

Multiple Knot B-Spline Representation of Incompressible Flow

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An efficient B-spline method for the construction of a piecewise polynomial velocity representation from a given set of normal fluid fluxes is described for steady incompressible flow in three dimensional rectangular regions. The fluxes should be defined across the face-centres of a cartesian tensor product mesh. The proposed spline representation interpolates the given fluxes exactly and also enables the normal fluid velocity to be set identically to zero across or around the surfaces of an arbitrary number of rectangular regions lying in specified planes.

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1 Introduction

Insight into many fluid flow situations is often gained by studying individual particle trajectories; in the case of steady flow these are the streamlines. To compute the path of a fluid particle through some given region we must be able to evaluate the velocity distribution at any point. Often a discrete representation of the velocity distribution is already available from a finite volume or finite element computation. This discrete information can be used to construct an interpolant which can be looked upon as an approximate reconstruction of the original velocity distribution. Integration of the interpolant enables the determination of specific particle trajectories. Usually the integration can be performed to any desired tolerance and so the overall accuracy of the resulting trajectories is limited by the accuracy of the reconstructed velocity representation.

1.1 Discrete Velocity Distribution

The given velocity distribution is usually represented by either a set of node-based velocity values or face-centred flux values depending on whether a finite element or finite volume method has been used. We assume throughout this paper that we are given a set of normal fluid fluxes defined across the face centers of some tensor product cartesian mesh. Other velocity data may require pre-processing to obtain the desired format.

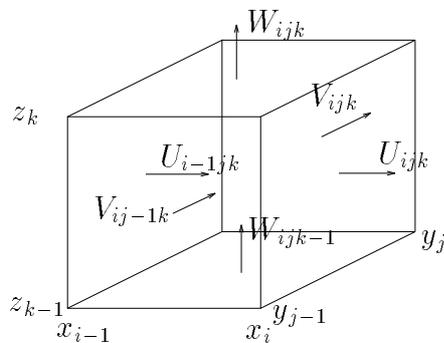


Figure 1: Face-centred normal fluid fluxes for a typical cell C_{ijk} .

Let $\Omega = [x_0, x_I] \otimes [y_0, y_J] \otimes [z_0, z_K]$ denote the region in which we require a velocity representation. The computational mesh Ω_h used to discretise Ω , across which the given fluxes are defined, must have the tensor product representation $\Omega_h = \Omega_h^x \otimes \Omega_h^y \otimes \Omega_h^z$ where $\Omega_h^x = \{x_0 < x_1 < \dots < x_I\}$, $\Omega_h^y = \{y_0 < y_1 < \dots < y_J\}$ and $\Omega_h^z = \{z_0 < z_1 < \dots < z_K\}$ for some I, J and K . Letting $\Omega_{ijk} : \{x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j, z_{k-1} \leq z \leq z_k\}$ denote a typical cell in the mesh, the

normal fluid fluxes should be given at the centre of each cell face as illustrated in figure (1).

Because we are assuming Ω is occupied by a steady incompressible fluid the fluxes should have the divergence-free property that,

$$(U_{ijk} - U_{i-1jk}) + (V_{ijk} - V_{ij-1k}) + (W_{ijk} - W_{ijk-1}) = 0 \quad (1.1)$$

for every cell (i, j, k) in the mesh. It is not essential that this condition be satisfied, but most finite volume packages generate flux values which satisfy (1.1) to some tolerance.

1.2 Simple Knot Spline Representation

Handscorn [9], showed that a smooth velocity representation can be constructed using tensor splines which satisfies the continuity equation at every point throughout Ω whenever the given flux values satisfy (1.1) exactly. The method can be applied to both two and three dimensional flows. In three dimensions, each velocity component is represented independently using anisotropic natural tensor product splines. The splines $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ are cubic in their coordinate direction and quadratic in the other two directions and are thus uniquely defined by the flux interpolation conditions,

$$\int_{y_{j-1}}^{y_j} \int_{z_{k-1}}^{z_k} u(x_i, y, z) dz dy = U_{ijk} \quad (i = 0 \text{ to } I, j = 1 \text{ to } J, k = 1 \text{ to } K) \quad (1.2)$$

$$\int_{x_{i-1}}^{x_i} \int_{z_{k-1}}^{z_k} v(x, y_j, z) dz dx = V_{ijk} \quad (i = 1 \text{ to } I, j = 0 \text{ to } J, k = 1 \text{ to } K) \quad (1.3)$$

$$\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} w(x, y, z_k) dy dx = W_{ijk} \quad (i = 1 \text{ to } I, j = 1 \text{ to } J, k = 0 \text{ to } K) \quad (1.4)$$

The region Ω , however, often contains internal boundaries, which may be either obstacles or the surrounding walls of apertures, across which the normal flow should be identically zero. The method proposed in [9] to satisfy these additional conditions involves making local corrections to the spline representation. These corrections take the form of multiple knot spline functions defined across planes containing internal boundaries. The correction functions are applied in such a way that the corrected velocity representation still satisfies the incompressible continuity equation. It is shown in [7], however, that for cases where adjustments have to be made to more than one velocity component the local correction functions become coupled and hence their construction becomes impractical.

2 B-Spline Theory, Definitions and Notation Conventions

Definition 1 *Simple Knot Spline Space [15]*

Let $a = x_0 < x_1 < \dots < x_K = b$, and write $\Delta = \{x_i\}_{i=0}^K$. The set Δ partitions the interval $[a, b]$ into the K subintervals, $I_i = [x_i, x_{i+1}]$ ($i = 0$ to $K - 2$) and $I_{K-1} = [x_{K-1}, x_K]$. Given a positive integer $m > 0$, we define

$$S_m(\Delta) = \{s : \exists s_0, s_1, \dots, s_{K-1} \in \mathcal{P}_m \text{ such that } s(x) = s_i \text{ for } x \in I_k \text{ (} k = 0 \text{ to } K - 1) \text{ and } D^j s_{i-1}(x_i) = D^j s_i(x_i) \text{ for } j = 0 \text{ to } m - 2, \\ i = 1, 2, \dots, K - 1\}$$

where \mathcal{P}_m denotes the space of polynomials of order m . We call $S_m(\Delta)$ the space of polynomial splines of order m with simple knots at the points $\{x_i\}_{i=1}^{K-1}$.

Definition 2 *Multiple Knot Spline Space [15]*

Let Δ be the partition of $[a, b]$ as in definition 1 and let m be a positive integer. We define

$$S(\mathcal{P}_m; \mathcal{M}; \Delta) = \{s : \exists s_0, s_1, \dots, s_{K-1} \in \mathcal{P}_m \text{ such that } s(x) = s_i \text{ for } x \in I_k \text{ (} k = 0 \text{ to } K - 1) \text{ and } D^j s_{i-1}(x_i) = D^j s_i(x_i) \text{ for } j = 0 \text{ to } m - 1 - m_i, \\ i = 1, 2, \dots, K - 1\}$$

where the integers $\{m_i\}_{i=1}^{K-1}$ are the components of the multiplicity vector \mathcal{M} . We call $S(\mathcal{P}_m; \mathcal{M}; \Delta)$ the space of polynomial splines of order m with knots $\{x_i\}_{i=1}^{K-1}$ of multiplicities $\{m_i\}_{i=1}^{K-1}$.

Note that in the special case where $\mathcal{M} = (1, 1, \dots, 1)$ we have that $S(\mathcal{P}_m; \mathcal{M}; \Delta) = S_m(\Delta)$.

2.1 B-Splines: Definition and Properties

We consider, for the moment, splines defined on the bi-infinite partition $\pi = \{t_i\}_{i=-\infty}^{\infty} ; t_i \leq t_{i+1} \forall i$. For some positive integer k define the function

$$g_m(s; t) = (s - t)_+^{m-1} = \begin{cases} (s - t)^{m-1} & s \geq t \\ 0 & s < t \end{cases} \quad (2.1)$$

It is shown in ([15] p.16) using dual Taylor expansions that (2.1) is the Green's function associated with the differential operator D^m . The B-spline of order m , $M_{m,i}(t)$ on $t_i, t_{i+1}, \dots, t_{i+m}$ is defined to be the m th divided difference of $g_m(s; t)$ (see for example [5], [15]) so that

$$M_{m,i}(t) = [t_i, \dots, t_{i+m}]g(s; t) \quad (2.2)$$

The Normalised B-spline of order m is then defined to be,

$$N_{m,i}(t) = (t_{i+m} - t_i)M_{m,i}(t) \quad (2.3)$$

To generate the order m B-splines $N_{m,i}(t)$ we use the following recursive relation,

$$\begin{aligned} N_{m,i}(t) &= \left(\frac{t - t_i}{t_{i+m-1} - t_i} \right) N_{m-1,i}(t) + \left(\frac{t_{i+m} - t}{t_{i+m} - t_{i+1}} \right) N_{m-1,i+1}(t) \\ N_{1,i}(t) &= \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (2.4)$$

For a proof of this result see for example [5] which makes use of Leibniz' formula. From identity (2.4) we can derive the following important properties of $N_{m,i}(t)$:

- 1 $N_{m,i}(t)$ is a piecewise polynomial of degree $m - 1$ changing its form only at the knots.
- 2 $N_{m,i}(t) = 0$ if $t \leq t_i$ or $t \geq t_{i+m}$.
- 3 $N_{m,i}(t) > 0$ if $t_i < t < t_{i+m}$.
- 4 $N_{m,i}^{(r)}(t)$ is continuous at the knots for $r = 0, 1, \dots, m - 2$.

Properties 1 and 4 ensure the necessary condition that for any subset Δ of $\{t_i\}$ containing at least $m + 1$ elements $N_{m,i}(t) \in S(\mathcal{P}_m; \mathcal{M}; \Delta)$. The computational benefits associated with property 2 (better known as the Minimal Support property) and property 3 are obvious.

2.2 Knot Insertion and Removal Algorithms

Consider the spline function $s(t)$ with representation,

$$s(t) = \sum_i c_i N_{m,i}(t) \quad (2.5)$$

where the $N_{m,i}(t)$ are the normalised B-splines over the partition π . We can equivalently represent $s(t)$ by the expansion,

$$s(t) = \sum_i \hat{c}_i \hat{N}_{m,i}(t) \quad (2.6)$$

where the B-splines $\hat{N}_{m,i}(t)$ are defined over a refined partition $\hat{\pi} = \{t_i\}_{i=-\infty}^{\infty}$ constructed from π by inserting an additional (not necessarily distinct) knot \hat{t} .

A proof of this result given in [1] makes use of the well known result that for any function $g(s)$,

$$(t_i - t_{i+m})[t_i, \dots, t_{i+m}]g(s) + (t_j - t_i)[t_i, \dots, t_{i+m-1}, t_j]g(s) + (t_{i+m} - t_j)[t_j, t_{i+1}, \dots, t_{i+m}]g(s) = 0 \quad (2.7)$$

noting that divided differences are independent of the ordering of their arguments. Now, for some l we must have $\hat{t}_i = t_i$ for $i \leq l$, $\hat{t}_{l+1} = \hat{t}$, $\hat{t}_i = t_{i-1}$ for $i \geq l+2$. Thus using (2.2) and setting $t_j = \hat{t}$ in (2.7) we obtain,

$$(t_{i+m} - t_i)M_{m,i}(t) = (\hat{t} - t_i)\hat{M}_{m,i}(t) + (t_{i+m} - \hat{t})\hat{M}_{m,i+1}(t) \quad (2.8)$$

$$(i = l - m + 1, \dots, l)$$

Therefore in terms of normalised B-splines,

$$N_{m,i}(t) = \begin{cases} \hat{N}_{m,i}(t) & i < l - m + 1 \\ \left(\frac{\hat{t} - \hat{t}_i}{\hat{t}_{i+m} - \hat{t}_i}\right) \hat{N}_{m,i}(t) + \left(\frac{\hat{t}_{i+1+m} - \hat{t}}{\hat{t}_{i+1+m} - \hat{t}_{i+1}}\right) \hat{N}_{m,i+1}(t) & l - m + 1 \leq i \leq l \\ \hat{N}_{m,i+1}(t) & i > l \end{cases} \quad (2.9)$$

So the $\{N_{m,i}\}$ defined on $\{t_i\}$ are spanned by the $\{\hat{N}_{m,i}\}$ defined on $\{\hat{t}_i\}$. Now,

$$\begin{aligned} s(t) &= \sum_i c_i N_{m,i}(t) \\ &= \sum_{i < l-m+2} c_i \hat{N}_{m,i}(t) + \sum_{i=l-m+2}^l \left[\left(\frac{\hat{t}_{i+m} - \hat{t}}{\hat{t}_{i+m} - \hat{t}_i} \right) c_{i-1} \right. \\ &\quad \left. + \left(\frac{\hat{t} - \hat{t}_i}{\hat{t}_{i+m} - \hat{t}_i} \right) c_i \right] \hat{N}_{m,i} + \sum_{i>l} c_{i-1} \hat{N}_{m,i}(t) \\ &= \sum_i \hat{c}_i \hat{N}_{m,i}(t) \end{aligned}$$

where we have used $\hat{t} - \hat{t}_{l+1} = 0$. Thus, due to the linear independence of the B-splines $\{N_{m,i}\}$ and $\{\hat{N}_{m,i}\}$ we have,

$$\hat{c}_i = \alpha_i c_i + (1 - \alpha_i) c_{i-1} \quad (2.10)$$

where,

$$\alpha_i = \begin{cases} 1 & i < l - m + 2 \\ \frac{\hat{t} - \hat{t}_i}{\hat{t}_{i+m} - \hat{t}_i} & l - m + 2 \leq i \leq l \\ 0 & i > l \end{cases}$$

Thus given a spline $s(t)$ defined on π we can find an equivalent representation on $\hat{\pi}$ whose coefficients $\{\hat{c}_i\}$ relative to the B-splines $\{N_{m,i}\}$ are given by (2.10).

However, if we wish to insert more than one additional knot, then the recursive use of (2.10) is inefficient. Let \mathbf{t} be a non-decreasing bi-infinite sequence

$\{t_j\}_{j=-\infty}^{\infty}$ as before and $\boldsymbol{\tau}$ a sub-sequence of \mathbf{t} containing at least $m + 1$ elements. Then, using a result first given in [13] we have for a sufficiently smooth function $f(t)$,

$$(\tau_{i+m} - \tau_i)[\tau_i, \dots, \tau_{i+m}]f = \sum_i \alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j)(t_{j+m} - t_j)[t_j, \dots, t_{j+m}]f \quad (2.11)$$

where we have used a slight variation of the notation of [11]. By definition, the Normalised B-spline $N_{m,\boldsymbol{\tau},i}$ defined on the knot set $\boldsymbol{\tau}$ is given by,

$$N_{m,\boldsymbol{\tau},i}(t) = (\tau_{i+m} - \tau_i)[\tau_i, \dots, \tau_{i+m}]g_m(s; t) \quad (2.12)$$

where $g_m(s; t)$ is the Green's function defined in (2.1). With this in mind we can now set $f(t) = g_m(s; t)$ in (2.11) to obtain,

$$N_{m,\boldsymbol{\tau},i}(t) = \sum_j \alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j)N_{m,\mathbf{t},j}(t) \quad (2.13)$$

The $\alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j)$ are known as discrete B-splines and are covered in [15]. Clearly $\text{Span}\{N_{m,\boldsymbol{\tau},i}\} \subseteq \text{Span}\{N_{m,\mathbf{t},i}\}$ and thus (2.13) represents a transformation from the basis $\{N_{m,\boldsymbol{\tau},i}\}$ for splines defined on $\boldsymbol{\tau}$ to the basis $\{N_{m,\mathbf{t},i}\}$ for splines on \mathbf{t} . So for some order m spline function $s(t)$ on $\boldsymbol{\tau}$ we have

$$\begin{aligned} s(t) &= \sum_i c_i N_{m,\boldsymbol{\tau},i}(t) \\ &= \sum_j d_j N_{m,\mathbf{t},j}(t) \end{aligned}$$

where,

$$d_j = \sum_i c_i \alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j) \quad (2.14)$$

Thus, to summarise, given a spline function $s(t)$ defined on $\boldsymbol{\tau}$ with coefficients $\{c_i\}$ relative to $\{N_{m,\boldsymbol{\tau},i}\}$ we can find an equivalent representation on \mathbf{t} with coefficients d_j relative to $\{N_{m,\mathbf{t},j}\}$ given by (2.14). The discrete B-splines $\{\alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j)\}$ can be efficiently computed using the Oslo algorithm [4] which goes as follows,

$$\begin{aligned} \alpha_{m,i}(j) &= \left(\frac{t_{j+m-1} - \tau_i}{\tau_{i+m-1} - \tau_i} \right) \alpha_{m-1,i}(j) + \left(\frac{\tau_{i+m} - t_{j+m-1}}{\tau_{i+m} - \tau_{i+1}} \right) \alpha_{m-1,i+1}(j) \\ \alpha_{1,i}(j) &= \begin{cases} 1, & \text{if } \tau_i \leq t_j < \tau_{i+1} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (2.15)$$

where we have set $\alpha_{m,i}(j) = \alpha_{m,i,\boldsymbol{\tau},\mathbf{t}}(j)$. There are many proofs of this relation other than the one in [4] which is rather long, see for example Prautzsch [14] (using recurrence relations, but assuming simple knots) and [10] where the discrete B-spline theory is simplified. Two efficient algorithms are presented in [4]

to determine the $\alpha_{m,i}(j)$ and the new spline coefficients $\{d_j\}$. An alternative algorithm for inserting knot sequences is given in [2]. This is known as the Insertion Algorithm and is based on a fast recursive implementation of the single knot insertion algorithm [1]. A detailed comparison of the speed of multiple knot insertion algorithms is given in [3], however since then Lyche and Mørken [11] have developed faster Oslo algorithms than those in [4] using further results from discrete B-spline theory. All the results in this paper have been computed using the improved Oslo algorithms of [11] although whether they are the fastest is debatable.

2.2.1 B-spline Basis Transformations

We now introduce some linear transformation operators for interchanging between B-spline representations defined on knot sets of varying multiplicity. We will make frequent use of these transformations in section 4.

Definition 3 *Transformation Operator for Interchanging between two Multiple Knot Spline Representations of differing Multiplicity*

Let \mathbf{X}_1 and \mathbf{X}_2 be the multiple knot sets associated with the spline spaces $S(\mathcal{P}_m; \mathcal{M}_1; \Delta)$ and $S(\mathcal{P}_m; \mathcal{M}_2; \Delta)$ where $\mathcal{M}_2 \geq \mathcal{M}_1$. Define the linear operator $Q(m, \mathbf{X}_1, \mathbf{X}_2) : S(\mathcal{P}_m; \mathcal{M}_1; \Delta) \rightarrow S(\mathcal{P}_m; \mathcal{M}_2; \Delta)$ with matrix $A(m, \mathbf{X}_1, \mathbf{X}_2)$ to be such that for any $s(x) \in S(\mathcal{P}_m; \mathcal{M}_1; \Delta)$,

$$\mathbf{a}_2 - A(\mathbf{X}_1, \mathbf{X}_2)\mathbf{a}_1 = 0$$

where \mathbf{a}_1 and \mathbf{a}_2 denote the B-spline representations of $s(x)$ relative to the bases $N_{m,i}^{\mathbf{X}_1}(x)$ and $N_{m,i}^{\mathbf{X}_2}(x)$ respectively, and write,

$$s(x; \mathbf{X}_2) = Q(m, \mathbf{X}_1, \mathbf{X}_2)s(x; \mathbf{X}_1)$$

to denote a transformation of $s(x) \in S(\mathcal{P}_m; \mathcal{M}_1; \Delta)$ from the basis $N_{m,i}^{\mathbf{X}_1}(x)$ to $N_{m,i}^{\mathbf{X}_2}(x)$. The inverse $Q^{-1}(m, \mathbf{X}_1, \mathbf{X}_2) : S(\mathcal{P}_m; \mathcal{M}_2; \Delta) \rightarrow S(\mathcal{P}_m; \mathcal{M}_1; \Delta)$ with matrix $A(m, \mathbf{X}_2, \mathbf{X}_1)$ is defined such that,

$$A(m, \mathbf{X}_2, \mathbf{X}_1)A(m, \mathbf{X}_1, \mathbf{X}_2) = I$$

Definition 4 *Projection of a Multiple Knot Spline Representation into a Simple Knot Spline Space*

Let \mathbf{x} and \mathbf{X} be the simple and multiple knot sets associated with $S_m(\Delta)$ and $S(\mathcal{P}_m; \mathcal{M}; \Delta)$ and let $A(m, \cdot, \cdot)$ be defined as in definition 3. Define the linear transformation $R(m, \mathbf{x}, \mathbf{X}) : S(\mathcal{P}_m; \mathcal{M}; \Delta) \rightarrow S_m(\Delta)$ with matrix $B(m, \mathbf{x}, \mathbf{X})$ to be such that for any $S(x) \in S(\mathcal{P}_m; \mathcal{M}; \Delta)$,

$$B(m, \mathbf{x}, \mathbf{X})A(m, \mathbf{x}, \mathbf{X}) = I$$

and for any $s(x) \in S(\mathcal{P}_m; \mathcal{M}; \Delta)$ over $\{N_{m,i}^{\mathbf{X}}(x)\}$ write,

$$g(x; \mathbf{x}) = R(m, \mathbf{x}, \mathbf{X})s(x; \mathbf{X})$$

to denote a transformation of $s(x)$ to $g(x) \in S_m(\Delta)$ over $\{N_{m,i}^{\mathbf{x}}(x)\}$. Note that $R(m, \mathbf{x}, \mathbf{X})$ does not have an inverse.

The matrix $A(m, \mathbf{X}_1, \mathbf{X}_2)$ of definition 3 is sometimes referred to as the B-spline Knot Insertion Matrix (see for example [12]) and its elements are given by (2.15) after replacing the elements of \mathbf{t} and $\boldsymbol{\tau}$ with those of \mathbf{X}_2 and \mathbf{X}_1 respectively.

