# A note on the Manin-Mumford conjecture

**Damian Roessler** †

**Abstract.** In [PR1], R. Pink and the author gave a short proof of the Manin-Mumford conjecture, which was inspired by an earlier model-theoretic proof by Hrushovski. The proof given in [PR1] uses a difficult unpublished ramification-theoretic result of Serre. It is the purpose of this note to show how the proof given in [PR1] can be modified so as to circumvent the reference to Serre's result. J. Oesterlé and R. Pink contributed several simplifications and shortcuts to this note.

## 0. Introduction.

Let $A$ be an abelian variety defined over an algebraically closed field $L$ of characteristic 0 and let $X$ be a closed subvariety. If $G$ is an abelian group, write $\mathrm{Tor}(G)$ for the group of elements of $G$ which are of finite order. A closed subvariety of $A$ whose irreducible components are translates of abelian subvarieties of $A$ by torsion points will be called a torsion subvariety. The Manin-Mumford conjecture is the following statement:

*The Zariski closure of* $\mathrm{Tor}(A(L))\cap X$ *is a torsion subvariety.*

This was first proved by Raynaud in [R]. In [PR1], R. Pink and the author gave a new proof of this statement, which was inspired by an earlier model-theoretic proof given by Hrushovski in [H]. The interest of this proof is the fact that it relies almost entirely on classical algebraic geometry and is quite short. Its only non elementary input is a ramification-theoretic result of Serre. The proof of this result is not published and relies (see [Se] (pp. 33–34, 56–59)) on deep theorems of Faltings, Nori and Raynaud. In this note, we show how the reference to Serre's result in [PR1] can be replaced by a reference to a classical result in the theory of formal groups (see Th. 4 (a)).

The structure of the paper is as follows. For the convenience of the reader, the text has been written so as to be logically independent of [PR1]. In particular, no knowledge of

† CNRS, Institut de mathématiques de Jussieu, Université Paris 7, Case Postale 7012, 2, place Jussieu, 75251 PARIS CEDEX 05, FRANCE, E-mail: dcr@math.jussieu.fr

[PR1] is necessary to read it. Section 1 recalls various classical results on abelian varieties and also contains two less well-known, but elementary propositions (Prop. 1 and Prop. 3) whose proofs can be found elsewhere but for which we have included short proofs to make the text more self-contained. The reader is encouraged to proceed directly to section 2, which contains a complete proof of the Manin-Mumford conjecture and to refer to the results listed in section 1 as needed.

**Notations.** w.r.o.g. is a shortening of *without restriction of generality*; if $X$ is closed subvariety of an abelian variety $A$ defined over an algebraically closed field $L$ of characteristic 0, then we write $\mathrm{Stab}(X)$ for the stabiliser of $X$; this is a closed subgroup of $A$ such that $\mathrm{Stab}(X)(L) := \{a \in A(L) | a + X = X\}$; it has the same field of definition as $X$ and $A$; if $p$ is a prime number and $G$ is an abelian group, we write $\mathrm{Tor}^p(G)$ for the set of elements of $\mathrm{Tor}(G)$ whose order is prime to $p$ and $\mathrm{Tor}_p(G)$ for the set of elements of $\mathrm{Tor}(G)$ whose order is a power of $p$.

**Acknowledgments.** We want to thank J. Oesterlé for his interest and for suggesting some simplifications in the proofs of [PR1] (see [Oes]) which have inspired some of the proofs given here. Also, the proof of Prop. 3 in its present form is due to him (see the explanations before the proof). I am also very grateful to R. Pink, who carefully read several versions of the text and suggested many improvements and simplifications. In particular, Prop. 6 was suggested by him. Many thanks as well to J. Boxall, who read the final version of the paper carefully and suggested generalizations. I am also grateful to T. Ito for his remarks and corrections. See his recent preprint *On the Manin-Mumford conjecture for abelian varieties with a prime of supersingular reduction* (ArXiv math.NT/0411291), which is partially inspired by this paper. Finally my thanks go to the referee, for a careful reading of the article.

