# Topics in General and Set-Theoretic Topology: slice sets, rigid subsets of $\mathbb{R}$, cleavability, Toronto spaces, and neight

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# Introduction

In his famous 2009 article [18], Freeman Dyson wrote that some mathematicians are birds and others are frogs. His birds fly high enough to see many things at one time; these mathematicians seek out and delight in unification and innovation. His frogs focus on much smaller pieces of the landscape; these mathematicians delight in detail, the trickier the better. A bird is interested in exploring new topics, while a frog is more interested in solving hard problems.

I certainly cannot claim to attain to Dyson's rather lofty description of a bird. I can assert, however, that my personality as a mathematician is more bird-like than frog-like. Although I may not yet fly very high, it is certainly my aim to fly higher and see more, and I do not identify very well with Dyson's description of a frog. For me, mathematics is more of an adventure than a challenge, although it certainly is challenging. Doing mathematics feels more like writing poetry than winning a wrestling match, although the latter feeling is not unfamiliar. The poetry is strange: arhythmic, unrhyming, and written in a language that most people will never learn to read. Nonetheless, having come far enough to see at least some of its beauty, I now aspire to become one of the poets.

This bird-like attitude (or "philosophy") is why this thesis is entitled "*Topics* etc." rather than the more traditional "*Problems* etc." Although some of the work below is problem-motivated or even problem-oriented, most of it is better described as exploratory. I have made it my work, during the two and a half years I have spent in Oxford, to explore the vast landscapes of set theory, topology, and their interaction. The five chapters of this thesis are a distilled report of that undertaking. In them I have recorded the parts of my exploration in which I abandoned the well-worn paths of textbooks and journal articles and attempted instead to find something new and unexplored.

Except for some interaction between Chapters 1 and 2, each chapter is self-contained and can be read like a paper. In fact, the last three chapters are papers, one of which is published and two of which are currently under review. Each of these chapters therefore contains its own small introduction, and each section of the first two chapters contains some introductory material as well. In this section, then, to avoid repeating myself later, I will avoid introducing notation, phrasing my questions technically, and listing my major results. Instead, in the next few pages I will explain briefly, for each chapter in its turn, its context and motivation, my guiding intuitions, and whatever persistent open questions may help to shape my future work.

In the first chapter I consider two-point sets and their generalizations. The work in Section 1.1 was done jointly with Jan van Mill and Rolf Suabedissen, and it began with conversations had by the three of us at the fourteenth Galway Topology Colloquium in Belfast in 2011. This work recently has been accepted for publication (see [8]), and Section 1.1 consists simply of the parts of our paper for which I was the main author. The basic idea of this section is to explore the generalizations of two-point sets that we have called *slice sets*. A slice set for a topological space $A$ is a subset $X$ of $\mathbb{R}^2$ such that, for any line $L$, $L \cap X$ is homeomorphic to $A$; thus, for example, a two-point set is just a slice set for the two-element discrete space. My work on this topic began by realizing how to prove Lemma 1.2 and noticing that this leads to an existence proof of slice sets for any $A \subseteq \mathbb{R}$ with $2 \leq |A| < \mathfrak{c}$ (Theorem 1.3). In fact, Lemma 1.2 and Theorem 1.3 preceded the definition of slice sets, which we invented later to give this result some context.

While Section 1.1 is purely topological, Sections 1.2 and 1.3 are more set-theoretic. Section 1.2 gives a consistency proof of a result similar in kind to those of Section 1.1: that every totally disconnected subspace of $\mathbb{R}$ has a slice set. The proof uses the axiom that the ideal of Lebesgue null sets is $\mathfrak{c}$-additive. Although Suabedissen was able to prove the same result in *ZFC*, and of course this stronger result is what will be published in our paper, I have included the original proof here for two reasons. The first and most obvious reason is that it is my own proof instead of Suabedissen's, but the second reason is that I think this original proof, though strictly weaker, has a more intuitive feel. The proof is very visual and geometric, and I hope it might be as much fun for someone else to read as it was for me to write.

The fundamental problem for two-point sets, the one posed by Erdös decades ago and attacked by Chad for several years in Oxford, is whether there is a Borel two-point set. This question may be intimately related to the question of whether the Axiom of Choice is necessary for the construction of two-point sets (we will catch glimpses of the relationship in Section 1.3). Both questions can be viewed as different ways of asking whether it is possible to have a two-point set with a simple description. I approach these questions in Section 1.3 and, unfortunately, am able to obtain only partial results. While a few open questions are stated in Section 1.1, the questions of Section 1.3 are the ones on which I honestly intend to concentrate future effort. Whether there is a Borel two-point set I do not know, but I do feel it should be possible to find a model for *ZF* containing no two-point sets at all. One of my goals in the coming years is to find such a model, and, if I am lucky, it may tell me also how to find a model for *ZFC* in which no two-point set is Borel.

The second chapter began with the question of whether it is possible, under *ZFC*, to have rigid subsets of $\mathbb{R}$ with cardinality smaller than $\mathfrak{c}$. This question arose in connection with the topic of Chapter 1: the existence of such a set implies the existence of a homogeneous subset of $\mathbb{R}$ that meets every line in a rigid set (see Proposition 1.9). In Section 2.1 I use forcing to answer this question in the affirmative. Section 2.2 explores the complementary question of whether $ZFC+\neg CH$ is consistent with every rigid subset of $\mathbb{R}$ having size $\mathfrak{c}$. By making use of a combinatorial axiom called Baumgartner's Axiom, I show that it is consistent with $ZFC+\mathfrak{c} = \aleph_2$ that every rigid subset of $\mathbb{R}$ has size $\mathfrak{c}$. Section 2.3 is a naturally corollary to these first two: in it I explore the question, for various properties $\mathcal{P}$, of how many subsets of $\mathbb{R}$ with size $\kappa$ have property $\mathcal{P}$.

While the results of Sections 2.1 and 2.2, taken together, seem to answer my original question thoroughly, there are several closely related questions that remain open. Foremost among them is whether it is possible to find a model for $ZFC+\mathfrak{c} > \aleph_2$ in which every rigid subset of $\mathbb{R}$ has size $\mathfrak{c}$. I will show in Section 2.2 that this follows from the natural extension of Baumgartner's Axiom to higher cardinals, but this natural extension is not known to be consistent with *ZFC*. Some very serious set theorists have worked on this problem (Shelah, Baumgartner, J. T. Moore), so its solution may be very difficult, but I do feel that the result should be consistent and that the appropriate forcing argument is merely awaiting discovery.

Another open question, one to which I have devoted a fair amount of energy already, is how to eliminate small rigid subsets of $\mathbb{R}$ directly, without factoring the problem through some version of Baumgartner's Axiom. Currently Baumgartner's original partial order (see [5]) is the only known (reasonably well-behaved) forcing partial order for killing small rigid sets. One motivation for finding another forcing is, of course, that it might give us a hint for how to generalize Baumgartner's Axiom to higher cardinals. Another is that we do not now know yet whether the existence of small rigid sets is consistent with $ZFC+MA+\neg CH$. An answer of "no" was conjectured by van Mill in a personal correspondence, but a proof of this would need to utilize some novel method for killing small rigid sets. This is because, by a recent result of Abraham and Shelah (see [2]), it is known that Martin's Axiom does not imply Baumgartner's Axiom.

As with Chapter 1, the work in Chapter 3 also began at the fourteenth Galway Topology Colloquium. Arhangel'skiĭ remarked to Levine, who later related the remark to me, that $K_5$ is a nice example of a space that is cleavable over, but not embeddable into, $\mathbb{R}^2$. I realized on the trip from Belfast back to Oxford that, while $K_5$ is cleavable over $\mathbb{R}^2$, it is possible to modify the definition of cleavability ever so slightly so that this is no longer the case. This alteration became the basis for my definition of cleavage, and Chapter 3 is the result of my exploration of this concept. The topic seems to involve more combinatorics than set theory; the reader will notice a liberal use of topological graphs in examples and constructions.

Many open questions are articulated throughout this chapter, and, while I do not feel the need to repeat them all here, I will say that I think the most important of them is Question 3.14: is there a finite list of compact spaces such that any compact space that does not embed in $\mathbb{R}^2$ contains something on this list as a subspace? In a sense, the question is whether Kuratowski's Theorem for finite graphs has an analogue in the category of compact spaces. The reader will see how this question relates to cleavage in Chapter 3, but I will say for now that if this question has a positive answer (even consistently), then it seems to be saying something deep about the class of compact spaces.

Chapter 3, almost verbatim as it is here, has been published recently in *Topology and its Applications* (see [7]), and while it is not the first paper I have had accepted for publication, it nonetheless represents my first published work.

Incidentally, the name "cleavage" also finds its origins in the fourteenth Galway Topology Colloquium. While at lunch one day, K. P. Hart quipped that someone should come up with a cardinal invariant related to cleavability and name it "cleavage"; this has now been done.

Chapter 4 is concerned with Toronto spaces, a Toronto space being a topological space $X$ such that, for any $Y \subseteq X$ with $|Y| = |X|$, $Y$ is homeomorphic to $X$. The major unanswered question for Toronto spaces, the so-called "Toronto problem", is whether any nontrivial examples can actually exist, where "nontrivial" means Hausdorff and non-discrete. The non-existence of any examples is implied by the *GCH* (see Proposition 4.6). The Toronto problem is likely very difficult: I find myself in the distinguished company of Shelah, Kunen, Steprāns, Dow, and many others for having worked on, and having failed to answer, this question.

The Toronto problem is explained near the beginning of *Open Problems in Topology* [28], and that is how I came to work on it. Rather than attempting to build nontrivial Toronto spaces in some generic extension and show that their existence is consistent with *ZFC*, I began with the intuition that surely such spaces are absurd and sought to prove that they cannot exist. While I have not been able to show that nontrivial Toronto spaces are inconsistent, my quest to do so has led me to several results of the form "if a nontrivial Toronto space does exist then it has property such and such". Sections 4.2 through 4.5 deal with results of this kind, including one result (Theorem 4.24) showing that if a nontrivial Toronto space fails to be $T_3$ then it must conform to one of two fairly specific types. Section 4.6 concludes the chapter with an exploration of non-Hausdorff Toronto spaces and a classification of all those that fail to be $T_1$ (there are exactly three for every infinite cardinality).

Of course the primary open question concerning Toronto spaces is whether any can exist, but questions likely more tractable than this one have arisen from my analysis. One such is whether Theorem 4.24 (the classification of non-regular nontrivial Toronto spaces) can be sharpened. The result outlines two possible types of non-regular Hausdorff Toronto spaces, and I would not be surprised if one or both of them can be eliminated as a possibility. If both types can be shown to be impossible, this would imply that any nontrivial Toronto space is $T_3$, thus answering part of Steprāns's question as posed in [28] (Steprāns lays particular emphasis on the separation properties of Toronto spaces). Another obvious question is how to extend the results of Section 4.4, which describes the convergence properties of a nontrivial Toronto space in terms of a specific fixed filter.

Another, necessarily more vague question is whether it is possible to construct a recipe for producing Toronto spaces similar to Gruenhage and Moore's construction in [26]. Their work considers a modification of the definition of Toronto spaces that applies to countable spaces, and they show how, given a filter $\mathcal{F}$, one can construct a certain scattered space whose convergence at each level is characterized by $\mathcal{F}$. They then go on to show that, for certain values of $\mathcal{F}$, their construction yields their modified Toronto spaces. The task, then, is to reproduce this sort of argument but for genuine, uncountable Toronto spaces. More specifically, the goal is (1) to define a construction of thin-tall scattered spaces with a single parameter, a filter $\mathcal{F}$ giving the convergence properties of the space (2) to show that, for a filter $\mathcal{F}$ with certain specific properties, this construction yields a Toronto space. If such a construction can be outlined, then the work of Section 4.4 could play an important role in finding the filter properties necessary for producing a Toronto space.

The fifth and final chapter explores a cardinal function called *neight* (which is short for "nested weight"). In short, the neight of a space $X$, denoted $\mathfrak{N}(X)$, is the least cardinal $\kappa$ such that $X$ has a subbasis consisting of a union of $\kappa$ nests in $X$, where a "nest in $X$" is a subset of $\mathcal{P}(X)$ totally ordered by $\subseteq$. In fact, I discovered only after submitting this work for publication that this cardinal function was noticed, but left largely unexplored, by E. Deák in [14]; he, along with J. Deák, focuses on variations of this definition, particularly one called the halfdirectional dimension, proving some quite strong results, one of which is discussed in Section 5.2. They consider neight (or, rather, the function $X \mapsto \mathfrak{N}(X) - 1$) to be a measure of dimension. The neight function was also explored by Yurovetskiĭ in [49], who proved that every metric space has countable neight.

The notion of neight came to me through Gareth Davies, a recent graduate of the Analytic Topology group at Oxford. He, in turn, got the idea from Chris Good and Kyriakos Papadopoulos at the University of Birmingham. Gareth, while unaware of the work of either E. Deák or J. Deák, was exploring the function $X \mapsto \mathfrak{N}(X) - 1$ as a possible measure of dimension; at the time he was trying to decide whether this function is additive under finite products. In particular, he was wondering if it is possible to have two spaces with neight 2 that have a product with neight 4, and he was exploring the Tychonoff Plank as a possible example. Although no one yet has been able to compute the neight of the Tychonoff Plank (it is trivial to show that it is either 3 or 4, but neither Gareth, nor I, nor my supervisor, nor anyone else seems to be able to prove which), I was able to find an example that does work; this example is the basis of Section 5.3.

As far as I am concerned, the primary open question concerning neight is whether Deák's results on the halfdirectional dimension can be extended to corresponding results for neight. More specifically, the goal is to prove or disprove the following conjecture: if $X$ is a separable metric space, then $\mathfrak{N}(X)$ is the least $n$ such that $X$ embeds in $\mathbb{R}^{n-1}$. This conjecture is equivalent to: if $X$ is a separable metric space with finite neight then $\mathfrak{N}(X) - 1 = \textsc{Dim}(X)$ (the $\textsc{Dim}$ function is defined in Section 5.2, where also is found a fuller discussion of this conjecture). We show below in Theorem 5.6 that this conjecture fails if the word "separable" is removed, but the conjecture as stated is open.

There are several reasons this conjecture, if proven true, would be valuable. First, it would give us an easy description of neight for separable metric spaces and would solidify the intuition that neight is, in some sense, a measure of dimension. Second, it would give us a way of computing neight for spaces such as the Klein bottle or projective plane. Currently the neight of such spaces has not been computed. Lastly, any proof that this conjecture holds would require the development of new techniques for finding lower bounds on $\mathfrak{N}(X)$, at least when $X$ is separable and metrizable. As we will see in Chapter 5, the search for lower bounds is fundamental in the study of neight.

# Chapter 1

# Slice sets and two-point sets

## 1.1 Slice sets

A **two-point set** is a subset of $\mathbb{R}^2$ that meets every line in exactly two points. Two-point sets were introduced by Mazurkiewicz in [37], where they are constructed by transfinite recursion from a well-ordering of the lines in $\mathbb{R}^2$. In this section we introduce and explore a generalization of two-point sets that we call slice sets.

**Definition 1.1.** *Let $A$ be a subset of $\mathbb{R}$. $X \subseteq \mathbb{R}^2$ is an $A$-**slice set** (or a slice set for $A$) if, for every line $L$ in $\mathbb{R}^2$, the intersection $X \cap L$ is homeomorphic to $A$. More generally, if $\mathcal{P}$ is a topological property then we say that $X \subseteq \mathbb{R}^2$ is a $\mathcal{P}$-slice set whenever $X \cap L$ has property $\mathcal{P}$ for every line $L$.*

**Lemma 1.2.** *Suppose that $A$ and $B$ are subsets of $\mathbb{R}$ with $2 \leq |A| < \mathfrak{c}$ and $|B| < \mathfrak{c}$, and suppose that $x_1, x_2 \in \mathbb{R} \setminus B$. There is a $C^\infty$ homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f[A] \cap B = \emptyset$ and $x_1, x_2 \in f[A]$.*

*Proof.* Assume that $x_1, x_2 \in A$ (if not, we can dilate and translate $\mathbb{R}$ so that this becomes true). Consider the three intervals $(-\infty, x_1)$, $(x_1, x_2)$, and $(x_2, \infty)$. We will find three $C^\infty$ automorphisms of $\mathbb{R}$, each of which is the identity off of one of these intervals, and, on the interval where it is not the identity, maps points of $A$ to $\mathbb{R} \setminus B$.

There is a $C^\infty$ bump function $\psi$ on $\mathbb{R}$ with the following properties:

- $\psi(x) = 0$ for all $x \notin (x_1, x_2)$.
- $\psi(x) > 0$ for all $x \in (x_1, x_2)$.
- There is a positive constant $h_0$ such that, for $0 < h < h_0$, $\left|\frac{d(h\psi)}{dx}\right| < 1$ at every point in $\mathbb{R}$.

If $0 < h < h_0$ then the map $\phi_h(x) = x + h\psi(x)$ is a $C^\infty$ automorphism of $\mathbb{R}$. We claim that there is a constant $h_1$ such that $0 < h_1 < h_0$ and, for all $x \in A \cap (x_1, x_2)$, $\phi_{h_1}(x) \notin B$. Suppose that this is not the case. Then, for every $h \in (0, h_0)$, there is (at least one) pair $(a_h, b_h) \in A \times B$ such that $a_h \in (x_1, x_2)$ and $\phi_h(a_h) = b_h$. If $h < h'$ and $a_h = a_{h'}$ then, since $\psi(a_h) > 0$,

$$b_h = a_h + h\psi(a_h) < a_h + h'\psi(a_h) = b_{h'}$$

It follows that $(a_h, b_h) \neq (a_{h'}, b_{h'})$ whenever $h \neq h'$. This is impossible since $|A \times B| < \mathfrak{c}$. Thus some such $h_1$ exists. $f_1 = \phi_{h_1}$ is a $C^\infty$ automorphism of $\mathbb{R}$ that is the identity on $\mathbb{R} \setminus (x_1, x_2)$ and that maps all $x \in (x_1, x_2)$ into $\mathbb{R} \setminus B$.

Similarly, there is a $C^\infty$ automorphism $f_2$ of $\mathbb{R}$ that is the identity on $\mathbb{R} \setminus (-\infty, x_1)$ and that maps all $x \in A \cap (-\infty, x_1)$ into $\mathbb{R} \setminus B$, and there is a $C^\infty$ automorphism $f_3$ of $\mathbb{R}$ that is the identity on $\mathbb{R} \setminus (x_2, \infty)$ and that maps all $x \in A \cap (x_2, \infty)$ into $\mathbb{R} \setminus B$. Take $f = f_3 \circ f_2 \circ f_1$. $\square$

**Theorem 1.3.** *If $A \subseteq \mathbb{R}$ and $2 \leq |A| < \mathfrak{c}$ then there is a slice set for $A$.*

*Proof.* Let $\langle L_\alpha \colon \alpha < \mathfrak{c} \rangle$ be an enumeration of all lines in $\mathbb{R}^2$. As with the original Mazurkiewicz construction of two-point sets, we build $X$ by transfinite recursion. Let $X^0 = \emptyset$. Let $\alpha < \mathfrak{c}$ and assume that we have constructed $\left\langle X^\beta \colon \beta < \alpha \right\rangle$ such that

- For $\gamma < \beta < \alpha$, $X^\beta \cap L_\gamma$ is homeomorphic to $A$
- For $\gamma \geq \beta < \alpha$, $\left| X^\beta \cap L_\gamma \right| \leq 2$
- If $\gamma < \beta < \alpha$ then $X^\gamma \subseteq X^\beta$

If $\alpha$ is a limit ordinal, take $X^\alpha = \bigcup_{\beta < \alpha} X^\beta$. If $\alpha = \beta + 1$ then, by assumption,

$X^\beta \cap L_\beta$ contains at most two points, say $x_1$ and $x_2$. Let

$$B = \left\{ x \in L_\beta \colon x \notin X^\beta \text{ but } x \in L_\gamma \text{ for some } \gamma < \beta \right\}$$

$$B' = \left\{ x \in L_\beta \colon \text{ for some } \gamma > \beta,\ \left| L_\gamma \cap X^\beta \right| = 2 \text{ and } x \in L_\gamma \right\}$$

It is straightforward to show that $|B| < \mathfrak{c}$ and $|B'| < \mathfrak{c}$. By Lemma 1.2, there is a subset $Y$ of $L_\beta$ that is homeomorphic to $A$, that includes both $x_1$ and $x_2$, and that is disjoint from $B \cup B'$. Setting $X^\alpha = X^\beta \cup Y$, it is clear that $X^\alpha$ satisfies the inductive hypotheses, so this completes the induction. $X = \bigcup_{\alpha<\mathfrak{c}} X^\alpha$ is the desired slice set. □

**Corollary 1.4.** *If $A \subseteq \mathbb{R}$ and $2 \leq |\mathbb{R} \setminus A| < \mathfrak{c}$, then there is a slice set for $A$.*

*Proof.* By Theorem 1.3 there is a slice set $X$ for $\mathbb{R} \setminus A$. Furthermore, because of our use of Lemma 1.2 in the induction step of the proof of Theorem 1.3, we may assume that $X$ has the additional property that for every line $L$ in $\mathbb{R}^2$ there is a homeomorphism $\mathbb{R} \to L$ that restricts to a homeomorphism from $\mathbb{R} \setminus A$ onto $X \cap L$. Taking complements, it follows that for every line $L$ in $\mathbb{R}^2$ there is a homeomorphism $\mathbb{R} \to L$ that restricts to a homeomorphism from $A$ onto $L \setminus (X \cap L) = L \cap (\mathbb{R}^2 \setminus X)$. Thus $\mathbb{R}^2 \setminus X$ is a slice set for $A$. □

Using the techniques found in [8] for the case $3 \leq |A| < \mathfrak{c}$, we may take our $A$-slice sets to be homogeneous:

**Corollary 1.5.** *If $A \subseteq \mathbb{R}$ and $3 \leq |A| < \mathfrak{c}$, then there is a homogeneous subset of $\mathbb{R}^2$ which is a slice set for $A$.*

*Proof.* The proof follows closely the proof of Theorem 2 in [8]. The only extra tool that is needed is a modification of lemma 1.2: suppose that $A$ and $B$ are subsets of $\mathbb{R}$ with $3 \leq |A| < \mathfrak{c}$ and $|B| < \mathfrak{c}$, and suppose that $x_1, x_2, x_3 \in \mathbb{R} \setminus B$; then there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f[A] \cap B = \emptyset$ and $x_1, x_2, x_3 \in f[A]$. The proof of this lemma is nearly identical to the proof of Lemma 1.2. $\square$

It is worth noting that homogeneous two-point sets were constructed by a different technique in [12], so Corollary 1.5 can be strengthened to include the case $|A| = 2$.

Unlike Theorem 1.3, Corollary 1.5 does not extend via complementation to the case $3 \leq |\mathbb{R} \setminus A| < \mathfrak{c}$, and it remains unknown whether homogeneous $A$-slice sets must exist for such $A$.

It is obvious that there are slice sets for $\emptyset$ and $\mathbb{R}$ and that there is not a slice set for a singleton. This observation, together with Theorem 1.3 and Corollary 1.4, nearly answers the question of the existence of slice sets for $A \subseteq \mathbb{R}$ with either $A$ or $\mathbb{R} \setminus A$ smaller than $\mathfrak{c}$. The one case that remains unsolved is $|\mathbb{R} \setminus A| = 1$, whose slice set would be a subset of the plane meeting every line in exactly two open intervals. The next natural question to ask is: for which subsets A of $\mathbb{R}$ with $|A| = |\mathbb{R} \setminus A| = \mathfrak{c}$ do $A$-slice sets exist? A general characterization has not been found, but in the next section we show that *MA* implies every totally disconnected subset of $\mathbb{R}$ has a slice set (Suabedissen strengthened this to a *ZFC* result in [8]). The following proposition summarizes a few results for various subsets of $\mathbb{R}$ that are not totally disconnected:

**Proposition 1.6.**

*(i) There is no slice set for* $[0,1]$ *or for* $[0,1)$*. Also,* $\mathbb{R}^2$ *is the unique slice set for* $(0,1)$*.*

*(ii) There are slice sets for a countable sum of closed intervals and for a countable sum of open intervals.*

*(iii) If* $A \subseteq \mathbb{R}$ *is such that* $A = -A$ *and, for any* $r \in \mathbb{R}$*,* $A$ *is homeomorphic to the image of* $A$ *in the quotient space* $\mathbb{R}/[-r,r]$*, then there is a slice set for* $A$*. Note that this property is not topological, so it is sufficient for* $A$ *to be homeomorphic to such a space.*

*Proof.*

(i) Suppose that $X$ is a slice set for $[0,1]$ or for $[0,1)$. Since either of $[0,1]$ or $[0,1)$ is connected, $X$ is convex. For each line $L$ in $\mathbb{R}^2$, there is an open ray in $L$ that does not belong to $X$, i.e., some $p \in L$ such that every point of $L$ on one side of $p$ does not belong to $X$; without loss of generality, we may take the origin to be in $X$ and, using polar coordinates, take the open ray $\{(r, \pi) \colon r > 0\}$ to be a subset of $\mathbb{R}^2 \setminus X$. Let $\theta_1 \leq \pi$ be the smallest and $\theta_2 \geq \pi$ the largest values for which $\mathbb{R}^2 \setminus X$ contains the open wedge

$$W = \{(r, \theta) \colon r > 0, \theta_1 < \theta < \theta_2\}$$

If $\theta_2 - \theta_1 < \pi$, consider the open ray $R = \left\{(r, \frac{\theta_1+\theta_2}{2}) \colon r > 0\right\}$; even in the degenerate case $\theta_1 = \theta_2 = \pi$, we have $R \subseteq \mathbb{R}^2 \setminus X$. Let

$$A = \left\{(r, \theta) \colon r > 0, \frac{\theta_1 + \theta_2}{2} - \frac{\pi}{2} < \theta < \frac{\theta_1 + \theta_2}{2}\right\}$$

$$B = \left\{(r, \theta) \colon r > 0, \frac{\theta_1 + \theta_2}{2} < \theta < \frac{\theta_1 + \theta_2}{2} + \frac{\pi}{2}\right\}$$

These are the two quadrants on either side of $R$. It must be that either $A \subseteq \mathbb{R}^2 \setminus X$ or $B \subseteq \mathbb{R}^2 \setminus X$; otherwise, by the convexity of $X$, we can find a point of $R$ in $X$. This contradicts either the minimality of $\theta_1$ or the maximality of $\theta_2$; thus we have $\theta_2 - \theta_1 \geq \pi$. However, if $\theta_2 - \theta_1 \geq \pi$, then there is a line in $\mathbb{R}^2$ that is completely contained in $\mathbb{R}^2 \setminus X$, contradicting the assumption that $X$ is a slice set for a nonempty set.

Notice that the only facts used here to prove that there is no slice set for $A = [0, 1], [0, 1)$ are that $A$ is convex and that some point of $\mathbb{R}^2$ is not in our slice set. Thus we have also proved the second assertion, that $\mathbb{R}^2$ is the only slice set for $(0,1)$.

(ii) Consider the hexagonal honeycomb packing of circles of radius 1 in the plane. Keeping the centers of the circles fixed, shrink the radius of each circle by some constant $\frac{2-\sqrt{3}}{2} < c < 1$. Now let $X$ be the union of the interiors of the circles. $X$ meets every line in a countable sum of open intervals and $\mathbb{R}^2 \setminus X$ meets every line in a countable sum of closed intervals.

(iii) For each $r > 0$, let $C_r$ denote the circle of radius $r$ centered at the origin and let $C_0 = \{(0,0)\}$. Take $X = \bigcup_{a \in A \cap [0,\infty)} C_a$. $\square$

Many open questions remain concerning slice sets for $A \subseteq \mathbb{R}$, $|A| = |\mathbb{R} \setminus A| = \mathfrak{c}$. For instance, it is unknown whether there is a slice set for $[0,1] \times \{0,1\}$, $[0,1] \times n$, or even whether there is a subset of $\mathbb{R}^2$ that meets every line in a finite union of closed intervals. Similarly, it is unknown whether there is a slice set for $(0,1) \times \{0,1\}$ (Corollary 1.4 covers the case of larger sums of open intervals).

Alternatively, we can ask for a subset of the plane meeting every line in a unique way:

**Proposition 1.7.** *There is a subset of the plane whose intersection with each line has unique homeomorphism type, i.e., any two such intersections* *are non-homeomorphic.*

*Proof.* Let $\langle A_\alpha : \alpha < \mathfrak{c}\rangle$ be a sequence of countable subsets of $\mathbb{R}$ such that if $\alpha \neq \beta$ then $A_\alpha$ is not homeomorphic to $A_\beta$. We will prove in Lemma 1.8 below that there are $\mathfrak{c}$ distinct homeomorphism classes of countable subsets of $\mathbb{R}$.

We now construct the desired set $X$ by transfinite recursion. The construction is exactly the same as in Theorem 1.3 except that, at the successor step $\alpha + 1$, we use Lemma 1.2 to guarantee that $L_\alpha \cap X$ is homeomorphic to $A_\alpha$. □

**Lemma 1.8.** *The number of distinct homeomorphism classes of countable subsets of* $\mathbb{R}$ *is* $\mathfrak{c}$.

*Proof.* The number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at most $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

Let $X \subseteq \mathbb{R}$. Let $P$ be the largest dense-in-itself subset of $X$ and let $S = X \setminus P$ be the scattered part of $X$. We define the **scattered signature** $H(X)$ of $X$ as follows. $H(X)$ is a set of ordinals, and $\alpha \in H(X)$ if and only if there is some $p \in P$ such that $p$ has Cantor-Bendixson rank $\alpha$ in $S \cup \{p\}$.

Let $A = \{\alpha_n \colon n \in \omega\}$ be a countable subset of $\omega_1$. We show that there is a countable subset of $\mathbb{R}$ with scattered signature $A$. On the interval

$[n+\frac{1}{4}, n+\frac{1}{2}]$, embed $\omega^{\alpha_n}+1$, making sure that the point $\omega^{\alpha_n}$ maps to the point $n+\frac{1}{2}$. Include the points $\mathbb{Q}\cap[n+\frac{1}{2}, n+\frac{3}{4}]$ and call the resulting set $X$. It is a routine exercise to show that $H(X)=A$.

As there are $\mathfrak{c}$ countable subsets of $\omega_1$, this proves that the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at least $\mathfrak{c}$. $\square$

## 1.2 Consistency results for slice sets

**Proposition 1.9.** *It is consistent with* ZFC *that there is a homogeneous subset $X$ of $\mathbb{R}^2$ such that, for every line $L$, $X\cap L$ is rigid (non-meagre, non-measurable).*

*Proof.* It is shown in the next chapter that there is a generic extension in which there is a rigid, non-meagre, non-measurable subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$. Applying Corollary 1.5, we obtain the desired result. $\square$

Recall that $add$(null) is the least cardinal $\kappa$ such that there exists a collection of $\kappa$ Lebesgue null subsets of $\mathbb{R}$ whose union is non-null. This number is one of the so-called "small cardinals" and, as with all small cardinals, Martin's Axiom ($MA$) implies $add(\text{null})=\mathfrak{c}$.

**Theorem 1.10.** *If $add(\text{null})=\mathfrak{c}$ then there is a Cantor-slice set. In particular, MA implies the existence of a Cantor-slice set.*

We prove this theorem by a sequence of lemmas. In the standard recursive construction of a two-point set, we are able to add points to a line at any given stage because there are fewer than $\mathfrak{c}$ forbidden points. The basic idea here is to reproduce this proof, but at any given stage we will ensure that a line has a Lebesgue null set of forbidden points on it rather than a small-in-cardinality set. In order for this to work, of course, we must first show that we can always add a Cantor set to the complement of our set of forbidden points:

**Lemma 1.11.** *Every compact subset of $\mathbb{R}$ with positive Lebesgue measure contains a Cantor set.*

*Proof.* If $K$ is a compact subset of $\mathbb{R}$ that does not contain a Cantor set then $K$ must be scattered. Every scattered subset of $\mathbb{R}$ is countable. □

**Lemma 1.12.** *Let $N$ be a subset of $\mathbb{R}$ with Lebesgue measure $0$ and let $x, y \in \mathbb{R} \backslash N$. Then there is a Cantor set $C$ such that $x, y \in C$ and $C \cap N = \emptyset$.*

*Proof.* Without loss of generality, suppose $|x - y| > 2$. For each $n > 0$, let $S_n = B(x, \frac{1}{n}) \setminus N$. $S_n$ is a set with positive Lebesgue measure, so there is a compact $K_n \subseteq S_n$ of positive Lebesgue measure. By Lemma 1.11, there is a Cantor set $C_n \subseteq K_n$. If, for some $n$, $x \in C_n$, then set $C_x = C_n$. If not, then pick a sequence $n_1, n_2, n_3, ...$ such that, for each $m$, $B(x, \frac{1}{n_{m+1}}) \cap C_{n_m} = \emptyset$ and set $C_x = \{x\} \cup \bigcup_{m>0} C_{n_m}$. Either way, we obtain a Cantor set $C_x$ such that $x \in C_x \subseteq B(x,1) \setminus N$. Similarly, there is a Cantor set $C_y$ such that $y \in C_y \subseteq B(y,1) \setminus N$. Take $C = C_x \cup C_y$. □

Define a **quick Cantor set** to be any Cantor set $C \subseteq \mathbb{R}$ such that, for every $n > 0$, there is a cover of $C$ consisting of $2^n$ open intervals, each of which meets $C$ in a clopen set and has diameter less than $\frac{1}{8^n}$.

**Lemma 1.13.** *If $C \subseteq \mathbb{R}$ is a Cantor set and $x, y \in C$, then there is a quick Cantor set $C' \subseteq C$ such that $x, y \in C'$.*

*Proof.* By induction, we may choose a set $\mathcal{S} = \{U_s \colon s \in 2^{<\omega}\}$ of clopen subsets of $C$ such that $U_s \subseteq U_t$ whenever $s$ extends $t$, $U_s \cap U_t = \emptyset$ if $s$ and $t$ are incompatible, and each $U_s$ has diameter less than $\frac{1}{8^{|s|}}$. Furthermore, we may do this in such a way that, for each $n$, there are $s, t$ with $|s| = |t| = n$ such that $x \in U_s$ and $y \in U_t$. Take

$$C' = \bigcup_{f \in 2^\omega} \bigcap_{n \in \mathbb{N}} U_{f \restriction n}$$

□

If $U$ and $V$ are distinct subsets of $\mathbb{R}^2$, we write $\langle U, V \rangle$ for the union of the set of all lines that contain both a point of $U$ and a point of $V$, and we write $\langle U, V \rangle_L$ for the union of the set of all lines other than $L$ that contain both a point of $U$ and a point of $V$. The following lemma is crucial to our iteration: it will allow us to claim, at stage $\alpha$ of our iteration, that the set of forbidden points on line $\alpha + 1$ is Lebesgue null.

**Lemma 1.14.** *Let $L_1, L_2, L_3$ be lines in $\mathbb{R}^2$ and let $C_1 \subseteq L_1$, $C_2 \subseteq L_2$ be quick Cantor sets. $L_3 \cap \langle C_1, C_2 \rangle_{L_3}$ has Lebesgue measure $0$ in $L_3$.*

*Proof.* Let $L_1, L_2$ be any two distinct lines in $\mathbb{R}^2$ and let $C_1 \subseteq L_1$, $C_2 \subseteq L_2$ be quick Cantor sets. Let $\varepsilon > 0$ and let $L_3$ be a line different from both $L_1$ and $L_2$. We will construct an open cover of $L_3 \cap \langle C_1, C_2 \rangle_{L_3}$ with measure less than $\varepsilon$. In what follows, if $U$ is a nonempty subset of some line $L$ then we will write $I(U)$ for the smallest closed interval in $L$ containing $U$.

*Case 1:* Suppose that $L_1$, $L_2$, and $L_3$ are all parallel and that $L_2$ is between $L_1$ and $L_3$. Without loss of generality, we take $L_1$ to be the line $y = 0$, $L_2$ to be the line $y = 1$, and $L_3$ to be the line $y = a$, $a > 1$. If $U = (a_1, b_1)$ is an open interval on $L_1$ and $V = (a_2, b_2)$ is an open interval on $L_2$, then $\langle U, V \rangle \cap L_3$ is an open interval on $L_3$. In fact, a simple computation shows that

$$\langle U, V \rangle \cap L_3 = (b_1 + a(a_2 - b_1), a_1 + a(b_2 - a_1))$$

That is, $\langle U, V \rangle \cap L_3$ is an open interval in $L_3$ of length $a(b_2 - a_2) + (a -$

$1)(b_1 - a_1)$. Thus, if $U$ and $V$ both have length less than $\delta$, $\langle U, V\rangle \cap L_3$ has length less than $(2a-1)\delta$. Now let $\mathcal{U} = \{U_1, ..., U_{2^n}\}$ and $\mathcal{V} = \{V_1, ...V_{2^n}\}$ be open covers of $C_1$ and $C_2$, respectively, such that each $U_i$ and each $V_j$ has length less than $\frac{1}{8^n}$. Let

$$\mathcal{W} = \{\langle U_i, V_j\rangle \cap L_3 \colon i = 1, ..., 2^n, j = 1, ..., 2^n\}$$

$\mathcal{W}$ is an open cover of $\langle C_1, C_2\rangle \cap L_3 = \langle C_1, C_2\rangle_{L_3} \cap L_3$, and the Lebesgue measure of $\bigcup \mathcal{W}$ is at most

$$\sum_{i=1}^{2^n}\sum_{j=1}^{2^n}(2a-1)\frac{1}{8^n} = (2^n)^2(2a-1)\frac{1}{8^n} = \frac{2a-1}{2^n}$$

Thus, for sufficiently large $n$, $\mathcal{W}$ is an open cover of $\langle C_1, C_2\rangle \cap L_3$ of size less than $\varepsilon$. Hence $\langle C_1, C_2\rangle \cap L_3$ has Lebesgue measure 0.