The right inverse of $A(m, \mathbf{X}_1, \mathbf{X}_2)$ can be computed once all the linear dependencies have been removed. The sparsity structures of $A(m, \mathbf{X}_1, \mathbf{X}_2)$, and in particular its inverse, are rather complex since they depend on the knot sets \mathbf{X}_1 and \mathbf{X}_2 . Since we will only require specific transformations for $m = 3$ and $m = 4$ we state the required matrix-vector operations in appendix A.

2.3 Notation Conventions

In the remainder of this paper we replace the definition of the Normalised B-Spline $N_{m,i}(x)$ given in (2.3) with,

$$N_{m,i}^{\mathbf{x}}(x) = (x_i - x_{i-m})[x_{i-m}, \dots, x_i]g_m(s; t)$$

where $g_m(s; t)$ is the Green's function defined in (2.1). Notice that here we have used a superscript to denote the defining knot set \mathbf{x} . The only effect of the above definition is to shift each basis function m intervals to the right, thus properties 2 and 3 now become,

- $N_{m,i}(x) = 0$ if $x \leq x_{i-m}$ or $x \geq x_i$
- $N_{m,i}(x) > 0$ if $x_{i-m} < x < x_i$

Properties 1 and 4 remain unchanged. Unless otherwise stated we will use lowercase letters such as \mathbf{x}, \mathbf{y} to denote simple knot sets (whose entries are all distinct) and uppercase letters such as \mathbf{X}, \mathbf{Y} to denote multiple knot sets. Furthermore it will sometimes be tacitly assumed that $\mathbf{x} \subseteq \mathbf{X}$ and $\mathbf{y} \subseteq \mathbf{Y}$.

3 Tensor Spline Formulation

3.1 Problem Specification

As in [9] we seek independent anisotropic spline representations for the u, v and w components which are cubic in the coordinate direction and quadratic in the other two directions. Consider the problem of finding a spline representation for the w component in the region $\Omega = [x_0, x_I] \otimes [y_0, y_J] \otimes [z_0, z_K]$ illustrated in

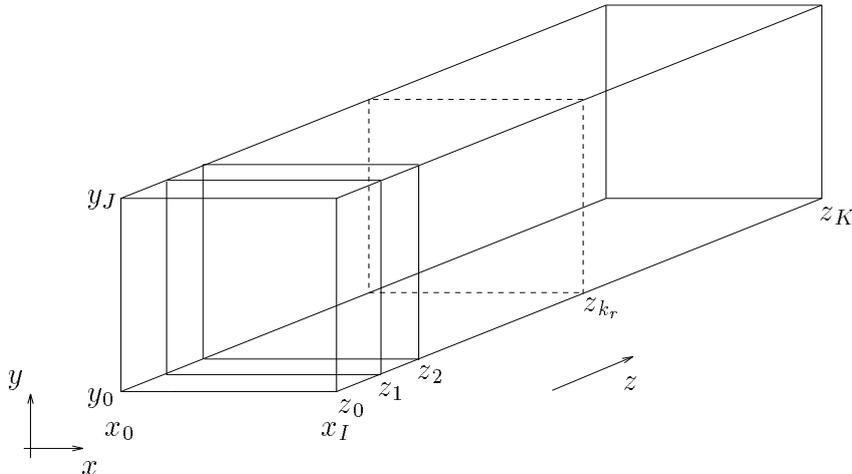


Figure 2: A rectangular region Ω with (x, y) planes at $z = z_{k_r}$ ($r = 1$ to R) containing sets of apertures or obstacles

figure (2). We assume that Ω has been discretised into a cartesian tensor product mesh of the form Ω_h defined in section 1.1 and that we have been provided with the normal fluxes W_{ijk} ($i = 1$ to $I, j = 1$ to J) across each (x, y) plane $z = z_k$ ($k = 0$ to K). We seek a spline representation $w(x, y, z)$ which satisfies condition (1.4) for each of these values. On a subset of the (x, y) planes $z = z_{k_r}$ ($r = 1$ to R) there are sets of non-overlapping rectangular regions $D_{r,m}$ whose edges coincide with the x, y lines forming the cell edges of the mesh. Suppose on the plane $z = z_{k_r}$ there are M_r of these regions, then $D_{r,m} = \{(x, y) | x_{i_a} \leq x \leq x_{i_b}, y_{j_a} \leq y \leq y_{j_b}, 0 \leq i_a < i_b \leq I, 0 \leq j_a < j_b \leq J\}$ and $\cap_{m=0}^{M_r} D_{r,m} = 0$. On the plane $z = z_{k_r}$ the regions $D_{r,m}$ ($m = 1$ to M_r) may represent either a set of obstacles or apertures. If $z = z_{k_r}$ is defined to be a set of obstacles then we require that the spline representation $w(x, y, z_{k_r})$ be identically zero whenever $(x, y) \in D_{r,m}$ ($m = 1$ to M_r) whereas if $z = z_{k_r}$ represents a plane of apertures then $w(x, y, z_{k_r})$ should be identically zero whenever $(x, y) \notin D_{r,m}$ ($m = 1$ to M_r).

In the following formulation we will find it convenient to introduce the lines $x = x_{i_{r,p}}$ ($p = 1$ to P_r) and $y = y_{j_{r,q}}$ ($q = 1$ to Q_r) to coincide with the edges of the regions $D_{r,m}$ ($m = 1$ to M_r) as illustrated in figure (3). Our basic approach will be to introduce multiple knots along these lines to reduce the spline continuity thus providing the extra degrees of freedom required to impose the zero conditions on the special planes $z = z_{k_r}$ ($r = 1$ to R). Planes across which multiple knots have been introduced will generally be referred to as multiple knot planes. However, we cannot just introduce multiple knots on the planes $z = z_{k_r}$ because such an approach does not yield a tensor-product representation.

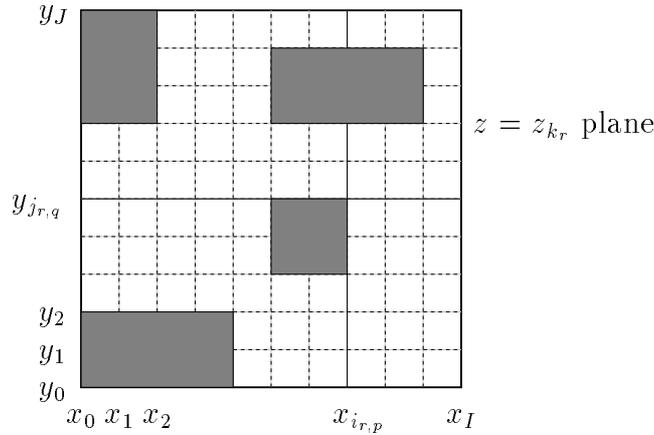


Figure 3: An (x, y) plane with multiple knots defined along the lines $x = x_{i_{r,p}}$ ($p = 1$ to P_r) and $y = y_{j_{r,q}}$ ($q = 1$ to Q_r) which are chosen to coincide with the mesh lines

3.2 Decomposition into Tensor Splines

Around each plane $z = z_{k_r}$ we define the sub-regions $\Omega_r = \{\mathbf{x} \in \Omega | z_{k_{r-1}} \leq z \leq z_{k_{r+1}}\}$ ($r = 1$ to R). In each Ω_r we can construct a tensor product spline defined

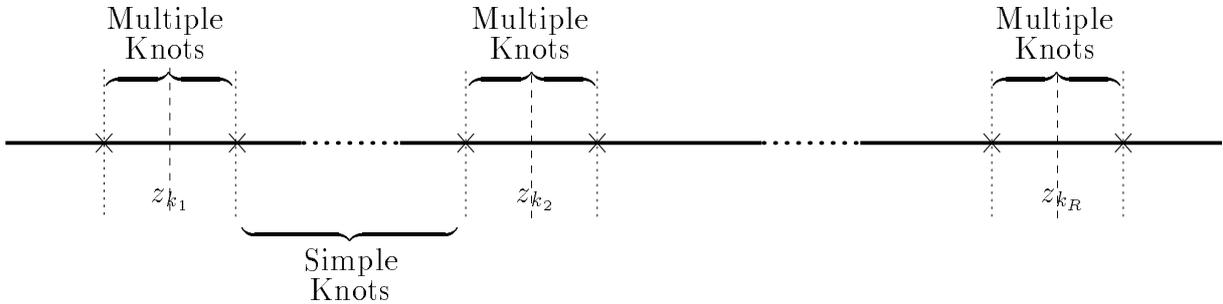


Figure 4: A partition of Ω into multiple and simple knot regions according to the location of the planes $z = z_{k_r}$ ($r = 1$ to R)

on a knot set determined by the multiplicity and location of the multiple knots introduced across the plane $z = z_{k_r}$. Consequently, the knot sets defined on the plane $z = z_{k_r}$ must also be defined on the planes $z = z_{k_{r-1}}$ and $z = z_{k_{r+1}}$. In the regions separating the Ω_r ($r = 1$ to R), we use simple knot tensor product splines, so that Ω is partitioned into simple and multiple knot regions according to the location of the planes $z = z_{k_r}$ ($r = 1$ to R), as illustrated in figure (4).

Although the simple knot regions are disjoint we find it convenient to refer to them collectively as Ω' where $\Omega' = \Omega \setminus \{\cup_{r=1}^R \Omega_r\}$. Consider the z planes $z = z_{k_{r-1}}$ and $z = z_{k_{r+1}}$ interfacing the regions Ω_r with Ω' . It is important that the

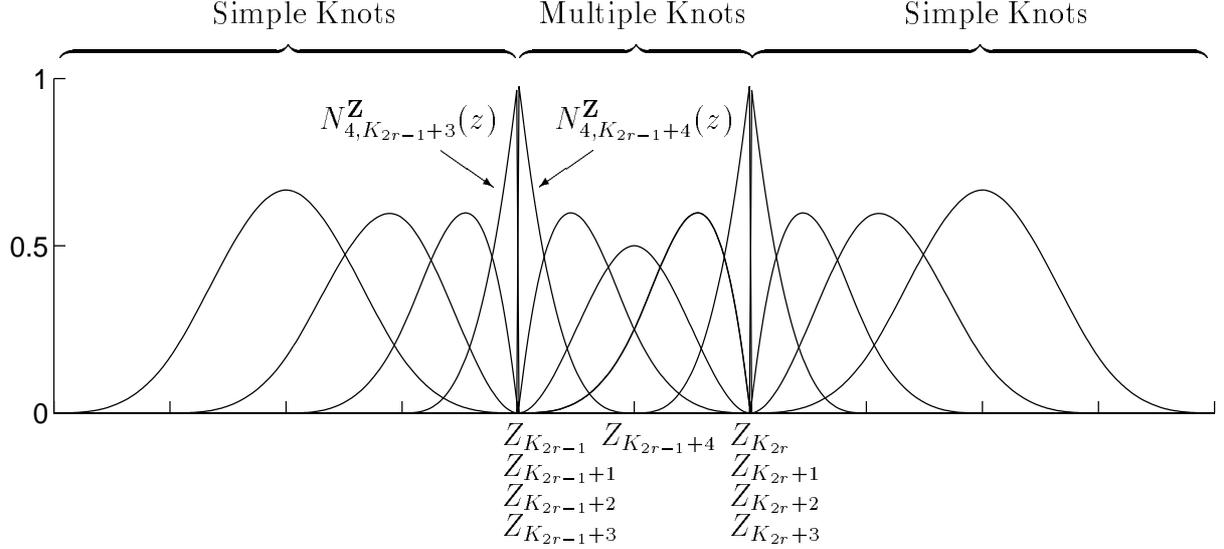


Figure 5: B-spline basis functions of order 4 defined across two quadruple knots at the points $z = z_{k_r \pm 1}$. The quadruple knot set \mathbf{Z} has been constructed so that $Z_{K(2r-1)} = z_{k_r-1}$ and $Z_{K(2r)} = z_{k_r+1}$

multiple knot spline representation used in Ω_r does not overlap into Ω' , and similarly that the simple knot representation defined in Ω' does not overlap into Ω_r . This problem arises in the context of B-spline expansions where the region, over which a spline representation is required, must be fictitiously extended to enable all the B-splines necessary for a complete basis to be fully defined.

Suppose at each interface plane $z = z_{k_r \pm 1}$ we introduce a quadruple knot in the z direction. The B-spline basis functions would have the form illustrated in figure (5). Notice that none of the basis functions illustrated in figure (5) actually cross the lines $z = z_{k_r \pm 1}$ with a non-zero support. The basis functions $N_{4, K(2r-1)+3}^{\mathbf{Z}}(z)$ and $N_{4, K(2r-1)+4}^{\mathbf{Z}}(z)$ each have jump discontinuities at $z = z_{k_r-1}$. Similarly, $N_{4, K(2r)+3}^{\mathbf{Z}}(z)$ and $N_{4, K(2r)+4}^{\mathbf{Z}}(z)$ have discontinuities at $z = z_{k_r+1}$. Thus the quadruple knots provide a means of disconnecting the multiple knot spline representations from the simple knot representations at the interfaces. Thus we can represent $w(x, y, z)$ as the sum of tensor splines,

$$w(x, y, z) = \sum_{r=1}^R w_r(x, y, z) + \sum_{r=0}^R w'_r(x, y, z) \quad (3.1)$$

where $w_r(x, y, z)$ ($r = 1$ to R) are multiple knot tensor splines defined in the regions Ω_r ($r = 1$ to R) and $w'_r(x, y, z)$ are simple knot splines defined in the regions Ω' ($r = 0$ to R) where $\Omega'_r = \{\mathbf{x} \in \Omega | z_{k_r+1} < z < z_{k_{r+1}-1}\}$ ($r = 1$ to $R-1$), $\Omega'_0 = \{\mathbf{x} \in \Omega | z_{-3} < z < z_{k_1-1}\}$ and $\Omega'_R = \{\mathbf{x} \in \Omega | z_{k_R+1} < z < z_{K+3}\}$.

So far, we have referred to the Ω_r ($r = 1$ to R) as multiple knot regions. However, since $w(x, y, z)$ is quadratic in the x and y directions the multiple

knots introduced across the planes $z = z_k$ ($k = k_r - 1$ to $k_r + 1$) within each region Ω_r , can only be of multiplicity two or three. From now on we will make the assumption that the multiple knots defined on the plane $z = z_{k_r}$ are all of the same multiplicity. That is, we will not permit double and triple knots to be mixed across any one plane. Consequently, since the multiplicities and locations of the multiple knots on the plane $z = z_{k_r}$ are duplicated on the adjacent planes $z = z_{k_r \pm 1}$, each Ω_r can be either a triple knot region or a double knot region. This assumption is not as restrictive as it may first appear since a knot of multiplicity M is capable of emulating knots of order ($m = 1$ to $M - 1$). So, for instance, a triple knot can take the place of either a double knot or a simple knot at the small cost of introducing a few extra spline coefficients. For this reason we use triple knot B-spline expansions in each of the regions Ω_r ($r = 1$ to R). Thus $w_r(x, y, z)$ and $w'_r(x, y, z)$ have B-spline expansions of the form,

$$w'_r(x, y, z) = \sum_{k=K(2r)+2}^{K(2r+1)+1} \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} \tilde{a}_{ijk} N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) N_{4,k+1}^{\mathbf{Z}}(z) \quad (r = 0 \text{ to } R) \quad (3.2)$$

$$w_r(x, y, z) = \sum_{k=K(2r-1)+2}^{K(2r)+1} \sum_{i=0}^{I+2P_r+1} \sum_{j=0}^{J+2Q_r+1} \tilde{a}_{ijk} N_{3,i+1}^{\mathbf{X}_r}(x) N_{3,j+1}^{\mathbf{Y}_r}(y) N_{4,k+1}^{\mathbf{Z}}(z) \quad (r = 1 \text{ to } R) \quad (3.3)$$

where $K(0) = -3$ and $K(2R+1) = K+6R$. The pointers $K(r)$ ($r = 1$ to $2R$) are defined so that $Z_{K(r)} = Z_{K(r)+1} = Z_{K(r)+2} = Z_{K(r)+3}$ ($r = 1$ to $2R$). Similarly, on the multiple knot planes $z = z_{k_r}, z_{k_r \pm 1}$ constituting the region Ω_r the triple knot sets \mathbf{X}_r and \mathbf{Y}_r are defined with the pointers $i_{r,p}, I_{r,p}, j_{r,q}$ and $J_{r,q}$ so that $x_{i_{r,p}} = X_{I_{r,p}} = X_{I_{r,p}+1} = X_{I_{r,p}+2}$ ($p = 1$ to P_r) and $y_{j_{r,q}} = Y_{J_{r,q}} = Y_{J_{r,q}+1} = Y_{J_{r,q}+2}$ ($q = 1$ to Q_r). We see from (3.2), (3.3) and figure (5) that $\text{Supp}\{w'_r(x, y, z)\} = \Omega'_r$ ($r = 0$ to R) and $\text{Supp}\{w_r(x, y, z)\} = \Omega_r$ ($r = 1$ to R) as required. The regions Ω'_r ($r = 0$ to R) and Ω_r ($r = 1$ to R) are not only disjoint, but they partition Ω so we can write (3.1) in the form,

$$w(x, y, z) = \begin{cases} w'_r(x, y, z) & \text{if } (x, y, z) \in \Omega'_r \quad 0 \leq r \leq R \\ w_r(x, y, z) & \text{if } (x, y, z) \in \Omega_r \quad 1 \leq r \leq R \end{cases} \quad (3.4)$$

Thus $w(x, y, z)$ always coincides with a tensor product spline.

The B-spline representation of $w(x, y, z)$ as given by (3.4) can be equivalently represented over the knot set $\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Z}$ when $\mathbf{X} = \cup_{r=1}^R \mathbf{X}_r$ and $\mathbf{Y} = \cup_{r=1}^R \mathbf{Y}_r$. Thus, using P and Q to denote the total number of triple knots in the sets \mathbf{X} and \mathbf{Y} respectively we can write,

$$w(x, y, z) = \sum_{k=-1}^{K+6R+1} \sum_{i=0}^{I+2P+1} \sum_{j=0}^{J+2Q+1} a_{ijk} N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) N_{4,k+2}^{\mathbf{Z}}(z) \quad (3.5)$$

We define the pointers i_p, I_p, j_q and J_q associated with the triple knot sets \mathbf{X} and \mathbf{Y} so that $x_{i_p} = X_{I_p} = X_{I_p+1} = X_{I_p+2}$ ($p = 1$ to P) and $y_{j_q} = Y_{J_q} = Y_{J_q+1} =$

Y_{J_q+2} ($q = 1$ to Q). The tensor product form of (3.5) enables the coefficients $\{a_{ijk}\}$ to be determined very efficiently. For this reason we will state the defining conditions on $w(x, y, z)$ over $\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Z}$ so as to uniquely determine (3.5); but in such a way that the representation will be reducible to the form given by (3.2) and (3.3).

3.3 Matching Conditions

Multiple Knot Regions

The spline representations within the regions Ω_r should be constructed so that at the interfacing planes $z = z_{k_r \pm 1}$ the splines $w(x, y, z_{k_r-1}^+)$ and $w(x, y, z_{k_r+1}^-)$ which have multiple knot B-spline expansions, can be equivalently represented with simple knot B-splines. Thus we require that the tangential derivatives of $w_r(x, y, z)$ and $\frac{\partial}{\partial z} w_r(x, y, z)$ at the planes $z = z_{k_r \pm 1}$ should be zeroth and first derivative continuous across the triple knot lines $x = x_{i_p}$ ($p = 1$ to P) and $y = y_{j_q}$ ($q = 1$ to Q). Thus we impose the following continuity conditions,

$$\frac{\partial^v}{\partial x^v} w_r(x_{i_p}^-, y, z) = \frac{\partial^v}{\partial x^v} w_r(x_{i_p}^+, y, z) \quad \forall y \quad (p = 1 \text{ to } P) \quad (3.6)$$

$$\frac{\partial^v}{\partial y^v} w_r(x, y_{j_q}^-, z) = \frac{\partial^v}{\partial y^v} w_r(x, y_{j_q}^+, z) \quad \forall x \quad (q = 1 \text{ to } Q) \quad (3.7)$$

$$\frac{\partial^{v+1}}{\partial x^v \partial z} w_r(x_{i_p}^-, y, z) = \frac{\partial^{v+1}}{\partial x^v \partial z} w_r(x_{i_p}^+, y, z) \quad \forall y \quad (p = 1 \text{ to } P) \quad (3.8)$$

$$\frac{\partial^{v+1}}{\partial y^v \partial z} w_r(x, y_{j_q}^-, z) = \frac{\partial^{v+1}}{\partial y^v \partial z} w_r(x, y_{j_q}^+, z) \quad \forall x \quad (q = 1 \text{ to } Q) \quad (3.9)$$

$$(z = z_{k_r-1}^+, z_{k_r+1}^-) \\ (v = 0 \text{ to } 1, r = 1 \text{ to } R)$$

Simple Knot Regions

In simple knot regions it is necessary that the spline functions $w'_r(x, y, z)$, $\frac{\partial}{\partial z} w'_r(x, y, z)$ and $\frac{\partial^2}{\partial z^2} w'_r(x, y, z)$ can be equivalently represented using simple knot B-spline basis functions. Thus we impose the following continuity conditions across the triple knot lines $x = x_{i_p}$ ($p = 1$ to P) and $y = y_{j_q}$ ($q = 1$ to Q),

$$\frac{\partial^v}{\partial z^v} w'_r(x_{i_p}^-, y, z) = \frac{\partial^v}{\partial z^v} w'_r(x_{i_p}^+, y, z) \quad \forall y, z \quad (p = 1 \text{ to } P) \quad (3.10)$$

$$\frac{\partial^v}{\partial z^v} w'_r(x, y_{j_q}^-, z) = \frac{\partial^v}{\partial z^v} w'_r(x, y_{j_q}^+, z) \quad \forall x, z \quad (q = 1 \text{ to } Q) \quad (3.11)$$

$$\frac{\partial^{v+1}}{\partial x \partial z^v} w'_r(x_{i_p}^-, y, z) = \frac{\partial^{v+1}}{\partial x \partial z^v} w'_r(x_{i_p}^+, y, z) \quad \forall y, z \quad (p = 1 \text{ to } P) \quad (3.12)$$

$$\frac{\partial^{v+1}}{\partial y \partial z^v} w'_r(x, y_{j_q}^-, z) = \frac{\partial^{v+1}}{\partial y \partial z^v} w'_r(x, y_{j_q}^+, z) \quad \forall x, z \quad (q = 1 \text{ to } Q) \quad (3.13)$$

$$(v = 0, r = 0 \text{ to } R)$$

Although we require conditions (3.10-3.13) to be true for $(v = 0 \text{ to } 2)$ using expansion (3.5) it is trivial to show that provided conditions (3.10-3.13) hold for $v = 0$ then they must also hold for the cases $v = 1$ and $v = 2$.