## 1. Preliminaries.

**Lemma 0.** *Let $L \subseteq L'$ be algebraically closed fields of characteristic 0. Let $A$ be an abelian variety defined over $L$ and let $X$ be a closed $L$-subvariety of $A$. Then:*

*(a) $X$ is a torsion subvariety of $A$ iff $X_{L'}$ is a torsion subvariety of $A_{L'}$;*

*(b) the Manin-Mumford conjecture holds for $X$ in $A$ iff it holds for $X_{L'}$ in $A_{L'}$.*

*Proof:* we first prove (a). To prove the equivalence of the two conditions, we only need to prove the sufficiency of the second one. The latter is a consequence of the fact that the morphism $\pi : A_{L'} \to A$ is faithfully flat and that any torsion point and any abelian subvariety of $A_{L'}$ has a model in $A$ (see [Mi] (Cor. 20.4, p. 146)). To prove (b), let $Z := \mathrm{Zar}(\mathrm{Tor}(A(L)) \cap X)$ (resp. $Z' := \mathrm{Zar}(\mathrm{Tor}(A(L')) \cap X_{L'})$). Using again the fact that any torsion point in $A_{L'}$ has a model in $A$ and that $\pi$ is faithfully flat, we see that $\pi^{-1}(\mathrm{Tor}(A(L)) \cap X) = \mathrm{Tor}(A(L')) \cap X_{L'}$. From this and the fact that the morphism $\pi$ is open ([EGA] (IV, 2.4.10)), we get a set-theoretic equality $\pi^{-1}(Z) = Z'$. Since $\pi$ is radicial, the underlying set of $\pi^*(Z) := Z_{L'}$ is $\pi^{-1}(Z)$ ([EGA] (I, 3.5.10)). Since $Z_{L'}$ is reduced ([EGA] (IV, 4.6.1)), we thus have an equality of closed subschemes $Z_{L'} = Z'$. Now, by (a), the closed subscheme $Z_{L'}$ is a torsion subvariety of $A_{L'}$ iff $Z$ is a torsion subvariety of $A$. •

**Proposition 1 (Pink-Roessler).** *Let $A$ be an abelian variety over $\mathbf{C}$ and let $F : A \to A$ be an isogeny. Suppose that the absolute value of all the eigenvalues of the pull-back map $F^*$ on the first singular cohomology group $H^1(A(\mathbf{C}), \mathbf{C})$ is larger than 1. Then any closed subvariety $Z$ of $A$ such that $F(Z) = Z$ is a torsion subvariety.*

The following proof can be found in [PR1] (Remark after Lemma 2.6).

*Proof:* w.r.o.g., we may replace $F$ by one of its powers and thus suppose that each irreducible component of $Z$ is stable under $F$. We may thus suppose that $Z$ is irreducible. Notice that $F(\mathrm{Stab}(Z)) \subseteq \mathrm{Stab}(Z)$. Let us first suppose that $\mathrm{Stab}(Z) = 0$.

Write $\mathrm{cl}(Z)$ for the cycle class of $Z$ in $H^*(A(\mathbf{C}), \mathbf{C})$. We list the following facts:

(1) the degree of $F$ is the determinant of the restriction of $F^*$ to $H^1(A(\mathbf{C}), \mathbf{C})$;

(2) each eigenvalue of $F^*$ on $H^i(A(\mathbf{C}), \mathbf{C})$ is the product of $i$ distinct zeroes (with multiplicities) of the characteristic polynomial of $F^*$ on $H^1(A(\mathbf{C}), \mathbf{C})$; Facts (1) and (2) follow from the fact that for all $i \geq 0$ there is a natural isomorphism $\Lambda^i(H^1(A(\mathbf{C}), \mathbf{C})) \simeq H^i(A(\mathbf{C}), \mathbf{C})$ (see [Mu] (p.3, Eq. (4))).