*Case 2:* Suppose that either $I(C_1) \cap L = \emptyset$ or $I(C_2) \cap L = \emptyset$ for any line $L$ that is parallel to $L_3$. This was also true in Case 1, but now we do not assume that any of $L_1$, $L_2$, and $L_3$ are parallel. Using the compactness of $I(C_1)$ and $I(C_2)$, we can find open intervals $W_1 \supseteq I(C_1)$ and $W_2 \supseteq I(C_2)$ such that $L \cap I(W_1) = \emptyset$ or $L \cap I(W_2) = \emptyset$ for any $L$ parallel to $L_3$. As in Case 1, there is a positive constant $M$ such that, whenever $U$ and $V$ are open intervals of length at most $\delta$, then $\langle U \cap W_1, V \cap W_2\rangle \cap L_3$ is an open interval of length at most $M\delta$. This $M$ is more difficult to compute than in Case 1 and depends on the exact positioning of these lines in the plane, but it is not too difficult to see that such a constant must exist. As in Case 1, we use this constant to obtain a suitably small open cover for $\langle C_1, C_2\rangle \cap L_3 = \langle C_1, C_2\rangle_{L_3} \cap L_3$.

*Case 3:* Suppose that $L_3$ is parallel to a line that contains a point of $I(C_1)$ and a point of $I(C_2)$. In what follows, we will write "$U \perp V$" for "no line parallel to $L_3$ contains both a point of $U$ and a point of $V$". Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of collections of clopen subsets of $C_1$ such that each $\mathcal{U}_n$ covers $C_1$, the members of $\mathcal{U}_n$ are disjoint, and each $U \in \mathcal{U}_n$ is the intersection of $C_1$ with an interval of length at most $\frac{1}{8^n}$. Find a similar sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ of covers of $C_2$. Let

$$\mathcal{S} = \left\{(U, V)\colon U \in \bigcup_{n\in\omega} \mathcal{U}_n, V \in \bigcup_{n\in\omega} \mathcal{V}_n, \text{ and } I(U) \perp I(V)\right\}$$

By Case 2, $\langle U, V\rangle \cap L_3$ has Lebesgue measure 0 for each $(U, V) \in \mathcal{S}$. As $\mathcal{S}$ is countable, the set

$$P = \bigcup_{(U,V)\in\mathcal{S}} \langle U, V\rangle_{L_3} \cap L_3$$

has Lebesgue measure 0. We claim that $P = \langle C_1, C_2\rangle_{L_3} \cap L_3$, which is enough to complete the proof.

If $U \subseteq C_1$ and $V \subseteq C_2$ then $\langle U, V\rangle_{L_3} \subseteq \langle C_1, C_2\rangle_{L_3}$; it follows that $P \subseteq \langle C_1, C_2\rangle_{L_3} \cap L_3$, and we need to prove the reverse inclusion. Let $x \in C_1, y \in C_2$ and, assuming $x \neq y$, let $L_{x,y}$ be the line containing $x$ and

$y$. If $L_{x,y}$ is parallel to (or equal to) $L_3$ then $L_{x,y}$ does not contribute to $\langle C_1, C_2\rangle_{L_3} \cap L_3$. If $L_{x,y}$ is not parallel to $L_3$, we must show that $L_{x,y} \cap L_3 \subseteq P$. But if $L_{x,y}$ is not parallel to $L_3$ then there are some $m, n$ and $U \in \mathcal{U}_m$, $V \in \mathcal{V}_n$ such that $I(U) \perp I(V)$. As $\langle U, V\rangle_{L_3} \cap L_3 \subseteq P$ by definition, $L_{x,y} \cap L_3 \subseteq P$. □

*Proof of Theorem 1.10.* Let $\langle L_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of all lines in $\mathbb{R}^2$. As above, we build our slice set $X$ by transfinite recursion. Let $X^0 = \emptyset$. Let $\alpha < \mathfrak{c}$ and assume that we have constructed $\langle X^\beta : \beta < \alpha\rangle$ such that

- For $\gamma < \beta < \alpha$, $X^\beta \cap L_\gamma$ is a quick Cantor set
- For $\gamma \geq \beta < \alpha$, $|X^\beta \cap L_\gamma| \leq 2$
- If $\gamma < \beta < \alpha$ then $X^\gamma \subseteq X^\beta$

If $\alpha$ is a limit ordinal, take $X^\alpha = \bigcup_{\beta<\alpha} X_A^\beta$. If $\alpha = \beta+1$ then, by assumption, $X^\beta \cap L_\beta$ contains at most two points, say $x_1$ and $x_2$. Let

$$B = \{x \in L_\beta : x \notin X^\beta \text{ but } x \in L_\gamma \text{ for some } \gamma < \beta\}$$

$$B' = \{x \in L_\beta : \text{ for some } \gamma \geq \alpha,\ \left|L_\gamma \cap X^\beta\right| = 2 \text{ and } x \in L_\gamma\}$$

Clearly $|B| < \mathfrak{c}$. Also, it is clear that

$$B' = \bigcup \left\{\langle X^\beta \cap L_\gamma, X^\beta \cap L_\delta\rangle_{L_\beta} \cap L_\beta \colon \gamma, \delta < \beta\right\}$$

By Lemma 1.14, $B'$ is the union of at most $|\beta \times \beta|$ sets of Lebesgue measure 0. Since we are assuming that the additivity of the ideal of null sets is $\mathfrak{c}$, $B \cup B'$ has Lebesgue measure 0. By Lemmas 1.12 and 1.13, there is a subset $C$ of $L_\beta$ that is a quick Cantor set, that includes both $x_1$ and $x_2$, and that is disjoint from $B \cup B'$. Setting $X^\alpha = X^\beta \cup C$, it is clear that $X^\alpha$ satisfies the inductive hypotheses, so this completes the recursion. $X = \bigcup_{\alpha<\mathfrak{c}} X^\alpha$ is the desired slice set. □

While Theorem 1.10 gives us only one new slice set, a slice set for a Cantor set, it can be easily extended, as we will see, to obtain a wide class of new slice sets.

**Lemma 1.15.** *If $A \subseteq \mathbb{R}$ is totally disconnected and $|A| \geq 2$, and if $x, y$ are two points of a (quick) Cantor set $C$, then there is an embedding $f : A \to C$ such that $x, y \in f[A]$.*

*Proof.* It suffices to show that $A$ embeds into the Cantor set. Since the Cantor set is homogeneous, we may compose any embedding of $A$ with an appropriate automorphism of the Cantor set to obtain an embedding with $x$ and $y$ in its image.

Let $S$ be a countable dense subset of $\mathbb{R} \setminus A$. $A$ embeds in $\mathbb{R} \setminus S$, so it is enough to embed $\mathbb{R} \setminus S$ in a Cantor set. Let $\{s_n : n \in \omega\}$ be an enumeration of $S$ and, for every $x \in \mathbb{R} \setminus S$, let $h(x) = x + \sum_{s_n < x} \frac{1}{2^n}$. Clearly $h$ is an embedding of $\mathbb{R} \setminus S$ into $\mathbb{R}$. Moreover, as $h[\mathbb{R} \setminus S]$ is nowhere dense and perfect, its closure is a Cantor set. $\square$

**Theorem 1.16.** *Assume add*(null) $= \mathfrak{c}$. *If $A \subseteq \mathbb{R}$ is totally disconnected and $|A| \geq 2$, then there is an $A$-slice set.*

*Proof.* The proof is the same as that of Theorem 1.10, this time using Lemma 1.15 at each successor stage to embed $A$ in such a way that it is contained within a quick Cantor set. $\square$

Intuitively, as mentioned above, this proof works by replacing the notion of small-in-cardinality with the notion of small-in-Lebesgue-measure and then proceeding with the usual recursive construction of slice sets. Suabedissen achieves the result of Theorem 1.16 from only *ZFC* in [8] by replacing the notion of small-in-Lebesgue-measure with the more algebraic notion that $X$ is small when $\mathbb{R}$ has transcendence degree $\mathfrak{c}$ over $X$. Interestingly, this is not the first time that a consistency result involving slice sets (two-point sets in particular) has been strengthened to a *ZFC* result by the use of algebraic techniques; see [12] for a second example.

## 1.3 Two-point sets in $HOD^{\mathbb{R}}$

The class $OD$ of ordinal-definable sets and the class $HOD$ of hereditarily ordinal-definable sets are well-established classical tools for inner model theory whose conception can be traced as far back as Gödel. A more recent variant is the class of **real-ordinal-definable sets**, denoted $OD^{\mathbb{R}}$. Formally, $A \in OD^{\mathbb{R}}$ if and only if there is a formula $\varphi(x, \alpha_1, ..., \alpha_m, r_1, ..., r_n)$ such that

$$A = \{u \colon \varphi(u, \alpha_1, ..., \alpha_m, r_1, ..., r_n)\}$$

where $\alpha_1, ..., \alpha_m \in ON$ and $r_1, ..., r_n \in \mathbb{R}$. $HOD^{\mathbb{R}}$ is the class of hereditarily real-ordinal-definable sets, i.e., the class of all sets $X$ such that $TC(X) \subseteq OD^{\mathbb{R}}$.

For a more formal treatment of $HOD^{\mathbb{R}}$, see [33] pp. 308-313. Here we will simply summarize a few results without proving them.

As models, $HOD$ and $HOD^{\mathbb{R}}$ have many similarities but a few crucial differences. For one thing, $HOD^{\mathbb{R}}$ is a model for *ZF* but is not necessarily a model for *ZFC* (whereas $HOD$ is always a model for *ZFC*). $HOD^{\mathbb{R}}$ will, however, always satisfy certain weakenings of $AC$, for example the Countable Axiom of Choice. For another thing, it is possible that $\mathbb{R}^{HOD} \neq \mathbb{R}$, but it is obvious from our definition that $\mathbb{R}^{HOD^{\mathbb{R}}} = \mathbb{R}^{OD^{\mathbb{R}}} = \mathbb{R}$. Since $\mathbb{R}$ is absolute for $HOD^{\mathbb{R}}$, so are many other notions defined in terms of $\mathbb{R}$: $\mathbb{R}^2$ and the set $\mathcal{L}$ of lines in $\mathbb{R}^2$ are both absolute for $HOD^{\mathbb{R}}$, and it is not too hard to see that the notion of a two-point set is also absolute. More explicitly, if $X \in HOD^{\mathbb{R}}$ then $HOD^{\mathbb{R}} \models (X$ is a two-point set) if and only if $X$ is a two-point set.

$HOD^{\mathbb{R}}$ can be viewed as a tool for constructing models for *ZF*, and the absoluteness of $\mathbb{R}$ for $HOD^{\mathbb{R}}$ indicates that $HOD^{\mathbb{R}}$ may be particularly useful for constructing models for *ZF* in which propositions about the real line, specifically those proved by the Axiom of Choice, fail. In particular, if we would like to find a model for *ZF* in which there are no two-point sets, it suffices to find a model $M$ for *ZFC* containing no real-ordinal-definable two-point sets: if we succeed in finding such an $M$, $(HOD^{\mathbb{R}})^M$ gives the required model for *ZF*. Better still, if such a model $M$ exists then $M$ will contain no Borel two-point sets; this is because every Borel set is (hereditarily) real-ordinal-definable, a fact rendered obvious by a brief consideration of the notion of Borel codes. This would give a partial answer to a longstanding question of Erdös: whether there is a Borel two-point set.

Unfortunately, it does not seem to be a trivial problem to find a model for *ZFC* with no real-ordinal-definable two-point sets. This section is devoted to illuminating this point. In the process we will prove a theorem (Theorem 1.19) that gives a "descriptively simple" two-point set, in a sense to be defined below, inside certain models of set theory. We will prove another theorem (Theorem 1.20) telling us that if real-ordinal-definable two-point sets do exist, at least in certain forcing extensions, then they must satisfy certain absoluteness properties.

If $V = L$ then $L = HOD = HOD^{\mathbb{R}}$, so the existence of sets that are not real-ordinal-definable can be proved only up to consistency and only by some use of forcing. The following lemma provides an essential tool for this:

**Lemma 1.17.** *Let $\mathbb{P}$ be a forcing partial order, and let $i$ be an automorphism of $\mathbb{P}$. By recursion, we may define a map $i_* : V^{\mathbb{P}} \to V^{\mathbb{P}}$ such that, for $\tau \in V^{\mathbb{P}}$, $i_*(\tau) = \{(i_*(\sigma), i(p)) \colon (\sigma, p) \in \tau\}$. If $\varphi(x_1, ..., x_n)$ is a formula of the language of set theory and $p \in \mathbb{P}$, then*

$$p \Vdash \varphi(\tau_1, ..., \tau_n) \qquad \Leftrightarrow \qquad i(p) \Vdash \varphi(i_*(\tau_1, ..., \tau_n))$$

*Proof.* See [33] pp. 269-271. □

Note that, by the definition of $i_*$ and a simple induction, $i_*(\check{b}) = \check{b}$ for every $b \in V$. That is, any automorphism of a forcing partial order fixes the ground model.

We will construct a specific model $M$ for $ZFC$ and a specific generic filter $G$ for $Fn(\kappa,2)^M$ such that $M[G]$ contains two-point sets that are definable from parameters in $M$. These two-point sets are not necessarily real-ordinal-definable, but the fact that they are definable from parameters in $M$ (together with the fact that $i_*(\check{b}) = b$ for any $b \in M$ and any automorphism $i$ of $Fn(\kappa,2)$) shows that definable two-point sets are not destroyed by taking a generic extension by Cohen reals.

The main idea for the following lemma is found in [40]:

**Lemma 1.18** (Miller)**.** *Let $M$ be a model for ZFC, let $\mathbb{P} = Fn(\omega_1,2)$, and let $G$ be $\mathbb{P}$-generic over $M$. In $M[G]$, there is a two-point set that is definable from $G$.*

*Proof.* In $M[G]$, consider the sequence $\langle r_\alpha \colon \alpha < \omega_1 \rangle$ of real numbers (which we will here identify with functions $\omega \to 2$) defined by $r_\alpha(n) = \bigcup G(\omega\cdot\alpha+n)$. Using a standard decomposition from the theory of product forcing, one can show that each $r_\alpha$ is Cohen over $M[\langle r_\beta \colon \beta < \alpha\rangle]$. In [40], Miller shows how to build a two-point set $X$ via recursion from such a sequence. Miller's recursion is of length $\omega_1$, and at stage $\alpha < \omega_1$ of the recursion he begins with a partial two-point set $X_\alpha$ and extends it to a partial two-point set $X_{\alpha+1}$ by adding points on a circle with radius determined by $r_\alpha$. The construction is carefully controlled at each stage and, in fact, it is clear from the proof that it would be possible to write down a formula $\psi(x, X, r)$ such that $x \in X_{\alpha+1} \backslash X_\alpha$ if and only if $\psi(x, X_\alpha, r_\alpha)$. Using this formula $\psi(x, X, r)$ and the definability of recursion, it is possible to write down a formula $\varphi(x, G)$ that captures the recursion, i.e., such that $X = \{x \colon \varphi(x, G)\}$. □

Intuitively, this lemma states that, at least consistently, a two-point set may in some sense contain only $\aleph_1$ bits of information, even though it is an object of size $\mathfrak{c}$, which may be large.[1] Miller used this idea in [40] to show that it is consistent with *ZF* to have a two-point set but no well-ordering of the real line: in his model $\mathbb{R}$ has size greater than $\aleph_1$ (and cannot be well-ordered) but still contains a length-$\omega_1$ sequence of Cohen reals from which to define a two-point set. We will use the idea to show that the technique of forcing poset symmetries cannot be applied naively to eliminate two-point sets. The following theorem shows that it is possible to have a forcing extension with a two-point set definable completely from parameters in the ground model.

**Theorem 1.19.** *Let $\lambda$ be an infinite cardinal. There is a model $M$ for* ZFC *and a $Fn(\lambda, 2)$-generic filter $G$ such that the following holds: there is*

[1] In fact, Miller proved elsewhere the consistency of a co-analytic two-point set. Since every co-analytic set can be coded by a real number, this shows that, consistently, there are two-point sets that contain only $\aleph_0$ bits of information.

*a formula $\varphi(x)$, all of whose parameters are in $M$, such that $\{x\colon \varphi(x)\}$ is a two-point set in $M[G]$.*

*Proof.* Let $N$ be any model for *ZFC* and let $\kappa = \max\{\lambda, \aleph_2\}$. Let $\mathbb{P} = Fn(\kappa, 2)$ and $\mathbb{Q} = Fn(\omega_1, 2)$. By well-known results in the theory of product forcing, forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{P}\times\mathbb{Q}$ or with $\mathbb{P}$ and then with $\mathbb{Q}$. By Lemma 1.18, forcing with $\mathbb{P}$ and then with $\mathbb{Q}$ adds a two-point set $X$ that is definable from the $\mathbb{Q}$-generic filter $G$ or, equivalently, from $\bigcup G$.

Let $\varphi(x, \bigcup G)$ be the sentence defining $X$ in the generic extension by $G$, i.e., $X = \{x\colon \varphi(x, \bigcup G)\}$. Now let $\psi$ be a sentence of the language of set theory expressing "there is a function $f : \omega_1 \to 2$ such that $\{x\colon \varphi(x, f)\}$ is a two-point set". We have seen that $\psi$ is true in any $\mathbb{P}\times\mathbb{Q}$-generic extension of $M$, so $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} \Vdash \psi$. Since forcing with $\mathbb{P}\times\mathbb{Q}$ is equivalent to forcing with $\mathbb{P}$, $\mathbb{1}_{\mathbb{P}} \Vdash \psi$. By the Maximal Principle (see [32], pp. 287), there is a name $\tau$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash (\tau \text{ is a function } \omega_1 \to 2 \ \wedge\ \{x\colon \varphi(x, \tau)\} \text{ is a two-point set})$$

Since $\tau$ names a function $\omega_1 \to 2$, we may assume without loss of generality that $\tau$ is hereditarily smaller than $\aleph_2$ and that there is some $S \subseteq \kappa$ with $|S| \leq \aleph_1$ such that $\tau \in M^{Fn(S,2)}$. Let $G$ be $\mathbb{P}$-generic over $M$. Using product forcing again, note that $\mathbb{P} \approx Fn(S,2) \times Fn(\kappa \setminus S, 2)$. Thus letting $G_S$ and $G_{\kappa\setminus S}$ denote, respectively, the restriction of $G$ to $Fn(S,2)$ and to $Fn(\kappa \setminus S, 2)$, we have $M[G] = M[G_S][G_{\kappa\setminus S}]$. In $M[G_S]$, $G_{\kappa\setminus S}$ is $Fn(\kappa \setminus S, 2) \approx Fn(\kappa, 2)$-generic over $M[G_S]$. Moreover, since $\tau \in M^{Fn(S,2)}$, $\tau_{G_S} \in M[G_S]$ and $\varphi(x, \tau)$ can be written in $M[G_S]$ using only the name $\check{\tau}_{G_S}$. Since $M[G] = M[G_S][G_{\kappa\setminus S}]$, $\{x \colon \varphi(x, \check{\tau}_{G_S})\}$ is a two-point set in $M[G_S][G_{\kappa\setminus S}]$. This finishes the proof in the case that $\lambda \geq \aleph_2$. For the other two cases, let $T \subseteq \kappa$ such that $|T| = \lambda$ and $T \cap S = \emptyset$. Then break the forcing into three parts: $M[G] = M[G_S][G_{\kappa\setminus(S\cup T)}][G_T]$. As before, we have that $M[G_S][G_{\kappa\setminus(S\cup T)}]$ is a transitive model, $G_T$ is $Fn(\lambda, 2)$-generic over $M[G_S][G_{\kappa\setminus(S\cup T)}]$, and $X$ is a two-point set in the extension. $\square$

This theorem shows that one cannot destroy definable two-point sets in an extension by Cohen reals using symmetry arguments alone. The next theorem shows that, nonetheless, it is still possible to say something about the real-ordinal-definable two-point sets in an extension by Cohen reals.

**Theorem 1.20.** *Let $M$ be a model for ZFC, let $\mathbb{P} = Fn(\kappa, 2)$ with $\kappa$ uncountable and $G$ $\mathbb{P}$-generic over $M$, and, for $S \subseteq \kappa$, $S \in M$, let $G_S$ denote the restriction of $G$ to $Fn(S, 2)$. If there is a two-point set $X$ in $(HOD^{\mathbb{R}})^{M[G]}$ then it is "almost absolute" in the following sense: there is some countable $D \subseteq \kappa$, $D \in M$ such that, for any $S \in M$ with $D \subseteq S \subseteq \kappa$,*

*$X \cap M[G_S]$ is a two-point set in $M[G_S]$.*

*Proof.* Let $\varphi(x)$ be a formula with one free variable such that every name mentioned in $\varphi(x)$ either is a nice name for a real number or is of the form $\check{b}$ for some $b \in M$ and such that, in $M[G]$, $\{x \colon \varphi(x)\}$ is a two-point set. Since $\mathbb{P}$ has the ccc, there is a countable $D \subseteq \kappa$ such that every forcing condition mentioned in a name in $\varphi(x)$ is in $D$.

An arbitrary line $L \in M[G]$ is defined by any two of its points: we write $L = \langle x, y \rangle$, where $x, y \in L$ and $x \neq y$. Clearly the sentence '$z \in \langle x, y \rangle$' is absolute for transitive models. Suppose $D \subseteq S \subseteq \kappa$ and $L$ is a line in $M[G_S] \cap \mathbb{R}^2$. Then there must be $a, b \in M[G_S] \cap \mathbb{R}^2$ such that $L = \langle a, b \rangle$. We must show that $L \cap X \subseteq M[G_S]$, where $X = \{x \colon \varphi(x)\} \in M[G]$.

Suppose that this is not the case. We may assume $|\kappa \setminus S| \geq \aleph_0$ since otherwise $M[G] = M[G_S]$ and $X$ is a two-point set. Note that $M[G] = M[G_S][G_{\kappa \setminus S}]$, where $G_{\kappa \setminus S}$ is generic over $M[G_S]$. We work in $M[G_S]$ to obtain a contradiction. Let $\mathbb{P}$ denote $Fn(\lambda, 2)$, where $\lambda = |\kappa \setminus S|$. Since $X$ is a two-point set in $M[G]$, there is some $\sigma \in M[G_S]^{Fn(\lambda,2)}$ and some $p \in \mathbb{P}$ such that

$$p \Vdash \sigma \in \mathbb{R} \ \wedge \ \varphi(a + \sigma(b-a)) \ \wedge \ \{x \colon \varphi(x)\} \text{ is a two-point set}$$

Let $i$ be any automorphism of $\mathbb{P}$ in $M[G_S]$ which fixes $\mathrm{dom}(p)$. We recall that $i$ induces an automorphism $i_*$ on the class of $\mathbb{P}$-names, and that, if $\psi(\tau)$ is a formula with $\mathbb{P}$-names as parameters, and $q \in \mathbb{P}$, then $q \Vdash \psi(\tau)$ if and only if $i(q) \Vdash \psi(i_*(\tau))$. Since $i$ fixes the domain of $p$, we have $i(p) = p$. Thus $p \Vdash \varphi(a + \sigma(b - a))$ implies $p \Vdash \varphi(a + i_*(\sigma)(b - a))$ (recall that, although $\varphi(x)$ may contain parameters, all of them can be written in the form $\check{b}$ for $b \in M[G_S]$, and all such names are fixed by any $i_*$). Since $p \Vdash \{x \colon \varphi(x)\}$ is a two-point set, we will obtain a contradiction if we can show that $i_*(\sigma)$ can have at least three different values in the generic extension. This follows from the fact that $\mathbb{P}$ is weakly homogeneous, that our only requirement on $i$ is that it fix the finite $\mathrm{dom}(p)$, and that $\sigma$ is assumed to name something not in $M[G_S]$: in particular, $i_*(\sigma)$ must take on infinitely many possible values as $i$ varies. $\square$

# Chapter 2

# Small rigid subsets of $\mathbb{R}$

## 2.1 Forcing small rigid subsets of $\mathbb{R}$

A topological space $X$ is **rigid** if the only homeomorphism $X \to X$ is the identity map. In this section we demonstrate two different methods for using forcing to add a rigid $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$. Sierpiński was able to construct a rigid subset of $\mathbb{R}$ of size $\mathfrak{c}$ as early as 1918 by using transfinite recursion (see [45]). However, his method is useful for producing only rigid sets of size $\mathfrak{c}$, and there do not seem to be any examples in the literature of small-in-cardinality rigid subsets of $\mathbb{R}$.

In the first of our proofs, which is concise and moderately well known, we will demonstrate that adding $\kappa$ Cohen reals adds a rigid subset of $\mathbb{R}$ of size $\lambda$

for every uncountable $\lambda \leq \kappa$. For the second proof, we will go back and prove this result again using a more involved iterated forcing argument. However, this new proof gives us much more because it generalizes to a large class of forcing partial orders. In particular, we will show that any non-trivial finite support iteration with cofinality $\kappa$ adds a rigid subset of the reals of size $\kappa$ (for uncountable $\kappa$). This result seems to say that small-in-cardinality rigid subsets of $\mathbb{R}$ are ubiquitous in forcing extensions.

Our notation in this section mostly follows Jech (see [31]). If $\mathbb{P}$ is a notion of forcing, $V^{\mathbb{P}}$ denotes the class of $\mathbb{P}$-names and $V[G]$ denotes the forcing extension of the ground model $V$ by a generic filter $G$. $\mathbb{P}_\alpha$ denotes the restriction of an iterated notion of forcing $\mathbb{P}$ to some ordinal $\alpha$, and $G_\alpha$ denotes the restriction of a $\mathbb{P}$-generic filter $G$ to $\mathbb{P}_\alpha$. We use $Fn(\kappa, 2)$ to denote the set of finite partial functions $\kappa \to 2$ (the Cohen forcing of size $\kappa$).

I am indebted to an anonymous referee for the following simple proof of the consistent existence of rigid subsets of $\mathbb{R}$ smaller than $\mathfrak{c}$. Although this simpler proof is apparently well known, it does not seem to appear anywhere in the literature, so we give it here. For the following theorem we will say that $X \subseteq \mathbb{R}$ is **uncountably perfect** if, for every open $U \subseteq \mathbb{R}$, either $X \cap U = \emptyset$ or $X \cap U$ is uncountable.

**Theorem 2.1.** *In an extension of $V$ by Cohen reals, any uncountably perfect* *set of Cohen reals is rigid.*

*Proof.* Let $X$ be an uncountable set of Cohen reals (that is, Cohen over $V$) and let $Y \subseteq X$ be uncountably perfect; we will show that $Y$ is rigid in $V[X]$. In fact, it is enough to show that $X$ is rigid in $V[X]$ (assuming $X$ uncountably perfect) since, if $Y$ is an uncountably perfect subset of $X$, then the same argument would show that $Y$ is rigid in $V[X \setminus Y][Y] = V[X]$. Suppose that $f \in V[X]$ is a nontrivial homeomorphism $X \to X$. Because $X$ is a separable Hausdorff space, $f$ is fully determined by its action on a countable dense subset $D$ of $X$. As $f \restriction D$ is an hereditarily countable set, there is some countable $Q \subseteq X$ such that $f \restriction D \in V[Q]$. Recall that the set of points moved by a continuous function on a Hausdorff space is always open, and every nonempty open subset of an uncountably perfect set is uncountable. Thus, because $f$ is not the identity and $X$ is uncountably perfect, we may find $x, y \in X \setminus Q$ with $x \neq y$ and $f(x) = y$. It is a standard result on Cohen reals that $x$ is generic over $V[Q]$ and $y$ is generic over $V[Q, x]$. However, we must have $y \in V[Q, x]$ since $x, f \restriction D \in V[Q, x]$ and $y = f(x)$. This is a contradiction. $\square$

This proof, though clear and concise, is too narrow because it tells us only about one particular forcing partial order. In order to get at the more general result, we will reprove Theorem 2.1 by a more difficult method.

The argument will include several non-essential features which we intend to abstract away later, but the proof will be easier to follow if given first for a finite support iteration of the Cohen partial order. We need a few lemmas which, in some variation, can be found in any text on forcing:

**Lemma 2.2.** *Let $\mathbb{P}$ be the finite support iteration of $\kappa$ copies of $Fn(\omega, 2)$. Then forcing with $\mathbb{P}$ is equivalent to forcing with $Fn(\kappa, 2)$.*

**Lemma 2.3.** *Let $\kappa$ be an infinite ordinal with* $\mathrm{cf}(\kappa) \geq \lambda$ *and let $H_\lambda$ denote the set of all sets which are hereditarily smaller than $\lambda$. Let $\mathbb{P}$ be a $\lambda$-c.c. iterated forcing of length $\kappa$ in which every condition has support smaller than $\kappa$; let $G$ be a generic filter on $\mathbb{P}$. If $X \in H_\lambda^{V[G]}$ then $X \in V[G_\alpha]$ for some $\alpha < \kappa$.*

*Proof.* This lemma is a generalization of Lemma 16.14 (p. 273) in [31]. Let $\Phi(X)$ be the statement "there is some $\alpha < \kappa$ such that $X$ has a name in $V^{\mathbb{P}_\alpha}$". We prove by $\in$-induction that $\Phi(X)$ holds for every $X \in H_\lambda^{V[G]}$. Clearly $\Phi(\emptyset)$ holds. Suppose that $|X| < \lambda$ and that, for each $Y \in X$, $\Phi(Y)$ holds. For each $Y \in X$, let $\alpha_Y < \kappa$ and let $\dot{Y}$ be a name for $Y$ in $V^{\mathbb{P}_{\alpha_Y}}$. $X$ has a name $\dot{X}$ which is determined by the set of Boolean values $\|\dot{Y} \in \dot{X}\|$. Each of these Boolean values is determined by an antichain in $P$ of size $\mu < \lambda$. Each of these $\mu$ conditions has a support included in some $\alpha < \kappa$;

since $\mu < \mathrm{cf}(\kappa)$, there is some $\alpha' < \kappa$ such that all of these $\mu$ conditions have support included in $\alpha'$. Setting $\alpha = \sup\{\alpha'\} \cup \{\alpha_Y : Y \in X\}$, it follows that $X$ has a name in $V^{\mathbb{P}_\alpha}$. This completes the induction and shows that every $X \in H_\lambda^{V[G]}$ has a name in $V^{\mathbb{P}_\alpha}$ for some $\alpha < \kappa$. If $X$ has a name in $V^{\mathbb{P}_\alpha}$ then $X \in V[G_\alpha]$. □

**Theorem 2.4.** *Let $\kappa$ be an infinite cardinal with* $\mathrm{cf}(\kappa) \neq \omega$. *Then forcing with $Fn(\kappa, 2)$ adds a rigid $X \subseteq \mathbb{R}$ with $|X| = \kappa$.*

*Proof.* Applying Lemma 2.2, we will show that the finite support iteration $\mathbb{P}$ of the sequence $\langle Fn(\omega, 2) \colon \alpha < \kappa \rangle$ adds a rigid $X \subseteq \mathbb{R}$ of size $\kappa$. It is well-known that $\mathbb{P}$ preserves cardinals. Let $G$ be a $\mathbb{P}$-generic filter.

For each $\alpha < \kappa$, let

$$\mathbb{R}_\alpha^{new} = \mathbb{R}^{V[G_\alpha]} \setminus \bigcup_{\beta < \alpha} \mathbb{R}^{V[G_\beta]}$$

$\mathbb{R}_\alpha^{new}$ is the set of all real numbers that appear at stage $\alpha$ of our extension. We will focus on successor $\alpha$ for convenience. Let $\beta < \kappa$ and $\alpha = \beta + 1$. In this case, $\mathbb{R}_\alpha^{new} = \mathbb{R}^{V[G_\alpha]} \setminus \mathbb{R}^{V[G_\beta]}$. Since $V[G_\alpha] = V[G_\beta][H_\beta]$ where $H_\beta$ is $Fn(\omega, 2)$-generic over $V[G_\beta]$, $\mathbb{R}_\alpha^{new}$ is nonempty. Moreover, since $\mathbb{Q}$ is absolute and addition is absolute, if $q \in \mathbb{Q}$ and $x \in \mathbb{R}_\alpha^{new}$ then $x + q \in \mathbb{R}_\alpha^{new}$. Thus $\mathbb{R}_\alpha^{new}$ is a dense subset of $\mathbb{R}$ for all successor $\alpha < \kappa$. [Nb: It is possible to prove the slightly stronger assertion that $\mathbb{R}_\alpha^{new}$ contains a size $\mathfrak{c}$ dense set of pairwise mutually generic reals, but we only need the fact that $\mathbb{R}^{new}_\alpha$ is dense.]

For each successor $\alpha < \kappa$, let $x_\alpha \in \mathbb{R}^{new}_\alpha$ and let $S_\alpha = \{x_\alpha + q \colon q \in \mathbb{Q}\} \subseteq \mathbb{R}^{new}_\alpha$. Let $\phi_\alpha : S_\alpha \to \omega$ be a bijection in $V[G_\alpha]$. Taking $H_\alpha$ as in the previous paragraph, we let $g_\beta = \bigcup H_\beta$ be the generic function $\omega \to 2$ added at stage $\alpha$ of our iteration.

We are now in a position to define our rigid subset $X$ of $\mathbb{R}$. For each $\beta < \kappa$, define

$$X_\beta = \{x \in S_{\beta+1} \colon g_{\beta+1} \circ \phi_{\beta+1}(x) = 1\} \, .$$

Let $X = \mathbb{Q} \cup \bigcup_{\alpha<\kappa} X_\alpha$.

It is not obvious from this definition that $X \in V[G]$. The problem is that our definition of the $\mathbb{R}^{new}_\alpha$ (and hence of the $X_\beta$ and of $X$) depends on the class $V$, and it is (presumably) possible that $V$ is not definable in $V[G]$. If there is a formula $\phi(x,p)$, $p \in V[G]$, such that $V[G] \models \phi(x,p) \Leftrightarrow x \in V$, then $V[G]$ can "see" $V$ as a subclass and it is possible to define the sequence of $V[G_\alpha]$, and consequently $X$, in $V[G]$ (from the parameter $G$). However, there is no reason to suppose that such a formula exists (although it will of course for special choices of the ground model, e.g. $V = L$). To sidestep this difficulty, we notice that, even if $V$ is not definable in $V[G]$, it is enough for $V[G]$ to "see" $V_\gamma$ for large enough $\gamma$. There is a $\gamma \in On$ such that each

$x \in \mathbb{R}^{V[G]}$ has a name $\dot{x} \in V_\gamma$, and we can assume that $V_\gamma$ contains a name for each real that is in $V^{\mathbb{P}_\alpha}$ for the smallest possible $\alpha$. $V_\gamma \in V[G]$, and we can define $V_\gamma[G]$ in $V[G]$ to be the evaluation under $G$ of every $\mathbb{P}$-name in $V_\gamma$. In $V_\gamma[G]$, it is possible to define the sequence $\left\langle \mathbb{R}^{V_\gamma[G_\alpha]} \colon \alpha < \kappa \right\rangle = \left\langle \mathbb{R}^{V[G_\alpha]} \colon \alpha < \kappa \right\rangle$, and from this sequence we may define $\langle S_{\beta+1} \colon \beta < \kappa \rangle$ and then $\langle X_\beta \colon \beta < \kappa \rangle$ and then $X$. Thus $X \in V[G]$.

Since $X$ is a union of $\kappa$ countable disjoint sets, $V[G] \models |X| = \kappa$. It remains to show that $X$ is rigid in $V[G]$.

If $F$ is an arbitrary continuous function $X \to X$ then, as $X$ is a Hausdorff space and $\mathbb{Q}$ is dense in $X$, $F$ is determined by its action on $\mathbb{Q}$. Let $f = F \restriction \mathbb{Q}$; $f$ is an hereditarily countable set so, by Lemma 2.3, $f \in V[G_\alpha]$ for some $\alpha < \kappa$. Thus every continuous function $X \to X$ is determined by a set in some $V[G_\alpha]$.

Let $\alpha_0 < \kappa$ and let $f$ be any non-identity continuous function $\mathbb{Q} \to \mathbb{R}$ in $V[G_{\alpha_0}]$. We will show using the genericity of $G$ that $f$ does not extend to a homeomorphism $X \to X$ in $V[G]$; by the argument of the previous paragraph, this is enough to prove the theorem.

If $F \in V[G]$, $F : X \to X$, we say that $F$ is **regressive** at $\beta < \kappa$ if there is some $\gamma < \beta$ and some $x \in X_\beta$ such that $f(x) \in X_\gamma$.

Let $\alpha = \beta + 1$ be a successor ordinal such that $\alpha_0 \leq \beta < \kappa$. We will argue in $V[G_\alpha]$ and examine the extension of $V[G_\alpha]$ to $V[G_{\alpha+1}]$ by the $Fn(\omega,2)$-generic filter $H_\alpha$. Let $p \in Fn(\omega,2)$ and let $S'_\alpha = S_\alpha \setminus \phi_\alpha^{-1}(\mathrm{dom}(p))$. There are four cases to consider:

*Case 1:* If there is some $x \in S'_\alpha$ such that $f$ does not extend continuously to $x$, then $q = p \cup \{(\phi_\alpha(x), 1)\}$ is an extension of $p$ such that, if $q \in H_\alpha$, then $f$ does not extend to an automorphism of $X$ in $V[G]$. If $f$ extends continuously to every $x \in S'_\alpha$, let $\tilde{f}$ denote this (unique) extension.

