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Randomness and probability: exploring student teachers' conceptions

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ABSTRACT

Understanding randomness is essential for modern life, as it underpins decisions under uncertainty. It is also an essential part of both the mathematics and science curricula in schools. Yet, research has shown that many people consider randomness difficult to perceive and argue about, with a number of different and contradictory views on the nature of randomness prevailing. This study explores beginning mathematics and science teachers' understanding of randomness. A questionnaire was used with student teachers in an initial teacher-education course to explore their understanding of and reasoning about randomness and random events. Results suggest that mathematics and science student teachers conceptualize and argue about randomness in a variety of ways. Furthermore, these different conceptualizations affect how they respond to both common classroom tasks and everyday contexts involving randomness. This raises important implications for the education of teachers who will themselves be teaching probability and statistical inference.

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Introduction

We experience randomness everywhere, in our daily lives and in the classroom. We need to understand randomness in order to make informed decisions under uncertainty about our lives (Batanero & Serrano, 1999), such as choosing investments, medical treatments or whether to take part in risky activities. Understanding random processes also underpins weather predictions, forecasting the economy, radioactive decay, molecular collisions, and evolutionary change in biological systems. The teaching of randomness is often subsumed within the teaching of other topics, such as probability in mathematics, statistical inference, or radioactive decay in physics. Furthermore, the language associated with randomness and probability can frequently have contrasting meanings in everyday life (Molnar, 2018), for example, an event is often called random if it is very rare, haphazard, or unusual, and hence unpredictable or uncertain (Bennett, 1993; Kaplan et al., 2009). Yet while unpredictability is often given in a definition of randomness, random events are not necessarily rare, haphazard, or unusual.

Research in teacher education related to probabilistic thinking and reasoning has been identified as scarce (Batanero et al., 2016). Furthermore, research focusing specifically on teachers' own understandings of randomness and probability is limited (Elbehary, 2020; Hourigan & Leavy, 2020). Even when teachers have studied mathematics to a high level, they have usually only studied theoretical probability (Kvatinsky & Even, 2002), meaning that many teachers have had limited formal exposure to situations involving randomness and uncertainty. Furthermore, randomness and probability are also often considered counterintuitive and difficult to teach (Batanero et al., 2014; Eichler & Vogel,

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2014; Kazak & Pratt, 2020). Therefore, this article explores student teachers' understandings and meanings of randomness in different contexts, and how these meanings influence their explanations of the probabilistic reasoning involved in interpreting common classroom tasks.

Defining randomness

Within mathematics education and beyond, there is no agreed definition of randomness and this is clearly visible in the various interpretations of randomness throughout history (Batanero, 2015; Borovcnik & Kapadia, 2014a). Whilst probability is the branch of mathematics that deals with randomness, risk and decision making, in the school-mathematics curriculum the focus is often on experiments with identifiable outcomes such as rolling a dice, tossing a coin, or in games of chance, which can often contrast with its use in the science curriculum where it is considered within topics such as radioactive decay and evolutionary change. Simulations of random processes are often more common in science than mathematics classrooms, despite the wide variety of technological resources now available (Watson et al., 2013). More subjective contexts, which rely on our personal judgments and experiences in assigning probabilities to potential outcomes are rarely considered in the school curriculum (Eichler, 2008). More broadly, school curricula often emphasize causal relations, which can lead to students looking for deterministic explanations (Greer, 2001). In school, randomness is often described as the absence of causal relationships, or the absence of patterns or predictability. As Piaget and Inhelder (1951) argue, the concept of randomness first emerges in contexts where events are not explainable in terms of reversible operations. Calude and Longo (2016) argue that the analysis of randomness depends on the theory that you are working with, contrasting classical dynamics, quantum randomness, and biological randomness. They also argue that there are often differences in the way people describe the quality of randomness, for example, where Brownian motion is perceived to be more random that tossing a coin.

Pratt (1998) identifies four ideas that school students draw upon when talking about randomness. These are *unsteerability, irregularity, unpredictability*, and *fairness*. Some of these ideas focus on the process of producing random outcomes, whilst others can focus on the outcome itself. Konold et al. (1993) describe four conceptions of randomness: randomness as equiprobability, randomness as opposed to causality, randomness as uncertainty, and randomness as a model. The idea of randomness as fairness or as equiprobability focuses more on the outcomes of an event and it is this idea that dominated mathematics before the 20th century (Batanero, 2015). Randomness as being something without a known cause (or unsteerability where "no known agent was involved in determining the result" (Pratt & Noss, 2002, p. 464)), or as a measure of what we do not (yet) know, was also prevalent in the early work on randomness (Bennett, 1993). The two latter conceptions, randomness as uncertainty or as a model, lie behind many of the statistical models and methods of statistical inference developed through understanding randomness. Following this range of ideas of randomness, Konold et al. (1991) argue that we should treat randomness as a family of concepts, including but not limited to, random phenomena, randomizing devices and random samples.

The word random is also frequently used in everyday language to describe an absence of pattern, a lack of uniformity in building work, something happening without a conscious decision (e.g., random acts of violence) or to describe interactions with people unknown (e.g., "I talked to some randoms in the park") or behaviors that are unexpected or unpredictable (e.g., that was a completely random comment). Even mathematics teachers and students can use the language of probability to mean different things (Ingram, 2021; Kaplan et al., 2009; Molnar, 2018).

