oplus \\ \mathcal{O}_i^{mnp}(-1,-3) \\ \oplus \\ \mathcal{O}^{mnpq}(0,-4) \end{pmatrix} \oplus \ldots \tag{4.17}$$

under the action of GL(V) ($\cong$ GL(N;$\mathbb{C}$)) into irreducibles, and then recombine wherever possible (preserving the $\mathbb{Z}_2$-grading) using infinitesimal neighbourhoods. We shall only write down explicit expressions for $\mathcal{O}_{[2]}$ and $\mathcal{O}_{[3]}$, as they will be needed later:

$$\mathcal{O}_{[2]} \cong \underbrace{\mathcal{O}_{(2)} \oplus \begin{pmatrix} \mathcal{O}_{ij}(-2,0) \\ \oplus \\ \mathcal{O}^{mn}(0,-2) \end{pmatrix} \oplus \text{tr.free}\,\mathcal{O}_i^m(-1,-1)}_{\bar{0}} \oplus \underbrace{\begin{pmatrix} \mathcal{O}_{(1)i}(-1,0) \\ \oplus \\ \mathcal{O}_{(1)}{}^m(0,-1) \end{pmatrix}}_{\bar{1}} \tag{4.16b}$$

$$\mathcal{O}_{[3]} \cong \underbrace{\mathcal{O}_{(3)} \oplus \begin{pmatrix} \mathcal{O}_{(1)ij}(-2,0) \\ \oplus \\ \mathcal{O}_{(1)}{}^{mn}(0,-2) \end{pmatrix} \oplus \text{tr free}\,\mathcal{O}^m_{i(1)}(-1,-1)}_{\bar{0}}$$

$$\oplus \underbrace{\begin{pmatrix} \mathcal{O}_{(2)i}(-1,0) \\ \oplus \\ \mathcal{O}_{(2)}{}^m(0,-1) \end{pmatrix} \oplus \begin{pmatrix} \text{tr free}\,\mathcal{O}_{ij}{}^m(-2,-1) \\ \oplus \\ \text{tr free}\,\mathcal{O}_i^{mn}(-1,-2) \end{pmatrix} \oplus \begin{pmatrix} \mathcal{O}_{ijk}(-3,0) \\ \oplus \\ \mathcal{O}^{mnp}(0,-3) \end{pmatrix}}_{\bar{1}} \tag{4.16c}$$

§4.2.5 Infinitesimal neighbourhoods of $\mathbb{A}_{[N]}$

Let $I_{[N]}$ denote the ideal generated by $W_\alpha Z^\alpha - 2\psi_j\zeta^j$, i.e. $I_{[N]} = \mathrm{Im}(W_\alpha Z^\alpha - 2\psi_j\zeta^j)$ in (4.14). We will define the *m^{th} infinitesimal neighbourhood of $\mathbb{A}_{[N]}$ in $\mathbb{P}T_{[N]} \times \mathbb{P}T^*_{[N]}$* as the supermanifold $\mathbb{A}_{[N](m)} = (\mathbb{A}, \mathcal{O}_{[N](m)})$, where

$$\mathcal{O}_{[N](m)} = \Lambda^\bullet(\mathcal{O}_j(-1,0) \oplus \mathcal{O}^k(0,-1))/I^{m+1}_{[N]}$$

with

$$\mathcal{O}_{[N](0)} = \mathcal{O}_{[N]} \;.$$

Informally, $\mathbb{A}_{[N](m)}$ may be regarded as the supermanifold with 'homogeneous supercoordinates' defined by

$[W_\alpha, Z^\alpha, \chi; \psi_j, \zeta^j]$ s.t. $[\mu W_\alpha, \lambda Z^\alpha, \mu\lambda\chi; \mu\psi_j, \lambda\zeta^j] = [W_\alpha, Z^\alpha, \chi; \psi_j, \zeta^j]$ $\forall\ \mu, \lambda \in \mathbb{C}^*$

with $\chi = W_\zeta Z^\zeta - 2\psi_j\zeta^j$ and $\chi^{m+1} = 0$ in $\mathcal{O}_{[N](m)}$.

The following two examples of $\mathcal{O}_{[N](m)}$ will be needed in later sections:

$$\mathcal{O}_{[1](2)} = \underbrace{\mathcal{O}_{(3)}}_{\bar{0}} \oplus \underbrace{\begin{pmatrix} \mathcal{O}_{(2)}(-1,0) \\ \oplus \\ \mathcal{O}_{(2)}(0,-1) \end{pmatrix}}_{\bar{1}} \tag{4.18a}$$

$$\mathcal{O}_{[2](1)} = \underbrace{\mathcal{O}_{(3)} \oplus \begin{pmatrix} \mathcal{O}_{(1)ij}(-2,0) \\ \oplus \\ \mathcal{O}_{(1)}{}^{ij}(0,-2) \end{pmatrix} \oplus \operatorname{tr f} \mathcal{O}_{(1)}{}^j{}_i(-1,-1)}_{\bar{0}} \oplus \underbrace{\begin{pmatrix} \mathcal{O}_{(2)i}(-1,0) \\ \oplus \\ \mathcal{O}_{(2)}{}^j(0,-1) \end{pmatrix}}_{\bar{1}} \tag{4.18b}$$

Note that there are inclusions

$$\mathcal{O}_{[3]} \hookleftarrow \mathcal{O}_{[2](1)} \hookleftarrow \mathcal{O}_{[1](2)} \hookleftarrow \mathcal{O}_{(3)} \;;$$

and that although the composition $\mathcal{O}_{(3)} \hookrightarrow \mathcal{O}_{[3]}$ is unique, the intermediate inclusions are not. This fact also shows up at the supercoordinate level:

$$A_{[3]} = A_{[3](0)} \twoheadrightarrow A_{[2](1)} \twoheadrightarrow A_{[1](2)} \twoheadrightarrow A_{[0](3)} = A_{(3)}$$

$$[\psi_1,\psi_2,\psi_3,\zeta^1,\zeta^2,\zeta^3] \mapsto [\chi_{(1)};\psi_1,\psi_2,\zeta^1,\zeta^2] \mapsto [\chi_{(2)};\psi_1,\zeta^1] \mapsto [\chi_{(3)}]\ ,$$

where we have made the following intermediate choices:

$$\chi_{(1)} = 2\psi_3\zeta^3\ ,\quad \chi_{(2)} = 2\psi_2\zeta^2 + \chi_{(1)}\ ,\quad \chi_{(3)} = 2\psi_1\zeta^1 + \chi_{(2)}\,.$$

§4.2.6 The superambitwistor correspondence

Let $\mathbb{G}$ denote the classical flag manifold $\mathbb{F}_{123} = \{L_1 \subset L_2 \subset L_3 \subset T\}$, i.e. the correspondence space for ambitwistors; and let $\mathbb{G}_{[N]}$ be the $(6|4N)$-dimensional supermanifold whose underlying classical manifold and vector bundle are $\mathbb{G}$ and $\{\mathcal{O}_{B'j} \oplus \mathcal{O}^k_B\}$ respectively. The superambitwistor correspondence is given by the following double fibration of supermanifolds

$$\begin{array}{ccc} & (x^{AA'},[\eta_A],[\pi_{A'}];\theta^A_j,\tilde\theta^{A'k}) & \\ & \Cap & \\ & \mathbb{G}_{[N]} & \\ \swarrow^{\rho} & & \searrow^{\sigma} \\ \mathbb{M}_{[N]} & & \mathbb{A}_{[N]} \\ \Cup & & \Cup \\ (x^{AA'};\theta^A_j,\tilde\theta^{A'k}) & & [W_\alpha,Z^\alpha;\psi_j,\zeta^j] \end{array} \qquad (4.19)$$

Fig. 4.1 below gives an indirect illustration of (4.19).

<u>Fig. 4.1</u>

$$
\begin{tikzcd}
 & & & (x_-^{AA'},[\pi_{A'}];\tilde{\theta}^{A'j})\,(x_+^{AA'},[\eta_A];\theta^A_j) \in {}^-\mathbb{F}_{[N]}\times{}^+\mathbb{F}_{[N]} \arrow[dl, dash] \arrow[dr, dash] & & & \\
(u^a,v^a)\in\mathbb{M}\times\mathbb{M} \arrow[rr, hook] & & (u^a,v^a;\tilde{\theta}^{A'j},\theta^A_j)\,((x_-^a;\tilde{\theta}^{A'j}),(x_+^a;\theta^A_j)) \in (\mathbb{M}\times\mathbb{M})_{[N]}\cong\mathbb{M}_{[N]}\times\mathbb{M}_{[N]} & & [[Z^\alpha;\zeta^j],[W_\alpha;\psi_j]] \in \mathbb{PT}_{[N]}\times\mathbb{PT}^*_{[N]} & & [[Z^\alpha],[W_\alpha]] \in \mathbb{PT}\times\mathbb{PT}^* \arrow[ll, hook'] \\
(x^a,\sigma^a)\in\mathbb{M}_{<N>} \arrow[u, hook] & & (x^a;\theta^A_j,\tilde{\theta}^{A'k})\in\mathbb{M}_{[N]} \arrow[ll, two heads, "\sigma^a=\theta^A_j\tilde{\theta}^{A'j}"'] \arrow[u, hook] & & [W_\alpha,Z^\alpha;\psi_j,\zeta^j]\in\mathbb{A}_{[N]} \arrow[u, hook] \arrow[rr, two heads, "\chi=2\psi_j\zeta^j"] & & [W_\alpha,Z^\alpha,\chi]\in\mathbb{A}_{(N)} \arrow[u, hook] \\
 & x^a\in\mathbb{M} \arrow[ul, hook] \arrow[ur, hook] & & & & [Z^\alpha,W_\alpha]\in\mathbb{A} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

The fibres of σ are $(1|2N)$-dimensional subsupermanifolds of $\mathbb{M}_{[N]}$ called *super light-rays*. On the other hand, the fibres of ρ are $(2|0)$-dimensional subsupermanifolds (of $\mathbb{A}_{[N]}$) $Q_{x,\theta,\tilde{\theta}} \cong (\mathbb{CP}_1 \times \mathbb{CP}_1, 0)$.

Lemma 4.i.c. Let H be a superfunction of $\mathbb{E}_{[N]}$. If $H(x^{AA'}, \eta_A, \pi_{A'}; \theta^A_j, \tilde{\theta}^{A'k})$ is of the form $h(-(x^{AA'}+\theta^A_j\tilde{\theta}^{A'j})\eta_A, \eta_A, (x^{AA'}-\theta^A_j\tilde{\theta}^{A'j})\pi_{A'}, \pi_{A'}; \theta^A_j\eta_A, \tilde{\theta}^{A'k}\pi_{A'})$, then

$$d_\sigma H = 0$$

where

$$d_\sigma \equiv (D, T^j, \tilde{T}_k) = (\pi^{A'}\eta^A\partial_{AA'}, \eta^A\partial^j_A, \pi^{A'}\tilde{\partial}_{A'k}) \ .$$

Proof: From (4.6) and (4.9):

$$\begin{aligned}
\frac{\partial H}{\partial x^{AA'}} &= \pi_{A'} \frac{\partial h}{\partial \omega^A} - \eta_A \frac{\partial h}{\partial \xi^{A'}} \\
\frac{\partial H}{\partial \theta^A_j} &= -\tilde{\theta}^{A'j}\pi_{A'} \frac{\partial h}{\partial \omega^A} - \tilde{\theta}^{A'j}\eta_A \frac{\partial h}{\partial \xi^{A'}} + \eta_A \frac{\partial h}{\partial \psi_j} \\
\frac{\partial H}{\partial \tilde{\theta}^{A'k}} &= \theta^A_k\pi_{A'} \frac{\partial h}{\partial \omega^A} + \pi_{A'}\frac{\partial h}{\partial \xi^k} + \theta^A_k\eta_A\frac{\partial h}{\partial \xi^{A'}} \ .
\end{aligned}$$

□

Consequently d_σ may be regarded as the relative exterior derivative along the fibres of σ in diagram (4.19). In this way, a super light-ray can be thought of as being spanned by D, T^j and $\tilde{T}_k$ [93].

§4.3 Contour integral representation of massless superfields

§4.3.1 Ferber's contour integral formulae

The Sparling-Ward splitting method [88] can be used to generate massless superfields from supertwistor and superambitwistor functions. Let $f(\pi_{A'})$ be homogeneous of degree -1, with singularities in two disjoint regions S_1 and S_2; so f splits as $f = f_1 - f_2$, where

$$f_j(\pi_{A'}) = \frac{1}{2\pi i}\oint_{\Gamma_j} (\xi_{B'}\pi^{B'})^{-1} f(\xi_{A'})\xi_{E'}d\xi^{E'} \quad (j = 1,2) \qquad (4.22)$$

for any closed curve Γ_j on the Riemann sphere separating the singularity sets of $f(\xi_{A'})$. Thus f_j is holomorphic on the complement of S_j, i.e. a Čech coboundary.

Suppose f is now a supertwistor function $f(Z^\alpha;\zeta^j)$ homogeneous of degree -m-2 (m ≥ 0). Set

$$F_{A'\ldots L'}(x_-^{AA'},\pi_{A'};\tilde{\theta}^{A'j}) = \underbrace{\pi_{A'}\ldots\pi_{L'}}_{m+1}\; f((x^{AA'}-\theta^A_j\tilde{\theta}^{A'j})\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'}).$$

Clearly $F_{A'\ldots L'}$ is homogeneous of degree -1 in $\pi_{A'}$, and so it splits, say as:

$$F_{1A'\ldots L'}(x_-^{AA'}\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'}) - F_{2A'\ldots L'}(x_-^{AA'}\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'}).$$

An antichiral superfield can now be obtained as follows. Set

$$\Psi_{\underbrace{A'\ldots K'}_{m}} := \pi^{L'}F_{1A'\ldots K'L'}, \qquad (4.23)$$

so that

$$\Psi_{A'\ldots K'} = \pi^{L'}F_{2A'\ldots K'L'}.$$

Since the left hand side and right hand side above can only be singular on S_1 and S_2 respectively, both of them must be globally holomorphic. $\Psi_{A'B'\ldots K'}$ is homogeneous of degree zero in $\pi_{A'}$ and therefore by Liouville's theorem it must be independent of $\pi_{A'}$, as required.

Ferber's integral formula [25] for superfields can be obtained by substituting (4.22), for j = 1, into (4.23):

$$\Psi_{A'\ldots K'}(x_-^{AA'};\tilde{\theta}^{A'j}) = \frac{\pi^{L'}}{2\pi i}\oint_{\Gamma_1} \xi_{A'}\ldots\xi_{K'}\xi_{L'}\frac{f(Z^\alpha;\zeta^j)}{(\xi_{M'}\pi^{M'})}\Big|_{L_{x_-,\tilde{\theta}}}\xi_{Q'}d\xi^{Q'}$$

$$= \frac{1}{2\pi i}\oint_{\Gamma_1} \pi_{A'}\ldots\pi_{K'}f(Z^\alpha;\zeta\ )\Big|_{L_{x_-,\tilde{\theta}}}\pi_{Q'}d\pi^{Q'}\ , \quad (4.24a)$$

where

$$f(Z^\alpha;\zeta^j)\Big|_{L_{x_-,\tilde{\theta}}} = f(x_-^{AA'}\pi_{A'},\pi_{A'};\tilde{\theta}^{A'j}\pi_{A'})\ , \quad (4.25a)$$

and the contour Γ_1 is a classical one on the Riemann sphere separating the singularities of the coefficient functions of $f(Z^\alpha;\zeta^j)$.

Similarly, given a dual supertwistor function $g(W_\alpha;\psi_j)$ homogeneous of degree -m-2 ($m \geq 0$), one can carry out the same construction and end up with a chiral superfield

$$\Phi_{A\ldots K}(x_+^{AA'};\theta_j^A)\ .$$

The contour integral formula in this case is

$$\Phi_{A\ldots K}(x_+^{AA'};\theta_j^A) = \frac{1}{2\pi i}\oint_{\tilde{\Gamma}_1}\eta_A\ldots\eta_K g(W_\alpha;\psi_j)\Big|_{L^\perp_{x_+,\theta}}\eta_Q d\eta^Q\ , \quad (4.24b)$$

where

$$g(W_\alpha;\psi_j)\Big|_{L^\perp_{x_+,\theta}} = g(-x_+^{AA'}\eta_A,\eta_A;\theta_j^A\eta_A)\ , \quad (4.25b)$$

and $\tilde{\Gamma}_1$ is a classical contour on the Riemann sphere separating the singularities of the coefficient functions of $g(W_\alpha;\psi_j)$.

A similar construction can be carried out for superambitwistors. Suppose we are given a superambitwistor function $h(Z^\alpha,W_\alpha;\zeta^j,\psi_j)$ homogeneous of degree $(-m-2,-m'-2)$ where $m,m'\geq 0$, i.e. a section of $\mathcal{O}(-m-2,-m'-2)\otimes_{\mathcal{O}}\mathcal{O}_{[N]}$.
Set

$$F_{A'\ldots L'A\ldots L}(x^{AA'},\pi^{A'},\eta^A;\tilde{\theta}^{A'k},\theta^A_j)$$

$$=\pi_{A'}\ldots\pi_{L'}\eta_A\ldots\eta_L f((x^{AA'}-\theta^A_j\tilde{\theta}^{A'j})\pi_{A'},\pi_{A'},-(x^{AA'}+\theta^A_j\tilde{\theta}^{A'j})\eta_A,\eta_A;\tilde{\theta}^{A'j}\pi_{A'},\theta^A_j\eta_A)$$

so that $F_{A'\ldots L'A\ldots L}$ is homogeneous of degree $(-1,-1)$ (in $(\pi_{A'},\eta_A)$), and hence it splits (once in π and once in η, as usual):

$$\begin{aligned}F_{A'\ldots L'A\ldots L} &= F_{11A'\ldots L'A\ldots L}-F_{12A'\ldots L'A\ldots L}\\ &\quad+F_{21A'\ldots L'A\ldots L}-F_{22A'\ldots L'A\ldots L}\,.\end{aligned}$$

A general superfield can now be defined by setting

$$\Psi_{\underbrace{A'\ldots K'}_{m}\underbrace{A\ldots K}_{m'}}:=\pi^{L'}\eta^L F_{11A'\ldots K'L'A\ldots KL}\,.$$

Since the left hand side above is globally defined in π and η, and homogeneous of degree $(0,0)$, it follows that $\Psi_{A'\ldots K'A\ldots K}$ must be independent of π and η, as required.