*Case 2:* If $\tilde{f}$ maps $S'_\alpha$ to $S'_\alpha$ but is not injective, pick any two distinct $x_1, x_2 \in S'_\alpha$ such that $\tilde{f}(x_1) = \tilde{f}(x_2)$. Then $q = p \cup \{(\phi_\alpha(x_1), 1), (\phi_\alpha(x_2), 1)\}$ is an extension of $p$ such that, if $q \in F_\alpha$, then $f$ does not extend to an automorphism of $X$ in $V[G]$.

*Case 3:* Suppose $\tilde{f}$ is injective but there is some $x \in S'_\alpha$ such that $\tilde{f}(x) \notin S'_\alpha$. Either (a) $\tilde{f}(x) \notin X_\gamma$ for any $\gamma \leq \beta$ and $\tilde{f}(x) \notin S_\alpha \setminus S'_\alpha$ (that is, $\Vdash_\alpha \tilde{f}(x) \notin \dot{X}$), in which case $q = p \cup \{(\phi_\alpha(x), 1)\}$ is an extension of $p$ such that, if $q \in F_\alpha$, then $f$ does not extend to an automorphism of $X$ in $V[G]$; or (b) $\tilde{f}(x) \in X_\gamma$ for some $\gamma < \beta$, in which case $q = p \cup \{(\phi_\alpha(x), 1)\}$ is an extension of $p$ such that, if $q \in F_\alpha$, then any extension of $f$ to $X$ is regressive at $\beta$; or (c) $\tilde{f}(x) \in S_\alpha \setminus S'_\alpha$; since $\tilde{f}$ is injective on $S'_\alpha$, this can be true of only finitely many points (recall that $S_\alpha \setminus S'_\alpha = \phi_\alpha^{-1}(\mathrm{dom}(p))$ is finite). Either we pick a different point $x \in S'_\alpha$ so that (a) or (b) applies, or we move on to Case 4.

*Case 4:* Suppose that $\tilde{f}$ is a continuous injection $S''_\alpha \to S'_\alpha$, where $S''_\alpha$ is some cofinite subset of $S'_\alpha$. Then $T = \left\{x \in S''_\alpha \colon x \neq \tilde{f}(x)\right\}$ is an open subset of $S''_\alpha$. As $f$ is not the identity on $\mathbb{Q}$ and $S''_\alpha$ is dense, $T$ is nonempty. If $x \in T$, then $q = p \cup \{(\phi_\alpha(x), 1), (\phi_\alpha(\tilde{f}(x)), 0)\}$ is an extension of $p$ such that, if $q \in F_\alpha$, then $f$ does not extend to an automorphism of $X$ in $V[G]$.

In $V[G_\alpha]$, $\tilde{f}$ is definable as the unique continuous extension of $f$ to $S'_\alpha$, if some such extension exists. Thus

$$\begin{aligned} D = \quad & \{q \leq p \colon f \text{ does not extend continuously to } x \text{ and } (\phi_\alpha(x), 1) \in q\} \\ & \cup \{q \leq p : \exists x_1, x_2 \in S'_\alpha \text{ such that } x_1 \neq x_2, \tilde{f}(x_1) = \tilde{f}(x_2), \\ & \qquad \text{and } (\phi_\alpha(x_1), 1), (\phi_\alpha(x_2), 1) \in q\} \\ & \cup \{q \leq p : \exists x \in S'_\alpha \text{ such that } \tilde{f}(x) \notin S_\alpha \text{ and } (\phi_\alpha(x), 1) \in q\} \\ & \cup \{q \leq p : \exists x \in S'_\alpha \text{ such that } x \neq \tilde{f}(x), \tilde{f}(x) \in S'_\alpha, \\ & \qquad \text{and } (\phi_\alpha(x), 1), (\phi_\alpha(\tilde{f}(x)), 0) \in q\} \end{aligned}$$

is definable in $V[G_\alpha]$ from $S_\alpha$ and $\phi_\alpha$; in particular, $D \in V[G_\alpha]$. Moreover, we have shown that $D$ is dense below $p$ and so, by the genericity of $H_\alpha$, $p \in H_\alpha$ implies $H_\alpha \cap D \neq \emptyset$. Furthermore, we have shown that, if $q \in \mathbb{P}$ and $q(\alpha) \in D$, then, taking $\dot{f}$, and $\dot{X}$ to be names for $f$ and $X$ respectively, and taking $\dot{F}$ to be a name for the unique continuous extension of $f$ to $X$ in $V[G]$, if it exists, then

$$q \Vdash (\dot{f} \text{ does not extend to an automorphism of } \dot{X} \text{ or } \dot{F} \text{ is regressive at } \beta)$$

It follows from the genericity of $G$ that if $f$ extends to an automorphism $F$ of $X$ in $V[G]$ then $F$ is regressive at $\beta$. As $\beta$ was an arbitrary ordinal with $\alpha_0 \leq \beta < \kappa$, $F$ is regressive on $[\alpha_0, \kappa)$. We may then use $F$ to define a regressive function $h$ on $[\alpha_0, \kappa)$ in $V[G]$: $h(\beta)$ is the least $\gamma$ such that, for some $x \in X_\beta$, $F(x) \in X_\gamma$. We now have two cases to consider:

*Case 1:* Suppose $\kappa$ is regular. By Fodor's Lemma, $h$ is constant on a stationary set $S$. If $\gamma$ is the common value of $h$ on $S$, then, as $S$ is uncountable, $F$ must map uncountably many elements of $X$ to $X_\gamma$, which is countable. This contradicts the injectivity of $F$, so $f$ does not extend to an automorphism of $X$ in $V[G]$.

*Case 2:* Suppose $\kappa$ is singular. Since $\kappa$ is a limit of regular cardinals, there is an uncountable regular cardinal $\lambda$ such that $\alpha_0 < \lambda < \kappa$. Apply Fodor's Lemma to $h \restriction \lambda$ to find a set of size $\lambda$ on which $h$ is constant. As in the previous paragraph, this contradicts the injectivity of $F$, so again $f$ does not extend to an automorphism of $X$ in $V[G]$.

As $f$ was an arbitrary non-identity continuous function on $\mathbb{Q}$, $X$ is rigid in $V[G]$. $\square$

We can recover part of Theorem 2.1 as a corollary to Theorem 2.4:

**Corollary 2.5.** *Let $\kappa$ and $\lambda$ be uncountable cardinals with $\lambda \leq \kappa$. Forcing with $Fn(\kappa, 2)$ adds a rigid $X \subseteq \mathbb{R}$ with $|X| = \lambda$.*

*Proof.* $Fn(\kappa, 2)$ has the ccc, so $\lambda$ remains a cardinal in the extension. If $\lambda \leq \kappa$ then $Fn(\kappa, 2) \cong Fn(\kappa, 2) \times Fn(\lambda, 2)$. Thus, by the Product Lemma, forcing with $Fn(\kappa, 2)$ is equivalent to forcing with $Fn(\kappa, 2)$ and then forcing with $Fn(\lambda, 2)$. If $\mathrm{cf}(\lambda) \neq \omega$ then, by Theorem 2.4, the end result will contain a rigid $X \subseteq \mathbb{R}$ with $|X| = \lambda$.

Now suppose $\mathrm{cf}(\lambda) = \omega$ and let $\langle \alpha_n : n < \omega \rangle$ be a sequence of distinct regular uncountable cardinals with limit $\lambda$. We work in $V[G]$. For each $\alpha_n$ there is, by the previous paragraph, a rigid $X_n \subseteq \mathbb{R}$ with $|X_n| = \alpha_n$. Furthermore, we may assume that, for each $n$, $X_n \subseteq (n, n + \frac{1}{2})$ and (by a simple genericity argument) that $U \cap X_n = \alpha_n$ for every open $U \subseteq (n, n + \frac{1}{2})$. Taking $X = \bigcup_{n<\omega} X_n$, it is not difficult to show that $X$ must be rigid. $\square$

**Corollary 2.6.** *If $\kappa \geq \omega_2$ and $G$ is generic over $Fn(\kappa, 2)$, then $V[G] \models$ there is a rigid $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$.*

As we have mentioned already, this second proof of the consistent existence of small rigid subsets of $\mathbb{R}$ does more than the first, because it generalizes to show that such subsets abound:

**Theorem 2.7.** *Let $\mathbb{P}$ be a ccc notion of forcing obtained by a finite support iteration of nontrivial partial orders. If $\mathbb{P}$ is of length $\kappa$ and $\mathrm{cf}(\kappa) \neq \omega$, then $\mathbb{P}$ adds a rigid subset of $\mathbb{R}$ of size $\lambda$ for every $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$.*

*Proof.* Observe that the proof of Theorem 2.4 does not depend in any essential way on the distribution of new reals in our iteration: it is enough that new Cohen reals are added at cofinally many stages. More formally, taking $\mathbb{P}$ to be the finite support iteration of a sequence $\left\langle \dot{Q}_\alpha \colon \alpha < \kappa \right\rangle$, it is enough that, for an iteration of length $\kappa$, there is an unbounded $S \subseteq \kappa$ such that, for all $\alpha \in S$, $\mathbb{R}^{new}_\alpha$ contains a Cohen real. We may use this set $S$ to define our sets $X_\alpha$ as before, and the rest of the proof of Theorem 2.4 remains essentially unchanged. Notice that it is the order type of $S$, not its particular elements, that allows us to write $X = \bigcup_{\beta<\kappa} X_\alpha$, and this, in turn, legitimizes our use of Fodor's Lemma.

To prove the theorem, then, it suffices to show that any ccc notion of finite support iterated forcing has such a set $S$ of size $\lambda$ for any $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$. This follows from the fact that, if each of the $\dot{Q}_\alpha$ is nontrivial, then a Cohen real is added at every limit stage of cofinality $\omega$. This is true for any finite support iteration of nontrivial partial orders (see [24], p. 306). $\square$

In the next section we point out a ccc forcing of length $\omega_2$ which, given a particular condition in the ground model, forces that there are no rigid subsets of $\mathbb{R}$ of size $\aleph_1$. Thus, in contrast to Corollary 2.5, Theorem 2.7 cannot be improved to say that any ccc notion of forcing obtained by a length-$\kappa$ finite support iteration of nontrivial partial orders adds a rigid subset of size $\lambda$ whenever $\aleph_0 < \lambda < \kappa$.

**Corollary 2.8.** *Let $\kappa \geq \omega_2$, $\mathrm{cf}(\kappa) \neq \omega$. If $\lambda \leq \kappa$ has uncountable cofinality, then forcing with $Fn(\kappa, 2)$ adds a rigid, non-meagre $X \subseteq \mathbb{R}$ with $|X| = \lambda$. Moreover, there is a ccc notion of forcing $\mathbb{P}$ such that forcing with $\mathbb{P}$ adds a rigid $X \subseteq \mathbb{R}$ such that $X$ is non-meagre, non-null, and $|X| = \kappa$.*

*Proof.* The first assertion is nearly proved, since we have shown that $Fn(\kappa, 2)$ adds a rigid $X \subseteq \mathbb{R}$ of size $\lambda$; we will finish the proof of the first assertion by showing that this same $X$ is non-meagre. This is accomplished by using Borel codes. Let $A \in BC^{V[G]}$ code a meagre set. $A$ is an hereditarily countable object so, by Lemma 2.3, $A \in V[G_\alpha]$ for some $\alpha < \kappa$. The extension of $V[G_\alpha]$ to $V[G_{\alpha+1}]$ is an extension by Cohen-generic reals, and it is well-known that a Cohen real misses any null set coded in the ground model. Thus $\mathbb{R}^{new}_{\alpha+1} \setminus A^* \neq \emptyset$, and we can choose $X_\beta$ such that

$X_\beta \cap A^* = \emptyset$. To do this (using the notation of the proof of Theorem 2.4) take $S_{\beta+1} = \{x_{\beta+1} + q \colon q \in \mathbb{Q}\}$, where $x_{\beta+1}$ is Cohen over $V[R_\beta]$; as before, we can determine in $V[G]$ whether $x$ is Cohen over $V[G_\beta]$ by looking at some sufficiently large $V_\gamma$. It follows from our construction that $V[G] \models X \not\subseteq A^*$. As $A$ was arbitrary, $X$ is not meagre.

For the second assertion, let $\mathbb{P}$ be the finite support iteration of the sequence $\langle \check{Q}_\alpha \colon \alpha < \kappa \rangle$, where $Q_{\omega\cdot\alpha+n}$ is the Cohen forcing for even $n$ and is the forcing that adds a random real for odd $n$. Using $\mathbb{P}$ instead of $Fn(\kappa, 2)$, the proof of Theorem 2.4 remains essentially the same. The only real difference is that we will use both Cohen reals and random reals to define our $S_\alpha$, but we will still need to define our $X_\alpha$ from $S_\alpha$ using Cohen reals. Since Cohen reals are added cofinally, this can be done with the simple use of an appropriate indexing function. In this way, we may ensure that $X_{\omega\cdot\alpha+n}$ contains a Cohen real from $\mathbb{R}^{new}_{\omega\cdot\alpha+n}$ for odd $n$ and a random real from $\mathbb{R}^{new}_{\omega\cdot\alpha+n}$ for even $n$. Repeating the argument of the previous paragraph, we now have that $X$ cannot be contained in any null set or in any meagre set. $\square$

Corollary 2.8 raises the question of whether a small rigid subset of $\mathbb{R}$ can be meagre, measurable, or both.

**Proposition 2.9.** *It is consistent with ZFC that there is a rigid, meagre, Lebesgue-null $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$. Moreover, it is consistent that there is* *a small rigid subset of $\mathbb{R}$ which has any of the four possible combinations of the properties null, non-null, meagre, and non-meagre.*

*Proof.* To obtain a set that is both null and meagre, consider a null and meagre Cantor subset of $\mathbb{R}$, e.g., the typical middle-thirds embedding. Up to a bit of fiddling and indexing, the Cantor set is essentially a copy of $\mathbb{R}$. Thus, up to some fiddling, we may use the Cohen forcing to add generic elements to our Cantor set. Without much modification, the proof of Theorem 2.4 may be used to show how these generic elements may be used to construct a small rigid subset of the reals inside of our null and meagre Cantor set. More formally, the proof is modified to show that, for a fixed Borel code $A$ for a (meagre, null) Cantor set in the ground model, we may add a small rigid set which is contained in $A*$. By well-known absoluteness arguments, $A*$ remains meagre and null in any generic extension. A similar technique is used for adding a small rigid set that is null and non-meagre or which is meagre and non-null. $\square$

## 2.2 Baumgartner's Axiom

In the previous section we showed that it is consistent to have a small rigid subset of $\mathbb{R}$, and in this section we show that the opposite is also consistent.

First, it follows almost trivially from *CH* that every $X \subseteq \mathbb{R}$ with $2 \leq |X| < \mathfrak{c}$ is non-rigid. This is because every countably infinite subset of $\mathbb{R}$ contains either a copy of $\mathbb{Q}$ or multiple isolated points.

A better result is that the non-existence of small rigid subsets of $\mathbb{R}$ is consistent with $\mathfrak{c} = \aleph_2$. This follows (with a bit of effort) from a result of Baumgartner, and most of this section will be devoted to proving it. Recall that $X \subseteq \mathbb{R}$ is **$\kappa$-dense** if, for any open $U \subseteq \mathbb{R}$, $|U \cap X| = \kappa$. **Baumgartner's Axiom**, henceforth abbreviated *BA*, states that *any two $\aleph_1$-dense subsets of $\mathbb{R}$ are order isomorphic.* Generalizing slightly, we will take $BA_\kappa$ to be the assertion that any two $\kappa$-dense subsets of $\mathbb{R}$ are order isomorphic. $BA_\kappa$ can be thought of as a generalization to higher cardinals of Cantor's theorem stating that any two countable dense total orders are isomorphic.

**Lemma 2.10.** *$BA_{\aleph_0}$ is true and* $\mathrm{BA}_\mathfrak{c}$ *is false.*

*Proof.* $BA_{\aleph_0}$ is a theorem of Cantor (see [10]), and it is easy to construct counterexamples to $BA_\mathfrak{c}$ (e.g., $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$). □

We will see in Corollary 2.19 that $BA_\kappa$ implies $2^\kappa = \mathfrak{c}$. This proves that *BA*, along with each $BA_\kappa$, is independent of *ZFC*. It also suggests that *MA* or *PFA* might be useful for proving the consistency of *BA*. As it turns out, *PFA* does imply *BA*, but Abraham and Shelah showed in [2] that *MA* does not.

**Theorem 2.11** (Baumgartner, [5])**.** *If ZFC is consistent then so is ZFC+ BA. Furthermore, PFA implies BA.*

*Proof.* In [5], Baumgartner proves that if *CH* holds then there is ccc notion of forcing that forces $BA + \mathfrak{c} = \aleph_2$.

For the second assertion, consider the partial order $P * Q$, where $P$ is the countably closed forcing that collapses $\mathfrak{c}$ to $\aleph_1$ and where $Q$ is Baumgartner's ccc forcing. $P * Q$ is a two-step iteration of proper forcings, hence itself proper, and $\mathbb{1}_{P*Q} \Vdash BA$. □

It is worth noting that the full strength of *PFA* is not needed to prove *BA*, as Baumgartner's argument in [5] does not involve any large cardinal assumptions. In [1], Abraham, Rubin, and Shelah strengthen Baumgartner's result by proving *BA* is consistent with an arbitrarily large continuum (again, this proof does not require large cardinals).

Nothing seems to be known about the consistency of $BA_\kappa$ for any $\kappa \geq \aleph_2$, and we do not attempt to solve this problem here. We will, however, prove some consequences of $BA_\kappa$.

**Lemma 2.12.** *$BA_\kappa$ implies that any two $\kappa$-dense subsets of $\mathbb{R}$ are homeomorphic.*

*Proof.* Let $X$ and $Y$ be $\kappa$-dense subsets of $\mathbb{R}$. Assuming $BA_\kappa$, there is an order isomorphism $h : X \to Y$. Because $X$ and $Y$ are dense, it is easy to see that both $h$ and $h^{-1}$ map open intervals to open intervals, and this shows that $h$ is a homeomorphism. □

Lemma 2.12 is a first step toward showing that $BA + \mathfrak{c} = \aleph_2$ implies there are no small rigid subsets of $\mathbb{R}$. However, Lemma 2.12 only talks about dense subsets of $\mathbb{R}$, and it is fairly obvious that not every small subset of $\mathbb{R}$ is order isomorphic to a dense subset of $\mathbb{R}$. In order to make $BA$ say something about arbitrary small subsets of $\mathbb{R}$, we have the following lemma:

**Lemma 2.13.** *Let $X$ be a nonempty subset of $\mathbb{R}$. If $X$ is totally disconnected and no open subset of $X$ is compact, then $X$ is homeomorphic to a dense subset of $\mathbb{R}$.*

*Proof.* Consider the usual middle-thirds embedding $C$ of the Cantor set in $\mathbb{R}$. We say that a point of $C$ is an *endpoint* if it is the first or last point of an open subset of $C$ (where the order on $C$ is just the restriction of the usual order on $\mathbb{R}$). Let $E$ denote the set of all these endpoints; note that $E$ is countable and dense in $C$.

Every totally disconnected set of reals $Y$ is homeomorphic to a nowhere dense set of reals. To see this, let $\{x_n \colon n \in \omega\}$ be dense in $\mathbb{R} \setminus Y$ and define $h : Y \to \mathbb{R}$ by $h(x) = x + \sum_{x_n < x} \frac{1}{2^n}$. It is easy to show that $h$ is a homeomorphism and that the image of $h$ is nowhere dense in $\mathbb{R}$.

Without loss of generality, assume $X$ is nowhere dense and bounded in $\mathbb{R}$. $X$ contains no isolated points (an isolated point would constitute a compact open subset), so $\overline{X}$ is a Cantor set. Without loss of generality, assume $\overline{X} = C$. Note that $C \setminus X$ is dense in $C$: this is because $X$ has no compact open subsets while $C$ has a basis consisting of compact sets.

Recall that a $C$ is *countable dense homogeneous*: for any countable dense $D_1, D_2 \subseteq C$ there is an automorphism $h : C \to C$ such that $h[D_1] = D_2$ (see [6] for a proof that the Cantor set is countable dense homogeneous). Let $D$ be any countable subset of $C \setminus Y$ that is dense in $C$. There is an automorphism $h$ of $C$ that maps $D$ onto $E$. Moreover, since $Y$ is dense in $C$ and $h$ is a homeomorphism, $h[Y]$ is dense in $h[C] = C$.

By the previous paragraph, we may assume that $X$ is dense in $C$ and contains no endpoints of $C$. Consider the continuous map $f : C \to [0,1]$ defined by

$$\sum_{n \in \omega} \frac{2 \cdot d_n}{3^n} \qquad \mapsto \qquad \sum_{n \in \omega} \frac{d_n}{2^n}$$

where $\langle d_n \colon n \in \omega \rangle \in 2^\omega$. It is not hard to see that $f$ restricts to a dense embedding on $C \setminus E$. Since $X$ contains no endpoints of $C$, $f[X]$ is homeomorphic to $X$ and dense in $(0,1)$. Applying another homeomorphism, we obtain a set that is homeomorphic to $X$ and dense in $\mathbb{R}$. □

A subset $X$ of $\mathbb{R}$ is **$\kappa$-perfect** if, for every open $U \subseteq \mathbb{R}$, either $U \cap X = \emptyset$ or $|U \cap X| = \kappa$.

**Corollary 2.14.** *$BA_\kappa$ implies that any two $\kappa$-perfect subsets of $\mathbb{R}$ are homeomorphic.*

*Proof.* If $\kappa < \mathfrak{c}$ then any $\kappa$-perfect subset of $\mathbb{R}$ is totally disconnected (recall that, for subsets of $\mathbb{R}$, being totally disconnected is equivalent to containing no intervals). Furthermore, every compact subset of $\mathbb{R}$ either contains isolated points or has size $\mathfrak{c}$. Thus, by Lemma 2.13, if $X$ and $Y$ are $\kappa$-perfect, then they are each homeomorphic to a dense subset of $\mathbb{R}$ and, by Lemma 2.12, homeomorphic to each other. □

**Lemma 2.15.** *$BA_\kappa$ implies that every $\kappa$-perfect subset of $\mathbb{R}$ is homogeneous. Furthermore, BA implies that every $\aleph_1$-perfect subset of $\mathbb{R}$ is countable dense homogeneous.*

*Proof.* The second assertion follows from Corollary 2.14 and a result of Farah, Hrušák, and Ranero, namely that there is an $\aleph_1$-dense subset of

$\mathbb{R}$ that is countable dense homogeneous (see [20]). Likewise, the first assertion follows from Corollary 2.14 and a result of van Mill, namely that for every $\kappa \leq \mathfrak{c}$ there is a homeogeneous subset of $\mathbb{R}$ with size $\kappa$.

In fact, there is a very simple proof of the first assertion of this theorem, and we give that here as well:

Let $X$ be a $\kappa$-perfect subset of $\mathbb{R}$. Since $BA_{\mathfrak{c}}$ is false, $X$ is small and hence totally disconnected. Every open subset of $X$ is perfect and every perfect compact subset of $\mathbb{R}$ has size $\mathfrak{c}$, so $X$ has no compact open subsets. By Lemma 2.13, we may assume that $X$ is $\kappa$-dense.

Let $x_1, x_2 \in X$. Since $|X| < \mathfrak{c}$, there are $a, b, c, d \in \mathbb{R} \setminus X$ such that $x_1 \in (a, b)$, $x_2 \in (c, d)$, and $(a, b) \cap (c, d) = \emptyset$. Since $\mathbb{R}$ is homeomorphic to an open interval by an order-preserving map, we may consider $X \cap (a, x_1)$, $X \cap (x_1, b)$, $X \cap (c, x_2)$, and $X \cap (x_2, d)$ to be $\kappa$-dense subsets of $\mathbb{R}$. By $BA_\kappa$, there are order isomorphisms $h_1 : (a, x_1) \cap X \to (c, x_2) \cap X$ and $h_2 : (x_1, b) \cap X \to (x_2, d) \cap X$. Define $h : (a, b) \cap X \to (c, d) \cap X$ by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in (a, x_1) \cap X \\ x_2 & \text{if } x = x_1 \\ h_2(x) & \text{if } x \in (x_1, b) \cap X \end{cases}$$

Clearly $h$ is an order-preserving injection $(a, b) \cap X \to (c, d) \cap X$ which maps $x_1$ to $x_2$. Since $X$ is dense in both $(a, b)$ and $(c, d)$, $h$ is a homeomorphism.

Furthermore, since $a, b, c, d \notin X$, $(a, b) \cap X$ and $(c, d) \cap X$ are clopen subsets of $X$. Thus we may extend $h$ to a homeomorphism of $X$ by setting $h(x) = x$ for $x \notin (a, b) \cup (c, d)$. $\square$

**Theorem 2.16.** *If $BA_\kappa$ holds for every $\kappa < \lambda \leq \mathfrak{c}$, then every $X \subseteq \mathbb{R}$ with $2 \leq |X| < \lambda$ is non-rigid.*

*Proof.* Let $\kappa$ be the smallest cardinal other than 0 or 1 such that, for some open $U \subseteq \mathbb{R}$, $|U \cap X| = \kappa$. If $2 \leq \kappa \leq \aleph_0$ then it is clear that $X$ cannot be rigid, so assume $\kappa$ is uncountable. If $X$ is rigid then $X$ can have at most one isolated point, so, by removing that point from $U$ if necessary, we may assume that $U$ contains no isolated points of $X$. Pick $a, b \in U$ such that $(a, b) \subseteq U$, $X \cap (a, b) \neq \emptyset$, and $a, b \notin X$; this is possible because $U$ is open and the complement of $X$ is dense. $X \cap (a, b)$ is a clopen subset of $X$ and, by our choice of $\kappa$, $X \cap (a, b)$ is $\kappa$-perfect. By Lemma 2.15, $X \cap (a, b)$ is homogeneous and, since $X \cap (a, b)$ is clopen in $X$, this implies that $X$ is not rigid. $\square$

**Corollary 2.17.** *BA implies that every $X \subseteq \mathbb{R}$ with $|X| \leq \aleph_1$ is non-rigid.*

By Theorem 2.11, it is consistent with $\mathfrak{c} = \aleph_2$ that all nontrivial rigid subsets of $\mathbb{R}$ have size $\mathfrak{c}$. Moreover, by Abraham, Rubin, and Shelah's result, it is consistent with an arbitrarily large continuum that all nontrivial rigid subsets of $\mathbb{R}$ have size at least $\aleph_2$. It is still unknown whether it is consistent with $\mathfrak{c} > \aleph_2$ that every rigid subset of $\mathbb{R}$ has size $\mathfrak{c}$, though we have shown that this would follow from the consistency of all $BA_\kappa$, $\kappa < \mathfrak{c}$, if $\mathfrak{c} > \aleph_2$.

Abraham and Shelah's proof that *MA* does not imply *BA* makes use of what they call an entangled set of reals and, although they show that such a set destroys *BA*, it is not clear that an entangled set should be topologically rigid. It remains an open question whether *MA* is consistent with a small rigid subset of $\mathbb{R}$. It is worth noting that Proposition 2.7 cannot be used to produce a model in which *MA* holds and there is a small rigid subset of $\mathbb{R}$. This is due to the following well known result:

**Proposition 2.18.** *Let $\mathbb{P}$ be a ccc notion of forcing obtained by a finite support iteration of nontrivial partial orders. If $\mathbb{P}$ is of length $\kappa$ and $\mathrm{cf}(\kappa) \neq \omega$, then $MA_\kappa$ fails in any extension by $\mathbb{P}$.*

*Proof.* As seen in Corollary 2.8, there is a non-meagre subset $X$ of $\mathbb{R}$ in $V[G]$ with $|X| = \kappa$. □

We conclude this section with the following observation:

**Proposition 2.19.** *$BA_\kappa$ implies $2^\kappa = \mathfrak{c}$.*

*Proof.* Let $\aleph_0 \leq \kappa < \mathfrak{c}$. Since $2^\kappa \geq 2^{\aleph_0} = \mathfrak{c}$, it suffices to show $2^\kappa \leq \mathfrak{c}$. We accomplish this by counting the $\kappa$-perfect subsets of $\mathbb{R}$ in two different ways.

First we show that there are at least $2^\kappa$ $\kappa$-dense subsets of $\mathbb{R}$. Let $X$ be any $\kappa$-perfect subset of $\mathbb{R}$ and let $Y$ be any set satisfying $Y \subseteq \mathbb{R} \setminus X$ and $|Y| = \kappa$. If $Z \subseteq Y$ then $X \cup Z$ is a $\kappa$-perfect subset of $\mathbb{R}$. Thus $Z \mapsto X \cup Z$ gives an injection from $\mathcal{P}(Y)$ into the set of $\kappa$-perfect subsets of $\mathbb{R}$.

Next we show (assuming $BA_\kappa$) that there are at most $\mathfrak{c}$ $\kappa$-perfect subsets of $\mathbb{R}$. Fix some $\kappa$-perfect $X \subseteq \mathbb{R}$ and let $D$ be a countable dense subset of $X$. Because $X$ is Hausdorff, any continuous function on $X$ is determined by its action on $D$. Let $Y$ be another $\kappa$-perfect subset of $\mathbb{R}$. By Corollary 2.14, $X$ and $Y$ are homeomorphic; in particular, there is a continuous function $f : X \to Y$ whose image is $Y$. Since $f$ is determined uniquely by $f \restriction D$, $Y$ is determined uniquely by $f \restriction D$. Thus there is an injection from the set of all $\kappa$-perfect subsets of $\mathbb{R}$ into the set of all continuous functions from $D$ into $\mathbb{R}$. Since $D$ is countable, the number of such functions is $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. □

## 2.3 The number of subspaces of $\mathbb{R}$

Up to homeomorphism, how many subspaces does $\mathbb{R}$ have? How many of these are smaller than $\mathfrak{c}$ or of size $\kappa$ for some $\kappa < \mathfrak{c}$? We will show in this section that the answer to all of these questions is the maximum possible answer allowed by cardinal arithmetic: there are as many subspaces as there are subsets.

We may also ask how many size-$\kappa$ subspaces of $\mathbb{R}$ are rigid, or how many are $\kappa$-perfect for $\kappa < \mathfrak{c}$. These questions are not answerable under *ZFC*. Under $BA + \mathfrak{c} = \aleph_2$ the answers are, respectively, 0 and 1 (up to homeomorphism). In this section we will obtain some consistency results that show that the answers to these questions can also be the largest possible.

We have already proved two results in this vein: Lemma 1.8 states that there are $\mathfrak{c}$ topologically distinct subsets of $\mathbb{Q}$ and Corollary 2.14 states that, assuming $BA_\kappa$ for all $\kappa < \mathfrak{c}$, there is for any $\kappa < \mathfrak{c}$ exactly one $\kappa$-perfect subspace of $\mathbb{R}$. Henceforth, we shall write $BA_{<\mathfrak{c}}$ to abbreviate the assertion that $BA_\kappa$ holds for every $\kappa < \mathfrak{c}$.

Recall that a perfect subset of $\mathbb{R}$ is one without any isolated points. We saw in Lemma 1.8 that there are $\mathfrak{c}$ homeomorphism classes of countable subsets of $\mathbb{R}$, and it follows that there are at least $\mathfrak{c}$ homeomorphism classes of size-$\kappa$ subsets of $\mathbb{R}$. However, the proof of Lemma 1.8 relies explicitly on sets of isolated points. If we restrict ourselves to nonempty perfect subsets of $\mathbb{R}$ then, by Cantor's theorem $BA_{\aleph_0}$, there is only one homeomorphism class of countable subsets of $\mathbb{R}$: any nonempty countable perfect subset of $\mathbb{R}$ is homeomorphic to $\mathbb{Q}$. Continuing our earlier usage, we will call a subset $X$ of $\mathbb{R}$ small if $|X| < \mathfrak{c}$. Assuming $BA_{<\mathfrak{c}}$, one might hope, especially in light of Theorem 2.14, that we might be able to put a small ($< \mathfrak{c}$) bound on the number of small perfect subsets of $\mathbb{R}$. The following lemma shows that this is not the case.

**Lemma 2.20.** *If $\aleph_1 \leq \kappa \leq \mathfrak{c}$ then there are at least $\mathfrak{c}$ distinct homeomorphism classes of size-$\kappa$ dense subsets of $\mathbb{R}$. In particular, if CH fails then there are at least $\mathfrak{c}$ distinct homeomorphism classes of small dense subsets of $\mathbb{R}$.*

*Proof.* Suppose $\kappa < \mathfrak{c}$. By Lemma 2.13, if $X \subseteq \mathbb{R}$ is perfect then $X$ is homeomorphic to a dense subset of $\mathbb{R}$. Thus it suffices to prove that there are at least $\mathfrak{c}$ size-$\kappa$ perfect subsets of $\mathbb{R}$ that are nowhere dense and have no compact open subsets.

Let $Q_0$ and $Q_\kappa$ be, respectively, $\aleph_0$-dense and $\kappa$-dense subsets of $\mathbb{R}$. For example, $Q_0$ might be the rational numbers and $Q_\kappa$ might be the union of $\kappa$ different translations of $Q_0$.

Consider the space

$$A = ((-1,0) \cap Q_0) \cup \{0\} \cup ((0,1) \cap Q_\kappa)$$

In the space $A$, every open neighborhood of 0 has an $\aleph_0$-perfect open subset and a $\kappa$-perfect open subset. In general, call a point $x$ of some $X \subseteq \mathbb{R}$ an $\aleph_0$-$\kappa$**-point** if every neighborhood of $x$ in $X$ contains an open $\aleph_0$-perfect set and an open $\kappa$-perfect set. Since being $\lambda$-perfect (for any infinite $\lambda$) is a topological property, so is being an $\aleph_0$-$\kappa$-point. Notice that 0 is the only $\aleph_0$-$\kappa$-point of $A$.

Let $Q_0$ be as before and let $C_\kappa$ be $\kappa$-dense in the Cantor set (i.e., every open subset of the Cantor set meets $C_\kappa$ in $\kappa$ points); scale $C_\kappa$ so that the leftmost and rightmost points of its closure are 0 and 1, respectively. Consider the space

$$B = \{0\} \cup ((0,1) \cap Q_0) \cup C_\kappa$$

In the space $B$, every open neighborhood of 0 contains $\kappa$ points but no neighborhood of 0 contains a $\kappa$-perfect open set. In general, we will say that $x$ is **$\kappa$-Cantorian** if every neighborhood of $x$ contains $\kappa$ points, and every neighborhood of $x$ contains an open subset $U$ such that $|U| = \kappa$ but there is no open $\kappa$-perfect $V \subseteq U$ (in general, we do not require that this $U$ is a neighborhood of $x$). Note that being $\kappa$-Cantorian is a topological property.

Let $S$ be a closed scattered subspace of $\mathbb{R}$ such that every point of $S$ has an immediate successor in $S$ (i.e., all convergence in $S$ happens "from the left"). We will give a "recipe" for modifying $S$ to obtain a space $M(S)$. If $x$ is an isolated point of $S$, let $\varepsilon_x = \sup(\{\varepsilon\colon (x-\varepsilon, x+\varepsilon) \cap S = \{x\}\} \cup \{1\})$. If $x$ is a limit point of $S$, let $\varepsilon_x = \sup(\{\varepsilon\colon [x, x+\varepsilon) \cap S = \{x\}\} \cup \{1\})$. In both cases $\varepsilon_x > 0$. Let $S'$ denote the set of limit points of $S$, and let $T$ be any subset of $S'$. Letting $aY+b$ denote $\{ay+b\colon y \in Y\}$, define

$$M(S)=\bigcup\left\{\frac{\varepsilon_x}{3}A+x\colon x\in S\setminus S'\right\}\cup\bigcup\left\{\frac{\varepsilon_x}{2}B+x\colon x\in T\right\}\cup S'\setminus T$$

Intuitively, $M(S)$ looks just like $S$ except that every isolated point of $S$ has been replaced by a small copy of $A$ and every point in $T$ has been made the leftmost point of a small copy of $B$. Notice that $M(S)$ is always a perfect subset of $\mathbb{R}$ of size $\kappa$. It is possible to recover $S$ from $M(S)$: $S$ is the set of all 0-$\kappa$-points of $M(S)$. Since being a 0-$\kappa$-point is a topological property, $M(S)$ and $M(V)$ are not homeomorphic if $S$ and $V$ are not homeomorphic. The converse is obvious, so $M(S)\approx M(V)$ if and only if $S\approx V$. Moreover, the set $T\subseteq S'$ may be recovered from $M(S)$ as the set of all $\kappa$-Cantorian points in $M(S)$.

Thus we have established an injection from the set of pairs of the form $(S,T)$, where $S\subseteq\mathbb{R}$ is scattered and closed, all convergence in $S$ is from the left, and $T\subseteq S'$, into the set of size-$\kappa$ perfect subspaces of $\mathbb{R}$. The proof of Lemma 1.8 shows that there are $\mathfrak{c}$ topologically distinct pairs of this form. This proves that there are at least $\mathfrak{c}$ topologically distinct subspaces of $\mathbb{R}$ with size $\kappa$.