Some researchers make a distinction between randomness as a property of an outcome and randomness as a process by which a string of outcomes is generated that cannot be predicted. The outcome of a random process may or may not be perceivable or identifiable as random. As the infinite monkey theory states, a monkey hitting keys on a keyboard at random for an infinite amount of time will almost surely type the complete works of Shakespeare. Frigg (2004, p. 431) offers an example where a random sequence may also be generated using a deterministic algorithm (the Champernowne sequence 01234567891011121314 . . .). Fiedler et al. (2017) use the word "random" to describe probabilistic or uncertain outcomes and "stochastic" to describe processes where the outcomes may include patterns, but a specific pattern cannot be predicted precisely. This distinction is perhaps one explanation for why the children in Pratt's (1998) would often articulate more than one idea about randomness as the idea of unsteerability focuses on the process, whereas unpredictability was often used to describe the outcome.

Randomness and probability consequently pose particular challenges for teacher education due to the range of interpretations, and contexts that occur in the discussion within the classroom. These challenges are further exacerbated by the conceptions held by the students they are teaching, but also by teachers themselves, which are considered next.

Understanding probability and randomness

Existing research documents the numerous misconceptions and fallacies related to randomness (see Bryant & Nunes, 2012; Chernoff & Sriraman, 2014; Pratt & Kazak, 2018 for recent reviews of this research). The most famous of these include the gambler's fallacy, the law of small numbers (as opposed to the law of large numbers), and the conjunction fallacy (Tversky & Kahneman, 1971, 1983). Other research has pointed to the specific topics within probability that students and teachers find particularly difficult (Elicer & Carrasco, 2017; Savard, 2014) or the implications of these misconceptions on statistical inference (Fiedler et al., 2019).

In mathematics classrooms, randomness is often considered in contexts where combinatorial reasoning is needed (Batanero, 2015) and where randomness is often treated as being related to equiprobability through the emphasis on games of chance (Batanero & Serrano, 1999). Within these games of Pratt (1998, 2000) argues that students can view randomness in two different ways, locally and globally. These two ways are revealed by examining students' descriptions of randomness when playing these games of chance. Students' understanding of local randomness was visible through their descriptions of the predictability of individual outcomes in each trial, whereas their understanding of global randomness focuses on patterns and distributions over many trials. In science classrooms this distinction is often made in contexts such as radioactive decay where one cannot predict when a single nucleus will decay but we can predict the time taken for half the nuclei to decay.

Research shows that people often also hold non-normative expectations about random sequences or strings of outcomes. For example, randomly produced sequences typically contain longer runs than people would expect (Konold et al., 1991). Random shuffles of music libraries can result in songs from the same artist or album appearing consecutively leading to concerns about whether the shuffles are indeed random, and subsequent changes to the algorithms so that random shuffles are perceived to be random rather than actually being random (Powers, 2014). Batanero and Serrano (1999) found that students overemphasized unpredictability when they gave explanations for why sequences were random or not, with students identifying runs or clusters as being associated with a lack of randomness. Difficulties with sequences of outcomes and the interpretation of runs are also prevalent in research into the understanding of conditional probability and independent events (Elbehary, 2020).

Research into adults' understanding of randomness often focuses on exploring different representations of randomness, to communicate risk (Han et al., 2012), and to explain the influence of context on interpretation of representations of randomness (Hahn & Warren, 2009). Studies from experimental psychology have found that adults believe that outcomes are generated by predictable processes rather than random processes even when this is not the case (Navarro et al., 2016; Szollosi et al., 2019). Much of this research also raises the issue of defining randomness in a way that can be used to make sense of how we make decisions under uncertainty or identify randomness in various contexts (e.g., Griffiths et al., 2018).

Much of the research to date has focused on students' conceptions of randomness and probability, or elementary teachers' conceptions. There is less research into those teachers who have studied mathematics to an advanced level and will themselves be teaching both probability and other mathematical topics that are underpinned by an understanding of randomness such as statistical modeling and inference. The variety of conceptions, misconceptions and contexts leads to the fundamental question of what mathematics teachers need to know about randomness and probability (Eichler & Vogel, 2014). This question is addressed in the next section and underpins the design of pedagogic activities, on which this article is based, and which are described in the methods section below.

Teachers' knowledge and understanding

Many authors have also stressed the importance of teachers themselves having a coherent understanding of randomness, probability, and statistics, as well as an awareness of the possible misconceptions that students may hold (e.g., Stohl, 2005), but also an understanding and awareness of how situations involving randomness can be interpreted differently (Liu & Thompson, 2007). Yet, research shows that teachers often have limited content knowledge within probability and statistics (Groth, 2013; Stohl, 2005; Watson, 2001). In particular Stohl (2005) describes a deterministic view where the emphasis is on procedures to calculate theoretical probabilities without considering the real-world application. Stohl then uses the avoidance of realistic interpretations and contexts as an explanation for why many teachers favor a classical approach to teaching probability, which focuses on predictable outcomes of random processes. Steinbring (1991) argues that many of the difficulties teachers face when teaching probability and contexts involving random processes lie in the definitions that cannot be understood using only a classical or experimental approach. However, there has been relatively little research into the subject knowledge of student mathematics teachers who will themselves be teaching more advanced probability topics, such as conditional probability and statistical inference and modeling. Most of the research instead has been with elementary or middle school pre-service teachers (e.g., Estrada & Batanero, 2019; Hourigan & Leavy, 2020).