3.4 Connectivity across Interfaces

At the interface planes $z = z_{k_r \pm 1}$ the simple knot splines $w'_r(x, y, z)$ should connect with the multiple knot splines $w_r(x, y, z)$ as smoothly as possible. We can only achieve zeroth and first derivative continuity at each interface plane, but a second derivative continuity condition can be imposed on the projection of the second derivative into a simple knot spline space. Thus, the conditions are,

$$\frac{\partial^v}{\partial z^v} w_r(x, y, z_{k_r+1}^-) = \frac{\partial^v}{\partial z^v} w'_r(x, y, z_{k_r+1}^+) \quad \forall x, y \quad (v = 0 \text{ to } 1) \quad (3.14)$$

$$[R^x(3, \mathbf{x}, \mathbf{X}) \circ R^y(3, \mathbf{y}, \mathbf{Y})] \frac{\partial^2}{\partial z^2} w_r(x, y, z_{k_r+1}^-) = \frac{\partial^2}{\partial z^2} w'_r(x, y, z_{k_r+1}^+) \quad \forall x, y \quad (3.15)$$

$(r = 1 \text{ to } R)$

$$\frac{\partial^v}{\partial z^v} w'_{r-1}(x, y, z_{k_r-1}^-) = \frac{\partial^v}{\partial z^v} w_r(x, y, z_{k_r-1}^+) \quad \forall x, y \quad (v = 0 \text{ to } 1) \quad (3.16)$$

$$\frac{\partial^2}{\partial z^2} w'_{r-1}(x, y, z_{k_r-1}^-) = [R^x(3, \mathbf{x}, \mathbf{X}) \circ R^y(3, \mathbf{y}, \mathbf{Y})] \frac{\partial^2}{\partial z^2} w_r(x, y, z_{k_r-1}^+) \quad \forall x, y \quad (3.17)$$

$(r = 1 \text{ to } R)$

3.5 Plane Interpolation Conditions

On each (x, y) plane $z = z_k$ ($k = 0 \text{ to } K$) we require that,

$$w(x, y, z_k) = W_k(x, y) \quad \forall x, y \quad (k = 0 \text{ to } K) \quad (3.18)$$

where $W_k(x, y)$ is a surface spline satisfying the flux interpolation conditions and any additional conditions arising from the possible presence of internal surfaces, apertures or fixed boundary conditions. On all (x, y) planes we impose the flux interpolation conditions,

$$\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} W_k(x, y) dy dx = W_{ijk} \quad (i = 1 \text{ to } I, j = 1 \text{ to } J) \quad (3.19)$$

and zero derivative boundary conditions,

$$\frac{\partial}{\partial x} W_k(x_0, y) = \frac{\partial}{\partial x} W_k(x_I, y) = 0 \quad (3.20)$$

$$\frac{\partial}{\partial y} W_k(x, y_0) = \frac{\partial}{\partial y} W_k(x, y_J) = 0 \quad (3.21)$$

3.5.1 Additional Conditions on Multiple Knot Planes

Although all multiple knot planes are represented over triple knot sets it is sometimes convenient to work with double knots in the formulation of the interpolation and continuity conditions and then make use of the B-spline basis transformations of section 1.3.1 to obtain equivalent triple knot representations. If, for some plane $z = z_{k_r}$, the associated triple knot surface spline $W_{k_r}(x, y; \mathbf{X}_r, \mathbf{Y}_r)$ can be equivalently represented over the double knot set $\tilde{\mathbf{X}}_r \otimes \tilde{\mathbf{Y}}_r$ where $\tilde{\mathbf{X}}_r$ and $\tilde{\mathbf{Y}}_r$ have been constructed from \mathbf{X}_r and \mathbf{Y}_r by replacing every triple knot with a double knot, then we refer to $z = z_{k_r}$ as a double knot plane and label it with the flag $m(r) = 2$. Planes on which the surface splines are actually discontinuous and thus do not have an equivalent double knot representation will be referred to as triple knot planes and we use $m(r) = 3$ to denote such planes. At this stage it may seem that double knot planes are always preferable to triple knot planes but sometimes, it is desirable to build discontinuities into a surface spline representation across prescribed lines to model for example a plug flow boundary condition or to reduce unwanted oscillations brought about by the inflexibility of the polynomials in satisfying strict interpolation conditions. We now state separately the conditions imposed on Double Knot and Triple Knot planes.

Double Knot Planes

These are suitable when we want the normal flux $W_k(x, y)$ to be identically zero either across or around the regions $D_{r,m}$ ($m = 1$ to M_r) but also want $W_k(x, y)$ to be continuous for all x and y . The triple knot pointers $i_{r,p}$ and $j_{r,q}$ are selected so that the lines $x = x_{i_{r,p}}$ ($p = 1$ to P_r) and $y = y_{j_{r,q}}$ ($q = 1$ to Q_r) coincide with the edges of the regions $D_{r,m}$ ($m = 1$ to M_r). Before we state the additional conditions at the multiple knots we define the following boolean function,

$$B_r(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \partial D_{r,m} \text{ for any } 1 \leq m \leq M_r \\ 0 & \text{otherwise} \end{cases}$$

which is unity whenever (x, y) coincides with the boundary of any of the regions $D_{r,m}$ ($m = 1$ to M_r) and zero otherwise. Using $B_r(x, y)$ the conditions are,

$$\int_{y_{j-1}}^{y_j} W_k(x_{i_{r,p}}^\pm, y) dy = 0 \quad \text{when} \quad B_r(x_{i_{r,p}}, y_{j-\frac{1}{2}}) = 1 \quad (3.22)$$

$$\int_{y_{j-1}}^{y_j} \left[\frac{\partial^v}{\partial x^v} W_k(x, y) \right]_{x_{i_{r,p}}^-}^{x_{i_{r,p}}^+} dy = 0 \quad \text{when} \quad B_r(x_{i_{r,p}}, y_{j-\frac{1}{2}}) = 0 \quad (3.23)$$

$$(v = 0 \text{ to } 1, p = 1 \text{ to } P_r, j = 1 \text{ to } J)$$

$$\int_{x_{i-1}}^{x_i} W_k(x, y_{j_{r,q}}^\pm) dx = 0 \quad \text{when} \quad B_r(x_{i-\frac{1}{2}}, y_{j_{r,q}}) = 1 \quad (3.24)$$

$$\int_{x_{i-1}}^{x_i} \left[\frac{\partial^v}{\partial y^v} W_k(x, y) \right]_{y_{j_r, q}^-}^{y_{j_r, q}^+} dx = 0 \quad \text{when} \quad B_r(x_{i-\frac{1}{2}}, y_{j_r, q}) = 0 \quad (3.25)$$

$$(v = 0 \text{ to } 1, q = 1 \text{ to } Q_r, i = 1 \text{ to } I)$$

$$W_k(x_{i_r, p}^\pm, y_{j_r, q}^\pm) = 0 \quad (p = 1 \text{ to } P_r, q = 1 \text{ to } Q_r) \quad (3.26)$$

Because each $W_k(x, y)$ spline is composed of quadratic polynomials in x and y , conditions (3.22-3.26) are clearly sufficient to ensure that $W_k(x, y)$ is identically zero around all the edges of the regions $D_{r,m}$ ($m = 1$ to M_r). If $D_{r,m}$ are solid surfaces representing internal boundaries and provided the flux data W_{ijk} is zero over all the cells included in $D_{r,m}$, then since $W_k(x, y)$ is identically zero on $\partial D_{r,m}$ it must also be identically zero within $D_{r,m}$. By a similar argument, if $D_{r,m}$ represent apertures in a solid wall and the W_{ijk} are thus zero across all the cells surrounding $D_{r,m}$, then the conditions that $W_k(x, y)$ be identically zero on $\partial D_{r,m}$, together with the boundary conditions (3.20) and (3.21) are sufficient to ensure that $W_k(x, y)$ is identically zero across all the wall segments.

Triple Knot Planes

Suppose, as for the double knot case, that we have a set of regions $D_{r,m}$ ($m = 1$ to M_r) either across or around which $W_k(x, y)$ should be identically zero. In this case, however, we require that $W_k(x, y)$ change discontinuously across certain lines coinciding with the boundaries of $D_{r,m}$ ($m = 1$ to M_r). This might be the case if for instance we wished to accurately impose a boundary condition. We will assume that some function $f_r(x, y)$ can be constructed from the W_{ijk_r} values which incorporates the desired discontinuities and is zero across or around the sub-domains $D_{r,m}$ ($m = 1$ to M_r). The only restriction we place on $f_r(x, y)$ is that,

$$\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f_r(x, y) dy dx = W_{ijk_r} \quad (i = 1 \text{ to } I, j = 1 \text{ to } J) \quad (3.27)$$

so there is no conflict with the flux interpolation conditions. We now wish to construct a surface spline $W_k(x, y)$ which approximates $f_r(x, y)$ and identically preserves any zero conditions which might be imposed across or around the regions $D_{r,m}$ ($m = 1$ to M_r). To achieve this we define the triple knot pointers $i_{r,p}$ and $j_{r,q}$ so that the lines $x = x_{i_{r,p}}$ ($p = 1$ to P_r) and $y = y_{j_{r,q}}$ ($q = 1$ to Q_r) coincide with the edges of the regions $D_{r,m}$ ($m = 1$ to M_r). Thus, in addition to (3.19-3.21) the $W_k(x, y)$ splines should satisfy,

$$\int_{y_{j-1}}^{y_j} W_k(x_{i_{r,p}}^\pm, y) dy = \int_{y_{j-1}}^{y_j} f_r(x_{i_{r,p}}^\pm, y) dy \quad (j = 1 \text{ to } J) \quad (3.28)$$

$$(p = 1 \text{ to } P_r)$$

$$\int_{x_{i-1}}^{x_i} W_k(x, y_{j_{r,q}}^\pm) dx = \int_{x_{i-1}}^{x_i} f_r(x, y_{j_{r,q}}^\pm) dx \quad (i = 1 \text{ to } I) \quad (3.29)$$

$$(q = 1 \text{ to } Q_r)$$

$$W_k(x_{i_r,p}^\pm, y_{j_r,q}^\pm) = f_r(x_{i_r,p}^\pm, y_{j_r,q}^\pm) \quad (3.30)$$

$$(p = 1 \text{ to } P_r, q = 1 \text{ to } Q_r)$$

3.6 Global Boundary Conditions

In standard spline theory, natural splines always have even order, that is they are constructed from polynomials of odd degree. Here we find it convenient to define a natural quadratic spline to be the derivative of a natural cubic spline. Thus, with this in mind, we can impose the following pseudo-natural boundary conditions,

$$\frac{\partial}{\partial x}w(x_0, y, z) = \frac{\partial}{\partial x}w(x_I, y, z) = 0 \quad (3.31)$$

$$\frac{\partial}{\partial y}w(x, y_0, z) = \frac{\partial}{\partial y}w(x, y_J, z) = 0 \quad (3.32)$$

$$\frac{\partial^2}{\partial z^2}w(x, y, z_0) = \frac{\partial^2}{\partial z^2}w(x, y, z_K) = 0 \quad (3.33)$$

to absorb the remaining degrees of freedom. Note that these conditions are compatible with those imposed on $W_k(x, y)$ (see (3.20) and (3.21)).

4 Determination of B-spline Coefficients

We now describe an efficient method for the determination of the coefficients $\{a_{ijk}\}$ of expansions (3.2) and (3.3). The conditions stated in section 3 will be imposed in such a way that the $\{a_{ijk}\}$ can be determined through the solution of sets of tridiagonal systems. The method makes frequent use of the B-spline basis transformations defined in section 2.2.1. There are three main stages:

- Construction of the surface splines $W_k(x, y)$ ($k = 0$ to K).
- Construction of the tensor product representation of $w(x, y, z)$ over $\mathbf{X} \otimes \mathbf{Y}$ given by (3.5).
- Transformation of $w(x, y, z)$ over $\mathbf{X} \otimes \mathbf{Y}$ to the non-tensor representations given by (3.2) and (3.3) over $\mathbf{x} \otimes \mathbf{y}$ and $\mathbf{X}_r \otimes \mathbf{Y}_r$ respectively.

In the first stage the plane $z = z_k$ over which $W_k(x, y)$ is defined can be either a simple knot plane, double knot plane or triple knot plane; we treat each case separately.

4.1 Surface Splines $W_k(x, y)$

All the $W_k(x, y)$ splines should satisfy conditions (3.19), (3.20) and (3.21). Introducing the natural quadratic splines $w_{i,k}(y)$ ($i = 1$ to I) we can re-state these conditions in the form,

$$\int_{x_{i-1}}^{x_i} W_k(x, y) = w_{i,k}(y) \quad (i = 1 \text{ to } I) \quad (4.1)$$

$$\frac{\partial}{\partial x} W_k(x_0, y) = \frac{\partial}{\partial x} W_k(x_I, y) = 0 \quad (4.2)$$

$$\int_{y_{j-1}}^{y_j} w_{i,k}(y) dy = W_{ijk} \quad (j = 1 \text{ to } J, i = 1 \text{ to } I) \quad (4.3)$$

$$\frac{\partial}{\partial y} w_{i,k}(y_0) = \frac{\partial}{\partial y} w_{i,k}(y_J) = 0 \quad (4.4)$$

4.1.1 Simple Knot Splines

On the planes $z = z_k$ ($k = 0$ to $K, k \neq k_r, 1 \leq r \leq R$) we seek simple knot B-spline expansions of the form,

$$W_k(x, y) = \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} b_{ijk} N_{3,i+1}^x(x) N_{3,j+1}^y(y) \quad (4.5)$$

$$w_{i,k}(y) = \sum_{j=0}^{J+1} c_{ijk} N_{3,j+1}^y(y) \quad (4.6)$$

Substitution of expansions (4.6) and (4.5) into (4.3), (4.4), (4.1) and (4.2) yields the following two sets of tridiagonal systems,

$$T_{i,k}^{(1)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(1)} \quad (i = 1 \text{ to } I, k = 0 \text{ to } K, k \neq k_r, 1 \leq r \leq R) \quad (4.7)$$

$$T_{j,k}^{(2)} \mathbf{b}_{j,k} = \mathbf{r}_{j,k}^{(2)} \quad (j = 0 \text{ to } J+1, k = 0 \text{ to } K, k \neq k_r, 1 \leq r \leq R) \quad (4.8)$$

which are given in Appendices B.1 and B.2. For each (i, k) ($k \neq k_r$) we solve system (4.7) for $\mathbf{c}_{i,k}$ to obtain $c_{i,j,k}$ ($i = 1$ to $I, j = 0$ to $J+1, k = 0$ to $K, k \neq k_r, 1 \leq r \leq R$). Then we use the $\{c_{ijk}\}$ to construct $\mathbf{r}_{j,k}^{(2)}$ of system (4.8). Now for each (j, k) we can solve system (4.8) for $\mathbf{b}_{j,k}$ ($k \neq k_r$) thus obtaining b_{ijk} ($i = 0$ to $I+1, j = 0$ to $J+1, k = 0$ to $K, k \neq k_r, 1 \leq r \leq R$).

4.1.2 Double Knot Planes

On the planes $z = z_{k_r}$ ($m(r) = 2, 1 \leq r \leq R$) we must impose additional conditions across the triple knot lines. The splines $w_{i,k}(y)$ should satisfy,

$$\frac{\partial^v}{\partial y^v} w_{i,k}(y_{j_r,q}^-) = \frac{\partial^v}{\partial y^v} w_{i,k}(y_{j_r,q}^+) \quad \text{when } B(x_{i-\frac{1}{2}}, y_{j_r,q}) = 1 \quad (4.9)$$

$$w_{i,k}(y_{j_r,q}^\pm) = 0 \quad \text{when } B(x_{i-\frac{1}{2}}, y_{j_r,q}) = 0 \quad (4.10)$$

$$(v = 0 \text{ to } 1, q = 1 \text{ to } Q_r)$$

In this case we use a double knot B-spline expansion of the form,

$$w_{i,k}(y) = \sum_{j=0}^{J+Q_r+1} \tilde{c}_{ijk} N_{3,j+1}^{\tilde{\mathbf{Y}}_r}(y) \quad (4.11)$$

where $\tilde{\mathbf{Y}}_r$ has been constructed from \mathbf{Y}_r by replacing every triple knot occurrence with a double knot and thus has the associated pointers $\tilde{J}_{r,q}$ defined so that $y_{j_r,q} = \tilde{Y}_{\tilde{J}_{r,q}} = \tilde{Y}_{\tilde{J}_{r,q+1}}$ ($q = 1$ to Q_r). Expansion (4.11) implicitly satisfies (4.9) for $v = 0$ and reduces (4.10) to $w_{i,k}(y_{j_r,q}) = 0$ when $B(x_{i-\frac{1}{2}}, y_{j_r,q}) = 1$. Substitution of (4.11) into (4.3), (4.4), (4.9) and (4.10) yields the following systems of tridiagonal equations,

$$T_{i,k}^{(3)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(3)} \quad (i = 1 \text{ to } I, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R) \quad (4.12)$$

given in Appendix B.3. Thus for each (i, k_r) we can solve for \mathbf{c}_{i,k_r} to obtain c_{ijk} ($i = 1$ to $I, j = 0$ to $J + Q_r + 1, k = k_r$ and $m(r) = 2, 1 \leq r \leq R$)

However, to obtain the appropriate $W_k(x, y)$ splines requires more work. We begin by transforming each $w_{i,k}(y; \tilde{\mathbf{Y}}_r)$ to the triple knot set \mathbf{Y}_r using,

$$\begin{aligned} w_{i,k}(y; \mathbf{Y}_r) &= Q(3, \tilde{\mathbf{Y}}_r, \mathbf{Y}_r) w_{i,k}(y; \tilde{\mathbf{Y}}_r) \\ \Rightarrow \mathbf{c}_{i,k} &= A(3, \tilde{\mathbf{Y}}_r, \mathbf{Y}_r) \tilde{\mathbf{c}}_{i,k} \end{aligned} \quad (4.13)$$

$$(i = 1 \text{ to } I, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R)$$

See Appendix A.2 system (A.3) for the coefficient operations. The reason for performing the above transformation at this stage will not become apparent until later. We now seek a B-spline expansion for $W_k(x, y)$ of the form,

$$W_k(x, y; \tilde{\mathbf{X}}_r, \mathbf{Y}_r) = \sum_{i=0}^{I+P+1} \sum_{j=0}^{J+2Q+1} \tilde{b}_{ijk} N_{3,i+1}^{\tilde{\mathbf{X}}_r}(x) N_{3,j+1}^{\mathbf{Y}_r}(y) \quad (4.14)$$

where $\tilde{\mathbf{X}}_r$ has been constructed from \mathbf{X}_r by replacing every triple knot with a double knot and hence $\tilde{I}_{r,p}$ is defined so that $x_{i_r,p} = \tilde{X}_{\tilde{I}_{r,p}} = \tilde{X}_{\tilde{I}_{r,p+1}}$ ($p = 1$ to P_r). Now since,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} W_k(x, y_{j_r,q}^{\pm}) dx &= w_{i,k}(y_{j_r,q}^{\pm}) = 0 \quad \text{when } B_r(x_{i-\frac{1}{2}}, y_{j_r,q}) = 1 \\ \int_{x_{i-1}}^{x_i} \left[\frac{\partial^v}{\partial y^v} W_k(x, y) \right]_{y_{j_r,q}^-}^{y_{j_r,q}^+} dx &= \left[\frac{\partial^v}{\partial y^v} w_{i,k}(y) \right]_{y_{j_r,q}^-}^{y_{j_r,q}^+} = 0 \quad \text{when } B_r(x_{i-\frac{1}{2}}, y_{j_r,q}) = 0 \\ &(v = 0 \text{ to } 1, q = 1 \text{ to } Q_r, i = 1 \text{ to } I) \end{aligned}$$

conditions (3.24) and (3.25) have already been satisfied. Thus it is only necessary to satisfy conditions (3.22), (3.23) and (3.26). Substituting (4.14) into (3.22) and (3.23) yields,

$$\sum_{q=j-1}^{j+1} \tilde{b}_{\tilde{I}_{r,p}+1,q,k} N_{3,\tilde{I}_{r,p}+2}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}) \int_{Y_{r,j-1}}^{Y_{r,j}} N_{3,q+1}^{\mathbf{Y}_r}(y) dy = 0 \quad \text{when } B_r(x_{i_{r,p}}, Y_{r,j-\frac{1}{2}}) = 1 \quad (4.15)$$

$$\sum_{q=j-1}^{j+1} d_q(\tilde{\mathbf{b}}, p, k) \int_{Y_{r,j-1}}^{Y_{r,j}} N_{3,q+1}^{\mathbf{Y}_r}(y) dy = 0 \quad \text{when } B_r(x_{i_{r,p}}, Y_{r,j-\frac{1}{2}}) = 0 \quad (4.16)$$

where,

$$\begin{aligned} d_q(\tilde{\mathbf{b}}, p, k) &= \tilde{b}_{\tilde{I}_{r,p},q,k} \frac{d}{dx} N_{3,\tilde{I}_{r,p}+1}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^-) + \tilde{b}_{\tilde{I}_{r,p}+1,q,k} \left[\frac{d}{dx} N_{3,\tilde{I}_{r,p}+2}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^-) \right. \\ &\quad \left. - \frac{d}{dx} N_{3,\tilde{I}_{r,p}+2}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^+) \right] - \tilde{b}_{\tilde{I}_{r,p}+2,q,k} \frac{d}{dx} N_{3,\tilde{I}_{r,p}+3}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^+) \end{aligned} \quad (4.17)$$

$$(j = 1 \text{ to } J_{r,1}, j = J_{r,q} + 3 \text{ to } J_{r,q+1}, q = 1 \text{ to } Q_r - 1, j = J_{r,Q_r} + 3 \text{ to } J + 2Q_r) \\ (p = 1 \text{ to } P_r)$$

By making use of (3.26), conditions (4.15) and (4.16) simplify to,

$$\tilde{b}_{\tilde{I}_{r,p}+1,j,k} = 0 \quad \text{when } B_r(x_{i_{r,p}}, \bar{Y}_j) = 1 \quad (4.18)$$

$$d_j(\tilde{\mathbf{b}}, p, k) = 0 \quad \text{when } B_r(x_{i_{r,p}}, \bar{Y}_j) = 0 \quad (4.19)$$

$$(p = 1 \text{ to } P_r, j = 0 \text{ to } J + 2Q_r + 1)$$

where $\bar{\mathbf{Y}}$ is defined by $\{\bar{\mathbf{Y}} | \bar{Y}_0 = Y_{r,0}, \bar{Y}_j = \frac{1}{2}(Y_{r,j} + Y_{r,j-1}) (j = 1 \text{ to } J_{r,1}, j = J_{r,q} + 3 \text{ to } J_{r,q+1}, q = 1 \text{ to } Q_r - 1, j = J_{r,Q_r} + 3 \text{ to } J + 2Q_r), \bar{Y}_{J_q+1} = \bar{Y}_{J_q} (q = 1 \text{ to } Q_r), \bar{Y}_{J_q+2} = \bar{Y}_{J_q+3} (q = 1 \text{ to } Q_r), \bar{Y}_{J+2Q_r+1} = Y_{r,J+2Q_r}\}$. Note that this simplification was only made possible through the use of the triple knot set \mathbf{Y}_r in expansion (4.14).