The integral (4.22) can be generalized to two spinor variables. Then $\Psi_{A'\ldots K'A\ldots K}(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k})$ may be expressed as the contour integral (cf. [40b])

$$\oint_{\Gamma_1\times\Gamma_2} \pi_{A'}\cdots\pi_{K'}\eta_A\cdots\eta_K h(Z^\alpha,W_\alpha;\zeta^j,\psi_j)|_{Q_{x,\theta,\tilde\theta}} d\pi_{Q'}d\pi^{Q'}\eta_p d\eta^p \ , \qquad (4.24c)$$

where

$$h(Z^\alpha,W_\alpha;\zeta^j,\psi_j)|_{Q_{x,\theta,\tilde\theta}} = h((x^{AA'}-\theta^A_j\tilde\theta^{A'j})\pi_{A'},\pi_{A'},-(x^{AA'}+\theta^A_j\tilde\theta^{A'j})\eta_A,\eta_A;\tilde\theta^{A'j}\pi_{A'},\theta^A_j\eta_A), \qquad (4.25c)$$

and $\Gamma_1\times\Gamma_2$ is a closed 2-dimensional torus on the quadric $\mathbb{CP}_1\times\mathbb{CP}_1$ avoiding the singularities of the coefficient functions of $h(Z^\alpha,W_\alpha;\zeta^j,\psi_j)$.

§4.3.2 The free massless superfield equations

It is not difficult to see that the contour integrals (4.24) will satisfy various differential equations. For example (4.24a) satisfies:

$$\begin{cases} \Box\Psi = 0 & (m = 0) \\ \partial^{A'}_A \Psi_{A'\ldots K'} = 0 & (m > 0) \end{cases} , \qquad (4.26a)$$

together with

$$\tilde\partial^2_{jk}\Psi_{A'\ldots K'} = 0 \qquad (\forall\ m \geq 0) \qquad (4.26b)$$

where

$$\Box \equiv \partial_a\partial^a \text{ and } \tilde\partial^2_{jk} \equiv \tilde\partial^-_{A'j}\tilde\partial^{-A'}_k .$$

Just as for the classical case (see for example [40a]), these superfield equations follow immediately by substituting (4.25a) in (4.24a) and differentiating under the integral sign.

Note:

$$\tilde{\partial}^{-}_{B'j}F = 2\theta^A_j\pi_{A'}\frac{\partial f}{\partial\omega^A} + \pi_{A'}\frac{\partial f}{\partial\zeta^j}$$

and

$$\begin{aligned}\tilde{\partial}^{-}_{B'k}(\tilde{\partial}^{-}_{A'j}f) &= \left(\frac{\partial}{\partial\tilde{\theta}^{B'k}} + \theta^B_k\frac{\partial}{\partial x_-^{BB'}}\right)\left(2\theta^A_j\pi_{A'}\frac{\partial f}{\partial\omega^A} + \pi_{A'}\frac{\partial f}{\partial\zeta^j}\right)\\ &= -2\theta^A_j\pi_{A'}\left(\theta^B_k\pi_{B'}\frac{\partial^2 f}{\partial\omega^B\partial\omega^A} + \pi_{B'}\frac{\partial^2 f}{\partial\zeta^k\partial\omega^A}\right)\\ &\quad + \pi_{A'}\left(\theta^B_k\pi_{B'}\frac{\partial^2 f}{\partial\omega^B\partial\zeta^j} + \pi_{B'}\frac{\partial^2 f}{\partial\zeta^k\partial\zeta^j}\right)\\ &\quad + \theta^B_k(2\theta^A_j\pi_{A'})\pi_{B'}\frac{\partial^2 f}{\partial\omega^B\partial\omega^A} + \pi_{A'}\pi_{B'}\frac{\partial^2 f}{\partial\omega^B\partial\zeta^j}\ . \quad \square\end{aligned}$$

Conversely, given for example a scalar N = 2 antichiral superfield

$$\Psi(x_-^{AA'};\tilde{\theta}^{A'j}) = \phi + \tilde{\theta}^{B'j}\psi_{B'j} + \tilde{\theta}^j_{A'}\tilde{\theta}^{A'k}\gamma_{jk} + \theta^{B'}_j\theta^{C'j}\gamma_{B'C'}\ ,$$

where $\gamma_{jk} = \gamma_{kj}$ and $\gamma_{B'C'} = \gamma_{C'B'}$; then

$$\begin{aligned}\tilde{\partial}^2_{jk}\Psi &= \left(\frac{\partial}{\partial\tilde{\theta}^{A'j}}\frac{\partial}{\partial\tilde{\theta}^{\ k}_{A'}} + \theta^A_j\theta_{Ak}\square + 2\theta^A_j\partial^{A'}_A\frac{\partial}{\partial\tilde{\theta}^{A'k}}\right)\Psi\\ &= \theta^A_j\theta_{Ak}\square\phi + 2\theta^A_j\partial^{A'}_A\psi_{A'k} + 4\theta^A_j\tilde{\theta}^{A'j}\partial^{B'}_A\gamma_{A'B'} + \gamma_{jk}\ ,\end{aligned}$$

and thus $\tilde{\partial}^2_{jk}\Psi = 0$ implies the following classical field equations

$$\square\phi = 0 \tag{4.27a}$$

$$\partial^{A'}_A\psi_{A'k} = 0 \tag{4.27b}$$

$$\partial^{A'}_A\gamma_{A'B'} = 0 \tag{4.27c}$$

$$\gamma_{jk} = 0\ . \tag{4.27d}$$

Equations (4.27) are precisely what one obtains by substituting the series expansion

$$f(Z^\alpha;\zeta^j) = f(Z^\alpha) + f_j(Z^\alpha)\zeta^j + f_{jk}(Z^\alpha)\zeta^j\zeta^k$$

(where for $N = 2{:}f_{jk}(Z^\alpha)=\varepsilon_{jk}f(Z^\alpha)$, with $\varepsilon_{jk} = -\varepsilon_{kj}$) term by term in the contour integral formula (4.24a).

It is not difficult to convince oneself that the constructions work in general for all N, and also for all $m \geq 0$. We expect that the positive homogeneity cases (which we have not considered) lead to potential modulo gauge descriptions, as in the classical case.

§4.4 The supertwistor scalar product

§4.4.1 A formal contour integral formula

We will use Einstein's summation convention and temporarily denote the series expansions of supertwistor and dual supertwistor functions as

$$f(Z^\alpha;\zeta^j) = f_{\sigma_1\ldots\sigma_N}(Z^\alpha)(\zeta^1)^{\sigma_1}\ldots(\zeta^N)^{\sigma_N} \qquad (\sigma_j = 0 \text{ or } 1,\quad 1 \le j \le N) \tag{4.28a}$$

$$g(W_\alpha;\psi_j) = g_{\mu_1\ldots\mu_N}(W_\alpha)(\psi_1)^{\mu_1}\ldots(\psi_N)^{\mu_N} \qquad (\mu_k = 0 \text{ or } 1,\quad 1 \le k \le N)\ . \tag{4.28b}$$

If $f(Z^\alpha;\zeta^j)$ and $g(W_\alpha;\psi_j)$ are homogeneous of degrees $-m-2$ and $-m'-2$ respectively, then

$$\begin{aligned}\langle\bar{g}|f\rangle &\equiv \oint \Delta ZW f(Z^\alpha;\zeta^j)\,g(W_\alpha;\psi_j)\,(W_\gamma Z^\gamma - 2\psi_\ell\zeta^\ell)_{m-N}\,d\zeta^N d\psi_N\ldots d\zeta^1 d\psi_1\\ &= \sum_{|\sigma|=0}^{N}(\mathrm{sgn})_{|\sigma|}(-2)^{N-|\sigma|}\oint \Delta ZW f_{\sigma_1\ldots\sigma_N}(Z^\alpha)\,g_{\mu_1\ldots\mu_N}(W_\alpha)\,\delta^{\sigma\mu}(W_\gamma Z^\gamma - \chi)_{m-|\sigma|}\end{aligned} \tag{4.29}$$

where:

(i) $\oint\Delta ZW$ denotes the usual twistor-dual twistor integration (see for example [65]).

(ii) $(\quad)_r$ denotes the usual twistor propagator (see for example [65]).

(iii) $\delta^{\sigma\mu} = \delta^{\sigma_1\mu_1}\ldots\delta^{\sigma_N\mu_N}$, where $\delta^{\sigma_j\mu_k} = \begin{cases}1 \text{ if } \sigma_j = \mu_k\\ 0 \text{ otherwise}\end{cases}$.

(iv) $\chi = 2\psi_j\zeta^j$.

(v) $|\sigma| = \sum_{j=1}^{N} \sigma_j$ (i.e. $0 \le |\sigma| \le N$)

$|\mu| = \sum_{k=1}^{N} \mu_k$ (i.e. $0 \le |\mu| \le N$)

(vi) $\text{đ}\zeta^j$ and $\text{đ}\psi_k$ are differential operators defined by

$$H\text{đ}\zeta^j := \frac{\partial H}{\partial \zeta^j} \quad \& \quad G\text{đ}\psi_k := \frac{\partial G}{\partial \psi_k} .$$

The following convention is adopted

$$\psi_1\zeta^1\psi_2\zeta^2 \ldots \psi_N\zeta^N \text{đ}\zeta^N \text{đ}\psi_N \ldots \text{đ}\zeta^2 \text{đ}\psi_2 \text{đ}\zeta^1 \text{đ}\psi_1 = +1$$

(vii) $(\text{sgn})_{|\sigma|}$ is an overall sign for each term in the sum. It is determined by the various sign changes required to bring a term such as $\zeta^1\zeta^2 \ldots \zeta^r\psi_1\psi_2 \ldots \psi_r$ into the form $\psi_1\zeta^1\psi_2\zeta^2 \ldots \psi_r\zeta^r$. For example

$$(\text{sgn})_0 = +, (\text{sng})_1 = -, (\text{sgn})_2 = -, (\text{sgn})_3 = +, (\text{sgn})_4 = +, (\text{sgn})_5 = -, (\text{sgn})_6 = -, \ldots$$

Note: The notation for the operators $\text{đ}\zeta^j$ and $\text{đ}\psi_k$ was chosen so as to coincide with the standard notation for 'Berezin Integration' [7a,7b], which we shall steer clear of (see [76]). For our purposes these are just differential operators with respect to the odd supercoordinates (defined in terms of their action on the series expansions of superfunctions).

The operators $\text{đ}\zeta^k$ and $\text{đ}\psi_k$ act on the propagator $(W_\gamma Z^\gamma - 2\psi_j\zeta^j)_{m-N}$ as follows:

$$(W_\gamma Z^\gamma - 2\psi_j\zeta^j)_p \text{đ}\zeta^k = -2\psi_k (W_\gamma Z^\gamma - 2\psi_j\zeta^j)_{p+1}$$

and

$$(W_\gamma Z^\gamma - 2\psi_j\zeta^j)_p \text{đ}\psi_k = +2\zeta^k (W_\gamma Z^\gamma - 2\psi_j\zeta^j)_{p+1} .$$

Thus, great care must be taken to keep track of the various sign changes. Consequently formal proof of (4.29) is very tedious and we shall only give some simple sample calculations.

§4.4.2 An example for N = 2

Substitute

$$f(Z^{\alpha};\zeta^1,\zeta^2) = f_{00} + f_{10}\zeta^1 + f_{01}\zeta^2 + f_{11}\zeta^1\zeta^2$$

$$g(W_{\alpha};\psi_1,\psi_2) = g_{00} + g_{10}\psi_1 + g_{01}\psi_2 + g_{11}\psi_1\psi_2 ,$$

in the left hand side of (4.29), to obtain

$$\begin{aligned}\oint \Delta ZW\big(&f_{11}g_{00} + \underline{f_{10}\zeta^1 g_{00}} + \underline{f_{01}\zeta^2 g_{00}} + \underline{f_{00}g_{10}\psi_1} + \underline{f_{00}g_{01}\psi_2}\\ &+ \underline{f_{11}\zeta^1\zeta^2 g_{00}} + \underline{f_{00}g_{11}\psi_1\psi_2} + \underline{f_{10}\zeta^1 g_{01}\psi_2} + \underline{f_{01}\zeta^2 g_{10}\psi_1}\\ &+ f_{10}\zeta^1 g_{10}\psi_1 + f_{01}\zeta^2 g_{01}\psi_2\\ &+ \underline{f_{11}\zeta^1\zeta^2 g_{10}\psi_1} + \underline{f_{11}\zeta^1\zeta^2 g_{01}\psi_2} + \underline{f_{10}\zeta^1 g_{11}\psi_1\psi_2} + \underline{f_{01}\zeta^2 g_{11}\psi_1\psi_2}\\ &+ f_{11}\zeta^1\zeta^2 g\psi_1\psi_2\big)(W_{\alpha}Z^{\alpha}-2\psi_1\zeta^1-2\psi_2\zeta^2)_{m-2}\,\text{đ}\zeta^2\,\text{đ}\psi_2\,\text{đ}\zeta^1\,\text{đ}\psi_1 \ . \qquad (3.30)\end{aligned}$$

All the underlined terms above vanish when applying the differential operators $\text{đ}\zeta^2\text{đ}\psi_2\text{đ}\zeta^1\text{đ}\psi_1$. For example, take the first underlined term

$$f_{10}\zeta^1 g_{00}(W_\alpha Z^\alpha - 2\phi_1\zeta^1 - 2\psi_2\zeta^2)_{m-2}\bar{d}\zeta^2\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= -2f_{10}\zeta^1 g_{00}\psi_2(\ldots)_{m-1}\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [-4f_{10}\zeta^1 g_{00}\psi_2\zeta^2(\ldots)_m - 2f_{10}\zeta^1 g_{00}(\ldots)_{m-1}]\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [+8f_{10}\zeta^1 g_{00}\psi_2\zeta^2\psi_1(\ldots)_{m+1} - 4f_{10}g_{00}\psi_2\zeta^2(\ldots)_m + 4f_{10}\zeta^1 g_{00}\psi_1(\ldots)_m$$
$$-2f_{10}g_{00}(\ldots)_{m-1}]\bar{d}\psi_1$$

$$= +16f_{10}\cancelto{0}{(\zeta^1)^2} g_{00}\psi_2\zeta^2\psi_1(\ldots)_{m+2} + 8f_{10}\zeta^1 g_{00}\psi_2\zeta^2(\ldots)_{m+1}$$
$$-8f_{10}g_{00}\psi_2\zeta^2\zeta^1(\ldots)_{m+1}$$
$$-8f_{10}\cancelto{0}{(\zeta^1)^2} g_{00}\psi_1(\ldots)_{m+1} + 4f_{10}\zeta^1 g_{00}(\ldots)_m - 4f_{10}g_{00}\zeta^1(\ldots)_m$$

$$= 0\ .$$

The vanishing of the other underlined terms can be similarly verified.

We will now consider each of the remaining terms:

$$f_{00}g_{00}(W_\alpha Z^\alpha - 2\psi_1\zeta^1 - 2\psi_2\zeta^2)_{m-2}\bar{d}\zeta^2\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= -2f_{00}g_{00}\psi_2(\ldots)_{m-2}\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [-2f_{00}g_{00}(\ldots)_{m-1} - 4f_{00}g_{00}\psi_2\zeta^2(\ldots)_m]\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [4f_{00}g_{00}\psi_1(\ldots)_m + 8f_{00}g_{00}\psi_2\zeta^2\psi_1(\ldots)_{m+1}]\bar{d}_1\psi_1$$

$$= 4f_{00}g_{00}(\ldots)_m + 8f_{00}g_{00}\psi_1\zeta^1(\ldots)_{m+1} + 8f_{00}g_{00}\psi_2\zeta^2(\ldots)_{m+1}$$
$$+16f_{00}g_{00}\psi_2\zeta^2\psi_1\zeta^1(\ldots)_{m+2}$$

$$= 4f_{00}g_{00}[(\ldots)_m + \chi(\ldots)_{m+1} + \frac{\chi}{2}(\ldots)_{m+2}]$$

$$\text{where } \chi = 2\psi_1\zeta^1 + 2\psi_2\zeta^2$$

$$f_{10}g_{10}\zeta^1\psi_1(W_\alpha Z^\alpha-2\psi_1\zeta^1-2\psi_2\zeta^2)_{m-2}\bar{d}\zeta^2\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= -2f_{10}g_{10}\zeta^1\psi_1\psi_2(\ldots)_{m-1}\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [-2f_{10}g_{10}\zeta^1\psi_1(\ldots)_{m-1}-4f_{10}g_{10}\zeta^1\psi_1\psi_2\zeta^2(\ldots)_m]\bar{d}\zeta^1\bar{d}\psi_1$$

$$= [+2f_{10}g_{10}\psi_1(\ldots)_{m-1}+4f_{10}g_{10}\psi_1\psi_2\zeta^2(\ldots)_m]\bar{d}\psi_1$$

$$= 2f_{10}g_{10}(\ldots)_{m-1}+4f_{10}g_{10}\psi_1\zeta^1(\ldots)_m+f_{10}g_{10}\psi_2\zeta^2(\ldots)_m$$
$$+8f_{10}g_{10}\psi_1\zeta^1\psi_2\zeta^2(\ldots)_{m+1}$$

$$= 2f_{10}g_{10}[(\ldots)_{m-1}+\chi(\ldots)_m+\frac{\chi^2}{2}(\ldots)_{m+1}] .$$

$$f_{01}g_{01}\zeta^2\psi_2(W\ Z-2\psi_1\zeta^1-2\psi_2\zeta^2)_{m-2}\bar{d}\zeta^2\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= 2f_{01}g_{01}[(\ldots)_{m-1}+\chi(\ldots)_m+\frac{\chi^2}{2}(\ldots)_{m+1}].$$

$$f_{11}g_{11}\zeta^1\zeta^2\psi_1\psi_2(W\ Z-2\psi_1\zeta^1-2\psi_2\zeta^2)_{m-2}\bar{d}\zeta^2\bar{d}\psi_2\bar{d}\zeta^1\bar{d}\psi_1$$

$$= -f_{11}g_{11}(\ldots)_{m-2} .$$

Hence

$$\begin{aligned}\langle\bar{g}|f\rangle = \oint\Delta ZW\{&4f_{00}g_{00}[(W_\alpha Z^\alpha-\chi)_m+\chi(W_\alpha Z^\alpha-\chi)_{m+1}+\frac{\chi}{2!}(W_\alpha Z^\alpha-\chi)_{m+2}]\\ &+2(f_{10}g_{10}+f_{01}g_{01})[(W_\alpha Z^\alpha-\chi)_{m-1}+\chi(W_\alpha Z^\alpha-\chi)_m+\frac{\chi}{2}(W_\alpha Z^\alpha-\chi)_{m+1}]\\ &-f_{11}g_{11}(W_\alpha Z^\alpha-\chi)_{m-2}\} \quad . \end{aligned} \tag{4.31}$$

Finally the terms in (4.31) involving χ (outside the propagator) will vanish when performing the Z and W contour integrals. This is because these integrands are of the 'wrong' homogeneity degree (i.e. they give rise to integrals not homogeneous of degree zero).

The quantity χ is reminiscent of that appearing in recent work by A.P. Hodges [38,39] concerning inhomogeneous twistor diagrams. It has always been part of the supersymmetry folklore that supersymmetric theories are typically 'less divergent' in perturbation theory than their classical counterparts. It might prove interesting to develop a 'supersymmetric scattering theory' via the natural generalization of twistor diagrams to the super category.