Now notice that since $\mathrm{Stab}(Z) = 0$, the varieties $Z + a$, where $a \in \mathrm{Ker(F)}(\mathbf{C})$, are pairwise distinct. These varieties are thus the irreducible components of $F^{-1}(Z)$. Now we compute

$$\mathrm{cl}(F^*(Z)) = \sum_{a \in \mathrm{Ker(F)}} \mathrm{cl}(Z + a) = \#\mathrm{Ker(F)}(\mathbf{C}) \cdot \mathrm{cl(Z)} = \mathrm{deg(F)}\,\mathrm{cl(Z)}$$

and thus $\mathrm{cl}(Z)$ belongs to the eigenspace of the eigenvalue $\deg(F)$ in $H^*(A(\mathbf{C}), \mathbf{C})$. Facts (1), (2) and the hypothesis on the eigenvalues imply that $\mathrm{cl}(Z) \in H^{2\dim(A)}(A(\mathbf{C}), \mathbf{C})$, which in turn implies that $Z$ is a point. This point is a torsion point since it lies in the kernel of $F - \mathrm{Id}$, which is an isogeny by construction.

If $\mathrm{Stab}(Z) \neq 0$, then replace $A$ by $A/\mathrm{Stab}(Z)$ and $Z$ by $Z/\mathrm{Stab}(Z)$. The isogeny $F$ then induces an isogeny on $A/\mathrm{Stab}(Z)$, which stabilises $Z/\mathrm{Stab}(Z)$. We deduce that $Z/\mathrm{Stab}(Z)$ is a torsion point. This implies that $Z$ is a translate of $\mathrm{Stab}(Z)$ by a torsion point and concludes the proof. •

**Corollary 2.** *Let $A$ be an abelian variety over an algebraically closed field $K$ of characteristic 0. Let $n \geq 1$ and let $M$ be an $n \times n$-matrix with integer coefficients. Suppose that the absolute value of all the eigenvalues of $M$ is larger than 1. Then any closed subvariety $Z$ of $A^n$ such that $M(Z) = Z$ is a torsion subvariety.*

*Proof:* Because of Lemma 0 (a), we may assume w.r.o.g. that $K$ is the algebraic closure of a field which is finitely generated as a field over $\mathbf{Q}$. We may thus also assume that $K \subseteq \mathbf{C}$. Prop. 1 then implies the result for $Z_{\mathbf{C}}$ in $A^n_{\mathbf{C}}$ and using Lemma 0 (a) again we can conclude. •

**Proposition 3 (Boxall).** *Let $A$ be an abelian variety over a field $K$ of characteristic 0. Let $p > 2$ be a prime number and let $L := K(A[p])$ be the extension of $K$ generated by* *the p-torsion points of A. Let* $P \in \mathrm{Tor}_p(A(\overline{K}))$ *and suppose that* $P \notin A(L)$. *Then there exists* $\sigma \in \mathrm{Gal}(\overline{L}|L)$ *such that* $\sigma(P) - P \in A[p] \setminus \{0\}$.

A proof of a variant of Prop. 3 can be found in [B]. For the convenience of the reader, we reproduce a proof, which is a simplification by Oesterlé (private communication) of a proof due to Coleman and Voloch (see [Vo]).

*Proof:* let $n \geq 1$ be the smallest natural number so that $p^n P \in A(L)$. For all $i \in \{1, \ldots, n\}$, let $P_i = p^{n-i}P$. Let also $\sigma_1$ be an element of $\mathrm{Gal}(\overline{L}|L)$ such that $\sigma_1(p^{n-1}P) \neq p^{n-1}P$. Furthermore, let $\sigma_i := \sigma_1^{p^{i-1}}$ and $Q_i := \sigma_i(P_i) - P_i$.

First, notice that we have $pQ_1 = \sigma_1(p^n P) - p^n P = 0$ and $Q_1 = \sigma_1(p^{n-1}P) - p^{n-1}P \neq 0$, hence $Q_1 \in A[p]\backslash\{0\}$. We shall prove by induction on $i \geq 1$ that $Q_i = Q_1$ if $i \leq n$. This will prove the proposition, since $Q_n = \sigma_n(P) - P$.