Now substituting the triple knot B-spline expansion of $w_{i,k}(y; \tilde{\mathbf{Y}}_r)$, $w_{i,k}(y; \mathbf{Y}_r) = \sum_{j=0}^{J+2Q_r+1} c_{ijk} N_{3,j+1}^{\mathbf{Y}_r}(y)$ and (4.14) into (4.1) gives,

$$\sum_{p=i-1}^{i+1} \tilde{b}_{p,j,k} \int_{\tilde{X}_{r,i-1}}^{\tilde{X}_{r,i}} N_{3,p+1}^{\tilde{\mathbf{X}}_r}(x) dx = c_{i-p,j,k} \quad (p = 0 \text{ to } P_r, i = \tilde{I}_{r,p} + 2 \text{ to } \tilde{I}_{r,p+1}) \quad (4.20)$$

$$(j = 0 \text{ to } J + 2Q_r + 1)$$

where $\tilde{I}_{r,0} = -1$ and $\tilde{I}_{r,P_r+1} = I + P_r$. The natural boundary condition (4.2) yields equations of the form,

$$\sum_{p=0}^2 \tilde{b}_{p,j,k} \frac{d}{dx} N_{3,p+1}^{\tilde{\mathbf{X}}_r}(x_0) = 0 \quad (4.21)$$

$$\sum_{p=I+P_r-1}^{I+P_r+1} \tilde{b}_{p,j,k} \frac{d}{dx} N_{3,p+1}^{\tilde{\mathbf{X}}_r}(x_I) = 0 \quad (4.22)$$

$$(j = 0 \text{ to } J + 2Q_r + 1)$$

Combining conditions (4.18-4.20) and (4.21-4.22) we obtain the sets of tridiagonal systems,

$$T_{j,k}^{(4)} \tilde{\mathbf{b}}_{j,k} = \mathbf{r}_{j,k}^{(4)} \quad (j = 0 \text{ to } J + 2Q_r + 1, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R) \quad (4.23)$$

given in Appendix B.4. Thus for each (j, k_r) we can solve for $\tilde{\mathbf{b}}_{j,k_r}$ to obtain \tilde{b}_{ijk_r} ($i = 0$ to $I + P_r + 1, j = 0$ to $J + 2Q_r + 1, k = k_r$ and $m(r) = 2, 1 \leq r \leq R$). Finally we transform to the triple knot basis $\mathbf{X}_r \otimes \mathbf{Y}_r$ using,

$$\begin{aligned} W_k(x, y; \mathbf{X}_r, \mathbf{Y}_r) &= Q(3, \tilde{\mathbf{X}}_r, \mathbf{X}_r) W_k(x, y; \tilde{\mathbf{X}}_r, \mathbf{Y}_r) \\ &\Rightarrow b_{j,k} = A(3, \tilde{\mathbf{X}}_r, \mathbf{X}_r) \tilde{b}_{j,k} \end{aligned} \quad (4.24)$$

$$(j = 0 \text{ to } J + 2Q_r + 1, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R)$$

See Appendix A.2 system (A.3) for the coefficient operations.

4.1.3 Triple Knot Planes

On the planes $z = z_{k_r}$ ($m(r) = 3, 1 \leq r \leq R$) the splines $w_{i,k}(y)$ should satisfy at the triple knot lines,

$$w_{i,k}(y_{j,r,q}^\pm) = \int_{x_{i-1}}^{x_i} f_r(x, y_{j,r,q}^\pm) \quad (i = 1 \text{ to } I, q = 1 \text{ to } Q_r) \quad (4.25)$$

This time we use a triple knot B-spline expansion of the form,

$$w_{i,k}(y) = \sum_{j=0}^{J+2Q_r+1} c_{ijk} N_{3,j+1}^{\mathbf{Y}_r}(y) \quad (4.26)$$

Substituting (4.26) into (4.25), (4.3) and (4.4) yields sets of tridiagonal systems of the form,

$$T_{i,k}^{(5)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(5)} \quad (i = 1 \text{ to } I, k = k_r \text{ and } m(r) = 3, 1 \leq r \leq R) \quad (4.27)$$

given in Appendix B.5. Thus for each (i, k) we can solve for $\mathbf{c}_{i,k}$ to obtain c_{ijk} ($i = 1$ to $I, j = 0$ to $J + 2Q_r + 1, k = k_r$ and $m(r) = 3, 1 \leq r \leq R$).

To obtain the $W_k(x, y)$ splines we use B-spline expansions of the form,

$$W_k(x, y; \mathbf{X}_r, \mathbf{Y}_r) = \sum_{i=0}^{I+2P_r+1} \sum_{j=0}^{J+2Q_r+1} b_{ijk} N_{3,i+1}^{\mathbf{X}_r}(x) N_{3,j+1}^{\mathbf{Y}_r}(y) \quad (4.28)$$

Using the function $f_r(x, y)$ defined earlier the conditions on $W_k(x, y)$ can now be stated in the form,

$$\begin{aligned} \int_{y_{j-1}}^{y_j} W_k(x, y) dy &= \int_{y_{j-1}}^{y_j} f_r(x, y) dy \quad (j = 1 \text{ to } J) \\ W_k(x, y_{j,r,q}^\pm) &= f_r(x, y_{j,r,q}^\pm) \quad (q = 1 \text{ to } Q_r) \\ \frac{\partial}{\partial y} W_k(x, y_0) &= \frac{\partial}{\partial y} W_k(x, y_J) = 0 \end{aligned} \quad (4.29)$$

$$(x = x_{i_r,p}^-, x_{i_r,p}^+, p = 1 \text{ to } P_r)$$

We see from conditions (4.29) that for each $x = x_{i_r,p}^-, x_{i_r,p}^+$ ($p = 1$ to P_r) there are $J + 2Q_r + 2$ conditions which enables us to determine the coefficient vectors $\mathbf{b}_{I_r,p+1,k}$ and $\mathbf{b}_{I_r,p+2,k}$ independently from the other $\{b_{ijk}\}$ coefficients. To determine the remaining b_{ijk} we substitute (4.28) and (4.26) into (4.1) and make use of the boundary conditions (4.2) to obtain,

$$\sum_{p=i-1}^{i+1} b_{p,j,k} \int_{X_{r,i-1}}^{X_{r,i}} N_{3,p+1}^{\mathbf{X}_r}(x) dx = c_{i-2p,j,k} \quad (p = 0 \text{ to } P_r, i = I_{r,p} + 3 \text{ to } I_{r,p+1}) \quad (4.30)$$

$$\sum_{i=0}^2 b_{ijk} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}_r}(x_0) = 0 \quad (4.31)$$

$$\sum_{i=I+2P_r-1}^{I+2P_r+1} b_{ijk} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}_r}(x_I) = 0 \quad (4.32)$$

$$(j = 0 \text{ to } J + 2Q_r + 1)$$

where $I_{r,0} = -3$ and $I_{r,P_r+1} = I + 2P_r + 1$. If we now insert the $b_{I_p+1,j,k}$ and $b_{I_p+2,j,k}$ into the matrix arising from (4.30-4.32) as known values (that is set the appropriate row of the coefficient matrix to that of the identity matrix) then we have the set of tridiagonal systems,

$$T_{j,k}^{(6)} \mathbf{b}_{j,k} = \mathbf{r}_{j,k}^{(6)} \quad (j = 1 \text{ to } J + 2Q_r + 1, k = k_r \text{ and } m(r) = 3, 1 \leq r \leq R) \quad (4.33)$$

given in Appendix B.6. Thus for each (j, k) we can solve for $\mathbf{b}_{j,k}$ to obtain b_{ijk} ($i = 0$ to $I + 2P_r + 1, j = 0$ to $J + 2Q_r + 1, k = k_r$ and $m(r) = 3, 1 \leq r \leq R$).

4.1.4 Transformation to a Common Basis

Combining (4.7), (4.8), (4.12), (4.23), (4.27), and (4.33) we have shown that all the coefficients for $W_k(x, y)$ ($k = 0$ to K) can be determined through the solution of sets of tridiagonal systems. Finally we transform each $W_k(x, y)$ to the triple knot sets $\mathbf{X} \otimes \mathbf{Y}$ so that each spline is compatible with the B-spline representation given in (3.5). On simple knot representations we apply,

$$W_k(x, y; \mathbf{X}, \mathbf{Y}) = [Q^x(3, \mathbf{x}, \mathbf{X}) \circ Q^y(3, \mathbf{y}, \mathbf{Y})] W_k(x, y; \mathbf{x}, \mathbf{y})$$

$$\Rightarrow \mathbf{b}_{i,k}^{(1)} = A^y(3, \mathbf{y}, \mathbf{Y}) \mathbf{b}_{i,k} \quad (i = 0 \text{ to } I + 1) \quad (4.34)$$

$$\mathbf{b}_{j,k}^{(2)} = A^x(3, \mathbf{x}, \mathbf{X}) \mathbf{b}_{j,k}^{(1)} \quad (j = 0 \text{ to } J + 2Q + 1) \quad (4.35)$$

$$(k = 0 \text{ to } K, k \neq k_r, 0 \leq r \leq R)$$

and on multiple knot planes we apply,

$$\begin{aligned} W_k(x, y; \mathbf{X}, \mathbf{Y}) &= [Q^x(3, \mathbf{X}_r, \mathbf{X}) \circ Q^y(3, \mathbf{Y}_r, \mathbf{Y})] W_k(x, y; \mathbf{X}_r, \mathbf{Y}_r) \\ \Rightarrow \mathbf{b}_{i,k}^{(1)} &= A^y(3, \mathbf{Y}_r, \mathbf{Y}) \mathbf{b}_{i,k} \quad (i = 0 \text{ to } I + 2P_r + 1) \end{aligned} \quad (4.36)$$

$$\begin{aligned} \mathbf{b}_{j,k}^{(2)} &= A^x(3, \mathbf{X}_r, \mathbf{X}) \mathbf{b}_{j,k}^{(1)} \quad (j = 0 \text{ to } J + 2Q + 1) \end{aligned} \quad (4.37)$$

($k = k_r, r = 1 \text{ to } R$)

See Appendix A.1 system (A.1) for the coefficient operations. So now we have the coefficient set $\{b_{ijk}^{(2)}\}$ ($i = 0 \text{ to } I + 2P + 1, j = 0 \text{ to } J + 2Q + 1$) of the expansions,

$$W_k(x, y; \mathbf{X}, \mathbf{Y}) = \sum_{i=0}^{I+2P+1} \sum_{j=0}^{J+2Q+1} b_{ij}^{(2)} N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) \quad (k = 0 \text{ to } K) \quad (4.38)$$

4.2 Tensor Product Construction of $w(\mathbf{x}, \mathbf{y}, \mathbf{z})$

We seek a spline representation satisfying $w(x, y, z_k) = W_k(x, y; \mathbf{X}, \mathbf{Y})$ ($k = 0 \text{ to } K$) and conditions (3.6-3.9), (3.10-3.13), (3.14-3.17) and (3.31-3.33). Substitution of expansion (3.5) into these conditions would yield a global set of linear equations for the unknowns $\{a_{i,j,k}\}$. However we would like to formulate the equations so that the $\{a_{i,j,k}\}$ can be determined through the solution of sets of tridiagonal systems in a similar manner to that described for the determination of the $\{b_{i,j,k}\}$ in section 3.1.

We begin by taking a closer look at the connectivity conditions at the quadruple knot interface planes given by (3.14-3.17). Using expansion (3.5) we can obtain the following representations of $\frac{\partial^2}{\partial z^2} w(x, y, z)$ at either side of the quadruple knot $z = z_{k_r+1}$,

$$\begin{aligned} \frac{\partial^2}{\partial z^2} w(x, y, z_{k_r+1}^-) &= \sum_{i=0}^{I+2P+1} \sum_{j=0}^{J+2Q+1} a_{i,j}^M(z_{k_r+1}^-) N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) \\ a_{i,j}^M(z) &= \sum_{k=K(2r)-1}^{K(2r)+1} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z) \\ \frac{\partial^2}{\partial z^2} w(x, y, z_{k_r+1}^+) &= \sum_{i=0}^{I+2P+1} \sum_{j=0}^{J+2Q+1} a_{i,j}^S(z_{k_r+1}^+) N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) \\ a_{i,j}^S(z) &= \sum_{k=K(2r)+2}^{K(2r)+4} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z) \end{aligned}$$

So the right interface condition (3.15) requires that,

$$\begin{aligned} [R^x(3, \mathbf{x}, \mathbf{X}) \circ R^y(3, \mathbf{y}, \mathbf{Y})] \left\{ \frac{\partial^2}{\partial z^2} w(x, y, z_{k_r+1}^-) - \frac{\partial^2}{\partial z^2} w(x, y, z_{k_r+1}^+) \right\} &= 0 \\ \Rightarrow B^x(3, \mathbf{x}, \mathbf{X}) \{A^M(z_{k_r+1}^-) - A^S(z_{k_r+1}^+)\} [B^y(3, \mathbf{y}, \mathbf{Y})]^T &= 0 \end{aligned}$$

where $[A^M(z)]_{ij} = a_{ij}^M(z)$ and $[A^S(z)]_{ij} = a_{ij}^S(z)$. Thus we must have,

$$a_{ij}^M - a_{ij}^S = 0 \quad \forall (i, j) \in \bar{I} \otimes \bar{J} \quad (4.39)$$

where we have defined $\bar{I} = \{i \in \mathcal{Z} | i \in [0, I_1] \cup \{\cup_{p=2}^{P-1} [I_p+3, I_{p+1}]\} \cup [I_P+3, I+2P+1]\}$ and $\bar{J} = \{j \in \mathcal{Z} | j \in [0, J_1] \cup \{\cup_{q=2}^{Q-1} [J_q+3, J_{q+1}]\} \cup [J_Q+3, J+2Q+1]\}$. The reason we have restricted the (i, j) range to $\bar{I} \otimes \bar{J}$ becomes clear after examining the coefficient operations given in Appendix A.1 system (A.2); there are no dependencies on the coefficients with indices $i = I_p + 1, I_p + 2$ and $j = J_q + 1, j = J_q + 1$. If we apply a similar procedure to the left interface $z = z_{k_r-1}$ and substitute the definitions of a_{ij}^M and a_{ij}^S into (4.39) then we obtain the conditions,

$$\sum_{k=K(2r)-1}^{K(2r)+1} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z_{k_r+1}^-) - \sum_{k=K(2r)+2}^{K(2r)+4} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z_{k_r+1}^+) = 0 \quad (4.40)$$

$$\sum_{k=K(2r-1)+2}^{K(2r-1)+4} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z_{k_r-1}^+) - \sum_{k=K(2r-1)-1}^{K(2r-1)+1} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{Z}}(z_{k_r-1}^-) = 0 \quad (4.41)$$

$$\forall (i, j) \in \bar{I} \otimes \bar{J} \\ (r = 1 \text{ to } R)$$

Now re-write expansion (3.5) in the form,

$$w(x, y, z) = \sum_{i=0}^{I+2P+1} \sum_{j=0}^{J+2Q+1} \tilde{w}_{ij}(z) N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) \quad (4.42)$$

$$\tilde{w}_{ij}(z) = \sum_{k=-1}^{K+6R+1} a_{ijk} N_{4,k+2}^{\mathbf{Z}}(z) \quad (4.43)$$

Conditions (3.14) and (3.16) require that $[\frac{d^v}{dz^v} \tilde{w}_{i,j}(z)]_{Z_{K_r}^+} = 0 \quad \forall i, j$ ($v = 0$ to $1, r = 1$ to $2R$). However conditions (4.40-4.41) and the fact that $z = z_k$ ($k = 0$ to $K, k \neq k_r \pm 1, 1 \leq r \leq R$) are simple knots enables us to write,

$$\left[\frac{d^v}{dz^v} \tilde{w}_{i,j}(z) \right]_{z_k^-}^{z_k^+} = 0 \quad (v = 0 \text{ to } 2, k = 0 \text{ to } K)$$

$$\forall (i, j) \in \bar{I} \otimes \bar{J}$$

Thus the cubic splines $\tilde{w}_{i,j}(z) \in S_4(\mathbf{z}) \quad \forall (i, j) \in \bar{I} \otimes \bar{J}$.

4.2.1 One Dimensional Connecting Splines $\tilde{w}_{ij}(z)$

We showed in the previous section that the splines $\tilde{w}_{ij}(z; \mathbf{Z})$ ($i, j \in \bar{I} \otimes \bar{J}$) could be equivalently represented using a simple knot spline space. Substituting expansions (4.42) and (4.38) into (3.18) and imposing the natural boundary conditions (3.33) gives,

$$\tilde{w}_{ij}(z_k) = b_{ijk}^{(2)} \quad (k = 0 \text{ to } K) \quad (4.44)$$

$$\frac{\partial^2}{\partial z^2} \tilde{w}_{ij}(z_0) = \frac{\partial^2}{\partial z^2} \tilde{w}_{ij}(z_K) = 0 \quad (4.45)$$

$$\forall (i, j) \in \bar{I} \otimes \bar{J}$$

To determine the $\tilde{w}_{ij}(z)$ splines we use the simple knot B-spline expansions,

$$\tilde{w}_{i,j}(z; \mathbf{z}) = \sum_{k=-1}^{K+1} a_{ijk}^{(1)} N_{4,k+2}^{\mathbf{z}}(z) \quad (4.46)$$

If we substitute expansion (4.46) into (4.44) and (4.45) then we obtain the set of tridiagonal systems (given in Appendix B.7),

$$T_{i,j}^{(7)} \mathbf{a}_{ij}^{(1)} = \mathbf{r}_{ij}^{(7)} \quad \forall (i, j) \in \bar{I} \otimes \bar{J} \quad (4.47)$$

which we can solve for the vectors $\mathbf{a}_{ij}^{(1)}$ ($i, j \in \bar{I} \otimes \bar{J}$). To obtain the $\{a_{ijk}\}$ ($i, j \in \bar{I} \otimes \bar{J}$) of expansion (4.43) and thus (3.5) we transform each $\tilde{w}_{i,j}(z; \mathbf{z})$ spline to the quadruple knot basis \mathbf{Z} to get,

$$\begin{aligned} \tilde{w}_{i,j}(z; \mathbf{Z}) &= Q^z(4, \mathbf{z}, \mathbf{Z}) \tilde{w}_{i,j}(z; \mathbf{z}) \\ \Rightarrow \mathbf{a}_{ij} &= A^z(4, \mathbf{z}, \mathbf{Z}) \mathbf{a}_{ij}^{(1)} \end{aligned} \quad (4.48)$$

$$\forall (i, j) \in \bar{I} \otimes \bar{J}$$

See Appendix A.3 system (A.5) for the coefficient operations. Thus we have shown how to efficiently determine $\{a_{ijk}\}$ ($i, j \in \bar{I} \otimes \bar{J}, (k = 0 \text{ to } K + 6R + 1)$). To calculate the remaining $\{a_{ijk}\}$ requires some more work.