§4.5 Cohomological description of massless free superfields

§4.5.1 Cohomology on projective supertwistor spaces

Let M be a Stein subset of $\mathbb{M}$ and let $P = \mu(\nu^{-1}(M))$ be the corresponding subset of $\mathbb{PT}$ under the classical twistor correspondence (4.4a). Suppose that $M \cap \Sigma_{[Z]}$ is connected and that $H^1(\Sigma_{[Z]};\mathbb{C}) = 0$ for all $[Z^\alpha] \in P$. Then the Penrose transform [21] gives the following isomorphism for $k \geq 0$

$$H^1(P,\mathcal{O}(-m-2)) \cong \begin{cases} \ker\partial_A^{A'} : \Gamma(M,\mathcal{O}_{\overbrace{(A'B'\ldots C')}^{m}}[-1]') \to \Gamma(M,\mathcal{O}_{A(B'\ldots C')}[-2]') & (m>0) \\ \ker\Box : \Gamma(M,\mathcal{O}[-1]') \to \Gamma(M,\mathcal{O}[-1][-2]') & (m=0) \end{cases}$$

Thus (for $m \geq 0$) we can immediately identify

$$H^1(P,\mathcal{O}(-m-2)\otimes_{\mathcal{O}}\Lambda^{\bullet}\mathcal{O}_j(-1)) = H^1(P,\mathcal{O}(-m-2)) \oplus H^1(P,\mathcal{O}_j(-m-3)) \oplus H^1(P,\mathcal{O}_{jk}(-m-4)) \oplus \ldots$$
$$\ldots \oplus H^1(P,\mathcal{O}_{jk\ldots\ell}(-m-2-N)) \qquad (4.32)$$

with the space of multiplets:

$$\Psi_{A'B'\ldots C'} \equiv [\phi_{A'B'\ldots C'},\psi_{A'B'\ldots C'D'j},\chi_{A'B'\ldots C'D'E'jk},\ldots$$
$$\ldots,\kappa_{A'B'\ldots C'\overbrace{D'E'\ldots L'}^{N}\overbrace{jk\ldots\ell}^{N}}] \qquad (4.33)$$

where each classical field (in the right hand side above) is symmetric in all its spinor indices and antisymmetric in all its latin indices, and satisfies the classical massless free field equations for the appropriate helicity.

Similarly,

$$H^1(P^\perp,\mathcal{O}(-m-2)) \cong \begin{cases} \ker\partial^A_{A'}:\Gamma(M,\mathcal{O}_{(AB\ldots C)}[-1]) \to \Gamma(M,\mathcal{O}_{A'(B\ldots C)}[-2]) & (m>0) \\ \ker\Box:\Gamma(M,\mathcal{O}[-1]) \to \Gamma(M,\mathcal{O}[-1]'[-2]) & (m=0) \end{cases}$$

where $P^\perp$ is the corresponding subset of $\mathbb{P}T^*$ such that $M \cap \Sigma^\perp_{[W]}$ is connected and $H^1(\Sigma^\perp_{[W]};\mathbb{C}) = 0$ for all $[W_\alpha] \in P^\perp$.

Hence (for $m \geq 0$)

$$\begin{aligned} H^1(P^\perp,\mathcal{O}(-m-2)) \otimes_{\mathcal{O}} \Lambda^\bullet \mathcal{O}^j(-1)) &\cong H^1(P^\perp,\mathcal{O}(-m-2)) \oplus H^1(P^\perp,\mathcal{O}^j(-m-3)) \\ &\oplus H^1(P^\perp,\mathcal{O}^{jk}(-m-4)) \oplus \ldots \oplus H^1(P^\perp,\mathcal{O}^{jk\ldots\ell}(-m-2-N)) \end{aligned} \quad (4.34)$$

can be identified with the space of multiplets:

$$\tilde{\Psi}_{AB\ldots C} \equiv [\tilde{\phi}_{AB\ldots C},\tilde{\psi}^{\,j}_{AB\ldots CD},\tilde{\chi}^{\,jk}_{AB\ldots CDE},\ldots,\tilde{\kappa}^{\overbrace{jk\ldots\ell}^{N}}_{AB\ldots CDE\ldots F}]. \quad (4.35)$$

§4.5.2 Cohomology on superambitwistor space

The situation for superambitwistors is more interesting. Let M be a Stein subset of $\mathbb{M}$ and $A = \sigma(\rho^{-1}(M))$ be the corresponding subset of $\mathbb{A}$ under the classical ambitwistor correspondence. Suppose that $M \cap \Sigma_{[Z,W]}$ is connected and that $H^1(\Sigma_{[Z,W]},\mathbb{C}) = 0$ for all $[Z^\alpha,W_\alpha] \in A$. Then $H^1(A,\mathcal{O}_{[N](m)})$ will be identified (for $N = 1,2,3$ and $m = 0,1,2$) with multiplets of classical massless free fields on M.

We state below (Fig. 4.2 and Fig. 4.3) the only non-vanishing cohomology groups on A with coefficients in $\mathcal{O}_{(m)}(-p,-q)$ for all the values of p,q and m which appear in $\mathcal{O}_{[3]}$. From Bott's [8] generalization of the Borel-Weil theorem, using the methods and notation of [17], the computation of these groups is reduced to a few trivial arithmetic calculations.

From Fig. 4.2 and Fig. 4.3 we can deduce the following:

$$H^1(A,\mathcal{O}_{[1]}) \cong \underbrace{G+\Gamma(M,\mathcal{O}[-1][-1]')}_{\bar{0}} \oplus \underbrace{\left\{\begin{matrix}\Gamma(M,\mathcal{O}_A[-1]') \\ \oplus \\ \Gamma(M,\mathcal{O}_{A'}[-1])\end{matrix}\right\}}_{\bar{1}} \tag{4.52}$$

$$H^1(A,\mathcal{O}_{[2]}) \cong G \oplus \underbrace{\left\{\begin{matrix}\Gamma(M,\varepsilon_{ij}\mathcal{O}[-1]') \\ \oplus \\ \Gamma(M,\varepsilon^{ij}\mathcal{O}[-1])\end{matrix}\right\}}_{\bar{0}} \oplus \underbrace{\left\{\begin{matrix}\Gamma(M,\mathcal{O}_{Aj}[-1]') \\ \oplus \\ \Gamma(M,\mathcal{O}^k_{A'}[-1])\end{matrix}\right\}}_{\bar{1}} \tag{4.53}$$

$$H^1(A,\mathcal{O}_{[3]}) \cong \underbrace{F \oplus \left\{\begin{matrix}W_{jk} \\ \oplus \\ \tilde{W}^{jk}\end{matrix}\right\}}_{\bar{0}} \oplus \underbrace{\left\{\begin{matrix}N_j \\ \oplus \\ \tilde{N}^j\end{matrix}\right\} \oplus \left\{\begin{matrix}\varepsilon_{ijk}Q \\ \oplus \\ \varepsilon^{ijk}\tilde{Q}\end{matrix}\right\}}_{\bar{1}} \tag{4.54}$$

$$H^1(A,\mathcal{O}_{[1](2)}) \cong \underbrace{F}_{\bar{0}} \oplus \underbrace{\left\{\begin{matrix}N \\ \oplus \\ \tilde{N}\end{matrix}\right\}}_{\bar{1}} \tag{4.55}$$

$$H^1(A,\mathcal{O}_{[2](1)}) \cong \underbrace{F \oplus \left\{\begin{matrix}\varepsilon_{ij}W \\ \oplus \\ \varepsilon^{ij}W\end{matrix}\right\}}_{\bar{0}} \oplus \underbrace{\left\{\begin{matrix}N \\ \oplus \\ \tilde{N}^j\end{matrix}\right\}}_{\bar{1}} , \tag{4.56}$$

where $\varepsilon_{ij}(\varepsilon^{ij})$ and $\varepsilon_{ijk}(\varepsilon^{ijk})$ are the totally antisymmetric tensors in two and three dimensions respectively, and:

$$W_{jk} \equiv \{\ker\Box : \Gamma(M,\mathcal{O}_{jk}[-1]') \to \Gamma(M,\mathcal{O}_{jk}[-2]'[-1])\} \tag{4.57a}$$

$$\tilde{W}^{jk} \equiv \{\ker\Box : \Gamma(M,\mathcal{O}^{jk}[-1]) \to \Gamma(M,\mathcal{O}^{jk}[-2][-1]')\} \tag{4.57b}$$

$$N_j \equiv \{\ker\partial^A_{A'} : \Gamma(M,\mathcal{O}_{Aj}[-1]') \to \Gamma(M,\mathcal{O}_{A'j}[-1]'[-1])\} \tag{4.58a}$$

$$\tilde{N}^j \equiv \{\ker\partial^{A'}_{A} : \Gamma(M,\mathcal{O}^j_{A'}[-1]) \to \Gamma(M,\mathcal{O}^j_{A}[-1][-1]')\} \tag{4.58b}$$

($W,\tilde{W},Q,\tilde{Q},N,\tilde{N},G$ and F are defined in Fig. 4.2 and Fig. 4.3.)

Thus $H^1(A,\mathcal{O}_{[1]})$ and $H^1(A,\mathcal{O}_{[2]})$ only involve G (usually referred to as the space of potentials modulo gauge freedom) together with fields which do not satisfy any equations. On the other hand $H^1(A,\mathcal{O}_{[3]})$ is isomorphic to a multiplet consisting of a Maxwell field $F_{ab} = \varepsilon_{AB}\tilde{\phi}_{A'B'} + \varepsilon_{A'B'}\phi_{AB}$, six scalar fields: W_{ij} $(= -W_{ji})$ and $\tilde{W}^{ij}$ $(= -\tilde{W}^{ji})$, and eight neutrino fields: λ_{Aj}, $\tilde{\lambda}^j_{A'}$, $Z_{A'}$ and $\tilde{Z}_A$ $(i,j,k = 1,2,3)$. $H^1(A,\mathcal{O}_{[1](2)})$ is isomorphic to a multiplet consisting of a Maxwell field F_{ab} and two neutrino fields λ_A and $\tilde{\lambda}_{A'}$. Finally, $H^1(A,\mathcal{O}_{[2](1)})$ is isomorphic to a multiplet consisting of a Maxwell field F_{ab}, two scalar fields W and $\tilde{W}$ and four neutrino fields λ_{Aj} and $\tilde{\lambda}^j_{A'}$ $(j = 1,2)$.

Note: The cohomology groups of Fig. 4.3 have also been computed by Henkin and Manin [36].

Fig. 4.2

$$H^1(A,\mathcal{O}(-3,0)) \cong \{\ker\partial_A^{A'}:\Gamma(M,\mathcal{O}_{A'}[-1]') \to \Gamma(M,\mathcal{O}_A[-2]')\} \equiv Q \qquad (4.36a)$$

$$H^1(A,\mathcal{O}(0,-3)) \cong \{\ker\partial_{A'}^{A}:\Gamma(M,\mathcal{O}_A[-1]) \to \Gamma(M,\mathcal{O}_{A'}[-2])\} \equiv \tilde{Q} \qquad (4.36b)$$

$$H^1(A,\mathcal{O}(-2,0)) \cong \Gamma(M,\mathcal{O}[-1]') \qquad (4.37a)$$

$$H^1(A,\mathcal{O}(0,-2)) \cong \Gamma(M,\mathcal{O}[-1]) \qquad (4.37b)$$

$$H^2(A,\mathcal{O}(-3,-1)) \cong \Gamma(M,\mathcal{O}[-2]'[-1]) \qquad (4.38a)$$

$$H^2(A,\mathcal{O}(-1,-3)) \cong \Gamma(M,\mathcal{O}[-2][-1]') \qquad (4.38b)$$

$$H^1(A,\mathcal{O}(-1,0)) \cong \Gamma(M,\mathcal{O}_A[-1]') \qquad (4.39a)$$

$$H^1(A,\mathcal{O}(0,-1)) \cong \Gamma(M,\mathcal{O}_{A'}[-1]) \qquad (4.39b)$$

$$H^2(A,\mathcal{O}(-3,-2)) \cong \Gamma(M,\mathcal{O}_{A'}[-1]'[-1]) \qquad (4.40a)$$

$$H^2(A,\mathcal{O}(-2,-3)) \cong \Gamma(M,\mathcal{O}_A[-1][-1]') \qquad (4.40b)$$

$$H^2(A,\mathcal{O}(-3,-3)) \cong \{\ker\partial^{AA'}:\Gamma(M,\mathcal{O}_{AA'}[-1][-1]') \to \Gamma(M,\mathcal{O}[-2][-2]') \equiv J \qquad (4.41)$$

$$H^2(A,\mathcal{O}(-2,-2)) \cong \Gamma(M,\mathcal{O}[-1][-1]') \qquad (4.42)$$

$$H^1(A,\mathcal{O}(-1,-1)) \cong \Gamma(M,\mathcal{O}[-1][-1]') \qquad (4.43)$$

$$H^1(A,\mathcal{O}) \cong \{\Gamma(M,\mathcal{O}_{AA'})/\{\mathrm{Im}\partial_{AA'}:\Gamma(M,\mathcal{O}) \to \Gamma(M,\mathcal{O}_{AA'})\}\} \equiv G \qquad (4.44)$$

Note:

$$H^k(A,\mathcal{O}(-2,-1)) = 0 \qquad (k = 1,2) \qquad (4.45a)$$

$$H^k(A,\mathcal{O}(-1,-2)) = 0 \qquad (k = 1,2) \qquad (4.45b)$$

Fig. 4.3

$$H^1(A,\mathcal{O}_{(1)}(-2,0)) \cong \{\ker\Box : \Gamma(M,\mathcal{O}[-1]') \to \Gamma(M,\mathcal{O}[-2]'[-1])\} \equiv W \tag{4.46a}$$

$$H^1(A,\mathcal{O}_{(1)}(0,-2)) \cong \{\ker\Box : \Gamma(M,\mathcal{O}[-1]) \to \Gamma(M,\mathcal{O}[-2][-1]')\} \equiv \tilde{W} \tag{4.46b}$$

$$H^1(A,\mathcal{O}_{(2)}(-1,0)) \cong \{\ker\partial^A_{A'} : \Gamma(M,\mathcal{O}_A[-1]') \to \Gamma(M,\mathcal{O}_{A'}[-1]'[-1])\} \equiv N \tag{4.47a}$$

$$H^1(A,\mathcal{O}_{(2)}(0,-1)) \cong \{\ker\partial^{A'}_{A} : \Gamma(M,\mathcal{O}_{A'}[-1]) \to \Gamma(M,\mathcal{O}_A[-1][-1]')\} \equiv N \tag{4.47b}$$

$$H^1(A,\mathcal{O}_{(1)}) \cong G + \Gamma(M,\mathcal{O}[-1][-1]') \tag{4.48}$$

$$H^1(A,\mathcal{O}_{(2)}) \cong G \tag{4.49}$$

$$H^1(A,\mathcal{O}_{(3)}) \cong \left\{\begin{matrix} \ker\partial^A_{A'} : \Gamma(M,\mathcal{O}_{AB}[-1]) \to \Gamma(M,\mathcal{O}_{A'B}[-2]) \\ \oplus \\ \ker\partial^{A'}_{A} : \Gamma(M,\mathcal{O}_{A'B'}[-1]') \to \Gamma(M,\mathcal{O}_{AB'}[-2]') \end{matrix}\right\} \equiv F \tag{4.50}$$

$$(H^1(A,\mathcal{O}_{(1)}(-1,-1)) = 0, \quad \text{for } k = 1 \ \&\ 2) \tag{4.51}$$

§4.6 Supertwistor description of super Yang-Mills theory

§4.6.1 Review of the description of classical Yang-Mills theory

The twistor description of the self-dual and full Yang-Mills equations will be summarized below (using the notation of this chapter) by three theorems.

Theorem 4.ii [90]. There is a one-to-one correspondence between vector bundles F, on M, equipped with a connection satisfying the self-dual Yang-Mills equations; and vector bundles E, on $P^{\perp}$, trivial on all $L_x^{\perp} \subset P^{\perp}$, i.e. such that $\mu^* E$ is a trivial vector bundle. □

Theorem 4.iii [45]. There is a one-to-one correspondence between vector bundles F, on M, equipped with a connection; and vector bundles E, on A, trivial on all $Q_x \subset A$, i.e. such that $\sigma^* E$ is a trivial vector bundle. □

Theorem 4.iv [45,93]†. There is a one-to-one correspondence between vector bundles F, on M, equipped with a connection satisfying the full Yang-Mills equations; and vector bundles E, on A, which extend to the third infinitesimal neighbourhood of $\mathbb{A}$ in $\mathbb{P} \times \mathbb{PT}^*$. □

Note: The subset $M \subset \mathbb{M}$, $P^{\perp} \subset \mathbb{PT}^*$ and $A \subset \mathbb{A}$ are assumed to satisfy the topological restrictions described in §4.5.1 and §4.5.2.

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† See also [12,16c,45a,70,53b].

§4.6.2 Integrability on super β-planes and super light-rays

In this section we give the analogues in the super category of the first two theorems of the previous subsection. They will be referred to as Theorem 4.v and Theorem 4.vi respectively (the latter was implicitly suggested by E. Witten in [93]). However, the correspondences will be proved for all N in one direction only. For the other direction we will only consider the N = 1 case. The calculations N > 1 rapidly get out of hand. We expect that a more complete theory of super differential geometry will eventually allow one to give a more abstract and elegant proof of these results.

A superconnection is said to be *integrable on super β-planes* if it satisfies the self-dual super Yang-Mills equations (cf. [90] for analogous statement in the classical case). The following theorem characterizes such superconnections in terms of supervector bundles on open subsupermanifolds of $\mathbb{P}T^*_{[N]}$.