Kvatinsky and Even (2002) focus on three areas of probability content that teachers need to understand. First, teachers need to understand what makes probability different from other mathematical topics with its focus on uncertainty and random processes. This coincides with the arguments of Burgess (2006) and Groth (2013) that there are fundamental differences between knowledge and understanding teachers need for teaching mathematics and the knowledge and understanding they need for teaching statistics. Second, teachers also need to understand what aspects of mathematics support or inhibit probabilistic thinking. Similarly, Greer (2001) stressed the importance of teachers' dealing with students' probabilistic intuitions such as those explored in Fischbein's work (e.g., Fischbein & Gazit, 1984; Fischbein et al., 1991), and the need to confront them in the classroom. Third, teachers need to understand the role of randomness and uncertainty in dealing with everyday situations.

Yet, teachers need more than a coherent understanding of randomness and probability and potential misconceptions their students may have. Teachers also need an awareness of the choices available to them about how to teach particular concepts such as randomness. This awareness needs to consider students' potential misconceptions and mistakes as well as useful contexts and representations. Teachers also need an awareness of how that what they teach now can influence learning later in the curriculum, an issue particularly relevant to early probability where the ideas underpin the understanding of statistical inference used in a range of disciplines (Burgess, 2009; Hannigan et al., 2013).

One such choice is the sequence in which to introduce probabilistic concepts which support a coherent understanding of the ideas (Foster et al., 2021). Batanero (2015) describes stages, which she argues teaching could go through to support the understanding of randomness. She argues that curricula and teachers should begin with games of chance with equiprobable outcomes that are observable through the symmetry of a dice or coin in primary school but including estimating probabilities at this stage by repeating experiments. Such an introduction is also supported by Borovcnik and Kapadia (2014b). The next stage would include experiments where the outcomes are not equiprobable using relative frequencies to estimate probabilities. Once students have developed this understanding of probability then real-life contexts, including scientific contexts, and subjective situations where only personal probabilities can be used can be introduced. These three stages overlap with the three approaches to teaching probability described by Eichler and Vogel's (2012) and by Liu and Thompson's (2007) stochastic conceptions of an event's probability. However, others challenge the practice of beginning with activities with known outcomes and existing models as "practically every real-life application of probability theory occurs when there is no possibility of repeating a process many times and counting outcomes" (Devlin, 2014, p. xii).

Teachers, therefore, also need to make choices relating to the use of contexts and representations within teaching. Even within the teaching of probabilities using relative frequencies and experimentation the contexts used by teachers can vary considerably, with many restricting themselves to contexts such as rolling dice or using random number generators rather than real-world contexts and data sets (Eichler & Vogel, 2012). Furthermore, Steinbring (1991) argues that successful learning of probability is dependent on the representations and activities teachers use. Kvatinsky and Even (2002) describe a variety of representations and models for teaching probability (e.g., tree diagrams, Venn diagrams, and tables) and argued that teachers need to know when each representation or model is appropriate and how they are connected, as well as a range of examples and contexts which can illustrate probabilistic concepts, properties, and theorems.

Research has shown that the classical approach beginning with equiprobable outcomes dominates the teaching of probability in schools (Eichler, 2008), and whilst the majority of mathematics teachers teach the frequentist approach that focuses on observed frequencies of events, only a minority deal with subjective situations or make any connections between the three approaches (e.g., Jones et al., 2007; Steinbring, 1991; Stohl, 2005). Yet the frequentist approach can still lead to difficulties in students making connections between theoretical and experimental probabilities (Konold et al., 2011) that underpins later statistical learning. Subjective situations are also those that they are more likely to meet outside the classroom and can have a direct and profound effect upon their later lives.

The challenge for teacher educators is, therefore, how to develop student teachers' understanding of the teaching and learning of probabilistic concepts such as randomness and reasoning whilst also confronting their own understandings. The questionnaire analyzed in this article is intended to meet this challenge in a sustainable way.

Methods

The data analyzed in this article arises from pedagogic materials rather than a research design, and in that sense, a retrospective analysis of secondary data was performed. It is also an example of insider research (Atkins & Wallace, 2012) and practitioner research (Menter et al., 2011). The data comes from a questionnaire that student teachers complete before a workshop on the teaching and learning of probability and was intended to provide a stimulus for rich and detailed discussions of multiple perspectives on randomness within different contexts. The questionnaire was developed in response to two pedagogic concerns: firstly, probability was a topic which, in previous years, a noticeable number of student teachers had shared and discussed their own misconceptions, when these misconceptions were described in the workshop and many of the student teachers expressed being uncomfortable with their own knowledge of probability and statistics; secondly, the workshop focused on considering probability, randomness and risk from multiple perspectives, such as those outlined above, and how each perspective suggests different tasks, activities, or even ways of approaching the teaching and learning of probability. These perspectives were often advocated by the student teachers within the workshops and in previous years the personal attachment to a particular perspective sometimes led to treating one perspective as "correct" and other perspectives as "incorrect," which did not always lead to an appreciation of the influence of the various perspectives on teaching and learning. Furthermore, these perspectives can also be illustrated through various tasks and contexts involving randomness drawn from both the school mathematics and science curricula. Thus, the questionnaire was designed to address an instructional problem (Cai et al., 2019).

The questionnaire varied slightly each year to reflect recent research and changes in the tasks being used in the schools where the student teachers were working. An adapted version of the questionnaire that includes the items discussed in this article is included in the Appendix. Consequently, the majority of the analysis below focuses on those items that were included across most of the 4 years the questionnaire was administered and where the data was retained. The questionnaire was designed for student teachers who were specializing in the teaching of mathematics to students aged 11–18; for the most recent two years the student teachers specializing in science also completed the questionnaire. In the ensuing workshops for the mathematics specialists, the participants analyzed the anonymized data generated by the questionnaire, worked on research articles examining the same tasks and questions, and discussed examples of classroom tasks used in local schools. They also worked on other tasks specifically designed to problematize didactical approaches to the teaching and learning of probability, random processes, and risk. The workshops for the science specialists focused on how that what pupils learnt in mathematics might influence their understanding of scientific ideas that involve randomness. Only those responses to the questionnaire are included where the participant gave their consent to use the data for research purposes. Ethical approval was given for the project by the Departmental Ethics Committee at the author's institution.