4.2.2 Tangential Continuity in Simple Knot Regions

To determine the $\{a_{ijk}\}$ ($i = I_p + 1, I_{p+2}, p = 1 \text{ to } P, j = J_q + 1, J_q + 2, q = 1 \text{ to } Q, k = K(2r) + 2 \text{ to } K(2r + 1) + 1, r = 0 \text{ to } R$) associated with the splines $w'_r(x, y, z)$ ($r = 0 \text{ to } R$) we make use of the tangential derivative continuity conditions (3.10 - 3.13). Substituting expansion (3.5) into conditions (3.10) and (3.12) gives,

$$\begin{aligned} a_{I_p+1,j,k} N_{3,I_p+2}^{\mathbf{X}}(x_{i_p}^-) &- a_{I_p+2,j,k} N_{3,I_p+3}^{\mathbf{X}}(x_{i_p}^+) = 0 \\ a_{I_p+1,j,k} \frac{d}{dx} N_{3,I_p+2}^{\mathbf{X}}(x_{i_p}^-) &- a_{I_p+2,j,k} \frac{d}{dx} N_{3,I_p+3}^{\mathbf{X}}(x_{i_p}^+) = \\ a_{I_p+3,j,k} \frac{d}{dx} N_{3,I_p+4}^{\mathbf{X}}(x_{i_p}^+) &- a_{I_p,j,k} \frac{d}{dx} N_{3,I_p+1}^{\mathbf{X}}(x_{i_p}^-) \end{aligned} \quad (4.49)$$

($p = 1$ to $P, j = 0$ to $J + 2Q + 1, k = K(2r) + 2$ to $K(2r + 1) + 1, r = 0$ to R)

Similarly conditions (3.11) and (3.13) yield,

$$\begin{aligned}
& a_{i,J_q+1,k} N_{3,J_q+2}^{\mathbf{Y}}(y_{j_q}^-) - a_{i,J_q+2,k} N_{3,J_q+3}^{\mathbf{Y}}(y_{j_q}^+) = 0 \\
& a_{i,J_q+1,k} \frac{d}{dy} N_{3,J_q+2}^{\mathbf{Y}}(y_{j_q}^-) - a_{i,J_q+2,k} \frac{d}{dy} N_{3,J_q+3}^{\mathbf{Y}}(y_{j_q}^+) = \\
& a_{i,J_q+3,k} \frac{d}{dy} N_{3,J_q+4}^{\mathbf{Y}}(y_{j_q}^+) - a_{i,J_q,k} \frac{d}{dy} N_{3,J_q+1}^{\mathbf{Y}}(y_{j_q}^-) \quad (4.50)
\end{aligned}$$

($q = 1$ to $Q, \forall i \in \bar{I}, k = K(2r) + 2$ to $K(2r + 1) + 1, r = 0$ to R)

System (4.49) is valid for ($j = 0$ to $J + 2Q + 1$) and system (4.50) is valid for $i \in \bar{I}$. But to fully exploit the fact that the $\{a_{ijk}\} \forall (i, j) \in \bar{I} \otimes \bar{J}$ ($k = 0$ to $K + 6R + 1$) are already known, we do the following:

- 1 Solve system (4.49) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to $P, \forall j \in \bar{J}$).
- 2 Solve system (4.50) for $a_{i,J_q+1,k}$ and $a_{i,J_q+2,k}$ ($q = 1$ to $Q, \forall i \in \bar{I}$).
- 3 Solve system (4.49) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to $P, j = J_q + 1, J_q + 2, q = 1$ to Q).

($k = K(2r) + 2$ to $K(2r + 1) + 1, r = 0$ to R)

Provided we loop through the (p, j, k) and (q, i, k) in the order stated in steps 1,2 and 3 then the right hand side terms of systems (4.49) and (4.50) are always known so we can determine all the remaining coefficients to fully construct the $w'(x, y, z)$ splines by solving sets of 2×2 systems.

4.2.3 Matching Conditions in Multiple Knot Regions

The remaining coefficients $\{a_{ijk}\}$ ($i = I_p+1, I_p+2, p = 1$ to $P, j = J_q+1, J_q+2, q = 1$ to $Q, k = K(2r - 1) + 2$ to $K(2r) + 1, r = 1$ to R) to completely construct the multiple knot splines $w_r(x, y, z)$ ($r = 1$ to R) are calculated by imposing conditions (3.6-3.9), using the known $\{a_{ijk}\}$ values and making use of the surface interpolants $W_k(x, y; \mathbf{X}, \mathbf{Y})$. Substituting expansion (3.5) into conditions (3.6-3.9) with $v = 0$ yields systems identical to those stated in (4.49) and (4.50) but this time we loop over the k values corresponding to the outermost interface planes within each triple knot region, that is ($k = K(2r - 1) + 2, K(2r) + 1, r = 1$ to R). By substituting (3.5) into conditions (3.6-3.9) with $v = 1$ we obtain the more complex systems,

$$\begin{aligned}
& a_{I_p+1,j,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) N_{3,I_p+2}^{\mathbf{X}}(x_{i_p}^-) - a_{I_p+2,j,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) N_{3,I_p+3}^{\mathbf{X}}(x_{i_p}^+) = \\
& a_{I_p+2,j,k+\alpha} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) N_{3,I_p+3}^{\mathbf{X}}(x_{i_p}^+) - a_{I_p+1,j,k+\alpha} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) N_{3,I_p+2}^{\mathbf{X}}(x_{i_p}^-)
\end{aligned}$$

$$\begin{aligned}
& a_{I_p+1,j,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) \frac{d}{dx} N_{3,I_p+2}^{\mathbf{X}}(x_{i_p}^-) - a_{I_p+2,j,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) \frac{d}{dx} N_{3,I_p+3}^{\mathbf{X}}(x_{i_p}^+) = \\
& \left\{ a_{I_p+3,j,k} \frac{d}{dx} N_{3,I_p+4}^{\mathbf{X}}(x_{i_p}^-) - a_{I_p,j,k} \frac{d}{dx} N_{3,I_p+1}^{\mathbf{X}}(x_{i_p}^+) \right\} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) + \\
& \left\{ \sum_{i=i_p+2}^{i_p+3} a_{i,j,k+\alpha} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}}(x_{i_p}^+) - \right. \\
& \left. \sum_{i=i_p}^{i_p+1} a_{i,j,k+\alpha} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}}(x_{i_p}^-) \right\} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) \tag{4.51}
\end{aligned}$$

$$\begin{aligned}
& (p = 1 \text{ to } P, j = 0 \text{ to } J + 2Q + 1) \\
& (k = K(2r - 1) + 3 \text{ and } \alpha = -1 \text{ and } z = z_{k_r-1}^+) \\
& (k = K(2r) \text{ and } \alpha = +1 \text{ and } z = z_{k_r+1}^-)
\end{aligned}$$

$$\begin{aligned}
& a_{i,J_q+1,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) N_{3,J_q+2}^{\mathbf{Y}}(y_{j_q}^-) - a_{i,J_q+2,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) N_{3,J_q+3}^{\mathbf{Y}}(y_{j_q}^+) = \\
& a_{i,J_q+2,k+\alpha} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) N_{3,J_q+3}^{\mathbf{Y}}(y_{j_q}^+) - a_{i,J_q+1,k+\alpha} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) N_{3,J_q+2}^{\mathbf{Y}}(y_{j_q}^-)
\end{aligned}$$

$$\begin{aligned}
& a_{i,J_q+1,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) \frac{d}{dy} N_{3,J_q+2}^{\mathbf{Y}}(y_{j_q}^-) - a_{i,J_q+2,k} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) \frac{d}{dy} N_{3,J_q+3}^{\mathbf{Y}}(y_{j_q}^+) = \\
& \left\{ a_{i,J_q+3,k} \frac{d}{dy} N_{3,J_q+4}^{\mathbf{Y}}(y_{j_q}^-) - a_{i,J_q,k} \frac{d}{dy} N_{3,J_q+1}^{\mathbf{Y}}(y_{j_q}^+) \right\} \frac{d}{dz} N_{4,k+2}^{\mathbf{Z}}(z) + \\
& \left\{ \sum_{j=J_q+2}^{J_q+3} a_{i,j,k+\alpha} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}}(y_{j_q}^+) - \right. \\
& \left. \sum_{j=J_q}^{J_q+1} a_{i,k+\alpha} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}}(y_{j_q}^-) \right\} \frac{d}{dz} N_{4,k+\alpha+2}^{\mathbf{Z}}(z) \tag{4.52}
\end{aligned}$$

$$\begin{aligned}
& (q = 1 \text{ to } Q, \forall i \in \bar{I}) \\
& (k = K(2r - 1) + 3 \text{ and } \alpha = -1 \text{ and } z = z_{k_r-1}^+) \\
& (k = K(2r) \text{ and } \alpha = +1 \text{ and } z = z_{k_r+1}^-)
\end{aligned}$$

required to obtain tangential continuity of $\frac{\partial}{\partial z} w_r(x, y, z)$ ($v = 0$ to 1) on the interface planes $z = z_{k_r+1}^-$ and $z = z_{k_r-1}^+$. We can solve systems (4.49-4.52) very efficiently in a similar manner to that used for the simple knot splines in section 3.2.2. We do the following:

- 1 Solve system (4.49) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to P , $\forall j \in \bar{J}$, $k = K(2r - 1) + 1$ and $k = K(2r) + 1$)
- 2 Solve system (4.50) for $a_{i,J_q+1,k}$ and $a_{i,J_q+2,k}$ ($q = 1$ to Q , $\forall i \in \bar{I}$, $k = K(2r - 1) + 1$ and $k = K(2r) + 1$)

- 3 Solve system (4.49) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to $P, j = J_q + 1, J_q + 2, q = 1$ to $Q, k = K(2r - 1) + 1$ and $k = K(2r) + 1$)
- 4 Solve system (4.51) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to $P, \forall j \in \bar{J}, k = K(2r - 1) + 3$ and $\alpha = -1$ and $z = z_{k_r-1}^+, k = K(2r)$ and $\alpha = +1$ and $z = z_{k_r+1}^-$)
- 5 Solve system (4.52) for $a_{i,J_q+1,k}$ and $a_{i,J_q+2,k}$ ($q = 1$ to $Q, \forall i \in \bar{I}, k = K(2r - 1) + 3$ and $\alpha = -1$ and $z = z_{k_r-1}^+, k = K(2r)$ and $\alpha = +1$ and $z = z_{k_r+1}^-$)
- 6 Solve system (4.51) for $a_{I_p+1,j,k}$ and $a_{I_p+2,j,k}$ ($p = 1$ to $P, j = J_q + 1, J_q + 2, q = 1$ to $Q, k = K(2r - 1) + 3$ and $\alpha = -1$ and $z = z_{k_r-1}^+, k = K(2r)$ and $\alpha = +1$ and $z = z_{k_r+1}^-$)

Steps (1-6) ensure that the indices (i, j, p, q, k) are looped through in such a way that when each of the 2×2 systems given by (4.49-4.52) is encountered all the coefficients in the right hand side terms have already been computed. Thus, again the solution procedure is highly efficient.

Each multiple knot region consists of five (x, y) planes $z = z_k$ ($k = K(2r - 1) + 2$ to $K(2r) + 1$). We have now shown how to compute the coefficients $\{a_{ijk}\}$ ($i = 0$ to $I + 2P + 1, j = 0$ to $J + 2Q + 1$) on the four planes $k = K(2r - 1) + 2, K(2r - 1) + 3, K(2r)$ and $K(2r) + 1$ so it only remains to compute the coefficient for the plane $k = K(2r - 1) + 4$ which is the interpolation plane $z = z_{k_r}$ across which the spline $W_{k_r}(x, y; \mathbf{x}, \mathbf{Y})$ is defined. With this in mind, we substitute expansions (3.5) and (4.38) into condition (3.18) to obtain,

$$\begin{aligned}
a_{i,j,K(2r-1)+4} N_{4,K(2r-1)+6}^{\mathbf{Z}}(z_{k_r}) &= b_{i,j,k_r}^{(2)} - a_{i,j,K(2r-1)+3} N_{4,K(2r-1)+5}^{\mathbf{Z}}(z_{k_r}) \\
&\quad - a_{i,j,K(2r)} N_{4,K(2r)+2}^{\mathbf{Z}}(z_{k_r}) \quad (4.53)
\end{aligned}$$

$$\begin{aligned}
(i = I_p + 1, I_p + 2, p = 1 \text{ to } P, j = 0 \text{ to } J + 2Q + 1) \\
(j = J_q + 1, J_q + 2, q = 1 \text{ to } Q, \forall i \in \bar{I})
\end{aligned}$$

All the coefficients on the right hand side of equation (4.53) have already been computed so we can use (4.53) explicitly to determine the remaining $\{a_{i,j,k}\}$ on the plane $k = K(2r - 1) + 4$.

4.3 Reverse Transformations for a Non-Tensor Representation

We have shown how to compute all the coefficients $\{a_{ijk}\}$ ($i = 0$ to $I + 2P + 1, j = 0$ to $J + 2Q + 1, k = 0$ to $K + 6R + 1$) of the tensor product representation of $w(x, y, z)$ given by expansion (3.5). However this representation is inefficient in terms of computational storage since we are using multiple knot B-spline

expansions in regions where the representation is in fact continuous and thus simple knot B-spline function could be equivalently used. We now state the transformations required to obtain the non-tensor product representation given by (3.2) and (3.3) from representation (3.5). In simple knot regions we apply,

$$\begin{aligned} W_k(x, y; \mathbf{x}, \mathbf{y}) &= [Q^x(3, \mathbf{x}, \mathbf{X}) \circ Q^y(3, \mathbf{y}, \mathbf{Y})]^{-1} W_k(x, y; \mathbf{X}, \mathbf{Y}) \\ \Rightarrow \mathbf{a}_{i,k}^{(1)} &= A^y(3, \mathbf{Y}, \mathbf{y}) \mathbf{a}_{i,k} \quad (i = 0 \text{ to } I + 2P + 1) \end{aligned} \quad (4.54)$$

$$\tilde{\mathbf{a}}_{j,k} = A^x(3, \mathbf{X}, \mathbf{x}) \mathbf{a}_{j,k}^{(1)} \quad (j = 0 \text{ to } J + 1) \quad (4.55)$$

$$(k = K(2r) + 2 \text{ to } K(2r + 1) + 1, r = 0 \text{ to } R)$$

and in multiple knot regions we apply,

$$\begin{aligned} W_k(x, y; \mathbf{X}_r, \mathbf{Y}_r) &= [Q^x(3, \mathbf{X}_r, \mathbf{X}) \circ Q^y(3, \mathbf{Y}_r, \mathbf{Y})]^{-1} W_k(x, y; \mathbf{X}, \mathbf{Y}) \\ \Rightarrow \mathbf{a}_{i,k}^{(1)} &= A^y(3, \mathbf{Y}, \mathbf{Y}_r) \mathbf{a}_{i,k} \quad (i = 0 \text{ to } I + 2P + 1) \end{aligned} \quad (4.56)$$

$$\tilde{\mathbf{a}}_{j,k} = A^x(3, \mathbf{X}, \mathbf{X}_r) \mathbf{a}_{j,k}^{(1)} \quad (j = 0 \text{ to } J + 2Q_r + 1) \quad (4.57)$$

$$(k = K(2r - 1) + 2 \text{ to } K(2r) + 1, r = 1 \text{ to } R)$$

See appendix A.1 system (A.2) for the coefficient operations. Thus, we have shown how to compute all the coefficients $\{\tilde{\mathbf{a}}_{ijk}\}$ of expansions (3.2) and (3.3) for the simple knot representations $w'_r(x, y, z)$ ($r = 0$ to R) and multiple knot representations $w_r(x, y, z)$ ($r = 1$ to R).

5 An Efficient Evaluation Scheme

Each of the splines $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ can be represented as a sum of simple and multiple knot tensor-product B-spline expansions. The expansions are defined in corresponding simple and multiple knot regions which partition Ω into a set of rectilinear sub-divisions which are different for each velocity spline. In (3.2) and (3.3) we stated these expansions for $w(x, y, z)$ where we denoted the simple knot splines by $w'_r(x, y, z)$ defined in the simple knot regions Ω'_r ($r = 1$ to R) and the multiple knot splines $w_r(x, y, z)$ defined in the multiple knot regions Ω_r ($r = 1$ to R). Using the same notation, we will now show how this property can be exploited to efficiently evaluate $w(x, y, z)$ at any point $(x', y', z') \in \Omega$.

In simple knot regions where $(x', y', z') \in \{\cup_{r=0}^R \Omega'_r\}$ we find $(i', j') \in \{0, 1, \dots, I\} \otimes \{0, 1, \dots, J\}$ such that $x_{i'-1} \leq x \leq x_{i'}$ and $y_{j'-1} \leq y \leq y_{j'}$. In multiple knot regions where $(x', y', z') \in \{\cup_{r=1}^R \Omega_r\}$ we find $(i', j') \in \bar{I}_r \otimes \bar{J}_r$ where $\bar{I}_r = \{i \in \mathcal{Z} | i \in [1, I_{r,1}] \cup \{\cup_{p=2}^{P_r-1} [I_{r,p} + 3, I_{r,p+1}]\} \cup [I_{r,p} + 3, I + 2P_r + 1]\}$ and $\bar{J}_r = \{j \in \mathcal{Z} | j \in [1, J_{r,1}] \cup \{\cup_{q=2}^{Q_r-1} [J_{r,q} + 3, J_{r,q+1}]\} \cup [J_{r,q} + 3, J + 2Q_r + 1]\}$ such that

$X_{r,i'-1} \leq x' \leq X_{r,i'}$ and $Y_{r,j'-1} \leq y' \leq Y_{r,j'}$. Using the minimal support property of B-splines and writing $\Omega' = \cup_{r=0}^R \Omega'_r$ we can combine (3.2) and (3.3) to get,

$$w(x, y, z) = \begin{cases} \sum_{k=K(2r)+2}^{K(2r+1)+1} \sum_{i=i'}^{i'+3} \sum_{j=j'}^{j'+3} \tilde{a}_{ijk} N_{3,i+1}^{\mathbf{X}}(x) N_{3,j+1}^{\mathbf{Y}}(y) N_{4,k+1}^{\mathbf{Z}}(z) & (x, y, z) \in \Omega' \\ \sum_{k=K(2r)+2}^{K(2r+1)+1} \sum_{i=i'}^{i'+3} \sum_{j=j'}^{j'+3} \tilde{a}_{ijk} N_{3,i+1}^{\mathbf{X}_r}(x) N_{3,j+1}^{\mathbf{Y}_r}(y) N_{4,k+1}^{\mathbf{Z}}(z) & (x, y, z) \in \Omega_r \end{cases} \quad (5.1)$$

Now define the splines $b_{j,k}(x)$ and $c_k(y, z)$ so that,

$$b_{j,k}(x) = \begin{cases} \sum_{i=i'}^{i'+2} \tilde{a}_{i-1,j-1,k-2} N_{3,i}^{\mathbf{X}}(x) & (x, y, z) \in \Omega' \\ \sum_{i=i'}^{i'+2} \tilde{a}_{i-1,j-1,k-2} N_{3,i}^{\mathbf{X}_r}(x) & (x, y, z) \in \Omega_r \end{cases} \quad (5.2)$$

$$(j = j' \text{ to } j' + 2, k = k' \text{ to } k' + 3)$$

$$c_k(y; x') = \begin{cases} \sum_{j=j'}^{j'+2} b_{j,k}(x') N_{3,j}^{\mathbf{Y}}(y) & (x, y, z) \in \Omega' \\ \sum_{j=j'}^{j'+2} b_{j,k}(x') N_{3,j}^{\mathbf{Y}_r}(y) & (x, y, z) \in \Omega_r \end{cases} \quad (5.3)$$

$$(k = k' \text{ to } k' + 3)$$

So to compute $w(x', y', z')$ we begin by evaluating the quadratic x splines $b_{j,k}(x)$ ($j = j' \text{ to } j' + 2, k = k' \text{ to } k' + 3$) at the point $x = x'$. Then we evaluate the quadratic y splines $c_k(y; x')$ ($k = k' \text{ to } k' + 3$) at the point $y = y'$. Finally, we find $k' \in \bar{K}$ where $\bar{K} = \{k \in \mathcal{Z} | k \in [1, K(1)] \cup \{\cup_{r=2}^{2R-1} [K(r) + 4, K(r+1)]\} \cup [K(2R) + 4, K + 6R]\}$ such that $Z_{k'-1} \leq z \leq Z_{k'}$. This enables us to write,

$$w(z; x', y') = \sum_{k=k'}^{k'+3} c_k(x', y') N_{4,k}^{\mathbf{Z}}(z) \quad (5.4)$$

which we can evaluate in the z direction at the point $z = z'$. Each of the splines $b_{j,k}(x)$, $c_k(y)$ and $w(x)$ can be efficiently evaluated using a stable algorithm attributable to Cox [6]. To evaluate a B-spline representation of the form $s(x) = \sum_i c_i N_{m,i}(x)$ the Cox algorithm requires $\frac{1}{2}(3m^2 - 3m)$ operations (multiplications and divisions). The standard approach of evaluating each non-zero basis function separately, multiplying it by the appropriate coefficient and finally computing the sum requires $\frac{1}{2}(3m^2 + 3m - 4)$ operations [15]. If we use the Cox algorithm to evaluate the splines given in (5.2), (5.3) and (5.4) then a total of 162 operations are required to compute $w(x', y', z')$ which is the same amount as for the simple knot representation given in [9].

6 Results and Conclusions

6.1 A Practical Problem

Figure (6) illustrates a simplified mixing tank used by Courtauld in their fibre production processes. The flux data $\{U_{ijk}\}$, $\{V_{ijk}\}$ and $\{W_{ijk}\}$ was obtained on the tensor-product mesh illustrated in figure (7) using the Phoenics fluid dynamics package. A plug flow boundary condition of 2.37×10^{-5} was imposed at the inlet. These computed values satisfy the divergence-free condition (1.1) to a relative tolerance of 10^{-4} . Figure (8) shows the planes on which we have introduced multiple knots to ensure that the normal flow is zero on the correct surfaces. On the inlet plane we have chosen to use triple knots to model the normal boundary condition exactly, but on the other planes double knots were sufficient. Figures (9a-f) show the spline representations of the normal component of velocity across a selection of the multiple knot planes. We see in each case that the flow is identically zero over all the required regions. In figures (10a-f) we show the velocity representations generated using the simple knot spline scheme described in [9] for the same planes. The flux data on which the multiple and simple knot representations were based for these planes is illustrated in figures (11a-f). We see from figure (9b) and (10b) that the spline representation of the u component across the inlet plane is particularly poor using the simple knot representation, but that the multiple knot representation has captured the boundary condition exactly. Comparing figures (9a,c-f) and (10a,c-f) it is clear that the introduction of double knots to force the flow to be zero across the internal surfaces $y = 1.23, y = 3.65, z = 10.27$ and $z = 10.87$ has had a much less dramatic effect on the representation across the regions over which the normal flow should be non-zero. In these cases the overall shape of the data appears to have been preserved satisfactorily by the multiple knot representation.