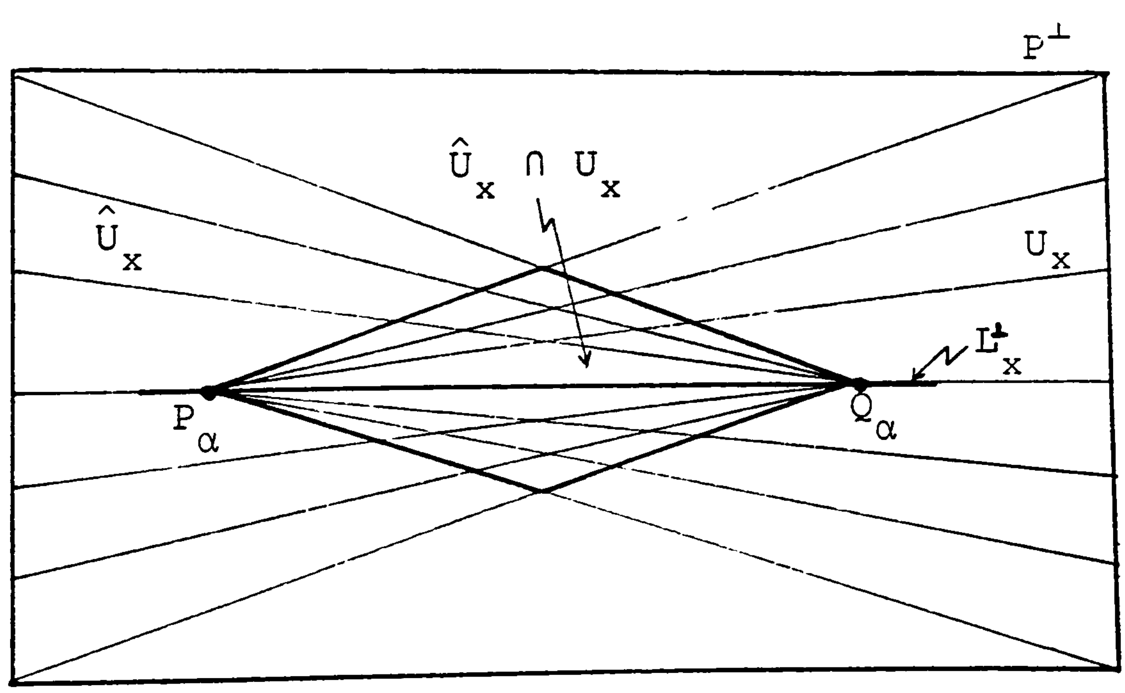
Theorem 4.v. There is a one-to-one correspondence between supervector bundles F, on ${}^{+}M_{[N]} \equiv (M, \Lambda^{\cdot}\mathcal{O}^{j}_{A})$, equipped with a superconnection ∇ satisfying the self-dual super Yang-Mills equations $F^{(-)} = 0$; and supervector bundles E, on $P^{\perp}_{[N]} \equiv (P^{\perp}, \Lambda^{\cdot}\mathcal{O}^{j}(-1))$, 'trivial on all $L^{\perp}_{x_{+},\theta} \subset P^{\perp}_{[N]}$', i.e. such that $\mu^{*}E$ is a trivial supervector bundle.

Proof

($\Rightarrow$ Construction of ∇ s.t. $F^{(-)} = 0$ given E s.t. μ^*E is trivial

The classical projective line $L_x^\perp \subset P^\perp$ can be covered in the usual way (see [89]) by taking two fixed points, say P_α and Q_α, on $L_x^\perp$, and then defining two open sets: $U_x = L_x^\perp \cap U$ and $\hat{U}_x = L_x^\perp \cap \hat{U}$, where U (respectively $\hat{U}$) is the open set consisting of all projective lines through P_α (respectively Q_α) lying entirely within $P^\perp$, as shown below

Fig. 4.4



Let g be a transition function of E, i.e. if we take E to be of rank $r|s$, then $g \in H^1(P^\perp, GL(r|s;E))$. Denote the series expansion of g by $g(W_\alpha;\psi_j)$, where $[W_\alpha;\psi_j]$ are homogeneous supercoordinates on $P^\perp_{[N]}$. Recall that the coefficients of such expansion are either even or odd elements of $\mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s}$, depending on whether they are contracted with an even or an odd number of ψ_i's (so as to make the overall grading of $g(W_\alpha;\psi_j)$ even). The procedure for constructing $g(W_\alpha;\psi_j)$ from $g \in H^1(P^\perp, GL(r|s;E))$ is completely analogous to that for the superpotentials (fully described in §3.2.4). Firstly, expand each entry of the $(r|s)\times(r|s)$-even supermatrix g in powers of ψ_j, and then remove the ψ_j's from the supermatrix according to the rules (2.20) of §2.2.3.

The corresponding series expansion of the pull-back of g to μ^*E, is

$$G(x_+^{AA'}, \eta_A; \theta_j^A) = g(-x_+^{AA'}\eta_A, \eta_A; \theta_j^A\eta_A) \quad . \tag{4.59}$$

Since $\mu^*E_{P^\perp}$ is trivial, G splits as

$$G = h\,\hat{h}^{-1} \tag{4.60}$$

where h and $\hat{h}$ are $(r|s)\times(r|s)$-even invertible supermatrices whose entries are holomorphic superfunctions on $\mu^{-1}U_x$ and $\mu^{-1}\hat{U}_x$ respectively.

Let $(D_{A'}, T^j, \tilde{T}_{A'k})$ denote the differential operators d^+_μ (given by $(\eta^A\partial^+_{AA'}, \eta^A\partial_A^{+j}, \tilde{\partial}^+_{A'k})$ in chiral supercoordinates) along the fibres of μ in diagram (4.8). From Lemma 4.ib and (4.59) it follows that

$$(D_{A'},T^j,\tilde{T}_{A'k})G = 0 \ .$$

Moreover, by (4.60) we have

$$[(D_{A'},T^j,\tilde{T}_{A'k})h]\hat{h}^{-1} - h[(D_{A'},T^j,T_{A'k})\hat{h}]\hat{h}^{-2} = 0 \ ,$$

i.e. $h^{-1}(D_{A'},T^j,\tilde{T}_{A'k})h = [(D_{A'},T^j,\tilde{T}_{A'k})\hat{h}]\hat{h}^{-1}$. (4.61)

The l.h.s. and r.ĥ.s. of (4.61) are holomorphic in $\mu^{-1}U_x$ and $\mu^{-1}\hat{U}_x$, hence each side must be globally holomorphic. Since h and ĥ are both homogeneous of degree zero in η^A, each side must be polynomials (in η^A). We denote such polynomials by

$$h^{-1}(D_{A'},T^j,\tilde{T}_{A'k})h = (\eta^A\Phi_{AA'},\eta^A\Psi^j_A,\tilde{\Psi}_{A'k}) \ , \tag{4.62}$$

i.e.

$$\tilde{T}_{A'j}h = h\tilde{\Psi}_{A'j} \tag{4.62a}$$

$$T^j h = h\eta^A\Psi^j_A \tag{4.62b}$$

$$D_{A'}h = h\eta^A\Phi_{AA'} \ . \tag{4.62c}$$

$\Phi_{AA'}$, Ψ^j_A and $\tilde{\Psi}_{A'k}$ are the superpotentials of the theory. Their series expansions in terms of the odd chiral supercoördinate are denoted by $\Phi_{AA'}(x_+^{AA'};\theta^A_j)$, $\Psi^j_A(x_+^{AA'};\theta^A_j)$ and $\tilde{\Psi}_{A'k}(x_+^{AA'};\theta^A_j)$. One can easily check that the $\mathbb{Z}_2$-grading of the coefficients of these expansions (determined by the grading of the coefficients of $h(W_\alpha;\psi_j)$ together with the grading of the differential operators ∂^+_a, ∂^{+j}_A and $\tilde{\partial}^+_{A'k}$) agrees with that described in §3.2.4.

Note: The superpotentials are defined on a neighbourhood of $x^{AA'}$ in M. A different choice of cover, obtained, say by taking two different fixed points on $L^\perp_x \subset P^\perp$, corresponds (as in the classical case) to choosing a different gauge.

The differential operators $(D_{A'}, T^j, \tilde{T}_{A'k})$ satisfy the same Lie superalgebra commutation relations as $(\partial_{AA'}, \partial^j_A, \tilde{\partial}_{A'k})$. From these (super) commutation relations together with (4.62) we will derive non-linear equations for the superpotentials, and show that these equations are precisely the self-dual super Yang-Mills equations for the superconnection

$$(\nabla_{AA'} = \partial_{AA'} + \Phi_{AA'}\ , \quad \nabla^j_A = \partial^j_A + \Psi^j_A\ , \quad \tilde{\nabla}_{A'k} = \tilde{\partial}_{A'k} + \tilde{\Psi}_{A'k}).$$

We start by applying $h^{-1}\tilde{T}_{B'k}$ to (4.62a) to obtain

$$h^{-1}T_{B'k}\{\tilde{T}_{A'j}h\} = h^{-1}\tilde{T}_{B'}\{h\tilde{\Psi}_{A'j}\} = \{h^{-1}\tilde{T}_{B'k}h\}\tilde{\Psi}_{A'j} + \{\tilde{T}_{B'k}\tilde{\Psi}_{A'j}\}\ .$$

Since $(\tilde{T}_{B'k}, \tilde{T}_{A'j}) = 0$, we have:

$$(\tilde{\Psi}_{B'}, \tilde{\Psi}_{A'}) + 2\tilde{\partial}_{(B'}\tilde{\Psi}_{A')} = 0 \tag{4.64a}$$

for N = 1, and

$$\begin{cases} (\tilde{\Psi}_{(B'k}, \tilde{\Psi}_{A')j}) + \tilde{\partial}_{k(B'}\tilde{\Psi}_{A')j} + \tilde{\partial}_{j(A'}\tilde{\Psi}_{B')k} = 0 & (4.64b) \\ (\tilde{\Psi}_{A'k}, \tilde{\Psi}^{A'}{}_{j}) + \tilde{\partial}_{A'k}\tilde{\Psi}^{A'}{}_{j} + \tilde{\partial}_{A'j}\tilde{\Psi}^{A'}{}_{k} = 0 & (4.65) \end{cases}$$

for N > 1.

Applying $h^{-1}T^k$ to (4.62b) gives

$$h^{-1}T^k\{T^jh\} = h^{-1}T^k\{h\Psi^j_A\}\eta^A = \{h^{-1}T^kh\}\eta^A\Psi^j_A + T^k\{\eta^A\Psi^j_A\}\ .$$

Since $(T^k, T^j) = 0$, we get:

$$(\Psi^k_B, \Psi^j_A) + 2\partial^k_{(B}\Psi^j_{A)} = 0 \tag{4.66a}$$

for N = 1, and

$$(\Psi^k_{(B}, \Psi^j_{A)}) + \partial^k_{(B}\Psi^j_{A)} + \partial^j_{(A}\Psi^k_{B)} = 0 \tag{4.66b}$$

for N > 1.

Applying $h^{-1}T^k$ to (4.62a) gives

$$h^{-1}T^k\{\tilde{T}_{A'j}h\} = h^{-1}T^k\{h\tilde{\Psi}_{A'j}\} = \{h^{-1}T^k h\}\tilde{\Psi}_{A'j} + T^j\tilde{\Psi}_{A'k} .$$

And since $(T^k, \tilde{T}_{A'j}) = 2\delta^k{}_j D_{A'}$, we obtain:

$$(\Psi^k_A, \tilde{\Psi}_{A'j}) + \partial^k_A\tilde{\Psi}_{A'j} + \tilde{\partial}_{A'j}\Psi^k_A = 2\delta^k{}_j\Phi_{AA'} \qquad (4.67)$$

for all N.

Finally, applying $h^{-1}D_{B'}$ to (4.62a) and (4.62b), gives respectively:

$$h^{-1}D_{B'}\{\tilde{T}_{A'j}h\} = h^{-1}D_{B'}\{h\tilde{\Psi}_{A'j}\} = \{h^{-1}D_{B'}h\}\tilde{\Psi}_{A'j} + D_{B'}\tilde{\Psi}_{A'j}$$

and

$$h^{-1}D_{B'}\{T^j h\} = h^{-1}D_{B'}\{h\eta^A\Psi^j_A\} = \{h^{-1}D_{B'}h\}\eta^A\Psi^j_A + \eta^A D_{B'}\Psi^j_A .$$

Since $[D_{B'}, T_{A'j}] = 0$ and $[D_{B'}, T^j] = 0$, we obtain:

$$\begin{cases} [\Phi_{AA'}, \tilde{\Psi}^{A'}_j] + \partial_{AA'}\tilde{\Psi}^{A'}_j - \tilde{\partial}^{A'}_j\Phi_{AA'} = 0 & (4.68) \\ [\Phi_{A(A'}, \tilde{\Psi}_{B')j}] + \partial_{A(A'}\tilde{\Psi}_{B')j} - \tilde{\partial}_{j(B'}\Phi_{A')A} = 0 & (4.69) \end{cases}$$

and

$$[\Phi_{A'(A}, \Psi^j_{B)}] + \partial_{A'(A}\Psi^j_{B)} - \partial^j_{(A}\Phi_{B)A'} = 0 \qquad (4.70)$$

respectively for all N.

For N = 1 equations (4.67), (4.64a) and (4.66a) are the constraint equations (3.36a); and equations (4.69) and (4.70) follow from them by using the Bianchi identities. Equation (4.68) is the self duality equation (3.47a).

For N > 1 equations (4.64b), (4.66b) and (4.67) are the constraint equations (3.36b). Again (4.69) and (4.70) follow from them by using the Bianchi identities. Equation (4.65) is the self-duality equation (3.47b); and equation (4.68) follows from (4.65) using the Bianchi identities.

($\Leftarrow$) *Construction of $\mathcal{E}$ s.t. $\mu^*\mathcal{E}$ is trivial given ∇ s.t. $F^{(-)} = 0$*

The construction will be carried out for the N = 1 case only. In order to facilitate comparison with the next theorem, we will carry out calculations in the supercoordinates $(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k})$ rather than in the chiral supercoordinates $(x^{AA}_+;\theta^A_j)$.

Just as for the classical construction (see [90,89]), $\mathcal{E}$ is constructed from the covariantly constant sections of $\mathcal{F}$. Let H be a section of $\mathcal{F}$ whose series expansion is denoted by $H(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k})$. Then H is said to be covariantly constant 'on a super β-plane' if:

$$\tilde{\nabla}_{A'k}H(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k}) = 0 \tag{4.72a}$$

$$\eta^A\nabla^j_A H(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k}) = 0 \tag{4.72b}$$

$$\eta^A\nabla_{AA'}H(x^{AA'};\theta^A_j,\tilde{\theta}^{A'k}) = 0 \tag{4.72c}$$

with ($\nabla_{AA'} = \partial_{AA'} + \Phi_{AA'}$, $\nabla^j_A = \partial^j_A + \Psi^j_A$, $\tilde{\nabla}_{A'k} = \tilde{\partial}_{A'k} + \tilde{\Psi}_{A'k}$) satisfying $F^{(-)} = 0$. (Recall that H may be regarded as an $(r|s)$-row supervector, and the superpotentials $(\Phi_{AA'},\Psi^j_A,\tilde{\Psi}_{A'k})$ as $(r|s)\times(r|s)$-supermatrices acting on H by supermatrix multiplication on the right.)

Our task will be to reduce the covariantly constancy equations (4.72) ('on the super β-plane') down to a system of classical equations (on the underlying β-plane), to which one can apply classical Frobenius' theorem to show that solutions exist. As in the classical case, this enables one to construct a basis of covariantly constant sections for the supervector bundle $\mathcal{E}$.

For N = 1 we can immediately read off (from Theorem 3.i and Theorem 3.ii) the solutions of $F^{(-)} = 0$:

$$\Psi_A(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = \tilde{\theta}^{A'}V_{AA'} + \theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\tilde{\lambda}_{B'} \tag{4.73a}$$

$$\tilde{\Psi}_{A'}(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = \theta^A V_{AA'} - \tilde{\theta}^2\tilde{\lambda}_{A'} + \theta^2\tilde{\theta}^{B'}\tilde{\phi}_{A'B'} \tag{4.73b}$$

$$\Phi_{AA'}(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = V_{AA'}+\theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\tilde{\phi}_{A'B'} - \frac{3}{2}\,\theta^B\varepsilon_{AB}\tilde{\lambda}_{A'} - \frac{\theta^2\tilde{\theta}^{B'}}{4}\,\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}, \tag{4.73c}$$

where

$$\tilde{\phi}_{A'B'} \equiv \frac{1}{2}\,\mathring{\nabla}_{A(A'}V^A_{B')} \tag{4.73d}$$

$$\phi_{AB} \equiv \frac{1}{2}\,\mathring{\nabla}_{A'(A}V^{A'}_{B)} = 0 \tag{4.73e}$$

$$\mathring{\nabla}_{AA'}\tilde{\lambda}^{A'} = 0 \quad . \tag{4.73f}$$

Substituting (4.73) in (4.72) we obtain:

$$\frac{\partial H}{\partial\tilde{\theta}^{A'}} + \theta^A\mathring{\nabla}_{AA'}H - \theta^2 H\tilde{\lambda}_{A'} + \theta^2\tilde{\theta}^{B'}H\tilde{\phi}_{A'B'} = 0 \tag{4.74a}$$

$$\eta^A\,\frac{\partial H}{\partial\theta^A} + \eta^A\tilde{\theta}^{A'}\mathring{\nabla}_{AA'}H + \eta^A\varepsilon_{AB}\theta^B\tilde{\theta}^{B'}H\tilde{\lambda}_{B'} = 0 \tag{4.74b}$$

$$\eta^A\mathring{\nabla}_{AA'}H + \eta^A\varepsilon_{AB}\theta^B\tilde{\theta}^{B'}H\tilde{\phi}_{A'B'} - \frac{3}{2}\,\eta^A\varepsilon_{AB}\theta^B H\tilde{\lambda}_{A'} - \frac{\theta^2\tilde{\theta}^{B'}}{4}\,H\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')} = 0\ . \tag{4.74c}$$

($\mathring{\nabla}_{AA}X := \partial_{AA'}X + XV_{AA'}$ for any $X \in \mathrm{End}_{\mathbb{C}}\mathbb{C}^{r|s}$.)