So assume that $Q_i = Q_1$ for some $i < n$. We have $p^2(\sigma_i - 1)(P_{i+1}) = p(\sigma_i - 1)(P_i) = pQ_i = 0$. Since any $p$-torsion point of $A$ is fixed by $\sigma$, and hence by $\sigma_i$, we also have $p(\sigma_i - 1)^2(P_{i+1}) = 0$ and $(\sigma_i - 1)^3(P_{i+1}) = 0$. The binomial formula shows that, in the ring of polynomials $\mathbf{Z}[T]$, $T^p$ is congruent to $1 + p(T-1)$ modulo the ideal generated by $p(T-1)^2$ and $(T-1)^3$ (notice that $p \neq 2$ !). We thus have $(\sigma_i^p - 1)(P_{i+1}) = p(\sigma_i - 1)(P_{i+1}) = (\sigma_i - 1)(P_i)$, id est $Q_{i+1} = Q_i$. This completes the induction on $i$. •

Suppose now that $K$ is a finite extension of $\mathbf{Q}_p$, for some prime number $p$ and let $K^{\mathrm{unr}}$ be its maximal unramified extension. Let $k$ be the residue field of $K$. Suppose that $A$ is an abelian variety over $K$ which has good reduction at the unique non-archimedean place of $K$. Denote by $A_0$ the corresponding special fiber, which is an abelian variety over $k$.

**Theorem 4.**

(a) *The kernel of the homomorphism*

$$\mathrm{Tor}(A(K^{\mathrm{unr}})) \to A_0(\overline{k})$$

*induced by the reduction map is a finite p-group.*

*(b) The equality* $\mathrm{Tor}^p(A(K^{\mathrm{unr}})) = \mathrm{Tor}^p(A(\overline{K}))$ *holds.*

*Proof:* for statement (b), see [Mi] (Cor. 20.8, p. 147). Statement (a), which is more difficult to prove, follows from general properties of formal groups over $K$. See [Oes2] (Prop. 2.3 (a)) for the proof. •

Let now $\phi \in \mathrm{Gal}(\overline{k}|k)$ be the arithmetic Frobenius map.

**Theorem 5 (Weil).** *There is a monic polynomial* $Q(T) \in \mathbf{Z}[T]$ *with the following properties:*

*(a)* $Q(\phi)(P) = 0$ *for all* $P \in A_0(\overline{k})$*;*

*(b) the complex roots of* $Q$ *have absolute value* $\sqrt{\#k}$*.*

*Proof:* see [We].•

## 2. Proof of the Manin-Mumford conjecture.

**Proposition 6.** *Let* $A$ *be an abelian variety over a field* $K_0$ *that is finitely generated as a field over* $\mathbf{Q}$*. Then for almost all prime numbers* $p$*, there exists an embedding of* $K_0$ *into a finite extension* $K$ *of* $\mathbf{Q}_p$*, such that* $A_K$ *has good reduction at the unique non-archimedean place of* $K$*.*

*Proof:* since by assumption $K_0$ has finite transcendence degree over $\mathbf{Q}$, there is a finite map

$$\mathrm{Spec}\ K_0 \to \mathrm{Spec}\ \mathbf{Q}(X_1, \dots, X_d),$$

for some $d \geq 0$ (notice that $d = 0$ is allowed). Let $V \to \mathbf{A}^d_{\mathbf{Z}}$ be the normalisation of the affine space $\mathbf{A}^d_{\mathbf{Z}}$ in $K_0$. The scheme $V$ is integral, normal and has $K_0$ as a field of rational functions. Furthermore, $V$ is finite and surjective onto $\mathbf{A}^d_{\mathbf{Z}}$. There is an open subset $B \subseteq V$ and an abelian scheme $\mathcal{A} \to B$, whose generic fiber is $A$. Choose $B$ sufficiently small so that its image $f(B)$ is open and so that $f^{-1}(f(B)) = B$ (this can be achieved by replacing $B$ by $f^{-1}(\mathbf{A}^d_{\mathbf{Z}} \backslash f(V \backslash B))$). Let $U := f(B)$. This accounts for the square on the left of the diagram (*) below.