Figures (12a-h) show the u component as computed by the multiple knot representation across 8 (y, z) planes equally spaced through the region $x \in [3.329, 4.11]$. The (y, z) plane $x = 3.329$ is significant in the sense that it represents the left interface of the multiple knot region $\Omega_2^u = \{(x, y, z) | x \in [3.329, 4.891], 0 \leq y \leq y_J, 0 \leq z \leq z_K\}$ used to fix the boundary condition at $x = 4.11$ with the neighbouring simple knot region $\Omega_1^{u'} = \{x \in [0.0412, 3.329], 0 \leq y \leq y_J, 0 \leq z \leq z_K\}$. We see from figure (12) that the discontinuities introduced by the triple knots at the lines $y = 0.8$ and $z = 0.8$ have been propagated throughout $\Omega_2^u = \{(x, y, z) | x \in [3.329, 4.891], 0 \leq y \leq y_J, 0 \leq z \leq z_K\}$ and only disappear at the interface $x = 3.329$. This is the penalty we pay for introducing triple knots. The left boundary $x = 3.329$ of $\Omega_1^{u'}$ cannot be brought any closer to the plane $x = 4.11$ since it is determined by the x -lines of the tensor-product mesh illustrated in figure (7). The multiple knot regions constructed around the planes $y = 1.23, 3.65$ and $z = 8.64, 9.24, 10.27, 10.87$ contain lines of double knots and thus the representations of the v and w components in these regions

Mesh Size	Number of Coefficients			Elapsed Time(s)
	u - spline	v - spline	w - spline	
$10 \times 15 \times 51$	23965	19560	16392	1.593
$20 \times 29 \times 98$	111160	97680	86130	6.147

Table 1: Elapsed time required on the SGI Power Challenge to compute the coefficients of the multiple knot spline representations for the u , v and w components

are continuous, but their tangential derivatives are discontinuous.

The connectivity conditions, stated in (3.14-3.17) for the w component, require the spline representations to be zeroth and first derivative continuous across planes interfacing simple and multiple knot regions. Figures (13a-f) show the spline representations of $u(x, y, z)$, $\frac{\partial}{\partial x}u$ and $\frac{\partial^2}{\partial x^2}u$ at either side of the interface plane $x = 3.329$. We see that u and $\frac{\partial}{\partial x}u$ are both identical at $x = 3.329^-, 3.329^+$ but that $\frac{\partial^2}{\partial x^2}u(3.329^-, y, z)$ and $\frac{\partial^2}{\partial x^2}u(3.329^+, y, z)$ differ. This is due to it only being possible to obtain a second derivative continuity condition on the projection of $u(x, y, 3.329^+)$ into a simple knot spline space. On the simple knot side of the interface the derivatives $\frac{\partial^v}{\partial x^v}u(3.329^-, y, z)$ ($v = 0$ to 2) are all continuous in the y and z directions, which is a condition that we must have to ensure that $u(x, y, z)$ remains continuous in y and z throughout $\Omega^{u'}$. Conversely, it is equally important that $\frac{\partial^2}{\partial x^2}u(3.329^+, y, z)$ be discontinuous across the lines $y = 0.8, z = 0.8$. The discontinuities in $\frac{\partial^2}{\partial x^2}u(3.329^+, y, z)$ generate discontinuities in $\frac{\partial}{\partial x}u(x, y, z)$ $x \in (3.329^+, 4.891]$, which in turn generate the discontinuities in $u(x, y, z)$ $x \in (3.329^+, 4.891]$ (which we see at sampled x values in figures (12a-h)) necessary to obtain the boundary condition at $x = 4.11$.

Table (1) shows the total amount of computing time required to calculate the coefficients for the proposed multiple knot spline velocity representation. The figures show the total elapsed time in seconds required by one processor of the SGI Power Challenge machine, which has a peak performance of 300Mflops. The calculations were performed for the tank illustrated in figure (6), but we have used two tensor product mesh discretisations: medium ($10 \times 15 \times 51$) and large ($20 \times 29 \times 98$). The medium sized mesh, illustrated in figure (7) is the one on which figures (9-13) were based. Clearly for this particular machine the method is extremely fast, requiring only 6.147s to determine all 294970 coefficients required for the tensor spline representations of u , v and w on the large mesh.

6.2 Conclusion

We have constructed a multiple knot spline representation which interpolates a given a set of face-centered fluid fluxes U_{ijk} , V_{ijk} and W_{ijk} exactly in accordance

with conditions (1.2-1.4) and enables the velocity components to be set identically to zero across sets of rectangular regions lying on planes normal to that components direction by relaxing certain continuity conditions connecting the piecewise polynomials. Because the spline representations $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ are constructed independently, the local arrangement of obstacles or apertures across these special planes can be of almost any complexity and there is no risk of, for example, conditions imposed on w interfering with the u and v splines as there was in the case of [9] where the components were coupled together by local correcting functions.

Although the resulting spline representation as given for the w component in (3.2) and (3.3) is not strictly speaking a tensor-product, by making use of an equivalent tensor representation we have developed an efficient scheme for the determination of the B-spline coefficients by making use of B-spline basis transformations. We showed in section 4 that because each velocity spline always coincides with a tensor-spline representation we can fully exploit the computational benefits associated with tensor-splines in the evaluation phases. This is particularly important for particle tracking applications where accurate integration to strict error tolerances may require the velocity representation to be evaluated at multiple points within each mesh cell [7, 8].

A drawback with the proposed spline method is that, unlike the simple knot approach described in [9], the final velocity representation is no longer divergence-free in a pointwise sense, that is $u_x + v_y + z_z \neq 0 \quad \forall(x, y, z) \in \Omega$. If we do not use any multiple knots then the resulting representation reduces identically to that generated by using [9] and thus in this case the representation is divergence-free. For a flow situation such as the one considered in section 5.1 for the tank illustrated in figure (6) we find that the divergence is only non-zero in the regions around which multiple knots have been introduced, and for this case the regions are relatively small. To determine these regions, however, we must consider the multiple knot regions relative to each velocity component, that is Ω_r^u ($r = 1$ to R^u) for the u component Ω_r^v ($r = 1$ to R^v) for the v component and Ω_r^w ($r = 1$ to R^w) for the w component. Thus, when we consider the divergence spline $u_x + v_y + z_z$ we are not just interested in the multiple knot lines but more importantly we would like to know what is happening in the regions where Ω_r^u , Ω_r^v and Ω_r^w overlap and at the interfaces between these regions. Such considerations could form the basis for the construction of a set of local correction functions which would force the divergence to be zero in problematic regions while also retaining any zero conditions imposed on either u , v or w and the flux interpolation conditions (1.2-1.4).

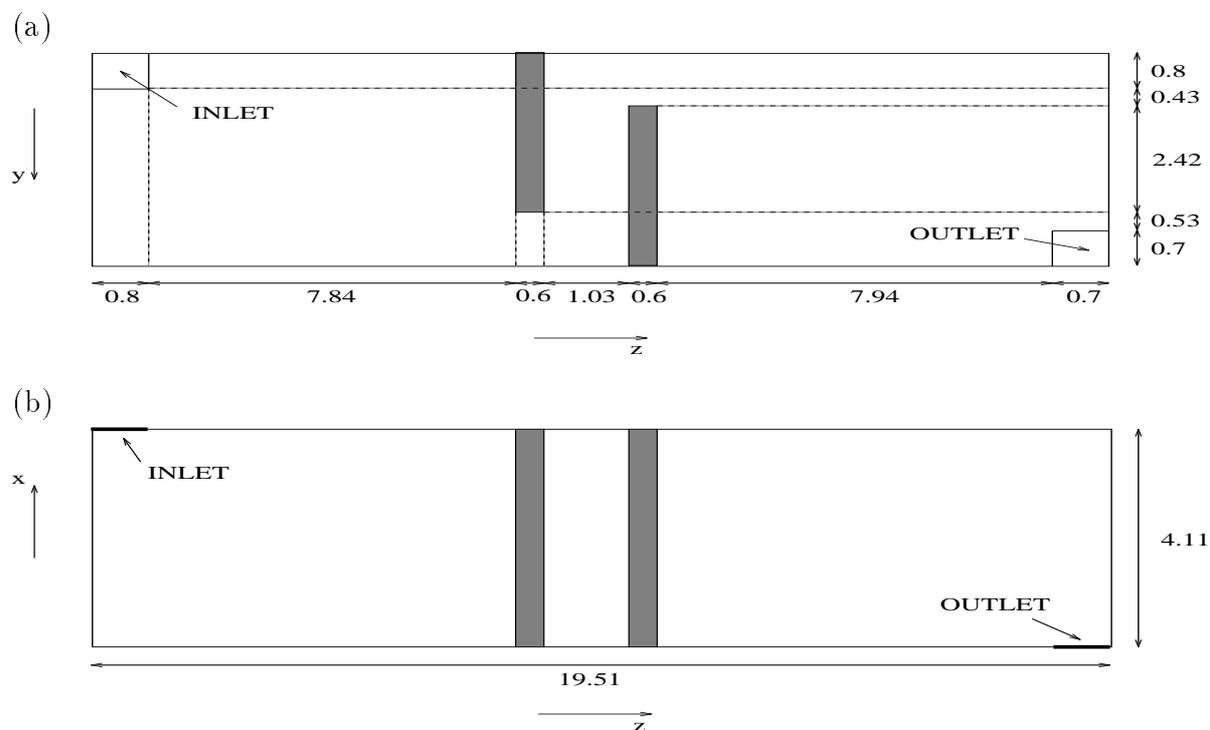


Figure 6: Simplified tank geometry: a) Plan View; b) Side View

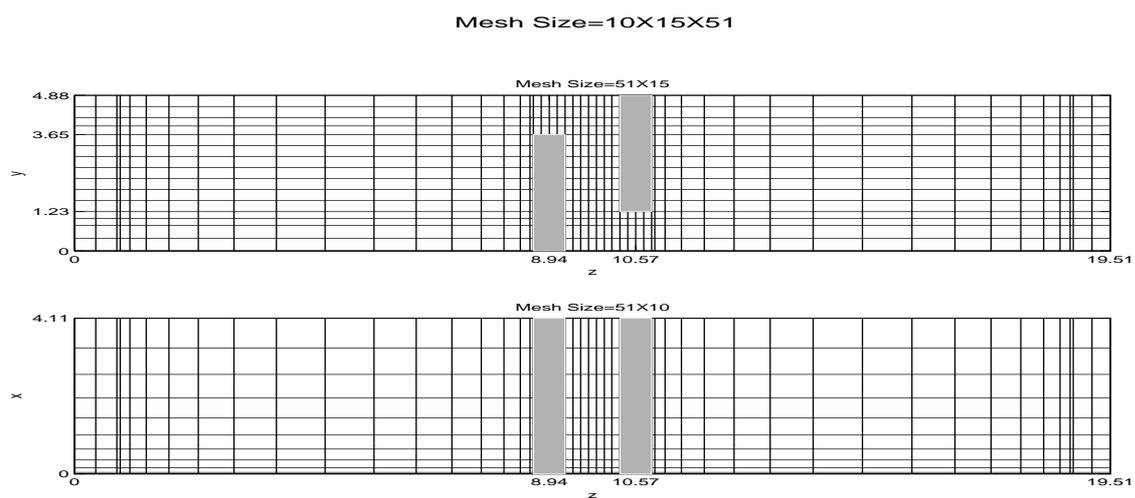


Figure 7: Tensor product mesh: $10 \times 15 \times 51$ cells

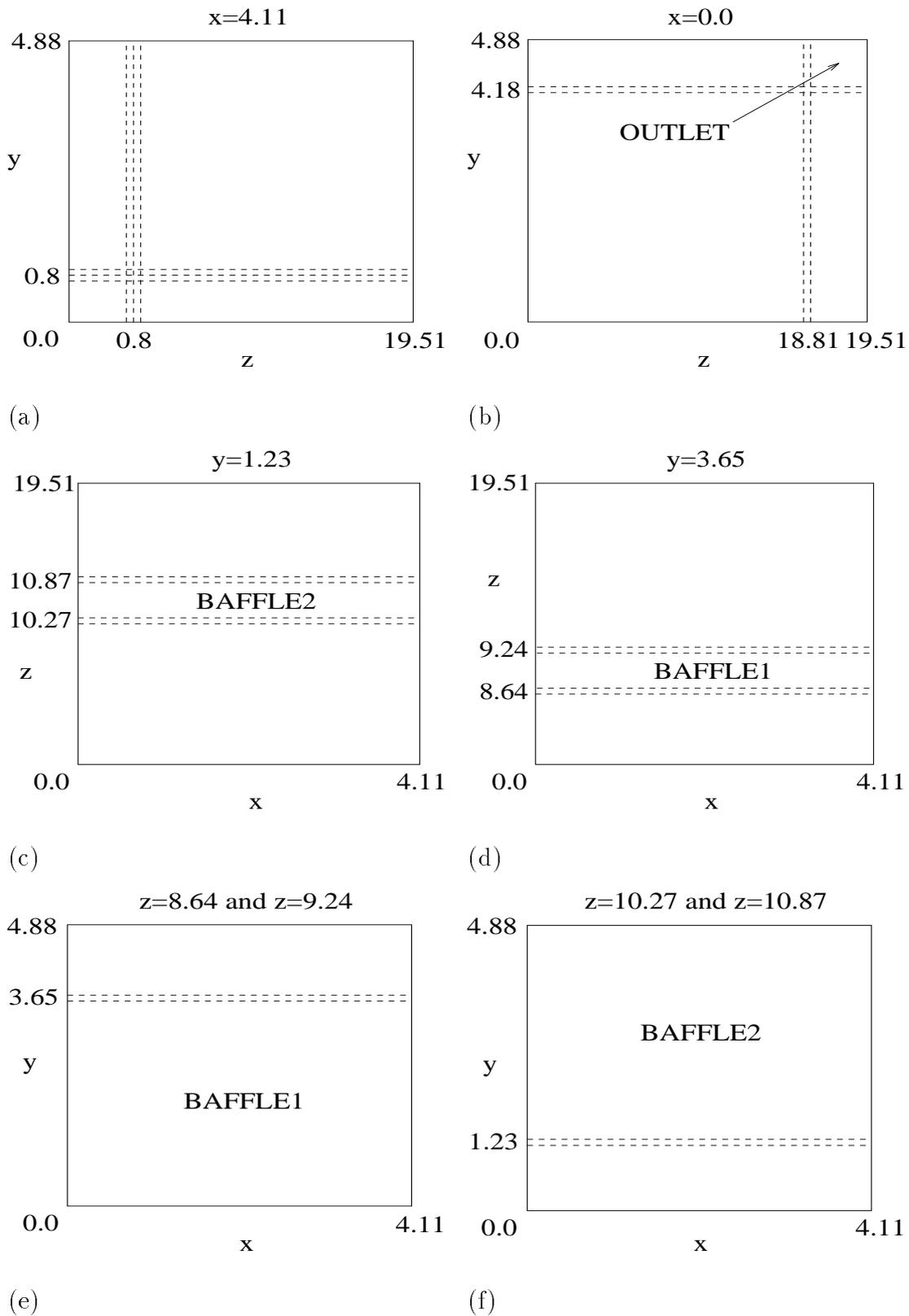
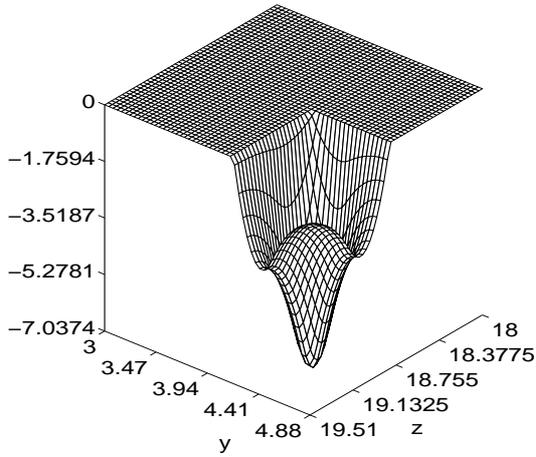
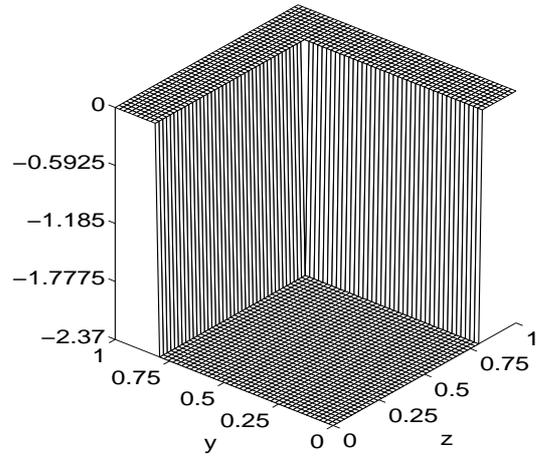


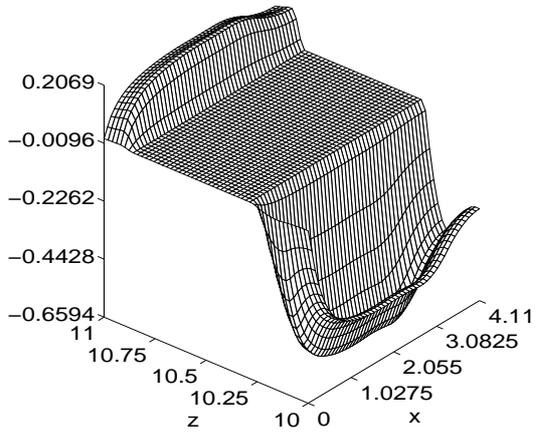
Figure 8: Multiple knot planes: a) Triple knots to correct u component at inlet; b) Double knots to correct u component at outlet; c) Double knots to correct v component on surface of second baffle; d) Double knots to correct v component on surface of first baffle; e) Double knots to correct w component on front and back surfaces of first baffle; f) Double knots to correct w component on front and back surfaces of second baffle.



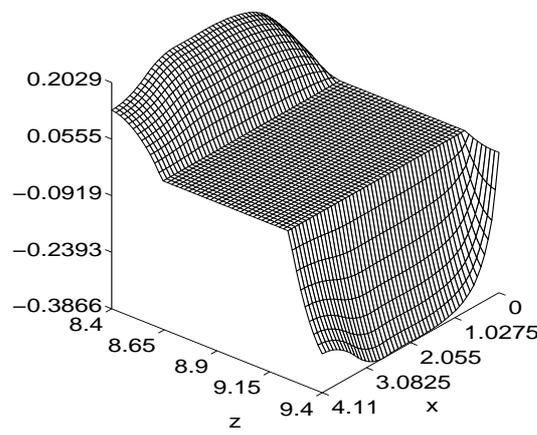
(a)



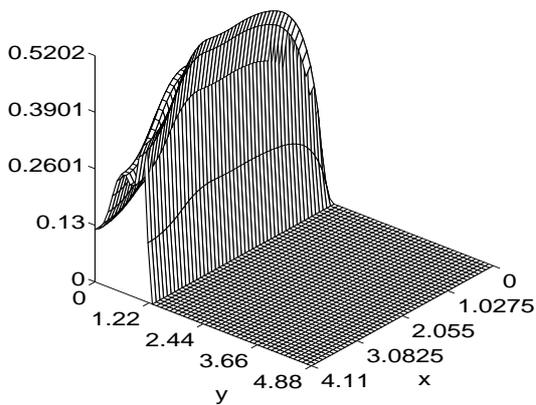
(b)



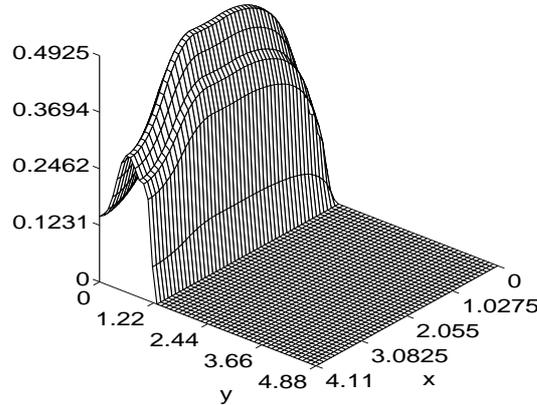
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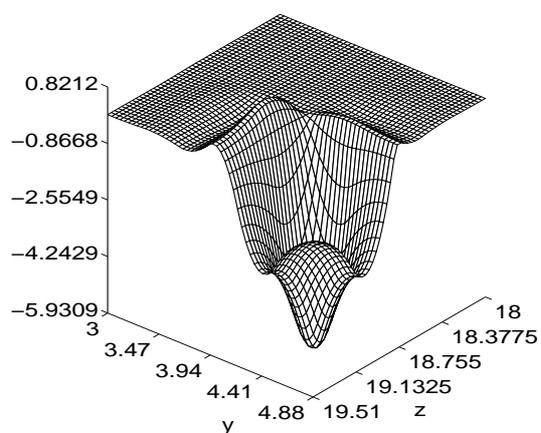


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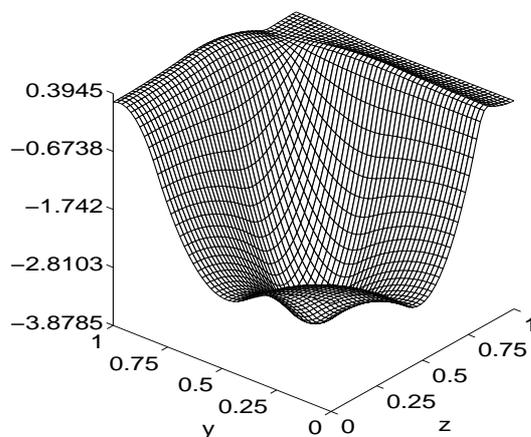


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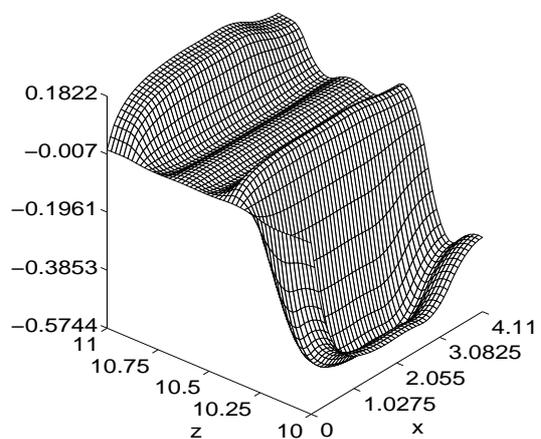
Figure 9: Velocity representation across planes containing surfaces/apertures: a) $u(0, y, z)$; b) $u(4.11, y, z)$; c) $v(x, 1.23, z)$; d) $v(x, 3.65, z)$; e) $w(x, y, 10.27)$; f) $w(x, y, 10.87)$.