If $\kappa=\mathfrak{c}=\aleph_1$, the argument is essentially the same; we simply restrict ourselves to those subsets of $\mathbb{R}$ that have no open compact subspaces. This allows us to consider subsets of $\mathbb{R}$ that are not dense, and the argument goes through as for $\kappa < \mathfrak{c}$. □

**Theorem 2.21.** *For any infinite $\kappa \leq \mathfrak{c}$, the number of distinct homeomorphism classes of size-$\kappa$ subsets of $\mathbb{R}$ is $2^\kappa$. If $\aleph_1 \leq \kappa \leq \mathfrak{c}$, the number of distinct homeomorphism classes of dense size-$\kappa$ subsets of $\mathbb{R}$ is $2^\kappa$.*

*Proof.* If $2^\kappa = \mathfrak{c}$ then this follows from Lemmas 1.8 and 2.20. Let $2^\kappa > \mathfrak{c}$ and suppose there are $\lambda < 2^\kappa$ homeomorphism classes of dense size-$\kappa$ subsets of $\mathbb{R}$. In this case, one of the homeomorphism classes must have size larger than $\mathfrak{c}$, since $\mathfrak{c} \cdot \lambda = \max\{\mathfrak{c}, \lambda\} < 2^\kappa$. If $X$ is $\kappa$-dense, take $D$ to be a countable dense subset of $X$; every homeomorphism of $X$ is determined by its action on $D$, and the number of maps from $D$ into $\mathbb{R}$ is $|\mathbb{R}|^{|D|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Thus there are at most $\mathfrak{c}$ subsets of $\mathbb{R}$ that are homeomorphic to $X$, contradicting our earlier assertion that some homeomorphism class must have size larger than $\mathfrak{c}$. □

In contrast to Theorem 2.21, Corollary 2.14 asserts that under $BA_\kappa$ all $\kappa$-perfect subsets of $\mathbb{R}$ are homeomorphic. Naturally, one might wonder what are the possibilities for the number of homeomorphism classes of $\kappa$-perfect subsets of $\mathbb{R}$. The next few results give partial answers to this question.

**Proposition 2.22.**

*(1) If $2^\kappa > \mathfrak{c}$ and $\kappa < \mathfrak{c}$, there are $2^\kappa$ topologically distinct $\kappa$-perfect subspaces of $\mathbb{R}$.*

*(2) If there is a rigid $\kappa$-perfect subset of $\mathbb{R}$ then there are at least $\mathfrak{c}$ topologically distinct rigid $\kappa$-perfect subspaces of $\mathbb{R}$.*

*(3) Suppose $X, Y \subseteq \mathbb{R}$ are $\kappa$-perfect and mutually generic, i.e., no clopen subspace of $X$ is homeomorphic to a clopen subspace of $Y$. Then there are at least $\aleph_1$ topologically distinct $\kappa$-perfect subspaces of $\mathbb{R}$. If there are three mutually generic spaces $X$, $Y$, and $Z$, then there are at least $\mathfrak{c}$ $\kappa$-perfect subspaces of $\mathbb{R}$.*

*Proof.*

(1) The proof is similar to that of Theorem 2.21: there are $2^\kappa$ distinct $\kappa$-perfect subsets of $\mathbb{R}$, and each equivalence class of these under the homeomorphism relation has size $\mathfrak{c}$.

(2) Let $X \subseteq \mathbb{R}$ be $\kappa$-perfect and rigid. By Lemma 2.13, we may assume that $X$ is $\kappa$-dense. Let

$$\mathcal{A} = \{(a, b) \cap X : a, b \in \mathbb{R} \setminus X \text{ and } a < b\}$$

Since $\kappa < \mathfrak{c}$, $|\mathcal{A}| = \mathfrak{c}$. We claim that any two distinct elements of $\mathcal{A}$ are not homeomorphic. Suppose $(a, b) \cap X, (c, d) \cap X \in \mathcal{A}$ and suppose that $h : (a, b) \cap X \to (c, d) \cap X$ is a homeomorphism. Since $(a, b) \cap X$ and $(c, d) \cap X$

are clopen in $X$, $X$ is not rigid if they are disjoint; suppose, without loss of generality, that $(a,b) \setminus (c,d) \neq \emptyset$. Since $X$ is totally disconnected and hence zero-dimensional, we may find a nonempty clopen $U \subseteq (a,b) \setminus (c,d) \cap X$. $h[U]$ is clopen in $(c,d) \cap X$ and hence in $X$, and $U \cap h[U] = \emptyset$. This shows that $X$ contains two homeomorphic disjoint clopen subsets, contradicting the fact that $X$ is rigid. Hence all of the elements of $\mathcal{A}$ are topologically distinct. Because it is a clopen subset of a rigid set, each element of $\mathcal{A}$ is rigid.

(3) For the second assertion, the proof is formally identical to the proof of Lemma 2.20, using $X$ and $Y$ in place of $Q_0$ and $Q_\kappa$, and $Z$ in place of $B$.

For the first assertion, let $A$ be defined as in the proof of Lemma 2.20, only using $X$ and $Y$ for $Q_0$ and $Q_\kappa$ respectively. Then, for any closed scattered $S \subseteq \mathbb{R}$, define $\varepsilon_x$ as before and let

$$M(S) = \left\{\frac{\varepsilon_x}{3}A + x \colon x \in S \setminus S'\right\} \cup S'$$

$M(S)$ is clearly $\kappa$-perfect. It is easy to see, from the given hypotheses on $X$ and $Y$, that the space $S$ can be recovered from the space $M(S)$ as (the closure of) all points $x$ such that every neighborhood of $x$ contains a clopen set homeomorphic to a clopen subset of $X$ and a clopen subset homeomorphic to a clopen subset of $Y$. This gives an injection from the set of all closed scattered subspaces of $\mathbb{R}$ into the set of all $\kappa$-perfect subsets of

$\mathbb{R}$. Every ordinal $\alpha < \omega_1$ is homeomorphic to a closed scattered subspace of $\mathbb{R}$, so this completes the proof. $\square$

Part (3) of the above proposition suggests the question of whether the value $\aleph_1$ is possible, under $\neg CH$, for the number of topologically distinct $\kappa$-perfect subsets of $\mathbb{R}$ (for some $\kappa < \mathfrak{c}$). Corollary 2.17 tells us that 1 is one possible value, and the next result tells us that, consistently, there may be as many size-$\kappa$ perfect subsets of $\mathbb{R}$ as possible, and that we may even take them all to be rigid.

**Proposition 2.23.** *Let $G$ be $Fn(\kappa, 2)$-generic over $V$. In $V[G]$ there are $2^\lambda$ homeomorphism classes of $\lambda$-dense rigid subsets of $\mathbb{R}$ for every uncountable $\lambda < \kappa$.*

*Proof.* For $\aleph_0 < \lambda \leq \kappa$, let $X$ be a $\lambda$-dense set of reals that are mutually generic and Cohen over $V$. If $Y, Z \subseteq R$, we will say that $Y$ and $Z$ are **uncountably distinct** if $|Y \Delta Z| > \aleph_0$. Using the method of Theorem 2.1, we will show that no two uncountably distinct subsets of $X$ are homeomorphic. Suppose $Y, Z \subseteq X$ are uncountably distinct and suppose $f \in V[X]$ is a homeomorphism $Y \to Z$. Because a homeomorphism between two subsets of $\mathbb{R}$ is an hereditarily countable object, there is some countable $Q \subseteq X$ such that $f \in V[Q]$. Because $X$ and $Y$ are uncountably distinct, there is some

$x \in X$ and some $y \in Y$ such that $x \neq y$, $f(x) = y$, and $x, y \notin Q$. Because $f(x) = y$, $y \in V[Q, x]$, contradicting the fact that all of the reals in $X$ are mutually generic.

To prove the theorem, then, it is enough to show that X has $2^\lambda$-many mutually uncountably distinct $\lambda$-dense subsets in $V[X]$ (we have just shown that no two of these are homeomorphic, and we know that each of them is rigid by Theorem 2.1). Using transfinite recursion in $V[X]$, it is easy to find a $Y, Z \subseteq X$ such that $|Y| = |Z| = \lambda$, $Z$ is $\lambda$-dense, and $Y \cap Z = \emptyset$. If $Y_1$ and $Y_2$ are any subsets of $Y$ such that $Y_1 \Delta Y_2$ is uncountable, then $Z \cup Y_1$ and $Z \cup Y_2$ are uncountably distinct $\lambda$-dense subsets of $X$. Thus we have reduced the problem to finding a set $\mathcal{A}$ of $2^\lambda$ subsets of Y (equivalently, of $\lambda$) such that, for any $X, Y \in \mathcal{A}$, $X \Delta Y$ is uncountable. Let $\{A_\alpha \colon \alpha < \lambda\}$ be a set of pairwise disjoint subsets of $\lambda$ such that, for each $\alpha < \lambda$, $|A_\alpha| = \lambda$. Take $\mathcal{A} = \{\bigcup \{A_\alpha \colon \alpha \in S\} : S \subseteq \lambda\}$. $\square$

# Chapter 3

# Cleavage

## 3.1 Introduction

If $X$ and $Y$ are topological spaces, we say that $X$ is **cleavable** over $Y$ if and only if, for any subset $A \subseteq X$, there is a continuous function $f : X \to Y$ such that $f[A] \cap f[X \setminus A] = \emptyset$. Intuitively, the image of $A$ under $f$ is a "reflection" of $A$ in $Y$, a near-copy of $A$ that sits inside of a different, possibly simpler space. This intuition is supported by the fact that there are many theorems of the form "if $Y$ has property $P$ and $X$ is cleavable over $Y$, then $X$ has property $P$"; see [3] for examples.

We can re-interpret the definition of cleavability as talking about partitions: $X$ is cleavable over $Y$ if and only if, for any partition of $X$ into two disjoint sets $A$ and $B$, there is a continuous function $f : X \to Y$ such that $f[A] \cap f[B] = \emptyset$. Generalizing to partitions of arbitrary size, we say that $X$ is **$\kappa$-cleavable** over $Y$ if and only if, for any partition $\mathcal{P}$ of $X$ into $\kappa$ disjoint sets, there is a continuous function $f : X \to Y$ such that the images of these sets under $f$ are pairwise disjoint. In this case we say that $f$ **respects** the partition $\mathcal{P}$. If $\lambda < \kappa$ then $\kappa$-cleavability implies $\lambda$-cleavability, and, if $\kappa$ is the least cardinal such that $X$ is not $\kappa$-cleavable over $Y$, we say that $\kappa$ is the **cleavability** of the pair $(X, Y)$ and write $\kappa = \clubsuit(X, Y)$. If $X$ is $\kappa$-cleavable over $Y$ for every cardinal $\kappa$, we write $\clubsuit(X, Y) = \infty$. Note that if $Y$ is nonempty then $\clubsuit(X, Y) \geq 2$ for any $X$.

**Proposition 3.1.**

*(i) $\clubsuit(X, Y) = \infty$ if and only if $X$ condenses into $Y$ (i.e., there is a continuous injection $X \to Y$).*

*(ii) If $\clubsuit(X, Y) \neq \infty$ then $\clubsuit(X, Y) \leq |Y|^+$.*

*(iii) For any space $Y$ and any class $\mathcal{C}$ of topological spaces, there is a cardinal $\kappa$ such that whenever $X \in \mathcal{C}$ is $\kappa$-cleavable over $Y$, $X$ condenses into $Y$.*

*Proof.*

$(i)$ If $\clubsuit(X, Y) = \infty$ then $X$ is $|X|$-cleavable over $Y$. Consider the partition of $X$ into singletons: any continuous function respecting this partition is an injection.

(*ii*) Suppose $X$ is $|Y|^+$-cleavable over $Y$. If $|X| \leq |Y|$, then the argument from (*i*) proves that $X$ condenses into $Y$, contrary to assumption. Thus $|X| > |Y|$ and we may partition $X$ into $|Y|^+$ nonempty subsets; the definition of $|Y|^+$-cleavability then implies that there are at least $|Y|^+$ disjoint nonempty subsets of $Y$.

(*iii*) This follows from (*i*) and (*ii*). $\square$

Part (*iii*) of this proposition defines a class of cardinal functions on $Y$: for any class $\mathcal{C}$ of topological spaces, let $\clubsuit_{\mathcal{C}}(Y)$ denote the least cardinal $\kappa$ such that if $X \in \mathcal{C}$ is $\kappa$-cleavable over $Y$ then $X$ condenses into $Y$. We will say that $\clubsuit_{\mathcal{C}}(Y)$ is the **cleavage** of $Y$ relative to the class $\mathcal{C}$.

For any space $Y$, the discrete space on $|Y|^+$ points is $|Y|$-cleavable over $Y$, yet there is no continuous injection from this space into $Y$. Thus the notion of cleavage becomes trivial if we allow the class $\mathcal{C}$ to be too large. Nonetheless, there are many examples of classes $\mathcal{C}$ and spaces $Y$ for which $\clubsuit_{\mathcal{C}}(Y)$ takes more interesting values. For example, Arhangel'skiĭ proved that the cleavage of $\mathbb{R}$ relative to compact Hausdorff spaces is 2 (see [4], Theorem 9). Buzyakova proved that the cleavage of any LOTS relative to compact continua is also 2 (see [9], Theorem 4.2). Motivated by these examples, we will prove in Section 3.3 that the cleavage of $\mathbb{R}^2$ relative to the class of $\sigma$-compact polyhedra is 6.

The notion of $\kappa$-cleavability was suggested to the author in a passing remark by Levine, to whom also is due our "$\clubsuit$" notation. Nonetheless this chapter (which has also been published as a paper, [7]) is, as far as we know, the literary debut of both $\kappa$-cleavability and cleavage (the latter concept being the author's own). As it is impossible to explore every avenue of inquiry for a new idea in a single thesis chapter, many open questions for the interested reader will be articulated along the way.

## 3.2 Nontriviality

In this section and throughout this chapter we will have occasion to talk about graphs. Although graphs can be considered purely combinatorial objects, for us they will be topological. If $G$ is a graph with vertex set $V$ and edge set $E$ then we will consider $G$ to be the topological space obtained from $|E|$ disjoint copies of $[0, 1]$ by an appropriate identification of endpoints.

The following theorem shows that the notion of $\kappa$-cleavability is nontrivial, i.e., strictly stronger than the notion of cleavability.

**Theorem 3.2.** *For every cardinal $\mu \geq 2$, there is a pair $(X, Y)$ of Hausdorff spaces such that $\clubsuit(X, Y) = \mu$.*

*Proof.* For any cardinal $\mu$, let $K_\mu$ denote the complete graph on $\mu$ points, i.e., a graph on $\mu$ points in which a single edge joins every pair of points. For any cardinal $\lambda$, let $K_\lambda^\mu$ denote the complete graph on $\lambda$ points, but with $\mu$ edges adjoining every pair of vertices and $\mu$ edges adjoining each vertex to itself (thus $K_\mu = K_\mu^1$).

Let $\mu$ be an infinite cardinal. Let $X = K_\mu$ and let $Y = \bigsqcup_{\lambda<\mu} K_\lambda^\mu$; the topology on $Y$ is the free union, i.e., each $K_\lambda^\mu$ is clopen in $Y$. We show that $\clubsuit(X,Y) = \mu$.

First we must argue that, for any $\lambda < \mu$, $X$ is $\lambda$-cleavable over $Y$. Let $\lambda < \mu$ and let $\{A_\alpha\}_{\alpha<\lambda}$ be a partition of $X$. We will show that there is a continuous map $f : X \to K_\lambda^\mu \subseteq Y$ such that $f[A_\alpha] \cap f[A_\beta] = \emptyset$ for distinct $\alpha, \beta < \lambda$. For simplicity, we will write the set of vertices of $K_\lambda^\mu$ as $V = \{v_\alpha\}_{\alpha<\lambda}$ and we will use the symbols $w$, $w'$, etc. when referring to vertices of $X$. If $w$ is a vertex of $X$ and $w \in A_\alpha$, we let $f$ map $w$ to $v_\alpha$. If $E$ is the edge of $X$ at that connects $w$ and $w'$, and if $w \in A_\alpha$, $w' \in A_\beta$, then we extend $f$ by mapping $E$ onto one of the $\mu$ edges connecting $v_\alpha$ and $v_\beta$; we do this in such a way that $f$ restricts to a homeomorphism on $E$ and such that distinct edges of $X$ map to distinct edges of $K_\lambda^\mu$. The resulting map $f : X \to K_\lambda^\mu$ is a witness to the fact that $X$ is $\lambda$-cleavable over $Y$: $f$ is injective off of the vertices of $X$, and it is clear that no two vertices in different partition elements map to the same point of $K_\lambda^\mu$.

Next we show that $X$ is not $\mu$-cleavable over $Y$. We partition $X$ by, for every vertex $w$ of $X$, declaring $\{w\}$ to be an equivalence class of our partition and declaring each edge (without the endpoints) to constitute an equivalence class of our partition. Suppose that $f : X \to Y$ is a continuous function that respects this partition, i.e., that maps points in different equivalence classes to different points. Since $f$ is continuous and $X$ is connected, $f[X]$ is a connected subset of $Y$ and hence $f[X] \subseteq K_\lambda^\mu$ for some $\lambda < \mu$. Now suppose that $f$ maps one of the vertices of $X$ to some point that is not a vertex of $K_\lambda^\mu$; it is clear from the continuity of $f$ that some two edges of $X$ will have overlapping images, contradicting the fact that each edge of $X$ constitutes an equivalence class of our partition. Thus the vertices of $X$ map to the vertices of $K_\lambda^\mu$. Since there are only $\lambda$ vertices of $K_\lambda^\mu$, some two vertices of $X$ must map to the same point, contradicting the fact that each vertex of $X$ constitutes an equivalence class of our partition.

This proves the theorem for infinite cardinals, and it remains to prove it for finite cardinals. A straightforward modification of the above argument suffices, and we leave the details to the reader. Alternatively, there is a second approach to proving the finite case, one that does not readily generalize to infinite cardinals, and we will see the details of this second approach in the proof of Theorem 3.4. $\square$

**Lemma 3.3.** *Let $X$ and $Y$ be topological spaces.*

*(i) If $A \subseteq X$ then $\clubsuit(A, Y) \geq \clubsuit(X, Y)$.*

*(ii) If $A \subseteq Y$ then $\clubsuit(X, A) \leq \clubsuit(X, Y)$.*

*(iii) If $\hat{X}$ is coarser than $X$ then $\clubsuit(\hat{X}, Y) \leq \clubsuit(X, Y)$.*

*(iv) If $\hat{Y}$ is coarser than $Y$ then $\clubsuit(X, \hat{Y}) \geq \clubsuit(X, Y)$.*

*Proof.* Straightforward. $\square$

The next theorem shows that the notion of cleavage is also nontrivial, and indeed can take on any finite value greater than 2.

**Theorem 3.4.** *Let $\mathcal{C}$ be the class of compact spaces. For every $n > 0$, $\clubsuit_{\mathcal{C}}(\omega \cdot n + 1) = n + 1$. Thus every finite $n \geq 2$ is represented as the cleavage of some space relative to the class of compact spaces.*

*Proof.* Fix $Y = \omega \cdot n + 1$. This theorem is proved in two parts: first we must find a compact space that is $n$-cleavable over $Y$ but that does not embed in it, and then we must prove that every compact space that is $(n+1)$-cleavable over $Y$ embeds in $Y$.

Let $X = \omega \cdot (n+1) + 1$ and let $\mathcal{P}$ be any partition of $X$ into $n$ subsets. $X$ has $n + 1$ limit points, so there is some element of $\mathcal{P}$ containing two distinct limit points of $X$; without loss of generality, suppose $\omega$ and $\omega \cdot 2$ are in the same $\mathcal{P}$-equivalence class. Define $f : X \to Y$ by

$$f(\alpha) = \begin{cases} n \cdot 2 & \text{if } \alpha = n < \omega \\ n \cdot 2 - 1 & \text{if } \alpha = \omega + n \text{ for some } 0 \neq n < \omega \\ \omega & \text{if } \alpha = \omega \text{ or } \alpha = \omega \cdot 2 \\ \omega \cdot (m-1) + n & \text{if } \alpha = \omega \cdot m + n > \omega \cdot 2 \end{cases}$$

Clearly, $f$ is a continuous function that respects the partition $\mathcal{P}$. Thus $X$ is $n$-cleavable over $Y$. This proves that $\clubsuit(X, Y) \geq n + 1$.

We now show that $X$ is not $(n + 1)$-cleavable over $Y$. The following argument is essentially due to Levine (see [35], Theorem 3.1) but we reproduce it here for completeness. Let $\mathcal{P}$ be the following $(n + 1)$-partition of $X$ (here, following standard notational conventions, we identify an ordinal number with the set of its predecessors):

$$\mathcal{P} = \{\omega \cdot 2 \setminus \omega, \omega \cdot 3 \setminus \omega \cdot 2, ..., \omega \cdot (n+1) \setminus \omega \cdot n, \omega \cup \{\omega \cdot (n+1)\}\}$$

Suppose $f : X \to Y$ is a continuous function respecting this partition. Notice that every neighborhood of every limit point of $x \in X$ contains points of a different equivalence class than that of $x$; this implies that $f$ cannot map any limit point of $X$ to an isolated point. But then there are only $n$ possible values of $f$ for the $n + 1$ different limit points of $X$, and no two of them belong to the same equivalence class; thus $f$ cannot respect this partition. This proves that $\clubsuit(X, Y) \leq n + 1$.

Now let $K$ be any compact space that is $(n{+}1)$-cleavable over $Y$. Then $K$ is cleavable over $Y$ and, by a recent result of Levine (see [35], Theorem 2.12), any compact space cleavable over a countable ordinal is homeomorphic to a countable ordinal. By the argument of the previous paragraph together with Lemma 3.3, $K$ is homeomorphic to some compact (i.e., non-limit) ordinal $\alpha < \omega \cdot (n + 1) + 1$. Every such ordinal is homeomorphic to a subset of $\omega \cdot n + 1$. $\square$

**Question 3.5.** *For a given infinite cardinal $\kappa$, is there a space whose cleavage relative to compact spaces is $\kappa$? Can such a space be Hausdorff or compact Hausdorff?*

## 3.3 The cleavage of $\mathbb{R}^2$

Recall that a CW-complex is constructed inductively: its 0-skeleton is a discrete set of points, its 1-skeleton is obtained by gluing copies of $[0, 1] = I$ onto its 0-skeleton, its 2-skeleton is obtained by gluing copies of $I^2$ onto its 1-skeleton, etc. Every CW-complex has a cellular structure, but, as with graphs, we will not be so concerned with the combinatorial structure of these spaces as with their topology. Following standard practice, we use the term **polyhedron** to refer to a finite-dimensional CW-complex considered as a topological space.

We do not give a precise definition of CW-complexes here. Although we prove several theorems below concerning polyhedra, we will not need to use their definition in these proofs: it will be enough to cite theorems from the literature. The uninitiated reader is referred to the appendix of [29] for a precise definition of CW-complexes and a survey of their topological properties.

In what follows we will denote the class of $\sigma$-compact polyhedra by $\sigma CP$.

Recall that $K_{3,3}$ is the graph on two groups of 3 vertices, with two vertices connected if and only if they belong to different groups (see picture below). Notice that $K_{3,3}$ is a compact 1-dimensional polyhedron: it is a set of 6 vertices to which copies of $[0,1]$ have been glued in the appropriate way. Likewise $K_5$, the complete graph on five points, is also a compact 1-dimensional polyhedron.

Kuratowski's Theorem, a fundamental result in graph theory, states that a finite graph $G$ embeds in $\mathbb{R}^2$ if and only if neither $K_{3,3}$ nor $K_5$ embeds in $G$ (see [27], Theorem 11.13, or [34]).

**Lemma 3.6.** *$\clubsuit(K_5, \mathbb{R}^2) = 5$ and $\clubsuit(K_{3,3}, \mathbb{R}^2) = 6$.*

*Proof.* First we show that $K_5$ is not 5-cleavable over $\mathbb{R}^2$. To do this we must find a 5-partition of $K_5$ that is not respected by any continuous function $f : K_5 \to \mathbb{R}^2$. Let the vertices of $K_5$ be denoted $v_1, v_2, v_3, v_4, v_5$ and let the edge connecting $v_i$ to $v_j$ (not including endpoints) be denoted $E_{i,j}$. We partition $X$ into the following 5 sets:

$$E_{1,2} \cup E_{1,4} \cup \{v_1\}, \qquad E_{2,3} \cup E_{2,5} \cup \{v_2\}, \qquad E_{3,4} \cup E_{1,3} \cup \{v_3\},$$

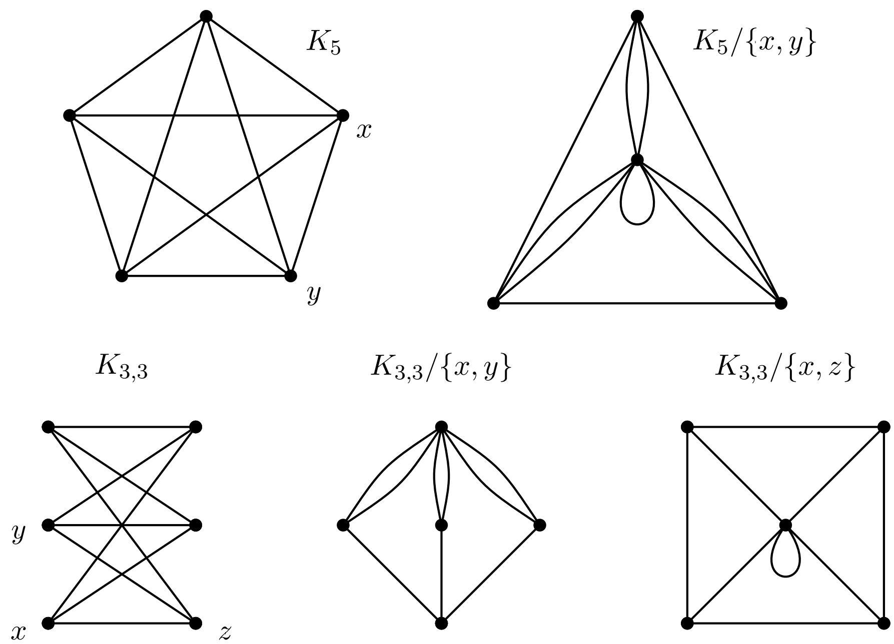
$$E_{4,5} \cup E_{2,4} \cup \{v_4\}, \qquad E_{1,5} \cup E_{4,5} \cup \{v_5\}$$

Suppose $f$ is a continuous map that respects this partition. Then $f$ maps each of the five vertices of $K_5$ to distinct points of $\mathbb{R}^2$. For each pair $v_i$, $v_j$, the set $f[E_{i,j} \cup \{v_i, v_j\}]$, although it may not itself be homeomorphic to $I$ (e.g., space-filling curves), nonetheless contains a homeomorphic copy of $I$, with $v_i$ and $v_j$ acting as the endpoints (this is a theorem of Hahn and Mazurkiewicz; see [48], Theorem 31.5). Thus the image of $K_5$ under $f$ contains a "picture" of $K_5$ (more precisely, $f[K_5]$ contains a union of 10 closed intervals, possibly overlapping, with endpoints identified in the appropriate way). Let $g$ be a map from $K_5$ onto this "picture" of $K_5$. It is clear that we may choose $g$ so that its restriction to any $E_{i,j}$ is a homeomorphism, so that $g[K_5] \subseteq f[K_5]$, and so that, for any element $A$ of our partition, $g[A] \subseteq f[A]$. In particular, $g$ respects our partition if $f$ does.

By Kuratowski's Theorem, $g$ cannot be an embedding. As $g$ is continuous and $K_5$ is compact, $g$ cannot be injective. As $g$ is injective on vertices and on individual edges, $g$ must map some two points of different edges to the same place. If $g$ mapped only points from adjacent edges to the same place then we could modify $g$ to obtain an embedding of $K_5$. Thus $g$ must map two points from non-adjacent edges to the same place. Since no two points from non-adjacent edges are in the same equivalence class, $g$ cannot respect our partition, and therefore neither can $f$. Thus $K_5$ is not 5-cleavable over $\mathbb{R}^2$, i.e., $\clubsuit(K_5, \mathbb{R}^2) \leq 5$.

The proof that $\clubsuit(K_{3,3}, \mathbb{R}^2) \leq 6$ is similar.

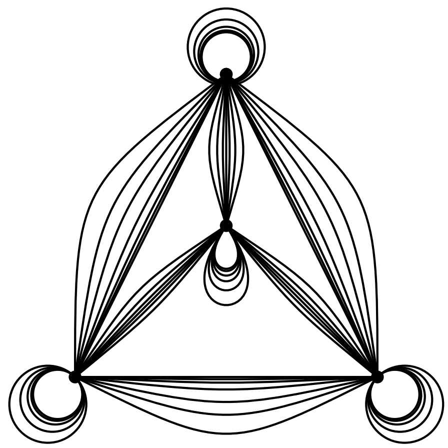
Next we show that $K_5$ is 4-cleavable over $\mathbb{R}^2$. Since $K_5$ has 5 vertices, any 4-partition of $K_5$ must have two vertices in the same equivalence class. The following picture shows how to obtain a continuous map $f : K_5 \to \mathbb{R}^2$ that identifies two vertices of $K_5$ but is otherwise injective; this map clearly can be used to respect any given 4-partition of $K_5$. Thus $\clubsuit(K_5, \mathbb{R}^2) \geq 5$. For $K_{3,3}$ there are, up to homeomorphism, two distinct ways to identify two distinct vertices. Both of these are shown below, and, as is apparent, each results in a planar graph, indicating by a similar argument that $\clubsuit(K_{3,3}, \mathbb{R}^2) \geq 6$.




□

**Theorem 3.7.** *If $G$ is a countable graph then $\clubsuit(G,\mathbb{R}^2)=5$, $\clubsuit(G,\mathbb{R}^2)=6$, or $\clubsuit(G,\mathbb{R}^2)=\infty$.*

*Proof.* If $G$ is a finite graph then by Kuratowski's Theorem either $G$ embeds in $\mathbb{R}^2$ or $G$ contains a homeomorphic copy of $K_5$ or $K_{3,3}$. The same, in fact, is true of countable graphs, by an extension of this result likely due to Erdös (see [17] or [47]). If $G$ can be embedded in $\mathbb{R}^2$ then $\clubsuit(G,\mathbb{R}^2)=\infty$. If $G$ contains a copy of $K_5$ or of $K_{3,3}$ then, by Lemmas 3.3 and 3.6, $\clubsuit(G,\mathbb{R}^2)\le 6$. Thus it suffices to show that any countable graph $G$ is 4-cleavable over $\mathbb{R}^2$. Given a 4-partition of $G$, we can map each of the vertices of $G$ to one of the four vertices in the picture shown below (while respecting our partition on vertices), and we can then map in the edges injectively as necessary:



$\square$

**Corollary 3.8.** *If $G$ is an arbitrary graph with edge set $E$ and $|E| \leq \mathfrak{c}$, then $\clubsuit(G, \mathbb{R}^2) > 4$.*

*Proof.* A variation of the above picture suffices for this proof. $\square$

We now turn our attention from 1-dimensional polyhedra (graphs) to $\sigma$-compact polyhedra.

**Lemma 3.9** (Arhangel'skiĭ)**.** *Let $X$ be a separable space cleavable over $\mathbb{R}^n$. Then every subspace $U$ of $X$ that is homeomorphic to $\mathbb{R}^n$ is open in $X$.*

*Proof.* See [4], Theorem 6. $\square$

Let the $Q$**-space** be the topological space obtained from $I = [0, 1]$ and $I^2$ by gluing an endpoint of $I$ to the middle of $I^2$; this space is sometimes also called the disk with feeler, the disk with sticker, or $F_2$.

**Lemma 3.10.** *Neither $S^2$ nor the $Q$-space is cleavable over $\mathbb{R}^2$.*

*Proof.* For the $Q$-space this follows directly from Lemma 3.9. For $S^2$, we recall the Borsuk-Ulam Theorem ([29], Theorem 1.10): every continuous function $S^2 \to \mathbb{R}^2$ maps some two antipodal points to the same place. It follows that, if $\mathcal{P}$ is any 2-partition of $S^2$ that separates into different equivalence classes every pair of antipodal points, there is no continuous function $S^2 \to \mathbb{R}^2$ that respects $\mathcal{P}$. □

**Lemma 3.11** (Márquez, Ayala, and Quintero)**.** *Let $X \in \sigma CP$. Then $X$ embeds in $\mathbb{R}^2$ if and only if $X$ does not contain a copy of $K_5$, $K_{3,3}$, $S^2$, or the $Q$-space.*

*Proof.* See [36], Theorem C'. □

**Theorem 3.12.** $\clubsuit_{\sigma CP}(\mathbb{R}^2) = 6$. *That is, any $\sigma$-compact polyhedron that is 6-cleavable over $\mathbb{R}^2$ embeds in $\mathbb{R}^2$.*

*Proof.* $K_{3,3}$ is a compact polyhedron that is 5-cleavable over $\mathbb{R}^2$ but that cannot be embedded in $\mathbb{R}^2$; thus $\clubsuit_{\sigma CP}(\mathbb{R}^2) \geq 6$. If $X$ is 6-cleavable over

$\mathbb{R}^2$ then, by Lemma 3.3, $X$ does not contain a copy of $K_5$, $K_{3,3}$, $S^2$, or the $Q$-space, so $X$ embeds in $\mathbb{R}^2$ by Lemma 3.11. Thus $\clubsuit_{\sigma CP}(\mathbb{R}^2) \leq 6$. $\square$

**Question 3.13.** *Can we compute the cleavage of* $\mathbb{R}^2$ *with respect to a broader class of spaces? Specifically, what is the cleavage of* $\mathbb{R}^2$ *with respect to compact spaces (*$\sigma$*-compact spaces, continua, etc.)?*

**Question 3.14.** *Lemma 3.11 is a Kuratowski-like theorem for* $\sigma CP$ *and* $\mathbb{R}^2$*: it gives a finite list of members of* $\sigma CP$ *such that any* $X \in \sigma CP$ *embeds in* $\mathbb{R}^2$ *if and only* $X$ *does not contain as a subspace anything on this list. Is there a Kuratowki-like theorem for compact spaces and* $\mathbb{R}^2$*? In other words, is there a finite list of compact spaces such that any compact space that does not embed in* $\mathbb{R}^2$ *contains something on this list as a subspace?*

A negative answer to Question 3.14 could imply that the cleavage of $\mathbb{R}^2$ with respect to the compact spaces is infinite, so this question is related to Question 3.5. A negative answer would also give contrast to the currently open problem of whether it is consistent with *ZFC* that every compact Hausdorff space contains a copy of either $\omega + 1$ or $\beta\omega$. See [28], pp. 113 for more on this problem.

In the same way that Theorem 3.12 translates a Kuratowski-like theorem into a theorem about cleavage, we now prove a theorem that translates into a theorem about cleavage a celebrated minimality result from graph theory. For the statement of this theorem, let $CG$ denote the class of compact graphs, which is the same as the class of graphs with finitely many vertices and edges.

**Theorem 3.15.** *If $Y$ is a surface (a 2-dimensional manifold) then $\clubsuit_{CG}(Y)$ is finite. That is, for every compact 2-manifold $Y$ there is a finite number $n_Y$ such that whenever a compact graph $G$ is $n_Y$-cleavable over $Y$ then $X$ embeds into $Y$. Moreover, if $Y$ is a surface of genus $\gamma$ then*

$$n_Y > \left[\frac{7 + \sqrt{48\gamma + 1}}{2}\right]$$

*(Here $[x]$ denotes the greatest integer $m$ such that $m \leq x$.) Thus arbitrarily high finite numbers are represented as the cleavage of 2-manifolds with respect to $CG$.*

*Proof.* If $Y$ is a surface then, by a result of Robertson and Seymour (see [44]), there is a finite list of graphs $G_0, \ . \ . \ . \ ,G_n$ such that an arbitrary graph $G$ embeds in $Y$ if and only if it does not contain a homeomorphic copy of one of $G_0, \ . \ . \ . \ ,G_n$. Moreover, for $i = 0, ..., n$, $\clubsuit(G_i, Y)$ is finite. To show this, it is enough to note that there is a finite partition of $G_i$ that is not respected by any continuous function. One such partition is the partition taking $\{v\}$ as an equivalence class for each vertex $v$ of $G$ and also taking each edge $E$ of $G$ (without the endpoints) to be its own equivalence class; the proof that no continuous function can respect this partition is analogous to (one direction of) the proof of Lemma 3.6 together with the fact that $G_i$ does not embed in $Y$. Evidently, $\clubsuit_{CG}(Y) \leq \max_{i\leq n}\{\clubsuit(G_i, Y)\}$.

To prove the stated bound for a surface $Y$ of genus $\gamma$, it is enough to prove that there is some compact graph that is $m$-cleavable over $Y$ but does not embed in $Y$ (where $m = \left[\frac{1}{2}(7+\sqrt{48\gamma+1})\right]$). This formula, taken from [27], gives the greatest $m$ such that $K_m$ embeds in $Y$; thus, by the same argument as in the proof of Theorem 3.7, every graph is $m$-cleavable over $Y$. As not every graph embeds in $Y$, this completes the proof. □

Note that this lower bound for $n_Y$ is not very strong because it only takes into account a single one of the excluded graphs for a surface of genus $\gamma$. The number of excluded graphs for a given surface may be quite large, despite the fact that there are only two excluded graphs for $\mathbb{R}^2$ (for example, the projective plane has 103 distinct excluded graphs). Even for the case of $\mathbb{R}^2$ the inequality given above is not optimal.