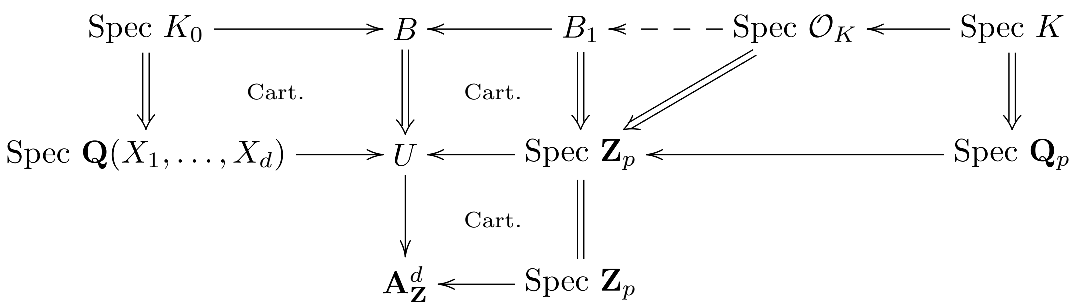
Now notice that $U(\mathbf{Q}) \neq \emptyset$, since $\mathbf{A}^d(\mathbf{Q})$ is dense in $\mathbf{A}^d_{\mathbf{Q}}$ and $U \cap \mathbf{A}^d_{\mathbf{Q}}$ is open and not empty. Thus, for almost all prime numbers $p$, we have $U(\mathbf{F}_p) \neq \emptyset$. Let $p$ be a prime number with this property. Let $P \in U(\mathbf{F}_p)$ and let $a_1, \dots, a_d \in \mathbf{F}_p$ be its coordinates. Choose as well elements $x_1, \dots, x_d \in \mathbf{Q}_p$ which are algebraically independent over $\mathbf{Q}$. The elements $x_1, \dots, x_d$ remain algebraically independent if we replace some $x_i$ by $\frac{1}{x_i}$ so we may suppose that $\{x_1, \dots, x_d\} \subseteq \mathbf{Z}_p$. Notice also that any element of the residue field $\mathbf{F}_p$ of $\mathbf{Z}_p$ is the reduction mod $p$ of an element of $\mathbf{Z} \subseteq \mathbf{Z}_p$. Furthermore, the elements $x_1, \dots, x_d$ remain algebraically independent if some $x_i$ is replaced by $x_i + m$, where $m$ is an integer. Hence, we may also suppose that $x_i$ mod $p = a_i$ for all $i \in \{1, \dots, d\}$. The choice of the $x_i$ induces a morphism $e : \text{Spec } \mathbf{Z}_p \to \mathbf{A}^d_{\mathbf{Z}}$, which by construction sends the generic point of Spec $\mathbf{Z}_p$ on the generic point of $\mathbf{A}^d_{\mathbf{Z}}$ and hence of $U$ and sends the special point of Spec $\mathbf{Z}_p$ on $P \in U(\mathbf{F}_p)$. Hence $e^{-1}(U) = \text{Spec } \mathbf{Z}_p$. This accounts for the lowest square in (*).

The middle square in (*) is obtained by taking the fibre product of $B \to U$ and Spec $\mathbf{Z}_p \to U$. The morphism $B_1 \to \text{Spec } \mathbf{Z}_p$ is then also finite and surjective.

To define the arrows in the triangle next to it, consider a reduced irreducible component $B_1'$ of $B_1$ which dominates Spec $\mathbf{Z}_p$. This exists, because the morphism $B_1 \to \text{Spec } \mathbf{Z}_p$ is dominant. The morphism $B_1' \to \text{Spec } \mathbf{Z}_p$ will then also be finite and will thus correspond to a finite (and hence integral) extension of integral rings. Let $K$ be the function field of $B_1'$, which is a finite extension of $\mathbf{Q}_p$; the ring associated to $B_1'$ is by construction included in the integral closure $\mathcal{O}_K$ of $\mathbf{Z}_p$ in $K$ and the arrow Spec $\mathcal{O}_K \ - - > \ B_1$ is defined by composing the morphism induced by this inclusion with the closed immersion $B_1' \to B_1$.