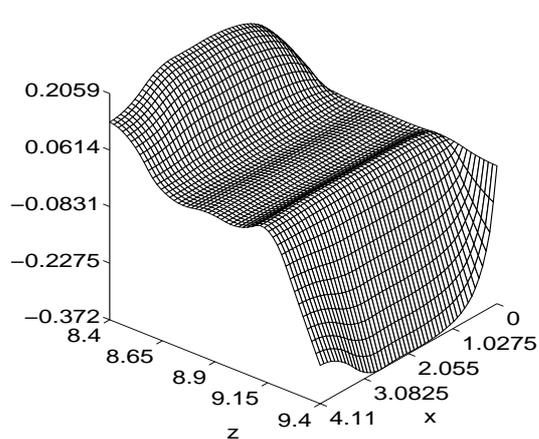
(a)



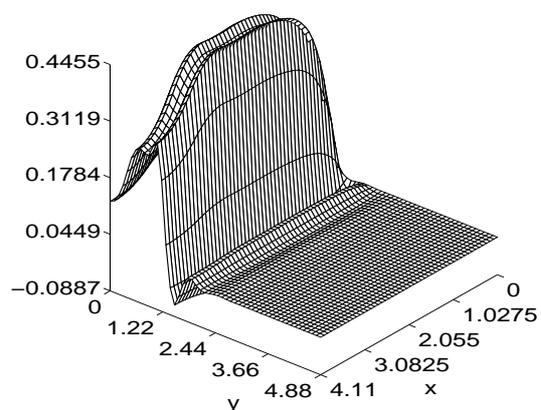
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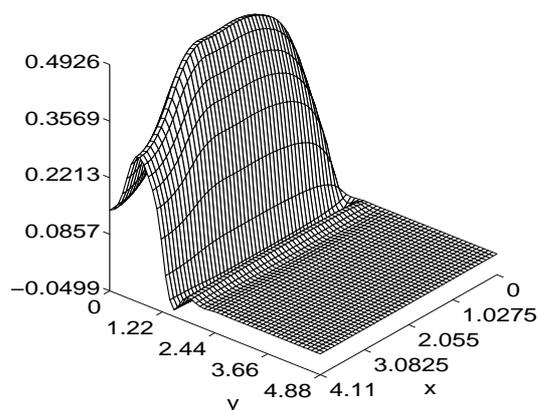
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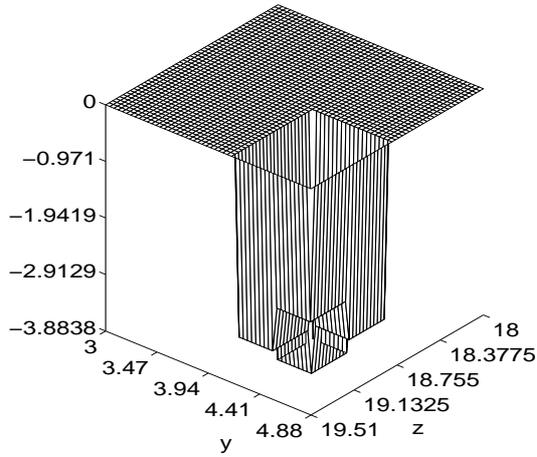


(e)

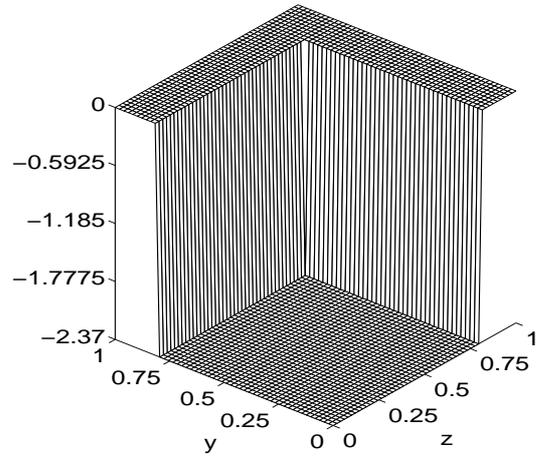


(f)

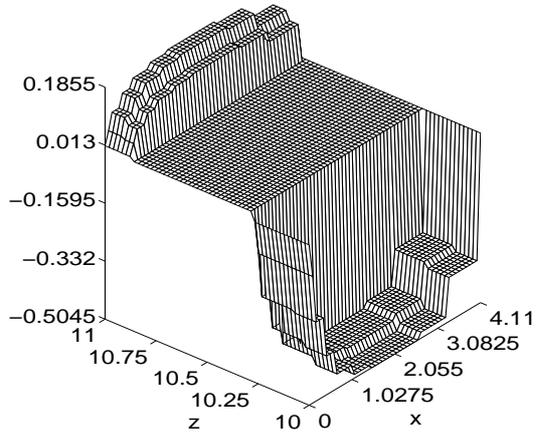
Figure 10: Simple knot velocity representation across planes containing surfaces/apertures: a) $u(0, y, z)$; b) $u(4.11, y, z)$; c) $v(x, 1.23, z)$; d) $v(x, 3.65, z)$; e) $w(x, y, 10.27)$; f) $w(x, y, 10.87)$.



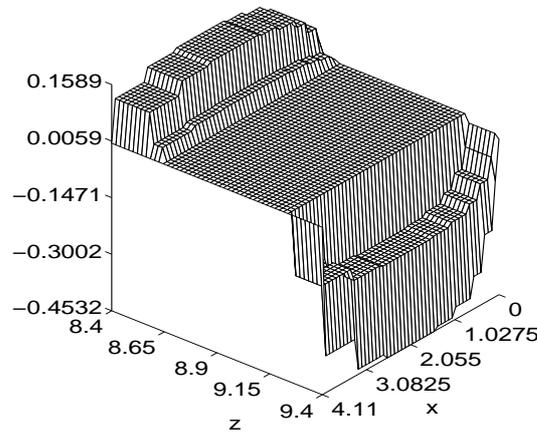
(a)



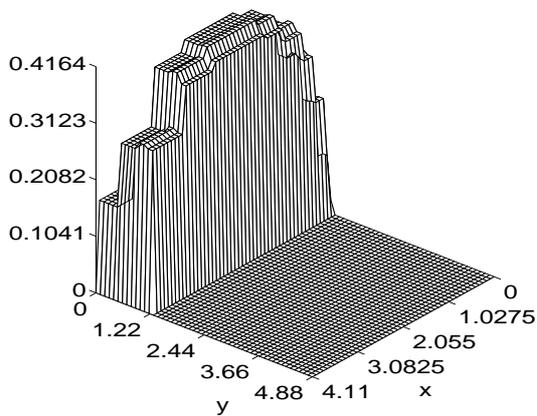
(b)



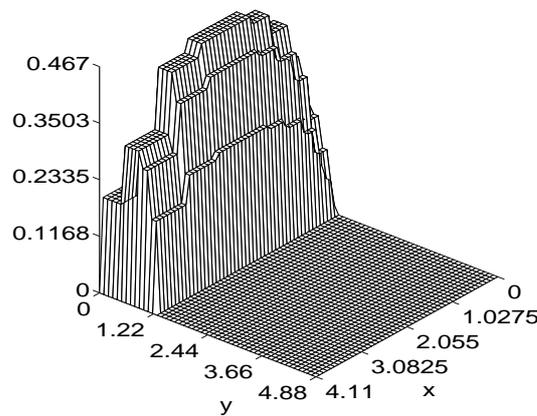
(c)



(d)



(e)



(f)

Figure 11: Given flux data across the planes $x = 0$, $x = 4.11$, $y = 1.23$, $y = 3.65$, $z = 10.27$ and $z = 10.87$ containing surfaces/apertures: a) $U_{0,j,k}$; b) $U_{10,j,k}$; c) $V_{i,4,k}$; d) $V_{i,11,k}$; e) $W_{i,j,29}$; f) $W_{i,j,33}$.

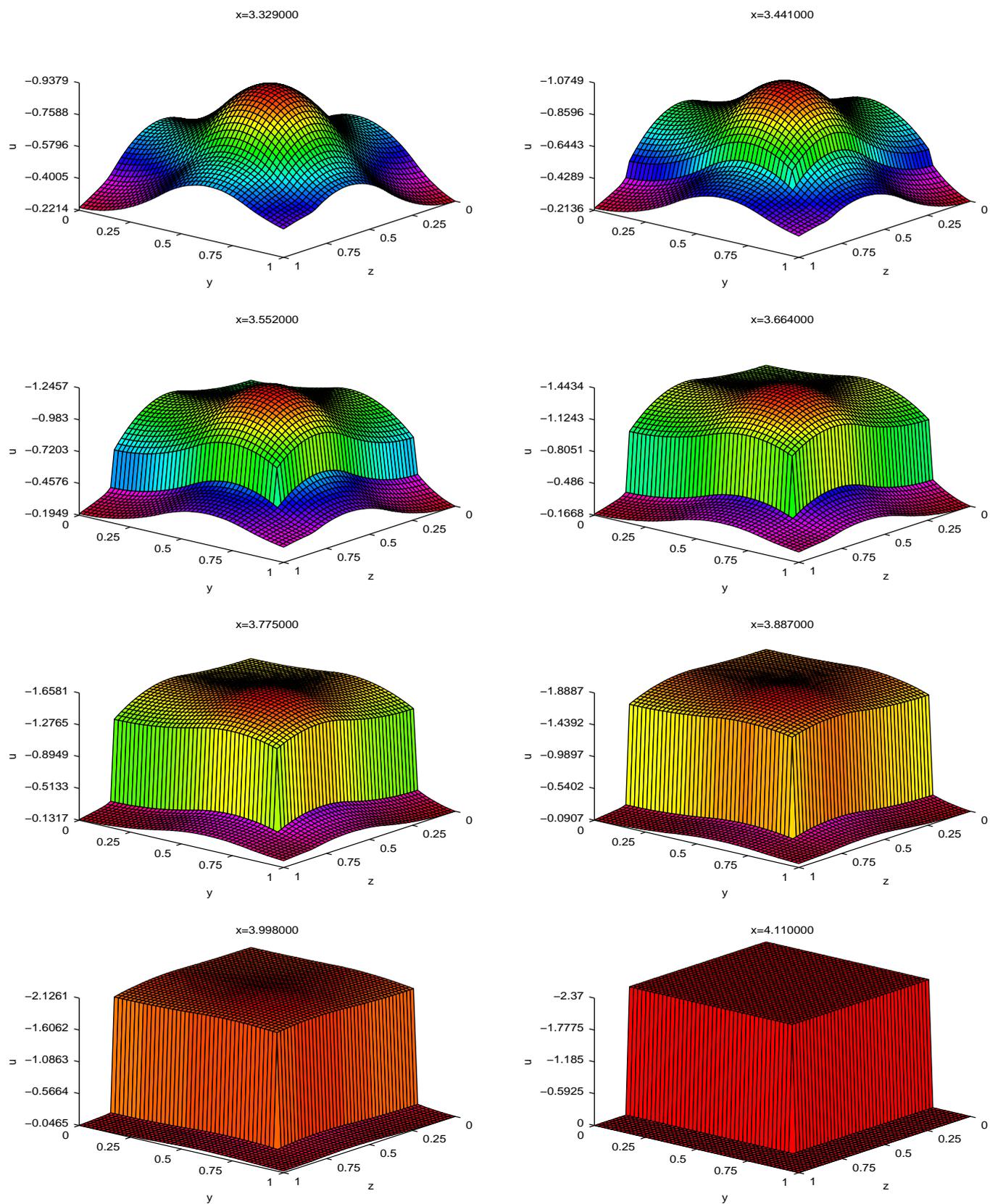


Figure 12: Velocity representations of $u(x, y, z)$ at: a) $x = 3.329$; b) $x = 3.441$; c) $x = 3.552$; d) $x = 3.664$; e) $x = 3.775$; f) $x = 3.887$; g) $x = 3.998$; h) $x = 4.11$.

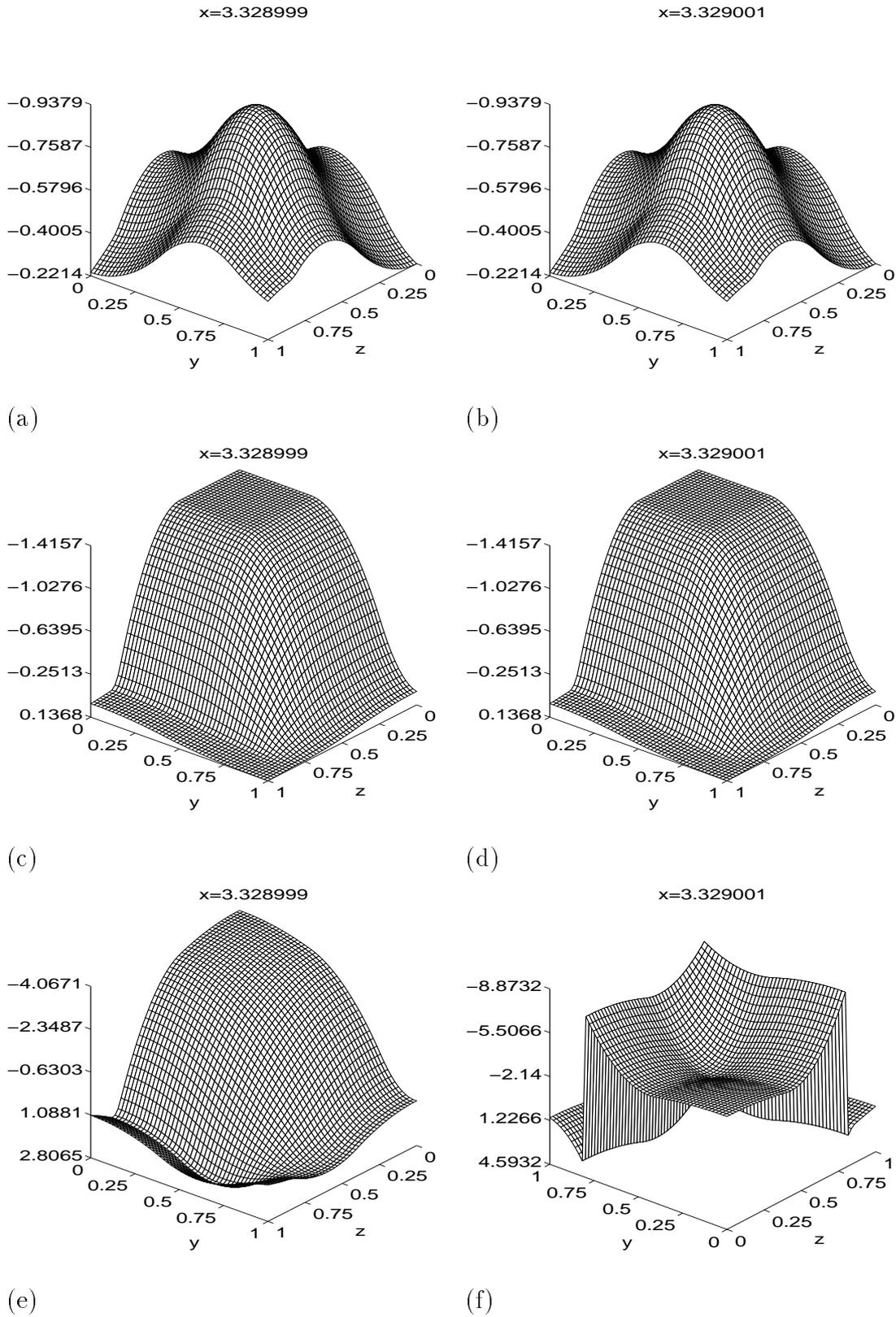


Figure 13: Velocity representations of u component at either side of the quadruple knot interface plane $x = 3.329$: a) $u(3.329^-, y, z)$; b) $u(3.329^+, y, z)$; c) $\frac{\partial}{\partial x}u(3.329^-, y, z)$; d) $\frac{\partial}{\partial x}u(3.329^+, y, z)$; e) $\frac{\partial^2}{\partial x^2}u(3.329^-, y, z)$; f) $\frac{\partial^2}{\partial x^2}u(3.329^+, y, z)$.

AppendixA B-Spline Basis Transformations

A.1 $Q(3, \mathbf{X}_1, \mathbf{X}_2)$: \mathbf{X}_1 and \mathbf{X}_2 are triple knot sets with the same distinct entries \mathbf{x} .

Let \mathbf{X}_1 and \mathbf{X}_2 be non-decreasing sequences of real numbers such that $\mathbf{x} \subseteq \mathbf{X}_1 \subseteq \mathbf{X}_2$ and $x_{i_1,p} = X_{1,I_1,p} = X_{1,I_1,p+1} = X_{1,I_1,p+2}$ ($p = 1$ to P_1) and $x_{i_2,p} = X_{2,I_2,p} = X_{2,I_2,p+1} = X_{2,I_2,p+2}$ ($p = 1$ to P_2) for some integer sets $\{i_1\}$, $\{i_2\}$, $\{I_1\}$ and $\{I_2\}$, the other elements of \mathbf{X}_1 and \mathbf{X}_2 being distinct. Now define the integer sets $\{s_p\}$ and $\{S_p\}$ to be such that $X_{1,s_p} = X_{2,s_p} = X_{2,s_p+1} = X_{2,s_p+2}$ ($p = 1$ to $P_2 - P_1$). Then given a spline $s(x; \mathbf{X}_1)$ on \mathbf{X}_1 with coefficients $\{a_{1,i}\}_{i=0}^{I+2P_1+1}$ relative to the basis $\{N_{3,i}^{\mathbf{X}_1}\}$ the equivalent representation on \mathbf{X}_2 , $s(x; \mathbf{X}_2)$ relative to $\{N_{3,i}^{\mathbf{X}_2}\}$ has coefficients $\{a_{2,i}\}_{i=0}^{I+2P_2+1}$ given by,

$$\begin{aligned}
a_{2,i} &= a_{1,i} \quad (i = 0 \text{ to } s_1 - 1) \\
a_{2,s_p+2(p-1)} &= a_{1,s_p+2(p-1)} + a_{1,s_p}d_{p,1,1} \quad (p = 1 \text{ to } P_2 - P_1) \\
a_{2,s_p+2(p-1)+1} &= a_{1,s_p}d_{p,1,2} + a_{1,s_p+1}d_{p,2,1} \quad (p = 1 \text{ to } P_2 - P_1) \\
a_{2,s_p+2(p-1)+2} &= a_{1,s_p}d_{p,1,3} + a_{1,s_p+1}d_{p,2,2} \quad (p = 1 \text{ to } P_2 - P_1) \\
a_{2,s_p+2(p-1)+3} &= a_{1,s_p+1}d_{p,2,3} \quad (p = 1 \text{ to } P_2 - P_1) \\
a_{2,i} &= a_{1,i-2(p-1)} \quad (i = s_{p-1} + 2p \text{ to } s_p + 2p - 3, p = 2 \text{ to } P_2 - P_1) \\
a_{2,i} &= a_{1,i-2(P_2-P_1)} \quad (i = s_{P_2-P_1} + 2(P_2 - P_1) + 2 \text{ to } I + 2P_2 + 1)
\end{aligned} \tag{A.1}$$

The inverse transformation from $\{N_{3,i}^{\mathbf{X}_2}\}$ to $\{N_{3,i}^{\mathbf{X}_1}\}$ is implemented using,

$$\begin{aligned}
a_{1,i} &= a_{2,i} \quad (i = 0 \text{ to } s_1 - 1) \\
a_{1,s_p} &= \begin{cases} \frac{a_{2,s_p+2(p-1)}}{(d_{p,1,1} + d_{p-1,2,3})} & \text{when } s_p - s_{p-1} = 1 \quad (p = 2 \text{ to } P_2 - P_1) \\ \frac{a_{2,s_p+2(p-1)}}{d_{p,1,1}} & \text{when } s_p - s_{p-1} > 1 \quad (p = 1 \text{ to } P_2 - P_1) \end{cases} \\
a_{1,s_p+1} &= \frac{a_{2,s_p+2(p-1)+3}}{d_{p,2,3}} \quad \text{when } s_p - s_{p-1} > 1 \quad (p = 1 \text{ to } P_2 - P_1) \\
a_{1,i} &= a_{2,i+2(p-1)} \quad (i = s_{p-1} + 2 \text{ to } s_p - 1, p = 2 \text{ to } P_2 - P_1) \\
a_{1,i} &= a_{2,i+2(P_2-P_1)} \quad (i = s_{P_2-P_1} + 2 \text{ to } I + 2P_1 + 1)
\end{aligned} \tag{A.2}$$

The coefficients $d_{p,1,i}$ and $d_{p,2,i}$ ($i = 1$ to $3, p = 1$ to $P_2 - P_1$) are given by the discrete B-splines which are computed using (2.15). Setting $\alpha_{3,i}(j) = \alpha_{3,i,\mathbf{X}_1,\mathbf{X}_2}(j)$ then,

$$\begin{aligned}
d_{p,1,j} &= \alpha_{3,i_p+1}(I_p + j) \\
d_{p,2,j} &= \alpha_{3,i_p+2}(I_p + j + 1) \\
&\quad (j = 1 \text{ to } 3, p = 1 \text{ to } P_2 - P_1)
\end{aligned}$$

A.2 $Q(3, \mathbf{X}_1, \mathbf{X}_2)$: \mathbf{X}_1 is a double knot set and \mathbf{X}_2 is a triple knot set.