**Question 3.16.** *Can the exact value of $\clubsuit_{CG}(Y)$ be computed for an arbitrary surface $Y$? What about the cleavage of surfaces with respect to a broader class of spaces, e.g., the compact spaces?*

## 3.4 The spectrum of $\mathbb{R}^2$

One general question we can ask about cleavage and cleavability is

**Question 3.17.** *For a fixed space $Y$, what are the possible values for $\clubsuit(X,Y)$? What if $X$ is restricted to be in some fixed class $\mathcal{C}$?*

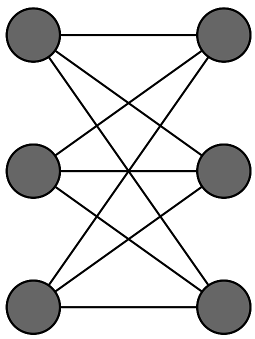
We call these sets of possible values the **spectrum** of $Y$ and the spectrum of $Y$ relative to $\mathcal{C}$, respectively. For example, Arhangel'skiĭ proved (see [4], Proposition 3) that the spectrum of $\mathbb{R}^\omega$ does not contain any values $\lambda$ such that $2 < \lambda \leq \mathfrak{c}$. Together with Proposition 3.1($ii$) and the fact that the discrete space on $\mathfrak{c}^+$ points is $\mathfrak{c}$-cleavable over $\mathbb{R}^\omega$, this shows that the spectrum of $\mathbb{R}^\omega$ is $\{2, \mathfrak{c}^+, \infty\}$. In fact, as Arhangel'skiĭ himself points out, his proof generalizes to show that the spectrum of $\mathbb{R}^\kappa$ is $\{2, (2^\kappa)^+, \infty\}$ for any infinite $\kappa$. We have already seen that this result does not extend to the case $\kappa = 2$. Theorem 3.7 tells us that the spectrum of $\mathbb{R}^2$ with respect to countable graphs is $\{5, 6, \infty\}$. We have also seen (Lemma 3.10) that there are compact polyhedra that are not cleavable over $\mathbb{R}^2$, i.e., spaces $X$ such that $\clubsuit(X, \mathbb{R}^2) = 2$, thus showing that the spectrum of $\mathbb{R}^2$ contains $\{2, 5, 6, \infty\}$. In this section we give two compact polyhedra, one that is 3-cleavable but not 4-cleavable over $\mathbb{R}^2$, and another that is cleavable but not 3-cleavable over $\mathbb{R}^2$. This will prove that the spectrum of $\mathbb{R}^2$ relative to $\sigma CP$

is $\{2,3,4,5,6,\infty\}$, and that the spectrum of $\mathbb{R}^2$ contains $\{2,3,4,5,6,\infty\}$.

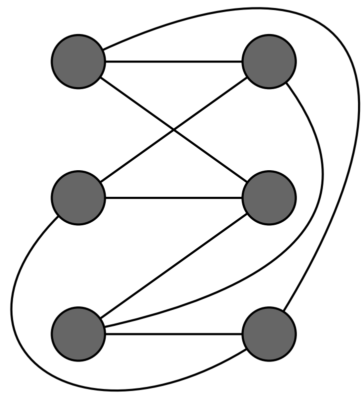
Before moving on, we note the contrast between Arhangel'skiĭ's result for $\mathbb{R}^\kappa$ and our result for $\mathbb{R}^2$. This contrast leads us naturally to the question of the intermediate cases:

**Question 3.18.** *For $n \geq 3$, what is the spectrum of $\mathbb{R}^n$? With respect to compact spaces, compact polyhedra, or $\sigma$-compact polyhedra? In particular, does the spectrum of $\mathbb{R}^n$ contain values other than 2, $\mathfrak{c}^+$, and $\infty$ for all finite $n \geq 3$?*

Define $D_{3,3}$ to be the topological space obtained by expanding each vertex of $K_{3,3}$ to a copy of the closed unit disc; the edges emanate from the boundary of the discs, so that the resulting space is locally embeddable in $\mathbb{R}^2$. Two pictures of this space are shown below. $D_{3,3}$ is a compact 2-dimensional polyhedron.



$D_{3,3}$



$D_{3,3}$

Let $f : X \to Y$ be any function. Define

$$M_f = \{x \in X : \exists y \in X \setminus \{x\} \text{ such that } f(x) = f(y)\}$$

That is, $M_f$ is the set of points on which $f$ is not injective. We will say that $f : X \to Y$ is **almost injective** if $|M_f| < |X|$.

**Lemma 3.19.** *Let $U$ denote the open unit disc. There exist $A, B, C \subseteq U$ such that, if $f : U \to \mathbb{R}^2$ is any continuous function satisfying*

$$f[A] \cap f[B] = f[A] \cap f[C] = f[B] \cap f[C] = \emptyset$$

*then $f$ is almost injective. Moreover, we may take $A$, $B$, and $C$ such that all of the following hold:*

- $A \cap B = A \cap C = B \cap C = \emptyset$

- $A \cup B \cup C = U$
- *none of* $A$, $B$, *or* $C$ *contains an arc*
- *if* $h_0, h_1$ *are embeddings* $U \to \mathbb{R}^2$ *with* $h_0 \neq h_1$, *then*

$$h_0[U] \cap h_1[U] = \emptyset \quad \Leftrightarrow \quad h_0[A] \cap h_1[B] = \emptyset \quad \Leftrightarrow$$

$$h_0[A] \cap h_1[C] = \emptyset \quad \Leftrightarrow \quad h_0[B] \cap h_1[C] = \emptyset$$

*Proof.* We build the sets $A$, $B$, and $C$ simultaneously by transfinite recursion. Let $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of all continuous functions $f : U \to \mathbb{R}^2$ that are not almost injective, i.e., such that

$$|\{x \in U : \exists y \in U \setminus \{x\} \text{ such that } f(x) = f(y)\}| = \mathfrak{c}$$

(Recall that every continuous function $U \to \mathbb{R}^2$ is determined by its action on a dense subset; it follows that there are only $\mathfrak{c}$-many such functions.) Let $\langle g_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of all embeddings $[0,1] \to U$. Lastly, let $\langle (h_0^\alpha, h_1^\alpha) : \alpha < \mathfrak{c} \rangle$ be an enumeration of all pairs $(h_0, h_1)$ of embeddings $U \to \mathbb{R}^2$ such that $h_0[U] \cap h_1[U] \neq \emptyset$. Since every subset of $\mathbb{R}^2$ that is homeomorphic to $\mathbb{R}^2$ is open, $h_0^\alpha[U] \cap h_1^\alpha[U]$ is open, and hence of cardinality $\mathfrak{c}$, for every $\alpha < \mathfrak{c}$.

As the base case for our recursion, take $A^0 = B^0 = C^0 = \emptyset$.

Let $\alpha < \mathfrak{c}$ and suppose that we have constructed three sets $A^\alpha$, $B^\alpha$, and $C^\alpha$ that are pairwise disjoint and of cardinality strictly less than $\mathfrak{c}$. First, let $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, and $z_2$ be any distinct points of $U \setminus (A^\alpha \cup B^\alpha \cup C^\alpha)$ such that

$$f_\alpha(x_1) = f_\alpha(y_1) \qquad f_\alpha(x_2) = f_\alpha(z_1) \qquad f_\alpha(y_2) = f_\alpha(z_2)$$

Some such points must exist since $f_\alpha$ is not almost injective. Next, let $x_3$, $y_3$, and $z_3$ be any three distinct points of $g_\alpha[0,1] \setminus (A^\alpha \cup B^\alpha \cup C^\alpha \cup \{x_1, x_2, y_1, y_2, z_1, z_2\})$. Lastly, set $V = h_0^\alpha[U] \cap h_1^\alpha[U]$ and let

$$x_4, y_4, z_4 \in (h_0^\alpha)^{-1}[V] \setminus (A^\alpha \cup B^\alpha \cup C^\alpha)$$

$$x_5, y_5, z_5 \in (h_1^\alpha)^{-1}[V] \setminus (A^\alpha \cup B^\alpha \cup C^\alpha)$$

such that all these points are distinct, both from each other and from $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$, $z_1$, $z_2$, and $z_3$, and such that

$$h_0^\alpha(x_4) = h_1^\alpha(y_4) \qquad h_0^\alpha(x_5) = h_1^\alpha(z_4) \qquad h_0^\alpha(y_5) = h_1^\alpha(z_5)$$

Some such points must exist because $|V| = \mathfrak{c}$. Let

$$A^{\alpha+1} = A^\alpha \cup \{x_1, x_2, x_3, x_4\} \qquad B^{\alpha+1} = B^\alpha \cup \{y_1, y_2, y_3, y_4\}$$

$$C^{\alpha+1} = C^\alpha \cup \{z_1, z_2, z_3, z_4\}$$

This defines our construction at successor stages.

If $\alpha$ is a limit ordinal, take $A^\alpha = \bigcup_{\beta<\alpha} A^\beta$, $B^\alpha = \bigcup_{\beta<\alpha} B^\beta$, and $C^\alpha = \bigcup_{\beta<\alpha} C^\beta$.

Taking $A = \bigcup_{\alpha<\mathfrak{c}} A^\alpha$, $B = \bigcup_{\alpha<\mathfrak{c}} B^\alpha$, and $C = U \setminus (A \cup B)$, it is clear from our construction that $A$, $B$, and $C$ satisfy the desired conclusions. □

**Lemma 3.20** (Arhangel'skiĭ)**.** *Let $n \in \mathbb{N}$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an almost injective continuous mapping. Then $f$ is an embedding.*

*Proof.* See [4], Lemma 3. □

**Theorem 3.21.** *$\clubsuit(D_{3,3}) = 3$.*

*Proof.* In what follows, we will denote the six (maximal) closed discs in $D_{3,3}$ by $D_1, ..., D_6$, and we label them such that $D_i$ and $D_j$ are connected by an edge if and only if $i \le 3$ and $j > 3$ or $j \le 3$ and $i > 3$. We will use the term "edge" in the obvious sense: an edge is a copy of $[0, 1]$ inside $D_{3,3}$, where one of the endpoints is identified with the boundary of one of the six closed discs, the other endpoint is identified with the boundary of a different one (thus $|D_i \cap E_{i,j}| = 1$), and, except for the endpoints, no point of any edge is contained in any of the $D_i$. The edge connecting $D_i$ and $D_j$ will be denoted $E_{i,j}$.

First we show that $D_{3,3}$ is cleavable over $\mathbb{R}^2$. Let $\mathcal{P}$ be any 2-partition of $D_{3,3}$; then there must be two non-adjacent edges, $E$ and $E'$, which contain points $p \in E, p' \in E'$, other than their endpoints, in the same equivalence class. It is easy to see that we can then find a continuous function $f : D_{3,3} \to \mathbb{R}^2$ injective at every point except that $f(p) = f(p')$ (one such near-embedding is given in the rightmost of the two pictures of $D_{3,3}$ shown above); $f$ is a continuous function that respects $\mathcal{P}$. Thus $\clubsuit(D_{3,3}, \mathbb{R}^2) \geq 3$.

Next we show that $D_{3,3}$ is not 3-cleavable over $\mathbb{R}^2$. To do this, it will be easier to show that a proper subset of $D_{3,3}$ is not 3-cleavable over $\mathbb{R}^2$, which is sufficient by Lemma 3.3. Let

$$U_{3,3} = \bigcup_{i=1}^{6}(D_i \setminus \partial D_i) \cup \bigcup_{i=1}^{3}\bigcup_{j=4}^{6} E_{i,j}$$

In other words, $U_{3,3}$ looks just like $D_{3,3}$ except that we have used six open discs, rather than six closed discs, in its construction. In what follows, we will refer to $D_i \setminus \partial D_i$ as $U_i$.

We construct a 3-partition of $U_{3,3}$ not respected by any continuous function into $\mathbb{R}^2$. For each $U_i$, let $A_i$, $B_i$, and $C_i$ denote the three subsets of $U_i$ described in Lemma 3.19. Partition $U_{3,3}$ into the following three sets:

$$E_{1,4} \cup E_{1,5} \cup E_{1,6} \cup A_1 \cup A_2 \cup B_3 \cup B_4 \cup C_5 \cup C_6$$

$$E_{2,4} \cup E_{2,5} \cup E_{2,6} \cup A_3 \cup A_5 \cup B_1 \cup B_6 \cup C_2 \cup C_4$$

$$E_{3,4} \cup E_{3,5} \cup E_{3,6} \cup A_4 \cup A_6 \cup B_2 \cup B_5 \cup C_1 \cup C_3$$

Suppose that $f : D_{3,3} \to \mathbb{R}^2$ is a continuous function that respects this partition. By Lemma 3.19 and our choice of partition, $f \restriction U_i$ is almost injective for each $i$, and it follows from Lemma 3.20 that $f \restriction U_i$ is a homeomorphism. It then follows from the last assertion of Lemma 3.19 and our choice of partition that, if $i \neq j$, $f[U_i] \cap f[U_j] = \emptyset$.

Following the argument of the proof of Lemma 3.6, we may assume that $f$ maps each $E_{i,j}$ either to a homeomorphic copy of $[0, 1]$ or to a single point. Now, reasoning as in the easy direction of Kuratowski's Theorem, we see that either two non-adjacent edges must cross or some edge must pass through one of the $U_i$. No two non-adjacent edges may cross since no two non-adjacent edges are in the same equivalence class of our partition. No edge may pass through any of the $U_i$ since none of the $A$, $B$, or $C$ contains an arc and since we have constructed our partition in such a way that, for any fixed $i$, a given edge $E$ is only in the same partition as $A_i$, $B_i$, or $C_i$, but never two of these. Thus it is impossible for $f$ to respect our partition, and we conclude that $\clubsuit(D_{3,3}, \mathbb{R}^2) \leq \clubsuit(U_{3,3}, \mathbb{R}^2) \leq 3$. □

**Theorem 3.22.** *There is a compact 2-dimensional polyhedron $X$ such that $\clubsuit(X, \mathbb{R}^2) = 4$.*

*Proof.* We construct $X$ in a manner analogous to the construction of $D_{3,3}$, except that only two vertices of $K_{3,3}$, rather than all six, are expanded to discs. Thus $X$ is composed of two discs, $D_1$ and $D_2$, four points, $v_1$, $v_2$, $v_3$, and $v_4$, and nine edges. (Nb: This description does not define $X$ uniquely up to homeomorphism, since we could have either an edge connecting $D_1$ and $D_2$ or not. It does not matter for our purposes which of these two spaces $X$ is chosen to be.)

If $\mathcal{P}$ is a 3-partition of $X$ then at least two of $v_1$, $v_2$, $v_3$, and $v_4$ are in the same equivalence class. It is then easy to show, as in the proof of Lemma 3.6, that there is a continuous function $f : X \to \mathbb{R}^2$ that identifies these two vertices but is otherwise injective. Thus $X$ is 3-cleavable over $\mathbb{R}^2$, i.e., $\clubsuit(X, \mathbb{R}^2) \geq 4$.

The proof that $\clubsuit(X, \mathbb{R}^2) \leq 4$ is essentially the same as the proof that $\clubsuit(D_{3,3}, \mathbb{R}^2) \leq 3$ (Theorem 3.21) and we omit the details. $\square$

**Corollary 3.23.** *The spectrum of* $\mathbb{R}^2$ *with respect to compact polyhedra and* $\sigma$*-compact polyhedra is* $\{2, 3, 4, 5, 6, \infty\}$.

**Question 3.24.** *Is there a compact space* $X$ *that does not embed in* $\mathbb{R}^2$ *but satisfies* $\clubsuit(X, \mathbb{R}^2) > 6$*?*

**Question 3.25.** *Under what circumstances is it the case that, if* $\kappa$ *is in the spectrum of* $Y$ *relative to compact polyhedra (or* $\sigma$*-compact polyhedra, or* *compact spaces generally), then so is any* $\lambda < \kappa$*?*

# Chapter 4

# Toronto spaces

## 4.1 Introduction

A **Toronto space** is a topological space $X$ such that $Y$ is homeomorphic to $X$ whenever $Y \subseteq X$ and $|Y| = |X|$. For a given infinite cardinal $\kappa$, there are a handful of easy examples of Toronto spaces of size $\kappa$: the discrete and indiscrete topologies, the co-finite topology, and more generally, the topology consisting of sets with complement smaller than $\lambda$ for some $\lambda \leq \kappa$. We will describe a few more examples in Section 4.6, but none of them, aside from the discrete space, is Hausdorff. It remains an open (and seemingly very difficult!) problem whether it is consistent with *ZFC* to have an infinite Hausdorff non-discrete Toronto space.

This problem has a straightforward solution for cardinality $\aleph_0$: every countable Hausdorff Toronto space is discrete. We will go a step further in Corollary 4.35 below and show that there are, up to homeomorphism, exactly five countable Toronto spaces. It is worth noting, however, that there is a variant of the definition of a Toronto space that turns out to be very interesting for countable spaces; these spaces have been explored extensively in [26] and will be mentioned again in Section 4.4.

The question of classifying the Toronto spaces of higher cardinality turns out to be much harder, and generalizes what has come to be known as the Toronto Problem. The original Toronto Problem, as posed in [46], asks whether every Hausdorff Toronto space of size $\aleph_1$ is discrete. This latter question is our primary focus here. We will have frequent occasion to refer to a non-discrete Hausdorff Toronto space of size $\aleph_1$; we will refer to such spaces as non-discrete HATS (Hausdorff Aleph-one Toronto Spaces).

Borrowing notation from model theory and set theory, if $S \subseteq X$ is some definable subset of $X$, say $S = \{x \in X : \phi(x, X, p)\}$, and if $Y \subseteq X$ is some subset of $X$ of full cardinality, then $Y \approx X$ and we will denote the set in $Y$ that corresponds to $S$, namely $\{y \in Y : \phi(y, Y, p)\}$, by $S^Y$. Equivalently, if $S \subseteq X$ and $h : X \to Y$ is a homeomorphism, we write $S^Y$ for $h[S]$ whenever $h[S]$ does not depend on our choice of homeomorphism.

## 4.2 Some basic folklore

In this section and the next two we will collect together some basic results about Toronto spaces. Many of these results are already known but their proofs have not been published elsewhere: instead they have become part of the "folklore" surrounding the Toronto problem. With the exception of Propositions 4.6 and 4.23 (which are due to the author), the results in sections 4.2 and 4.4 are a group effort of Toronto-based topologists, most notably Alan Dow, Juris Steprāns, and Bill Weiss. The material in Section 4.3 is original except for Theorem 4.7, which is a rediscovery of an unpublished result of Ken Kunen.

For any topological space $X$, we can define simultaneously by transfinite induction two sequences of subsets of $X$:

$$X^0 = X$$

$$I^X_\alpha = \{x\colon x \text{ is isolated in } X^\alpha\}$$

$$X^{\alpha+1} = X^\alpha \setminus I^X_\alpha$$

$$X^\alpha = \bigcap_{\beta\in\alpha} X^\beta \text{ for limit } \alpha$$

The sequence $\langle X^\alpha\colon \alpha \in Ord\rangle$ is a decreasing sequence of closed subsets of $X$. Since $X$ is not a proper class, there will be some $\beta$ such that, for all $\gamma \geq \beta$, $X^\gamma = X^\beta$; the least such $\beta$ is called the **Cantor-Bendixson rank** of $X$ and $X^\beta$ is the **perfect kernel** of $X$. This $X^\beta$ is the largest subset of $X$ with no isolated points. If the perfect kernel of $X$ is empty, i.e., if every nonempty subspace of $X$ contains an isolated point, then $X$ is said to be **scattered**. Equivalently, $X$ is scattered if and only if $X = \bigcup_{\alpha<\beta} I^X_\alpha$. Where no confusion can result, we will simply write $I_\alpha$ for $I^X_\alpha$. If $X$ is scattered, we will refer to the Cantor-Bendixson rank of $X$ as the **height** of $X$, denoted $h(X)$. Also, the **width** of $X$, denoted $w(X)$, is the supremum of the cardinalities of the $I_\alpha$. $X$ is called **thin-tall** if $w(X) < h(X)$. It is a nontrivial problem to show that thin-tall Hausdorff spaces exist, but *ZFC* examples have been constructed; see, for instance, [43]. If $X$ is scattered and $x \in X$, the **rank** of $x$ in $X$, denoted $\mathrm{rk}^X(x)$, is the least $\alpha$ such that $x \notin X^{\alpha+1}$, or, equivalently, the rank of $x$ is the unique $\alpha$ such that $x \in I_\alpha$. When no confusion will result, we will write $\mathrm{rk}(x)$ for $\mathrm{rk}^X(x)$.

**Lemma 4.1.** *If $Y$ is a subspace of a scattered space $X$ then $Y$ is scattered and, for every $x \in Y$, $\mathrm{rk}^Y(x) \le \mathrm{rk}^X(x)$. If $Y$ is open in $X$ then $\mathrm{rk}^Y(x) = \mathrm{rk}^X(x)$ for every $x \in Y$.*

*Proof.* If every subset of $X$ contains an isolated point then every subset of $Y$ contains an isolated point, so $Y$ is scattered if $X$ is.

Suppose it is not true that $\mathrm{rk}^Y(x) \le \mathrm{rk}^X(x)$ for all $x \in Y$. Then there is some $x$ of minimal $X$-rank $\alpha$ such that $\mathrm{rk}^Y(x) > \mathrm{rk}^X(x)$. $x$ is isolated in

$X^\alpha$, so there is a neighborhood $U$ of $x$ that only intersects points of strictly smaller $X$-rank. But then $U \cap Y$ is a neighborhood of $x$ in $Y$ which, by our assumption on the minimality of $\alpha$, only intersects points of $Y$-rank less than $\alpha$. It follows that $x$ is isolated in $Y^\alpha$ and that $\mathrm{rk}^Y(x) \leq \alpha$.

Now suppose $Y$ is open in $X$ and let $x$ be of minimal $Y$-rank $\alpha$ such that $\mathrm{rk}^Y(x) < \mathrm{rk}^X(x)$. Since $\mathrm{rk}^Y(x) = \alpha$, there is an open set $U \subseteq Y$ such that $U \cap Y^\alpha = \{x\}$. By the assumption of minimality, we have $U \cap X^\alpha = \{x\}$. But $U$ is also an open set in $X$, and it follows that $x$ is isolated in $X^\alpha$, i.e., $\mathrm{rk}^X(x) = \alpha$. $\square$

Continuing the analogy with model theory and set theory, Lemma 4.1 is something of an absoluteness result: it tells us how $\mathrm{rk}^X(x)$ and $\mathrm{rk}^Y(x)$ relate to each other when $Y \subseteq X$ and, in particular, that rank is "absolute" for open subsets.

**Proposition 4.2.** *Let $X$ be a non-discrete HATS. Then $X$ is a thin-tall scattered space with $h(X) = \omega_1$, and $w(X) = \omega$.*

*Proof.* Let $x$ and $y$ be distinct points of $X$. There are disjoint open sets $U_x$ and $U_y$ separating these points, so at least one of them, say $U_x$, must have uncountable complement. Then $(X \setminus U_x) \cup \{x\}$ is uncountable and has an isolated point. Hence $X$ contains an isolated point. But $X$ cannot have only a finite set $I_0$ of isolated points, because then $X \setminus I_0 \approx X$ would have no isolated points. If $X$ is not discrete, $X$ cannot have an uncountable set $I_0$ of isolated points, since then $X \approx I_0$. Thus $|I_0| = \aleph_0$. Since $X^1 = X \setminus I_0 \approx X$, we have $\left|I_1^X\right| = \left|I_0^{X^1}\right| = \aleph_0$, and $X^2 = X \setminus (I_0 \cup I_1) \approx X$. In general, it follows by transfinite induction that if $\alpha < \omega_1$ then $X^\alpha \approx X$ and $|I_\alpha| = \aleph_0$: the successor step of this induction is the same as for the case $\alpha = 1$ shown above, and when $\alpha$ is a limit ordinal, $X^\alpha = X \setminus \bigcup_{\beta<\alpha} I_\beta$ is uncountable and hence homeomorphic to $X$. Now, as the $I_\alpha$ are disjoint, $\bigcup_{\alpha<\omega_1} I_\alpha$ is an uncountable subset of $X$, and hence homeomorphic to $X$. Since $\bigcup_{\alpha<\omega_1} I_\alpha$ is a scattered space of width $\omega$ and height $\omega_1$, the result is proved. □

**Proposition 4.3.** *If $X$ is a non-discrete HATS then $X$ is hereditarily separable but not Lindelöf.*

*Proof.* $X$ is separable because $I_0$ is dense in $X$. Every subset of $X$ is either homeomorphic to $X$ or countable, so $X$ is hereditarily separable. $X$ is not Lindelöf because the open cover $\{X \setminus X^\alpha \colon \alpha < \omega_1\}$ has no countable subcover. □

In fact, any non-discrete HATS $X$ is hereditarily separable and anti-Lindelöf, i.e., every Lindelöf subset of $X$ is countable. If in addition $X$ is regular, then $X$ is an $S$-space. The existence of $S$-spaces is known to be independent of *ZFC*, so it is consistent that there are no regular non-discrete HATS. The next theorem shows that, even without the requirement of regularity, it is consistent with *ZFC* that all HATS are discrete.

**Theorem 4.4.** *Suppose there is a non-discrete HATS. Then* $2^{\aleph_0} = 2^{\aleph_1}$.

*Proof.* Suppose $X$ is a non-discrete HATS and let

$$S = \{f\colon f \text{ is a homeomorphism } X \to Y,\ Y \subseteq X, \text{ and } I_0^X \subseteq Y\}$$

Let $Y$ be any uncountable subset of $X$ containing $I_0$. Clearly, there are $2^{\aleph_1}$ such sets. Each such $Y$ is homeomorphic to $X$ under some map $f \in S$, so $|S| \geq 2^{\aleph_1}$.

Each $f \in S$ is a homeomorphism and therefore must map isolated points to isolated points: $f[I_0^X] = I_0^{f[X]}$. Thus $f$ acts as a permutation on $I_0$. Furthermore, since $I_0$ is dense in $X$ and $X$ is Hausdorff, $f$ is determined uniquely by its action on $I_0$. In other words, every $f \in S$ is determined uniquely by a permutation of $I_0$. As there are $2^{\aleph_0}$ such permutations, $|S| \leq 2^{\aleph_0}$. □

**Corollary 4.5.** *It is consistent with ZFC that every HATS is discrete.*

Although our focus here is on Toronto spaces of size $\aleph_1$, the following observation is worth noting:

**Proposition 4.6.** *If the GCH holds then every Hausdorff Toronto space (of any infinite size) is discrete.*

*Proof.* Let $\kappa$ be an infinite cardinal and let $X$ be a non-discrete Hausdorff Toronto space of size $\kappa$. The case $\kappa = \aleph_0$ will be covered in Corollary 4.35, so assume $\kappa$ is uncountable. As in the proof of Proposition 4.2, $X$ must have a nonempty set $I_0^X$ of isolated points, and this set must have size $\lambda$ for some infinite $\lambda < \kappa$. Working by induction just as in the proof of Proposition 4.2, one may show that $X$ is a thin-tall scattered space with width $\lambda$ and height $\kappa$. Following the proof of Theorem 4.4, one may then show that if such a space exists then $2^\lambda = 2^\kappa$. □

## 4.3 Compactness properties

In the last section we saw that non-discrete HATS do not necessarily exist. In the next three sections we will explore the question of what a non-discrete HATS would look like if it did exist. We begin by looking at some compactness properties. For what follows, take $X$ to be a non-discrete HATS.

**Theorem 4.7** (Kunen)**.** $\omega + 1$ *cannot be embedded in* $X$*, i.e.,* $X$ *contains no convergent sequences.*

*Proof.* Suppose that $f : \omega+1 \to X$ is an embedding; we will find a contradiction. Let $\alpha = \sup\{\mathrm{rk}(x) \colon x \in f[\omega+1]\}$ and consider $Y = f[\omega+1] \cup X^{\alpha+1}$. $f[\omega+1]$ is open in $Y$ because $f[\omega+1] = Y \cap \bigcup_{\beta\leq\alpha} I^X_\beta$, and $\bigcup_{\beta\leq\alpha} I^X_\beta$ is open in $X$. $f[\omega+1]$ is closed in $Y$ because it is compact and $Y$ is Hausdorff. Thus $f[\omega+1]$ is a clopen subset of $Y$ homeomorphic to $\omega+1$, and it is clear that $f(n) \in I^Y_0$ for each $n$ while $f(\omega) \in I^Y_1$. Since $X \approx Y$, we have shown that some $x \in I_1$ has a clopen neighborhood homeomorphic to $\omega+1$.

Next we will show that every $x \in I_1$ has a clopen neighborhood homeomorphic to $\omega+1$. Let us say that $P$ is the property of having such a neighborhood. We have shown that at least one $x \in I_1$ has property $P$. For each $x \in I_1$ with property $P$, choose a clopen neighborhood $U_x$ of $x$ such that $U_x \approx \omega+1$. Let

$$A = \{x \in I_1 \colon x \text{ has property } P\} \neq \emptyset$$

Suppose $\overline{A}$ is countable. Then $Y = X \setminus \overline{A} \approx X$. Let $x \in I^Y_1$ have a clopen neighborhood $U \approx \omega+1$ (some such point must exist since $X \approx Y$). $I^Y_1 \subseteq I^X_1$ by Lemma 4.1, so $x \in I^X_1$. Viewed as a neighborhood in $X$ rather than $Y$, $U$ is open (because $Y$ is open in $X$) and closed (because $U$ is compact and $X$ is Hausdorff). Thus $x$ has property $P$ and is in $A$, a contradiction. Thus $\overline{A}$ is uncountable; set $Y = I_0 \cup \overline{A}$. Clearly $I^Y_1 = A$. If $x \in A = I^Y_1$ and $U_x$ is a clopen neighborhood of $x$ in $X$ that is homeomorphic to $\omega+1$, then $U_x \cap Y$ is a clopen neighborhood of $x$ in $Y$ homeomorphic to $\omega+1$. Thus every point of $I_1^Y$ has property $P$ in $Y$. Since $Y \approx X$, every point of $I_1^X$ has property $P$.

Let $x \in I_2$. $X \approx X^1$, so $x \in I_2^X = I_1^{X^1}$ has a clopen neighborhood (in $X^1$) that is homeomorphic to $\omega + 1$. Let $U$ be a neighborhood of $x$ in $X$ such that $U \cap X^1$ is clopen in $X^1$ and $U \cap X^1 \approx \omega + 1$. Since $I_2$ is relatively discrete and since any $V \subseteq U$ with $x \in V$ has the property that $V \cap X^1 \approx \omega + 1$, we may assume that $U \cap X^2 = \{x\}$.

Write $U \cap X^1 = \{x_n : n \in \omega\} \cup \{x\}$; note that the map sending $n$ to $x_n$ and $\omega$ to $x$ is an embedding of $\omega+1$. For each $n$, let $U_n$ be a clopen neighborhood (in $X$) of $x_n$ that isolates $x_n$ in $I_1$ and is homeomorphic to $\omega + 1$. Set $V_n = U_n \setminus \bigcup_{m<n} U_m$. Then each $V_n$ is clopen and is homeomorphic to $\omega + 1$ (it is an open subset of $U_n$ containing the limit point) and, furthermore, the $V_n$ are pairwise disjoint.

$Y = X \setminus I_1 \approx X$, so $x$ ($\in I_1^Y$) has a neighborhood $V'$ (in $Y$) that is homeomorphic to $\omega + 1$. Let $V$ be open in $X$ such that $V' = V \cap Y$ and consider $W = U \cap V$.

*Case 1:* For some $n \in \omega$, $W \cap V_n$ is infinite. As $V_n$ is clopen in $X$, $W \setminus V_n$ is an open neighborhood of $x$. In $Y$, however, every neighborhood of $x$ contains cofinitely many points of $V'$, and hence of $W \cap Y \subseteq V'$, contradicting the fact that $W \cap Y \setminus V_n$ is an open neighborhood of $x$ in $Y$.

*Case 2:* $W \cap V_n$ is finite for all $n \in \omega$. Since $W$ is a neighborhood of $x$, $W$ must contain some (in fact cofinitely many) $x_n$. But $W$ is also a neighborhood of $x_n$, and any neighborhood of $x_n$ contains cofinitely many points of $V_n$, a contradiction. □

**Corollary 4.8.** *$X$ is not first countable. In fact, no limit point of $X$ has a countable neighborhood basis.*

*Proof.* It suffices to show that if some limit $x \in X$ has a countable neighborhood basis then $\omega + 1$ can be embedded in $X$. As $X$ is Hausdorff and not discrete, this is obvious. □

**Corollary 4.9.** *$X$ is not metrizable.*

**Corollary 4.10.** *$X$ is not a GO space (recall that a GO space, or Generalized Order space, is any topology that is homeomorphic to a subspace of a linearly ordered space).*

*Proof.* $X$ is locally countable, meaning that every $x \in X$ has a countable open neighborhood, e.g. $X \setminus X^{\mathrm{rk}(x)+1}$. By a theorem of van Dalen and Wattel (see [25] for a streamlined proof), a $T_1$ space is a GO space if and only if it has a subbasis consisting of a union of two nests of open sets. Using this characterization, it is straightforward to show that any locally countable GO space must also be first countable.

This result will be strengthened significantly in Corollary 5.26. □

**Corollary 4.11.** *$X$ is anti-compact, i.e., every compact subset of $X$ is finite.*

*Proof.* Suppose $K$ is an infinite compact subset of $X$. Note that $K$ is an infinite scattered Hausdorff space. If $K = I_0^K$ then $K$ would be an infinite discrete space, hence not compact. Let $x \in I_1^K$ and let $H$ be a compact neighborhood of $x$ such that $H \cap K^1 = \{x\}$ (recall that, as a compact Hausdorff space, $K$ is locally compact). $H$ must contain infinitely many points of $I_0^K$ since otherwise $x$ would be isolated in $H$ and hence in $K$. Moreover, $I_0^K$ is countable because is it a discrete subset of $X$. Thus $H$ is a countable compact Hausdorff space consisting solely of isolated points except for a single limit point. Any such space is homeomorphic to $\omega + 1$, contradicting Theorem 4.7. □

**Corollary 4.12.** *$X$ is not locally compact.*

**Lemma 4.13.** *Every paracompact space with a dense Lindelöf subspace is Lindelöf.*

*Proof.* This is an exercise from [48], pp. 152.

Let $Y$ be paracompact and suppose $D$ is Lindelöf and dense in $Y$. Let $\mathcal{U}$ be an open cover of $Y$. Applying paracompactness, we obtain an open locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. Every paracompact space is normal, and every locally finite open cover of a normal space has a shrinking. Let $\mathcal{W}$ be a shrinking of $\mathcal{V}$. $\{W \cap D \colon W \in \mathcal{W}\}$ covers $D$ and thus has a countable subcover $\{W_n \cap D \colon n \in \omega\}$. For each $n$, there is a $U_n \in \mathcal{U}$ such that $\overline{W_n} \subseteq U_n$ because $\mathcal{W}$ is a shrinking of $\mathcal{V}$ which refines $\mathcal{U}$. We are done if we can show that $\{U_n \colon n \in \omega\}$ covers $Y$. Since $\mathcal{W}$ is locally finite, $\bigcup_{n\in\omega} \overline{W_n} = \overline{\bigcup_{n\in\omega} W_n}$. Thus

$$X = \overline{D} \subseteq \overline{\bigcup_{n\in\omega} W_n} = \bigcup_{n\in\omega} \overline{W_n} \subseteq \bigcup_{n\in\omega} U_n$$

□

**Proposition 4.14.** *$X$ is not paracompact.*

*Proof.* We have already shown that $X$ is separable but is not Lindelöf; apply the above lemma. □

We say that a subset $A$ of $X$ is **almost dense** if $\overline{A}$ is cocountable, or, equivalently, if there is some $\alpha < \omega_1$ such that $X^\alpha \subseteq \overline{A}$. Recall that a **countably compact** Hausdorff space is one in which every infinite subset has a limit point.

**Proposition 4.15.** *If $X$ is countably compact then every infinite subset of $X$ is almost dense.*

*Proof.* Suppose that $X$ is countably compact and let $A \subseteq X$ be countable. $\overline{A}$ is a scattered space and thus has a relatively discrete subspace $I_0^{\overline{A}}$. Furthermore, $I_0^{\overline{A}}$ is dense in $\overline{A}$ and $I_0^{\overline{A}} \subseteq A$. Thus, without loss of generality, we may assume that $A$ is relatively discrete. If $\overline{A}$ is not cocountable, $Y = (X \setminus \overline{A}) \cup A \approx X$. But $Y$ is not countably compact, since $A$ has no limit points in $Y$. $\square$

**Corollary 4.16.** *If $X$ is countably compact then every open subset of $X$ is either countable or cofinite.*

**Proposition 4.17.** *If $X$ is regular then $X$ is not countably compact.*

*Proof.* Let $x \in I_1$ and let $U$ be any neighborhood of $x$ that isolates $x$ in $X^1$. Using regularity, take V to be an open neighborhood of $x$ such that $\overline{V} \subseteq U$. Let $Y = X \setminus \{x\}$. Then $Y \approx X$, but $Y$ is not countably compact, since $V \cap Y$ is an infinite subset of $Y$ with no limit points. $\square$

## 4.4 Filters on $I_0$

We now take a closer look at the "bottom" of $X$. Specifically, we examine what convergence looks like in $I_0 \cup I_1$.

Let $U \subseteq X$ be a relatively discrete set and let $x \in \overline{U}$. Define

$$\mathcal{F}_x(U) = \{V \cap U : V \text{ is open and } x \in V\}$$

It is clear that

- $U \in \mathcal{F}_x(U)$, $\emptyset \notin \mathcal{F}(U)$
- If $A \in \mathcal{F}_x(U)$ and $B \in \mathcal{F}_x(U)$ then $A \cap B \in \mathcal{F}_x(U)$
- If $A \in \mathcal{F}_x(U)$ and $A \subseteq B \subseteq U$, $B \in \mathcal{F}_x(U)$ (because $U$ is discrete)
- $U \setminus \{y\} \in \mathcal{F}_x(U)$ for any $y \in U$ (because $X$ is $T_1$)

In other words, $\mathcal{F}_x(U)$ is a non-principal filter on the countable set $U$. In general, there is a duality between non-principle filters on $\omega$ and countable Hausdorff topologies with a single non-isolated point.