The morphism Spec $K \to \text{Spec } \mathbf{Q}_p$ has been implicitly defined in the last paragraph and the morphisms Spec $\mathbf{Q}_p \to \text{Spec } \mathbf{Z}_p$ and Spec $K \to \text{Spec } \mathcal{O}_K$ are the obvious ones.

We have a commutative diagram (*):




*The single-barreled continuous arrows ($\rightarrow$) represent dominant maps; the double-barreled continuous ones ($\Rightarrow$) represent finite and dominant maps; all the schemes in the diagram apart from $B_1$ are integral; the cartesian squares carry the label "Cart.".*

Now notice that the map Spec $K \rightarrow B$ obtained by composing the connecting morphisms sends Spec $K$ on the generic point of $B$; to see this notice that the maps Spec $K \rightarrow$ Spec $\mathcal{O}_K$, Spec $\mathcal{O}_K \Rightarrow$ Spec $\mathbf{Z}_p$ and Spec $\mathbf{Z}_p \rightarrow U$ are all dominant; hence Spec $K$ is sent on the generic point of $U$; since $B \rightarrow U$ is a finite map, this implies that Spec $K$ is sent on the generic point of $B$.

Thus the map Spec $K \rightarrow B$ induces a field extension $K|K_0$. Furthermore, as we have seen, $K$ is a finite extension of $\mathbf{Q}_p$ and by construction, the abelian variety $A_K$ is the generic fiber of the abelian scheme $\mathcal{A} \times_B$ Spec $\mathcal{O}_K$. In other words $A_K$ is an abelian variety defined over $K$ which has good reduction at the unique non-archimedean place of $K$.$\bullet$

Next, we shall consider the following situation. Let $p > 2$ be a prime number and let $K$ be a finite extension of $\mathbf{Q}_p$. Let $k$ be its residue field. Let $A$ be an abelian variety over $K$. Suppose that $A$ has good reduction at the unique non-archimedean place of $K$. Let $A_0$ be the corresponding special fiber, which is an abelian variety over $k$.

Recall that $K^{\rm unr}$ refers to the maximal unramified extension of $K$. Let $\phi \in {\rm Gal}(\overline{k}|k)$ be the arithmetic Frobenius map and let $\tau \in {\rm Gal}(K^{\rm unr}|K)$ be its canonical lift.

**Proposition 7.** *Let $X$ be a closed $K$-subvariety of $A$. Then the Zariski closure of $X_{\overline{K}} \cap$ ${\rm Tor}(A(K^{\rm unr}))$ is a torsion subvariety.*

*Proof:* w.r.o.g. we may suppose that ${\rm Tor}(A(K^{\rm unr}))$ is dense in $X_{\overline{K}}$ (otherwise, replace $X$

by the natural model of $\mathrm{Zar}(X_{\overline{K}} \cap \mathrm{Tor}(A(K^{\mathrm{unr}})))$ over $K$). By Th. 4 (a), the kernel of the reduction homomorphism $\mathrm{Tor}(A(K^{\mathrm{unr}})) \to A_0(\overline{k})$ is a finite $p$-group. Let $p^r$ be its cardinality and let $Y := p^r \cdot X$. Let $Q(T) := T^n - (a_n T^{n-1} + \ldots + a_0) \in \mathbf{Z}[T]$ be the polynomial provided by Th. 5 (i.e. the characteristic polynomial of $\phi$ on $A_0(\overline{k})$). Let $F$ be the matrix

$$\begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \end{pmatrix}$$

For any $a \in A(K^{\mathrm{unr}})$, write $u(x) := (x, \tau(x), \tau^2(x), \ldots, \tau^{n-1}(x)) \in A^n(K^{\mathrm{unr}})$. Let $\widetilde{Y} := \mathrm{Zar}(\{u(a) | a \in (p^r \cdot \mathrm{Tor}(A(K^{\mathrm{unr}}))) \cap Y_{\overline{K}}\})$. Th. 5 (a) and Th. 4 (a) imply that

$$F(u(a)) = u(\tau(a))$$

for all $a \in p^r \cdot \mathrm{Tor}(A(K^{\mathrm{unr}}))$. Furthermore, by construction,

$$\tau(p^r \cdot \mathrm{Tor}(A(K^{\mathrm{unr}}))) \subseteq p^r \cdot \mathrm{Tor}(A(K^{\mathrm{unr}})).$$