The analysis below focused on two research questions:

- What are the various understandings and meanings student teachers have for randomness in different contexts?
- How do these meanings influence their explanations of the probabilistic reasoning involved in interpreting common classroom tasks?

Questionnaire design

The questionnaire always began with a question asking the student teacher to define "random" and "fair" and to give an example of something that was "random." The questionnaire then posed a series of questions about whether different scenarios would produce a random outcome or not, drawn from:

- classical contexts such as rolling dice and spinning spinners, both in terms of individual events and sequences of events, and
- scientific contexts such as the movement of molecules between two vessels.

These questions were chosen to address topics which would arise in the workshop such as using a continuous area model rather than a discrete set model (Baturo et al., 2007), common misconceptions (Kahneman et al., 1982), and distinctions between local and global randomness (Pratt & Noss, 2002). For example, there was a series of questions about random sequences and random two-dimensional distributions taken from Batanero and Serrano (1999). All the questions considered in this article were taken from questions in national guidance on teaching probability in secondary schools or from the research discussed in the workshop.

Workshops

One aim of the workshops was to enable the student teachers to confront their own conceptions and misconceptions and encounter different perspectives in ways that supported a focus on the implications for teaching and learning rather than evaluative judgments. Although the workshops are not part of the research presented here, the questionnaire was designed to provide a stimulus for these considerations of different conceptions and perspectives. Consequently, a brief description of the workshops is offered here to illustrate the principles drawn upon in the design of the questionnaire.

The pedagogy behind the workshop takes an interactional view of teaching and learning mathematics, whereby meaning is co-constructed through discussion and negotiation (Eckert & Nilsson, 2017), within a framework of "practical theorising" (Burn & Mutton, 2015; Ellis & Childs, 2019) with "genuine theory-practice integration on the basis of parity of academic and professional experiential perspectives" (Alexander, 1984, p. 148). The topic of probability in particular often resulted in conflicting reactions to both the questions focused on content and on the different pedagogic approaches considered within the workshop, provoking the need for the student teachers not only to try to convince others of their thinking, but also to consider the mathematics/statistics from different perspectives. A theme underpinning the workshops, and the program overall, is that teachers need to develop a sensitivity to alternative understandings their students might have (Liu & Thompson, 2007) and around choices for how to respond to these understandings that are contingent upon the context in which they arise (Mason, 2002; Mason & Spence, 1999).

The completion of the questionnaire is part of the student teachers engaging as if they are learners in school within their teacher education course so that they "can personally experience the power and drawbacks of such activities and thus be in a better position to evaluate the potential value of the activities for their own students" (Borasi et al., 1999, p. 50). Batanero (2015) recommends that student teachers are first confronted with their own previous ideas before performing and discussing experiments to increase both probabilistic and didactic knowledge. The analysis of the results of the questionnaire takes this a step further by offering insights into how different student teachers have engaged as learners with the same tasks and making more explicit how different perspectives and different understandings can affect the learning of probabilistic reasoning. This then supports the student teachers engaging in decision making in their roles as teachers in schools (Schoenfeld, 2010).

The workshops were not recorded as this would be too intrusive; it was important that the environment made the participants feel comfortable talking about various perspectives, and sharing their own uncertainties or misunderstandings, and offering contrasting and sometimes conflicting arguments. Whilst some student teachers are happy to talk about their own misconceptions, others are not. Therefore, the pedagogic aims of the workshop and the professional development of the student teachers were prioritized over a potential contribution to research.

Participants

All the student teachers completing the questionnaire were enrolled in a 1-year university-led initial teacher education course in the Uk leading to Qualified Teacher Status with a focus on teaching students aged 11–18 years. As part of this course the student teachers spend the majority of their time on teaching placements. Variants of the questionnaire were completed each year for 6 consecutive years by student teachers who were learning to teach mathematics, though data from the questionnaire is only available for the 4 most recent years when the questionnaire moved from being paper based to electronic. For the last 2 years, student teachers who were learning to teach science (chemistry, physics, or biology) were also invited to complete the questionnaire but for a different workshop lead by the science education tutors at the same institution. All student teachers are post-graduates with a first degree in mathematics, science, or mathematics-related studies such as engineering or economics. Whilst the questionnaire was an essential part of the workshops, the student teachers are given the option on the questionnaire of whether their work, including their responses to the questionnaire, can also be used for research purposes. All questions were optional, and some student teachers omitted some questions. In total, 82 mathematics student teachers and 45 science student teachers consented to be included in this study. The study received ethical approval from the institution's departmental ethics committee.