Let $\mathcal{F}$ be a filter on a set $A$ and let $\mathcal{G}$ be a filter on a set $B$. We say that $\mathcal{F}$ and $\mathcal{G}$ are **isomorphic** (denoted $\mathcal{F} \approx \mathcal{G}$) if there is a bijection $f : A \to B$ such that

$$C \in \mathcal{F} \Leftrightarrow f[C] \in \mathcal{G}$$

It is straightforward to show that, whenever $U$ and $V$ are neighborhoods of $x \in I_1$ isolating $x$ in $X^1$, $\mathcal{F}_x(U \cap I_0) \approx \mathcal{F}_x(V \cap I_0)$ if and only if $U \approx V$. In other words, the duality between countably infinite Hausdorff spaces with a single non-isolated point and non-principle filters on $\omega$ extends to their respective isomorphisms.

Recall that the *Fréchet filter* on an infinite set $A$ is the set of all cofinite subsets of $A$. This filter is dual to the topology of $\omega + 1$. By Theorem 4.7, there is no relatively discrete $U \subseteq X$ and $x \in \overline{U}$ such that $\mathcal{F}_x(U)$ is the Fréchet filter.

If $\mathcal{F}$ is a filter on $A$ and $B \subseteq A$, define

$$\mathcal{F}\#B = \{F \cap B : F \in \mathcal{F}\}$$

to be the **restriction** of $\mathcal{F}$ to $B$. It is clear that this set will be a filter if and only if $B$ is in some ultrafilter extending $\mathcal{F}$. We denote the set of all such $B$ (alternatively, the union of all such ultrafilters) by $\mathcal{F}^+$:

$$\mathcal{F}^+ = \{B : \mathcal{F}\#B \text{ is a filter}\}$$

As in [28], a filter $\mathcal{F}$ is called **homogeneous** if $\mathcal{F} \approx \mathcal{F}\#B$ whenever $B \in \mathcal{F}^+$.

**Lemma 4.18.** *Let $\mathcal{F}$ be a non-principal filter on a countably infinite set $A$ and let $B \in \mathcal{F}$. If $\mathcal{F}\#B$ is not the Fréchet filter, then $\mathcal{F} \approx \mathcal{F}\#B$.*

*Proof.* Since $\mathcal{F}\#B$ is not the Fréchet filter, there is some $F \in \mathcal{F}$ such that $|B \setminus F| = \aleph_0$. Since $B \setminus F \subseteq A \setminus (B \cap F)$, we also have $|A \setminus (B \cap F)| = \aleph_0$.

Thus there is a bijection $f : A \setminus (B \cap F) \to B \setminus F$. Define $\phi : A \to B$ by

$$\phi(x) = \begin{cases} f(x) & \text{if } x \in A \setminus (B \cap F) \\ x & \text{if } x \in B \cap F \end{cases}$$

If $C \subseteq A$ then

$$C \in \mathcal{F} \Leftrightarrow C \cap B \cap F = \phi[C \cap B \cap F] \in \mathcal{F} \Leftrightarrow$$

$$\phi[C \cap B \cap F] = \phi[C] \cap B \cap F \in \mathcal{F}\#B \Leftrightarrow \phi[C] \in \mathcal{F}\#B$$

Hence $\phi$ is a filter isomorphism. □

**Theorem 4.19.** *Let $U$ and $V$ be open subsets of $X$ such that $|U \cap X^1| = |V \cap X^1| = 1$. Then $U \approx V$.*

*Proof.* It suffices to show that $\mathcal{F}_x(U \cap I_0) \approx \mathcal{F}_y(V \cap I_0)$ whenever $U \cap X^1 = \{x\}$ and $V \cap X^1 = \{y\}$, where $U, V$ are open and $x, y \in I_1$. First we will show that $\mathcal{F}_x(I_0) = \mathcal{F}_y(I_0)$ for every $x, y \in I_1$. In order to prove this, pick $x \in I_1$ and set

$$A = \{y \in I_1 : \mathcal{F}_x(I_0) \approx \mathcal{F}_y(I_0)\}$$

If $\overline{A}$ were countable, $Y = X \setminus \overline{A}$ would have the property that for no $y \in I_1^Y = I_1^X \setminus Y$ is $\mathcal{F}_y^Y(I_0)$ isomorphic to $\mathcal{F}_x^X(I_0)$; as $\mathcal{F}_y^Y(I_0) \approx \mathcal{F}_y^X(I_0)$ and $X \approx Y$, this is a contradiction. Thus $\overline{A}$ is uncountable and $X \approx I_0 \cup \overline{A}$. But it is clear that $I_1^{I_0 \cup \overline{A}} = A$, so every $y \in I_1^{I_0 \cup \overline{A}}$ satisfies $\mathcal{F}_y^{I_0 \cup \overline{A}}(I_0) \approx \mathcal{F}_x^{I_0 \cup \overline{A}}(I_0)$. Hence all the $\mathcal{F}_x(I_0)$ are isomorphic.

Now let $U, V$ be as above, and let $x$ denote the unique element of $U \cap I_1$ and let $y$ denote the unique element of $V \cap I_1$. By Theorem 4.7, neither $U$ nor $V$ is homeomorphic to $\omega + 1$. In the language of filters, this means that neither $\mathcal{F}(U)$ nor $\mathcal{F}(V)$ is the Fréchet filter. Furthermore, since $X$ is $T_1$, neither $\mathcal{F}_x(U \cap I_0)$ nor $\mathcal{F}_y(V \cap I_0)$ is a principal filter. Thus, using the previous paragraph and Lemma 4.18, we have

$$\mathcal{F}_x(U \cap I_0) = \mathcal{F}_x(I_0)\#(U \cap I_0) \approx \mathcal{F}_x(I_0) \approx$$

$$\mathcal{F}_y(I_0) \approx \mathcal{F}_y(I_0)\#(V \cap I_0) = \mathcal{F}_y(V \cap I_0)$$

□

Thus there is a single distinguished filter (up to isomorphism) given by $X$ describing how $I_0$ converges to each $x \in I_1$; henceforth this filter will be denoted by $\mathcal{F}$. $\mathcal{F}$ also tells us something about convergence in higher levels:

**Corollary 4.20.** *Let $A$ be any countable relatively discrete subset of $X$ and let $x \in \overline{A}$. Then $\mathcal{F}_x(A) \approx \mathcal{F}$.*

*Proof.* Let $\alpha = \sup\{\mathrm{rk}(y) + 1 \colon y \in A \text{ or } y = x\}$ and consider $Y = A \cup \{x\} \cup X^\alpha$. In $Y$, it is clear that $A \cup \{x\}$ is an open set and that $(A \cup \{x\}) \cap Y^1 = \{x\}$. $X \approx Y$, so it follows from Theorem 4.19 that $\mathcal{F}_x(A) \approx \mathcal{F}$. □

**Corollary 4.21.** *$\mathcal{F}$ is homogeneous, does not have a countable base, is not the Fréchet filter, and is not principal.*

*Proof.* That $\mathcal{F}$ is homogeneous follows from the previous corollary, and that $\mathcal{F}$ does not have a countable base follows from the fact that no limit point of $X$ has a countable neighborhood basis. That $\mathcal{F}$ is neither the Fréchet filter nor a principal filter follows from the fact that it does not have a countable base. □

**Theorem 4.22.** *$\mathcal{F}$ is not an ultrafilter.*

*Proof.* The following definition is due to Frolík (see [22]): if $\{\mathcal{G}_n\colon n \in \omega\}$ is a collection of ultrafilters on $\omega$ and if $\mathcal{G}$ is another ultrafilter on $\omega$, $\sum_{\mathcal{G}} \{\mathcal{G}_n\colon n \in \omega\}$ consists of all sets of the form $\cup\{M_n\colon n \in X\}$ where $X \in \mathcal{G}$ and $M_n \in \mathcal{G}_n$ for each $n \in X$.

Assume that $\mathcal{F}$ is an ultrafilter; we show that $\mathcal{F} = \sum_{\mathcal{G}} \{\mathcal{G}_n\colon n \in \omega\}$ where $\mathcal{G} \approx \mathcal{G}_1 \approx \mathcal{G}_2 \approx ... \approx \mathcal{F}$. By a Theorem of Frolík (see [21]), no such ultrafilter exists, so this is enough to prove the theorem. (In fact, a contradiction to Frolík's theorem does not even require $\mathcal{G}_1 \approx \mathcal{G}_2 \approx ... \approx \mathcal{F}$; it is enough to show merely that $\mathcal{G} \approx \mathcal{F}$.)

Let $x \in I_2$ and let $U$ be an open neighborhood of $x$ such that $U \cap X^2 = \{x\}$. Let $U \cap I_1 = \{x_n\colon n \in \omega\}$ and let $U_n$ be a neighborhood of $x_n$ such that $U_n \subseteq U$ and $U_n \cap X^1 = \{x_n\}$. Let $Z = U \cap I_0$. Let $\mathcal{H} = \mathcal{F}_x(Z)$, let $\mathcal{G}_n = \mathcal{F}_{x_n}(Z)$, and let $\mathcal{G}$ be the filter defined on $\omega$ as follows: $A \in \mathcal{G}$ if and only if there is some open neighborhood $V$ of $x$ such that $V \subseteq U$ and $V \cap I_1 = \{x_n : n \in A\}$. In other words, $\mathcal{G}$ is the induced image of $\mathcal{F}_x(\{x_n : n \in \omega\})$ under the natural isomorphism $\{x_n : n \in \omega\} \to \omega$.

Let $A \subseteq Z$ be an element of $\mathcal{H}$. By definition, there is some open subset $V$ of $U$ such that $x \in V$ and $V \cap Z = A$. Since $V$ is open, $\{n : x_n \in V\} \in \mathcal{G}$ and, for each $n \in \mathcal{G}$, $V$ is a neighborhood of $x_n$ and so $A = V \cap Z \in \mathcal{F}_{x_n}(Z)$. Thus $A \in \sum_{\mathcal{G}} \{\mathcal{G}_n : n \in \omega\}$. As $A$ was arbitrary, $\mathcal{H} \subseteq \sum_{\mathcal{G}} \{\mathcal{G}_n : n \in \omega\}$. By Corollary 4.20, $\mathcal{H} \approx \mathcal{G} \approx \mathcal{G}_1 \approx \mathcal{G}_2 \approx ... \approx \mathcal{F}$. Since $\mathcal{F}$ (hence $\mathcal{G}, \mathcal{G}_n$) is an ultrafilter, $\sum_{\mathcal{G}} \{\mathcal{G}_n : n \in \omega\}$ is also an ultrafilter. Hence $\mathcal{H} \subseteq \sum_{\mathcal{G}} \{\mathcal{G}_n : n \in \omega\}$ implies $\mathcal{H} = \sum_{\mathcal{G}} \{\mathcal{G}_n : n \in \omega\}$, which proves the theorem. $\square$

Given a filter $\mathcal{G}$ on a set $A$, define the filter $\mathcal{G}^2$ on $A \times A$ by

$$B \in \mathcal{G}^2 \Leftrightarrow \{n : \{m : (m, n) \in B\} \in \mathcal{G}\} \in \mathcal{G}$$

If $\mathcal{G} \approx \mathcal{G}^2$, we say that $\mathcal{G}$ is **idempotent**. If there is a filter $\mathcal{H}$ on $A \times A$ such that $\mathcal{H} \subseteq \mathcal{G}^2$ and $\mathcal{G} \approx \mathcal{H}$, then we say that $\mathcal{G}$ is **sub-idempotent**.

In the original statement of the Toronto problem it was asserted that the existence of a Hausdorff Toronto space implies the existence of a homogeneous idempotent filter on $\omega$, the implication being that the characteristic filter $\mathcal{F}$ described above is idempotent. This has not been proved, however, and the work of Gruenhage and Moore leads us to believe that it might be incorrect, as explained in the next paragraph. Of course we cannot say definitively that it is incorrect until a non-discrete HATS has been constructed.

In [46], Steprāns defines an $\alpha$-Toronto space, $\alpha < \omega_1$, to be a scattered Hausdorff space of height $\alpha$ that is homeomorphic to any of its subspaces of height $\alpha$. The analogy with HATS is clear and, in fact, with this terminology a HATS is just an $\omega_1$-Toronto space. Gruenhage and Moore, in [26], prove that $\alpha$-Toronto spaces exist for $\alpha \leq \omega$ and exist consistently for $\omega < \alpha < \omega_1$. Their examples of $\alpha$-Toronto spaces also satisfy Theorem 4.19: convergence in the bottom levels depends on a characteristic filter. However, the characteristic filters of the spaces they construct are not idempotent. As we have no compelling reason to believe that the bottom few levels of a Toronto space should necessarily look very different from the bottom few levels of Gruenhage and Moore's $\alpha$-Toronto spaces, this leads us to believe that $\mathcal{F}$ need not be idempotent.

Nonetheless, in the case that $X$ is regular, we can prove something close to idempotence for $\mathcal{F}$:

**Proposition 4.23.** *If $X$ is regular then $\mathcal{F}$ is sub-idempotent.*

*Proof.* Assume $X$ is regular. We first observe that every $x \in I_1$ has a neighborhood basis of clopen sets. Indeed, if we take any neighborhood $U$ of $x$ such that $U \cap X^1 = \{x\}$ and then take an open $V \subseteq U$ with $\overline{V} \subseteq U$, $V$ must be clopen. This is because $\overline{V} \setminus V \subseteq I_0$, and as every subset of $I_0$ is open, this implies $\overline{V} = V$. Furthermore, since $x$ is the only non-isolated point of $V$, any neighborhood of $x$ that is a subset of $V$ is also clopen.

Let $x \in I_2$ and let $U$ be a neighborhood of $x$ such that $U \cap X^2 = \{x\}$. Let $V$ be a neighborhood of $x$ such that $\overline{V} \subseteq U$. Write $I_1 \cap V = \{y_n \colon n \in \omega\}$. We can choose a clopen neighborhood $W_0$ of $y_0$ isolating $y_0$ in $X^1$. Proceeding inductively, we can choose $W_{n+1}$ to be a clopen neighborhood of $y_{n+1}$ that isolates $y_{n+1}$ in $X^1$ and is disjoint from $W_1, ..., W_n$. Furthermore, because $W_n \setminus \{y_n\} \subseteq I_0$, every open subset of $W_n$ containing $y_n$ is also clopen. In particular, we may assume each $W_n \subseteq V$.

Let $W = \bigcup_{n\in\omega} W_n \cup \{x\}$. To obtain the idempotence of $\mathcal{F}$, we would like for $W$ to be an open neighborhood of $x$, but this may not be the case. Nonetheless, if we take $Y = X \setminus (U \setminus W)$, then $Y \approx X$, $x \in I_2^Y$, and (in $Y$) $W$ is a clopen neighborhood of $x$. Thus, without loss of generality, we may assume that $W$ is clopen. We argue that the filter

$$\mathcal{G} = \{W' \cap W \cap I_0 \mid W' \text{ is a neighborhood of } x\}$$

is isomorphic to $\mathcal{F}$ and is weaker than a filter isomorphic to $\mathcal{F}^2$. It follows from this that $\mathcal{F}$ is sub-idempotent.

Since $x$ is a limit point of $W \cap I_0$, Corollary 4.20 implies $\mathcal{G} \approx \mathcal{F}$.

To see that $\mathcal{G}$ is (isomorphic to a filter that is) weaker than $\mathcal{F}^2$, note that $W \cap I_0$ is partitioned into countably many countable subsets, namely the $W_n \cap I_0$. We may think of these sets as the different "rows" of $(W \cap I_0) \times (W \cap I_0)$. Let $\mathcal{F}_n = \mathcal{F}_{y_n}(W_n \cap I_0)$ and $\mathcal{H} = \mathcal{F}_x(\{y_n \colon n \in \omega\})$. By Corollary 4.20, $\mathcal{H} \approx \mathcal{F}_1 \approx \mathcal{F}_2 \approx ... \approx \mathcal{F}$. If $A \in \mathcal{G}$ then there is some neighborhood $W' \subseteq W$ of $x$ such that $W' \cap I_0 = A$. But then, because $W'$ is open, $A_1 = \{y_n \colon y_n \in W'\} \in \mathcal{H}$ and, for each $y_n \in A_1$, $\{z \in W_n \colon z \in W\} \in \mathcal{F}_n$ (this is because $W \cap W'$ is a neighborhood of $y_n$). Thus

$$A \in \mathcal{G} \ \Rightarrow \ \{n \colon A \cap W_n \in \mathcal{F}_n\} \in \mathcal{H}$$

Since $\mathcal{H} \approx \mathcal{F}_1 \approx \mathcal{F}_2 \approx ... \approx \mathcal{F}$, we conclude that $\mathcal{G}$ is weaker than $\mathcal{F}^2$. $\square$

Notice that this proposition does not rule out the idempotence of $\mathcal{F}$: "sub-idempotent" does not necessarily mean strictly sub-idempotent.

## 4.5 Irregularity

We say that $x$ is a **regular point** of $X$ if every open neighborhood of $x$ contains a closed neighborhood of $x$; otherwise $x$ is called an irregular point. $X$ is a regular space if and only if all of its points are regular points. $x \in X$

is a **strongly irregular point** if every neighborhood of $x$ is almost dense. We say that $x$ is a **weakly irregular point** of $X$ if $x$ has a neighborhood whose closure is contained in $\bigcup_{\alpha \leq \mathrm{rk}(x)} I_\alpha$. The main result of this section is:

**Theorem 4.24.** *If $X$ is a non-discrete HATS then one of the following must hold:*

*(1) $X$ is regular.*

*(2) There is an infinite closed $R \subseteq I_1$ such that every $x \in R$ is a regular point of $X$, and every other limit point of $X$ is weakly irregular.*

*(3) Every limit point of $X$ is strongly irregular.*

*Proof.* We will prove Theorem 4.24 through a sequence of lemmas.

**Lemma 4.25.** *If $Y$ is any topological space and $x \in Z \subseteq Y$, and if $x$ is a regular point of $Y$, then $x$ is a regular point of $Z$. In other words, regularity at a point is an hereditary property.*

*Proof.* Let $x \in Z \subseteq Y$ be a regular point of $Y$ and let $U'$ be a neighborhood of $x$ in $Z$. $U' = U \cap Z$ for some neighborhood $U$ of $x$ in $Y$. Applying the fact that $x$ is a regular point of $Y$, there is some neighborhood $V$ of $y$ such that $\overline{V}^Y \subseteq U$. Setting $V' = V \cap Z$, $\overline{V'}^Z = \overline{V}^Y \cap Z \subseteq U \cap Z = U'$. As $U$ was arbitrary, it follows that $x$ is a regular point of $Z$. $\square$

**Lemma 4.26.** *If $Y$ is any topological space and $D$ is dense in $Y$, then $D \cap U$ is dense in $\overline{U}$ for any open subset $U$ of $Y$.*

*Proof.* $D \cap U$ is dense in $U$, so $U \subseteq \overline{D \cap U} \subseteq \overline{U}$. Hence $\overline{D \cap U} = \overline{U}$. □

**Lemma 4.27.** *If $X$ contains an irregular point that is not weakly irregular, then every limit point of $X$ is strongly irregular.*

*Proof.* Assume that $X$ contains an irregular point $x$ that is not weakly irregular. Let $\alpha = \mathrm{rk}(x) > 0$ and consider $Y = X \setminus \bigcup_{1 \leq \beta < \alpha} I_\beta \approx X$. Because $I_0^X \subseteq Y$ and $I_0^X$ is dense in $X$, it follows from Lemma 4.26 that, for any open neighborhood $U$ of $x$ in $Y$, $\overline{U}^Y = \overline{U}^X \cap Y$. Hence $x$ is still an irregular point in $Y$ and $x$ is still not weakly irregular in $Y$. Furthermore, it is clear that $\mathrm{rk}^Y(x) = 1$. Thus, if $X$ contains an irregular point that is not weakly irregular, it contains such a point in $I_1$.

Within the scope of this proof, we will say that a point $x$ of $X$ is **moderately irregular** if every neighborhood of $x$ has uncountable closure. (After this proof we will have no need of this new notion since, as we shall see shortly, every irregular point of $X$ is either weakly irregular or strongly irregular.)

Next we show that some point of $I_1$ is moderately irregular. Suppose that this is not the case. Then every point $x$ of $I_1$ has a neighborhood $U_x$

such that $\overline{U_x}$ is countable. Set $Y = (X \setminus \bigcup_{x \in I_1} \overline{U_x}) \cup I_0 \cup I_1$. Then $X \approx Y$ and every point of $I_1^Y = I_1^X$ is weakly irregular. This contradicts what we have shown so far, so $I_1$ contains moderately irregular points.

Next we show that every point of $I_1$ is moderately irregular. Let $A$ denote the set of moderately irregular points of $I_1$. If $\overline{A}$ is not cocountable, then $Y = X \setminus \overline{A} \approx X$ has no moderately irregular points in $I_1^Y$, and we have already shown that this is impossible. Thus $\overline{A}$ is cocountable. Set $Y = I_0 \cup \overline{A}$. Because $X \setminus Y$ is countable and because $I_0 \subseteq Y$, it follows from Lemma 4.26 that every moderately irregular point of $X$ remains moderately irregular in $Y$. Hence every point of $I_1^Y$ is moderately irregular in $Y$. As $Y \approx X$, every point of $I_1^X$ is moderately irregular in $X$. In particular, $I_1$ contains no weakly irregular points.

Finally, we show that every limit point of $X$ is strongly irregular. Suppose this is not the case. Then, for some $x \in I_\alpha$, $\alpha \geq 1$, there is a neighborhood $U$ of $x$ that is not almost dense, i.e., such that $\overline{U}$ is not cocountable. Set $Y = (X \setminus \overline{U}) \cup I_0 \cup \{x\}$. Since $U \cap Y$ is a neighborhood of $x$ in $Y$ containing only $x$ and points of $I_0^Y$, $\mathrm{rk}^Y(x) = 1$. $\overline{U \cap Y}^Y = U \cap Y \subseteq I_0^Y \cup \{x\}$, so $x$ is not moderately irregular in $Y$. As $Y \approx X$, this contradicts the conclusion of the previous paragraph. Thus every limit point of $X$ is strongly irregular. $\square$

**Lemma 4.28.** *If $x \in I_1$ then $x$ is a regular point if and only if $x$ has a clopen neighborhood $U$ such that $U \cap X^1 = \{x\}$. Moreover, this holds if and only if $x$ has a neighborhood basis of clopen sets.*

*Proof.* If $x \in I_1$, there is an open set $V$ such that $V \cap X^1 = \{x\}$. Using the regularity of $x$, let $U \subseteq V$ be an open neighborhood of $x$ such that $\overline{U} \subseteq V$. Then $\overline{U} \setminus U \subseteq I_0$, and every subset of $I_0$ is open; thus $\overline{U} = U$, i.e., $U$ is clopen. Conversely, suppose $U$ is clopen and $U \cap I_1 = \{x\}$. Any open subset of $U$ containing $x$ is also clopen, so $x$ has a neighborhood basis of clopen sets. □

**Lemma 4.29.** *Suppose that $X$ is not regular and that not every limit point of $X$ is strongly irregular. Then $X$ has countably many regular points, and all other points of $X$ are weakly irregular. Furthermore, the closure of the set of regular limit points of $X$ is countable, and these points are arranged in $X$ as follows: there is some $1 < \alpha < \omega_1$ such that there are no regular points of rank $\alpha$ or higher, and, for every $\beta < \alpha$, there are infinitely many regular points of rank $\beta$.*

*Proof.* If $X$ is not regular and not every limit point of $X$ is strongly irregular, then by Lemma 4.27 every limit point of $X$ is either regular or weakly irregular. Regularity at a point is an hereditary property, so the collection of regular points of $X$ is a regular subspace of $X$. If this collection is uncountable then it is homeomorphic to $X$, making $X$ regular. Thus $X$ has countably many regular points. Let

$$\alpha = \sup\{\mathrm{rk}(x)+1 \colon x \text{ is a regular point of X}\} < \omega_1$$

Clearly $X$ has no regular points of rank $\alpha$ or higher.

Next we show that $\alpha > 1$, i.e., that $X$ has regular limit points. We know that $X$ has weakly irregular points: let $x$ be one such and let $U$ be a neighborhood of $X$ such that $\overline{U}$ is countable. Set $Y = (X \setminus \overline{U}) \cup (U \cap I_0) \cup \{x\}$. $I_0^X = I_0^Y$ is dense in $Y$, so $x \notin I_0$ is a limit point of $Y$. $U \cap Y$ is a neighborhood of $x$ in $Y$ that contains only $x$ and points of $I_0^Y$, so $\mathrm{rk}^Y(x) = 1$. Furthermore, $\overline{U \cap Y}^Y = \overline{U}^X \cap Y = U$, so $U \cap Y$ is a clopen neighborhood of $x$ in $Y$. It follows from Lemma 4.28 that $x$ is a regular limit point of $Y$. Since $Y \approx X$, $X$ has regular limit points.

Let $R$ denote the set of regular limit points of $X$. It remains to show that $|\overline{R}| = \aleph_0$ and that, for every $1 \leq \beta < \alpha$, $|R \cap I_\beta| = \aleph_0$.

Let $\beta < \alpha$ and let $R_\beta = R \cap I_\beta$ denote the set of regular points of rank $\beta$. If $R_\beta$ is finite, $Y = X \setminus R_\beta$ has no regular points of rank $\beta$; as $Y \approx X$, $R_\beta = \emptyset$. Thus $R_\beta$ is either empty or infinite. Since $\beta < \alpha$, there is some $\gamma$ such that $\beta \leq \gamma < \alpha$ and $R_\gamma \neq \emptyset$. Let $Y = \bigcup_{\zeta<\beta} I_\zeta \cup X^\gamma$. If a point has rank $\gamma$ in $X$ then that same point has rank $\beta$ in $Y$. Since the property of being a regular point is an hereditary property (Lemma 4.25), $Y$ has regular points of rank $\beta$. As $Y \approx X$, $R_\beta \neq \emptyset$. Since, as we have already argued, $R_\beta$ cannot be finite and nonempty, $|R_\beta| = \aleph_0$, and this holds for arbitrary $1 \leq \beta < \alpha$.

In order to show that $\overline{R}$ is countable, we must first show that some point of $I_1$ is irregular in $X$. Let $x$ be any irregular point of $X$. There is an open neighborhood $U$ of $x$ such that, for any neighborhood $V$ of $x$, $\overline{V} \nsubseteq U$. Consider $Y = (X \setminus U) \cup (I_0 \cap U) \cup \{x\}$. Clearly $Y \approx X$ and $\mathrm{rk}^Y(x) = 1$, so it suffices to show that $x$ is irregular in $Y$. Let $V \cap Y$ be an arbitrary neighborhood of $x$ in $Y$. By Lemma 4.26 and the fact that $I_0^X = I_0^Y$ is dense in both $X$ and $Y$, $\overline{V \cap Y}^Y = \overline{V}^X \cap Y$. Hence

$$\overline{V \cap Y}^Y \setminus (U \cap Y) = (\overline{V}^X \cap Y) \setminus (U \cap Y) = (\overline{V}^X \setminus U) \cap Y = \overline{V}^X \setminus U \neq \emptyset$$

Since $V \cap Y$ was arbitrary, $U \cap Y$ bears witness to the fact that $x$ is not a regular point in $Y$. Thus $X$ has irregular points in $I_1$.

Suppose $\overline{R}$ is uncountable. Setting $Y = \overline{R} \cup I_0$, we have $I_1^Y \subseteq R$. It follows from Lemma 4.25 that every point of $I_1^Y$ is a regular point of $Y$. Since $Y \approx X$, this contradicts the conclusion of the previous paragraph, and it follows that $\overline{R}$ is countable. $\square$

All that remains for the proof of Theorem 4.24 is to show that in case

(2) the set of regular points of $X$ is a closed subset of $I_1$. For the remainder of the proof, we assume that $X$ is not regular and does not have strongly irregular points (i.e., that neither case (1) nor case (3) holds). As above, we use $R$ to denote the set of regular points of $X$, $\alpha$ to denote the least ordinal such that $R \cap I_\alpha = \emptyset$, and $R_\beta$ to denote $R \cap I_\beta$.

**Lemma 4.30.** *Each $R_\beta$, $0 < \beta < \alpha$, is closed in $X$. In particular, as $R_1 \subseteq I_1$, $R_1$ is closed and relatively discrete.*

*Proof.* $\overline{R_1}$ is countable because $\overline{R}$ is countable. Let $Y = X \setminus (\overline{R_1} \setminus R_1) \approx X$.

We claim that $R_1^Y = R_1^X$. $R_1^X \subseteq R_1^Y$ because regularity at a point is an hereditary property. Suppose $x \in R_1^Y \setminus R_1^X$. Since $X \setminus Y \subseteq X^2$, $I_0^X = I_0^Y$ and $I_1^X = I_1^Y$. Thus $x \in I_1^X$. As $x$ is weakly irregular in $X$, $x$ has a neighborhood $U$ in $X$ with $\overline{U}^X \subseteq I_0^Y \cup I_1^Y$. As $x \in R_1^Y$, $x$ has a clopen neighborhood $V \subseteq I_0^Y \cup \{x\}$ in $Y$. As $V \subseteq I_0^X \cup I_1^X$ and $I_0^Y \cup I_1^Y = I_0^X \cup I_1^X$ is open in $X$, we have $V \subseteq X$, $V$ is open in $X$, and $U \cap V$ is a neighborhood of $x$ in $X$. Furthermore,

$$\overline{U \cap V}^X \subseteq \overline{U}^X \cap \overline{V}^X \subseteq \overline{U}^X \cap \left(\overline{V}^Y \cup (Y \setminus X)\right) \subseteq \overline{V}^Y = V$$

Since $V \cap I_1^X = \{x\}$, $\overline{U \cap V}^X \subseteq I_0 \cup \{x\}$. Since every subset of $I_0$ is open, $\overline{U \cap V}^X$ is open, hence clopen, in $X$. It follows from Lemma 4.28 that $x$ is a regular point of $X$. Thus $x \in R_1^X$ and $R_1^X = R_1^Y$.

$\overline{R_1^Y}^Y = \overline{R_1^X}^Y = \overline{R_1^X}^X \cap Y = R_1^X = R_1^Y$. Thus $R_1^Y$ is closed in $Y$ and, as $Y \approx X$, $R_1$ is closed in $X$.

Now let $0 < \beta < \alpha$ and let $Y = X \setminus \bigcup_{0<\gamma<\beta} I_\gamma$. $Y \approx X$ and $R_\beta^X = R_1^Y$. Thus $R_\beta^X$ is closed in $Y$. Since $Y$ is the union of a closed subset of $X$ (namely $X^\beta$) and $I_0$, and since $R_\beta \cap I_0 = \emptyset$, $R_\beta$ is closed in $X$. $\square$

**Lemma 4.31.** *$\alpha = 2$, i.e., $R \subseteq I_1$.*

*Proof.* Suppose $\alpha > 2$ and let $X_0 = X$. Take $X_1 = (X_0 \setminus I_1^{X_0}) \cup R_1^{X_0}$. Clearly $X_1$ is cocountable in $X$. $R_1^{X_0} \subseteq R_1^{X_1}$: this is because regularity is hereditary, $R_1^{X_0} \subseteq X_1$, and, as $I_0^{X_0} \subseteq X_1$, every rank-1 point of $X_0$ remains a rank-1 point in $X_1$. Also, $R_1^{X_0} \neq R_1^{X_1}$: this is because $R_1$ is closed, which implies that every $x \in R_2^{X_0}$ becomes a rank-1 point in $X_1$ and, as regularity is hereditary, a member of $R_1^{X_1}$.

Using transfinite recursion, we now define a sequence $\langle X_\beta \colon \beta < \omega_1 \rangle$ of subsets of $X$. Let $X_0 = X$. Given $X_\beta$, we let $X_{\beta+1} = (X_\beta \setminus I_1^{X_\beta}) \cup R_1^{X_\beta}$. For limit $\beta$, we take $X_\beta = \bigcap_{\gamma<\beta} X_\gamma$. It follows from induction and the argument of the previous paragraph that, for every $\beta < \omega_1$, $X_\beta$ is cocountable in $X$, hence homeomorphic to $X$, and $R_1^{X_\beta} \subseteq R_1^{X_{\beta+1}} \neq R_1^{X_\beta}$.

Take $Y = \bigcup_{\beta<\omega_1} R_1^{X_\beta}$. Because $R_1^{X_\beta} \subsetneq R_1^{X_\beta+1}$ for every $\beta$, $Y$ is uncountable and hence homeomorphic to $X$. There is some $\gamma < \omega_1$ such that $I_0^Y \subseteq \bigcup_{\beta<\gamma} R_1^{X_\beta}$. Thus $\bigcup_{\beta<\gamma+1} R_1^{X_\beta}$ is not relatively discrete. On the other hand, $\bigcup_{\beta<\gamma+1} R_1^{X_\beta} \subseteq R_1^{X_{\gamma+1}}$, so, by Lemma 4.30 and the fact that $X_{\gamma+1} \approx X$, $\bigcup_{\beta<\gamma+1} R_1^{X_\beta}$ is relatively discrete. This is a contradiction, so $\alpha = 2$. □

This completes the proof of Theorem 4.24. □

The method of the proof of Lemma 4.31 can be abstracted into a more general result:

**Proposition 4.32.** *Assume $X$ satisfies (2) from Theorem 4.24. If $D$ is a countable subset of $X$ containing no regular points, then $R^X = R^{X\setminus D}$.*

*Proof.* Stretching our notation slightly, if $Y \subseteq X$ and $h : X \to Y$ is a homeomorphism, we take $D^Y$ to mean $h[D]$. When a homeomorphism $h$ is not mentioned explicitly, it is assumed that $h$ has been chosen arbitrarily and fixed.

Assume that $D$ contains no regular points and that $R^X \neq R^{X\setminus D}$. Because every isolated point is a regular point, $D \cap I_0 = \emptyset$, so if $\mathrm{rk}^X(x) = 1$ then $\mathrm{rk}^{X\setminus D}(x) = 1$. Since regularity is hereditary, this implies $R^X \subseteq R^{X\setminus D}$. Thus $R^X \subsetneq R^{X\setminus D}$. As in the proof of Lemma 4.31, we define a transfinite sequence of uncountable subsets of $X$ by taking $X_0 = X$, $X_{\beta+1} = X_\beta \setminus D^{X_\beta}$, and, for limit $\alpha$, $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. For each $\beta < \omega_1$, $X_\beta \approx X$ and $R^{X_\beta} \subsetneq R^{X_{\beta+1}}$.

Let $Y = \bigcup_{\beta<\omega_1} R^{X_\beta}$. Since $R^{X_\beta} \subsetneq R^{X_\beta+1}$ for all $\beta < \omega_1$, $Y$ is uncountable and $Y \approx X$. As in the proof of Lemma 4.31, there is some $\gamma < \omega_1$ such that $I_0^Y \subseteq \bigcup_{\beta<\gamma} R^{X_\beta}$. Thus $\bigcup_{\beta<\gamma+1} R^{X_\beta}$ is not relatively discrete. On the other hand, $\bigcup_{\beta<\gamma+1} R^{X_\beta} \subseteq R^{X_{\gamma+1}}$, so, by Lemma 4.30 and the fact that $X_{\gamma+1} \approx X$, $\bigcup_{\beta<\gamma+1} R_1^{X_\beta}$ is relatively discrete, a contradiction. □

**Corollary 4.33.** *Assume $X$ satisfies (2) from Theorem 4.24. If $U$ is an open subset of $X$ such that $R \subseteq U$ then $X^1 \subseteq \overline{U}$. In other words, it is impossible to separate any limit point of $X$ from $R$ by open sets.*

*Proof.* Let $U$ be open, $R \subseteq U$, and suppose there is some $x \in X^1 \setminus \overline{U}$. Because $I_1^X = I_0^{X^1}$ is dense in $X^1$, we may assume $x \in I_1$. Let $V$ be an open neighborhood of $x$ such that $V \subseteq X^1 \setminus \overline{U}$, such that $V \subseteq \{x\} \cup I_0$, and such that $\overline{V} \subseteq I_0 \cup I_1$ (for this last requirement, recall that $x$ is weakly irregular). Clearly $\overline{V} \cap U = \emptyset$ so, in particular, $V \cap R = \emptyset$. Let $\partial V = \overline{V} \setminus V$. By Proposition 4.32, $R^X = R^{X \setminus \partial V}$; thus $x$ is irregular in $X \setminus \partial V$. On the other hand, $V$ is a clopen neighborhood of $x$ in $X \setminus \partial V$, and it follows from Lemma 4.28 that $x$ is regular in $X \setminus \partial V$. □

## 4.6 Toronto spaces that are not Hausdorff

We now abandon HATS and consider Toronto spaces that are not Hausdorff. The **lower topology** on a totally ordered set $X$ is the topology with sets of the form $(-\infty, x]$ as a basis, and the **upper topology** on $X$ is the topology with sets of the form $[x, \infty)$ as a basis. Alternatively, a set is open in the lower topology if and only if it is an initial segment, and a set is open in the upper topology if and only if it is a final segment. Notice that, if $\kappa$ is a cardinal, the lower and upper topologies on $\kappa$ are examples of Toronto spaces. This is because every full-cardinality subset $Y$ of $\kappa$ has the same order type as $\kappa$; since a subset of $Y$ will be open in $Y$ if and only if it is an initial or a final segment, respectively, the natural order-isomorphism from $Y$ to $\kappa$ is a homeomorphism. The next theorem tells us that, other than the trivial topology on $\kappa$ points, these are the only Toronto spaces of size $\kappa$ that are not $T_1$.