We now proceed to find solutions to the above three equations order by order. Substituting

$$H(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = \mathring{H} + \theta^A F_A + \tilde{\theta}^{A'}\tilde{F}_{A'} + \theta^A\tilde{\theta}^{A'}J_{AA'}$$
$$+ \theta^2 G + \tilde{\theta}^2\tilde{G} + \theta^2\tilde{\theta}^{A'}X_{A'} + \tilde{\theta}^2\theta^A\tilde{X}_A + \theta^2\tilde{\theta}^2 Y \qquad (4.75)$$

in (4.74), we obtain the following equations to each order:

$o(1)$:
$$\tilde{F}_{A'} = 0$$
$$\eta^A F_A = 0$$
$$\eta^A\mathring{\nabla}_{AA'}\mathring{H} = 0$$

$o(\theta)$:
$$\mathring{\nabla}_{AA'}\mathring{H} - J_{AA'} = 0$$
$$G = 0$$
$$\eta^A(\mathring{\nabla}_{AA'}F_B - \frac{3}{2}\varepsilon_{AB}\mathring{H}\tilde{\lambda}_{A'}) = 0$$

$o(\tilde{\theta})$:
$$\tilde{G} = 0$$
$$\eta^A(\mathring{\nabla}_{AA'}\mathring{H} + J_{AA'}) = 0$$
$$\eta^A\mathring{\nabla}_{AA'}\tilde{F}_{B'} = 0$$

$o(\theta^2)$:
$$X_{A'} + \frac{1}{2}\mathring{\nabla}_{AA'}F^A - \mathring{H}\tilde{\lambda}_{A'} = 0$$
$$\eta^A(\mathring{\nabla}_{AA'}G + \frac{3}{2}F_A\tilde{\lambda}_{A'}) = 0$$

$o(\theta\tilde{\theta})$:
$$-2\varepsilon_{A'B'}\tilde{X}_B + \mathring{\nabla}_{BA'}\tilde{F}_{B'} = 0$$
$$\eta^A(2\varepsilon_{AB}X_{B'} - \mathring{\nabla}_{AB'}F_B + \varepsilon_{AB}\mathring{H}\tilde{\lambda}_{B'}) = 0$$
$$\eta^A(\mathring{\nabla}_{AA'}J_{BB'} + \varepsilon_{AB}\mathring{H}\tilde{\phi}_{A'B'} - \frac{3}{2}\varepsilon_{AB}\tilde{F}_{B'}\tilde{\lambda}_{A'}) = 0$$

$o(\tilde{\theta}^2)$:
$$\eta^A(\tilde{X}_A + \frac{1}{2}\mathring{\nabla}_{AA'}\tilde{F}^{A'}) = 0$$
$$\eta^A\mathring{\nabla}_{AA'}\tilde{G} = 0$$

$$o(\theta^2\tilde{\theta}):\quad 2\varepsilon_{A'B'}Y + \frac{1}{2}\mathring{\nabla}_{AA'}J^{A}{}_{B'} - \tilde{F}_{B'}\tilde{\lambda}_{A'} + \mathring{H}\tilde{\phi}_{A'B'} = 0$$

$$\eta^{A}(\mathring{\nabla}_{AA'}G - F_{A}\tilde{\lambda}_{A'}) = 0$$

$$\eta^{A}(\mathring{\nabla}_{AA'}X_{B'} - \frac{1}{2}F_{A}\tilde{\phi}_{A'B'} + \frac{3}{2}J_{AB'}\tilde{\lambda}_{A'} - \frac{1}{4}\mathring{H}\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}) = 0$$

$$o(\theta\tilde{\theta}^2):\quad \mathring{\nabla}_{AA'}\tilde{G} = 0$$

$$\eta^{A}(2\varepsilon_{AB}Y - \frac{1}{2}\mathring{\nabla}_{AA'}J^{A'}_{B} - \frac{1}{2}\varepsilon_{AB}\tilde{F}_{A'}\tilde{\lambda}^{A'}) = 0$$

$$\eta^{A}(\mathring{\nabla}_{AA'}\tilde{X}_{B} + \frac{1}{2}\varepsilon_{AB}\tilde{F}^{B'}\tilde{\phi}_{A'B'} - \frac{3}{2}\varepsilon_{AB}\tilde{G}\tilde{\lambda}_{A'} = 0$$

$$o(\theta^2\tilde{\theta}^2):\quad \frac{1}{2}\mathring{\nabla}_{AA'}\tilde{X}^{A} - \tilde{G}\tilde{\lambda}_{A'} - \frac{1}{2}\tilde{F}^{B'}\tilde{\phi}_{A'B'} = 0$$

$$\eta^{A}(\frac{1}{2}\mathring{\nabla}_{AA'}X^{A'} + \frac{1}{4}J_{AA'}\tilde{\lambda}^{A'}) = 0$$

$$\eta^{A}(\mathring{\nabla}_{AA'}Y - \frac{1}{4}J^{B'}_{A}\tilde{\phi}_{B'A'} - \frac{3}{4}\tilde{X}_{A}\tilde{\lambda}_{A'} + \frac{1}{8}\tilde{F}^{B'}\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}) = 0$$

The above equations can be rewritten as follows:

$$\tilde{F}_{A'} = 0\,,\qquad G = 0\,,\qquad \tilde{G} = 0\,,\qquad \tilde{X}_{A} = 0 \tag{4.76}$$

$$\eta^{A}\mathring{\nabla}_{AA'}\mathring{H} = 0 \tag{4.77}$$

$$J_{AA'} = \mathring{\nabla}_{AA'}H \tag{4.78}$$

$$\eta^{A}F_{A} = 0\,,\qquad \eta^{A}J_{AA'} = 0\,,\qquad \eta^{A}\mathring{\nabla}_{A(A'}\lambda_{B')} = 0 \tag{4.79}$$

$$\eta^{A}\mathring{\nabla}_{AA'}F_{B} = 3\eta^{A}\varepsilon_{AB}\mathring{H}\tilde{\lambda}_{A'} \tag{4.80}$$

$$X_{A'} = \mathring{H}\tilde{\lambda}_{A'} - \frac{1}{2}\mathring{\nabla}_{AA'}F^{A} \tag{4.81}$$

$$\eta^{A}\mathring{\nabla}_{A(A'}J_{B')B} = -\varepsilon_{AB}\mathring{H}\tilde{\phi}_{A'B'} \tag{4.82}$$

$$Y = -\frac{1}{8}\mathring{\nabla}_{AA'}J^{AA'} = 0 \tag{4.83}$$

$$\eta^{A}\mathring{\nabla}_{AA'}X^{A'} = 0 \tag{4.84}$$

$$\eta^{A}\mathring{\nabla}_{AA'}J^{A'}{}_{B} = 0\quad . \tag{4.85}$$

Further analysis reveals that equations (4.76) to (4.86) can be summarized by stating that the solution of the covariantly constancy equations (4.74) is given by

$$H(x^{AA'};\theta^{A},\tilde{\theta}^{A'}) = \mathring{H} + \theta^{A}\tilde{\theta}^{A'}J_{AA'} + \theta^{A}F_{A} + \frac{\tilde{\theta}^{A'}\theta^{2}}{12}\mathring{\nabla}_{AA'}F^{A} , \qquad (4.86)$$

where $J_{AA'} = \mathring{\nabla}_{AA'}\mathring{H}$, $F_{A} = \eta_{A}F$, and $\mathring{H}$ and F satisfy

$$\eta^{A}\mathring{\nabla}_{AA'}\mathring{H} = 0 \qquad (4.87)$$

$$\eta^{A}\mathring{\nabla}_{AA'}F = 3\mathring{H}\lambda_{A'} . \qquad (4.88)$$

By Frobenius' theorem, solutions of equations (4.87) and (4.88) exist if $V_{AA'}$ and $\tilde{\lambda}_{A'}$ satisfy the self-dual super Yang-Mills equations (4.73e) and (4.73f) respectively, as required. □

We will now restrict ourselves to the super abelian case and construct the transition functions of the superline bundle by analogy with the classical construction [89].

Recall (from §2.3.7 and §3.2.1) that, in the super abelian case, (4.86) can be rewritten in the form

$$\check{H}(x^{AA'};\theta^{A},\tilde{\theta}^{A'}) = \check{\mathring{H}} + \theta^{A}\tilde{\theta}^{A'}\check{J}_{AA'} + [\theta^{A}\check{F}_{A} + \frac{\tilde{\theta}^{A'}\theta^{2}}{12}\check{\mathring{\nabla}}_{AA'}\check{F}^{A}]\tau \qquad (4.92)$$

where each of the coefficients is now an ordinary function on M, and τ is the auxiliary quantity introduced in §2.3.7 and §3.2.1 in order to define the exponential map *exp*. In this way, equations (4.76) to (4.85) become differential equations for ordinary functions on M.

Suppose $\check{\mathring{H}}$ has the value $\check{\mathring{H}}_{p}$ at a point $p^{AA'}$ on the β-plane $\Sigma^{\perp}_{[P_{\alpha}]} \subset M$ corresponding to P_{α} in U (a member of the cover of $P^{\perp} \subset \mathbb{P}T^{*}$). Then it follows from (4.77), that at some other point (say $x^{AA'}$) on the same β-plane, $\overset{\circ}{\check{H}}$ is given by:

$$\overset{\circ}{\check{H}} = \overset{\circ}{\check{H}}_p e^{-\int_p^x \check{V}_{AA'}(z)\,dz^{AA'}} , \tag{4.93}$$

for any path from $p^{AA'}$ to $x^{AA'}$ within $\Sigma^{\perp}_{[P_\alpha]}$. Furthermore, solutions (at $x^{AA'}$) of (4.82) and (4.80) are given respectively by

$$\check{J}_{BB'} = -\overset{\circ}{\check{H}} \int_p^x \varepsilon_{AB}\tilde{\check{\phi}}_{A'B'}(z)\,dz^{AA'} \tag{4.94}$$

and

$$\check{F}_B = \frac{3}{2}\,\overset{\circ}{\check{H}} \int_p^x \varepsilon_{AB}\tilde{\check{\lambda}}_{A'}(z)\,dz^{AA'} . \tag{4.95}$$

Substituting (4.93), (4.94) and (4.95) in (4.92) we obtain

$$\begin{aligned}\check{H}(x^{AA'};\theta^A,\tilde{\theta}^{A'}) &= \overset{\circ}{\check{H}}_p e^{-\int_p^x \check{V}_{AA'}(z)\,dz^{AA'}} \Big\{1-\theta^B\tilde{\theta}^{B'}\int_p^x \varepsilon_{AB}\tilde{\check{\phi}}_{A'B'}(z)\,dz^{AA'} \\ &\qquad + [\tfrac{3}{2}\,\theta^B\int_p^x \varepsilon_{AB}\tilde{\lambda}_{A'}(z)\,dz^{AA'} + \tilde{\theta}^{A'}\theta^2\tilde{\check{\lambda}}_{A'}(x)]\tau\Big\} \\ &\equiv \overset{\circ}{\check{H}}_p \exp\{-f_p(x^{AA'};\theta^A,\tilde{\theta}^{A'})\}\end{aligned} \tag{4.96}$$

(See (2.50) in §2.3.7 for definition of *exp*.)

Similarly, starting with $\overset{\circ}{\check{H}}_q$ at $q^{AA'}$ on the β-plane (through $x^{AA'}$) $\Sigma^{\perp}_{[Q_\alpha]} \subset M$ corresponding to Q_α in $\hat{U}$ (another member of the cover of $P^{\perp} \subset \mathbb{P}T^*$), we obtain (at $x^{AA'}$)

$$\overset{\circ}{\check{H}}_q \exp\{-f_q(x^{AA'};\theta^A,\tilde{\theta}^{A'})\} . \tag{4.97}$$

On the overlap $U \cap \hat{U}$, (4.96) and (4.97) must agree; and so the two answers are related by

$$\mathring{H}_p = \mathring{H}_q \exp\{f_p(x^{AA'};\theta^A,\tilde{\theta}^{A'})-f_q(x^{AA'};\theta^A,\tilde{\theta}^{A'})\}$$

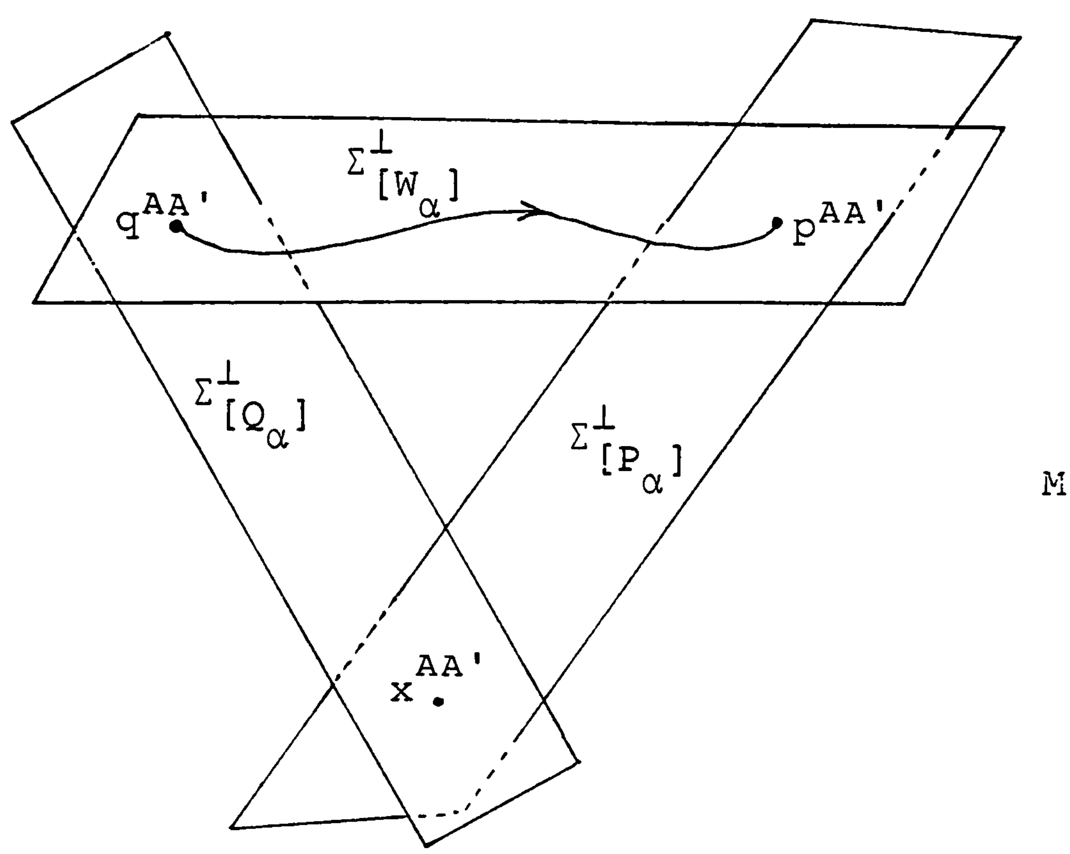
$$= \mathring{H}_q \exp\{-f(W_\alpha;\psi_j)\} \;, \qquad (4.98)$$

where

$$f(W_\alpha;\psi_j) = \int_q^p \{\check{V}_{AA'}(z) + \theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\check{\tilde{\phi}}_{A'B'}(z) - \frac{3}{2}\,\varepsilon_{AB}\check{\tilde{\lambda}}_{A'}(z)\tau\}dz^{AA'}$$

for some path from $q^{AA'}$ to $p^{AA'}$ on the β-plane $\Sigma^{\perp}_{[W_\alpha]}$ (see Fig. 4.5).

<u>Fig. 4.5</u>



Transition functions of a supervector bundle can be given in terms of path ordered integrals, as in the classical case.

A superconnection is said to be *integrable on a super light-ray* if it satisfies the constraint equations (3.36) (cf. light rays in $\mathbb{M}$ are 1-dimensional so a connection is always integrable on them). The following theorem characterizes such superconnections in terms of supervector bundles on open subsupermanifolds of $\mathbb{A}_{[N]}$.

Theorem 4.vi. There is a one-to-one correspondence between super vector bundles F, on $M_{[N]} \equiv (M, \mathcal{O}_{[N]})$, equipped with a superconnection ∇ satisfying the constraint equations; and supervector bundles E, on $A_{[N]} \equiv (A, \mathcal{O}_{[N]})$, 'trivial on all $Q_{x,\theta,\tilde{\theta}} \subset A_{[N]}$', i.e. such that $\sigma^* E$ is a trivial supervector bundle.

Proof

($\Rightarrow$) Construction of ∇ satisfying (3.36) given E s.t. $\sigma^* E$ is trivial

This construction is very similar to that of the previous theorem except that now a three set cover ($\{U_\alpha\}$, $\alpha = 1,2,3$) is required for the quadric $Q_x = \mathbb{CP}_1 \times \mathbb{CP}_1$ in A. For a detailed description of these sets, see for example [69].

Let $g(Z^\alpha, W_\alpha; \zeta^j, \psi_j)$ denote the expansion of a transition function of E. Then the expansion of the pull-back of g to $\sigma^* E$ can be written as

$$G(x^{AA'}, \pi_{A'}, \eta_A; \theta^A_j, \tilde{\theta}^{A'j}) = g((x^{AA'} - \theta^A_j \tilde{\theta}^{A'j})\pi_{A'}, \pi_{A'}, -(x^{AA'} + \theta^A_j \tilde{\theta}^{A'j})\eta_A, \eta_A; \tilde{\theta}^{A'j}\pi_{A'}, \theta^A_j \eta_A). \tag{4.100}$$

Since $\sigma^* E$ is trivial, G splits as

$$G = h_\alpha h_\beta^{-1} \qquad (\alpha = 1,2,3; \quad \beta = 1,2,3) \tag{4.101}$$

where h_α and h_β are $(r|s)\times(r|s)$ even invertible supermatrices whose entries are holomorphic superfunctions on U_α and U_β respectively.

Let

$$(D,T^j,\tilde{T}_k) = (\eta^A\pi^{A'}\partial_{AA'},\eta^A\partial^j_A,\pi^{A'}\tilde{\partial}_{A'k}) .$$

Then from Lemma 4.ic and (4.100)

$$(D,T^j,\tilde{T}_k)G = 0 .$$

Hence by (4.101)

$$[(D,T^j,\tilde{T}_k)h_\alpha]h^{-1}_\beta - h_\alpha[(D,T^j,\tilde{T}_k)h_\beta]h^{-2}_\beta = 0 ,$$

i.e. $$h^{-1}_\alpha(D,T^j,\tilde{T}_k)h_\alpha = [(D,T^j,\tilde{T}_k)h_\beta]h^{-1}_\beta . \qquad (4.102)$$

We can now apply once again the generalized Sparling-Ward splitting method. Both sides of (4.102) are globally defined in $\pi_{A'}$ and η_A, and h_α & h_β are both homogeneous of degree zero in $\pi_{A'}$ and η_A. Hence each side must be polynomials. That is

$$h^{-1}_\alpha(D,T^j,\tilde{T}_k)h_\beta = (\eta^A\pi^{A'}\Phi_{AA'},\eta^A\Psi^j_A,\pi^{A'}\tilde{\Psi}_{A'k}) , \qquad (4.103)$$

where $\Phi_{AA'}$, Ψ^j_A and $\tilde{\Psi}_{A'k}$ are the superpotentials.

The constraint equations (3.36) now follow from (4.103) and the fact that the differential operators $(D,T^j,\tilde{T}_k)$ satisfy the usual super commutation relations

$$\begin{aligned} &(T^j,T^k) = 0 , \quad (\tilde{T}_j,\tilde{T}_k) = 0 , \quad (T^j,\tilde{T}_k) = 2\delta^j{}_k \\ &[D,D] = 0 , \quad [D,T^j] = 0 , \quad [D,\tilde{T}_k] = 0 . \end{aligned} \qquad (4.104)$$

The procedure for deriving the constraint equations from (4.103) and (4.104) is identical to that used in Theorem 4.v, and it will not be reproduced; the only difference being that equations (4.65) and (4.68) do not arise here because of the different symmetries of $(D,T^j,\tilde{T}_k)$.