Hence $F(\widetilde{Y}) = \widetilde{Y}$. Now Th. 5 (b) implies that the absolute value of the eigenvalues of the matrix $F$ are larger than 1 and Cor. 2 then implies that $\widetilde{Y}$ is a torsion subvariety of $A_{\overline{K}}$. The variety $Y_{\overline{K}}$ is the projection of $\widetilde{Y}$ on the first factor and is thus also a torsion subvariety. Finally, this implies that $X_{\overline{K}}$ is a torsion subvariety. •

**Proposition 8.** *Let $X$ be a closed $K$-subvariety of $A$. Then the Zariski closure of $X_{\overline{K}} \cap \mathrm{Tor}(A(\overline{K}))$ is a torsion subvariety.*

*Proof:* we may suppose w.r.o.g. that $K = K(A[p])$, that $X$ is geometrically irreducible and that $X_{\overline{K}} \cap \mathrm{Tor}(A(\overline{K}))$ is dense in $X_{\overline{K}}$. We shall first suppose that $\mathrm{Stab}(X) = 0$. Let $x \in X_{\overline{K}} \cap \mathrm{Tor}(A(\overline{K}))$ and suppose that $x \notin A(K^{\mathrm{unr}})$. Write $x = x^p + x_p$, where $x^p \in \mathrm{Tor}^p(A(\overline{K}))$ and $x_p \in \mathrm{Tor}_p(A(\overline{K}))$. By Th. 4 (b) $x^p \in A(K^{\mathrm{unr}})$ and thus $x_p \notin A(K^{\mathrm{unr}})$. By Prop. 3, there exists $\sigma \in \mathrm{Gal}(\overline{K}|K^{\mathrm{unr}})$ such that

$$\sigma(x_p) - x_p = \sigma(x) - x \in A[p] \setminus \{0\}.$$

Now notice that for all $y \in X(\overline{K})$ and all $\tau \in \mathrm{Gal}(\overline{K}|K^{\mathrm{unr}})$, we have $\tau(y) \in X(\overline{K})$. Hence if the set $\{x \in X_{\overline{K}} \cap \mathrm{Tor}(A(\overline{K}))|x \notin A(K^{\mathrm{unr}})\}$ is dense in $X_{\overline{K}}$ then $\mathrm{Stab}(X)(\overline{K})$ contains a element of $A[p]\setminus\{0\}$. Since $\mathrm{Stab}(X) = 0$, we deduce that the set $\{x \in X_{\overline{K}} \cap \mathrm{Tor}(A(\overline{K}))|x \notin A(K^{\mathrm{unr}})\}$ is not dense in $X_{\overline{K}}$ and thus the set $X_{\overline{K}} \cap \mathrm{Tor}(A(K^{\mathrm{unr}}))$ is dense in $X_{\overline{K}}$. Prop. 7 then implies that $X_{\overline{K}}$ is a torsion point. If $\mathrm{Stab}(X) \neq 0$, then we may apply the same reasoning to $X/\mathrm{Stab}(X)$ and $A/\mathrm{Stab}(A)$ to conclude that $X_{\overline{K}}$ is a translate of $\mathrm{Stab}(X)_{\overline{K}}$ by a torsion point. •

We shall now prove the Manin-Mumford conjecture. Let the terminology of the introduction hold. By Lemma 0 (b), we may assume w.r.o.g. that $L$ is the algebraic closure of a field $K_0$ that is finitely generated as a field over $\mathbf{Q}$ and that $A$ (resp. $X$) has a model $\mathbf{A}$ (resp. $\mathbf{X}$) over $K_0$. By Prop. 6, there is an embedding of $K_0$ into a field $K$, with the following properties: $K$ is a finite extension of $\mathbf{Q}_p$, where $p$ is a prime number larger than 2 and $\mathbf{A}_K$ has good reduction at the unique non-archimedean place of $K$. Prop. 8 now implies that the Manin-Mumford conjecture holds for $\mathbf{X}_{\overline{K}}$ in $\mathbf{A}_{\overline{K}}$ and using Lemma 0 (b) we deduce that it holds for $X$ in $A$.