**Theorem 4.34.** *Let $\kappa$ be an infinite cardinal. Up to homeomorphism, there are exactly three Toronto spaces of size $\kappa$ that are not $T_1$: the indiscrete topology and the lower and upper topologies on $\kappa$.*

*Proof.* Let $X$ be a Toronto space of size $\kappa$ that is not $T_1$. We distinguish three cases: either $X$ is indiscrete, $X$ has an open set of size $\kappa$ other than

$X$, or $X$ has a closed set of size $\kappa$ other than $X$. It is easy to see that these three cases are exhaustive. We will show that in the second case $X$ is homeomorphic to the upper topology on $\kappa$, and in the third case $X$ is homeomorphic to the lower topology on $\kappa$.

Consider the case that $X$ has a nontrivial open subset of size $\kappa$. We prove first the following claim: *there is some $x_0 \in X$ such that $\overline{\{x_0\}} = \{x_0\}$ and, for any $x \in X$, $x_0 \in \overline{\{x\}}$.* Let $U$ be an open subset of $X$ of size $\kappa$, $x \in X \setminus U$, and set $Y = U \cup \{x\}$. $X \approx Y$ and $Y$ contains a closed singleton, namely $\{x\}$. Let $C$ denote the set of closed singletons in $X$. We have just shown that $C \neq \emptyset$. Also, $|C| < \kappa$: if $|C| = \kappa$ then $X \approx C$ and, as every singleton in $C$ is closed, $X$ is $T_1$, contrary to assumption.

Let $\tilde{C} = \left\{x \in X : \overline{\{x\}} \cap C \neq \emptyset\right\}$; we claim that $X = \tilde{C}$. Set $Y = X \setminus \tilde{C}$ and suppose that $|Y| = \kappa$. Since $X$ is a Toronto space, $Y \approx X$ and there is a nonempty set $C^Y$ of closed singletons in $Y$. Let $y \in C^Y$. Since $y \notin C^X$ and $\overline{\{y\}}^Y = Y \cap \overline{\{y\}}^X$, we must have $\overline{\{y\}}^X \cap \tilde{C}^X \neq \emptyset$. But the relation $y \in \overline{\{x\}}$ is transitive, and it follows from the definition of $\tilde{C}$ that $\overline{\{y\}}^X \cap C^X \neq \emptyset$. Thus $y \in \tilde{C}^X$, a contradiction. Thus $|Y| < \kappa$ and, consequently, $\left|\tilde{C}\right| = \kappa$. It is straightforward to show that $C^X = C^{\tilde{C}}$ and that $\tilde{C}^X = \tilde{C}^{\tilde{C}}$. Since $X \approx \tilde{C}$ and $\tilde{C}^{\tilde{C}} = \tilde{C}$, we have $X = \tilde{C}^X$.

For each $x \in C$, let $K_x = \left\{y \in X : x \in \overline{\{y\}}\right\}$. Suppose there is some

$x_0 \in C$ such that $|K_{x_0}| = \kappa$. Then $X \approx K_{x_0}$ and, for every point $y \in K_{x_0}$, $x_0 \in \overline{\{y\}}^{K_{x_0}}$; in this case the claim is proved. Now suppose that, for every $x \in C$, $|K_x| < \kappa$ (note that, since $|C| < \kappa$ and $X = \tilde{C} = \bigcup_{x \in C} K_x$, this is a concern only for singular $\kappa$). Let $x \in C$ and consider $Y = (X \setminus K_x) \cup \{x\}$. $Y \approx X$ and it is clear that $K_x^Y = \{x\}$. Setting $C' = \{x \in C \colon K_x = \{x\}\}$, this shows that $C' \neq \emptyset$. Setting $Y = X \setminus C'$, we have $Y \approx X$ and $C'^Y = \emptyset$. This contradiction establishes that for some (every) $x_0 \in C$, $|K_{x_0}| = \kappa$. As above, then, we have $C = \{x_0\}$ and $X = \left\{y \in X \colon x_0 \in \overline{\{y\}}\right\}$. In either case the claim is proved.

We now will use transfinite recursion to find a sequence $\langle x_\alpha \colon \alpha < \kappa \rangle$ of points in $X$ such that $\{x_{\alpha+1} \colon \alpha < \kappa\}$ is homeomorphic to the upper topology on $\kappa$. Let $x_0$ be the unique point of $X$ such that $\overline{\{x_0\}} = \{x_0\}$. Assuming $\langle x_\beta \colon \beta < \alpha \rangle$ has already been defined (for $\alpha$ a successor or a limit), take $x_\alpha$ to be the unique point of $X_\alpha = X \setminus \{x_\beta \colon \beta < \alpha\}$ such that $\overline{\{x_\alpha\}}^{X_\alpha} = \{x_\alpha\}$. Let $Y = \{x_\alpha \colon \alpha < \kappa\}$. It follows from a straightforward transfinite induction that $\overline{\{x_\alpha\}}^Y = \{x_\beta \colon \beta \leq \alpha\}$ for every $\alpha < \kappa$. Taking $Z = \{x_{\alpha+1} \colon \alpha < \kappa\}$, we may conclude that $A$ is a closed subset of $Z$ if and only if $A$ is an initial segment of $Z$ (i.e., $A = \{x_{\beta+1} \colon \beta < \alpha\}$ for some $\alpha \leq \kappa$). Thus $Z$ is homeomorphic to the upper topology on $\kappa$, and $X \approx Z$.

Now suppose that $X$ has a nontrivial closed subset of size $\kappa$. Whereas before $X$ contained a closed singleton, now $X$ must contain an open singleton. For $x \in X$, define $x^\circ$ to be the intersection of all open sets containing $x$; this mimics the idea of $\overline{\{x\}}$ for open sets, except that $x^\circ$ is not in general open. As before, set $C = \{x \in X : x^\circ = \{x\}\}$. $C \neq \emptyset$ because $X$ contains an open singleton and, as a subspace of $X$, $C$ is $T_1$ (in fact discrete), so $|C| < \kappa$. Arguing as before, it is possible to show that there is a unique $x_0 \in X$ such that $x_0^\circ = \{x_0\}$ and, for every $y \in X$, $x_0 \in y^\circ$. Also, $\{x_0\}$ must be open in $X$ because $X$ must contain an open singleton. Using transfinite recursion as before, we obtain a subset $Y = \{x_\alpha : \alpha < \kappa\}$ of $X$ such that $A \subseteq Y$ is open if and only if $A$ is an initial segment of $Y$. Thus $X$ is homeomorphic to the lower topology on $\kappa$. $\square$

**Corollary 4.35.** *Up to homeomorphism, there are exactly five Toronto spaces of size $\aleph_0$: the discrete topology, the indiscrete topology, the co-finite topology, and the lower and upper topologies on $\omega$.*

*Proof.* Let $X$ be a countable Toronto space and suppose that $X$ is $T_1$, i.e., that every co-finite set is open in $X$. If $X$ has also some open co-infinite subset A then, taking some $x \in A$, $Y = \{x\} \cup X \setminus A$ is an infinite subspace of X that contains a clopen singleton $\{x\}$. Since $Y \approx X$, $X$ contains a clopen singleton. Let $I \neq \emptyset$ denote the set of clopen singletons in $X$. If $I$ is finite then $X \setminus I$ is a space without clopen singletons that is homeomorphic to $X$, a contradiction. If $I$ is infinite, $X \approx I$ and $X$ is discrete. Thus if $X$ is $T_1$ then $X$ has either the co-finite topology or the discrete topology, and the result follows from Theorem 4.34. □

In order to state the next result succinctly, let $LT$ be the lower topology on $\omega_1$, let $UT$ be the upper topology on $\omega_1$, let $CF$ be the topology of co-finite subsets of $\omega_1$, and let $CC$ be the topology of co-countable subsets of $\omega_1$. If $\sigma$ and $\tau$ are two different topologies on some set $X$, let $\langle \sigma, \tau \rangle$ denote the topology on $X$ generated by $\sigma \cup \tau$.

**Proposition 4.36.** *There are at least eight Toronto spaces of size $\aleph_1$: the discrete and indiscrete topologies, $LT$, $UT$, $CF$, $CC$, $\langle LT, CF \rangle$, and $\langle UT, CF \rangle$. Any HATS is a refinement of $\langle LT, CF \rangle$, and any other Toronto space of size $\aleph_1$ either is a refinement of $\langle LT, CF \rangle$ or is strictly between $CF$ and $CC$.*

*Proof.* It is easy to check that each of these eight spaces is Toronto. Notice that $\langle LT, UT \rangle$ and $\langle LT, CC \rangle$ are both the discrete topology, and $\langle UT, CF \rangle = \langle CF, CC \rangle = CC$. Thus any subcollection of these eight topologies generates one of these eight topologies.

We have seen that if $X$ is a HATS then $X$ is a scattered space with height

$\omega_1$ and width $\omega$. Thus we may assume that $X = \omega_1$ and that $X^\alpha = \omega_1 \setminus \omega \cdot \alpha$. It is then easy to see that every initial segment of $\omega_1$ is open and, since $X$ is Hausdorff, so is every cofinite set. Hence $X$ refines $\langle LT, CF \rangle$.

Examining the proof of Proposition 4.2, we see that the Hausdorff property is not used fully: in general, a non-discrete Toronto space $X$ is a scattered space of height $\omega_1$ and width $\omega$ if (and only if) $X$ contains a clopen singleton. Then, by the previous paragraph, $X$ is a refinement of $\langle LT, CF \rangle$. Thus, to finish the proof, it suffices to show that any Toronto space not on our list either contains a clopen singleton or is strictly between $CF$ and $CC$.

Suppose $X$ is a Toronto space other than the eight listed above. By Theorem 4.34, $X$ refines $CF$. If $X$ is not strictly coarser than $CC$, $X$ contains a nonempty open set $A$ with uncountable complement. Taking $x \in A$ and $Y = \{x\} \cup (X \setminus A)$, $\{x\}$ is a clopen singleton in $Y$ and $Y \approx X$. $\square$

# Chapter 5

# On neight

## 5.1 Definitions and preliminaries

Let $X$ be a topological space. A set $\mathcal{N}$ of subsets of $X$ is called a **nest** if $\mathcal{N}$ is totally ordered by $\subseteq$. In this chapter we consider subbases for topological spaces consisting of unions of nests. For example, if $X$ is a LOTS (Linearly Ordered Topological Space) then

$$\mathcal{N}_L = \{(-\infty, a)\colon a \in X\} \quad \text{and} \quad \mathcal{N}_R = \{(a, \infty)\colon a \in X\}$$

are two nests in $X$ and $\mathcal{N}_L \cup \mathcal{N}_R$ provides a subbasis for $X$.

It is trivially true that every topological space $X$ has a subbasis that can be written as a union of nests since, for each open subset $U$ of $X$, $\{U\}$ is a nest in $X$. Thus there is a least cardinal $\kappa$ such that the topology on $X$ is generated by a subbasis that can be written as the union of $\kappa$ nests in $X$. This cardinal is called the **nested weight** or **neight** of $X$, which we denote $\mathfrak{N}(X)$. If $X \approx Y$ then $\mathfrak{N}(X) = \mathfrak{N}(Y)$, so $\mathfrak{N}$ is a cardinal function in the usual sense of the term.

As mentioned already, if $X$ is a LOTS then $\mathfrak{N}(X) = 2$. A stronger result, found in [13], is that if $X$ is a Hausdorff space then $\mathfrak{N}(X) \leq 2$ if and only if $X$ is a GO space (recall that a GO space, or Generalized Order space, is any space homeomorphic to a subspace of a LOTS). In an intuitive sense, any GO space is at most one-dimensional. Also, we will show in Section 5.2 that $\mathfrak{N}(\mathbb{R}^n) = n + 1$ for every $n$. These two facts together might lead one to believe that the function $X \mapsto \mathfrak{N}(X) - 1$ is in some sense a measure of dimension. Even more suggestive is the so-called halfdirectional dimension of Deák, which we will consider in Section 5.2: as we will see, it is a measure of dimension whose definition is very similar to that of neight.

In exploring this idea we will focus mostly on spaces with finite neight. Such spaces will be called **FUN** spaces, since their topologies are generated by a **F**inite **U**nion of **N**ests. We will show that the number $\mathfrak{N}(X) - 1$ acts in some ways like a measure of dimension for FUN spaces, but in other important ways it does not. We will use the shorthand

$$\mathfrak{N}^{-}(X) = \mathfrak{N}(X) - 1$$

whenever $X$ is a FUN space.

In some intuitive sense, it seems that the neight of a space tells us the complexity of convergence in that space. This is borne out by the connections between neight and dimension discussed in Section 5.2, but also by the connections between neight and other cardinal functions like weight, character, and density; these connections will be discussed in Section 5.4.

The word "neight" was coined by Yurovetskiĭ in [49], who explored some of the basic properties of this function (his results are outlined below). Similar notions have been defined and explored in detail by E. Deák and J. Deák (see [14], [15], and [16]).

We begin with a lemma summarizing the work of Yurovetskiĭ on the basic properties of $\mathfrak{N}$:

**Lemma 5.1** (Yurovetskiĭ)**.**

*(i)* $\mathfrak{N}(X) = 0$ *if and only if* $X = \emptyset$*, and, if* $X$ *is* $T_1$, $\mathfrak{N}(X) = 1$ *if and only if* $X$ *consists of a single point.*

*(ii)* *If* $Y \subseteq X$ *then* $\mathfrak{N}(Y) \leq \mathfrak{N}(X)$.

*(iii)* $\mathfrak{N}^-(X \times Y) \leq \mathfrak{N}^-(X) + \mathfrak{N}^-(Y) + 1$.

*(iv)* $\mathfrak{N}(\prod_{\alpha \in I} X_\alpha) \leq \sum_{\alpha \in I} \mathfrak{N}(X_\alpha)$.

*(v)* *If* $Y = \bigsqcup_{\alpha \in I} X_\alpha$*, and some some* $X_\alpha$ *has* $\mathfrak{N}(X_\alpha) \geq 2$*, then* $\mathfrak{N}(Y) =$

$\sup_{\alpha \in I} \mathfrak{N}(X_\alpha)$.

*(vi) If $X$ is metrizable then $\mathfrak{N}(X) \leq \aleph_0$.*

*Proof.*

(i) $\mathfrak{N}(X) = 0$ if and only if $\emptyset$ is a (sub)basis for the topology on $X$ if and only if $X = \emptyset$. Now suppose that $X$ is $T_1$ and that there is some nest $\mathcal{N}$ that provides a subbasis for the topology on $X$. If $a, b$ are two distinct points of $X$ then there are three possibilities:

*Case 1:* There is some $U \in \mathcal{N}$ such that $a \in U$ and $b \notin U$. Since $\mathcal{N}$ is a nest, there is no member of $\mathcal{N}$ that contains $b$ but does not contain $a$. It follows that any finite intersection of members of $\mathcal{N}$ that contains $b$ must also contain $a$. Since $\mathcal{N}$ is a subbasis for $X$, this contradicts the fact that $X$ is $T_1$.

*Case 2:* There is some $U \in \mathcal{N}$ such that $b \in U$ and $a \notin U$. This case is essentially the same as Case 1.

*Case 3:* For each $U \in \mathcal{N}$, either $a \in U$, $b \in U$ or $a \notin U$, $b \notin U$. In this case, any finite intersection of members of $\mathcal{N}$ either contains both $a$ and $b$ or it contains neither $a$ nor $b$. Since $\mathcal{N}$ is a subbasis for $X$, this contradicts the fact that $X$ is $T_0$.

Thus any $T_1$ space $X$ with $\mathfrak{N}(X) = 1$ consists of at most one point. For the opposite inequality, note that the nest $\{\emptyset, \{a\}\}$ provides a subbasis for the topology on $\{a\}$, so that $\mathfrak{N}(\{a\}) = 1$.

(ii)-(vi) See [49]. □

Although $(iii)$ is a special case of $(iv)$, we have listed it separately to emphasize that $\mathfrak{N}^-$ is not necessarily additive under products. In Section 5.3 we will exhibit two fairly well-behaved spaces such that the equality in $(iii)$ holds. Part $(i)$ of this lemma gives us our first way in which $\mathfrak{N}^-$ fails to behave like a measure of dimension: if $X$ is a zero-dimensional $T_1$ space with more than one point, then $\mathfrak{N}^-(X) \geq 1$.

In what follows, we will say, whenever $\mathcal{C}$ is a collection of nests in $X$ such that $\bigcup \mathcal{C}$ is a subbasis for $X$, that $\mathcal{C}$ **generates** the topology on $X$. Also, if $\mathcal{C}$ is any collection of subsets of $X$, we define

$$\#\mathcal{C} = \left\{ \bigcap_{i=0}^{n} U_i \colon n \in \mathbb{N}, U_0, ..., U_n \in \bigcup \mathcal{C} \right\}.$$

Thus, when $\mathcal{C}$ is a collection of nests, $\mathcal{C}$ generates the topology on $X$ if and only if $\#\mathcal{C}$ is a basis for $X$.

## 5.2 DIM, ind, and $\mathfrak{N}^-$

J. Deák defines a **halfdirection** on $X$ to be a set $\mathcal{H}$ of open subsets of $X$ such that $(i)$ $\mathcal{H}$ is totally ordered by the relation $U \leq V \Leftrightarrow \overline{U} \subseteq V$

and (*ii*) If $\mathcal{H}' \subseteq \mathcal{H}$ then $\bigcup \mathcal{H}' \in \mathcal{H}$. (see [15]). Evidently, a halfdirection is a special kind of nest that is closed under supremums and in which, intuitively speaking, the "below" order has been replaced by a "way below" order. When $X$ has a topology that is generated by a finite collection of halfdirections, the **halfdirectional dimension** of $X$, denoted $\text{Dim}(X)$, is defined so that $\text{Dim}(X) + 1$ is the least $n$ such that some collection of $n$ halfdirections on $X$ generates its topology; otherwise $\text{Dim}(X) = \infty$. This definition is essentially the same as our definition of $\mathfrak{N}^-$, only with "nest" replaced by "halfdirection"; thus we have

**Proposition 5.2.** $\mathfrak{N}^-(X) \leq \text{Dim}(X)$ *for every topological space* $X$*.*

One of the most fundamental theorems concerning the halfdirectional dimension, indeed the one justifying its name, is the following:

**Theorem 5.3** (Deák)**.** *If* $X$ *is a separable metrizable space then* $X$ *embeds in* $\mathbb{R}^n$ *if and only if* $\text{Dim}(X) \leq n$*.*

*Proof.* See [15], pp. 255-256. ☐

In addition to the halfdirectional dimension, Deák studies several other related measures of dimension, most notably Dim, the directional dimension, and O-Dim, the orderly directional dimension (see [16] for an overview of these and more). The directional dimension was defined originally by E.

Deák and satisfies a theorem identical to Theorem 5.3 (so that DIM and Dim coincide for separable metric spaces). All of these dimensions are similar in that they are all defined in terms of subbases. Moreover, for any topological space $X$, we have $\mathfrak{N}^-(X) \leq \mathrm{DIM}(X) \leq \mathrm{Dim}(X) \leq \text{O-Dim}(X)$.

$\mathfrak{N}^-$, DIM, Dim, and O-Dim all have in common that it is in principle very easy to find an upper bound for these functions on a particular space (one only needs to produce a subbasis witnessing the upper bound) and more difficult in general to find a lower bound. In a sense, our study of neight is justified by this fact and by the inequality given in the previous paragraph. In this section and the two following, we will prove and apply three theorems, each of which gives a lower bound for $\mathfrak{N}$ under certain circumstances. Since $\mathfrak{N}^-$ is in turn a lower bound for DIM, Dim, and O-Dim, each our three main theorems also gives lower bounds for these.

Before moving on to these three lower bound theorems, we point out that, at least for $T_1$ spaces, DIM and $\mathfrak{N}^-$ agree on which spaces have dimension 1:

**Proposition 5.4.** *If $X$ is a $T_1$ space then $\mathfrak{N}(X) = 1$ if and only if $\mathrm{DIM}(X) = 1$ if and only if $X$ is a GO space.*

*Proof.* Assume $X$ is $T_1$. $\mathfrak{N}^-(X) \leq \mathrm{DIM}(X)$. Also, $\mathfrak{N}^-(X) < 1$ implies $X$ is either empty or a singleton; in either case, $\mathrm{DIM}(X) < 1$. It suffices, then, to prove that if $\mathfrak{N}^{-}(X) = 1$ then $\textsc{Dim}(X) \leq 1$. For this it suffices to prove that if two nests generate the topology on $X$ then these two nests are halfdirections.

Let $\mathcal{L}$, $\mathcal{R}$ be two nests on $X$ such that $\mathcal{L} \cup \mathcal{R}$ is a subbasis for $X$. Begin by expanding $\mathcal{L}$ and $\mathcal{R}$, if necessary, so that they are closed under supremums, i.e., so that $\mathcal{L}$ and $\mathcal{R}$ each contain all unions of subsets of themselves. This does not change the topology generated by $\mathcal{L} \cup \mathcal{R}$. By results of van Dalen and Wattel (see [13]), $\mathcal{L}$ and $\mathcal{R}$ induce an order $\leq$ on $X$ given by $x \leq y \Leftrightarrow \exists U \in \mathcal{L}(x \in U \wedge y \notin U)$; moreover, the topology of $X$ is at least as fine as the topology induced by this order. Using this fact, it is easy to see that if $U \in \mathcal{L}$ then $\overline{U}^{X}$ is either $U$ itself or $U$ together with one extra point. This implies that if $U, V \in \mathcal{L}$ with $U \subseteq V$ and $U \neq V$ then $\overline{U} \subseteq V$. Thus $\mathcal{L}$ is a halfdirection on $X$ and, by a similar argument, so is $\mathcal{R}$. $\square$

The following example shows that the condition in proposition 5.4 that $X$ be $T_1$ is necessary:

**Proposition 5.5.** *There is a $T_0$ space $X$ such that $\mathfrak{N}^{-}(X) = 1$ and $\textsc{Dim}(X) = \infty$.*

*Proof.* Let $X = \omega + 1$ with the following basis: $\{\omega\}$ is open in $X$ and, for each $n$, $\{n, \omega\}$ is open in $X$.

As $\omega$ is an infinite $T_1$ subspace of $X$, $\mathfrak{N}^-(X) \geq 1$ by Lemma 5.1 (*i*) and (*ii*). The two nests $\{n \cup \{\omega\} \colon n \in \omega\}$ and $\{X \setminus \alpha \colon \alpha \in \omega + 1\}$ together provide a subbasis for $X$, so $\mathfrak{N}^-(X) = 1$. (Here, as elsewhere, we identify an ordinal number with the set of its predecessors.)

Any neighborhood of any $n \in \omega$ is dense in $X$. Thus any halfdirection in $X$ can contain at most one set other than $\emptyset$, $\{\omega\}$, and $X$. From this fact it is easy to see that $\textsc{Dim}(X) = \infty$. □

The argument of Proposition 5.4 does not extend to higher-dimensional spaces, even if the condition that $X$ is $T_1$ is replaced with the much stronger condition of metrizability:

**Theorem 5.6.** *There is a metric space $X$ such that $\mathfrak{N}^-(X) = 2$ and $\textsc{Dim}(X) = \infty$.*

The proof of Theorem 5.6 is rather long and requires several auxiliary definitions and lemmas that we do not use elsewhere, so we have postponed the proof until Section 5.5. The metric space used in the proof of Theorem 5.6 is non-separable (its density is $\aleph_1$). It remains an open problem either to find a separable metric space on which $\mathfrak{N}^-$ and $\textsc{Dim}$ disagree, or to prove that there is none. In particular, it remains an open problem to show whether Deák's Theorem 5.3 holds under the weaker hypothesis that replaces DIM with $\mathfrak{N}^-$.

Recall the definition of the **small inductive dimension**: a space is $n$-dimensional if it has a basis of sets with $(n-1)$-dimensional boundaries. Formally, we begin by defining $\operatorname{ind}(\emptyset) = -1$. A space $X$ satisfies $\operatorname{ind}(X) \leq n$ if and only if there is a basis $\mathcal{B}$ for $X$ such that each $U \in \mathcal{B}$ satisfies $\operatorname{ind}(\partial U) \leq n-1$. We say that the small inductive dimension of $X$ is equal to $n$, and write $\operatorname{ind}(X) = n$, whenever it is true that $\operatorname{ind}(X) \leq n$ but it is false that $\operatorname{ind}(X) \leq m$ for $m < n$. If it is not true for any $n$ that $\operatorname{ind}(X) \leq n$, then we write $\operatorname{ind}(X) = \infty$.

Deák proves that ind is a lower bound for DIM (see [15]); for separable metric spaces, this is just a corollary of Theorem 5.3. In this section we prove this result for $\mathfrak{N}^-$ as well, but with a few additional conditions.

**Lemma 5.7.** *Let $X$ be a separable metric space.*

*(i) If $Y \subseteq X$ then $\operatorname{ind}(Y) \leq \operatorname{ind}(X)$.*

*(ii) If $C_1, ..., C_n$ are closed subsets of $X$, then*

$$\operatorname{ind}\left(\bigcup_{i=1}^{n} C_i\right) = \max\{\operatorname{ind}(C_i) : i = 1, ..., n\}.$$

*Proof.* See any reference on dimension theory, e.g. [41]. □

**Lemma 5.8.** *Let $X$ be a regular space and let $\mathcal{C}$ be a collection of nests that generates the topology on $X$. If $\mathcal{N} \in \mathcal{C}$ and $U \in \mathcal{N}$, then $\mathcal{C} \setminus \{\mathcal{N}\}$ generates the topology on $\partial U$, that is,*

$$\mathcal{S} = \{V \cap \partial U \colon V \in \mathcal{M} \in \mathcal{C} \setminus \{\mathcal{N}\}\}$$

*is a subbasis for $\partial U$. In particular, if $\mathcal{C}$ is a collection of nests in $X$ of minimal cardinality such that $\bigcup \mathcal{C}$ is a subbasis for $X$, and if $U \in \bigcup \mathcal{C}$, then $\mathfrak{N}(\partial U) \leq \mathfrak{N}(X) - 1$.*

*Proof.* Let $X$ be a regular space and let $\mathcal{C}$ be a collection of nests in $X$ that generates the topology on $X$. Let $U \in \mathcal{N} \in \mathcal{C}$ and $x \in \partial U$. We show that $x$ has a neighborhood basis (in $\partial U$) consisting of finite intersections of elements of $\mathcal{S}$. Since $\#\mathcal{C}$ is a basis for $X$, it suffices to show that, for any $V \in \#\mathcal{C}$ such that $x \in V$, there is some $W \in \#\mathcal{S}$ such that $x \in W \subseteq V \cap \partial U$.

Since $X$ is regular and $\#\mathcal{C}$ is a basis for $X$, there is some $V' \in \#\mathcal{C}$ such that $V'$ is a neighborhood of $x$ and $\overline{V'} \subseteq V$. Now

$$V' = U_0 \cap U_1 \cap ... \cap U_n$$

where, without loss of generality, $U_0 \in \mathcal{N} \cup \{X\}$ and $U_1, ..., U_n \notin \mathcal{N}$ (recall that $\mathcal{N}$, hence $\mathcal{N} \cup \{X\}$, is a nest and thus is closed under finite intersections). We claim that

$$(U_1 \cap ... \cap U_n) \cap \partial U \subseteq V \cap \partial U.$$

Suppose this is not the case and let $z \in (U_1 \cap ... \cap U_n \cap \partial U) \setminus V$. Since $\mathcal{N}$ is a nest, $U, U_0 \in \mathcal{N}$, $x \in U_0$, and $x \notin U$, we have $U \subseteq U_0$. Thus, because $z \in \partial U$, either $z \in U_0$ or $z \in \partial U_0$. On the one hand, if $z \in U_0$ then $z \in U_0 \cap U_1 \cap ... \cap U_n = V'$ and $z \notin V$, contradicting the fact that $V' \subseteq V$. On the other hand, if $z \in \partial U_0$ and $z \in U_1 \cap ... \cap U_n$, then $z \in \partial(U_0 \cap U_1 \cap ... \cap U_n) = \partial V'$. Since $z \notin V$, this contradicts the fact that $\overline{V'} \subseteq V$. □

**Theorem 5.9.** *If $X$ is a FUN separable metric space then*

$$\mathrm{ind}(X) \leq \mathfrak{N}^-(X).$$

*Proof.* The proof is by induction on $\mathfrak{N}^-(X)$. If $\mathfrak{N}^-(X) = -1$ then, by Lemma 5.1($i$), $X = \emptyset$, in which case $\mathrm{ind}(X) = -1$ as well. Now assume that the result is true for spaces $Y$ such that $\mathfrak{N}^-(Y) < \mathfrak{N}^-(X)$. Let $\mathcal{C}$ be a collection of nests that generates the topology on $X$ and is such that $|\mathcal{C}| = \mathfrak{N}(X)$. We show that, for each $U \in \#\mathcal{C}$, $\mathfrak{N}^-(\partial U) \leq \mathfrak{N}^-(X) - 1$. Let $U = U_0 \cap ... \cap U_n$, where $U_0, ..., U_n \in \mathcal{C}$. Clearly

$$\partial U \subseteq \partial U_0 \cup ... \cup \partial U_n.$$

By Lemma 5.8, $\mathfrak{N}^-(\partial U_i) \leq \mathfrak{N}^-(X) - 1$ for each $i = 0, ..., n$. By the inductive hypothesis, $\mathrm{ind}(\partial U_i) \leq \mathfrak{N}^-(X) - 1$ for each $i = 0, ..., n$. It now follows from

Lemma 5.7 that $\operatorname{ind}(\partial U) \leq \mathfrak{N}^-(X) - 1$. Since this is true for an arbitrary $U \in \#\mathcal{C}$, it follows from the definition of the small inductive dimension that $\operatorname{ind}(X) \leq \mathfrak{N}^-(X)$. □

**Corollary 5.10.** *For each* $n \in \mathbb{N}$, $\mathfrak{N}^-(\mathbb{R}^n) = n$.

*Proof.* It follows from Theorem 5.9 that $\mathfrak{N}^-(\mathbb{R}^n) \geq n$. Therefore it suffices to find, for each $n$, a collection of $n+1$ nests whose union is a subbasis for $\mathbb{R}^n$. This was done in [49]; even before that, J. Deák showed in [15] that we may use $n+1$ halfdirections to generate the topology of $\mathbb{R}^n$. □

In general, we say that the sum theorem holds for $X$ if, whenever $A$ and $B$ are closed subsets of $X$,

$$\operatorname{ind}(A \cup B) = \max\{\operatorname{ind}(A), \operatorname{ind}(B)\}.$$

Lemma 5.7($i$) says that the sum theorem holds for separable metric spaces. However, the sum theorem holds in many spaces that are not separable metric spaces. This, along with the observation that Theorem 5.9 really only uses the fact that $X$ satisfies the sum theorem (not that $X$ is a separable metric space), leads to the following strengthening of Theorem 5.9:

**Corollary 5.11.** *Let $X$ be a regular FUN space and suppose that the sum theorem holds in $X$. Then* $\operatorname{ind}(X) \leq \mathfrak{N}^-(X)$.

## 5.3 Pathologies of $\mathfrak{N}^-$

Although the notion of cofinality is most commonly applied to ordinals and cardinals, it works just as well for arbitrary linear orders. Specifically, if $\langle X, \leq \rangle$ is a linear order and $\kappa$ is an ordinal, a **cofinal** map $f : \kappa \to X$ has the property that, for all $x \in X$, there is some $\alpha < \kappa$ such that $x \leq f(\alpha)$. For any linear order $X$, the **cofinality** of $X$, denoted $\mathrm{cf}(X)$, is the least cardinal $\kappa$ such that there is a cofinal map $\kappa \to X$.

**Lemma 5.12.** *Let $\langle X, \leq \rangle$ be a linear order.*

*(i) There is a strictly increasing cofinal map* $\mathrm{cf}(X) \to X$.

*(ii) Suppose $X$ has no largest element. If there is a nondecreasing cofinal map $\lambda \to X$ then* $\mathrm{cf}(\lambda) = \mathrm{cf}(X)$.

*(iii)* $\mathrm{cf}(\mathrm{cf}(X)) = \mathrm{cf}(X)$.

*(iv) If $f : \lambda \to X$ is a cofinal map and $\mathrm{cf}(\lambda) \neq \mathrm{cf}(X)$ then there is some $\alpha < \lambda$ such that $f \upharpoonright \alpha$ is cofinal.*

*Proof.* The proof is easily modified from the proof of the analogous results for ordinals and cardinals. These can be found in any standard reference on set theory, e.g. [31]. □

We now prove that, even for fairly well-behaved spaces, the inequality in

Lemma 5.1(*iii*) cannot be improved. Our example is similar to an example used by Gerlits in the study of Dim (see [23]), but the result we will extract from it, Theorem 5.14, is more general.

**Theorem 5.13.** *There are compact LOTS $X$ and $Y$ such that*

$$\mathfrak{N}^-(X \times Y) = \mathfrak{N}^-(X) + \mathfrak{N}^-(Y) + 1$$

*Proof.* Let $X = \{0\} \times (\omega + 1) \ \cup \ \{1\} \times \omega_1$ be ordered by

$$(i, \alpha) \leq (j, \beta) \quad \text{if and only if} \quad \begin{array}{l} i < j \ \text{ or} \\ i = j = 0 \text{ and } \alpha < \beta \ \text{ or} \\ i = j = 1 \text{ and } \beta < \alpha \end{array}$$

and let $X$ have the topology induced by $\leq$. In other words, $X$ is the ordered space obtained by concatenating a copy of $\omega + 1$ with a reversed copy of $\omega_1$; this is sometimes denoted $X = (\omega + 1)^\frown \omega_1^*$. In a similar fashion, let $Y = (\omega_2 + 1)^\frown \omega_3^*$. It is easy to see that both $X$ and $Y$ are compact LOTS. Furthermore, we have already seen that every LOTS has neight 2; thus $\mathfrak{N}^-(X) = \mathfrak{N}^-(Y) = 1$.

We now show that $\mathfrak{N}^-(X \times Y) = 3$. We already know from Lemma 5.1(*iii*) that $\mathfrak{N}^-(X \times Y) \leq 3$, so it suffices to show that no collection of only 3 nests can produce a subbasis for $X \times Y$. Suppose that $\mathcal{N}_0, \ . \ . \ . \ , \mathcal{N}_n$ are nests in

$X \times Y$ such that $\bigcup_{i=0}^{n} \mathcal{N}_i$ is a subbasis for $X \times Y$. Let $p = ((0, \omega), (0, \omega_2))$ be the pit point of $X \times Y$ and, for each $i \leq n$, let

$$\mathcal{A}_i = \langle \{U \in \mathcal{N}_i \colon p \in U\}, \supseteq \rangle.$$

Each $\mathcal{A}_i$ is a linear order, and we show $n \geq 3$ by examining the cofinalities of these orders.

For each $n \in \omega$, let $O_m = ((0, m), \infty) \times Y$ (as is customary with a LOTS, we take $((0, m), \infty) = \{x \in X : (0, m) < x\}$). $\langle O_m \rangle_{m<\omega}$ is a nested sequence of open subsets of $X \times Y$. We now define simultaneously by induction a sequence $\left\langle U_m^i \colon m < \omega \right\rangle$ in $\mathcal{A}_i$ for each $i \leq n$. Take $U_0^i \in \mathcal{A}_i$ arbitrarily. Suppose we have $U_{m-1}^i \in \mathcal{A}_i$ for each $i$; since $\bigcup_{i \leq n} \mathcal{N}_i$ is a subbasis for $X \times Y$, there are $U_m^i \in \mathcal{A}_i$ such that $U_m^i \subseteq U_{m-1}^i$ for each $i$ and $p \in \bigcap_{i \leq n} U_m^i \subseteq O_m$.

For each $i \leq n$, $m \mapsto U_m^i$ is a nondecreasing map $\omega \to \mathcal{A}_i$. We claim that, for at least one value of $i$, this map must be cofinal. If not, there is for each $i$ some $U_\infty^i \in \mathcal{A}_i$ such that $U_\infty^i \supseteq U_m^i$ for each $m \in \omega$. But then $\bigcap_{i \leq n} U_\infty^m$ is an open neighborhood of $p$ contained in each $O_m$. This is impossible because $p$ is not in the interior of $\bigcap_{m \in \omega} O_m$. Thus one of these maps is nondecreasing and cofinal and, by Lemma 5.12$(ii)$, one of the $\mathcal{A}_i$ has cofinality $\omega$.

Next, for each $\alpha \in \omega_1$, let $O_\alpha = (-\infty, (1, \alpha)) \times Y$. As before, $\langle O_\alpha \rangle_{\alpha < \omega_1}$ is a nested sequence of open subsets of $X \times Y$. We will once again pick sequences of sets from our subbasis to get inside the $O_\alpha$, but now we must do so in two stages.

Let $I = \{i \leq n\colon \mathrm{cf}(\mathcal{A}_i) < \omega_1\}$ and $J = \{i \leq n\colon \mathrm{cf}(\mathcal{A}_i) \geq \omega_1\}$. For each $i \in I$, fix some cofinal sequence $\langle W^i_m\colon m < \omega\rangle$ in $\mathcal{A}_i$ (if $\mathrm{cf}(\mathcal{A}_i) < \omega$ then just take each $W^i_m$ to be the $\supseteq$-maximal element of $\mathcal{A}_i$). For each $\alpha < \omega_1$, choose a tuple $t_\alpha \in \omega^I$ such that there exist sets $U^j_\alpha$, $j \in J$, with

$$\bigcap_{i\in I} W^i_{t_\alpha(i)} \cap \bigcap_{j\in J} U^j_\alpha \subseteq O_\alpha.$$

Because $\omega^I$ is countable there is some $t \in \omega^I$ such that $t = t_\alpha$ for uncountably many $\alpha$. But then, for any $\alpha < \omega_1$, there exist sets $U^j_\alpha$, $j \in J$, such that $\bigcap_{i\in I} W^i_{t(i)} \cap \bigcap_{j\in J} U^j_\alpha \subseteq O_\alpha$. For all $\alpha < \omega_1$ and $i \in I$, set $V^i_\alpha = W^i_{t(i)}$.