<u>(⇐) *Construction of* E *s.t.* σ^*E *is trivial given* ∇ *satisfying* *3.36* (Case N = 1 only).</u>

This construction is very similar to that of Theorem 4.v. Let H be a section of F, of the general form (4.75), and substitute the solutions of the N = 1 constraint equations (3.50) in

$$\pi^{A'}\tilde{\nabla}_{A'}H(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = 0 \tag{4.105a}$$

$$\eta^A\nabla_A H(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = 0 \tag{4.105b}$$

$$\pi^{A'}\eta^A\nabla_{AA'}H(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = 0 \ . \tag{4.105c}$$

The following equations are then obtained to each order:

$o(1)$:
$$\pi^{A'}\tilde{F}_{A'} = 0$$
$$\eta^A F_A = 0$$
$$\eta^A\pi^{A'}\mathring{\nabla}_{AA'}\mathring{H} = 0$$

$o(\theta)$:
$$\pi^{A'}(\mathring{\nabla}_{AA'}\mathring{H} - J_{AA'}) = 0$$
$$G = 0$$
$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}F_B - \tfrac{3}{2}\,\varepsilon_{AB}\mathring{H}\tilde{\lambda}_{A'}) = 0$$

$o(\tilde{\theta})$:
$$\tilde{G} = 0$$
$$\eta^A(\mathring{\nabla}_{AA'}\mathring{H} + J_{AA'}) = 0$$
$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}\tilde{F}_{B'} - \tfrac{3}{2}\,\varepsilon_{A'B'}\mathring{H}\lambda_A) = 0$$

$o(\theta^2)$:
$$\pi^{A'}(X_{A'} + \tfrac{1}{2}\,\mathring{\nabla}_{AA'}F^A - \mathring{H}\tilde{\lambda}_{A'}) = 0$$
$$\eta^A\tilde{X}_A = 0$$
$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}G + \tfrac{3}{2}\,F_A\,\tilde{\lambda}_{A'}) = 0$$

$o(\tilde{\theta}^2)$:
$$\pi^{A'}X_{A'} = 0$$
$$\eta^A(\tilde{X}_A + \tfrac{1}{2}\,\mathring{\nabla}_{AA'}\tilde{F}^{A'} - \mathring{H}\lambda_A) = 0$$
$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}\tilde{G} + \tfrac{3}{2}\,\tilde{F}_{A'}\lambda_A\ ) = 0$$

$$o(\theta\tilde{\theta}):\quad \pi^{A'}(-2\varepsilon_{A'B'}\tilde{X}_A + \mathring{\nabla}_{AA'}\tilde{F}_{B'} - \varepsilon_{A'B'}\mathring{H}\lambda_A) = 0$$

$$\eta^A(2\varepsilon_{AB}X_{A'} - \mathring{\nabla}_{AA'}F_B + \varepsilon_{AB}\mathring{H}\tilde{\lambda}_{A'}) = 0$$

$$\eta^A\pi^{A'}(\mathring{\nabla}_{AA'}J_{BB'} - 2\mathring{H}\{G_{ab} + \varepsilon_{AB}\varepsilon_{A'B'}f\} - \frac{3}{2}\varepsilon_{A'B'}F_B\lambda_A - \frac{3}{2}\varepsilon_{AB}\tilde{F}_{B'}\tilde{\lambda}_{A'}) = 0$$

$$o(\theta\tilde{\theta}^2):\quad \pi^{A'}(\mathring{\nabla}_{AA'}\tilde{G} - \frac{1}{2}\tilde{F}_{A'}\lambda_A) = 0$$

$$\eta^A(2\varepsilon_{AB}Y - \frac{1}{2}\mathring{\nabla}_{AA'}J_B^{A'} - \frac{1}{2}\varepsilon_{AB}\tilde{F}_{A'}\tilde{\lambda}^{A'} - F_B\lambda_A + \mathring{H}\{\phi_{AB} + \varepsilon_{AB}f\}) = 0$$

$$\eta^A\pi^{A'}(\mathring{\nabla}_{AA'}\tilde{X}_B - \tilde{F}^{B'}\{G_{ab} + \varepsilon_{AB}\varepsilon_{A'B'}f) + \frac{1}{2}\mathring{H}\{\varepsilon_{AB}\tilde{H}_{A'} - \frac{1}{2}\mathring{\nabla}_{A'(A}\lambda_{B)}\} + \frac{3}{2}J_{AA'}\lambda_B - \frac{3}{2}\varepsilon_{AB}\tilde{G}\tilde{\lambda}_{A'}) = 0$$

$$o(\theta^2\tilde{\theta}):\quad \pi^{A'}(2\varepsilon_{A'B'}Y + \frac{1}{2}\mathring{\nabla}_{AA'}J^A_{B'} - \frac{1}{2}\varepsilon_{A'B'}F_A\lambda^A - \tilde{F}_{B'}\tilde{\lambda}_{A'} + \mathring{H}\{\tilde{\phi}_{A'B'} - \varepsilon_{A'B'}f\}) = 0$$

$$\eta^A(\nabla_{AA'}G - F_A\tilde{\lambda}_{A'}) = 0$$

$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}X_{B'} + F^B\{G_{ab} + \varepsilon_{AB}\varepsilon_{A'B'}f\} + \frac{1}{2}\mathring{H}\{\varepsilon_{A'B'}H_A - \frac{1}{2}\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}\} + \frac{3}{2}J_{AA'}\tilde{\lambda}_{B'} - \frac{3}{2}\varepsilon_{A'B'}G\lambda_A) = 0$$

$$o(\theta^2\tilde{\theta}^2):\quad \pi^{A'}(\tilde{H}_{A'} - \frac{1}{2}\tilde{F}^{B'}\{\tilde{\phi}_{A'B'} - \varepsilon_{A'B'}f\} - \tilde{G}\tilde{\lambda}_{A'} + \frac{1}{4}J_{AA'}\lambda^A + \frac{1}{2}\mathring{\nabla}_{AA'}\tilde{X}^A) = 0$$

$$\eta^A(H_A - \frac{1}{2}F^B\{\phi_{AB} + \varepsilon_{AB}f\} - G\lambda_A + \frac{1}{4}J_{AA'}\tilde{\lambda}^{A'} + \frac{1}{2}\mathring{\nabla}_{AA'}X^{A'}) = 0$$

$$\pi^{A'}\eta^A(\mathring{\nabla}_{AA'}Y - \frac{\mathring{H}}{4}\{\mathring{\nabla}^b{*}G_{ab} + \frac{3}{2}(\lambda_A\tilde{\lambda}_{A'})\} + \frac{1}{2}J^b\{G_{ab} + \varepsilon_{AB}\varepsilon_{A'B'}f\} - \frac{1}{4}F^B\{\varepsilon_{AB}\tilde{H}_{A'} - \frac{1}{2}\mathring{\nabla}_{A'(A}\lambda_{B)}\} - \frac{1}{4}\tilde{F}^{B'}\{\varepsilon_{A'B'}H_A - \frac{1}{2}\mathring{\nabla}_{A(A'}\tilde{\lambda}_{B')}\} - \frac{3}{4}\tilde{X}_A\tilde{\lambda}_{A'} - \frac{3}{4}X_{A'}\lambda_A) = 0$$

An analysis of the above equations shows that we only need to solve the following system:

$$\mathring{\nabla}\mathring{H} = 0 \tag{4.106a}$$

$$\mathring{\nabla}F = 3\mathring{H}\lambda \tag{4.106b}$$

$$\mathring{\nabla}\tilde{F} = 3\mathring{H}\tilde{\lambda} \tag{4.106c}$$

$$4Y = F\lambda + \tilde{F}\tilde{\lambda} \tag{4.106d}$$

where

$$\mathring{\nabla} = \eta^A\pi^{A'}\mathring{\nabla}_{AA'}$$

$$F_A = \eta_A F$$

$$\tilde{F}_{A'} = \pi_{A'}\tilde{F}$$

$$\lambda = \eta_A\lambda^A$$

$$\tilde{\lambda} = \pi_{A'}\tilde{\lambda}^{A'}$$

Equations (4.106) are always integrable, as required. □

§4.6.3 Comments concerning the Classical Yang-Mills equations

We have seen in Chapter 2 that the classical Yang-Mills equations are included in the N = 3 constraint equations (see 3.54g). That is, solutions of (3.54) with the scalar and spinor fields set to zero correspond to solutions of the classical Yang-Mills equations. Assuming that Theorem 4.vi holds for N = 3 (which is inevitably true), the constraint equations are precisely the conditions that enable one to introduce a supervector bundle 'trivial on all $L_{x,\theta,\tilde{\theta}}$' on $A_{[3]}$. Consequently (as predicted by Witten [93]), we expect the classical Yang-Mills equations to be in one-to-one correspondence with the above mentioned supervector bundles in which the supercoordinate expansions of their transition functions depend on the odd supercoordinates, ψ_j and ζ^j, only through their product $\psi_j\zeta^j$. These supervector bundles are equivalent to vector bundles on $A_{(3)}$. The detailed calculations have been carried out by M.G. Eastwood [16b] in the abelian case. His method and computational lemmata can also be used to compute all the cohomology groups of Fig. 4.3, page 174.

§4.6.4 The 'superanalogue' of Green-Isenberg-Yasskin-Witten theorem

We will state a conjecture, attributed to Manin [82], and give a discussion based on the previous results stated in this chapter and Chapter 3.

For N = 1 and 2 we expect the conjecture to hold. However, in the N = 3 case, via Theorem 4.vi, the conjecture is equivalent to the statement that the constraint equations are the super Yang-Mills equations. In §3.2.6 we have seen that our definition of super Yang-Mills for N = 3 implies equations stronger than the constraint equations. Consequently further research is required in order to clarify the situation in this case.

The case N = 4 will not be discussed. In fact we expect more serious difficulties here because the additional constraint equation (3.37) does not seem to have a natural interpretation within supertwistor theory (see footnote in [93]).

Conjecture 4.vii. Let E be the supervector bundle over $A_{[N]}$ which corresponds (according to Theorem 4.vi) to a superconnection (satisfying the constraint equations) on a supervector bundle F over $M_{[N]}$. There is a one-to-one correspondence between solutions of the super Yang-Mills equations on $M_{[N]}$ for N = 1,2,3; and extensions of the supervector bundle E to the (3-N)-infinitesimal neighbourhoods of $\mathbb{A}_{[N]}$ in $\mathbb{PT}_{[N]} \times \mathbb{PT}^*_{[N]}$.

Preliminary discussion. Let F and E be of rank $r|s$. Recall that setting N=0 corresponds to replacing F and E by classical vector bundles $\mathring{F}$ and $\mathring{E}$ of rank r+s, i.e. locally, this corresponds to replacing $\mathcal{O}_{[N]}$ by $\mathcal{O}$ in $\mathcal{O}^{r|s}_{[N]} \equiv \mathcal{O}_{[N]} \otimes \mathbb{C}^{r|s}$. On the other hand, the super Yang-Mills equations for N = 0 become the classical Yang-Mills equations. Hence 4.vii for N = 0 becomes Theorem 4.iv (i.e. Green-Isenberg-Yasskin-Witten Theorem [45,93]). We will briefly review this theorem from the obstruction theory viewpoint (see for example [12]).

The obstruction and freedom to extending a vector bundle $\mathring{E}$ on $A_{(m-1)}$ to $A_{(m)} \equiv (A, \mathcal{O}_{(m)})$, lie in $H^2(A, End\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-m,-m))$ and $H^1(A, End\,\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-m,-m))$ respectively. This cohomology may be interpreted (see for example [17]) in terms of background coupled fields on M, roughly speaking as follows. First evaluate the cohomology groups in the absence of $End\mathring{E}$, (the results are stated in §4.5. , Fig. 4.3), and then replace $\partial_{AA'}$ by the connection $\mathring{\nabla}_{AA'}$ on the corresponding vector bundle $End\,\mathring{F}$.

Let us consider the obstruction and freedom to extending $\mathring{E}$ to the first, second and third infinitesimal neighbourhoods. Since $H^2(A, End\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-1,-1)) = 0$, $\mathring{E}$ extends to $A_{(1)}$. Moreover, if $\mathring{E}_{(1)}$ is a choice of extension then $H^1(A, End\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-1,-1)) \cong \Gamma(M, End\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-1][-1]')$ acts on $\mathring{E}_{(1)}$ to produce all possible first order extensions. The obstructions to the second order extensions lie in $H^2(A, End\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-2,-2)) \cong \Gamma(M, End\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-1][-1]')$, and it turns out (see for example [69]) that there is one choice of $\mathring{E}_{(1)}$ which admits a unique (since $H^1(A, End\,\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-2,-2)) = 0$) extension, $\mathring{E}_{(2)}$, to $A_{(2)}$. The third order obstructions must lie in $H^2(A, End\,\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-3,-3))$ $\cong \{\ker\nabla^{AA'}: \Gamma(M, End\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_{AA'}[-1][-1]') \to \Gamma(M, End\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-2][-2]')\}$; and (if they exist) they must be unique, since $H^1(A, End\mathring{E} \otimes_{\mathcal{O}} \mathcal{O}(-3,-3)) = 0$. Theorem 4.iv is the statement that the third order obstruction is the Yang-Mills current $j_a = \nabla^b * F_{ab}$.

Discussion of conjecture Guided by the previous discussion we expect the obstruction to extending E (on $A_{[N](m-1)}$) to $A_{[N](m)}$, to lie in $H^2(A, End\ E \otimes_{\mathcal{O}_{[N]}} \mathcal{O}_{[N]}(-m,-m))$; and $H^1(A, End\ E \otimes_{\mathcal{O}_{[N]}} \mathcal{O}_{[N]}(-m,-m))$ to act on the choice of extension $E_{(m)}$.

Case $N = 1$. From the definition of $\mathcal{O}_{[1]}$ (see 4.16a) together with (4.45) and (4.51) we know that

$$H^k(A,\mathcal{O}_{[1]}(-1,-1)) \cong \left\{\begin{matrix} H^k(A,\mathcal{O}(-2,-1)) \\ \oplus \\ H^k(A,\mathcal{O}(-1,-2)) \end{matrix}\right\} \oplus H^k(A,\mathcal{O}_{(1)}(-1,-1)) = 0,$$

for k = 1 and 2. Thus we expect $H^k(A, End\, E \otimes_{\mathcal{O}_{[1]}} \mathcal{O}_{[1]}(-1,-1)) = 0$, for k = 1 and 2. Hence E uniquely extends to $A_{[1](1)}$.

Let us now consider extensions to second order.

$$H^1(A,\mathcal{O}_{[1]}(-2,-2)) \cong \left\{\begin{matrix} H^1(A,\mathcal{O}(-3,-2)) \\ \oplus \\ H^1(A,\mathcal{O}(-2,-3)) \end{matrix}\right\} \oplus H^1(A,\mathcal{O}_{(1)}(-2,-2))$$

vanishes, and therefore we expect no freedom. The second order obstructions must lie in $H^2(A, End E \otimes_{\mathcal{O}_{[1]}} \mathcal{O}_{[1]}(-2,-2))$. Now consider this cohomology in the absence of $End E$:

$$H^2(A,\mathcal{O}_{[1]}(-2,-2)) \cong \underbrace{\left\{\begin{matrix} \Gamma(M,\mathcal{O}_A[-1][-1]') \\ \\ \Gamma(M,\mathcal{O}_{A'}[-1]'[-1] \end{matrix}\right\}}_{\bar{1}} \oplus \underbrace{R}_{\bar{0}} \qquad (4.107)$$

where $R \equiv H^2(A,\mathcal{O}_{(1)}(-2,-2))$ is determined by the exact sequence

$$0 \to \{\ker\partial^{AA'} : \Gamma(M,\mathcal{O}_{AA'}[-1][-1]') \to \Gamma(M,\mathcal{O}[-2][-2]')\} \to R \to \Gamma(M,\mathcal{O}[-1][-1]') \to 0 \qquad (4.108)$$

(see (4.41) in Fig. 4.2).

In the presence of $End\ E$ we suspect that one no longer has the direct sum symbol between the odd and even parts of (4.107). Instead we expect the following exact sequence

$$0 \to \{\ker \mathring{\nabla}^{AA'} : \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_{AA'}[-1][-1]') \to \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-2][-2]')\} \xrightarrow{\beta} H^2(A, \mathit{End}\,E \otimes_{\mathcal{O}_{[1]}} \mathcal{O}_{[1]}(-2,-2)) \xrightarrow{\alpha} \left\{\begin{matrix} \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_A[-1][-1]') \\ \oplus \\ \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-1][-1]') \\ \oplus \\ \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_{A'}[-1][-1]') \end{matrix}\right\} \to 0 \qquad (4.109)$$

To see this, suppose that $\omega \in H^2(A, \mathit{End}\,E \otimes_{\mathcal{O}_{[1]}} \mathcal{O}_{[1]}(-2,-2))$ is the obstruction. Then the image of ω under α may be regarded as the *primary obstruction*; and the obvious candidates (cf.(3.52)) are

$$\mathring{\nabla}_A^{A'}\mathring{\tilde{W}}_{A'} \in \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_A[-1][-1]')$$

$$\mathring{f} \in \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}[-1]'[-1])$$

$$\mathring{\nabla}_{A'}^{A}\mathring{W}_A \in \Gamma(M, \mathit{End}\,\mathring{F} \otimes_{\mathcal{O}} \mathcal{O}_{A'}[-1][-1]') \;.$$

If these primary obstruction candidates vanish, then

$$j_a \equiv \mathring{\nabla}^b {*}\mathring{F}_{ab} + \tfrac{1}{2}(\mathring{W}_A, \mathring{\tilde{W}}_{A'})$$

is annihilated by $\mathring{\nabla}^b$ (this can be deduced from the super Jacobi identities); and so j_a is the obvious candidate for the *secondary obstruction*.

Case $N = 2$ Consider $H^k(A, \mathit{End}\,E \otimes_{\mathcal{O}_{[2]}} \mathcal{O}_{[2]}(-1,-1))$ in the absence of $\mathit{End}\,E$:

$$\underbrace{\left\{\begin{matrix} H^k(A, \mathcal{O}_{(1)j}(-2,-1)) \\ \oplus \\ H^k(A, \mathcal{O}^{\,j}_{(1)}(-1,-2)) \end{matrix}\right\}}_{\bar{1}} \oplus \underbrace{\left\{\begin{matrix} H^k(A, \mathcal{O}_{jk}(-3,-1)) \\ \oplus \\ H^k(A, \text{tr free } \mathcal{O}^j_k(-2,-2)) \\ \oplus \\ H^k(A, \mathcal{O}^{jk}(-1,-3)) \end{matrix}\right\} \oplus H^k(A, \mathcal{O}_{(2)}(-1,-1))}_{\bar{0}} .$$

For k = 1 all the above cohomology groups vanish; while for k = 2 they are isomorphic to

$$\underbrace{\left\{\begin{matrix}\Gamma(M,\mathcal{O}_{jA}[-1][-1]') \\ \oplus \\ \Gamma(M,\mathcal{O}^{j}_{A'}[-1][-1]')\end{matrix}\right\}}_{\bar{1}} \oplus \underbrace{\left\{\begin{matrix}\Gamma(M,\varepsilon_{jk}\mathcal{O}[-2][-2]') \\ \oplus \\ \Gamma(M,\text{tr free } \mathcal{O}^{k}_{j}[-1][-1]') \\ \oplus \\ \Gamma(M,\varepsilon^{jk}\mathcal{O}[-2][-2]')\end{matrix}\right\}}_{\bar{0}} \oplus R \quad ,$$

where R is also given by (4.108).