Table 1. Number (row percentage) of student teachers including specific meanings of random in their descriptions.

	n	No planning, or not conscious	Unpredictable	No pattern	Equiprobable	Other
Mathematics	82	12 (15%)	32 (39%)	25 (31%)	13 (16%)	15 (18%)
Science	45	9 (20%)	16 (36%)	16 (36%)	5 (11%)	6 (13%)
Total	127	21 (17%)	48 (38%)	41 (32%)	18 (14%)	21(17%)

Codes are not mutually exclusive

Analysis

The analysis was completed in two stages. The first stage was the analysis needed for the workshop which involved descriptive reporting of the multiple-choice questions and the coding of the descriptions and examples given for randomness and fairness. The descriptions were coded in two different ways, of which only the first is included in the analysis presented in this article. Firstly, the author coded the descriptions using the four ideas employed by Pratt and Noss (2002), in order to present the frequencies of the categories of descriptions. These four ideas are specifically referred to in the national guidance on teaching probability that some of the questionnaire items were drawn from. The labels for the codes are altered slightly in this article and in the workshops to reflect the language used by the student teachers themselves. In one of the years, where the science student teachers also completed the questionnaire, the descriptions and examples of randomness and fairness were also coded independently by a science teacher educator. Whilst there was agreement between the two coders on the categories drawn from Pratt and Noss, for the purposes of the workshops, a further distinction was made between no planning or decisions and other types of human influence within the category of *unsteerability*. In the second stage, the student teachers worked in small groups and sorted the anonymized descriptions into categories during the workshop according to the similarities and differences they identified, which in some years were revisited after reading a brief extract from the Pratt and Noss article explaining the various conceptions. Any disagreements between the student teachers over the categorization of descriptions were discussed until agreement was reached within a small group. The categorization of the researchers agreed with the most common categorizations of the student teachers.

In addition to the descriptive statistics that were shared with the student teachers in the workshops, further additional statistical analysis was undertaken to explore relationships between the subject specialisms of the student teachers, their descriptions of randomness and their subsequent responses to the remaining questions on the questionnaire where the requirements of the statistical analysis methods were satisfied and there was no evidence of any major violation in the assumptions. The descriptions of randomness by the student teachers occasionally included multiple categories so where the statistical analysis requires codes to be mutually exclusive (e.g., chi-square tests) an additional code of "multiple definitions" was created. Logistic regression models were used to examine relationships between descriptions of randomness and the participants' responses to the other questions. The statistical analysis was completed using R version 4.1.2 (R Core Team, 2021).

Table 2. Number (row percentage) of student teachers including unpredictable, no pattern and/or equiprobable in their description of random stating that a dice or spinner without equiprobable outcomes would not generate a random number.

	Total stating that it is not a random device	Unpredictable ($n = 48$)	No pattern ($n = 41$)	Equiprobable ($n = 18$)
Dice	45 (42%)	24 (50%)	21 (51%)	14 (78%)
Spinner	42 (39%)	24 (50%)	18 (44%)	15 (83%)
Asking someone	71 (56%)	29 (60%)	24 (59%)	12 (67%)

Findings

Meanings for randomness

Questions 1 and 2 on the questionnaire focus on the student teachers' meanings for randomness. Question 1 asked what the word "random" means. Some common themes emerged in the student teachers' responses, which are very similar to those identified by Pratt and Noss (2002) in children. "No pattern" (e.g., "Without pattern or structure") was mentioned by 32% ($n = 41$) of the participants in their description, 38% described random as "unpredictable" ($n = 48$) (e.g., Something that can't be determined or predicted precisely"), with 8% ($n = 10$) including both "no pattern" and "unpredictable" in their description. Furthermore, 14% ($n = 18$) described random as "equiprobable" and 8% ($n = 10$) as "independent." A further 17% ($n = 21$) described random as involving "no planning" or "not conscious" (which Pratt and Noss describe as "unsteerability") or other type of influence (e.g., "Something occurring without external influence, following the natural entropy of the universe"). Table 1 shows how the number and proportion of participants who gave each type of description varied depending upon whether they were learning to teach mathematics or science, with 15 mathematics participants and 7 science participants giving more than one definition.

A larger proportion of science teachers described random as not involving planning or consciousness, and similarly a larger proportion of science teachers described random as involving no patterns. In contrast, a larger proportion of mathematics teachers described random as "unpredictable" or as "equiprobable," however none of these differences were statistically significant (using chi-squared tests for each category that met the assumptions with a Bonferroni correction).

In response to Question 2, the student teachers gave examples of something that is random ($n = 109$), 55% ($n = 24$) of the science student teachers and 37% ($n = 24$) of the mathematics teachers gave a context related to the science curriculum such as radioactive decay or the movement of particles in a gas. These were the most common examples given with an overall proportion of 44%. Contexts including dice were the next most common type of example with 27% ($n = 29$) of student teachers, and contexts with lotteries accounting for a further 20% ($n = 22$).

Random devices

A sequence of seven questions were specifically about ways in which a random number could be selected (i.e., a random device), such as by rolling a dice or spinning a spinner (Question 3 in the Appendix). In the two scenarios where the outcomes were not equiprobable, 42% (n= 45) of all the student teachers said that the die would not select a random number and 39% ($n = 42$) said the spinner would not. Similar proportions can be seen with student teachers who included "unpredictable" in their description of random (50% ($n = 24$) for both the dice question and the spinner question), and who described random as meaning "no pattern" (51% ($n = 21$) for the dice and 44% ($n = 18$) for the spinner). However, for the student teachers who described random as including equiprobable outcomes 78% ($n = 14$) stated that the die would not select a number at random and 83% ($n = 15$) said that the spinner would not select a number at random. These results are summarized in Table 2.

Including "equiprobable" in a description of the meaning of random meant that student teacher was 79% less likely to say that the dice would select a random number (b = −1.55, $z = -2.48$, $p < .013$) and 88% less likely to say that the spinner would select a random number ($b = -2.11$, $z= -3.094$, $p< .002$) than student teachers who did not include equiprobable. Almost all student teachers identified the other dice and spinner questions, which had equiprobable outcomes, as methods for selecting random numbers. Asking a person to pick a random number between 1 and 10, in contrast, was identified as not selecting a random number by 56% ($n = 71$) of the student teachers, with 60% ($n = 29$) of those who described random as "unpredictable," 59% ($n = 24$) of those who described random as having "no pattern," and 67% ($n = 12$) of those who described random as "equiprobable," but only 43% ($n = 9$) of the student teachers who described random as involving no planning or not conscious.