**Remark.** Let the notation of the introduction hold. Prop. 3. *alone* implies the statement of the Manin-Mumford conjecture, with $\mathrm{Tor}(A(L))$ replaced by $\mathrm{Tor}_p(A(L))$, for any prime number $p > 2$. To see this, we may w.r.o.g. assume that $X$ is irreducible and that $\mathrm{Tor}_p(A(L)) \cap X$ is dense in $X$. By an easy variant of Lemma 0 (b), we may w.r.o.g. assume that $L$ is the algebraic closure of a field $K$ that is finitely generated as a field over $\mathbf{Q}$ and that $A$ (resp. $X$) has a model $\mathbf{A}$ (resp. $\mathbf{X}$) over $K$. Finally, we may assume w.r.o.g. that $K = K(\mathbf{A}[p])$. Suppose first that $\mathrm{Stab}(X) = 0$. By the same argument as above, the set $\{a \in \mathrm{Tor}_p(A(L))|a \notin \mathbf{A}(K),\ a \in X\}$ is not dense in $X$. Hence the set $\{a \in \mathrm{Tor}_p(A(L))|a \in \mathbf{A}(K),\ a \in X\}$ must be dense in $X$; the theorem of Mordell-Weil (for instance) implies that this set is finite and thus $X$ consists of a single torsion point. If $\mathrm{Stab}(X) \neq 0$, then we deduce by the same reasoning that $X/\mathrm{Stab}(X)$ is a torsion point in $A/\mathrm{Stab}(X)$ and hence $X$ is a translate of $\mathrm{Stab}(X)$ by a torsion point. This proof of a special case of the Manin-Mumford conjecture is outlined in [B] (Remarque 3, p. 75).

**References.**


[B] Boxall, J. Sous-variétés algébriques de variétés semi-abéliennes sur un corps fini in *Number Theory, Paris 1992-3*, S. David, ed., London Math. Soc. lecture note series **215**, 69–89, Cambridge Univ. Press, 1995.

[EGA] Grothendieck, A. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.* **4, 8, 11, 17, 20, 24, 28, 32** (1960-1967).

[H] Hrushovski, E. The Manin-Mumford conjecture and the model theory of difference fields. *Ann. Pure Appl. Logic* **112** (2001), no. 1, 43–115.

[Mi] Milne, J. S. Abelian varieties. *Arithmetic geometry (Storrs, Conn., 1984)*, 103–150, Springer, New York, 1986.

[Mu] Mumford, D. Abelian varieties. *Tata Institute of Fundamental Research Studies in Mathematics, No. 5*, Oxford University Press, London, 1970.

[Oes] Oesterlé, J. Lettre à l'auteur (20/12/2002).

[Oes2] Oesterlé, J. Courbes sur une variété abélienne (d'après M. Raynaud). Séminaire Bourbaki, Vol. 1983/84. *Astérisque* No. **121-122** (1985), 213–224.

[PR1] Pink, R., Roessler, D. On Hrushovski's proof of the Manin-Mumford conjecture. *Proceedings of the International Congress of Mathematicians*, Vol. I (Beijing, 2002), 539–546, Higher Ed. Press, Beijing, 2002.

[R] Raynaud, M. Sous-variétés d'une variété abélienne et points de torsion. *Arithmetic and geometry*, Vol. I, 327–352, Progr. Math. 35, Birkhäuser Boston, Boston, MA, 1983.

[Se] Serre, J.-P. Oeuvres, vol. IV (1985-1998). Springer 2000.

[Vo] Voloch, J.-F. Integrality of torsion points on abelian varieties over $p$-adic fields. *Math. Res. Lett.* **3** (1996), no. 6, 787–791.

[We] Weil, A. Variétés abéliennes et courbes algébriques. Hermann 1948.