Thus we expect the obstructions to extending E on $A_{[2]}$ to $A_{[2](1)}$ to be unique, and lie in $H^2(A, End E \otimes_{\mathcal{O}_{[2]}} \mathcal{O}_{[2]}(-1,-1))$. From (3.56) one can immediately see which are the obvious candidates for the obstructions. However, it is not altogether clear how to classify these obstructions since the situation here is much more complicated than for N = 1. We shall not speculate any further in this case.

Case N = 3 Recall that $H^1(A,\mathcal{O}_{[3]})$ is already isomorphic to a decoupled system of field equations (see (4.54)). Hence we expect the presence of the supervector bundle E to yield coupled field equations at the zeroth order. □

This completes our attempt to present a full supertwistor description of super Yang-Mills theory.

APPENDIX

Proof of Theorem 3.1

Firstly we show that the gauge condition

$$\theta^A\Psi_A + \tilde{\theta}^{A'}\tilde{\Psi}_{A'} = 0 \tag{A1}$$

can always be imposed. To do this write the θ-expansion of an arbitrary $g \in GL(r|s;\mathcal{O}_{[1]})$ as

$$g(x^{AA'};\theta^A_j,\tilde{\theta}^{A'}) \equiv \phi + \theta^B p_B + \tilde{\theta}^{B'}\tilde{p}_{B'} + \theta^B\tilde{\theta}^{B'}m_{BB'} + \theta^2 q + \tilde{\theta}^2\tilde{q}$$
$$+ \tilde{\theta}^2\theta^B\tilde{r}_B + \tilde{\theta}^{B'}\theta^2 r_{B'} + \theta^2\tilde{\theta}^2 s\ . \tag{A2}$$

Then

$$\partial_A g = p_A + \tilde{\theta}^{B'}m_{AB'} + 2\theta^B\varepsilon_{AB}q + \tilde{\theta}^2\tilde{r}_A - 2\theta^B\tilde{\theta}^{B'}\varepsilon_{AB}r_{B'} + 2\theta^B\tilde{\theta}^2\varepsilon_{AB}s$$
$$+ \tilde{\theta}^{B'}\partial_{AB'}\phi + \frac{\tilde{\theta}^2\theta^B}{2}\,\partial_{B'A}m_B^{B'} - \theta^B\tilde{\theta}^{B'}\partial_{B'A}p_B + \frac{\tilde{\theta}^2}{2}\,\partial_{B'A}\tilde{p}^{B'}$$
$$+ \theta^2\tilde{\theta}^{B'}\partial_{B'A}q + \frac{\tilde{\theta}^2\theta^2}{2}\,\partial_{B'A}r^{B'}\ .$$

$$= p_A + \tilde{\theta}^{B'}\{m_{AB'} + \partial_{AB'}\phi\} + \theta^B\{2\varepsilon_{AB}q\} - \theta^B\theta^{B'}\{2\varepsilon_{AB}r_{B'} + \partial_{B'A}p_B\}$$
$$+ \tilde{\theta}^2\{\tilde{r}_A + \tfrac{1}{2}\,\partial_{AA'}\tilde{p}^{A'}\} + \tilde{\theta}^2\theta^B\{2\varepsilon_{AB}s - \tfrac{1}{2}\,\partial_{B'A}m_B^{B'}\}$$
$$+ \theta^2\tilde{\theta}^{B'}\partial_{B'A}q + \frac{\theta^2\tilde{\theta}^2}{2}\,\partial_{B'A}r^{B'}\ . \tag{A3}$$

$$\tilde{\partial}_{A'}g = \tilde{p}_{A'} - \theta^B m_{BA'} + 2\tilde{\theta}^{B'}\varepsilon_{A'B'}\tilde{q} - 2\theta^B\tilde{\theta}^{B'}\varepsilon_{A'B'}\tilde{r}_B$$
$$+ \theta^2 r_{A'} + 2\theta^2\theta^{B'}\varepsilon_{A'B'}s + \theta^B\partial_{BA'}\phi + \frac{\theta^2}{2}\,\partial_{AA'}p^A$$
$$+ \theta^B\theta^{B'}\partial_{BA'}\tilde{p}_{B'} + \frac{\theta^2\theta^{B'}}{2}\,\partial_{A'A}m^A_{B'} + \tilde{\theta}^2\theta^B\partial_{BA'}\tilde{q} + \tilde{\theta}^2\theta^2\partial_{A'A}\tilde{r}^A$$

$$= \tilde{p}_{B'} - \theta^B\{m_{BA'} - \partial_{BA'}\phi\} + \tilde{\theta}^{B'}\{2\varepsilon_{A'B'}\tilde{q}\} - \theta^B\tilde{\theta}^{B'}\{2\varepsilon_{A'B'}\tilde{r}_B - \partial_{BA'}\tilde{p}_{B'}\}$$

$$+ \theta^2\{r_{A'} + \tfrac{1}{2}\partial_{AA'}p^A\} + \theta^2\tilde{\theta}^{B'}\{2\varepsilon_{A'B'}s + \tfrac{1}{2}\partial_{A'A}m^A_{B'}\}$$

$$+ \tilde{\theta}^2\theta^B\partial_{BA'}\tilde{q} + \frac{\theta^2\tilde{\theta}^2}{2}\partial_{A'A}\tilde{r}^A . \qquad \text{(A4)}$$

Now impose the gauge condition (A1) order by order to the following θ-expansions of Ψ_A and $\tilde{\Psi}_{A'}$:

$$\Psi_A(x^{AA'};\theta^A,\tilde{\theta}^{A'}) \equiv \pi_A + \theta^B\{p_{AB}+\varepsilon_{AB}h\} + \tilde{\theta}^{B'}V_{AB'} + \theta^B\tilde{\theta}^{B'}\{\Theta_{ABB'}+\varepsilon_{AB}\lambda_{B'}\}$$

$$+\theta^2G_A + \tilde{\theta}^2F_A + \theta^2\tilde{\theta}^{B'}G_{AB'} + \tilde{\theta}^2\theta^B\{\phi_{AB}+\varepsilon_{AB}f\} + \tilde{\theta}^2\theta^2H_A \qquad \text{(A5)}$$

($p_{AB} = p_{BA}$ and $\Theta_{ABB'} = \Theta_{BAB'}$)

$$\tilde{\Psi}_{A'}(x^{AA'};\theta^A,\tilde{\theta}^{A'}) \equiv \tilde{\pi}_{A'} + \tilde{\theta}^{B'}\{\tilde{p}_{A'B'}+\varepsilon_{A'B'}\tilde{h}\} + \theta^B\tilde{V}_{BA'}$$

$$+\theta^B\theta^{B'}\{\tilde{\Theta}_{A'B'B}+\varepsilon_{A'B'}\tilde{\lambda}_B\} + \tilde{\theta}^2\tilde{G}_{A'} + \theta^2\tilde{F}_{A'}$$

$$+\tilde{\theta}^2\theta^B\tilde{G}_{A'B} + \theta^2\tilde{\theta}^{B'}\{\tilde{\phi}_{A'B'}+\varepsilon_{A'B'}\tilde{f}\} + \theta^2\tilde{\theta}^2\tilde{H}_A \qquad \text{(A6)}$$

($\tilde{p}_{A'B'} = \tilde{p}_{B'A'}$ and $\tilde{\Theta}_{A'B'A} = \tilde{\Theta}_{B'A'A}$).

The equations implied by (A1) on the coefficients of (A5) and (A6) are stated on the left hand column of (A7). The right hand column of (A7) shows the coefficients of (A2) which must be fixed to each order.

$\pi_A = 0 = \tilde{\pi}_{A'}$	$p_A,\ \tilde{p}_{A'}$	$o(\theta,\tilde{\theta})$	
$h = 0 = \tilde{h}$	$q,\ \tilde{q}$	$o(\theta^2,\tilde{\theta}^2)$	
$V_{AA'} = \tilde{V}_{AA'}$	$m_{AA'} = \partial_{AA'}\phi$	$o(\theta\tilde{\theta})$	(A7)
$\lambda_{A'} = -\tilde{F}_{A'}$	$r_{A'}$	$o(\theta^2\tilde{\theta})$	
$F_A = -\tilde{\lambda}_A$	$\tilde{r}_A$	$o(\theta\tilde{\theta}^2)$	
$f = -\tilde{f}$	$s = -\frac{1}{8}\partial_a m^a$	$o(\theta^2\tilde{\theta}^2)$	

Hence the effect of this gauge condition is to restrict all further gauge freedom to elements of $GL(r|s;\mathcal{O}_{[1]})$ whose θ-expansion is of the form

$$\phi + \theta^B\tilde{\theta}^{B'} m_{BB'} + \theta^2\tilde{\theta}^2 s \ ,$$

where $m_{BB'} = \partial_{BB'}\phi$ and $s = -\frac{1}{8}\,\partial^{BB'}\partial_{BB'}\phi$.

We now proceed to solve the constraint equations in this gauge. From (A7) one can rewrite the Θ-expansions (A5) and (A6) as follows

$$\begin{aligned}\Psi_A(x^{AA'};\theta^A,\tilde{\theta}^{A'}) = {}& \theta^B p_{AB} + \tilde{\theta}^{B'} V_{AB'} + \theta^B\tilde{\theta}^{B'}\{\Theta_{ABB'} + \varepsilon_{AB}\tilde{\lambda}_{B'}\} \\ & + \theta^2 G_A - \tilde{\theta}^2\lambda_A + \theta^2\tilde{\theta}^{B'} G_{AB'} \\ & + \theta^B\tilde{\theta}^2\{\phi_{AB} + \varepsilon_{AB} f\} + \theta^2\tilde{\theta}^2 H_A \end{aligned} \tag{A8}$$

$$\begin{aligned}\tilde{\Psi}_{A'}(x^{AA'};\theta^A,\theta^{A'}) = {}& \tilde{\theta}^{B'}\tilde{p}_{A'B'} + \theta^B V_{A'B'} - \theta^B\tilde{\theta}^{B'}\{-\tilde{\Theta}_{A'B'B} + \varepsilon_{A'B'}\lambda_B\} \\ & + \tilde{\theta}^2\tilde{G}_{A'} - \theta^2\tilde{\lambda}_{A'} + \tilde{\theta}^2\theta^B\tilde{G}_{A'B} \\ & + \theta^2\tilde{\theta}^{B'}\{\tilde{\phi}_{A'B'} - \varepsilon_{A'B'} f\} + \theta^2\tilde{\theta}^2\tilde{H}_{A'} \ . \end{aligned} \tag{A9}$$

Consider the first constraint equation (3.49a)

$$\begin{aligned}\partial_{(C}\Psi_{A)} = {}& \left\{\frac{\partial}{\partial\theta^C} + \tilde{\theta}^{C'}\partial_{C'C}\right\}\Psi_A \\ = {}& p_{AC} - \theta^B\tilde{\theta}^{C'}\partial_{C'(C}p_{A)B} + \tilde{\theta}^{C'}\tilde{\theta}^{B'}\partial_{C'(C}V_{A)B'} + \tilde{\theta}^{B'}\Theta_{ACB'} \\ & - \theta^B\tilde{\theta}^{C'}\tilde{\theta}^{B'}\{\partial_{C'(C}\Theta_{A)BB'} + \partial_{C'(C}\varepsilon_{A)B}\tilde{\lambda}_{B'}\} - 2\theta^B\varepsilon_{B(C}G_{A)} \\ & + \theta^2\tilde{\theta}^{C'}\partial_{C'(C}G_{A)} - 2\theta^B\tilde{\theta}^{B'}\varepsilon_{B(C}G_{A)B'} + \theta^2\tilde{\theta}^{C'}\tilde{\theta}^{B'}\partial_{C'(C}G_{A)B'} \\ & + \tilde{\theta}^2\phi_{AC} - 2\theta^B\tilde{\theta}^2\varepsilon_{B(C}H_{A)} \ . \end{aligned} \tag{A10}$$

$$(\Psi_C,\Psi_A) = 2(\theta^B P_{BC},\tilde\theta^{B'}V_{AB'})+(\theta^B P_{BC},\theta^D P_{DA})+(\tilde\theta^{B'}V_{B'C},\tilde\theta^{D'}V_{AD'})$$
$$-2(\theta^B P_{B(C},\tilde\theta^2\lambda_{A)})+2(\tilde\theta^{B'}V_{B'(C},\theta^B\tilde\theta^{D'}\{\Theta_{A)BD'}+\varepsilon_{A)B}\tilde\lambda_{D'}\})$$
$$+2(\theta^B P_{B(C},\theta^C\tilde\theta^{C'}\{\Theta_{A)BB'}+\varepsilon_{A)B}\tilde\lambda_{B'}\})+2(\tilde\theta^{B'}V_{B'(C},\theta^2 G_{A)})$$
$$+2(\theta^B P_{B(C},\theta^D\tilde\theta^2\{\phi_{A)B}+\varepsilon_{A)B}f\})+2(\tilde\theta^{B'}V_{B'(C},\theta^2\tilde\theta^{D'}G_{A)D'})$$
$$+(\theta^B\tilde\theta^{B'}\{\Theta_{B'CB}+\varepsilon_{CB}\tilde\lambda_{B'}\},\theta^D\tilde\theta^{D'}\{\Theta_{ADD'}+\varepsilon_{AD}\tilde\lambda_{D'}\})-2(\theta^2 G_C,\tilde\theta^2 G_A) \tag{A11}$$

Note: $(\tilde\theta^{B'}V_{B'C},\tilde\theta^{D'}V_{AD'}) = \tilde\theta^{B'}\tilde\theta^{D'}[V_{B'C},V_{AD'}]$

$(\tilde\theta^{B'}V_{B'C},\theta^B\tilde\theta^{D'}\varepsilon_{AB}\tilde\lambda_{D'}) = -\theta^B\tilde\theta^{B'}\tilde\theta^{D'}[V_{CB'},\varepsilon_{AB}\tilde\lambda_{D'}]$

etc. (see §3.24).

Adding (A10) to half of (A11) and equating to zero gives

$P_{AC} = 0$ $\qquad o(1)$

$G_A = 0$ $\qquad o(\theta)$

$\Theta_{ABB'} = 0$ $\qquad o(\tilde\theta)$

$G_{AA'} = 0$ $\qquad o(\theta\tilde\theta)$

$\phi_{AC} + \frac{1}{2}\partial_{C'(C}V^{C'}_{A)} + \frac{1}{4}[V_{C'C},V^{C'}_A] = 0$ $\qquad o(\tilde\theta^2)$

$-\frac{1}{2}\partial_{C'(C}\varepsilon_{A)B}\tilde\lambda^{C'} - \frac{1}{2}[V_{B'(C},\varepsilon_{A)B}\tilde\lambda^{B'}]-2\varepsilon_{B(C}H_{A)} = 0.$ $\qquad o(\theta\tilde\theta^2)$

This establishes (3.50a).

Similarly for the second constraint equation (3.49b). We shall only write the terms which in the end will not be forced to vanish:

$$\tilde{\partial}_{(C'}\tilde{\Psi}_{A')} = \left\{\frac{\partial}{\partial\tilde{\theta}^{C'}} + \theta^C \frac{\partial}{\partial x^{CC'}}\right\}\Psi_{A'}$$

$$= \theta^C\theta^B\partial_{C(C'}V_{A')B} + \theta^2\tilde{\phi}_{A'B'}$$

$$+\theta^C\theta^B\tilde{\theta}^{B'}\varepsilon_{B'(A'}\partial_{C')C}\lambda_B - 2\theta^2\tilde{\theta}^{B'}\varepsilon_{B'(A'}\tilde{H}_{C')}$$

$$+ \ldots \tag{A12}$$

$$(\tilde{\Psi}_{B'},\tilde{\Psi}_{A'}) = \frac{\theta^2}{2}[V_{CC'},V^C_{A'}]$$

$$-\theta^2\tilde{\theta}^{B'}[V_{C(C'},\lambda^C]\varepsilon_{A')B'}$$

$$+ \ldots \quad . \tag{A13}$$

Finally we construct $\Phi_{AA'}(x^b;\theta^B,\tilde{\theta}^{B'})$ from (3.49c):

$$\partial_A\tilde{\Psi}_{A'} = \left\{\frac{\partial}{\partial\theta^A} + \tilde{\theta}^{C'}\partial_{C'A}\right\}\tilde{\Psi}_{A'}$$

$$= V_{AA'} - \theta^B\tilde{\theta}^{B'}\partial_{B'A}V_{A'B} - \tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A + \theta^B\frac{\tilde{\theta}^2}{2}\partial_{A'A}\lambda_B - 2\theta^B\varepsilon_{AB}\tilde{\lambda}_{A'}$$

$$-\tilde{\theta}^{B'}\theta^2\partial_{B'A}\tilde{\lambda}_{A'} + 2\theta^B\tilde{\theta}^{B'}\varepsilon_{AB}(\tilde{\phi}_{A'B'} - \varepsilon_{A'B'}f)$$

$$+ \frac{\theta^2\tilde{\theta}^2}{2}(-\partial^{B'}_A\tilde{\phi}_{A'B'} - \partial_{AA'}f) + 2\tilde{\theta}^2\theta^B\varepsilon_{AB}\tilde{H}_{A'} \tag{A14}$$

$$\tilde{\partial}_{A'}\Psi_A = \left\{\frac{\partial}{\partial\tilde{\theta}^{A'}} + \theta^C\partial_{CA'}\right\}\Psi_A$$

$$= V_{AA'} + \theta^B\tilde{\theta}^{B'}\partial_{BA'}V_{AB'} - \theta^B\varepsilon_{AB}\tilde{\lambda}_{A'} + \frac{\tilde{\theta}^{B'}\theta^2}{2}\partial_{AA'}\tilde{\lambda}_{B'} - 2\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A$$

$$-\theta^B\tilde{\theta}^2\partial_{BA'}\lambda_A - 2\theta^B\tilde{\theta}^{B'}\varepsilon_{A'B'}(\phi_{AB} + \varepsilon_{AB}f)$$

$$+ \frac{\tilde{\theta}^2\theta^2}{2}(-\partial^B_{A'}\phi_{AB} + \partial_{AA'}f) + 2\tilde{\theta}^{B'}\theta^2\varepsilon_{A'B'}H_A \tag{A15}$$

$$(\Psi_A,\tilde{\Psi}_{A'}) = (\theta^{B'}V_{AB'},\theta^C V_{A'C}) + (-\theta^2\tilde{\lambda}_{A'},\tilde{\theta}^{B'}V_{AB'})$$
$$+ (\theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\tilde{\lambda}_{B'},\theta^C V_{A'C}) + (\theta^{B'}V_{AB'},-\theta^B\tilde{\theta}^{C'}\varepsilon_{A'C'}\lambda_B)$$
$$+ (-\tilde{\theta}^2\lambda_A,\theta^B V_{A'B}) + (\theta^B\tilde{\theta}^2\{\phi_{AB}+\varepsilon_{AB}f\},\theta^C V_{A'C})$$
$$+ (\theta^{B'}V_{AB'},\theta^2\tilde{\theta}^{C'}\{\phi_{A'C'}-\varepsilon_{A'C'}f\}) + (-\theta^2\lambda_A,-\tilde{\theta}^2\tilde{\lambda}_{A'})$$
$$+ (\theta^B\tilde{\theta}^{B'}\varepsilon_{AB}\tilde{\lambda}_{B'},\theta^C\tilde{\theta}^{C'}\varepsilon_{A'C'}\lambda_C)$$
$$= -\theta^B\tilde{\theta}^{B'}[V_{AB'},V_{A'B}] + \theta^2\tilde{\theta}^{B'}[\tilde{\lambda}_{A'},V_{AB'}] - \frac{\theta^2\tilde{\theta}^{B'}}{2}\varepsilon_A{}^C[\lambda_{B'},V_{A'C}]$$
$$+ \frac{\theta^B\tilde{\theta}^2}{2}[V_{AA'},\lambda_B] + \tilde{\theta}^2\theta^B[\lambda_A,V_{A'B}] + \frac{\tilde{\theta}^2\theta^2}{2}[\phi_{AB}+\varepsilon_{AB}f,V^B_{A'}]$$
$$+ \frac{\theta^2\tilde{\theta}^2}{2}[V_{AB'},\phi^{B'}_{A'}-\varepsilon_{A'}{}^{B'}f] + \tilde{\theta}^2\theta^2(\lambda_A,\tilde{\lambda}_{A'}) - \frac{\theta^2\tilde{\theta}^2}{4}(\lambda_A,\tilde{\lambda}_{A'})$$
$$= -\theta^B\tilde{\theta}^{B'}[V_{AB'},V_{A'B}] + \theta^2\tilde{\theta}^{B'}\{-[V_{AB'},\tilde{\lambda}_{A'}] + [\tfrac{1}{2}V_{AA'},\tilde{\lambda}_{B'}]\}$$
$$+\theta^B\tilde{\theta}^2\{-[V_{A'B},\lambda_A] + \tfrac{1}{2}[V_{A'A},\lambda_B]\}$$
$$- \frac{\theta^2\tilde{\theta}^2}{2}\{[V^B_{A'},\phi_{AB}+\varepsilon_{AB}f] + [V^{B'}_A,\phi_{A'B'}-\varepsilon_{A'B'}f] - \tfrac{3}{2}(\lambda_A,\tilde{\lambda}_{A'})\}. \tag{A16}$$