The student teachers were then asked if any of these scenarios were "more" random than others and why. Of those that answered (n = 98), 32% said that none of them were more random than others, but 15% (n = 15) said that the situations where there were more possible outcomes were more random, and 22% (n = 22) said that the situations where there was no human involvement were more random, which for some meant that the dice were more random than the spinner as a person can influence a spinner more than a die.

The last question about each of these scenarios asked which of the situations were "fair" and why. Here, 61% (n = 59) of the student teachers who answered stated that the scenarios where the outcomes were equiprobable were the ones that were fair whilst 19% (n = 18) stated that it was the scenarios where there was no human influence, with 7 student teachers saying it was both. The remaining answers mentioned neither equiprobability or human influence and there were no further patterns in these answers.

Random samples

Question 4 included four questions focused on a sample of Heads and Tails which asked the student teachers whether the sequence was random or not, but in terms of playing a game and whether someone had cheated or not. These questions were based on Batanero and Serrano (1999). These four questions involved a sequence of Heads and Tails from a coin being tossed 40 times. In these random sequences the proportion of Heads was varied so that the sequences of Ahmed, Bernard and Claire sequences had a proportion of Heads that was close to the theoretical probability of ½, but Diana's sequence had a much lower proportion (P(H) = .3). The length of runs was also varied between the four sequences so that Bernard's sequence had no runs of length greater than 2 making it less likely to be randomly generated.

For the questions about the sequences of coin tosses, between 36% and 39% of the teachers stated that it was not possible to tell whether the sequence was produced randomly or not for each of the four sequences, with many stating that this was because each of the sequences was possible even when it was unlikely to occur. Of the remaining responses, the most frequent were that Ahmed's sequence (43%, n = 45) and Claire's sequence (43%, n = 45) were not made up, and were therefore generated randomly, but Bernard's (39%, n= 41) and Diana's (39%, n = 40) were. Mathematics specialist teachers were more likely to say that Ahmed's sequence was not made up ($\chi^2(2) = 7.949$, $p < .05$) and that Diana's sequence was made up ($\chi^2(2) = 7.2292$, $p < .05$) than science specialist teachers who were more likely to say you cannot tell.

Around half the teachers gave reasons for their answers (n = 59). For Ahmed's sequence the most frequent reason given was that it was random because there were similar numbers of Heads and Tails (n = 30). The difference in the number of Heads and Tails was also the most frequent reason given as for why Diana's sequence was not random (n = 41), as there were too many Tails. For Bernard's sequence the most frequent reasons for it not being random was because of the runs of Heads or runs of Tails within the sequence (n = 20) or because there was a pattern in the sequence (n = 25); in this case no runs were longer than 2. For Claire's sequence, 15 student teachers considered runs in their reasons for whether Claire had cheated or not, and 22 considered patterns in the sequence.

Random phenomena

The remaining three questions focused on probability and randomness in a range of contexts and only a few of these questions were used in all the years where the data from the questionnaire was retained.

Question 4 was similar to the first question from the first activity in Liu and Thompson (2007), but the weather forecast is a 30% chance of rain. The student teachers were asked which one of three statements they most agreed with and then explain why it was not the other two statements: 21% agreed with the statement "It's just random – it might rain, it might not," 52% agreed with "I don't think it's going to rain," while 27% agreed that "That's quite likely – I'll take an umbrella." The relative log odds of saying "It's just random – it might rain, it might not" verses saying "I don't think it's going to rain" will increase by 1.36 if the student teacher described randomness as "unpredictable" compared to student teachers who did not (b = 1.36, $\chi^2(1) = 2$, $p < .05$), meaning that student teachers who described random as "unpredictable" were 3.89 times more likely to say "It's just random – it might rain, it might not" than those who did not describe random as "unpredictable." There were no other significant relationships between the student teachers' descriptions of randomness and which statement they agreed with.

In the reasons given for why they did not consider the other two statements, 64% referred directly to the likelihood of rain. For some students the 30% was used as a justification for rain being unlikely, or not raining being more likely than raining, e.g., "It's more likely to not rain than rain," whilst others considered 30% to be a relatively high chance of rain, e.g., "When looking at the chance of rain on a weather forecast, whilst 30% is a low percentage, it actually means there is quite a high chance of rain." The next most frequent explanation related to the data that weather predictions are based on, with 21% of the explanations, e.g., "It's not completely random – good predictions can be made given data that has come before." Finally, 11% drew upon their personal experiences in their explanations, e.g., "My experience with weather predictions (made on the day) is that more than about 15% usually means that it will rain at some point in that hour."

Question 6 is drawn from questions looking at the use of the representativeness heuristic described by Kahneman et al. (1982) (also known as the Linda Problem) in a way that can lead to erroneous judgment, also described as the conjunction fallacy (Pratt & Kazak, 2018). For this question, 31% of the students did indeed give an erroneous answer, which is considerably fewer than those asked by Tversky and Kahneman (1983).

The final question analyzed in this article focuses on the problem of insensitivity to sample size and is related to the law of large numbers from Kahneman et al. (1982). The question is which hospital recorded the most days on which more than 60% of babies born were boys by comparing a larger hospital, where 45 babies are born each day, and a smaller hospital, where 15 babies are born each day. For this question, 89% of student teachers stated correctly that the smaller hospital would report more of these days, which is considerably more than the 27% in Watson and Callingham (2013) and the 22% in Kahneman et al. (1982).