Let \mathbf{X}_1 and \mathbf{X}_2 be non-decreasing sequences of real numbers such that $\mathbf{X}_1 \subseteq \mathbf{X}_2$ and $X_{1,i_p} = X_{1,i_p+1} = X_{2,I_p} = X_{2,I_p+1} = X_{2,I_p+2}$ ($p = 1$ to P) for some integer sets $\{i_p\}$ and $\{I_p\}$, the other entries of \mathbf{X}_1 and \mathbf{X}_2 being distinct. Then given a spline $s(x; \mathbf{X}_1)$ on \mathbf{X}_1 with coefficients $\{a_{1,i}\}_{i=0}^{I+P+1}$ relative to the basis $\{N_{3,i}^{\mathbf{X}_1}\}$ the equivalent representation on \mathbf{X}_2 , $s(x; \mathbf{X}_2)$ relative to $\{N_{3,i}^{\mathbf{X}_2}\}$ has coefficients $\{a_{2,i}\}_{i=0}^{I+2P+1}$ given by,

$$\begin{aligned}
a_{2,i} &= a_{1,i} \quad (i = 0 \text{ to } i_1) \\
a_{2,i_p+p} &= a_{1,i_p+1} d_{p,1} \quad (p = 1 \text{ to } P) \\
a_{2,i_p+p+1} &= a_{1,i_p+1} d_{p,2} \quad (p = 1 \text{ to } P) \\
a_{2,i} &= a_{1,i-p+1} \quad (i = i_{p-1} + p + 1 \text{ to } i_p + p - 1, p = 2 \text{ to } P) \\
a_{2,i} &= a_{1,i-P} \quad (i = i_P + P + 2 \text{ to } I + 2P + 1)
\end{aligned} \tag{A.3}$$

The inverse transformation from $\{N_{3,i}^{\mathbf{X}_2}\}$ to $\{N_{3,i}^{\mathbf{X}_1}\}$ is implemented using,

$$\begin{aligned}
a_{1,i} &= a_{2,i} \quad (i = 0 \text{ to } i_1) \\
a_{1,i_p+1} &= \frac{a_{2,i_p+p}}{d_{p,1}} \quad (p = 1 \text{ to } P) \\
a_{1,i} &= a_{2,i+p-1} \quad (i = i_{p-1} + 2 \text{ to } i_p, p = 2 \text{ to } P) \\
a_{1,i} &= a_{2,i+P} \quad (i = i_P + 2 \text{ to } I + P + 1)
\end{aligned} \tag{A.4}$$

The coefficients $d_{p,i}$ ($i = 1$ to $2, p = 1$ to P) are given by the appropriate discrete B-splines which are computed using (2.15). Setting $\alpha_{3,i}(j) = \alpha_{3,i,\mathbf{X}_1,\mathbf{X}_2}(j)$ then,

$$\begin{aligned}
d_{p,j} &= \alpha_{3,i_p+2}(I_p + j + 1) \\
&\quad (j = 1 \text{ to } 2, p = 1 \text{ to } P)
\end{aligned}$$

A.3 $Q(4, \mathbf{X}_1 = \mathbf{x}, \mathbf{X}_2 = \mathbf{X})$: \mathbf{x} is a simple knot set and \mathbf{X} is a quadruple knot set whose distinct entries are equal to \mathbf{x} .

Let \mathbf{x} and \mathbf{X} be non-decreasing sequences of real numbers such that $\mathbf{x} \subseteq \mathbf{X}$ and $x_{i_p} = X_{I_p} = X_{I_p+1} = X_{I_p+2} = X_{I_p+3}$ ($p = 1$ to P) for some integer sets $\{i_p\}$ and $\{I_p\}$, the other entries of \mathbf{X} being distinct. Then given a spline $s(x; \mathbf{x})$ on \mathbf{x} with coefficients $\{b_i\}_{i=-1}^{I+1}$ relative to the basis $\{N_{4,i}^{\mathbf{x}}\}$ the equivalent representation on \mathbf{X} , $s(x; \mathbf{X})$ relative to $\{N_{4,i}^{\mathbf{X}}\}$ has coefficients $\{a_i\}_{i=-1}^{I+2P+1}$ given by,

$$a_i = b_i \quad (i = -1 \text{ to } i_1 - 2)$$

$$\begin{aligned}
a_{i_p+3(p-1)-1} &= a_{i_p+3(p-1)-1} + b_{i_p-1}d_{p,1,1} \quad (p = 1 \text{ to } P) \\
a_{i_p+3(p-1)} &= b_{i_p-1}d_{p,1,2} + b_{i_p}d_{p,2,1} \quad (p = 1 \text{ to } P) \\
a_{i_p+3(p-1)+1} &= b_{i_p-1}d_{p,1,3} + b_{i_p}d_{p,2,2} + b_{i_p+1}d_{p,3,1} \quad (p = 1 \text{ to } P) \\
a_{i_p+3(p-1)+2} &= b_{i_p-1}d_{p,1,4} + b_{i_p}d_{p,2,3} + b_{i_p+1}d_{p,3,2} \quad (p = 1 \text{ to } P) \\
a_{i_p+3(p-1)+3} &= b_{i_p}d_{p,2,4} + b_{i_p+1}d_{p,3,3} \quad (p = 1 \text{ to } P) \\
a_{i_p+3(p-1)+4} &= b_{i_p+1}d_{p,3,4} \quad (p = 1 \text{ to } P) \\
a_i &= b_{i-3(p-1)} \quad (i = i_{p-1} + 3p - 1 \text{ to } i_p + 3p - 5, p = 2 \text{ to } P) \\
a_i &= b_{i-3P} \quad (i = i_P + 3P + 2 \text{ to } I + 3P + 1)
\end{aligned} \tag{A.5}$$

The inverse transformation from $\{N_{4,i}^{\mathbf{X}}\}$ to $\{N_{4,i}^{\mathbf{x}}\}$ is implemented using,

$$\begin{aligned}
b_i &= a_i \quad (i = -1 \text{ to } i_1 - 2) \\
b_{i_p-1} &= \begin{cases} \frac{a_{i_p+3(p-1)-1}}{(d_{p,1,1}+d_{p-1,3,4})} & \text{when } i_p - i_{p-1} = 2 \quad (p = 2 \text{ to } P) \\ \frac{a_{i_p+3(p-1)-1}}{d_{p,1,1}} & \text{when } i_p - i_{p-1} > 2 \quad (p = 1 \text{ to } P) \end{cases} \\
b_{i_p} &= \frac{a_{i_p+3(p-1)} - b_{i_p-1}d_{p,1,2}}{d_{p,2,1}} \quad (p = 1 \text{ to } P) \\
b_{i_p+1} &= \frac{a_{i_p+3(p-1)+4}}{d_{p,2,4}} \text{when } i_p - i_{p-1} > 2 \quad (p = 1 \text{ to } P) \\
b_i &= a_{i+3(p-1)} \quad (i = i_{p-1} + 2 \text{ to } i_p - 2, p = 2 \text{ to } P) \\
b_i &= a_{i+3P} \quad (i = i_P + 2 \text{ to } I + 1)
\end{aligned} \tag{A.6}$$

The coefficients $d_{p,1,i}$, $d_{p,2,i}$, $d_{p,3,i}$ ($i = 1 \text{ to } 4, p = 1 \text{ to } P$) are given by the appropriate discrete B-splines which are computed using (2.15). Setting $\alpha_{4,i}(j) = \alpha_{4,i,\mathbf{x},\mathbf{X}}(j)$ then,

$$\begin{aligned}
d_{p,1,j} &= \alpha_{4,i_p+1}(I_p + j) \\
d_{p,2,j} &= \alpha_{4,i_p+2}(I_p + j + 1) \\
d_{p,3,j} &= \alpha_{4,i_p+3}(I_p + j + 2)
\end{aligned} \tag{j = 1 to 4, p = 1 to P}$$

AppendixB Tridiagonal Systems

B.1 $T_{i,k}^{(1)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(1)}$

$$\sum_{q=j-1}^{j+1} c_{i,q,k} \int_{y_{j-1}}^{y_j} N_{3,q+1}^{\mathbf{y}}(y) dy = W_{ijk} \quad (j = 1 \text{ to } J) \quad (\text{B.1})$$

$$\sum_{j=0}^2 c_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{y}}(y_0) = 0 \quad (\text{B.2})$$

$$\sum_{j=J-1}^{J+1} c_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{y}}(y_J) = 0 \quad (\text{B.3})$$

$$(i = 1 \text{ to } I, k = 0 \text{ to } K, k \neq k_r, 0 \leq r \leq R)$$

Combining (B.1), (B.2) and (B.3) gives $(J+2)$ equations in the $(J+2)$ unknowns $\{c_{i,j,k}\}$ ($j = 0$ to $J+1$) for each fixed (i, k) .

B.2 $T_{j,k}^{(2)} \mathbf{b}_{j,k} = \mathbf{r}_{j,k}^{(2)}$

$$\sum_{p=i-1}^{i+1} b_{p,j,k} \int_{x_{i-1}}^{x_i} N_{3,p+1}^{\mathbf{x}}(x) dy = c_{ijk} \quad (i = 1 \text{ to } I) \quad (\text{B.4})$$

$$\sum_{i=0}^2 b_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\mathbf{x}}(x_0) = 0 \quad (\text{B.5})$$

$$\sum_{i=I-1}^{I+1} b_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\mathbf{x}}(x_I) = 0 \quad (\text{B.6})$$

$$(j = 0 \text{ to } J+1, k = 0 \text{ to } K, k \neq k_r, 0 \leq r \leq R)$$

Combining (B.4), (B.5) and (B.6) gives $(I+2)$ equations in the $(I+2)$ unknowns $\{c_{i,j,k}\}$ ($i = 0$ to $I+1$) for each fixed (j, k) .

B.3 $T_{i,k}^{(3)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(3)}$

$$\sum_{q=j-1}^{j+1} \tilde{c}_{i,q,k} \int_{\tilde{Y}_{r,j-1}}^{\tilde{Y}_{r,j}} N_{3,q+1}^{\tilde{\mathbf{Y}}_r}(y) dy = W_{ijk} \quad (j = \tilde{J}_{r,q}+2 \text{ to } \tilde{J}_{r,q+1}, q = 0 \text{ to } Q_r) \quad (\text{B.7})$$

$$\begin{aligned} & \tilde{c}_{i,\tilde{J}_{r,q},k} \frac{d}{dy} N_{3,\tilde{J}_{r,q}+1}^{\tilde{\mathbf{Y}}_r}(y_{j_r,q}^-) + \tilde{c}_{\tilde{J}_{r,q}+1,k} \left[\frac{d}{dx} N_{3,\tilde{J}_{r,q}+2}^{\tilde{\mathbf{Y}}_r}(y_{j_r,q}^-) - \right. \\ & \left. \frac{d}{dy} N_{3,\tilde{J}_{r,q}+2}^{\tilde{\mathbf{Y}}_r}(y_{j_r,q}^+) \right] - \tilde{c}_{\tilde{J}_{r,q}+2,k} \frac{d}{dy} N_{3,\tilde{J}_{r,q}+3}^{\tilde{\mathbf{Y}}_r}(y_{j_r,q}^+) = 0 \quad \text{when } B_r(x_{i-\frac{1}{2}}, y_{j_r,q}) = 1 \\ & c_{i,\tilde{J}_{r,q}+1,k} N_{3,\tilde{J}_{r,q}+2}(y_{j_r,q}) = 0 \quad \text{when } B_r(x_{i-\frac{1}{2}}, y_{j_r,q}) = 0 \end{aligned} \quad (\text{B.8})$$

$$(q = 1 \text{ to } Q_r)$$

$$\sum_{j=0}^2 \tilde{c}_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\tilde{\mathbf{Y}}_r}(y_0) = 0 \quad (\text{B.9})$$

$$\sum_{j=J+Q_r-1}^{J+Q_r+1} \tilde{c}_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\tilde{\mathbf{Y}}_r}(y_J) = 0 \quad (\text{B.10})$$

$$(i = 1 \text{ to } I, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R)$$

Combining (B.7), (B.8),(B.9) and (B.10) gives $(J + Q_r + 2)$ equations in the $(J + Q_r + 2)$ unknowns $\{\tilde{c}_{i,j,k}\}$ ($j = 0$ to $J + Q_r + 1$) for each (i, k) .

B.4 $T_{j,k}^{(4)} \mathbf{c}_{j,k} = \mathbf{r}_{j,k}^{(4)}$

$$\sum_{p=i-1}^{i+1} \tilde{b}_{p,j,k} \int_{\tilde{X}_{r,i-1}}^{\tilde{X}_{r,i}} N_{3,p+1}^{\tilde{\mathbf{X}}_r}(x) dx = c_{i-p,j,k} \quad (i = \tilde{I}_{r,p} + 2 \text{ to } \tilde{I}_{r,p+1}, p = 0 \text{ to } P_r) \quad (\text{B.11})$$

$$\begin{aligned} & \tilde{b}_{\tilde{I}_{r,p},j,k} \frac{d}{dx} N_{3,\tilde{I}_{r,p}+1}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^-) + \tilde{b}_{\tilde{I}_{r,p}+1,j,k} \left[\frac{d}{dx} N_{3,\tilde{I}_{r,p}+2}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^-) \right. \\ & \left. - \frac{d}{dx} N_{3,\tilde{I}_{r,p}+2}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^+) \right] - \tilde{b}_{\tilde{I}_{r,p}+2,j,k} \frac{d}{dx} N_{3,\tilde{I}_{r,p}+3}^{\tilde{\mathbf{X}}_r}(x_{i_{r,p}}^+) = 0 \quad \text{when } B_r(x_{i_{r,p}}, \bar{Y}_j) = 0 \end{aligned}$$

$$\tilde{b}_{\tilde{I}_{r,p}+1,j,k} = 0 \quad \text{when } B_r(x_{i_{r,p}}, \bar{Y}_j) = 1 \quad (\text{B.12})$$

$$(p = 1 \text{ to } P_r)$$

$$\sum_{i=0}^2 \tilde{b}_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\tilde{\mathbf{X}}_r}(x_0) = 0 \quad (\text{B.13})$$

$$\sum_{i=I+P_r-1}^{I+P_r+1} \tilde{b}_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\tilde{\mathbf{X}}_r}(x_I) = 0 \quad (\text{B.14})$$

$$(j = 0 \text{ to } J + 2Q_r + 1, k = k_r \text{ and } m(r) = 2, 1 \leq r \leq R)$$

Combining (B.11), (B.12),(B.13) and (B.14) gives $(I + P_r + 2)$ equations in the $(I + P_r + 2)$ unknowns $\{\tilde{b}_{i,j,k}\}$ ($i = 0$ to $I + P_r + 1$) for each (j, k) .

B.5 $T_{i,k}^{(5)} \mathbf{c}_{i,k} = \mathbf{r}_{i,k}^{(5)}$

$$\sum_{q=j-1}^{j+1} c_{i,q,k} \int_{Y_{r,j-1}}^{Y_{r,j}} N_{3,q+1}^{\mathbf{Y}_r}(y) dy = W_{ijk} \quad (j = J_{r,q} + 3 \text{ to } J_{r,q+1}, q = 0 \text{ to } Q_r) \quad (\text{B.15})$$

$$c_{i,J_{r,q}+1,k} N_{3,J_{r,q}+2}^{\mathbf{Y}_r}(y_{j_{r,q}}^-) = \int_{x_{i-1}}^{x_i} f_r(x, y_{j_{r,q}}^-) dx \quad (\text{B.16})$$

$$c_{i,J_{r,q}+2,k} N_{3,J_{r,q}+3}^{\mathbf{Y}_r}(y_{j_{r,q}}^+) = \int_{x_{i-1}}^{x_i} f_r(x, y_{j_{r,q}}^+) dx \quad (\text{B.17})$$

$$(q = 1 \text{ to } Q_r)$$

$$\sum_{j=0}^2 c_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}_r}(y_0) = 0 \quad (\text{B.18})$$

$$\sum_{j=J+2Q_r-1}^{J+2Q_r+1} c_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}_r}(y_J) = 0 \quad (\text{B.19})$$

$$(i = 1 \text{ to } I, k = k_r \text{ and } m(r) = 3, 1 \leq r \leq R)$$

Combining (B.15), (B.16), (B.17), (B.18) and (B.19) gives $(J + 2Q_r + 2)$ equations in the $(J + 2Q_r + 2)$ unknowns $\{c_{i,j,k}\}$ ($j = 0$ to $J + 2Q_r + 1$) for each (i, k) .

B.6 $T_{j,k}^{(6)} \mathbf{b}_{j,k} = \mathbf{r}_{j,k}^{(6)}$

$$\sum_{q=j-1}^{j+1} b_{i,q,k} \int_{Y_{r,j-1}}^{Y_{r,j}} N_{3,q+1}^{\mathbf{Y}_r}(y) dy = \int_{Y_{r,j-1}}^{Y_{r,j}} f_r(x, y) dy \quad (j = J_{r,q} + 3 \text{ to } J_{r,q+1}, q = 0 \text{ to } Q_r) \quad (\text{B.20})$$

$$b_{i,J_{r,q}+1,k} N_{3,J_{r,q}+2}^{\mathbf{Y}_r}(y_{j_{r,q}}^-) = f_r(x, y_{j_{r,q}}^-) \quad (\text{B.21})$$

$$b_{i,J_{r,q}+2,k} N_{3,J_{r,q}+3}^{\mathbf{Y}_r}(y_{j_{r,q}}^+) = f_r(x, y_{j_{r,q}}^+) \quad (\text{B.22})$$

$$(q = 1 \text{ to } Q_r)$$

$$\sum_{j=0}^2 b_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}_r}(y_0) = 0 \quad (\text{B.23})$$

$$\sum_{j=J+2Q_r-1}^{J+2Q_r+1} b_{i,j,k} \frac{d}{dy} N_{3,j+1}^{\mathbf{Y}_r}(y_J) = 0 \quad (\text{B.24})$$

$$(i = I_{r,p} + 1, I_{r,p} + 2, p = 1 \text{ to } P_r, k = k_r \text{ and } m(r) = 3, 1 \leq r \leq R)$$

Combining (B.20), (B.21),(B.22),(B.23) and (B.24) gives $(J + 2Q_r + 2)$ equations in the $(J + 2Q_r + 2)$ unknowns $\{b_{i,j,k}\}$ ($j = 0$ to $J + 2Q_r + 1$) for each (i, k) .

$$\sum_{p=i-1}^{i+1} b_{p,j,k} \int_{X_{r,i-1}}^{X_{r,i}} N_{3,p+1}^{\mathbf{X}_r}(x) dx = c_{i-2p,j,k} \quad (i = I_{r,p} + 3 \text{ to } I_{r,p+1}, p = 0 \text{ to } P_r) \quad (\text{B.25})$$

$$\sum_{i=0}^2 b_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}_r}(x_0) = 0 \quad (\text{B.26})$$

$$\sum_{i=I+2P_r-1}^{I+2P_r+1} b_{i,j,k} \frac{d}{dx} N_{3,i+1}^{\mathbf{X}_r}(x_I) = 0 \quad (\text{B.27})$$

$$(j = 0 \text{ to } J + 2Q_r + 1, k = k_r \text{ and } m(r) = 3, 1 \leq r \leq R)$$

Combining (B.25), (B.26) and (B.27) and inserting the values $\{b_{I_{r,p+1},j,k}\}, \{b_{I_{r,p+2},j,k}\}$ obtained from the solution of (B.20-B.24) into $T_{j,k}^{(6)}$ and $\mathbf{r}_{j,k}$ as known values gives $(I + 2P_r + 2)$ equations in the $(I + 2P_r + 2)$ unknowns $\{b_{i,j,k}\}$ ($i = 0$ to $I + 2P_r + 1$) for each (j, k) .

B.7 $T_{j,k}^{(7)} \mathbf{b}_{j,k} = \mathbf{r}_{j,k}^{(7)}$

$$\sum_{r=k-1}^{k+1} a_{ijk}^{(1)} N_{4,r+2}^{\mathbf{z}}(z_k) = b_{ijk}^{(2)} \quad (k = 0 \text{ to } K) \quad (\text{B.28})$$

$$\sum_{k=0}^2 a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{z}}(z_0) = 0 \quad (\text{B.29})$$

$$\sum_{k=K-1}^{K+1} a_{ijk} \frac{d^2}{dz^2} N_{4,k+2}^{\mathbf{z}}(z_K) = 0 \quad (\text{B.30})$$

$$(i, j) \in \bar{I} \otimes \bar{J}$$

Combining (B.28), (B.29) and (B.30) gives $(K + 3)$ equations in the $(K + 3)$ unknowns $\{a_{i,j,k}\}$ ($k = -1$ to $K + 1$) for each $(i, j) \in \bar{I} \otimes \bar{J}$.

References

- [1] W. Boehm, 1980. *Inserting New Knots into B-Spline Curves*. Computer Aided Design. **12**, 199-201.
- [2] W. Boehm and H. Prautzsch, 1985. *The Insertion Algorithm*. Computer-Aided Design. **17** No. 2, 58-59.
- [3] W. Boehm and H. Prautzsch, 1985. *On the Efficiency of Knot Insertion Algorithms*. Comput. Aided Geometric Des. 141-143.
- [4] E. Cohen, T. Lyche and R. Riesenfeld, 1980. *Discrete B-Splines and Subdivision Techniques in Computer-Aided Geometric Design and Computer Graphics*. Computer Graphics and Image Processing. **14**, 87-111.
- [5] C. De Boor, 1972. *On Calculating with B-Splines*. Journal of Approximation Theory. **6**, 50-62.
- [6] M. G. Cox, 1978. *The Numerical Evaluation of a Spline from its B - Spline Representation*. J. Inst. Maths. Applics. **21**, 135-143.
- [7] C. Glasgow, 1995. *Residence Time Calculations in Incompressible Flow*. D.Phil. Qualifying Dissertation, Oxford University Computing Lab.
- [8] C. Glasgow, A. K. Parrott and D. C. Handscomb, 1996. *Particle Tracking Methods for Residence Time Calculations in Incompressible Flow*. Technical Report, Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford.
- [9] D. C. Handscomb, 1984. *Spline Representation of Incompressible Flow*. IMA J. Numer. Anal. **4**, 491-502.
- [10] T. Lyche, 1988. *Note on the Oslo Algorithm*. Computer-Aided Design. **20** No.6 353-355.
- [11] T. Lyche and K. Mørken, 1986. *Making the Oslo Algorithm More Efficient*. SIAM J. Numer. Anal. **23**, No. 3, 663-675
- [12] T. Lyche and K. Mørken, 1987. *Knot Removal for Parametric B-spline Curves and Surfaces*. Comput. Aided Geometric Des. **4**, 217-230.
- [13] T. Popoviciu, 1934. *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*. Mathematica, **8**, 1-85
- [14] H. Prautzsch, 1984. *A Short Proof of the Oslo Algorithm*. Comput. Aided Geometric Des. **1**, 95-96
- [15] L. L. Schumaker, 1981. *Spline Functions, Basic Theory*. Wiley.