Adding (A12), (A13) and (A14) one obtains

$$\Phi_{AA'}(x^{BB'};\theta^B,\tilde{\theta}^{B'}) = V_{AA'} - \tfrac{3}{2}\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A - \tfrac{3}{2}\theta^B\varepsilon_{AB}\tilde{\lambda}_{A'}$$
$$+2\theta^B\tilde{\theta}^{B'}\{\varepsilon_{AB}\tilde{\phi}_{A'B'}-\varepsilon_{A'B'}\phi_{AB}-\varepsilon_{AB}\varepsilon_{A'B'}f\}$$
$$+ \frac{\theta^2\tilde{\theta}^{B'}}{2}\{-\mathring{\nabla}_{AB'}\tilde{\lambda}_{A'} + \tfrac{1}{2}\mathring{\nabla}_{AA'}\tilde{\lambda}_{B'} + 2\varepsilon_{A'B'}H_A\}$$
$$+ \frac{\tilde{\theta}^2\theta^B}{2}\{-\mathring{\nabla}_{BA'}\lambda_A + \tfrac{1}{2}\mathring{\nabla}_{AA'}\lambda_B + 2\varepsilon_{AB}\tilde{H}_{A'}\}$$
$$+ \frac{\theta^2\tilde{\theta}^2}{4}\{-\partial^{B'}_A\tilde{\phi}_{A'B'}-\partial^B_{A'}\phi_{AB} - [V^{B'}_A,\tilde{\phi}_{A'B'}] - [V^B_{A'},\phi_{AB}]$$
$$+ \tfrac{3}{2}(\lambda_A,\tilde{\lambda}_{A'})\}, \tag{A17}$$

establishing (3.49c). □

Proof of Theorem 3.ii. Construction of $W_A(x^{BB'};\theta^B,\tilde{\theta}^{B'})$

$$
\begin{aligned}
\tilde{\partial}_{A'}\Phi_A^{A'} &= \{\frac{\partial}{\partial\tilde{\theta}^{A'}} + \theta^B\partial/\partial x^{BA'}\}\Phi_A^{A'} \\
&= \theta^B\partial_{BA'}V_A^{A'} + \theta^B\{-4\phi_{AB} - 4\varepsilon_{AB}f\} \\
&\quad + \theta^2\tilde{\theta}^{B'}\{-\partial^B_{A'}\phi_{AB} - \partial_A^{A'}\tilde{\phi}_{A'B'} + \partial_{AB'}f\} \\
&\quad + \frac{\theta^2\tilde{\theta}^{B'}}{2}\{-\mathring{\nabla}^B_{B'}\phi_{AB} - \mathring{\nabla}_A^{A'}\tilde{\phi}_{A'B'} + \frac{3}{2}(\lambda_B,\tilde{\lambda}_{A'})\} \\
&\quad + 3\lambda_A + \frac{3}{2}\theta^B\tilde{\theta}^{B'}\partial_{BB'}\lambda_A \\
&\quad - \theta^B\tilde{\theta}^{B'}\{\varepsilon_{AB}\tilde{H}_{B'} - \frac{1}{2}\mathring{\nabla}_{B'(A}\lambda_{B)}\} \\
&\quad + \frac{\theta^2\tilde{\theta}^2}{4}\{\partial_{AA'}\tilde{H}^{A'} + \frac{1}{2}\partial^B_{A'}\mathring{\nabla}^{A'}_{(A}\lambda_{B)}\} \\
&\quad - \frac{3}{4}\theta^2\partial_{AA'}\tilde{\lambda}^{A'} - \theta^2 H_A
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
\partial_{AA'}\tilde{\Psi}^{A'} &= \theta^B\partial_{AA'}V_B^{A'} + \theta^B\tilde{\theta}^{B'}\partial_{B'A}\lambda_B - \theta^2\partial_{A'A}\tilde{\lambda}^{A'} \\
&\quad + \theta^2\tilde{\theta}^{B'}\{-\partial_A^{A'}\tilde{\phi}_{A'B'} + \partial_{B'A}f\} + \theta^2\tilde{\theta}^2\partial_{AA'}\tilde{H}^{A'}
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
[\Phi_{AA'},\tilde{\Psi}^{A'}] &= \theta^B[V_{AA'},V_B^{A'}] \\
&\quad +\theta^B\tilde{\theta}^{B'}[V_{AB'},\lambda_B]+[-\frac{3}{2}\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A,\theta^C V_C^{A'}] \\
&\quad -\theta^2[V_{AA'},\tilde{\lambda}^{A'}]+[-\frac{3}{2}\theta^B\varepsilon_{AB}\lambda_{A'},\theta^C V_C^{A'}] \\
&\quad +\theta^2\tilde{\theta}^{B'}[V_{AA'},\phi_{B'}^{A'}-\varepsilon^{A'}{}_{B'}f] \\
&\quad +2[\theta^B\tilde{\theta}^B\{*F_{ab}-\varepsilon_{AB}\varepsilon_{A'B'}[f,V_{A'B}]\},\theta^C V_C^{A'}] \\
&\quad +[-\frac{3}{2}\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_A,-\theta^2\tilde{\lambda}^{A'}]+[-\frac{3}{2}\theta^B\varepsilon_{AB}\tilde{\lambda}_{A'},-\theta^C\tilde{\theta}^{B'}\varepsilon^{A'}{}_{B'}\lambda_C] \\
&\quad +[-\frac{3}{2}\tilde{\theta}^{C'}\varepsilon_{A'C'}\lambda_A,-\theta^B\tilde{\theta}^{B'}\varepsilon^{A'}{}_{B'}\lambda_B] \\
&\quad +\theta^2\tilde{\theta}^2[V_{AA'},\tilde{H}^{A'}]+2[\theta^B\tilde{\theta}^{B'}\{*F_{ab}-\varepsilon_{AB}\varepsilon_{A'B'}f\},-\theta^C\tilde{\theta}^{C'}\varepsilon^{A'}{}_{C'}\lambda_C]
\end{aligned}
$$

$$+[-\frac{3}{2}\tilde{\theta}^{B'}\varepsilon_{A'B'}\lambda_{A'},\theta^2\tilde{\theta}^C\{\phi^{A'}_{C'}-\varepsilon^{A'}\varepsilon^{A'}{}_{C'}f]\}$$

$$+[\frac{\tilde{\theta}^2\theta^B}{2}\{\varepsilon_{AB}\tilde{H}_{A'}-\frac{1}{8}\mathring{\nabla}_{A'(A}\lambda_{B)}\},\theta^C V^{A'}_C]$$

$$=\theta^B[V_{AA'},V^{A'}_B]+\theta^B\tilde{\theta}^B\{[V_{AB'},\lambda_B]+\frac{3}{2}[V_{BB'},\lambda_A]\}$$

$$+\theta^2\{-\frac{7}{4}[V_{AA'},\tilde{\lambda}^{A'}]\}-\frac{3}{2}\tilde{\theta}^2\theta^B(\lambda_A,\lambda_B)$$

$$+\theta^2\tilde{\theta}^{B'}\{-\frac{3}{2}[V^{A'}_A,\tilde{\phi}_{A'B'}]-\frac{1}{2}[V^B_{B'},\phi_{AB}]+\frac{9}{4}(\lambda_A,\tilde{\lambda}_{B'})\}$$

$$+\theta^2\tilde{\theta}^2\{\frac{5}{4}[V_{AA'},\tilde{H}^{A'}]-\frac{1}{8}[V^{A'B},\mathring{\nabla}_{A'(A}\lambda_{B)}]+[\lambda_A,f]-[\phi_{AB},\lambda^B]\}$$

$$-\frac{3}{2}\tilde{\theta}^2\theta^B(\lambda_A,\lambda_B)\,. \qquad \text{(A20)}$$

Adding (A18), (A19) and (A20)

$$W_A(x^{BB'};\theta^B,\tilde{\theta}^{B'})=3\lambda_A+\theta^B\{2\partial_{A'(B}V^{A'}_{A)}+[V_{A'A},V^{A'}_B]-4\phi_{AB}-\varepsilon_{AB}f\}$$

$$+\theta^B\tilde{\theta}^{B'}\{-2\varepsilon_{AB}\tilde{H}_{B'}+3\mathring{\nabla}_{B'(A}\lambda_{B)}\}-8\theta^2 H_A-\frac{3}{2}\tilde{\theta}^2\theta^B(\lambda_A,\lambda_B)\}$$

$$+\theta^2\tilde{\theta}^{B'}\{-\frac{5}{2}\mathring{\nabla}^{A'}_A\tilde{\phi}_{A'B'}-\frac{3}{2}\mathring{\nabla}^B_{B'}\phi_{AB}+\mathring{\nabla}_{B'A}f+3(\lambda_A,\tilde{\lambda}_{B'})\}$$

$$+\frac{\theta^2\tilde{\theta}^2}{4}\{5\mathring{\nabla}_{AA'}\tilde{H}^{A'}+\frac{1}{2}\mathring{\nabla}^B_{A'}\nabla^{A'}_{(A}\lambda_{B)}+4[\lambda_A,f]-4[\phi_{AB},\lambda^B]\}. \qquad \text{(A21)}$$

Noting that

$$2\partial_{A'(B}V^{A'}_{A)}+[V_{A'A},V^{A'}_B]=-4\phi_{AB}$$

and setting (A21) to zero order by order we obtain the equations (3.51) as required. □

<u>*Proof of Theorem 3.iii.*</u>

$$\partial_A W^A = \left\{\frac{\partial}{\partial\theta^A} + \tilde{\theta}^{B'}\frac{\partial}{\partial x^{B'A}}\right\}W^A$$

$$= 8f - \frac{3}{2}\tilde{\theta}^2(\lambda^A,\lambda_A) - 2\tilde{\theta}^2\theta^{B'}[\phi_{CB},\lambda^C]$$

$$+ \theta^B\tilde{\theta}^{B'}\{-8\partial^A_{B'}\phi_{AB} - 4\partial_{BB'}f$$

$$-5\mathring{\nabla}^{A'}_B\tilde{\phi}_{A'B'} - 3\mathring{\nabla}^A_{A'}\phi_{AB} + \mathring{\nabla}_{BB'}f + 6(\lambda_B,\tilde{\lambda}_{B'})\}$$

$$+ \frac{\theta^2\tilde{\theta}^2}{2}\{-\frac{5}{2}\partial^{AB'}\mathring{\nabla}^{A'}_A\tilde{\phi}_{A'B'} - \frac{3}{2}\partial^{AB'}\mathring{\nabla}^B_{A'}\phi_{AB}$$

$$+ \partial^{B'A}\mathring{\nabla}_{B'A}f + 3\partial^{AB'}(\lambda_A,\tilde{\lambda}_{B'})\}$$

$$- 8\theta^2\tilde{\theta}^{B'}\partial_{B'A}H^A - 16\theta^B H_B$$

$$+ 3\theta^{B'}\partial_{B'A}\lambda^A + 4\theta^{B'}\tilde{H}_{B'}$$

$$+ \frac{\tilde{\theta}^2\theta^B}{2}\{5\nabla_{BA'}\tilde{H}^{A'} + 4[\lambda_B,f] - 4[\phi_{BC},\lambda^C]\}$$

$$- \tilde{\theta}^2\theta^B\partial_{BA'}\tilde{H}^{A'} \qquad \text{(A22)}$$

$$(\Psi_A,W^A) = 3\tilde{\theta}^{B'}[V_{B'A},\lambda^A] + \tilde{\theta}^2\theta^B[-5\phi_{AB} - \varepsilon_{AB}f,\lambda^A]$$

$$- 8\theta^2\tilde{\theta}^{B'}\{[V_{AB'},H^A] + [\tilde{\lambda}_{B'},f]\}$$

$$+ \theta^B\tilde{\theta}^{B'}\{-8[V^A_{B'},\phi_{AB}] - 4[V_{BB'},f] + 3(\tilde{\lambda}_{B'},\lambda_B)\}$$

$$+ \theta^2\tilde{\theta}^2\{-5(H_A,\lambda^A) + (\tilde{\lambda}_{A'},\tilde{H}^{A'}) - 4[\phi_{AB},\phi^{AB}] - 4[f,f]$$

$$-\frac{1}{2}[V^{AB'},-\frac{5}{2}\mathring{\nabla}^{A'}_A\tilde{\phi}_{A'B'} - \frac{3}{2}\mathring{\nabla}^B_{B'}\phi_{AB} + \mathring{\nabla}_{B'A}f + 3(\lambda_A,\tilde{\lambda}_{B'})]\} \qquad \text{(A23)}$$

Adding (A22) and (A23):

$$\begin{aligned}
\nabla_A W^A = 8f & \\
& + \theta^B\tilde{\theta}^{B'}\{-11\mathring{\nabla}^A_{B'}\phi_{AB} - 5\mathring{\nabla}^{A'}_B\tilde{\phi}_{A'B'} + 9(\lambda_B,\tilde{\lambda}_{B'}) - 3\mathring{\nabla}_{BB'}f) \\
& + \theta^2\tilde{\theta}^2\{\frac{\mathring{\nabla}^{AB'}}{4}\{-5\mathring{\nabla}^{A'}_A\tilde{\phi}_{A'B'} - 3\mathring{\nabla}^B_{A'}\phi_{AB} + 2\mathring{\nabla}_{B'A}f + 6(\lambda_A,\tilde{\lambda}_{B'})\} \\
& \qquad -5(H_A,\lambda^A)+(\tilde{H}_{B'},\tilde{\lambda}^{B'})-4[\phi_{AB},\phi^{AB}]-4[f,f]\} \\
& + \tilde{\theta}^{B'}\{3\mathring{\nabla}_{B'A}\lambda^A + 4\tilde{H}_{B'}\} \\
& + \tilde{\theta}^2\theta^B\{\frac{3}{2}\mathring{\nabla}_{BA'}\tilde{H}^{A'} - [\lambda_B,f]-7[\phi_{AB},\lambda^A]\} \\
& - 16\theta^B H_B - \frac{3}{2}\tilde{\theta}^2(\lambda^A,\lambda_A) \\
& - 8\theta^2\tilde{\theta}^{B'}\{\mathring{\nabla}_{B'B}H^B + [\tilde{\lambda}_{B'},f]\} .
\end{aligned}$$

The vanishing of (A24) order by order implies

$$f = 0 \qquad o(1) \qquad \text{(A25a)}$$

$$H_B = 0, \text{ i.e. } \mathring{\nabla}_{AA'}\tilde{\lambda}^{A'} = 0 \qquad o(\theta) \qquad \text{(A25b)}$$

$$\tilde{H}_{B'} = 0, \text{ i.e. } \mathring{\nabla}_{AA'}\lambda^A = 0 \qquad o(\tilde{\theta}) \qquad \text{(A25c)}$$

$$-11\mathring{\nabla}^A_{B'}\phi_{AB} - 5\mathring{\nabla}^{A'}_B\tilde{\phi}_{A'B'} + 9(\lambda_B,\tilde{\lambda}_{B'}) = 0 \qquad o(\theta\tilde{\theta}) \qquad \text{(A25d)}$$

$$(\phi^A,\lambda_A) = 0 \qquad o(\tilde{\theta}^2) \qquad \text{(A25e)}$$

$$[\phi_{AB},\lambda^A] = 0 \qquad o(\tilde{\theta}^2\theta) \qquad \text{(A25f)}$$

$$\frac{\mathring{\nabla}^{AB'}}{4}\{-5\mathring{\nabla}^{A'}_A\phi_{A'B'} - 3\mathring{\nabla}^B_{A'}\phi_{AB} + 6(\lambda_A,\tilde{\lambda}_{B'})\} - 4[\phi_{AB},\phi^{AB}] = 0 . \qquad o(\theta\tilde{\theta}^2) \qquad \text{(A25g)}$$

From the above equations together with the Bianchi identity (3.8e") it is not difficult to deduce that

$$\mathring{\nabla}^A_{B'}\phi_{AB} - \mathring{\nabla}^{A'}_B\tilde{\phi}_{A'B'} = 0 , \qquad \text{(A26)}$$

so equation (A25d) may be rewritten as

$$-16\mathring{\nabla}^{A}_{B'}\phi_{AB} + 9(\lambda_{B},\tilde{\lambda}_{B'}) = 0,$$

or

$$8\mathring{\nabla}^{a}{*G}_{ab} + 9(\lambda_{B},\tilde{\lambda}_{B'}) = 0 .$$

Equations (A25e), (A25f) and (A25g) are identities. □

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