We now proceed to define $V^j_\alpha$ for each $j \in J$ using transfinite recursion. Pick $V^j_0$ arbitrarily for each $j \in J$. Assuming $V^j_\beta$ has been chosen for each $\beta < \alpha$ and $j \in J$, choose $V^j_\alpha$ such that

$$U^j_\alpha \subseteq \bigcap_{\beta<\alpha} V^j_\beta \text{ for each } j \text{ and } \bigcap_{j\leq n} V^j_\alpha \subseteq O_\alpha.$$

This is possible by our choice of the $V^i_\alpha$ for $i \in I$ and because, by our definition of $J$, $\langle V_\beta\colon \beta < \alpha\rangle$ cannot be cofinal in $\mathcal{A}_j$ if $\alpha < \omega_1$.

For each $j \in J$, $\alpha \mapsto V^j_\alpha$ is a nondecreasing map $\omega_1 \to \mathcal{A}_i$. We claim that, for at least one value of $j$, this map must be cofinal. If not, there is for each $j \in J$ some $U^j_\infty \in \mathcal{A}_j$ such that $U^j_\alpha \supseteq U^j_\infty$ for each $\alpha < \omega_1$. Then

$$\bigcap_{i\in I} V^i_0 \cap \bigcap_{j\in J} V^j_\infty \subseteq O_\alpha$$

for each $\alpha$, which implies $p \in \mathrm{Int}(\bigcap_{\alpha<\omega_1} O_\alpha)$, a contradiction. Thus some $\mathcal{A}_i$ has cofinality $\omega_1$.

Similarly, we can prove that some $\mathcal{A}_i$ has cofinality $\omega_2$ and that some $\mathcal{A}_i$ has cofinality $\omega_3$. Thus $n \geq 3$. □

The proof of Theorem 5.13 generalizes to a proof of the following more general result:

**Theorem 5.14.** *Let $p \in X$ and suppose that, for $i = 0, ..., n$, $\mathcal{A}_i$ is a nest of open sets in $X$ such that, for each $i$, $p$ is in $\bigcap \mathcal{A}_i$ but is not in the interior of $\bigcap \mathcal{A}_i$. If $\mathrm{cf}(\mathcal{A}_i) \neq \mathrm{cf}(\mathcal{A}_j)$ whenever $i \neq j$ then $\mathfrak{N}^-(X) \geq n$.*

*Proof.* Let $\kappa_i = \mathrm{cf}(\mathcal{A}_i)$ for each $i$ and suppose that $\kappa_0 < \kappa_1 < ... < \kappa_{n-1} < \kappa_n$.

Suppose that $\mathcal{N}_0, \ . \ . \ . \ , \mathcal{N}_m$ are nests in $X$ such that $\bigcup_{i=0}^{n} \mathcal{N}_i$ is a subbasis for $X$. For each $i \leq m$, let

$$\mathcal{J}_i = \langle \{U \in \mathcal{N}_i \colon p \in U\}, \supseteq \rangle.$$

Each $\mathcal{J}_i$ is a linear order. We showed in the proof of Theorem 5.13 that some $\mathcal{A}_i$ must have cofinality $\omega$ (here we refer to the $\mathcal{A}_i$ defined in that proof); the exact same argument (replacing $\omega$ with $\kappa_0$, of course) shows that some $\mathcal{J}_i$ must have cofinality $\kappa_0$. Next in the proof of Theorem 5.13, we showed by a slightly more involved technique that some $\mathcal{A}_i$ must have cofinality $\omega_1$. The exact same argument, replacing $\omega_1$ with $\kappa_1$, shows that some $\mathcal{J}_i$ must have cofinality $\kappa_1$. Once this is done, this argument may be repeated to show that some $\mathcal{J}_i$ has cofinality $\kappa_2$, then repeated again to show that some $\mathcal{J}_i$ has cofinality $\kappa_3$, and so forth. Ultimately, we see that some $\mathcal{J}_i$ must have cofinality $\kappa_i$ for every $i \leq n$. This proves that $m \geq n$; i.e., there are at least $n+1$ of the $\mathcal{J}_i$. This proves that the neight of $X$ is at least $n+1$. □

Theorem 5.13 says that, if $\mathfrak{N}^-$ is to be viewed as a measure of dimension, then it is, in an intuitive sense, pathological with respect to products, at least for compact LOTS. The following few corollaries show other ways in which the function $\mathfrak{N}^-$ is pathological as a measure of dimension for compact Hausdorff spaces.

**Corollary 5.15.** *For each odd $n \geq 1$ there is a compact Hausdorff space $X$ with $\mathfrak{N}^-(X) = n$ and a point $x \in X$ such that $\mathfrak{N}^-(X \setminus \{x\}) = 1$.*

*Proof.* Let $n = 2m - 1$, $m \geq 1$, and let $Y$ be the disjoint union:

$$Y = \bigsqcup_{i=0}^{n} (\omega_i + 1) \times \{i\}$$

Let $X$ be the quotient space obtained from $Y$ by identifying the $n+1$ points $(\omega_0, 0), (\omega_1, 1), ..., (\omega_n, n)$. $X$ is clearly a compact Hausdorff space, and it follows from Theorem 5.14 that $\mathfrak{N}^-(X) \geq n$. By pairing off the various $\omega_i+1$

into $m$ different LOTS (one ordinal going forward and one going backward to form a copy of $\omega_i + 1^\frown\omega_j^*$), it is not difficult to construct a collection of $2m = n+1$ nests on $X$ that generates the topology on $X$. Thus $\mathfrak{N}^-(X) = n$. It remains to show that, for some $x \in X$, $\mathfrak{N}^-(X\backslash\{x\}) = 1$. Let $x$ be the point in $X$ whose pre-image in $Y$ is the $n+1$-point set $\{(\omega_0, 0), (\omega_1, 1), ..., (\omega_n, n)\}$. It is clear that $X \setminus \{x\}$ is (homeomorphic to) a disjoint union of LOTS, and hence is a GO space. Thus $\mathfrak{N}^-(X \setminus \{x\}) = 1$. □

Note that, by replacing each $\omega_n$ with a copy of the "long line" of length $\omega_n$, we may assume in the previous example that $X$ is connected. The same is true of the following few examples as well.

**Corollary 5.16.** *The $\mathfrak{N}^-$ function does not satisfy the sum theorem for compact Hausdorff spaces. That is, there is a compact Hausdorff space $X$, and closed subsets $C_0, ..., C_n$ of $X$, such that*

$$\mathfrak{N}^-(X) > \max\{\mathfrak{N}^-(C_0), ..., \mathfrak{N}^-(C_n)\}$$

*Proof.* The space $X$ described in the previous proof can be divided into a finite number of compact LOTS (overlapping only at the point $x$), each of which is closed in $X$. □

The following corollary says that $\mathfrak{N}^-$ not only fails to be additive under products, but for arbitrary finite powers it fails as badly as Lemma 5.1(*iii*) allows.

**Corollary 5.17.** *There is a compact LOTS $X$ such that, for every $n \in \omega$, $\mathfrak{N}^-(X^n)$ has the maximum possible value of $2n-1$.*

*Proof.* Let $X = \omega_\omega + 1$. In $X^n$, consider the point $(\omega, \omega_1, ..., \omega_{n-1})$ and apply Theorem 5.14. □

## 5.4 Spaces that are not FUN

Theorems 5.9 and 5.14 give two different ways for finding a lower bound for $\mathfrak{N}(X)$ when $X$ is a FUN space. In this section we will prove a theorem that relates the neight of a space to its weight. In particular, this theorem will allow us to get a lower bound on $\mathfrak{N}$ for spaces with infinite neight and to prove that certain spaces are not FUN. Recall that $w(X)$ denotes the **weight** of a space $X$, the smallest cardinal $\kappa$ such that $X$ has a base of size $\kappa$, and $\chi(x)$ denotes the **character** of a point $x$, the smallest cardinal $\kappa$ such that $x$ has a local base of size $\kappa$.

**Lemma 5.18** (Yurovetskiĭ)**.** $\mathfrak{N}(X) \leq w(X)$.

*Proof.* If $\mathcal{B}$ is a basis for $X$ then $\{\{B\}\colon B \in \mathcal{B}\}$ is a collection of nests in $X$ whose union is a (sub)basis for $X$. □

**Theorem 5.19.** *If* $|X| < w(X)$ *then* $\mathfrak{N}(X) = w(X)$.

*Proof.* Let $x \in X$. We will begin by showing that $\chi(x) \leq \mathfrak{N}(X) \cdot |X|$.

By assumption, there is a collection $\mathcal{C} = \{\mathcal{N}_\alpha\colon \alpha < \mathfrak{N}(X)\}$ of nests that generates the topology on $X$. For each $\mathcal{N}_\alpha \in \mathcal{C}$, let $\mathcal{N}'_\alpha$ be the restriction of $\mathcal{N}_\alpha$ to sets containing $x$: $\mathcal{N}'_\alpha = \{U\colon x \in U \in \mathcal{N}_\alpha\}$. Each $\mathcal{N}'_\alpha$ is totally ordered by $\supseteq$. As a totally ordered set, the cofinality of $\mathcal{N}'_\alpha$ is at most $|X|$. To see this, note that, by Lemma 5.12($i$), there must be a well-ordered, strictly decreasing sequence $\langle U_\beta\colon \beta < \mathrm{cf}(\mathcal{N}'_\alpha)\rangle$ of subsets of $X$, and (choosing $x_\beta \in U_\beta \backslash U_{\beta+1}$) we may use this sequence to find a subset of $X$ of size $\mathrm{cf}(\mathcal{N}'_\alpha)$.

For each $\alpha$, let $\mathcal{N}''_\alpha$ be a decreasing sequence of subsets of $X$ that is cofinal in $\mathcal{N}'_\alpha$ and has order type $\mathrm{cf}(\mathcal{N}'_\alpha)$. Because these sequences are cofinal,

$$\left\{\bigcap_{i=0}^{n} U_i\colon n \in \omega, \alpha_0, \alpha_1, ..., \alpha_n < \mathfrak{N}(X), \text{ and } U_i \in \mathcal{N}''_{\alpha_i} \text{ for each } i\right\}$$

is a neighborhood basis for $x$. Since the cardinality of each $\mathcal{N}''_\alpha$ is at most $|X|$, the cardinality of this neighborhood basis for $x$ is at most $\mathfrak{N}(X) \cdot |X|$.

For every $x \in X$, let $\mathcal{N}_x$ be a neighborhood basis for $x$ of size at most $\mathfrak{N}(X) \cdot |X|$. Then $\bigcup_{x \in X} \mathcal{N}_x$ is a basis for $X$ of size at most $\mathfrak{N}(X) \cdot |X|$. Hence $w(X) \leq \mathfrak{N}(X) \cdot |X|$. As we are assuming $|X| < w(X)$, this proves

$w(X) \leq \mathfrak{N}(X)$.

Lemma 5.18 provides the opposite inequality and completes the proof of the theorem. $\square$

Combining this result with Lemma 5.1($ii$), we have:

**Corollary 5.20.** *If $Y \subseteq X$ and $|Y| < w(Y)$ then $\mathfrak{N}(X) \geq w(Y)$.*

**Corollary 5.21.** *$\beta\omega$ is not FUN. In fact, $\mathfrak{N}(\beta\omega) = \mathfrak{c}$.*

*Proof.* Let $p \in \beta\omega \setminus \omega$ be a point with character $\mathfrak{c}$ (see [42] for a proof that some such $p$ exists). Consider the space $X = \omega \cup \{p\} \subseteq \beta\omega$. $X$ is countable, but the fact that $\chi(p) = \mathfrak{c}$ implies that $w(X) = \mathfrak{c}$. By Corollary 5.20, $\mathfrak{N}(\beta\omega) \geq \mathfrak{c}$. Since $w(\beta\omega) = \mathfrak{c}$, the result now follows from Lemma 5.18. $\square$

A similar argument shows that, for any discrete space $A$, $\beta A$ has neight $2^{|A|}$. A more general form of this argument will be given in Corollary 5.23, but first we need a lemma:

**Lemma 5.22.** *Let $X$ be regular and let $D$ be dense in $X$. If $x \in X$ then the character of $x$ in $X$ is the same as the character of $x$ in $D \cup \{x\}$.*

*Proof.* Let $\chi(x)$ be the character of $x$ in $X$ and suppose $\mathcal{U}$ is a family of open neighborhoods of $x$ in $D \cup \{x\}$ with $|\mathcal{U}| < \chi(x)$. It suffices to prove that there is some open neighborhood $V$ of $x$ in $D \cup \{x\}$ such that, for any $U \in \mathcal{U}$, $U \nsubseteq V$. For each $U \in \mathcal{U}$ let

$$\tilde{U} = \bigcup \{V \subseteq X : V \text{ is open and } V \cap (D \cup \{x\}) = U\}.$$

In other words, $\tilde{U}$ is the largest open subset of $X$ that restricts to $U$ on $D \cup \{x\}$. Let $\tilde{\mathcal{U}} = \left\{\tilde{U} : U \in \mathcal{U}\right\}$. Since $\left|\tilde{\mathcal{U}}\right| = |\mathcal{U}| < \chi(x)$, there is an open subset $W$ of $X$ such that $x \in W$ and, for any $\tilde{U} \in \tilde{\mathcal{U}}$, $\tilde{U} \nsubseteq W$. Using the regularity of $X$, take an open $W' \subseteq W$ such that $x \in W' \subseteq \overline{W'} \subseteq W$. By our choice of $W$, $W'$ has the property that $\tilde{U} \nsubseteq \overline{W'}$ for every $\tilde{U} \in \tilde{\mathcal{U}}$. Set $V = W' \cap (D \cup \{x\})$. If $U \in \mathcal{U}$ then, because $D$ is dense, there is some $d \in D \cap (\tilde{U} \setminus \overline{W'})$. But then $d \in U \setminus V$, so $U \nsubseteq V$. $\square$

**Corollary 5.23.** *If $X$ is regular and separable but not first countable, then $\mathfrak{N}(X)$ is uncountable. More generally, if $X$ is regular, $D$ is dense in $X$, and $x \in X$ with $\chi(x) > |D|$, then $\mathfrak{N}(X) \geq \chi(x)$.*

*Proof.* By Lemma 5.22, the character of $x$ in $D \cup \{x\}$ is the same as its character in $X$. Applying Corollary 5.20 to the space $D \cup \{x\}$, we have $\mathfrak{N}(X) \geq \chi(x)$. $\square$

The following lemma is a generalization of an exercise from [32] (see ex. 3, pp. 86):

**Lemma 5.24.** *Let $\kappa, \lambda$ be infinite cardinals with $\kappa \leq 2^{\lambda}$. Then $[0,1]^{\kappa}$ has a dense subset of size $2^{<\lambda}$.*

*Proof.* Let $I \subseteq {}^{\lambda}2$ (the set of all functions from $\lambda$ into $2 = \{0,1\}$). We will show that $[0,1]^{I}$ has a dense subset of size $2^{<\lambda}$. Let $D = \mathbb{Q} \cap [0,1]$ and let $E$ be the set of all $\varphi \in {}^{I}D$ such that for some $\alpha < \lambda$ and some finite $F \subseteq {}^{\alpha}D$,

$$\forall f, g \in I \ (f \restriction \alpha = g \restriction \alpha \ \Rightarrow \ \varphi(f) = \varphi(g))$$

$$\text{and } \varphi(f) = 0 \text{ unless } f \restriction \alpha \in F.$$

For each $\alpha < \lambda$ there are $2^{|\alpha|}$ possibilities for $f \restriction \alpha$. Choosing finitely many of these to have a nonzero value under $\varphi$ provides $2^{|\alpha|}$ choices, and choosing a value for $\varphi$ from $D$ for each of these still provides $2^{|\alpha|}$ choices. Thus $E$ has size $\sup_{\alpha<\lambda} 2^{|\alpha|} = 2^{<\lambda}$.

Let $U$ be an arbitrary basic open subset of $[0,1]^{I}$. Specifically, there are $f_0, ..., f_n \in I$ and $(a_0, b_0), ..., (a_n, b_n) \subseteq [0,1]$ such that

$$U = \left\{\varphi \in {}^{I}[0,1] \colon \varphi(f_i) \in (a_i, b_i) \text{ for all } i \leq n\right\}$$

Assuming without loss of generality that all the $f_i$ are different, there is some $\alpha < \lambda$ such that if $i \neq j$ then $f_i(\beta) \neq f_j(\beta)$ for some $\beta < \alpha$. For each $i \leq n$ pick some $c_i \in (a_i, b_i)$. Let

$$\varphi(f) = \begin{cases} c_i & \text{if } f \restriction \alpha = f_i \restriction \alpha \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\varphi \in E \cap U$. As $U$ was arbitrary, this proves $E$ is dense in $[0,1]^I$. $\square$

The following result complements Corollary 5.10:

**Corollary 5.25.** *$\mathfrak{N}([0,1]^\kappa) = \kappa$ for every infinite $\kappa$.*

*Proof.* That $\mathfrak{N}([0,1]^\kappa) \leq \kappa$ follows from Lemma 5.1($iv$), so we must show that $\mathfrak{N}([0,1]^\kappa) \geq \kappa$.

First suppose $\kappa$ is a successor cardinal, say $\kappa = \mu^+$. Let $\lambda$ be the least cardinal such that $2^\lambda \geq \kappa$; then $2^{<\lambda} \leq \mu < \kappa$. By Lemma 5.24, $[0,1]^\kappa$ has a dense subset $D$ of size $2^{<\lambda}$. However, every point of $[0,1]^\kappa$ has character $\kappa$. That $\mathfrak{N}([0,1]^\kappa) \geq \kappa$ now follows from Corollary 5.23.

If $\kappa = \omega$ then, for each $n$, $\mathbb{R}^n$ embeds in $[0,1]^\omega$ and (by Corollary 5.10) has neight $n+1$. It follows that $\mathfrak{N}([0,1]^\omega) > n$ for all $n < \omega$, i.e., $\mathfrak{N}([0,1]^\omega) \geq \aleph_0$.

Finally, suppose $\kappa$ is an uncountable limit cardinal. Let $\mu$ be any successor cardinal with $\mu < \kappa$. $[0,1]^\mu$ embeds naturally in $[0,1]^\kappa$. By Lemma 5.1($ii$), $\mathfrak{N}([0,1]^\kappa) \geq \mathfrak{N}([0,1]^\mu) = \mu$. Because every limit cardinal is a limit of successor cardinals, this proves $\mathfrak{N}([0,1]^\kappa) \geq \kappa$. $\square$

**Corollary 5.26.** *If $X$ is a non-discrete Hausdorff Toronto space of size $\aleph_1$ then $X$ is not FUN. In fact, $\mathfrak{N}(X)$ is uncountable.*

*Proof.* It is shown in Chapter 4 that, if $X$ is such a space, then every limit point $x$ of $X$ has a countable neighborhood but uncountable character. Corollary 5.20 thus implies $\mathfrak{N}(X) \geq \chi(x)$. □

## 5.5 Proof of Theorem 5.6

Let $X$ be the hedgehog with $\aleph_1$ spines. That is, let $X = (0,1] \times \omega_1 \cup \{*\}$ with the topology generated by the following metric: for every $x$, $y$, $\alpha$, and $\beta$, we take $d((x,\alpha),(y,\beta)) = x + y$ if $\alpha \neq \beta$, $d((x,\alpha),(y,\alpha)) = |x - y|$ for any fixed $\alpha$, and $d((x,\alpha),*) = x$.

We will prove through a sequence of lemmas that $\mathfrak{N}^-(X) = 2$ and $\textsc{Dim}(X) = \infty$.

**Lemma 5.27.** $\mathfrak{N}^-(X) = 2$.

*Proof.* The proof that $\mathfrak{N}^-(X) \leq 2$, i.e. that there are three nests that together generate the topology on $X$, relies on the fact that $X \setminus \{*\}$ is open in $X$ and is a GO space. This allows us to generate the topology of $X \setminus \{*\}$ with two nests, and (since $X$ if first countable) we may take as our third nest any nested neighborhood basis for $\{*\}$. More formally, consider the following three open nests in $X$:

$$\mathcal{N}_0 = \big\{L_{(x,\alpha)} = \{(y,\beta)\colon \beta < \alpha \text{ or } \beta = \alpha \wedge y < x\} : x \in (0,1], \alpha \in \omega_1\big\}$$

$$\mathcal{N}_1 = \left\{R_{(x,\alpha)} = \{(y,\beta)\colon \beta > \alpha \text{ or } \beta = \alpha \wedge y > x\} : x \in (0,1], \alpha \in \omega_1\right\}$$

$$\mathcal{N}_2 = \left\{B_{\frac{1}{n}}(*)\colon n \in \omega\right\}$$

It is straightforward to check that $\mathcal{N}_0 \cup \mathcal{N}_1$ is a subbasis for $X \setminus \{*\}$, and it follows that $\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2$ is a subbasis for $X$. This proves that $\mathfrak{N}^-(X) \leq 2$.

For the opposite inequality, recall the theorem of van Dalen and Wattel: that a Hausdorff space $X$ is a GO space if and only if it has neight 2. Thus it is enough to show that $X$ is not a GO space. This follows from the well-known fact that the letter $Y$ (i.e., the space obtained from three copies of $[0,1]$ by identifying the three left-hand endpoints) is not a GO space, since the letter $Y$ embeds in $X$.

Alternatively, that $\mathfrak{N}^-(X) \geq 2$ follows from Proposition 5.4 and the fact (proved below) that $\textsc{Dim}(X) = \infty$. □

Recall that a halfdirection, or a nest more generally, is totally ordered by $\subseteq$. Before continuing with the proof of Theorem 5.6, we will need a result from the general theory of total orders.

In general, a total order $(X, \leq)$ is separable if there is some countable $Q \subseteq X$ such that every nonempty interval of the form $(a, b)$ contains a point of $Q$. We will say that $(X, \leq)$ is **strongly separable** if there is some countable $Q \subseteq X$ such that every nonempty interval of the form $(a, b]$ or

$[a, b)$ contains a point of $Q$; such a $Q$ will be called **strongly dense**. Note that the notion of strong separability is strictly stronger than the notion of separability (for strictness, consider the "double arrow" space).

**Lemma 5.28.** *The following are equivalent for a total order $(X, \leq)$:*

*(i)* $(X, \leq)$ *is strongly separable.*

*(ii)* $(X, \leq)$ *is order isomorphic to a subset of* $\mathbb{R}$*.*

*(iii)* *When endowed with the order topology,* $(X, \leq)$ *is second countable.*

*Proof.*

$(i) \Rightarrow (ii)$: Let $X$ be strongly separable and let $Q$ be a countable strongly dense subset of $X$ that, without loss of generality, contains the least and greatest points of $X$ (if they exist). Since $Q$ is countable, $Q$ is order isomorphic to a subset of $\mathbb{R}$. Let $\phi : Q \to \mathbb{R}$ be an order embedding, and extend $\phi$ to all of $X$ by left Dedekind cuts: for any $x \in X$, let $\phi(x) = \sup\{\phi(q) \colon q \in Q \text{ and } q \leq x\}$. Suppose $a, b \in X$ with $a < b$. Since $Q$ is strongly dense, there is some $q_1 \in [a, b) \cap Q$ and some $q_2 \in (q_1, b] \cap Q$. We then have $\phi(a) \leq \phi(q_1) < \phi(q_2) \leq \phi(b)$, so $\phi(a) < \phi(b)$. This shows that $\phi$ is injective and order preserving. Thus $\phi$ is an order isomorphism onto its image.

$(ii) \Rightarrow (iii)$: If $X \subseteq \mathbb{R}$, then $X$, endowed with the subspace topology, has a countable subbasis $\mathcal{S} = \{(q, \infty) \cap X : q \in \mathbb{Q}\} \cup \{(-\infty, q) \cap X : q \in \mathbb{Q}\}$. The order topology on $X$ may be coarser than this subspace topology, but it will have some subset of $\mathcal{S}$ as a subbasis.

$(iii) \Rightarrow (i)$: If $X$ is second countable, then for any subbasis $\mathcal{S}$ for $X$ there is a countable $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}'$ is a subbasis for $X$ (the proof of this is only a slight modification of the proof of Theorem 1.1.15 in [19], pp. 17-18). Thus there is some countable $Q \subseteq X$ such that $\{(q, \infty) : q \in Q\} \cup \{(-\infty, q) : q \in Q\}$ is a subbasis for the order topology on $X$. Let $Q'$ consist of the points of $Q$ together with any points in $X$ that have an immediate successor or an immediate predecessor. If there were uncountably many points of $X$ with an immediate successor or predecessor then the order topology on $X$ would not be second countable, so $Q'$ is countable. It is easy to check that $Q'$ is also strongly dense in $X$. $\square$

The equivalence of $(i)$ and $(ii)$ in the above lemma makes it clear that strong separability is an hereditary property:

**Corollary 5.29.** *If $X$ is a strongly separable linear order then so is any subset of $X$.*

Let us call a nest **complete** if it is closed under arbitrary unions. By definition, every halfdirection is complete. Notice that every nest $\mathcal{N}$ has a completion $\tilde{\mathcal{N}} = \{\bigcup \mathcal{A} \colon \mathcal{A} \subseteq \mathcal{N}\}$. This idea of completion extends to collections of nests as well: if $\mathcal{C}$ is a collection of nests then its completion is $\tilde{\mathcal{C}} = \left\{\tilde{\mathcal{N}} \colon \mathcal{N} \in \mathcal{C}\right\}$. Moreover, if $\mathcal{C}$ is a collection of nests in some space $Y$, then $\cup\mathcal{C}$ is a subbasis for $Y$ if and only if $\cup\tilde{\mathcal{C}}$ is.

**Lemma 5.30.** *Let $Y$ be a topological space and suppose that $\mathcal{C}$ is an at most countable collection of complete, strongly separable open nests in $Y$. If $\mathcal{C}$ generates the topology on $Y$, then $Y$ is second countable.*

*Proof.* For each $\mathcal{N} \in \mathcal{C}$ fix some $\mathcal{Q}_{\mathcal{N}} \subseteq \mathcal{N}$ such that $\mathcal{Q}_{\mathcal{N}}$ is countable and strongly dense in $\mathcal{N}$. We claim that $\mathcal{S} = \bigcup \{\mathcal{Q}_{\mathcal{N}} \colon \mathcal{N} \in \mathcal{C}\}$ is a subbasis for $Y$. The lemma follows directly from this claim: $\mathcal{S}$ is countable, so the set of all finite intersections of elements of $\mathcal{S}$ is also countable and is a basis for $Y$ if $\mathcal{S}$ is a subbasis for $Y$.

Let $y \in Y$ and let $N \subseteq Y$ be an arbitrary neighborhood of $y$. $\bigcup\mathcal{C}$ is a subbasis for $Y$, so we may find $\mathcal{N}_0, ..., \mathcal{N}_n \in \mathcal{C}$ and $U_0 \in \mathcal{N}_0, ..., U_n \in \mathcal{N}_n$ such that $y \in \bigcap_{i \leq n} U_i \subseteq N$. For each $i \leq n$, let $V_i = \bigcup \{U \in \mathcal{N}_i \colon y \notin U\}$; by completeness, $V_i \in \mathcal{N}_i$. Since $y \in U_i \setminus V_i$ and $\mathcal{N}_i$ is a nest, $V_i \subseteq U_i$. As $\mathcal{Q}_{\mathcal{N}_i}$ is strongly dense in $\mathcal{N}_i$, there is some $W_i \in \mathcal{Q}_{\mathcal{N}_i}$ such that $V_i \subsetneq W_i \subseteq U_i$ (using the notation above, $W_i \in (V_i, U_i])$. By our choice of $V_i$ and $W_i$, $y \in W_i$, $W_i \in \mathcal{Q}_{\mathcal{N}_i}$, and $W_i \subseteq U_i$. This holds for all $i \leq n$, so $y \in \bigcap_{i \leq n} W_i \subseteq \bigcap_{i \leq n} U_i \subseteq N$, proving that $\mathcal{S}$ is a subbasis for $Y$. $\square$

**Lemma 5.31.** *Every complete open nest on a second countable space is strongly separable.*

*Proof.* Let $Y$ be a second countable space and let $\mathcal{B} = \{B_n : n \in \omega\}$ be a countable basis for $Y$. Let $\mathcal{N}$ be an open nest in $Y$. For each $U \in \mathcal{N}$, let $A_U = \{n \in \omega : B_n \subseteq U\}$. Clearly $U = \bigcup\{B_n : n \in A_U\}$ for every $U \in \mathcal{N}$, so the map $U \mapsto A_U$ is an injection from $\mathcal{N}$ into $2^\omega$. Moreover, if we order $2^\omega$ lexicographically, then this map is order preserving: $U \subseteq V$ implies $A_U \subseteq A_V$.

Note that $2^\omega$ with the lexicographic order is order isomorphic to a subset of $\mathbb{R}$, namely to the Cantor set. Thus $\mathcal{N}$ is order isomorphic to a subset of $\mathbb{R}$. Its strong separability now follows from Lemma 5.28. $\square$

If $Y \subseteq X$ and $\mathcal{N}$ is a nest in $X$, we let $\mathcal{N} \restriction Y$ denote $\{U \cap Y : U \in \mathcal{N}\}$.

**Lemma 5.32.** *If $\mathcal{N}$ is a strongly separable halfdirection on $Y$ and $Z \subseteq Y$ then $\mathcal{N} \restriction Z$ is a strongly separable halfdirection on $Z$.*

*Proof.* It is obvious that if $\mathcal{N}$ is a halfdirection then so is $\mathcal{N} \restriction Z$. It suffices, then, to show that $\mathcal{N} \restriction Z$ is strongly separable if $\mathcal{N}$ is. For every $U \in \mathcal{N} \restriction Z$, pick some $f(U) \in \mathcal{N}$ such that $f(U) \cap Z = U$. Then $\{f(U) : U \in \mathcal{N} \restriction Z\}$ is a subset of $\mathcal{N}$ that is clearly order isomorphic to $\mathcal{N}\upharpoonright Z$. By Corollary 5.29, $\mathcal{N}\upharpoonright Z$ is strongly separable. $\square$

**Lemma 5.33.** *If $\mathcal{N}$ is a halfdirection on $X$ then there is an open neighborhood $V$ of $*$ such that $\mathcal{N}\upharpoonright V$ is a strongly separable halfdirection on $V$.*

*Proof.* Let $\mathcal{N}$ be a halfdirection on $X$. If $V$ is any subset of $X$ then $\mathcal{N}\upharpoonright V$ is a halfdirection by Lemma 5.32, so we must find an open $V \ni *$ for which $\mathcal{N}\upharpoonright V$ is strongly separable. There are three cases to consider:

*Case* 1: Suppose $* \notin U$ for every $U \in \mathcal{N}$. The cases $|\mathcal{N}| = 0, 1$ are trivial, so assume there is some nonempty, nonmaximal $U' \in \mathcal{N}$. Since $\mathcal{N}$ is a halfdirection and $U'$ is not maximal, there is some $U \in \mathcal{N}$ such that $\overline{U'} \subseteq U$. Since $* \notin U$, we have $* \notin \overline{U'}$. Take $V = X \setminus \overline{U'}$.

Let $\alpha \in \omega_1$ such that $U' \cap (0,1] \times \{\alpha\} \neq \emptyset$ (some such $\alpha$ must exist as $U'$ is nonempty and $* \notin U'$). Let $Y = \{*\} \cup (0,1] \times \{\alpha\}$; note $Y \approx [0,1]$. $\mathcal{N}\upharpoonright Y$ is strongly separable by Lemma 5.31 because $Y$ is second countable. Consider the map $\pi : \mathcal{N} \to \mathcal{N}\upharpoonright Y$ given by $U \mapsto U \cap Y$. This map is clearly an order preserving surjection. We now find a condition under which $\pi$ is also an injection. If $\pi$ is not injective then there exist $U, W \in \mathcal{N}$ such that $\overline{U} \subseteq W$ and $U \cap Y = W \cap Y$. Then $U \cap Y \subseteq \overline{U \cap Y}^{Y} \subseteq \overline{U}^{X} \cap Y \subseteq W \cap Y = U \cap Y$. Thus $U \cap Y = \overline{U \cap Y}^{Y}$. Since $Y$ is connected and $U$ is open, this is only possible if $U \cap Y = \emptyset$ or $U \cap Y = Y$. The latter is impossible by the assumption $* \notin U$, so we have $U \cap Y = \emptyset$. This implies $W \cap Y = \emptyset$ as well. Thus $\pi(U) = \pi(W)$ if and only if $U \cap Y = W \cap Y = \emptyset$. That is, $\pi$ fails to be injective only on those sets in $\mathcal{N}$ that miss $Y$.

Consider $\mathcal{N} \restriction V$ and recall that there is some $U' \in \mathcal{N}$ such that $V = X \setminus \overline{U'}$. Since $\mathcal{N}$ is a halfdirection, $\mathcal{N} \restriction V = \{U \cap V : U \in \mathcal{N}\} = \left\{U \setminus \overline{U'} : U \in \mathcal{N}\right\} = \left\{U \setminus \overline{U'} : U \in \mathcal{N} \text{ and } U' \subseteq U\right\}$. This shows us that $\mathcal{N} \restriction V$ is order isomorphic to a final segment of $\mathcal{N}$, namely to $\mathcal{N}' = \{U \in \mathcal{N} : U' \subseteq U\}$.

By our choice of $U'$ and of $Y$, $U \cap Y \neq \emptyset$ for every $U \in \mathcal{N}'$. Thus, by the argument above, $\pi : \mathcal{N}' \to \mathcal{N} \restriction Y$ is an order preserving bijection. Hence $\mathcal{N}'$ is order isomorphic to $\mathcal{N} \restriction Y$ and is thus strongly separable. As $\mathcal{N}'$ and $\mathcal{N} \restriction V$ are order isomorphic, $\mathcal{N} \restriction V$ is strongly separable.

*Case* 2: Suppose $* \in U$ for every $U \in \mathcal{N}$. The case $\mathcal{N} = \{X\}$ is trivial, so assume there is some $V \in \mathcal{N}$ such that $V \neq X$. As $* \in V$, so there is some $(x, \alpha) \in (0, 1] \times \omega_1$ such that $(x, \alpha) \notin V$.

As in Case 1, take $Y = \{*\} \cup (0, 1] \times \{\alpha\}$ and consider the map $\pi : \mathcal{N} \to \mathcal{N} \restriction Y$ defined by $\pi(U) = U \cap Y$. As in Case 1, if $U, W \in \mathcal{N}$ then $\pi(U) = \pi(W)$ if and only if $U \cap Y = W \cap Y = \emptyset$ or $U \cap Y = W \cap Y = Y$. The former case is impossible since $* \in U \cap Y$ for any $U \in \mathcal{N}$. Thus $\pi$ is an injection from $\{U \in \mathcal{N} : U \cap Y \neq Y\}$ into $\mathcal{N} \restriction Y$. This proves, as above, that

$\{U \in \mathcal{N} : U \cap Y \neq Y\}$ is strongly separable; moreover, $\mathcal{N} \restriction V$ is isomorphic to an initial segment of this order, and thus is itself strongly separable.

*Case* 3: Suppose $* \in U$ for some $U \in \mathcal{N}$ and $* \notin U'$ for some $U' \in \mathcal{N}$. Let $\mathcal{N}_0 = \{U \notin \mathcal{N} : * \in U\}$ and $\mathcal{N}_1 = \{U \in \mathcal{N} : * \in U\}$. $\mathcal{N}_0$ and $\mathcal{N}_1$ are both complete halfdirections in $X$ and, by Cases 1 and 2, are each strongly separable. Let $\mathcal{Q}_0$ and $\mathcal{Q}_1$ be countable strongly dense subsets of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively. It is easy to check that $\mathcal{Q}_0 \cup \mathcal{Q}_1$, together with the maximal element of $\mathcal{N}_0$ and the minimal element of $\mathcal{N}_1$ (if it exists), is a countable strongly dense subset of $\mathcal{N}$. □

**Lemma 5.34.** *$X$ is not second countable. Furthermore, no neighborhood of $*$ in $X$ is second countable.*

*Proof.* It is obvious that $X$ is not second countable. If $U$ is a neighborhood of $*$ then there is some $\varepsilon > 0$ such that $B_\varepsilon(*) \subseteq U$. But then $\overline{B_{\varepsilon/2}(*)}$ is a subspace of $U$ and is homeomorphic to $X$. Since second countability is an hereditary property, this fact, together with the first assertion, proves the second assertion. □

Suppose now that $\textsc{Dim}(X)$ is finite and let $\mathcal{N}_0, ..., \mathcal{N}_n$ be a collection of halfdirections on $X$ whose union is a subbasis for $X$. By Lemma 5.33 we may, for each $i$, find some open neighborhood $V_i$ of $*$ such that $\mathcal{N}_i \restriction V_i$ is a strongly separable halfdirection on $V_i$. Moreover, setting $V = \bigcap_{i \leq n} V_i$, $V$ is an open neighborhood of $*$ such that, for each $i \leq n$, $\mathcal{N}_i \restriction V$ is a strongly separable halfdirection on $V$. Recalling that a halfdirection is a special kind of nest, it follows from Lemma 5.30 that $V$ is second countable, contradicting Lemma 5.34. $\square$

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