Discussion

The results from this questionnaire illustrate the variety of understandings of randomness even within a sample of people with more advanced qualifications in mathematics and science. Whilst the descriptions given often align with normatively accepted definitions of randomness, the student teachers' answers to tasks involving random processes or outcomes reveal that these descriptions are not necessarily applied in contexts, or even held as an understanding of meaning. For example, some of the student teachers provided descriptions for random as equiprobable outcomes stated that rolling a dice or spinning a spinner where the outcomes were not equiprobable was an example of a random process. Furthermore, the student teachers drew on different conceptions of randomness depending upon the context, with some drawing on "unpredictability" in their examples, "patterns" or the absence of patterns in their analysis of a string of outcomes, and subjective experiences when interpreting a weather forecast. This flexibility that some student teachers demonstrated on the one hand could show a nuanced understanding of randomness that is essential for teaching randomness to others. On the other hand, this variation could also suggest a more limited understanding of randomness, that could be problematic if the student teachers are not aware of this variation and how context can influence the meaning and interpretation of terms such as randomness or probability (Aven, 2020). Individual student teachers may hold a range of conceptions concurrently and use different ones depending on their interpretation of the context in a similar way to the students in the studies of Prediger (2008) and Konold et al. (1993). Furthermore, the association some of these student teachers held between randomness and equiprobable outcomes suggests that their experiences with probabilistic concepts in school that involve games of chance with equiprobable outcomes, as recommended by Batanero (2015), continue to influence their reasoning around random processes and outcomes.

The science curriculum emphasizes naturally occurring contexts such as the motion of particles, evolutionary change, and radioactive decay. In contrast, the mathematics curriculum often emphasizes theoretical probabilities with known and predictable outcomes (Eichler, 2008). This difference may explain why a larger proportion of science student teachers described random as not involving planning or consciousness than the mathematics teachers whereas a larger proportion of mathematics student teachers focused on unpredictability or equiprobability in their descriptions. It may also explain the differences in the types of examples the different student teachers gave of something that is random with the majority of science student teachers giving an example from the science curriculum and the majority of mathematics student teachers giving an example from the mathematics curriculum.

Human influence in contexts affected how the student teachers responded to several of the items but in different ways. Around half of the student teachers said that asking a person to pick a random number was not a way of selecting a random number and a quarter stating that situations where there was no human influence were more random than others. Consideration of human influence focuses on the process of generating a random number, rather than the outcomes of this process. In contrast descriptions that consider patterns and unpredictability focus on the outcomes.

Deciding whether a string is generated randomly or not is more complicated (Batanero & Serrano, 1999; Konold et al., 1991). The majority of the student teachers explicitly drew on the fact that any of the strings of heads and tails in the questionnaire were possible, even if they were unlikely, to occur. A third of them then used this reasoning to say it was not possible to say if a sequence was randomly generated or not. Others did make a decision as to whether the sequence was randomly generated or not but accompanied this with an explanation like "No about equal H and T, but you can't really tell" indicating that they were aware of the uncertainty. Mathematics student teachers were more likely to make a decision in these situations than the science mathematics student teachers and these decisions considered both the frequencies of Heads and Tails as well as the runs, in contrast to Batanero and Serrano (1999) where both the 14-year-olds and 17-year-olds largely focused on just the frequencies of Head and Tails.

The additional mathematical and scientific studies that these student teachers do seem to influence the responses to the items included in the questionnaires. The student teachers were less likely to make the conjunction fallacy and were more sensitive to sample size that the participants in other research that focuses on these fallacies and heuristics (e.g., Kahneman et al., 1982; Tversky & Kahneman, 1983; Watson & Callingham, 2013). They also drew on theoretical ideas when identifying random sequencies more often than school students. However, whilst fewer of these student teachers conveyed these fallacies, there were still a high number who made them. Many of these contexts may not be encountered explicitly within the school curriculum so that the question remains as to whether they directly influence the teaching of probability in school.

The focus of the questionnaire, however, was neither on identifying these student teachers' knowledge nor on their misconceptions but on stimulating discussion around how the various perspectives might influence their teaching and learning of probabilistic reasoning in their own classrooms. Similar to Konold et al. (2011), the student teachers found it difficult to categorize events as random or not and were not always consistent in their description of randomness and how they categorized events as random or not. This may be a consequence of the emphasis on probabilistic situations in school where the outcomes are known and are often equiprobable. Alternatively, it may also be a consequence of how definitions are used in the teaching and learning of mathematics as the descriptions given may or may not reflect a student teacher's own understanding of randomness.

Conclusion

The results from the questionnaire suggest that student teachers with a strong mathematical background may still be unfamiliar with the various meanings of randomness and probability. Randomness and ideas of unpredictability, fairness and independence are not intuitive (Batanero, 2015; Pratt, 2000). However, Hourigan and Leavy (2020) showed that student teachers can demonstrate more sophisticated understandings of probabilistic concepts when *designing* tasks than when *asked questions* used to identify specific misconceptions such as the ones included in the questionnaire. This resonates with our experience within the workshops where the discussions between student teachers demonstrated not only more sophisticated understandings but also an ability to consider various perspectives on randomness.

Probabilistic reasoning and statistical inference are growing in importance within mathematics curricula around the world, but also have a strong influence on other curriculum areas such as science. Scientific and real-world contexts that involve random processes or outcomes have not been considered in much depth in this article. This is partly due to the challenges faced in designing questions where the student teachers' responses would reflect their understanding of randomness rather than, for example, their knowledge and understanding of the scientific context. In two of the years, the questionnaire included questions taken from science education research on evolution and the movement of molecules but the number of student teachers who answered these questions and gave an explanation for their answers was too small to be valid. Further research and curriculum development is needed so that student teachers can learn to teach in ways that support the learning of the reasoning needed both in mathematics but also in scientific and statistical contexts.

There is still much to learn both about student teachers' knowledge and understanding of randomness, and how this influences their teaching of probability and statistics. Yet it can be professionally and ethically challenging to collect the data needed to address these research gaps. Investigating student teachers' learning often involves altering the pedagogical design of the activities so that they suit the aims of such a study, yet a teacher educator's "first commitment is to teach" (Tabach, 2011, p. 33). Research conducted outside of their normal courses, often through interviews or questionnaires such as the one reported in this article, only offer a limited insight into a student teachers' knowledge and understanding and misses the influences of this knowledge on their decision making and practice. Research on student teachers always affects their learning and experiences, and not always in positive or supportive ways. There are also tensions resulting from the relationships between teacher educator and student teacher, between researcher and participant, and the range of institutional contexts and constraints within which teacher education occurs.

The data analyzed in this article does not include the activities from the workshops that focused on thinking about the teaching and learning of probability or statistical inference. Discussions in the workshops were rich and enabled the three perspectives to be drawn upon in a way that focuses on the implications for teaching and learning. As Batanero (2015) argues, teachers need to be aware of different interpretations of randomness due to their influence on students' reasoning when working with chance. I would add that they also need to be aware of how these influences on reasoning suggest and affect the pedagogical choices they may make.

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No potential conflict of interest was reported by the author(s).

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Appendix: Items from the questionnaire reported in this article

(1) What does “random” mean?
(2) Please give an example of something that happens in a “random” way.
(3) Which of these events do you think will select a number at random?

a. Rolling a six-sided dice numbered 1 to 6

Yes □ No □

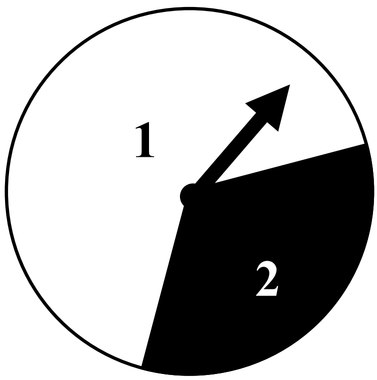
b. Rolling an eight-sided dice numbered 1 to 8

Yes □ No □

c. Rolling an eight-sided dice with three 1s and five 2s

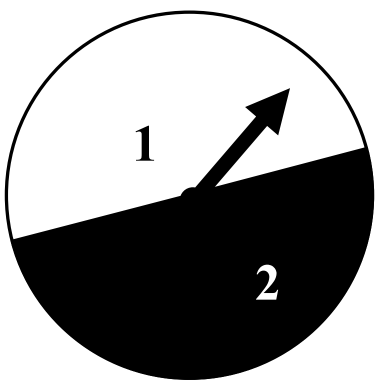
Yes □ No □

d. Spinning this spinner



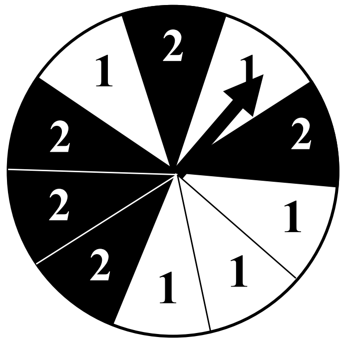
Yes □ No □

e. Spinning this spinner



Yes □ No □

f. Spinning this spinner



Yes □ No □

g. Asking the person sitting next to you to pick a whole number between 1 and 10.

Yes □ No □

h. Do you think that any of the above are ‘more random than the others? Explain why.

(4) Some children were each told to toss a coin 40 times. Some tossed the coin as asked, others made up a list of Heads (H) and Tails (T).

Ahmed:	Bernard:
TTTHTHHTTT	HTHTTHHTHT
HTHHHHHTHT	HHTTHTTHHT
HTTHTHHTTT	THTHHTTHTH
THHHTHHTHH	THTHTHTTHT
Claire:	Diana:
HTTTHTTHHH	HTTTHTTHTH
THTTTTTHTH	TTTHTTTTHH
THHTHTTHHH	TTTHTTHTTH
HTTTHTTHHH	TTTTHTTTHT

a. Did Ahmed make it up?

Yes □ No □

How can you tell?

b. Did Bernard make it up?

Yes □ No □

How can you tell?

c. Did Claire make it up?

Yes □ No □

How can you tell?

d. Did Diana make it up?

Yes □ No □

How can you tell?

(Adapted from Batanero and Serrano (1999))

5) The weather forecast says that there is a 30% chance of rain.

Elaine says “That’s quite likely, I will take an umbrella with me”

Fahira says “I don’t think it’s going to rain”
Gordon says “It’s just random – it might rain, it might not.”
Who do you most agree with?
Elaine □ Fahira □ Gordon □
Explain why you think the others are wrong
(Adapted from Liu and Thompson (2007))

6) Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which of the two statements about Linda is more probable?

□ Linda is a bank teller
□ Linda is a bank teller who is active in the feminist movement

7) A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50% of babies are boys. However, the exact percentage varies from day-to-day. Sometimes it may be higher than 50%, sometimes lower. For a period of 1 year, each hospital recorded the number of days on which more than 60% of babies born were boys. Which hospital do you think recorded more such days?

□ larger hospital
□ smaller hospital